Chapter 3

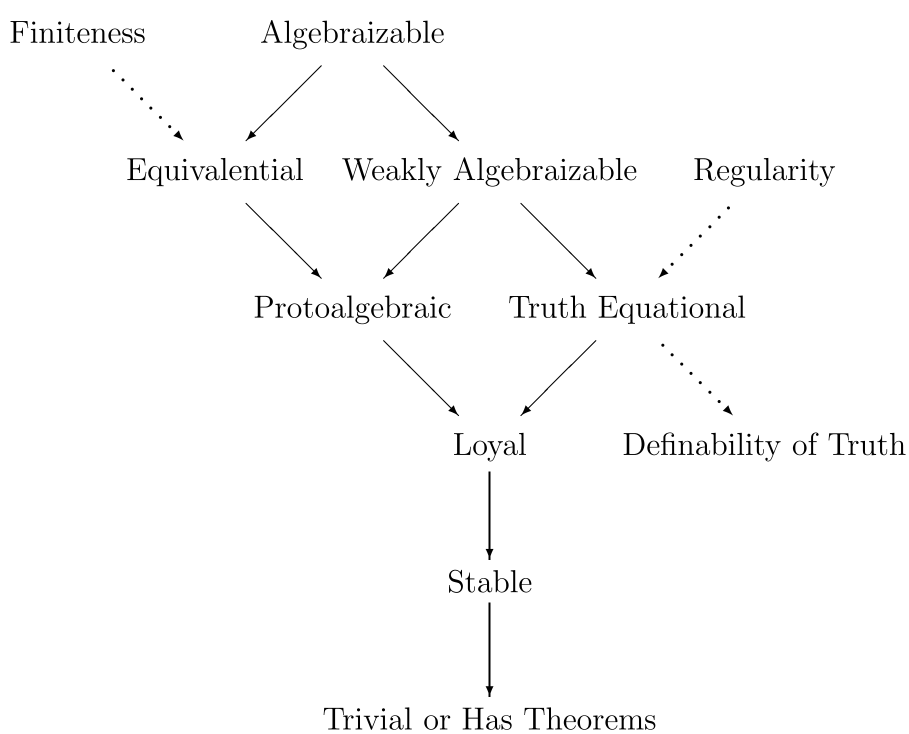
The Semantic Leibniz Hierarchy: Bottom Half

3.1 Introduction

In this chapter we give the main definitions and some key properties of the main classes of the *semantic Leibniz hierarchy*. The term *Leibniz hierarchy* refers to the classification of logical systems according to the strength of their relation to classes of algebraic systems. The term *semantic* refers to the definition of those classes by means of the *Leibniz operator* as applied to the theory families/systems of the corresponding π-institution or, via available so called *transfer theorems*, to the filter families/systems of their Leibniz reduced matrix systems.

As we will see later, there is also a *syntactic Leibniz hierarchy* whose classes do not coincide with those of the semantic hierarchy in general. However, we will show that under certain conditions, there is a correspondence between the classes in the two hierarchies.

A *very rough idea* of the main classes of the semantic Leibniz hierarchy, with the inclusion relations between them, is given in the following diagram. The hierarchy parallels that of sentential logics, established in the classical theory (see, e.g., Figure 9 on Page 316 of [86]). However, as we will see in this and subsequent chapters, in the case of logics formalized as π-institutions, various refinements of these classes are possible. One of the main goals of the monograph is to study those refinements and their interrelations. In the diagram we also give an idea of how this hierarchy is extended with other classes that are "attached" via dotted links to the main classes. Some of these extensions will also be studied later.



In Section 3.2, we introduce three fundamental classes of π-institutions, namely *systemic*, *stable* and *loyal* π-institutions. *Systemicity* and *stability* play a very important role throughout the monograph and facilitate discussions about the refinements of the various classes alluded to previously. *Loyalty* does not play a comparable role, but it constitutes at the same time a relaxation of *order preservation* and of *order reflectivity* and, as such, defines an important class close to the bottom of the hierarchy.

A π-institution is called *systemic* if all of its theory families are theory systems, i.e., invariant under the action of signature morphisms. In the context of π-institutions, the importance of systemicity was brought to the fore in the study of protoalgebraicity [105, 104]. Generally speaking, however, preservation of relations or, more concretely, of distinguished sets, is an important property in the model theory of first-order logics (see, e.g., page 71 of [17], page 5 of [43] or page 8 of [66]) and, hence, also in the theory of logical matrices serving as models of sentential logics (see, e.g., page 31 of [64] or page 200 of [86]). Systemicity is characterized by asserting that the closure family generated by any sentence of the given π-institution includes all translates of that sentence via signature morphisms. Systemicity also affords the chance to introduce the first of a host of so-called *transfer theorems*. This term refers to a property holding on the lattice of theory families of a π-institution *transferring* to the lattice of its filter families on arbitrary algebraic systems. This paradigm follows an oft-encountered situation in the theory of sentential logics (see, e.g., Section 3.6 of [86]). In this specific instance, it is shown that a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is systemic if and only if every filter family on every $\mathbf{F}$-algebraic system is actually a filter system.

Recall from Chapter 2 that, given a π-institution $\mathcal{I}$ and a theory family T of $\mathcal{I}$, $\overleftarrow{T}$ is the largest theory system of $\mathcal{I}$ included in T. In Section 2.3 it was shown (Proposition 20) that the Leibniz congruence system associated with T is included in the one associated with $\overleftarrow{T}$. We call the π-institution $\mathcal{I}$ *stable* if, for all theory families T of $\mathcal{I}$, these two Leibniz congruence systems coincide. Since systemicity directly yields that, for all T, $\overleftarrow{T} = T$, it clearly implies stability. Moreover, this implication is proper. As was the case with systemicity, stability also transfers.

The last concept introduced in Section 3.2 is that of *loyalty*. There are four possible flavors, analogs of which also arise and are pursued for many other properties considered in the monograph. The general idea of the property is, perhaps, best conveyed by *family loyalty*. The property holds for a π-institution $\mathcal{I}$ if, for no pair T and T' of theory families of $\mathcal{I}$ is it the case that $T < T'$ and $\Omega(T) > \Omega(T')$, i.e., $\mathcal{I}$ is *family loyal* if proper inclusion between theory families is never reversed when passing to corresponding Leibniz congruence systems. As in most other properties that we study, the other three versions are obtained from the family version as follows:

- *left loyalty* by replacing on the theory family side T and T' by their arrow counterparts $\overleftarrow{T}$, $\overleftarrow{T'}$, respectively;

- *right loyalty* by replacing on the congruence system side T and T' by $\overleftarrow{T}$ and $\overleftarrow{T'}$, respectively, i.e., by considering the inequality $\Omega(\overleftarrow{T}) \geq \Omega(\overleftarrow{T'})$ in place of $\Omega(T) \geq \Omega(T')$;

- *system loyalty* by applying the defining condition only on the collection of theory systems of $\mathcal{I}$, instead of considering arbitrary pairs of theory families.

It turns out that family loyalty implies stability. Moreover, family loyalty is the strongest of the four properties, followed by left loyalty, which, in turn, implies system loyalty, which is equivalent to right loyalty. Both implications are proper. Another feature of family loyalty is that, apart from trivial π-institutions, all family loyal ones must possess theorems. In closing the section, it is shown that all flavors of loyalty also transfer from theory families/systems to filter families/systems over arbitrary algebraic systems.

In Section 3.3, we introduce versions of the *monotonicity property*. This property is very important historically, since one of the first major classes of sentential logics to be studied in detail in the context of abstract algebraic logic was that of protoalgebraic logics [28] (see, also, Chapter 1 of [64] and Section 6.2 of [86]). They are characterized by the monotonicity of the Leibniz operator on their theory lattices. In the context of π-institutions, *family monotonicity* asserts that, for every pair T, T' of theory families, if T is included in T', then the Leibniz congruence system of T is also dominated by that of T'. *Left monotonicity* results by replacing, on the theory family side (hypothesis), T and T' by $\overleftarrow{T}$ and $\overleftarrow{T'}$, respectively. Similarly, *right monotonicity* ensues when the same is done on the congruence system side (conclusion). Finally, *system monotonicity* is monotonicity restricted to the collection of theory systems. It is shown that family and left monotonicity coincide, as do right and system monotonicity. In agreement with the terminology inherited by the sentential framework, we call a π-institution satisfying family monotonicity *protoalgebraic*, whereas one satisfying system monotonicity is termed *prealgebraic* [105, 104]. Since prealgebraicity is defined by the same monotonicity condition as protoalgebraicity, but restricted to theory systems, protoalgebraic π-institutions form a subclass of the class of prealgebraic ones. Moreover, it turns out that a π-institution is protoalgebraic if and only if it is prealgebraic and stable. Protoalgebraicity actually implies family loyalty, a condition stronger than stability, and similarly, prealgebraicity implies system loyalty. Finally, it is shown that both monotonicity properties transfer.

In Sections 3.4 and 3.5, we undertake the study of properties that may be referred to, collectively, as *complete monotonicity* properties. The reason for studying these properties can be traced back to the work of Raftery

[77] in an indirect way, but they are also loosely related, especially in the study of finitary deductive systems, to the property of continuity, whose importance was already apparent in [35]. Raftery used a property termed complete order reflectivity to characterize truth equationality of sentential logics. The property asserts that, given a sentential logic $\mathcal{S}$, for all collections $\mathcal{T} \cup \{T'\}$ of theories of $\mathcal{S}$, $\bigcap_{T \in \mathcal{T}} \Omega(T) \subseteq \Omega(T')$ implies $\bigcap \mathcal{T} \subseteq T'$. Noting that in both the lattice of theories and the lattice of congruences on the formula algebra intersection coincides with meet, this property may be rewritten as $\bigwedge_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$ implies $\bigwedge \mathcal{T} \leq T'$. Since, however, join and union of both theories and congruences differ, depending on whether one adopts a set-theoretic or a lattice-theoretic point of view, the dual property of complete order reflectivity may take one of two possible forms. The first asserts that, for all $\mathcal{T} \cup \{T'\}$, $T' \subseteq \bigcup \mathcal{T}$ implies $\Omega(T') \subseteq \bigcup_{T \in \mathcal{T}} \Omega(T)$. The second stipulates that, for all $\mathcal{T} \cup \{T'\}$, $T' \leq \bigvee \mathcal{T}$ implies $\Omega(T') \leq \bigvee_{T \in \mathcal{T}} \Omega(T)$. The translation of complete order reflectivity in the categorical context was first introduced in [107]. Various flavors of it are studied in detail in Section 3.8. In Sections 3.4 and 3.5, we study the properties corresponding to the two aforementioned duals.

In Section 3.4, we look at the various flavors of *complete* $\bigcup$*-monotonicity*. Again, the simplest one is *family complete* $\bigcup$*-monotonicity*. A π-institution is *family completely* $\bigcup$*-monotone* if, for all collections $\mathcal{T} \cup \{T'\}$ of theory families, $T' \leq \bigcup_{T \in \mathcal{T}} T$ implies $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. *Left complete* $\bigcup$*-monotonicity* results by replacing all theory families appearing in the hypothesis by their arrow counterparts. Similarly, *right complete* $\bigcup$*-monotonicity* arises by performing the same replacement in the conclusion. Finally, *system complete* $\bigcup$*-monotonicity* is the property resulting by applying the same condition defining the family version to collections of theory systems only. We use the abbreviation *c*$^\cup$*-monotonicity* to refer to complete $\bigcup$-monotonicity. Moreover, when we drop the $\bigcup$ (or $^\cup$) from the notation, it is to this version of complete monotonicity that we refer to. Family or left c$^\cup$-monotonicity are strong enough to imply stability. Moreover, family c$^\cup$-monotonicity is equivalent to possessing both left and right c$^\cup$-monotonicity. Either left or right c$^\cup$-monotonicity on its own implies system c$^\cup$-monotonicity. For these four properties, it is also the case that they transfer from theory families/systems to filter families/systems on arbitrary algebraic systems. In closing, it is established that left c$^\cup$-monotonicity implies protoalgebraicity, whereas system c$^\cup$-monotonicity is sufficient for prealgebraicity.

In Section 3.5, we undertake a similar study of the complete monotonicity properties involving the join instead of the union operation. We say that a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is *family completely* $\bigvee$*-monotone*, abbreviated *family c*$^\vee$*-monotone*, if, for every collection $\mathcal{T} \cup \{T'\}$ of theory families, $T' \leq \bigvee^{\mathcal{I}} \mathcal{T}$ implies $\Omega(T') \leq \bigvee^{\mathbf{F}}_{T \in \mathcal{T}} \Omega(T)$, where $\bigvee^{\mathcal{I}}$ denotes the join in the complete lattice of theory families of $\mathcal{I}$ and $\bigvee^{\mathbf{F}}$ the join in the complete lattice of congruence systems on $\mathbf{F}$. As in Section 3.4, *left c*$^\vee$*-monotonicity* results by replacing in the hypothesis every theory family by its arrow counterpart, *right* $c^{\vee}$*-monotonicity* by doing the same in the conclusion and *system* $c^{\vee}$*-monotonicity* by restricting the defining implication to all collections of theory systems, instead of insisting that it hold for arbitrary collections of theory families. Working with join instead of union leaves most of our conclusions intact. It is still the case that family $c^{\vee}$-monotonicity and left $c^{\vee}$-monotonicity are each sufficient for stability. Family $c^{\vee}$ monotonicity is equivalent to the combination of left and right $c^{\vee}$-monotonicities and each of the latter implies system $c^{\vee}$-monotonicity. And it is still the case that left $c^{\vee}$-monotonicity implies protoalgebraicity and system $c^{\vee}$-monotonicity implies prealgebraicity. There are, however, some differences between $c^{\vee}$-monotonicity properties and their counterparts using union. One example is that $c^{\vee}$-monotonicity properties, unlike $c^{\cup}$-monotonicity properties, do not transfer in general. This is due to the fact that, unlike union, the join operation does not commute with inverse surjective morphisms between algebraic systems. Another difference, which may also be viewed as a partial justification for considering both properties, is that the corresponding classes in the two hierarchies are incomparable. For instance, there exists a family $c^{\cup}$-monotone π-institution which is not family $c^{\vee}$-monotone and vice-versa.

In Section 3.6 we switch from the study of monotonicity properties to the study of *injectivity properties*. The importance of injectivity in the context of sentential logics was already apparent in the work of Blok and Pigozzi [35], but it was brought more in focus following its generalizations, first by Herrmann [43] and, ultimately, with the work of Czelakowski and Jansana [62] on weakly algebraizable logics. A π-institution is *family injective* if, for all theory families T and T', $\Omega(T) = \Omega(T')$ implies $T = T'$, i.e., when the Leibniz operator on theory families is injective. *Left injectivity* is obtained by replacing T and T' on the theory family side (conclusion) by $\overleftarrow{T}$ and $\overleftarrow{T'}$, respectively, while *right injectivity* by doing the same on the congruence system side (hypothesis). Finally, *system injectivity* imposes injectivity of the Leibniz operator on theory systems only. Here, right injectivity turns out to be the most potent of the four properties and it implies systemicity. Then comes family injectivity, followed by left injectivity, which, in turn, implies system injectivity. Right injectivity is equivalent to system injectivity coupled with systemicity, whereas, if system injectivity is combined with stability, they imply left injectivity. All injectivity properties transfer.

In the last two sections of the chapter, Sections 3.7 and 3.8, we study *reflectivity properties*, which are dual to the monotonicity properties delved into in Sections 3.3, 3.4 and 3.5.

In Section 3.7, we study the various flavors of *reflectivity*. In the context of algebraizable sentential logics the importance of this property was at least implicit, if not apparent, in the work of Blok and Pigozzi [35]. And, as was the case with injectivity, it kept its central role in the generalizations to infinitary algebraizable [43] and weakly algebraizable logics [62]. A π-institution $\mathcal{I}$ is *family reflective* if, for all theory families T and T', $\Omega(T) \le \Omega(T')$ implies $T \le T'$, i.e., if the Leibniz operator is order reflecting on the theory families of $\mathcal{I}$. *Left reflectivity* replaces T and T' on the theory family side (conclusion) by their arrow counterparts, while *right reflectivity* applies the same replacement on the congruence system side (hypothesis). Finally, *system reflectivity* imposes order reflectivity of the Leibniz operator on theory systems only. Each of family and right reflectivity implies systemicity. This allows proving that these two versions of reflectivity are actually equivalent. They imply left reflectivity, which dominates system reflectivity. System reflectivity together with stability imply left reflectivity. System reflectivity, coupled with systemicity, is equivalent to family reflectivity. All these properties transfer. Section 3.7 ends by establishing some relations between reflectivity and properties introduced in preceding sections. More precisely, it is shown that family, left and system reflectivity imply, respectively, right, left and system injectivity. Additionally, family, left and system reflectivity imply, respectively, family, left and system loyalty.

In Section 3.8, we study versions of *complete reflectivity*. As was mentioned previously, in the context of sentential logics, the property was introduced by Raftery in [77], where it was used to characterize truth equationality of sentential logics. A logic is truth equational if the filters of its Leibniz reduced matrix models are equationally definable. This is equivalent to the assertion that the filters of arbitrary matrix models are definable using equations via the corresponding Leibniz congruences. Raftery showed that truth equationality is equivalent to the complete order reflectivity of the Leibniz operator on the theories of the logic, i.e., the property that, for every collection $\mathcal{T} \cup \{T'\}$ of theories, $\bigcap_{T \in \mathcal{T}} \Omega(T) \subseteq \Omega(T')$ implies $\bigcap \mathcal{T} \subseteq T'$ (see, e.g., pages 371-382 of [86]; in particular, Theorem 6.101). Given a π-institution $\mathcal{I}$, *family complete reflectivity* asserts that, for every collection $\mathcal{T} \cup \{T'\}$ of theory families of $\mathcal{I}$, $\bigcap_{T \in \mathcal{T}} \Omega(T) \le \Omega(T')$ implies $\bigcap \mathcal{T} \le T'$. *Left complete reflectivity* is obtained by replacing on the theory family side (conclusion) each theory appearing by its arrow version. Similarly, *right complete reflectivity* results by performing the same replacement on the congruence system side (hypothesis). *System complete reflectivity* is the restriction of the condition defining family complete reflectivity on collections of theory systems. We abbreviate complete reflectivity by *c-reflectivity*. On their own, family and right c-reflectivity each implies systemicity, and this allows showing that they are equivalent properties. They imply left c-reflectivity, which, in turn, implies system c-reflectivity. System c-reflectivity, coupled with stability implies left c-reflectivity. Moreover, together with systemicity, it turns out to be equivalent to family c-reflectivity. All three different properties transfer and it is fairly obvious that they generalize the corresponding reflectivity properties, since the latter are special cases of the former in which the collection $\mathcal{T}$ is a singleton.

The properties of systemicity, stability, loyalty, monotonicity, c$^{\cup}$-monotonicity, c$^{\vee}$-monotonicity, injectivity, reflectivity and c-reflectivity constitute the building blocks of the hierarchies of π-institutions that will be presented and studied in subsequent chapters of the monograph.

3.2 Systemicity, Stability and Loyalty

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system. Recall that, by Proposition 42, for every theory family $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T}$ is the largest theory system included in T. Moreover, recall that, by Proposition 20, $\Omega(T) \leq \Omega(\overleftarrow{T})$.

Definition 148 (Systemicity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is called* **systemic** *if* $\mathrm{ThFam}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$*, or, equivalently, if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\overleftarrow{T} = T.$$

Another interesting characterization is the following. Recall that, given an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$, a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$, based on $\mathbf{F}$, and a sentence family $X \in \mathrm{SenFam}(\mathbf{F})$, we denote by

$$C(X) = \{C_\Sigma(X)\}_{\Sigma\in|\mathbf{Sign}^\flat|}$$

the least theory family of $\mathcal{I}$ including X. Moreover if $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, we use $C(\Phi)$ and $C(\phi) := C(\{\phi\})$ to denote $C(X)$, where $X = \{X_{\Sigma'}\}_{\Sigma'\in|\mathbf{Sign}^\flat|}$ is such that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, $X_{\Sigma'} = \begin{cases} \Phi, & \text{if } \Sigma' = \Sigma \\ \varnothing, & \text{if } \Sigma' \neq \Sigma \end{cases}$.

Proposition 149 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is systemic if and only if, for all* $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$*, all* $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\mathrm{SEN}^\flat(f)(\phi) \in C_{\Sigma'}(\phi).$$

Proof: Suppose, first, that $\mathcal{I}$ is systemic. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Since, by hypothesis, for all $T \in \mathrm{ThFam}(\mathcal{I})$, such that $\phi \in T_\Sigma$, we have, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, $\mathrm{SEN}^\flat(f)(\phi) \in T_{\Sigma'}$, we conclude that for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, $\mathrm{SEN}^\flat(f)(\phi) \in C_{\Sigma'}(\phi)$.

Assume, conversely, that the displayed condition in the statement holds and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$. Consider $\Sigma' \in |\mathbf{Sign}^\flat|$ and $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$. Then, by hypothesis,

$$\begin{array}{rcl}\mathrm{SEN}^\flat(f)(\phi) & \in & C_{\Sigma'}(\phi) \\ & = & \bigcap\{T'_{\Sigma'} : T' \in \mathrm{ThFam}(\mathcal{I}), \phi \in T'_\Sigma\} \\ & \subseteq & T_{\Sigma'}.\end{array}$$

Thus, $T \in \mathrm{ThSys}(\mathcal{I})$ and $\mathcal{I}$ is systemic. ∎

The following is one of many typical transfer theorems that we will encounter for various properties regarding π-institutions.

Theorem 150 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is systemic if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$*.*

Proof: The right to left implication follows by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$, where $\langle I, \iota \rangle : \mathbf{F} \to \mathbf{F}$ is the identity morphism, and taking into account the fact that, by Lemma 51, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}) = \mathrm{ThFam}(\mathcal{I})$ and $\mathrm{FiSys}^{\mathcal{I}}(\mathcal{F}) = \mathrm{ThSys}(\mathcal{I})$.

For the left to right implication, suppose that $\mathcal{I}$ is systemic and assume that $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ is an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$. Thus, by hypothesis, $\alpha^{-1}(T) \in \mathrm{ThSys}(\mathcal{I})$. Hence, using again Lemma 51, $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$. Therefore $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$. ∎

Now we introduce another important class of π-institutions in the semantic Leibniz hierarchy.

Definition 151 (Stability) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is called* **stable** *if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\Omega(\overleftarrow{T}) = \Omega(T).$$

Since, by Proposition 20, it always holds that $\Omega(T) \le \Omega(\overleftarrow{T})$, we have that

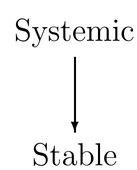
$\mathcal{I}$ is stable if and only if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(\overleftarrow{T}) \le \Omega(T)$.

The following obvious relation holds between these two classes.

