Chapter 4

The Semantic Leibniz Hierarchy: Top Half

4.1 Introduction

In this chapter, we study the classes of π-institutions that result when combining monotonicity properties of the Leibniz operator with injectivity, reflectivity or complete reflectivity. As such, all those classes correspond, in the categorical framework, to the class of weakly algebraizable sentential logics [62], which is obtained by combining protoalgebraicity [28] (see, also, [26]) with truth equationality [77] (see, also, Section 6.4 of [86]). It should also be mentioned that algebraizable logics, as introduced in [35] and generalized in [43, 53, 54], form subclasses of weakly algebraizable ones obtained by strengthening protoalgebraicity to equivalentiality [19, 23, 24]. The analogs of equivalentiality for π-institutions and the corresponding subclasses of algebraizable π-institutions will be considered in Chapters 5 and 5.

In Section 4.2, we study the hierarchy that results when combining prealgebraicity, i.e., monotonicity of the Leibniz operator on theory systems (Section 3.3) with each of the various flavors of injectivity (Section 3.6), reflectivity (Section 3.7) or complete reflectivity (Section 3.8). Since there are four different flavors of injectivity, three of reflectivity and three of complete reflectivity, we get, a priori, ten classes of *weakly prealgebraizable π-institutions*. The qualifier "weakly" suggests the use of monotonicity rather than equivalentiality, and the prefix "pre" in prealgebraizable that prealgebraicity, i.e., system monotonicity, rather than protoalgebraicity, i.e., family monotonicity, is used in the definition of these ten classes. Since prealgebraicity is common to all ten, the differentiating factor is the type of injectivity, reflectivity or c-reflectivity being imposed. Accordingly, the following ten classes are obtained, all named "weakly X prealgebraizable", or "WX Prealgebraizable" for short, where the string X stands for one of the following:

- SI for system injective, LI for left injective, FI for family injective, RI for right injective; or

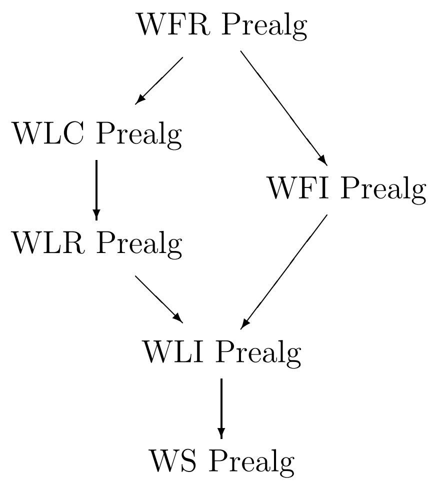
- SR for system reflective, LR for left reflective, FR for family reflective; or

- SC for system c-reflective, LC for left c-reflective, FC for family c-reflective.

A fundamental result is that, under prealgebraicity, all three system properties (SI, SR and SC) coincide. Thus, WSI, WSR and WSC prealgebraizability are identical properties. We call π-institutions belonging to this class *WS prealgebraizable*. It is shown that WS prealgebraizability transfers. Moreover, WS prealgebraizable π-institutions $\mathcal{I} = \langle \mathbf{F}, C \rangle$ are characterized by the property that $\Omega^{\mathcal{A}}$ on $\mathcal{I}$-filter systems is an order embedding, for every $\mathbf{F}$-algebraic system $\mathcal{A}$. As prealgebraicity identifies also family reflectivity with family c-reflectivity, the classes of WFR prealgebraizable and WFC prealgebraizable

π-institutions coincide. Finally, both WFR and WRI prealgebraizability turn out to be equivalent, as they are both equivalent to WFI prealgebraizability plus systemicity. Hence, at the top of the weak prealgebraizability hierarchy, only two of the four classes are potentially different. We refer to them as *WFR* and *WFI prealgebraizability*. Both properties transfer. Moreover, both have characterizations in terms of the Leibniz operator viewed as a mapping between ordered sets. Namely, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is WFR prealgebraizable iff $\Omega^{\mathcal{A}}$ is an order isomorphism and it is WFI prealgebraizable iff $\Omega^{\mathcal{A}}$ is a bijection on $\mathcal{I}$-filter families, which restricts to an order embedding on $\mathcal{I}$-filter systems, for every $\mathbf{F}$-algebraic system $\mathcal{A}$.

As no further identifications seem possible, one obtains the hierarchy

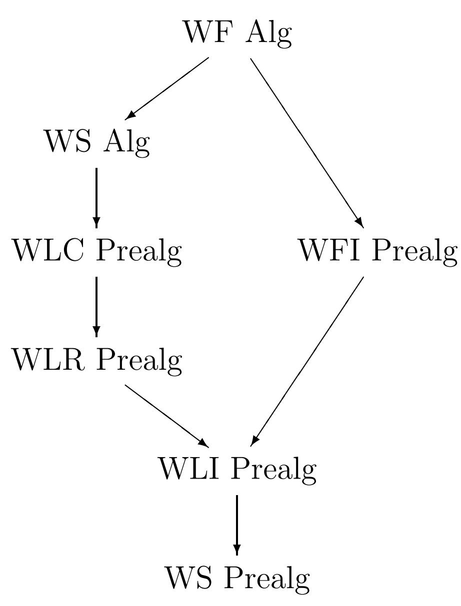


Some specialized results reduce the hierarchy further under additional provisos. First, under systemicity, the entire hierarchy collapses to a single class. Second, it is shown that, under stability, the two family properties coincide, as do all four remaining properties. Thus, under stability, the hierarchy reduces to only two distinct classes.

The section focuses, next, to the three left properties. More precisely, it is shown that all three of WLI, WLR and WLC prealgebraizability versions transfer and that each is characterized via theorems perceiving the Leibniz operator as a mapping from filter families to congruence systems over arbitrary algebraic systems. Briefly, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, it turns out that $\mathcal{I}$ is WLI (WLR, WLC, respectively) prealgebraizable iff, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}(\mathcal{A})$ is a left injective (left order reflecting, left completely order reflecting, respectively) surjection, which restricts to an order embedding on filter systems.

In Section 4.3, we study those classes that are formed by combining protoalgebraicity (family monotonicity) with each of the ten versions of injectivity, reflectivity or complete reflectivity properties. So, once more, a priori, before any detailed study, one obtains ten potentially different classes of weakly algebraizable π-institutions. However, since protoalgebraicity is a stronger condition than prealgebraicity, one obtains immediately at least those identifications that apply to the weak prealgebraizability hierarchy. So, e.g., we get that WFI, WRI, WFR and WFC algebraizable π-institutions coincide. We term the corresponding property *WF algebraizability*. It turns out to be equivalent to the conjunction of WS prealgebraizability and systemicity. WF algebraizability transfers and, moreover, it can be characterized by $\Omega^{\mathcal{A}}$ being an order isomorphism on every algebraic system $\mathcal{A}$. It follows that this class of π-institutions is actually identical to the class of WFR prealgebraizable ones, i.e., those belonging to the top class in the weak prealgebraizability hierarchy. What is a massive collapse, however, results from showing that the lowest class in the weak algebraizability hierarchy, WSI algebraizability, can be characterized as the conjunction of stability with $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ being an order isomorphism. This allows showing that all classes of WS, WLI, WLR, WLC and WFI algebraizable π-institutions are identical. We term the corresponding property *WS algebraizability*. It is shown that WS algebraizability also transfers.

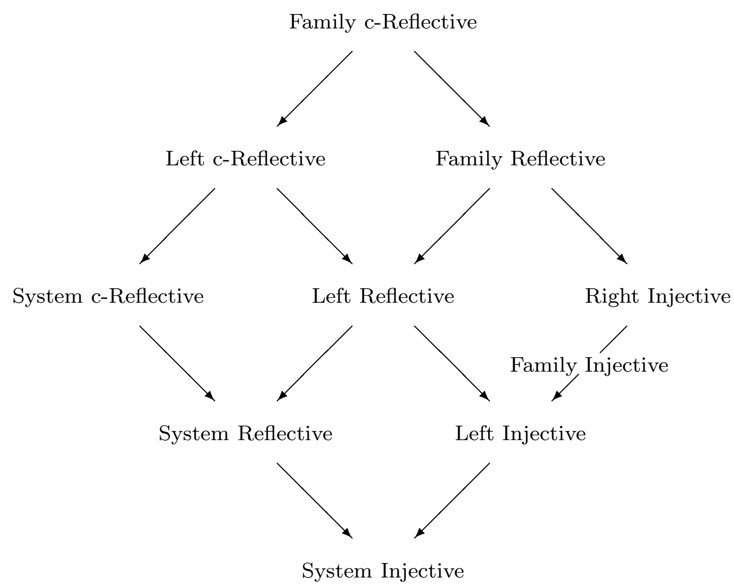
Having reduced the weak algebraizability hierarchy down to two classes, we conclude Section 4.3 (and Chapter 4) by merging it with the weak prealgebraizability hierarchy to obtain the following refinement of the classes that correspond, in the categorical framework, to the class of weakly algebraizable deductive systems.



4.2 Weak PreAlgebraizability

We now shift attention to classes of π-institutions that are defined as a result of interactions between the various kinds of injectivity, reflectivity and complete reflectivity, on the one hand, and prealgebraicity and protoalgebraicity, on the other.

Recall that the hierarchy that was established in the preceding chapter as regards the various versions of injectivity, reflectivity and complete reflectivity has the following form:



Thus, a priori, based on the preceding hierarchy, and combining with prealgebraicity, we obtain a mimicking hierarchy of ten classes which are defined, and whose hierarchy is shown, below.

Definition 245 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is* **weakly system injective prealgebraizable** *or* **WSI Prealgebraizable***, for short, if it is system injective and prealgebraic.*
- $\mathcal{I}$ *is* **weakly left injective prealgebraizable** *or* **WLI Prealgebraizable***, for short, if it is left injective and prealgebraic.*
- $\mathcal{I}$ *is* **weakly family injective prealgebraizable** *or* **WFI Prealgebraizable***, for short, if it is family injective and prealgebraic.*
- $\mathcal{I}$ *is* **weakly right injective prealgebraizable** *or* **WRI Prealgebraizable***, for short, if it is right injective and prealgebraic.*

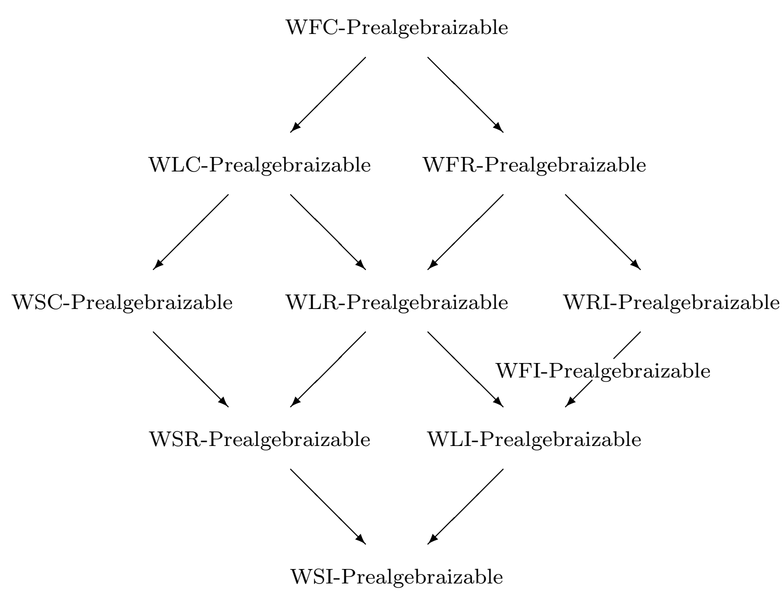
Definition 246 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is* **weakly system reflective prealgebraizable** *or* **WSR Prealgebraizable***, for short, if it is system reflective and prealgebraic.*

- *$\mathcal{I}$ is* **weakly left reflective prealgebraizable** *or* **WLR Prealgebraizable***, for short, if it is left reflective and prealgebraic.*
- *$\mathcal{I}$ is* **weakly family reflective prealgebraizable** *or* **WFR Prealgebraizable***, for short, if it is family reflective and prealgebraic.*

Definition 247 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$.

- *$\mathcal{I}$ is* **weakly system completely reflective prealgebraizable** *or* **WSC Prealgebraizable***, for short, if it is system completely reflective and prealgebraic.*
- *$\mathcal{I}$ is* **weakly left completely reflective prealgebraizable** *or* **WLC Prealgebraizable***, for short, if it is left completely reflective and prealgebraic.*
- *$\mathcal{I}$ is* **weakly family completely reflective prealgebraizable** *or* **WFC Prealgebraizable***, for short, if it is family completely reflective and prealgebraic.*



A few words in the nomenclature used in this diagram are in order.

- W stands for "weakly" which refers to the fact that these classes are defined using forms of monotonicity of the Leibniz operator without any stipulation as to commutativity of the Leibniz operator with inverse special endomorphisms (to be studied later in the chapter). If one adds that condition (using essentially (pre)equivalentiality instead of pre- or protoalgebraicity), then the letter is dropped.

