Chapter 2

Algebra and Logic

2.1 Introduction

In Section 2.2, we introduce the basic algebraic machinery that underlies all structures considered in the monograph. We start with *sentence functors*, which are arbitrary **Set**-valued functors on a category of signatures. *Sentence families* are families of sets over sentence functors. They are called *systems* in case they are invariant under signature morphisms. Associated with a sentence family T is the largest sentence system $\overleftarrow{T}$ included in T and the smallest sentence system $\overrightarrow{T}$ which includes T. We also introduce and discuss *morphisms* between sentence functors and, in particular, distinguish the key class of *surjective morphisms*. By analogy to sentence families, one may also consider *relation families* over sentence functors, i.e., families of relations on sentences. Relation families satisfying the requisite properties constitute *equivalence families*. A fundamental notion, pervasive throughout our treatise, is that of *compatibility* of an equivalence family with a given sentence family. The importance of compatibility was exemplified in [35] (see, e.g., Section 1.4 of [35], where the notion is defined). Whereas sentence functors capture the underlying carriers of all algebraic and logical structures we consider, the earnest algebraic treatment begins when they get endowed with *categories of natural transformations* which correspond to clones of algebraic operations [31, 44]. These enriched structures are termed *algebraic systems*. Appropriate mappings, preserving the relevant features, are also called *morphisms* (of algebraic systems). In most contexts, it is required that all algebraic systems under consideration are over the same algebraic signature. This is ensured by adopting a base algebraic system $\mathbf{F}$, which fixes the signature, and, then, considering only algebraic systems whose sentences and clones of operations are, in a certain sense, interpretations of the basic one. These play an important role and are termed *interpreted algebraic systems* or $\mathbf{F}$-*algebraic systems*.

In Section 2.3, we introduce and study *congruence systems*. These are equivalence systems on an underlying algebraic system that satisfy a suitably adapted version of the congruence (sometimes also called compatibility or replacement) property. They play in this context the role that congruences play in universal algebra [22, 13, 21, 30, 84]. The collection of congruence systems on a given algebraic system forms a complete lattice. Of utmost importance is the process of constructing the *quotient* of an algebraic system by a congruence system and of the accompanying *canonical quotient morphism*. Equally important, in fact indispensable for the development of the theory, is the fact that the collection of congruence systems on a given algebraic system $\mathbf{A}$ that are compatible with a given sentence family T of $\mathbf{A}$ form a complete lattice. This fact allows considering the largest congruence system on $\mathbf{A}$ compatible with T, which is denoted by $\Omega^{\mathbf{A}}(T)$ and termed the *Leibniz congruence system of T on $\mathbf{A}$* [35]. A property that is worth mentioning, since it plays a critical role in establishing pieces of the various hierarchies considered in subsequent chapters, is that the Leibniz congruence system of a sentence family T is always included in that of the largest sentence system contained in T, i.e., $\Omega^{\mathbf{A}}(T) \leq \Omega^{\mathbf{A}}(\overleftarrow{T})$.

In Section 2.4, we look at a special class of congruence systems whose definition presupposes fixed in the background a class K of algebraic systems. Given an arbitrary algebraic system, a congruence system on it is said to be a K-*congruence system* or a *congruence system relative to* K if the quotient algebraic system it induces belongs to the class K (see, e.g., Chapter Q of [64]). Two important concepts in this context are *closure of a class under morphic images* and *closure under subdirect intersections*. If the class K is closed under morphic images, then, for every algebraic system in K, the absolute and relative concepts of congruence system coincide. On the other hand, if K happens to be closed under subdirect intersections and contains a trivial algebraic system, then the collection of all K-congruence systems on any algebraic system forms a complete lattice. In this case, it makes sense to consider, given a relation system X on an algebraic system $\mathcal{A}$, the *least* K-*congruence system on* $\mathcal{A}$ *including* X, also known as the K- *congruence system generated by* X, and denoted by $\Theta^{\mathsf{K},\mathcal{A}}(X)$. In the main result of the section, it is shown that this congruence system coincides with the equational closure of X relative to the class K.

In Section 2.5, we introduce *semantic* and *syntactic varieties* of algebraic systems. These play the role that varieties play in universal algebra (see, e.g., [21, 30, 84]). All algebraic systems are understood to be over a fixed signature specified by a base algebraic system $\mathbf{F}$. To define the two types of varieties, we look at *equations*, consisting of pairs of sentences, and at *natural equations*, which are pairs of natural transformations. Given a class K of algebraic systems, the *semantic variety* generated by K is the class of all algebraic systems satisfying all equations valid in all members of K. The *syntactic variety* generated by K is defined analogously with reference to natural equations. It turns out that the semantic variety generated by K is subsumed by the corresponding syntactic one. A technical definition, that of a *transformational algebraic system*, is introduced as a way to establish a sufficient condition for semantic and syntactic varieties to coincide.

In Section 2.6, we switch from purely algebraic to logical considerations. We define *systems of closure operators* on algebraic systems, which give rise to π-*institutions* [33] (see, also, [25, 41]). Those constitute the basic underlying logical structures on which all subsequent studies will be founded. Many well-known fundamental logical concepts are adapted to this framework, among them, *theorem systems*, *theory families* and *inconsistent*, *almost inconsistent* and *trivial* π-*institutions* (see, e.g., [64, 86] for the counterparts in abstract algebraic logic). Concerning theory families, it is worth mentioning that in case T is a theory family of a given π-institution, the construction of $\overleftarrow{T}$ gives rise to a theory system, and not merely a sentence system, but this is not the case for $\overrightarrow{T}$. Therefore, $\overleftarrow{T}$ does constitute the largest theory system included in T, but to construct the smallest theory system including T, one has to apply the closure operator and obtain $C(\overrightarrow{T})$. Comparing closure systems over the same underlying algebraic system, the notions of *extension* and *weakening* are introduced, as well as that of the closure system obtained as the intersection of a family of closure systems. Given a closure system C and one of its theory systems T, we also consider the extension C^T of C that is induced by adopting the given theory system as a system of axioms. Finally, we look at *logical morphisms* between π-institutions. These are morphisms that preserve the logical structure, i.e., map closures into closures in a formal sense, or, what turns out to be equivalent, morphisms whose inverses preserve theory families.

In Section 2.7, after having discussed the algebraic and logical prerequisites, we turn into developing the first rudiments of their interaction. We look at *matrix families* which serve both to define closure systems, and, hence, also, π-institutions, but also as algebraically based models of given π-institutions. They are pairs consisting of an underlying algebraic system together with a sentence family over it and correspond to the ordinary logical matrices of abstract algebraic logic [64, 86]. For a given π-institution $\mathcal{I}$, its matrix family models are termed *$\mathcal{I}$-matrix families* and the corresponding sentence families are called *$\mathcal{I}$-filter families*. Some characterizations of these families are provided along with the observation that the collection of all $\mathcal{I}$-filter families on a given algebraic system forms a complete lattice. A discussion follows on when and under which conditions morphisms between the underlying algebraic systems preserve, under taking direct or inverse images, $\mathcal{I}$-filter families. In closing the Section, we look at *quotients of matrix families* under the Leibniz congruence systems of their filter families. These are referred to as *Leibniz reductions* (see, e.g., Section 4.3 of [86]). We say that a matrix family is *Leibniz reduced* when the Leibniz congruence system of its filter family is the identity. Leibniz reductions give rise to the fundamental collection of *Leibniz reduced $\mathcal{I}$-matrix families* and the accompanying collection of their algebraic system reducts. Two more related subcollections are obtained if one restricts attention to *$\mathcal{I}$-filter systems* and *$\mathcal{I}$-matrix systems*, i.e., those that consist of filter families that are invariant under the action of signature morphisms.

In Section 2.8, continuing the study of filter families and matrix families, we introduce *axiomatic extensions*, or *axiomatic strengthenings*, and the closely related concept of *filter extension* (see Section 0.8 of [64] and Sections 1.3 and 1.4 of [86]). We provide characterizations and study interactions with morphisms, looking, in particular, into some preservation properties.

In Section 2.9, a generalization of matrix families and filter families is introduced. Namely, we consider structures consisting of an underlying algebraic system together with a collection of sentence families over it. These are called *generalized matrix families* or *gmatrix families*, for short. They play the role that generalized matrices play in the traditional treatment [52] (see, also, Chapter 5 of [86]). As was the case with matrix families, gmatrix families serve a dual purpose. They may be used to define closure systems, but they also serve as models of π-institutions. In the latter case, if a gmatrix family is a model of a given π-institution $\mathcal{I}$, we say that it is an *$\mathcal{I}$-gmatrix family*. By analogy with $\mathcal{I}$-matrix families, one may consider reductions of gmatrix families. The *Tarski congruence system* of a gmatrix family is the largest congruence system on its underlying algebraic system which is compatible with all filter families of the gmatrix family [52]. Equivalently, it may be characterized as the intersection of all Leibniz congruence systems of its constituent filter families. The process of taking the quotient of a gmatrix family by its Tarski congruence system is called *Tarski reduction*. We say that a gmatrix family is *Tarski reduced* if its Tarski congruence system in the identity. The construction gives rise to the class of all Tarski reduced $\mathcal{I}$-gmatrix families and the class of the corresponding algebraic system reducts. Both are of critical importance in the study of algebraization of π-institutional logics. Very intimately related to Tarski congruence systems is the notion of *Suszko congruence systems* [67] (see, also, Section 1.5 of [64] and Section 5.3 of [86]). Here, one considers the filter family subcollection $\mathcal{T}^T$ of a filter family collection $\mathcal{T}$ by keeping only those filter families containing a fixed filter family $T \in \mathcal{T}$. The *Suszko congruence system of T relative to $\mathcal{T}$* is the Tarski congruence system of $\mathcal{T}^T$. Conversely, assuming that $\mathcal{T}$ has a smallest filter family T, the Tarski congruence system of $\mathcal{T}$ coincides with the Susko congruence system of T in $\mathcal{T}$. As before, one may consider *Suszko reductions* and *Suszko reduced $\mathcal{I}$-matrix families*, where the reductions are taken relative to the collection of all $\mathcal{I}$-filter families. Even though, given a π-institution $\mathcal{I}$, this process results in the new class of Suszko reduced $\mathcal{I}$-matrix families, the class of corresponding algebraic system reducts turns out to be identical with that obtained from the process of Tarski reduction.

In Section 2.10, we continue the study of classes of algebraic systems associated with a given π-institution $\mathcal{I}$. In Section 2.7, we introduced the class of all algebraic system reducts of all Leibniz reduced $\mathcal{I}$-matrix families. This class is known as the class of *$\mathcal{I}^*$-algebraic systems*. In Section 2.9, we looked at the class of all algebraic system reducts of all Tarski reduced $\mathcal{I}$-gmatrix families. These are known as *$\mathcal{I}$-algebraic systems*. The two classes correspond, respectively, to the classes $\mathrm{Alg}^*\mathcal{S}$ and $\mathrm{Alg}\mathcal{S}$ in the case of a sentential logic $\mathcal{S}$ [52]. On top of these two classes of algebraic systems, two more classes considered in relation to a π-institution $\mathcal{I}$ are the semantic and syntactic varieties generated by the underlying algebraic system of the Tarski reduction of the $\mathcal{I}$-gmatrix system consisting of the collection of all theory families of $\mathcal{I}$. The first is termed the *semantic* and the second the *syntactic variety of $\mathcal{I}$*. It turns out that, in general, the class of $\mathcal{I}^*$-algebraic systems forms the smallest class, followed by the class of $\mathcal{I}$-algebraic systems, followed by the semantic variety of $\mathcal{I}$, while the syntactic variety of $\mathcal{I}$ constitutes the largest of these four classes. An interesting result is that any of these four classes generates the same syntactic variety, namely, the syntactic variety of $\mathcal{I}$. The section concludes with the observation that the class of all $\mathcal{I}$-algebraic systems is closed under subdirect intersections and contains a trivial algebraic system. Consequently, one is justified in considering congruence systems generated by any relation system on any given algebraic system relative to this class.

In Section 2.11, we switch from the study of congruence systems associated with a given π-institution and of their quotients to the study of equivalence families and systems resulting by considering mutual membership or non-membership in theory families. The reader is warned that the terminology here deviates from the standard one for sentential logics (Section 2.4 of [52] and Section 1.3 of [86]). This is done in an attempt to streamline the theory of these equivalence families with the theory based on the Leibniz, Tarski and Suszko congruence systems. The most basic equivalence family is the *Frege equivalence family* of a given theory family, which identifies sentences if they are both inside or both outside the given theory family. Sometimes, this is expressed by saying that the sentences are *equivalent modulo the theory family*. The *Frege relation system* is the largest equivalence system included in the Frege equivalence family. There is a close connection between Leibniz congruence systems and Frege relation families/systems. The Leibniz congruence system of a given theory family is the largest congruence system contained in the Frege equivalence family or system associated with the theory family. In a way analogous to the passage from Leibniz congruence systems of single theory families to the Tarski congruence systems of collections of theory families, one transitions from Frege equivalence families to *Carnap equivalence families*. These express equivalence of sentences modulo collections of theory families. The Carnap equivalence family turns out to be the intersection of the Frege equivalence families of all theory families in the collection. Here, again, the *Carnap equivalence system* is the largest equivalence system included in the Carnap equivalence family. Further, extending the relation between Leibniz congruence systems and Frege equivalence families, the Tarski congruence system of a collection is the largest congruence system included in either the Carnap equivalence family or the Carnap equivalence system of the same collection. The same paradigm gives rise to *Lindenbaum equivalence families/systems*, which formalize the equivalence of sentences modulo a theory family, relative to a given collection of theory families. This is identical to the intersection of all Frege equivalence families/systems of those theory families in the collection including the given one. Similar relations as before hold in this case as well, with the role of Leibniz and Tarski congruence systems played by Suszko congruence systems. A small table at the end of the section summarizes the three congruence systems and the three corresponding pairs of equivalence families/systems that are considered in this context. Hopefully, the analogies outlined between congruence systems and equivalence families/systems provide some justification for introducing distinct names for the Carnap equivalences and the Lindenbaum equivalences, which are all referred to as Frege equivalences in the literature.

In Section 2.12, we look at *subsystems* of algebraic systems and induced *π-subsinstitutions*. Given an algebraic system, a *universe* is a sentence subfunctor over the same category of signatures that is also closed under the algebraic operations. In a natural way, a universe gives rise to an *algebraic subsystem*. With each subsystem, there is associated a *natural injection morphism*. Given a sentence family of an algebraic system, by closing successively under the action of signature morphisms and under the action of natural transformations, one obtains the *universe* of the algebraic system *generated by* the given sentence family. If the given algebraic system happens to be the underlying system of a π-institution, which is a case of central interest, then, by restricting the action of the closure system of the π-institution on sentences of the universe, we obtain a *π-subinstitution*. Its theory families turn out to be exactly the restrictions of the theory families of the original π-institution on the universe. The section concludes by establishing some connections between the Leibniz congruences of theory families of the original π-institution and those of the induced theory families of the subinstitution. These relations extend in a natural way to filter families of the two institutions.

Up to Section 2.12, only cursory attention is paid to natural transformations. They are used in establishing syntactic varieties of algebraic systems via natural equations, but they are not thoroughly studied as "syntactic" objects of interest in their own right. This deficiency is rectified by devoting Sections 2.13-2.15 to their study and to particular aspects of their properties and behavior that are of interest for subsequent considerations.

In Section 2.13, we consider the role played by collections of natural transformations. In general, in the context of collections of natural transformations, a number of arguments is fixed and they are considered as *primary* or *distinguished arguments*. The remaining positions play an auxiliary role and are perceived as *parametric* (see, e.g., Section 1.2 of [64] and Section 6.2 of [86]). In accordance with this paradigm, if E is a collection of natural transformations, of which k positions are considered distinguished, then, for any k-tuple of sentences $\vec{\phi}$ over a signature Σ, $E_\Sigma[\vec{\phi}]$ denotes the sentence family consisting of all sentences of the form $\varepsilon_{\Sigma'}(\mathrm{SEN}(f)(\vec{\phi}),\vec{\chi})$, for $\varepsilon \in E$, $f : \Sigma \to \Sigma'$ a signature morphism and $\vec{\chi}$ an arbitrary tuple of sentences over Σ'. In this way a tuple, or a collection of tuples, of sentences gives rise to a sentence family. Dually, given a sentence family T, one may consider the family of all k-tuples $\vec{\phi}$, such that $\varepsilon_{\Sigma'}(\mathrm{SEN}(f)(\vec{\phi}),\vec{\chi}) \in T_{\Sigma'}$, for all ε, f and $\vec{\chi}$. This gives rise to a k-ary relation system, depending on both E and T, denoted $\overleftarrow{E}(T)$. $\overleftarrow{E}$, viewed as an operator from sentence families to relation systems, is monotone and commutes with inverse surjective morphisms. For the purposes of relating logical with algebraic systems, critical is the role played by $\overleftarrow{E}$ as a potential means of defining Leibniz congruence systems of theory families. Along those lines, it is shown that, if $k = 2$ and $\overleftarrow{E}(T)$ defines a reflexive relation system, then this includes the Leibniz congruence system of T. Consequently, if $\overleftarrow{E}(T)$ is itself a congruence system compatible with T, then it necessarily coincides with the Leibniz congruence system of T (see, e.g., Theorem 1.6 of [35]).

In Section 2.14, taking a cue from the definition of the operator $\overleftarrow{E}$ in Section 2.13, we investigate membership relations of k-tuples of sentences in theory families of a π-institution induced by a fixed set E of natural transformations, taken to possess k distinguished arguments. Four modes are considered, namely, *E-local*, *E-global*, *left E-local* and *left E-global* membership. It is shown that E-global and left E-global memberships are equivalent, that they imply left E-local membership, which, in turn, implies E-local membership. Both implications are shown to be strict in general. If a membership property holds for all k-tuples of sentences (for the same E), then that property is attributed to the set E itself. It turns out that, in that case, all three resulting modes of membership of E in a theory family T are actually equivalent properties.

Section 2.15 is the last of the three sections that are devoted exclusively to the analysis of syntactic definability properties via sets of natural transformations. In this section, we consider two possible ways which may be used to obtain, starting from a parametric collection S of natural transformations, a related one that is parameter-free. The first is effectuated by replacing all parametric arguments by k-ary terms, where k is the number of distinguished arguments of S. This process gives rise to a new collection $\dot{S}$ of natural transformations with k arguments altogether and, therefore, without parameters. The second process is more abstract. It is defined via the use of, so called, *anti-monotone global properties* of natural transformations. These are properties that satisfy a technical anti-monotonicity condition. Given such a property P, by slightly abusing notation, we also denote by P the collection of all natural transformations (possibly with parameters) satisfying P. Then $\widehat{P}$ denotes the subcollection of P of parameter-free natural transformations satisfying P. In the main result of Section 2.15, it is shown that, given such a property P, both constructors $\dot{P}$ and $\widehat{P}$ result in identical de-parameterizations of the collection P.

The last three sections of Chapter 2 deal with more specialized topics. Section 2.16 addresses the special case of π-institutions whose closure systems are *finitary*. Most applied logical systems encountered in the literature fall under this case. Section 2.17 deals with *equational π-institutions*. These are π-institutions whose sets of sentences are pairs of sentences drawn from a base algebraic system and whose closure operators reflect the equational consequence determined by a class of algebraic systems. Finally, Section 2.18 adapts some of the rudiments pertaining to *varieties*, *quasivarieties* and *generalized quasivarieties* of universal algebra and their generation to the context of algebraic systems.

In Section 2.16, we study *finitarity* (see, e.g., Section 0.1 of [64] and Section 1.4 of [86]). This is the property of a closure system (or π-institution) that holds when every sentence which is a consequence of a set of sentences is also a consequence of some finite subset of that set. Some characterizations of finitarity are provided based on the properties of *local continuity* and *continuity* of a π-institution which, in turn, are defined using *local directedness* and *directedness* of collections of theory families. The last part of the section provides a step-wise, inductive construction of the filter family of a finitary π-institution on an arbitrary algebraic system generated by a give sentence family of the algebraic system.

In Section 2.17, we introduce *equational consequences* based on fixed classes of algebraic systems and show that all their theory families happen to be theory systems and that, moreover, they coincide with the congruence systems relative to the class of algebraic systems inducing the equational consequence. Then, as in Section 2.16, we present a step-wise construction of the equational consequence generated by a given family of equations, considered as axioms. We show that, if this defining family of equations is taken to be the family of equations that holds in a class K of algebraic systems, then the equational consequence they generate, according to this step-wise process, coincides with the equational consequence induced by the class K.

The final section, Section 2.18, translates some of the classical results of universal algebra pertaining to *varieties*, *quasivarieties* and *generalized quasivarieties* [21, 30, 84] (see, also, Chapter Q of [64]) to the context of classes of algebraic systems. We revisit *equations* and, in addition, consider *quasiequations* and *generalized quasiequations*, referred to as *guasiequations*. *Satisfaction* of an equation, quasiequation or guasiequation by a given algebraic system is defined. These relations give rise to Galois connections (see, e.g., Chapter 11 of [36]). The closed sets on the algebraic side form, respectively, *equational*, *quasiequational* and *guasiequational classes* of algebraic systems. Equivalently, these are the classes of algebraic systems defined by equations, quasiequations and guasiequations. When they are thought of as classes generated by given collections of algebraic systems, they are termed *varieties*, *quasivarieties* and *guasivarieties*, respectively. The second part of Section 2.18 is dedicated to proving Birkhoff [4] and Mal'cev [18] style characterization theorems of these classes using closures under class operators (see, also, [21, 30, 84]). The four operators considered are taking *certifications*, *directed certifications*, *subdirect intersections* and *morphic images*. It is shown that a given class of algebraic systems is a variety if it is closed under subdirect intersections and morphic images, it is a quasivariety if it is closed under directed certifications and subdirect intersections and it is a guasivariety if it is closed under certifications and subdirect intersections. In the last part of the section, we translate the conditions of closure under subdirect intersections and morphic images into properties of the subcollection of the collection of all congruence systems on the base algebraic system relative to the class under consideration. On the other hand, certifications and directed certifications are abstraction conditions (akin to closure under isomorphisms) and do not seem to have such intrinsic equivalent formalizations.

Chapter 2, in a nutshell, includes the majority of the very basic concepts and results that constitute the prerequisites for following the developments recounted in subsequent chapters of the monograph.

2.2 Algebraic Systems

A **sentence functor** $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is a **Set**-valued functor, with the property that, for every $\Sigma \in |\mathbf{Sign}|$, $\mathrm{SEN}(\Sigma) \neq \varnothing$. We say that a sentence functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is **trivial** if $|\mathrm{SEN}(\Sigma)| = 1$, for all $\Sigma \in |\mathbf{Sign}|$.

A **sentence family** of SEN is a collection $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, such that $T_\Sigma \subseteq \mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$. The collection of all sentence families of SEN is denoted by SenFam(SEN). Sentence families can be ordered by signature-wise inclusion. More precisely, given $T, T' \in \mathrm{SenFam}(\mathrm{SEN})$, we define

$$T \leq T' \text{ iff } \quad T_\Sigma \subseteq T'_\Sigma, \text{ for all } \Sigma \in |\mathbf{Sign}|.$$

Under this ordering sentence families form a complete lattice which is denoted by $\mathbf{SenFam}(\mathrm{SEN}) = \langle \mathrm{SenFam}(\mathrm{SEN}), \leq \rangle$.

A sentence family T of SEN is called a **sentence system** if it is invariant under signature morphisms, i.e., if, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, we have

$$\mathrm{SEN}(f)(T_\Sigma) \subseteq T_{\Sigma'}.$$

The collection of all sentence systems of SEN is denoted by SenSys(SEN). It forms a complete sublattice of the lattice of sentence families under $\leq$, denoted by $\mathbf{SenSys}(\mathrm{SEN}) = \langle \mathrm{SenSys}(\mathrm{SEN}), \leq \rangle$.

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a sentence functor and $T \in \mathrm{SenFam}(\mathrm{SEN})$. We define, based on T, two important sentence families of SEN:

- $\overleftarrow{T} = \{\overleftarrow{T}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\begin{aligned} \overleftarrow{T}_\Sigma \;=\; \{\phi \in \mathrm{SEN}(\Sigma) : & \text{ for all } \Sigma' \in |\mathbf{Sign}| \text{ and all } f \in \mathbf{Sign}(\Sigma, \Sigma'), \\ & \mathrm{SEN}(f)(\phi) \in T_{\Sigma'}\}. \end{aligned}$$

Sometimes, we abbreviate this using the notation

$$\overleftarrow{T}_\Sigma = \{\phi \in \mathrm{SEN}(\Sigma) : (\forall f)(\mathrm{SEN}(f)(\phi) \in T_{\Sigma'})\}$$

- $\overrightarrow{T} = \{\overrightarrow{T}_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\overrightarrow{T}_\Sigma = \{\mathrm{SEN}(f)(\phi) : \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma', \Sigma), \phi \in T_{\Sigma'}\}$$

First, it is clear that both operators on sentence families are monotone.

Lemma 1 *Let* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a sentence functor and consider* $T, T' \in \mathrm{SenFam}(\mathrm{SEN})$. *If* $T \leq T'$, *then* $\overleftarrow{T} \leq \overleftarrow{T'}$ *and* $\overrightarrow{T} \leq \overrightarrow{T'}$.

Proof: Both implications are quite obvious. For the second, e.g., consider $\Sigma \in |\mathbf{Sign}|$, $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in \overrightarrow{T}_\Sigma$. Thus, there exists $\Sigma_0 \in |\mathbf{Sign}|$, $\phi_0 \in T_{\Sigma_0}$ and $f_0 \in \mathbf{Sign}(\Sigma_0, \Sigma)$ such that $\phi = \mathrm{SEN}(f_0)(\phi_0)$.

$$\begin{array}{ccc} \Sigma_0 & \xrightarrow{f_0} & \Sigma \\ T'_{\Sigma_0} \supseteq T_{\Sigma_0} \ni \phi_0 & \longmapsto & \phi \end{array}$$

Since $T_{\Sigma_0} \subseteq T'_{\Sigma_0}$, $\phi_0 \in T'_{\Sigma_0}$ and we conclude that $\phi \in \overrightarrow{T'}_\Sigma$. ■

The importance of $\overleftarrow{T}$ and $\overrightarrow{T}$ stems, in part, from their relationship with T, which is described in the following proposition, but also from the critical role they play in the theory presented here.

Proposition 2 *Let* $\mathbf{Sign}$ *be a category,* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *a sentence functor and suppose that* $T \in \mathrm{SenFam}(\mathrm{SEN})$.

(a) $\overleftarrow{T}$ *is the largest sentence system of* SEN *included in* T;

(b) $\overrightarrow{T}$ *is the smallest sentence system of* SEN *that contains* T.

Proof:

(a) It is obvious that $\overleftarrow{T} \leq T$. We must show that $\overleftarrow{T}$ is a sentence system and that it is the largest one included in T.

To show that it is a sentence system, consider $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\phi \in \overleftarrow{T}_\Sigma$. We must show that $\mathrm{SEN}(f)(\phi) \in \overleftarrow{T}_{\Sigma'}$. To this end, let $\Sigma'' \in |\mathbf{Sign}|$ and $g \in \mathbf{Sign}(\Sigma', \Sigma'')$.

$$\Sigma \xrightarrow{f} \Sigma' \xrightarrow{g} \Sigma''$$

Then we have

$$\mathrm{SEN}(g)(\mathrm{SEN}(f)(\phi)) = \mathrm{SEN}(gf)(\phi) \stackrel{\phi\in\overleftarrow{T}_\Sigma}{\in} T_{\Sigma''}.$$

Since this holds for all $\Sigma'' \in |\mathbf{Sign}|$ and all $g \in \mathbf{Sign}(\Sigma', \Sigma'')$, we conclude that $\mathrm{SEN}(f)(\phi) \in \overleftarrow{T}_{\Sigma'}$.

To show that $\overleftarrow{T}$ is the largest sentence system in T, consider $T' \in \mathrm{SenSys}(\mathrm{SEN})$, such that $T' \leq T$ and let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in T'_\Sigma$. Since T' is a sentence system, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, we get $\mathrm{SEN}(f)(\phi) \in T'_{\Sigma'}$. Now since $T' \leq T$, we get that, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\mathrm{SEN}(f)(\phi) \in T_{\Sigma'}$. But this shows that $\phi \in \overleftarrow{T}_\Sigma$. Thus, $T' \leq \overleftarrow{T}$ and $\overleftarrow{T}$ is the largest sentence system included in T.

(b) It is obvious that $T \leq \overrightarrow{T}$. We must show that $\overrightarrow{T}$ is a sentence system and that it is the smallest one containing T.

To show that $\overrightarrow{T}$ is a sentence system, consider $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in \overrightarrow{T}_\Sigma$. Let $\Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma, \Sigma')$. We must show that $\mathrm{SEN}(f)(\phi) \in \overrightarrow{T}_{\Sigma'}$. Since $\phi \in \overrightarrow{T}_\Sigma$, there exists $\Sigma_0 \in |\mathbf{Sign}|$, $f_0 \in \mathbf{Sign}(\Sigma_0, \Sigma)$ and $\phi_0 \in T_{\Sigma_0}$, such that $\mathrm{SEN}(f_0)(\phi_0) = \phi$.

$$\Sigma_0 \xrightarrow{\ f_0\ } \Sigma \xrightarrow{\ f\ } \Sigma'$$

Thus, we get

$$\mathrm{SEN}(f)(\phi) = \mathrm{SEN}(f)(\mathrm{SEN}(f_0)(\phi_0)) = \mathrm{SEN}(ff_0)(\phi_0) \in \overrightarrow{T}_{\Sigma'}.$$

Finally, we must show that $\overrightarrow{T}$ is the smallest sentence system that contains T. To this end, suppose that $T' \in \mathrm{SenSys}(\mathrm{SEN})$, such that $T \leq T'$. Let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in \overrightarrow{T}_\Sigma$. Then, there exist $\Sigma_0 \in |\mathbf{Sign}|$, $f_0 \in \mathbf{Sign}(\Sigma_0, \Sigma)$ and $\phi_0 \in T_{\Sigma_0}$, such that $\phi = \mathrm{SEN}(f_0)(\phi_0)$. Now, since $T \leq T'$, we get $\phi_0 \in T'_{\Sigma_0}$. Moreover, since T' is a sentence system, we get $\mathrm{SEN}(f_0)(\phi_0) \in T'_\Sigma$. But this means $\phi = \mathrm{SEN}(f_0)(\phi_0) \in T'_\Sigma$. This proves that $\overrightarrow{T} \leq T'$ and, hence, $\overrightarrow{T}$ is the least sentence system that contains T.

■

It is also of interest to observe that the back arrow operator commutes with intersections:

Lemma 3 *Let* **Sign** *be a category,* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *a sentence functor and consider* $\mathcal{T} \subseteq \mathrm{SenFam}(\mathrm{SEN})$*. Then*

$$\overleftarrow{\bigcap_{T \in \mathcal{T}} T} = \bigcap_{T \in \mathcal{T}} \overleftarrow{T}.$$

Proof: First, by Lemma 1, we have, for all $T \in \mathcal{T}$, $\overleftarrow{\bigcap_{T\in\mathcal{T}} T} \leq \overleftarrow{T}$. Therefore, we conclude that $\overleftarrow{\bigcap_{T\in\mathcal{T}} T} \leq \bigcap_{T\in\mathcal{T}} \overleftarrow{T}$.

On the other hand, we have, by Proposition 2, $\overleftarrow{T} \leq T$, for all $T \in \mathcal{T}$. Therefore $\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \leq \bigcap_{T\in\mathcal{T}} T$. Now, since $\bigcap_{T\in\mathcal{T}} \overleftarrow{T}$ is a sentence system (Proposition 2) included in $\bigcap_{T\in\mathcal{T}} T$, it must lie below the largest such, which, by Proposition 2, is $\overleftarrow{\bigcap_{T\in\mathcal{T}} T}$. Thus, we have $\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \leq \overleftarrow{\bigcap_{T\in\mathcal{T}} T}$. ■

On the other hand, the back arrow does not commute, in general, with unions. We first prove a lemma showing the there is an inclusion relation governing the interaction between the back arrow and unions and, then, provide an example to show that this inclusion may be proper.

Lemma 4 *Let* **Sign** *be a category,* SEN : **Sign** → **Set** *a sentence functor and consider* $\mathcal{T} \subseteq \mathrm{SenFam}(\mathrm{SEN})$*. Then*

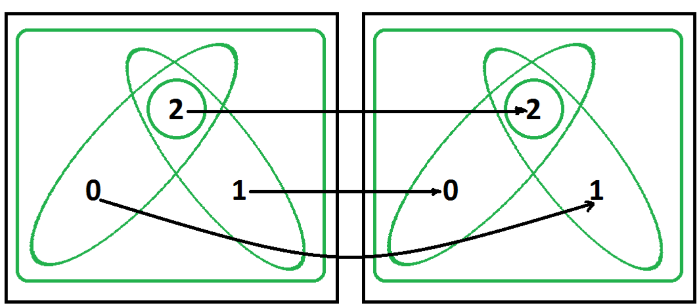
$$\bigcup_{T\in\mathcal{T}} \overleftarrow{T} \leq \overleftarrow{\bigcup_{T\in\mathcal{T}} T}.$$

Proof: Since, for all $T \in \mathcal{T}$, $T \leq \bigcup_{T\in\mathcal{T}} T$, we get, by Lemma 1, $\overleftarrow{T} \leq \overleftarrow{\bigcup_{T\in\mathcal{T}} T}$. Since this holds for all $T \in \mathcal{T}$, we conclude that $\bigcup_{T\in\mathcal{T}} \overleftarrow{T} \leq \overleftarrow{\bigcup_{T\in\mathcal{T}} T}$. ■

That the inclusion of Lemma 4 is, in general, a proper inclusion is showcased by the following example.

Example 5 *Let* **Sign** *be the category with a single object* Σ *and a single (non-identity) arrow* $f : \Sigma \to \Sigma$*, such that* $f \circ f = i_\Sigma$*.*

Let SEN : **Sign** → **Set** *be the functor defined by setting* $\mathrm{SEN}(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}(f)(0) = 1$, $\mathrm{SEN}(f)(1) = 0$ *and* $\mathrm{SEN}(f)(2) = 2$*. Consider the col-*



lection $\{T, T'\} \subseteq \mathrm{SenFam}(\mathrm{SEN})$*, with* $T_\Sigma = \{0, 2\}$ *and* $T'_\Sigma = \{1, 2\}$*. Then we have* $\overleftarrow{T}_\Sigma = \{2\} = \overleftarrow{T}_{\Sigma'}$ *and, therefore*

$$\overleftarrow{T}_\Sigma \cup \overleftarrow{T'}_\Sigma = \{2\} \cup \{2\} = \{2\}.$$

On the other hand,

$$\overleftarrow{T \cup T'}_\Sigma = \overleftarrow{\{\{0,1,2\}\}}_\Sigma = \{0,1,2\}.$$

Thus, we get $\bigcup_{T \in \mathcal{T}} \overleftarrow{T} \lneqq \overleftarrow{\bigcup_{T \in \mathcal{T}} T}$.

Let **Sign**, **Sign′** be categories and $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ and $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ be two sentence functors. A **morphism** (of sentence functors) $\langle F, \alpha \rangle : \mathrm{SEN} \to \mathrm{SEN}'$ consists of:

- A functor $F : \mathbf{Sign} \to \mathbf{Sign}'$;
- A natural transformation $\alpha : \mathrm{SEN} \to \mathrm{SEN}' \circ F$.

We will make heavy use of the following particular types of morphisms:

- A morphism $\langle F, \alpha \rangle : \mathrm{SEN} \to \mathrm{SEN}'$ is **special** if $F : \mathbf{Sign} \to \mathbf{Sign}'$ is surjective on objects and full.
- A morphism $\langle F, \alpha \rangle : \mathrm{SEN} \to \mathrm{SEN}'$ is **surjective** if it is special and $\alpha_\Sigma : \mathrm{SEN}(\Sigma) \to \mathrm{SEN}'(F(\Sigma))$ is surjective, for all $\Sigma \in |\mathbf{Sign}|$.

Let **Sign**, **Sign′** be categories, $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ and $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ be two sentence functors and $\langle F, \alpha \rangle : \mathrm{SEN} \to \mathrm{SEN}'$ be a morphism. Given a sentence family $T \in \mathrm{SenFam}(\mathrm{SEN}')$, define the sentence family $\alpha^{-1}(T) = \{\alpha^{-1}(T)_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} \in \mathrm{SenFam}(\mathrm{SEN})$ by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\alpha^{-1}(T)_\Sigma = \alpha_\Sigma^{-1}(T_{F(\Sigma)}).$$

In the next lemma, we prove some useful properties concerning this operator.

Lemma 6 *Let* **Sign**, **Sign′** *be categories,* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *and* $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ *be two sentence functors,* $\langle F, \alpha \rangle : \mathrm{SEN} \to \mathrm{SEN}'$ *be a morphism and* $T \in \mathrm{SenFam}(\mathrm{SEN}')$.

(a) If $T \in \mathrm{SenSys}(\mathrm{SEN}')$, *then* $\alpha^{-1}(T) \in \mathrm{SenSys}(\mathrm{SEN})$, *with equivalence holding if* $\langle F, \alpha \rangle : \mathrm{SEN} \to \mathrm{SEN}'$ *is surjective;*

(b) $\alpha^{-1}(\overleftarrow{T}) \leq \overleftarrow{\alpha^{-1}(T)}$, *with equality holding if* $\langle F, \alpha \rangle$ *is special;*

(c) $\overrightarrow{\alpha^{-1}(T)} \leq \alpha^{-1}(\overrightarrow{T})$, *with equality holding if* $\langle F, \alpha \rangle$ *is surjective.*

Proof:

(a) Let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})$. Then, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, we have

$$\begin{array}{rcl}\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) & = & \mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi)) \\ & & (\alpha \text{ natural transformation}) \\ & \in & \mathrm{SEN}'(F(f))(T_{F(\Sigma)}) \quad (\phi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})) \\ & \subseteq & T_{F(\Sigma')} \quad (T \in \mathrm{SenSys}(\mathrm{SEN}')).\end{array}$$

This shows that $\mathrm{SEN}(f)(\phi) \in \alpha_{\Sigma'}^{-1}(T_{F(\Sigma')})$. We now conclude that $\alpha^{-1}(T) \in \mathrm{SenSys}(\mathrm{SEN})$.

Suppose, next, that $\langle F, \alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$ is surjective and $\alpha^{-1}(T) \in \mathrm{SenSys}(\mathrm{SEN})$. Let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$. Note that this implies that $\phi \in \alpha^{-1}(T_{F(\Sigma)})$. So, by hypothesis, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}(f)(\phi) \in \alpha_{\Sigma'}^{-1}(T_{F(\Sigma')}).$$

Therefore,

$$\begin{array}{rcl}\mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi)) & = & \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \\ & \in & \alpha_{\Sigma'}(\alpha_{\Sigma'}^{-1}(T_{F(\Sigma')})) \\ & \subseteq & T_{F(\Sigma')}.\end{array}$$

Since $\langle F, \alpha\rangle$ is surjective, we conclude that, for all $\Sigma, \Sigma' \in |\mathbf{Sign}'|$ and all $f \in \mathbf{Sign}'(\Sigma, \Sigma')$,

$$\mathrm{SEN}'(f)(T_\Sigma) \subseteq T_{\Sigma'}.$$

Therefore, $T \in \mathrm{SenSys}(\mathrm{SEN}')$.

(b) Let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in \alpha_\Sigma^{-1}(\overleftarrow{T}_{F(\Sigma)})$. Then we get that $\alpha_\Sigma(\phi) \in \overleftarrow{T}_{F(\Sigma)}$. Thus, by definition of $\overleftarrow{T}$, for all $\Sigma' \in |\mathbf{Sign}'|$ and all $f \in \mathbf{Sign}'(F(\Sigma), \Sigma')$,

$$\mathrm{SEN}'(f)(\alpha_\Sigma(\phi)) \in T_{\Sigma'}.$$

This implies, in particular, that, for all $\Sigma'' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma'')$, $\mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi)) \in T_{F(\Sigma'')}$. So we get $\alpha_{\Sigma''}(\mathrm{SEN}(f)(\phi)) \in T_{F(\Sigma'')}$, i.e., $\mathrm{SEN}(f)(\phi) \in \alpha_{\Sigma''}^{-1}(T_{F(\Sigma'')})$. Since Σ'' and f were arbitrary, we finally obtain $\phi \in \overleftarrow{\alpha_\Sigma^{-1}(T_{F(\Sigma)})}$.

It is straightforward to see that, if $\langle F, \alpha\rangle$ is special, then the above chain of implications is reversible and, by following it, we get the reverse inclusion.

(c) Let $\Sigma \in |\mathbf{Sign}|$, Then we have

$$\begin{array}{l}\alpha_\Sigma(\overrightarrow{\alpha_\Sigma^{-1}(T_{F(\Sigma)})})\\ = \alpha_\Sigma(\{\mathrm{SEN}(f_0)(\phi_0) : f_0 \in \mathbf{Sign}(\Sigma_0,\Sigma), \phi_0 \in \alpha_{\Sigma_0}^{-1}(T_{F(\Sigma_0)})\})\\ = \{\alpha_\Sigma(\mathrm{SEN}(f_0)(\phi_0)) : f_0 \in \mathbf{Sign}(\Sigma_0,\Sigma), \phi_0 \in \alpha_{\Sigma_0}^{-1}(T_{F(\Sigma_0)})\}\\ = \{\mathrm{SEN}'(F(f_0))(\alpha_{\Sigma_0}(\phi_0)) : f_0 \in \mathbf{Sign}(\Sigma_0,\Sigma), \phi_0 \in \alpha_{\Sigma_0}^{-1}(T_{F(\Sigma_0)})\}\\ \subseteq \{\mathrm{SEN}'(f_0')(\phi_0') : f_0' \in \mathbf{Sign}'(\Sigma_0', F(\Sigma)), \phi_0' \in T_{\Sigma_0'}\}\\ = \overrightarrow{T}_{F(\Sigma)}.\end{array}$$

Again, it is easy to see that the only inclusion becomes an equality in case $\langle F,\alpha\rangle$ is a surjective morphism.

■

Let **Sign** be a category and $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a sentence functor. A **relation family on** SEN is a collection $R = \{R_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$, such that $R_\Sigma \subseteq \mathrm{SEN}(\Sigma)^2$, for all $\Sigma \in |\mathbf{Sign}|$. A relation family is a **relation system** if it is invariant under **Sign**-morphisms, i.e., if for all $\Sigma,\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma,\Sigma')$,

$$\mathrm{SEN}(f)(R_\Sigma) \subseteq R_{\Sigma'}.$$

The collection of all relation families on SEN is denoted by RelFam(SEN) and, similarly, the collection of all relation systems by RelSys(SEN). A relation family/system on SEN is an **equivalence family/system on** SEN if, for all $\Sigma \in |\mathbf{Sign}|$, R_Σ is an equivalence relation on SEN(Σ). As with relation families/systems, we denote the collection of all equivalence families on SEN by EqvFam(SEN) and the collection of all equivalence systems on SEN by EqvSys(SEN).

Given a sentence family $T \in \mathrm{SenFam(SEN)}$, we say that the equivalence family R on SEN is **compatible with** T, if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi,\psi \in \mathrm{SEN}(\Sigma)$,

$$\langle\phi,\psi\rangle \in R_\Sigma \quad \text{and} \quad \phi \in T_\Sigma \quad \text{imply} \quad \psi \in T_\Sigma.$$

Lemma 7 *Let* **Sign** *be a category,* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *a sentence functor,* $T \in \mathrm{SenFam(SEN)}$ *and* θ *a relation system on* SEN*. If* θ *is compatible with* T*, then it is also compatible with* $\overleftarrow{T}$*.*

Proof: Suppose that θ is compatible with T. Let $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle \in \theta_\Sigma$ and $\phi \in \overleftarrow{T}_\Sigma$. Let $\Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma,\Sigma')$. Since θ is a relation system, we get $\langle\mathrm{SEN}(f)(\phi),\mathrm{SEN}(f)(\psi)\rangle \in \theta_{\Sigma'}$. Since $\phi \in \overleftarrow{T}_\Sigma$, $\mathrm{SEN}(f)(\phi) \in T_{\Sigma'}$. Thus, by compatibility, we get $\mathrm{SEN}(f)(\psi) \in T_{\Sigma'}$. Since $\Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma,\Sigma')$ were arbitrary, we conclude that $\psi \in \overleftarrow{T}_\Sigma$, showing that θ is also compatible with $\overleftarrow{T}$. ■

Let **Sign**, **Sign′** be categories, $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ and $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ be sentence functors and $\langle F,\alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$ be a morphism. Define the

kernel system of $\langle F,\alpha\rangle$, denoted $\mathrm{Ker}(\langle F,\alpha\rangle)=\{\mathrm{Ker}_\Sigma(\langle F,\alpha\rangle)\}_{\Sigma\in|\mathbf{Sign}|}$, by letting, for all $\Sigma\in|\mathbf{Sign}|$ and all $\phi,\psi\in\mathrm{SEN}(\Sigma)$,

$$\langle\phi,\psi\rangle\in\mathrm{Ker}_\Sigma(\langle F,\alpha\rangle)\quad\text{iff}\quad\alpha_\Sigma(\phi)=\alpha_\Sigma(\psi).$$

The kernel system $\mathrm{Ker}(\langle F,\alpha\rangle)$ is sometimes denoted more compactly by $\theta^{\langle F,\alpha\rangle}=\{\theta^{\langle F,\alpha\rangle}_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$.

Lemma 8 *Let* **Sign**, **Sign**′ *be categories,* $\mathrm{SEN}:\mathbf{Sign}\to\mathbf{Set}$, $\mathrm{SEN}':\mathbf{Sign}'\to\mathbf{Set}$ *be sentence functors and* $\langle F,\alpha\rangle:\mathrm{SEN}\to\mathrm{SEN}'$ *a morphism. Then* $\mathrm{Ker}(\langle F,\alpha\rangle)$ *is an equivalence system on* SEN.

Proof: It is obvious from the definition that $\mathrm{Ker}(\langle F,\alpha\rangle)$ is an equivalence family of SEN. The system property follows from the fact that α is a natural transformation. Let $\Sigma\in|\mathbf{Sign}|$ and $\phi,\psi\in\mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle\in\mathrm{Ker}_\Sigma(\langle F,\alpha\rangle)$. Then, for all $\Sigma'\in|\mathbf{Sign}|$ and all $f\in\mathbf{Sign}(\Sigma,\Sigma')$,

$$\begin{array}{rcll}\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) & = & \mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi)) & (\text{naturality of }\alpha)\\ & = & \mathrm{SEN}'(F(f))(\alpha_\Sigma(\psi)) & (\langle\phi,\psi\rangle\in\mathrm{Ker}_\Sigma(\langle F,\alpha\rangle))\\ & = & \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi)) & (\text{naturality of }\alpha).\end{array}$$

Therefore, we get that $\langle\mathrm{SEN}(f)(\phi),\mathrm{SEN}(f)(\psi)\rangle\in\mathrm{Ker}_{\Sigma'}(\langle F,\alpha\rangle)$, showing that $\mathrm{Ker}(\langle F,\alpha\rangle)$ is an equivalence system. ■

Let **Sign**, **Sign**′ be categories, $\mathrm{SEN}:\mathbf{Sign}\to\mathbf{Set}$ and $\mathrm{SEN}':\mathbf{Sign}'\to\mathbf{Set}$ be sentence functors and $\langle F,\alpha\rangle:\mathrm{SEN}\to\mathrm{SEN}'$ be a morphism, with F an isomorphism. Given a sentence family $T\in\mathrm{SenFam}(\mathrm{SEN})$, define the sentence family $\alpha(T)=\{\alpha(T)_{F(\Sigma)}\}_{\Sigma\in|\mathbf{Sign}|}\in\mathrm{SenFam}(\mathrm{SEN}')$ by setting, for all $\Sigma\in|\mathbf{Sign}|$,

$$\alpha(T)_{F(\Sigma)}=\alpha_\Sigma(T_\Sigma).$$

In the next lemma, we prove some useful properties concerning this operator.

Lemma 9 *Let* **Sign**, **Sign**′ *be categories,* $\mathrm{SEN}:\mathbf{Sign}\to\mathbf{Set}$ *and* $\mathrm{SEN}':\mathbf{Sign}'\to\mathbf{Set}$ *be sentence functors,* $\langle F,\alpha\rangle:\mathrm{SEN}\to\mathrm{SEN}'$ *a surjective morphism, with* F *an isomorphism, and* $T\in\mathrm{SenFam}(\mathrm{SEN})$*, such that the kernel* $\mathrm{Ker}(\langle F,\alpha\rangle)$ *of* $\langle F,\alpha\rangle$ *is compatible with* T.

(a) $\alpha(T)\in\mathrm{SenSys}(\mathrm{SEN}')$ *iff* $T\in\mathrm{SenSys}(\mathrm{SEN})$*;*

(b) $\alpha(\overleftarrow{T})=\overleftarrow{\alpha(T)}$*;*

(c) $\overrightarrow{\alpha(T)}=\alpha(\overrightarrow{T})$.

Proof:

(a) Exploiting the surjectivity of $\langle F,\alpha\rangle$, $\alpha(T)\in\mathrm{SenSys}(\mathrm{SEN}')$ holds if and only if, for all $\Sigma\in|\mathbf{Sign}|$, all $\phi\in\mathrm{SEN}(\Sigma)$, all $\Sigma'\in|\mathbf{Sign}|$ and all $f\in\mathbf{Sign}(\Sigma,\Sigma')$,

$$\mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi))\in\alpha_{\Sigma'}(T_{\Sigma'}).$$

By the naturality of α, the latter is equivalent to

$$\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi))\in\alpha_{\Sigma'}(T_{\Sigma'}).$$

Finally, by compatibility of $\mathrm{Ker}(\langle F,\alpha\rangle)$ with T, this is equivalent to $\mathrm{SEN}(f)(\phi)\in T_{\Sigma'}$. But this holds for all $\Sigma\in|\mathbf{Sign}|$, all $\phi\in\mathrm{SEN}(\Sigma)$, all $\Sigma'\in|\mathbf{Sign}|$ and all $f\in\mathbf{Sign}(\Sigma,\Sigma')$ if and only if $T\in\mathrm{SenSys}(\mathrm{SEN})$.

(b) Again we exploit the surjectivity of $\langle F,\alpha\rangle$. We have, for all $\Sigma\in|\mathbf{Sign}|$ and all $\phi\in\mathrm{SEN}(\Sigma)$, $\alpha_\Sigma(\phi)\in\overleftarrow{\alpha_\Sigma(T_\Sigma)}$ iff, for all $\Sigma'\in|\mathbf{Sign}|$ and all $f\in\mathbf{Sign}(\Sigma,\Sigma')$, $\mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi))\in\alpha_{\Sigma'}(T_{\Sigma'})$ iff, by the naturality of α, $\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi))\in\alpha_{\Sigma'}(T_{\Sigma'})$ iff, by the compatibility of $\mathrm{Ker}(\langle F,\alpha\rangle)$ with T, $\mathrm{SEN}(f)(\phi)\in T_{\Sigma'}$ iff, by the definition of $\overleftarrow{T}$, $\phi\in\overleftarrow{T}_\Sigma$ iff, by the compatibility of $\mathrm{Ker}(\langle F,\alpha\rangle)$ with $\overleftarrow{T}$, which follows from Lemmas 7 and 8, $\alpha_\Sigma(\phi)\in\alpha_\Sigma(\overleftarrow{T}_\Sigma)$. Thus, we conclude that $\alpha(\overleftarrow{T})=\overleftarrow{\alpha(T)}$.

(c) Suppose, first, that $\alpha_\Sigma(\phi)\in\overrightarrow{\alpha_\Sigma(T_\Sigma)}$. Then, there exist, by surjectivity, $\Sigma_0\in|\mathbf{Sign}|$, $f_0\in\mathbf{Sign}(\Sigma_0,\Sigma)$ and $\phi_0\in T_{\Sigma_0}$, such that

$$\begin{array}{rcl}\alpha_\Sigma(\phi) & = & \mathrm{SEN}'(F(f_0))(\alpha_{\Sigma_0}(\phi_0))\\ & = & \alpha_\Sigma(\mathrm{SEN}(f_0)(\phi_0))\\ & \in & \alpha_\Sigma(\overrightarrow{T}_\Sigma).\end{array}$$

Suppose, conversely, that $\alpha_\Sigma(\phi)\in\alpha_\Sigma(\overrightarrow{T}_\Sigma)$. Then, there exist $\Sigma_0\in|\mathbf{Sign}|$, $f_0\in\mathbf{Sign}(\Sigma_0,\Sigma)$ and $\phi_0\in T_{\Sigma_0}$, such that

$$\begin{array}{rcl}\alpha_\Sigma(\phi) & = & \alpha_\Sigma(\mathrm{SEN}(f_0)(\phi_0))\\ & = & \mathrm{SEN}'(F(f_0))(\alpha_{\Sigma_0}(\phi_0))\\ & \in & \overrightarrow{\alpha_\Sigma(T_\Sigma)}.\end{array}$$

■

By analogy to the case of sentence families, we may also define the inverse of a relation family under a morphism of sentence functors. Let $\mathbf{Sign}$, $\mathbf{Sign}'$ be categories, $\mathrm{SEN}:\mathbf{Sign}\to\mathbf{Set}$ and $\mathrm{SEN}':\mathbf{Sign}'\to\mathbf{Set}$ be sentence functors and $\langle F,\alpha\rangle:\mathrm{SEN}\to\mathrm{SEN}'$ a morphism. Let, also, $R=\{R_\Sigma\}_{\Sigma\in|\mathbf{Sign}'|}$ be a relation family on SEN'. Define the relation family $\alpha^{-1}(R)=\{\alpha^{-1}(R)_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ on SEN by setting, for all $\Sigma\in|\mathbf{Sign}|$,

$$\alpha^{-1}(R)_\Sigma=\alpha_\Sigma^{-1}(R_{F(\Sigma)}).$$

Proposition 10 *Let* **Sign**, **Sign**$'$ *be categories,* SEN : **Sign** → **Set** *and* SEN$'$: **Sign**$'$ → **Set** *be sentence functors,* $\langle F, \alpha\rangle$: SEN → SEN$'$ *a morphism and* R *a relation family on* SEN$'$.

(a) If R *is a relation system, then* $\alpha^{-1}(R)$ *is also a relation system;*

(b) If R *is an equivalence family, then* α^{-1} *is also an equivalence family.*

Proof:

(a) Let $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi\rangle \in \alpha_\Sigma^{-1}(R_{F(\Sigma)})$. Then, we have $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in R_{F(\Sigma)}$. Since R is a relation system, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, we get

$$\langle \mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi)), \mathrm{SEN}'(F(f))(\alpha_\Sigma(\psi))\rangle \in R_{F(\Sigma')}.$$

Thus, by the naturality of α,

$$\langle \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)), \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi))\rangle \in R_{F(\Sigma')}.$$

Now we get $\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)\rangle \in \alpha_{\Sigma'}^{-1}(R_{F(\Sigma')})$. This proves that $\alpha^{-1}(R)$ is a relation system on SEN.

(b) Let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \chi, \psi \in \mathrm{SEN}(\Sigma)$ be arbitrary. Then we have:

Reflexivity By the reflexivity of R, $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\phi)\rangle \in R_{F(\Sigma)}$. Therefore, $\langle \phi, \phi\rangle \in \alpha_\Sigma^{-1}(R_{F(\Sigma)})$.

Symmetry If $\langle \phi, \psi\rangle \in \alpha_\Sigma^{-1}(R_{F(\Sigma)})$, then $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in R_{F(\Sigma)}$, whence, by the symmetry of R, $\langle \alpha_\Sigma(\psi), \alpha_\Sigma(\phi)\rangle \in R_{F(\Sigma)}$, showing that $\langle \psi, \phi\rangle \in \alpha_\Sigma^{-1}(R_{F(\Sigma)})$.

Transitivity If $\langle \phi, \chi\rangle, \langle \chi, \psi\rangle \in \alpha_\Sigma^{-1}(R_{F(\Sigma)})$, then, we get

$$\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\chi)\rangle, \langle \alpha_\Sigma(\chi), \alpha_\Sigma(\psi)\rangle \in R_{F(\Sigma)},$$

whence, by the transitivity of R, we get $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in R_{F(\Sigma)}$, showing that $\langle \phi, \psi\rangle \in \alpha_\Sigma^{-1}(R_{F(\Sigma)})$.

■

Let **Sign** be a category and SEN : **Sign** → **Set** a sentence functor. The **clone of all natural transformations** on SEN is the category **Cln**(SEN) with collection of objects $\{\mathrm{SEN}^\alpha : \alpha \text{ an ordinal}\}$ and collection of morphisms $\tau : \mathrm{SEN}^\alpha \to \mathrm{SEN}^\beta$ β-sequences of natural transformations $\tau^i : \mathrm{SEN}^\alpha \to \mathrm{SEN}$, $i < \beta$. Composition of $\langle \tau^i : i < \beta\rangle : \mathrm{SEN}^\alpha \to \mathrm{SEN}^\beta$ with $\langle \sigma^j : j < \gamma\rangle : \mathrm{SEN}^\beta \to \mathrm{SEN}^\gamma$

$$\mathrm{SEN}^\alpha \xrightarrow{\langle \tau^i : i < \beta\rangle} \mathrm{SEN}^\beta \xrightarrow{\langle \sigma^j : j < \gamma\rangle} \mathrm{SEN}^\gamma$$

is defined by

$$\langle \sigma^j : j < \gamma\rangle \circ \langle \tau^i : i < \beta\rangle = \langle \sigma^j(\langle \tau^i : i < \beta) : j < \gamma\rangle.$$

A **clone** (or a **category**) **of natural transformations** on SEN is a subcategory N of the category **Cln**(SEN), such that:

- Its objects are those in $\{\mathrm{SEN}^k : k < \omega\}$;
- Its morphisms include all projection natural transformations

  $$p^{k,i} : \mathrm{SEN}^k \to \mathrm{SEN}, i < k, k < \omega,$$

  with $p^{k,i}_\Sigma : \mathrm{SEN}(\Sigma)^k \to \mathrm{SEN}(\Sigma)$ given by

  $$p^{k,i}_\Sigma(\vec{\phi}) = \phi_i, \text{ for all } \vec{\phi} \in \mathrm{SEN}(\Sigma)^k,$$

  and are such that, for every family $\{\tau^i : \mathrm{SEN}^k \to \mathrm{SEN} : i < \ell\}$ of natural transformations in N, $\langle \tau^i : i < \ell\rangle : \mathrm{SEN}^k \to \mathrm{SEN}^\ell$ is also in N.

This definition has two important consequences that we now make explicit. Let **Sign** be a category, SEN : **Sign** → **Set** a sentence functor and $k \in \omega$. Consider a function

$$\pi : \{0, 1, \ldots, k-1\} \to \{0, 1, \ldots, k-1\}.$$

Given $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi} = \langle \phi_0, \phi_1, \ldots, \phi_{k-1}\rangle \in \mathrm{SEN}(\Sigma)^k$, we define

$$\vec{\phi}^\pi = \langle \phi_{\pi(0)}, \phi_{\pi(1)}, \ldots, \phi_{\pi(k-1)}\rangle.$$

Now, consider, in addition, a clone N of natural transformations on SEN and $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N. Define the natural transformation

$$\sigma^\pi : \mathrm{SEN}^k \to \mathrm{SEN}$$

by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)$,

$$\sigma^\pi_\Sigma(\vec{\phi}) = \sigma_\Sigma(\vec{\phi}^\pi).$$

That this is a natural transformation is easy to see: For all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)$, we have

$$\begin{array}{ccc}
\mathrm{SEN}(\Sigma)^k & \xrightarrow{\sigma^\pi_\Sigma} & \mathrm{SEN}(\Sigma) \\
{\scriptstyle \mathrm{SEN}(f)^k}\downarrow & & \downarrow{\scriptstyle \mathrm{SEN}(f)} \\
\mathrm{SEN}(\Sigma')^k & \xrightarrow[\sigma^\pi_{\Sigma'}]{} & \mathrm{SEN}(\Sigma')
\end{array}$$

$$\begin{aligned}
\mathrm{SEN}(f)(\sigma^\pi_\Sigma(\vec{\phi})) &= \mathrm{SEN}(f)(\sigma_\Sigma(\vec{\phi}^\pi)) \\
&= \sigma_{\Sigma'}(\mathrm{SEN}(f)^k(\vec{\phi}^\pi)) \\
&= \sigma_{\Sigma'}(\mathrm{SEN}(f)^k(\vec{\phi})^\pi) \\
&= \sigma^\pi_{\Sigma'}(\mathrm{SEN}(f)^k(\vec{\phi})).
\end{aligned}$$

Proposition 11 *Let* **Sign** *be a category,* SEN : **Sign** → **Set** *a sentence functor and* N *a clone of natural transformations on* SEN*. If* $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ *is in* N*, then, for all functions* $\pi : \{0,\dots,k-1\} \to \{0,\dots,k-1\}$*,* $\sigma^\pi : \mathrm{SEN}^k \to \mathrm{SEN}$ *is also in* N*.*

Proof: The key is to observe that

$$\sigma^\pi = \sigma \circ \langle p^{k,\pi(0)},\dots,p^{k,\pi(k-1)}\rangle.$$

Since all projections are in N and N is closed under formation of tuples, we get that $\langle p^{k,\pi(0)},\dots,p^{k,\pi(k-1)}\rangle : \mathrm{SEN}^k \to \mathrm{SEN}^k$ is in N. Therefore, since N is a category and, by hypothesis, σ is in N, we get that σ^π is also in N. ■

The following is a very useful consequence that allows simplifying quantifications.