Proposition 152 *Let* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution. If* $\mathcal{I}$ *is systemic, then it is stable.*

Proof: If $\mathcal{I}$ is systemic, then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} = T$, whence $\Omega(\overleftarrow{T}) = \Omega(T)$. Thus, $\mathcal{I}$ is stable. ∎

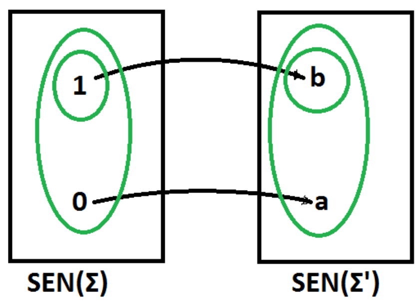
We denote this relation by the following diagram, where the arrow represents inclusion.



We show that this is a proper inclusion, i.e., there are stable π-institutions that are not systemic.

Example 153 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is a category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$;
- $N^\flat$ *is the trivial clone (consisting only of the projection natural transformations).*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\},\{a,b\}\}.$$

There is a single theory family which is not a theory system, namely $T = \{\{0,1\},\{b\}\}$. *So* $\mathcal{I}$ *is not systemic. On the other hand, we have* $\Omega(\overleftarrow{T}) = \Omega(\{\{1\},\{b\}\}) = \Delta^{\mathbf{F}} = \Omega(T)$. *Therefore,* $\mathcal{I}$ *is stable.*

The stability property transfers from the theory families of a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ to all $\mathcal{I}$-filter families on all $\mathbf{F}$-algebraic systems.

Theorem 154 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is stable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(\overleftarrow{T}) = \Omega^{\mathcal{A}}(T)$.

Proof: The "if" part is trivial, since stability is defined by the given condition on all theory families of the π-institution, which, by Lemma 51, are exactly the $\mathcal{I}$-filter families on $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$.

For the "only if" assume that $\mathcal{I}$ is stable and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then we have:

$$\begin{array}{rcl} \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T})) & = & \Omega(\alpha^{-1}(\overleftarrow{T})) \quad \text{(by Proposition 24)} \\ & = & \Omega(\overleftarrow{\alpha^{-1}(T)}) \quad \text{(by Lemma 6)} \\ & = & \Omega(\alpha^{-1}(T)) \\ & & \text{(by Lemma 51, Proposition 42 and stability)} \\ & = & \alpha^{-1}(\Omega^{\mathcal{A}}(T)) \quad \text{(by Proposition 24).} \end{array}$$

By surjectivity of $\langle F,\alpha\rangle$, we get that $\Omega^{\mathcal{A}}(\overleftarrow{T}) = \Omega^{\mathcal{A}}(T)$. ■

Next, we introduce various versions of the loyalty property and the corresponding classes in the loyalty hierarchy of π-institutions.

Definition 155 (Loyalty) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is called* **family loyal** *if, for all* $T,T' \in \mathrm{ThFam}(\mathcal{I})$,
$$T \nleq T' \quad or \quad \Omega(T) \ngeq \Omega(T').$$
- $\mathcal{I}$ *is called* **left loyal** *if, for all* $T,T' \in \mathrm{ThFam}(\mathcal{I})$,
$$\overleftarrow{T} \nleq \overleftarrow{T'} \quad or \quad \Omega(T) \ngeq \Omega(T').$$
- $\mathcal{I}$ *is called* **right loyal** *if, for all* $T,T' \in \mathrm{ThFam}(\mathcal{I})$,
$$T \nleq T' \quad or \quad \Omega(\overleftarrow{T}) \ngeq \Omega(\overleftarrow{T'}).$$
- $\mathcal{I}$ *is called* **system loyal** *if, for all* $T,T' \in \mathrm{ThSys}(\mathcal{I})$,
$$T \nleq T' \quad or \quad \Omega(T) \ngeq \Omega(T').$$

We establish relationships between these properties that lead to a loyalty hierarchy of π-institutions.

We show, first, that family loyalty implies stability.

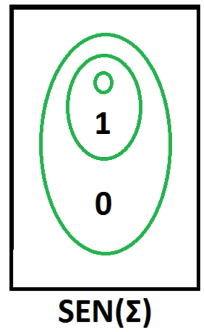
Lemma 156 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is family loyal, then it is stable.*

Proof: Suppose $\mathcal{I}$ is family loyal and let $T \in \mathrm{ThFam}(\mathcal{I})$. We must show that $\Omega(\overleftarrow{T}) = \Omega(T)$. If $\overleftarrow{T} = T$, then the conclusion is obvious. So suppose $\overleftarrow{T} \neq T$. Then, by Proposition 42, we have $\overleftarrow{T} < T$. Using family loyalty, we get $\Omega(\overleftarrow{T}) \ngeq \Omega(T)$. Hence, by Proposition 20, we have $\Omega(\overleftarrow{T}) = \Omega(T)$. We conclude that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(\overleftarrow{T}) = \Omega(T)$, whence $\mathcal{I}$ is stable. ■

There are π-institutions that are stable but not family loyal. The following example shows that family loyal π-institutions form a proper subclass of the class of stable π-institutions.

Example 157 *Consider the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category, with object* Σ*;*

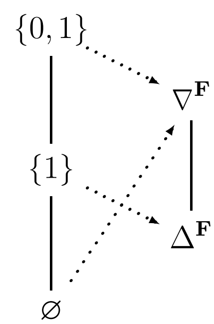


- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by setting* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*;*
- $N^\flat$ *is the trivial clone (consisting only of the projections).*

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by setting*

$$C_\Sigma = \{\varnothing, \{1\}, \{1,0\}\}.$$

The lattice of theory families (which are all systems) and the corresponding Leibniz congruence systems are given in the diagram.



Since all theory families are theory systems, $\mathcal{I}$ *is clearly stable. On the other hand, letting* $T = \{\varnothing\}$ *and* $T' = \{\{1\}\}$*, we have* $T < T'$ *and* $\Omega(T) > \Omega(T')$*. Therefore,* $\mathcal{I}$ *is not family loyal.*

Now we can establish the following relationships between the four loyalty properties.

Proposition 158 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is family loyal then it is left loyal;*

(b) If $\mathcal{I}$ *is left loyal, then it is system loyal;*

(c) $\mathcal{I}$ *is system loyal if and only if it is right loyal;*

(b) $\mathcal{I}$ *is family loyal if and only if it is system loyal and stable.*

Proof:

(a) Suppose $\mathcal{I}$ is family loyal. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\overleftarrow{T} < \overleftarrow{T'}$. Then, by family loyalty $\Omega(\overleftarrow{T}) \not> \Omega(\overleftarrow{T'})$. But, by Lemma 156, $\mathcal{I}$ is stable. So we get $\Omega(T) \not> \Omega(T')$. We conclude that $\mathcal{I}$ is left loyal.

(b) Suppose that $\mathcal{I}$ is left loyal and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\Omega(T) > \Omega(T')$. Then, by left loyalty, $\overleftarrow{T} \not< \overleftarrow{T'}$. But, since T, T' are theory systems, $\overleftarrow{T} = T$ and $\overleftarrow{T'} = T'$. Hence $T \not< T'$. Therefore $\mathcal{I}$ is system loyal.

(c) Suppose, now, that $\mathcal{I}$ is right loyal and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $T < T'$. Then, by right loyalty, $\Omega(\overleftarrow{T}) \not> \Omega(\overleftarrow{T'})$. Thus, since T, T' are theory systems, $\Omega(T) \not> \Omega(T')$. We conclude that $\mathcal{I}$ is system loyal.

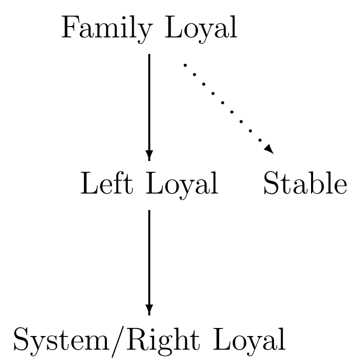
Suppose, conversely, that $\mathcal{I}$ is system loyal and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T < T'$. Then, by Lemma 1, $\overleftarrow{T} \leq \overleftarrow{T'}$. If $\overleftarrow{T} = \overleftarrow{T'}$, then $\Omega(\overleftarrow{T}) \not> \Omega(\overleftarrow{T'})$. On the other hand, if $\overleftarrow{T} < \overleftarrow{T'}$, then, by system loyalty, $\Omega(\overleftarrow{T}) \not> \Omega(\overleftarrow{T'})$. We conclude that $\mathcal{I}$ is right loyal.

(d) Suppose, first, that $\mathcal{I}$ is family loyal. Then, by Lemma 156, it is stable and it is, a fortiori, system loyal.

Suppose, conversely, that $\mathcal{I}$ is system loyal and stable. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) > \Omega(T')$. By stability, $\Omega(\overleftarrow{T}) > \Omega(\overleftarrow{T'})$. Therefore, by system loyalty, $\overleftarrow{T} \not< \overleftarrow{T'}$. Since $\Omega(\overleftarrow{T}) \neq \Omega(\overleftarrow{T'})$, we also have, $\overleftarrow{T} \not\leq \overleftarrow{T'}$. Therefore, by Lemma 1, $T \not< T'$. We conclude that $\mathcal{I}$ is family loyal.

■

By Proposition 158, the following **loyalty hierarchy** arises.



Example 157 brings to the fore another interesting point, namely, that family loyal π-institutions must have theorems, unless they are trivial.

Proposition 159 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ is trivial, then it is family loyal.

(b) If $\mathcal{I}$ is family loyal and non-trivial, then it has theorems.

Proof:

(a) If $\mathcal{I}$ is trivial, then the only Leibniz congruence system is $\nabla^{\mathbf{F}}$. So $\mathcal{I}$ is family loyal.

(b) Suppose $\mathcal{I}$ is family loyal and non-trivial. By non-triviality, it has a theory family T, such that, for some $\Sigma \in |\mathbf{Sign}^\flat|$, $T_\Sigma \neq \varnothing$ and $T_\Sigma \neq \mathrm{SEN}(\Sigma)$. Therefore, we have $\overline{\varnothing} < T$. So, by loyalty, $\Omega(\overline{\varnothing}) \not> \Omega(T)$. But $\Omega(\overline{\varnothing}) = \nabla^{\mathbf{F}}$. So $\nabla^{\mathbf{F}} \not> \Omega(T)$. This shows that $\Omega(T) = \nabla^{\mathbf{F}}$, which is a contradiction, since $\nabla^{\mathbf{F}}$ cannot be compatible with any theory family T, with a component $T_\Sigma \neq \varnothing, \mathrm{SEN}(\Sigma)$.

■

We also have the following straightforward relationship.

Proposition 160 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is systemic and system loyal, then it is family loyal.*

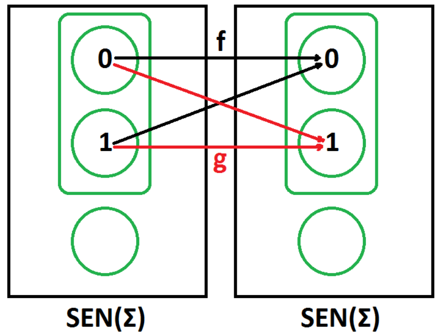
Proof: If $\mathcal{I}$ is systemic, then $\mathrm{ThFam}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$ and, as a result, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} = T$. Thus, under this hypothesis, all loyalty properties coincide. ■

The next example serves many purposes:

- It shows a π-institution that is left loyal, but not family loyal.
- It shows a π-institution that is system loyal, but not stable, and, hence, by Proposition 158, not family loyal.
- It shows an example of a nontrivial π-institution without theorems that is system loyal, but not family loyal, illustrating Proposition 159.

Example 161 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- *$\mathbf{Sign}^\flat$ is the category with a single object Σ and two non-identity morphisms $f, g : \Sigma \to \Sigma$, such that $f \circ f = f$, $g \circ g = g$, $g \circ f = g$ and $f \circ g = f$;*
- *$\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ is defined by $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$, $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = 0$, $\mathrm{SEN}^\flat(g)(0) = \mathrm{SEN}^\flat(g)(1) = 1$;*
- *$N^\flat$ is the trivial clone.*



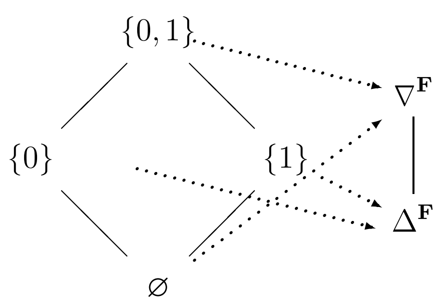
Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by setting

$$\mathcal{C}_\Sigma = \{\varnothing, \{0\}, \{1\}, \{0,1\}\}.$$

The following table gives the theory families and the theory systems of the π-institution $\mathcal{I}$:

T	$\overleftarrow{T}$
$\varnothing$	$\varnothing$
$\{0\}$	$\varnothing$
$\{1\}$	$\varnothing$
$\{0,1\}$	$\{0,1\}$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.

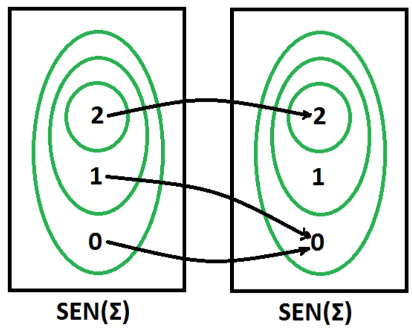


Note that, since $\overleftarrow{\{\{0\}\}} = \{\varnothing\}$ and these theory families map to different congruence systems, $\mathcal{I}$ is not stable. $\mathcal{I}$ is not family loyal, since $\{\varnothing\} < \{\{0\}\}$ and $\Omega(\{\varnothing\}) = \nabla^{\mathbf{F}} > \Delta^{\mathbf{F}} = \Omega(\{\{0\}\})$. However, $\mathcal{I}$ is left loyal, since, if $\overleftarrow{T} < \overleftarrow{T'}$, then $T' = \{\{0,1\}\}$ and, therefore, since $\Omega(T') = \nabla^{\mathbf{F}}$, $\Omega(T) \not> \Omega(T')$.

Now we provide a variety of additional examples, all showcasing π-institutions that are left loyal (and, hence, also system loyal), but fail to be family loyal.

Example 162 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial clone.*

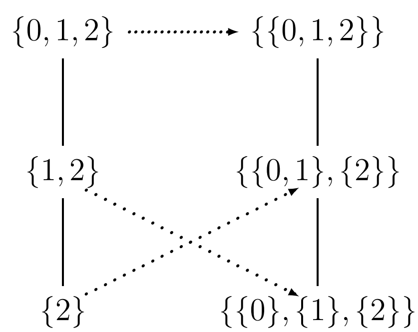


Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}$*.*

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$*:*

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1, 2\}$	$\{2\}$
$\{0, 1, 2\}$	$\{0, 1, 2\}$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



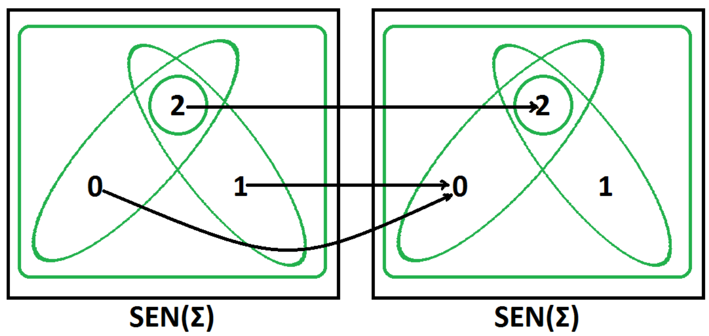
Taking into account the fact that $\{\{1, 2\}\}$ *is a theory family that is not a theory system, it is easy to see that this* π*-institution is left loyal (and, hence, system loyal), but fails to be family loyal.*

Example 163 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*

- $N^\flat$ *is the trivial clone.*

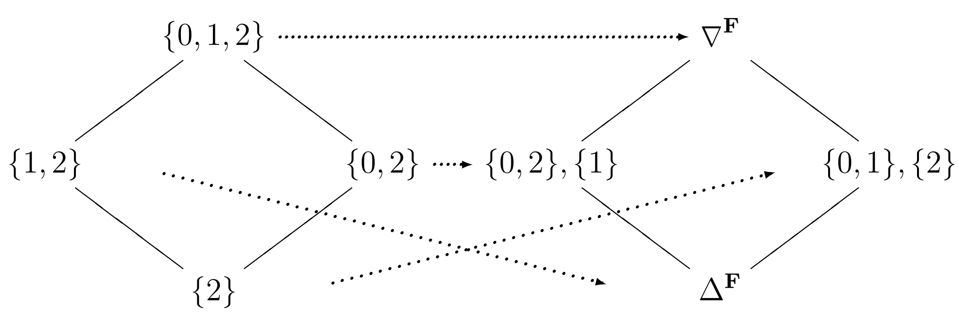


Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

$$\mathcal{C}_\Sigma = \{\{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}.$$

Note that $\{\{1,2\}\}$ *is the only theory family that is not a theory system.*

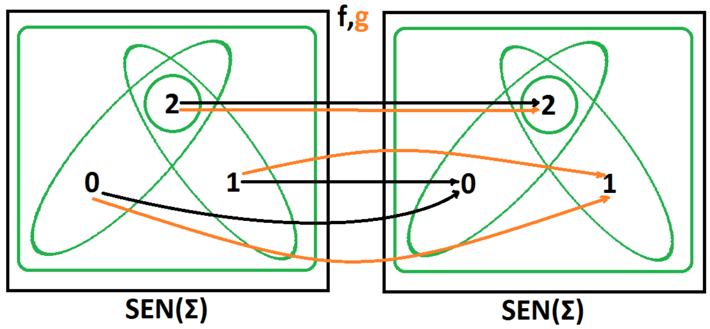
The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



Again it is not difficult to see that $\mathcal{I}$ *is left and right loyal, but fails to be family loyal.*

Example 164 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and two non-identity morphisms* $f, g : \Sigma \to \Sigma$*, such that* $f \circ f = f$*,* $g \circ g = g$*,* $g \circ f = g$ *and* $f \circ g = f$*;*



- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$, $\mathrm{SEN}^\flat(f)(2) = 2$ *and* $\mathrm{SEN}^\flat(g)(0) = 1$, $\mathrm{SEN}^\flat(g)(1) = 1$, $\mathrm{SEN}^\flat(g)(2) = 2$*;*

- $N^\flat$ *is the trivial clone.*

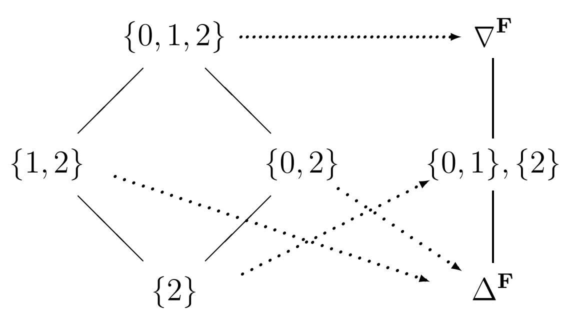
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

$$\mathcal{C}_\Sigma = \{\{2\},\{0,2\},\{1,2\},\{0,1,2\}\}.$$

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$*:*

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{0,2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



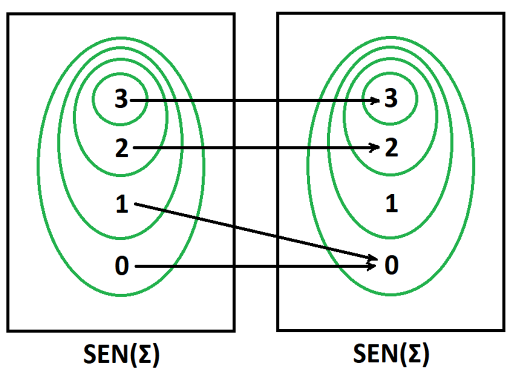
Again it is easy to check, keeping in mind that $\{\{2\}\}$ *and* $\{\{0,1,2\}\}$ *are the only theory systems, that* $\mathcal{I}$ *is left loyal (and, hence, system loyal), but not family loyal.*

Finally, an example of a system loyal π-institution that is not left loyal.

Example 165 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$*,* $\mathrm{SEN}^\flat(f)(2) = 2$ *and* $\mathrm{SEN}^\flat(f)(3) = 3$*;*
- $N^\flat$ *is the clone generated by the following two unary natural transformations* $\sigma^\flat, \tau^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*:*

x	$\sigma^\flat_\Sigma(x)$	$\tau^\flat_\Sigma(x)$
0	0	0
1	1	1
2	0	3
3	3	3



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

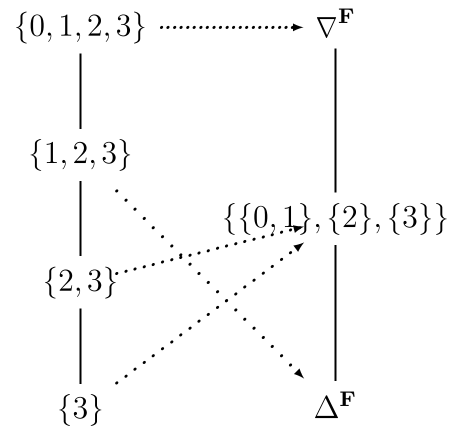
$$\mathcal{C}_\Sigma = \{\{3\}, \{2,3\}, \{1,2,3\}, \{0,1,2,3\}\}.$$

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$*:*

T	$\overleftarrow{T}$
$\{3\}$	$\{3\}$
$\{2,3\}$	$\{2,3\}$
$\{1,2,3\}$	$\{2,3\}$
$\{0,1,2,3\}$	$\{0,1,2,3\}$

The lattice of theory families and the corresponding Leibniz congruence sys-

tems are shown in the diagram.



$\mathcal{I}$ has three theory systems, $\mathrm{Thm}(\mathcal{I}) = \{\{3\}\}$, $T = \{\{2,3\}\}$ *and* $\mathrm{SEN} = \{\{0,1,2,3\}\}$*. An inspection of the diagram shows that $\mathcal{I}$ is system loyal. On the other hand, setting $T' = \{\{1,2,3\}\}$, we get that*

$$\overleftarrow{T'} = \overleftarrow{\{\{1,2,3\}\}} = \{\{2,3\}\} > \{\{3\}\} = \overleftarrow{\{\{3\}\}} = \overleftarrow{\mathrm{Thm}(\mathcal{I})},$$

whereas

$$\Omega(T') = \Delta^{\mathbf{F}} < \{\{0,1\},\{2\},\{3\}\} = \Omega(\mathrm{Thm}(\mathcal{I})).$$

Therefore, $\mathcal{I}$ is not left loyal.

For all loyalty properties, we have transfer theorems, detailed in the various parts of the following result.