- The letters S for "system", L for "left", R for "right" and F for "family" have obvious meanings referring to which of the four versions (family, left, right or system) of injectivity (I), reflectivity (R) or complete reflectivity (C) conditions are used in the definition.
- Finally, the term "prealgebraizable" is associated with application of monotonicity to theory systems only (as in "prealgebraic"), as opposed to the term "algebraizable", which stipulates monotonicity for all theory families.

We start by proving that under prealgebraicity, system injectivity, system reflectivity and system complete reflectivity turn out to be equivalent properties.

Theorem 248 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is prealgebraic, then the following statements are equivalent:*

(a) $\mathcal{I}$ *is system injective;*

(b) $\mathcal{I}$ *is system reflective;*

(c) $\mathcal{I}$ *is system completely reflective.*

Proof:

(a)⇒(b) Suppose that $\mathcal{I}$ is system injective. Let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\Omega(T) \le \Omega(T')$. Then we get $\Omega(T) = \Omega(T) \cap \Omega(T')$. Moreover, by Lemma 23, $\Omega(T) \cap \Omega(T') \le \Omega(T \cap T')$. On the other hand, by prealgebraicity, we have $\Omega(T \cap T') \le \Omega(T)$ and $\Omega(T \cap T') \le \Omega(T')$, whence $\Omega(T \cap T') \le \Omega(T) \cap \Omega(T')$. We conclude that

$$\Omega(T) = \Omega(T) \cap \Omega(T') = \Omega(T \cap T').$$

Now we use system injectivity to get $T = T \cap T'$. Therefore, $T \le T'$. So $\mathcal{I}$ is also system reflective.

(b)⇒(c) Suppose, next, that $\mathcal{I}$ is system reflective. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \le \Omega(T')$. Since $\mathcal{I}$ is prealgebraic, i.e., Ω is monotone on theory systems, we have, for all $T \in \mathcal{T}$, $\Omega(\bigcap \mathcal{T}) \le \Omega(T)$. Therefore, we get

$$\Omega(\bigcap_{T \in \mathcal{T}} T) \le \bigcap_{T \in \mathcal{T}} \Omega(T) \le \Omega(T').$$

Since, by hypothesis, $\mathcal{I}$ is system reflective, we get $\bigcap_{T \in \mathcal{T}} T \le T'$. Thus, $\mathcal{I}$ is system completely reflective.

(c)⇒(a) Suppose, finally, that $\mathcal{I}$ is system completely reflective. By Proposition 243, it is system reflective, and, then, by Proposition 228, it is system injective.

■

Theorem 248 shows that three of the classes in the previous diagram coincide.

Corollary 249 *The classes of WSI prealgebraizable, WSR prealgebraizable, and WSC prealgebraizable π-institutions coincide.*

Taking advantage of Corollary 249 we define:

Definition 250 (WS Prealgebraizable) *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is called* **weakly system prealgebraizable** *(or* **WS prealgebraizable** *for short) if it is prealgebraic and system injective, i.e., if the Leibniz operator is monotone and injective on theory systems: For all $T, T' \in \mathrm{ThSys}(\mathcal{I})$,*

$$\begin{array}{rcl} T \leq T' & \textit{implies} & \Omega(T) \leq \Omega(T'); \\ \Omega(T) = \Omega(T') & \textit{implies} & T = T'. \end{array}$$

We present two examples of WS prealgebraizable π-institutions. They are crafted to provide a sneak preview of the state of affairs in the case of systemic and non-systemic π-institutions with regards to weak prealgebraizability. The reader will notice that, in both examples, there is an order isomorphism between the lattice of theory systems of the π-institution and that of the associated Leibniz congruence systems. On the other hand, for this isomorphism to extend to an isomorphism between the lattice of all theory families and the corresponding Leibniz congruence systems, the condition of systemicity on the π-institution under consideration seems to be required (and is, as we shall see later).

Example 251 *Define the algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ as follows:*

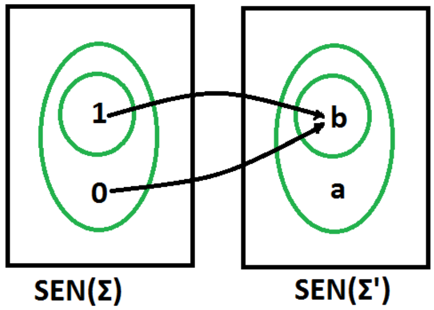
- $\mathbf{Sign}^\flat$ *is the category with objects Σ and Σ' and a unique (non-identity) morphism $f : \Sigma \to \Sigma'$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = b = \mathrm{SEN}^\flat(f)(1)$;
- $N^\flat$ *is the trivial clone.*

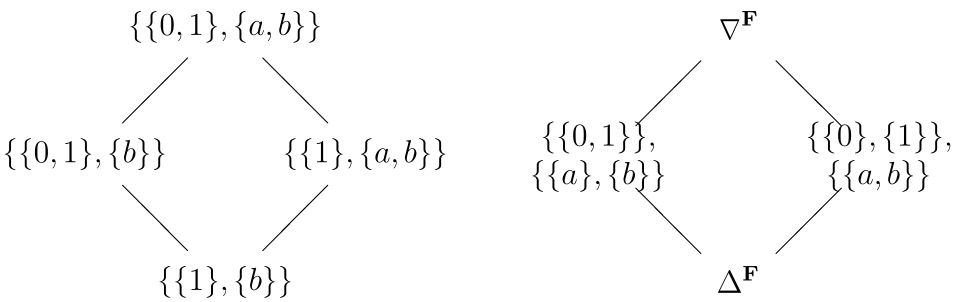
Specify the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by setting

$$\mathcal{C}_\Sigma = \{\{1\}, \{0, 1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\}, \{a, b\}\}.$$

Notice that every theory family is a theory system, whence $\mathcal{I}$ is systemic.

The following diagrams show the lattices of theory families and of the corresponding Leibniz congruence systems:

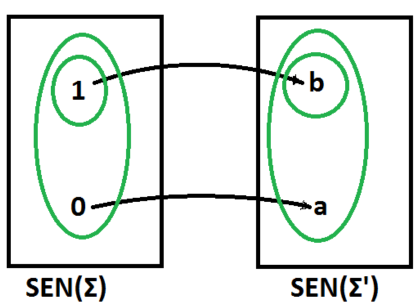




Note that the π-institution $\mathcal{I}$ is WS prealgebraizable and that the two lattices are clearly isomorphic.

Example 252 *Define the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *as follows:*

- $\mathbf{Sign}^\flat$ *is the category with two objects* Σ, Σ' *and a single (non-identity) morphism* $f : \Sigma \to \Sigma'$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by setting* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$, $\mathrm{SEN}^\flat(f)(0) = a$ *and* $\mathrm{SEN}^\flat(f)(1) = b$;
- $N^\flat$ *is the trivial clone.*



Specify the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

$$\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\},\{a,b\}\}.$$

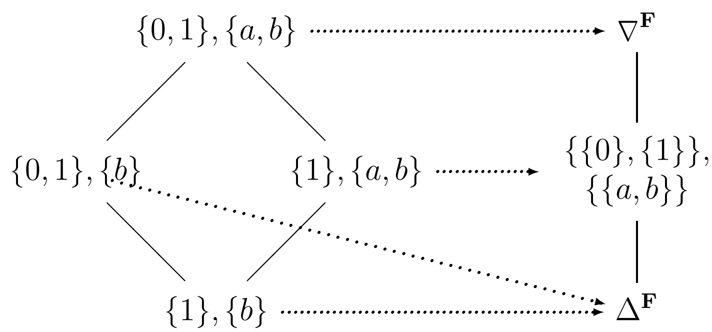
Notice that the theory family $T = \{\{0,1\},\{b\}\}$ *is not a theory system, whence* $\mathcal{I}$ *is not systemic. In fact,* $\overleftarrow{\ } : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ThSys}(\mathcal{I})$ *is given by the following table:*

	$\{b\}$	$\{a,b\}$
$\{1\}$	$\{1\},\{b\}$	$\{1\},\{a,b\}$
$\{0,1\}$	$\{1\},\{b\}$	$\{0,1\},\{a,b\}$

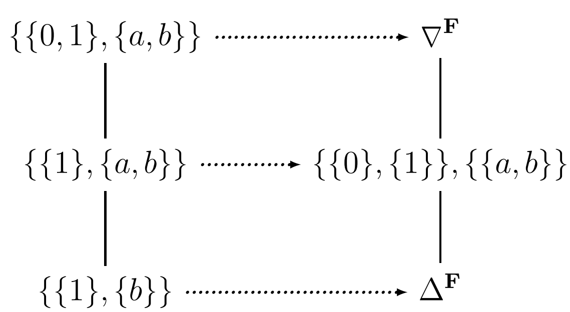
The next table gives the theory families and the associated Leibniz congruence systems:

T	$\Omega(T)$
$\{\{1\},\{b\}\}$	$\{\{0\},\{1\}\},\{\{a\},\{b\}\}$
$\{\{0,1\},\{b\}\}$	$\{\{0\},\{1\}\},\{\{a\},\{b\}\}$
$\{\{1\},\{a,b\}\}$	$\{\{0\},\{1\}\},\{\{a,b\}\}$
$\{\{0,1\},\{a,b\}\}$	$\{\{0,1\}\},\{\{a,b\}\}$

So, even though the lattices of theory families and of the corresponding Leibniz congruence systems are not isomorphic,



the lattices of theory systems and of the corresponding Leibniz congruence systems are indeed isomorphic:



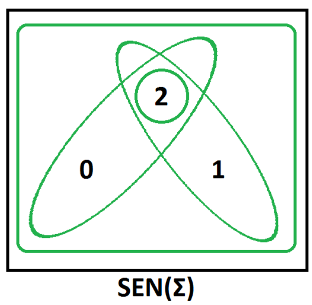
This π*-institution is also WS prealgebraizable.*

We present next examples to show that the class of weakly system prealgebraizable π-institutions is properly included in both the class of prealgebraic and that of system completely reflective π-institutions.

Example 253 *Consider the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the clone of natural transformations on* $\mathrm{SEN}^\flat$ *generated by the the two unary natural transformations* $\sigma^\flat, \tau^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, given by the following table:*

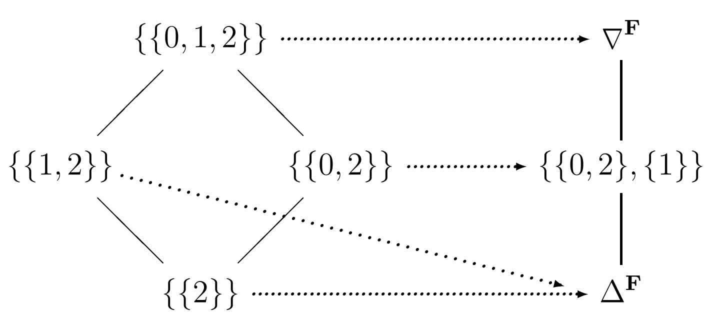
$x \in \mathrm{SEN}^\flat(\Sigma)$	$\sigma^\flat_\Sigma(x)$	$\tau^\flat_\Sigma(x)$
0	0	0
1	1	2
2	0	2



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$*, by setting*

$$C_\Sigma = \{\{2\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

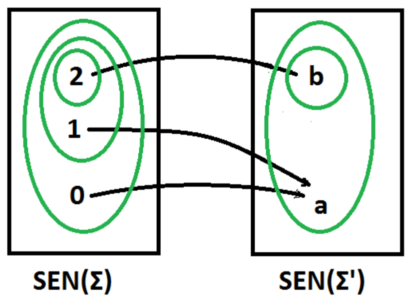
The lattice of the theory systems of $\mathcal{I}$ *and that of the associated Leibniz congruence systems are shown in the following diagrams*



It is clear that the Leibniz operator is monotone. On the other hand, the Leibniz operator is not injective on theory systems. Therefore, we conclude that $\mathcal{I}$ *is prealgebraic but that it fails to be WS prealgebraizable.*

Example 254 *Define the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *as follows:*

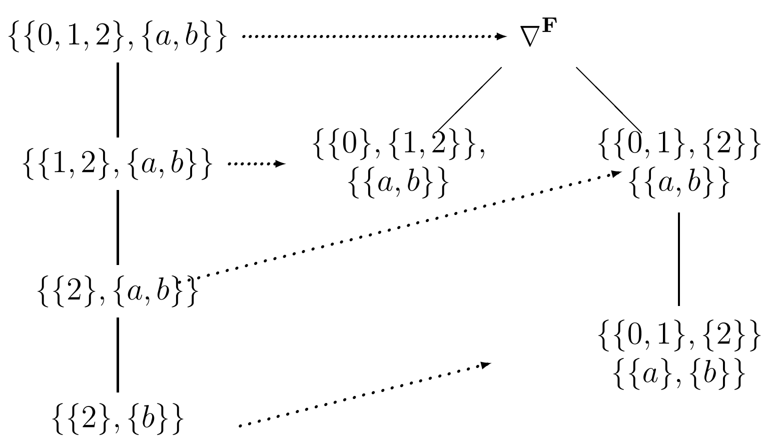
- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*,* $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$*,* $\mathrm{SEN}^\flat(f)(1) = a$ *and* $\mathrm{SEN}^\flat(f)(2) = b$*;*
- $N^\flat$ *is the trivial clone.*



We consider the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *defined by*

$$\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\}, \{a, b\}\}.$$

This π*-institution has six theory families, but only four theory systems. The lattice of theory systems and the associated congruence systems are shown below.*



It is clear from these that the Leibniz operator is completely order reflecting on the theory systems of $\mathcal{I}$*, but it is not monotonic. It follows that* $\mathcal{I}$ *is system c-reflective but not prealgebraic. Therefore, it is system c-reflective, but fails to be weakly system prealgebraizable.*

The defining properties of weak system prealgebraizability transfer from theory systems to filter systems over arbitrary algebraic systems. This result follows naturally from corresponding constituent pieces that have already been put in place when studying monotonicity and injectivity.

