ABSTRACT ALGEBRAIC LOGIC

A Collection of Open Problems®

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I have used various sources to compile a collection of open problems in Abstract Algebraic Logic. If you are aware of any open problems worthy of investigation that are not included in this list or would like to update me on the status of any of these problems, please, do not hesitate to contact me. The more input I receive the better and more reliable this list will become!!

1 Universal Abstract Algebraic Logic

1.1 Specific Problems

1. The class of fully Fregean logics is included in both the class of Fregean logics and of fully selfextensional logics. In turn, both of these classes are included in the class of selfextensional logics. The question asks to show that the class of fully Fregean logics equals the intersection of the classes of Fregean and of fully selfextensional logics or to provide a counterexample. (Existence of a counterexample more likely.)

   Update: Note that in Theorem 26 of

   > Albuquerque, H., Font, J.M., Jansana, R., and Moraschini, T., *Assertional Logics, Truth-Equational Logics, and the Hierarchies of Abstract Algebraic Logic*, "Don Pigozzi on Abstract Algebraic Logic, Universal Algebra, and Computer Science", J. Czelakowski (ed.), Outstanding Contributions to Logic, Volume 16, Springer 2018, pp. 53-79

   Albuquerque et al. show that a fully selfextensional Fregean logic *with theorems* is fully Fregean. This partially solves this problem leaving open the question of *whether the additional hypothesis of having theorems is necessary.*

2. Let K be a quasivariety and $\tau = \{\langle \delta_i, \epsilon_i \rangle : i < n\}$ a parameterized translation, X a set of terms and z a variable. Consider the set R of all pairs $\langle G, t \rangle$, where $G \cup \{t\}$ is a finite set of terms and, for every variable z, $\tau_z(G) \models_{\mathsf{K}} \tau_z(t)$. Define $\mathcal{S}(\mathsf{K}, \tau)$ to be the deductive system axiomatized by R. K is an (X, z)-**algebraic semantics** for a deductive system $\mathcal{S}$ if there exists a parameterized translation τ, such that, for any $G \cup \{t\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, $X, G \vdash_{\mathcal{S}} t$ iff $\tau_z(G) \models_{\mathsf{K}} \tau_z(t)$. K is τ-**deductive** if K is a $(\tau_z/\bot_{\mathsf{K}}, z)$-algebraic semantics for $\mathcal{S}(\mathsf{K}, \tau)$ with defining equations τ, where $\tau_z/\bot_{\mathsf{K}} = \{t \in \mathrm{Fm}_{\mathcal{L}}(V) : \tau_z(t) \subseteq \bot_{\mathsf{K}}\}$. An algebra $\mathbf{A}$ is **relatively τ-regular with respect to** K, or (K, τ)-**regular** if, for all $a \in A$ and $\alpha, \beta \in \mathrm{Con}_{\mathsf{K}}(\mathbf{A})$, $\tau^{\mathbf{A}}(a)/\alpha = \tau^{\mathbf{A}}(a)/\beta$ implies $\alpha = \beta$. K is τ-**regular** if every algebra $\mathbf{A} \in \mathsf{K}$ is (K, τ)-regular. This problem asks to prove that every τ-regular quasivariety is τ-deductive or to provide a counterexample.

3. Let $L_{\mathrm{int}} = \{\wedge, \vee, \to, \bot\}$. Let M be a class of Kripke models. The global consequence associated with M is the relation $\models_{g\mathsf{M}}$ between sets of L_{int}-formulas and L_{int}-formulas defined by $\Gamma \models_{g\mathsf{M}} \phi$ iff, for every $\langle W, R, V \rangle \in \mathsf{M}$, $V(\Gamma) = W$ implies $V(\phi) = W$. Let $\mathbb{M} = \{\langle W, R, V \rangle : \langle W, R, V \rangle$ is a Kripke model$\}$ and $\mathbb{T} = \{\{\langle W, R, V \rangle : \langle W, R, V \rangle \in \mathbb{M}$ and R is transitive$\}$. The

problem asks to provide characterizations for the reduced full g-models of the logics $\models_{g\mathbb{M}}$ and $\models_{g\mathbb{T}}$.

4. Does the property of "commuting with substitutions" transfer? More explicitly, is the property that the Leibniz operator $\boldsymbol{\Omega}$ on the theories of an $\mathcal{L}$-logic $\mathcal{S}$ commutes with substitutions equivalent to the property that, for every $\mathcal{L}$-algebra $\boldsymbol{A}$, the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}}$ on $\mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}$ commutes with substitutions?