Theorem 166 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is family loyal if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$T \not\leq T' \quad or \quad \Omega^{\mathcal{A}}(T) \not\geq \Omega^{\mathcal{A}}(T');$$

(b) $\mathcal{I}$ is left loyal if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$\overleftarrow{T} \not\leq \overleftarrow{T'} \quad or \quad \Omega^{\mathcal{A}}(T) \not\geq \Omega^{\mathcal{A}}(T');$$

(c) $\mathcal{I}$ is system loyal if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$,

$$T \not\leq T' \quad or \quad \Omega^{\mathcal{A}}(T) \not\geq \Omega^{\mathcal{A}}(T').$$

Proof:

(a) For the "if", suppose that the loyalty condition holds for the $\mathcal{I}$-filter families of every $\mathbf{F}$-algebraic system. Then it holds, in particular, for the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$, where $\langle I, \iota \rangle : \mathbf{F} \to \mathbf{F}$ is the identity morphism. The fact that, by Lemma 51, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}) = \mathrm{ThFam}(\mathcal{I})$, concludes the proof.

Suppose, conversely, that $\mathcal{I}$ is family loyal. Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T < T'$. We must show that $\Omega^{\mathcal{A}}(T) \not> \Omega^{\mathcal{A}}(T')$. Since $T < T'$, we must have $\alpha^{-1}(T) \le \alpha^{-1}(T')$. However, by surjectivity of $\langle F, \alpha \rangle$, if $\alpha^{-1}(T) = \alpha^{-1}(T')$, we get $T = T'$. Thus, we must have $\alpha^{-1}(T) < \alpha^{-1}(T')$. By Lemma 51, $\alpha^{-1}(T), \alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$. Thus, by loyalty, we get $\Omega(\alpha^{-1}(T)) \not> \Omega(\alpha^{-1}(T'))$. Thus, by Proposition 24,

$$\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \not> \alpha^{-1}(\Omega^{\mathcal{A}}(T')).$$

The following claim now completes the proof:

Claim: $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \not> \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$ implies $\Omega^{\mathcal{A}}(T) \not> \Omega^{\mathcal{A}}(T')$.

We work by contraposition. Assume $\Omega^{\mathcal{A}}(T) > \Omega^{\mathcal{A}}(T')$. Then, clearly, $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \ge \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. Moreover, by surjectivity, $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \ne \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. Thus, we conclude that $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) > \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$.

(b) If the left loyalty condition holds for the $\mathcal{I}$-filter families of every $\mathbf{F}$-algebraic system, it holds, in particular, for the $\mathbf{F}$-algebraic system $\mathcal{F}$. Since, by Lemma 51, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}) = \mathrm{ThFam}(\mathcal{I})$, $\mathcal{I}$ is left loyal.

Suppose, conversely, that $\mathcal{I}$ is left loyal. Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\overleftarrow{T} < \overleftarrow{T'}$. We must show that $\Omega^{\mathcal{A}}(T) \not> \Omega^{\mathcal{A}}(T')$. Since $\overleftarrow{T} < \overleftarrow{T'}$, we must have $\alpha^{-1}(\overleftarrow{T}) \le \alpha^{-1}(\overleftarrow{T'})$. However, by surjectivity of $\langle F, \alpha \rangle$, if $\alpha^{-1}(\overleftarrow{T}) = \alpha^{-1}(\overleftarrow{T'})$, we get $\overleftarrow{T} = \overleftarrow{T'}$. Thus, we must have $\alpha^{-1}(\overleftarrow{T}) < \alpha^{-1}(\overleftarrow{T'})$. By Lemma 51, $\alpha^{-1}(T), \alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$. By Lemma 6, $\overleftarrow{\alpha^{-1}(T)} < \overleftarrow{\alpha^{-1}(T')}$. Thus, by left loyalty, we get $\Omega(\alpha^{-1}(T)) \not> \Omega(\alpha^{-1}(T'))$. Thus, by Proposition 24, $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \not> \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. The claim used in Part (a) is used once more to complete the proof of Part (b).

(c) The proof follows along the lines of that of Part (a). ■

As a concluding remark, we rephrase the definitions of family and of system loyalty in terms of mappings between partially ordered sets.

Given two posets $\mathbf{P} = \langle P, \le \rangle$ and $\mathbf{Q} = \langle Q, \preccurlyeq \rangle$, we call a mapping $f : P \to Q$ **loyal** if, for all $p_1, p_2 \in P$,

$$p_1 < p_2 \quad \text{implies} \quad f(p_1) \not\succ f(p_2).$$

Then we have the following easy consequence (or, rather, reformulation) of the definition, combined with Theorem 166.

For a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ and an $\mathbf{F}$-algebraic system $\mathcal{A}$, we define

$$\begin{array}{rcl}\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}) & := & \mathrm{ConSys}^{\mathrm{AlgSys}^*(\mathcal{I})}(\mathcal{A});\\ \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}) & := & \mathrm{ConSys}^{\mathrm{AlgSys}(\mathcal{I})}(\mathcal{A}).\end{array}$$

Moreover, we set

$$\begin{array}{rcl}\mathrm{ConSys}^*(\mathcal{I}) & := & \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{F}) = \mathrm{ConSys}^{\mathrm{AlgSys}^*(\mathcal{I})}(\mathcal{F});\\ \mathrm{ConSys}(\mathcal{I}) & := & \mathrm{ConSys}^{\mathcal{I}}(\mathcal{F}) = \mathrm{ConSys}^{\mathrm{AlgSys}(\mathcal{I})}(\mathcal{F}).\end{array}$$

Proposition 167 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is family loyal;*

(b) $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is loyal;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is loyal, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

Similarly, we get

Proposition 168 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is system loyal;*

(b) $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is loyal;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is loyal, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

3.3 Monotonicity

In this section we define and study classes of π-institutions that are defined using monotonicity properties of the Leibniz operator.

Definition 169 (Monotonicity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is called* **family monotone** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$*,*

$$T \le T' \quad \textit{implies} \quad \Omega(T) \le \Omega(T').$$

- $\mathcal{I}$ *is called* **left monotone** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,
$$\overleftarrow{T} \leq \overleftarrow{T'} \quad \textit{implies} \quad \Omega(T) \leq \Omega(T').$$
- $\mathcal{I}$ *is called* **right monotone** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,
$$T \leq T' \quad \textit{implies} \quad \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'}).$$
- $\mathcal{I}$ *is called* **system monotone** *if, for all* $T, T' \in \mathrm{ThSys}(\mathcal{I})$,
$$T \leq T' \quad \textit{implies} \quad \Omega(T) \leq \Omega(T').$$

First, we show a very useful lemma to the effect that family monotonicity implies stability.

Lemma 170 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is family monotone, then* $\mathcal{I}$ *is stable.*

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$. Then we have, by Proposition 42, that $T, \overleftarrow{T} \in \mathrm{ThFam}(\mathcal{I})$, such that $\overleftarrow{T} \leq T$. Therefore, by family monotonicity, $\Omega(\overleftarrow{T}) \leq \Omega(T)$. However, by Proposition 20, $\Omega(T) \leq \Omega(\overleftarrow{T})$. Therefore, we get that $\Omega(\overleftarrow{T}) = \Omega(T)$. So $\mathcal{I}$ is stable. ■

Using Lemma 170, we can now show that family and left monotonicity are equivalent properties.

Proposition 171 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is family monotone if and only if it is left monotone.*

Proof: Suppose, first, that $\mathcal{I}$ is left monotone. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$. Then, by Lemma 1, $\overleftarrow{T} \leq \overleftarrow{T'}$. Therefore, by hypothesis, $\Omega(T) \leq \Omega(T')$. Hence $\mathcal{I}$ is family monotone.

Suppose, conversely, that $\mathcal{I}$ is family monotone. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\overleftarrow{T} \leq \overleftarrow{T'}$. Then, by hypothesis, $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. But, by Lemma 170, $\mathcal{I}$ is stable, whence $\Omega(\overleftarrow{T}) = \Omega(T)$ and $\Omega(\overleftarrow{T'}) = \Omega(T')$. Thus, we conclude that $\Omega(T) \leq \Omega(T')$. Therefore, $\mathcal{I}$ is left monotone. ■

An interesting observation is that system monotonicity may also be defined by using arbitrary theory families, but modifying the application of monotonicy to that of "arrow monotonicity". Formally speaking, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, we say that $\mathcal{I}$ is **arrow monotone** if, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,
$$\overleftarrow{T} \leq \overleftarrow{T'} \quad \text{implies} \quad \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'}).$$

Lemma 172 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is system monotone if and only if it is arrow monotone.*

Proof: Suppose, first, that $\mathcal{I}$ is system monotone and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\overleftarrow{T} \le \overleftarrow{T'}$. Since, by Proposition 42, $\overleftarrow{T}, \overleftarrow{T'} \in \mathrm{ThSys}(\mathcal{I})$, we get, by system monotonicity, that $\Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'})$. Thus, $\mathcal{I}$ is arrow monotone.

Suppose, conversely, that $\mathcal{I}$ is arrow monotone and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $T \le T'$. Then, again by Proposition 42, we get that $\overleftarrow{T} = T \le T' = \overleftarrow{T'}$. Therefore, by arrow monotonicity, $\Omega(T) = \Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'}) = \Omega(T')$. So $\mathcal{I}$ is system monotone. ■

Next, we show that the two properties of right monotonicity and system monotonicity also coincide.

Proposition 173 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is system monotone if and only if it is right monotone.*

Proof: Suppose, first, that $\mathcal{I}$ is right monotone and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $T \le T'$. Then, by right monotonicity, $\Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'})$. But, since T, T' are theory systems, we have $\overleftarrow{T} = T$ and $\overleftarrow{T'} = T'$. Hence, $\Omega(T) \le \Omega(T')$. Therefore, $\mathcal{I}$ is system monotone.

Suppose, conversely, that $\mathcal{I}$ is system monotone and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \le T'$. Then by Lemma 1, $\overleftarrow{T} \le \overleftarrow{T'}$. Since $\overleftarrow{T}, \overleftarrow{T'} \in \mathrm{ThSys}(\mathcal{I})$, we can apply system monotonicity to get $\Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'})$. Thus, $\mathcal{I}$ is right monotone. ■

Because of Propositions 171 and 173, we make the following definitions:

Definition 174 (Pre- and Protoalgebraicity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is called* **protoalgebraic** *if it is family monotone;*
- $\mathcal{I}$ *is called* **prealgebraic** *if it is system monotone.*

We show now that stability is exactly the separating property between prealgebraicity and protoalgebraicity.

Theorem 175 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is protoalgebraic if and only if it is prealgebraic and stable.*

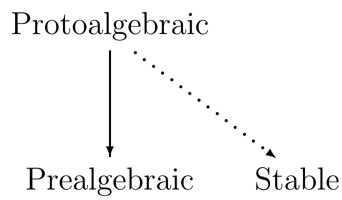
Proof: Suppose, first, that $\mathcal{I}$ is protoalgebraic. Then it is clearly prealgebraic and, by Lemma 170, it is stable.

Suppose, conversely, that $\mathcal{I}$ is stable and prealgebraic and consider $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$. Then, using stability, Proposition 42 and prealgebraicity, we get

$$\begin{array}{rcll} \Omega(T) & = & \Omega(\overleftarrow{T}) & \text{(stability)} \\ & \leq & \Omega(\overleftarrow{T'}) & \text{(Proposition 42 and prealgebraicity)} \\ & = & \Omega(T') & \text{(stability).} \end{array}$$

So Ω is monotone on theory families and $\mathcal{I}$ is protoalgebraic. ■

In terms of monotonicity, we have established the following **monotonicity hierarchy**:



Now we give examples of π-institutions to show that the two inclusions depicted in this diagram are proper. Moreover, we show that there are π-institutions that are neither prealgebraic nor stable. In other words, we show the following

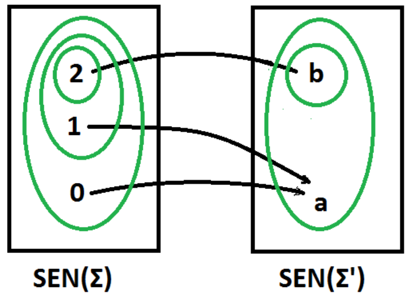
- There exist π-institutions that are neither prealgebraic nor stable.
- There exist π-institutions that are prealgebraic but not stable and, hence, not protoalgebraic.
- There exist π-institutions that are stable but not prealgebraic and, hence, not protoalgebraic.

Example 176 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = a$ *and* $\mathrm{SEN}^\flat(f)(2) = b$*;*
- $N^\flat$ *is the trivial clone.*

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\}, \{a, b\}\}.$$



It is easy to see that $\mathcal{I}$ is not stable: Consider the theory family $T = \{\{1,2\},\{b\}\}$. *Then we have* $\overleftarrow{T} = \{\{2\},\{b\}\}$ *and*

$$\Omega(\overleftarrow{T}) = \{\{\{0,1\},\{2\}\},\{\{a\},\{b\}\}\} \neq \Delta^{\mathbf{F}} = \Omega(T).$$

As a consequence, we get that $\mathcal{I}$ is not protoalgebraic.

We now show that it is not prealgebraic either. We use the two theory systems

$$T = \{\{2\},\{a,b\}\} \leq \{\{1,2\},\{a,b\}\} = T'.$$

We have

$$\Omega_\Sigma(T) = \{\{0,1\},\{2\}\}, \quad \Omega_{\Sigma'}(T) = \{\{a,b\}\};$$
$$\Omega_\Sigma(T') = \{\{0\},\{1,2\}\}, \quad \Omega_{\Sigma'}(T') = \{\{a,b\}\}.$$

Since $T \leq T'$ *but* $\Omega(T) \nleq \Omega(T')$, *we conclude that $\mathcal{I}$ is not prealgebraic.*

Example 177 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

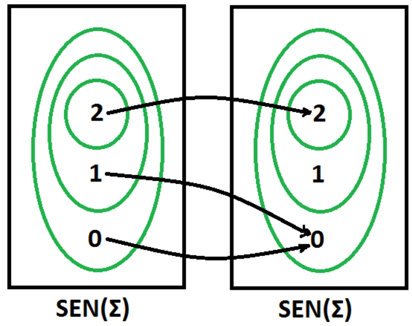
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$;
- $N^\flat$ *is the trivial clone.*

Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}$.

The following table gives the theory families and the theory systems of the π-institution $\mathcal{I}$:

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

First, observe that the only two theory systems are $T = \{\{2\}\}$ *and* $T' = \{\{0,1,2\}\}$. *Further, we have* $\Omega_\Sigma(T) = \{\{0,1\},\{2\}\}$ *and* $\Omega_\Sigma(T') = \{\{0,1,2\}\}$. *So $\mathcal{I}$ is prealgebraic.*



On the other hand, for $T'' = \{\{1,2\}\} \in \mathrm{ThFam}(\mathcal{I})$, *we have* $\overleftarrow{T''} = \{\{2\}\}$. *Moreover* $\Omega_\Sigma(T'') = \{\{0\},\{1\},\{2\}\} \not\supseteq \{\{0,1\},\{2\}\} = \Omega_\Sigma(\overleftarrow{T''})$. *Therefore, we conclude that* $\mathcal{I}$ *is not stable. As a consequence, it is not protoalgebraic either.*

Example 178 *Take any non-protoalgebraic deductive system* $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ *or, in closure system notation,* $\mathcal{S} = \langle \mathcal{L}, C_{\mathcal{S}} \rangle$. *Consider the discrete* π*-institution* $\mathcal{I}^{\mathcal{S}} = \langle \mathbf{F}^{\mathcal{L}}, C^{\mathcal{S}} \rangle$ *corresponding to the deductive system* $\mathcal{S}$ *(see Section 1.1 for details). This* π*-institution is not protoalgebraic (since the deductive system is not), but it is certainly stable (since its only signature morphism is the identity). Therefore, we conclude that it is not prealgebraic either.*

The monotonicity property transfers from the theory families/systems of a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ to all $\mathcal{I}$-filter families/systems on an arbitrary $\mathbf{F}$-algebraic system.

Theorem 179 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is prealgebraic if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$,

$$T \leq T' \quad \text{implies} \quad \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T');$$

(b) $\mathcal{I}$ *is protoalgebraic if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$T \leq T' \quad \text{implies} \quad \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T').$$

Proof:

(a) The "if" follows by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account that $\mathrm{ThSys}(\mathcal{I}) = \mathrm{FiSys}^{\mathcal{I}}(\mathcal{F})$, by Lemma 51.

For the "only if", suppose that $\mathcal{I}$ is prealgebraic and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, such that $T \leq T'$.

Then, clearly, $\alpha^{-1}(T) \leq \alpha^{-1}(T')$. Since, by Lemma 51, we have $\alpha^{-1}(T)$, $\alpha^{-1}(T') \in \mathrm{ThSys}(\mathcal{I})$, we get, by applying prealgebraicity, $\Omega(\alpha^{-1}(T)) \leq \Omega(\alpha^{-1}(T'))$. But, then, by Proposition 24, we get that $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. Finally, surjectivity yields that $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$.

(b) The "if" is obtained by considering the **F**-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account that $\mathrm{ThFam}(\mathcal{I}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F})$, by Lemma 51.

For the "only if", assume $\mathcal{I}$ is protoalgebraic and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an **F**-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \leq T'$. Then, clearly, $\alpha^{-1}(T) \leq \alpha^{-1}(T')$. Since, by Lemma 51, $\alpha^{-1}(T), \alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$, we get, by protoalgebraicity, $\Omega(\alpha^{-1}(T)) \leq \Omega(\alpha^{-1}(T'))$. By Proposition 24, $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$ and, hence, using surjectivity of $\langle F, \alpha \rangle$, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$.

■

As we did for loyalty, we may recast the two monotonicity classes in terms of the monotonicity of mappings from posets of theory or filter families/systems into posets of congruence systems.

Given two posets $\mathbf{P} = \langle P, \leq \rangle$ and $\mathbf{Q} = \langle Q, \preccurlyeq \rangle$, we call a mapping $f : P \to Q$ **monotone** or **order preserving** if, for all $p_1, p_2 \in P$,

$$p_1 \leq p_2 \quad \text{implies} \quad f(p_1) \preccurlyeq f(p_2).$$

Proposition 180 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is protoalgebraic;*

(b) $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is monotone;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is monotone, for every* **F***-algebraic system* $\mathcal{A}$*.*

Similarly, for prealgebraicity, we get

Proposition 181 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is prealgebraic;*

(b) $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is monotone;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is monotone, for every* **F***-algebraic system* $\mathcal{A}$*.*

Now we turn into exploring some of the relationships that hold between protoalgebraicity and prealgebraicity, on the one hand, and the various loyalty properties, on the other. Namely, we show that protoalgebraicity implies family loyalty and that prealgebraicity implies system loyalty. Note that, since family loyalty implies stability, the first part of the following theorem is a strengthening of Lemma 170.

Theorem 182 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is protoalgebraic, then it is family loyal;*

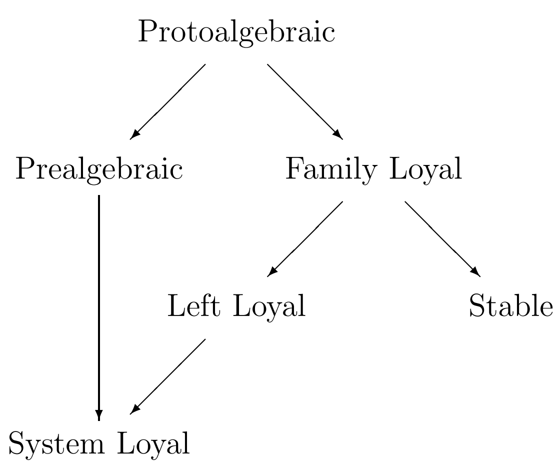
(b) If $\mathcal{I}$ *is prealgebraic, then it is system loyal.*

Proof:

(a) Suppose $\mathcal{I}$ is protoalgebraic and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T < T'$. Then, we have $T \leq T'$, whence, by protoalgebraicity, $\Omega(T) \leq \Omega(T')$. But this implies that $\Omega(T) \not\geq \Omega(T')$. We conclude that $\mathcal{I}$ is family loyal.

(b) Suppose that $\mathcal{I}$ is prealgebraic and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $T < T'$. Then, by prealgebraicity, $\Omega(T) \leq \Omega(T')$. This implies that $\Omega(T) \not\geq \Omega(T')$. We conclude that $\mathcal{I}$ is system loyal. ■

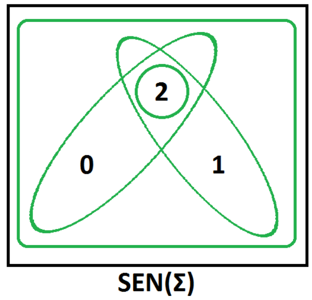
We have now established the following hierarchies:



Finally, we provide an example to show that the loyalty classes are proper subclasses of the classes defined using monotonicity.

Example 183 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be defined as follows:*

- $\mathbf{Sign}^\flat$ *is a trivial one object category, with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by setting* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*

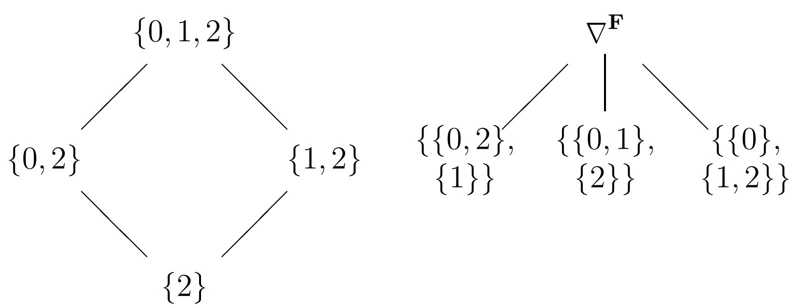


- *$N^\flat$ is the trivial clone, consisting of the projections only.*

Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by setting

$$\mathcal{C}_\Sigma = \{\{2\}, \{0,2\}, \{1,2\}, \{\{0,1,2\}\}.$$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



First, note that, since the category $\mathbf{Sign}^\flat$ is trivial, $\mathcal{I}$ is systemic, i.e., every theory family is also a theory system.

By considering, for instance, $T = \{\{2\}\}$ and $T' = \{\{0,2\}\}$, we see that $T \leq T'$, but $\Omega(T) \nleq \Omega(T')$. Thus, $\mathcal{I}$ is not prealgebraic.

On the other hand, it is clear that there do not exist $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T < T'$ and $\Omega(T) > \Omega(T')$. Hence $\mathcal{I}$ is family loyal.

3.4 Complete $\bigcup$-Monotonicity

We now define classes of π-institutions that are based on various versions of a property called *complete monotonicity*. These properties are strengthened versions of the monotonicity properties and the purpose for introducing them is that they are, in some sense, the dual properties of complete order reflectivity, which strengthens order reflectivity, which, in turn, is, in this same sense, the property dual to monotonicity. This property is somehow related to a property known as *continuity* in the context of sentential logics.