Theorem 255 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is WS prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, the Leibniz operator on* $\mathcal{A}$ *is monotone and injective on the* $\mathcal{I}$*-filter systems of* $\mathcal{A}$*, i.e., for all* $T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$*,*

$$\begin{array}{rcl} T \leq T' & \textit{implies} & \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T'); \\ \Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T') & \textit{implies} & T = T'. \end{array}$$

Proof: Suppose, first, that the displayed implications hold for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $\mathcal{I}$-filter systems T, T' on $\mathcal{A}$. By taking $\mathcal{A} = \mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and keeping in mind Lemma 51, we conclude that the Leibniz operator is monotone and injective on all theory systems of $\mathcal{I}$. Thus, by definition, $\mathcal{I}$ is WS prealgebraizable.

Suppose, conversely, that $\mathcal{I}$ is WS prealgebraizable. Then, by definition, it is prealgebraic and system injective. Thus, by Theorems 179 and 214, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, the Leibniz operator $\Omega^{\mathcal{A}}$ is monotone and injective on the $\mathcal{I}$-filter systems of $\mathcal{A}$. ∎

We finally establish the result that we alluded to before presenting Examples 251 and 252. Namely, we show that WS prealgebraizability can be equivalently characterized by the fact that the Leibniz operator $\Omega^{\mathcal{A}}$ over an arbitrary $\mathbf{F}$-algebraic system $\mathcal{A}$ establishes an order embedding from the lattice of filter systems on $\mathcal{A}$ into the poset of all relative congruence systems on $\mathcal{A}$ with respect to the class $\mathrm{AlgSys}^*(\mathcal{I})$.

Consider an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ and a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ based on $\mathbf{F}$. Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and let $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, observe that the $\mathcal{I}$-matrix family $\langle \mathcal{A}^{\Omega^{\mathcal{A}}(T)}, T/\Omega^{\mathcal{A}}(T) \rangle$ is Leibniz reduced. Hence, the $\mathbf{F}$-algebraic system $\mathcal{A}^{\Omega^{\mathcal{A}}(T)}$ is in $\mathrm{AlgSys}^*(\mathcal{I})$. Equivalently, we have that $\Omega^{\mathcal{A}}(T) \in \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. Thus, the Leibniz operator is always a well defined function

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

In particular, it restricts to a well-defined function

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Additionally, by definition of $\mathrm{AlgSys}^\bullet(\mathcal{I})$, this may be perceived also as a function

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A}),$$

where, we set

$$\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A}) = \{\theta \in \mathrm{ConSys}(\mathcal{A}) : \mathcal{A}/\theta \in \mathrm{AlgSys}^{\bullet}(\mathcal{I})\}.$$

We keep these remarks in mind in the formulation of several of the following results.

Theorem 256 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is WS prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order embedding.

Proof: Suppose, first, that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, $\Omega^{\mathcal{A}}$ is an order embedding. In particular, because of Lemma 51,

$$\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^{*}(\mathcal{I})$$

is an order embedding. This implies that the Leibniz operator is monotone and injective on theory systems. Thus $\mathcal{I}$ is WS prealgebraizable.

Suppose, conversely, that $\mathcal{I}$ is WS prealgebraizable. Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system. Consider the mapping

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

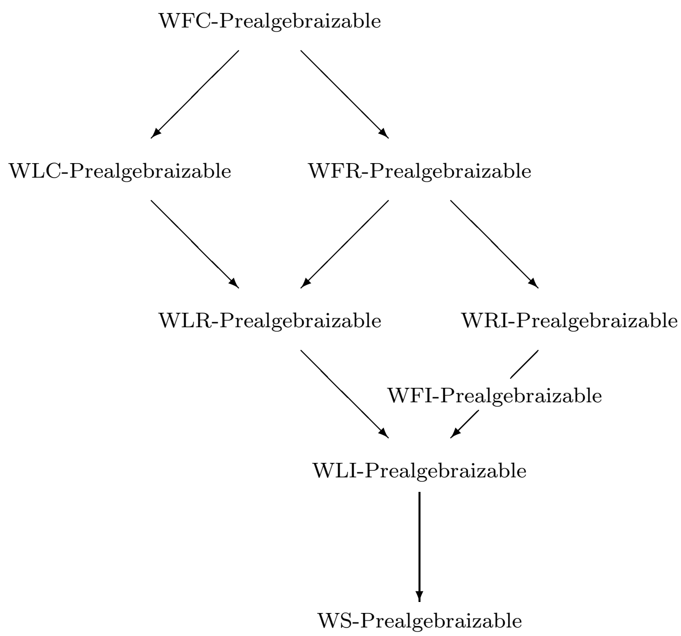
By Theorem 255, this mapping is monotone and injective. To show that it is an order embedding, we must show that it is also order reflecting. By Theorem 225, it suffices to show that $\mathcal{I}$ is system reflective. But this was accomplished in Theorem 248. ■

Corollary 257 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is WS prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$$

is an order isomorphism.

At this point in our studies we have the following hierarchy, which results from the preceding one by the identification established in Corollary 249.



We continue our study by showing that all three upper diagonal classes, namely those of WFC, WFR and WRI prealgebraizable π-institutions also coincide. To accomplish this for WFC and WFR prealgebraizability, we prove a partial analog of Theorem 248 that under prealgebraicity, family reflectivity and family complete reflectivity turn out to be equivalent properties. The crucial observation is that, as shown in Lemma 218, family reflectivity implies systemicity.

Theorem 258 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is prealgebraic and family reflective, then it is family completely reflective.*

Proof: Since $\mathcal{I}$ is family reflective, by Lemma 218, it is systemic. Since it is prealgebraic and system reflective, by Theorem 248, it is also system completely reflective. Hence, by systemicity, it is also family completely reflective. ■

Theorem 258 shows that two of the top classes in the previous diagram coincide.

Corollary 259 *The classes of WFR prealgebraizable and WFC prealgebraizable* π*-institutions coincide.*

Next we show that the classes of WFR and WRI prealgebraizable π-institutions coincide. We do this indirectly by providing identical characterizations of both classes involving WFI prealgebraizability and systemicity.

First, we need a result which will also prove useful later in our investigations. Namely, we look at an interesting and useful connection between family injectivity and family reflectivity, by means of protoalgebraicity, that forms a partial analog of Theorem 248, which related system injectivity with system reflectivity in the presence of prealgebraicity.

Proposition 260 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is protoalgebraic and family injective, then it is family reflective.*

Proof: Suppose that $\mathcal{I}$ is protoalgebraic and family injective. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then we get $\Omega(T) = \Omega(T) \cap \Omega(T')$. Moreover, by Lemma 23, $\Omega(T) \cap \Omega(T') \leq \Omega(T \cap T')$. On the other hand, by protoalgebraicity, we have $\Omega(T \cap T') \leq \Omega(T)$ and $\Omega(T \cap T') \leq \Omega(T')$, whence $\Omega(T \cap T') \leq \Omega(T) \cap \Omega(T')$. We conclude that

$$\Omega(T) = \Omega(T) \cap \Omega(T') = \Omega(T \cap T').$$

Now we use family injectivity to get $T = T \cap T'$. Therefore, $T \leq T'$. So $\mathcal{I}$ is also family reflective. ■

Now we characterize WFR prealgebraizability.

Theorem 261 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. Then* $\mathcal{I}$ *is WFR prealgebraizable if and only if it is WFI prealgebraizable and systemic.*

Proof: Suppose, first, that $\mathcal{I}$ is WFR prealgebraizable. Then it is, by definition, prealgebraic, it is, by Lemma 218, systemic and, by Proposition 228, it is family injective. Thus, it is WFI prealgebraizable and systemic.

Suppose, conversely, that $\mathcal{I}$ is WFI prealgebraizable and systemic. Then, it is, by definition, prealgebraic and family injective, which, by systemicity, imply that it is protoalgebraic and family injective. Thus, by Proposition 260, it is protoalgebraic and family reflective. Hence, it is, a fortiori, WFR prealgebraizable. ■

But it is easy to show also that WRI prealgebraizability has exactly the same characterization as WFR prealgebraizability.

Theorem 262 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. Then* $\mathcal{I}$ *is WRI prealgebraizable if and only if it is WFI prealgebraizable and systemic.*

Proof: This follows directly from Proposition 209. ■

Corollary 263 *The classes of WFR prealgebraizable and WRI prealgebraizable* π*-institutions coincide.*

Proof: The conclusion follows from Theorems 261 and 262. ■

Corollaries 259 and 263 show that, among the top four classes of the hierarchy in the preceding diagram, only two may be (and are, as we show in the following example) different. We keep the names WFR prealgebraizable and WFI prealgebraizable for the π-institutions in each of these classes. Thus, given an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ and a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ based on $\mathbf{F}$:

- $\mathcal{I}$ is *WFR prealgebraizable* if it prealgebraic and family reflective (or, equivalently, family c-reflective or right injective), i.e., if

$$\begin{array}{rll} T \le T' & \text{implies} & \Omega(T) \le \Omega(T'), \text{ for all } T, T' \in \mathrm{ThSys}(\mathcal{I}); \\ \Omega(T) \le \Omega(T') & \text{implies} & T \le T', \text{ for all } T, T' \in \mathrm{ThFam}(\mathcal{I}); \end{array}$$

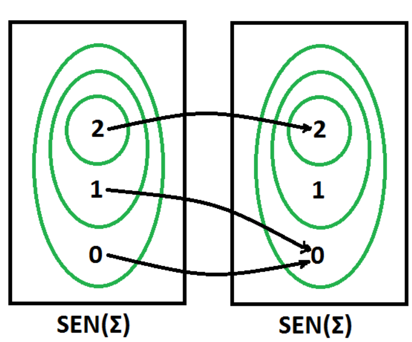
- $\mathcal{I}$ is *WFI prealgebraizable* if it is prealgebraic and family injective, i.e., if

$$\begin{array}{rll} T \le T' & \text{implies} & \Omega(T) \le \Omega(T'), \text{ for all } T, T' \in \mathrm{ThSys}(\mathcal{I}); \\ \Omega(T) = \Omega(T') & \text{implies} & T = T', \text{ for all } T, T' \in \mathrm{ThFam}(\mathcal{I}). \end{array}$$

We provide an example to show that these two classes of π-institutions are indeed different, i.e., we exhibit a WFI prealgebraizable π-institution which is not WFR prealgebraizable.

Example 264 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial clone.*

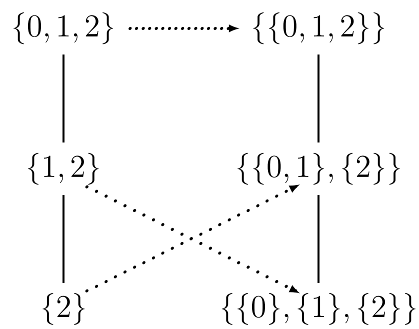


Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\}, \{1,2\}, \{0,1,2\}\}$.

The table giving the action of $\overleftarrow{\ }$ *on theory families is shown below. It is clear that* $\mathcal{I}$ *is not systemic.*

T_Σ	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$
$\overleftarrow{T}_\Sigma$	$\{2\}$	$\{2\}$	$\{0,1,2\}$

The following diagram gives the lattice of theory families and the corresponding Leibniz congruence systems.



We can see that $\mathcal{I}$ *is prealgebraic and family injective. Since it is not systemic, by Theorem 261, it follows that it is not family reflective, a fact that can also be directly verified by the diagram. We conclude that* $\mathcal{I}$ *is a WFI prealgebraizable* π*-institution, which is not WFR prealgebraizable.*

We now provide a theorem to the effect that both classes are characterized by theorems asserting that their properties transfer from theory systems/families to filter systems/families on arbitrary algebraic systems.

Theorem 265 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is WFI prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, the Leibniz operator on* $\mathcal{A}$ *is monotone on* $\mathcal{I}$*-filter systems and injective on* $\mathcal{I}$*-filter families, i.e.,*

$$T \le T' \text{ implies } \Omega^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T'), \text{ for all } T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A});$$
$$\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T') \text{ implies } T = T', \text{ for all } T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}).$$

Proof: The "if" direction follows by specializing to $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account Lemma 51.