Corollary 12 *Let* **Sign** *be a category,* SEN : **Sign** → **Set** *a sentence functor and* N *a clone of natural transformations on* SEN*. The statement*

$$\text{For all } \sigma : \mathrm{SEN}^k \to \mathrm{SEN} \text{ in } N, \text{ all } i < k \text{ and all } \Sigma \in |\mathbf{Sign}|,\ \phi,\vec{\chi} \in \mathrm{SEN}(\Sigma),$$
$$\mathrm{Property}(\sigma_\Sigma(\chi_0,\dots,\chi_{i-1},\phi,\chi_{i+1},\dots,\chi_{k-1}))$$

is equivalent to the simpler statement

$$\text{For all } \sigma : \mathrm{SEN}^k \to \mathrm{SEN} \text{ in } N \text{ and all } \Sigma \in |\mathbf{Sign}|,\ \phi,\vec{\chi} \in \mathrm{SEN}(\Sigma),$$
$$\mathrm{Property}(\sigma_\Sigma(\phi,\vec{\chi})).$$

Proof: The left-to-right implication is trivial. The right-to-left implication follows from Proposition 11, since $\sigma^{\pi_i} : \mathrm{SEN}^k \to \mathrm{SEN}$, with π_i being the permutation

$$\begin{pmatrix} 0 & 1 & \cdots & i-1 & i & i+1 & \cdots & k-1 \\ 1 & 2 & \cdots & i & 0 & i+1 & \cdots & k-1 \end{pmatrix},$$

is also in N, for every $i < k$. ■

An **algebraic system** is a triple $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, where:

- **Sign** is an arbitrary category;
- SEN : **Sign** → **Set** is a sentence functor;
- N is a clone on SEN.

An algebraic system $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ is said to be **trivial** if its underlying sentence functor SEN : **Sign** → **Set** is trivial, i.e., if all its sets of sentences are singletons.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system. An $N^\flat$**-algebraic system** $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ is an algebraic system, such that there exists a surjective functor $\Xi : N^\flat \to N$ that preserves all projection natural transformations, i.e., such that, for all $k < \omega$ and all $i < k$, if $p^{k,i^\flat} : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ denotes the i-th projection natural transformation on $(\mathrm{SEN}^\flat)^k$, then $\Xi(p^{k,i^\flat}) : \mathrm{SEN}^k \to \mathrm{SEN}$ is the i-th projection $p^{k,i}$ on SEN^k.

This condition implies that Ξ also preserves the arities of all natural transformations involved. Given $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, the image $\Xi(\sigma^\flat) : \mathrm{SEN}^k \to \mathrm{SEN}$ in N will be denoted by σ, keeping the same lowercase Greek letter, but adjusting superscripts, subscripts, primes, etc., as demanded by context. Occasionally, to simplify notation, we might drop superscripts, subscripts, etc., overloading the notation of the lowercase Greek letter, allowing the context to make the interpretation of each occurrence clear (and hoping that, because of this, confusion can be avoided).

In the context where $N^\flat$-algebraic systems are under consideration, the algebraic system $\mathbf{F}$ will be referred to as the **base algebraic system**, since the clones on all other systems under consideration are images of the clone of $\mathbf{F}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system and $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ be $N^\flat$-algebraic systems. A **morphism** (of $N^\flat$-algebraic systems) $\langle F, \alpha \rangle : \mathbf{A} \to \mathbf{A}'$ is a morphism of sentence functors $\langle F, \alpha \rangle : \mathrm{SEN} \to \mathrm{SEN}'$, such that, for all $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)$ (meaning $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$),

$$\begin{array}{ccc}
\mathrm{SEN}(\Sigma)^k & \xrightarrow{\sigma_\Sigma} & \mathrm{SEN}(\Sigma) \\
\Big\downarrow{\scriptstyle \alpha_\Sigma^k} & & \Big\downarrow{\scriptstyle \alpha_\Sigma} \\
\mathrm{SEN}'(F(\Sigma))^k & \xrightarrow[\sigma'_{F(\Sigma)}]{} & \mathrm{SEN}'(F(\Sigma))
\end{array}$$

$$\alpha_\Sigma(\sigma_\Sigma(\vec{\phi})) = \sigma'_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})).$$

We call this the **morphism property**.

Concerning algebraic systems, we will have occasion to make use of the following useful construction and properties.

Let again $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system and $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ and $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ be $N^\flat$-algebraic systems and $\langle F, \alpha \rangle : \mathbf{A} \to \mathbf{A}'$ an algebraic system morphism, with $F : \mathbf{Sign} \to \mathbf{Sign}'$ an isomorphism. We define the algebraic system $\alpha(\mathbf{A}) = \langle \mathbf{Sign}', \mathrm{SEN}'^\alpha, N'^\alpha \rangle$ as follows:

- For all $\Sigma \in |\mathbf{Sign}|$,

$$\mathrm{SEN}'^\alpha(F(\Sigma)) = \alpha_\Sigma(\mathrm{SEN}(\Sigma));$$

For all $\Sigma, \Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}'^{\alpha}(F(f)) : \mathrm{SEN}'^{\alpha}(F(\Sigma)) \to \mathrm{SEN}'^{\alpha}(F(\Sigma'))$$

is given by setting, for all $\phi \in \mathrm{SEN}'^{\alpha}(F(\Sigma))$,

$$\mathrm{SEN}'^{\alpha}(F(f))(\phi) = \mathrm{SEN}'(F(f))(\phi).$$

- For every $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, we let $\sigma'^{\alpha} : (\mathrm{SEN}'^{\alpha})^k \to \mathrm{SEN}'^{\alpha}$ be the restriction of $\sigma' : \mathrm{SEN}'^k \to \mathrm{SEN}'$ to SEN'^{α}.

  Composition works as expected, i.e., for all $\tau^\flat : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^\ell$ and all $\sigma^\flat : (\mathrm{SEN}^\flat)^\ell \to (\mathrm{SEN}^\flat)^m$ in $N^\flat$,

$$\sigma'^{\alpha} \circ \tau'^{\alpha} = (\sigma' \circ \tau')^{\alpha}.$$

It is not difficult to see that $\alpha(\mathbf{A})$, thus defined, is an $N^\flat$-algebraic system.

Lemma 13 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system, $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ and $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ be $N^\flat$-algebraic systems and $\langle F, \alpha \rangle : \mathbf{A} \to \mathbf{A}'$ an algebraic system morphism, with $F : \mathbf{Sign} \to \mathbf{Sign}'$ an isomorphism. Then $\alpha(\mathbf{A}) = \langle \mathbf{Sign}', \mathrm{SEN}'^{\alpha}, N'^{\alpha} \rangle$ is an $N^\flat$-algebraic system.*

Proof: The critical step is to show that $\mathrm{SEN}'^{\alpha} : \mathbf{Sign}' \to \mathbf{Set}$ is a well-defined functor and that N'^{α} consists in fact of natural transformations on SEN'^{α}.

For the first, let $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\phi \in \mathrm{SEN}(\Sigma)$. Then we have

$$\begin{array}{rcl}\mathrm{SEN}'^{\alpha}(F(f))(\alpha_\Sigma(\phi)) & = & \mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi)) \\ & = & \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \\ & \in & \mathrm{SEN}'^{\alpha}(F(\Sigma')).\end{array}$$

So SEN'^{α} is a well-defined functor.

Similarly, for $\sigma^\flat \in N^\flat$, $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\vec{\phi} \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{ccc}\mathrm{SEN}'^{\alpha}(F(\Sigma))^k & \xrightarrow{\sigma'^{\alpha}_{F(\Sigma)}} & \mathrm{SEN}'^{\alpha}(F(\Sigma)) \\ {\scriptstyle \mathrm{SEN}'^{\alpha}(F(f))^k}\downarrow & & \downarrow{\scriptstyle \mathrm{SEN}'^{\alpha}(F(f))} \\ \mathrm{SEN}'^{\alpha}(F(\Sigma'))^k & \xrightarrow[\sigma'^{\alpha}_{F(\Sigma')}]{} & \mathrm{SEN}'^{\alpha}(F(\Sigma'))\end{array}$$

$$\begin{array}{rcl}\mathrm{SEN}'^{\alpha}(F(f))(\sigma'^{\alpha}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi}))) & = & \mathrm{SEN}'^{\alpha}(F(f))(\sigma'_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi}))) \\ & = & \mathrm{SEN}'^{\alpha}(F(f))(\alpha_\Sigma(\sigma_\Sigma(\vec{\phi}))) \\ & = & \mathrm{SEN}'(F(f))(\alpha_\Sigma(\sigma_\Sigma(\vec{\phi}))) \\ & = & \alpha_{\Sigma'}(\mathrm{SEN}(f)(\sigma_\Sigma(\vec{\phi}))) \\ & = & \alpha_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\vec{\phi}))) \\ & = & \sigma'_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}(f)(\vec{\phi}))) \\ & = & \sigma'^{\alpha}_{F(\Sigma')}(\mathrm{SEN}'(F(f))(\alpha_\Sigma(\vec{\phi}))) \\ & = & \sigma'^{\alpha}_{F(\Sigma')}(\mathrm{SEN}'^{\alpha}(F(f))(\alpha_\Sigma(\vec{\phi}))).\end{array}$$

Thus, $\sigma'^{\alpha} : (\mathrm{SEN}'^{\alpha})^k \to (\mathrm{SEN}'^{\alpha})$ is a well-defined natural transformation on SEN'^{α}. ∎

We call $\alpha(\mathbf{A})$ the **image algebraic system** of $\mathbf{A}$ under $\langle F, \alpha\rangle$.

It is not difficult to see that, additionally, one may construct a surjective morphism from $\mathbf{A}$ to $\alpha(\mathbf{A})$. In fact, we define $\langle F, \alpha'\rangle : \mathbf{A} \to \alpha(\mathbf{A})$ by letting $\alpha' : \mathrm{SEN} \to \mathrm{SEN}'^{\alpha} \circ F$ be given, for all $\Sigma \in |\mathbf{Sign}|$, by

$$\alpha'_\Sigma(\phi) = \alpha_\Sigma(\phi), \text{ for all } \phi \in \mathrm{SEN}(\Sigma).$$

Lemma 14 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *and* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *be* $N^\flat$*-algebraic systems and* $\langle F, \alpha\rangle : \mathbf{A} \to \mathbf{A}'$ *an algebraic system morphism, with* $F : \mathbf{Sign} \to \mathbf{Sign}'$ *an isomorphism. Then* $\langle F, \alpha'\rangle : \mathbf{A} \to \alpha(\mathbf{A})$ *is a surjective algebraic system morphism.*

Proof: The fact that $\alpha' : \mathrm{SEN} \to \mathrm{SEN}'^{\alpha} \circ F$ is a natural transformation follows from the corresponding property of α. Moreover, the fact that $\langle F, \alpha'\rangle$ has the morphism property also follows from the corresponding property of $\langle F, \alpha\rangle$. Finally, surjectivity of $\alpha'_\Sigma : \mathrm{SEN}(\Sigma) \to \mathrm{SEN}'^{\alpha}(F(\Sigma))$, for all $\Sigma \in |\mathbf{Sign}|$, follows by the definition of SEN'^{α}. ∎

Let again $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system. An **F-algebraic system** (or an **interpreted algebraic system**) $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ consists of:

- An $N^\flat$-algebraic system $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$;
- A surjective algebraic system morphism $\langle F, \alpha\rangle : \mathbf{F} \to \mathbf{A}$.

We denote the class of all $\mathbf{F}$-algebraic systems by $\mathrm{AlgSys}(\mathbf{F})$.

Given two $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ and $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha'\rangle\rangle$, a **morphism** (of $\mathbf{F}$-algebraic systems) $\langle\langle G, \gamma\rangle, \langle H, \delta\rangle\rangle : \mathcal{A} \to \mathcal{A}'$ consists of:

- A morphism of $N^\flat$-algebraic systems $\langle G, \gamma\rangle : \mathbf{F} \to \mathbf{F}$;
- A morphism of $N^\flat$-algebraic systems $\langle H, \delta\rangle : \mathbf{A} \to \mathbf{A}'$

such that the following diagram commutes

$$\begin{array}{ccc} \mathbf{F} & \xrightarrow{\langle G, \gamma\rangle} & \mathbf{F} \\ {\scriptstyle \langle F, \alpha\rangle}\downarrow & & \downarrow{\scriptstyle \langle F', \alpha'\rangle} \\ \mathbf{A} & \xrightarrow[\langle H, \delta\rangle]{} & \mathbf{A}' \end{array}$$

We call a morphism $\langle\langle G, \gamma\rangle, \langle H, \delta\rangle\rangle : \mathcal{A} \to \mathcal{A}'$ **special** if $\langle G, \gamma\rangle : \mathbf{F} \to \mathbf{F}$ is special and we call it **surjective** if $\langle G, \gamma\rangle : \mathbf{F} \to \mathbf{F}$ is surjective.

We show that these properties propagate to $\langle H, \delta\rangle : \mathbf{A} \to \mathbf{A}'$.

Lemma 15 *Consider a base algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$. *Let* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and* $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$ *be* $\mathbf{F}$*-algebraic systems and* $\langle \langle G, \gamma \rangle, \langle H, \delta \rangle \rangle : \mathcal{A} \to \mathcal{A}'$ *a morphism.*

(a) *If* $\langle \langle G, \gamma \rangle, \langle H, \delta \rangle \rangle : \mathcal{A} \to \mathcal{A}'$ *is special, then* $\langle H, \delta \rangle : \mathbf{A} \to \mathbf{A}'$ *is special;*

(b) *If* $\langle \langle G, \gamma \rangle, \langle H, \delta \rangle \rangle : \mathcal{A} \to \mathcal{A}'$ *is surjective, then* $\langle H, \delta \rangle : \mathbf{A} \to \mathbf{A}'$ *is surjective.*

Proof:

(a) Suppose $\langle G, \gamma \rangle$ is special. We show, first, that H is surjective on objects and, then, that it is full. Surjectivity on objects is easy. Since F' and G are surjective on objects, $H \circ F = F' \circ G$ is also surjective on objects. This implies that H is surjective on objects.

For fullness, recall that it suffices to show that, for all $Y, Y' \in |\mathbf{Sign}|$,

$$H : \mathbf{Sign}(Y, Y') \to \mathbf{Sign}'(H(Y), H(Y'))$$

is surjective. So let $k \in \mathbf{Sign}'(H(Y), H(Y'))$. Then, by the surjectivity of F, there exist $X, X' \in |\mathbf{Sign}^\flat|$, such that $F(X) = Y$ and $F(X') = Y'$. Thus, we get

$$k \in \mathbf{Sign}'(H(F(X)), H(F(X'))) = \mathbf{Sign}'(F'(G(X)), F'(G(X'))).$$

Since G and F' are full, there exists $f \in \mathbf{Sign}^\flat(X, X')$, such that $F'(G(f)) = k$. So we have that $H(F(f)) = k$ and $F(f) \in \mathbf{Sign}(F(X), F(X')) = \mathbf{Sign}(Y, Y')$. Therefore H is full.

(b) By Part (a), it suffices to show that, for all $Y \in |\mathbf{Sign}|$, $\delta_Y : \mathrm{SEN}(Y) \to \mathrm{SEN}'(H(Y))$ is surjective. Let $\chi \in \mathrm{SEN}'(H(Y))$. Since F is surjective, there exists $X \in |\mathbf{Sign}^\flat|$, such that $F(X) = Y$. So we get $\chi \in \mathrm{SEN}'(H(F(X))) = \mathrm{SEN}'(F'(G(X)))$. Since both $\gamma_X : \mathrm{SEN}^\flat(X) \to \mathrm{SEN}^\flat(G(X))$ and $\alpha'_{G(X)} : \mathrm{SEN}^\flat(G(X)) \to \mathrm{SEN}'(F'(G(X)))$ are surjective, we get that $\alpha'_{G(X)} \circ \gamma_X : \mathrm{SEN}^\flat(X) \to \mathrm{SEN}'(F'(G(X)))$ is surjective. Thus, there exists $\phi \in \mathrm{SEN}^\flat(X)$, such that

$$\chi = \alpha'_{G(X)}(\gamma_X(\phi)) = \delta_{F(X)}(\alpha_X(\phi)) = \delta_Y(\alpha_X(\phi)).$$

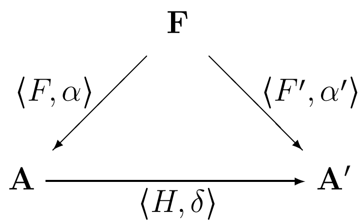
So $\delta_Y : \mathrm{SEN}(Y) \to \mathrm{SEN}'(H(Y))$ is also surjective.

■

In the future, we will restrict attention mostly to $\mathbf{F}$-algebraic system morphisms $\langle \langle G, \gamma \rangle, \langle H, \delta \rangle \rangle : \mathcal{A} \to \mathcal{A}'$, with

$$\langle G, \gamma \rangle = \langle I, \iota \rangle : \mathbf{F} \to \mathbf{F},$$

where $\langle I, \iota\rangle : \mathbf{F} \to \mathbf{F}$ denotes the identity morphism on $\mathbf{F}$. Since this morphism is surjective, by Lemma 15, this property will automatically hold for $\langle H, \delta\rangle : \mathbf{A} \to \mathbf{A}'$ as well. In this case, we also use the simplified notation $\langle H, \delta\rangle : \mathcal{A} \to \mathcal{A}'$



and even though we might say a "surjective" morphism $\langle H, \delta\rangle : \mathcal{A} \to \mathcal{A}'$ for emphasis, it is understood that this will always be the case, even without this specification.

2.3 Congruence Systems

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system. A **relation family on A** is a relation family on SEN, i.e., a collection $R = \{R_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, such that $R_\Sigma \subseteq \mathrm{SEN}(\Sigma)^2$, for all $\Sigma \in |\mathbf{Sign}|$. A relation family on $\mathbf{A}$ is a **relation system** if it is a relation system on SEN, i.e., if it is invariant under **Sign**-morphisms; that is, if for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}(f)(R_\Sigma) \subseteq R_{\Sigma'}.$$

A relation family/system on $\mathbf{A}$ is an **equivalence family/system on A** if it is an equivalence family/system on SEN, i.e., for all $\Sigma \in |\mathbf{Sign}|$, R_Σ is an equivalence relation on $\mathrm{SEN}(\Sigma)$. Finally, an equivalence system is called a **congruence system on A** if, for all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi_i, \psi_i\rangle \in R_\Sigma, i < k, \text{ implies } \langle \sigma_\Sigma(\vec{\phi}), \sigma_\Sigma(\vec{\psi})\rangle \in R_\Sigma.$$

We call this the **congruence property**.

The collection of all congruence systems on the algebraic system $\mathbf{A}$ will be denoted by $\mathrm{ConSys}(\mathbf{A})$. Ordered under signature-wise inclusion $\leq$, it forms a complete lattice, which is denoted by $\mathbf{ConSys}(\mathbf{A}) = \langle \mathrm{ConSys}(\mathbf{A}), \leq\rangle$.

The least congruence system on $\mathbf{A}$ is the **identity congruence system**, which denoted by $\Delta^{\mathbf{A}} = \{\Delta^{\mathbf{A}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, where, for all $\Sigma \in |\mathbf{Sign}|$,

$$\Delta^{\mathbf{A}}_\Sigma = \{\langle \phi, \phi\rangle : \phi \in \mathrm{SEN}(\Sigma)\}.$$

The largest congruence system is the **nabla congruence system**, denoted $\nabla^{\mathbf{A}}$ or SEN^2, and defined by $\nabla^{\mathbf{A}} = \{\nabla^{\mathbf{A}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, such that, for all $\Sigma \in |\mathbf{Sign}|$,

$$\nabla^{\mathbf{A}}_\Sigma = \{\langle \phi, \psi\rangle : \phi, \psi \in \mathrm{SEN}(\Sigma)\} = \mathrm{SEN}(\Sigma)^2.$$

The infimum of a family $\{\theta^i : i \in I\} \subseteq \mathrm{ConSys}(\mathbf{A})$ is given by signature-wise intersection $\bigcap_{i\in I}\theta^i$, while the supremum is the congruence system generated by the signature-wise union of the θ^i, $\bigvee_{i\in I}\theta^i = \{\Theta(\bigcup_{i\in I}\theta^i_\Sigma)\}_{\Sigma\in|\mathbf{Sign}|}$, where $\Theta(\bigcup_{i\in I}\theta^i_\Sigma)$ denotes the congruence on $\mathrm{SEN}(\Sigma)$ (viewed as an ordinary algebra with operations $\sigma_\Sigma : \mathrm{SEN}(\Sigma)^k \to \mathrm{SEN}(\Sigma)$, for $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N) generated by $\bigcup_{i\in I}\theta^i$.

Proposition 16 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *two* $N^\flat$*-algebraic systems and* $\langle F,\alpha\rangle : \mathbf{A} \to \mathbf{A}'$ *a morphism of* $N^\flat$*-algebraic systems. If* $\theta \in \mathrm{ConSys}(\mathbf{A}')$*, then* $\alpha^{-1}(\theta) \in \mathrm{ConSys}(\mathbf{A})$*.*

Proof: By Proposition 10 it suffices to show that, if θ has the congruence property, then $\alpha^{-1}(\theta)$ also has the congruence property. To see this, consider $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi},\vec{\psi} \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi_i,\psi_i\rangle \in \alpha_\Sigma^{-1}(\theta_{F(\Sigma)})$, for all $i < k$. Then we get $\langle \alpha_\Sigma(\phi_i), \alpha_\Sigma(\psi_i)\rangle \in \theta_{F(\Sigma)}$, for all $i < k$. Thus, by the congruence property of θ,

$$\langle \sigma'_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})), \sigma'_{F(\Sigma)}(\alpha_\Sigma(\vec{\psi}))\rangle \in \theta_{F(\Sigma)}.$$

By the morphism property, we get

$$\langle \alpha_\Sigma(\sigma_\Sigma(\vec{\phi})), \alpha_\Sigma(\sigma_\Sigma(\vec{\psi}))\rangle \in \theta_{F(\Sigma)}.$$

Hence $\langle \sigma_\Sigma(\vec{\phi}), \sigma_\Sigma(\vec{\psi})\rangle \in \alpha_\Sigma^{-1}(\theta_{F(\Sigma)})$, showing that $\alpha^{-1}(\theta)$ also satisfies the congruence property. ■

As a special case of Proposition 16, we obtain that kernels of morphisms are congruence systems.

Corollary 17 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *two* $N^\flat$*-algebraic systems and* $\langle F,\alpha\rangle : \mathbf{A} \to \mathbf{A}'$ *a morphism of* $N^\flat$*-algebraic systems. Then* $\mathrm{Ker}(\langle F,\alpha\rangle) \in \mathrm{ConSys}(\mathbf{A})$*.*

Proof: This follows by Proposition 16 by taking $\theta = \Delta^{\mathcal{A}'}$. Then, obviously, $\alpha^{-1}(\theta) = \mathrm{Ker}(\langle F,\alpha\rangle)$. ■

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system and $\theta \in \mathrm{ConSys}(\mathbf{A})$. The **quotient** $\mathbf{A}^\theta$ (or $\mathbf{A}/\theta$) **of A by** θ is the algebraic system $\mathbf{A}^\theta = \langle \mathbf{Sign}, \mathrm{SEN}^\theta, N^\theta\rangle$, defined as follows:

- For all $\Sigma \in |\mathbf{Sign}|$,

$$\mathrm{SEN}^\theta(\Sigma) = \mathrm{SEN}(\Sigma)/\theta_\Sigma = \{\phi/\theta_\Sigma : \phi \in \mathrm{SEN}(\Sigma)\}.$$

  For all $\Sigma,\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma,\Sigma')$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\mathrm{SEN}^\theta(f)(\phi/\theta_\Sigma) = \mathrm{SEN}(f)(\phi)/\theta_{\Sigma'}.$$

- N^θ is the category of natural transformations on SEN^θ of the form $\sigma^\theta : (\mathrm{SEN}^\theta)^k \to \mathrm{SEN}^\theta$, where $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ is in N, defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)$, by
$$\sigma^\theta_\Sigma(\vec{\phi}/\theta_\Sigma) = \sigma_\Sigma(\vec{\phi})/\theta_\Sigma.$$

The fact that θ is an equivalence system makes the functor SEN^θ well-defined at both the object and the morphism level. Moreover, the fact that θ is a congruence system makes the definition of each natural transformation in N^θ sound. Finally, the identities, projections and the composition in N^θ are the images of the corresponding operations and of the composition in N under $\cdot \mapsto \cdot^\theta$: For all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $\vec{\phi} \in \mathrm{SEN}(\Sigma)$,

- $i^\theta_\Sigma(\phi/\theta_\Sigma) = \phi/\theta_\Sigma = i_\Sigma(\phi)/\theta_\Sigma$;
- $p^{k,i^\theta}_\Sigma(\vec{\phi}/\theta_\Sigma) = \phi_i/\theta_\Sigma = p^{k,i}_\Sigma(\vec{\phi})/\theta_\Sigma$;
- $\tau^\theta_\Sigma(\sigma^{0^\theta}_\Sigma(\vec{\phi}/\theta_\Sigma), \dots, \sigma^{k-1^\theta}_\Sigma(\vec{\phi}/\theta_\Sigma)) = \tau^\theta_\Sigma(\sigma^0_\Sigma(\vec{\phi})/\theta_\Sigma, \dots, \sigma^{k-1}_\Sigma(\vec{\phi})/\theta_\Sigma)$
$= \tau_\Sigma(\sigma^0_\Sigma(\vec{\phi}), \dots, \sigma^{k-1}_\Sigma(\vec{\phi}))/\theta_\Sigma$.

We denote by $\langle I, \pi^\theta\rangle : \mathbf{A} \to \mathbf{A}^\theta$ the **quotient morphism**, defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, by
$$\pi^\theta_\Sigma(\phi) = \phi/\theta_\Sigma.$$
To see that it is well-defined, we must show that $\pi^\theta : \mathrm{SEN} \to \mathrm{SEN}^\theta$ is a natural transformation and that it satisfies the morphism property. In fact, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{ccc}
\mathrm{SEN}(\Sigma) & \xrightarrow{\pi^\theta_\Sigma} & \mathrm{SEN}^\theta(\Sigma) \\
{\scriptstyle \mathrm{SEN}(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{SEN}^\theta(f)} \\
\mathrm{SEN}(\Sigma') & \xrightarrow[\pi^\theta_{\Sigma'}]{} & \mathrm{SEN}^\theta(\Sigma')
\end{array}$$

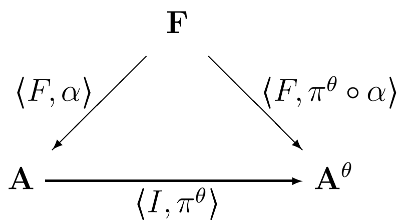
$$\begin{array}{rcl}
\pi^\theta_{\Sigma'}(\mathrm{SEN}(f)(\phi)) & = & \mathrm{SEN}(f)(\phi)/\theta_{\Sigma'} \\
& = & \mathrm{SEN}^\theta(f)(\phi/\theta_\Sigma) \\
& = & \mathrm{SEN}^\theta(f)(\pi^\theta_\Sigma(\phi)).
\end{array}$$

And for all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{ccc}
\mathrm{SEN}(\Sigma)^k & \xrightarrow{\pi^{\theta^k}_\Sigma} & \mathrm{SEN}^\theta(\Sigma)^k \\
{\scriptstyle \sigma_\Sigma}\downarrow & & \downarrow{\scriptstyle \sigma^\theta_\Sigma} \\
\mathrm{SEN}(\Sigma) & \xrightarrow[\pi^\theta_\Sigma]{} & \mathrm{SEN}^\theta(\Sigma)
\end{array}$$

$$\pi^\theta_\Sigma(\sigma_\Sigma(\vec{\phi})) = \sigma_\Sigma(\vec{\phi})/\theta_\Sigma = \sigma^\theta_\Sigma(\vec{\phi}/\theta_\Sigma) = \sigma^\theta_\Sigma(\pi^\theta_\Sigma(\vec{\phi})).$$

Note that this construction allows us to discuss also quotients of **F**-algebraic systems. More precisely, consider a base algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$. Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an **F**-algebraic system and let $\theta \in \mathrm{ConSys}(\mathcal{A}) \coloneqq \mathrm{ConSys}(\mathbf{A})$. The **quotient F-algebraic system of $\mathcal{A}$ by θ** is defined as $\mathcal{A}^\theta = \langle \mathbf{A}^\theta, \langle F, \pi^\theta \circ \alpha\rangle\rangle$.



Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system and let $T \in \mathrm{SenFam}(\mathbf{A})$. We say that a congruence system θ on $\mathbf{A}$ is **compatible with** T if it is compatible with T as an equivalence system on SEN, i.e., if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi, \psi\rangle \in \theta_\Sigma \quad \text{and} \quad \phi \in T_\Sigma \quad \text{imply} \quad \psi \in T_\Sigma.$$

Note that, for every $T \in \mathrm{SenFam}(\mathbf{A})$, $\Delta^{\mathbf{A}}$ is compatible with T. We denote the collection of all congruence systems on $\mathbf{A}$ that are compatible with T by $\mathrm{ConSys}^{\mathbf{A}}(T)$.

Proposition 18 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system and* $T \in \mathrm{SenFam}(\mathbf{A})$. *The collection* $\mathrm{ConSys}^{\mathbf{A}}(T)$, *of all congruence systems on* $\mathbf{A}$ *that are compatible with* T, *forms a complete lattice*

$$\mathbf{ConSys}^{\mathbf{A}}(T) = \langle \mathrm{ConSys}^{\mathbf{A}}(T), \leq\rangle$$

under signature-wise inclusion.

Proof: First, the collection $\mathrm{ConSys}^{\mathbf{A}}(T)$ is closed under arbitrary intersections: Let θ^i, $i \in I$, be in $\mathrm{ConSys}^{\mathbf{A}}(T)$. Suppose that $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi\rangle \in \bigcap_{i\in I} \theta^i_\Sigma$ and $\phi \in T_\Sigma$. Then $\langle \phi, \psi\rangle \in \theta^i_\Sigma$, for all $i \in I$. Since θ^i is compatible with T, we get $\psi \in T_\Sigma$. This shows that $\bigcap_{i\in I} \theta^i$ is compatible with T.

It suffices, therefore, to show that $\mathrm{ConSys}^{\mathbf{A}}(T)$ has a greatest element. The signature-wise union of every directed subset of $\mathrm{ConSys}^{\mathbf{A}}(T)$ is an upper bound for the subset in $\mathrm{ConSys}(\mathbf{A})$. Moreover, it is in $\mathrm{ConSys}^{\mathbf{A}}(T)$ since every member of the subset is. So, by Zorn's Lemma, $\mathrm{ConSys}^{\mathbf{A}}(T)$ has a maximal element.

Suppose, for the sake of obtaining a contradiction, that $\theta \neq \theta'$ are two such maximal elements. Recall that their join $\theta \vee \theta'$ is given by $\theta \vee \theta' =$

$\{\theta_\Sigma \vee \theta'_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$, where

$$\theta_\Sigma \vee \theta'_\Sigma = \bigcup_{k=0}^{\infty} \underbrace{\theta_\Sigma \circ \theta'_\Sigma \circ \cdots \circ \theta_\Sigma}_{k \text{ factors}}.$$

Thus, their join $\theta \vee \theta'$ as congruence systems on $\mathbf{A}$ is also compatible with T. This, however, contradicts the maximality of θ and θ', since, clearly, $\theta < \theta \vee \theta'$ and $\theta' < \theta \vee \theta'$. Therefore, the unique maximal element of $\mathrm{ConSys}^{\mathbf{A}}(T)$ is a largest element. ∎

The largest congruence system on an algebraic system $\mathbf{A}$ compatible with $T \in \mathrm{SenFam}(\mathbf{A})$ is called the **Leibniz congruence system** of T on $\mathbf{A}$ and is denoted by $\Omega^{\mathbf{A}}(T)$.

The following theorem provides an explicit characterization of the Leibniz congruence system of a sentence family T on an algebraic system $\mathbf{A}$.

Theorem 19 *Suppose that* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *is an algebraic system and* $T \in \mathrm{SenFam}(\mathbf{A})$. *Then, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll}\langle\phi,\psi\rangle \in \Omega^{\mathbf{A}}_\Sigma(T) & \textit{iff} & \textit{for all } \sigma : \mathrm{SEN}^k \to \mathrm{SEN} \textit{ in } N, \textit{ all } \Sigma' \in |\mathbf{Sign}|, \\ & & \textit{all } f \in \mathbf{Sign}(\Sigma,\Sigma') \textit{ and all } \vec{\chi} \in \mathrm{SEN}(\Sigma'), \textit{ we have} \\ & & \sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}) \in T_{\Sigma'} \textit{ iff } \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi}) \in T_{\Sigma'}.\end{array}$$

Proof: Let $R = \{R_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ be the relation system on $\mathbf{A}$ defined by the given condition, i.e., for all $\Sigma \in |\mathbf{Sign}|$,

$$\begin{array}{ll}R_\Sigma = \{\langle\phi,\psi\rangle \in \mathrm{SEN}(\Sigma)^2 : & \\ & \text{for all } \sigma : \mathrm{SEN}^k \to \mathrm{SEN} \text{ in } N, \text{ all } \Sigma' \in |\mathbf{Sign}|, \\ & \text{all } f \in \mathbf{Sign}(\Sigma,\Sigma') \text{ and all } \vec{\chi} \in \mathrm{SEN}(\Sigma'), \\ & \sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}) \in T_{\Sigma'} \text{ iff } \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi}) \in T_{\Sigma'}\}.\end{array}$$

First, we show that $\Omega^{\mathbf{A}}(T) \le R$. Let $\Sigma \in |\mathbf{Sign}|$, $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle \in \Omega^{\mathbf{A}}_\Sigma(T)$. Since $\Omega^{\mathbf{A}}(T)$ is a congruence system, we get that, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma,\Sigma')$, $\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)\rangle \in \Omega^{\mathbf{A}}_{\Sigma'}(T)$. Now, since $\Omega^{\mathbf{A}}(T)$ is a congruence system, we get that, for all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

$$\langle \sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}), \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi})\rangle \in \Omega^{\mathbf{A}}_{\Sigma'}(T).$$

Finally, since $\Omega^{\mathbf{A}}(T)$ is compatible with T, we get that

$$\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}) \in T_{\Sigma'} \quad \text{iff} \quad \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi}) \in T_{\Sigma'}.$$

But the last condition, being universally quantified on $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma,\Sigma')$, σ in N and $\vec{\chi} \in \mathrm{SEN}(\Sigma')$, yields $\langle\phi,\psi\rangle \in R_\Sigma$. Therefore, we get that $\Omega^{\mathbf{A}}(T) \le R$.

Finally, we show that $R \leq \Omega^{\mathbf{A}}(T)$. For this inclusion, it suffices to show that R is a congruence system on $\mathbf{A}$ that is compatible with T. Then the conclusion would follow from the fact that $\Omega^{\mathbf{A}}(T)$ is, by definition, the largest congruence system on $\mathbf{A}$ that is compatible with T.

It is clear from its definition that R is an equivalence family on $\mathbf{A}$.

To see that it is an equivalence system, let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in R_\Sigma$. Consider $\Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma, \Sigma')$. Then, for all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, all $\Sigma'' \in |\mathbf{Sign}|$, all $g \in \mathbf{Sign}(\Sigma', \Sigma'')$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma'')$,

$$\Sigma \xrightarrow{f} \Sigma' \xrightarrow{g} \Sigma''$$

we have

$$\begin{array}{ll} \sigma_{\Sigma''}(\mathrm{SEN}(g)(\mathrm{SEN}(f)(\phi)), \vec{\chi}) \in T_{\Sigma''} & \\ \quad \text{iff} \quad \sigma_{\Sigma''}(\mathrm{SEN}(gf)(\phi), \vec{\chi}) \in T_{\Sigma''} & \\ \quad \text{iff} \quad \sigma_{\Sigma''}(\mathrm{SEN}(gf)(\psi), \vec{\chi}) \in T_{\Sigma''} & (\text{since } \langle \phi, \psi \rangle \in R_\Sigma) \\ \quad \text{iff} \quad \sigma_{\Sigma''}(\mathrm{SEN}(g)(\mathrm{SEN}(f)(\psi)), \vec{\chi}) \in T_{\Sigma''}. & \end{array}$$

Thus, we conclude that $\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi) \rangle \in R_{\Sigma'}$, showing that R is an equivalence system.

Next, to see that R is a congruence system, consider $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, $\Sigma \in |\mathbf{Sign}|$, and $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi_i, \psi_i \rangle \in R_\Sigma$, $i < k$. Let $\tau : \mathrm{SEN}^\ell \to \mathrm{SEN}$ be in N, $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\vec{\chi} \in \mathrm{SEN}(\Sigma')$. Then, we have

$$\begin{array}{ll} \tau_{\Sigma'}(\mathrm{SEN}(f)(\sigma_\Sigma(\vec{\phi})), \vec{\chi}) \in T_{\Sigma'} \\ \text{iff} \quad \tau_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\vec{\phi})), \vec{\chi}) \in T_{\Sigma'} \\ \text{iff} \quad \tau_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\vec{\psi})), \vec{\chi}) \in T_{\Sigma'} \\ \qquad (\tau \circ \langle \sigma \circ \langle p^{k+\ell-1,0}, \ldots, p^{k+\ell-1,k} \rangle, p^{k+\ell-1,k+1}, \ldots, p^{k+\ell-1,k+\ell-2} \rangle \text{ in } N \\ \qquad \text{together with Corollary 12, applied } k \text{ times}) \\ \text{iff} \quad \tau_{\Sigma'}(\mathrm{SEN}(f)(\sigma_\Sigma(\vec{\psi})), \vec{\chi}) \in T_{\Sigma'}. \end{array}$$

This shows that $\langle \sigma_\Sigma(\vec{\phi}), \sigma_\Sigma(\vec{\psi}) \rangle \in R_\Sigma$, whence R is a congruence system.

Finally, upon setting in the defining condition $\sigma = p^{1,0} : \mathrm{SEN} \to \mathrm{SEN}$ in N, $\Sigma' = \Sigma$, $f = i_\Sigma$, the identity **Sign**-morphism, we get that for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, with $\langle \phi, \psi \rangle \in R_\Sigma$

$$\phi \in T_\Sigma \quad \text{iff} \quad \psi \in T_\Sigma.$$

Thus, R is compatible with T. ■

The characterization of the Leibniz congruence system, presented in Theorem 19, provides a justification for an alternative name that is sometimes attributed to the Leibniz congruence system of a sentence family T on an algebraic system $\mathbf{A}$. Given $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, we say that ϕ and ψ are **indiscernible modulo** T if

$$\langle \phi, \psi \rangle \in \Omega^{\mathbf{A}}_{\Sigma}(T).$$

Therefore $\Omega^{\mathbf{A}}(T)$ is also referred to as the **indiscernibility relation on A modulo** T.

We can now prove a proposition asserting that the Leibniz congruence system of a sentence family T is included in that of the sentence system $\overleftarrow{T}$.

Proposition 20 *Suppose that* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *is an algebraic system and* $T \in \mathrm{SenFam}(\mathbf{A})$*. Then*

$$\Omega^{\mathbf{A}}(T) \leq \Omega^{\mathbf{A}}(\overleftarrow{T}).$$

Proof: To prove this inclusion, it suffices to show that $\Omega^{\mathbf{A}}(T)$ is compatible with $\overleftarrow{T}$. We can invoke Lemma 7, but we also give a direct proof due to the heavy significance of this result. Let $\Sigma \in |\mathbf{Sign}|$, and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \Omega^{\mathbf{A}}_{\Sigma}(T)$ and $\phi \in \overleftarrow{T}_{\Sigma}$. Since $\Omega^{\mathbf{A}}(T)$ is a congruence system and by the definition of $\overleftarrow{T}$, we get that, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi) \rangle \in \Omega^{\mathbf{A}}_{\Sigma'}(T) \quad \text{and} \quad \mathrm{SEN}(f)(\phi) \in T_{\Sigma'}.$$

Thus, by the compatibility of $\Omega^{\mathbf{A}}(T)$ with T, we obtain $\mathrm{SEN}(f)(\psi) \in T_{\Sigma'}$. Since this holds for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, we get $\psi \in \overleftarrow{T}_{\Sigma}$. Thus, $\Omega^{\mathbf{A}}(T)$ is compatible with $\overleftarrow{T}$, showing that $\Omega^{\mathbf{A}}(T) \leq \Omega^{\mathbf{A}}(\overleftarrow{T})$. ■

We exhibit, next, an algebraic system $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ together with a sentence family $T \in \mathrm{SenFam}(\mathbf{A})$, such that $\Omega^{\mathbf{A}}(T) \lneq \Omega^{\mathbf{A}}(\overleftarrow{T})$.

Example 21 *Define* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *as follows:*

- $\mathbf{Sign}$ *is a category with two objects* Σ, Σ' *and a single (non-identity) morphism* $f : \Sigma \to \Sigma'$.
- $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *is defined by setting* $\mathrm{SEN}(\Sigma) = \{0, 1\}$, $\mathrm{SEN}(\Sigma') = \{a, b\}$, $\mathrm{SEN}(f)(0) = a$ *and* $\mathrm{SEN}(f)(1) = b$.
- *The clone* N *of natural transformations is trivial, i.e., consists of the projection natural transformations only.*

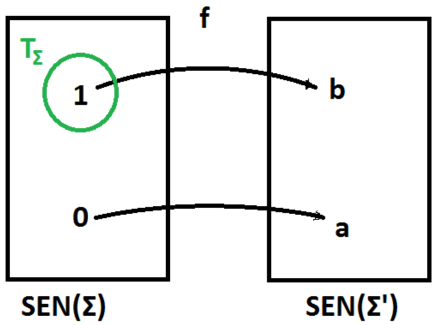
Finally, let $T = \{T_{\Sigma}, T_{\Sigma'}\}$ *be specified by setting* $T_{\Sigma} = \{1\}$ *and* $T_{\Sigma'} = \varnothing$*. Then it is not difficult to see that* $\overleftarrow{T}_{\Sigma} = \varnothing = \overleftarrow{T}_{\Sigma'}$ *and, therefore, that*

$$\Omega^{\mathbf{A}}_{\Sigma}(\overleftarrow{T}) = \nabla^{\mathbf{A}}_{\Sigma} \quad \text{and} \quad \Omega^{\mathbf{A}}_{\Sigma'}(\overleftarrow{T}) = \nabla^{\mathbf{A}}_{\Sigma'},$$

whereas

$$\Omega^{\mathbf{A}}_{\Sigma}(T) = \Delta^{\mathbf{A}}_{\Sigma} \quad \text{and} \quad \Omega^{\mathbf{A}}_{\Sigma'}(T) = \nabla^{\mathbf{A}}_{\Sigma'}.$$

Hence, we have $\Omega^{\mathbf{A}}(T) \lneq \Omega^{\mathbf{A}}(\overleftarrow{T})$.



Proposition 20 and Example 21 have important consequences. We give a brief account here, as is proper after proving these facts, but postpone further treatment for subsequent chapters.

1. Note that, given an algebraic system $\mathbf{A}$, for any sentence family T of $\mathbf{A}$, both $T, \overleftarrow{T}$ are sentence families of $\mathbf{A}$, such that, in general,

$$\overleftarrow{T} \le T \quad \text{and} \quad \Omega^{\mathbf{A}}(T) \le \Omega^{\mathbf{A}}(\overleftarrow{T}).$$

   But it is an accepted wisdom in abstract algebraic logic that a logic is amenable to a meaningful algebraic treatment and, thus, deserves a place in the algebraic (Leibniz) hierarchy, if it is at least *protoalgebraic* or *truth-equational*, meaning that the Leibniz operator on its collection of theories is at least monotone of completely order reflecting. The displayed relations between T and $\overleftarrow{T}$, therefore, force us to define a new class of π-institutional logics, fulfilling a minimum, in some sense, condition for amenability to algebraic treatment and techniques, which we shall call **stable**, if their Leibniz operator satisfies, for all theory families T of the π-institution,

$$\Omega(T) = \Omega(\overleftarrow{T}).$$

   The term "stable" is adopted to insinuate contrast to *inverting* or *changing* the order, since, given that $\overleftarrow{T} \le T$ and that $\Omega(T) \le \Omega(\overleftarrow{T})$, for all theory families T, an inversion in the order would occur in case $\Omega(T) \neq \Omega(\overleftarrow{T})$ for some theory family T.

2. Now note the remarkable fact that, for a stable π-institution, the range of the Leibniz operator is entirely covered by its values on theory systems of the π-institution, since, given a theory family T, one can work with its Leibniz congruence system by working with the congruence system $\Omega(\overleftarrow{T})$ of the theory system $\overleftarrow{T}$.

These two remarkable facts form an enticement, a preview and a justification for some of the upcoming definitions and concepts regarding classes of π-institutions, forming the semantic Leibniz hierarchy, in subsequent chapters.

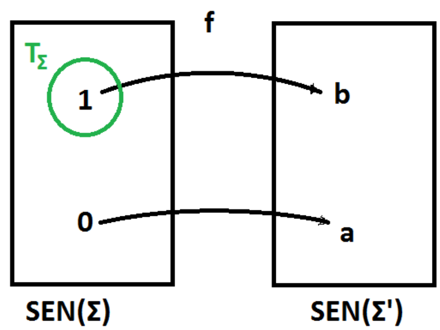
In the next example it is shown that the Leibniz congruence system of a sentence family T of an algebraic system $\mathbf{A}$ does not stand in a definite relationship with that of the sentence system $\overrightarrow{T}$.

Example 22 *We exhibit, first, an algebraic system* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *together with a sentence family* $T \in \mathrm{SenFam}(\mathbf{A})$, *such that* $\Omega^{\mathbf{A}}(\overrightarrow{T}) \subsetneq \Omega^{\mathbf{A}}(T)$.

We use the same algebraic system and the same sentence family as in Example 21. Define $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *as follows:*

- $\mathbf{Sign}$ *is a category with two objects* Σ, Σ' *and a single (non-identity) morphism* $f : \Sigma \to \Sigma'$.
- $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *is defined by setting* $\mathrm{SEN}(\Sigma) = \{0,1\}$, $\mathrm{SEN}(\Sigma') = \{a, b\}$, $\mathrm{SEN}(f)(0) = a$ *and* $\mathrm{SEN}(f)(1) = b$.
- *The clone* N *of natural transformations is trivial, i.e., consists of the projection natural transformations only.*

Finally, let $T = \{T_\Sigma, T_{\Sigma'}\}$ *be specified by setting* $T_\Sigma = \{1\}$ *and* $T_{\Sigma'} = \varnothing$.



Note that $\overrightarrow{T}_\Sigma = \{1\}$ *and* $\overrightarrow{T}_{\Sigma'} = \{b\}$. *So in this case we have*

$$\Omega^{\mathbf{A}}_{\Sigma}(\overrightarrow{T}) = \Delta^{\mathbf{A}}_{\Sigma} \quad \textit{and} \quad \Omega^{\mathbf{A}}_{\Sigma'}(\overrightarrow{T}) = \Delta^{\mathbf{A}}_{\Sigma'},$$

whereas, as pointed out in Example 21,

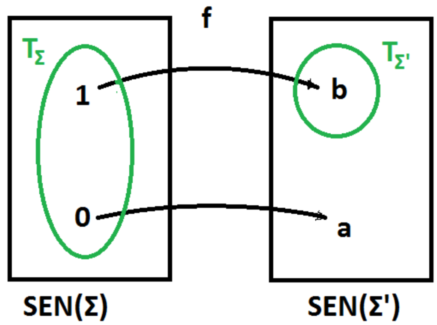
$$\Omega^{\mathbf{A}}_{\Sigma}(T) = \Delta^{\mathbf{A}}_{\Sigma} \quad \textit{and} \quad \Omega^{\mathbf{A}}_{\Sigma'}(T) = \nabla^{\mathbf{A}}_{\Sigma'}.$$

So we see that $\Omega^{\mathbf{A}}(\overrightarrow{T}) \subsetneq \Omega^{\mathbf{A}}(T)$.

Finally, we construct an algebraic system $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *together with a sentence family* $T \in \mathrm{SenFam}(T)$, *such that* $\Omega^{\mathbf{A}}(T) \subsetneq \Omega^{\mathbf{A}}(\overrightarrow{T})$.

The algebraic system is the same algebraic system $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, *defined above. But now the sentence family* $T = \{T_\Sigma, T_{\Sigma'}\}$ *is defined by*

$$T_\Sigma = \{0,1\} \quad \textit{and} \quad T_{\Sigma'} = \{b\}.$$



It is clear that $\overrightarrow{T}_\Sigma = \{0,1\}$ *and* $\overrightarrow{T}_{\Sigma'} = \{a,b\}$. *Thus, we have*

$$\Omega^{\mathbf{A}}(T) = \Delta^{\mathbf{A}}_\Sigma \quad \textit{and} \quad \Omega^{\mathbf{A}}_{\Sigma'}(T) = \Delta^{\mathbf{A}}_{\Sigma'},$$

whereas

$$\Omega^{\mathbf{A}}_\Sigma(\overrightarrow{T}) = \nabla^{\mathbf{A}}_\Sigma \quad \textit{and} \quad \Omega^{\mathbf{A}}_{\Sigma'}(\overrightarrow{T}) = \nabla^{\mathbf{A}}_{\Sigma'}.$$

Thus we see that, in this case, $\Omega^{\mathbf{A}}(T) \lneq \Omega^{\mathbf{A}}(\overrightarrow{T})$.

It turns out that the Leibniz congruence system of the intersection of two sentence families of an algebraic system is at least as large as the intersection of the corresponding Leibniz congruence systems.

Lemma 23 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system and let* $\mathcal{T} \subseteq \mathrm{SenFam}(\mathbf{A})$. *Then*

$$\bigcap_{T\in\mathcal{T}} \Omega^{\mathbf{A}}(T) \le \Omega^{\mathbf{A}}(\bigcap_{T\in\mathcal{T}} T).$$

Proof: The Leibniz congruence system of $\bigcap_{T\in\mathcal{T}} T$ is, by definition, the largest congruence system on $\mathbf{A}$ that is compatible with $\bigcap_{T\in\mathcal{T}} T \in \mathrm{SenFam}(\mathbf{A})$. So to prove the conclusion it suffices to show that $\bigcap_{T\in\mathcal{T}} \Omega^{\mathbf{A}}(T)$ is a congruence system on $\mathbf{A}$ that is compatible with $\bigcap_{T\in\mathcal{T}} T$. That it is a congruence system follows from the fact that $\mathbf{ConSys}(\mathbf{A})$ has the structure of a complete lattice with signature-wise intersection as its infimum. For compatibility, Let $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi\rangle \in \bigcap_{T\in\mathcal{T}} \Omega^{\mathbf{A}}_\Sigma(T)$ and $\phi \in \bigcap_{T\in\mathcal{T}} T_\Sigma$. These two imply the following relations:

$$\langle \phi, \psi\rangle \in \Omega^{\mathbf{A}}_\Sigma(T), \quad \phi \in T_\Sigma, \quad \text{for all } T \in \mathcal{T}.$$

Now, using the compatibility property of $\Omega^{\mathbf{A}}(T)$, $T \in \mathcal{T}$, we get $\psi \in T_\Sigma$, for all $T \in \mathcal{T}$. So $\psi \in \bigcap_{T\in\mathcal{T}} T_\Sigma$ and, therefore, $\bigcap_{T\in\mathcal{T}} \Omega^{\mathbf{A}}(T)$ is compatible with $\bigcap_{T\in\mathcal{T}} T$. ■

An important property of the Leibniz operator is that it commutes with inverse surjective morphisms of $N^\flat$-algebraic systems.

Proposition 24 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ *two* $N^\flat$*-algebraic systems,* $\langle F, \alpha \rangle : \mathbf{A} \to \mathbf{A}'$ *an algebraic system morphism and* $T \in \mathrm{SenFam}(\mathbf{A}')$*. We have:*

(a) $\alpha^{-1}(\Omega^{\mathbf{A}'}(T)) \le \Omega^{\mathbf{A}}(\alpha^{-1}(T))$*;*

(b) If $\langle F, \alpha \rangle$ *is surjective,* $\alpha^{-1}(\Omega^{\mathbf{A}'}(T)) = \Omega^{\mathbf{A}}(\alpha^{-1}(T))$*.*

Proof:

(a) Since $\Omega^{\mathbf{A}}(\alpha^{-1}(T))$ is the largest congruence system that is compatible with $\alpha^{-1}(T)$, it suffices to show that $\alpha^{-1}(\Omega^{\mathbf{A}'}(T))$ is a congruence system on $\mathbf{A}$ that is compatible with $\alpha^{-1}(T)$. The fact that it is a congruence system on $\mathbf{A}$ is guaranteed by Proposition 16. So it suffices to show its compatibility with $\alpha^{-1}(T)$. Let $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \alpha_\Sigma^{-1}(\Omega^{\mathbf{A}'}_{F(\Sigma)}(T))$ and $\phi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})$. Now we get $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \Omega^{\mathbf{A}'}_{F(\Sigma)}(T)$ and $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$. By compatibility of $\Omega^{\mathbf{A}'}(T)$ with T, we get $\alpha_\Sigma(\psi) \in T_{F(\Sigma)}$. Therefore $\psi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})$, which proves compatibility of $\alpha^{-1}(\Omega^{\mathbf{A}'}(T))$ with $\alpha^{-1}(T)$.

(b) By Part (a), it suffices to prove, under the hypothesis that $\langle F, \alpha \rangle : \mathbf{A} \to \mathbf{A}'$ is surjective, the inclusion $\Omega^{\mathbf{A}}(\alpha^{-1}(T)) \le \alpha^{-1}(\Omega^{\mathbf{A}'}(T))$. Let $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \Omega^{\mathbf{A}}_\Sigma(\alpha^{-1}(T))$. Then, by Theorem 19, we get that, for all $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

$$\begin{aligned} &\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi}) \in \alpha_{\Sigma'}^{-1}(T_{F(\Sigma')}) \\ &\qquad \text{iff} \quad \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}) \in \alpha_{\Sigma'}^{-1}(T_{F(\Sigma')}). \end{aligned}$$

Equivalently,

$$\begin{aligned} &\alpha_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi})) \in T_{F(\Sigma')} \\ &\qquad \text{iff} \quad \alpha_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})) \in T_{F(\Sigma')}. \end{aligned}$$

Equivalently, by the morphism property,

$$\begin{aligned} &\sigma'_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)), \alpha_{\Sigma'}(\vec{\chi})) \in T_{F(\Sigma')} \\ &\qquad \text{iff} \quad \sigma'_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi)), \alpha_{\Sigma'}(\vec{\chi})) \in T_{F(\Sigma')}. \end{aligned}$$

Equivalently, by the naturality of α,

$$\begin{aligned} &\sigma'_{F(\Sigma')}(\mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi)), \alpha_{\Sigma'}(\vec{\chi})) \in T_{F(\Sigma')} \\ &\qquad \text{iff} \quad \sigma'_{F(\Sigma')}(\mathrm{SEN}'(F(f))(\alpha_\Sigma(\psi)), \alpha_{\Sigma'}(\vec{\chi})) \in T_{F(\Sigma')}. \end{aligned}$$

Equivalently, by Theorem 19 and the surjectivity of $\langle F, \alpha \rangle$, we get that

$$\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \Omega^{\mathbf{A}'}_{F(\Sigma)}(T),$$

which finishes the proof.

■

2.4 Relative Congruence Systems

We look at a variety of results related to congruence systems in this section. First, we give a condition that ensures that, given a morphism $\langle H,\gamma\rangle : \mathbf{A} \to \mathbf{A}'$ of $N^\flat$-algebraic systems, with an isomorphic functor component, and an equivalence family θ on $\mathbf{A}$, we have $\gamma^{-1}(\gamma(\theta)) = \theta$.

Lemma 25 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *be* $N^\flat$*-algebraic systems,* $\langle F,\alpha\rangle : \mathbf{A} \to \mathbf{A}'$ *a morphism, with* F *an isomorphism, and* $\theta \in \mathrm{EqvFam}(\mathbf{A})$*. Then*

$$\mathrm{Ker}(\langle F,\alpha\rangle) \le \theta \quad \textit{iff} \quad \alpha^{-1}(\alpha(\theta)) = \theta.$$

Proof: Suppose, first, that $\mathrm{Ker}(\langle F,\alpha\rangle) \le \theta$ and let $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi,\psi\rangle \in \alpha_\Sigma^{-1}(\alpha_\Sigma(\theta_\Sigma))$. Then, by definition, we get

$$\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in \alpha_\Sigma(\theta_\Sigma).$$

Thus, there exist $\phi',\psi' \in \mathrm{SEN}(\Sigma)$, such that

$$\langle \phi',\psi'\rangle \in \theta_\Sigma \quad \text{and} \quad \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle = \langle \alpha_\Sigma(\phi'), \alpha_\Sigma(\psi')\rangle.$$

Thus, we get

$$\langle \phi',\psi'\rangle \in \theta_\Sigma \quad \text{and} \quad \langle \phi,\phi'\rangle, \langle \psi,\psi'\rangle \in \mathrm{Ker}_\Sigma(\langle F,\alpha\rangle).$$

Since $\mathrm{Ker}(\langle F,\alpha\rangle) \le \theta$ and θ is an equivalence family, we get that $\langle \phi,\psi\rangle \in \theta_\Sigma$. Thus, we conclude that $\alpha^{-1}(\alpha(\theta)) \le \theta$. Since the reverse inclusion always holds, $\alpha^{-1}(\alpha(\theta)) = \theta$.

Assume, conversely, that $\alpha^{-1}(\alpha(\theta)) = \theta$ and let $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi,\psi\rangle \in \mathrm{Ker}_\Sigma(\langle F,\alpha\rangle)$. Then, by definition, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. Therefore, since θ is an equivalence family, we get

$$\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle = \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\phi)\rangle \in \alpha_\Sigma(\theta).$$

Now we get $\langle \phi,\psi\rangle \in \alpha_\Sigma^{-1}(\alpha_\Sigma(\theta_\Sigma))$ and, by hypothesis, $\langle \phi,\psi\rangle \in \theta_\Sigma$. We conclude that $\mathrm{Ker}(\langle F,\alpha\rangle) \le \theta$. ∎

Next we show that, given algebraic systems $\mathbf{A}$ and $\mathbf{A}'$, a surjective morphism $\langle F,\alpha\rangle : \mathbf{A} \to \mathbf{A}'$, with an isomorphic functor component, and a congruence system θ on $\mathbf{A}$, its image under $\langle F,\alpha\rangle$ is a congruence system on $\mathbf{A}'$, provided that θ contains the kernel system of $\langle F,\alpha\rangle$.

Lemma 26 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *be* $N^\flat$*-algebraic systems,* $\langle F,\alpha\rangle : \mathbf{A} \to \mathbf{A}'$ *a surjective morphism, with* F *an isomorphism, and* $\theta \in \mathrm{ConSys}(\mathbf{A})$*, such that* $\mathrm{Ker}(\langle F,\alpha\rangle) \le \theta$*. Then* $\alpha(\theta) \in \mathrm{ConSys}(\mathbf{A}')$*.*

Proof: We first show that $\alpha(\theta)$ is an equivalence family on $\mathbf{A}'$. To this end, let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi, \psi', \chi \in \mathrm{SEN}(\Sigma)$.

- By hypothesis, $\theta \in \mathrm{ConSys}(\mathbf{A})$. Hence, $\langle \phi, \phi \rangle \in \theta_\Sigma$. Thus, $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\phi) \rangle \in \alpha_\Sigma(\theta_\Sigma)$. Since $\langle F, \alpha \rangle$ is surjective, $\alpha(\theta)$ is reflexive.

- Suppose $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \alpha_\Sigma(\theta_\Sigma)$. Then $\langle \phi, \psi \rangle \in \alpha_\Sigma^{-1}(\alpha_\Sigma(\theta_\Sigma))$. By Lemma 25, $\langle \phi, \psi \rangle \in \theta_\Sigma$. Since $\theta \in \mathrm{ConSys}(\mathbf{A})$, $\langle \psi, \phi \rangle \in \theta_\Sigma$. Hence, $\langle \alpha_\Sigma(\psi), \alpha_\Sigma(\phi) \rangle \in \alpha_\Sigma(\theta_\Sigma)$. Thus, by the surjectivity of $\langle F, \alpha \rangle$, $\alpha(\theta)$ is also symmetric.