In the case of sentential logics, the property of complete reflectivity asserts that, given a sentential logic $\mathcal{S}$ and a collection of theories $\mathcal{T} \cup \{T'\}$ of $\mathcal{S}$,

$$\bigcap_{T \in \mathcal{T}} \Omega(T) \subseteq \Omega(T') \quad \text{implies} \quad \bigcap \mathcal{T} \subseteq T.$$

Note that meet and intersection coincide both in the lattice of theories of $\mathcal{S}$ and in the lattice of congruences on the formula algebra. Since, however, join and union differ, depending on the point of view, either lattice- or set-theoretic, one may perceive two different properties as dual properties of complete reflectivity. One, which we refer to as complete $\cup$-monotonicity, asserts that, for every collection $\mathcal{T} \cup \{T'\}$ of theories,

$$T' \subseteq \bigcup \mathcal{T} \quad \text{implies} \quad \Omega(T') \subseteq \bigcup_{T \in \mathcal{T}} \Omega(T).$$

The other, which may be termed complete $\vee$-monotonicity, asserts that, for every collection $\mathcal{T} \cup \{T'\}$ of theories,

$$T' \subseteq \bigvee \mathcal{T} \quad \text{implies} \quad \Omega(T') \subseteq \bigvee_{T \in \mathcal{T}} \Omega(T).$$

In this section, we deal with an analog of the first property for π-institutions. In the next section, we look at the second property.

Definition 184 (Complete $\bigcup$-Monotonicity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is* **family completely $\bigcup$-monotone** *or, simply,* **family completely monotone** *if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,

$$T' \leq \bigcup_{T \in \mathcal{T}} T \quad \textit{implies} \quad \Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T).$$

- $\mathcal{I}$ *is* **left completely $\bigcup$-monotone** *or, simply,* **left completely monotone** *if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,

$$\overleftarrow{T'} \leq \bigcup_{T \in \mathcal{T}} \overleftarrow{T} \quad \textit{implies} \quad \Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T).$$

- $\mathcal{I}$ *is* **right completely $\bigcup$-monotone** *or, simply,* **right completely monotone** *if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,

$$T' \leq \bigcup_{T \in \mathcal{T}} T \quad \textit{implies} \quad \Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T}).$$

- $\mathcal{I}$ *is* **system completely $\bigcup$-monotone** *or, simply,* **system completely monotone** *if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$,

$$T' \leq \bigcup_{T \in \mathcal{T}} T \quad \textit{implies} \quad \Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T).$$

Sometimes we will use the abbreviated form **c$^\cup$-monotonicity** or **c-monotonicity** to refer to complete $\bigcup$-monotonicity.

We have seen in Lemma 170 that family monotonicity (protoalgebraicity) implies stability. Since family complete monotonicity is a stronger property than family monotonicity, we get Part (a) of the following:

Lemma 185 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is family completely monotone, then it is stable.*

(b) If $\mathcal{I}$ *is left completely monotone, it is stable.*

Proof:

(a) If $\mathcal{I}$ is family completely monotone, then it is, a fortiori, family monotone. Thus, the result follows from Lemma 170.

(b) Suppose that $\mathcal{I}$ is left c-monotone and let $T \in \mathrm{ThFam}(\mathcal{I})$. By Proposition 42, $\overleftarrow{\overleftarrow{T}} = \overleftarrow{T}$. Applying left c-monotonicity, we get that $\Omega(\overleftarrow{T}) = \Omega(T)$. Hence $\mathcal{I}$ is stable.

■

Family completely monotone π-institutions are both left and right completely monotone. And, conversely, if a π-institution is both left and right c-monotone, then it is family c-monotone.

Proposition 186 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is family completely monotone if and only if it is both left and right completely monotone.*

Proof: Suppose, first, that $\mathcal{I}$ is family completely monotone.

- Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\overleftarrow{T'} \leq \bigcup_{T \in \mathcal{T}} \overleftarrow{T}$. Applying family c-monotonicity, we get $\Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$. However, by Lemma 185, $\mathcal{I}$ is stable. Hence we get $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. We conclude that $\mathcal{I}$ is left completely monotone.
- Next, let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $T' \leq \bigcup_{T \in \mathcal{T}} T$. Applying family c-monotonicity, we get $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. Once more, by Lemma 185, $\mathcal{I}$ is stable. Hence we get $\Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$. We conclude that $\mathcal{I}$ is right completely monotone.

Suppose, conversely, that $\mathcal{I}$ is both left and right completely monotone and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $T' \leq \bigcup_{T \in \mathcal{T}} T$. Then, by right c-monotonicity, we get that $\Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$. But since $\mathcal{I}$ is left completely monotone, by Lemma 185, it is stable, whence we get $\Omega(T') \leq \bigcup_{T\in\mathcal{T}}\Omega(T)$. Therefore, $\mathcal{I}$ is family completely monotone. ■

If a π-institution $\mathcal{I}$ is left or right completely monotone, then it is also system completely monotone.

Proposition 187 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is left c-monotone, then it is system c-monotone;*

(b) If $\mathcal{I}$ *is right c-monotone, then it is system c-monotone.*

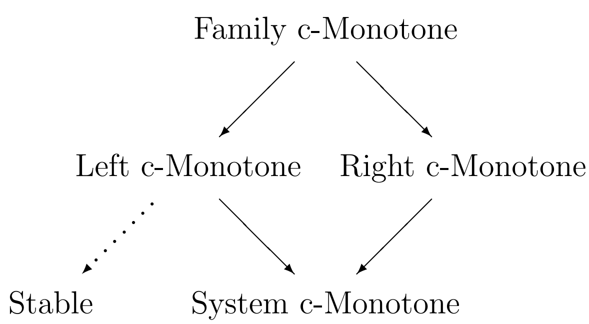
Proof:

(a) Suppose $\mathcal{I}$ is left c-monotone and let $\mathcal{T}\cup\{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $T' \leq \bigcup_{T\in\mathcal{T}} T$. Since $\mathcal{T}\cup\{T'\}$ is a collection of theory systems, we get $\overleftarrow{T'} \leq \bigcup_{T\in\mathcal{T}} \overleftarrow{T}$. Hence, applying left c-monotonicity, we get $\Omega(T') \leq \bigcup_{T\in\mathcal{T}}\Omega(T)$. Thus, $\mathcal{I}$ is system c-monotone.

(b) Suppose $\mathcal{I}$ is right c-monotone and let $\mathcal{T}\cup\{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $T' \leq \bigcup_{T\in\mathcal{T}} T$. Applying right c-monotonicity, we get $\Omega(\overleftarrow{T'}) \leq \bigcup_{T\in\mathcal{T}}\Omega(\overleftarrow{T})$. Since $\mathcal{T}\cup\{T'\}$ is a collection of theory systems, we now get $\Omega(T') \leq \bigcup_{T\in\mathcal{T}}\Omega(T)$. Thus, $\mathcal{I}$ is system c-monotone.

■

In terms of complete monotonicity, we have established the following hierarchy:



Now we give examples of π-institution to show that the inclusions depicted in this diagram are proper. We first give an example of a π-institution that is left c-monotone but not right c-monotone. This shows that:

- The class of family c-monotone π-institutions is properly contained in the class of all left c-monotone π-institutions;

- The class of all system c-monotone π-institutions properly includes the class of all right c-monotone π-institutions.

Example 188 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

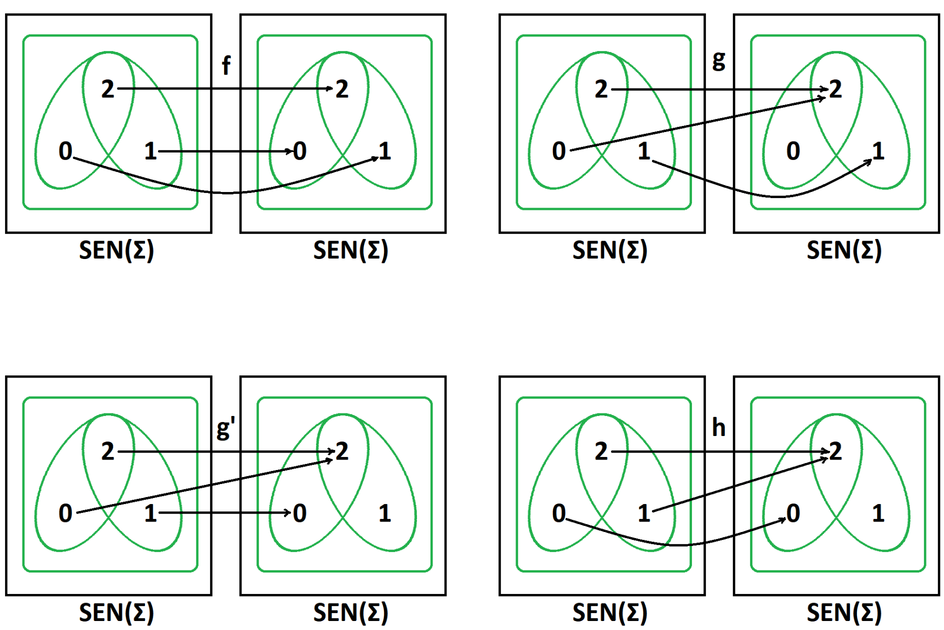
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and six non-identity morphisms* $f, g, g', h, h', t : \Sigma \to \Sigma$*, in which composition is defined by the following table, whose entry in row* k *and column* ℓ *is the result of the composition* $\ell \circ k$*:*

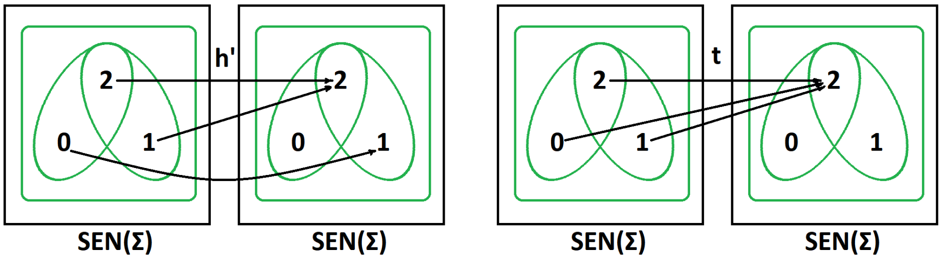
$\circ$	f	g	g'	h	h'	t
f	f	h'	h	g'	g	t
g	g'	g	g'	t	t	t
g'	g	t	t	g'	g	t
h	h'	t	t	h	h'	t
h'	h	h'	h	t	t	t
t	t	t	t	t	t	t

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given, on objects, by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and, on morphisms, by the following table, whose entries in column* k *give the values of the function* $\mathrm{SEN}^\flat(k) : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SEN}^\flat(\Sigma)$*:*

x	f	g	g'	h	h'	t
0	1	2	2	0	1	2
1	0	1	0	2	2	2
2	2	2	2	2	2	2

- $N^\flat$ *is the trivial clone.*





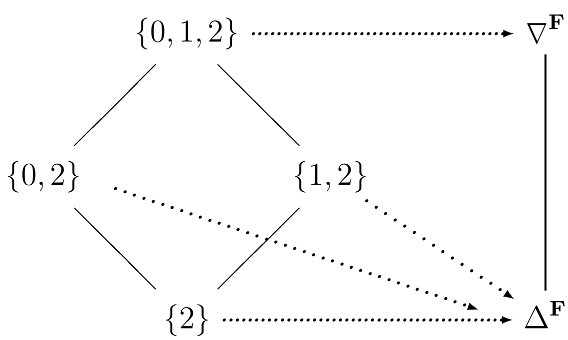
Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by setting

$$\mathcal{C}_\Sigma = \{\{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}.$$

The following table gives the theory families and the theory systems of the π-institution $\mathcal{I}$:

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{0,2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



$\mathcal{I}$ has only two theory systems, $\mathrm{Thm}(\mathcal{I}) = \{\{2\}\}$, *and* $\mathrm{SEN} = \{\{0,1,2\}\}$.

To show that $\mathcal{I}$ is left completely monotone, assume that, for some $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T'} \leq \bigcup_{T \in \mathcal{T}} \overleftarrow{T}$.

- *If $\bigcup_{T \in \mathcal{T}} \overleftarrow{T} = \{\{0,1,2\}\}$, then $\{\{0,1,2\}\} \in \mathcal{T}$ and, hence,*

$$\Omega(T') \leq \nabla^{\mathbf{F}} = \Omega(\{\{0,1,2\}\}) \leq \bigcup_{T \in \mathcal{T}} \Omega(T);$$

- *If $\bigcup_{T \in \mathcal{T}} \overleftarrow{T} = \{\{2\}\}$, then $T' \neq \{\{0,1,2\}\}$, whence*

$$\Omega(T') = \Delta^{\mathbf{F}} \leq \bigcup_{T \in \mathcal{T}} \Omega(T).$$

Thus, in any case, $\Omega(T') \le \bigcup_{T\in\mathcal{T}} \Omega(T)$ *and* $\mathcal{I}$ *is left completely monotone.*

On the other hand, we have

$$\{\{0,1,2\}\} \le \{\{0,2\}\} \cup \{\{1,2\}\},$$

whereas

$$\begin{array}{rcl}\Omega(\overleftarrow{\{\{0,1,2\}\}}) & = & \Omega(\{\{0,1,2\}\}) = \nabla^{\mathbf{F}} \\ & \not\le & \Delta^{\mathbf{F}} \\ & = & \Omega(\{\{2\}\}) \cup \Omega(\{\{2\}\}) \\ & = & \Omega(\overleftarrow{\{\{0,2\}\}}) \cup \Omega(\overleftarrow{\{\{1,2\}\}}).\end{array}$$

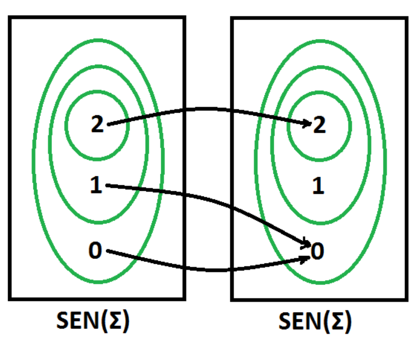
Therefore, $\mathcal{I}$ *is not right completely monotone.*

We now give an example of a right c-monotone π-institution that fails to be left c-monotone. This will show that:

- The class of family c-monotone π-institutions is properly contained in the class of right c-monotone π-institutions;
- The class of left c-monotone π-institutions is a proper subclass of the class of system c-monotone π-institutions.

Example 189 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial clone.*

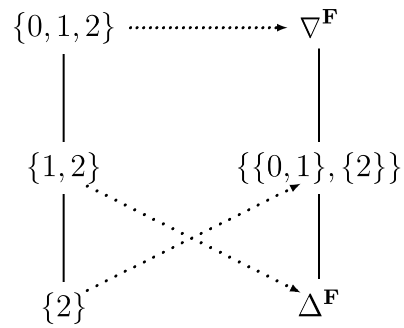


Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}$*.*

The following table gives the theory families and the theory systems of the π-institution $\mathcal{I}$:

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

The structure of the lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



Taking into account that $\overleftarrow{\{\{1,2\}\}} = \{\{2\}\}$, we can see that $\mathcal{I}$ is right c-monotone.

On the other hand, for $T = \{\{1,2\}\}$ and $T' = \{\{2\}\}$, we have $\overleftarrow{T'} = \{\{2\}\} \leq \overleftarrow{T}$, but $\Omega(T') = \{\{\{0,1\},\{2\}\}\} \nleq \Delta^{\mathbf{F}} = \Omega(T)$. Hence $\mathcal{I}$ is not left c-monotone.

As we saw in Theorem 179 for the various monotonicity properties, all versions of complete monotonicity transfer from the theory families/systems of a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ to the $\mathcal{I}$-filter families/systems on arbitrary $\mathbf{F}$-algebraic systems.

Theorem 190 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is family c-monotone if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ and all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T' \leq \bigcup_{T\in\mathcal{T}} T$ implies $\Omega^{\mathcal{A}}(T') \leq \bigcup_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(T)$.

(b) $\mathcal{I}$ is left c-monotone if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ and all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\overleftarrow{T'} \leq \bigcup_{T\in\mathcal{T}} \overleftarrow{T}$ implies $\Omega^{\mathcal{A}}(T') \leq \bigcup_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(T)$.

(c) $\mathcal{I}$ is right c-monotone if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ and all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T' \leq \bigcup_{T\in\mathcal{T}} T$ implies $\Omega^{\mathcal{A}}(\overleftarrow{T'}) \leq \bigcup_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(\overleftarrow{T})$.

(d) $\mathcal{I}$ is system c-monotone if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ and all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, $T' \leq \bigcup_{T\in\mathcal{T}} T$ implies $\Omega^{\mathcal{A}}(T') \leq \bigcup_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(T)$.

Proof: We shall prove Parts (b) and (c). Parts (a) and (d) follow along the same lines and are slightly easier.

(b) The "if" follows by considering the **F**-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$ and taking into account that $\mathrm{ThFam}(\mathcal{I}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F})$, by Lemma 51.

For the "only if", suppose that $\mathcal{I}$ is left c-monotone and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an **F**-algebraic system and $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\overleftarrow{T'} \leq \bigcup_{T\in\mathcal{T}} \overleftarrow{T}$. Apply the inverse morphism $\langle F, \alpha\rangle$ to get $\alpha^{-1}(\overleftarrow{T'}) \leq \alpha^{-1}(\bigcup_{T\in\mathcal{T}} \overleftarrow{T})$, or, equivalently, $\alpha^{-1}(\overleftarrow{T'}) \leq \bigcup_{T\in\mathcal{T}} \alpha^{-1}(\overleftarrow{T})$. Now apply Lemma 6 to get $\overleftarrow{\alpha^{-1}(T')} \leq \bigcup_{T\in\mathcal{T}} \overleftarrow{\alpha^{-1}(T)}$. But, by Lemma 51, $\{\alpha^{-1}(T) : T \in \mathcal{T}\} \cup \{\alpha^{-1}(T')\} \subseteq \mathrm{ThFam}(\mathcal{I})$. Therefore, applying left c-monotonicity, we get $\Omega(\alpha^{-1}(T')) \leq \bigcup_{T\in\mathcal{T}} \Omega(\alpha^{-1}(T))$. Hence, by Proposition 24, $\alpha^{-1}(\Omega^{\mathcal{A}}(T')) \leq \bigcup_{T\in\mathcal{T}} \alpha^{-1}(\Omega^{\mathcal{A}}(T))$, i.e.,

$$\alpha^{-1}(\Omega^{\mathcal{A}}(T')) \leq \alpha^{-1}(\bigcup_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(T)).$$

Finally, taking into account the surjectivity of $\langle F, \alpha\rangle$, we conclude that $\Omega^{\mathcal{A}}(T') \leq \bigcup_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(T)$.

(c) The "if" is obtained by considering the **F**-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$ and taking into account that $\mathrm{ThFam}(\mathcal{I}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F})$, by Lemma 51.

For the "only if", suppose that $\mathcal{I}$ is right c-monotone and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an **F**-algebraic system and $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T' \leq \bigcup_{T\in\mathcal{T}} T$. Apply the inverse morphism $\langle F, \alpha\rangle$ to get $\alpha^{-1}(T') \leq \alpha^{-1}(\bigcup_{T\in\mathcal{T}} T)$, or, equivalently, $\alpha^{-1}(T') \leq \bigcup_{T\in\mathcal{T}} \alpha^{-1}(T)$. By Lemma 51, $\{\alpha^{-1}(T) : T \in \mathcal{T}\} \cup \{\alpha^{-1}(T')\} \subseteq \mathrm{ThFam}(\mathcal{I})$. Therefore, applying right c-monotonicity, we get $\Omega(\overleftarrow{\alpha^{-1}(T')}) \leq \bigcup_{T\in\mathcal{T}} \Omega(\overleftarrow{\alpha^{-1}(T)})$. By Lemma 6, $\Omega(\alpha^{-1}(\overleftarrow{T'})) \leq \bigcup_{T\in\mathcal{T}} \Omega(\alpha^{-1}(\overleftarrow{T}))$. Hence, by Proposition 24, $\alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'})) \leq \bigcup_{T\in\mathcal{T}} \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T}))$. This is equivalent to $\alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'})) \leq \alpha^{-1}(\bigcup_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(\overleftarrow{T}))$. Finally, taking into account the surjectivity of $\langle F, \alpha\rangle$, we conclude that $\Omega^{\mathcal{A}}(\overleftarrow{T'}) \leq \bigcup_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(\overleftarrow{T})$.

■

Next we look at the relationships that hold between protoalgebraicity and prealgebraicity, on the one hand, and the various c-monotonicity properties, on the other. More precisely, we show that left complete monotonicity implies protoalgebraicity and that system complete monotonicity implies prealgebraicity.

Theorem 191 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* **F**.

(a) If $\mathcal{I}$ *is left c-monotone, then it is protoalgebraic;*

(b) If $\mathcal{I}$ is system c-monotone, then it is prealgebraic.

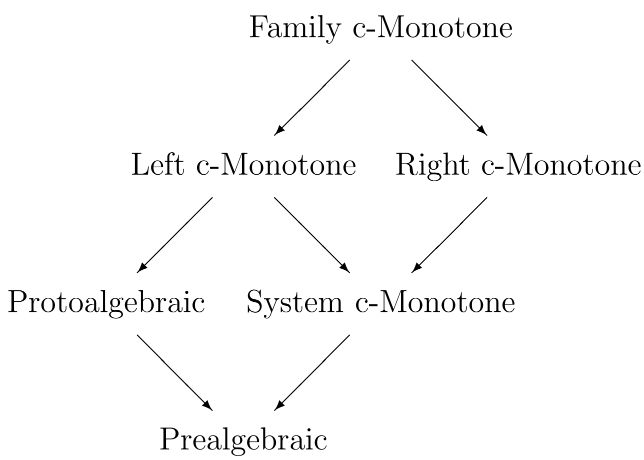
Proof:

(a) Suppose $\mathcal{I}$ is left c-monotone and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$. Then, by Lemma 1, we get $\overleftarrow{T} \leq \overleftarrow{T'}$. Thus, by left c-monotonicity, $\Omega(T) \leq \Omega(T')$. Hence $\mathcal{I}$ is protoalgebraic.

(b) Suppose that $\mathcal{I}$ is system c-monotone and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $T \leq T'$. Then we get right away from system c-monotonicity that $\Omega(T) \leq \Omega(T')$ and, therefore, $\mathcal{I}$ is prealgebraic.

■

We have now established the following hierarchy of **monotonicity** and **complete monotonicity** properties:

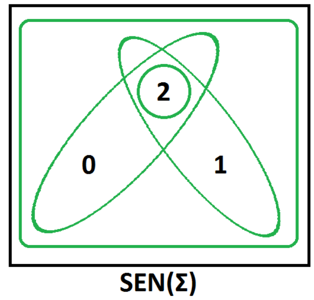


Finally, we provide an example to show that the c-monotonicity classes are proper subclasses of the monotonicity classes. Namely, we construct a protoalgebraic π-institution that fails to be system c-monotone.