For the "only if" suppose that $\mathcal{I}$ is WFI prealgebraizable and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system. By definition, $\mathcal{I}$ is prealgebraic and family injective. Thus, by Theorem 179, the Leibniz operator on the $\mathcal{I}$-filter systems of $\mathcal{A}$ is monotone and, by Theorem 214, the Leibniz operator on the $\mathcal{I}$-filter families of $\mathcal{A}$ is injective. ∎

Theorem 266 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is WFR prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, the Leibniz operator on* $\mathcal{A}$ *is monotone on* $\mathcal{I}$*-filter systems and order reflecting on* $\mathcal{I}$*-filter families, i.e.,*

$$T \leq T' \text{ implies } \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T'), \text{ for all } T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A});$$
$$\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T') \text{ implies } T \leq T', \text{ for all } T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}).$$

Proof: The "if" direction follows by specializing to $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account Lemma 51.

For the "only if" suppose that $\mathcal{I}$ is WFR prealgebraizable and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system. By definition, $\mathcal{I}$ is prealgebraic and family reflective. Thus, by Theorem 179, the Leibniz operator on the $\mathcal{I}$-filter systems of $\mathcal{A}$ is monotone and, by Theorem 225, the Leibniz operator on the $\mathcal{I}$-filter families of $\mathcal{A}$ is injective. ■

Next we give two important results, along the lines of the characterization Theorem 256 for WS prealgebraizability, characterizing the classes of WFI prealgebraizable and WFR prealgebraizable π-institutions.

Theorem 267 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is WFI prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a bijection which restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: Suppose, first, that $\mathcal{I}$ is WFI prealgebraizable. Then, it is a fortiori WS prealgebraizable. Thus, by Theorem 256, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order embedding. So it suffices to show that it extends to a bijection from $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ onto $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. Since $\mathcal{I}$ is family injective, this mapping is injective. It is also surjective: Given $\theta \in \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$, we have by definition, $\mathcal{A}^\theta \in \mathrm{AlgSys}^*(\mathcal{I})$. Thus, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta)$, such that $\Omega^{\mathcal{A}^\theta}(T) = \Delta^{\mathcal{A}^\theta}$. Now applying the inverse of the canonical quotient morphism $\langle I, \pi^\theta \rangle : \mathcal{A} \to \mathcal{A}^\theta$, we get $\pi^{\theta^{-1}}(\Omega^{\mathcal{A}^\theta}(T)) = \pi^{\theta^{-1}}(\Delta^{\mathcal{A}^\theta})$, whence, by Proposition 24, $\Omega^{\mathcal{A}}(\pi^{\theta^{-1}}(T)) = \theta$. Since, by Corollary 55, $\pi^{\theta^{-1}}(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get that the Leibniz operator is also surjective.

Suppose, conversely, that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a bijection which restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Then, by Theorem 256, $\mathcal{I}$ is WS prealgebraizable. Thus, in particular, it is prealgebraic. The fact that $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is a bijection ensures that the Leibniz operator on $\mathrm{ThFam}(\mathcal{I})$ is injective. Thus $\mathcal{I}$ is also family injective and, therefore, it is WFI prealgebraizable. ■

And now an analogous characterization for WFR prealgebraizability.

Theorem 268 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is WFR prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism.

Proof: Suppose, first, that $\mathcal{I}$ is WFR prealgebraizable. Then, it is a fortiori WFI prealgebraizable. So by Theorem 267

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order embedding which extends to a bijection

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

But, by Theorem 261, $\mathcal{I}$ is systemic. Therefore, we get an order isomorphism

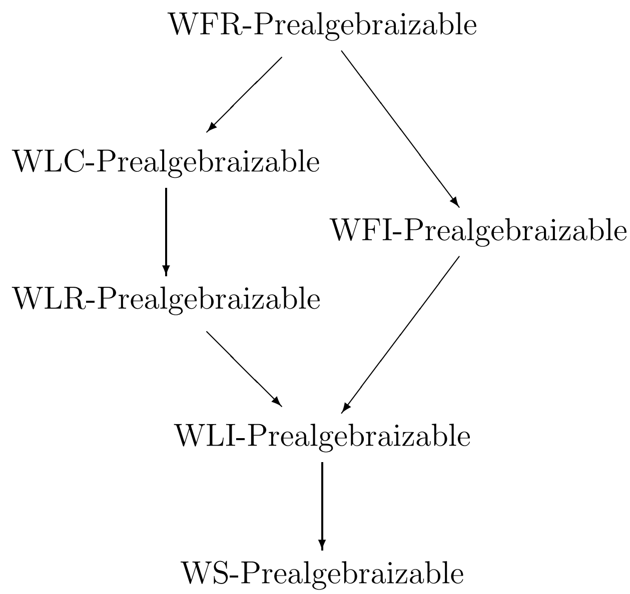
$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Suppose, conversely, that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism. In particular, $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{F})$ is an order isomorphism. This ensures that the Leibniz operator is monotone on theory families, hence on theory systems, and, moreover, that it is reflective on theory families. Thus, $\mathcal{I}$ is prealgebraic and family reflective, i.e., it is a WFR prealgebraizable π-institution. ■

We take a break again to draw the hierarchy incorporating the information that we have currently available.



Recall again the formal definitions of the three classes that have not yet been at the focus of our investigations, namely those of WLC, WLR and WLI prealgebraizable π-institutions:

- $\mathcal{I}$ is *WLI Prealgebraizable* if it is prealgebraic and left injective, i.e., if

$$T \leq T' \text{ implies } \Omega(T) \leq \Omega(T'), \text{ for all } T, T' \in \text{ThSys}(\mathcal{I});$$
$$\Omega(T) = \Omega(T') \text{ implies } \overleftarrow{T} = \overleftarrow{T'}, \text{ for all } T, T' \in \text{ThFam}(\mathcal{I});$$

- $\mathcal{I}$ is *WLR Prealgebraizable* if it is prealgebraic and left reflective, i.e., if

$$T \leq T' \text{ implies } \Omega(T) \leq \Omega(T'), \text{ for all } T, T' \in \text{ThSys}(\mathcal{I});$$
$$\Omega(T) \leq \Omega(T') \text{ implies } \overleftarrow{T} \leq \overleftarrow{T'}, \text{ for all } T, T' \in \text{ThFam}(\mathcal{I});$$

- $\mathcal{I}$ is *WLC Prealgebraizable* if it is prealgebraic and left completely reflective, i.e., if

$$T \leq T' \text{ implies } \Omega(T) \leq \Omega(T'), \text{ for all } T, T' \in \text{ThSys}(\mathcal{I});$$
$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \text{ implies } \bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}, \text{ for all } \mathcal{T} \cup \{T'\} \subseteq \text{ThFam}(\mathcal{I}).$$

We showed in Example 264 that the top right arrow in the preceding diagram represents a proper inclusion. Moreover, we showed in Theorem 261 that the two classes are separated by systemicity. Now we study the remaining five inclusions to reveal relationships between them and to verify that they are also proper.

We look, first, at the top left arrow, i.e., at the inclusion of the class of WFR prealgebraizable into that of WLC prealgebraizable π-institutions. We have the following extension of Theorem 261, which shows that systemicity is actually the property that separates the top class from every other class in this hierarchy.

Theorem 269 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is WFR prealgebraizable if and only if it is WLC, WLR, WFI, WLI or WS prealgebraizable and systemic.*

Proof: Suppose that $\mathcal{I}$ is WFR prealgebraizable. We showed in Theorem 261 that it is systemic. Moreover, it belongs, a fortiori, to all other classes in the hierarchy, since the conditions defining them are weaker than prealgebraicity and family complete reflectivity (which was showed to be equivalent to family reflectivity under prealgebraicity in Theorem 261).

Suppose, conversely, that $\mathcal{I}$ is WS prealgebraizable and systemic. This implies, by definition, that it is prealgebraic and system completely reflective. Thus, by systemicity, it is also family completely reflective. Therefore, since it is prealgebraic and family completely reflective, it is, by definition, WFR prealgebraizable. ■

A more interesting, perhaps, view is the status of this hierarchy under the milder assumption of stability. Even though systemicity leads to a total collapse of the hierarchy into a single class, it turns out that stability allows for a two-class hierarchy.

Proposition 270 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is WFI prealgebraizable and stable, then it is systemic.*

Proof: Suppose that $\mathcal{I}$ is WFI prealgebraizable and stable and let $T \in \mathrm{ThFam}(\mathcal{I})$. Since $\mathcal{I}$ is stable, we have $\Omega(T) = \Omega(\overleftarrow{T})$. Thus, using family injectivity, we get $T = \overleftarrow{T}$. It follows that $T \in \mathrm{ThSys}(\mathcal{I})$. We now conclude that $\mathrm{ThFam}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$ and, therefore, $\mathcal{I}$ is systemic. ■

Theorem 271 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is WFR prealgebraizable if and only if it is WFI prealgebraizable and stable.*

Proof: If $\mathcal{I}$ is WFR prealgebraizable, then it is, a fortiori, WFI prealgebraizable and, by Theorem 261, systemic and, therefore, stable. On the other hand, if $\mathcal{I}$ is WFI prealgebraizable and stable, then, by Proposition 270, it is systemic and, hence, by Theorem 269, it is WFR prealgebraizable. ■

Proposition 272 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is WS prealgebraizable and stable, then it is WLC prealgebraizable.*

Proof: Suppose that $\mathcal{I}$ is WS prealgebraizable and stable and consider $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. By stability, we get that $\bigcap_{T\in\mathcal{T}} \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Since $\{\overleftarrow{T} : T \in \mathcal{T}\} \cup \{\overleftarrow{T'}\} \subseteq \mathrm{ThSys}(\mathcal{I})$, we get, by WS prealgebraizability, that $\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$. This proves that $\mathcal{I}$ is left c-reflective and, hence, that it is WLC prealgebraizable. ∎

Theorem 273 *For stable π-institutions the weak prealgebraizability hierarchy collapses to the classes of weakly family prealgebraizable and weakly system/left prealgebraizable classes that are related as follows*

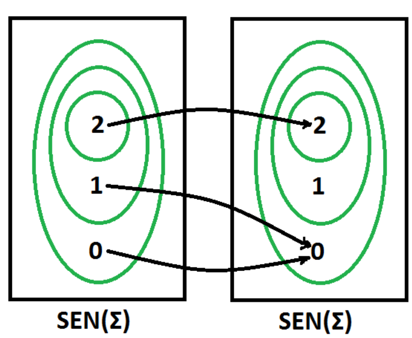
$$\begin{array}{c} \textit{WF Prealgebraizable} \\ \downarrow \\ \textit{WS Prealgebraizable} \end{array}$$

Proof: This follows by Theorem 271 and Proposition 272. ∎

Now we look at an example to verify that WFR prealgebraizable π-institutions form a proper subclass of WLC prealgberaizable π-institutions.

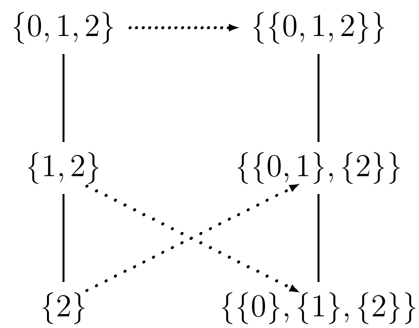
Example 274 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object Σ and a single non-identity morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial clone.*



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by setting $\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}$.

It has three theory families $T := \mathrm{Thm}(\mathcal{I})$, T' and $T'' := \mathrm{SEN}$, with $T_\Sigma = \{2\}$, $T'_\Sigma = \{1, 2\}$ and $T''_\Sigma = \{0, 1, 2\}$, but only two theory systems T and T'', *since* $\overleftarrow{T'} = T$. *A look at the lattice of theory families and the corresponding Leibniz congruence systems shows that it is prealgebraic and left completely reflective.*



On the other hand, it is not family reflective, since $\Omega(T') \leq \Omega(T)$, *but* $T' \nleq T$. *So* $\mathcal{I}$ *it is WLC prealgebraizable, but not WFR prealgebraizable.*

For WLC prealgebraizable π-institutions we have the following transfer theorem.

Theorem 275 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is WLC prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, the Leibniz operator on* $\mathcal{A}$ *is monotone on* $\mathcal{I}$*-filter systems and left completely order reflecting on* $\mathcal{I}$*-filter families, i.e.,*

$$T \leq T' \text{ implies } \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T'), \text{ for all } T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A});$$
$$\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T') \text{ implies } \bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}, \text{ for all } \mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}).$$

Proof: The "if" direction follows by specializing to $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account Lemma 51.

For the "only if" suppose that $\mathcal{I}$ is WLC prealgebraizable and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system. By definition, $\mathcal{I}$ is prealgebraic and left completely reflective. Thus, by Theorem 179, the Leibniz operator on the $\mathcal{I}$-filter systems of $\mathcal{A}$ is monotone and, by Theorem 240, the Leibniz operator on the $\mathcal{I}$-filter families of $\mathcal{A}$ is left completely order reflecting. ∎

Moreover, we obtain the following characterization theorem:

Theorem 276 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is WLC prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left completely order reflecting surjection that restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: Suppose, first, that $\mathcal{I}$ is WLC prealgebraizable. Then, it is a fortiori WS prealgebraizable. Thus, by Theorem 256, for every **F**-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order embedding. So it suffices to show that it extends to a left completely order reflecting surjection from $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ onto $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. Since $\mathcal{I}$ is left completely order reflective, by Theorem 275, this mapping is left completely order reflecting. That it is also surjective may be seen by the same argument used in the proof of Theorem 267.