- Finally, suppose that $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \alpha_\Sigma(\theta_\Sigma)$ and $\langle \alpha_\Sigma(\psi'), \alpha_\Sigma(\chi) \rangle \in \alpha_\Sigma(\theta_\Sigma)$, with $\alpha_\Sigma(\psi) = \alpha_\Sigma(\psi')$. Then, by Lemma 25, $\langle \phi, \psi \rangle \in \theta_\Sigma$ and $\langle \psi', \chi \rangle \in \theta_\Sigma$. Moreover, by hypothesis, $\langle \psi, \psi' \rangle \in \mathrm{Ker}_\Sigma(\langle F, \alpha \rangle) \subseteq \theta_\Sigma$. Since $\theta \in \mathrm{ConSys}(\mathbf{A})$, we get $\langle \phi, \chi \rangle \in \theta_\Sigma$ and, therefore, $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\chi) \rangle \in \alpha_\Sigma(\theta_\Sigma)$. Taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that $\alpha(\theta)$ is also transitive.

We showed that $\alpha(\theta) \in \mathrm{EqvFam}(\mathbf{A}')$.

Next, we show that $\alpha(\theta)$ is also a system. To this end, suppose $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \alpha_\Sigma(\theta_\Sigma)$. Then, by Lemma 25, $\langle \phi, \psi \rangle \in \theta_\Sigma$. Since $\theta \in \mathrm{ConSys}(\mathbf{A})$, we get $\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi) \rangle \in \theta_{\Sigma'}$. Thus,

$$\begin{aligned}&\langle \mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi)), \mathrm{SEN}'(F(f))(\alpha_\Sigma(\psi)) \rangle \\ &\quad = \langle \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)), \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi)) \rangle \in \alpha_{\Sigma'}(\theta_{\Sigma'}).\end{aligned}$$

Since $\langle F, \alpha \rangle$ is surjective, we get that $\alpha(\theta)$ is invariant under $\mathbf{Sign}'$-morphisms. Now we have that $\alpha(\theta) \in \mathrm{EqvSys}(\mathbf{A}')$.

Finally, it remains to see that it is also a congruence system. To this end, let $\sigma^\flat$ be a natural transformation in $N^\flat$, $\Sigma \in |\mathbf{Sign}|$, $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)$, such that $\langle \alpha_\Sigma(\phi_i), \alpha_\Sigma(\psi_i) \rangle \in \alpha_\Sigma(\theta_\Sigma)$, for all $i < k$. We get, by Lemma 25, $\langle \phi_i, \psi_i \rangle \in \theta_\Sigma$, whence, since $\theta \in \mathrm{ConSys}(\mathbf{A})$, $\langle \sigma_\Sigma^{\mathbf{A}}(\vec{\phi}), \sigma_\Sigma^{\mathbf{A}}(\vec{\psi}) \rangle \in \theta_\Sigma$. Now, applying the morphism property, we get

$$\langle \sigma_{F(\Sigma)}^{\mathbf{A}'}(\alpha_\Sigma(\vec{\phi})), \sigma_{F(\Sigma)}^{\mathbf{A}'}(\alpha_\Sigma(\vec{\psi})) \rangle = \langle \alpha_\Sigma(\sigma_\Sigma^{\mathbf{A}}(\vec{\phi})), \alpha_\Sigma(\sigma_\Sigma^{\mathbf{A}}(\vec{\psi})) \rangle \in \alpha_\Sigma(\theta_\Sigma).$$

Again, taking into account the surjectivity of $\langle F, \alpha \rangle$, we get that $\alpha(\theta)$ has the congruence property. We conclude that $\alpha(\theta) \in \mathrm{ConSys}(\mathbf{A}')$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system, K be a class of $\mathbf{F}$-algebraic systems and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system. A congruence system θ on $\mathbf{A}$ is called a K-**congruence system**, or a **congruence system relative to** K, if the quotient algebraic system $\mathcal{A}/\theta$ is a member of the class K, i.e., $\mathcal{A}/\theta = \mathcal{A}^\theta \in \mathsf{K}$. Given an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, we denote by $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ the collection of all K-congruence systems on $\mathcal{A}$:

$$\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A}) = \{\theta \in \mathrm{ConSys}(\mathcal{A}) : \mathcal{A}/\theta \in \mathsf{K}\}.$$

Let K be a class of $\mathbf{F}$-algebraic systems. We write $\mathbb{H}(\mathsf{K})$ for the class of all $\mathbf{F}$-algebraic systems $\mathcal{B}$, such that, there exists $\mathcal{A} \in \mathsf{K}$ and a (surjective) $\mathbf{F}$-algebraic system morphism $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$:

$$\mathbb{H}(\mathsf{K}) = \{\mathcal{B} \in \mathrm{AlgSys}(\mathbf{F}) : (\exists \mathcal{A} \in \mathsf{K})(\exists \ \langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B})\}.$$

We show that, if K is a class that is closed under the operator $\mathbb{H}$, then the K-congruence systems on any $\mathbf{F}$-algebraic system in K coincide with the ordinary congruence systems on $\mathcal{A}$.

Proposition 27 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems, such that* $\mathbb{H}(\mathsf{K}) \subseteq \mathsf{K}$*. Then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} \in \mathsf{K}$*,* $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A}) = \mathrm{ConSys}(\mathcal{A})$*.*

Proof: Clearly, $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A}) \subseteq \mathrm{ConSys}(\mathcal{A})$. Suppose $\theta \in \mathrm{ConSys}(\mathcal{A})$. Consider the quotient morphism

$$\langle I, \pi^\theta \rangle : \mathcal{A} \to \mathcal{A}/\theta.$$

Since $\mathcal{A} \in \mathsf{K}$, $\mathcal{A}/\theta \in \mathbb{H}(\mathsf{K}) \subseteq \mathsf{K}$. Thus, by definition, $\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and $\mathcal{A}^i = \langle \mathbf{A}^i, \langle F^i, \alpha^i \rangle \rangle$, $i \in I$, be $\mathbf{F}$-algebraic systems and, for all $i \in I$,

$$\langle H^i, \gamma^i \rangle : \mathcal{A} \to \mathcal{A}^i$$

a surjective morphism. We say that $\{\langle H^i, \gamma^i \rangle : \mathcal{A} \to \mathcal{A}^i : i \in I\}$ is a **subdirect intersection** if

$$\bigcap_{i \in I} \mathrm{Ker}(\langle H^i, \gamma^i \rangle) = \Delta^{\mathcal{A}}.$$

Given a class K of $\mathbf{F}$-algebraic systems, we write $\overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$ to denote the class of all $\mathbf{F}$-algebraic systems $\mathcal{A}$, for which there exists a subdirect intersection $\{\langle H^i, \gamma^i \rangle : \mathcal{A} \to \mathcal{A}^i : i \in I\}$, with $\mathcal{A}^i \in \mathsf{K}$, for all $i \in I$.

We show that if a class K is closed under subdirect intersections, then the collection of all K-congruence systems on any $\mathbf{F}$-algebraic system $\mathcal{A}$ is closed under intersections. If, in addition, K contains a trivial $\mathbf{F}$-algebraic system, then $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ becomes a closure family on $\mathcal{A}^2$, for every $\mathbf{F}$-algebraic system $\mathcal{A}$.

Proposition 28 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *be a class of* $\mathbf{F}$*-algebraic systems, such that* $\overset{\triangleleft}{\mathbb{P}}(\mathsf{K}) \subseteq \mathsf{K}$*.*

(a) For every $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ *is closed under signature-wise intersections;*

(b) If, in addition, K *contains a trivial* $\mathbf{F}$*-algebraic system, then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ *is a closure family on* $\mathcal{A}^2$*.*

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $\{\theta^i : i \in I\} \subseteq \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$. Then $\mathcal{A}/\theta^i \in \mathsf{K}$, for all $i \in I$. Consider the canonical morphisms

$$\langle I, \pi^i \rangle : \mathcal{A}/\bigcap_{i \in I} \theta^i \to \mathcal{A}/\theta^i, \quad i \in I.$$

Clearly, we have

$$\bigcap_{i \in I} \mathrm{Ker}(\langle I, \pi^i \rangle) = \bigcap_{i \in I}(\theta^i / \bigcap_{i \in I} \theta^i) = \bigcap_{i \in I} \theta^i / \bigcap_{i \in I} \theta^i = \Delta^{\mathcal{A}/\bigcap_{i \in I} \theta^i}.$$

Thus, $\{\langle I, \pi^i \rangle : \mathcal{A}/\bigcap_{i \in I} \theta^i \to \mathcal{A}/\theta^i : i \in I\}$ is a subdirect intersection. Since $\mathcal{A}/\theta^i \in \mathsf{K}$, for all $i \in I$, we get $\mathcal{A}/\bigcap_{i \in I} \theta^i \in \overset{\triangleleft}{\mathbb{P}}(\mathsf{K}) \subseteq \mathsf{K}$. Therefore, $\bigcap_{i \in I} \theta^i \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$.

Suppose, in addition, that K contains a trivial $\mathbf{F}$-algebraic system. Then $\nabla^{\mathcal{A}} \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$, whence $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ is a closure family on $\mathcal{A}^2$. ■

By Proposition 28, for a class K of $\mathbf{F}$-algebraic systems closed under $\overset{\triangleleft}{\mathbb{P}}$ and containing a trivial $\mathbf{F}$-algebraic system, it makes sense to define, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $X \in \mathrm{SenFam}(\mathcal{A}^2)$,

$$\Theta^{\mathsf{K},\mathcal{A}}(X) = \bigcap\{\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A}) : X \leq \theta\}.$$

When $\mathcal{A}$ coincides with the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$, where $\langle I, \iota \rangle : \mathbf{F} \to \mathbf{F}$ is the identity morphism, we write simply Θ^{K}.

We now provide a different characterization of the operator $\Theta^{\mathsf{K},\mathcal{A}}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system and K a class of $\mathbf{F}$-algebraic systems. Define the operator $D^{\mathsf{K}} : \mathcal{P}(\mathrm{SEN}^\flat)^2 \to \mathcal{P}(\mathrm{SEN}^\flat)^2$, by letting, for all $X \leq (\mathrm{SEN}^\flat)^2$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\langle \phi, \psi \rangle \in \mathrm{SEN}^\flat(\Sigma)^2$,

$$\langle \phi, \psi \rangle \in D^{\mathsf{K}}_{\Sigma}(X) \quad \text{iff} \quad \begin{array}{l} \text{for all } \mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle \in \mathsf{K}, \\ \alpha(X) \leq \Delta^{\mathcal{A}} \text{ implies } \alpha_{\Sigma}(\phi) = \alpha_{\Sigma}(\psi). \end{array}$$

We show that D^{K} is a closure family on $(\mathrm{SEN}^\flat)^2$.

Proposition 29 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems.* D^{K} *is a closure family on* $(\mathrm{SEN}^\flat)^2$.

Proof: We must show that D^{K} is inflationary, monotone and idempotent.

Let $X \leq (\mathrm{SEN}^\flat)^2$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\langle \phi, \psi \rangle \in X_\Sigma$. Then, for all $\mathcal{A} \in \mathsf{K}$, if $\alpha(X) \leq \Delta^{\mathcal{A}}$, we clearly have $\alpha(\phi) = \alpha(\psi)$. Hence, $\langle \phi, \psi \rangle \in D^{\mathsf{K}}_{\Sigma}(X)$ and D^{K} is inflationary.

Suppose $X \leq Y \leq (\mathrm{SEN}^\flat)^2$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\langle \phi, \psi \rangle \in \mathrm{SEN}^\flat(\Sigma)^2$, such that $\langle \phi, \psi \rangle \in D^{\mathsf{K}}_{\Sigma}(X)$. Let $\mathcal{A} \in \mathsf{K}$, such that $\alpha(Y) \leq \Delta^{\mathcal{A}}$. Then, we get $\alpha(X) \leq \alpha(Y) \leq \Delta^{\mathcal{A}}$, whence, by hypothesis, $\alpha(\phi) = \alpha(\psi)$. Therefore, $\langle \phi, \psi \rangle \in D^{\mathsf{K}}_{\Sigma}(Y)$ and D^{K} is also monotone.

Finally, suppose $X \le (\mathrm{SEN}^\flat)^2$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\langle\phi,\psi\rangle \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle\phi,\psi\rangle \in D^{\mathsf{K}}_\Sigma(D^{\mathsf{K}}(X))$. Let $\mathcal{A} \in \mathsf{K}$, such that $\alpha(X) \le \Delta^{\mathcal{A}}$. Then, by definition, $\alpha(D^{\mathsf{K}}(X)) \le \Delta^{\mathcal{A}}$, whence, by hypothesis, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. Thus, $\langle\phi,\psi\rangle \in D^{\mathsf{K}}_\Sigma(X)$ and D^{K} is also idempotent.

We conclude that D^{K} is a closure family on $(\mathrm{SEN}^\flat)^2$. ■

We show, next, that, for all $X \le (\mathrm{SEN}^\flat)^2$, the sentence family $D^{\mathsf{K}}(X)$ is a congruence system on the algebraic system $\mathbf{F}$ and that, moreover, it is a congruence system relative to the class K.

Proposition 30 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system and K a class of $\mathbf{F}$-algebraic systems. For all $X \le (\mathrm{SEN}^\flat)^2$, $D^{\mathsf{K}}(X) \in \mathrm{ConSys}(\mathbf{F})$.*

Proof: We first show that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $D^{\mathsf{K}}_\Sigma(X)$ is an equivalence family.

- Let $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Since, for all $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle \in \mathsf{K}$, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\phi)$, we get that $\langle\phi,\phi\rangle \in D^{\mathsf{K}}_\Sigma(X)$, whence $D^{\mathsf{K}}_\Sigma(X)$ is reflexive.
- Suppose $\langle\phi,\psi\rangle \in D^{\mathsf{K}}_\Sigma(X)$ and let $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle \in \mathsf{K}$, such that $\alpha(X) \le \Delta^{\mathcal{A}}$. Then, by hypothesis, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$, giving $\alpha_\Sigma(\psi) = \alpha_\Sigma(\phi)$. Hence, $\langle\psi,\phi\rangle \in D^{\mathsf{K}}_\Sigma(X)$, showing that $D^{\mathsf{K}}_\Sigma(X)$ is also symmetric.
- Finally, suppose $\langle\phi,\psi\rangle$, $\langle\psi,\chi\rangle \in D^{\mathsf{K}}_\Sigma(X)$. Let $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle \in \mathsf{K}$, such that $\alpha(X) \le \Delta^{\mathcal{A}}$. By hypothesis, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$ and $\alpha_\Sigma(\psi) = \alpha_\Sigma(\chi)$. Therefore, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\chi)$, showing that $\langle\phi,\chi\rangle \in D^{\mathsf{K}}_\Sigma(X)$. Hence, $D^{\mathsf{K}}_\Sigma(X)$ is also transitive.

We show, next, that $D^{\mathsf{K}}(X)$ is an equivalence system, i.e., invariant under signature morphisms. Let $\Sigma,\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle\phi,\psi\rangle \in D^{\mathsf{K}}_\Sigma(X)$. Let $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle \in \mathsf{K}$, such that $\alpha(X) \le \Delta^{\mathcal{A}}$. Then, by hypothesis, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. Thus, we get

$$\begin{array}{rcl}\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) & = & \mathrm{SEN}(F(f))(\alpha_\Sigma(\phi))\\ & = & \mathrm{SEN}(F(f))(\alpha_\Sigma(\psi))\\ & = & \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi)).\end{array}$$

Hence, $\langle\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi)\rangle \in D^{\mathsf{K}}_{\Sigma'}(X)$.

Finally, to see that it also satisfies the congruence property, let $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ be in $N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi},\vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle\phi_i,\psi_i\rangle \in D^{\mathsf{K}}_\Sigma(X)$, for all $i < k$. Let $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle \in \mathsf{K}$, such that $\alpha(X) \le \Delta^{\mathcal{A}}$. then, by hypothesis, $\alpha_\Sigma(\phi_i) = \alpha_\Sigma(\psi_i)$, for all $i < k$. Therefore,

$$\begin{array}{rcl}\alpha_\Sigma(\sigma^\flat_\Sigma(\vec{\phi})) & = & \sigma^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi}))\\ & = & \sigma^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\psi}))\\ & = & \alpha_\Sigma(\sigma^\flat_\Sigma(\vec{\psi})).\end{array}$$

We conclude that $\langle\sigma^\flat_\Sigma(\vec{\phi}),\sigma^\flat_\Sigma(\vec{\psi})\rangle\in D^{\mathsf{K}}_\Sigma(X)$ and, therefore, $D^{\mathsf{K}}(X)$ is indeed a congruence system on $\mathbf{F}$. ■

Furthermore, if K happens to contain a trivial $\mathbf{F}$-algebraic system and be closed under subdirect intersections, we can show that $D^{\mathsf{K}}(X)$ is a K-congruence system on $\mathbf{F}$.

Proposition 31 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be a base algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems, containing a trivial* $\mathbf{F}$*-algebraic system and closed under* $\overset{\triangleleft}{\mathrm{III}}$*. For all* $X\leq(\mathrm{SEN}^\flat)^2$*,* $D^{\mathsf{K}}(X)\in\mathrm{ConSys}^{\mathsf{K}}(\mathbf{F})$.

Proof: By Proposition 30, we know that $D^{\mathsf{K}}(X)$ is a congruence system on $\mathbf{F}$. Therefore, it suffices to show that it is a congruence system relative to K. For this, let $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle\in\mathsf{K}$, such that $X\leq\mathrm{Ker}(\langle F,\alpha\rangle)$. Define the morphism

$$\langle F,\alpha^{\mathsf{K}}\rangle:\mathcal{F}/D^{\mathsf{K}}(X)\to\mathcal{A}$$

by setting, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\phi\in\mathrm{SEN}^\flat(\Sigma)/D^{\mathsf{K}}_\Sigma(X)$,

$$\alpha^{\mathsf{K}}_\Sigma(\phi/D^{\mathsf{K}}_\Sigma(X))=\alpha_\Sigma(\phi).$$

This morphism is well defined, since, if $\mathcal{A}\in\mathsf{K}$, with $X\leq\mathrm{Ker}(\langle F,\alpha\rangle)$, then, for all $\Sigma\in|\mathbf{Sign}^\flat|$, $\phi,\psi\in\mathrm{SEN}^\flat(\Sigma)$,

$$\langle\phi,\psi\rangle\in D^{\mathsf{K}}_\Sigma(X)\quad\text{implies}\quad\alpha_\Sigma(\phi)=\alpha_\Sigma(\psi).$$

Now consider the collection

$$\langle F,\alpha^{\mathsf{K}}\rangle:\mathcal{F}/D^{\mathsf{K}}(X)\to\mathcal{A},\ \mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle\in\mathsf{K},\ X\leq\mathrm{Ker}(\langle F,\alpha\rangle).$$

We have, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\phi,\psi\in\mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{ll}\multicolumn{2}{l}{\langle\phi/D^{\mathsf{K}}_\Sigma(X),\psi/D^{\mathsf{K}}_\Sigma(X)\rangle\in\bigcap_{\langle F,\alpha^{\mathsf{K}}\rangle}\mathrm{Ker}_\Sigma(\langle F,\alpha^{\mathsf{K}}\rangle)}\\ \text{iff} & \alpha^{\mathsf{K}}_\Sigma(\phi/D^{\mathsf{K}}_\Sigma(X))=\alpha^{\mathsf{K}}_\Sigma(\psi/D^{\mathsf{K}}_\Sigma(X)),\ \text{for all }\langle F,\alpha^{\mathsf{K}}\rangle\\ \text{iff} & \alpha_\Sigma(\phi)=\alpha_\Sigma(\psi)\ \text{for all }\langle F,\alpha^{\mathsf{K}}\rangle\\ \text{iff} & \langle\phi,\psi\rangle\in D^{\mathsf{K}}_\Sigma(X).\end{array}$$

Therefore, the displayed collection above constitutes a subdirect intersection and, since $\mathcal{A}\in\mathsf{K}$, for all $\langle F,\alpha^{\mathsf{K}}\rangle$, and K is closed under subdirect intersections, we get that $\mathcal{F}/D^{\mathsf{K}}(X)\in\mathsf{K}$, and, therefore, $D^{\mathsf{K}}(X)\in\mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$. ■

We are now in a position to show the promised alternative characterization of the operator Θ^{K}. It turns out that it coincides with D^{K}.

Theorem 32 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems, containing a trivial* $\mathbf{F}$*-algebraic system and closed under subdirect intersections. For all* $X\leq(\mathrm{SEN}^\flat)^2$*,* $\Theta^{\mathsf{K}}(X)=D^{\mathsf{K}}(X)$.

Proof: Let $X \leq (\mathrm{SEN}^\flat)^2$. By Proposition 31, $D^{\mathsf{K}}(X) \in \mathrm{ConSys}^{\mathsf{K}}(X)$ and, by Proposition 29, $X \leq D^{\mathsf{K}}(X)$. Therefore, by the minimality of $\Theta^{\mathsf{K}}(X)$, we get that $\Theta^{\mathsf{K}}(X) \leq D^{\mathsf{K}}(X)$. To show the reverse inclusion, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in D^{\mathsf{K}}_\Sigma(X)$. Consider $\mathcal{F}/\Theta^{\mathsf{K}}(X) \in \mathsf{K}$. Since $\pi^{\Theta^{\mathsf{K}}(X)}(X) \leq \Delta^{\mathcal{F}/\Theta^{\mathsf{K}}(X)}$, we get, by hypothesis, $\pi^{\Theta^{\mathsf{K}}(X)}_\Sigma(\phi) = \pi^{\Theta^{\mathsf{K}}(X)}_\Sigma(\psi)$, i.e., $\langle \phi, \psi \rangle \in \Theta^{\mathsf{K}}_\Sigma(X)$. We conclude that $D^{\mathsf{K}}(X) \leq \Theta^{\mathsf{K}}(X)$. ∎

We look, next, at how the operator Θ^{K} interacts with morphisms.

Proposition 33 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *be a class of* $\mathbf{F}$*-algebraic systems, containing a trivial* $\mathbf{F}$*-algebraic system and such that* $\overset{\triangleleft}{\mathbb{IP}}(\mathsf{K}) \subseteq \mathsf{K}$. *Let also* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ *be* $\mathbf{F}$*-algebraic systems and* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism.*

(a) If $\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{B})$*, then* $\gamma^{-1}(\theta) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$*;*

(b) If H *is an isomorphism,* $\mathrm{Ker}(\langle H, \gamma \rangle) \leq \theta$ *and* $\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$*, then* $\gamma(\theta) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{B})$*.*

Proof:

(a) By Proposition 16, $\gamma^{-1}(\theta) \in \mathrm{ConSys}(\mathcal{A})$. Consider the morphism

$$\langle H, \gamma^* \rangle : \mathcal{A}/\gamma^{-1}(\theta) \to \mathcal{B}/\theta,$$

defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, by

$$\gamma^*_\Sigma(\phi/\gamma^{-1}_\Sigma(\theta_{H(\Sigma)})) = \gamma_\Sigma(\phi)/\theta_{H(\Sigma)}.$$

This is well-defined, since, if $\langle \phi, \psi \rangle \in \gamma^{-1}_\Sigma(\theta_{H(\Sigma)})$, then $\langle \gamma_\Sigma(\phi), \gamma_\Sigma(\psi) \rangle \in \theta_{H(\Sigma)}$. Moreover,

$$\mathrm{Ker}(\langle H, \gamma^* \rangle) = {\gamma^*}^{-1}(\Delta^{\mathcal{B}/\theta}) = \Delta^{\mathcal{A}/\gamma^{-1}(\theta)}.$$

Thus, $\{\langle H, \gamma^* \rangle : \mathcal{A}/\gamma^{-1}(\theta) \to \mathcal{B}/\theta\}$ is a subdirect intersection and, since, by hypothesis, $\mathcal{B}/\theta \in \mathsf{K}$, $\mathcal{A}/\gamma^{-1}(\theta) \in \overset{\triangleleft}{\mathbb{IP}}(\mathsf{K}) \subseteq \mathsf{K}$. Therefore, $\gamma^{-1}(\theta) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$.

(b) By Lemma 26, $\gamma(\theta) \in \mathrm{ConSys}(\mathcal{B})$. Moreover, it is not difficult to see that

$$\langle H, \gamma^* \rangle : \mathcal{A}/\theta \to \mathcal{B}/\gamma(\theta),$$

defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, by

$$\gamma^*_\Sigma(\phi/\theta_\Sigma) = \gamma_\Sigma(\phi)/\gamma_\Sigma(\theta_\Sigma)$$

is an isomorphism of $\mathbf{F}$-algebraic systems, since, by Lemma 25, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi, \psi \rangle \in \theta_\Sigma \quad \text{iff} \quad \langle \gamma_\Sigma(\phi), \gamma_\Sigma(\psi) \rangle \in \gamma_\Sigma(\theta_\Sigma).$$

Therefore, $\{\langle H,\gamma\rangle^{-1}:\mathcal{B}/\gamma(\theta)\to\mathcal{A}/\theta\}$ is a subdirect intersection. Since $\mathcal{A}/\theta\in\mathsf{K}$, it follows that $\mathcal{B}/\gamma(\theta)\in\overset{\triangleleft}{\mathbb{II}}(\mathsf{K})\subseteq\mathsf{K}$. Therefore, $\gamma(\theta)\in\mathrm{ConSys}^{\mathsf{K}}(\mathcal{B})$.

■

Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be a base algebraic system, $\tau^\flat:(\mathrm{SEN}^\flat)^\omega\to(\mathrm{SEN}^\flat)^2$ be a collection of natural transformations in $N^\flat$ and $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$, with $\mathbf{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$ an $\mathbf{F}$-algebraic system. If $\tau^\flat$ is perceived as having a single distinguished argument, with the remaining arguments as parameters, we define, for all $\Sigma\in|\mathbf{Sign}|$ and all $\phi\in\mathrm{SEN}(\Sigma)$, the sentence family

$$\tau_\Sigma^{\mathcal{A}}[\phi]=\{\tau_{\Sigma,\Sigma'}^{\mathcal{A}}[\phi]\}_{\Sigma'\in|\mathbf{Sign}|},$$

by setting, for all $\Sigma'\in|\mathbf{Sign}|$,

$$\tau_{\Sigma,\Sigma'}^{\mathcal{A}}[\phi]=\bigcup\{\tau_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}(f)(\phi),\vec{\chi}):f\in\mathbf{Sign}(\Sigma,\Sigma'),\vec{\chi}\in\mathrm{SEN}(\Sigma')\}.$$

Given $\Phi\subseteq\mathrm{SEN}(\Sigma)$, we set

$$\tau_\Sigma^{\mathcal{A}}[\Phi]=\bigcup\{\tau_\Sigma^{\mathcal{A}}[\phi]:\phi\in\Phi\}$$

and, given a sentence family $X\in\mathrm{SenFam}(\mathcal{A})$, we set

$$\tau^{\mathcal{A}}[X]=\bigcup\{\tau_\Sigma^{\mathcal{A}}[X_\Sigma]:\Sigma\in|\mathbf{Sign}|\}.$$

We will revisit these and similar definitions in more depth in Section 2.13. For now, we only use them to establish a result that involves the relative congruence system operator Θ^{K}, introduced in this section, and direct images under morphisms of $\mathbf{F}$-algebraic systems with isomorphic functor components.

Proposition 34 *Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, with $\tau^\flat:(\mathrm{SEN}^\flat)^\omega\to(\mathrm{SEN}^\flat)^2$ in $N^\flat$ and K be a class of $\mathbf{F}$-algebraic systems, containing a trivial $\mathbf{F}$-algebraic system and such that $\overset{\triangleleft}{\mathbb{II}}(\mathsf{K})\subseteq\mathsf{K}$. Let also $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$, $\mathcal{B}=\langle\mathbf{B},\langle G,\beta\rangle\rangle$ be $\mathbf{F}$-algebraic systems and $\langle H,\gamma\rangle:\mathcal{A}\to\mathcal{B}$ a surjective morphism, with H an isomorphism. Then, for all $X\in\mathrm{SenFam}(\mathcal{A})$,*

$$\Theta^{\mathsf{K},\mathcal{B}}(\gamma(\Theta^{\mathsf{K},\mathcal{A}}(\tau^{\mathcal{A}}[X])))=\Theta^{\mathsf{K},\mathcal{B}}(\tau^{\mathcal{B}}[\gamma(X)]).$$

Proof: Taking into account the surjectivity of $\langle H,\gamma\rangle$, we have $\tau^{\mathcal{B}}[\gamma(X)]=\gamma(\tau^{\mathcal{A}}[X])\leq\gamma(\Theta^{\mathsf{K},\mathcal{A}}(\tau^{\mathcal{A}}[X]))$. Hence

$$\Theta^{\mathsf{K},\mathcal{B}}(\tau^{\mathcal{B}}[\gamma(X)])\leq\Theta^{\mathsf{K},\mathcal{B}}(\gamma(\Theta^{\mathsf{K},\mathcal{A}}(\tau^{\mathcal{A}}[X]))).$$

On the other hand, $\gamma^{-1}(\Theta^{\mathsf{K},\mathcal{B}}(\tau^{\mathcal{B}}[\gamma(X)]))$ is, by Proposition 33, a K-congruence system on $\mathcal{A}$, and, moreover, it contains $\tau^{\mathcal{A}}[X]$, since

$$\gamma(\tau^{\mathcal{A}}[X])=\tau^{\mathcal{B}}[\gamma(X)]\leq\Theta^{\mathsf{K},\mathcal{B}}(\tau^{\mathcal{B}}[\gamma(X)]).$$

Hence, $\Theta^{\mathsf{K},\mathcal{A}}(\tau^{\mathcal{A}}[X]) \leq \gamma^{-1}(\Theta^{\mathsf{K},\mathcal{B}}(\tau^{\mathcal{B}}[\gamma(X)]))$, i.e.,

$$\gamma(\Theta^{\mathsf{K},\mathcal{A}}(\tau^{\mathcal{A}}[X])) \leq \Theta^{\mathsf{K},\mathcal{B}}(\tau^{\mathcal{B}}[\gamma(X)]).$$

This yields $\Theta^{\mathsf{K},\mathcal{B}}(\gamma(\Theta^{\mathsf{K},\mathcal{A}}(\tau^{\mathcal{A}}[X]))) \leq \Theta^{\mathsf{K},\mathcal{B}}(\tau^{\mathcal{B}}[\gamma(X)])$. ∎

We conclude the section by showing that the relative congruence system generated by a family of pairs may be expressed as the join in the complete lattice of relative congruence systems of those relative congruence systems generated by the single pairs of elements in the generating family.

Proposition 35 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *be a class of* $\mathbf{F}$*-algebraic systems, containing a trivial* $\mathbf{F}$*-algebraic system and such that* $\overset{\triangleleft}{\mathbb{P}}(\mathsf{K}) \subseteq \mathsf{K}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $X \in \mathrm{SenFam}(\mathcal{A}^2)$,

$$\Theta^{\mathsf{K},\mathcal{A}}(X) = \bigvee \{\Theta^{\mathsf{K},\mathcal{A}}(\phi,\psi) : \langle \phi,\psi \rangle \in X_\Sigma, \Sigma \in |\mathbf{Sign}|\}.$$

Proof: Set

$$\theta := \bigvee \{\Theta^{\mathsf{K},\mathcal{A}}(\phi,\psi) : \langle \phi,\psi \rangle \in X_\Sigma, \Sigma \in |\mathbf{Sign}|\}.$$

For all $\Sigma \in |\mathbf{Sign}|$ and all $\langle \phi,\psi \rangle \in X_\Sigma$, we have $\langle \phi,\psi \rangle \in \Theta^{\mathsf{K},\mathcal{A}}_\Sigma(X)$. So $\Theta^{\mathsf{K},\mathcal{A}}(\phi,\psi) \leq \Theta^{\mathsf{K},\mathcal{A}}(X)$ and, therefore, $\theta \leq \Theta^{\mathsf{K},\mathcal{A}}(X)$. Conversely, for all $\Sigma \in |\mathbf{Sign}|$ and all $\langle \phi,\psi \rangle \in X_\Sigma$, we have $\langle \phi,\psi \rangle \in \Theta^{\mathsf{K},\mathcal{A}}_\Sigma(\phi,\psi) \subseteq \theta_\Sigma$. Hence, $X \leq \theta$, which implies that $\Theta^{\mathsf{K},\mathcal{A}}(X) \leq \theta$. ∎

2.5 Varieties of F-Algebraic Systems

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system.

An **natural F-equation** (sometimes, referred to, simply, as **natural equation**, **F-equation** or just **equation**, if the meaning is made clear from context) is a pair $\langle \sigma^\flat, \tau^\flat \rangle$, where $\sigma^\flat, \tau^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ are natural transformations in $N^\flat$. The $\mathbf{F}$-equation $\langle \sigma^\flat, \tau^\flat \rangle$ will be denoted also by $\sigma^\flat \approx \tau^\flat$. Sometimes notation such as $\tau^\flat := \tau^{0\,\flat} \approx \tau^{1\,\flat}$ may also become handy. We denote by $\mathrm{NEq}(\mathbf{F})$ the collection of all natural $\mathbf{F}$-equations.

Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, be an $\mathbf{F}$-algebraic system. Then, given $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$, we write $\mathcal{A} \vDash_\Sigma \sigma^\flat \approx \tau^\flat[\vec{\phi}]$ and say that $\vec{\phi}$ **Σ-satisfies $\sigma^\flat \approx \tau^\flat$ in $\mathcal{A}$** if

$$\alpha_\Sigma(\sigma^\flat_\Sigma(\vec{\phi})) = \alpha_\Sigma(\tau^\flat_\Sigma(\vec{\phi})).$$

The following is a useful lemma concerning satisfiability of an equation.

Lemma 36 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\sigma^\flat \approx \tau^\flat$ *a natural* $\mathbf{F}$*-equation and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system. The following statements are equivalent:*

(a) $\mathcal{A} \models_\Sigma \sigma^\flat \approx \tau^\flat[\vec{\phi}]$;

(b) $\sigma^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})) = \tau^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi}))$;

(c) For all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}))) = \alpha_{\Sigma'}(\tau^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}))).$$

Proof:

(a)⇔(b) By the homomorphism property,

$$\alpha_\Sigma(\sigma^\flat_\Sigma(\vec{\phi})) = \sigma^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})) \quad \text{and} \quad \alpha_\Sigma(\tau^\flat_\Sigma(\vec{\phi})) = \tau^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})).$$

So we get

$$\begin{array}{lll} \mathcal{A} \models_\Sigma \sigma^\flat \approx \tau^\flat[\vec{\phi}] & \text{iff} & \alpha_\Sigma(\sigma^\flat_\Sigma(\vec{\phi})) = \alpha_\Sigma(\tau^\flat_\Sigma(\vec{\phi})) \\ & \text{iff} & \sigma^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})) = \tau^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})). \end{array}$$

(c)⇒(a) This implication is trivial by taking $\Sigma' = \Sigma$ and $f = i_\Sigma$.

(b)⇒(c) We have

$$\begin{array}{l} \sigma^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})) = \tau^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})) \\ \text{implies} \quad \alpha_\Sigma(\sigma^\flat_\Sigma(\vec{\phi})) = \alpha_\Sigma(\tau^\flat_\Sigma(\vec{\phi})) \\ \text{implies, for all } \Sigma' \in |\mathbf{Sign}^\flat| \text{ and all } f \in \mathbf{Sign}^\flat(\Sigma, \Sigma'), \\ \qquad \mathrm{SEN}(F(f))(\alpha_\Sigma(\sigma^\flat_\Sigma(\vec{\phi}))) = \mathrm{SEN}(F(f))(\alpha_\Sigma(\tau^\flat_\Sigma(\vec{\phi}))) \\ \text{implies, for all } \Sigma' \in |\mathbf{Sign}^\flat| \text{ and all } f \in \mathbf{Sign}^\flat(\Sigma, \Sigma'), \\ \qquad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\sigma^\flat_\Sigma(\vec{\phi}))) = \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\tau^\flat_\Sigma(\vec{\phi}))) \\ \text{implies, for all } \Sigma' \in |\mathbf{Sign}^\flat| \text{ and all } f \in \mathbf{Sign}^\flat(\Sigma, \Sigma'), \\ \qquad \alpha_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}))) = \alpha_{\Sigma'}(\tau^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}))). \end{array}$$

■

Given a natural **F**-equation $\sigma^\flat \approx \tau^\flat$ and an **F**-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ we write

$$\mathcal{A} \models \sigma^\flat \approx \tau^\flat$$

and say that $\mathcal{A}$ **satisfies** $\sigma^\flat \approx \tau^\flat$ or that $\sigma^\flat \approx \tau^\flat$ is **satisfied in** $\mathcal{A}$ or is **valid in** $\mathcal{A}$, if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$, $\mathcal{A} \models_\Sigma \sigma^\flat \approx \tau^\flat[\vec{\phi}]$.

Let K be a class of **F**-algebraic systems and $E^\flat$ a set of natural **F**-equations. We write $\mathsf{K} \models E^\flat$ for

$$\mathcal{A} \models \sigma^\flat \approx \tau^\flat, \text{ for all } \mathcal{A} \in \mathsf{K} \text{ and all } \sigma^\flat \approx \tau^\flat \in E^\flat.$$

Given an **F**-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, we define the **kernel** $\mathrm{Ker}(\mathcal{A})$ **of** $\mathcal{A}$ to be the kernel of the morphism $\langle F, \alpha \rangle : \mathbf{F} \to \mathbf{A}$, i.e., we let

$$\mathrm{Ker}(\mathcal{A}) \coloneqq \mathrm{Ker}(\langle F, \alpha \rangle).$$

Moreover, given a class K of $\mathbf{F}$-algebraic systems, we let

$$\mathrm{Ker}(\mathsf{K}) = \bigcap_{\mathcal{A}\in\mathsf{K}} \mathrm{Ker}(\mathcal{A}).$$

Now we are in a position to define two kinds of classes of $\mathbf{F}$-algebraic systems generated by a given class K of $\mathbf{F}$-algebraic systems. In other words, we introduce two class operators on classes of $\mathbf{F}$-algebraic systems.

Definition 37 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system and* K *be a class of* $\mathbf{F}$*-algebraic systems.*

- *The* **semantic variety** $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$ **generated by** K *is defined by*

$$\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : \mathrm{Ker}(\mathsf{K}) \le \mathrm{Ker}(\mathcal{A})\};$$

- *The* **syntactic variety** $\mathbb{V}^{\mathsf{Syn}}(\mathsf{K})$ **generated by** K *is defined by*

$$\mathbb{V}^{\mathsf{Syn}}(\mathsf{K}) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : (\forall \sigma^\flat \approx \tau^\flat)(\mathsf{K} \vDash \sigma^\flat \approx \tau^\flat \;\Rightarrow\; \mathcal{A} \vDash \sigma^\flat \approx \tau^\flat)\}.$$

It is relatively easy to see that both $\mathbb{V}^{\mathsf{Sem}}$ and $\mathbb{V}^{\mathsf{Syn}}$ are closure operators on the class of $\mathbf{F}$-algebraic systems.

Proposition 38 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system. Then* $\mathbb{V}^{\mathsf{Sem}}$ *and* $\mathbb{V}^{\mathsf{Syn}}$ *are closure operators on* $\mathrm{AlgSys}(\mathbf{F})$.

Proof: We work, first, with $\mathbb{V}^{\mathsf{Sem}}$.

- If $\mathcal{A} \in \mathsf{K}$, then, by definition, we have $\mathrm{Ker}(\mathsf{K}) \le \mathrm{Ker}(\mathcal{A})$. Thus, $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. So $\mathsf{K} \subseteq \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$.

- Suppose $\mathsf{K} \subseteq \mathsf{L}$ and $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. Then we have

$$\mathrm{Ker}(\mathsf{L}) \le \mathrm{Ker}(\mathsf{K}) \le \mathrm{Ker}(\mathcal{A}).$$

  So $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{L})$. Hence, if $\mathsf{K} \subseteq \mathsf{L}$ then $\mathbb{V}^{\mathsf{Sem}}(\mathrm{K}) \subseteq \mathbb{V}^{\mathsf{Sem}}(\mathrm{L})$.

- Finally, suppose $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}))$. Then $\mathrm{Ker}(\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})) \le \mathrm{Ker}(\mathcal{A})$. But, note that, for all $\mathcal{B} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$, we have $\mathrm{Ker}(\mathsf{K}) \le \mathrm{Ker}(\mathcal{B})$, whence $\mathrm{Ker}(\mathsf{K}) \le \mathrm{Ker}(\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}))$. Combining the two inclusions, we get

$$\mathrm{Ker}(\mathsf{K}) \le \mathrm{Ker}(\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})) \le \mathrm{Ker}(\mathcal{A}).$$

  Thus, $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. We conclude that $\mathbb{V}^{\mathsf{Sem}}(\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})) \subseteq \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$.

We work, next, with $\mathbb{V}^{\mathsf{Syn}}$. Consider the two mappings

$$\begin{array}{c}\mathrm{NEq}:\mathcal{P}(\mathrm{AlgSys}(\mathbf{F}))\to\mathcal{P}(\mathrm{NEq}(\mathbf{F})),\\ \mathrm{NMod}:\mathcal{P}(\mathrm{NEq}(\mathbf{F}))\to\mathcal{P}(\mathrm{AlgSys}(\mathbf{F})),\end{array}$$

defined by

$$\begin{array}{rcl}\mathrm{NEq}(\mathsf{K}) & = & \{\sigma^\flat\approx\tau^\flat\in\mathrm{NEq}(\mathbf{F}):\mathsf{K}\vDash\sigma^\flat\approx\tau^\flat\},\quad \mathsf{K}\subseteq\mathrm{AlgSys}(\mathbf{F});\\ \mathrm{NMod}(E) & = & \{\mathcal{A}\in\mathrm{AlgSys}(\mathbf{F}):\mathcal{A}\vDash E\},\quad E\subseteq\mathrm{NEq}(\mathbf{F}).\end{array}$$

It is not difficult to see that NEq and NMod form a Galois connection. Thus, $\mathbb{V}^{\mathsf{Syn}}=\mathrm{NMod}\circ\mathrm{NEq}$ is a closure operator on $\mathrm{AlgSys}(\mathbf{F})$. ∎

We prove that the semantic variety is always included in the syntactic variety generated by the same class of **F**-algebraic systems.

Theorem 39 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be a base algebraic system and* K *a class of* **F***-algebraic systems. Then*

$$\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})\subseteq\mathbb{V}^{\mathsf{Syn}}(\mathsf{K}).$$

Proof: Suppose that $\mathcal{A}\in\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. Let $\sigma^\flat\approx\tau^\flat$ be a natural **F**-equation, such that $\mathsf{K}\vDash\sigma^\flat\approx\tau^\flat$. We must show that $\mathcal{A}\vDash\sigma^\flat\approx\tau^\flat$. To this end, let $\Sigma\in|\mathbf{Sign}^\flat|$ and $\vec{\phi}\in\mathrm{SEN}^\flat(\Sigma)$. Since $\mathsf{K}\vDash\sigma^\flat\approx\tau^\flat$, we have, for all $\mathcal{K}=\langle\mathbf{K},\langle K,\kappa\rangle\rangle\in\mathsf{K}$,

$$\kappa_\Sigma(\sigma^\flat_\Sigma(\vec{\phi}))=\kappa_\Sigma(\tau^\flat_\Sigma(\vec{\phi})).$$

This means that $\langle\sigma^\flat_\Sigma(\vec{\phi}),\tau^\flat_\Sigma(\vec{\phi})\rangle\in\mathrm{Ker}_\Sigma(\mathcal{K})$. Since this holds for all $\mathcal{K}\in\mathsf{K}$, we conclude that $\langle\sigma^\flat_\Sigma(\vec{\phi}),\tau^\flat_\Sigma(\vec{\phi})\rangle\in\mathrm{Ker}_\Sigma(\mathsf{K})$. But, by hypothesis, $\mathrm{Ker}(\mathsf{K})\leq\mathrm{Ker}(\mathcal{A})$. Therefore, we get $\langle\sigma^\flat_\Sigma(\vec{\phi}),\tau^\flat_\Sigma(\vec{\phi})\rangle\in\mathrm{Ker}_\Sigma(\mathcal{A})$. This means that

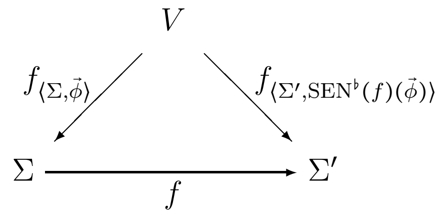
$$\alpha_\Sigma(\sigma^\flat_\Sigma(\vec{\phi}))=\alpha_\Sigma(\tau^\flat_\Sigma(\vec{\phi})).$$

Since $\Sigma\in|\mathbf{Sign}^\flat|$ and $\vec{\phi}\in\mathrm{SEN}^\flat(\Sigma)$ were arbitrary, we get that $\mathcal{A}\vDash\sigma^\flat\approx\tau^\flat$. Now we conclude that $\mathcal{A}\in\mathbb{V}^{\mathsf{Syn}}(\mathsf{K})$. Thus, $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})\subseteq\mathbb{V}^{\mathsf{Syn}}(\mathsf{K})$. ∎

Now we look at some sufficient conditions that ensure that these two variety operators generate the same class of **F**-algebraic systems. However, the terminology, methodology and work presented in the rest of the section have proven very useful in many contexts and can be used to reconcile results that hold in more restricted contexts with partial analogs that hold in this very abstract setting.

Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be a base algebraic system and consider a cardinal κ (which will usually be taken to be either finite or ω). A **source signature κ-variable pair** (**ssv**$^\kappa$ for short) $\langle V,\vec{v}\rangle$ consists of a signature $V\in|\mathbf{Sign}^\flat|$ and a vector $\vec{v}\in\mathrm{SEN}^\flat(V)^\kappa$, satisfying the following conditions:

1. For all $\Sigma \in |\mathbf{Sign}^\flat|$, $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^\kappa$, there exists $f_{\langle \Sigma, \vec{\phi} \rangle} \in \mathbf{Sign}^\flat(V, \Sigma)$, such that
$$\mathrm{SEN}^\flat(f_{\langle \Sigma, \vec{\phi} \rangle})(\vec{v}) = \vec{\phi};$$
2. For all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^\kappa$ and $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,



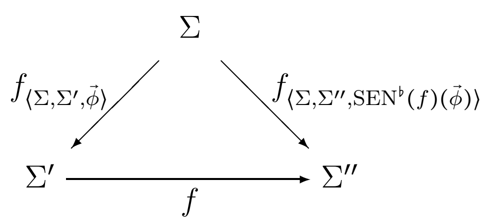
$$f \circ f_{\langle \Sigma, \vec{\phi} \rangle} = f_{\langle \Sigma', \mathrm{SEN}^\flat(f)(\vec{\phi}) \rangle}.$$

An algebraic system $\mathbf{F}$ is called **κ-term** if it has an ssv$^\kappa$. The morphisms $f_{\langle \Sigma, \vec{\phi} \rangle} : V \to \Sigma$ are referred to as the **ssv$^\kappa$ maps**.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system. We say that $\mathbf{F}$ **has κ-variables** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, there exists $\vec{v}^\Sigma \in \mathrm{SEN}^\flat(\Sigma)^\kappa$, such that $\langle \Sigma, \vec{v}^\Sigma \rangle$ is an ssv$^\kappa$, with ssv$^\kappa$ maps $f_{\langle \Sigma, \Sigma', \vec{\phi} \rangle} : \Sigma \to \Sigma'$, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$ and $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma')^\kappa$. The algebraic system $\mathbf{F}$ is called **κ-formulaic** if it has κ-variables.

It follows, according to the preceding definitions, that $\mathbf{F}$ is κ-formulaic, with Σ-κ-variables $\vec{v}^\Sigma$ and ssv$^\kappa$ maps $f_{\langle \Sigma, \Sigma', \vec{\phi} \rangle}$ if:

- For all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma')^\kappa$,
$$f_{\langle \Sigma, \Sigma', \vec{\phi} \rangle}(\vec{v}^\Sigma) = \vec{\phi};$$
- For all $\Sigma, \Sigma', \Sigma'' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma', \Sigma'')$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma')^\kappa$,



$$f \circ f_{\langle \Sigma, \Sigma', \vec{\phi} \rangle} = f_{\langle \Sigma, \Sigma'', \mathrm{SEN}^\flat(f)(\vec{\phi}) \rangle}.$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a κ-formulaic algebraic system, with κ-variables $\vec{v}^\Sigma$, $\Sigma \in |\mathbf{Sign}^\flat|$. $\mathbf{F}$ will be called **κ-transformational** (**modulo the given κ-variables**) if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, there exists $\sigma^{\langle \Sigma, \phi \rangle} : (\mathrm{SEN}^\flat)^\kappa \to \mathrm{SEN}^\flat$, such that:

- $\sigma^{\langle \Sigma, \phi \rangle}$ depends on only finitely may variables;

- $\phi = \sigma_{\Sigma}^{\langle\Sigma,\phi\rangle}(\vec{v}^{\Sigma})$.

We have the following relation now that serves, so to speak, in bridging the gap between the semantical and syntactical definitions of varieties of algebraic systems.

Lemma 40 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a transformational algebraic system and* $\mathcal{A} = \langle\mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system. Then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle\phi,\psi\rangle \in \mathrm{Ker}_\Sigma(\mathcal{A}) \quad \textit{iff} \quad \mathcal{A} \vDash \sigma^{\langle\Sigma,\phi\rangle} \approx \sigma^{\langle\Sigma,\psi\rangle}.$$

Proof: Suppose, first, that $\mathcal{A} \vDash \sigma^{\langle\Sigma,\phi\rangle} \approx \sigma^{\langle\Sigma,\psi\rangle}$. This means that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\alpha_{\Sigma'}(\sigma_{\Sigma'}^{\langle\Sigma,\phi\rangle}(\vec{\phi})) = \alpha_{\Sigma'}(\sigma_{\Sigma'}^{\langle\Sigma,\psi\rangle}(\vec{\phi})).$$

Taking $\Sigma' = \Sigma$ and $\vec{\phi} = \vec{v}^\Sigma$, we get $\alpha_\Sigma(\sigma_\Sigma^{\langle\Sigma,\phi\rangle}(\vec{v}^\Sigma)) = \alpha_\Sigma(\sigma_\Sigma^{\langle\Sigma,\psi\rangle}(\vec{v}^\Sigma))$, or, what amounts to the same, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. Hence, $\langle\phi,\psi\rangle \in \mathrm{Ker}_\Sigma(\mathcal{A})$.

Suppose, conversely, that $\langle\phi,\psi\rangle \in \mathrm{Ker}_\Sigma(\mathcal{A})$. This means that $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. Since $\mathbf{F}$ is assumed to be transformational, there exist $\sigma^{\langle\Sigma,\phi\rangle}$ and $\sigma^{\langle\Sigma,\psi\rangle}$ in $N^\flat$, such that $\sigma_\Sigma^{\langle\Sigma,\phi\rangle}(\vec{v}^\Sigma) = \phi$ and $\sigma_\Sigma^{\langle\Sigma,\psi\rangle}(\vec{v}^\Sigma) = \psi$. Thus, we get

$$\alpha_\Sigma(\sigma_\Sigma^{\langle\Sigma,\phi\rangle}(\vec{v}^\Sigma)) = \alpha_\Sigma(\sigma_\Sigma^{\langle\Sigma,\psi\rangle}(\vec{v}^\Sigma)).$$

Now, by formulaicity, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma')$, we get an ssv$^\kappa$ map $f_{\langle\Sigma,\Sigma',\vec{\phi}\rangle} : \Sigma \to \Sigma'$, for which we have

$$\mathrm{SEN}(F(f_{\langle\Sigma,\Sigma',\vec{\phi}\rangle}))(\alpha_\Sigma(\sigma_\Sigma^{\langle\Sigma,\phi\rangle}(\vec{v}^\Sigma))) = \mathrm{SEN}(F(f_{\langle\Sigma,\Sigma',\vec{\phi}\rangle}))(\alpha_\Sigma(\sigma_\Sigma^{\langle\Sigma,\psi\rangle}(\vec{v}^\Sigma))).$$

Hence, since α is a natural transformation,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f_{\langle\Sigma,\Sigma',\vec{\phi}\rangle})(\sigma_\Sigma^{\langle\Sigma,\phi\rangle}(\vec{v}^\Sigma))) = \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f_{\langle\Sigma,\Sigma',\vec{\phi}\rangle})(\sigma_\Sigma^{\langle\Sigma,\psi\rangle}(\vec{v}^\Sigma))).$$

And since $\sigma^{\langle\Sigma,\phi\rangle}$, $\sigma^{\langle\Sigma,\psi\rangle}$ are also natural transformations, we get

$$\alpha_{\Sigma'}(\sigma_{\Sigma'}^{\langle\Sigma,\phi\rangle}(\mathrm{SEN}^\flat(f_{\langle\Sigma,\Sigma',\vec{\phi}\rangle})(\vec{v}^\Sigma))) = \alpha_{\Sigma'}(\sigma_{\Sigma'}^{\langle\Sigma,\psi\rangle}(\mathrm{SEN}^\flat(f_{\langle\Sigma,\Sigma',\vec{\phi}\rangle})(\vec{v}^\Sigma))).$$

Finally, by the κ-variable property, we get

$$\alpha_{\Sigma'}(\sigma_{\Sigma'}^{\langle\Sigma,\phi\rangle}(\vec{\phi})) = \alpha_{\Sigma'}(\sigma_{\Sigma'}^{\langle\Sigma,\psi\rangle}(\vec{\phi})).$$

Since $\Sigma' \in |\mathbf{Sign}^\flat|$ and $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma')$ were arbitrary, we conclude that $\mathcal{A} \vDash \sigma^{\langle\Sigma,\phi\rangle} \approx \sigma^{\langle\Sigma,\psi\rangle}$. ■

Now we are in a position to prove that, for algebraic systems over transformational base algebraic systems, the semantic and syntactic variety operators coincide.

Theorem 41 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a transformational algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems. Then*

$$\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathbb{V}^{\mathsf{Syn}}(\mathsf{K}).$$

Proof: By Theorem 39, $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{V}^{\mathsf{Syn}}(\mathsf{K})$ always holds. For the reverse inclusion, suppose that $\mathcal{A} \in \mathbb{V}^{\mathsf{Syn}}(\mathsf{K})$. We must show that $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$, i.e., that $\mathrm{Ker}(\mathsf{K}) \leq \mathrm{Ker}(\mathcal{A})$. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in \mathrm{Ker}_\Sigma(\mathsf{K})$. Then, by Lemma 40, $\mathsf{K} \vDash \sigma^{\langle \Sigma, \phi \rangle} \approx \sigma^{\langle \Sigma, \psi \rangle}$. Since $\mathcal{A} \in \mathbb{V}^{\mathsf{Syn}}(\mathsf{K})$, we get that $\mathcal{A} \vDash \sigma^{\langle \Sigma, \phi \rangle} \approx \sigma^{\langle \Sigma, \psi \rangle}$. Using Lemma 40 again, we infer that $\langle \phi, \psi \rangle \in \mathrm{Ker}_\Sigma(\mathcal{A})$. Thus, $\mathrm{Ker}(\mathsf{K}) \leq \mathrm{Ker}(\mathcal{A})$. Hence, $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. ■

2.6 π-Institutions

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. A **closure** (**operator**) **system** on $\mathbf{F}$ is a collection $C = \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, such that

$$C_\Sigma : \mathcal{P}(\mathrm{SEN}^\flat(\Sigma)) \to \mathcal{P}(\mathrm{SEN}^\flat(\Sigma))$$

is a closure operator on $\mathrm{SEN}^\flat(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}^\flat|$, and, moreover, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, and all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\mathrm{SEN}^\flat(f)(C_\Sigma(\Phi)) \subseteq C_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)).$$

This condition is often referred to as **structurality**.

A π**-institution** is a pair $\mathcal{I} = \langle \mathbf{F}, C \rangle$, where $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ is an algebraic system and C is a closure system on $\mathbf{F}$. We say that the π-institution $\mathcal{I}$ is **based on** the algebraic system $\mathbf{F}$. The following assumption is adopted throughout our treatise, unless explicitly stated otherwise:

> **Global Assumption**: If, for some $\Sigma \in |\mathbf{Sign}^\flat|$, $C_\Sigma(\varnothing) \neq \varnothing$, then, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $C_\Sigma(\varnothing) \neq \varnothing$. (2.1)

The set of Σ-**theorems**, denoted $\mathrm{Thm}_\Sigma(\mathcal{I})$, is defined by

$$\mathrm{Thm}_\Sigma(\mathcal{I}) = C_\Sigma(\varnothing).$$

We then set $\mathrm{Thm}(\mathcal{I}) = \{\mathrm{Thm}_\Sigma(\mathcal{I})\}_{\Sigma \in |\mathbf{Sign}|}$. We denote by $\overline{\varnothing}$ the $|\mathbf{Sign}^\flat|$-indexed collection $\overline{\varnothing} = \{\varnothing\}_{\Sigma \in |\mathbf{Sign}^\flat|}$. The Global Assumption (2.1), adopted above, says that, if a π-institution has Σ-theorems, for some signature Σ, then it has Σ-theorems, for every signature Σ.

A **natural theorem of** $\mathcal{I}$ is a natural transformation

$$\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$$

in $N^\flat$, for some $k \geq 0$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^k$,

$$\top^\flat_\Sigma(\vec{\phi}) \in \mathrm{Thm}_\Sigma(\mathcal{I}).$$

That is, a natural theorem of $\mathcal{I}$ is a natural transformation in $N^\flat$ all of whose values are theorems. We denote by $\mathrm{NThm}(\mathcal{I})$ the collection of all natural theorems of a π-institution $\mathcal{I}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution, based on $\mathbf{F}$, and $\Sigma \in |\mathbf{Sign}^\flat|$. A subset $T_\Sigma \subseteq \mathrm{SEN}^\flat(\Sigma)$ is called a **Σ-theory** if

$$C_\Sigma(T_\Sigma) = T_\Sigma.$$

We use $\mathrm{Th}_\Sigma(\mathcal{I})$ to denote the collection of all Σ-theories of the π-institution $\mathcal{I}$. A **theory family** of $\mathcal{I}$ is a sentence family $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ of $\mathbf{F}$, such that $T_\Sigma \in \mathrm{Th}_\Sigma(\mathcal{I})$, for all $\Sigma \in |\mathbf{Sign}^\flat|$. The collection of all theory families of $\mathcal{I}$ will be denoted by $\mathrm{ThFam}(\mathcal{I})$. Ordered by signature-wise inclusion $\leq$, it forms a complete lattice, denoted $\mathbf{ThFam}(\mathcal{I}) = \langle \mathrm{ThFam}(\mathcal{I}), \leq \rangle$.

A theory family of $\mathcal{I}$ is called a **theory system** of $\mathcal{I}$ if it is a sentence system, i.e., if it is invariant under signature morphisms. We denote by $\mathrm{ThSys}(\mathcal{I})$, the collection of all theory systems of $\mathcal{I}$. This collection forms a complete sublattice $\mathbf{ThSys}(\mathcal{I}) = \langle \mathrm{ThSys}(\mathcal{I}), \leq \rangle$ of the complete lattice $\mathbf{ThFam}(\mathcal{I})$.

Note that the minimum element of both $\mathbf{ThFam}(\mathcal{I})$ and $\mathbf{ThSys}(\mathcal{I})$ is $\mathrm{Thm}(\mathcal{I})$, the **theorem system** of $\mathcal{I}$, and the maximum element is

$$\mathrm{SEN}^\flat = \{\mathrm{SEN}^\flat(\Sigma)\}_{\Sigma \in |\mathbf{Sign}^\flat|}.$$

Thus, $\mathrm{SEN}^\flat$ is used to denote both the sentence functor of the base algebraic system $\mathbf{F}$ of the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ and the maximum theory family (system) $\mathrm{SEN}^\flat = \{\mathrm{SEN}^\flat(\Sigma)\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ of $\mathcal{I}$. This overloading will not, hopefully, cause any confusion, since the context can be used to clarify the meaning.