Example 192 *Define the algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ as follows:*

- $\mathbf{Sign}^\flat$ *is a trivial category with object Σ;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the clone generated by the unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, given by*

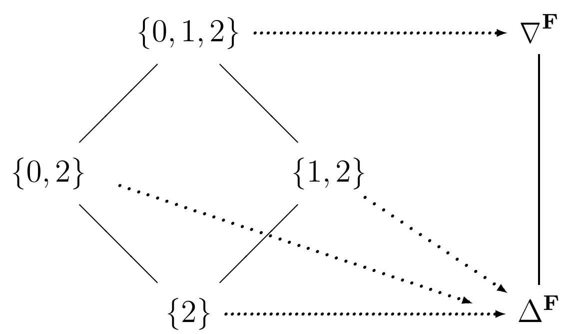
$x \in \mathrm{SEN}^\flat(\Sigma)$	$\sigma^\flat_\Sigma(x)$
0	1
1	2
2	0



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by setting

$$\mathcal{C}_\Sigma = \{\{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}.$$

It is easy to see that the lattices of theory families and corresponding Leibniz congruence systems are as given in the diagram.



From the diagram one can verify immediately that $\mathcal{I}$ is protoalgebraic, On the other hand, we have $\{\{0,1,2\}\} \leq \{\{0,2\}\} \cup \{\{1,2\}\}$, but, obviously, $\Omega(\{\{0,1,2\}\}) \nleq \Omega(\{\{0,2\}\}) \cup \Omega(\{\{1,2\}\})$. Taking into account that $\mathcal{I}$ is systemic, we conclude that $\mathcal{I}$ fails to be system c-monotone.

3.5 Complete $\bigvee$-Monotonicity

We now define classes of π-institutions that are based on the corresponding versions of the property of *complete monotonicity* using the join operation. These properties are also strengthened versions of the monotonicity properties.

To define these complete monotonicity properties, let us introduce the following notation. Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Denote by $\bigvee^{\mathcal{I}} \mathcal{T} = \bigvee^{\mathcal{I}}_{T \in \mathcal{T}} T$ the join of a collection $\mathcal{T}$ of theory families of $\mathcal{I}$ in the complete lattice $\mathbf{ThFam}(\mathcal{I})$ of theory families of $\mathcal{I}$. Analogously, denote by $\bigvee^{\mathbf{F}} \Theta = \bigvee^{\mathbf{F}}_{\theta \in \Theta} \theta$ the join of a collection Θ of congruence systems on $\mathbf{F}$ in the complete lattice $\mathbf{ConSys}(\mathbf{F})$ of congruence systems on $\mathbf{F}$.

Definition 193 (Complete $\bigvee$-Monotonicity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is* **family completely $\bigvee$-monotone** *if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,

$$T' \leq \bigvee_{T \in \mathcal{T}}^{\mathcal{I}} T \quad \textit{implies} \quad \Omega(T') \leq \bigvee_{T \in \mathcal{T}}^{\mathbf{F}} \Omega(T).$$

- $\mathcal{I}$ *is* **left completely $\bigvee$-monotone** *if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,

$$\overleftarrow{T'} \leq \bigvee_{T \in \mathcal{T}}^{\mathcal{I}} \overleftarrow{T} \quad \textit{implies} \quad \Omega(T') \leq \bigvee_{T \in \mathcal{T}}^{\mathbf{F}} \Omega(T).$$

- $\mathcal{I}$ *is* **right completely $\bigvee$-monotone** *if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,

$$T' \leq \bigvee_{T \in \mathcal{T}}^{\mathcal{I}} T \quad \textit{implies} \quad \Omega(\overleftarrow{T'}) \leq \bigvee_{T \in \mathcal{T}}^{\mathbf{F}} \Omega(\overleftarrow{T}).$$

- $\mathcal{I}$ *is* **system completely $\bigvee$-monotone** *if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$,

$$T' \leq \bigvee_{T \in \mathcal{T}}^{\mathcal{I}} T \quad \textit{implies} \quad \Omega(T') \leq \bigvee_{T \in \mathcal{T}}^{\mathbf{F}} \Omega(T).$$

Sometimes we will use the abbreviated form **$\mathbf{c}^\vee$-monotonicity** to refer to complete $\bigvee$-monotonicity.

We have seen in Lemma 170 that family monotonicity (protoalgebraicity) implies stability. Since family complete monotonicity is a stronger property than family monotonicity, we get Part (a) of the following:

Lemma 194 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is family completely* $\bigvee$*-monotone, then it is stable.*

(b) If $\mathcal{I}$ *is left completely* $\bigvee$*-monotone, it is stable.*

Proof:

(a) If $\mathcal{I}$ is family completely $\bigvee$-monotone, then it is, a fortiori, family monotone. Thus, the result follows from Lemma 170.

(b) Suppose that $\mathcal{I}$ is left $\mathrm{c}^\vee$-monotone and let $T \in \mathrm{ThFam}(\mathcal{I})$. By Proposition 42, $\overleftarrow{\overleftarrow{T}} = \overleftarrow{T}$. Applying left $\mathrm{c}^\vee$-monotonicity, we get that $\Omega(\overleftarrow{T}) = \Omega(T)$. Hence $\mathcal{I}$ is stable.

■

Family completely $\bigvee$-monotone π-institutions are both left and right completely $\bigvee$-monotone. And, conversely, if a π-institution is both left and right $c^\vee$-monotone, then it is family $c^\vee$-monotone. This parallels Proposition 186, which concerned the case of $c^\cup$-monotonicity properties.

Proposition 195 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is family completely* $\bigvee$*-monotone if and only if it is both left and right completely* $\bigvee$*-monotone.*

Proof: Suppose, first, that $\mathcal{I}$ is family completely $\bigvee$-monotone.

- Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\overleftarrow{T'} \leq \bigvee^{\mathcal{I}}_{T \in \mathcal{T}} \overleftarrow{T}$. Applying family $c^\vee$-monotonicity, we get $\Omega(\overleftarrow{T'}) \leq \bigvee^{\mathbf{F}}_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$. However, by Lemma 185, $\mathcal{I}$ is stable. Hence we get $\Omega(T') \leq \bigvee^{\mathbf{F}}_{T \in \mathcal{T}} \Omega(T)$. We conclude that $\mathcal{I}$ is left completely $\bigvee$-monotone.

- Next, let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $T' \leq \bigvee^{\mathcal{I}}_{T \in \mathcal{T}} T$. Applying family $c^\vee$-monotonicity, we get $\Omega(T') \leq \bigvee^{\mathbf{F}}_{T \in \mathcal{T}} \Omega(T)$. Once more, by Lemma 185, $\mathcal{I}$ is stable. Hence we get $\Omega(\overleftarrow{T'}) \leq \bigvee^{\mathbf{F}}_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$. We conclude that $\mathcal{I}$ is right completely $\bigvee$-monotone.

Suppose, conversely, that $\mathcal{I}$ is both left and right completely $\bigvee$-monotone and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $T' \leq \bigvee^{\mathcal{I}}_{T \in \mathcal{T}} T$. Then, by right $c^\vee$-monotonicity, we get that $\Omega(\overleftarrow{T'}) \leq \bigvee^{\mathbf{F}}_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$. But since $\mathcal{I}$ is left completely $\bigvee$-monotone, by Lemma 185, it is stable, whence we get $\Omega(T') \leq \bigvee^{\mathbf{F}}_{T \in \mathcal{T}} \Omega(T)$. Therefore, $\mathcal{I}$ is family completely $\bigvee$-monotone. ■

If a π-institution $\mathcal{I}$ is left or right completely $\bigvee$-monotone, then it is also system completely $\bigvee$-monotone. This is an analog of Proposition 187, which addressed the case of $c^\cup$-monotonicity.

Proposition 196 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is left* $c^\vee$*-monotone, then it is system* $c^\vee$*-monotone;*

(b) If $\mathcal{I}$ *is right* $c^\vee$*-monotone, then it is system* $c^\vee$*-monotone.*

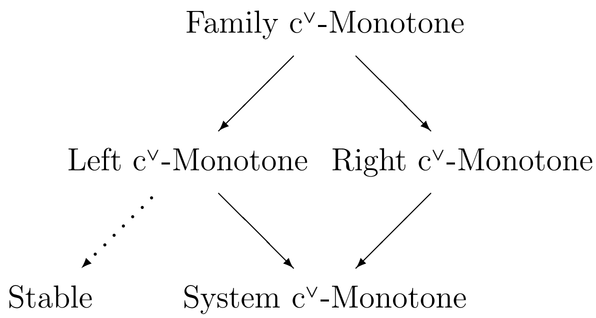
Proof:

(a) Suppose $\mathcal{I}$ is left $c^\vee$-monotone and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $T' \leq \bigvee^{\mathcal{I}}_{T \in \mathcal{T}} T$. Since $\mathcal{T} \cup \{T'\}$ is a collection of theory systems, we get $\overleftarrow{T'} \leq \bigvee^{\mathcal{I}}_{T \in \mathcal{T}} \overleftarrow{T}$. Hence, applying left $c^\vee$-monotonicity, we get $\Omega(T') \leq \bigvee^{\mathbf{F}}_{T \in \mathcal{T}} \Omega(T)$. Thus, $\mathcal{I}$ is system $c^\vee$-monotone.

(b) Suppose $\mathcal{I}$ is right $c^\vee$-monotone and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $T' \leq \bigvee^{\mathcal{I}}_{T\in\mathcal{T}} T$. Applying right $c^\vee$-monotonicity, we get $\Omega(\overleftarrow{T'}) \leq \bigvee^{\mathbf{F}}_{T\in\mathcal{T}} \Omega(\overleftarrow{T})$. Since $\mathcal{T} \cup \{T'\}$ is a collection of theory systems, we now get $\Omega(T') \leq \bigvee^{\mathbf{F}}_{T\in\mathcal{T}} \Omega(T)$. Thus, $\mathcal{I}$ is system $c^\vee$-monotone.

■

In terms of complete $\bigvee$-monotonicity, we have established the following hierarchy, which exactly mirrors the hierarchy of $c^\cup$-monotonicity classes:



Now we give examples of π-institutions to show that the inclusions depicted in this diagram are proper. We first give an example of a π-institution that is left $c^\vee$-monotone but not right $c^\vee$-monotone. This shows that:

- The class of family $c^\vee$-monotone π-institutions is properly contained in the class of all left $c^\vee$-monotone π-institutions;
- The class of all system $c^\vee$-monotone π-institutions properly includes the class of all right $c^\vee$-monotone π-institutions.

Example 197 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

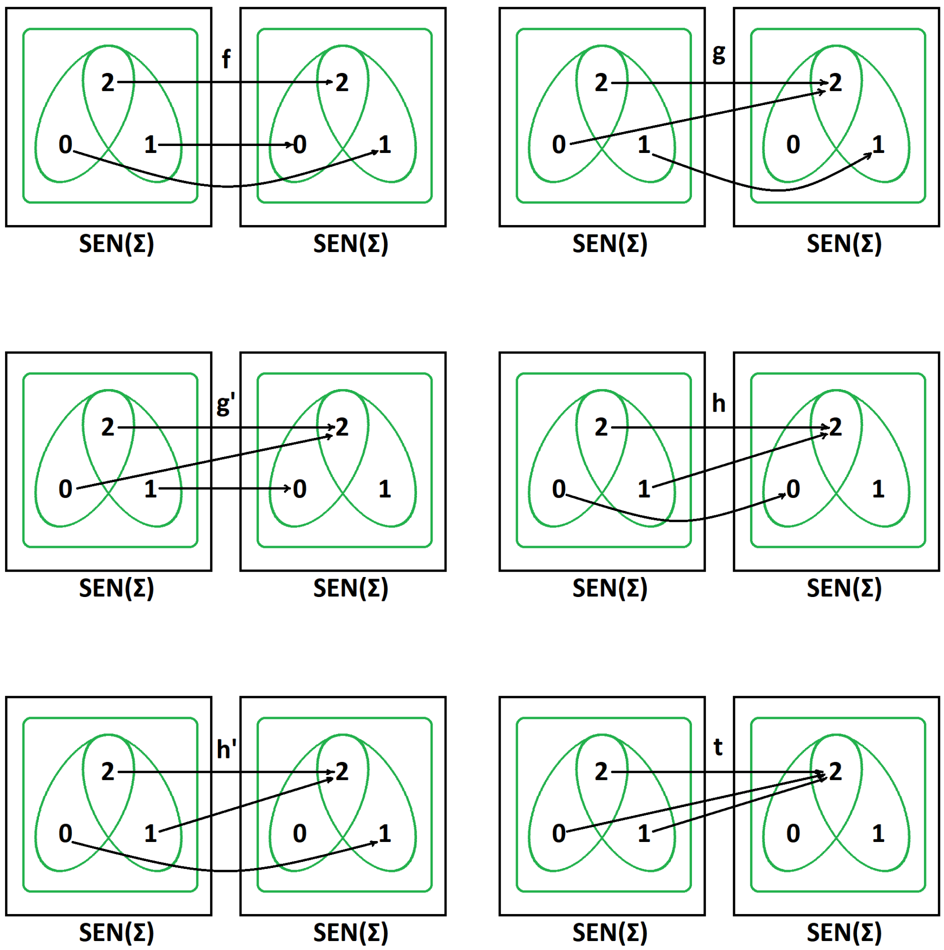
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and six non-identity morphisms* $f, g, g', h, h', t : \Sigma \to \Sigma$, *in which composition is defined by the following table, whose entry in row* k *and column* ℓ *is the result of the composition* $\ell \circ k$:

$\circ$	f	g	g'	h	h'	t
f	f	h'	h	g'	g	t
g	g'	g	g'	t	t	t
g'	g	t	t	g'	g	t
h	h'	t	t	h	h'	t
h'	h	h'	h	t	t	t
t	t	t	t	t	t	t

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given, on objects, by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and, on morphisms, by the following table, whose entries in column* k *give* *the values of the function* $\mathrm{SEN}^\flat(k) : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SEN}^\flat(\Sigma)$*:*

x	f	g	g'	h	h'	t
0	1	2	2	0	1	2
1	0	1	0	2	2	2
2	2	2	2	2	2	2

- $N^\flat$ *is the trivial clone.*



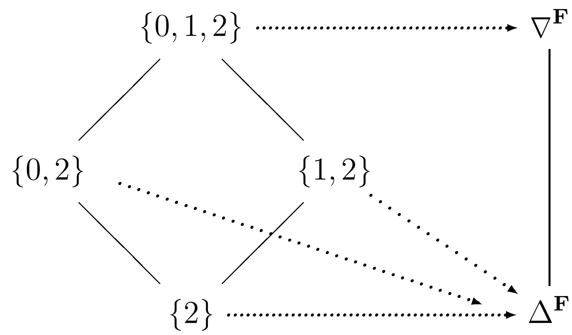
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

$$\mathcal{C}_\Sigma = \{\{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}.$$

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$*:*

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{0,2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



$\mathcal{I}$ *has only two theory systems,* $\mathrm{Thm}(\mathcal{I}) = \{\{2\}\}$, *and* $\mathrm{SEN} = \{\{0,1,2\}\}$.

To show that $\mathcal{I}$ *is left completely monotone, assume that, for some* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T'} \leq \bigvee^{\mathcal{I}}_{T \in \mathcal{T}} \overleftarrow{T}$.

- *If* $\bigvee^{\mathcal{I}}_{T\in\mathcal{T}} \overleftarrow{T} = \{\{0,1,2\}\}$, *then* $\{\{0,1,2\}\} \in \mathcal{T}$ *and, hence,*

$$\Omega(T') \leq \nabla^{\mathbf{F}} = \Omega(\{\{0,1,2\}\}) \leq \bigvee_{T\in\mathcal{T}}^{\mathbf{F}} \Omega(T);$$

- *If* $\bigvee^{\mathcal{I}}_{T\in\mathcal{T}} \overleftarrow{T} = \{\{2\}\}$, *then* $T' \neq \{\{0,1,2\}\}$, *whence*

$$\Omega(T') = \Delta^{\mathbf{F}} \leq \bigvee_{T\in\mathcal{T}}^{\mathbf{F}} \Omega(T).$$

Thus, in any case, $\Omega(T') \leq \bigvee^{\mathbf{F}}_{T\in\mathcal{T}} \Omega(T)$ *and* $\mathcal{I}$ *is left completely* $\bigvee$*-monotone.*

On the other hand, we have

$$\{\{0,1,2\}\} \leq \{\{0,2\}\} \vee^{\mathcal{I}} \{\{1,2\}\},$$

whereas

$$\begin{array}{rcl}\Omega(\overleftarrow{\{\{0,1,2\}\}}) & = & \Omega(\{\{0,1,2\}\}) = \nabla^{\mathbf{F}} \\ & \nleq & \Delta^{\mathbf{F}} \\ & = & \Omega(\{\{2\}\}) \vee^{\mathbf{F}} \Omega(\{\{2\}\}) \\ & = & \Omega(\overleftarrow{\{\{0,2\}\}}) \vee^{\mathbf{F}} \Omega(\overleftarrow{\{\{1,2\}\}}).\end{array}$$

Therefore, $\mathcal{I}$ *is not right completely* $\bigvee$*-monotone.*

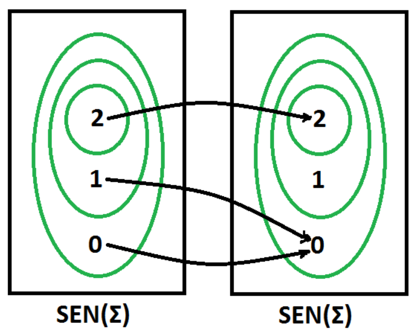
We now give an example of a right $c^\vee$-monotone π-institution that fails to be left $c^\vee$-monotone. This will show that:

- The class of family $c^\vee$-monotone π-institutions is properly contained in the class of right $c^\vee$-monotone π-institutions;

- The class of left $c^\vee$-monotone π-institutions is a proper subclass of the class of system $c^\vee$-monotone π-institutions.

Example 198 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial clone.*

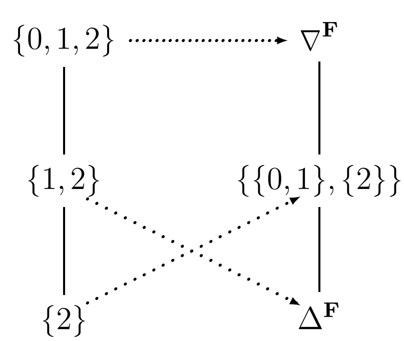


Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}$.

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$*:*

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

The structure of the lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



Taking into account that $\overleftarrow{\{\{1,2\}\}} = \{\{2\}\}$*, we can see that* $\mathcal{I}$ *is right* $c^\vee$*-monotone.*

On the other hand, for $T = \{\{1,2\}\}$ *and* $T' = \{\{2\}\}$*, we have* $\overleftarrow{T'} = \{\{2\}\} \leq \overleftarrow{T}$*, but* $\Omega(T') = \{\{\{0,1\},\{2\}\}\} \nleq \Delta^{\mathbf{F}} = \Omega(T)$*. Hence* $\mathcal{I}$ *is not left* $c^{\vee}$*-monotone.*

As we saw in Theorem 190, the various complete $\bigcup$-monotonicity properties defined based on the union operation transfer from the theory families/systems of a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ to the $\mathcal{I}$-filter families/systems on arbitrary $\mathbf{F}$-algebraic systems. On the other hand, the $\bigvee$-monotonicity versions introduced in this section do not transfer in general. The problem appears to be that the commutativity of unions with inverse surjective morphisms, i.e., $\bigcup_{T\in\mathcal{T}}\alpha^{-1}(T) = \alpha^{-1}(\bigcup_{T\in\mathcal{T}}T)$, ceases to hold when unions are replaced by joins. In that case, one has, in general, a proper inclusion instead of an equality. To describe it, let us again introduce some notation. Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Let, also $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system. Denote by $\bigvee^{\mathcal{I},\mathcal{A}}\mathcal{T} = \bigvee^{\mathcal{I},\mathcal{A}}_{T\in\mathcal{T}}T$ the join of a collection $\mathcal{T}$ of $\mathcal{I}$-filter families of $\mathcal{A}$ in the complete lattice $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})$ of $\mathcal{I}$-filter families of $\mathcal{A}$. Analogously, denote by $\bigvee^{\mathcal{A}}\Theta = \bigvee^{\mathcal{A}}_{\theta\in\Theta}\theta$ the join of a collection Θ of congruence systems on $\mathcal{A}$ in the complete lattice $\mathbf{ConSys}(\mathcal{A})$ of congruence systems on $\mathcal{A}$. According to this notation, we get, in general, that

$$\bigvee_{T\in\mathcal{T}}^{\mathcal{I}}\alpha^{-1}(T) \subsetneq \alpha^{-1}\left(\bigvee_{T\in\mathcal{T}}^{\mathcal{I},\mathcal{A}}T\right).$$

The following example showcases this proper inclusion.

Example 199 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

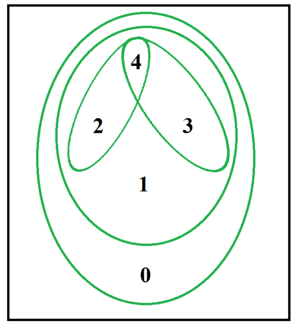
- $\mathbf{Sign}^\flat$ *is the trivial category, with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3,4\}$*;*
- $N^\flat$ *is the trivial clone.*

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by setting*

$$\mathcal{C}_\Sigma = \{\{4\},\{2,4\},\{3,4\},\{1,2,3,4\},\{0,1,2,3,4\}\}.$$

Next, consider the $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*, defined as follows:*

- *The algebraic system* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *is specified by the following data:*
  - $\mathbf{Sign}$ *is the trivial category, with object* Σ'*;*

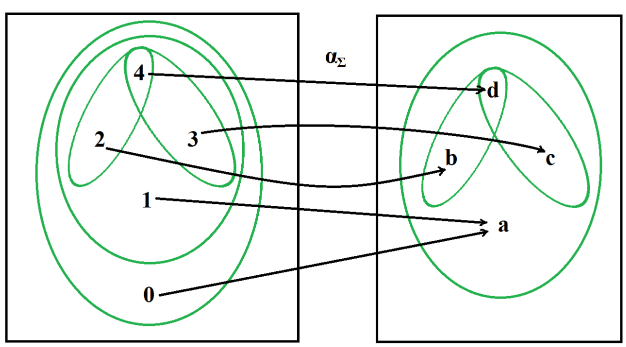


– SEN : **Sign** → **Set** *is given by* $\mathrm{SEN}(\Sigma') = \{a, b, c, d\}$*;*

– N *is the trivial clone.*

- $F : \mathbf{Sign}^\flat \to \mathbf{Sign}$ *is the trivial functor taking* Σ *to* Σ'*;*
- $\alpha : \mathrm{SEN}^\flat \to \mathrm{SEN} \circ F$ *is determined by letting* $\alpha_\Sigma : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SEN}(\Sigma')$ *be given by*

$x \in \mathrm{SEN}^\flat(\Sigma)$	$\alpha_\Sigma(x)$
0	a
1	a
2	b
3	c
4	d



Based on Lemma 51, we get that

$$\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \{\{\{d\}\}, \{\{b, d\}\}, \{\{c, d\}\}, \{\{a, b, c, d\}\}\}.$$

Now we can easily verify that

$$\alpha^{-1}(\{\{b, d\}\}) \vee^{\mathcal{I}} \alpha^{-1}(\{\{c, d\}\}) = \{\{2, 4\}\} \vee^{\mathcal{I}} \{\{3, 4\}\} = \{\{1, 2, 3, 4\}\},$$

whereas

$$\alpha^{-1}(\{\{b,d\}\} \vee^{\mathcal{I},\mathcal{A}} \{\{c,d\}\}) = \alpha^{-1}(\{\{a,b,c,d\}\}) = \{\{0,1,2,3,4\}\}.$$

Thus, for $\mathcal{T} = \{\{\{b,d\}\},\{\{c,d\}\}\}$, *we get* $\bigvee^{\mathcal{I}}_{T\in\mathcal{T}} \alpha^{-1}(T) \lneq \alpha^{-1}(\bigvee^{\mathcal{I},\mathcal{A}}_{T\in\mathcal{T}} T)$.