Suppose, conversely, that, for every **F**-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left completely order reflecting surjection that restricts to an order embedding

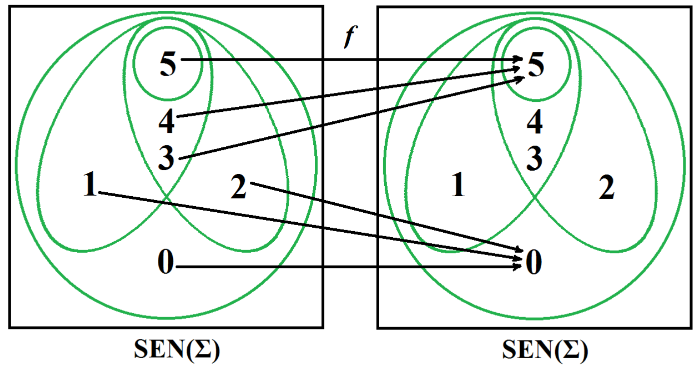
$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Then, by Theorem 256, $\mathcal{I}$ is WS prealgebraizable. Thus, in particular, it is prealgebraic. The fact that $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{F})$ is left completely order reflecting ensures that the Leibniz operator on $\mathrm{ThFam}(\mathcal{I})$ is left completely order reflecting. Thus $\mathcal{I}$ is also completely order reflective and, therefore, it is WLC prealgebraizable. ■

We switch to the left vertical arrow in the diagram. We present an example to verify that WLC prealgebraizable π-institutions form a proper subclass of WLR prealgebraizable π-institutions.

Example 277 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2, 3, 4, 5\}$ *and*
$$\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = \mathrm{SEN}^\flat(f)(2) = 0,$$
$$\mathrm{SEN}^\flat(f)(3) = \mathrm{SEN}^\flat(f)(4) = \mathrm{SEN}^\flat(f)(5) = 5;$$
- $N^\flat$ *is the category of natural transformations generated by the two unary natural transformations* $\sigma^\flat, \tau^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, with*
$$\sigma^\flat_\Sigma, \tau^\flat_\Sigma : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SEN}^\flat(\Sigma)$$
defined by
  - $\sigma^\flat_\Sigma(3) = 1$ *and* $\sigma^\flat_\Sigma(x) = 0$*, for all* $x \in \{0, 1, 2, 4, 5\}$*;*
  - $\sigma^\flat_\Sigma(4) = 2$ *and* $\sigma^\flat_\Sigma(x) = 0$*, for all* $x \in \{0, 1, 2, 3, 5\}$*.*



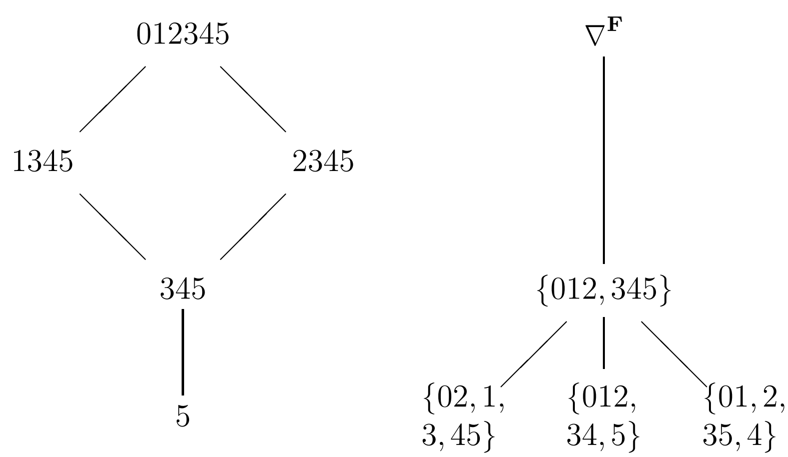
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

$$\mathcal{C}_\Sigma = \{\{5\}, \{3,4,5\}, \{1,3,4,5\}, \{2,3,4,5\}, \{0,1,2,3,4,5\}\}.$$

$\mathcal{I}$ *has five theory families but only three theory systems. The action of* $\overleftarrow{\ }$ *on theory families is given by the following table.*

T	$\overleftarrow{T}$
$\{5\}$	$\{5\}$
$\{3,4,5\}$	$\{3,4,5\}$
$\{1,3,4,5\}$	$\{3,4,5\}$
$\{2,3,4,5\}$	$\{3,4,5\}$
$\{0,1,2,3,4,5\}$	$\{0,1,2,3,4,5\}$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



From the diagram, it is clear that $\mathcal{I}$ *is prealgebraic, i.e., that, for all* $T, T' \in \mathrm{ThSys}(\mathcal{I})$, $T \le T'$ *implies* $\Omega(T) \le \Omega(T')$. *Moreover, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$, *if* $\Omega(T) \le \Omega(T')$, *then* $\overleftarrow{T} \le \overleftarrow{T'}$, *i.e.,* $\mathcal{I}$ *is left reflective. Therefore,*

$\mathcal{I}$ if WLR prealgebraizable. On the other hand, setting, $T^1 = \{\{1,3,4,5\}\}$, $T^2 = \{\{2,3,4,5\}\}$ and $T' = \{\{5\}\}$, we get

$$\begin{array}{rcl}\Omega(T^1) \cap \Omega(T^2) & = & \{\{02,1,3,45\}\} \cap \{\{01,2,35,4\}\} \\ & = & \Delta^{\mathbf{F}} \\ & \leq & \{\{012,34,5\}\} = \Omega(T'),\end{array}$$

whereas

$$\overleftarrow{T^1} \cap \overleftarrow{T^2} = \{\{3,4,5\}\} \cap \{\{3,4,5\}\} = \{\{3,4,5\}\} \nleq \{\{5\}\} = \overleftarrow{T'}.$$

Hence, $\mathcal{I}$ is not left completely reflective and, thus, a fortiori, it fails to be WLC prealgebraizable.

For WLR prealgebraizable π-institutions we have the following transfer theorem.

Theorem 278 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is WLR prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, the Leibniz operator on $\mathcal{A}$ is monotone on $\mathcal{I}$-filter systems and left order reflecting on $\mathcal{I}$-filter families, i.e.,*

$$\begin{array}{c} T \leq T' \text{ implies } \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T'), \text{ for all } T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}); \\ \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T') \text{ implies } \overleftarrow{T} \leq \overleftarrow{T'}, \text{ for all } T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}). \end{array}$$

Proof: The "if" direction follows by specializing to $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account Lemma 51.

For the "only if" suppose that $\mathcal{I}$ is WLR prealgebraizable and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system. By definition, $\mathcal{I}$ is prealgebraic and left reflective. Thus, by Theorem 179, the Leibniz operator on the $\mathcal{I}$-filter systems of $\mathcal{A}$ is monotone and, by Theorem 225, the Leibniz operator on the $\mathcal{I}$-filter families of $\mathcal{A}$ is left order reflecting. ■

Moreover, we obtain the following characterization theorem:

Theorem 279 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is WLR prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left order reflecting surjection that restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: Suppose, first, that $\mathcal{I}$ is WLR prealgebraizable. Then, it is, a fortiori, WS prealgebraizable. Thus, by Theorem 256, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order embedding. So it suffices to show that it extends to a left order reflecting surjection from $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ onto $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. Since $\mathcal{I}$ is left reflective, by Theorem 278, this mapping is left order reflecting. That it is also surjective may be seen by the same argument used in the proof of Theorem 267.

Suppose, conversely, that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left order reflecting surjection that restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Then, by Theorem 256, $\mathcal{I}$ is WS prealgebraizable. So, it is prealgebraic. The fact that $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is left order reflecting ensures that the Leibniz operator on $\mathrm{ThFam}(\mathcal{I})$ is left order reflecting. Thus $\mathcal{I}$ is also left reflective and, hence, WLR prealgebraizable. ■

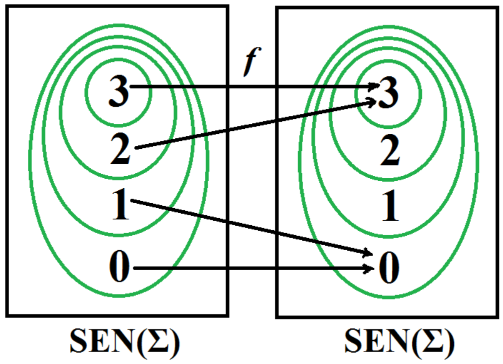
We switch to the bottom left arrow in the diagram. We present an example to verify that WLR prealgebraizable π-institutions form a proper subclass of WLI prealgebraizable π-institutions.

Example 280 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2, 3\}$ *and*
$$\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = 0,$$
$$\mathrm{SEN}^\flat(f)(2) = \mathrm{SEN}^\flat(f)(3) = 3;$$
- $N^\flat$ *is the category of natural transformations generated by the single unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined by* $\sigma^\flat_\Sigma(0) = \sigma^\flat_\Sigma(1) = \sigma^\flat_\Sigma(3) = 0$ *and* $\sigma^\flat_\Sigma(2) = 1$*.*

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

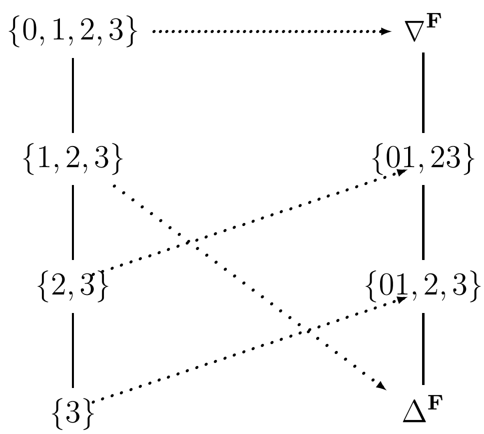
$$\mathcal{C}_\Sigma = \{\{3\}, \{2, 3\}, \{1, 2, 3\}, \{0, 1, 2, 3\}\}.$$



$\mathcal{I}$ has four theory families but only three theory systems. The action of $\overleftarrow{\ }$ *on theory families is given by the following table.*

T	$\overleftarrow{T}$
$\{3\}$	$\{3\}$
$\{2,3\}$	$\{2,3\}$
$\{1,2,3\}$	$\{2,3\}$
$\{0,1,2,3\}$	$\{0,1,2,3\}$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



From the diagram, it is clear that $\mathcal{I}$ is prealgebraic, i.e., that, for all $T,T' \in \mathrm{ThSys}(\mathcal{I})$, *$T \le T'$ implies $\Omega(T) \le \Omega(T')$. Moreover, for all $T,T' \in$* $\mathrm{ThFam}(\mathcal{I})$, *the implication $\Omega(T) = \Omega(T')$ implies $\overleftarrow{T} = \overleftarrow{T'}$ holds trivially, since no two different theory families share a common Leibniz congruence system. Hence, $\mathcal{I}$ is left injective. We conclude that $\mathcal{I}$ if WLI prealgebraizable. On the other hand, setting, $T = \{\{1,2,3\}\}$ and $T' = \{\{3\}\}$, we get*

$$\Omega(T) = \Delta^{\mathbf{F}} \le \{\{01,2,3\}\} = \Omega(T'),$$

whereas

$$\overleftarrow{T} = \{\{2,3\}\} \nleq \{\{3\}\} = \overleftarrow{T'}.$$

Hence, $\mathcal{I}$ is not left reflective and, therefore, a fortiori, it is not WLR prealgebraizable.

For WLI prealgebraizable π-institutions we have the following transfer theorem.

Theorem 281 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is WLI prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, the Leibniz operator on $\mathcal{A}$ is monotone on $\mathcal{I}$-filter systems and left injective on $\mathcal{I}$-filter families, i.e.,*

$$T \leq T' \text{ implies } \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T'), \text{ for all } T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A});$$
$$\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T') \text{ implies } \overleftarrow{T} = \overleftarrow{T'}, \text{ for all } T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}).$$

Proof: The "if" direction follows by specializing to $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account Lemma 51.

For the "only if" suppose that $\mathcal{I}$ is WLI prealgebraizable and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system. By definition, $\mathcal{I}$ is prealgebraic and left injective. Thus, by Theorem 179, the Leibniz operator on the $\mathcal{I}$-filter systems of $\mathcal{A}$ is monotone and, by Theorem 214, the Leibniz operator on the $\mathcal{I}$-filter families of $\mathcal{A}$ is left injective. ■

For WLI prealgebraizable π-institutions, we obtain the following characterization theorem:

Theorem 282 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is WLI prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left injective surjection that restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: Suppose, first, that $\mathcal{I}$ is WLI prealgebraizable. Then, it is, a fortiori, WS prealgebraizable. Thus, by Theorem 256, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a lattice embedding. So it suffices to show that it extends to a left injective surjection from $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ onto $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. Since $\mathcal{I}$ is left injective, by Theorem 281, this mapping is left injective. That it is also surjective may be seen by the same argument used in the proof of Theorem 267.