Proposition 42 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution and $T \in \mathrm{ThFam}(\mathcal{I})$. Then $\overleftarrow{T}$ is the largest theory system of $\mathcal{I}$ included in T.*

Proof: Since, by Proposition 2, $\overleftarrow{T}$ is the largest sentence system included in T, it suffices to show that $\overleftarrow{T}$ is a theory system. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\overleftarrow{T}_\Sigma)$. We must show that $\phi \in \overleftarrow{T}_\Sigma$. So let $\Sigma' \in |\mathbf{Sign}^\flat|$ and $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$. Then we have

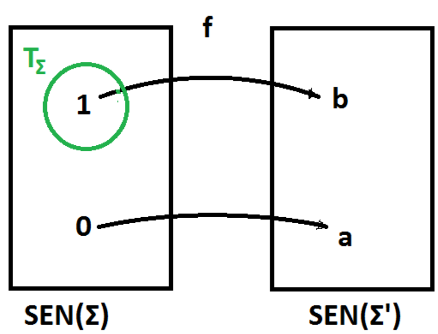
$$\begin{array}{lll}
\mathrm{SEN}^\flat(f)(\phi) & \in & \mathrm{SEN}(f)(C_\Sigma(\overleftarrow{T}_\Sigma)) \quad \text{(hypothesis)} \\
& \subseteq & C_{\Sigma'}(\mathrm{SEN}(f)(\overleftarrow{T}_\Sigma)) \quad \text{(structurality)} \\
& \subseteq & C_{\Sigma'}(T_{\Sigma'}) \quad \text{(definition of } \overleftarrow{T}) \\
& = & T_{\Sigma'} \quad (T \in \mathrm{ThFam}(\mathcal{I})).
\end{array}$$

We now conclude, by the definition of $\overleftarrow{T}$, that $\phi \in \overleftarrow{T}_\Sigma$. ■

On the negative side, it is not true, in general, that, given a theory family T of a π-institution $\mathcal{I}$, the least sentence system $\overrightarrow{T}$, containing T, is a theory system. We show that this is the case via an example.

Example 43 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *consists of two signatures* Σ *and* Σ' *and the only (non-identity) morphism is* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign} \to \mathbf{Set}$ *is defined by setting*
$$\mathrm{SEN}^\flat(\Sigma) = \{0,1\}, \quad \mathrm{SEN}^\flat(\Sigma') = \{a,b\}$$
and $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$*;*
- $N^\flat$ *consists of only the projection natural transformations.*



Consider the closure system C *on* $\mathbf{F}$ *defined by setting*
$$\mathcal{C}_\Sigma = \{\varnothing, \{0\}, \{1\}, \{0,1\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{a,b\}\}$$
and let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the associated* π*-institution.*

Finally, take $T = \{T_\Sigma, T_{\Sigma'}\} \in \mathrm{ThFam}(\mathcal{I})$ *to be the theory family specified by*
$$T_\Sigma = \{1\} \quad \text{and} \quad T_{\Sigma'} = \varnothing.$$
Then we have
$$\overrightarrow{T}_\Sigma = \{1\} \quad \text{and} \quad \overrightarrow{T}_{\Sigma'} = \{b\}.$$
Since clearly
$$C_{\Sigma'}(\overrightarrow{T}_{\Sigma'}) = C_{\Sigma'}(\{b\}) = \{a,b\} \neq \overrightarrow{T}_{\Sigma'},$$
it follows that $\overrightarrow{T}$ *is not a theory system of* $\mathcal{I}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We define two operators

$$\begin{array}{rrcl} C: & \mathrm{SenFam}(\mathbf{F}) & \to & \mathrm{ThFam}(\mathcal{I}); \\ \overrightarrow{C}: & \mathrm{SenFam}(\mathbf{F}) & \to & \mathrm{ThSys}(\mathcal{I}); \end{array}$$

as follows. Consider a sentence family $T \in \mathrm{SenFam}(\mathbf{F})$.

- $C(T) = \{C(T)_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,
$$C(T)_\Sigma = C_\Sigma(T_\Sigma);$$
- $\overrightarrow{C}(T) = \{\overrightarrow{C}(T)_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,
$$\overrightarrow{C}(T)_\Sigma = C_\Sigma(\overrightarrow{T}_\Sigma).$$

It is clear that $C(T)$ is the smallest theory family of $\mathcal{I}$ containing T. We show in the next proposition that $\overrightarrow{C}(T)$ is the smallest theory system of $\mathcal{I}$ that contains the sentence family T. Note that this implies, in particular, that $\overrightarrow{C}(T)$ is the smallest theory system of $\mathcal{I}$ that contains a given theory family T of $\mathcal{I}$. Note, also, that $\overrightarrow{C}(T) = C(\overrightarrow{T})$ should not be confused with $\overrightarrow{C(T)}$, which, as shown in Example 43, may not be a theory family of $\mathcal{I}$.

Proposition 44 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution, based on $\mathbf{F}$, and $T \in \mathrm{SenFam}(\mathbf{F})$. Then $\overrightarrow{C}(T)$ is the smallest theory system of $\mathcal{I}$ that includes T.*

Proof: It is clear by the definition that $\overrightarrow{C}(T) = C(\overrightarrow{T}) \in \mathrm{ThFam}(\mathcal{I})$. We show that it is a theory system. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\overrightarrow{T}_\Sigma)$. Consider $\Sigma' \in |\mathbf{Sign}^\flat|$ and $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$. Then we have

$$\begin{array}{rcll} \mathrm{SEN}^\flat(f)(\phi) & \in & \mathrm{SEN}^\flat(f)(C_\Sigma(\overrightarrow{T}_\Sigma)) & \text{(definition of } \overrightarrow{C}(T)) \\ & \subseteq & C_{\Sigma'}(\mathrm{SEN}^\flat(f)(\overrightarrow{T}_\Sigma)) & \text{(structurality)} \\ & \subseteq & C_{\Sigma'}(\overrightarrow{T}_{\Sigma'}) & \text{(definition of } \overrightarrow{T}) \\ & = & \overrightarrow{C}(T)_{\Sigma'} & \text{(definition of } \overrightarrow{C}(T)). \end{array}$$

It remains to show that $C(\overrightarrow{T})$ is the smallest theory system containing T. To this end, let $T' \in \mathrm{ThSys}(\mathcal{I})$, such that $T \le T'$. Since, by Proposition 2, $\overrightarrow{T}$ is the least sentence system containing T, we get $\overrightarrow{T} \le T'$. Therefore, since $C(\overrightarrow{T})$ is the least theory family containing $\overrightarrow{T}$, $C(\overrightarrow{T}) \le T'$. Thus, we conclude that $\overrightarrow{C}(T) = C(\overrightarrow{T}) \le T'$ and $\overrightarrow{C}(T)$ is the least theory system of $\mathcal{I}$ that includes T. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. We say that $\mathcal{I}$ is:

- **inconsistent** if $\mathrm{ThFam}(\mathcal{I}) = \{\mathrm{SEN}^\flat\}$, i.e., if, for all $\Sigma \in |\mathbf{Sign}^\flat|$,
$$C_\Sigma(\varnothing) = \mathrm{SEN}^\flat(\Sigma);$$
- **almost inconsistent** if
$$\mathrm{ThFam}(\mathcal{I}) = \{T : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(T_\Sigma = \varnothing \text{ or } T_\Sigma = \mathrm{SEN}^\flat(\Sigma))\};$$
- **trivial** if it is either inconsistent or almost inconsistent.

Lemma 45 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is trivial if and only if, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,* $\psi \in C_\Sigma(\phi)$*.*

Proof: Suppose, first, that $\mathcal{I}$ is trivial and let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$. Since $\phi \in C_\Sigma(\phi)$, we have $C_\Sigma(\phi) \neq \varnothing$, which implies that $C_\Sigma(\phi) = \mathrm{SEN}^\flat(\Sigma)$. Therefore, $\psi \in C_\Sigma(\phi)$.

Suppose, conversely, that the given condition holds. Let $T \in \mathrm{ThFam}(\mathcal{I})$ and $\Sigma \in |\mathbf{Sign}^\flat|$, such that $T_\Sigma \neq \varnothing$. Then, there exists $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$. But then, by hypothesis, for all $\psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\psi \in C_\Sigma(\phi) \subseteq C_\Sigma(T_\Sigma) = T_\Sigma.$$
Therefore, we get that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $T_\Sigma = \varnothing$ or $T_\Sigma = \mathrm{SEN}^\flat(\Sigma)$, showing that T is almost inconsistent. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system. We can order π-institutions based on $\mathbf{F}$ by comparing their closure systems. Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ and $\mathcal{I}' = \langle \mathbf{F}, C'\rangle$ two π-institutions based on $\mathbf{F}$. We say that $\mathcal{I}'$ is an **extension** of $\mathcal{I}$ and that $\mathcal{I}$ is **weaker** than $\mathcal{I}'$, written $\mathcal{I} \leq \mathcal{I}'$ (or $C \leq C'$) if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$,
$$C_\Sigma(\Phi) \subseteq C'_\Sigma(\Phi).$$
Given a collection $\mathcal{I}^i = \langle \mathbf{F}, C^i\rangle$, $i \in I$, of π-institutions based on the same algebraic system $\mathbf{F}$, the **intersection** $\bigcap_{i\in I} \mathcal{I}^i = \langle \mathbf{F}, \bigcap_{i\in I} C^i\rangle$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$,
$$(\bigcap_{i\in I} C^i)_\Sigma(\Phi) = \bigcap_{i\in I} C^i_\Sigma(\Phi).$$
It can be shown that $\bigcap_{i\in I} C^i$ is a closure system on $\mathbf{F}$ and that it forms the meet with respect to the $\leq$ order of the closure systems C^i, $i \in I$, on $\mathbf{F}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution. Given a theory system $T \in \mathrm{ThSys}(\mathcal{I})$, we define the family $C^T = \{C^T_\Sigma\}_{\Sigma\in|\mathbf{Sign}^\flat|}$ of operators $C^T_\Sigma : \mathcal{P}(\mathrm{SEN}^\flat(\Sigma)) \to \mathcal{P}(\mathrm{SEN}^\flat(\Sigma))$ by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$,
$$C^T_\Sigma(\Phi) = C_\Sigma(T_\Sigma \cup \Phi).$$
We show that C^T is a closure system on $\mathbf{F}$.

Proposition 46 *Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution and $T \in \mathrm{ThSys}(\mathcal{I})$. Then C^T is a closure system on $\mathbf{F}$.*

Proof: We must first show that $C^T_\Sigma : \mathcal{P}(\mathrm{SEN}^\flat(\Sigma)) \to \mathcal{P}(\mathrm{SEN}^\flat(\Sigma))$ is a closure operator. That it is inflationary and monotone follows directly from the corresponding properties of C_Σ. To see that it is idempotent, let $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{rcl} C^T_\Sigma(C^T_\Sigma(\Phi)) & = & C_\Sigma(T_\Sigma \cup C_\Sigma(T_\Sigma \cup \Phi)) \quad \text{(by definition)} \\ & = & C_\Sigma(C_\Sigma(T_\Sigma \cup \Phi)) \quad (\text{since } T_\Sigma \subseteq C_\Sigma(T_\Sigma \cup \Phi)) \\ & = & C_\Sigma(T_\Sigma \cup \Phi) \quad (\text{idempotency of } C) \\ & = & C^T_\Sigma(\Phi) \quad \text{(by definition).} \end{array}$$

Finally, we must show that C^T is structural. To this end, let $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$. We have

$$\begin{array}{rcl} \mathrm{SEN}^\flat(f)(C^T_\Sigma(\Phi)) & = & \mathrm{SEN}^\flat(f)(C_\Sigma(T_\Sigma \cup \Phi)) \quad \text{(by definition)} \\ & \subseteq & C_{\Sigma'}(\mathrm{SEN}^\flat(f)(T_\Sigma) \cup \mathrm{SEN}^\flat(f)(\Phi)) \\ & & \qquad\qquad (\text{by the structurality of } C) \\ & \subseteq & C_{\Sigma'}(T_{\Sigma'} \cup \mathrm{SEN}^\flat(f)(\Phi)) \quad (T \in \mathrm{ThSys}(\mathcal{I})) \\ & = & C^T_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \quad \text{(by definition).} \end{array}$$

We conclude that $C^T = \{C^T_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ is a closure system on $\mathbf{F}$. ∎

Since C^T is a closure system on $\mathbf{F}$, we get, by definition, that the structure $\langle \mathbf{F}, C^T\rangle$ is a π-institution. We use the notation $\mathcal{I}^T = \langle \mathbf{F}, C^T\rangle$ to denote this π-institution.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$. An $\mathcal{I}$-**logical morphism** (or simply **logical morphism** if $\mathcal{I}$ is clear from context) is a morphism $\langle F, \alpha\rangle : \mathbf{F} \to \mathbf{F}$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\alpha_\Sigma(C_\Sigma(\Phi)) \subseteq C_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

More generally, let $\mathbf{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ and $\mathbf{F}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ be two algebraic systems and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ and $\mathcal{I}' = \langle \mathbf{F}', C'\rangle$ be π-institutions based on $\mathbf{F}$ and $\mathbf{F}'$, respectively. A **logical morphism** $\langle F, \alpha\rangle : \mathcal{I} \to \mathcal{I}'$ is an algebraic system morphism $\langle F, \alpha\rangle : \mathbf{F} \to \mathbf{F}'$, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)$,

$$\alpha_\Sigma(C_\Sigma(\Phi)) \subseteq C_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

The following lemma characterizes logical morphisms:

Lemma 47 *Let $\mathbf{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{F}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ be two algebraic systems and $\mathcal{I} = \langle \mathbf{F}, C\rangle$, $\mathcal{I}' = \langle \mathbf{F}', C'\rangle$ be π-institutions, based on $\mathbf{F}$, $\mathbf{F}'$, respectively. Suppose $\langle F, \alpha\rangle : \mathbf{F} \to \mathbf{F}$ is an algebraic system morphism. Then the following conditions are equivalent:*

(a) $\langle F,\alpha\rangle : \mathcal{I} \to \mathcal{I}'$ *is a logical morphism;*

(b) For all $\Sigma \in |\mathbf{Sign}|$ *and all* $\Psi \subseteq \mathrm{SEN}'(F(\Sigma))$,

$$C_\Sigma(\alpha_\Sigma^{-1}(\Psi)) \leq \alpha_\Sigma^{-1}(C'_{F(\Sigma)}(\Psi));$$

(c) For all $T' \in \mathrm{ThFam}(\mathcal{I}')$, $\alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$.

Proof:

(a)⇒(b) Let $\Sigma \in |\mathbf{Sign}|$ and $\Psi \subseteq \mathrm{SEN}'(F(\Sigma))$. Then, we have

$$\begin{array}{rcll}\alpha_\Sigma(C_\Sigma(\alpha_\Sigma^{-1}(\Psi))) & \subseteq & C_{F(\Sigma)}(\alpha_\Sigma(\alpha_\Sigma^{-1}(\Psi))) & \text{(hypothesis)}\\ & \subseteq & C_{F(\Sigma)}(\Psi). & \text{(set theory)}\end{array}$$

We conclude that $C_\Sigma(\alpha_\Sigma^{-1}(\Psi)) \subseteq \alpha_\Sigma^{-1}(C_{F(\Sigma)}(\Psi))$.

(b)⇒(c) Suppose that $T' \in \mathrm{ThFam}(\mathcal{I}')$. Then we have

$$\begin{array}{rcll}C(\alpha^{-1}(T')) & \leq & \alpha^{-1}(C'(T')) & \text{(hypothesis)}\\ & = & \alpha^{-1}(T'). & (T' \in \mathrm{ThFam}(\mathcal{I}'))\end{array}$$

Therefore, $\alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$.

(c)⇒(a) Let $\Sigma \in |\mathbf{Sign}|$ and $\Phi \subseteq \mathrm{SEN}(\Sigma)$. Then, we have, for all $T \in \mathrm{ThFam}(\mathcal{I}')$,

$$\begin{array}{rcll}\alpha_\Sigma(\Phi) \subseteq T_{F(\Sigma)} & \text{iff} & \Phi \subseteq \alpha_\Sigma^{-1}(T_{F(\Sigma)}) & \text{(set theory)}\\ & \text{implies} & C_\Sigma(\Phi) \subseteq \alpha_\Sigma^{-1}(T_{F(\Sigma)}) & \text{(hypothesis)}\\ & \text{iff} & \alpha_\Sigma(C_\Sigma(\Phi)) \subseteq T_{F(\Sigma)}. & \text{(set theory)}\end{array}$$

Since $T \in \mathrm{ThFam}(\mathcal{I}')$ was arbitrary, we get that

$$\alpha_\Sigma(C_\Sigma(\Phi)) \subseteq C'_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

So $\langle F,\alpha\rangle$ is a logical morphism.

■

In the special case of $\mathcal{I}$-logical morphisms, we obtain the following

Corollary 48 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution, based on* $\mathbf{F}$*, and* $\langle F,\alpha\rangle : \mathbf{F} \to \mathbf{F}$ *an algebraic system morphism. Then* $\langle F,\alpha\rangle$ *is an* $\mathcal{I}$*-logical morphism if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$.

Proof: Directly from Lemma 47. ■

2.7 Matrix Families and Systems

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. An **F-matrix family** is a pair $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, where $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ is an $\mathbf{F}$-algebraic system and $T \in \mathrm{SenFam}(\mathcal{A})$. The collection of all $\mathbf{F}$-matrix families is denoted by $\mathrm{MatFam}(\mathbf{F})$. An **F-matrix system** is an $\mathbf{F}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, such that $T \in \mathrm{SenSys}(\mathcal{A})$. The collection of all $\mathbf{F}$-matrix systems is denoted by $\mathrm{MatSys}(\mathbf{F})$.

An $\mathbf{F}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ defines a closure system $C^{\mathfrak{A}} = \{C^{\mathfrak{A}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ on $\mathbf{F}$ by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{l}\phi \in C^{\mathfrak{A}}_\Sigma(\Phi) \text{ if and only if, for all } \Sigma' \in |\mathbf{Sign}^\flat| \text{ and all } f \in \mathbf{Sign}^\flat(\Sigma, \Sigma'),\\ \qquad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq T_{F(\Sigma')} \text{ imlpies } \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \in T_{F(\Sigma')}.\end{array}$$

Let, now, M be a class of $\mathbf{F}$-matrix families. We denote by

$$C^{\mathsf{M}} = \{C^{\mathsf{M}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$$

the closure system on $\mathbf{F}$ that is the signature-wise intersection of the closure systems $C^{\mathfrak{A}}$, $\mathfrak{A} \in \mathsf{M}$, i.e.,

$$C^{\mathsf{M}} = \bigcap_{\mathfrak{A} \in \mathsf{M}} C^{\mathfrak{A}}.$$

We use the notation $\mathcal{I}^{\mathsf{M}} = \langle \mathbf{F}, C^{\mathsf{M}} \rangle$ to denote the associated π-institution based on $\mathbf{F}$.

We give a characterization of the closure system C^{M} on $\mathbf{F}$ generated by a class M of matrix families which shows how that closure system is constructed using the generating matrix families.

Proposition 49 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and M a class of $\mathbf{F}$-matrix families. Then $\mathcal{C}^{\mathsf{M}}$ is the least closure system on $\mathbf{F}$ containing the family*

$$\mathcal{T} = \{\alpha^{-1}(T) : \mathfrak{A} = \langle \langle \mathbf{A}, \langle F, \alpha \rangle \rangle, T \rangle \in \mathsf{M}\}.$$

Proof: First we show that $\mathcal{T} \subseteq \mathcal{C}^{\mathsf{M}}$. To this end, let $\mathfrak{A} = \langle \langle \mathbf{A}, \langle F, \alpha \rangle \rangle, T \rangle \in \mathsf{M}$. We must show that $\alpha^{-1}(T) \in \mathcal{C}^{\mathsf{M}}$. Suppose $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C^{\mathsf{M}}_\Sigma(\alpha^{-1}_\Sigma(T_{F(\Sigma)}))$. Then, by the definition of C^{M} and the fact that $\mathfrak{A} \in \mathsf{M}$, we get

$$\alpha_\Sigma(\alpha^{-1}_\Sigma(T_{F(\Sigma)})) \subseteq T_{F(\Sigma)} \quad \text{implies} \quad \alpha_\Sigma(\phi) \in T_{F(\Sigma)}.$$

Note, however, that the antecedent of the displayed implication always holds. So the consequent $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$ holds. Hence, $\phi \in \alpha^{-1}_\Sigma(T_{F(\Sigma)})$. Therefore, $C^{\mathsf{M}}(\alpha^{-1}(T)) \leq \alpha^{-1}(T)$, showing that $\alpha^{-1}(T) \in \mathcal{C}^{\mathsf{M}}$.

Next, we show that, if $\mathcal{C}$ is a closure system on $\mathbf{F}$, such that $\mathcal{T} \subseteq \mathcal{C}$, then $\mathcal{C}^{\mathsf{M}} \subseteq \mathcal{C}$. Equivalently, it suffices to show that $C \leq C^{\mathsf{M}}$. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$. Since $\mathcal{C}$ is a closure system on $\mathbf{F}$, we get, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$, $\mathrm{SEN}^\flat(f)(\phi) \in C_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))$. Thus, since $\mathcal{T} \subseteq \mathcal{C}$, we get, for all $\langle\mathcal{A},T\rangle \in \mathsf{M}$, with $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle$,

$$\mathrm{SEN}^\flat(f)(\Phi) \subseteq \alpha_{\Sigma'}^{-1}(T_{F(\Sigma')}) \quad \text{implies} \quad \mathrm{SEN}^\flat(f)(\phi) \in \alpha_{\Sigma'}^{-1}(T_{F(\Sigma')}),$$

i.e., for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq T_{F(\Sigma')} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \in T_{F(\Sigma')}.$$

Hence, for all $\mathfrak{A} = \langle\mathcal{A},T\rangle \in \mathsf{M}$, $\phi \in C_\Sigma^{\mathfrak{A}}(\Phi)$. We conclude that $\phi \in C_\Sigma^{\mathsf{M}}(\Phi)$. Therefore, $C \leq C^{\mathsf{M}}$, as was to be shown. ■

Let again $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F},C\rangle$ be a π-institution based on $\mathbf{F}$. Let $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system, with $\mathbf{A} = \langle\mathbf{Sign},\mathrm{SEN},N\rangle$. A sentence family $T \in \mathrm{SenFam}(\mathcal{A})$ is called an **$\mathcal{I}$-filter family** and the $\mathbf{F}$-matrix family $\mathfrak{A} = \langle\mathcal{A},T\rangle$ an **$\mathcal{I}$-matrix family** if

$$C \leq C^{\mathfrak{A}}.$$

If T happens to be a sentence system, then we refer to T as an **$\mathcal{I}$-filter system** and to $\mathfrak{A} = \langle\mathcal{A},T\rangle$ as an **$\mathcal{I}$-matrix system**.

We have the following simpler characterization of $\mathcal{I}$-filter families, which follows from the structurality of the closure system of a π-institution.

Lemma 50 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle$ an $\mathbf{F}$-algebraic system, with $\mathbf{A} = \langle\mathbf{Sign},\mathrm{SEN},N\rangle$, and $T \in \mathrm{SenFam}(\mathcal{A})$. T is an $\mathcal{I}$-filter family if and only if, for every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$,*

$$\alpha_\Sigma(\Phi) \subseteq T_{F(\Sigma)} \quad \textit{implies} \quad \alpha_\Sigma(\phi) \in T_{F(\Sigma)}.$$

Proof: Suppose, first, that T is an $\mathcal{I}$-filter family and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi\cup\{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$ and $\alpha_\Sigma(\Phi) \subseteq T_{F(\Sigma)}$. Since T is an $\mathcal{I}$-filter family, $C \leq C^{\langle\mathcal{A},T\rangle}$. Therefore, by taking in the definition of $C^{\langle\mathcal{A},C\rangle}$, $\Sigma' = \Sigma$ and $f : \Sigma \to \Sigma$ to be the identity morphism, we get that $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$.

Suppose, conversely, that the given condition holds and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$. Consider $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$, such that $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq T_{F(\Sigma')}$. Note, that, by structurality, $\mathrm{SEN}^\flat(f)(\phi) \in C_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))$. Therefore, by the assumption and the hypothesis, $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \in T_{F(\Sigma')}$. We conclude that T is an $\mathcal{I}$-filter family. ■

The next lemma shows that the inverse image under an interpretation of an $\mathcal{I}$-filter family or system is a theory family or system, respectively, of $\mathcal{I}$. Moreover this property also characterizes $\mathcal{I}$-filter families/systems.

Lemma 51 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution, based on* $\mathbf{F}$*, and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *be an* $\mathbf{F}$*-algebraic system.*

(a) $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *if and only if* $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$*;*

(b) $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ *if and only if* $\alpha^{-1}(T) \in \mathrm{ThSys}(\mathcal{I})$*.*

Proof:

(a) Suppose, first, that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. We must show that $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$. To this end, suppose $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\alpha_\Sigma^{-1}(T_{F(\Sigma)})$. Since $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we have, by definition,

$$\alpha_\Sigma(\alpha_\Sigma^{-1}(T_{F(\Sigma)})) \subseteq T_{F(\Sigma)} \quad \text{implies} \quad \alpha_\Sigma(\phi) \in T_{F(\Sigma)}.$$

But the hypothesis of this implication holds, whence the conclusion is also true and we get $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$ or, equivalently, $\phi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})$. Thus $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$.

Suppose, conversely, that $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$. To show that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$, and assume that $\alpha_\Sigma(\Phi) \subseteq T_{F(\Sigma)}$. Then, we have $\Phi \subseteq \alpha_\Sigma^{-1}(T_{F(\Sigma)})$. Since $\phi \in C_\Sigma(\Phi)$ and $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, we get that $\phi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})$ or, equivalently, $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$. This proves, by Lemma 50, that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

(b) This follows from Part (a) and from Part (a) of Lemma 6.

∎

We denote by $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and by $\mathrm{MatFam}(\mathcal{I})$, respectively, the collection of all $\mathcal{I}$-filter families on $\mathcal{A}$ and the collection of all $\mathcal{I}$-matrix families. Note that $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is a complete lattice $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A}) = \langle \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}), \leq \rangle$, with the order $\leq$ inherited by the corresponding order on sentence families.

Similarly, we denote by $\mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ and by $\mathrm{MatSys}(\mathcal{I})$, respectively, the collection of all $\mathcal{I}$-filter systems on $\mathcal{A}$ and the collection of all $\mathcal{I}$-matrix systems. Note that $\mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ forms a complete lattice

$$\mathbf{FiSys}^{\mathcal{I}}(\mathcal{A}) = \langle \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}), \leq \rangle,$$

which is a complete sublattice of the complete lattice $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})$.

Moreover, given a $\mathbf{F}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, we say that T' is a **sentence family of** $\mathfrak{A}$, written $T' \in \mathrm{SenFam}(\mathfrak{A})$, if $T \leq T'$. Similarly, given an $\mathcal{I}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, we say that $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is an $\mathcal{I}$**-filter family of** $\mathfrak{A}$, written $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$, if $T \leq T'$.

Since $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $\mathbf{FiSys}^{\mathcal{I}}(\mathcal{A})$ are both complete lattices, it makes sense to define associated closure operators on $\mathrm{SenFam}(\mathcal{A})$.

- Denote by $C^{\mathcal{I},\mathcal{A}} : \mathrm{SenFam}(\mathcal{I}) \to \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ the operator that maps a given sentence family T of $\mathcal{A}$ to the least $\mathcal{I}$-filter family of $\mathcal{A}$ that includes T;

- Denote by $\overrightarrow{C}^{\mathcal{I},\mathcal{A}} : \mathrm{SenFam}(\mathcal{A}) \to \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ the operator that maps a given sentence family T of $\mathcal{A}$ to the least $\mathcal{I}$-theory system of $\mathcal{A}$ that includes T.

We look now at some relations between the pairs of operators $C^{\mathcal{I},\mathcal{A}}$, C on the one hand, and $\overrightarrow{C}^{\mathcal{I},\mathcal{A}}$, $\overrightarrow{C}$ on the other, established via the inverse interpretation α^{-1} of the $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$.

Proposition 52 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution, based on* $\mathbf{F}$*, and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *be an* $\mathbf{F}$*-algebraic system. Then, for all* $T \in \mathrm{SenFam}(\mathcal{A})$*, we have:*

(a) $C(\alpha^{-1}(T)) \leq \alpha^{-1}(C^{\mathcal{I},\mathcal{A}}(T))$*;*

(b) $\overrightarrow{C}(\alpha^{-1}(T)) \leq \alpha^{-1}(\overrightarrow{C}^{\mathcal{I},\mathcal{A}}(T))$*.*

Proof:

(a) Suppose $T \in \mathrm{SenFam}(\mathcal{A})$. We have $T \leq C^{\mathcal{I},\mathcal{A}}(T)$, whence $\alpha^{-1}(T) \leq \alpha^{-1}(C^{\mathcal{I},\mathcal{A}}(T))$. By Lemma 51, $\alpha^{-1}(C^{\mathcal{I},\mathcal{A}}(T))$ is a theory family of $\mathcal{I}$ and it includes $\alpha^{-1}(T)$. Therefore, by the definition of C, $C(\alpha^{-1}(T)) \leq \alpha^{-1}(C^{\mathcal{I},\mathcal{A}}(T))$.

(b) We have $T \leq \overrightarrow{C}^{\mathcal{I},\mathcal{A}}(T)$. Therefore, since, by Proposition 2, $\overrightarrow{T}$ is the least sentence system containing T, we get $\overrightarrow{T} \leq \overrightarrow{C}^{\mathcal{I},\mathcal{A}}(T)$. Now, taking into account Lemma 6, we get $\overrightarrow{\alpha^{-1}(T)} = \alpha^{-1}(\overrightarrow{T}) \leq \alpha^{-1}(\overrightarrow{C}^{\mathcal{I},\mathcal{A}}(T))$. By Lemma 51, $\alpha^{-1}(\overrightarrow{C}^{\mathcal{I},\mathcal{A}}(T))$ is a theory system of $\mathcal{I}$ including $\overrightarrow{\alpha^{-1}(T)}$ and, therefore, $C(\overrightarrow{\alpha^{-1}(T)}) \leq \alpha^{-1}(\overrightarrow{C}^{\mathcal{I},\mathcal{A}}(T))$, i.e., $\overrightarrow{C}(\alpha^{-1}(T)) \leq \alpha^{-1}(\overrightarrow{C}^{\mathcal{I},\mathcal{A}}(T))$.

■

We now exhibit a relation between the closure operators $C^{\mathcal{I},\mathcal{A}}$ and $\overrightarrow{C}^{\mathcal{I},\mathcal{A}}$ and the arrow operators, as applied to $\mathcal{I}$-filter families on an $\mathbf{F}$-algebraic system $\mathcal{A}$.

Proposition 53 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution, based on* $\mathbf{F}$*, and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*. Consider* $T \in \mathrm{SenFam}(\mathcal{A})$*. Then, we have:*

(a) If $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, then* $\overleftarrow{T} \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ *and it is the largest* $\mathcal{I}$*-filter system on* $\mathcal{A}$ *included in* T*;*

(b) $\overrightarrow{C}^{\mathcal{I},\mathcal{A}}(T) = \overrightarrow{C}^{\mathcal{I},\mathcal{A}}(\overrightarrow{T})$.

Proof:

(a) By Proposition 2, we know that $\overleftarrow{T}$ is a sentence system of $\mathcal{A}$ and that it is the largest one included in T. It suffices, thus, to show that $\overleftarrow{T}$ is an $\mathcal{I}$-filter system. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that

$$\phi \in C_\Sigma(\Phi) \quad \text{and} \quad \alpha_\Sigma(\Phi) \subseteq \overleftarrow{T}_{F(\Sigma)}.$$

Then, by definition of $\overleftarrow{T}$, we get that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$, $\mathrm{SEN}(F(f))(\alpha_\Sigma(\Phi)) \subseteq T_{F(\Sigma')}$. Since α is a natural transformation, $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq T_{F(\Sigma')}$. Since $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $\phi \in C_\Sigma(\Phi)$, we get that $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \in T_{F(\Sigma')}$. Therefore, $\mathrm{SEN}(F(f))(\alpha_\Sigma(\phi)) \in T_{F(\Sigma')}$. Now, noting that this holds for all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and that F is surjective, we conclude that $\alpha_\Sigma(\phi) \in \overleftarrow{T}_{F(\Sigma)}$. Therefore, we get that $\overleftarrow{T} \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$.

(b) The inclusion from left to right is clear, since $T \le \overrightarrow{T}$. On the other hand, since, by Proposition 2, $\overrightarrow{T}$ is the least sentence system including T, we have that every $\mathcal{I}$-filter system including T, also includes $\overrightarrow{T}$. Therefore,

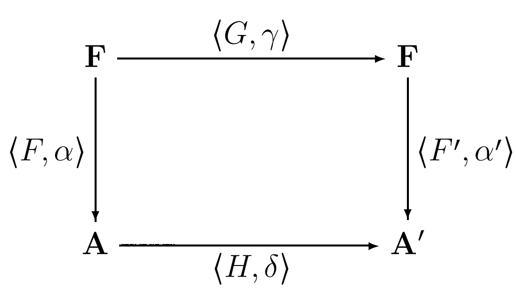
$$\begin{aligned} \overrightarrow{C}^{\mathcal{I},\mathcal{A}}(T) &= \bigcap\{T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) : T \le T'\} \\ &= \bigcap\{T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) : \overrightarrow{T} \le T'\} \\ &= \overrightarrow{C}^{\mathcal{I},\mathcal{A}}(\overrightarrow{T}). \end{aligned}$$

■

We extend the definition of logical morphism to morphisms between **F**-algebraic systems.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. An $\mathcal{I}$-**logical morphism** is a morphism

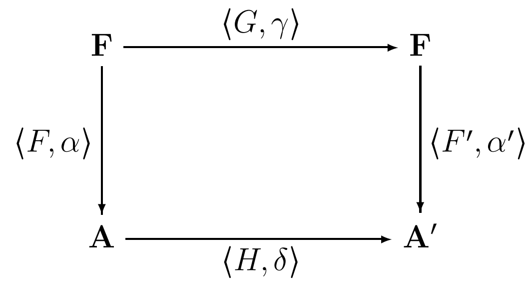
$$\langle\langle G,\gamma\rangle,\langle H,\delta\rangle\rangle : \mathcal{A} \to \mathcal{A}',$$



such that $\langle G,\gamma\rangle : \mathbf{F} \to \mathbf{F}$ is an $\mathcal{I}$-logical morphism $\langle G,\gamma\rangle : \mathcal{I} \to \mathcal{I}$.

Next, we prove a result relating $\mathcal{I}$-filter families/systems on algebraic systems related by morphisms. This result generalizes Lemma 51.

Proposition 54 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. Consider two* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and* $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$ *and a logical morphism* $\langle \langle G, \gamma \rangle, \langle H, \delta \rangle \rangle : \mathcal{A} \to \mathcal{A}'$*.*



(a) If $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathbf{A}')$*, then* $\delta^{-1}(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*;*

(b) If $\langle \langle G, \gamma \rangle, \langle H, \delta \rangle \rangle : \mathcal{A} \to \mathcal{A}'$ *is surjective and* $\delta^{-1}(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, then* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$*;*

(c) If $\langle \langle G, \gamma \rangle, \langle H, \delta \rangle \rangle : \mathcal{A} \to \mathcal{A}'$ *is surjective, with* G, H *isomorphisms and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *is such that* $\delta^{-1}(\delta(T)) = T$*, then* $\delta(T) \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A}')$*.*

Proof:

(a) Let $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$. Then, by Lemma 51, $\alpha'^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$. Thus, by Corollary 48, $\gamma^{-1}(\alpha'^{-1}(T)) \in \mathrm{ThFam}(\mathcal{I})$. Therefore, by the commutativity of the rectangle, $\alpha^{-1}(\delta^{-1}(T)) \in \mathrm{ThFam}(\mathcal{I})$. So, again by Lemma 51, we get that $\delta^{-1}(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

(b) Because of the surjectivity of $\langle G, \gamma \rangle$ and Lemma 50, it suffices to show that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$, we have

$$\alpha'_{G(\Sigma)}(\gamma_\Sigma(\Phi)) \subseteq T_{F'(G(\Sigma))} \quad \text{implies} \quad \alpha'_{G(\Sigma)}(\gamma_\Sigma(\phi)) \in T_{F'(G(\Sigma))}.$$

We have the following:

$$\begin{array}{ll} & \alpha'_{G(\Sigma)}(\gamma_\Sigma(\Phi)) \subseteq T_{F'(G(\Sigma))} \\ \Rightarrow & \delta_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \subseteq T_{H(F(\Sigma))} \\ \Rightarrow & \alpha_\Sigma(\Phi) \subseteq \delta^{-1}_{F(\Sigma)}(T_{H(F(\Sigma))}) \\ \Rightarrow & \alpha_\Sigma(\phi) \in \delta^{-1}_{F(\Sigma)}(T_{H(F(\Sigma))}) \\ \Rightarrow & \delta_{F(\Sigma)}(\alpha_\Sigma(\phi)) \in T_{H(F(\Sigma'))} \\ \Rightarrow & \alpha'_{G(\Sigma)}(\gamma_\Sigma(\phi)) \in T_{F'(G(\Sigma))}. \end{array}$$

(c) As in Part (b) because of the surjectivity of $\langle G, \gamma \rangle$ and Lemma 50, it suffices to show that for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$, we have

$$\alpha'_{G(\Sigma)}(\gamma_\Sigma(\Phi)) \subseteq \delta_{F(\Sigma)}(T_{F(\Sigma)}) \quad \text{implies} \quad \alpha'_{G(\Sigma)}(\gamma_\Sigma(\phi)) \in \delta_{F(\Sigma)}(T_{F(\Sigma)}).$$

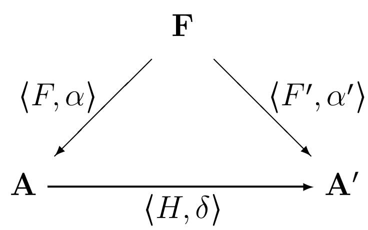
We have

$$\begin{array}{ll} & \alpha'_{G(\Sigma)}(\gamma_\Sigma(\Phi)) \subseteq \delta_{F(\Sigma)}(T_{F(\Sigma)}) \\ \Rightarrow & \delta_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \subseteq \delta_{F(\Sigma)}(T_{F(\Sigma)}) \\ \Rightarrow & \alpha_\Sigma(\Phi) \subseteq \delta^{-1}_{F(\Sigma)}(\delta_{F(\Sigma)}(T_{F(\Sigma)})) = T_{F(\Sigma)} \\ \Rightarrow & \alpha_\Sigma(\phi) \in \delta^{-1}_{F(\Sigma)}(\delta_{F(\Sigma)}(T_{F(\Sigma)})) \\ \Rightarrow & \delta_{F(\Sigma)}(\alpha_\Sigma(\phi)) \in \delta_{F(\Sigma)}(T_{F(\Sigma)}) \\ \Rightarrow & \alpha'_{G(\Sigma)}(\gamma_\Sigma(\phi)) \in \delta_{F(\Sigma)}(T_{F(\Sigma)}). \end{array}$$

■

This proposition has the following significant consequences.

Corollary 55 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. Let, also,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and* $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$ *be two* $\mathbf{F}$*-algebraic systems and* $\langle H, \delta \rangle : \mathcal{A} \to \mathcal{A}'$ *a (surjective) morphism (making the following diagram commute):*

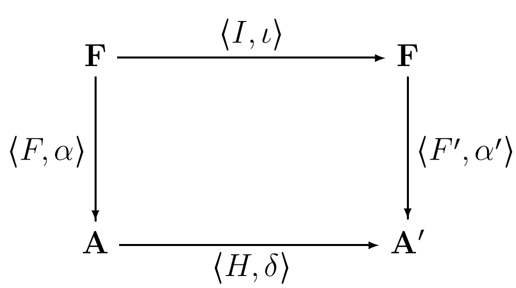


Consider $T \in \mathrm{SenFam}(\mathbf{A}')$.

(a) $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$ *iff* $\delta^{-1}(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*;*

(b) $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}')$ *iff* $\delta^{-1}(T) \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$*.*

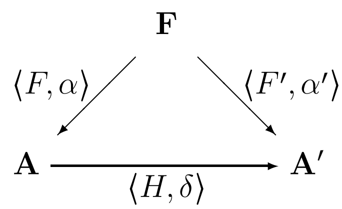
Proof: Follows immediately from Proposition 54 upon considering the commutative square,



where $\langle I, \iota \rangle : \mathbf{F} \to \mathbf{F}$ is the identity morphism. ■

Corollary 56 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. Let, also,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and* $\mathcal{A}' =$

$\langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$ *be two* $\mathbf{F}$*-algebraic systems and* $\langle H, \delta \rangle : \mathcal{A} \to \mathcal{A}'$ *a morphism, with* H *an isomorphism:*



Suppose $T \in \mathrm{SenFam}(\mathbf{A})$ *and* $\mathrm{Ker}(\langle H, \delta \rangle)$ *is compatible with* T*.*

(a) $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *iff* $\delta(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$*;*

(b) $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ *iff* $\delta(T) \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}')$*.*

Proof: First we show that $\delta^{-1}(\delta(T)) = T$: The right to left inclusion is obvious. For the left to right inclusion, consider $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in \delta_\Sigma^{-1}(\delta_\Sigma(T_\Sigma))$. Then, we have $\delta_\Sigma(\phi) \in \delta_\Sigma(T_\Sigma)$. Thus, there exists $\psi \in T_\Sigma$, such that $\delta_\Sigma(\phi) = \delta_\Sigma(\psi)$. By hypothesis, $\mathrm{Ker}(\langle H, \delta \rangle)$ is compatible with T. Therefore, $\phi \in T_\Sigma$. Thus, we get $\delta^{-1}(\delta(T)) \leq T$.

Now the conclusion follows from Proposition 54, since $\delta^{-1}(\delta(T)) = T$. ∎

Corollary 57 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. Let, also,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *be an* $\mathbf{F}$*-algebraic system and* $\theta \in \mathrm{ConSys}(\mathbf{A})$*. Consider* $T \in \mathrm{SenFam}(\mathbf{A}^\theta)$*.*

(a) $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta)$ *iff* $\pi^{\theta^{-1}}(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*;*

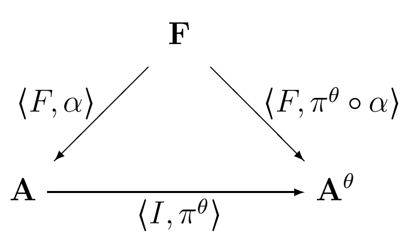
(b) $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}^\theta)$ *iff* $\pi^{\theta^{-1}}(T) \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$*.*

On the other hand, if $T \in \mathrm{SenFam}(\mathbf{A})$ *and* θ *is compatible with* T*, then we have:*

(c) $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *iff* $\pi^\theta(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta)$*;*

(d) $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ *iff* $\pi^\theta(T) \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}^\theta)$*.*

Proof: Parts (a) and (b) follow immediately from Corollary 55 upon considering the commutative diagram



Parts (c) and (d) follow from Corollary 56 upon noticing that $I : \mathbf{Sign} \to \mathbf{Sign}$ is an isomorphism and that, by hypothesis, $\mathrm{Ker}(\langle I, \pi^\theta \rangle) = \theta$ is compatible with T. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. Consider two $\mathcal{I}$-matrix families $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ and $\mathfrak{A}' = \langle \mathcal{A}', T' \rangle$, where $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$. A morphism $\langle \langle G, \gamma \rangle, \langle H, \delta \rangle \rangle : \mathcal{A} \to \mathcal{A}'$, is called a **matrix family morphism** $\langle \langle G, \gamma \rangle, \langle H, \delta \rangle \rangle : \mathfrak{A} \to \mathfrak{A}'$ if, for all $\Sigma \in |\mathbf{Sign}|$,

$$\delta_\Sigma(T_\Sigma) \subseteq T'_{H(\Sigma)}.$$

This matrix family morphism is said to be **strict** if, for all $\Sigma \in |\mathbf{Sign}|$,

$$\delta_\Sigma(T_\Sigma) \subseteq T'_{H(\Sigma)} \quad \text{and} \quad \delta_\Sigma(\mathrm{SEN}(\Sigma) \backslash T_\Sigma) \subseteq \mathrm{SEN}'(H(\Sigma)) \backslash T'_{H(\Sigma)}.$$

These conditions can be equivalently expressed by saying that, for all $\Sigma \in |\mathbf{Sign}|$,

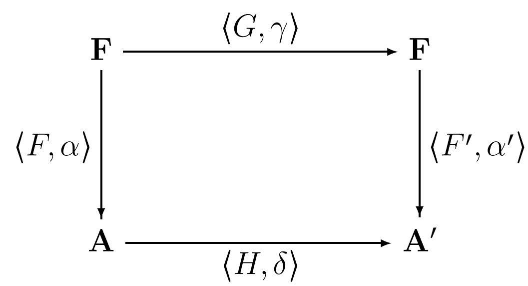
$$\delta_\Sigma^{-1}(T'_{H(\Sigma)}) = T_\Sigma.$$

They are also equivalent to the statement that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{if and only if} \quad \delta_\Sigma(\phi) \in T'_{H(\Sigma)}.$$

We have the following result relating strict morphisms between matrix families with strict morphisms between matrix families based on $\mathcal{F}$.

Lemma 58 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution, based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$ be two $\mathbf{F}$-algebraic systems and $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ and $\mathfrak{A}' = \langle \mathcal{A}', T' \rangle$ two $\mathcal{I}$-matrix families. A matrix morphism $\langle \langle G, \gamma \rangle, \langle H, \delta \rangle \rangle : \mathfrak{A} \to \mathfrak{A}'$*



is strict if and only if $\langle G, \gamma \rangle : \langle \mathcal{F}, \alpha^{-1}(T) \rangle \to \langle \mathcal{F}, \alpha'^{-1}(T') \rangle$ is strict.

Proof: The statement follows by noticing that

$$\begin{array}{rll} \delta^{-1}(T') = T & \text{iff} & \alpha^{-1}(\delta^{-1}(T')) = \alpha^{-1}(T) \quad (\text{by the surjectivity of } \langle F, \alpha \rangle) \\ & \text{iff} & \gamma^{-1}(\alpha'^{-1}(T')) = \alpha^{-1}(T) \\ & & \quad (\text{by the commutativity of the square}). \end{array}$$

Therefore $\langle\langle G,\gamma\rangle,\langle H,\delta\rangle\rangle:\mathfrak{A}\to\mathfrak{A}'$ is strict if and only if $\langle G,\gamma\rangle:\langle\mathcal{F},\alpha^{-1}(T)\rangle\to\langle\mathcal{F},\alpha'^{-1}(T')\rangle$ is strict. ■

Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and $\mathcal{I}=\langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$. Given an $\mathbf{F}$-matrix family $\mathfrak{A}=\langle\mathcal{A},T\rangle$, the **Leibniz reduction** of $\mathfrak{A}$, denoted $\mathfrak{A}^*$, is defined as

$$\mathfrak{A}^*=\langle\mathcal{A}^*,T^*\rangle=\langle\mathcal{A}^{\Omega^{\mathcal{A}}(T)},T/\Omega^{\mathcal{A}}(T)\rangle,$$

where $\mathcal{A}^{\Omega^{\mathcal{A}}(T)}$ is the quotient $\mathbf{F}$-algebraic system of $\mathcal{A}$ by the congruence system $\Omega^{\mathcal{A}}(T)$ and $T/\Omega^{\mathcal{A}}(T)=\{T_\Sigma/\Omega^{\mathcal{A}}_\Sigma(T)\}_{\Sigma\in|\mathbf{Sign}|}$, with

$$T_\Sigma/\Omega^{\mathcal{A}}_\Sigma(T)=\{\phi/\Omega^{\mathcal{A}}_\Sigma(T):\phi\in T_\Sigma\}.$$

An $\mathcal{I}$-matrix family $\mathfrak{A}=\langle\mathcal{A},T\rangle$ is **Leibniz reduced** if

$$\Omega^{\mathcal{A}}(T)=\Delta^{\mathcal{A}}.$$

An $\mathbf{F}$-algebraic system $\mathcal{A}$ is **Leibniz reduced** if it is the algebraic system reduct of a Leibniz reduced $\mathcal{I}$-matrix family.

We denote:

- the class of all Leibniz reduced $\mathcal{I}$-matrix families by $\mathrm{MatFam}^*(\mathcal{I})$;
- the class of all Leibniz reduced $\mathcal{I}$-matrix systems by $\mathrm{MatSys}^*(\mathcal{I})$;
- the class of all reduced $\mathbf{F}$-algebraic systems by $\mathrm{AlgSys}^*(\mathcal{I})$;
- the class of all system reduced $\mathbf{F}$-algebraic systems by $\mathrm{AlgSys}^\bullet(\mathcal{I})$;

i.e., we have:

$$\begin{array}{rcl}\mathrm{MatFam}^*(\mathcal{I}) & = & \{\langle\mathcal{A},T\rangle:T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\text{ and }\Omega^{\mathcal{A}}(T)=\Delta^{\mathcal{A}}\};\\ \mathrm{MatSys}^*(\mathcal{I}) & = & \{\langle\mathcal{A},T\rangle:T\in\mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})\text{ and }\Omega^{\mathcal{A}}(T)=\Delta^{\mathcal{A}}\};\\ \mathrm{AlgSys}^*(\mathcal{I}) & = & \{\mathcal{A}:(\exists T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(\Omega^{\mathcal{A}}(T)=\Delta^{\mathcal{A}})\};\\ \mathrm{AlgSys}^\bullet(\mathcal{I}) & = & \{\mathcal{A}:(\exists T\in\mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}))(\Omega^{\mathcal{A}}(T)=\Delta^{\mathcal{A}})\}.\end{array}$$

2.8 Axiomatic and Filter Extensions

Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and $\mathcal{I}=\langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$. Given an $\mathbf{F}$-algebraic system $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$ and an $\mathbf{F}$-matrix family $\mathfrak{A}=\langle\mathcal{A},T\rangle$, we set

$$\mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})=\{T'\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}):T\le T'\}.$$

$\mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$ is a complete sublattice of $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and we have $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$ if and only if $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. We call $\mathfrak{A}'=\langle\mathcal{A},T'\rangle$ a **filter extension of**

$\mathfrak{A}$ if $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$. Sometimes, by slightly abusing notation, we write $\mathfrak{A}' \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$ in this case.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$, $\mathcal{I}' = \langle \mathbf{F}, C' \rangle$ be two π-institutions based on $\mathbf{F}$. $\mathcal{I}'$ is an **axiomatic extension** (or **axiomatic strengthening**) of $\mathcal{I}$ if there exists $X \in \mathrm{SenSys}(\mathbf{F})$, such that, for all $\Phi \in \mathrm{SenFam}(\mathbf{F})$,

$$C'(\Phi) = C(X \cup \Phi).$$

If this is the case, X is said to be a **system of axioms witnessing** the extension.

We provide now a characterization of axiomatic extensions.

Lemma 59 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$, $\mathcal{I}' = \langle \mathbf{F}, C' \rangle$ *be two π-institutions based on* $\mathbf{F}$. $\mathcal{I}'$ *is an axiomatic extension of* $\mathcal{I}$ *if and only if, for all* $\Phi \in \mathrm{SenFam}(\mathbf{F})$,

$$C'(\Phi) = C(\mathrm{Thm}(\mathcal{I}') \cup \Phi).$$

Proof: Assume, first, that $\mathcal{I}'$ is an axiomatic extension of $\mathcal{I}$, with witnessing system of axioms X. Then, we have $\mathrm{Thm}(\mathcal{I}') = C'(\varnothing) = C(X \cup \varnothing) = C(X)$. Therefore, for all $\Phi \in \mathrm{SenFam}(\mathbf{F})$,

$$C'(\Phi) = C(X \cup \Phi) = C(C(X) \cup \Phi) = C(\mathrm{Thm}(\mathcal{I}') \cup \Phi).$$

Assume conversely, that, for all $\Phi \in \mathrm{SenFam}(\mathbf{F})$, $C'(\Phi) = C(\mathrm{Thm}(\mathcal{I}') \cup \Phi)$. Then $X = \mathrm{Thm}(\mathcal{I}')$ is a system of axioms witnessing the fact that $\mathcal{I}'$ is an axiomatic extension of $\mathcal{I}$. ■

We also have the following characterization in terms of $\mathcal{I}$- and $\mathcal{I}'$-filter families and corresponding theory families.

Proposition 60 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$, $\mathcal{I}' = \langle \mathbf{F}, C' \rangle$ *be two π-institutions based on* $\mathbf{F}$. *The following statements are equivalent:*

(i) $\mathcal{I}'$ *is an axiomatic extension of* $\mathcal{I}$;

(ii) For all $\mathfrak{A} \in \mathrm{MatFam}(\mathcal{I}')$, $\mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}) = \mathrm{FiFam}^{\mathcal{I}'}(\mathfrak{A})$;

(iii) For all $T' \in \mathrm{ThFam}(\mathcal{I}')$ *and* $T' \leq T \in \mathrm{SenFam}(\mathbf{F})$,

$$T \in \mathrm{ThFam}(\mathcal{I}) \quad \textit{if and only if} \quad T \in \mathrm{ThFam}(\mathcal{I}').$$

Proof:

(i)⇒(ii) Suppose that $\mathcal{I}'$ is an axiomatic extension of $\mathcal{I}$ and let $\mathfrak{A} = \langle \mathcal{A}, T' \rangle \in \mathrm{MatFam}(\mathcal{I}')$. Since $C \leq C'$, we have $\mathrm{FiFam}^{\mathcal{I}'}(\mathfrak{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$. So suppose that $T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$, i.e., $T' \leq T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C'_\Sigma(\Phi)$ and $\alpha_\Sigma(\Phi) \subseteq T''_{F(\Sigma)}$. Since $\phi \in C'_\Sigma(\Phi)$, by Lemma 59, $\phi \in C_\Sigma(\mathrm{Thm}_\Sigma(\mathcal{I}') \cup \Phi)$. Now observe that $\alpha_\Sigma(\mathrm{Thm}_\Sigma(\mathcal{I}')) \subseteq T'_{F(\Sigma)} \subseteq T''_{F(\Sigma)}$, since $T' \in \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A})$. Thus, we get

$$\alpha_\Sigma(\mathrm{Thm}_\Sigma(\mathcal{I}') \cup \Phi)) \subseteq T''_{F(\Sigma)}.$$

Hence, since $T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get that $\alpha_\Sigma(\phi) \in T''_{F(\Sigma)}$. So $T'' \in \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A})$. And, since $T' \leq T''$, $T'' \in \mathrm{FiFam}^{\mathcal{I}'}(\mathfrak{A})$.

(ii)⇒(iii) Let $\mathfrak{A} = \langle \mathcal{F}, T' \rangle \in \mathrm{MatFam}(\mathcal{I}')$. Then, by hypothesis, for all $T' \leq T$, we have $\langle \mathcal{F}, T \rangle \in \mathrm{MatFam}^{\mathcal{I}}(\mathfrak{A})$ iff $\langle \mathcal{F}, T \rangle \in \mathrm{MatFam}^{\mathcal{I}'}(\mathfrak{A})$, i.e., $T \in \mathrm{ThFam}(\mathcal{I})$ iff $T \in \mathrm{ThFam}(\mathcal{I}')$.

(iii)⇒(i) First, note that (iii) implies that $\mathrm{ThFam}(\mathcal{I}') \subseteq \mathrm{ThFam}(\mathcal{I})$ and, therefore, $C \leq C'$. We use this to show that, for all $X \in \mathrm{SenFam}(\mathbf{F})$,

$$C'(X) = C(\mathrm{Thm}(\mathcal{I}') \cup X).$$

From left to right, note that $\mathrm{Thm}(\mathcal{I}') \subseteq C(\mathrm{Thm}(\mathcal{I}')) \subseteq C(\mathrm{Thm}(\mathcal{I}') \cup X)$. So, by hypothesis, $C(\mathrm{Thm}(\mathcal{I}') \cup X) \in \mathrm{ThFam}(\mathcal{I}')$. Thus, we get

$$C'(X) \subseteq C'(C(\mathrm{Thm}(\mathcal{I}') \cup X)) = C(\mathrm{Thm}(\mathcal{I}') \cup X).$$

On the other hand, $C(\mathrm{Thm}(\mathcal{I}') \cup X) \subseteq C'(\mathrm{Thm}(\mathcal{I}') \cup X) = C'(X)$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and $\mathfrak{A} = \langle \mathcal{A}, T \rangle \in \mathrm{MatFam}(\mathcal{I})$. Define, for all $\Phi \in \mathrm{SenFam}(\mathcal{A})$,

$$C^{\mathcal{I},\mathfrak{A}}(\Phi) = C^{\mathcal{I},\mathcal{A}}(T \cup \Phi).$$

$C^{\mathcal{I},\mathfrak{A}}(\Phi)$ is the **$\mathcal{I}$-filter family of $\mathfrak{A}$ generated by** Φ.

We have, for all $\Phi \in \mathrm{SenFam}(\mathcal{A})$, $T \leq C^{\mathcal{I},\mathfrak{A}}(\Phi)$. In the special case where $\mathcal{A} = \mathcal{F}$ and $\mathfrak{A} = \mathfrak{F} = \langle \mathcal{F}, T \rangle \in \mathrm{MatFam}(\mathcal{I})$, we get, for all $\Phi \in \mathrm{SenFam}(\mathbf{F})$,

$$C^{\mathfrak{F}}(\Phi) = C(T \cup \Phi).$$

The following proposition gives many properties governing filter family generation and the interaction with surjective morphisms between $\mathcal{I}$-matrix families.

Proposition 61 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$*,* $\mathfrak{A} = \langle \mathcal{A}, T \rangle$*,* $\mathfrak{A}' = \langle \mathcal{A}', T' \rangle$ *be* $\mathbf{F}$*-matrix families,* $\langle H, \gamma \rangle : \mathfrak{A} \to \mathfrak{A}'$ *a surjective morphism and* $X \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$*,* $Y, Y' \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}')$.

(a) $\gamma^{-1}(Y) \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$;

(b) *If* H *is an isomorphism,* $C^{\mathcal{I},\mathfrak{A}'}(\gamma(X)) \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}')$;

(c) *If* H *is an isomorphism,* $C^{\mathcal{I},\mathfrak{A}'}(\gamma(\gamma^{-1}(Y))) = \gamma(\gamma^{-1}(Y)) = Y$;

(d) *If* H *is an isomorphism,* $\gamma^{-1}(C^{\mathcal{I},\mathfrak{A}'}(\gamma(X))) = \gamma^{-1}(\gamma(X)) = X$ *if and only if* $\gamma^{-1}(T') \leq X$ *and* $\mathrm{Ker}(\langle H,\gamma\rangle)$ *is compatible with* X;

(e) $\gamma^{-1}(Y \cap Y') = \gamma^{-1}(Y) \cap \gamma^{-1}(Y')$;

(f) *If* H *is an isomorphism, for all* $\Phi \in \mathrm{SenFam}(\mathcal{A})$, $C^{\mathcal{I},\mathfrak{A}'}(\gamma(C^{\mathcal{I},\mathfrak{A}}(\Phi))) = C^{\mathcal{I},\mathfrak{A}'}(\gamma(\Phi))$.

Proof:

(a) We know, by Corollary 55, that $\gamma^{-1}(Y) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. In addition, $T \leq \gamma^{-1}(T') \leq \gamma^{-1}(Y)$. So we get $\gamma^{-1}(Y) \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$.

(b) It is obvious that $C^{\mathcal{I},\mathfrak{A}'}(\gamma(X)) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$. Moreover, by definition, $T' \leq C^{\mathcal{I},\mathfrak{A}'}(\gamma(X))$. So, we get $C^{\mathcal{I},\mathfrak{A}'}(\gamma(X)) \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}')$.

(c) We have

$$\begin{array}{rcll} C^{\mathcal{I},\mathfrak{A}'}(\gamma(\gamma^{-1}(Y))) & = & C^{\mathcal{I},\mathfrak{A}'}(Y) & (\langle H,\gamma\rangle \text{ surjective}) \\ & = & Y. & (Y \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}')) \end{array}$$

(d) Assume, first, that $\gamma^{-1}(C^{\mathcal{I},\mathfrak{A}'}(\gamma(X))) = \gamma^{-1}(\gamma(X)) = X$. Then, by surjectivity of $\langle H,\gamma\rangle$, $C^{\mathcal{I},\mathfrak{A}'}(\gamma(X)) = \gamma(X)$. This implies that $T' \leq \gamma(X)$, whence $\gamma^{-1}(T') \leq X$. To show compatibility, suppose $\Sigma \in |\mathbf{Sign}|$, $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\gamma_\Sigma(\phi) = \gamma_\Sigma(\psi)$ and $\phi \in X_\Sigma$. Then, we have

$$\psi \in \gamma_\Sigma^{-1}(\gamma_\Sigma(\phi)) \subseteq \gamma_\Sigma^{-1}(\gamma_\Sigma(X_\Sigma)) = X_\Sigma.$$

So $\mathrm{Ker}(\langle H,\gamma\rangle)$ is compatible with X.

Assume, conversely, that $\gamma^{-1}(T') \leq X$ and that $\mathrm{Ker}(\langle H,\gamma\rangle)$ is compatible with X. Then, by compatibility, $\gamma^{-1}(\gamma(X)) = X \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$. Thus, by Corollary 55, $\gamma(X) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$. But we also have $T' \leq \gamma(X)$, whence $\gamma(X) \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}')$. Now we get $\gamma^{-1}(C^{\mathcal{I},\mathfrak{A}'}(\gamma(X))) = \gamma^{-1}(\gamma(X)) = X$.