We may characterize family and system c$^\vee$-monotonicity in terms of the complete order preservation of mappings from posets of theory families/ systems into posets of congruence systems. Given complete lattices $\mathbf{P} = \langle P,\le\rangle$ and $\mathbf{Q} = \langle Q,\preccurlyeq\rangle$, call a mapping $f : P \to Q$ **completely order preserving** if, for all $\{x\} \cup Y \subseteq P$,

$$x \le \bigvee_{y\in Y}^{\mathbf{P}} y \quad \text{implies} \quad f(x) \preccurlyeq \bigvee_{y\in Y}^{\mathbf{Q}} f(y).$$

Proposition 200 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *The following statements are equivalent:*

(a) $\mathcal{I}$ *is family completely* $\bigvee$*-monotone;*

(b) $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}(\mathcal{I})$ *is completely order preserving.*

Similarly, for system c-monotonicity, we have

Proposition 201 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *The following statements are equivalent:*

(a) $\mathcal{I}$ *is system completely* $\bigvee$*-monotone;*

(b) $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}(\mathcal{I})$ *is completely order preserving.*

Next we look at the relationships that hold between protoalgebraicity and prealgebraicity, on the one hand, and the various c$^\vee$-monotonicity properties, on the other. More precisely, we show that left complete $\bigvee$-monotonicity implies protoalgebraicity and that system complete $\bigvee$-monotonicity implies prealgebraicity. Once more, this theorem parallels Theorem 191, which dealt with $\bigcup$-monotonicity.

Theorem 202 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is left* c$^\vee$*-monotone, then it is protoalgebraic;*

(b) If $\mathcal{I}$ *is system* c$^\vee$*-monotone, then it is prealgebraic.*

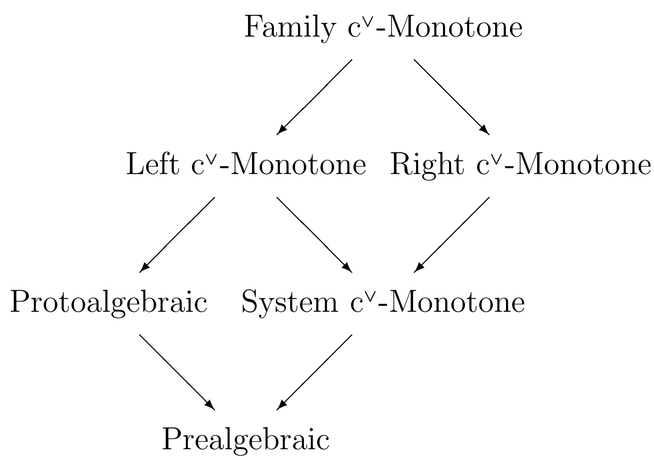
Proof:

(a) Suppose $\mathcal{I}$ is left $c^\vee$-monotone and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$. Then, by Lemma 1, we get $\overleftarrow{T} \leq \overleftarrow{T'}$. Thus, by left $c^\vee$-monotonicity, $\Omega(T) \leq \Omega(T')$. Hence $\mathcal{I}$ is protoalgebraic.

(b) Suppose that $\mathcal{I}$ is system $c^\vee$-monotone and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $T \leq T'$. Then we get right away from system $c^\vee$-monotonicity that $\Omega(T) \leq \Omega(T')$ and, therefore, $\mathcal{I}$ is prealgebraic.

■

We have now established the following hierarchy of **monotonicity** and **complete $\bigvee$-monotonicity** properties, which also mirrors the combined hierarchy of monotonicity and complete $\bigcup$-monotonicity properties:



We now provide an example to show that the $c^\vee$-monotonicity classes are proper subclasses of the monotonicity classes. Namely, we construct a protoalgebraic π-institution that fails to be system $c^\vee$-monotone.

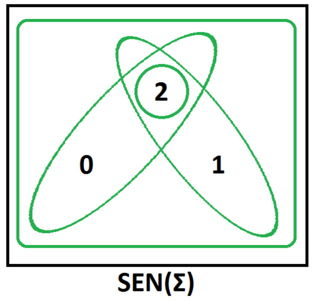
Example 203 *Define the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *as follows:*

- $\mathbf{Sign}^\flat$ *is a trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$*;*
- $N^\flat$ *is the clone generated by the unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, given by*

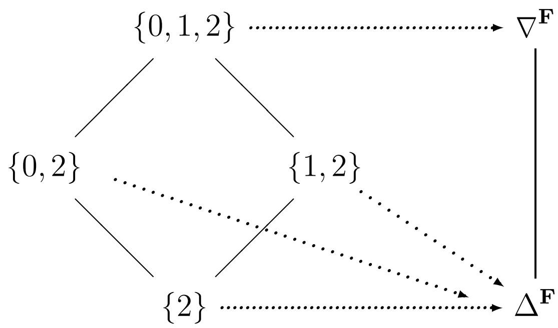
$x \in \mathrm{SEN}^\flat(\Sigma)$	$\sigma^\flat_\Sigma(x)$
0	1
1	2
2	0

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

$$\mathcal{C}_\Sigma = \{\{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}.$$



It is easy to see that the lattices of theory families and corresponding Leibniz congruence systems are as given in the diagram.



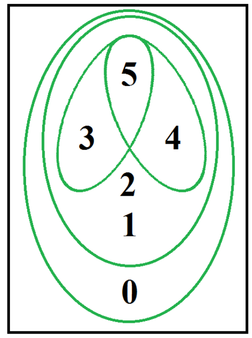
From the diagram one can verify immediately that $\mathcal{I}$ is protoalgebraic, On the other hand, we have $\{\{0,1,2\}\} \leq \{\{0,2\}\} \vee^{\mathcal{I}} \{\{1,2\}\}$, but, obviously, $\Omega(\{\{0,1,2\}\}) \nleq \Omega(\{\{0,2\}\}) \vee^{\mathbf{F}} \Omega(\{\{1,2\}\})$. Taking into account that $\mathcal{I}$ is systemic, we conclude that $\mathcal{I}$ fails to be system $c^{\vee}$-monotone.

We conclude this section with two examples showing that the classes of $c^{\cup}$-monotone and $c^{\vee}$-monotone π-institutions are incomparable. The first example shows that there exists a $c^{\cup}$-monotone π-institution which is not $c^{\vee}$-monotone.

Example 204 *Define the algebraic system $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat} \rangle$ as follows:*

- $\mathbf{Sign}^{\flat}$ *is a trivial category with object* Σ*;*
- $\mathrm{SEN}^{\flat} : \mathbf{Sign}^{\flat} \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^{\flat}(\Sigma) = \{0,1,2,3,4,5\}$*;*
- $N^{\flat}$ *is the clone generated by the unary natural transformations* $\rho^{\flat}, \sigma^{\flat}, \tau^{\flat} : \mathrm{SEN}^{\flat} \to \mathrm{SEN}^{\flat}$*, given by*

$x \in \mathrm{SEN}^{\flat}(\Sigma)$	$\rho^{\flat}_{\Sigma}(x)$	$\sigma^{\flat}_{\Sigma}(x)$	$\tau^{\flat}_{\Sigma}(x)$
0	5	0	0
1	1	1	1
2	2	2	2
3	3	5	3
4	4	4	5
5	5	5	5

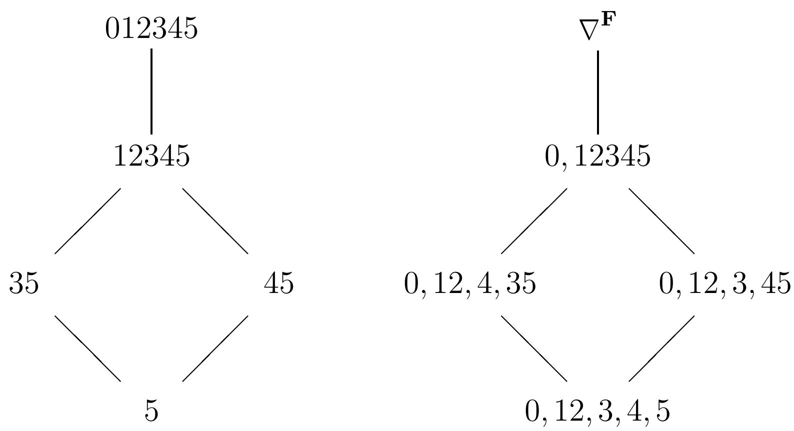


SEN(Σ)

Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by setting

$$\mathcal{C}_\Sigma = \{\{5\}, \{3,5\}, \{4,5\}, \{1,2,3,4,5\}, \{0,1,2,3,4,5\}\}.$$

The lattice of theory families and the corresponding Leibniz congruence systems (in block form) are given in the following diagram.



From the diagram one can verify that, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $T' \le \bigcup_{T \in \mathcal{T}} T$ implies $T' \le T$, for some $T \in \mathcal{T}$. Therefore, we get $\Omega(T') \le \bigcup_{T \in \mathcal{T}} \Omega(T)$ and, hence, $\mathcal{I}$ is $c^\cup$-monotone. On the other hand,

$$\{\{1,2,,3,4,5\}\} \le \{\{3,5\}\} \vee^{\mathcal{I}} \{\{4,5\}\},$$

whereas

$$\begin{array}{rcl} \Omega(\{\{1,2,3,4,5\}\}) & = & \{\{1,12345\}\} \\ & \nleq & \{\{0,12,345\}\} \\ & = & \{\{0,12,4,35\}\} \vee^{\mathbf{F}} \{\{0,12,3,45\}\} \\ & = & \Omega(\{\{3,5\}\}) \vee^{\mathbf{F}} \Omega(\{\{4,5\}\}). \end{array}$$

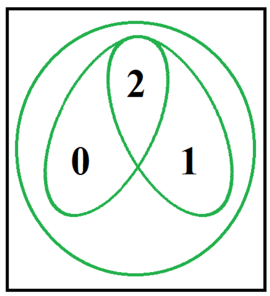
Thus, $\mathcal{I}$ is not $c^\vee$-monotone.

The second example exhibits a $c^\vee$-monotone π-institution which is not $c^\cup$-monotone.

Example 205 *Define the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *as follows:*

- $\mathbf{Sign}^\flat$ *is a trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the clone generated by the unary natural transformations* $\sigma^\flat, \tau^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$, *given by*

$x \in \mathrm{SEN}^\flat(\Sigma)$	$\sigma^\flat_\Sigma(x)$	$\tau^\flat_\Sigma(x)$
0	2	0
1	1	2
2	2	2

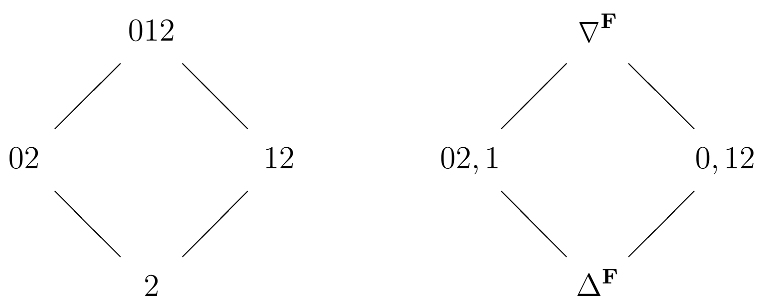


SEN(Σ)

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

$$C_\Sigma = \{\{2\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

The lattice of theory families and the corresponding Leibniz congruence systems (in block form) are given in the following diagram.



Note that

$$\{\{0,1,2\}\} = \{\{0,2\}\} \cup \{\{1,2\}\} = \{\{0,2\}\} \vee^{\mathcal{I}} \{\{1,2\}\}.$$

But, even though

$$\begin{array}{rcl} \Omega(\{\{0,1,2\}\}) & = & \nabla^{\mathbf{F}} \\ & = & \{\{02,1\}\} \vee^{\mathbf{F}} \{\{0,12\}\} \\ & = & \Omega(\{\{0,2\}\}) \vee^{\mathbf{F}} \Omega(\{\{1,2\}\}), \end{array}$$

we have

$$\begin{array}{rcl}\Omega(\{\{0,1,2\}\}) & = & \nabla^{\mathbf{F}} \\ & \nleq & \{\{02,1\}\} \cup \{\{0,12\}\} \\ & = & \Omega(\{\{0,2\}\}) \cup \Omega(\{\{1,2\}\}),\end{array}$$

since $\langle 0,1\rangle \in \nabla^{\mathbf{F}}_\Sigma$, *whereas* $\langle 0,1\rangle \notin \{\{02,1\}\} \cup \{\{0,12\}\}$. *Thus,* $\mathcal{I}$ *is* $c^\vee$*-monotone but not* $c^\cup$*-monotone.*

3.6 Injectivity

In this section we study classes of π-institutions defined using injectivity properties of the Leibniz operator.

Definition 206 (Injectivity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is called* **family injective** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,
$$\Omega(T) = \Omega(T') \quad \textit{implies} \quad T = T'.$$
- $\mathcal{I}$ *is called* **left injective** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,
$$\Omega(T) = \Omega(T') \quad \textit{implies} \quad \overleftarrow{T} = \overleftarrow{T'}.$$
- $\mathcal{I}$ *is called* **right injective** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,
$$\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'}) \quad \textit{implies} \quad T = T'.$$
- $\mathcal{I}$ *is called* **system injective** *if, for all* $T, T' \in \mathrm{ThSys}(\mathcal{I})$,
$$\Omega(T) = \Omega(T') \quad \textit{implies} \quad T = T'.$$

First, we show that right injectivity is so strong that it implies systemicity and, hence, a fortiori, stability.

Lemma 207 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is right injective, then it is systemic.*

Proof: Suppose that $\mathcal{I}$ is right injective and let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, we have, by Proposition 42, that $\overleftarrow{\overleftarrow{T}} = \overleftarrow{T}$. Therefore, we get $\Omega(\overleftarrow{\overleftarrow{T}}) = \Omega(\overleftarrow{T})$. Hence, by right injectivity, we get that $\overleftarrow{T} = T$. Thus, $T \in \mathrm{ThSys}(\mathcal{I})$. This proves that $\mathrm{ThFam}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$, whence $\mathcal{I}$ is systemic. ∎

Next we look into establishing the *injectivity hierarchy* of π-institutions. The following relationships can be established between the four injectivity classes.

Proposition 208 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is right injective, then it is family injective;*

(b) If $\mathcal{I}$ *is family injective, then it is left injective;*

(c) If $\mathcal{I}$ *is left injective, then it is system injective.*

Proof:

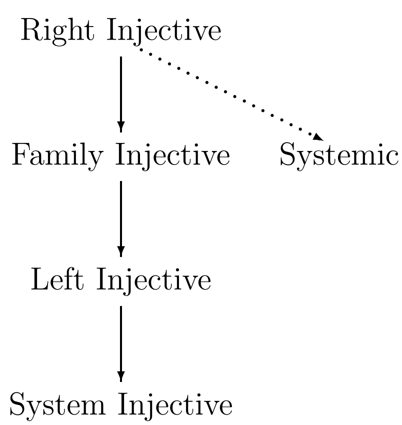
(a) Suppose that $\mathcal{I}$ is right injective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. By Lemma 207, $\mathcal{I}$ is systemic, whence $\overleftarrow{T} = T$ and $\overleftarrow{T'} = T'$. Thus, we get $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$. Now applying right injectivity, we get $T = T'$. This proves that $\mathcal{I}$ is family injective.

(b) Suppose that $\mathcal{I}$ is family injective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, by family injectivity, we get $T = T'$, whence $\overleftarrow{T} = \overleftarrow{T'}$. Therefore $\mathcal{I}$ is left injective.

(c) Suppose that $\mathcal{I}$ is left injective and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. By left injectivity, we conclude that $\overleftarrow{T} = \overleftarrow{T'}$. However, since T, T' are theory systems, we have $\overleftarrow{T} = T$ and $\overleftarrow{T'} = T'$. Hence we get $T = T'$ and $\mathcal{I}$ is system injective.

■

We have now established the following **injectivity hierarchy** of π-institutions.



There are two additional properties that can be formulated concerning the relationships between these classes. First, it turns out that the separating property between system injectivity and right injectivity is exactly systemicity.

Proposition 209 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is right injective if and only if it is system injective and systemic.*

Proof: Suppose, first, that $\mathcal{I}$ is right injective. Then, by Lemma 207, it is systemic and by Proposition 208 it is system injective.

Suppose conversely, that $\mathcal{I}$ is system injective and systemic and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$. By system injectivity we get $\overleftarrow{T} = \overleftarrow{T'}$. Hence, by systemicity, $T = T'$. Thus, $\mathcal{I}$ is right injective. ■

Second, we show that system injectivity reinforced with stability implies left injectivity.

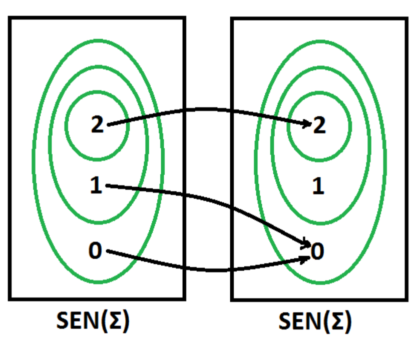
Proposition 210 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is system injective and stable, then it is left injective.*

Proof: Suppose that $\mathcal{I}$ is system injective and stable and consider $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, by stability $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$. Hence, since $\overleftarrow{T}, \overleftarrow{T'} \in \mathrm{ThSys}(\mathcal{I})$, by system injectivity, $\overleftarrow{T} = \overleftarrow{T'}$. This shows that $\mathcal{I}$ is left injective. ■

We now present three examples to show that all inclusions established between injectivity classes and depicted in the diagram above are proper inclusions. The first example will show that the class of right injective π-institutions is a proper subclass of the class of family injective π-institutions.

Example 211 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial clone.*



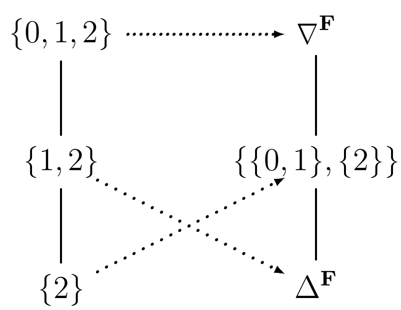
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}$.

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$*:*

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

Since $\{\{1,2\}\}$ *is a theory family that is not a theory system,* $\mathcal{I}$ *is not systemic.*

The lattice of theory families and that of the corresponding Leibniz congruence systems are depicted below:



It is obvious from the diagram that $\mathcal{I}$ *is family injective, since each of the three theory families has a different Leibniz congruence system.*

On the other hand, $\mathcal{I}$ *is not right injective. This can be seen either by applying Proposition 209 or directly. Take* $T = \{\{2\}\}$ *and* $T' = \{\{1,2\}\}$*. Then, we have* $\overleftarrow{T} = \overleftarrow{T'} = T$*, whence* $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$*, whereas, obviously,* $T \neq T'$*.*

The second example shows that the class of family injective π-institutions is properly included in the class of left injective π-institutions.

Example 212 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

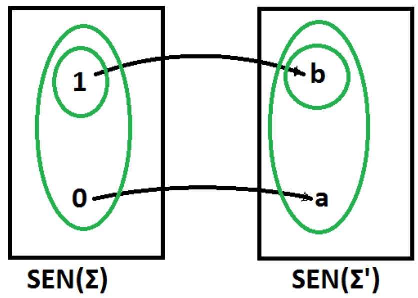
- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*,* $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$*,* $\mathrm{SEN}^\flat(f)(1) = b$*;*
- $N^\flat$ *is the trivial clone.*

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

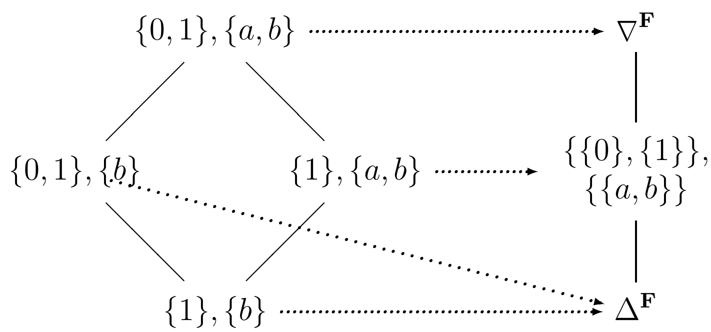
$$\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\},\{a,b\}\}.$$

The following table shows the action of $\overleftarrow{\ }$ *on theory families, where rows correspond to* T_Σ *and columns to* $T_{\Sigma'}$ *and each entry is written as* $\overleftarrow{T}_\Sigma, \overleftarrow{T}_{\Sigma'}$*.*

$\leftarrow$	$\{b\}$	$\{a,b\}$
$\{1\}$	$\{1\},\{b\}$	$\{1\},\{a,b\}$
$\{0,1\}$	$\{1\},\{b\}$	$\{0,1\},\{a,b\}$



The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right:



Since the only two theory families that have the same Leibniz congruence system are $\{\{0,1\},\{b\}\}$ *and* $\{\{1\},\{b\}\}$ *and it holds that*

$$\overleftarrow{\{\{0,1\},\{b\}\}} = \overleftarrow{\{\{1\},\{b\}\}} = \{\{1\},\{b\}\},$$

we conclude that $\mathcal{I}$ *is left injective.*

From the diagram, it is also clear that $\mathcal{I}$ *is not family injective, since the two theory families* $\{\{0,1\},\{b\}\}$ *and* $\{\{1\},\{b\}\}$ *have the same Leibniz congruence system.*

In conclusion we have shown that $\mathcal{I}$ *is left injective but not family injective.*

The next example shows that the class of left injective π-institutions is properly included in the class of all system injective π-institutions.