5. The problem (due to Font) asks for an example of an algebraizable logic with a finite set of defining equations which, however, cannot be taken to be unitary.

1.2 General Problems

1. A deductive system **possesses an algebraic semantics** when there exists a class of algebras K and a finite set of equations in a single variable that interprets the consequence of the deductive system in the equational consequence of the class K. The following facts have been proven by Blok and Rebagliato concerning algebraic semantics:

   - All extensions of a deductive system that possesses an algebraic semantics themselves possess an algebraic semantics.
   - Necessary conditions for the existence of an algebraic semantics are given.
   - The mono-unary deductive systems possessing an algebraic semantics are characterized.
   - Conditions on a deductive system are formulated that guarantee the existence of an algebraic semantics.

   The problem asks for elegant and useful necessary and sufficient conditions for a deductive system to possess an algebraic semantics.

2. A generalized matrix $\mathcal{A} = \langle \mathbf{A}, C \rangle$ is a **model of a Gentzen-style rule** $\frac{\{\Gamma_i \vdash \phi_i : i < n\}}{\Gamma_n \vdash \phi_n}$ if $h(\phi_i) \in C(h(\Gamma_i)), i < n$, imply that $h(\phi_n) \in C(h(\Gamma_n))$, for every interpretation h in $\mathbf{A}$. $\mathcal{A}$ is a **model of the Gentzen system** $\mathcal{G}$ if it is a model of all its derivable rules. The class of all such models is denoted by $\mathsf{Mod}\mathcal{G}$. A Gentzen system $\mathcal{G}$ is **fully adequate** for a sentential logic $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ if $\mathsf{FGMod}\mathcal{S} = \mathsf{Mod}\mathcal{G}$, i.e., if the class of all full models of $\mathcal{S}$ coincides with the class of all models of $\mathcal{G}$. (In case $\mathcal{S}$ has no theorems, it must be assumed that all generalized matrices have no theorems either.) Font, Jansana and Pigozzi have shown that

   (a) $\mathcal{S}$ has a fully adequate Gentzen system if and only if $\mathsf{FGMod}\mathcal{S}$ is closed under sub-generalized matrices and reduced products of generalized matrices if and only if the lattice $\mathsf{FGFi}_{\mathcal{S}}(\mathbf{A})$ is closed under arbitrary intersections, for every algebra A.

   (b) For a finitely algebraizable logic $\mathcal{S}$, $\mathsf{FGMod}\mathcal{S}$ is closed under sub-generalized matrices if and only if S has the Local Deduction Theorem.

   (c) For a finitely algebraizable logic $\mathcal{S}$, $\mathsf{FGMod}\mathcal{S}$ has a fully adequate Gentzen system if and only if $\mathcal{S}$ has the Deducton Theorem.

   This problem asks for an investigation into the exact relationship between these results. It also calls for the discovery of alternative elegant and useful characterizations of the property of having a fully adequate Gentzen system for an arbitrary sentential logic $\mathcal{S}$.

3. Suppose that a sentential logic S has a fully adequate Gentzen system. The problem asks to find a calculus or presentation for it in terms of Gentzen-style rules that have a relevant metalogical significance.

4. Czelakowski and Pigozzi have obtained several results that reveal general conditions under which a class of properties transfers from a sentential logic to the class of all its full models. The problem calls for discovering additional elegant and useful conditions under which properties transfer. A complete solution would provide necessary and sufficient condition(s) for metalogical properties to transfer that would contain all known previous results as special cases.

5. The general theory of finitely algebraizable (finitary) sentential logics guarantees that the class $\mathsf{Alg}\mathcal{S}$ of algebraic reducts of reduced generalized matrix models of $\mathcal{S}$ is a quasivariety. However, experience shows that for many logics (called strongly algebraizable) this class is actually a variety. For instance, Font and Jansana have shown that, for every selfextensional logic $\mathcal{S}$ having the Uniterm Deduction Theorem or a Conjunction, $\mathsf{Alg}\mathcal{S}$ is a variety. This implies that, if $\mathcal{S}$ is selfextensional and finitely algebraizable with the Uniterm Deduction Theorem or a Conjunction, then it is strongly algebraizable. The problem asks to find a characterization of strongly algebraizable logics inside the class of algebraizable logics.