(e) This follows from set theory.

(f) Let $\Phi \in \mathrm{SenFam}(\mathcal{A})$. Clearly, $C^{\mathcal{I},\mathfrak{A}'}(\gamma(\Phi)) \leq C^{\mathcal{I},\mathfrak{A}'}(\gamma(C^{\mathcal{I},\mathfrak{A}}(\Phi)))$, since $C^{\mathcal{I},\mathfrak{A}'}(\gamma(C^{\mathcal{I},\mathfrak{A}}(\Phi)))$ is an $\mathcal{I}$-filter family of $\mathfrak{A}'$ including $\gamma(\Phi)$.

To show the reverse inclusion, assume $Y \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}')$, such that $\gamma(\Phi) \le Y$. Then $\Phi \le \gamma^{-1}(Y)$. Thus, $C^{\mathcal{I},\mathfrak{A}}(\Phi) \le C^{\mathcal{I},\mathfrak{A}}(\gamma^{-1}(Y)) = \gamma^{-1}(Y)$, the equality following by Part (a). Hence, we get

$$C^{\mathcal{I},\mathfrak{A}'}(\gamma(C^{\mathcal{I},\mathfrak{A}}(\Phi))) \le C^{\mathcal{I},\mathfrak{A}'}(\gamma(\gamma^{-1}(Y))) = C^{\mathcal{I},\mathfrak{A}'}(Y) = Y.$$

Since $C^{\mathcal{I},\mathfrak{A}'}(\gamma(C^{\mathcal{I},\mathfrak{A}}(\Phi))) \le Y$ holds, for all $Y \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}')$, such that $\gamma(\Phi) \le Y$, we get, in particular, $C^{\mathcal{I},\mathfrak{A}'}(\gamma(C^{\mathcal{I},\mathfrak{A}}(\Phi))) \le C^{\mathcal{I},\mathfrak{A}'}(\gamma(\Phi))$. ■

2.9 Generalized Matrix Families and Systems

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. A **generalized F-matrix family**, or **F-gmatrix family** for short, is a pair $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$, where $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ is an $\mathbf{F}$-algebraic system and $\mathcal{T} \subseteq \mathrm{SenFam}(\mathcal{A})$ is a collection of sentence families of $\mathcal{A}$.

An $\mathbf{F}$-gmatrix family $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$ is said to be an **F-gmatrix system** if $\mathcal{T} \subseteq \mathrm{SenSys}(\mathcal{A})$.

Given an $\mathbf{F}$-gmatrix family $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$, the **Tarski congruence system of** $\mathbb{A}$ (or **of** $\mathcal{T}$ **on** $\mathcal{A}$), denoted $\widetilde{\Omega}(\mathbb{A})$ or $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})$, is the largest congruence system on $\mathcal{A}$ that is compatible with all sentence families in $\mathcal{T}$.

Lemma 62 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system. Then, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $\mathcal{T} \subseteq \mathrm{SenFam}(\mathcal{A})$*,*

$$\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T).$$

Proof: Note that, by definition, $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})$ is compatible with every $T \in \mathcal{T}$. Therefore, since $\Omega^{\mathcal{A}}(T)$ is the largest congruence system on $\mathcal{A}$ compatible with T, we get that

$$\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \le \Omega^{\mathcal{A}}(T), \text{ for all } T \in \mathcal{T}.$$

Thus, $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \le \bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)$.

For the reverse inclusion, note that $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)$ is a congruence system on $\mathcal{A}$ that is compatible with every $T \in \mathcal{T}$. Therefore, since $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})$ is the largest such congruence system, we get that $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \le \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. An $\mathbf{F}$-gmatrix family $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$ is called a **generalized** $\mathcal{I}$**-matrix family**, or $\mathcal{I}$**-gmatrix family** for short, if $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

We have a special notation for the Tarski congruence systems, when applied and/or relativized to the collection of all $\mathcal{I}$-filter families:

$$\widetilde{\Omega}^{\mathcal{A}}(\mathcal{I}) \coloneqq \widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})).$$

Recall that $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$, where $\langle I, \iota \rangle : \mathbf{F} \to \mathbf{F}$ is the identity morphism. We set

$$\widetilde{\Omega}(\mathcal{I}) := \widetilde{\Omega}^{\mathcal{F}}(\mathcal{I}) = \widetilde{\Omega}^{\mathcal{F}}(\mathrm{ThFam}(\mathcal{I})).$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution, based on $\mathbf{F}$, and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system. Given an $\mathbf{F}$-gmatrix family $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$, the **Tarski reduction** of $\mathbb{A}$, denoted $\mathbb{A}^*$, is defined as

$$\mathbb{A}^* = \langle \mathcal{A}^*, \mathcal{T}^* \rangle = \langle \mathcal{A}^{\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})}, \mathcal{T}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \rangle,$$

where $\mathcal{A}^{\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})}$ is the quotient $\mathbf{F}$-algebraic system of $\mathcal{A}$ by the Tarski congruence system $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})$ and

$$\mathcal{T}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \{T/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) : T \in \mathcal{T}\},$$

with $T/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \{T_\Sigma/\widetilde{\Omega}^{\mathcal{A}}_\Sigma(\mathcal{T})\}_{\Sigma \in |\mathbf{Sign}|}$ such that, for all $\Sigma \in |\mathbf{Sign}|$,

$$T_\Sigma/\widetilde{\Omega}^{\mathcal{A}}_\Sigma(\mathcal{T}) = \{\phi/\widetilde{\Omega}^{\mathcal{A}}_\Sigma(\mathcal{T}) : \phi \in T_\Sigma\}.$$

An $\mathbf{F}$-gmatrix family $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$ is **Tarski reduced** if $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathcal{A}}$. An $\mathbf{F}$-algebraic system $\mathcal{A}$ is **Tarski reduced** if it is the algebraic system reduct of a Tarski reduced $\mathbf{F}$-gmatrix family.

We denote:

- the class of all Tarski reduced $\mathcal{I}$-gmatrix families by $\mathrm{GMatFam}^*(\mathcal{I})$;
- the corresponding class of all Tarski reduced $\mathbf{F}$-algebraic systems by $\mathrm{AlgSys}(\mathcal{I})$,

i.e., we have:

$$\begin{array}{rcl} \mathrm{GMatFam}^*(\mathcal{I}) & = & \{\langle \mathcal{A}, \mathcal{T} \rangle : \mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \text{ and } \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathcal{A}}\}; \\ \mathrm{AlgSys}(\mathcal{I}) & = & \{\mathcal{A} : (\exists \mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathcal{A}})\}. \end{array}$$

Consider again an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$, an $\mathbf{F}$-algebraic system $\mathcal{A}$ and $\mathcal{T} \subseteq \mathrm{SenFam}(\mathcal{A})$. The **Suszko congruence system of** $T \in \mathcal{T}$ (**relative to** $\mathcal{T}$), denoted by $\widetilde{\Omega}^{\mathcal{A},\mathcal{T}}(T)$, is the largest congruence system on $\mathcal{A}$ that is compatible with all $T' \in \mathcal{T}$, such that $T \leq T'$.

Lemma 63 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system. Then, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, all* $\mathcal{T} \subseteq \mathrm{SenFam}(\mathcal{A})$ *and all* $T \in \mathcal{T}$*,*

$$\widetilde{\Omega}^{\mathcal{A},\mathcal{T}}(T) = \bigcap_{T \leq T' \in \mathcal{T}} \Omega^{\mathcal{A}}(T').$$

Proof: The proof is similar to that of Lemma 62. ∎

We note also the following relation between the Suszko congruence system of T relative to $\mathcal{T}$ and the Tarski congruence system of $\mathcal{T}^T = \{T' \in \mathcal{T} : T \leq T'\}$:

$$\widetilde{\Omega}^{\mathcal{A},\mathcal{T}}(T) = \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}^T).$$

We also have some special notations reserved for the Suszko congruence systems, when applied and/or relativized to $\mathrm{ThFam}(\mathcal{I})$ and to all $\mathcal{I}$-filter families.

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathcal{I}}(T) & := & \widetilde{\Omega}^{\mathcal{F},\mathrm{ThFam}(\mathcal{I})}(T), \text{ for all } T \in \mathrm{ThFam}(\mathcal{I});\\ \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) & := & \widetilde{\Omega}^{\mathcal{A},\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})}(T), \text{ for all } T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}).\end{array}$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Given an $\mathcal{I}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, the **Suszko reduction** of $\mathfrak{A}$, denoted $\mathfrak{A}^{\mathsf{Su}}$, is defined as

$$\mathfrak{A}^{\mathsf{Su}} = \langle \mathcal{A}^{\mathsf{Su}}, T^{\mathsf{Su}} \rangle = \langle \mathcal{A}^{\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}, T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \rangle,$$

where $\mathcal{A}^{\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}$ is the quotient $\mathbf{F}$-algebraic system of $\mathcal{A}$ by the Suszko congruence system $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$ and $T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \{T_\Sigma/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}_\Sigma(T)\}_{\Sigma \in |\mathbf{Sign}|}$, with

$$T_\Sigma/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}_\Sigma(T) = \{\phi/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}_\Sigma(T) : \phi \in T_\Sigma\}.$$

An $\mathcal{I}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ is **Suszko reduced** if $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. An $\mathbf{F}$-algebraic system $\mathcal{A}$ is **Suszko reduced** if it is the algebraic system reduct of a Suszko reduced $\mathcal{I}$-matrix family.

It turns out that, relative to a given π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, the classes of Tarski reduced $\mathbf{F}$-algebraic systems and of Suszko reduced $\mathbf{F}$-algebraic systems coincide.

Proposition 64 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution, based on* $\mathbf{F}$*, and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system.* $\mathcal{A}$ *is Suszko reduced if and only if* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*.*

Proof: Suppose, first, that $\mathcal{A}$ is a Suszko reduced $\mathbf{F}$-algebraic system. Then, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. But then we have

$$\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \subseteq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Delta^{\mathcal{A}}.$$

Hence $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle \in \mathrm{GMatFam}^*(\mathcal{I})$ and, consequently, $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$.

Suppose, conversely, that $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$. Thus, by definition, there exists $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathcal{A}}$. Now we get

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(\bigcap \mathcal{T}) & = & \widetilde{\Omega}^{\mathcal{A}}(\{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \bigcap \mathcal{T} \leq T\})\\ & \leq & \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathcal{A}}.\end{array}$$

Since $\bigcap\mathcal{T} \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get that $\langle\mathcal{A},\bigcap\mathcal{T}\rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})$ and, consequently, $\mathcal{A}$ is Suszko reduced. ■

We let $\mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})$ be the class of all Suszko reduced $\mathcal{I}$-matrix families, i.e., we have

$$\mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I}) = \{\langle\mathcal{A},T\rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \text{ and } \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Delta^{\mathcal{A}}\},$$

whereas, because of Proposition 64, there is no reason for introducing fresh notation for the class of all Suszko reduced **F**-algebraic systems, that class being $\mathrm{AlgSys}(\mathcal{I})$.

2.10 The Algebraic Systems of a π-Institution

We have introduced in Sections 2.7 and 2.9 two of the most important classes of **F**-algebraic systems associated to a π-institution $\mathcal{I} = \langle\mathbf{F},C\rangle$, namely, the classes $\mathrm{AlgSys}^*(\mathcal{I})$ and $\mathrm{AlgSys}(\mathcal{I})$. In this section, we introduce two more classes and consider some of the relationships that hold between them.

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F},C\rangle$ a π-institution based on **F**. The **semantic variety of** $\mathcal{I}$ is the semantic variety generated by the algebraic system $\mathcal{F}/\widetilde{\Omega}(\mathcal{I})$, i.e., the class

$$\begin{array}{rcl}\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}) & := & \mathbb{V}^{\mathsf{Sem}}(\mathcal{F}/\widetilde{\Omega}(\mathcal{I}))\\ & = & \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : \widetilde{\Omega}(\mathcal{I}) \leq \mathrm{Ker}(\mathcal{A})\}.\end{array}$$

The **syntactic variety of** $\mathcal{I}$ is the syntactic variety generated by $\mathcal{F}/\widetilde{\Omega}(\mathcal{I})$, i.e., the class defined by

$$\begin{array}{rcl}\mathbb{V}^{\mathsf{Syn}}(\mathcal{I}) & := & \mathbb{V}^{\mathsf{Syn}}(\mathcal{F}/\widetilde{\Omega}(\mathcal{I}))\\ & = & \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : (\forall\sigma^\flat \approx \tau^\flat \in \mathrm{NEq}(\mathbf{F}))\\ & & \quad(\mathcal{F}/\widetilde{\Omega}(\mathcal{I}) \vDash \sigma^\flat \approx \tau^\flat \ \Rightarrow\ \mathcal{A} \vDash \sigma^\flat \approx \tau^\flat)\}.\end{array}$$

We can say a few things about the relationships governing the four classes of **F**-algebraic systems associated with a given π-institution $\mathcal{I} = \langle\mathbf{F},C\rangle$.

Proposition 65 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F},C\rangle$ *a* π*-institution based on* **F**. *Then we have*

$$\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I}) \subseteq \mathbb{V}^{\mathsf{Sem}}(\mathcal{I}) \subseteq \mathbb{V}^{\mathsf{Syn}}(\mathcal{I}).$$

Proof: Suppose, first, that $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I})$. Then there exists an $\mathcal{I}$-filter family $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. Hence, we get

$$\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \leq \Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}.$$

It follows that $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$.

Suppose, next, that $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$. Thus, there exists $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathcal{A}}$. This implies that $\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \leq \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathcal{A}}$, i.e., that $\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}$. Applying the inverse of the surjective morphism $\langle F, \alpha\rangle$, we get $\alpha^{-1}(\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \alpha^{-1}(\Delta^{\mathcal{A}}) = \mathrm{Ker}(\mathcal{A})$. Therefore, we obtain

$$\begin{array}{rcll}
\widetilde{\Omega}(\mathcal{I}) & = & \bigcap_{T\in\mathrm{ThFam}(\mathcal{I})}\Omega(T) & \text{(by Lemma 62)}\\
& \leq & \bigcap_{T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})}\Omega(\alpha^{-1}(T)) & \text{(by Lemma 51)}\\
& = & \bigcap_{T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})}\alpha^{-1}(\Omega^{\mathcal{A}}(T)) & \text{(by Proposition 24)}\\
& = & \alpha^{-1}(\bigcap_{T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})}\Omega^{\mathcal{A}}(T)) & \text{(set theory))}\\
& = & \alpha^{-1}(\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))) & \text{(by Lemma 62)}\\
& = & \mathrm{Ker}(\mathcal{A}). & \text{(as shown above)}
\end{array}$$

Hence $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathcal{I})$.

The last inclusion follows from Theorem 39. ∎

Finally, it can be shown that all four classes generate the same syntactic variety. We first prove a technical lemma that simplifies some algebraic computations.

Lemma 66 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*, an* $\mathbf{F}$*-algebraic system,* $\theta \in \mathrm{ConSys}(\mathcal{A})$ *and* $\sigma^\flat \approx \tau^\flat$ *an* $\mathbf{F}$*-equation. Then*

$$\mathcal{A}/\theta \vDash \sigma^\flat \approx \tau^\flat \quad \textit{iff} \quad \textit{for all } \Sigma \in |\mathbf{Sign}^\flat|,\ \vec{\phi} \in \mathrm{SEN}^\flat(\Sigma), \ \langle \alpha_\Sigma(\sigma^\flat_\Sigma(\vec{\phi})), \alpha_\Sigma(\tau^\flat_\Sigma(\vec{\phi}))\rangle \in \theta_{F(\Sigma)}.$$

Proof: We have, by definition, $\mathcal{A}/\theta \vDash \sigma^\flat \approx \tau^\flat$ iff, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\alpha^\theta_\Sigma(\sigma^\flat_\Sigma(\vec{\phi})) = \alpha^\theta_\Sigma(\tau^\flat_\Sigma(\vec{\phi}))$$

iff, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\alpha_\Sigma(\sigma^\flat_\Sigma(\vec{\phi}))/\theta_{F(\Sigma)} = \alpha_\Sigma(\tau^\flat_\Sigma(\vec{\phi}))/\theta_{F(\Sigma)}$$

iff, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \alpha_\Sigma(\sigma^\flat_\Sigma(\vec{\phi})), \alpha_\Sigma(\tau^\flat_\Sigma(\vec{\phi}))\rangle \in \theta_{F(\Sigma)}.$$

∎

Theorem 67 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then*

$$\mathbb{V}^{\mathsf{Syn}}(\mathrm{AlgSys}^*(\mathcal{I})) = \mathbb{V}^{\mathsf{Syn}}(\mathrm{AlgSys}(\mathcal{I})) = \mathbb{V}^{\mathsf{Syn}}(\mathbb{V}^{\mathsf{Sem}}(\mathcal{I})) = \mathbb{V}^{\mathsf{Syn}}(\mathcal{I}).$$

Proof: By Propositions 65 and 38, we have

$$\mathbb{V}^{\mathsf{Syn}}(\mathrm{AlgSys}^*(\mathcal{I})) \subseteq \mathbb{V}^{\mathsf{Syn}}(\mathrm{AlgSys}(\mathcal{I})) \subseteq \mathbb{V}^{\mathsf{Syn}}(\mathbb{V}^{\mathsf{Sem}}(\mathcal{I})) \subseteq \mathbb{V}^{\mathsf{Syn}}(\mathcal{I}).$$

To conclude the proof we need to show that

$$\mathbb{V}^{\mathsf{Syn}}(\mathcal{I}) \subseteq \mathbb{V}^{\mathsf{Syn}}(\mathrm{AlgSys}^*(\mathcal{I})).$$

To this end, suppose $\mathcal{A} \in \mathbb{V}^{\mathsf{Syn}}(\mathcal{I})$, i.e., that, for every natural **F**-equation $\sigma^\flat \approx \tau^\flat$,

$$\mathcal{F}/\widetilde{\Omega}(\mathcal{I}) \vDash \sigma^\flat \approx \tau^\flat \quad \text{implies} \quad \mathcal{A} \vDash \sigma^\flat \approx \tau^\flat.$$

To show that $\mathcal{A} \in \mathbb{V}^{\mathsf{Syn}}(\mathrm{AlgSys}^*(\mathcal{I}))$, suppose that $\sigma^\flat \approx \tau^\flat$ is an **F**-equation, such that $\mathrm{AlgSys}^*(\mathcal{I}) \vDash \sigma^\flat \approx \tau^\flat$. In particular, for all $T \in \mathrm{ThFam}(\mathcal{I})$, we have that $\mathcal{F}/\Omega(T) \vDash \sigma^\flat \approx \tau^\flat$. This means, by Lemma 66, that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$, $\langle \sigma^\flat_\Sigma(\vec{\phi}), \tau^\flat_\Sigma(\vec{\phi})\rangle \in \Omega_\Sigma(T)$. Since this holds for all $T \in \mathrm{ThFam}(\mathcal{I})$, we get that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \sigma^\flat_\Sigma(\vec{\phi}), \tau^\flat_\Sigma(\vec{\phi})\rangle \in \bigcap_{T \in \mathrm{ThFam}(\mathcal{I})} \Omega_\Sigma(T) = \widetilde{\Omega}_\Sigma(\mathcal{I}).$$

Thus, again by Lemma 66, $\mathcal{F}/\widetilde{\Omega}(\mathcal{I}) \vDash \sigma^\flat \approx \tau^\flat$. Therefore, by hypothesis, $\mathcal{A} \vDash \sigma^\flat \approx \tau^\flat$. We conclude that $\mathcal{A} \in \mathbb{V}^{\mathsf{Syn}}(\mathrm{AlgSys}^*(\mathcal{I}))$ and, hence, $\mathbb{V}^{\mathsf{Syn}}(\mathcal{I}) \subseteq \mathbb{V}^{\mathsf{Syn}}(\mathrm{AlgSys}^*(\mathcal{I}))$. ■

We close this section by showing that, given a π-institution $\mathcal{I}$, the class of Tarski reduced algebraic systems $\mathrm{AlgSys}(\mathcal{I})$ is closed under the operator $\overset{\triangleleft}{\mathbb{P}}$ and contains a trivial **F**-algebraic system and, therefore, by Proposition 28, it makes sense, for every **F**-algebraic system $\mathcal{A}$, to consider the relative congruence system $\Theta^{\mathrm{AlgSys}(\mathcal{I}),\mathcal{A}}(X)$ on $\mathcal{A}$ generated by a relation family $X \in \mathrm{RelFam}(\mathcal{A})$.

Proposition 68 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. The class of* **F***-algebraic systems* $\mathrm{AlgSys}(\mathcal{I})$ *is closed under subdirect intersections and contains a trivial* **F***-algebraic system.*

Proof: It is clear that $\mathrm{AlgSys}(\mathcal{I})$ contains a trivial **F**-algebraic system $\mathcal{A}$, since $\Delta^{\mathcal{A}} = \nabla^{\mathcal{A}}$ is the only congruence system on $\mathcal{A}$. So it suffices to show that $\mathrm{AlgSys}(\mathcal{I})$ is closed under subdirect intersections. To this end, let

$$\langle H^i, \gamma^i\rangle : \mathcal{A} \to \mathcal{A}^i, \quad i \in I,$$

be a subdirect intersection, with $\mathcal{A}^i \in \mathrm{AlgSys}(\mathcal{I})$, for all $i \in I$. Thus, by definition, we have, on the one hand, that

$$\bigcap_{i \in I} \mathrm{Ker}(\langle H^i, \gamma^i\rangle) = \Delta^{\mathcal{A}},$$

and on the other, that, for all $i \in I$, there exists $\mathcal{T}^i \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^i)$, such that

$$\widetilde{\Omega}^{\mathcal{A}^i}(\mathcal{T}^i) = \Delta^{\mathcal{A}^i}.$$

Now we obtain

$$\begin{array}{rcl}
\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) & \leq & \widetilde{\Omega}^{\mathcal{A}}(\bigcup_{i\in I}(\gamma^i)^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^i))) \\
& = & \bigcap_{i\in I}\widetilde{\Omega}^{\mathcal{A}}((\gamma^i)^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^i))) \\
& = & \bigcap_{i\in I}(\gamma^i)^{-1}(\widetilde{\Omega}^{\mathcal{A}^i}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^i))) \\
& \leq & \bigcap_{i\in I}(\gamma^i)^{-1}(\widetilde{\Omega}^{\mathcal{A}^i}(\mathcal{T}^i)) \\
& = & \bigcap_{i\in I}(\gamma^i)^{-1}(\Delta^{\mathcal{A}^i}) \\
& = & \bigcap_{i\in I}\mathrm{Ker}(\langle H^i,\gamma^i\rangle) \\
& = & \Delta^{\mathcal{A}}.
\end{array}$$

Therefore, $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, showing that $\overset{\triangleleft}{\mathbb{I}\mathrm{P}}(\mathrm{AlgSys}(\mathcal{I})) \subseteq \mathrm{AlgSys}(\mathcal{I})$. ∎

Based on Proposition 68 and Proposition 28, we define, for every **F**-algebraic system $\mathcal{A}$, and all $X \in \mathrm{RelFam}(\mathcal{A})$,

$$\Theta^{\mathcal{I},\mathcal{A}}(X) := \Theta^{\mathrm{AlgSys}(\mathcal{I}),\mathcal{A}}(X).$$

2.11 Frege Relations

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a sentence functor and $T \in \mathrm{SenFam}(\mathrm{SEN})$. We define:

- The **Frege relation system** $\Lambda(T) = \{\Lambda_\Sigma(T)\}_{\Sigma\in|\mathbf{Sign}|}$ of T on SEN by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\begin{array}{rcl}
\Lambda_\Sigma(T) & = & \{\langle\phi,\psi\rangle \in \mathrm{SEN}(\Sigma)^2 : \text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma'), \\
& & \mathrm{SEN}^\flat(f)(\phi) \in T_{\Sigma'} \iff \mathrm{SEN}^\flat(f)(\psi) \in T_{\Sigma'}\};
\end{array}$$

- The **Frege relation family** $\lambda(T) = \{\lambda_\Sigma(T)\}_{\Sigma\in|\mathbf{Sign}|}$ of T on SEN by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\lambda_\Sigma(T) = \{\langle\phi,\psi\rangle \in \mathrm{SEN}(\Sigma)^2 : (\phi \in T_\Sigma \iff \psi \in T_\Sigma)\}.$$

It turns out that the Frege relation system of T on SEN is an equivalence system, the Frege relation family of T on SEN is an equivalence family and that the former is the largest equivalence system included in the latter.

Proposition 69 *Let* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a sentence functor and* $T \in \mathrm{SenFam}(\mathrm{SEN})$.

(a) $\Lambda(T)$ *is an equivalence system on* SEN*;*

(b) $\lambda(T)$ *is an equivalence family on* SEN*;*

(c) $\Lambda(T)$ *is the largest equivalence system included in* $\lambda(T)$*.*

Proof:

(a) That $\Lambda(T)$ is an equivalence family, i.e., that, for all $\Sigma \in |\mathbf{Sign}|$, $\Lambda_\Sigma(T)$ is an equivalence relation, is straightforward. To see that it is a system, i.e., invariant under signature morphisms, let $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle \in \Lambda_\Sigma(T)$, and $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma,\Sigma')$. Then, we have, for all $\Sigma'' \in |\mathbf{Sign}|$ and all $g \in \mathbf{Sign}(\Sigma',\Sigma'')$,

$$\Sigma \xrightarrow{\ f\ } \Sigma' \xrightarrow{\ g\ } \Sigma''$$

$\mathrm{SEN}(gf)(\phi) \in T_{\Sigma''}$ iff $\mathrm{SEN}(gf)(\psi) \in T_{\Sigma''}$, whence, we derive that , for all $g \in \mathbf{Sign}(\Sigma',\Sigma'')$,

$$\mathrm{SEN}(g)(\mathrm{SEN}(f)(\phi)) \in T_{\Sigma''} \quad \text{iff} \quad \mathrm{SEN}(g)(\mathrm{SEN}(f)(\psi)) \in T_{\Sigma''}.$$

This shows that $\langle\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)\rangle \in T_{\Sigma'}$. Thus, $\Lambda(T)$ is an equivalence system.

(b) This part is straightforward.

(c) It is clear that $\Lambda(T) \le \lambda(T)$, simply by considering, in the definition of $\Lambda(T)$, the particular case where $\Sigma' = \Sigma$ and $f = i_\Sigma : \Sigma \to \Sigma$ is the identity signature morphism. Suppose, next, that θ is an equivalence system, such that $\theta \le \lambda(T)$. We must show that $\theta \le \Lambda(T)$. To this end, let $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle \in \theta_\Sigma$. Since θ is a system, we get, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma,\Sigma')$, $\langle\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)\rangle \in \theta_{\Sigma'}$. Hence, since $\theta \le \lambda(T)$, we conclude that $\mathrm{SEN}(f)(\phi) \in T_{\Sigma'}$ iff $\mathrm{SEN}(f)(\psi) \in T_{\Sigma'}$. Therefore, by definition, $\langle\phi,\psi\rangle \in \Lambda_\Sigma(T)$. Thus, $\theta \le \Lambda(T)$ and $\Lambda(T)$ is indeed the largest equivalence system included in $\lambda(T)$.

■

There is also a close relationship between the two Frege equivalence families and the Leibniz congruence system of a sentence family. In case SEN is the underlying sentence functor of an algebraic system $\mathbf{A} = \langle\mathbf{Sign}, \mathrm{SEN}, N\rangle$, we sometimes write $\Lambda^{\mathbf{A}}(T)$ and $\lambda^{\mathbf{A}}(T)$ for the relation families $\Lambda(T)$ and $\lambda(T)$, respectively.

Proposition 70 *Let* $\mathbf{A} = \langle\mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system and* $T \in \mathrm{SenFam}(\mathbf{A})$*.*

(a) $\Omega^{\mathbf{A}}(T)$ *is the largest congruence system contained in* $\lambda^{\mathbf{A}}(T)$*;*

(b) *$\Omega^{\mathbf{A}}(T)$ is the largest congruence system contained in $\Lambda^{\mathbf{A}}(T)$.*

Proof:

(a) By definition $\Omega^{\mathbf{A}}(T)$ is a congruence system on $\mathbf{A}$. So we must show that $\Omega^{\mathbf{A}}(T) \leq \lambda^{\mathbf{A}}(T)$ and that, moreover, it is the largest congruence system that satisfies this inclusion property.

To see that $\Omega^{\mathbf{A}}(T) \leq \lambda^{\mathbf{A}}(T)$, let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \Omega^{\mathbf{A}}_{\Sigma}(T)$. Then, by compatibility of $\Omega^{\mathbf{A}}(T)$ with T, we get that, $\phi \in T_\Sigma$ iff $\psi \in T_\Sigma$. So, by definition $\langle \phi, \psi \rangle \in \lambda^{\mathbf{A}}_{\Sigma}(T)$.

Finally, suppose, that $\theta \in \mathrm{ConSys}(\mathbf{A})$, such that $\theta \leq \lambda^{\mathbf{A}}(T)$. We must show that $\theta \leq \Omega^{\mathbf{A}}(T)$. Since, by definition $\Omega^{\mathbf{A}}(T)$ is the largest congruence system compatible with T, it suffices to show that θ is compatible with T. To this end, let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \theta_\Sigma$ and $\phi \in T_\Sigma$. Since $\theta \leq \lambda^{\mathbf{A}}(T)$, we get that $\langle \phi, \psi \rangle \in \lambda^{\mathbf{A}}_{\Sigma}(T)$ and $\phi \in T_\Sigma$. By the definition of $\lambda^{\mathbf{A}}(T)$, we conclude that $\psi \in T_\Sigma$. Therefore, θ is compatible with T and, hence, $\theta \leq \Omega^{\mathbf{A}}(T)$.

(b) Since $\Omega^{\mathbf{A}}(T)$ is, in particular, an equivalence system, we get, by Part (a) and Part (c) of Proposition 69, that $\Omega^{\mathbf{A}}(T) \leq \Lambda^{\mathbf{A}}(T)$. It is the largest congruence system satisfying this property, since $\Lambda^{\mathbf{A}}(T) \leq \lambda^{\mathbf{A}}(T)$ and, by Part (a), it is the largest congruence system in $\lambda^{\mathbf{A}}(T)$.

■

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a sentence functor and $\mathcal{T} \subseteq \mathrm{SenFam}(\mathrm{SEN})$ a collection of sentence families of SEN. The following relation systems are also known by the name of Frege in the literature, but we use the name "Carnap" instead to differentiate the two. In the present context, they have the same relation with Frege relation systems as Tarski congruence systems have with Leibniz congruence systems. We define:

- The **Carnap relation system** $\widetilde{\Lambda}(\mathcal{T}) = \{\widetilde{\Lambda}_\Sigma(\mathcal{T})\}_{\Sigma \in |\mathbf{Sign}|}$ of $\mathcal{T}$ on SEN, by
$$\widetilde{\Lambda}(\mathcal{T}) = \bigcap_{T \in \mathcal{T}} \Lambda(T),$$
where the intersection is taken signature-wise;

- The **Carnap relation family** $\widetilde{\lambda}(\mathcal{T}) = \{\widetilde{\lambda}_\Sigma(\mathcal{T})\}_{\Sigma \in |\mathbf{Sign}|}$ of $\mathcal{T}$ on SEN, by
$$\widetilde{\lambda}(\mathcal{T}) = \bigcap_{T \in \mathcal{T}} \lambda(T),$$
where the intersection is taken signature-wise.

That is, we have, for all $\Sigma \in |\mathbf{Sign}|$,

$$\begin{array}{rcl}\widetilde{\Lambda}_\Sigma(\mathcal{T}) & = & \{\langle \phi, \psi \rangle \in \mathrm{SEN}(\Sigma)^2 : \text{for all } T \in \mathcal{T}, \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma'), \\ & & \quad \mathrm{SEN}(f)(\phi) \in T_{\Sigma'} \iff \mathrm{SEN}(f)(\psi) \in T_{\Sigma'}\}\end{array}$$

and, similarly,

$$\widetilde{\lambda}_\Sigma(\mathcal{T}) = \{\langle\phi,\psi\rangle \in \mathrm{SEN}(\Sigma)^2 : \text{for all } T \in \mathcal{T}, \\ \phi \in T_\Sigma \Leftrightarrow \psi \in T_\Sigma\}$$

We have analogs of Propositions 69 and 1420 for the case of $\widetilde{\Lambda}$ and $\widetilde{\lambda}$. The analog of Proposition 69 asserts that $\widetilde{\Lambda}(\mathcal{T})$ is an equivalence system on SEN, $\widetilde{\lambda}(\mathcal{T})$ is an equivalence family on SEN and that $\widetilde{\Lambda}(\mathcal{T})$ is the largest equivalence system included in $\widetilde{\lambda}(\mathcal{T})$.

Corollary 71 *Let* SEN : **Sign** → **Set** *be a sentence functor and consider* $\mathcal{T} \subseteq \mathrm{SenFam}(\mathrm{SEN})$.

- *(a)* $\widetilde{\Lambda}(\mathcal{T})$ *is an equivalence system on* SEN*;*
- *(b)* $\widetilde{\lambda}(\mathcal{T})$ *is an equivalence family on* SEN*;*
- *(c)* $\widetilde{\Lambda}(\mathcal{T})$ *is the largest equivalence system on* SEN *included in* $\widetilde{\lambda}(T)$.

Proof:

- (a) Since the intersection of equivalence relations is an equivalence relation, we get, by definition, that $\widetilde{\Lambda}(\mathcal{T})$ is an *equivalence* family. Moreover, since the intersection of relation systems is a relation system, we get, by Proposition 69, that $\widetilde{\Lambda}(\mathcal{T})$ is an equivalence *system*.
- (b) As in Part (a), Part (b) follows from the fact that $\lambda(T)$ is an equivalence family, for all $T \in \mathcal{T}$.
- (c) By Proposition 69, we get $\widetilde{\Lambda}(\mathcal{T}) = \bigcap_{T\in\mathcal{T}} \Lambda(T) \leq \bigcap_{T\in\mathcal{T}} \lambda(T) = \widetilde{\lambda}(\mathcal{T})$. Let, now, θ be an equivalence system on SEN, such that $\theta \leq \widetilde{\lambda}(\mathcal{T})$. We must show that $\theta \leq \widetilde{\Lambda}(\mathcal{T})$. By hypothesis, $\theta \leq \lambda(T)$, for all $T \in \mathcal{T}$. Therefore, by Proposition 69, $\theta \leq \Lambda(T)$, for all $T \in \mathcal{T}$. Hence, $\theta \leq \bigcap_{T\in\mathcal{T}} \Lambda(T) = \widetilde{\Lambda}(\mathcal{T})$. Thus, $\widetilde{\Lambda}(\mathcal{T})$ is indeed the largest equivalence system included in $\widetilde{\lambda}(\mathcal{T})$.

■

Once more, if SEN happens to be the underlying sentence functor of an algebraic system $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, we sometimes write $\widetilde{\Lambda}^{\mathbf{A}}(\mathcal{T})$ and $\widetilde{\lambda}^{\mathbf{A}}(\mathcal{T})$ for $\widetilde{\Lambda}(\mathcal{T})$ and $\widetilde{\lambda}(\mathcal{T})$, respectively.

The analog of Proposition 1420 asserts that both $\widetilde{\Lambda}^{\mathbf{A}}(\mathcal{T})$ and $\widetilde{\lambda}^{\mathbf{A}}(\mathcal{T})$ are in the same relation with $\widetilde{\Omega}^{\mathbf{A}}(\mathcal{T})$ as $\Lambda^{\mathbf{A}}(T)$ and $\lambda^{\mathbf{A}}(T)$ are with $\Omega^{\mathbf{A}}(T)$, i.e., that the Tarski congruence system of a collection of sentence families is the largest congruence system included in either the Carnap equivalence system or the Carnap equivalence family of the collection.

Proposition 72 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system and* $\mathcal{T} \subseteq \mathrm{SenFam}(\mathbf{A})$.

(a) $\widetilde{\Omega}^{\mathbf{A}}(\mathcal{T})$ *is the largest congruence system on* $\mathbf{A}$ *included in* $\widetilde{\lambda}^{\mathbf{A}}(\mathcal{T})$*;*

(b) $\widetilde{\Omega}^{\mathbf{A}}(\mathcal{T})$ *is the largest congruence system on* $\mathbf{A}$ *included in* $\widetilde{\Lambda}^{\mathbf{A}}(\mathcal{T})$*.*

Proof:

(a) To see that $\widetilde{\Omega}^{\mathbf{A}}(\mathcal{T}) \le \widetilde{\lambda}^{\mathbf{A}}(\mathcal{T})$, let $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \widetilde{\Omega}^{\mathbf{A}}_{\Sigma}(\mathcal{T})$. Since $\widetilde{\Omega}^{\mathbf{A}}(\mathcal{T})$ is compatible with every $T \in \mathcal{T}$, we get that, for all $T \in \mathcal{T}$, $\phi \in T_\Sigma$ if and only if $\psi \in T_\Sigma$. Thus, $\langle \phi, \psi \rangle \in \widetilde{\lambda}^{\mathbf{A}}_{\Sigma}(\mathcal{T})$.

Suppose, next, that θ is a congruence system on $\mathbf{A}$, such that $\theta \le \widetilde{\lambda}^{\mathbf{A}}(\mathcal{T})$. We must show that $\theta \le \widetilde{\Omega}^{\mathbf{A}}(\mathcal{T})$. For this it suffices to show that θ is compatible with every $T \in \mathcal{T}$. Let $T \in \mathcal{T}$, $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \theta_\Sigma$ and $\phi \in T_\Sigma$. By hypothesis, $\langle \phi, \psi \rangle \in \widetilde{\lambda}^{\mathbf{A}}_{\Sigma}(\mathcal{T})$. By definition, $\langle \phi, \psi \rangle \in \lambda^{\mathbf{A}}_{\Sigma}(T)$. Therefore, since $\phi \in T_\Sigma$ we get $\psi \in T_\Sigma$ and, hence, θ is compatible with T.

(b) Since $\widetilde{\Omega}^{\mathbf{A}}(\mathcal{T})$ is an equivalence system and, by Part (a), $\widetilde{\Omega}^{\mathbf{A}}(\mathcal{T}) \le \widetilde{\lambda}^{\mathbf{A}}(\mathcal{T})$, we get, by Corollary 71, $\widetilde{\Omega}^{\mathbf{A}}(\mathcal{T}) \le \widetilde{\Lambda}^{\mathbf{A}}(\mathcal{T})$. Moreover, since, by Corollary 71, $\widetilde{\Lambda}^{\mathbf{A}}(\mathcal{T}) \le \widetilde{\lambda}^{\mathbf{A}}(\mathcal{T})$ and, by Part (a), $\widetilde{\Omega}^{\mathbf{A}}(\mathcal{T})$ is the largest congruence system in $\widetilde{\lambda}^{\mathbf{A}}(\mathcal{T})$, it must also be the largest one in $\widetilde{\Lambda}^{\mathbf{A}}(\mathcal{T})$. ■

Finally, consider a sentence functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, a collection $\mathcal{T}$ of sentence families of SEN and a sentence family $X \in \mathcal{T}$. The following is sometimes also termed Frege relation family, but, once more, to differentiate it from the preceding notions, we use the term "Lindenbaum" instead. we define:

- The **Lindenbaum relation system** $\widetilde{\Lambda}^{\mathcal{T}}(X) = \{\widetilde{\Lambda}^{\mathcal{T}}_{\Sigma}(X)\}_{\Sigma \in |\mathbf{Sign}|}$ **of** X **relative to** $\mathcal{T}$ by instantiating the definition of $\widetilde{\Lambda}$, given above, to the collection $\mathcal{T}^X$ of sentence families in $\mathcal{T}$ that include X, i.e.,
$$\widetilde{\Lambda}^{\mathcal{T}}(X) \coloneqq \widetilde{\Lambda}(\mathcal{T}^X) = \bigcap\{\Lambda(T) : T \in \mathcal{T}, X \le T\}.$$
- The **Lindenbaum relation family** $\widetilde{\lambda}^{\mathcal{T}}(X) = \{\widetilde{\lambda}^{\mathcal{T}}_{\Sigma}(X)\}_{\Sigma \in |\mathbf{Sign}|}$ **of** X **relative to** $\mathcal{T}$ by instantiating the definition of $\widetilde{\lambda}$ to the collection $\mathcal{T}^X$ of sentence families in $\mathcal{T}$ that include X, i.e.,
$$\widetilde{\lambda}^{\mathcal{T}}(X) \coloneqq \widetilde{\lambda}(\mathcal{T}^X) = \bigcap\{\lambda(T) : T \in \mathcal{T}, X \le T\}.$$

Using Corollary 71, we get immediately

Corollary 73 *Let* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a sentence functor,* $\mathcal{T} \subseteq \mathrm{SenFam}(\mathrm{SEN})$ *and* $X \in \mathcal{T}$*.*

(a) $\widetilde{\Lambda}^{\mathcal{T}}(X)$ *is an equivalence system on* SEN*;*

(b) $\widetilde{\lambda}^{\mathcal{T}}(X)$ *is an equivalence family on* SEN*;*

(c) $\widetilde{\Lambda}^{\mathcal{T}}(X)$ *is the largest equivalence system on* SEN *included in* $\widetilde{\lambda}^{\mathcal{T}}(X)$*.*

Proof: Directly by Corollary 71. ∎

When SEN happens to be the underlying sentence functor of an algebraic system $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, we sometimes write $\widetilde{\Lambda}^{\mathbf{A},\mathcal{T}}(X)$ and $\widetilde{\lambda}^{\mathbf{A},\mathcal{T}}(X)$ for the equivalence system $\widetilde{\Lambda}^{\mathcal{T}}(X)$ and the equivalence family $\widetilde{\lambda}^{\mathcal{T}}(X)$, respectively. Proposition 72 allows us to derive a relation between the Lindenbaum equivalence system $\widetilde{\Lambda}^{\mathbf{A},\mathcal{T}}(X)$ or the Lindenbaum equivalence family $\widetilde{\lambda}^{\mathbf{A},\mathcal{T}}(X)$ of a sentence family X relative to the collection $\mathcal{T}$ of sentence families and the Suszko congruence system $\widetilde{\Omega}^{\mathbf{A},\mathcal{T}}(X)$ of the family relative to the same collection.

Corollary 74 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system, consider* $\mathcal{T} \subseteq \mathrm{SenFam}(\mathbf{A})$ *and* $X \in \mathcal{T}$*.*

(a) $\widetilde{\Omega}^{\mathbf{A},\mathcal{T}}(X)$ *is the largest congruence system on* $\mathbf{A}$ *included in* $\widetilde{\lambda}^{\mathcal{T}}(X)$*;*

(b) $\widetilde{\Omega}^{\mathbf{A},\mathcal{T}}(X)$ *is the largest congruence system on* $\mathbf{A}$ *included in* $\widetilde{\Lambda}^{\mathcal{T}}(X)$*.*

Proof: We apply Proposition 72 to the collection $\mathcal{T}^X$. We get that $\widetilde{\Omega}^{\mathbf{A}}(\mathcal{T}^X)$ is the largest congruence system on $\mathbf{A}$ that is included in either $\widetilde{\lambda}(\mathcal{T}^X)$ or $\widetilde{\Lambda}(\mathcal{T}^X)$. The former is, by definition, equal to $\widetilde{\Omega}^{\mathbf{A},\mathcal{T}}(X)$ and the latter ones to $\widetilde{\lambda}^{\mathbf{A},\mathcal{T}}(X)$ and $\widetilde{\Lambda}^{\mathbf{A},\mathcal{T}}(X)$, respectively. So we get the conclusion. ∎

Consider now a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$, with $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$, and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system. The most common application of the Carnap operator will be to the collection $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ of all $\mathcal{I}$-filter families and that of the Lindenbaum operator to an $\mathcal{I}$-filter family T of $\mathcal{A}$ relative to $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. So we set the following notation:

$$\widetilde{\Lambda}^{\mathcal{A}}(\mathcal{I}) \coloneqq \widetilde{\Lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \quad \text{and} \quad \widetilde{\lambda}^{\mathcal{A}}(\mathcal{I}) \coloneqq \widetilde{\lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})).$$

Moreover, given $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we set

$$\widetilde{\Lambda}^{\mathcal{I},\mathcal{A}}(T) \coloneqq \widetilde{\Lambda}^{\mathcal{A},\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})}(T) \quad \text{and} \quad \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \coloneqq \widetilde{\lambda}^{\mathcal{A},\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})}(T).$$

When those notions specialize to the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$, the superscript referring to the algebraic system is often omitted. Thus, we have

$$\widetilde{\Lambda}(\mathcal{I}) = \widetilde{\Lambda}^{\mathcal{F}}(\mathcal{I}) \quad \text{and} \quad \widetilde{\lambda}(\mathcal{I}) = \widetilde{\lambda}^{\mathcal{F}}(\mathcal{I})$$

and, for $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\widetilde{\Lambda}^{\mathcal{I}}(T) = \widetilde{\Lambda}^{\mathcal{I},\mathcal{F}}(T) \quad \text{and} \quad \widetilde{\lambda}^{\mathcal{I}}(T) = \widetilde{\lambda}^{\mathcal{I},\mathcal{F}}(T).$$

We have the following characterizations of Lindenbaum equivalence systems and Lindenbaum equivalence families. We use those to derive characterizations of other relation families/systems as corollaries.

Theorem 75 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. Then, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,*

(a) $\langle \phi, \psi \rangle \in \widetilde{\Lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T)$ *if and only if, for all* $\Sigma' \in |\mathbf{Sign}|$*,* $f \in \mathbf{Sign}(\Sigma, \Sigma')$*,*

$$C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(T_{\Sigma'}, \mathrm{SEN}(f)(\phi)) = C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(T_{\Sigma'}, \mathrm{SEN}(f)(\psi));$$

(b) $\langle \phi, \psi \rangle \in \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T)$ *if and only if* $C^{\mathcal{I},\mathcal{A}}_\Sigma(T_\Sigma, \phi) = C^{\mathcal{I},\mathcal{A}}_\Sigma(T_\Sigma, \psi)$*.*

In particular, if $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$*, then* $\widetilde{\Lambda}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T)$*.*

Proof:

(a) We have, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $\langle \phi, \psi \rangle \in \widetilde{\Lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T)$ iff, for all $T \le T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}(f)(\phi) \in T'_{\Sigma'} \quad \text{iff} \quad \mathrm{SEN}(f)(\psi) \in T'_{\Sigma'}$$

iff, for all $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$T_\Sigma \cup \{\mathrm{SEN}(f)(\phi)\} \subseteq T'_{\Sigma'} \quad \text{iff} \quad T_\Sigma \cup \{\mathrm{SEN}(f)(\psi)\} \subseteq T'_{\Sigma'}$$

iff, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(T_\Sigma, \mathrm{SEN}(f)(\phi)) = C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(T_\Sigma, \mathrm{SEN}(f)(\psi)).$$

(b) We have, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $\langle \phi, \psi \rangle \in \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T)$ iff, for all $T \le T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\phi \in T'_\Sigma \Leftrightarrow \psi \in T'_\Sigma$ iff, for all $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T_\Sigma \cup \{\phi\} \subseteq T'_\Sigma \Leftrightarrow T_\Sigma \cup \{\psi\} \subseteq T'_\Sigma$ iff $C^{\mathcal{I},\mathcal{A}}_\Sigma(T_\Sigma, \phi) = C^{\mathcal{I},\mathcal{A}}_\Sigma(T_\Sigma, \psi)$.

The last statement follows from Parts (a) and (b) and the structurality property of $C^{\mathcal{I},\mathcal{A}}$. ■

Specializing to the least $\mathcal{I}$-filter family on $\mathcal{A}$, which happens to be a theory system, we get

Corollary 76 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system. Then* $\widetilde{\Lambda}^{\mathcal{A}}(\mathcal{I}) = \widetilde{\lambda}^{\mathcal{A}}(\mathcal{I})$ *and, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,*

$$\langle \phi, \psi \rangle \in \widetilde{\lambda}^{\mathcal{A}}_\Sigma(\mathcal{I}) \quad \textit{iff} \quad C^{\mathcal{I},\mathcal{A}}_\Sigma(\phi) = C^{\mathcal{I},\mathcal{A}}_\Sigma(\psi).$$

Proof: Directly by Theorem 75, by taking $T = C^{\mathcal{I},\mathcal{A}}(\varnothing)$. ■

Specializing to theory families, we get the following

Theorem 77 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $T \in \mathrm{ThFam}(\mathcal{I})$*. Then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

(a) $\langle \phi, \psi \rangle \in \widetilde{\Lambda}^{\mathcal{I}}_\Sigma(T)$ *if and only if, for all* $\Sigma' \in |\mathbf{Sign}^\flat|$*,* $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$*,*

$$C_{\Sigma'}(T_{\Sigma'}, \mathrm{SEN}^\flat(f)(\phi)) = C_{\Sigma'}(T_{\Sigma'}, \mathrm{SEN}^\flat(f)(\psi));$$

(b) $\langle \phi, \psi \rangle \in \widetilde{\lambda}^{\mathcal{I}}_\Sigma(T)$ *if and only if* $C_\Sigma(T_\Sigma, \phi) = C_\Sigma(T_\Sigma, \psi)$*.*

In particular, if $T \in \mathrm{ThSys}(\mathcal{I})$*, then* $\widetilde{\Lambda}^{\mathcal{I}}(T) = \widetilde{\lambda}^{\mathcal{I}}(T)$*.*

Proof: We apply Theorem 75 to the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. ∎

As a corollary, we also get

Corollary 78 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. Then* $\widetilde{\Lambda}(\mathcal{I}) = \widetilde{\lambda}(\mathcal{I})$ *and, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\langle \phi, \psi \rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}) \quad \textit{iff} \quad C_\Sigma(\phi) = C_\Sigma(\psi).$$

Proof: Apply Theorem 77 to $T = \mathrm{Thm}(\mathcal{I})$, which happens to be a theory system. ∎

We record, finally, a couple of relatively straightforward monotonicity properties of the Carnap and Lindenbaum operators. The following theorem refers to collections of filter families and individual filter families and the subsequent corollary specializes this to theory families.

Theorem 79 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$*,* $\mathcal{I}' = \langle \mathbf{F}, C' \rangle$ π*-institutions based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

(a) If $\mathcal{I} \le \mathcal{I}'$*, then* $\widetilde{\Lambda}^{\mathcal{A}}(\mathcal{I}) \le \widetilde{\Lambda}^{\mathcal{A}}(\mathcal{I}')$ *and* $\widetilde{\lambda}^{\mathcal{A}}(\mathcal{I}) \le \widetilde{\lambda}^{\mathcal{A}}(\mathcal{I}')$*;*

(b) If $T \le T'$*, then* $\widetilde{\Lambda}^{\mathcal{A},\mathcal{I}}(T) \le \widetilde{\Lambda}^{\mathcal{A},\mathcal{I}}(T')$ *and* $\widetilde{\lambda}^{\mathcal{A},\mathcal{I}}(T) \le \widetilde{\lambda}^{\mathcal{A},\mathcal{I}}(T')$*.*

Proof:

(a) Since $\mathcal{I} \le \mathcal{I}'$, we have $\mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Hence,

$$\begin{array}{rcl}\widetilde{\Lambda}^{\mathcal{A}}(\mathcal{I}) & = & \bigcap\{\Lambda^{\mathcal{A}}(X) : X \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \\ & \le & \bigcap\{\Lambda^{\mathcal{A}}(X) : X \in \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A})\} \\ & = & \widetilde{\Lambda}^{\mathcal{A}}(\mathcal{I}').\end{array}$$

An almost identical reasoning yields the second inclusion.

(b) Since $T \leq T'$, we get

$$\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T'} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T},$$

whence we have

$$\begin{array}{rcl}\widetilde{\Lambda}^{\mathcal{I},\mathcal{A}}(T) & = & \bigcap\{\Lambda^{\mathcal{A}}(X) : T \leq X \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \\ & \leq & \bigcap\{\Lambda^{\mathcal{A}}(X) : T' \leq X \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \\ & = & \widetilde{\Lambda}^{\mathcal{I},\mathcal{A}}(T')\end{array}$$

and, similarly, $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \leq \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T')$.

■

Corollary 80 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$, $\mathcal{I}' = \langle \mathbf{F}, C' \rangle$ π*-institutions, based on* $\mathbf{F}$*, and* $T, T' \in \mathrm{ThFam}(\mathcal{I})$.

(a) If $\mathcal{I} \leq \mathcal{I}'$*, then* $\widetilde{\Lambda}(\mathcal{I}) \leq \widetilde{\Lambda}(\mathcal{I}')$ *and* $\widetilde{\lambda}(\mathcal{I}) \leq \widetilde{\lambda}(\mathcal{I}')$*;*

(b) If $T \leq T'$*, then* $\widetilde{\Lambda}^{\mathcal{I}}(T) \leq \widetilde{\Lambda}^{\mathcal{I}}(T')$ *and* $\widetilde{\lambda}^{\mathcal{I}}(T) \leq \widetilde{\lambda}^{\mathcal{I}}(T')$*.*

Proof: Apply Theorem 79 to $\mathcal{A} = \mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. ■

In closing, we provide the following table summarizing the correspondences between notions giving rise to congruence systems and notions giving rise to equivalence families and systems:

	$T \in \mathrm{SenFam}(\mathbf{A})$	$\mathcal{T} \subseteq \mathrm{SenFam}(\mathbf{A})$	$T \in \mathcal{T} \subseteq \mathrm{SenFam}(\mathbf{A})$
Congrunece Systems	Leibniz $\Omega^{\mathbf{A}}(T)$	Tarski $\widetilde{\Omega}^{\mathbf{A}}(\mathcal{T})$	Suszko $\widetilde{\Omega}^{\mathbf{A},\mathcal{T}}(T)$
Equivalence Families/Systems	Frege $\Lambda^{\mathbf{A}}(T), \lambda^{\mathbf{A}}(T)$	Carnap $\widetilde{\Lambda}^{\mathbf{A}}(\mathcal{T}), \widetilde{\lambda}^{\mathbf{A}}(\mathcal{T})$	Lindenbaum $\widetilde{\Lambda}^{\mathbf{A},\mathcal{T}}(T), \widetilde{\lambda}^{\mathbf{A},\mathcal{T}}(T)$

2.12 Subsystems and π-Subinstitutions

In this section, we look at $N^\flat$-algebraic subsystems. Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system and let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an $N^\flat$-algebraic system. A **universe** U **of** $\mathbf{A}$ is a sentence system of $\mathbf{A}$ that is closed under the operations in N, i.e., such that, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in U_\Sigma$,

$$\sigma_\Sigma(\vec{\phi}) \in U_\Sigma.$$

We denote by $\mathrm{Unv}(\mathbf{A})$ the collection of all universes of $\mathbf{A}$.

Given a universe $U \in \mathrm{Unv}(\mathbf{A})$, we may define a functor $\mathrm{SEN}' : \mathbf{Sign} \to \mathbf{Set}$, as follows:

- For all $\Sigma \in |\mathbf{Sign}|$, $\mathrm{SEN}'(\Sigma) = U_\Sigma$;

- For all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\phi \in \mathrm{SEN}'(\Sigma)$,

$$\mathrm{SEN}'(f)(\phi) = \mathrm{SEN}(f)(\phi).$$

Moreover, given a natural transformation $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, we may define the natural transformation $\sigma' : \mathrm{SEN}'^k \to \mathrm{SEN}'$ by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}'(\Sigma)$,

$$\sigma'_\Sigma(\vec{\phi}) = \sigma_\Sigma(\vec{\phi}).$$

In other words $\sigma' = \sigma \upharpoonright_U = \sigma \upharpoonright_{\mathrm{SEN}'}$.

We denote by N' the category of natural transformations on SEN' consisting of the restrictions $\sigma' = \sigma \upharpoonright_{\mathrm{SEN}'}$, with the composition operation inherited by that of N, i.e., such that

$$\sigma' \circ \tau' = \sigma \upharpoonright_{\mathrm{SEN}'} \circ \tau \upharpoonright_{\mathrm{SEN}'} = (\sigma \circ \tau) \upharpoonright_{\mathrm{SEN}'} = (\sigma \circ \tau)'.$$

Finally, we set $\mathbf{A}' = \langle \mathbf{Sign}, \mathrm{SEN}', N' \rangle$ and call $\mathbf{A}'$ the **algebraic subsystem of A** on the universe U or on the functor SEN'. We write $\mathbf{A}' \leq \mathbf{A}$ to signify that $\mathbf{A}'$ is an algebraic subsystem of $\mathbf{A}$.

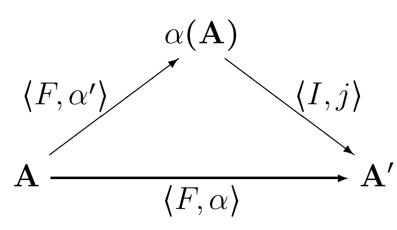
Note that the pair $\langle I, j \rangle : \mathbf{A}' \to \mathbf{A}$, where $I : \mathbf{Sign} \to \mathbf{Sign}$ and $j : \mathrm{SEN}' \to \mathrm{SEN}$, defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}'(\Sigma)$, by

$$j_\Sigma(\phi) = \phi,$$

becomes a morphism of $N^\flat$-algebraic systems, called the **injection morphism of A′ into A**.

Now we relate injection morphisms with the construction of the image algebraic system outlined in Lemma 13.

Proposition 81 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *and* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ $N^\flat$*-algebraic systems and* $\langle F, \alpha \rangle : \mathbf{A} \to \mathbf{A}'$ *an algebraic system morphism, with* $F : \mathbf{Sign} \to \mathbf{Sign}'$ *an isomorphism. Then, we have* $\langle F, \alpha \rangle = \langle I, j \rangle \circ \langle F, \alpha' \rangle$*,*



where $\langle F, \alpha' \rangle : \mathbf{A} \to \alpha(\mathbf{A})$ *is the surjective morphism defined in Lemma 14 and* $\langle I, j \rangle : \alpha(\mathbf{A}) \to \mathbf{A}'$ *is the injection morphism.*

Proof: We have, using the definitions, that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$j_{F(\Sigma)}(\alpha'_\Sigma(\phi)) = j_{F(\Sigma)}(\alpha_\Sigma(\phi)) = \alpha_\Sigma(\phi).$$

This proves the commutativity of the triangle. ■

We call the decomposition of $\langle F, \alpha\rangle : \mathbf{A} \to \mathbf{A}'$ established in Proposition 81, the **(natural) epi-mono factorization of** $\langle F, \alpha\rangle$.

Of particular interest are the subuniverses of an algebraic system that are generated by a given sentence family X of the algebraic system. We detail this construction here and introduce some relevant notation.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system and let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an $N^\flat$-algebraic system. Consider a sentence family

$$X \in \mathrm{SenFam}(\mathbf{A}).$$

Of course, it is very likely that X is neither a system (i.e., invariant under signature morphisms) nor closed under the operations in N. But we have pertinent constructions that can be employed to obtain a closure of X with respect to those operations.

Recall, first, that, by Proposition 2, $\overrightarrow{X}$ is the least sentence system of $\mathbf{A}$ containing X.

Second, define $\nu^{\mathbf{A}}(X) = \{\nu^{\mathbf{A}}_\Sigma(X)\}_{\Sigma\in|\mathbf{Sign}|}$, by letting, for all $\Sigma \in |\mathbf{Sign}|$, $\nu^{\mathbf{A}}_\Sigma(X)$ be given by

$$\nu^{\mathbf{A}}_\Sigma(X) = \{\sigma^{\mathbf{A}}_\Sigma(\vec{\phi}) : \sigma \in N, \vec{\phi} \in X_\Sigma\}.$$

We can show that $\nu^{\mathbf{A}}(X)$ is the least sentence family of $\mathbf{A}$ containing X and closed under the operations in N and that, moreover, it happens to be a sentence system in case X is a sentence system. As a consequence, we obtain that $\nu^{\mathbf{A}}(\overrightarrow{X})$ is the least universe of $\mathbf{A}$ including X. These results are detailed in the following proposition and theorem.

Proposition 82 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system and* $X \in \mathrm{SenFam}(\mathbf{A})$.