Example 213 *Define the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *as follows:*

- $\mathbf{Sign}^\flat$ *is the two object category with objects* Σ, Σ' *and two (non-identity) morphisms*

$$\Sigma \underset{g}{\overset{f}{\rightleftarrows}} \Sigma'$$

such that $g \circ f = i_\Sigma$ *and* $f \circ g = i_{\Sigma'}$*;*

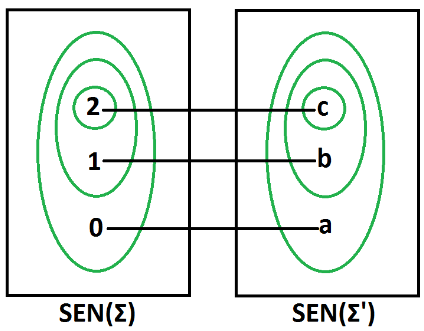
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b,c\}$ *and*

$$\begin{array}{lll} \mathrm{SEN}^\flat(f)(0) = a, & \mathrm{SEN}^\flat(f)(1) = b, & \mathrm{SEN}^\flat(f)(2) = c; \\ \mathrm{SEN}^\flat(g)(a) = 0, & \mathrm{SEN}^\flat(g)(b) = 1, & \mathrm{SEN}^\flat(g)(c) = 2; \end{array}$$

- $N^\flat$ *is the clone on* $\mathrm{SEN}^\flat$ *generated by the natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *specified by*

$x \in \mathrm{SEN}^\flat(\Sigma)$	$\sigma^\flat_\Sigma(x)$
0	0
1	1
2	0

$y \in \mathrm{SEN}^\flat(\Sigma')$	$\sigma^\flat_{\Sigma'}(y)$
a	a
b	b
c	a



Next, define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

$$\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\{c\},\{b,c\},\{a,b,c\}\}.$$

The table giving the action of $\overleftarrow{\ }$ *on theory families is shown below:*

$\leftarrow$	$\{c\}$	$\{b,c\}$	$\{a,b,c\}$
$\{2\}$	$\{2\},\{c\}$	$\{2\},\{c\}$	$\{2\},\{c\}$
$\{1,2\}$	$\{2\},\{c\}$	$\{1,2\},\{b,c\}$	$\{1,2\},\{b,c\}$
$\{0,1,2\}$	$\{2\},\{c\}$	$\{1,2\},\{b,c\}$	$\{0,1,2\},\{a,b,c\}$

The following table gives the Leibniz congruence systems associated with each of the nine theory families of $\mathcal{I}$*, where we have denoted by* θ *the Leibniz congruence system with* $\theta_\Sigma = \{\{0,1\},\{2\}\}$ *and* $\theta_{\Sigma'} = \{\{a,b\},\{c\}\}$*:*

$\Omega(\{T_\Sigma, T_{\Sigma'}\})$	$\{c\}$	$\{b,c\}$	$\{a,b,c\}$
$\{2\}$	θ	$\Delta^{\mathbf{F}}$	θ
$\{1,2\}$	$\Delta^{\mathbf{F}}$	$\Delta^{\mathbf{F}}$	$\Delta^{\mathbf{F}}$
$\{0,1,2\}$	θ	$\Delta^{\mathbf{F}}$	$\nabla^{\mathbf{F}}$

Observe, first, that there are only three theory systems $\{\{2\},\{c\}\}$, $\{\{1,2\},\{b,c\}\}$ *and* $\{\{0,1,2\},\{a,b,c\}\}$. *To each of these corresponds a different Leibniz congruence system. It follows that* $\mathcal{I}$ *is system injective.*

On the other hand, for $T = \{\{2\},\{b,c\}\}$ *and* $T' = \{\{1,2\},\{b,c\}\}$, *we have* $\Omega(T) = \Omega(T') = \Delta^{\mathbf{F}}$, *but* $\overleftarrow{T} = \{\{2\},\{c\}\} \neq \{\{1,2\},\{b,c\}\} = \overleftarrow{T'}$. *Therefore* $\mathcal{I}$ *is not left injective.*

The injectivity properties transfer from the theory families/systems of a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ to the collections of all $\mathcal{I}$-filter families/systems on arbitrary $\mathbf{F}$-algebraic systems.

Theorem 214 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$.

- *(a)* $\mathcal{I}$ *is family injective if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$ *implies* $T = T'$.
- *(b)* $\mathcal{I}$ *is left injective if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$ *implies* $\overleftarrow{T} = \overleftarrow{T'}$.
- *(c)* $\mathcal{I}$ *is right injective if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(\overleftarrow{T}) = \Omega^{\mathcal{A}}(\overleftarrow{T'})$ *implies* $T = T'$.
- *(d)* $\mathcal{I}$ *is system injective if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$ *implies* $T = T'$.

Proof: We will prove Parts (a) and (b) to establish the method.

(a) The "if" follows by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$ and taking into account that $\mathrm{ThFam}(\mathcal{I}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F})$, by Lemma 51.

For the "only if", suppose that $\mathcal{I}$ is family injective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$. Then $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) = \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. So, by Proposition 24, $\Omega(\alpha^{-1}(T)) = \Omega(\alpha^{-1}(T'))$. Since, by Lemma 51, we have that $\alpha^{-1}(T), \alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$, we get, by applying family injectivity, $\alpha^{-1}(T) = \alpha^{-1}(T')$. Finally, the surjectivity of $\langle F, \alpha\rangle$ yields that $T = T'$.

(b) The "if" follows as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is left injective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$. Then $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) = \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. So, by Proposition 24, $\Omega(\alpha^{-1}(T)) = \Omega(\alpha^{-1}(T'))$. Since, by Lemma 51, we have that $\alpha^{-1}(T), \alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$, we get, by applying left injectivity, $\overleftarrow{\alpha^{-1}(T)} = \overleftarrow{\alpha^{-1}(T')}$. Thus, by Lemma 6, $\alpha^{-1}(\overleftarrow{T}) = \alpha^{-1}(\overleftarrow{T'})$. Finally, the surjectivity of $\langle F, \alpha\rangle$ yields that $\overleftarrow{T} = \overleftarrow{T'}$.

■

Finally, we may recast the injectivity classes in terms of the injectivity of mappings from posets of theory or filter families/systems into posets of congruence systems.

Given two posets $\mathbf{P} = \langle P, \leq \rangle$ and $\mathbf{Q} = \langle Q, \preccurlyeq \rangle$, we call a mapping $f : P \to Q$ **injective** if it is injective as a set map, i.e., if, for all $p_1, p_2 \in P$,

$$f(p_1) = f(p_2) \quad \text{implies} \quad p_1 = p_2.$$

Proposition 215 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is family injective;*

(b) $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is injective;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is injective, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

Similarly, for system injectivity, we have

Proposition 216 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is system injective;*

(b) $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is injective;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is injective, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

3.7 Reflectivity

In this section we look at classes of π-institutions defined using the order reflectivity of the Leibniz operator.

Definition 217 (Reflectivity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is* **family reflective** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\Omega(T) \leq \Omega(T') \quad \textit{implies} \quad T \leq T'.$$

- $\mathcal{I}$ *is* **left reflective** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\Omega(T) \leq \Omega(T') \quad \textit{implies} \quad \overleftarrow{T} \leq \overleftarrow{T'}.$$

- $\mathcal{I}$ *is* **right reflective** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'}) \quad \textit{implies} \quad T \leq T'.$$

- $\mathcal{I}$ *is* **system reflective** *if, for all* $T, T' \in \mathrm{ThSys}(\mathcal{I})$,

$$\Omega(T) \leq \Omega(T') \quad \textit{implies} \quad T \leq T'.$$

We first establish the fact that both family and right reflectivity imply systemicity.

Lemma 218 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is family reflective, then it is systemic;*

(b) If $\mathcal{I}$ *is right reflective, then it is systemic.*

Proof:

(a) Suppose $\mathcal{I}$ is family reflective and let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, we have, by Proposition 20, $\Omega(T) \leq \Omega(\overleftarrow{T})$. Applying family reflectivity, we get $T \leq \overleftarrow{T}$. Since, by Proposition 42, it always holds that $\overleftarrow{T} \leq T$, we get $\overleftarrow{T} = T$. This shows that $\mathrm{ThFam}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$ and, thus, $\mathcal{I}$ is systemic.

(b) Suppose $\mathcal{I}$ is right reflective and let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, we have, by Proposition 42, $\overleftarrow{\overleftarrow{T}} = \overleftarrow{T}$. Thus, we get $\Omega(\overleftarrow{\overleftarrow{T}}) = \Omega(\overleftarrow{T})$ and, hence $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{\overleftarrow{T}})$. Applying right reflectivity, we get $T \leq \overleftarrow{T}$. Since, by Proposition 42, it always holds that $\overleftarrow{T} \leq T$, we get $\overleftarrow{T} = T$. This shows that $\mathrm{ThFam}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$ and, thus, $\mathcal{I}$ is systemic.

■

Lemma 218 enables us to prove that family reflectivity and right reflectivity are actually equivalent properties.

Proposition 219 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is family reflective if and only if it is right reflective.*

Proof: Suppose, first, that $\mathcal{I}$ is family reflective. Then, by Lemma 218, it is systemic, and, by Proposition 152, it is stable. To see that it is right reflective, let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Then, by stability, $\Omega(T) \leq \Omega(T')$. Hence, by family reflectivity, $T \leq T'$. Thus, $\mathcal{I}$ is right reflective.

Suppose, conversely, that $\mathcal{I}$ is right reflective. Then, by Lemma 218, it is systemic, and, by Proposition 152, it is stable. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then by stability, we get $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Now we use right reflectivity to get $T \leq T'$. Thus, $\mathcal{I}$ is family reflective. ∎

Now we establish several relationships among the three reflectivity classes. First, we show that, if a π-institution is family reflective, then it is left reflective.

Proposition 220 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is family reflective, then it is left reflective.*

Proof: Suppose $\mathcal{I}$ if family reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. By family reflectivity, $T \leq T'$. But, by Lemma 218, $\mathcal{I}$ is systemic, whence we get $\overleftarrow{T} \leq \overleftarrow{T'}$ and, hence, $\mathcal{I}$ is left reflective. ∎

Next, we show that left reflectivity implies system reflectivity and, moreover system reflectivity supplied with stability implies left reflectivity.

Proposition 221 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is left reflective, then it is system reflective;*

(b) If $\mathcal{I}$ *is system reflective and stable, then it is left reflective.*

Proof:

(a) Suppose that $\mathcal{I}$ is left reflective and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then, by left reflectivity, $\overleftarrow{T} \leq \overleftarrow{T'}$. But, as T, T' are theory systems, we have $\overleftarrow{T} = T$ and $\overleftarrow{T'} = T'$, whence $T \leq T'$ and $\mathcal{I}$ is system reflective.

(b) Suppose that $\mathcal{I}$ is system reflective and stable. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then, by stability, $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Since $\overleftarrow{T}, \overleftarrow{T'} \in \mathrm{ThSys}(\mathcal{I})$, we apply system reflectivity to get $\overleftarrow{T} \leq \overleftarrow{T'}$. Thus, $\mathcal{I}$ is left reflective.

∎

Finally, we show that systemicity is exactly the separating property between family reflectivity and system reflectivity.

Proposition 222 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is family reflective if and only if it is system reflective and systemic.*

Proof: Suppose, first, that $\mathcal{I}$ is family reflective. Then, by Lemma 218, it is systemic. Moreover, by hypothesis, for all $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$, we get, $T \leq T'$. So $\mathcal{I}$ is also system reflective.

Suppose, conversely, that $\mathcal{I}$ is system reflective and systemic and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. By systemicity, $T, T' \in \mathrm{ThSys}(\mathcal{I})$, whence, by hypothesis, we get $T \leq T'$. Thus, $\mathcal{I}$ is family reflective. ∎

We have now established the following **reflectivity hierarchy**:

Family/Right Reflective

↓

Left Reflective

↓

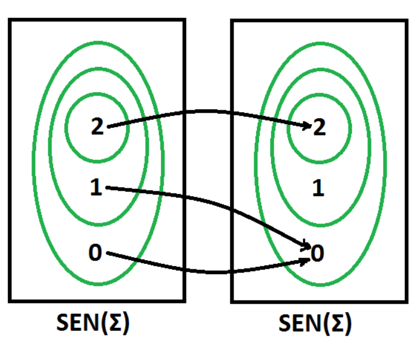
System Reflective

We present now some examples to show that the inclusions shown in the diagram are indeed proper inclusions.

The first example showcases a π-institution which is left reflective, but not systemic, and, hence, according to Proposition 220, not family reflective.

Example 223 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial clone.*



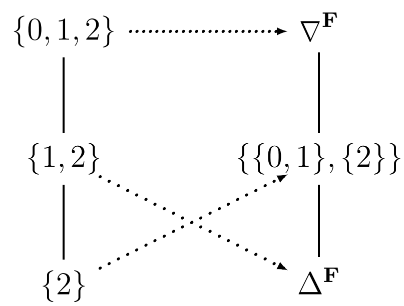
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}$.

The following table gives the theory families and the theory systems of the π-institution $\mathcal{I}$:

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

Since $\{\{1,2\}\}$ is a theory family that is not a theory system, $\mathcal{I}$ is not systemic.

The lattice of theory families and the corresponding Leibniz congruence systems are depicted below:



By the diagram, keeping in mind that $\overleftarrow{\{\{1,2\}\}} = \{\{2\}\}$, one can see that $\mathcal{I}$ is left reflective. But, clearly, it is not family reflective, as $\Omega(\{\{1,2\}\}) \leq \Omega(\{\{2\}\})$, whereas, obviously, $\{\{1,2\}\} \nleq \{\{2\}\}$.

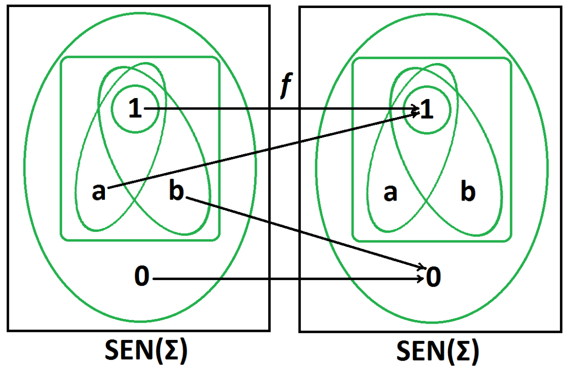
The second example presents a π-institution which is system reflective, but fails to be left reflective and, hence, by Proposition 221, is not stable.

Example 224 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object Σ and a single non-identity morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,a,b,1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(a) = 1$, $\mathrm{SEN}^\flat(f)(b) = 0$ *and* $\mathrm{SEN}^\flat(f)(1) = 1$;
- $N^\flat$ *is the category of natural transformations generated by the two binary natural transformations $\wedge, \vee : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ defined by the following tables:*

$\wedge_\Sigma$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee_\Sigma$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1



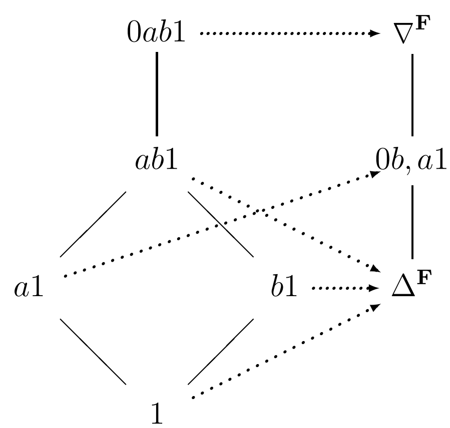
Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution defined by setting*

$$\mathcal{C}_\Sigma = \{\{1\}, \{a,1\}, \{b,1\}, \{a,b,1\}, \{0,a,b,1\}\}.$$

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$*:*

T	$\overleftarrow{T}$
$\{1\}$	$\{1\}$
$\{a,1\}$	$\{a,1\}$
$\{b,1\}$	$\{1\}$
$\{a,b,1\}$	$\{a,1\}$
$\{0,a,b,1\}$	$\{0,a,b,1\}$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



Since $\Omega(\overleftarrow{\{\{a,b,1\}\}}) = \Omega(\{\{a,1\}\}) = \{\{0,b\},\{a,1\}\} \neq \Delta^{\mathbf{F}} = \Omega(\{\{a,b,1\}\})$*, we conclude that* $\mathcal{I}$ *is not stable.*

Since $\{\{1\}\}$*,* $\{\{a,1\}\}$ *and* $\mathrm{SEN}^\flat$ *are the only theory systems of* $\mathcal{I}$*, it is clear from the diagram above that* $\mathcal{I}$ *is system reflective. On the other hand, we have*

$$\Omega(\{\{a,b,1\}\}) = \Delta^{\mathbf{F}} = \Omega(\{\{b,1\}\}),$$

but
$$\overleftarrow{\{\{a,b,1\}\}} = \{\{a,1\}\} \nleq \{\{1\}\} = \overleftarrow{\{\{b,1\}\}},$$
whence, $\mathcal{I}$ is not left reflective.

We turn now to transfer theorems regarding the reflectivity properties studied in this section.

Theorem 225 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is family reflective if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ implies $T \leq T'$.

(b) $\mathcal{I}$ is left reflective if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T, T' \in \mathrm{FiFam}^{\mathcal{A}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ implies $\overleftarrow{T} \leq \overleftarrow{T'}$.

(c) $\mathcal{I}$ is system reflective if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T, T' \in \mathrm{FiSys}^{\mathcal{A}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ implies $T \leq T'$.

Proof: We prove Part (b).

The "if" is obtained by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account that $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}) = \mathrm{ThFam}(\mathcal{I})$, by Lemma 51.

For the "only if" suppose that $\mathcal{I}$ is left reflective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. Apply the inverse of $\langle F, \alpha \rangle$ to get $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. Thus, by Proposition 24, we get $\Omega(\alpha^{-1}(T)) \leq \Omega(\alpha^{-1}(T'))$. Take into account the fact that, by Lemma 51, $\alpha^{-1}(T), \alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$ and apply left reflectivity to get $\overleftarrow{\alpha^{-1}(T)} \leq \overleftarrow{\alpha^{-1}(T')}$. Hence, by Lemma 6, we get $\alpha^{-1}(\overleftarrow{T}) \leq \alpha^{-1}(\overleftarrow{T'})$. Finally, by the surjectivity of $\langle F, \alpha \rangle$, we conclude that $\overleftarrow{T} \leq \overleftarrow{T'}$. ∎

Turning to characterizations in terms of mappings between posets, we get the following characterizations of family and system reflectivity in terms of the order reflectivity of mappings from posets of theory or filter families/systems into posets of congruence systems. Given posets $\mathbf{P} = \langle P, \leq \rangle$ and $\mathbf{Q} = \langle Q, \preccurlyeq \rangle$, call a mapping $f : P \to Q$ **order reflecting** if, for all $p_1, p_2 \in P$,
$$f(p_1) \preccurlyeq f(p_2) \quad \text{implies} \quad p_1 \leq p_2.$$

Proposition 226 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The following statements are equivalent:*

(a) $\mathcal{I}$ is family reflective;

(b) $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is order reflecting;

(c) $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is order reflecting, for every* **F***-algebraic system* $\mathcal{A}$*.*

Similarly, for system reflectivity, we have

Proposition 227 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* **F***. The following statements are equivalent:*

(a) $\mathcal{I}$ *is system reflective;*

(b) $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is order reflecting;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is order reflecting, for every* **F***-algebraic system* $\mathcal{A}$*.*

We continue our studies of reflectivity properties by looking at the relationships governing classes defined using reflectivity with corresponding classes defined using injectivity. We have the following result:

Proposition 228 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* **F***.*

(a) *If* $\mathcal{I}$ *is family/right reflective, then it is right injective;*

(b) *If* $\mathcal{I}$ *is left reflective, then it is left injective;*

(c) *If* $\mathcal{I}$ *is system reflective, then it is system injective.*

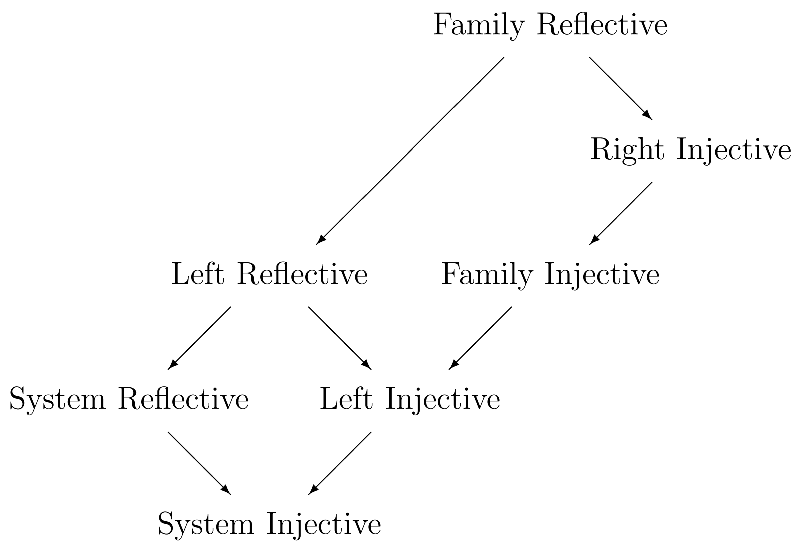
Proof:

(a) Suppose $\mathcal{I}$ is right reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$. Then, a fortiori, $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$ and $\Omega(\overleftarrow{T'}) \leq \Omega(\overleftarrow{T})$. Thus, by right reflectivity, $T \leq T'$ and $T' \leq T$. It follows that $T = T'$. Therefore, $\mathcal{I}$ is right injective.

(b),(c) Very similar to Part (a).

■

Proposition 228 has established the following combined hierarchy of injectivity and reflectivity properties.

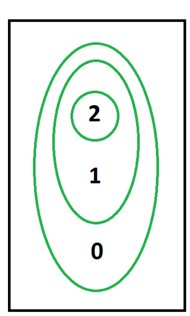


Now we turn to an example that will show that all three fresh inclusions depicted in the diagram, i.e., those established in Proposition 228, are actually proper inclusions.

Example 229 *Define the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor determined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$*;*
- $N^\flat$ *is the clone generated by the natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, defined by the following table:*

$x \in \mathrm{SEN}^\flat(\Sigma)$	$\sigma^\flat_\Sigma(x)$
0	0
1	0
2	2

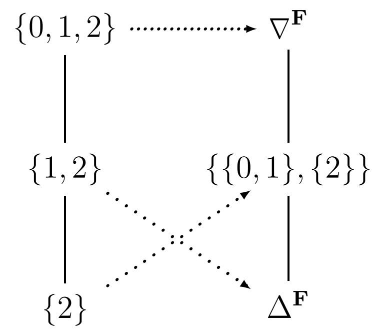


SEN(Σ)

The π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is defined by setting

$$\mathcal{C}_\Sigma = \{\{2\}, \{1,2\}, \{0,1,2\}\}.$$

Note that $\mathcal{I}$ is systemic, since the category $\mathbf{Sign}^\flat$ is trivial. The lattice of theory families and that of the corresponding Leibniz congruence systems are shown in the diagram.



Since all three theory families/systems have distinct Leibniz congruence systems, $\mathcal{I}$ is right injective.

On the other hand, $\Omega(\{\{1,2\}\}) \leq \Omega(\{\{2\}\})$, whereas, obviously, $\{\{1,2\}\} \nleq \{\{2\}\}$, whence $\mathcal{I}$ is not system reflective.