6. Rebagliato and Verdu defined the notion of an algebraizable Gentzen system. A sentential logic $\mathcal{S}$ is G-algebraizable if it has a fully adequate Gentzen system that is algebraizable. It is now known that there are logics that are algebraizable but not G-algebraizable and logics that are G-algebraizable but not algebraizable. This problem calls for further exploration of the notion of G-algebraizability and of its connections with the classes of the Leibniz and the Frege hierarchies.

7. Provide a characterization of the classes of algebras that are algebraic counterparts of a G-algebraizable logic, i.e., those classes of algebras that form equivalent algebraic semantics of a fully adequate Gentzen system associated with some sentential logic.

8. Provide a characterization of those classes of algebras that are of the form $\mathsf{Alg}\mathcal{S}$, for some arbitrary sentential logic $\mathcal{S}$. This has been done for specific classes of sentential logics; most notably, finitary, finitely algebraizable logics and protoalgebraic Fregean logics.

   Update: Let V be a variety of algebras of a type $\mathcal{L}$. Consider the class M_V of matrices defined by
   $$\mathsf{M}_V = \{\langle \mathbf{A}, F\rangle : \mathbf{A} \in V, F \subseteq A\}.$$
   Let $\mathcal{S}_V$ be the logic determined by M_V. In Theorem 3.5 of

   > Font, J.M., and Moraschini, T., *Logics of Varieties, Logics of Semilattices, and Conjunction*, Logic Journal of the IGPL, Vol. 22, No. 6 (2014), pp. 818-843

   Font and Moraschini show that $V = \mathsf{Alg}\mathcal{S}_V$, i.e., every variety of algebras is the algebraic counterpart of some sentential logic of the appropriate type $\mathcal{L}$.

   This provides a large family of well-studied classes of algebras, namely varieties, that are of the form $\mathsf{Alg}\mathcal{S}$ for some sentential logic $\mathcal{S}$, yielding a partial answer to this open problem.

Solved Problems

1. (a) (Previously Problem 1.1.1) A generalized matrix $\mathbb{L} = \langle \mathbf{A}, C\rangle$ has the **congruence property** when the Frege relation $\Lambda_{\mathbf{A}}(C) = \{\langle a, b\rangle \in A : C(a) = C(b)\}$ is a congruence on $\mathbf{A}$. $\mathbb{L}$, on the other hand, has the **Fregean property** if, for every $F \in \mathcal{C}$, $\mathbb{L}^F = \langle \mathbf{A}, C^F\rangle$ has the congruence property. A sentential logic $\mathcal{S}$ is **selfextensional** if it has the congruence property. It is **Fregean** if it has the Fregean property. It is **fully selfextensional** if all its full models have the congruence property and **fully Fregean** if all its full models have the

Fregean property. Independently, Babyonyshev and Bou have shown that the congruence property and the Fregean property do not transfer, i.e., that there exist selfextensional logics that are not fully selfextensional and Fregean logics that are not fully Fregean. The examples devised consist of non-protoalgebraic logics. On the positive side, Czelakowski and Pigozzi and Font and Jansana, respectively, have shown

   – If S is a protoalgebraic logic, then the Fregean property transfers to all its full models.
   – If S is a logic satisfying the uniterm Deduction Theorem or having a Conjunction, then the congruence property transfers to all its full models.

   The problem asks to prove the transfer of the congruence property for protoalgebraic logics or provide a counterexample. (Existence of a counterexample more likely.)

(b) (Previously Problem 1.1.4) The two main hierarchies of sentential logics considered in the abstract algebraic logic literature are the Leibniz hierarchy and the Frege hierarchy. There exist a few known results connecting various classes of logics across these hierarchies. The first one below is due to Czelakowski and Pigozzi and the last two to Font and Jansana.

   – Every protoalgebraic Fregean logic with theorems is regularly algebraizable.
   – A weakly algebraizable logic is Fregean if and only if it is fully selfextensional.
   – A fully selfextensional logic is algebraizable if and only if it is weakly algebraizable.

   The problem asks to prove that every selfextensional weakly algebraizable sentential logic is Fregean or provide a counterexample to this statement. (Existence of a counterexample more likely.)