(a) $\nu^{\mathbf{A}}(X)$ *is the least sentence family of* $\mathbf{A}$ *including* X *and closed under the operations in* N*;*

(b) If $X \in \mathrm{SenSys}(\mathbf{A})$*, the* $\nu^{\mathbf{A}}(X)$ *is also a sentence system.*

Proof: Note, first, that, since the identity $\iota : \mathrm{SEN} \to \mathrm{SEN}$ is a natural transformation in N, we have, by definition, that $X \le \nu^{\mathbf{A}}(X)$. Suppose, next, that $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ is in N, $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi} \in \nu^{\mathbf{A}}_\Sigma(X)$. Thus, for all $i < k$, there exists $\tau^i : \mathrm{SEN}^{n_i} \to \mathrm{SEN}$ and $\vec{\chi}^i \in X_\Sigma$, such that

$$\phi_i = \tau^i_\Sigma(\vec{\chi}^i).$$

Let $n = n_0 + n_1 + \cdots + n_{k-1}$ and $\vec{\chi} = \langle \vec{\chi}^0, \vec{\chi}^1, \dots, \vec{\chi}^{k-1} \rangle$ be the vector of length n resulting from the concatenation of the elements of the $\vec{\chi}^i$'s. Then we get that

$$\begin{array}{rcl}\sigma_\Sigma(\vec{\phi}) & = & \sigma_\Sigma(\tau^0_\Sigma(\vec{\chi}^0), \dots, \tau^{k-1}_\Sigma(\vec{\chi}^{k-1})) \\ & = & [\sigma \circ \langle \tau^0 \circ \langle p^{n,0}, \dots, p^{n,n_0-1} \rangle, \tau^1 \circ \langle p^{n,n_0}, \dots, p^{n,n_0+n_1-1} \rangle, \dots, \\ & & \qquad \tau^{k-1} \circ \langle p^{n,n_0+\cdots+n_{k-1}}, \dots, p^{n,n_0+\cdots+n_k-1} \rangle \rangle](\vec{\chi}).\end{array}$$

Since the natural transformation above is in N and $\vec{\chi} \in X_\Sigma$, we conclude that $\sigma_\Sigma(\vec{\phi}) \in \nu^{\mathbf{A}}_\Sigma(X)$, whence $\nu^{\mathbf{A}}(\Sigma)$ is closed under the operations in N.

To show minimality, suppose that $Y \in \mathrm{SenFam}(\mathbf{A})$, such that $X \leq Y$ and Y is closed under the operations in N. Consider $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \nu^{\mathbf{A}}_\Sigma(X)$. By definition, there exists σ in N and $\vec{\phi} \in X_\Sigma$, such that $\phi = \sigma_\Sigma(\vec{\phi})$. But, then, since $\vec{\phi} \in X_\Sigma \subseteq Y_\Sigma$ and Y is closed under the operations in N, we get that $\phi = \sigma_\Sigma(\vec{\phi}) \in Y_\Sigma$. Since this holds for all $\Sigma \in |\mathbf{Sign}|$, we get that $\nu^{\mathbf{A}}(X) \leq Y$ and, hence, $\nu^{\mathbf{A}}(X)$ is the least sentence family of $\mathbf{A}$ including X and closed under the operations in N.

Finally, let $X \in \mathrm{SenSys}(\mathbf{A})$. Suppose $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\phi \in \nu^{\mathbf{A}}_\Sigma(X)$. Then, there exists σ in N and $\vec{\phi} \in X_\Sigma$, such that $\phi = \sigma_\Sigma(\vec{\phi})$. We now get

$$\begin{array}{rcl}\mathrm{SEN}(f)(\phi) & = & \mathrm{SEN}(f)(\sigma_\Sigma(\vec{\phi})) \\ & = & \sigma_{\Sigma'}(\mathrm{SEN}(f)(\vec{\phi})) \quad (\sigma \text{ in } N) \\ & \in & \nu^{\mathbf{A}}_{\Sigma'}(X). \quad (\sigma \text{ in } N,\ \phi \in X_\Sigma,\ X \in \mathrm{SenSys}(\mathbf{A}))\end{array}$$

Therefore $\nu^{\mathbf{A}}(X) \in \mathrm{SenSys}(\mathbf{A})$. ■

Theorem 83 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *be an algebraic system and consider* $X \in \mathrm{SenFam}(\mathbf{A})$. *Then* $\nu^{\mathbf{A}}(\overrightarrow{X})$ *is the least universe of* $\mathbf{A}$ *including* X.

Proof: By Proposition 2, $\overrightarrow{X} \in \mathrm{SenSys}(\mathbf{A})$. Therefore, by Proposition 82, $\nu^{\mathbf{A}}(\overrightarrow{X}) \in \mathrm{Unv}(\mathbf{A})$. Suppose, that $U \in \mathrm{Unv}(\mathbf{A})$, such that $X \leq U$. Since U is a universe, it is a sentence system. Thus, by Proposition 2, $\overrightarrow{X} \leq U$. Moreover, since U is a universe, it is closed under the operations in N, whence, by Proposition 82, $\nu^{\mathbf{A}}(\overrightarrow{X}) \leq U$. We conclude that $\nu^{\mathbf{A}}(\overrightarrow{X})$ is the least universe of $\mathbf{A}$ containing X. ■

Based on Theorem 83, given $X \in \mathrm{SenFam}(\mathbf{A})$, we call $\nu^{\mathbf{A}}(\overrightarrow{X})$ the **universe of A generated by** X and sometimes denote it by

$$\langle X \rangle = \{\langle X \rangle_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}.$$

We adopt many simplifying notations such as writing $\langle \Phi \rangle$, $\Phi \subseteq \mathrm{SEN}(\Sigma)$, for the universe $\langle T \rangle$, generated by $T \in \mathrm{SenFam}(\mathbf{A})$, with

$$T_{\Sigma'} = \begin{cases} \Phi, & \text{if } \Sigma' = \Sigma \\ \varnothing, & \text{if } \Sigma' \neq \Sigma \end{cases}$$

and $\langle\phi,\psi\rangle$ for $\langle\{\phi,\psi\}\rangle$, $\phi,\psi\in\mathrm{SEN}(\Sigma)$, if such overloading is unlikely to result into major mayhem.

Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system. Consider an $\mathbf{F}$-algebraic system $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$, with $\mathbf{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$ and let $\mathbf{A}'=\langle\mathbf{Sign},\mathrm{SEN}',N'\rangle$ be an algebraic subsystem of $\mathbf{A}$. Define $\alpha^{-1}(\mathrm{SEN}')=\{\alpha_\Sigma^{-1}(\mathrm{SEN}')\}_{\Sigma\in|\mathbf{Sign}^\flat|}$ by letting $\alpha_\Sigma^{-1}(\mathrm{SEN}')$ be given, for all $\Sigma\in|\mathbf{Sign}^\flat|$, by

$$\alpha_\Sigma^{-1}(\mathrm{SEN}')=\alpha_\Sigma^{-1}(\mathrm{SEN}'(F(\Sigma))).$$

Lemma 84 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$*, with* $\mathbf{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$*, be an* $\mathbf{F}$*-algebraic system. If* $\mathbf{A}'=\langle\mathbf{Sign},\mathrm{SEN}',N'\rangle\le\mathbf{A}$ *is an algebraic subsystem of* $\mathbf{A}$*, then* $\alpha^{-1}(\mathrm{SEN}')$ *is a universe of* $\mathbf{F}$*.*

Proof: Since SEN' is a sentence system of $\mathcal{A}$, by Lemma 6, we get that $\alpha^{-1}(\mathrm{SEN}')$ is a sentence system of $\mathbf{F}$. So it suffices to show that $\alpha^{-1}(\mathrm{SEN}')$ is closed under the operations in $N^\flat$. To this end, let $\sigma^\flat\in N^\flat$, $\Sigma\in|\mathbf{Sign}^\flat|$ and $\vec{\phi}\in\alpha^{-1}(\mathrm{SEN}'(F(\Sigma))$. Then we have

$$\begin{array}{rcl}\alpha_\Sigma(\sigma_\Sigma^\flat(\vec{\phi})) &=& \sigma_{F(\Sigma)}^{\mathcal{A}}(\alpha_\Sigma(\vec{\phi}))\\ &\in& \mathrm{SEN}'(F(\Sigma)),\end{array}$$

since $\alpha_\Sigma(\vec{\phi})\in\mathrm{SEN}'(F(\Sigma))$, by hypothesis, and $\mathrm{SEN}'(F(\Sigma))$ is a universe of $\mathbf{A}$. Thus $\alpha^{-1}(\mathrm{SEN}')$ is indeed a universe of $\mathbf{F}$. ∎

We define the triple $\alpha^{-1}(\mathbf{A}')=\langle\mathbf{Sign}^\flat,\mathrm{SEN}'^\flat,N'^\flat\rangle$ as the algebraic subsystem of $\mathbf{F}$ determined by the universe $\alpha^{-1}(\mathrm{SEN}')$ of $\mathbf{F}$.

Let, again, $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$ an $\mathbf{F}$-algebraic system, with $\mathbf{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$, and $\mathbf{A}'=\langle\mathbf{Sign},\mathrm{SEN}',N'\rangle$ be an algebraic subsystem of $\mathbf{A}$. We define the pair $\langle F,\alpha'\rangle:\alpha^{-1}(\mathbf{A}')\to\mathbf{A}'$ by letting, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\phi\in\alpha_\Sigma^{-1}(\mathrm{SEN}'(F(\Sigma))$,

$$\alpha'_\Sigma(\phi)=\alpha_\Sigma(\phi).$$

Then $\langle F,\alpha'\rangle$ turns out to be a surjective morphism from $\alpha^{-1}(\mathbf{A}')$ to $\mathbf{A}'$.

Lemma 85 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$*, with* $\mathbf{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$*, be an* $\mathbf{F}$*-algebraic system. If* $\mathbf{A}'=\langle\mathbf{Sign},\mathrm{SEN}',N'\rangle\le\mathbf{A}$ *is an algebraic subsystem of* $\mathbf{A}$*, then* $\langle F,\alpha'\rangle:\alpha^{-1}(\mathbf{A}')\to\mathbf{A}'$ *is a surjective morphism.*

Proof: Since $F:\mathbf{Sign}^\flat\to\mathbf{Sign}$ is surjective and full, by hypothesis, it suffices to show that, for all $\Sigma\in|\mathbf{Sign}^\flat|$, $\alpha'_\Sigma:\alpha^{-1}(\mathrm{SEN}'(F(\Sigma)))\to\mathrm{SEN}'(F(\Sigma))$ is also surjective. But this follows by the definition of $\alpha^{-1}(\mathrm{SEN}')$ and the surjectivity of $\langle F,\alpha\rangle:\mathbf{F}\to\mathbf{A}$. ∎

Lemma 85 shows that $\mathcal{A}' = \langle \mathbf{A}', \langle F, \alpha' \rangle \rangle$ may be viewed as an $\alpha^{-1}(\mathbf{A}')$-algebraic system.

Consider now an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ and a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ based on $\mathbf{F}$. Given an algebraic subsystem $\mathbf{F}' = \langle \mathbf{Sign}^\flat, \mathrm{SEN}'^\flat, N'^\flat \rangle$ of $\mathbf{F}$, we define the π-**subinstitution induced by**, or **associated with** $\mathbf{F}'$, to be the pair $\mathcal{I}' = \langle \mathbf{F}', C' \rangle$, where $C' : \mathcal{P}\mathrm{SEN}'^\flat \to \mathcal{P}\mathrm{SEN}'^\flat$ is defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \mathrm{SEN}'^\flat(\Sigma)$, by

$$C'_\Sigma(\Phi) = C_\Sigma(\Phi) \cap \mathrm{SEN}'^\flat(\Sigma).$$

Proposition 86 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution, based on* $\mathbf{F}$*, and* $\mathbf{F}' = \langle \mathbf{Sign}^\flat, \mathrm{SEN}'^\flat, N'^\flat \rangle$ *an algebraic subsystem of* $\mathbf{F}$*. Then* $\mathcal{I}' = \langle \mathbf{F}', C' \rangle$ *is a* π*-institution.*

Proof: We must show that $C' : \mathcal{P}\mathrm{SEN}'^\flat \to \mathcal{P}\mathrm{SEN}'^\flat$ is a closure system on $\mathbf{F}'$. The inflation property is clear, since, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \mathrm{SEN}'^\flat(\Sigma)$,

$$\Phi \subseteq C_\Sigma(\Phi) \cap \mathrm{SEN}'^\flat(\Sigma) = C'_\Sigma(\Phi).$$

Monotonicity is also clear, since, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi, \Psi \subseteq \mathrm{SEN}'^\flat(\Sigma)$, such that $\Phi \subseteq \Psi$,

$$C'_\Sigma(\Phi) = C_\Sigma(\Phi) \cap \mathrm{SEN}'^\flat(\Sigma) \subseteq C_\Sigma(\Psi) \cap \mathrm{SEN}'^\flat(\Sigma) = C'_\Sigma(\Psi).$$

For idempotency, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}'^\flat(\Sigma)$, such that $\phi \in C'_\Sigma(C'_\Sigma(\Phi))$. Then we have

$$\begin{array}{rcl}\phi & \in & C_\Sigma(C_\Sigma(\Phi) \cap \mathrm{SEN}'^\flat(\Sigma)) \cap \mathrm{SEN}'^\flat(\Sigma) \\ & \subseteq & C_\Sigma(C_\Sigma(\Phi)) \cap \mathrm{SEN}'^\flat(\Sigma) \\ & = & C_\Sigma(\Phi) \cap \mathrm{SEN}'^\flat(\Sigma) \\ & = & C'_\Sigma(\Phi).\end{array}$$

It now only remains to show that C' is also structural. Let $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\Phi \subseteq \mathrm{SEN}'^\flat(\Sigma)$. Then, we have

$$\begin{array}{rcl}\mathrm{SEN}'^\flat(f)(C'_\Sigma(\Phi)) & = & \mathrm{SEN}'^\flat(f)(C_\Sigma(\Phi) \cap \mathrm{SEN}'^\flat(\Sigma)) \\ & \subseteq & \mathrm{SEN}^\flat(f)(C_\Sigma(\Phi)) \cap \mathrm{SEN}'^\flat(\Sigma') \\ & \subseteq & C_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \cap \mathrm{SEN}'^\flat(\Sigma') \\ & = & C_{\Sigma'}(\mathrm{SEN}'^\flat(f)(\Phi)) \cap \mathrm{SEN}'^\flat(\Sigma') \\ & = & C'_{\Sigma'}(\mathrm{SEN}'^\flat(f)(\Phi)).\end{array}$$

We conclude that C' is a closure system on $\mathbf{F}'$ and, therefore, $\mathcal{I}'$ is a π-institution. ■

We also give a characterization of the theory families and the theory systems of the induced subinstitution in terms of those of its parent.

Proposition 87 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution, based on* $\mathbf{F}$*, and* $\mathbf{F}' = \langle \mathbf{Sign}^\flat, \mathrm{SEN}'^\flat, N'^\flat \rangle$ *an algebraic subsystem of* $\mathbf{F}$*. Then*

$$\mathrm{ThFam}(\mathcal{I}') = \{T \cap \mathrm{SEN}'^\flat : T \in \mathrm{ThFam}(\mathcal{I})\}$$
$$\textit{and} \quad \mathrm{ThSys}(\mathcal{I}') = \{T \cap \mathrm{SEN}'^\flat : T \in \mathrm{ThSys}(\mathcal{I})\}.$$

Proof: We show the first equality. The second may be proved similarly.

Suppose, first, that $T' \in \mathrm{ThFam}(\mathcal{I}')$. Then we have $C'(T') = T'$. By definition, $C'(T') = C(T') \cap \mathrm{SEN}'^\flat$. Thus, we get $T' = C(T') \cap \mathrm{SEN}'^\flat$. Since $C(T') \in \mathrm{ThFam}(\mathcal{I})$, we get that $\mathrm{ThFam}(\mathcal{I}') \subseteq \{T \cap \mathrm{SEN}'^\flat : T \in \mathrm{ThFam}(\mathcal{I})\}$.

Suppose, conversely, that $T \in \mathrm{ThFam}(\mathcal{I})$. Then, we have

$$\begin{array}{rcl} C'(T \cap \mathrm{SEN}'^\flat) & = & C(T \cap \mathrm{SEN}'^\flat) \cap \mathrm{SEN}'^\flat \\ & \subseteq & C(T) \cap \mathrm{SEN}'^\flat \\ & = & T \cap \mathrm{SEN}'^\flat. \end{array}$$

So $T \cap \mathrm{SEN}'^\flat \in \mathrm{ThFam}(\mathcal{I}')$ and we conclude that

$$\{T \cap \mathrm{SEN}'^\flat : T \in \mathrm{ThFam}(\mathcal{I})\} \subseteq \mathrm{ThFam}(\mathcal{I}').$$

Equality now follows. ∎

Proposition 87 implies that the property of all theory families being theory systems (which shall be used in the next chapter as the defining property of a *systemic* π*-institution*) is inherited by all π-subinstitutions of a π-institution:

Corollary 88 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution, based on* $\mathbf{F}$*, and* $\mathbf{F}' = \langle \mathbf{Sign}^\flat, \mathrm{SEN}'^\flat, N'^\flat \rangle$ *an algebraic subsystem of* $\mathbf{F}$*. If* $\mathcal{I}$ *is such that* $\mathrm{ThFam}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$*, then* $\mathcal{I}' = \langle \mathbf{F}', C' \rangle$ *satisfies the same property.*

Proof: If $T' \in \mathrm{ThFam}(\mathcal{I}')$, then, by Proposition 87, there exists a theory family $T \in \mathrm{ThFam}(\mathcal{I})$, such that $T' = T \cap \mathrm{SEN}'^\flat$. By hypothesis, we have $T \in \mathrm{ThSys}(\mathcal{I})$, whence $T' = T \cap \mathrm{SEN}'^\flat \in \mathrm{ThSys}(\mathcal{I}')$. It follows that $\mathrm{ThFam}(\mathcal{I}') = \mathrm{ThSys}(\mathcal{I}')$. ∎

We now look at a relationship between Leibniz congruence systems of theory families in institutions and of Leibniz congruence systems of corresponding theory families in subinstitutions associated with given universes.

Proposition 89 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution, based on* $\mathbf{F}$*, and* $\mathcal{I}' = \langle \mathbf{F}', C' \rangle$ *a* π*-subinstitution of* $\mathcal{I}$*, associated with* $\mathbf{F}' = \langle \mathbf{Sign}, \mathrm{SEN}'^\flat, N'^\flat \rangle \leq \mathbf{F}$*. Then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\Omega^{\mathbf{F}}(T) \cap (\mathrm{SEN}'^\flat)^2 \leq \Omega^{\mathbf{F}'}(T \cap \mathrm{SEN}'^\flat).$$

Proof: By the maximality property of $\Omega^{\mathbf{F}'}(T \cap \mathrm{SEN}'^\flat)$, it suffices to show that $\Omega^{\mathbf{F}}(T) \cap (\mathrm{SEN}'^\flat)^2$ is a congruence system on $\mathbf{F}'$ that is compatible with the theory family $T \cap \mathrm{SEN}'^\flat$.

The reflexivity, symmetry, transitivity and congruence properties of

$$\Omega^{\mathbf{F}}(T) \cap (\mathrm{SEN}'^\flat)^2$$

are inherited by those of $\Omega^{\mathbf{F}}(T)$. Moreover, we have, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\begin{array}{lll}\mathrm{SEN}'^\flat(f)(\Omega^{\mathbf{F}}_\Sigma(T) \cap \mathrm{SEN}'^\flat(\Sigma)^2) & \subseteq & \mathrm{SEN}^\flat(f)(\Omega^{\mathbf{F}}_\Sigma(T)) \cap \mathrm{SEN}'^\flat(\Sigma')^2 \\ & \subseteq & \Omega^{\mathbf{F}}_{\Sigma'}(T) \cap \mathrm{SEN}'^\flat(\Sigma')^2.\end{array}$$

So $\Omega^{\mathbf{F}}(T) \cap (\mathrm{SEN}'^\flat)^2$ is indeed a congruence system on $\mathbf{F}'$. Finally, assume that $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}'^\flat(\Sigma)$, such that

$$\langle \phi, \psi \rangle \in \Omega^{\mathbf{F}}_\Sigma(T) \cap \mathrm{SEN}'^\flat(\Sigma)^2 \quad \text{and} \quad \phi \in T_\Sigma \cap \mathrm{SEN}'^\flat(\Sigma).$$

Then, by the compatibility of $\Omega^{\mathbf{F}}(T)$ with T, we get that $\psi \in T_\Sigma \cap \mathrm{SEN}'^\flat(\Sigma)$. We conclude that $\Omega^{\mathbf{F}}(T) \cap (\mathrm{SEN}'^\flat)^2$ is indeed compatible with $T \cap \mathrm{SEN}'^\flat$ and, therefore, $\Omega^{\mathbf{F}}(T) \cap (\mathrm{SEN}'^\flat)^2 \leq \Omega^{\mathbf{F}'}(T \cap \mathrm{SEN}'^\flat)$. ■

In particular, we have the following, where, recall that $\langle \phi, \psi \rangle$ denotes the universe of $\mathbf{F}$ generated by $\{\phi, \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$.

Corollary 90 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$. *Then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(T) \quad \textit{implies} \quad \langle \phi, \psi \rangle \in \Omega^{\langle \phi, \psi \rangle}_\Sigma(T \cap \langle \phi, \psi \rangle).$$

Proof: This follows directly by Proposition 89 by considering the universe $\langle \phi, \psi \rangle$ of $\mathbf{F}$ generated by the sentence family T, with $T_\Sigma = \{\phi, \psi\}$ and $T_{\Sigma'} = \varnothing$, for all $\Sigma' \neq \Sigma$. ■

We turn now to the examination of the relation between π-institutions and their models, on the one hand, and π-subinstitutions and their models, on the other.

We show first that, for every π-institution $\mathcal{I}$, every $\mathcal{I}$-filter family on an $\mathbf{F}$-algebraic system $\mathcal{A}$ gives rise naturally to an $\mathcal{I}'$-filter family on an $\mathbf{F}'$-algebraic subsystem of $\mathcal{A}$.

Proposition 91 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. *Let, also* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle \leq \mathbf{A}$ *be an algebraic subsystem of* $\mathbf{A}$. *Then* $T \cap \mathrm{SEN}' \in \mathrm{FiFam}^{\mathcal{I}'}(\langle \mathbf{A}', \langle F, \alpha' \rangle \rangle)$, *where* $\mathcal{I}' = \langle \alpha^{-1}(\mathbf{A}'), C' \rangle$ *is the* π*-subinstitution of* $\mathcal{I}$ *induced by* $\alpha^{-1}(\mathbf{A}')$.

Proof: By Lemma 84, $\alpha^{-1}(\mathbf{A}')$ is an algebraic subsystem of $\mathbf{F}$. Therefore, the pair $\mathcal{I}' = \langle \alpha^{-1}(\mathbf{A}'), C'\rangle$ is a well defined π-subinstitution of $\mathcal{I}$. So it suffices to show, by Lemma 51, that $\alpha^{-1}(T \cap \mathrm{SEN}') \in \mathrm{ThFam}(\mathcal{I}')$. But this is easy, since we have

$$\alpha^{-1}(T \cap \mathrm{SEN}') = \alpha^{-1}(T) \cap \alpha^{-1}(\mathrm{SEN}') \in \mathrm{ThFam}(\mathcal{I}'),$$

membership following by Lemma 51 and Proposition 87. ■

As a corollary, we obtain the fact that inverse images of Leibniz congruence systems of filter families on algebraic subsystems equal Leibniz congruence systems of the corresponding theory families of π-subinstitutions.

Corollary 92 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. Let, also* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle \leq \mathbf{A}$ *be an algebraic subsystem of* $\mathbf{A}$*. Then*

$$\alpha^{-1}(\Omega^{\mathbf{A}'}(T \cap \mathrm{SEN}')) = \Omega^{\alpha^{-1}(\mathbf{A}')}(\alpha^{-1}(T) \cap \alpha^{-1}(\mathrm{SEN}')).$$

Proof: This follows by Proposition 91 and Proposition 24. ■

2.13 Syntax

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system and consider a set $E \subseteq N$ of natural transformations in N. All natural transformations in E are, therefore, finitary. Since, however, there may be an infinite number of them, they may be collectively of unbounded arity. As a consequence, we write, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^\omega$,

$$E_\Sigma(\vec{\phi}) = \{\sigma_\Sigma(\phi_0, \dots, \phi_{k-1}) : \sigma \in E\}$$

to denote the values of E on the tuple $\vec{\phi}$, where, for each $\sigma \in E$ k-ary, only the first k components of $\vec{\phi}$ are actually used.

In certain contexts, we will view the first k positions of each natural transformation in E as **distinguished**, while treating all remaining positions as **parametric**. In that case we have to exercise meticulous care when we employ the following notation. Given $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$, we write

$$E_\Sigma[\vec{\phi}] = \{E_{\Sigma,\Sigma'}[\vec{\phi}]\}_{\Sigma' \in |\mathbf{Sign}|},$$

where, for all $\Sigma' \in |\mathbf{Sign}|$, we define

$$E_{\Sigma,\Sigma'}[\vec{\phi}] \quad = \quad \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\vec{\phi}), \vec{\chi}) : \sigma \in E, f \in \mathbf{Sign}(\Sigma, \Sigma'), \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')\}.$$

Let, again, $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an algebraic system and $E \subseteq N$. For $T \in \mathrm{SenFam}(\mathbf{A})$, we set

$$\overleftarrow{E}(T) = \{\overleftarrow{E}_\Sigma(T)\}_{\Sigma \in |\mathbf{Sign}|},$$

where, for all $\Sigma \in |\mathbf{Sign}|$,

$$\overleftarrow{E}_\Sigma(T) = \{\vec{\phi} \in \mathrm{SEN}(\Sigma) : E_\Sigma[\vec{\phi}] \leq T\}.$$

We show that $\overleftarrow{E}(T)$ is a relation system on $\mathbf{A}$.

Lemma 93 *Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an algebraic system, $E \subseteq N$ and $T \in \mathrm{SenFam}(\mathbf{A})$. Then $\overleftarrow{E}(T)$ is a relation system on $\mathbf{A}$.*

Proof: Let $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi} \in \mathrm{SEN}(\Sigma)$, such that $\vec{\phi} \in \overleftarrow{E}_\Sigma(T)$. Our goal is to show that, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}(f)(\vec{\phi}) \in \overleftarrow{E}_{\Sigma'}(T).$$

So we fix $\Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma, \Sigma')$. By hypothesis, we have that, $E_\Sigma[\vec{\phi}] \leq T$. Thus, for all $\Sigma'' \in |\mathbf{Sign}|$, $g \in \mathbf{Sign}(\Sigma', \Sigma'')$ and $\vec{\chi} \in \mathrm{SEN}(\Sigma'')$,

$$\Sigma \xrightarrow{\ f\ } \Sigma' \xrightarrow{\ g\ } \Sigma''$$

$$E_{\Sigma''}(\mathrm{SEN}(gf)(\vec{\phi}), \vec{\chi}) \subseteq T_{\Sigma''},$$

or, equivalently,

$$E_{\Sigma''}(\mathrm{SEN}(g)(\mathrm{SEN}(f)(\vec{\phi})), \vec{\chi}) \subseteq T_{\Sigma''}.$$

By definition, this means that $E_{\Sigma'}[\mathrm{SEN}(f)(\vec{\phi})] \leq T$, i.e., that $\mathrm{SEN}(f)(\vec{\phi}) \in \overleftarrow{E}_{\Sigma'}(T)$. Therefore $\overleftarrow{E}(T)$ is a relation system. ■

We show, next, that $\overleftarrow{E}$ is a monotone operator on $\mathrm{SenFam}(\mathbf{A})$.

Lemma 94 *Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an algebraic system and $E \subseteq N$. Then, for all $T, T' \in \mathrm{SenFam}(\mathbf{A})$,*

$$T \leq T' \quad \textit{implies} \quad \overleftarrow{E}(T) \leq \overleftarrow{E}(T').$$

Proof: Suppose that $T, T' \in \mathrm{SenFam}(\mathcal{I})$, with $T \leq T'$. Then, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)$, we have

$$\begin{array}{rcl}\vec{\phi} \in \overleftarrow{E}_\Sigma(T) & \text{iff} & E_\Sigma[\vec{\phi}] \leq T \\ & \text{implies} & E_\Sigma[\vec{\phi}] \leq T' \\ & \text{iff} & \vec{\phi} \in \overleftarrow{E}_\Sigma(T').\end{array}$$

So $\overleftarrow{E}(T) \leq \overleftarrow{E}(T')$. ■

A very useful property of the $\overleftarrow{E}$ operator on sentence families is that it commutes with inverse surjective morphisms.

Lemma 95 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ *be* $N^\flat$*-algebraic systems and* $\langle F, \alpha \rangle : \mathbf{A} \to \mathbf{A}'$ *be a surjective morphism. Then, for all* $E \subseteq N^\flat$*, we have*

$$\alpha^{-1}(\overleftarrow{E^{\mathbf{A}'}}(T)) = \overleftarrow{E^{\mathbf{A}}}(\alpha^{-1}(T)), \quad \textit{for all } T \in \mathrm{SenFam}(\mathbf{A}').$$

Proof: Let $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi} \in \mathrm{SEN}(\Sigma)$. Then we have $\vec{\phi} \in \alpha_\Sigma^{-1}(\overleftarrow{E^{\mathbf{A}'}_{F(\Sigma)}}(T))$ iff $\alpha_\Sigma(\vec{\phi}) \in \overleftarrow{E^{\mathbf{A}'}_{F(\Sigma)}}(T)$ iff $E^{\mathbf{A}'}_{F(\Sigma)}[\alpha_\Sigma(\vec{\phi})] \leq T$ iff, by surjectivity, for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

$$E^{\mathbf{A}'}_{F(\Sigma')}(\mathrm{SEN}'(F(f))(\alpha_\Sigma(\vec{\phi})), \alpha_{\Sigma'}(\vec{\chi})) \subseteq T_{F(\Sigma')}$$

iff for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

$$E^{\mathbf{A}'}_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}(f)(\vec{\phi})), \alpha_{\Sigma'}(\vec{\chi})) \subseteq T_{F(\Sigma')}$$

iff for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

$$\alpha_{\Sigma'}(E^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\vec{\phi}), \vec{\chi})) \subseteq T_{F(\Sigma')}$$

iff for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

$$E^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\vec{\phi}), \vec{\chi}) \subseteq \alpha_{\Sigma'}^{-1}(T_{F(\Sigma')})$$

iff $E^{\mathbf{A}}_\Sigma[\vec{\phi}] \leq \alpha^{-1}(T)$ iff $\vec{\phi} \in \overleftarrow{E^{\mathbf{A}}_\Sigma}(\alpha^{-1}(T))$. ■

On the other hand, there is also a relationship between the operator $\overleftarrow{E}$ and images under morphisms.

Lemma 96 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ *be* $N^\flat$*-algebraic systems,* $\langle F, \alpha \rangle : \mathbf{A} \to \mathbf{A}'$ *be a morphism and* $E \subseteq N$*. Then, for all* $\Sigma \in |\mathbf{Sign}|$*, all* $\vec{\phi} \in \mathrm{SEN}(\Sigma)$ *and all* $\Sigma' \in |\mathbf{Sign}|$*, we have*

$$\alpha_{\Sigma'}(E_{\Sigma,\Sigma'}[\vec{\phi}]) \subseteq E'_{F(\Sigma),F(\Sigma')}[\alpha_\Sigma(\vec{\phi})],$$

with equality holding in case $\langle F, \alpha \rangle$ *is surjective.*

Proof: Let $\varepsilon \in E$ and $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\vec{\chi} \in \mathrm{SEN}(\Sigma')$. Then, we have

$$\begin{aligned}
&\alpha_{\Sigma'}(\varepsilon_{\Sigma'}(\mathrm{SEN}(f)(\vec{\phi}), \vec{\chi})) \\
&\quad = \varepsilon'_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}(f)(\vec{\phi})), \alpha_{\Sigma'}(\vec{\chi})) \\
&\quad = \varepsilon'_{F(\Sigma')}(\mathrm{SEN}'(F(f))(\alpha_\Sigma(\vec{\phi})), \alpha_{\Sigma'}(\vec{\chi})) \\
&\quad \in E'_{F(\Sigma),F(\Sigma')}[\alpha_\Sigma(\vec{\phi})].
\end{aligned}$$

If $\langle F, \alpha \rangle$ is surjective, then every element in $E'_{F(\Sigma),F(\Sigma')}[\alpha_\Sigma(\vec{\phi})]$ is of the form $\varepsilon'_{F(\Sigma')}(\mathrm{SEN}'(F(f))(\alpha_\Sigma(\vec{\phi})), \alpha_{\Sigma'}(\vec{\chi}))$, for some $\varepsilon \in E$, $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\vec{\chi} \in \mathrm{SEN}(\Sigma')$. Thus, by following the preceding equalities bottom-up, we get the reverse inclusion. ■

Finally, we prove a close relationship between $\overleftarrow{E^{\mathbf{A}}}$, where E is a collection of natural transformations, with two distinguished arguments, and the Leibniz operator on the algebraic system $\mathbf{A}$.

Proposition 97 *Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an algebraic system, $E \subseteq N$, with two distinguished arguments, and $T \in \mathrm{SenFam}(\mathbf{A})$. If $\overleftarrow{E}(T)$ is a reflexive relation system on $\mathbf{A}$, then*

$$\Omega^{\mathbf{A}}(T) \leq \overleftarrow{E}(T).$$

Proof: Suppose that $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \Omega^{\mathbf{A}}_\Sigma(T)$. Since $\Omega^{\mathbf{A}}(T)$ is a congruence system, we have, for all $\sigma \in E \subseteq N$ and all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

$$\begin{aligned} &\langle \sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi), \vec{\chi}), \\ &\qquad \sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\phi), \vec{\chi}) \rangle \in \Omega^{\mathbf{A}}_{\Sigma'}(T). \end{aligned}$$

By the assumption of reflexivity, we get that, for all $\sigma \in E$, all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\vec{\chi} \in \mathrm{SEN}(\Sigma')$, $\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\phi), \vec{\chi}) \in T_{\Sigma'}$. Therefore, by the compatibility of $\Omega^{\mathbf{A}}(T)$ with T, we conclude that, for all $\sigma \in E$, all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

$$\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi), \vec{\chi}) \in T_{\Sigma'}.$$

This means that $E_\Sigma[\phi, \psi] \leq T$, or equivalently, $\langle \phi, \psi \rangle \in \overleftarrow{E_\Sigma}(T)$. Therefore, $\Omega^{\mathbf{A}}(T) \leq \overleftarrow{E}(T)$. ■

Proposition 97 allows us to conclude that in cases where $\overleftarrow{E}(T)$ is actually a congruence system compatible with the sentence family T, it coincides with the Leibniz congruence system of T on $\mathbf{A}$.

Corollary 98 *Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an algebraic system, $E \subseteq N$, with two distinguished arguments, and $T \in \mathrm{SenFam}(\mathbf{A})$. If $\overleftarrow{E}(T)$ is a congruence system on $\mathbf{A}$ compatible with T, then*

$$\overleftarrow{E}(T) = \Omega^{\mathbf{A}}(T).$$

Proof: Since, by hypothesis, $\overleftarrow{E}(T)$ is a congruence system on $\mathbf{A}$, it is reflexive. So, by Proposition 97, we have $\Omega^{\mathbf{A}}(T) \leq \overleftarrow{E}(T)$. On the other hand, since it is a congruence system on $\mathbf{A}$ compatible with T and, by definition, $\Omega^{\mathbf{A}}(T)$ is the largest such, we get that $\overleftarrow{E}(T) \leq \Omega^{\mathbf{A}}(T)$. We conclude that $\overleftarrow{E}(T) = \Omega^{\mathbf{A}}(T)$. ■

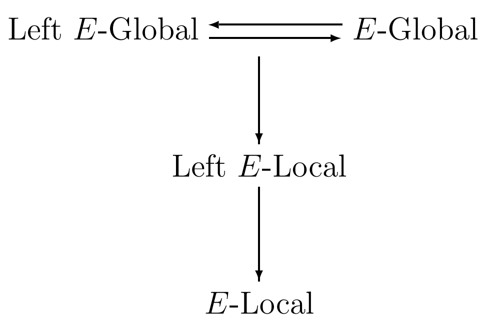
2.14 Global versus Local Membership

We turn now to exploring some syntactic conditions with respect to morphisms, parameters and theory families in a π-institution. We consider the following setting: Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $E \subseteq N^\flat$ a collection of natural transformations in $N^\flat$, with k distinguished arguments, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^k$.

- We say $\vec{\phi}$ is E-**locally in** T if, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,
$$E_\Sigma(\vec{\phi}, \vec{\chi}) \subseteq T_\Sigma.$$
- We say that $\vec{\phi}$ is E-**globally in** T if
$$E_\Sigma[\vec{\phi}] \leq T.$$
- We say $\vec{\phi}$ is **left** E-**locally in** T if it is E-locally in $\overleftarrow{T}$.
- Similarly, $\vec{\phi}$ is **left** E-**globally in** T if it is E-globally in $\overleftarrow{T}$.

We show next that these properties satisfy the following diagram, where arrows are implications pointing from the stronger to the weaker property. After the lemma proving this result, we construct some examples showing that all implications are proper (i.e., none of them are equivalences in general for arbitrary π-institutions).



Proposition 99 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $E \subseteq N^\flat$, with k distinguished variables, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^k$.*

(a) $\vec{\phi}$ is left E-globally in T if and only if it is E-globally in T.

(b) If $\vec{\phi}$ is E-globally in T, then it is left E-locally in T. The implication becomes an equivalence if all arguments in E are distinguished (i.e., there are no parameters).

(c) If $\vec{\phi}$ is left E-locally in T, then it is E-locally in T. The implication becomes an equivalence if $T \in \mathrm{ThSys}(\mathcal{I})$.

Proof:

(a) If $\vec{\phi}$ is left E-globally in T, then $E_\Sigma[\vec{\phi}] \leq \overleftarrow{T}$. But, by Proposition 2, $\overleftarrow{T} \leq T$, whence $E_\Sigma[\vec{\phi}] \leq T$. Thus, $\vec{\phi}$ is E-globally in T.

Suppose, conversely, that $\vec{\phi}$ is E-globally in T. Then, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$E_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}), \vec{\chi}) \subseteq T_{\Sigma'}.$$

As a special case, we get that, for all $\Sigma', \Sigma'' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, $g \in \mathbf{Sign}^\flat(\Sigma', \Sigma'')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\begin{array}{ccccc}
\Sigma & \xrightarrow{\quad f \quad} & \Sigma' & \xrightarrow{\quad g \quad} & \Sigma'' \\
\vec{\phi} & \longmapsto & \mathrm{SEN}^\flat(f)(\vec{\phi}) & \longmapsto & \mathrm{SEN}^\flat(g \circ f)(\vec{\phi}) \\
 & & \vec{\chi} & \longmapsto & \mathrm{SEN}^\flat(g)(\vec{\chi})
\end{array}$$

$$E_{\Sigma''}(\mathrm{SEN}^\flat(g \circ f)(\vec{\phi}), \mathrm{SEN}^\flat(g)(\vec{\chi})) \subseteq T_{\Sigma''}.$$

So $\mathrm{SEN}^\flat(g)(E_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}), \vec{\chi})) \subseteq T_{\Sigma''}$. Since this holds for all $\Sigma'' \in |\mathbf{Sign}^\flat|$ and all $g \in \mathbf{Sign}^\flat(\Sigma', \Sigma'')$, we get $E_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}), \vec{\chi}) \subseteq \overleftarrow{T}_{\Sigma'}$. Since this holds for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, we get $E_\Sigma[\vec{\phi}] \leq \overleftarrow{T}$. We now conclude that $\vec{\phi}$ is left E-globally in T.

(b) Suppose $\vec{\phi}$ is E-globally in T. Then, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$E_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}), \vec{\chi}) \subseteq T_{\Sigma'}.$$

Thus, in particular, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $E_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}), \mathrm{SEN}^\flat(f)(\vec{\chi})) \subseteq T_{\Sigma'}$. Therefore,

$$\mathrm{SEN}^\flat(f)(E_\Sigma(\vec{\phi}, \vec{\chi})) \subseteq T_{\Sigma'},$$

which shows that $E_\Sigma(\vec{\phi}, \vec{\chi}) \subseteq \overleftarrow{T}_\Sigma$. So $\vec{\phi}$ is left E-locally in T.

Finally, assume all arguments in E are distinguished. Then, if $\vec{\phi}$ is left E-locally in T, we have $E_\Sigma(\vec{\phi}) \subseteq \overleftarrow{T}_\Sigma$, whence, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$E_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi})) = \mathrm{SEN}^\flat(f)(E_\Sigma(\vec{\phi})) \subseteq T_{\Sigma'}.$$

Hence $E_\Sigma[\vec{\phi}] \leq T$ and $\vec{\phi}$ is E-globally in T.

(c) The implication holds, exactly as the left to right implication in Part (a), because $\overleftarrow{T} \leq T$, for all $T \in \mathrm{ThFam}(\mathcal{I})$. The equivalence statement holds because, by Proposition 2, $\overleftarrow{T} = T$, whenever $T \in \mathrm{ThSys}(\mathcal{I})$. ■

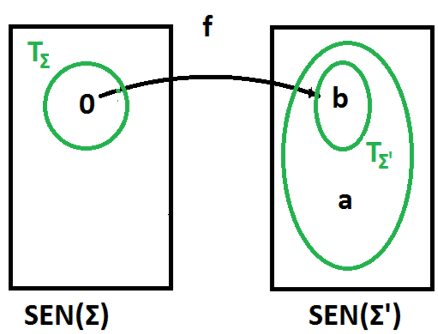
We provide examples to show that the implications in Proposition 99 are proper in general, i.e., they are not equivalences for arbitrary π-institutions, arbitrary sets of natural transformations E and arbitrary theory families T.

Example 100 *Define the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *as follows:*

- $\mathbf{Sign}^\flat$ *is the category with two objects* Σ, Σ' *and a single non-identity morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is determined by* $\mathrm{SEN}^\flat(\Sigma) = \{0\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ *and* $\mathrm{SEN}^\flat(f) : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SEN}^\flat(\Sigma')$, *given by* $\mathrm{SEN}^\flat(f)(0) = b$*;*
- $N^\flat$ *is the category of natural transformations generated by the binary transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$, *determined by the following tables:*

$\sigma^\flat_\Sigma$	0
0	0

$\sigma^\flat_{\Sigma'}$	a	b
a	b	b
b	a	b



Consider the closure system C on $\mathbf{F}$ *defined by setting*

$$\mathcal{C}_\Sigma = \{\{0\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\}, \{a, b\}\}$$

and let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the associated* π*-institution.*

Finally, take $T = \{T_\Sigma, T_{\Sigma'}\} \in \mathrm{ThFam}(\mathcal{I})$ *to be the theory family specified by*

$$T_\Sigma = \{0\} \quad \text{and} \quad T_{\Sigma'} = \{b\}$$

and consider $E = \{\sigma^\flat\} \subseteq N^\flat$*, with one distinguished argument. Notice that* $\overleftarrow{T} = T$.

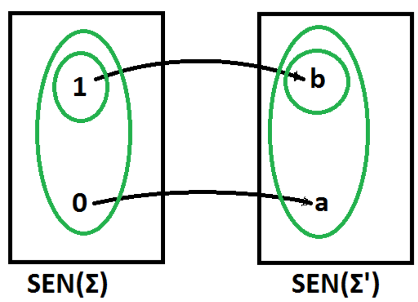
We now have $\sigma^\flat_\Sigma(0, 0) \in T_\Sigma = \overleftarrow{T}_\Sigma$*. Thus, 0 is E-left locally in T. On the other hand* $\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(0), a) = \sigma^\flat_{\Sigma'}(b, a) = a \notin T_{\Sigma'}$*. Therefore 0 is not E-globally in T.*

Example 101 *Define the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *as follows:*

- $\mathbf{Sign}^\flat$ *is the category with two objects* Σ, Σ' *and a single non-identity morphism* $f : \Sigma \to \Sigma'$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is determined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f) : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SEN}^\flat(\Sigma')$, *given by* $\mathrm{SEN}^\flat(f)(0) = a$ *and* $\mathrm{SEN}^\flat(f)(1) = b$;
- $N^\flat$ *is the category of natural transformations generated by the binary transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$, *determined by the following tables:*

$\sigma^\flat_\Sigma$	0	1
0	1	1
1	0	1

$\sigma^\flat_{\Sigma'}$	a	b
a	b	b
b	a	b



Consider the closure system C *on* $\mathbf{F}$ *defined by setting*

$$\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\},\{a,b\}\}$$

and let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the associated* π*-institution.*

Finally, take $T = \{T_\Sigma, T_{\Sigma'}\} \in \mathrm{ThFam}(\mathcal{I})$ *to be the theory family specified by*

$$T_\Sigma = \{0,1\} \quad \textit{and} \quad T_{\Sigma'} = \{b\}$$

and consider $E = \{\sigma^\flat\} \subseteq N^\flat$, *with one distinguished argument. Notice that we have* $\overleftarrow{T} = \{\{1\},\{b\}\}$.

Since $\sigma^\flat_\Sigma(1,0) = 0$ *and* $\sigma^\flat_\Sigma(1,1) = 1$ *are both in* T_Σ, *we conclude that* 1 *is* E*-locally in* T. *On the other hand,* $\sigma^\flat_\Sigma(1,0) = 0 \notin \overleftarrow{T}_\Sigma$. *Thus* 1 *is not left* E*-locally in* T.

Let again $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $E \subseteq N^\flat$ a set of natural transformations, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $T \in \mathrm{ThFam}(\mathcal{I})$. Quantifying over all signatures and all sentences, we get the following definitions:

- We say E is **locally in** T if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,
$$E_\Sigma(\vec{\phi}, \vec{\chi}) \subseteq T_\Sigma.$$
- We say E is **left locally in** T if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,
$$E_\Sigma(\vec{\phi}, \vec{\chi}) \subseteq \overleftarrow{T}_\Sigma.$$
- We say E is **globally in** T if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,
$$E_\Sigma[\vec{\phi}] \leq T.$$

Of course, we have, taking into account Proposition 99:

Corollary 102 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $E \subseteq N^\flat$*, with* k *distinguished arguments,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution, based on* $\mathbf{F}$*, and* $T \in \mathrm{ThFam}(\mathcal{I})$.

(a) If E *is globally in* T*, then it is left locally in* T*;*

(b) If E *is left locally in* T*, then it is locally in* T.

Proof: Directly by Proposition 99. ■

But Corollary 102 gives only half the true story. It turns out all three universal properties are equivalent.

Proposition 103 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $E \subseteq N^\flat$*, with* k *distinguished arguments,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution, based on* $\mathbf{F}$*, and* $T \in \mathrm{ThFam}(\mathcal{I})$*.* E *is globally in* T *if and only if it is locally in* T.

Proof: By Corollary 102, it suffices to show that, if E is locally in T, then it is also globally in T. To this end, suppose E is locally in T, i.e., that for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,
$$E_\Sigma(\vec{\phi}, \vec{\psi}) \subseteq T_\Sigma.$$
Thus, in particular, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,
$$\begin{array}{ccc} \Sigma & \xrightarrow{\ f\ } & \Sigma' \\ \vec{\phi} & \longmapsto & \mathrm{SEN}^\flat(f)(\vec{\phi}) \\ & & \vec{\chi} \end{array}$$
$$E_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}), \vec{\chi}) \subseteq T_{\Sigma'}.$$
But this is equivalent to $E_\Sigma[\vec{\phi}] \leq T$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$. Thus, E is globally in T. ■

2.15 Global Properties and Parameters

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. Every $\sigma^\flat \in N^\flat$ has finite arity, but, when the exact arity is unimportant, we will write

$$\sigma^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell.$$

As already mentioned at the beginning of Section 2.13, this is also convenient in case we are dealing with a set $S^\flat \subseteq N^\flat$. In that case the set of arities of the natural transformations in $S^\flat$ may be unbounded and we write

$$S^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell,$$

even though, again, the arity of each member of $S^\flat$ is finite. Finally, we denote

$$p^k := \langle p^{k,0}, p^{k,1}, \dots, p^{k,k-1} \rangle : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^k$$

the identity natural transformation, being a tuple of the appropriate k-ary projections.

For all $\sigma^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$ in $N^\flat$, with k distinguished arguments, we denote by

$$\dot{\sigma}^\flat : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^\ell$$

the collection of k-ary natural transformations in $N^\flat$, defined by

$$\dot{\sigma}^\flat = \{\sigma^\flat \circ \langle p^k, \tau^\flat \rangle : \tau^\flat : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^\omega \in N^\flat\}.$$

More generally, given a collection $S^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$ in $N^\flat$, with k distinguished arguments, we denote by

$$\dot{S}^\flat : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^\ell$$

the collection of all k-ary natural transformations in $N^\flat$ defined by

$$\dot{S}^\flat = \bigcup\{\dot{\sigma}^\flat : \sigma^\flat \in S^\flat\}.$$

Concerning these definitions, we adopt the following conventions:

1. If $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^\ell$ is k-ary, with k distinguished arguments, i.e., is thought of as parameter free, then $\dot{\sigma}^\flat = \{\sigma^\flat\}$. In this case, we identify the singleton $\dot{\sigma}^\flat$ with $\sigma^\flat$, the unique element that it contains. Similarly, for a parameterless collection $S^\flat : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^\ell$ in $N^\flat$, we identify $\dot{S}^\flat$ with $S^\flat$.

2. If $\sigma^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$ has 2 distinguished arguments, we write $\ddot{\sigma}^\flat : (\mathrm{SEN}^\flat)^2 \to (\mathrm{SEN}^\flat)^\ell$ for the collection $\dot{\sigma}^\flat$ to emphasize the binary character of $\ddot{\sigma}^\flat$. More generally, $\ddot{S}^\flat : (\mathrm{SEN}^\flat)^2 \to (\mathrm{SEN}^\flat)^\ell$ stands for the collection $\dot{S}^\flat$, when $S^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$ has 2 distinguished arguments.

We have the following relation concerning global membership based on a set of natural transformations and membership based on the corresponding parameter free counterpart.

Lemma 104 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and consider a collection* $S^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$ *of natural transformations in* $N^\flat$*, with* k *distinguished arguments. Then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^k$*,*

$$\dot{S}^\flat_\Sigma[\vec{\phi}] \leq S^\flat_\Sigma[\vec{\phi}].$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^k$. Then, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, we have

$$\begin{array}{rcl}
\dot{S}^\flat_{\Sigma,\Sigma'}[\vec{\phi}] & = & \bigcup_{f\in\mathbf{Sign}^\flat(\Sigma,\Sigma')} \{\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}), \tau^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}))) : \\
& & \qquad \sigma^\flat \in S^\flat, \tau^\flat \in N^\flat\} \quad \text{(by definition)} \\
& \subseteq & \bigcup_{f\in\mathbf{Sign}^\flat(\Sigma,\Sigma')} \{\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}), \vec{\chi}) : \sigma^\flat \in S^\flat, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')\} \\
& & \text{(set theoretic)} \\
& = & S^\flat_{\Sigma,\Sigma'}[\vec{\phi}]. \quad \text{(by definition)}
\end{array}$$

Since $\Sigma' \in |\mathbf{Sign}^\flat|$ was arbitrary, we conclude that $\dot{S}^\flat_\Sigma[\vec{\phi}] \leq S^\flat_\Sigma[\vec{\phi}]$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $S^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$ a collection of natural transformations in $N^\flat$, with k distinguished arguments, and $T \in \mathrm{SenFam}(\mathbf{F}^\ell)$. Recall that by $\overleftarrow{S^\flat}(T)$ is denoted the k-ary relation system $\overleftarrow{S^\flat}(T) = \{\overleftarrow{S^\flat_\Sigma}(T)\}_{\Sigma\in|\mathbf{Sign}^\flat|}$ on $\mathbf{F}$, given, for all $\Sigma \in |\mathbf{Sign}^\flat|$, by

$$\overleftarrow{S^\flat_\Sigma}(T) = \{\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^k : S^\flat_\Sigma[\vec{\phi}] \leq T\}.$$

Then we obtain

Corollary 105 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and consider a collection* $S^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$ *of natural transformations in* $N^\flat$*, with* k *distinguished arguments. Then, for all* $T \in \mathrm{SenFam}(\mathbf{F}^\ell)$*,*

$$\overleftarrow{S^\flat}(T) \leq \overleftarrow{\dot{S}^\flat}(T).$$

Proof: We have, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\begin{array}{rcll}
\overleftarrow{S^\flat_\Sigma}(T) & = & \{\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^k : S^\flat_\Sigma[\vec{\phi}] \leq T\} & \text{(definition)} \\
& \subseteq & \{\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^k : \dot{S}^\flat_\Sigma[\vec{\phi}] \leq T\} & \text{(Lemma 104)} \\
& = & \overleftarrow{\dot{S}^\flat_\Sigma}(T). \quad \text{(definition)} &
\end{array}$$

We conclude that $\overleftarrow{S^\flat}(T) \leq \overleftarrow{\dot{S}^\flat}(T)$. ■

We now turn to collections of natural transformations satisfying certain properties globally.

For fixed k, we assume P is a **(antimonotone) global property** of natural transformations $\sigma^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$ in $N^\flat$, with k distinguished arguments. That is:

(a) For $\sigma^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$ in $N^\flat$, with k distinguished arguments, $\sigma^\flat$ either does or does not satisfy P;

(b) For every $\sigma^\flat, \tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$ in $N^\flat$, with k distinguished arguments, if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^k$, $\sigma^\flat_\Sigma[\vec{\phi}] \leq \tau^\flat_\Sigma[\vec{\phi}]$, then, if $\tau^\flat$ satisfies P, then $\sigma^\flat$ also satisfies P.

For instance, given $T \in \mathrm{SenFam}(\mathbf{F}^\ell)$,

$$P^T(\sigma): \quad \text{for all } \Sigma \in |\mathbf{Sign}^\flat| \text{ and all } \vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^k, \\ \sigma_\Sigma[\vec{\phi}] \leq T$$

is a global property of natural transformations in $N^\flat$, with k distinguished arguments.

Given such a global property P, we denote by $P^\flat \subseteq N^\flat$ the collection of all $\sigma^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$ in $N^\flat$, with k distinguished arguments, that satisfy property P:

$$P^\flat = \{\sigma^\flat \in N^\flat : P(\sigma^\flat)\}.$$

We call $P^\flat$ the P-**core** of $N^\flat$.

As an example, for the property P^T introduced above, based on a fixed sentence family $T \in \mathrm{SenFam}(\mathbf{F}^\ell)$, we have

$$P^{T\flat} = \{\sigma^\flat \in N^\flat : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^k)(\sigma^\flat_\Sigma[\vec{\phi}] \leq T)\}.$$

Along the same lines, given a global property P of natural transformations in $N^\flat$, with k distinguished arguments, we may consider the restriction $\hat{P}$ of P to the collection of parameter free k-ary natural transformations in $N^\flat$:

$$\hat{P}: \quad \sigma : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^\ell \in N^\flat \text{ and } P(\sigma).$$

Then we define

$$\hat{P}^\flat = \{\sigma^\flat \in N^\flat : \hat{P}(\sigma^\flat)\}.$$

We call $\hat{P}^\flat$ the k-**ary** P-**core** of $N^\flat$ or the **parameter free** P-**core** of $N^\flat$. The following inclusion is straightforward:

Lemma 106 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and consider a global property* P *of natural transformations* $\sigma^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$ *in* $N^\flat$*, with* k *distinguished arguments. Then*

$$\hat{P}^\flat \subseteq P^\flat.$$

Proof: Straightforward by the definition of $\hat{P}$, since any parameter free k-ary natural transformation is a natural transformation with k distinguished arguments and no parameters. ∎

Let P be a global property of natural transformations in $N^\flat$, with k distinguished arguments. We have now defined two sets of k-ary natural transformations in $N^\flat$ associated with P:

- The first set is $\dot{P}^\flat$, obtained by $P^\flat$ by applying the dot operator;
- The second is the set $\hat{P}^\flat$ obtained by restricting the property P on the subfamily of parameter free k-ary natural transformations in $N^\flat$.

In the main theorem of the section we show that, for any global property P of natural transformations in $N^\flat$, with k distinguished arguments, these two sets are identical, i.e., $\hat{P}^\flat = \dot{P}^\flat$.

Theorem 107 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and consider a global property* P *of natural transformations* $\sigma^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$ *in* $N^\flat$*, with* k *distinguished arguments. Then*

$$\hat{P}^\flat = \dot{P}^\flat.$$

Proof: Suppose, first, that $\sigma^\flat \in \hat{P}^\flat$. Then, by definition, $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^\ell$ is parameter free and satisfies P. Thus, we have $\sigma^\flat \in P^\flat$ and $\sigma^\flat = \dot{\sigma}^\flat \in \dot{P}^\flat$. Therefore, $\hat{P}^\flat \subseteq \dot{P}^\flat$.

Suppose, conversely, that $\rho^\flat \in \dot{P}^\flat$. Then, by definition, there exists $\sigma^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$, with k distinguished arguments, in $P^\flat$ and $\tau^\flat : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^\omega$ in $N^\flat$, such that

$$\rho^\flat = \sigma^\flat \circ \langle p^k, \tau^\flat \rangle : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^\ell.$$

Noting that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$, $\rho^\flat_\Sigma[\vec{\phi}] \le \sigma^\flat_\Sigma[\vec{\phi}]$, and taking into account that P is global and that $\sigma^\flat \in P^\flat$, we obtain that $\rho^\flat \in P^\flat$. But $\rho^\flat : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^\ell$ is also parameter free. Therefore $\rho^\flat \in \hat{P}^\flat$. We conclude that $\dot{P}^\flat \subseteq \hat{P}^\flat$. ∎

2.16 Finitarity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

$\mathcal{I}$ is **finitary** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$, if $\phi \in C_\Sigma(\Phi)$, then, there exists finite $\Psi \subseteq \Phi$, such that $\phi \in C_\Sigma(\Psi)$.

Equivalently, $\mathcal{I}$ is finitary if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$C_\Sigma(\Phi) = \bigcup \{C_\Sigma(\Psi) : \Psi \subseteq_\omega \Phi\},$$

where $\Psi \subseteq_\omega \Phi$ denotes the finite subset relation.

Yet another well-known equivalent characterization of finitarity asserts that $\mathcal{I}$ is finitary if and only if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and every upward directed collection $\{T^i_\Sigma : i \in I\}$ of Σ-theories, i.e., a collection, such that:

- $C_\Sigma(T^i_\Sigma) = T^i_\Sigma$, for all $i \in I$;
- for every $i, j \in I$, there exists $k \in I$, such that $T^i_\Sigma, T^j_\Sigma \subseteq T^k_\Sigma$,

the union $\bigcup_{i\in I} T^i_\Sigma$ is also a Σ-theory.

We formulate next some versions of these properties with reference to theory families.

Let **Sign** be a category and SEN : **Sign** → **Set** a functor. A sentence family $X \in \mathrm{SenFam}(\mathrm{SEN})$ is called **locally finite** if, for all $\Sigma \in |\mathbf{Sign}|$, X_Σ is finite. In this case we write $|X| <_l \omega$. Given sentence families $X, Y \in \mathrm{SenFam}(\mathrm{SEN})$, we use the notation $X \leq_{lf} Y$ to suggest that X is a locally finite subfamily of Y.

We say that a collection $\{X^i : i \in I\} \subseteq \mathrm{SenFam}(\mathrm{SEN})$ is:

- **locally directed** if, for all $\Sigma \in |\mathbf{Sign}|$ and all finite $J \subseteq I$, there exists $k \in I$, such that $X^j_\Sigma \leq X^k_\Sigma$, for all $j \in J$;
- **directed** if, for all finite $J \subseteq I$, there exists $k \in I$, such that $X^j \leq X^k$, for all $j \in J$.

Directedness is a stronger property than local directedness.

Lemma 108 *Let* **Sign** *be a category,* SEN : **Sign** → **Set** *a sentence functor and* $\{T^i : i \in I\} \subseteq \mathrm{SenFam}(\mathrm{SEN})$*. If* $\{T^i\}$ *is directed, then it is locally directed.*

Proof: Suppose $\{T^i\}$ is directed. Let $\Sigma \in |\mathbf{Sign}|$ and $i, j \in I$. Since $\{T^i\}$ is directed, there exists a $k \in I$, such that $T^i, T^j \leq T^k$. In particular, $T^i_\Sigma, T^j_\Sigma \subseteq T^k_\Sigma$. Therefore, $\{T^i\}$ is also locally directed. ■

The opposite implication patently fails, i.e., in general, local directedness does not imply directedness.

Example 109 *Let* **Sign** *be the discrete category with objects* Σ *and* Σ'*. Let* $\mathrm{SEN}^\flat : \mathbf{Sign} \to \mathbf{Set}$ *be defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$ *and* $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$*.*

Consider the sentence families $T = \{\{1\}, \{a, b\}\}$ *and* $T' = \{\{0, 1\}, \{b\}\}$ *and the collection* $\mathcal{T} = \{T, T'\}$*.*

$\mathcal{T}$ *is locally directed, since* $T_\Sigma \subseteq T'_\Sigma$ *and* $T'_{\Sigma'} \subseteq T_{\Sigma'}$*.*

On the other hand, $\mathcal{T}$ *is not directed since there does not exist* $X \in \mathcal{T}$*, such that* $T, T' \leq X$*.*

Given an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ and a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, based on $\mathbf{F}$, we say that $\mathcal{I}$ is:

- **locally continuous** if, for every locally directed family $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$,
$$\bigcup_{i\in I} T^i \in \mathrm{ThFam}(\mathcal{I});$$
- **continuous** if, for every directed family $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$,
$$\bigcup_{i\in I} T^i \in \mathrm{ThFam}(\mathcal{I}).$$

Since, by Lemma 108, directedness implies local directedness, we get the following straightforward relationship between local continuity and continuity.