Suppose, conversely, that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left injective surjection that restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

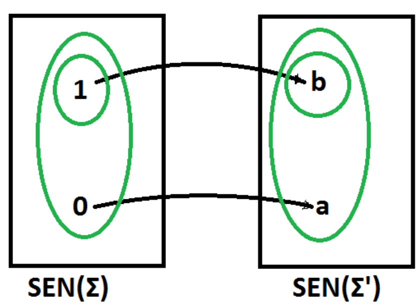
Then, by Theorem 256, $\mathcal{I}$ is WS prealgebraizable. Thus, in particular, it is prealgebraic. The fact that $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is left injective ensures that the Leibniz operator on $\mathrm{ThFam}(\mathcal{I})$ is left injective. Thus $\mathcal{I}$ is also left injective and, hence, WLI prealgebraizable. ■

We turn next to the bottom right arrow in the diagram.

We know by Proposition 208 that WFI π-institutions form a subclass of the class of WLI π-institutions. Moreover, we know by Theorem 269 that, if $\mathcal{I}$ is WLI prealgebraizable and systemic, then it is WFI prealgebraizable. We give now an example showing that the inclusion of the class of WFI prealgebraizable π-institutions into the class of WLI prealgebraizable π-institutions is proper.

Example 283 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$*;*
- $N^\flat$ *is the trivial clone.*



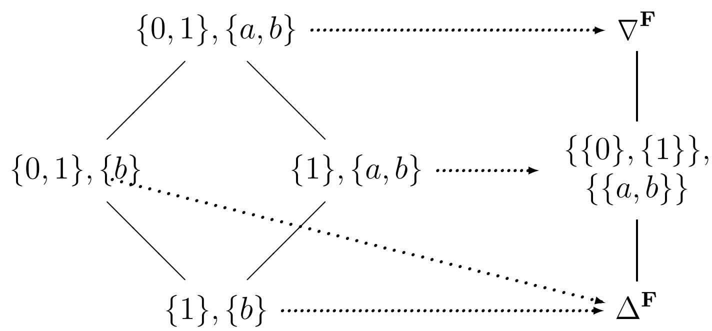
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$C_\Sigma = \{\{1\}, \{0, 1\}\} \quad \textit{and} \quad C_{\Sigma'} = \{\{b\}, \{a, b\}\}.$$

The table yielding the action of $\overleftarrow{\ }$ *on theory families is shown below.*

$\leftarrow$	$\{b\}$	$\{a, b\}$
$\{1\}$	$\{1\}, \{b\}$	$\{1\}, \{a, b\}$
$\{0, 1\}$	$\{1\}, \{b\}$	$\{0, 1\}, \{a, b\}$

The accompanying diagram gives the structure of the lattice of theory families and the corresponding Leibniz congruence systems.



From the diagram one can check that the Leibniz operator is monotone on theory systems and left injective on theory families. Thus, the π-institution is prealgebraic and left injective, i.e., WLI prealgebraizable.

On the other hand, letting $T = \{\{1\},\{b\}\}$ and $T' = \{\{0,1\},\{b\}\}$, we have $\Omega(T) = \Omega(T')$, but $T \neq T'$, whence $\mathcal{I}$ is not family injective and, therefore, it is not WFI prealgebraizable.

We look now at the very bottom arrow of the diagram. By Theorem 273, if $\mathcal{I}$ is a WS prealgebraizble and stable π-institution, then it is WLI prealgebraizable. We provide, next, an example to show that these two classes are different, i.e., the class of WLI prealgebraizable π-institutions is properly included in that of WS prealgebraizble π-institutions.

Example 284 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be the algebraic system defined as follows:*

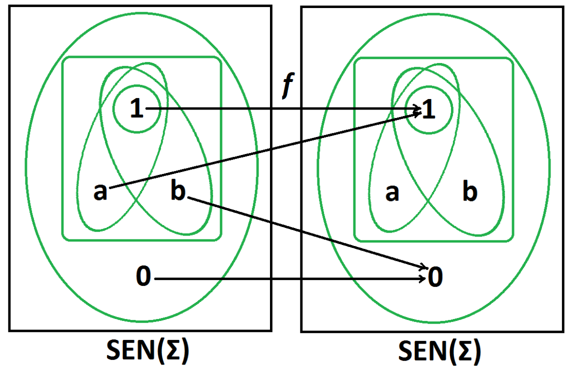
- $\mathbf{Sign}^\flat$ *is the category with a single object Σ and a single non-identity morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, a, b, 1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(a) = 1$, $\mathrm{SEN}^\flat(f)(b) = 0$ *and* $\mathrm{SEN}^\flat(f)(1) = 1$*;*
- $N^\flat$ *is the category of natural transformations generated by the two binary natural transformations* $\wedge, \vee : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by the following tables:*

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be the π-institution, defined by setting

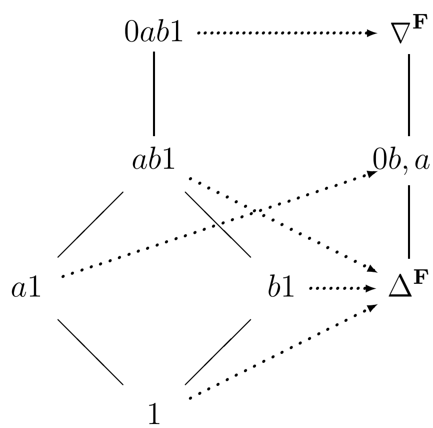
$$\mathcal{C}_\Sigma = \{\{1\},\{a,1\},\{b,1\},\{a,b,1\},\{0,a,b,1\}\}.$$



The following table gives the theory families and the theory systems of the π-institution $\mathcal{I}$:

T	$\overleftarrow{T}$
$\{1\}$	$\{1\}$
$\{a,1\}$	$\{a,1\}$
$\{b,1\}$	$\{1\}$
$\{a,b,1\}$	$\{a,1\}$
$\{0,a,b,1\}$	$\{0,a,b,1\}$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



Since $\Omega(\overleftarrow{\{\{a,b,1\}\}}) = \Omega(\{\{a,1\}\}) = \{\{0,b\},\{a,1\}\} \neq \Delta^{\mathbf{F}} = \Omega(\{\{a,b,1\}\})$, *we conclude that* $\mathcal{I}$ *is not stable.*

Note that, since $\{\{1\}\}$, $\{\{a,1\}\}$ *and* $\mathrm{SEN}^\flat$ *are the only theory systems of* $\mathcal{I}$, *the Leibniz operator* $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is an order isomorphism. Hence,* $\mathcal{I}$ *is both prealgebraic and system injective, i.e., it is WS prealgebraizable. On the other hand, we have*

$$\Omega(\{\{a,b,1\}\}) = \Delta^{\mathbf{F}} = \Omega(\{\{b,1\}\}),$$

but

$$\overleftarrow{\{\{a,b,1\}\}} = \{\{a,1\}\} \neq \{\{1\}\} = \overleftarrow{\{\{b,1\}\}},$$

whence, $\mathcal{I}$ is not left injective and, hence, it fails to be WLI prealgebraizable.

4.3 Weak Algebraizability

We now shift attention to classes of π-institutions that are defined as a result of interactions between the various kinds of injectivity, reflectivity and complete reflectivity, on the one hand, and protoalgebraicity on the other. A priori, based on the ordering of the various injectivity, reflectivity and complete reflectivity properties, we have ten classes, which are defined below and whose hierarchy is shown in the accompanying diagram.

Definition 285 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

- *$\mathcal{I}$ is* **weakly system injective algebraizable** *or* **WSI Algebraizable**, *for short, if it is system injective and protoalgebraic.*
- *$\mathcal{I}$ is* **weakly left injective algebraizable** *or* **WLI Algebraizable**, *for short, if it is left injective and protoalgebraic.*
- *$\mathcal{I}$ is* **weakly family injective algebraizable** *or* **WFI Algebraizable**, *for short, if it is family injective and protoalgebraic.*
- *$\mathcal{I}$ is* **weakly right injective algebraizable** *or* **WRI Algebraizable**, *for short, if it is right injective and protoalgebraic.*

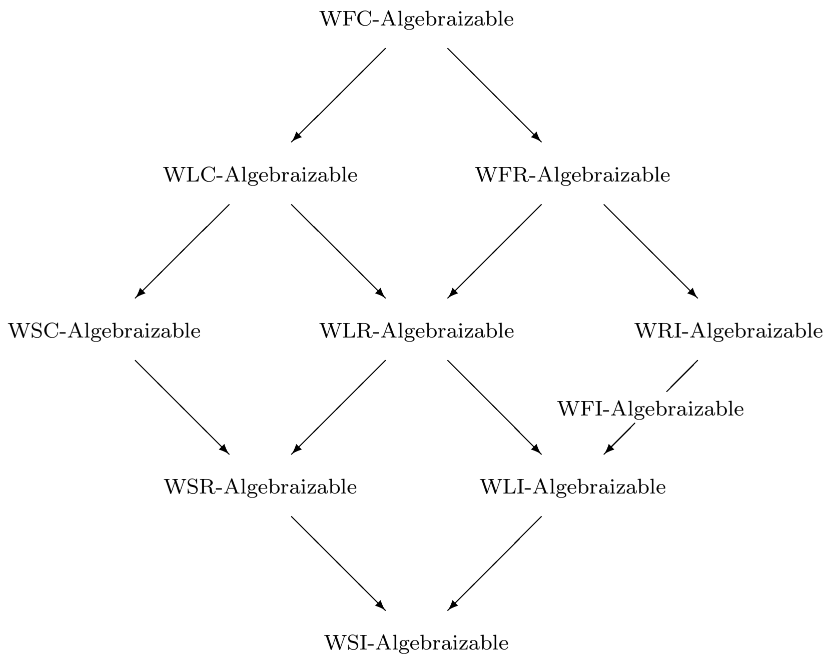
Definition 286 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

- *$\mathcal{I}$ is* **weakly system reflective algebraizable** *or* **WSR Algebraizable**, *for short, if it is system reflective and protoalgebraic.*
- *$\mathcal{I}$ is* **weakly left reflective algebraizable** *or* **WLR Algebraizable**, *for short, if it is left reflective and protoalgebraic.*
- *$\mathcal{I}$ is* **weakly family reflective algebraizable** *or* **WFR Algebraizable**, *for short, if it is family reflective and protoalgebraic.*

Definition 287 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

- *$\mathcal{I}$ is* **weakly system completely reflective algebraizable** *or* **WSC Algebraizable**, *for short, if it is system completely reflective and protoalgebraic.*

- *$\mathcal{I}$ is* **weakly left completely reflective algebraizable** *or* **WLC Algebraizable***, for short, if it is left completely reflective and protoalgebraic.*
- *$\mathcal{I}$ is* **weakly family completely reflective algebraizable** *or* **WFC Algebraizable***, for short, if it is family completely reflective and protoalgebraic.*



In view of the remarks made about terminology at the beginning of Section 4.2, the naming conventions here should be fairly obvious. The only difference is that the term "prealgebraizable" has been replaced by the term "algebraizable" to reflect the fact that the condition that the π-institution be prealgebraic is being replaced in the definitions by that of being protoalgebraic.

Recall from Theorem 248 that, under prealgebraicity, the properties of being system injective, system reflective and system completely reflective coincide. A similar result holds for the properties of family injectivity, right injectivity, family reflectivity and family complete reflectivity under the assumption of protoalgebraicity.

Theorem 288 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$*. If $\mathcal{I}$ is protoalgebraic, then the following statements are equivalent:*

(a) $\mathcal{I}$ is family injective;

(b) $\mathcal{I}$ is family reflective;

(c) $\mathcal{I}$ *is family completely reflective;*

(d) $\mathcal{I}$ *is right injective.*

Proof:

(a)⇒(b) Suppose that $\mathcal{I}$ is fanily injective. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then we get $\Omega(T) = \Omega(T) \cap \Omega(T')$. Moreover, by Lemma 23, $\Omega(T) \cap \Omega(T') \leq \Omega(T \cap T')$. On the other hand, by protoalgebraicity, we have $\Omega(T \cap T') \leq \Omega(T)$ and $\Omega(T \cap T') \leq \Omega(T')$, whence $\Omega(T \cap T') \leq \Omega(T) \cap \Omega(T')$. We conclude that

$$\Omega(T) = \Omega(T) \cap \Omega(T') = \Omega(T \cap T').$$

Now we use family injectivity to get $T = T \cap T'$. Therefore, $T \leq T'$. So $\mathcal{I}$ is also family reflective.

(b)⇒(c) Suppose, next, that $\mathcal{I}$ is family reflective. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. Since $\mathcal{I}$ is protoalgebraic, i.e., Ω is monotone on theory families, we have, for all $T \in \mathcal{T}$, $\Omega(\bigcap \mathcal{T}) \leq \Omega(T)$. Therefore, we get

$$\Omega(\bigcap_{T \in \mathcal{T}} T) \leq \bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T').$$

Since, by hypothesis, $\mathcal{I}$ is family reflective, we get $\bigcap_{T \in \mathcal{T}} T \leq T'$. Thus, $\mathcal{I}$ is family completely reflective.

(c)⇒(d) Suppose that $\mathcal{I}$ is family completely reflective. By Proposition 243, it is family reflective, and, then, by Proposition 228, it is right injective.