Solution: In Example 25 of

> Albuquerque, H., Font, J.M., Jansana, R., and Moraschini, T., *Assertional Logics, Truth-Equational Logics, and the Hierarchies of Abstract Algebraic Logic*, "Don Pigozzi on Abstract Algebraic Logic, Universal Algebra, and Computer Science", J. Czelakowski (ed.), Outstanding Contributions to Logic, Volume 16, Springer 2018, pp. 53-79

Albuquerque et al. construct a logic over a language with 1 binary, 1 unary and 4 nullary connectives determined by a 4-element generalized matrix. They show that it is finitary and finitely regularly algebraizable and that, moreover, it is selfextensional but not Fregean (and, hence, neither fully selfextensional nor fully Fregean, since they also show that for finitary weakly algebraizable logics the notions of Fregeanity, full selfextensionality and full Fregeanity coincide). This example shows that:

- The congruence property does not transfer even for finitely regularly algebraizable logics.
- Not every weakly algebraizable selfextensional logic is Fregean.

2. (Previously Problem 1.1.3) The class of fully Fregean logics is included in both the class of Fregean logics and of fully selfextensional logics. In turn, both of these classes are included in the class of selfextensional logics. The question asks to show that the class of selfextensional logics equals the union of the classes of Fregean and of fully selfextensional logics or to provide a counterexample. (Existence of a counterexample more likely.)

   Solution: In Example 23 of

   > Albuquerque, H., Font, J.M., Jansana, R., and Moraschini, T., *Assertional Logics, Truth-Equational Logics, and the Hierarchies of Abstract Algebraic Logic*, "Don Pigozzi on Abstract Algebraic Logic, Universal Algebra, and Computer Science", J. Czelakowski (ed.), Outstanding Contributions to Logic, Volume 16, Springer 2018, pp. 53-79

Albuquerque et al. construct a logic over a language consisting of two unary connectives determined by a 3-element matrix. They prove that it is assertional (and, hence, truth equational), but not protoalgebraic and that, moreover, it is selfextensional but not Fregean (and, thus, not fully selfextensional, since they also show that a truth equational logic is fully selfextensional if and only if it is fully Fregean). This example shows that:

- The class of selfextensional logics does not equal the union of the classes of fully selfextensional and of Fregean logics.

3. (Previously Problem 1.1.7) If the Leibniz operator of a (not necessarily finitary) deductive system $\mathcal{S}$ is injective on the theories of S, must it be injective on the lattices of $\mathcal{S}$-filters of every algebra of the same similarity type as $\mathcal{S}$? Equivalently, if truth is implicitly definable in $\mathsf{LMod}^*\mathcal{S}$, the class of reduced formula matrix models of $\mathcal{S}$, must it be implicitly definable in $\mathsf{Mod}^*\mathcal{S}$, the class of reduced matrix models of $\mathcal{S}$?

   Solution: In Theorem 5.6 of

   > Moraschini, T., *Study of the Truth Predicates of Matrix Semantics*, To appear in the Review of Symbolic Logic

   Moraschini shows that for logics *expressed in a countable language* the injectivity of the Leibniz operator transfers from theories to filters over arbitrary algebras. However, in Section 6, he provides an example of a logic $\mathcal{S}$ over a language with continuum many binary connectives, continuum many constants and 1 unary connective determined by two matrices $\langle \mathbf{A}, F\rangle$ and $\langle \mathbf{A}, G\rangle$ over an algebra $\mathbf{A}$. Moraschini shows that the Leibniz operator of $\mathcal{S}$ is order reflecting (and, hence, injective) on theories, but the Leibniz operator $\Omega^{\mathbf{A}}$ is not injective over $\mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$, thus proving, as is stated in Theorem 6.7, that the injectivity of the Leibniz operator does not transfer in general.

   We note that this work dates back to Tommaso's Ph.D. Dissertation:

   > Moraschini, T., *Investigations into the Role of Translations in Abstract Algebraic Logic*, Ph.D. Dissertation, University of Barcelona, 2016

2 Categorical Abstract Algebraic Logic

2.1 Specific Problems

2.2 General Problems

Lists will appear at some point, when time allows.

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I express my heartfelt gratitude to Josep Maria Font for informing me of developments in the area, contributing to the updates posted, as well as to the exposition of the solutions to previously Open Problems that had appeared in a previous edition of this list.

I apologize for having delayed the update and, of course, I assume sole responsibility should potential inaccuracies in the account given be revealed. I will try to correct them, if and when notified.