Corollary 110 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is locally continuous, then it is continuous.*

Proof: Assume $\mathcal{I}$ is locally continuous and let $\mathcal{T} \subseteq \mathrm{ThFam}(\mathcal{I})$ be directed. By Lemma 108, $\mathcal{T}$ is locally directed. Thus, by local continuity, $\bigcup \mathcal{T} \in \mathrm{ThFam}(\mathcal{I})$. Hence, $\mathcal{I}$ is continuous. ■

However, more is true. In fact, continuity turns out to be equivalent to the seemingly stronger notion of local continuity.

Theorem 111 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is locally continuous if and only if it is continuous.*

Proof: The "only if" is by Corollary 110. Suppose, conversely, that $\mathcal{I}$ is continuous. Let $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$ be locally directed. We construct the collection

$$\mathcal{T} = \{T' \in \mathrm{ThFam}(\mathcal{I}) : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\exists i \in I)(T'_\Sigma = T^i_\Sigma)\}.$$

First, note that $\mathcal{T}$ is directed. In fact, let $T, T' \in \mathcal{T}$ and $\Sigma \in |\mathbf{Sign}^\flat|$. By the definition of $\mathcal{T}$, there exist $i(\Sigma), j(\Sigma) \in I$, such that $T_\Sigma = T^{i(\Sigma)}_\Sigma$ and $T'_\Sigma = T^{j(\Sigma)}_\Sigma$. Since $\{T^i : i \in I\}$ is locally directed, there exists $k(\Sigma) \in I$, such that $T^{i(\Sigma)}_\Sigma, T^{j(\Sigma)}_\Sigma \subseteq T^{k(\Sigma)}_\Sigma$. Consider $T'' = \{T''_\Sigma\}_{\Sigma\in|\mathbf{Sign}^\flat|}$, where, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$T''_\Sigma = T^{k(\Sigma)}_\Sigma.$$

Then $T'' \in \mathcal{T}$ and, moreover, $T, T' \leq T''$. Thus, $\mathcal{T}$ is indeed directed. Second, notice that $\bigcup \mathcal{T} = \bigcup_{i\in I} T^i$. Thus, taking into account the continuity of $\mathcal{I}$, we get

$$\bigcup_{i\in I} T^i = \bigcup \mathcal{T} \in \mathrm{ThFam}(\mathcal{I}).$$

Therefore, $\mathcal{I}$ is locally continuous. ■

Now we get the following characterizations of finitarity.

Proposition 112 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *Then the following conditions are equivalent:*

(i) $\mathcal{I}$ *is finitary;*

(ii) For every $X \in \mathrm{SenFam}(\mathbf{F})$,

$$C(X) = \bigcup \{C(Y) : Y \leq_{lf} X\};$$

(iii) $\mathcal{I}$ *is locally continuous.*

(iv) $\mathcal{I}$ *is continuous.*

Proof:

(i)⇒(ii) Suppose $\mathcal{I}$ is finitary and let $X \in \mathrm{SenFam}(\mathbf{F})$. Clearly, by the monotonicity of C, $\bigcup\{C(Y) : Y \leq_{lf} X\} \leq C(X)$. To prove the converse, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(X_\Sigma)$. By finitarity, there exists $Y_\Sigma \subseteq_f X_\Sigma$, such that $\phi \in C_\Sigma(Y_\Sigma)$. Now set $Y = \{Y_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}^\flat|}$, where, for all $\Sigma' \in |\mathbf{Sign}^\flat|$,

$$Y_{\Sigma'} = \begin{cases} Y_\Sigma, & \text{if } \Sigma' = \Sigma \\ \varnothing, & \text{if } \Sigma' \neq \Sigma \end{cases}$$

Clearly, $Y \leq_{lf} X$ and, moreover, $\phi \in C_\Sigma(Y)$. Thus, we get $C(X) \leq \bigcup\{C(Y) : Y \leq_{lf} X\}$.

(ii)⇒(iii) Suppose that, for every $X \in \mathrm{SenFam}(\mathbf{F})$, $C(X) = \bigcup\{C(Y) : Y \leq_{lf} X\}$ and let $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$ be locally directed. Consider $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\bigcup_{i \in I} T^i_\Sigma)$. By hypothesis, there exists locally finite $Y \leq \bigcup_{i \in I} T^i$, such that $\phi \in C_\Sigma(Y_\Sigma)$. Since $\{T^i : i \in I\}$ is locally directed, there exists $i \in I$, such $Y_\Sigma \subseteq T^i_\Sigma$. Now we get $\phi \in C_\Sigma(T^i_\Sigma) = T^i_\Sigma \subseteq \bigcup_{i \in I} T^i_\Sigma$. We conclude that $\bigcup_{i \in I} T^i$ is a theory family and, therefore, $\mathcal{I}$ is locally continuous.

(iii)⇒(i) Assume that $\mathcal{I}$ is locally continuous and let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$. We define a collection of theory families of $\mathcal{I}$ as follows: For every finite subset $\Psi \subseteq_f \Phi$, let $T^\Psi = \{T^\Psi_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}^\flat|}$ be given, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, by setting

$$T^\Psi_{\Sigma'} = \begin{cases} C_\Sigma(\Psi), & \text{if } \Sigma' = \Sigma \\ C_{\Sigma'}(\varnothing), & \text{if } \Sigma' \neq \Sigma \end{cases}$$

Clearly, $\{T^\Psi : \Psi \subseteq_f \Phi\}$ is a locally directed. Therefore, by hypothesis

$$C(\bigcup\{T^\Psi : \Psi \subseteq_f \Phi\}) = \bigcup\{T^\Psi : \Psi \subseteq_f \Phi\}.$$

Now we get

$$\phi \in C_\Sigma(\Phi) = C_\Sigma(\bigcup_{\Psi \subseteq_f \Phi} T^\Psi_\Sigma) = \bigcup_{\Psi \subseteq_f \Phi} T^\Psi_\Sigma,$$

whence $\phi \in T^\Psi_\Sigma = C_\Sigma(\Psi)$, for some $\Psi \subseteq_f \Phi$. We conclude that $\mathcal{I}$ is finitary.

(iii)⇔(iv) This is the content of Theorem 111.

■

We now prove a lemma concerning $\mathcal{I}$-filter generation to the effect that, for a finitary π-institution, the $\mathcal{I}$-filter generated by a certain sentence family can be built inductively by "closing under consequences" in a structured way.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a finitary π-institution based on $\mathbf{F}$. Then, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $X \in \mathrm{SenFam}(\mathcal{A})$, we define

$$\Xi^{\mathcal{I},\mathcal{A},n}(X) = \{\Xi^{\mathcal{I},\mathcal{A},n}_\Sigma(X)\}_{\Sigma \in |\mathbf{Sign}|}, \quad n < \omega,$$

by induction on n, as follows:

- If $n = 0$, $\Xi^{\mathcal{I},\mathcal{A},0}(X) = X$;
- Assume $\Xi^{\mathcal{I},\mathcal{A},i}(X)$ has been defined, for all $i < n$. We define

  $$\Xi^{\mathcal{I},\mathcal{A},n}(X) = \{\Xi^{\mathcal{I},\mathcal{A},n}_\Sigma(X)\}_{\Sigma \in |\mathbf{Sign}|},$$

  by setting, for all $\Sigma' \in |\mathbf{Sign}|$,

  $$\begin{array}{rcl}\Xi^{\mathcal{I},\mathcal{A},n}_{\Sigma'}(X) & = & \{\alpha_\Sigma(\phi) : \Sigma \in |\mathbf{Sign}^\flat|, \text{ such that } F(\Sigma) = \Sigma', \\ & & \text{and } \Phi \cup \{\phi\} \subseteq_\omega \mathrm{SEN}^\flat(\Sigma), \text{ such that} \\ & & \phi \in C_\Sigma(\Phi) \text{ and } \alpha_\Sigma(\Phi) \subseteq \Xi^{\mathcal{I},\mathcal{A},n-1}_{\Sigma'}(X)\}.\end{array}$$

We may write the latter set more concisely as

$$\Xi^{\mathcal{I},\mathcal{A},n}_{\Sigma'}(X) = \bigcup_{\Sigma : F(\Sigma) = \Sigma'} \{\alpha_\Sigma(\phi) : \phi \in C_\Sigma(\Phi), \alpha_\Sigma(\Phi) \subseteq \Xi^{\mathcal{I},\mathcal{A},n-1}_{\Sigma'}(X)\}.$$

We prove some basic properties of this set.

Lemma 113 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a finitary* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $X \in \mathrm{SenFam}(\mathcal{A})$.

(a) For all $n < \omega$*,* $\Xi^{\mathcal{I},\mathcal{A},n}(X) \leq \Xi^{\mathcal{I},\mathcal{A},n+1}(X)$*;*

(b) $\bigcup_{n<\omega} \Xi^{\mathcal{I},\mathcal{A},n}(X) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*;*

(c) $\bigcup_{n<\omega} \Xi^{\mathcal{I},\mathcal{A},n}(X) \leq C^{\mathcal{I},\mathcal{A}}(X)$.

Proof:

(a) Let $\Sigma' \in |\mathbf{Sign}|$, $\phi' \in \mathrm{SEN}(\Sigma)$, such that $\phi' \in \Xi^{\mathcal{I},\mathcal{A},n}_{\Sigma'}(X)$. By surjectivity of $\langle F, \alpha\rangle$, there exists $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $F(\Sigma) = \Sigma'$ and $\alpha_\Sigma(\phi) = \phi'$. But $\phi \in C_\Sigma(\phi)$ and $\alpha_\Sigma(\phi) = \phi' \in \Xi^{\mathcal{I},\mathcal{A},n}_{\Sigma'}(X)$ imply that $\phi' \in \alpha_\Sigma(\phi) \in \Xi^{\mathcal{I},\mathcal{A},n+1}_{\Sigma'}(X)$. So, we get $\Xi^{\mathcal{I},\mathcal{A},n}(X) \leq \Xi^{\mathcal{I},\mathcal{A},n+1}(X)$.

(b) Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq_\omega \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$ and assume that $\alpha_\Sigma(\Phi) \subseteq \bigcup_{n<\omega} \Xi^{\mathcal{I},\mathcal{A},n}_{F(\Sigma)}(X)$. Then, since $\Phi \subseteq_\omega \mathrm{SEN}^\flat(\Sigma)$, there exists $n < \omega$, such that $\alpha_\Sigma(\Phi) \subseteq \Xi^{\mathcal{I},\mathcal{A},n}_{F(\Sigma)}(X)$. Since $\phi \in C_\Sigma(\Phi)$, we get, by the definition of $\Xi^{\mathcal{I},\mathcal{A},n+1}(X)$,

$$\alpha_\Sigma(\phi) \in \Xi^{\mathcal{I},\mathcal{A},n+1}_{F(\Sigma)}(X) \subseteq \bigcup_{n<\omega} \Xi^{\mathcal{I},\mathcal{A},n}_{F(\Sigma')}(X).$$

We conclude that $\bigcup_{n<\omega} \Xi^{\mathcal{I},\mathcal{A},n}(X) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

(c) We prove this by induction on $n < \omega$.

For $n = 0$, $\Xi^{\mathcal{I},\mathcal{A},0}(X) = X \leq C^{\mathcal{I},\mathcal{A}}(X)$.

Suppose that $\Xi^{\mathcal{I},\mathcal{A},i}(X) \leq C^{\mathcal{I},\mathcal{A}}(X)$, for all $i < n$.

Let $\Sigma' \in |\mathbf{Sign}|$, $\phi' \in \Xi^{\mathcal{I},\mathcal{A},n}_{\Sigma'}(X)$. Thus, there exists $\Sigma \in |\mathbf{Sign}^\flat|$, such that $F(\Sigma) = \Sigma'$, and $\Phi \cup \{\phi\} \subseteq_\omega \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$, $\alpha_\Sigma(\Phi) \subseteq \Xi^{\mathcal{I},\mathcal{A},n-1}_{\Sigma'}(X)$ and $\alpha_\Sigma(\phi) = \phi'$. By the induction hypothesis, $\alpha_\Sigma(\Phi) \subseteq C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(X)$. Hence, since $\phi \in C_\Sigma(\Phi)$ and $C^{\mathcal{I},\mathcal{A}}(X) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, it follows that $\phi' = \alpha_\Sigma(\phi) \in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(X)$. We conclude that $\Xi^{\mathcal{I},\mathcal{A},n}_{\Sigma'}(X) \subseteq C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(X)$.

It now follows that $\bigcup_{n<\omega} \Xi^{\mathcal{I},\mathcal{A},n}(X) \leq C^{\mathcal{I},\mathcal{A}}(X)$.

■

We set

$$\Xi^{\mathcal{I},\mathcal{A}}(X) := \bigcup_{n<\omega} \Xi^{\mathcal{I},\mathcal{A},n}(X).$$

Proposition 114 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a finitary* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $X \in \mathrm{SenFam}(\mathcal{A})$*. Then*

$$C^{\mathcal{I},\mathcal{A}}(X) = \Xi^{\mathcal{I},\mathcal{A}}(X).$$

Proof: Since by Lemma 113, $\Xi^{\mathcal{I},\mathcal{A}}(X) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $X \leq \Xi^{\mathcal{I},\mathcal{A}}(X)$, we get, by the minimality of $C^{\mathcal{I},\mathcal{A}}(X)$, that $C^{\mathcal{I},\mathcal{A}}(X) \leq \Xi^{\mathcal{I},\mathcal{A}}(X)$. On the other hand, by Lemma 113, $\Xi^{\mathcal{I},\mathcal{A}}(X) \leq C^{\mathcal{I},\mathcal{A}}(X)$. Thus, we conclude that $C^{\mathcal{I},\mathcal{A}}(X) = \Xi^{\mathcal{I},\mathcal{A}}(X)$. ■

2.17 Equational π-Institutions

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and K be a class of $\mathbf{F}$-algebraic systems. Denote by $\mathrm{Eq}(\mathbf{F}) = \{\mathrm{Eq}_\Sigma(\mathbf{F})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ the family of **$\mathbf{F}$-equations**, i.e., for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{Eq}_\Sigma(\mathbf{F}) = \mathrm{SEN}^\flat(\Sigma)^2.$$

The **equational consequence relative to K** or **K-equational consequence** is the closure family $D^{\mathsf{K}} : \mathcal{P}\mathrm{Eq}(\mathbf{F}) \to \mathcal{P}\mathrm{Eq}(\mathbf{F})$, defined by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$D^{\mathsf{K}}_\Sigma : \mathcal{P}(\mathrm{Eq}_\Sigma(\mathbf{F})) \to \mathcal{P}(\mathrm{Eq}_\Sigma(\mathbf{F}))$$

be given, for all $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$, by

$$\begin{array}{lll}\phi \approx \psi \in D^{\mathsf{K}}_\Sigma(E) & \text{iff} & \text{for all } \mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle \in \mathsf{K}, \\ & & \alpha_\Sigma(E) \subseteq \Delta^{\mathcal{A}}_{F(\Sigma)} \text{ implies } \alpha_\Sigma(\phi) = \alpha_\Sigma(\psi).\end{array}$$

This closure operator appeared, for the first time, in Section 2.4 as a means to characterize the relative congruence system generated by a family of equations, with respect to the class K of $\mathbf{F}$-algebraic systems. In Proposition 29, it was shown that, given an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ and a class K of $\mathbf{F}$-algebraic systems, $D^{\mathsf{K}} : \mathcal{P}\mathrm{Eq}(\mathbf{F}) \to \mathcal{P}\mathrm{Eq}(\mathbf{F})$ is a (not necessarily structural) closure family on $\mathrm{Eq}(\mathbf{F})$.

Moreover, it turns out that the closure family D^{K} satisfies the properties of reflexivity, symmetry, transitivity, congruence and invariance, detailed in the following

Proposition 115 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and let K be a class of $\mathbf{F}$-algebraic systems. For all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, all $\sigma^\flat$ in $N^\flat$, all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$, all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,*

(Reflexivity) $\phi \approx \phi \in D^{\mathsf{K}}_\Sigma(\varnothing)$;

(Symmetry) $\psi \approx \phi \in D^{\mathsf{K}}_\Sigma(\phi \approx \psi)$;

(Transitivity) $\phi \approx \chi \in D^{\mathsf{K}}_\Sigma(\phi \approx \psi, \psi \approx \chi)$;

(Congruence) $\sigma^\flat_\Sigma(\vec{\phi}) \approx \sigma^\flat_\Sigma(\vec{\psi}) \in D^{\mathsf{K}}_\Sigma(\{\phi_i \approx \psi_i : i \in I\})$;

(Invariance) $\mathrm{SEN}^\flat(f)(\phi) \approx \mathrm{SEN}^\flat(f)(\psi) \in D^{\mathsf{K}}_{\Sigma'}(\phi \approx \psi)$.

Proof: All properties follow directly by applying Proposition 30. ■

Corollary 116 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and K a class of $\mathbf{F}$-algebraic systems. Then $\mathrm{ThFam}(D^{\mathsf{K}}) = \mathrm{ThSys}(D^{\mathsf{K}}) = \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$.*

Proof: The first equality is a direct consequence of Invariance, given in Proposition 115, while the second follows directly from Theorem 32. ■

Assume, next, that $Q = \{Q_\Sigma\}_{\Sigma\in|\mathbf{Sign}^\flat|} \leq \mathrm{Eq}(\mathbf{F})$ is an $\mathbf{F}$-equation system, i.e., a family of $\mathbf{F}$-equations that is invariant under signature morphisms in the sense that, for all $\Sigma,\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$,

$$\mathrm{SEN}^\flat(f)(Q_\Sigma) \subseteq Q_{\Sigma'}.$$

Let, also $E = \{E_\Sigma\}_{\Sigma\in|\mathbf{Sign}^\flat|} \leq \mathrm{Eq}(\mathbf{F})$ be an $\mathbf{F}$-equation family (not necessarily invariant under signature morphisms). We define, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $n < \omega$,

$$\Xi_\Sigma^{Q,n}(E) : \mathcal{P}(\mathrm{Eq}_\Sigma(\mathbf{F})) \to \mathcal{P}(\mathrm{Eq}_\Sigma(\mathbf{F})),$$

by induction on $n < \omega$, as follows:

- $\Xi_\Sigma^{Q,0}(E) = \{\phi \approx \phi : \phi \in \mathrm{SEN}^\flat(\Sigma)\} \cup Q_\Sigma \cup E_\Sigma$;
- Assuming that $\Xi_\Sigma^{Q,n}(E)$ has been defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$, we set

$$\begin{array}{rcl}\Xi_\Sigma^{Q,n+1}(E) & = & \{\psi \approx \phi : \phi \approx \psi \in \Xi_\Sigma^{Q,n}(E)\}\\ & & \cup\{\phi \approx \chi : \phi \approx \psi, \psi \approx \chi \in \Xi_\Sigma^{Q,n}(E)\}\\ & & \cup\{\sigma_\Sigma^\flat(\vec{\phi}) \approx \sigma_\Sigma^\flat(\vec{\psi}) : \phi_i \approx \psi_i \in \Xi_\Sigma^{Q,n}(E), i < k\}\\ & & \cup\{\mathrm{SEN}^\flat(f)(\phi \approx \psi) : \phi \approx \psi \in \Xi_{\Sigma'}^{Q,n}(E),\\ & & \qquad \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma',\Sigma)\}.\end{array}$$

Now, set, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\Xi_\Sigma^Q(E) = \bigcup_{n<\omega} \Xi_\Sigma^{Q,n}(E)$$

and, finally,

$$\Xi^Q(E) = \{\Xi_\Sigma^Q(E)\}_{\Sigma\in|\mathbf{Sign}^\flat|}.$$

We show that $\Xi^Q : \mathcal{P}(\mathrm{Eq}(\mathbf{F})) \to \mathcal{P}(\mathrm{Eq}(\mathbf{F}))$ is a closure family on $\mathrm{Eq}(\mathbf{F})$ that satisfies Reflexivity, Symmetry, Transitivity, Congruence and Invariance.

Proposition 117 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $Q \leq \mathrm{Eq}(\mathbf{F})$ *an* $\mathbf{F}$*-equation system. Then* $\Xi^Q : \mathcal{P}\mathrm{Eq}(\mathbf{F}) \to \mathcal{P}\mathrm{Eq}(\mathbf{F})$ *is a closure family on* $\mathrm{Eq}(\mathbf{F})$*, that satisfies Reflexivity, Symmetry, Transitivity, Congruence and Invariance.*

Proof: We start by showing that Ξ^Q is a closure family.

- Let $\Sigma \in |\mathbf{Sign}^\flat|$, $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$, such that $\phi \approx \psi \in E$. Then, by definition, $\phi \approx \psi \in \Xi_\Sigma^{Q,0}(E) \subseteq \Xi_\Sigma^Q(E)$. So Ξ^Q is inflationary.

- Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $E \cup F \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$, such that $\phi \approx \psi \in \Xi^Q_\Sigma(E)$ and $E \subseteq F$. Then, for some $n < \omega$, $\phi \approx \psi \in \Xi^{Q,n}_\Sigma(E)$. We show by induction on $n < \omega$ that, for all $n < \omega$,

$$\phi \approx \psi \in \Xi^{Q,n}_\Sigma(E) \quad \text{implies} \quad \phi \approx \psi \in \Xi^{Q,n}_\Sigma(F).$$

  - For $n = 0$, we have $\phi = \psi$ or $\phi \approx \psi \in Q_\Sigma$ or $\phi \approx \psi \in E$. In the first two cases, $\phi \approx \psi \in \Xi^{Q,0}_\Sigma(F)$ by definition, and, in the last case, $\phi \approx \psi \in \Xi^{Q,0}_\Sigma(F)$, since $E \subseteq F$, by hypothesis.
  - Assume, next, that the statement holds for $n > 0$. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \approx \psi \in \Xi^{Q,n+1}_\Sigma(E)$.

    If $\psi \approx \phi \in \Xi^{Q,n}_\Sigma(E)$, then, by the induction hypothesis, $\psi \approx \phi \in \Xi^{Q,n}_\Sigma(F)$, whence, by definition, $\phi \approx \psi \in \Xi^{Q,n+1}_\Sigma(F)$.

    If $\phi \approx \chi, \chi \approx \psi \in \Xi^{Q,n}_\Sigma(E)$, then, by the induction hypothesis, $\phi \approx \chi, \chi \approx \psi \in \Xi^{Q,n}_\Sigma(F)$, whence, by definition, $\phi \approx \psi \in \Xi^{Q,n+1}_\Sigma(F)$.

    If $\phi \approx \psi$ is of the form $\sigma^\flat_\Sigma(\vec{\phi}) \approx \sigma^\flat_\Sigma(\vec{\psi})$, with $\phi_i \approx \psi_i \in \Xi^{Q,n}_\Sigma(E)$, $i < k$, then, by the induction hypothesis, $\phi_i \approx \psi_i \in \Xi^{Q,n}_\Sigma(F)$, whence, by definition, $\sigma^\flat_\Sigma(\vec{\phi}) \approx \sigma^\flat_\Sigma(\vec{\psi}) \in \Xi^{Q,n+1}_\Sigma(F)$.

    Finally, if $\phi \approx \psi$ is of form $\mathrm{SEN}^\flat(f)(\phi' \approx \psi')$, with $\phi' \approx \psi' \in \Xi^{Q,n}_{\Sigma'}(E)$, then, by the induction hypothesis, $\phi' \approx \psi' \in \Xi^{Q,n}_{\Sigma'}(F)$, and, therefore, by definition, $\mathrm{SEN}^\flat(f)(\phi' \approx \psi') \in \Xi^{Q,n+1}_\Sigma(F)$.

  Thus, if $\phi \approx \psi \in \Xi^Q_\Sigma(E)$, then $\phi \approx \psi \in \Xi^Q_\Sigma(F)$ and Ξ^Q is monotone.

- Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$, such that $\phi \approx \psi \in \Xi^Q_\Sigma(\Xi^Q_\Sigma(E))$. Then, for some $n < \omega$, $\phi \approx \psi \in \Xi^{Q,n}_\Sigma(\Xi^Q_\Sigma(E))$. We show by induction on $n < \omega$ that, for all $n < \omega$,

$$\phi \approx \psi \in \Xi^{Q,n}_\Sigma(\Xi^Q_\Sigma(E)) \quad \text{implies} \quad \phi \approx \psi \in \Xi^Q_\Sigma(E).$$

  - For $n = 0$, $\phi = \psi$ or $\phi \approx \psi \in Q_\Sigma$ or $\phi \approx \psi \in \Xi^Q_\Sigma(E)$. In the first two cases $\phi \approx \psi \in \Xi^{Q,0}_\Sigma(E) \subseteq \Xi^Q_\Sigma(E)$, by definition, and in the last the implication is trivial.
  - Suppose that the statement holds for $n > 0$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \approx \psi \in \Xi^{Q,n+1}_\Sigma(\Xi^Q_\Sigma(E))$.

    If $\psi \approx \phi \in \Xi^{Q,n}_\Sigma(\Xi^Q_\Sigma(E))$, then, by the induction hypothesis, $\psi \approx \phi \in \Xi^Q_\Sigma(E)$, i.e., $\psi \approx \phi \in \Xi^{Q,m}_\Sigma(E)$, for some $m < \omega$. Thus, by definition, $\phi \approx \psi \in \Xi^{Q,m+1}_\Sigma(E) \subseteq \Xi^Q_\Sigma(E)$.

    If $\phi \approx \chi, \chi \approx \psi \in \Xi^{Q,n}_\Sigma(\Xi^Q_\Sigma(E))$, then, by the induction hypothesis, $\phi \approx \chi, \chi \approx \psi \in \Xi^Q_\Sigma(E)$, i.e., for some $m < \omega$, $\phi \approx \chi, \chi \approx \psi \in \Xi^{Q,m}_\Sigma(E)$. Thus, by definition, $\phi \approx \psi \in \Xi^{Q,m+1}_\Sigma(E) \subseteq \Xi^Q_\Sigma(E)$.

If $\phi \approx \psi$ is of the form $\sigma^\flat_\Sigma(\vec{\phi}) \approx \sigma^\flat_\Sigma(\vec{\psi})$, with $\phi_i \approx \psi_i \in \Xi^{Q,n}_\Sigma(\Xi^Q_\Sigma(E))$, $i < k$, then, by the induction hypothesis, $\phi_i \approx \psi_i \in \Xi^Q_\Sigma(E)$, for all $i < k$. Thus, there exists $m < \omega$, such that $\phi_i \approx \psi_i \in \Xi^{Q,m}_\Sigma(E)$, for all $i < k$, and, consequently, by definition, $\sigma^\flat_\Sigma(\vec{\phi}) \approx \sigma^\flat_\Sigma(\vec{\psi}) \in \Xi^{Q,m+1}_\Sigma(E) \subseteq \Xi^Q_\Sigma(E)$.

Finally, suppose that $\phi \approx \psi$ is of the form $\mathrm{SEN}^\flat(f)(\phi' \approx \psi')$, where $\phi' \approx \psi' \in \Xi^{Q,n}_{\Sigma'}(E)$. Then, by the induction hypothesis, $\phi' \approx \psi' \in \Xi^Q_{\Sigma'}(E)$, whence, there exists $m < \omega$, such that $\phi' \approx \psi' \in \Xi^{Q,m}_{\Sigma'}(E)$. But, then, by definition, $\mathrm{SEN}^\flat(f)(\phi' \approx \psi') \in \Xi^{Q,m+1}_\Sigma(E) \subseteq \Xi^Q_\Sigma(E)$.

So Ξ^Q is a closure family. Finally, we show that it satisfies the five extra rules.

- For Reflexivity, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, by definition, $\phi \approx \phi \in \Xi^{Q,0}_\Sigma(\varnothing) \subseteq \Xi^Q_\Sigma(\varnothing)$, whence Ξ^Q is Reflexive.

- For Symmetry, let $E \le \mathrm{Eq}(\mathbf{F})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \approx \psi \in \Xi^Q_\Sigma(E)$. Then, there exists $n < \omega$, such that $\phi \approx \psi \in \Xi^{Q,n}_\Sigma(E)$. By definition, $\psi \approx \phi \in \Xi^{Q,n+1}_\Sigma(E) \subseteq \Xi^Q_\Sigma(E)$. We conclude that $\psi \approx \phi \in \Xi^Q_\Sigma(\phi \approx \psi)$ and, hence Ξ^Q is Symmetric.

- For Transitivity, let $E \le \mathrm{Eq}(\mathbf{F})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \approx \psi, \psi \approx \chi \in \Xi^Q_\Sigma(E)$. Then, there exists $n < \omega$, such that $\phi \approx \psi, \psi \approx \chi \in \Xi^{Q,n}_\Sigma(E)$. By definition, $\phi \approx \chi \in \Xi^{Q,n+1}_\Sigma(E) \subseteq \Xi^Q_\Sigma(E)$. So Ξ^Q is Transitive.

- For Congruence, let $E \le \mathrm{Eq}(\mathbf{F})$, $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi_i, \psi_i \in \mathrm{SEN}^\flat(\Sigma)$, $i < k$, such that $\phi_i \approx \psi_i \in \Xi^Q_\Sigma(E)$. Then, there exists $n < \omega$, such that $\phi_i \approx \psi_i \in \Xi^{Q,n}_\Sigma(E)$, for all $i < k$, whence, by definition, $\sigma^\flat_\Sigma(\vec{\phi}) \approx \sigma^\flat_\Sigma(\vec{\psi}) \in \Xi^{Q,n+1}_\Sigma(E) \subseteq \Xi^Q_\Sigma(E)$. Thus Ξ^Q satisfies Congruence.

- Finally, for Invariance, let $E \le \mathrm{Eq}(\mathbf{F})$, $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \approx \psi \in \Xi^Q_\Sigma(E)$. Then, there exists $n < \omega$, such that $\phi \approx \psi \in \Xi^{Q,n}_\Sigma(E)$, and, hence, by definition,

$$\mathrm{SEN}^\flat(f)(\phi \approx \psi) \in \Xi^{Q,n+1}_{\Sigma'}(E) \subseteq \Xi^Q_{\Sigma'}(E).$$

  We conclude that Ξ^Q satisfies Invariance as well.

This shows that $\Xi^Q : \mathcal{P}\mathrm{Eq}(\mathbf{F}) \to \mathcal{P}\mathrm{Eq}(\mathbf{F})$ is a closure family that satisfies Reflexivity, Symmetry, Transitivity, Congruence and Invariance. ■

We show that, given a semantic variety, i.e., a class K of $\mathbf{F}$-algebraic systems, such that $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$, we have $D^{\mathsf{K}} = \Xi^{\mathrm{Ker}(\mathsf{K})}$.

We prove first a lemma.

Lemma 118 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a semantic variety of* $\mathbf{F}$*-algebraic systems, i.e., a class of* $\mathbf{F}$*-algebraic systems, such that* $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$*. Then, for all* $E \subseteq \mathrm{Eq}(\mathbf{F})$*,* $\Xi^{\mathrm{Ker}(\mathsf{K})}(E) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$*.*

Proof: By Proposition 117, $\Xi^{\mathrm{Ker}(\mathsf{K})}(E)$ is a congruence system on $\mathbf{F}$. Moreover, by definition, $\mathrm{Ker}(\mathsf{K}) \leq \Xi^{\mathrm{Ker}(\mathsf{K})}(E)$. But, note that

$$\mathrm{Ker}(\mathcal{F}/\Xi^{\mathrm{Ker}(\mathsf{K})}(E)) = \Xi^{\mathrm{Ker}(\mathsf{K})}(E).$$

Thus, we have $\mathrm{Ker}(\mathsf{K}) \leq \mathrm{Ker}(\mathcal{F}/\Xi^{\mathrm{Ker}(\mathsf{K})}(E))$. Thus, by definition and the hypothesis,

$$\mathcal{F}/\Xi^{\mathrm{Ker}(\mathsf{K})}(E) \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}.$$

We conclude that $\Xi^{\mathrm{Ker}(\mathsf{K})}(E) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$. ∎

Theorem 119 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems, such that* $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$*. Then* $D^{\mathsf{K}} = \Xi^Q$*, where* $Q = \mathrm{Ker}(\mathsf{K})$*.*

Proof: Assume, first, that $\Sigma \in |\mathbf{Sign}^\flat|$, $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$, such that $\phi \approx \psi \in D^{\mathsf{K}}_\Sigma(E)$. Thus, by definition, for all $\mathcal{A} \in \mathsf{K}$,

$$E \subseteq \mathrm{Ker}_\Sigma(\mathcal{A}) \quad \text{implies} \quad \langle \phi, \psi \rangle \in \mathrm{Ker}_\Sigma(\mathcal{A}).$$

In particular, by Lemma 118,

$$E \subseteq \mathrm{Ker}_\Sigma(\mathcal{F}/\Xi^Q(E)) \quad \text{implies} \quad \langle \phi, \psi \rangle \in \mathrm{Ker}_\Sigma(\mathcal{F}/\Xi^Q(E)).$$

Equivalently, we have $E \subseteq \Xi^Q_\Sigma(E)$ implies $\phi \approx \psi \in \Xi^Q_\Sigma(E)$. Since the first inclusion holds by the definition of Ξ^Q, we have $\phi \approx \psi \in \Xi^Q_\Sigma(E)$. We conclude that $D^{\mathsf{K}} \leq \Xi^Q$.

Assume, conversely, that $\Sigma \in |\mathbf{Sign}^\flat|$, $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$, such that $\phi \approx \psi \in \Xi^Q_\Sigma(E)$. Then, by definition, there exists an $n < \omega$, such that $\phi \approx \psi \in \Xi^{Q,n}_\Sigma(E)$. We show, by induction on $n < \omega$, that, for all $n < \omega$,

$$\phi \approx \psi \in \Xi^{Q,n}_\Sigma(E) \quad \text{implies} \quad \phi \approx \psi \in D^{\mathsf{K}}_\Sigma(E).$$

- If $n = 0$, then $\phi = \psi$ or $\phi \approx \psi \in \mathrm{Ker}_\Sigma(\mathsf{K})$ or $\phi \approx \psi \in E$.

  In the first case, the conclusion follows by Proposition 115, and in the last, by Proposition 29.

  In the second case, we have, for all $\mathcal{A} \in \mathsf{K}$, $\mathrm{Ker}(\mathsf{K}) \leq \mathrm{Ker}(\mathcal{A})$, whence $\phi \approx \psi \in \mathrm{Ker}_\Sigma(\mathcal{A})$. So $\phi \approx \psi \in D^{\mathsf{K}}_\Sigma(\varnothing) \subseteq D^{\mathsf{K}}_\Sigma(E)$.

- Assume, now, that the conclusion holds for $n > 0$. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$, such that $\phi \approx \psi \in \Xi_\Sigma^{Q,n+1}(E)$.

  If $\psi \approx \phi \in \Xi_\Sigma^{Q,n}(E)$, then, by the induction hypothesis, $\psi \approx \phi \in D_\Sigma^{\mathsf{K}}(E)$, whence, by Proposition 115, $\phi \approx \psi \in D_\Sigma^{\mathsf{K}}(E)$.

  If $\phi \approx \chi, \chi \approx \psi \in \Xi_\Sigma^{Q,n}(E)$, then , by the induction hypothesis, $\phi \approx \chi, \chi \approx \psi \in D_\Sigma^{\mathsf{K}}(E)$. So, by Proposition 115, we have $\phi \approx \psi \in D_\Sigma^{\mathsf{K}}(E)$.

  If $\phi \approx \psi$ is of the form $\sigma_\Sigma^\flat(\vec{\phi}) \approx \sigma_\Sigma^\flat(\vec{\psi})$, for some $\sigma^\flat$ in $N^\flat$ and $\phi_i \approx \psi_i \in \Xi_\Sigma^{Q,n}(E)$, $i < k$, then, by the induction hypothesis, $\phi_i \approx \psi_i \in D_\Sigma^{\mathsf{K}}(E)$, for all $i < k$, whence, once more by Proposition 115, $\sigma_\Sigma^\flat(\vec{\phi}) \approx \sigma_\Sigma^\flat(\vec{\psi}) \in D_\Sigma^{\mathsf{K}}(E)$.

  Finally, if $\phi \approx \psi$ is of the form $\mathrm{SEN}^\flat(f)(\phi' \approx \psi')$, for some $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma', \Sigma)$ and $\phi' \approx \psi' \in \Xi_{\Sigma'}^{Q,n}(E)$, then, by the induction hypothesis, $\phi' \approx \psi' \in D_{\Sigma'}^{\mathsf{K}}(E)$, whence, by Proposition 115, $\mathrm{SEN}^\flat(f)(\phi' \approx \psi') \in D_\Sigma^{\mathsf{K}}(E)$.

Thus, we get $\Xi^Q \leq D^{\mathsf{K}}$ and, therefore, $D^{\mathsf{K}} = \Xi^Q$. ■

2.18 Categorical Universal Algebra

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. We define **F**-equations, **F**-quasiequations and **F**-guasiequations (standing for generalized **F**-quasiequations). Recall that **F**-equations have already been introduced in Section 2.17, but the definition is repeated here for the sake of completeness.

- The family $\mathrm{Eq}(\mathbf{F}) = \{\mathrm{Eq}_\Sigma(\mathbf{F})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ of **F-equations** is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{Eq}_\Sigma(\mathbf{F}) = \mathrm{SEN}^\flat(\Sigma)^2 = \{\phi \approx \psi : \phi, \psi \in \mathrm{SEN}^\flat(\Sigma)\};$$

- The family $\mathrm{QEq}(\mathbf{F}) = \{\mathrm{QEq}_\Sigma(\mathbf{F})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ of **F-quasiequations** is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{QEq}_\Sigma(\mathbf{F}) = \{\langle \{\phi_i \approx \psi_i : i < k\}, \phi \approx \psi \rangle : k \in \omega, \vec{\phi}, \vec{\psi}, \phi, \psi \in \mathrm{SEN}^\flat(\Sigma)\};$$

- The family $\mathrm{GEq}(\mathbf{F}) = \{\mathrm{GEq}_\Sigma(\mathbf{F})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ of **F-guasiequations** is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{GEq}_\Sigma(\mathbf{F}) = \{\langle \{\phi_i \approx \psi_i : i \in I\}, \phi \approx \psi \rangle : \vec{\phi}, \vec{\psi}, \phi, \psi \in \mathrm{SEN}^\flat(\Sigma)\}.$$

Sometimes we write $\langle \phi, \psi \rangle$ in place of $\phi \approx \psi$. Moreover, we use the notation

$$\vec{\phi} \approx \vec{\psi} := \{\phi_i \approx \psi_i : i \in I\}.$$

Thus, the **F**-guasiequation $\langle\{\phi_i \approx \psi_i : i \in I\}, \phi \approx \psi\rangle$ may be written more compactly $\langle\vec{\phi} \approx \vec{\psi}, \phi \approx \psi\rangle$ and, sometimes, also $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi$. Note that

$$\mathrm{Eq}(\mathbf{F}) \leq \mathrm{QEq}(\mathbf{F}) \leq \mathrm{GEq}(\mathbf{F}).$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\Sigma \in |\mathbf{Sign}^\flat|$, $\langle\vec{\phi} \approx \vec{\psi}, \phi \approx \psi\rangle \in \mathrm{GEq}_\Sigma(\mathbf{F})$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ an **F**-algebraic system. We say that $\mathcal{A}$ **satisfies** $\langle\vec{\phi} \approx \vec{\psi}, \phi \approx \psi\rangle$ or that the guasiequation $\langle\vec{\phi} \approx \vec{\psi}, \phi \approx \psi\rangle$ **is true in**, or **is satisfied in**, or **holds in** $\mathcal{A}$, written

$$\mathcal{A} \vDash_\Sigma \langle\vec{\phi} \approx \vec{\psi}, \phi \approx \psi\rangle,$$

if

$$\alpha_\Sigma(\phi_i) = \alpha_\Sigma(\psi_i),\ i \in I,\ \text{imply}\ \alpha_\Sigma(\phi) = \alpha_\Sigma(\psi).$$

Since **F**-quasiequations and **F**-equations are special cases of **F**-guasiequations, the definition covers them as well. Thus, we have

- $\mathcal{A}$ satisfies the **F**-quasiequation $\langle\vec{\phi} \approx \vec{\psi}, \phi \approx \psi\rangle$ if $\alpha_\Sigma(\phi_i) = \alpha_\Sigma(\psi_i)$, for all $i < k$, imply $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$;
- $\mathcal{A}$ satisfies the **F**-equation $\phi \approx \psi$ if $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$.

Now we define the following families:

- The family $\mathrm{Eq}(\mathcal{A}) = \{\mathrm{Eq}_\Sigma(\mathcal{A})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ of **F**-equations satisfied by $\mathcal{A}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{Eq}_\Sigma(\mathcal{A}) = \{e \in \mathrm{Eq}_\Sigma(\mathbf{F}) : \mathcal{A} \vDash_\Sigma e\};$$

- The family $\mathrm{QEq}(\mathcal{A}) = \{\mathrm{QEq}_\Sigma(\mathcal{A})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ of **F**-quasiequations satisfied by $\mathcal{A}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{QEq}_\Sigma(\mathcal{A}) = \{q \in \mathrm{QEq}_\Sigma(\mathbf{F}) : \mathcal{A} \vDash_\Sigma q\};$$

- The family $\mathrm{GEq}(\mathcal{A}) = \{\mathrm{GEq}_\Sigma(\mathcal{A})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ of **F**-guasiequations satisfied by $\mathcal{A}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{GEq}_\Sigma(\mathcal{A}) = \{g \in \mathrm{GEq}_\Sigma(\mathbf{F}) : \mathcal{A} \vDash_\Sigma g\}.$$

Finally, given a class K of **F**-algebraic systems, we define:

- $\mathrm{Eq}(\mathsf{K}) = \bigcap\{\mathrm{Eq}(\mathcal{A}) : \mathcal{A} \in \mathsf{K}\}$;
- $\mathrm{QEq}(\mathsf{K}) = \bigcap\{\mathrm{QEq}(\mathcal{A}) : \mathcal{A} \in \mathsf{K}\}$;
- $\mathrm{GEq}(\mathsf{K}) = \bigcap\{\mathrm{GEq}(\mathcal{A}) : \mathcal{A} \in \mathsf{K}\}$.

Note, again, that

$$\mathrm{Eq}(\mathcal{A}) \le \mathrm{QEq}(\mathcal{A}) \le \mathrm{GEq}(\mathcal{A}) \quad \text{and} \quad \mathrm{Eq}(\mathsf{K}) \le \mathrm{QEq}(\mathsf{K}) \le \mathrm{GEq}(\mathsf{K}).$$

Given a class $G \le \mathrm{GEq}(\mathbf{F})$ of $\mathbf{F}$-guasiequations (which includes the case of quasiequations or equations), we define $\mathrm{AlgSys}(G)$ or, sometimes, $\mathrm{Mod}(G)$, to be the collection of all $\mathbf{F}$-algebraic systems that satisfy all the $\mathbf{F}$-guasiequations in G:

$$\mathrm{AlgSys}(G) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : \mathcal{A} \vDash G\}.$$

As is well-known, based on an underlying Galois connection, we get the following, for all $G, G' \le \mathrm{GEq}(\mathbf{F})$ and all $\mathsf{K}, \mathsf{K}' \subseteq \mathrm{AlgSys}(\mathbf{F})$,

- If $\mathsf{K} \subseteq \mathsf{K}'$, then $\mathrm{GEq}(\mathsf{K}') \le \mathrm{GEq}(\mathsf{K})$;
- If $G \le G'$, then $\mathrm{AlgSys}(G') \subseteq \mathrm{AlgSys}(G)$;
- $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathrm{GEq}(\mathsf{K}))$ and $\mathrm{GEq}(\mathsf{K}) = \mathrm{GEq}(\mathrm{AlgSys}(\mathrm{GEq}(\mathsf{K})))$;
- $G \subseteq \mathrm{GEq}(\mathrm{AlgSys}(G))$ and $\mathrm{AlgSys}(G) = \mathrm{AlgSys}(\mathrm{GEq}(\mathrm{AlgSys}(G)))$.

Similar relations hold with the GEq operator replaced by either the Eq or the QEq operator. We may apply some of these either without providing explicit justification or, simply, by saying "by the Galois connection".

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and K a class of $\mathbf{F}$-algebraic systems.

- K is called an **equational class** if there exists $E \le \mathrm{Eq}(\mathbf{F})$, such that $\mathsf{K} = \mathrm{AlgSys}(E)$;
- K is called a **quasiequational class** if there exists $Q \le \mathrm{QEq}(\mathbf{F})$, such that $\mathsf{K} = \mathrm{AlgSys}(Q)$;
- K is called a **guasiequational class** if there exists $G \le \mathrm{GEq}(\mathbf{F})$, such that $\mathsf{K} = \mathrm{AlgSys}(G)$.

Clearly, by definition, if K is an equational class, then it is a quasiequational class and, if it is a quasiequational class, then it is a guasiequational class.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and K a class of $\mathbf{F}$-algebraic systems. We define:

- The **semantic variety generated by** K

$$\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : \mathrm{Eq}(\mathsf{K}) \le \mathrm{Eq}(\mathcal{A})\};$$

- The **semantic quasivariety generated by** K

$$\mathbb{Q}^{\mathsf{Sem}}(\mathsf{K}) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : \mathrm{QEq}(\mathsf{K}) \le \mathrm{QEq}(\mathcal{A})\};$$

- The **semantic guasivariety generated by** K

$$\mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : \mathrm{GEq}(\mathsf{K}) \leq \mathrm{GEq}(\mathcal{A})\}.$$

We have the following straightforward relationships between these classes.

Lemma 120 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems. Then*

$$\mathsf{K} \subseteq \mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{Q}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{V}^{\mathsf{Sem}}(\mathsf{K}).$$

Proof: The essential observation we use, which has been discussed before, is that

$$\mathrm{Eq}(\mathsf{K}) \leq \mathrm{QEq}(\mathsf{K}) \leq \mathrm{GEq}(\mathsf{K}).$$

Thus, we get

$$\begin{aligned} \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : (\forall g \in \mathrm{GEq}(\mathsf{K}))(\mathcal{A} \vDash g)\} \\ \subseteq \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : (\forall q \in \mathrm{QEq}(\mathsf{K}))(\mathcal{A} \vDash q)\} \\ \subseteq \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : (\forall e \in \mathrm{Eq}(\mathsf{K}))(\mathcal{A} \vDash e)\}. \end{aligned}$$

In other words, $\mathsf{K} \subseteq \mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{Q}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. ■

Given a class K of $\mathbf{F}$-algebraic systems

- K is a **semantic variety** if $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$;
- K is a **semantic quasivariety** if $\mathbb{Q}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$;
- K is a **semantic guasivariety** if $\mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$.

We have the following result identifying equational classes with semantic varieties, quasiequational classes with semantic quasivarieties and guasiequational classes with semantic guasivarieties.

Proposition 121 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems.*

(a) K *is an equational class iff it is a semantic variety;*

(b) K *is a quasiequational class iff it is a semantic quasivariety;*

(c) K *is a guasiequational class iff it is a semantic guasivariety.*

Proof:

(a) Suppose, first, that K is an equational class. Then, there exists $E \leq \mathrm{Eq}(\mathbf{F})$, such that $\mathsf{K} = \mathrm{AlgSys}(E)$. Let $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, such that $\mathrm{Eq}(\mathsf{K}) \leq \mathrm{Eq}(\mathcal{A})$. Then we have

$$\begin{array}{rcl}\mathcal{A} & \in & \mathrm{AlgSys}(\mathrm{Eq}(\mathcal{A})) \\ & \subseteq & \mathrm{AlgSys}(\mathrm{Eq}(\mathsf{K})) \\ & = & \mathrm{AlgSys}(\mathrm{Eq}(\mathrm{AlgSys}(E))) \\ & = & \mathrm{AlgSys}(E) = \mathsf{K}.\end{array}$$

Therefore, K is a semantic variety.

Suppose, conversely, that K is a semantic variety. Set $E = \mathrm{Eq}(\mathsf{K})$. Then $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathrm{Eq}(\mathsf{K})) = \mathrm{AlgSys}(E)$. On the other hand, if $\mathcal{A} \in \mathrm{AlgSys}(E)$, then

$$\mathrm{Eq}(\mathsf{K}) = \mathrm{Eq}(\mathrm{AlgSys}(\mathrm{Eq}(\mathsf{K}))) = \mathrm{Eq}(\mathrm{AlgSys}(E)) \leq \mathrm{Eq}(\mathcal{A}),$$

whence, by hypothesis, $\mathcal{A} \in \mathsf{K}$. Therefore, $\mathsf{K} = \mathrm{AlgSys}(E)$ and K is an equational class.

(b) Suppose, first, that K is a quasiequational class. Then, there exists $Q \leq \mathrm{QEq}(\mathbf{F})$, such that $\mathsf{K} = \mathrm{AlgSys}(Q)$. Let $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, such that $\mathrm{QEq}(\mathsf{K}) \leq \mathrm{QEq}(\mathcal{A})$. Then we have

$$\begin{array}{rcl}\mathcal{A} & \in & \mathrm{AlgSys}(\mathrm{QEq}(\mathcal{A})) \\ & \subseteq & \mathrm{AlgSys}(\mathrm{QEq}(\mathsf{K})) \\ & = & \mathrm{AlgSys}(\mathrm{QEq}(\mathrm{AlgSys}(Q))) \\ & = & \mathrm{AlgSys}(Q) = \mathsf{K}.\end{array}$$

Therefore, K is a semantic quasivariety.

Suppose, conversely, that K is a semantic quasivariety. Set $Q = \mathrm{QEq}(\mathsf{K})$. Then $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathrm{QEq}(\mathsf{K})) = \mathrm{AlgSys}(Q)$. On the other hand, if $\mathcal{A} \in \mathrm{AlgSys}(Q)$, then

$$\mathrm{QEq}(\mathsf{K}) = \mathrm{QEq}(\mathrm{AlgSys}(\mathrm{QEq}(\mathsf{K}))) = \mathrm{QEq}(\mathrm{AlgSys}(Q)) \leq \mathrm{QEq}(\mathcal{A}),$$

whence, by hypothesis, $\mathcal{A} \in \mathsf{K}$. Therefore, $\mathsf{K} = \mathrm{AlgSys}(Q)$ and K is a quasiequational class.

(c) Very similar to Part (b).

■

We define or revisit, next, some operators on classes of **F**-algebraic systems that will serve to provide different characterizations to the equational, quasi-equational and guasiequational classes of **F**-algebraic systems.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, K a class of **F**-algebraic systems and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an **F**-algebraic system.

- Given $\Sigma \in |\mathbf{Sign}^\flat|$, we say that $\mathcal{A}$ is Σ-K-**certified** if there exists $\mathcal{A}^\Sigma \in \mathsf{K}$, such that $\mathrm{Eq}_\Sigma(\mathcal{A}) = \mathrm{Eq}_\Sigma(\mathcal{A}^\Sigma)$. In this case $\mathcal{A}^\Sigma$ is called the Σ-K-**certificate** of $\mathcal{A}$.

- We say that $\mathcal{A}$ is K-**certified** if it is Σ-K-certified, for all $\Sigma \in |\mathbf{Sign}^\flat|$. This, of course, means that
$$(\forall \Sigma \in |\mathbf{Sign}^\flat|)(\exists \mathcal{A}^\Sigma \in \mathsf{K})(\mathrm{Eq}_\Sigma(\mathcal{A}) = \mathrm{Eq}_\Sigma(\mathcal{A}^\Sigma)).$$

We write $\mathbb{C}(\mathsf{K})$ for the class of all $\mathbf{F}$-algebraic systems that are K-certified. We say that K is an **abstract class** whenever every K-certified $\mathbf{F}$-algebraic system belongs to K, i.e., when $\mathbb{C}(\mathsf{K}) = \mathsf{K}$.

It is not difficult to show that $\mathbb{C}$ is a closure operator on classes of $\mathbf{F}$-algebraic systems.

Proposition 122 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. Then the operator $\mathbb{C}$ on classes of $\mathbf{F}$-algebraic systems is a closure operator.*

Proof: Suppose K is a class of $\mathbf{F}$-algebraic systems.

- Let $\mathcal{A} \in \mathsf{K}$. Then, since for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathcal{A}^\Sigma = \mathcal{A} \in \mathsf{K}$ is a Σ-K-certificate for $\mathcal{A}$, we get that $\mathcal{A} \in \mathbb{C}(\mathsf{K})$. Thus, $\mathsf{K} \subseteq \mathbb{C}(\mathsf{K})$ and $\mathbb{C}$ is inflationary.

- If $\mathsf{K} \subseteq \mathsf{K}'$ and $\mathcal{A} \in \mathbb{C}(\mathsf{K})$, then, by definition, for every $\Sigma \in |\mathbf{Sign}^\flat|$, there exists a Σ-K-certificate $\mathcal{A}^\Sigma$. Since $\mathsf{K} \subseteq \mathsf{K}'$, $\mathcal{A}^\Sigma \in \mathsf{K}'$ is also a Σ-K'-certificate. Thus, $\mathcal{A} \in \mathbb{C}(\mathsf{K}')$ and $\mathbb{C}$ is also monotone.

- Finally, suppose that $\mathcal{A} \in \mathbb{C}(\mathbb{C}(\mathsf{K}))$. Then, there exists, for all $\Sigma \in |\mathbf{Sign}^\flat|$, a Σ-$\mathbb{C}(\mathsf{K})$-certificate $\mathcal{A}^\Sigma$ for $\mathcal{A}$. Therefore, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Sigma' \in |\mathbf{Sign}^\flat|$, there exists a Σ'-K-certificate $\mathcal{A}^{\langle \Sigma, \Sigma' \rangle}$ for $\mathcal{A}^\Sigma$. But, then, for every $\Sigma \in |\mathbf{Sign}^\flat|$,
$$\mathrm{Eq}_\Sigma(\mathcal{A}) = \mathrm{Eq}_\Sigma(\mathcal{A}^\Sigma) = \mathrm{Eq}_\Sigma(\mathcal{A}^{\langle \Sigma, \Sigma \rangle}).$$
Thus, for every $\Sigma \in |\mathbf{Sign}^\flat|$, there exists a Σ-K-certificate $\mathcal{A}^{\langle \Sigma, \Sigma \rangle}$ for $\mathcal{A}$, i.e., $\mathcal{A} \in \mathbb{C}(\mathsf{K})$ and $\mathbb{C}$ is also idempotent.

Thus $\mathbb{C}$ is a closure operator on classes of $\mathbf{F}$-algebraic systems. ■

The importance of abstract classes of $\mathbf{F}$-algebraic systems here, and the reason why they will be our exclusive focus in this section, rests on the following observation to the effect that the validity of a guasiequation transfers from K-certificates of an $\mathbf{F}$-algebraic system to the $\mathbf{F}$-algebraic system itself.

Lemma 123 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, K a class of $\mathbf{F}$-algebraic systems and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system. If $\mathcal{A} \in \mathbb{C}(\mathsf{K})$, then $\mathrm{GEq}(\mathsf{K}) \leq \mathrm{GEq}(\mathcal{A})$.*

Proof: Suppose $\mathcal{A} \in \mathbb{C}(\mathsf{K})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\langle \vec{\phi} \approx \vec{\psi}, \phi \approx \psi \rangle \in \mathrm{GEq}_\Sigma(\mathsf{K})$, such that $\vec{\phi} \approx \vec{\psi} \subseteq \mathrm{Eq}_\Sigma(\mathcal{A})$. Let $\mathcal{A}^\Sigma \in \mathsf{K}$ be a Σ-K-certificate for $\mathcal{A}$. Then, by definition $\vec{\phi} \approx \vec{\psi} \subseteq \mathrm{Eq}_\Sigma(\mathcal{A}^\Sigma)$. Since $\mathcal{A}^\Sigma \in \mathsf{K}$ and $\langle \vec{\phi} \approx \vec{\psi}, \phi \approx \psi \rangle \in \mathrm{GEq}_\Sigma(\mathsf{K})$, we get $\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathcal{A}^\Sigma) = \mathrm{Eq}_\Sigma(\mathcal{A})$. Therefore, $\langle \vec{\phi} \approx \vec{\psi}, \phi \approx \psi \rangle \in \mathrm{GEq}_\Sigma(\mathcal{A})$. We conclude that $\mathrm{GEq}(\mathsf{K}) \leq \mathrm{GEq}(\mathcal{A})$. ■

Using Lemma 123, we get the following corollary to the effect that all semantically defined classes of algebraic systems are abstract.

Corollary 124 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems. If* K *is a guasiequational class (and, hence, a fortiori, if it is a quasiequational class or an equational class), then it is abstract.*

Proof: Suppose K is a guasiequational class defined by the family of $\mathbf{F}$-guasiequations $G \leq \mathrm{GEq}(\mathbf{F})$ and let $\mathcal{A} \in \mathbb{C}(\mathsf{K})$. Then, by Lemma 123, $\mathrm{GEq}(\mathsf{K}) \leq \mathrm{GEq}(\mathcal{A})$, whence

$$\begin{array}{rcl} \mathcal{A} & \in & \mathrm{AlgSys}(\mathrm{GEq}(\mathcal{A})) \\ & \subseteq & \mathrm{AlgSys}(\mathrm{GEq}(\mathsf{K})) \\ & = & \mathrm{AlgSys}(\mathrm{GEq}(\mathrm{AlgSys}(G))) \\ & = & \mathrm{AlgSys}(G) \\ & = & \mathsf{K}. \end{array}$$

Thus, K is an abstract class. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, K a class of $\mathbf{F}$-algebraic systems and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system.

- Given $\Sigma \in |\mathbf{Sign}^\flat|$, we say that $\mathcal{A}$ is **directedly** Σ-K-**certified** if there exists a collection of $\mathbf{F}$-algebraic systems $\{\mathcal{A}^{\Sigma,i} : i \in I\} \subseteq \mathsf{K}$, such that:
  - $\bigcup_{i \in I} \mathrm{Eq}_\Sigma^\omega(\mathcal{A}^{\Sigma,i})$ is directed, where, for all $i \in I$, $\mathrm{Eq}_\Sigma^\omega(\mathcal{A}^{\Sigma,i})$ denotes the collection of all finite subsets of $\mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,i})$, and
  - $\mathrm{Ker}_\Sigma(\mathcal{A}) = \bigcup_{i \in I} \mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,i})$.

  We call $\{\mathcal{A}^{\Sigma,i} : i \in I\}$ the **directed** Σ-K-**certificate** of $\mathcal{A}$.

- We say that $\mathcal{A}$ is **directedly** K-**certified** if it is directedly Σ-K-certified, for all $\Sigma \in |\mathbf{Sign}^\flat|$.

We write $\mathbb{C}^*(\mathsf{K})$ for the class of all $\mathbf{F}$-algebraic systems that are directedly K-certified. We say that K is a **directedly abstract class** whenever every directedly K-certified $\mathbf{F}$-algebraic system belongs to K, i.e., when $\mathbb{C}^*(\mathsf{K}) = \mathsf{K}$.

We show that, like $\mathbb{C}$, $\mathbb{C}^*$ is a closure operator on classes of $\mathbf{F}$-algebraic systems.

Proposition 125 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system. Then the operator* $\mathbb{C}^*$ *on classes of* $\mathbf{F}$*-algebraic systems is a closure operator.*

Proof: Suppose K is a class of $\mathbf{F}$-algebraic systems.

- Let $\mathcal{A} \in \mathsf{K}$. Then, since, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\{\mathcal{A}\} \subseteq \mathsf{K}$ is a directed Σ-K-certificate for $\mathcal{A}$, we get that $\mathcal{A} \in \mathbb{C}^*(\mathsf{K})$. Thus, $\mathsf{K} \subseteq \mathbb{C}^*(\mathsf{K})$ and $\mathbb{C}^*$ is inflationary.

- If $\mathsf{K} \subseteq \mathsf{K}'$ and $\mathcal{A} \in \mathbb{C}^*(\mathsf{K})$, then, by definition, for every $\Sigma \in |\mathbf{Sign}^\flat|$, there exists a directed Σ-K-certificate $\{\mathcal{A}^{\Sigma,i} : i \in I_\Sigma\} \subseteq \mathsf{K}$. Since $\mathsf{K} \subseteq \mathsf{K}'$, $\{\mathcal{A}^{\Sigma,i} : i \in I_\Sigma\} \subseteq \mathsf{K}'$ is also a directed Σ-K'-certificate. Thus, $\mathcal{A} \in \mathbb{C}^*(\mathsf{K}')$ and $\mathbb{C}^*$ is also monotone.