We close this section on reflectivity by looking at the relationships between the various reflectivity classes and the classes in the loyalty hierarchy.

Proposition 230 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) If $\mathcal{I}$ is family reflective, then it is family loyal;

(b) If $\mathcal{I}$ is left reflective, then it is left loyal;

(c) If $\mathcal{I}$ is system reflective, then it is system loyal.

Proof:

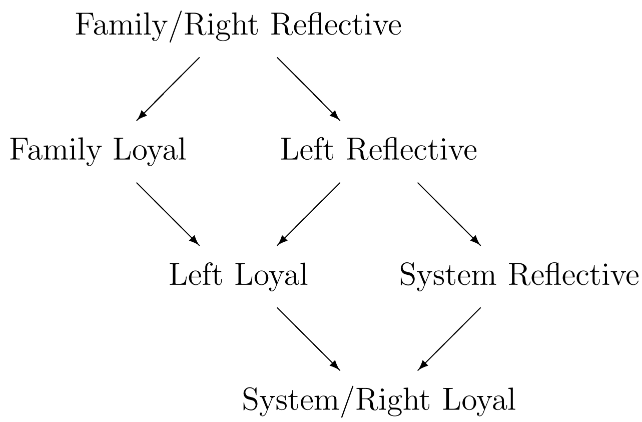
(a) Suppose that $\mathcal{I}$ is family reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) > \Omega(T')$. Then, a fortiori, $\Omega(T) \geq \Omega(T')$. Therefore, by family reflectivity, $T \geq T'$ and, hence $T \nless T'$. We conclude that $\mathcal{I}$ is family loyal.

(b) Suppose that $\mathcal{I}$ is left reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) > \Omega(T')$. Then, a fortiori, $\Omega(T) \geq \Omega(T')$. Hence, by left reflectivity, $\overleftarrow{T} \geq \overleftarrow{T'}$. But this implies that $\overleftarrow{T} \nless \overleftarrow{T'}$. Therefore, $\mathcal{I}$ is left loyal.

(c) Very similar to Parts (a) and (b).

■

Proposition 230 establishes the following combined hierarchy of reflectivity and loyalty classes of π-institutions.

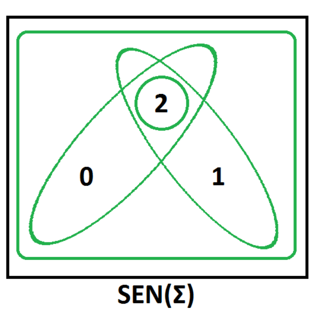


We reinforce the picture by constructing a π-institution that is family loyal but fails to be system reflective. This demonstrates that all three inclusions established in Proposition 230 are proper.

Example 231 *Define the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category, with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor determined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the clone of natural transformations generated by the three unary natural transformations* $\rho^\flat, \sigma^\flat, \tau^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, defined by the following table:*

$x \in \mathrm{SEN}^\flat(\Sigma)$	$\rho^\flat_\Sigma(x)$	$\sigma^\flat_\Sigma(x)$	$\tau^\flat_\Sigma(x)$
0	0	0	1
1	2	0	1
2	2	2	2

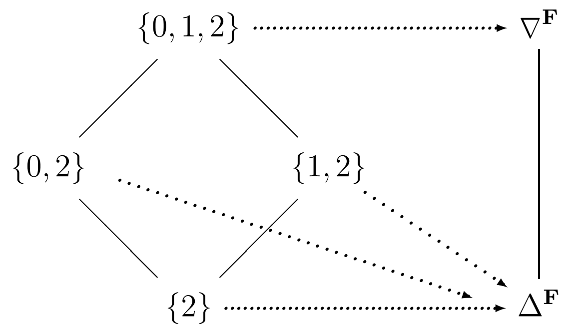


The π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is defined by setting

$$\mathcal{C}_\Sigma = \{\{2\},\{0,2\},\{1,2\},\{0,1,2\}\}.$$

Note that, since $\mathbf{Sign}^\flat$ is trivial, all theory families are theory systems.

The lattice of theory families and the corresponding Leibniz congruence systems are given in the following diagram.



From the diagram it is clear that there are no theory families T, T', such that $T < T'$ and $\Omega(T) > \Omega(T')$. Thus, $\mathcal{I}$ is family loyal.

On the other hand, setting $T = \{\{0,2\}\}$ and $T' = \{\{1,2\}\}$, we have $\Omega(T) \leq \Omega(T')$, whereas, obviously, $T \nleq T'$. Therefore $\mathcal{I}$ is not system reflective.

3.8 Complete Reflectivity

In this section we define the classes arising by imposing the various versions of the property of complete order reflectivity.

Definition 232 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.*

- *$\mathcal{I}$ is* **family completely reflective** *if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,*

$$\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T') \quad \textit{implies} \quad \bigcap_{T\in\mathcal{T}} T \leq T'.$$

- *$\mathcal{I}$ is* **left completely reflective** *if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,*

$$\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T') \quad \textit{implies} \quad \bigcap_{T\in\mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}.$$

- *$\mathcal{I}$ is* **right completely reflective** *if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,*

$$\bigcap_{T\in\mathcal{T}} \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'}) \quad \textit{implies} \quad \bigcap_{T\in\mathcal{T}} T \leq T'.$$

- $\mathcal{I}$ *is* **system completely reflective** *if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$,

$$\bigcap_{T \in \mathcal{T}} \Omega(T) \le \Omega(T') \quad \textit{implies} \quad \bigcap_{T \in \mathcal{T}} T \le T'.$$

For the sake of brevity, we sometimes shorten "complete reflectivity" to **c-reflectivity**.

Lemma 218 allows us to obtain easily the fact that both family c-reflectivity and right c-reflectivity imply systemicity.

Lemma 233 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is family completely reflective, then it is systemic;*

(b) If $\mathcal{I}$ *is right completely reflective, then it is systemic.*

Proof: Note that, if $\mathcal{I}$ is family completely reflective, then it is family reflective and that, if $\mathcal{I}$ is right completely reflective, then it is right reflective. Therefore, the conclusion is obtained by applying Lemma 218. ■

Now we show in an analog of Proposition 219 for complete reflectivity that family complete reflectivity and right complete reflectivity are equivalent properties.

Proposition 234 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is family completely reflective if and only if it is right completely reflective.*

Proof: Suppose, first, that $\mathcal{I}$ is family completely reflective and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'})$. By Lemma 233, $\mathcal{I}$ is systemic, whence we get $\bigcap_{T \in \mathcal{T}} \Omega(T) \le \Omega(T')$. This allows us to apply family c-reflectivity to obtain $\bigcap \mathcal{T} \le T'$. Hence $\mathcal{I}$ is right completely reflective.

Suppose, conversely, that $\mathcal{I}$ is right completely reflective and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \le \Omega(T')$. By Lemma 233, $\mathcal{I}$ is systemic, whence, we get $\bigcap_{T \in \mathcal{T}} \Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'})$. Applying right c-reflectivity, we get that $\bigcap \mathcal{T} \le T'$. Therefore, $\mathcal{I}$ is family completely reflective. ■

We now look at the relationships that govern the three complete reflectivity classes, which also parallel the ones established in Propositions 220, 221 and 222 for the various classes defined using reflectivity.

We show, first, that family complete reflectivity implies left complete reflectivity.

Proposition 235 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is family completely reflective then it is left completely reflective.*

Proof: Suppose that $\mathcal{I}$ is family completely reflective and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. By family complete reflectivity, we get $\bigcap_{T\in\mathcal{T}} T \leq T'$, whence, by Lemma 233, $\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$. Thus, $\mathcal{I}$ is left completely reflective. ■

Next, we show that left complete reflectivity implies system complete reflectivity and, moreover, system complete reflectivity, combined with stability, implies left complete reflectivity.

Proposition 236 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is left completely reflective, then it is system completely reflective;*

(b) If $\mathcal{I}$ *is system completely reflective and stable, then it is left completely reflective.*

Proof:

(a) Suppose that $\mathcal{I}$ is left c-reflective and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. Then, by left c-reflectivity, $\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$. But, as $\mathcal{T} \cup \{T'\}$ consists of theory systems, we get $\bigcap \mathcal{T} \leq T'$. So $\mathcal{I}$ is system c-reflective.

(b) Suppose that $\mathcal{I}$ is system c-reflective and stable. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. Then, by stability, we get $\bigcap_{T\in\mathcal{T}} \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Since $\{\overleftarrow{T} : T \in \mathcal{T}\} \cup \{\overleftarrow{T'}\} \in \mathrm{ThSys}(\mathcal{I})$, we apply system c-reflectivity to get $\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$. Thus, $\mathcal{I}$ is left c-reflective.

■

Finally, it is not difficult to see that family complete reflectivity is tantamount to system complete reflectivity plus systemicity.

Proposition 237 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is family completely reflective if and only if it is system completely reflective and systemic.*

Proof: Suppose, first, that $\mathcal{I}$ is family completely reflective. Then, by Lemma 233, it is systemic. Moreover, by Propositions 235 and 236, it is system completely reflective.

Suppose, conversely, that $\mathcal{I}$ is system completely reflective and systemic and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. Since, by systemicity, we get $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, we get, by system complete reflectivity, $\bigcap_{T\in\mathcal{T}} T \leq T'$. Thus, $\mathcal{I}$ is family completely reflective. ■

We have now established the following **complete reflectivity hierarchy**:

Family/Right c-Reflective

↓

Left c-Reflective

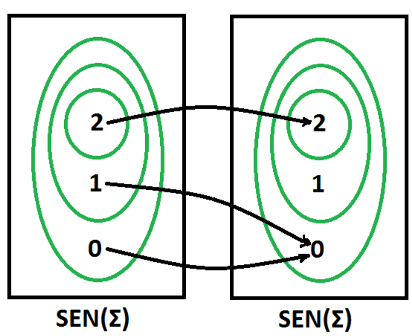
↓

System c-Reflective

We now provide two examples to show that the inclusions depicted in the diagram between the three complete reflectivity classes introduced in this section are proper. The first example presents a π-institution which is left completely reflective, but not systemic, and, hence, according to Proposition 237, not family completely reflective.

Example 238 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial clone.*



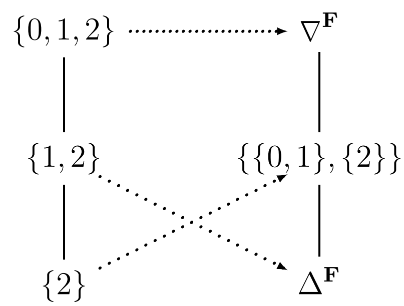
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}$.

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$*:*

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

Since $\{\{1,2\}\}$ is a theory family that is not a theory system, $\mathcal{I}$ is not systemic.

The lattice of theory families and that of the corresponding Leibniz congruence systems are depicted below:



By the diagram, keeping in mind that $\overleftarrow{\{\{1,2\}\}} = \{\{2\}\}$, one can see that $\mathcal{I}$ is left c-reflective. But, clearly, it is not family c-reflective, as $\Omega(\{\{1,2\}\}) \leq \Omega(\{\{2\}\})$, whereas, obviously, $\{\{1,2\}\} \nleq \{\{2\}\}$, giving, as we have already seen in Example 223, that $\mathcal{I}$ is not even family reflective.

The second example presents a π-institution which is system c-reflective, but fails to be left c-reflective and, hence, by Proposition 236, is not stable.

Example 239 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

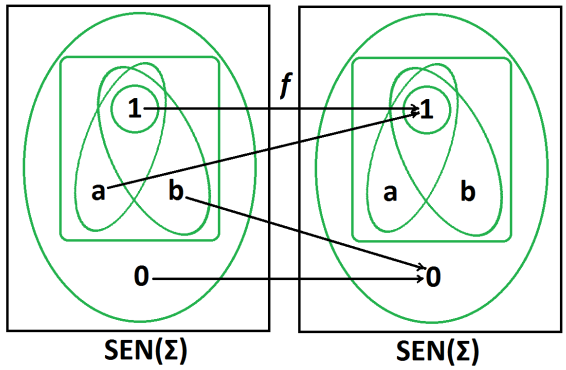
- *$\mathbf{Sign}^\flat$ is the category with a single object Σ and a single non-identity morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- *$\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ is the functor specified by $\mathrm{SEN}^\flat(\Sigma) = \{0,a,b,1\}$ and $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(a) = 1$, $\mathrm{SEN}^\flat(f)(b) = 0$ and $\mathrm{SEN}^\flat(f)(1) = 1$;*
- *$N^\flat$ is the category of natural transformations generated by the two binary natural transformations $\wedge, \vee : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ defined by the following tables:*

$\wedge_\Sigma$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee_\Sigma$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be the π-institution defined by setting

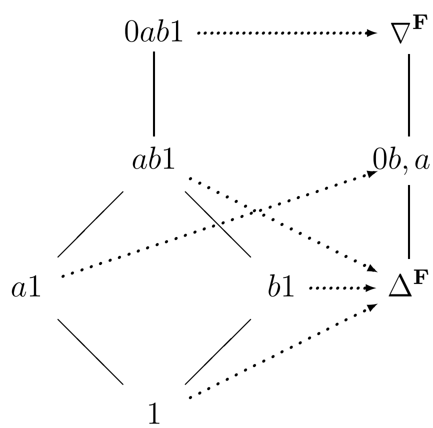
$$\mathcal{C}_\Sigma = \{\{1\},\{a,1\},\{b,1\},\{a,b,1\},\{0,a,b,1\}\}.$$



The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$*:*

T	$\overleftarrow{T}$
$\{1\}$	$\{1\}$
$\{a,1\}$	$\{a,1\}$
$\{b,1\}$	$\{1\}$
$\{a,b,1\}$	$\{a,1\}$
$\{0,a,b,1\}$	$\{0,a,b,1\}$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



Since $\Omega(\overleftarrow{\{\{a,b,1\}\}}) = \Omega(\{\{a,1\}\}) = \{\{0,b\},\{a,1\}\} \neq \Delta^{\mathbf{F}} = \Omega(\{\{a,b,1\}\})$*, we conclude that* $\mathcal{I}$ *is not stable.*

Note that, since $\{\{1\}\}$*,* $\{\{a,1\}\}$ *and* $\mathrm{SEN}^\flat$ *are the only theory systems of* $\mathcal{I}$*, the Leibniz operator* $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is an order isomorphism. Hence,* $\mathcal{I}$ *is system completely reflective. On the other hand, we have*

$$\Omega(\{\{a,b,1\}\}) = \Delta^{\mathbf{F}} = \Omega(\{\{b,1\}\}),$$

but

$$\overleftarrow{\{\{a,b,1\}\}} = \{\{a,1\}\} \nleq \{\{1\}\} = \overleftarrow{\{\{b,1\}\}},$$

whence, $\mathcal{I}$ is not left reflective and, a fortiori, it is not left completely reflective either.

We turn now to transfer theorems regarding the reflectivity properties studied in this section.

Theorem 240 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is family completely reflective if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ implies $\bigcap \mathcal{T} \leq T'$.

(b) $\mathcal{I}$ is left completely reflective if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{A}}(\mathcal{A})$, $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ implies $\bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$.

(c) $\mathcal{I}$ is system completely reflective if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiSys}^{\mathcal{A}}(\mathcal{A})$, $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ implies $\bigcap_{T \in \mathcal{T}} T \leq T'$.

Proof: We prove Part (b).

The "if" is obtained by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account that $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}) = \mathrm{ThFam}(\mathcal{I})$, by Lemma 51.

For the "only if" suppose that $\mathcal{I}$ is left c-reflective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. Apply the inverse of $\langle F, \alpha \rangle$ to get $\alpha^{-1}(\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. This yields that $\bigcap_{T \in \mathcal{T}} \alpha^{-1}(\Omega^{\mathcal{A}}(T)) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. Thus, by Proposition 24, we get $\bigcap_{T \in \mathcal{T}} \Omega(\alpha^{-1}(T)) \leq \Omega(\alpha^{-1}(T'))$. Take into account the fact that, by Lemma 51, $\{\alpha^{-1}(T) : T \in \mathcal{T}\} \cup \{\alpha^{-1}(T')\} \subseteq \mathrm{ThFam}(\mathcal{I})$ and apply left c-reflectivity to get $\bigcap_{T \in \mathcal{T}} \overleftarrow{\alpha^{-1}(T)} \leq \overleftarrow{\alpha^{-1}(T')}$. Hence, by Lemma 6, we get $\bigcap_{T \in \mathcal{T}} \alpha^{-1}(\overleftarrow{T}) \leq \alpha^{-1}(\overleftarrow{T'})$. Hence, $\alpha^{-1}(\bigcap_{T \in \mathcal{T}} \overleftarrow{T}) \leq \alpha^{-1}(\overleftarrow{T'})$. Finally, by the surjectivity of $\langle F, \alpha \rangle$, we conclude that $\bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$. ■

Next, we obtain characterizations of family and system c-reflectivity in terms of the complete order reflectivity of mappings from posets of theory or filter families/systems into posets of congruence systems. Given complete lattices $\mathbf{P} = \langle P, \leq \rangle$ and $\mathbf{Q} = \langle Q, \preccurlyeq \rangle$, call a mapping $f : P \to Q$ **completely order reflecting** if, for all $X \cup \{y\} \subseteq P$,

$$\bigwedge_{x \in X}^{\mathbf{Q}} f(x) \preccurlyeq f(y) \quad \text{implies} \quad \bigwedge_{x \in X}^{\mathbf{P}} x \leq y.$$

Proposition 241 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The following statements are equivalent:*

(a) *$\mathcal{I}$ is family completely reflective;*

(b) $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}(\mathcal{I})$ *is completely order reflecting;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}(\mathcal{A})$ *is completely order reflecting, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$.

Similarly, for system c-reflectivity, we have

Proposition 242 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) *$\mathcal{I}$ is system completely reflective;*

(b) $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}(\mathcal{I})$ *is completely order reflecting;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}(\mathcal{A})$ *is completely order reflecting, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$.

We look now at the relationships governing classes defined using complete reflectivity with corresponding classes defined using reflectivity. We have referred to this straightforward relationships already in the proof of Lemma 233.

Proposition 243 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

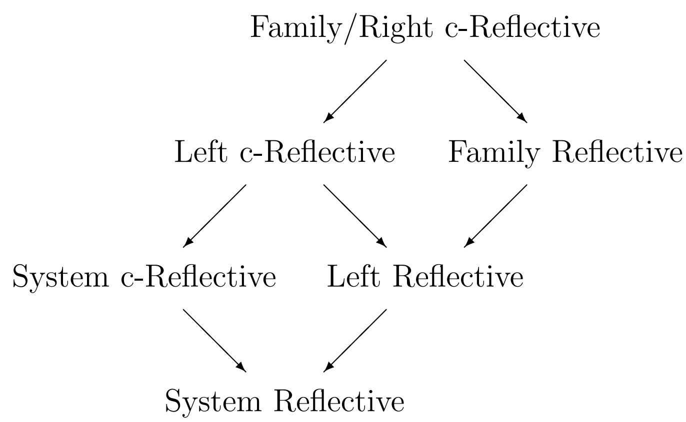
(a) *If $\mathcal{I}$ is family/right completely reflective, then it is family/right reflective;*

(b) *If $\mathcal{I}$ is left completely reflective, then it is left reflective;*

(c) *If $\mathcal{I}$ is system completely reflective, then it is system reflective.*

Proof: All three parts are based on the observation that the reflectivity conditions are specializations of the corresponding complete reflectivity conditions to the special case of singleton collections of theory families/systems on the left hand sides of the relevant inequalities. ■

Proposition 243 has established the following combined hierarchy of injectivity and reflectivity properties.



Now we turn to an example that will show that all three inclusions established in Proposition 243 and depicted in the diagram are proper. Namely, we construct a family reflective π-institution that is not system completely reflective.

Example 244 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category, with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2, 3, 4\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the two unary natural transformations* $\sigma^\flat, \tau^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined by the following table:*

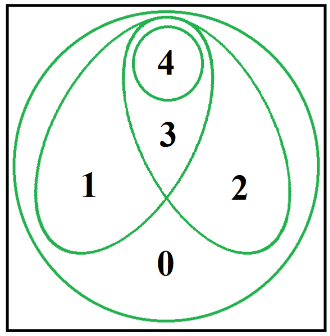
$x \in \mathrm{SEN}^\flat(\Sigma)$	$\sigma^\flat_\Sigma(x)$	$\tau^\flat_\Sigma(x)$
0	2	0
1	2	0
2	2	0
3	1	2
4	2	0

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

$$\mathcal{C}_\Sigma = \{\{4\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{0, 1, 2, 3, 4\}\}.$$

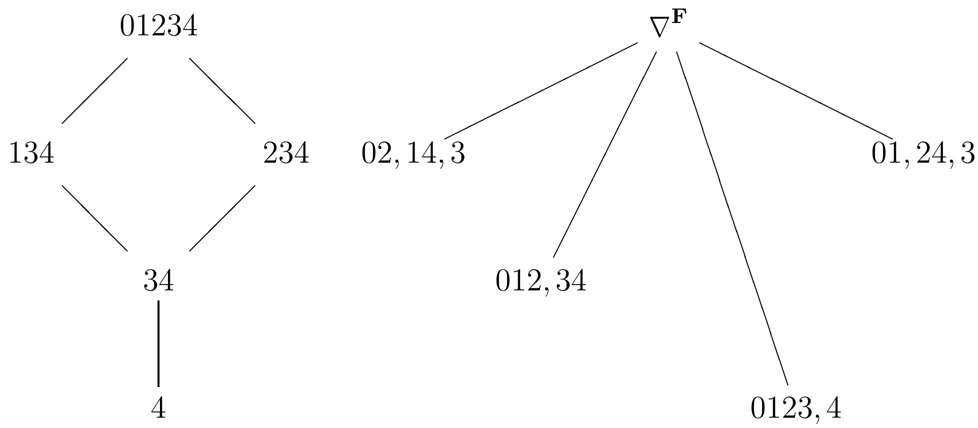
Clearly, since $\mathbf{Sign}^\flat$ *is trivial,* $\mathcal{I}$ *is systemic.*

The lattice of theory families and the corresponding Leibniz congruence



SEN(Σ)

systems are shown in the diagram.



From the diagram, it is clear that, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) \leq \Omega(T')$ *implies* $T \leq T'$, *i.e.,* $\mathcal{I}$ *is family reflective. On the other hand, setting* $T^1 = \{\{1,3,4\}\}$, $T^2 = \{\{2,3,4\}\}$ *and* $T' = \{\{4\}\}$, *we get*

$$\begin{array}{rcl}\Omega(T^1) \cap \Omega(T^2) & = & \{\{02, 14, 3\}\} \cap \{\{01, 24, 3\}\} \\ & = & \Delta^{\mathbf{F}} \\ & \leq & \{\{0123, 4\}\} = \Omega(T'),\end{array}$$

whereas $T^1 \cap T^2 = \{\{3,4\}\} \nleq \{\{4\}\} = T'$. *Hence,* $\mathcal{I}$ *is not system completely reflective.*