(d)⇒(a) Suppose, finally, that $\mathcal{I}$ is right injective. Then, by Proposition 208, it is family injective.

■

Theorem 288 shows that four of the classes in the previous diagram coincide.

Corollary 289 *The classes of WFI, WRI, WFR and WFC algebraizable* π*-institutions coincide.*

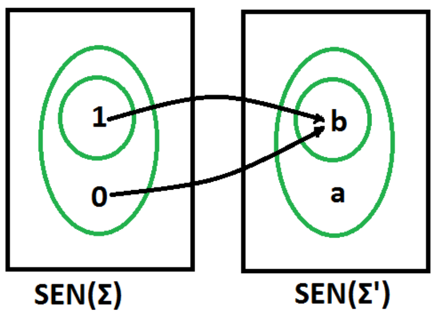
Given an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ and a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ based on $\mathbf{F}$, we use the term **weakly family algebraizable** (ot **WF algebraizable** for short) for $\mathcal{I}$ if it is protoalgebraic and family injective (or, equivalently, right injective or family reflective or family completely reflective), i.e., if the Leibniz operator is monotone and injective on theory families: For all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$\begin{array}{rcl} T \leq T' & \text{implies} & \Omega(T) \leq \Omega(T'); \\ \Omega(T) = \Omega(T') & \text{implies} & T = T'. \end{array}$$

We revisit a previously constructed example to give a WF algebraizable π-institution. Note that the π-institution in question is systemic. As we will see in Theorem 291, this is no coincidence!

Example 290 *Define the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = b = \mathrm{SEN}^\flat(f)(1)$*;*
- $N^\flat$ *is the trivial clone.*

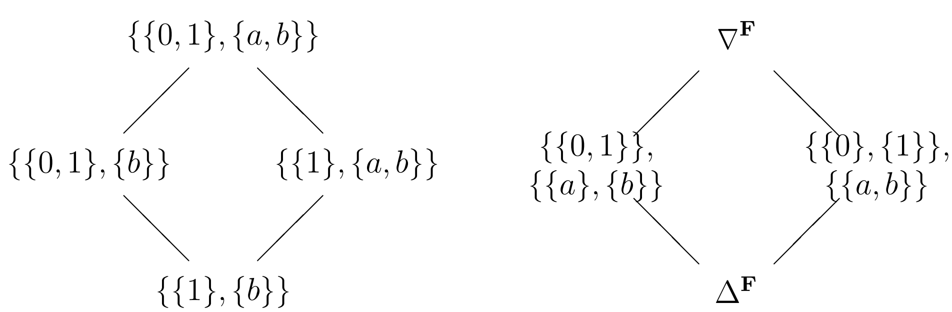


Specify the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

$$\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\},\{a,b\}\}.$$

Notice that every theory family is a theory system, whence $\mathcal{I}$ *is systemic.*

The following diagrams show the lattices of theory families and of the corresponding Leibniz congruence systems:



$\mathcal{I}$ *is protoalgebraic and family injective. Therefore, it is a WF algebraizable* π*-institution.*

We show that a π-institution that is WF algebraizable is necessarily systemic.

Theorem 291 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is WF algebraizable, then it is systemic.*

Proof: Suppose that $\mathcal{I}$ is WF algebraizable. Let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, by Proposition 42, $T, \overleftarrow{T} \in \mathrm{ThFam}(\mathcal{I})$, such that $\overleftarrow{T} \le T$. Thus, by protoalgebraicity, we get $\Omega(\overleftarrow{T}) \le \Omega(T)$. But, by Proposition 20, it is always the case that $\Omega(T) \le \Omega(\overleftarrow{T})$. Therefore, we have $\Omega(\overleftarrow{T}) = \Omega(T)$. Thus, by family injectivity, we conclude that $\overleftarrow{T} = T$. Therefore $T \in \mathrm{ThSys}(\mathcal{I})$. We conclude that $\mathrm{ThFam}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$ and, hence, $\mathcal{I}$ is systemic. ■

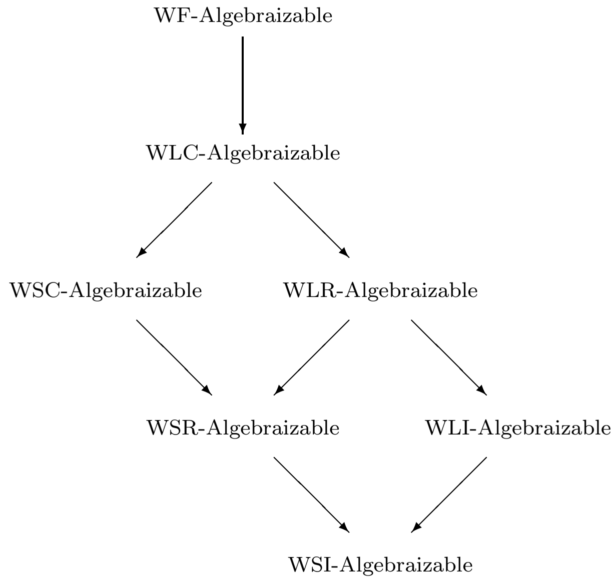
An interesting consequence of Theorem 291 is an exact characterization of those WS prealgebraizable π-institutions that are WF algebraizable.

Corollary 292 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then* $\mathcal{I}$ *is WF algebraizable if and only if it is WS prealgebraizable and systemic.*

Proof: Suppose $\mathcal{I}$ is WF algebraizable. Then, by Theorem 291, it is systemic. Moreover, by definition, its Leibniz operator is monotone and injective on theory families. Thus, it is also monotone and injective on theory systems. So $\mathcal{I}$ is WS prealgebraizable.

Suppose conversely, that $\mathcal{I}$ is WS prealgebraizable and systemic. Then, by definition, its Leibniz operator is monotone and injective on theory systems. But, by systemicity, the collection of theory systems coincides with the collection of theory families. Therefore, the Leibniz operator is monotone and injective on theory families. It follows, by definition, that $\mathcal{I}$ is WF algebraizable. ■

We pause to give an updated version of the hierarchical diagram regarding weak algebraizability classes:



We present examples to show that the class of weakly family algebraizable π-institutions is properly included in both the class of protoalgebraic and that of family completely reflective π-institutions.

Example 293 *Consider the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *defined as follows:*

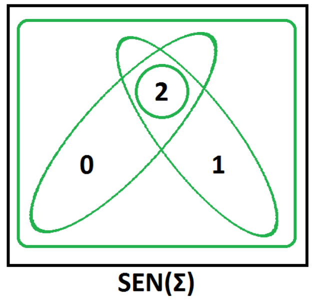
- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the clone of natural transformations on* $\mathrm{SEN}^\flat$ *generated by the the two unary natural transformations* $\sigma^\flat, \tau^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *given by the following table:*

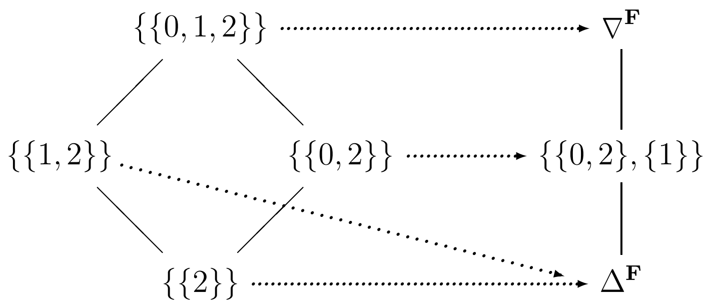
$x \in \mathrm{SEN}^\flat(\Sigma)$	$\sigma^\flat_\Sigma(x)$	$\tau^\flat_\Sigma(x)$
0	0	0
1	1	2
2	0	2

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$*, by setting*

$$\mathcal{C}_\Sigma = \{\{2\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

The lattice of theory families of $\mathcal{I}$ *and the associated Leibniz congruence systems are shown in the diagram.*

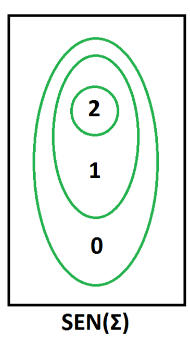




The Leibniz operator is monotone, but not injective on theory families. Therefore, we conclude that $\mathcal{I}$ is protoalgebraic but that it fails to be WF algebraizable.

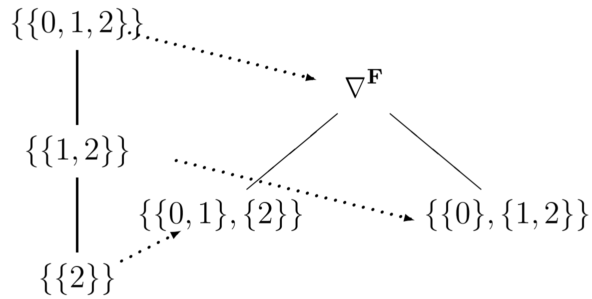
Example 294 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by setting* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}$.

The lattice of theory families and the associated Leibniz congruence systems (in block form) are shown in the diagram.



It is clear from these that the Leibniz operator is completely order reflecting on the theory families of $\mathcal{I}$, but it is not monotone. It follows that $\mathcal{I}$ is family completely reflective but not protoalgebraic. Therefore, $\mathcal{I}$ is family completely reflective, but fails to be WF algebraizable.

The properties defining weak family algebraizability transfer from theory families to filter families over arbitrary algebraic systems.

Theorem 295 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is WF algebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, the Leibniz operator on* $\mathcal{A}$ *is monotone and injective on* $\mathcal{I}$*-filter families, i.e., for all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$\begin{array}{rcl} T \le T' & \textit{implies} & \Omega^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T'); \\ \Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T') & \textit{implies} & T = T'. \end{array}$$

Proof: Suppose, first, that the displayed implications hold for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $\mathcal{I}$-filter families T, T' on $\mathcal{A}$. By taking $\mathcal{A} = \mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and keeping in mind Lemma 51, we conclude that the Leibniz operator is monotone and injective on all theory families of $\mathcal{I}$. Thus, by definition, $\mathcal{I}$ is WF algebraizable.

Suppose, conversely, that $\mathcal{I}$ is WF algebraizable. Then, by Theorem 288, it is protoalgebraic and family completely reflective. Thus, by Theorems 179 and 240, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, the Leibniz operator $\Omega^{\mathcal{A}}$ is monotone and completely order reflecting on the $\mathcal{I}$-filter families of $\mathcal{A}$. Thus, by Propositions 243 and 228, the Leibniz operator is monotone and injective on the $\mathcal{I}$-filter families of $\mathcal{A}$. ■

We showed in Theorem 256 that WS prealgebraizability is equivalent to the Leibniz operator $\Omega^{\mathcal{A}}$ over an arbitrary $\mathbf{F}$-algebraic system $\mathcal{A}$ establishing an order embedding from the lattice of $\mathcal{I}$-filter systems on $\mathcal{A}$ into the poset of all $\mathrm{AlgSys}^*(\mathcal{I})$-congruence systems on $\mathcal{A}$. We show, next, that WF algebraizability has a similar characterization. Namely, it can be characterized by the fact that the Leibniz operator $\Omega^{\mathcal{A}}$ over an arbitrary **F**-algebraic system $\mathcal{A}$ establishes an order isomorphism from the lattice of $\mathcal{I}$-filter families of $\mathcal{A}$ into the lattice of all $\mathrm{AlgSys}^*(\mathcal{I})$-congruence systems on $\mathcal{A}$.

Recall the function

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

that we have introduced before Theorem 256, that restricts to a well-defined function from $\mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ into $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$.

Theorem 296 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is WF algebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism.

Proof: Suppose, first, that, for every **F**-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, $\Omega^{\mathcal{A}}$ is an order isomorphism. Then, by Theorem 268, $\mathcal{I}$ is WFR prealgebraizable. To show that it is WF algebraizable, it suffices, by Corollary 292, to show that it is systemic. But, by Theorem 261, every WFR prealgebraizable π-institution is systemic.

Suppose, conversely, that $\mathcal{I}$ is WF algebraizable. Then it is, a fortiori, WFR prealgebraizable. Therefore, by Theorem 268, for every **F**-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism. ∎

We now have the following corollary to the effect that the classes of WF algebraizable π-institutions and of WFR prealgebraizable π-institutions coincide.

Corollary 297 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is WF algebraizable if and only if it is WFR prealgebraizable.*

Proof: By Theorems 268 and 296, membership in each of these two classes is characterized by $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ being an order isomorphism, for every **F**-algebraic system $\mathcal{A}$. ∎

In light of Corollary 297, we shall call both the class of WF algebraizable π-institutions and the class of WFR prealgebraizable π-institutions by the term **weakly family algebraizable** or **WF algebraizable**, for short.

We now work towards a sweeping contraction of the classes appearing in the weak algebraizability hierarchy. To accomplish this, we provide, first, a characterization of the class of WSI algebraizable π-institutions. Namely, we show that a π-institution is WSI algebraizable if and only if it is stable and $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism, for all **F**-algebraic systems $\mathcal{A}$.

Theorem 298 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is WSI algebraizable if and only if* $\mathcal{I}$ *is stable and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism.