- Finally, suppose that $\mathcal{A} \in \mathbb{C}^*(\mathbb{C}^*(\mathsf{K}))$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$, there exists $\{\mathcal{A}^{\Sigma,i} : i \in I_\Sigma\} \subseteq \mathbb{C}^*(\mathsf{K})$, such that $\bigcup_{i \in I_\Sigma} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i})$ is directed and $\mathrm{Ker}_\Sigma(\mathcal{A}) = \bigcup_{i \in I_\Sigma} \mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,i})$. Thus, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$ and all $i \in I_\Sigma$, there exists $\{\mathcal{A}^{\Sigma,i,\Sigma',j} : j \in J^{\Sigma,i}_{\Sigma'}\} \subseteq \mathsf{K}$, such that $\bigcup_{j \in J^{\Sigma,i}_{\Sigma'}} \mathrm{Eq}^\omega_{\Sigma'}(\mathcal{A}^{\Sigma,i,\Sigma',j})$ is directed and, moreover, $\mathrm{Ker}_{\Sigma'}(\mathcal{A}^{\Sigma,i}) = \bigcup_{j \in J^{\Sigma,i}_{\Sigma'}} \mathrm{Ker}_{\Sigma'}(\mathcal{A}^{\Sigma,i,\Sigma',j})$. Now notice that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, the collection
$$\{\mathcal{A}^{\Sigma,i,\Sigma,j} : i \in I_\Sigma, j \in J^{\Sigma,i}_\Sigma\} \subseteq \mathsf{K}$$
satisfies
$$\mathrm{Ker}_\Sigma(\mathcal{A}) = \bigcup_{i \in I_\Sigma} \mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,i}) = \bigcup_{i \in I_\Sigma} \bigcup_{j \in J^{\Sigma,i}_\Sigma} \mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,i,\Sigma,j}).$$
Thus, to see that $\mathcal{A} \in \mathbb{C}^*(\mathsf{K})$, it suffices to show that the collection
$$\bigcup_{i \in I_\Sigma} \bigcup_{j \in J^{\Sigma,i}_\Sigma} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i,\Sigma,j})$$
is directed. Consider $X \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i,\Sigma,j})$ and $X' \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i',\Sigma,j'})$. Then, as
$$\begin{array}{rcl}\mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,i}) & = & \bigcup_{j \in J^{\Sigma,i}_\Sigma} \mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,i,\Sigma,j}),\\ \mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,i'}) & = & \bigcup_{j \in J^{\Sigma,i'}_\Sigma} \mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,i',\Sigma,j}),\end{array}$$
we get that $X \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i})$ and $X' \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i'})$. As $\bigcup_{i \in I_\Sigma} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i})$ is directed, there exists $k \in I_\Sigma$ and $Y \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,k})$, such that $X, X' \subseteq Y$. Now, from $\mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,k}) = \bigcup_{j \in J^{\Sigma,k}_\Sigma} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,k,\Sigma,j})$, the finiteness of Y and the fact that the union is directed, there must exist $\ell \in J^{\Sigma,k}_\Sigma$, such that $Y \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,k,\Sigma,\ell})$. This establishes the directedness of the collection $\bigcup_{i \in I_\Sigma} \bigcup_{j \in J^{\Sigma,i}_\Sigma} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i,\Sigma,j})$.

Thus $\mathbb{C}^*$ is a closure operator on classes of $\mathbf{F}$-algebraic systems. ∎

The importance of directedly abstract classes of $\mathbf{F}$-algebraic systems stems from the fact that the validity of a quasiequation transfers from directed K-certificates of an $\mathbf{F}$-algebraic system to the $\mathbf{F}$-algebraic system itself.

Lemma 126 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* K *a class of* $\mathbf{F}$*-algebraic systems and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system. If* $\mathcal{A} \in \mathbb{C}^*(\mathsf{K})$*, then* $\mathrm{QEq}(\mathsf{K}) \leq \mathrm{QEq}(\mathcal{A})$.

Proof: Suppose $\mathcal{A} \in \mathbb{C}^*(\mathsf{K})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\langle \vec{\phi} \approx \vec{\psi}, \phi \approx \psi \rangle \in \mathrm{QEq}_\Sigma(\mathsf{K})$, such that $\vec{\phi} \approx \vec{\psi} \subseteq \mathrm{Eq}_\Sigma(\mathcal{A})$. Let $\{\mathcal{A}^{\Sigma,i} : i \in I\} \subseteq \mathsf{K}$ be a directed Σ-K-certificate for $\mathcal{A}$. Then, by definition $\vec{\phi} \approx \vec{\psi} \subseteq \bigcup_{i \in I} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i})$. Since $\vec{\phi} \approx \vec{\psi}$ is finite and the union is directed, there exists $i \in I$, such that $\vec{\phi} \approx \vec{\psi} \subseteq \mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,i})$. But $\mathcal{A}^{\Sigma,i} \in \mathsf{K}$ and $\langle \vec{\phi} \approx \vec{\psi}, \phi \approx \psi \rangle \in \mathrm{QEq}_\Sigma(\mathsf{K})$, whence

$$\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,i}) \subseteq \bigcup_{i \in I} \mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,i}) = \mathrm{Eq}_\Sigma(\mathcal{A}).$$

Therefore, $\langle \vec{\phi} \approx \vec{\psi}, \phi \approx \psi \rangle \in \mathrm{QEq}_\Sigma(\mathcal{A})$ and $\mathrm{QEq}(\mathsf{K}) \leq \mathrm{QEq}(\mathcal{A})$. ∎

Using Lemma 126, we get that all semantic quasivarieties of algebraic systems are directedly abstract.

Corollary 127 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems. If* K *is a quasiequational class (and, hence, a fortiori, if it is an equational class), then it is directedly abstract.*

Proof: Suppose K is a quasiequational class defined by the family of $\mathbf{F}$-quasiequations $Q \leq \mathrm{QEq}(\mathbf{F})$ and let $\mathcal{A} \in \mathbb{C}^*(\mathsf{K})$. Then, by Lemma 126, $\mathrm{QEq}(\mathsf{K}) \leq \mathrm{QEq}(\mathcal{A})$, whence

$$\begin{array}{lll}
\mathcal{A} & \in & \mathrm{AlgSys}(\mathrm{QEq}(\mathcal{A})) \\
 & \subseteq & \mathrm{AlgSys}(\mathrm{QEq}(\mathsf{K})) \\
 & = & \mathrm{AlgSys}(\mathrm{QEq}(\mathrm{AlgSys}(Q))) \\
 & = & \mathrm{AlgSys}(Q) \\
 & = & \mathsf{K}.
\end{array}$$

Thus, K is a directedly abstract class. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{A}^i = \langle \mathbf{A}^i, \langle F^i, \alpha^i \rangle \rangle$, $i \in I$, $\mathbf{F}$-algebraic systems and $\langle H^i, \gamma^i \rangle : \mathcal{A} \to \mathcal{A}^i$, $i \in I$, surjective morphisms. Recall from Section 2.4 that we say that the collection

$$\langle H^i, \gamma^i \rangle : \mathcal{A} \to \mathcal{A}^i, \quad i \in I,$$

is a **subdirect intersection** if

$$\bigcap_{i \in I} \mathrm{Ker}(\langle H^i, \gamma^i \rangle) = \Delta^{\mathcal{A}}.$$

Given a class K of $\mathbf{F}$-algebraic systems, we write $\mathcal{A} \in \overset{\triangleleft}{\Pi}(\mathsf{K})$ in case there exists a subdirect intersection $\{\langle H^i, \gamma^i\rangle : \mathcal{A} \to \mathcal{A}^i, i \in I\}$, with $\mathcal{A}^i \in \mathsf{K}$, for all $i \in I$. If $\overset{\triangleleft}{\Pi}(\mathsf{K}) = \mathsf{K}$, we say that K is **closed under subdirect intersections**.

The following lemma provides an alternative characterization of the concept of subdirect intersection.

Lemma 128 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, $\mathcal{A}^i = \langle \mathbf{A}^i, \langle F^i, \alpha^i\rangle\rangle$, $i \in I$, $\mathbf{F}$*-algebraic systems and* $\{\langle H^i, \gamma^i\rangle : \mathcal{A} \to \mathcal{A}^i : i \in I\}$ *a collection of morphisms. The collection* $\{\langle H^i, \gamma^i\rangle : i \in I\}$ *is a subdirect intersection if and only if* $\mathrm{Ker}(\langle F, \alpha\rangle) = \bigcap_{i\in I} \mathrm{Ker}(\langle F^i, \alpha^i\rangle)$.

Proof: Suppose, first, that $\{\langle H^i, \gamma^i\rangle : i \in I\}$ is a subdirect intersection and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{lll}\langle \phi, \psi\rangle \in \mathrm{Ker}_\Sigma(\langle F, \alpha\rangle) & \text{iff} & \alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)\\ & \text{iff} & \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in \Delta^{\mathcal{A}}_{F(\Sigma)}\\ & \text{iff} & \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in \bigcap_{i\in I} \mathrm{Ker}_\Sigma(\langle H^i, \gamma^i\rangle)\\ & \text{iff} & \gamma^i_{F(\Sigma)}(\alpha_\Sigma(\phi)) = \gamma^i_{F(\Sigma)}(\alpha_\Sigma(\psi)), i \in I\\ & \text{iff} & \alpha^i_\Sigma(\phi) = \alpha^i_\Sigma(\psi), i \in I\\ & \text{iff} & \langle \phi, \psi\rangle \in \bigcap_{i\in I} \mathrm{Ker}_\Sigma(\langle F^i, \alpha^i\rangle).\end{array}$$

The reverse relies on the surjectivity of $\langle F, \alpha\rangle$. Suppose $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then we get

$$\begin{array}{lll}\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in \Delta^{\mathcal{A}}_{F(\Sigma)} & \text{iff} & \langle \phi, \psi\rangle \in \mathrm{Ker}_\Sigma(\langle F, \alpha\rangle)\\ & \text{iff} & \langle \phi, \psi\rangle \in \bigcap_{i\in I} \mathrm{Ker}_\Sigma(\langle F^i, \alpha^i\rangle)\\ & \text{iff} & \alpha^i_\Sigma(\phi) = \alpha^i_\Sigma(\psi), i \in I\\ & \text{iff} & \gamma^i_{F(\Sigma)}(\alpha_\Sigma(\phi)) = \gamma^i_{F(\Sigma)}(\alpha_\Sigma(\psi)), i \in I\\ & \text{iff} & \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in \bigcap_{i\in I} \mathrm{Ker}_{F(\Sigma)}(\langle H^i, \gamma^i\rangle).\end{array}$$

Thus, by the surjectivity of $\langle F, \alpha\rangle$ we get that $\Delta^{\mathcal{A}} = \bigcap_{i\in I} \mathrm{Ker}(\langle H^i, \gamma^i\rangle)$. ∎

It is not difficult to verify that the subdirect intersection operator is also a closure operator on classes of $\mathbf{F}$-algebraic systems.

Proposition 129 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system. Then the operator* $\overset{\triangleleft}{\Pi}$ *on classes of* $\mathbf{F}$*-algebraic systems is a closure operator.*

Proof: Suppose K is a class of $\mathbf{F}$-algebraic systems.

- If $\mathcal{A} \in \mathsf{K}$, then $\{\langle I, \iota\rangle : \mathcal{A} \to \mathcal{A}\}$, where $\langle I, \iota\rangle : \mathcal{A} \to \mathcal{A}$ is the identity morphism, is a subdirect intersection family. Thus, we get that $\mathcal{A} \in \overset{\triangleleft}{\Pi}(\mathsf{K})$. Hence $\mathsf{K} \subseteq \overset{\triangleleft}{\Pi}(\mathsf{K})$ and $\overset{\triangleleft}{\Pi}$ is inflationary;

- It is obvious that $\overset{\triangleleft}{\mathbb{\Pi}}$ is monotonic;

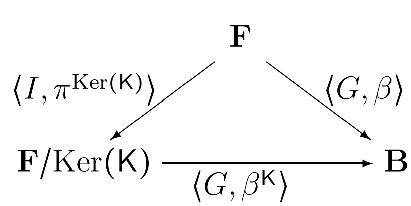
- Suppose that $\mathcal{A} \in \overset{\triangleleft}{\mathbb{\Pi}}(\overset{\triangleleft}{\mathbb{\Pi}}(\mathsf{K}))$. Then, there exists a subdirect intersection family $\{\langle H^i, \gamma^i\rangle : \mathcal{A} \to \mathcal{A}^i, i \in I\}$, with $\mathcal{A}^i \in \overset{\triangleleft}{\mathbb{\Pi}}(\mathsf{K})$, for all $i \in I$. Therefore, for each $i \in I$, there exists a sibdirect intersection family $\{\langle H^{ij}, \gamma^{ij}\rangle : \mathcal{A}^i \to \mathcal{A}^{ij}, j \in J_i\}$, with $\mathcal{A}^{ij} \in \mathsf{K}$, for all $i \in I$ and all $j \in J_i$. Consider
$$\{\langle H^{ij}, \gamma^{ij}\rangle \circ \langle H^i, \gamma^i\rangle : \mathcal{A} \to \mathcal{A}^{ij}, i \in I, j \in J_i\}.$$
It is a subdirect intersection family, since
$$\begin{aligned}
&\textstyle\bigcap_{i\in I, j\in J_i} \mathrm{Ker}(\langle H^{ij}, \gamma^{ij}\rangle \circ \langle H^i, \gamma^i\rangle)\\
&\quad = \textstyle\bigcap_{i\in I, j\in J_i} (\gamma^{ij} \circ \gamma^i)^{-1}(\Delta^{\mathcal{A}^{ij}})\\
&\quad = \textstyle\bigcap_{i\in I, j\in J_i} (\gamma^i)^{-1}((\gamma^{ij})^{-1}(\Delta^{\mathcal{A}^{ij}}))\\
&\quad = \textstyle\bigcap_{i\in I} (\gamma^i)^{-1}(\bigcap_{j\in J_i} (\gamma^{ij})^{-1}(\Delta^{\mathcal{A}^{ij}}))\\
&\quad = \textstyle\bigcap_{i\in I} (\gamma^i)^{-1}(\Delta^{\mathcal{A}^i})\\
&\quad = \Delta^{\mathcal{A}}.
\end{aligned}$$
Since $\mathcal{A}^{ij} \in \mathsf{K}$, for all $i \in I$, $j \in J_i$, we get that $\overset{\triangleleft}{\mathbb{\Pi}}(\overset{\triangleleft}{\mathbb{\Pi}}(\mathsf{K})) \subseteq \overset{\triangleleft}{\mathbb{\Pi}}(\mathsf{K})$ and $\overset{\triangleleft}{\mathbb{\Pi}}$ is idempotent.

Thus, $\overset{\triangleleft}{\mathbb{\Pi}}$ is a closure operator. ∎

A key property concerning subdirect intersections, which is very useful in applying the concept, is given in the following

Lemma 130 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and consider a class* $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$. *The class of morphisms*
$$\langle G, \beta^{\mathsf{K}}\rangle : \mathcal{F}/\bigcap_{\mathcal{B}\in\mathsf{K}} \mathrm{Ker}(\langle G, \beta\rangle) \to \mathcal{B}, \quad \mathcal{B} = \langle \mathbf{B}, \langle G, \beta\rangle\rangle \in \mathsf{K},$$
where, for all $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\beta^{\mathsf{K}}_\Sigma(\phi/\bigcap_{\mathcal{B}\in\mathsf{K}} \mathrm{Ker}_\Sigma(\langle G, \beta\rangle)) = \beta_\Sigma(\phi),$$
forms a subdirect intersection.

Proof: It is not difficult to see that β^{K} is well defined and forms a natural transformation. Moreover, $\langle G, \beta^{\mathsf{K}}\rangle$ is an $\mathbf{F}$-morphism. Letting $\mathrm{Ker}(\mathsf{K}) = \bigcap_{\mathcal{B}\in\mathsf{K}} \mathrm{Ker}(\langle G, \beta\rangle)$, we have, by definition, the following commutative triangle.



To show that the displayed family forms a subdirect intersection, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then, we get

$$\begin{array}{ll}\langle \phi/\mathrm{Ker}_\Sigma(\mathsf{K}), \psi/\mathrm{Ker}_\Sigma(\mathsf{K})\rangle \in \bigcap_{\mathcal{B}\in\mathsf{K}} \mathrm{Ker}_\Sigma(\langle G, \beta^{\mathsf{K}}\rangle) & \\ \quad\text{iff} \quad \beta^{\mathsf{K}}_\Sigma(\phi/\mathrm{Ker}_\Sigma(\mathsf{K})) = \beta^{\mathsf{K}}_\Sigma(\psi/\mathrm{Ker}_\Sigma(\mathsf{K})), \quad \mathcal{B} \in \mathsf{K}, & \\ \quad\text{iff} \quad \beta_\Sigma(\phi) = \beta_\Sigma(\psi), \quad \mathcal{B} \in \mathsf{K}, & \\ \quad\text{iff} \quad \phi/\mathrm{Ker}_\Sigma(\mathsf{K}) = \psi/\mathrm{Ker}_\Sigma(\mathsf{K}). & \end{array}$$

Thus, $\bigcap_{\mathcal{B}\in\mathsf{K}} \mathrm{Ker}(\langle G, \beta^{\mathsf{K}}\rangle) = \Delta^{\mathcal{F}/\mathrm{Ker}(\mathsf{K})}$, showing that

$$\langle G, \beta^{\mathsf{K}}\rangle : \mathcal{F}/\bigcap_{\mathcal{B}\in\mathsf{K}} \mathrm{Ker}(\langle G, \beta\rangle) \to \mathcal{B}, \quad \mathcal{B} = \langle \mathbf{B}, \langle G, \beta\rangle\rangle \in \mathsf{K},$$

constitutes indeed a subdirect intersection. ■

Finally, we show that every semantic guasivariety of $\mathbf{F}$-algebraic systems is closed under subdirect intersections.

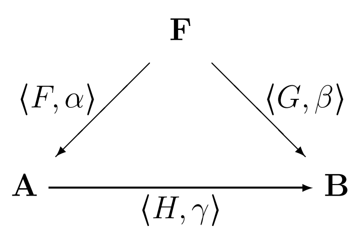
Proposition 131 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$. *If* $\mathsf{K} = \mathbb{G}^{\mathsf{Sem}}(\mathsf{K})$, *then* $\overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K}) \subseteq \mathsf{K}$.

Proof: Assume that $\mathsf{K} = \mathbb{G}^{\mathsf{Sem}}(\mathsf{K})$. Let $X = \mathrm{GEq}(\mathsf{K})$. Assume that $\mathcal{A} \in \overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K})$ and $\Sigma \in |\mathbf{Sign}^\flat|$, $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in X_\Sigma$, such that $\mathcal{A} \vDash_\Sigma \vec{\phi} \approx \vec{\psi}$, i.e., $\vec{\phi} \approx \vec{\psi} \subseteq \mathrm{Eq}_\Sigma(\mathcal{A})$. Since $\mathcal{A} \in \overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K})$, there exists a subdirect intersection

$$\langle H^i, \gamma^i\rangle : \mathcal{A} \to \mathcal{A}^i, \quad i \in I,$$

such that $\mathcal{A}^i \in \mathsf{K}$, for all $i \in I$. Hence, we get $\vec{\phi} \approx \vec{\psi} \subseteq \mathrm{Eq}_\Sigma(\mathcal{A}^i)$, $i \in I$. Now, since $\mathcal{A}^i \in \mathsf{K}$ and $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in X_\Sigma = \mathrm{GEq}_\Sigma(\mathsf{K})$, we conclude that $\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathcal{A}^i)$, for all $i \in I$. Therefore, $\phi \approx \psi \in \bigcap_{i\in I} \mathrm{Eq}_\Sigma(\mathcal{A}^i) = \mathrm{Eq}_\Sigma(\mathcal{A})$, the latter by the definition of subdirect intersection and Lemma 128. Therefore, $\mathcal{A} \vDash_\Sigma \vec{\phi} \approx \vec{\psi} \to \phi \approx \psi$. This shows that $\mathcal{A} \in \mathrm{AlgSys}(X) = \mathrm{AlgSys}(\mathrm{GEq}(\mathsf{K})) = \mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$. We conclude that $\overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K}) \subseteq \mathsf{K}$, i.e., K is closed under subdirect intersections. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta\rangle\rangle$ $\mathbf{F}$-algebraic systems and $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism.



In this case we say $\mathcal{B}$ is a **morphic image** of $\mathcal{A}$. Given a class K of $\mathbf{F}$-algebraic systems, we write $\mathcal{B} \in \mathbb{H}(\mathsf{K})$ in case there exists a surjective morphism $\langle H, \gamma\rangle$:

$\mathcal{A} \to \mathcal{B}$, with $\mathcal{A} \in \mathsf{K}$. If $\mathbb{H}(\mathsf{K}) = \mathsf{K}$, we say that K is **closed under morphic images**.

It is straightforward to verify that $\mathbb{H}$ is a closure operator on classes of $\mathbf{F}$-algebraic systems.

Proposition 132 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system. Then the operator* $\mathbb{H}$ *on classes of* $\mathbf{F}$*-algebraic systems is a closure operator.*

Proof: Let K be a class of $\mathbf{F}$-algebraic systems. If $\mathcal{A} \in \mathsf{K}$, then, using again the identity $\langle I, \iota\rangle : \mathcal{A} \to \mathcal{A}$, we see that $\mathcal{A} \in \mathbb{H}(\mathsf{K})$, and, hence, $\mathbb{H}$ is inflationary. It is again obvious that it is monotonic. Finally, if $\mathcal{A} \in \mathbb{H}(\mathbb{H}(\mathsf{K}))$, then, there exists a surjective morphism $\langle G, \beta\rangle : \mathcal{A}' \to \mathcal{A}$, with $\mathcal{A}' \in \mathbb{H}(\mathsf{K})$, whence, there also exists a surjective morphism $\langle H, \gamma\rangle : \mathcal{A}'' \to \mathcal{A}'$, with $\mathcal{A}'' \in \mathsf{K}$. Now the surjective morphism $\langle G, \beta\rangle \circ \langle H, \gamma\rangle : \mathcal{A}'' \to \mathcal{A}$ witnesses the fact that $\mathcal{A} \in \mathbb{H}(\mathsf{K})$. Therefore, $\mathbb{H}(\mathbb{H}(\mathsf{K})) \subseteq \mathbb{H}(\mathsf{K})$, and $\mathbb{H}$ is idempotent. Thus, $\mathbb{H}$ is a closure operator. ■

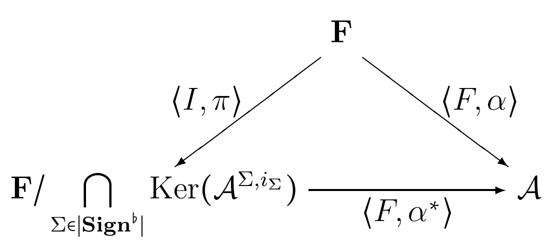
We show, next, that, if a class K of $\mathbf{F}$-algebraic systems is closed under subdirect intersections and morphic images, then it is also closed under directed K-certifications.

Proposition 133 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* K *be a class of* $\mathbf{F}$*-algebraic systems. Then* $\mathbb{C}^*(\mathsf{K}) \subseteq \mathbb{H}(\overset{\triangleleft}{\mathbb{IP}}(\mathsf{K}))$.

Proof: Suppose $\mathcal{A} \in \mathbb{C}^*(\mathsf{K})$. Then, by definition, for all $\Sigma \in |\mathbf{Sign}^\flat|$, there exists a collection $\{\mathcal{A}^{\Sigma,i} : i \in I_\Sigma\} \subseteq \mathsf{K}$, such that $\bigcup_{i\in I_\Sigma} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i})$ is directed and $\mathrm{Ker}_\Sigma(\mathcal{A}) = \bigcup_{i\in I_\Sigma} \mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,i})$. Fix, for every $\Sigma \in |\mathbf{Sign}^\flat|$, an $i_\Sigma \in I_\Sigma$ and consider the family of morphisms

$$\langle H^{\Sigma,i_\Sigma}, \gamma^{\Sigma,i_\Sigma}\rangle : \mathcal{F}/\bigcap_{\Sigma\in|\mathbf{Sign}^\flat|} \mathrm{Ker}(\mathcal{A}^{\Sigma,i_\Sigma}) \to \mathcal{A}^{\Sigma,i_\Sigma}, \quad \Sigma \in |\mathbf{Sign}^\flat|.$$

By Lemma 130, it constitutes a subdirect intersection, whence, since $\mathcal{A}^{\Sigma,i_\Sigma} \in \mathsf{K}$, for all $\Sigma \in |\mathbf{Sign}^\flat|$, we get $\mathcal{F}/\bigcap_{\Sigma\in|\mathbf{Sign}^\flat|} \mathrm{Ker}(\mathcal{A}^{\Sigma,i_\Sigma}) \in \overset{\triangleleft}{\mathbb{IP}}(\mathsf{K})$. Now it is not difficult to see that there exists a morphism $\langle F, \alpha^*\rangle : \mathcal{F}/\bigcap_{\Sigma\in|\mathbf{Sign}^\flat|} \mathrm{Ker}(\mathcal{A}^{\Sigma,i_\Sigma}) \to \mathcal{A}$, such that the following diagram commutes



The natural transformation α^* is defined, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma')$, by

$$\alpha^*_{\Sigma'}(\phi/\bigcap_{\Sigma\in|\mathbf{Sign}^\flat|} \mathrm{Ker}_{\Sigma'}(\mathcal{A}^{\Sigma,i_\Sigma})) = \alpha_{\Sigma'}(\phi).$$

It is well-defined, since, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma')$, we have

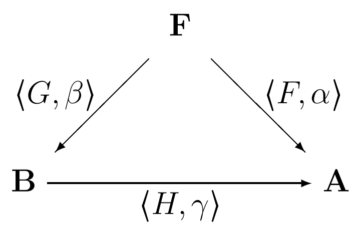
$$\begin{array}{lll}\langle\phi,\psi\rangle \in \bigcap_{\Sigma\in|\mathbf{Sign}^\flat|}\mathrm{Ker}_{\Sigma'}(\mathcal{A}^{\Sigma,i_\Sigma}) & \text{implies} & \langle\phi,\psi\rangle \in \mathrm{Ker}_{\Sigma'}(\mathcal{A}^{\Sigma',i_{\Sigma'}}) \\ & \text{implies} & \langle\phi,\psi\rangle \in \bigcup_{i\in I_{\Sigma'}}\mathrm{Ker}_{\Sigma'}(\mathcal{A}^{\Sigma',i}) \\ & \text{implies} & \langle\phi,\psi\rangle \in \mathrm{Ker}_{\Sigma'}(\mathcal{A}).\end{array}$$

Taking into account the surjectivity of $\langle F,\alpha\rangle$, we conclude that $\mathcal{A} \in \mathbb{H}(\overset{\triangleleft}{\mathbb{P}}(\mathsf{K}))$. Therefore, $\mathbb{C}^*(\mathsf{K}) \subseteq \mathbb{H}(\overset{\triangleleft}{\mathbb{P}}(\mathsf{K}))$. ■

Finally, it is not difficult to see that semantic varieties are closed under morphic images.

Proposition 134 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$*. If* $\mathsf{K} = \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$*, then* $\mathbb{H}(\mathsf{K}) \subseteq \mathsf{K}$*.*

Proof: Assume that $\mathsf{K} = \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. Let $X = \mathrm{Eq}(\mathsf{K})$. Assume that $\mathcal{A} = \langle\mathbf{A}, \langle F,\alpha\rangle\rangle \in \mathbb{H}(\mathsf{K})$ and $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \approx \psi \in X_\Sigma$. Since $\mathcal{A} \in \mathbb{H}(\mathsf{K})$, there exists $\mathcal{B} = \langle\mathbf{B}, \langle G,\beta\rangle\rangle \in K$ and $\langle H,\gamma\rangle : \mathcal{B} \to \mathcal{A}$:



Since $\mathcal{B} \in \mathsf{K}$ and $\phi \approx \psi \in X_\Sigma = \mathrm{Eq}_\Sigma(\mathsf{K})$, we conclude that $\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathcal{B})$. Therefore, $\beta_\Sigma(\phi) = \beta_\Sigma(\psi)$. But this gives $\gamma_{G(\Sigma)}(\beta_\Sigma(\phi)) = \gamma_{G(\Sigma)}(\beta_\Sigma(\psi))$ or, equivalently, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. Therefore, $\mathcal{A} \vDash_\Sigma \phi \approx \psi$. This shows that $\mathcal{A} \in \mathrm{AlgSys}(X) = \mathrm{AlgSys}(\mathrm{Eq}(\mathsf{K})) = \mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$. We conclude that $\mathbb{H}(\mathsf{K}) \subseteq \mathsf{K}$, i.e., K is closed under morphic images. ■

We are now ready to provide alternative characterizations of equational, quasiequational and guasiequational classes of **F**-algebraic systems. Namely, we show that a class of **F**-algebraic systems is:

- a guasiequational class if and only if it is abstract and closed under subdirect intersections;
- a quasiequational class if and only if it is directedly abstract and closed under subdirect intersections;
- an equational class if and only if it is closed under subdirect intersections and morphic images.

Theorem 135 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* K *a class of* **F***-algebraic systems.* K *is a guasiequational class if and only if it is abstract and closed under subdirect intersections.*

Proof: If K is a guasiequational class, then it is abstract by Corollary 124 and it is closed under subdirect intersections by Proposition 131.

Assume, conversely, that K is abstract and closed under subdirect intersections and set $G = \mathrm{GEq}(\mathsf{K})$. Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle \in \mathrm{AlgSys}(\mathbf{F})$, such that $G \leq \mathrm{GEq}(\mathcal{A})$. Let $\Sigma \in |\mathbf{Sign}^\flat|$, and $\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathbf{F})$, such that $\phi \approx \psi \notin \mathrm{Eq}_\Sigma(\mathcal{A})$, i.e., such that $\alpha_\Sigma(\phi) \neq \alpha_\Sigma(\psi)$. Thus, by definition, the guasiequation

$$\langle \mathrm{Eq}_\Sigma(\mathcal{A}), \phi \approx \psi\rangle \notin \mathrm{GEq}_\Sigma(\mathcal{A}).$$

Therefore, since $G \leq \mathrm{GEq}(\mathcal{A})$, $\langle \mathrm{Eq}_\Sigma(\mathcal{A}), \phi \approx \psi\rangle \notin \mathrm{GEq}_\Sigma(\mathsf{K})$. Hence, for every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \approx \psi \notin \mathrm{Eq}_\Sigma(\mathcal{A})$, there exists $\mathcal{K}^{\langle \Sigma, \phi\approx\psi\rangle} \in \mathsf{K}$, such that $\mathrm{Eq}_\Sigma(\mathcal{A}) \subseteq \mathrm{Eq}_\Sigma(\mathcal{K}^{\langle \Sigma, \phi\approx\psi\rangle})$, but $\phi \approx \psi \notin \mathrm{Eq}_\Sigma(\mathcal{K}^{\langle \Sigma, \phi\approx\psi\rangle})$. We conclude that

$$\mathrm{Eq}_\Sigma(\mathcal{A}) = \bigcap\{\mathrm{Eq}_\Sigma(\mathcal{K}^{\langle \Sigma, \phi\approx\psi\rangle}) : \phi \approx \psi \notin \mathrm{Eq}_\Sigma(\mathcal{A})\}.$$

Let, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathsf{K}^\Sigma = \{\mathcal{K}^{\langle \Sigma, \phi\approx\psi\rangle} : \phi \approx \psi \notin \mathrm{Eq}_\Sigma(\mathcal{A})\}.$$

- Since, by hypothesis, K is closed under subdirect intersections, and, by Lemma 130,
$$\{\langle F^{\mathcal{K}}, \alpha^{\mathcal{K}}\rangle : \mathcal{F}/\mathrm{Ker}(\mathsf{K}^\Sigma) \to \mathcal{K}, \mathcal{K} \in \mathsf{K}^\Sigma\}$$
is a subdirect intersection, we get that $\mathcal{F}/\mathrm{Ker}(\mathsf{K}^\Sigma) \in \mathsf{K}$.

- Since, for all $\Sigma \in |\mathbf{Sign}^\flat|$,
$$\mathrm{Ker}_\Sigma(\mathcal{A}) = \mathrm{Ker}_\Sigma(\mathsf{K}^\Sigma) = \mathrm{Ker}_\Sigma(\mathcal{F}/\mathrm{Ker}(\mathsf{K}^\Sigma))$$
and $\mathcal{F}/\mathrm{Ker}(\mathsf{K}^\Sigma) \in \mathsf{K}$, $\mathcal{A} \in \mathbb{C}(\mathsf{K})$. Since K is abstract, we conclude that $\mathcal{A} \in \mathsf{K}$.

Therefore, K is a guasiequational class of $\mathbf{F}$-algebraic systems. ■

Theorem 136 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems.* K *is a quasiequational class if and only if it is directedly abstract and closed under subdirect intersections.*

Proof: If K is a quasiequational class, then it is directedly abstract by Corollary 127 and it is closed under subdirect intersections by Proposition 131.

Conversely, suppose that $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$, such that $\mathbb{C}^*(\mathsf{K}) \subseteq \mathsf{K}$ and $\overset{\triangleleft}{\mathbb{I}\mathbb{I}}(\mathsf{K}) \subseteq \mathsf{K}$. It suffices to show that $\mathsf{K} = \mathrm{AlgSys}(\mathrm{QEq}(\mathsf{K}))$. The left to right inclusion always holds. For the converse, consider $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle \in \mathrm{AlgSys}(\mathrm{QEq}(\mathsf{K}))$. For all $\Sigma \in |\mathbf{Sign}^\flat|$, all $X \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A})$ and all $\phi \approx \psi \notin \mathrm{Eq}_\Sigma(\mathcal{A})$, we consider the $\mathbf{F}$-quasiequation

$$q^{\Sigma, X, \phi\approx\psi} := X \to \phi \approx \psi.$$

Since $\mathcal{A} \vDash_\Sigma \mathrm{Eq}_\Sigma(\mathcal{A})$ and $\mathcal{A} \nvDash_\Sigma \phi \approx \psi$, we get that $q^{\Sigma,X,\phi\approx\psi} \notin \mathrm{QEq}_\Sigma(\mathcal{A})$. Thus, since $\mathcal{A} \in \mathrm{AlgSys}(\mathrm{QEq}(\mathsf{K}))$, we infer that $q^{\Sigma,X,\phi\approx\psi} \notin \mathrm{QEq}_\Sigma(\mathsf{K})$. Therefore, there exists $\mathcal{A}^{\Sigma,X,\phi\approx\psi} \in \mathsf{K}$, such that $\mathcal{A}^{\Sigma,X,\phi\approx\psi} \nvDash_\Sigma q^{\Sigma,X,\phi\approx\psi}$, i.e.,

$$\mathcal{A}^{\Sigma,X,\phi\approx\psi} \vDash_\Sigma X \quad \text{and} \quad \mathcal{A}^{\Sigma,X,\phi\approx\psi} \nvDash_\Sigma \phi \approx \psi.$$

Let, for all $X \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A})$,

$$\mathsf{A}^{\Sigma,X} = \{\mathcal{A}^{\Sigma,X,\phi\approx\psi} : \phi \approx \psi \notin \mathrm{Eq}_\Sigma(\mathcal{A})\}.$$

Define, for all $X \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A})$,

$$\mathcal{A}^{\Sigma,X} := \mathcal{F}/\mathrm{Ker}(\mathsf{A}^{\Sigma,X}) = \mathcal{F}/\bigcap_{\phi\approx\psi\notin\mathrm{Eq}_\Sigma(\mathcal{A})} \mathrm{Ker}(\mathcal{A}^{\Sigma,X,\phi\approx\psi}).$$

By Proposition 130, for all $X \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A})$, $\mathcal{A}^{\Sigma,X} \in \overset{\triangleleft}{\mathbb{P}}(\mathsf{K}) = \mathsf{K}$. Consequently, it suffices to show the following:

- $\bigcup_{X\in\mathrm{Eq}^\omega_\Sigma(\mathcal{A})} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,X})$ is directed;
- $\mathrm{Ker}_\Sigma(\mathcal{A}) = \bigcup_{X\in\mathrm{Eq}^\omega_\Sigma(\mathcal{A})} \mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,X})$.

Suppose, first, that $E \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,X})$ and $E' \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,X'})$, for some $X, X' \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A})$. Then, by construction of $\mathcal{A}^{\Sigma,X}$ and $\mathcal{A}^{\Sigma,X'}$, we get that $E, E' \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A})$. Therefore, $E \cup E' \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,E\cup E'})$ and, hence, $\bigcup_{X\in\mathrm{Eq}^\omega_\Sigma(\mathcal{A})} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,X})$ is indeed directed.

Finally, note that, by construction, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{Ker}_\Sigma(\mathcal{A}) = \bigcup_{X\in\mathrm{Eq}^\omega_\Sigma(\mathcal{A})} \mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,X}).$$

Indeed, for all $\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathbf{F})$,

- if $\phi \approx \psi \in \mathrm{Ker}_\Sigma(\mathcal{A})$, then, $\phi \approx \psi \in \mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,\{\phi\approx\psi\}})$, whence $\phi \approx \psi \in \bigcup_{X\in\mathrm{Eq}^\omega_\Sigma(\mathcal{A})} \mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,X})$.
- if $\phi \approx \psi \notin \mathrm{Ker}_\Sigma(\mathcal{A})$, then, by construction, for all $X \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A})$, $\phi \approx \psi \notin \mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,X})$. Therefore, $\phi \approx \psi \notin \bigcup_{X\in\mathrm{Eq}^\omega_\Sigma(\mathcal{A})} \mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,X})$.

Since, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $X \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A})$, $\mathcal{A}^{\Sigma,X} \in \mathsf{K}$, we get, by the definition of $\mathbb{C}^*$ and the two properties just proven, that $\mathcal{A} \in \mathbb{C}^*(\mathsf{K}) = \mathsf{K}$. Thus, K is a quasiequational class of $\mathbf{F}$-algebraic systems. ∎

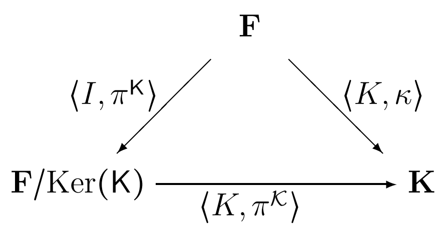
Theorem 137 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and K a class of $\mathbf{F}$-algebraic systems. K is an equational class if and only if it is closed under subdirect intersections and morphic images.*

Proof: If K is an equational class, then it is closed under subdirect intersections by Proposition 131 and under morphic images by Proposition 134.

Suppose, conversely, that K is a class of $\mathbf{F}$-algebraic systems that is closed under subdirect intersections and morphic images. Set $E = \mathrm{Eq}(\mathsf{K})$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system, such that $E \leq \mathrm{Eq}(\mathcal{A})$. Consider, for all $\mathcal{K} = \langle \mathbf{K}, \langle K, \kappa \rangle \rangle \in \mathsf{K}$, the mapping

$$\langle K, \pi^{\mathcal{K}} \rangle : \mathbf{F}/\mathrm{Ker}(\mathsf{K}) \to \mathbf{K}$$

defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,



$$\pi^{\mathcal{K}}_\Sigma(\phi/\mathrm{Ker}_\Sigma(\mathsf{K})) = \kappa_\Sigma(\phi).$$

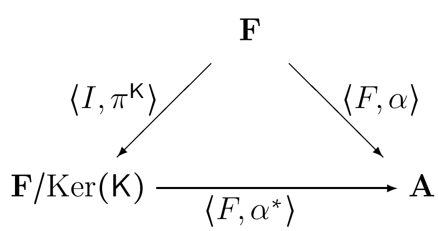
By Lemma 130, the collection

$$\{\langle K, \pi^{\mathcal{K}} \rangle : \mathcal{F}/\mathrm{Ker}(\mathsf{K}) \to \mathcal{K}, \mathcal{K} = \langle \mathbf{K}, \langle K, \kappa \rangle \rangle \in \mathsf{K}\}$$

forms a subdirect intersection. Since all codomains are in K and K is closed under subdirect intersections, we get $\mathcal{F}/\mathrm{Ker}(\mathsf{K}) \in \mathsf{K}$. Now consider the morphism

$$\langle F, \alpha^* \rangle : \mathcal{F}/\mathrm{Ker}(\mathsf{K}) \to \mathcal{A},$$

given, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, by



$$\alpha^*_\Sigma(\phi/\mathrm{Ker}_\Sigma(\mathsf{K})) = \alpha_\Sigma(\phi).$$

It is well defined, since, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, if $\langle \phi, \psi \rangle \in \mathrm{Ker}_\Sigma(\mathsf{K})$, then, by hypothesis, $\langle \phi, \psi \rangle \in \mathrm{Eq}_\Sigma(\mathcal{A})$ and, hence, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. Moreover, since $\langle F, \alpha \rangle$ is surjective, so is $\langle F, \alpha^* \rangle$. Since $\mathcal{F}/\mathrm{Ker}(\mathsf{K}) \in \mathsf{K}$ and K is closed under morphic images, we conclude that $\mathcal{A} \in \mathsf{K}$. Therefore, K is an equational class of $\mathbf{F}$-algebraic systems. ■

We prove, next, the following result to the effect that, for any guasiequational class K of $\mathbf{F}$-algebraic systems, the theory families of the equational structure $\mathcal{Q}^{\mathsf{K}} = \langle \mathbf{F}, D^{\mathsf{K}} \rangle$ coincide with the K-congruence systems on $\mathcal{F}$.

Corollary 138 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems. If* K *is a guasiequational class, then*

$$\mathrm{ThFam}(\mathcal{Q}^{\mathsf{K}}) = \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F}).$$

Proof: We can rely on preceding results, but we also give a direct proof.

Since K is a guasiequational class, by Theorem 135, it is abstract and closed under subdirect intersections. Since any guasiequational class also contains a trivial $\mathbf{F}$-algebraic system, we conclude, by Theorem 32, that $\mathrm{ThFam}(\mathcal{Q}^{\mathsf{K}}) = \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$.

Next, we provide a direct proof of the same result. Suppose K is a guasiequational class of $\mathbf{F}$-algebraic systems.

Let $\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ and $\phi \approx \psi \in D^{\mathsf{K}}_\Sigma(\theta_\Sigma)$. Then $\langle \theta_\Sigma, \phi \approx \psi \rangle \in \mathrm{GEq}_\Sigma(\mathsf{K})$. Thus, since, by hypothesis, $\mathcal{F}/\theta \in \mathsf{K}$, $\mathcal{F}/\theta \vDash_\Sigma \langle \theta_\Sigma, \phi \approx \psi \rangle$. But, obviously, $\mathcal{F}/\theta \vDash_\Sigma \theta_\Sigma$. Therefore, we get $\mathcal{F}/\theta \vDash_\Sigma \phi \approx \psi$, or, equivalently, $\langle \phi, \psi \rangle \in \theta_\Sigma$. We conclude that $D^{\mathsf{K}}(\theta) = \theta$ and, hence, $\theta \in \mathrm{ThFam}(\mathcal{Q}^{\mathsf{K}})$.

Assume, conversely, that $\theta \in \mathrm{ThFam}(\mathcal{Q}^{\mathsf{K}})$ and consider $\Sigma \in |\mathbf{Sign}^\flat|$, $\vec{\phi}$, $\vec{\psi}$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \vec{\phi} \approx \vec{\psi}, \phi \approx \psi \rangle \in \mathrm{GEq}_\Sigma(\mathsf{K})$ and $\mathcal{F}/\theta \vDash_\Sigma \vec{\phi} \approx \vec{\psi}$. Then, $\langle \phi_i, \psi_i \rangle \in \theta_\Sigma$, for all $i \in I$. Since $\langle \vec{\phi} \approx \vec{\psi}, \phi \approx \psi \rangle \in \mathrm{GEq}_\Sigma(\mathsf{K})$ and $\theta \in \mathrm{ThFam}(\mathcal{Q}^{\mathsf{K}})$, we get $\langle \phi, \psi \rangle \in \theta_\Sigma$. Hence, $\mathcal{F}/\theta \vDash_\Sigma \phi \approx \psi$. We conclude, taking into account the fact that K is a guasiequational class, that $\mathcal{F}/\theta \in \mathrm{AlgSys}(\mathrm{GEq}(\mathsf{K})) = \mathsf{K}$. Thus, $\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$. ∎

We obtain, as a corollary, that, if the relative equational consequences of two semantic guasivarieties are identical, then the two guasivarieties coincide.

Proposition 139 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *and* K' *semantic guasivarieties of* $\mathbf{F}$*-algebraic systems, such that* $D^{\mathsf{K}} = D^{\mathsf{K}'}$. *Then* $\mathsf{K} = \mathsf{K}'$.

Proof: Let $\mathcal{A} \in \mathsf{K}$ and consider $\Sigma \in |\mathbf{Sign}^\flat|$, $\vec{\phi}, \vec{\psi}, \phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$\mathsf{K}' \vDash_\Sigma \langle \vec{\phi} \approx \vec{\psi}, \phi \approx \psi \rangle.$$

This is equivalent to $\phi \approx \psi \in D^{\mathsf{K}'}_\Sigma(\vec{\phi} \approx \vec{\psi})$. By hypothesis, we get $\phi \approx \psi \in D^{\mathsf{K}}_\Sigma(\vec{\phi} \approx \vec{\psi})$, i.e., $\mathsf{K} \vDash_\Sigma \langle \vec{\phi} \approx \vec{\psi}, \phi \approx \psi \rangle$. Since $\mathcal{A} \in \mathsf{K}$, $\mathcal{A} \vDash_\Sigma \langle \vec{\phi} \approx \vec{\psi}, \phi \approx \psi \rangle$. This shows that $\mathrm{GEq}(\mathsf{K}') \leq \mathrm{GEq}(\mathcal{A})$ and, hence $\mathcal{A} \in \mathbb{G}^{\mathsf{Sem}}(\mathsf{K}') = \mathsf{K}'$, the latter equation by the assumption that K' is a semantic guasivariety. We conclude that $\mathsf{K} \subseteq \mathsf{K}'$. By symmetry, we get $\mathsf{K} = \mathsf{K}'$. ∎

These results allow us to obtain another round of different characterizations of equational, quasiequational and guasiequational classes of $\mathbf{F}$-algebraic systems.

To provide the characterization of guasiequational classes, we need, first, some technical lemmas.

Lemma 140 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* K *an abstract class of* $\mathbf{F}$*-algebraic systems.* K *is closed under subdirect intersections if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ *is closed under intersection.*

Proof: Let K be an abstract class of $\mathbf{F}$-algebraic systems. Suppose, first, that K is closed under subdirect intersections and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $\{\theta^i : i \in I\} \subseteq \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$. Then, by definition, $\mathcal{A}/\theta^i \in \mathsf{K}$, for all $i \in I$. Let, for all $i \in I$,

$$\langle I, \rho^i \rangle : \mathcal{A}/\bigcap_{i \in I} \theta^i \to \mathcal{A}/\theta^i$$

be defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, by

$$\begin{array}{ccc} & \mathbf{F} & \\ \langle F, \pi \circ \alpha \rangle \swarrow & & \searrow \langle F, \pi^i \circ \alpha \rangle \\ \mathcal{A}/\bigcap_{i \in I} \theta^i & \xrightarrow[\langle I, \rho^i \rangle]{} & \mathcal{A}/\theta^i \end{array}$$

$$\rho^i_\Sigma(\phi/\bigcap_{i \in I} \theta^i_\Sigma) = \phi/\theta^i_\Sigma.$$

Then, we have $\bigcap_{i \in I} \mathrm{Ker}(\langle I, \rho^i \rangle) = \Delta^{\mathcal{A}/\bigcap_{i \in I} \theta^i}$. Hence, the family $\{\langle I, \rho^i \rangle : i \in I\}$ forms a subdirect intersection. Thus, by hypothesis, since $\mathcal{A}/\theta^i \in \mathsf{K}$, for all $i \in I$, $\mathcal{A}/\bigcap_{i \in I} \theta^i \in \mathsf{K}$ and, therefore, $\bigcap_{i \in I} \theta^i \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$. We conclude that $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ is closed under intersections.

Suppose, conversely, that, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ is closed under intersection and let

$$\langle H^i, \gamma^i \rangle : \mathcal{A} \to \mathcal{A}^i, \quad i \in I,$$

be a subdirect intersection, such that $\mathcal{A}^i \in \mathsf{K}$, for all $i \in I$. For every $i \in I$, consider the morphism

$$\langle H^i, \delta^i \rangle : \mathcal{A}/\mathrm{Ker}(\langle H^i, \gamma^i \rangle) \to \mathcal{A}^i,$$

defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, by

$$\begin{array}{ccc} & \mathbf{F} & \\ \langle F, \pi^i \circ \alpha \rangle \swarrow & & \searrow \langle F^i, \alpha^i \rangle \\ \mathbf{A}/\mathrm{Ker}(\langle H^i, \gamma^i \rangle) & \xrightarrow[\langle H^i, \delta^i \rangle]{} & \mathbf{A}^i \end{array}$$

$$\delta^i_\Sigma(\phi/\mathrm{Ker}_\Sigma(\langle H^i, \gamma^i \rangle)) = \gamma^i_\Sigma(\phi).$$

It is clearly well-defined and, moreover, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathcal{A}/\mathrm{Ker}(\langle H^i, \gamma^i\rangle)) & \text{iff} & \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in \mathrm{Ker}_{F(\Sigma)}(\langle H^i, \gamma^i\rangle) \\ & \text{iff} & \gamma^i_{F(\Sigma)}(\alpha_\Sigma(\phi)) = \gamma^i_{F(\Sigma)}(\alpha_\Sigma(\psi)) \\ & \text{iff} & \alpha^i_\Sigma(\phi) = \alpha^i_\Sigma(\psi) \\ & \text{iff} & \phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathcal{A}^i).\end{array}$$

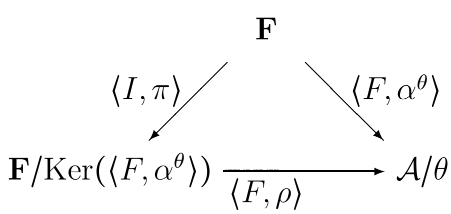
Thus, $\mathcal{A}^i$ is a Σ-K-certificate for $\mathcal{A}/\mathrm{Ker}(\langle H^i, \gamma^i\rangle)$, for all $\Sigma \in |\mathbf{Sign}^\flat|$. Since K is abstract, we get that $\mathcal{A}/\mathrm{Ker}(\langle H^i, \gamma^i\rangle) \in \mathsf{K}$ and, hence, $\mathrm{Ker}(\langle H^i, \gamma^i\rangle) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$. Thus, by hypothesis, $\Delta^{\mathcal{A}} = \bigcap_{i\in I} \mathrm{Ker}(\langle H^i, \gamma^i\rangle) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$, showing that $\mathcal{A} \in \mathsf{K}$. We conclude that K is closed under subdirect intersections. ■

Lemma 141 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* K *be an abstract class of* $\mathbf{F}$*-algebraic systems,* $\mathcal{A}$ *an* $\mathbf{F}$*-algebraic system and* $\theta \in \mathrm{ConSys}(\mathcal{A})$. *Then* $\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ *if and only if* $\mathrm{Ker}(\langle F, \alpha^\theta\rangle) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$,

$$\mathbf{F} \xrightarrow{\langle F, \alpha\rangle} \mathbf{A} \xrightarrow{\langle I, \pi^\theta\rangle} \mathbf{A}/\theta$$

where $\langle F, \alpha^\theta\rangle = \langle I, \pi^\theta\rangle \circ \langle F, \alpha\rangle$.

Proof: Consider the diagram,



where $\langle F, \rho\rangle : \mathbf{F}/\mathrm{Ker}(\langle F, \alpha^\theta\rangle) \to \mathcal{A}/\theta$ is defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, by

$$\rho_\Sigma(\phi/\mathrm{Ker}_\Sigma(\langle F, \alpha^\theta\rangle)) = \alpha_\Sigma(\phi)/\theta_{F(\Sigma)}.$$

This is well-defined, since, if $\langle \phi, \psi\rangle \in \mathrm{Ker}_\Sigma(\langle F, \alpha^\theta\rangle)$, then $\alpha^\theta_\Sigma(\phi) = \alpha^\theta_\Sigma(\psi)$, i.e., by definition, $\alpha_\Sigma(\phi)/\theta_{F(\Sigma)} = \alpha_\Sigma(\psi)/\theta_{F(\Sigma)}$. Moreover, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, we have

$$\begin{array}{lll}\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathcal{F}/\mathrm{Ker}(\langle F, \alpha^\theta\rangle)) & \text{iff} & \langle \phi, \psi\rangle \in \mathrm{Ker}_\Sigma(\langle I, \pi\rangle) \\ & \text{iff} & \langle \phi, \psi\rangle \in \mathrm{Ker}_\Sigma(\langle F, \alpha^\theta\rangle) \\ & \text{iff} & \phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathcal{A}/\theta).\end{array}$$

Thus, $\mathrm{Eq}(\mathcal{F}/\mathrm{Ker}(\langle F, \alpha^\theta\rangle)) = \mathrm{Eq}(\mathcal{A}/\theta)$. Since K is abstract, we conclude that $\mathbf{F}/\mathrm{Ker}(\langle F, \alpha^\theta\rangle) \in \mathsf{K}$ if and only if $\mathcal{A}/\theta \in \mathsf{K}$. Therefore, $\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ if and only if $\mathrm{Ker}(\langle F, \alpha^\theta\rangle) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$. ■

Lemma 142 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* K *an abstract class of* $\mathbf{F}$*-algebraic systems.* $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ *is closed under intersection if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ *is closed under intersection.*

Proof: The "if" direction is obvious. For the only if, suppose $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ is closed under intersection and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $\{\theta^i : i \in I\} \subseteq \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$. Note that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll} \langle \phi, \psi \rangle \in \mathrm{Ker}_\Sigma(\langle F, \alpha^{\bigcap_{i\in I}\theta^i} \rangle) & \text{iff} & \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \bigcap_{i\in I} \theta^i_{F(\Sigma)} \\ & \text{iff} & \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \theta^i_{F(\Sigma)}, \text{ all } i \in I, \\ & \text{iff} & \langle \phi, \psi \rangle \in \bigcap_{i\in I} \mathrm{Ker}_\Sigma(\langle F, \alpha^{\theta^i} \rangle). \end{array}$$

Thus, $\mathrm{Ker}(\langle F, \alpha^{\bigcap_{i\in I}\theta^i} \rangle) = \bigcap_{i\in I} \mathrm{Ker}(\langle F, \alpha^{\theta^i} \rangle)$. Using Lemma 141, we now get

$$\begin{array}{lcl} \theta^i \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A}), i \in I, & \text{iff} & \mathrm{Ker}(\langle F, \alpha^{\theta^i} \rangle) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F}), i \in I, \\ & \text{implies} & \bigcap_{i\in I} \mathrm{Ker}(\langle F, \alpha^{\theta^i} \rangle) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F}) \\ & \text{iff} & \mathrm{Ker}(\langle F, \alpha^{\bigcap_{i\in I}\theta^i} \rangle) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F}) \\ & \text{iff} & \bigcap_{i\in I} \theta^i \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A}). \end{array}$$

Therefore, $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ is closed under intersection. ■

Now we formulate our first characterization theorem.

Theorem 143 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems.* K *is a guasiequational class if and only if it is abstract and* $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ *is closed under intersection.*

Proof: We have K is a guasiequational class if and only if, by Theorem 135, it is abstract and closed under subdirect intersections, if and only if, by Lemma 140, it is abstract and, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ is closed under intersection, if and only if, by Lemma 142, it is abstract and $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ is closed under intersection. ■

A similar characterization can be obtained for quasiequational classes.

Theorem 144 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems.* K *is a quasiequational class if and only if it is directedly abstract and* $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ *is closed under intersection.*

Proof: We have K is a quasiequational class if and only if, by Theorem 136, it is directedly abstract and closed under subdirect intersections, if and only if, by Lemma 140 (taking into account that directed abstraction implies abstraction), it is directedly abstract and, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ is closed under intersections, if and only if, by Lemma 142, it is directedly abstract and $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ is closed under intersection. ■

Finally, we work with equational classes. Again, to provide an analogous characterization, we go through a couple of technical lemmas.

The first is an analog of Lemma 140, but instead of addressing subdirect intersections and intersections of relative congruence systems, it addresses morphic images and shows that closure of an abstract class under morphic images amounts to the collection of all relative congruence systems on every algebraic system being an up-set in the lattice of congruence systems.

Lemma 145 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* K *an abstract class of* $\mathbf{F}$*-algebraic systems.* K *is closed under morphic images if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ *is an up-set in* $\mathrm{ConSys}(\mathcal{A})$*.*

Proof: Let K be an abstract class of $\mathbf{F}$-algebraic systems.

Assume, first, that K is closed under morphic images and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $\theta, \theta' \in \mathrm{ConSys}(\mathcal{A})$, such that $\theta \leq \theta'$ and $\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$. We consider the morphism $\langle I, \rho \rangle : \mathcal{A}/\theta \to \mathcal{A}/\theta'$, given, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, by

$$\begin{array}{ccc} & \mathbf{F} & \\ {\scriptstyle\langle F, \alpha^\theta \rangle} \swarrow & & \searrow {\scriptstyle\langle F, \alpha^{\theta'} \rangle} \\ \mathbf{A}/\theta & \xrightarrow[\langle I, \rho \rangle]{} & \mathbf{A}/\theta' \end{array}$$

$$\rho_\Sigma(\phi/\theta_\Sigma) = \phi/\theta'_\Sigma.$$

It is clearly, well-defined, since $\theta \leq \theta'$. Since $\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$, $\mathcal{A}/\theta \in \mathsf{K}$, whence, since K is closed under morphic images, $\mathcal{A}/\theta' \in \mathsf{K}$, giving $\theta' \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$. Therefore, $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ is an up-set in $\mathrm{ConSys}(\mathcal{A})$.

Suppose, conversely, that $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ is an up-set in $\mathrm{ConSys}(\mathcal{A})$, for every $\mathbf{F}$-algebraic system $\mathcal{A}$. Consider $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$ and a surjective morphism $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{A}'$

$$\begin{array}{ccc} & \mathbf{F} & \\ {\scriptstyle\langle F, \alpha \rangle} \swarrow & & \searrow {\scriptstyle\langle F', \alpha' \rangle} \\ \mathbf{A} & \xrightarrow[\langle H, \gamma \rangle]{} & \mathbf{A}' \end{array}$$

and assume that $\mathcal{A} \in \mathsf{K}$. Then, we have $\Delta^{\mathcal{A}} \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ iff, by Lemma 141, $\mathrm{Ker}(\langle F, \alpha \rangle) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ implies, by the hypothesis and the commutativity of the triangle, $\mathrm{Ker}(\langle F', \alpha' \rangle) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ iff, again by Lemma 141, $\Delta^{\mathcal{A}'} \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A}')$ iff $\mathcal{A}' \in \mathsf{K}$. Therefore, K is closed under morphic images. ■

The second is an analog of Lemma 142, but instead of addressing closure of the collections of relative congruence systems under intersection, it deals with their upward closure under the signature-wise ordering in the lattices of congruence systems.

Lemma 146 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* K *an abstract class of* $\mathbf{F}$*-algebraic systems.* $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ *is an upset in* $\mathrm{ConSys}(\mathcal{F})$ *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ *is an upset in* $\mathrm{ConSys}(\mathcal{A})$*.*

Proof: The "if" direction is obvious.

For the "only if" assume that $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ is an up-set in $\mathrm{ConSys}(\mathcal{F})$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $\theta, \theta' \in \mathrm{ConSys}(\mathcal{A})$, such that $\theta \leq \theta'$ and $\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$. Then, taking into account the fact that $\mathrm{Ker}(\langle F, \alpha^\theta \rangle) \leq \mathrm{Ker}(\langle F, \alpha^{\theta'} \rangle)$ and that $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ is an upset and using Lemma 141, we have

$$\begin{array}{rcl}
\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A}) & \text{iff} & \mathrm{Ker}(\langle F, \alpha^\theta \rangle) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F}) \\
& \text{implies} & \mathrm{Ker}(\langle F, \alpha^{\theta'} \rangle) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F}) \\
& \text{iff} & \theta' \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A}).
\end{array}$$

Therefore, $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ is an up-set in $\mathrm{ConSys}(\mathcal{A})$. ∎

Now we get the following theorem characterizing equational classes of $\mathbf{F}$-algebraic systems.

Theorem 147 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems.* K *is an equational class if and only if it is abstract and* $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ *is an upset in* $\mathrm{ConSys}(\mathcal{F})$*, closed under intersections.*

Proof: We have K is an equational class if and only if, by Theorem 137, it is closed under subdirect intersections and morphic images, if and only if, by Proposition 133 and Lemmas 140 and 145, it is abstract and, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ is an upset in $\mathrm{ConSys}(\mathcal{A})$, closed under intersections, if and only if, by Lemmas 142 and 146, it is abstract and $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ is an upset in $\mathrm{ConSys}(\mathcal{F})$, closed under intersections. ∎