Proof: Suppose, first, that $\mathcal{I}$ is WSI algebraizable. Then it is, by definition, protoalgebraic and, hence, by Lemma 170, it is stable. Also, it is, a fortiori, WS prealgebraizable. Thus, by Theorem 256, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order embedding. So it suffices to show that $\Omega^{\mathcal{A}}$ on $\mathcal{I}$-filter systems on $\mathcal{A}$ is surjective. To this end, consider $\theta \in \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. Then $\mathcal{A}^\theta \in \mathrm{AlgSys}^*(\mathcal{I})$. Thus, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta)$, such that $\Omega^{\mathcal{A}^\theta}(T) = \Delta^{\mathcal{A}^\theta}$. Applying the inverse of $\langle I, \pi^\theta \rangle : \mathcal{A} \to \mathcal{A}^\theta$, we get $\pi^{\theta^{-1}}(\Omega^{\mathcal{A}^\theta}(T)) = \pi^{\theta^{-1}}(\Delta^{\mathcal{A}^\theta})$. So, by Proposition 24, $\Omega^{\mathcal{A}}(\pi^{\theta^{-1}}(T)) = \theta$. By stability and Theorem 154, we get that $\Omega^{\mathcal{A}}(\overleftarrow{\pi^{\theta^{-1}}(T)}) = \theta$. Hence, by Lemma 6, $\Omega^{\mathcal{A}}(\pi^{\theta^{-1}}(\overleftarrow{T})) = \theta$. Now, by Proposition 53 and Lemma 51, $\pi^{\theta^{-1}}(\overleftarrow{T}) \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$. Therefore, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is surjective, as was to be shown.

Suppose, conversely, that $\mathcal{I}$ is stable and for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism. In particular, we have that $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is an order isomorphism. This yields immediately that the Leibniz operator is injective on theory systems and, hence $\mathcal{I}$ is system injective. The isomorphism also yields that the Leibniz operator is monotone on theory systems, i.e., that $\mathcal{I}$ is prealgebraic. So it suffices to show that it is monotone on all theory families. To this end, let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$. Then, by Lemma 1, $\overleftarrow{T} \leq \overleftarrow{T'}$. Thus, taking into account Proposition 42, by prealgebraicity, $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Now using the postulated stability of $\mathcal{I}$, we get $\Omega(T) \leq \Omega(T')$. Therefore, $\mathcal{I}$ is protoalgebraic. ■

Using this characterization of WSI algebraizable π-institutions, we now show that the class of WSI algebraizable π-institutions and that of WLC algebraizable π-institutions coincide. This causes a collapse of both squares of the diagram describing the weak algebraizability hierarchy (i.e., of all six bottom classes) into a single class.

Theorem 299 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is WSI algebraizable, then it is WLC algebraizable.*

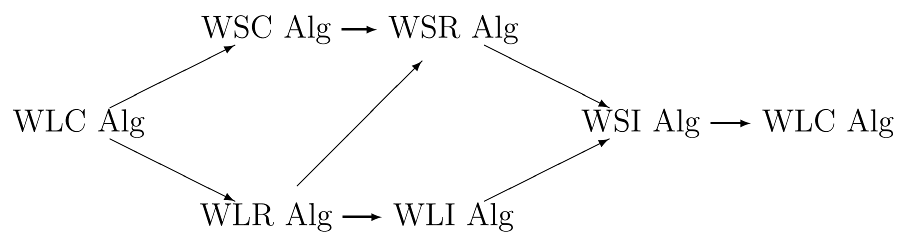
Proof: Suppose $\mathcal{I}$ is WSI algebraizable. Then it is, by definition, protoalgebraic. Moreover, by Theorem 298, it is stable and, for every **F**-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism. To see that it is WLC algebraizable, it suffices to show that it is left completely reflective. So consider $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. By protoalgebraicity, we get $\bigcap_{T\in\mathcal{T}} \Omega(T) = \Omega(\bigcap_{T\in\mathcal{T}} T)$. Thus, $\Omega(\bigcap_{T\in\mathcal{T}} T) \leq \Omega(T')$. By stability, $\Omega(\overleftarrow{\bigcap_{T\in\mathcal{T}} T}) \leq \Omega(\overleftarrow{T'})$. By Proposition 42 and the hypothesis, $\overleftarrow{\bigcap_{T\in\mathcal{T}} T} \leq \overleftarrow{T'}$. By Lemma 3, $\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$. We conclude that the Leibniz operator is left completely order reflecting on theory families and, therefore, $\mathcal{I}$ is WLC algebraizable. ■

Corollary 300 *The classes of WLC, WSC, WLR, WSR, WLI and WSI algebraizable π-institutions coincide.*

Proof: According to Theorem 299 and because of the hierarchy of the defining properties, we get the following diagram, where the arrows denote inclusions.



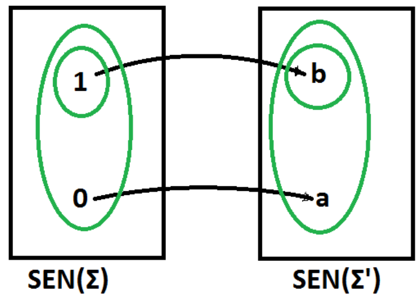
The conclusion readily follows. ■

Because of Corollary 300, we shall call a π-institution belonging to any of these six classes **weakly** (**system**) **algebraizable**, or **W algebraizable** (sometimes **WS algebraizable**) for short.

We revisit an example showing that the inclusion of the class of WF algebraizable π-institutions into the class of WS algebraizable π-institutions is proper.

Example 301 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$*;*
- $N^\flat$ *is the trivial clone.*



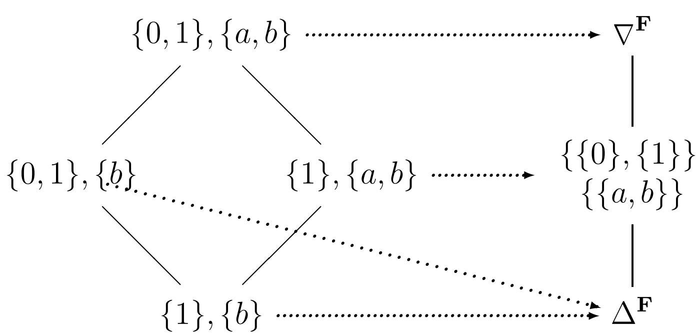
Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\{1\}, \{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\}, \{a,b\}\}.$$

The table yielding the action of $\overleftarrow{\ }$ *on theory families is shown below.*

$\leftarrow$	$\{b\}$	$\{a,b\}$
$\{1\}$	$\{1\},\{b\}$	$\{1\},\{a,b\}$
$\{0,1\}$	$\{1\},\{b\}$	$\{0,1\},\{a,b\}$

The accompanying diagram gives the structure of the lattice of theory families and the corresponding Leibniz congruence systems.



From the diagram one can check that the Leibniz operator is monotone on theory families and left injective on theory families (or injective on theory systems). Thus, the π-institution is protoalgebraic and system injective, i.e., WS algebraizable. On the other hand, $\mathcal{I}$ is, obviously, not family injective and, therefore, it is not WF algebraizable.

As with other classes in the hierarchy, we have a number of transfer theorems for weakly algebraizable π-institutions. We choose here to formalize the result by providing the two most powerful implications:

Theorem 302 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Then the following statements are equivalent:*

(a) *$\mathcal{I}$ is weakly system algebraizable;*

(b) *For every $\mathbf{F}$-algebraic system $\mathcal{A}$, the Leibniz operator on $\mathcal{A}$ is monotone and left completely order reflecting on $\mathcal{I}$-filter families, i.e., for all $\mathcal{T} \cup \{T, T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,*

$$T \le T' \quad \textit{implies} \quad \Omega^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T');$$
$$\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T') \quad \textit{implies} \quad \bigcap_{T \in \mathcal{T}} \overleftarrow{T} \le \overleftarrow{T'}.$$

(c) *For every $\mathbf{F}$-algebraic system $\mathcal{A}$, the Leibniz operator on $\mathcal{A}$ is monotone on $\mathcal{I}$-filter families and injective on $\mathcal{I}$-filter systems.*

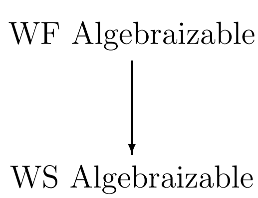
Proof:

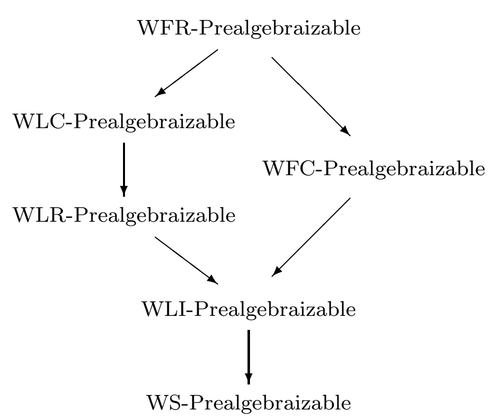
(a)⇒(b) Suppose that $\mathcal{I}$ is weakly system algebraizable. Then, by Theorem 300, it is protoalgebraic and family completely reflective. Thus, by Theorems 179 and 240, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, the Leibniz operator $\Omega^{\mathcal{A}}$ is monotone and left completely order reflecting on $\mathcal{I}$-filter families.

(b)⇒(c) Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system. By hypothesis, the Leibniz operator is monotone and left completely order reflecting on the $\mathcal{I}$-filter families of $\mathcal{A}$. By Propositions 243 and 228, the Leibniz operator is monotone on the $\mathcal{I}$-filter families and injective on the $\mathcal{I}$-filter systems of $\mathcal{A}$.

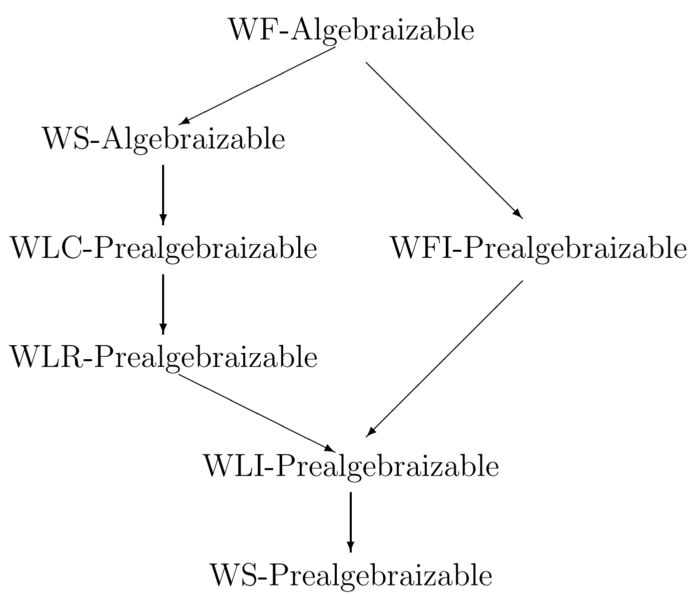
(c)⇒(a) Suppose that, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, the Leibniz operator is monotone on the $\mathcal{I}$-filter families and injective on the $\mathcal{I}$-filter systems of $\mathcal{A}$. By taking $\mathcal{A} = \mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and keeping in mind Lemma 51, we conclude that the Leibniz operator is monotone on all theory families and injective on all theory systems of $\mathcal{I}$. Thus, by definition, $\mathcal{I}$ is weakly system algebraizable. ■

We are left now with a weak algebraizability hierarchy consisting of only two classes as shown on the left below. On the right is reprinted the weak prealgebraizability hierarchy, as revealed in the previous section.





Recalling that, by Corollary 297, the classes of WF algebraizable π-institutions and WFR prealgebraizable π-institutions coincide and noting that the class of weakly algebraizable π-institutions (coinciding with WLC algebraizable π-institutions) is included in the class of WLC prealgebraizable π-institutions, we get the following complete picture of weak (pre)algebraizability.



We close with an example that shows that the class of weakly system algebraizable π-institutions is properly contained in the class of WLC prealgebraizable π-institutions.

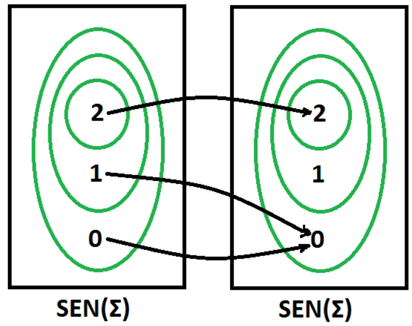
Example 303 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial clone.*

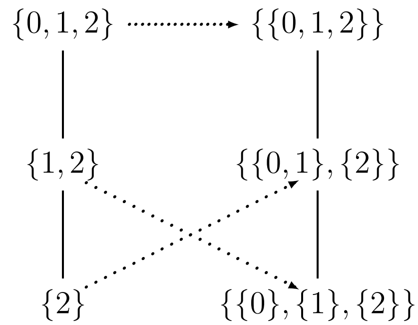
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}$.

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$*:*

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1, 2\}$	$\{2\}$
$\{0, 1, 2\}$	$\{0, 1, 2\}$



The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



$\mathcal{I}$ is prealgebraic, but not protoalgebraic. Moreover, it is left completely reflective. Thus, it is WLC prealgebraizable but not WS algebraizable.