Chapter 5

The Semantic Leibniz Hierarchy: Extensionality

5.1 Introduction

Protoalgebraic sentential logics were introduced by Czelakowski in [26, 29] and studied by Blok and Pigozzi [28]. Perhaps the best known among several existing characterizations of protoalgebraicity is the property of monotonicity of the Leibniz operator on the filters of a logic over arbitrary algebras (of the same algebraic type). Equivalential logics were introduced by Prucnal and Wroński [19] and studied by Czelakowski [22, 24]. A Leibniz characterization asserts that a sentential logic is equivalential iff, for every algebra, the Leibniz operator on its filters is monotone and commutes with inverse endomorphisms. More details may be found in Section 3.4 of [69], Sections 6.1-6.3 of [86] and Chapters 1-3 of [64]. In addition, whereas protoalgebraicity, in conjunction with injectivity of the Leibniz operator, is used to define weakly algebraizable logics [62], the stronger condition of equivalentiality, coupled with injectivity of the Leibniz operator, is used to define algebraizable logics [35, 54]. Section 3.4 of [69], Sections 6.4 and 6.5 of [86] and Chapter 4 of [64] provide detailed information about these classes of sentential logics.

In Section 3.3, we studied classes of π-institutions defined using monotonicity properties of the Leibniz operator. In Chapter 4, we used monotonicity to define the weak algebraizability hierarchy of π-institutions. The present chapter introduces analogs of the property of equivalentiality for π-institutions, strengthening monotonicity. Further, by replacing monotonicity by equivalentiality, one obtains from the weak algebraizability hierarchy the hierarchy of algebraizable π-institutions.

Strengthening protoalgebraicity to equivalentiality involves adding, on top of monotonicity properties, some property that emulates (or forms an analog of) the property of commutativity of the Leibniz operator with inverse endomorphisms. This desideratum informs the structure of the current chapter. In Sections 5.2 and 5.3, properties that can be used as analogs of commutativity with inverse endomorphisms in the framework of π-institutions are discussed and some of their interrelationships are explored. These are combined with monotonicity in Section 5.4 to define equivalentiality. Finally, in Sections 5.5 and 5.6, we obtain the (pre)algebraizability hierarchy of π-institutions, based on the weak (pre)algebraizability hierarchy, studied in Chapter 4. More details, by section, follow.

In Section 5.2, we study *extensionality*. Recall that, given an algebraic system $\mathbf{F}$ and a sentence family X of $\mathbf{F}$, one may determine the subsystem $\langle X\rangle$ of $\mathbf{F}$ generated by X. Moreover, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ and a subsystem $\mathbf{F}'$ of $\mathbf{F}$, $\mathbf{F}'$ determines a π-subinstitution $\mathcal{I}' = \langle \mathbf{F}', C'\rangle$ of $\mathcal{I}$ which is obtained by restricting the action of C on $\mathbf{F}'$. For details on these constructions, see Section 2.12. A π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is said to be *family extensional* if, roughly speaking, the action of the Leibniz operator on theory families of subinstitutions can be obtained as the restriction of the Leibniz operator of $\mathcal{I}$ on the universe corresponding to the subinstitution. More precisely, $\mathcal{I}$ is *family extensional* if, for every sentence family X of $\mathbf{F}$ and every theory family T of $\mathcal{I}$, $\Omega^{\langle X\rangle}(T \cap \langle X\rangle) = \Omega(T) \cap \langle X\rangle^2$. *System extensionality* is defined similarly, except that T is allowed to range over theory systems only, instead of over arbitrary theory families. By definition, family extensionality implies system extensionality. Further, system extensionality, combined with stability, implies family extensionality. The significance of extensionality stems, in part, from allowing important properties of a π-institution to be inherited by its subinstitutions. Indicative of this phenomenon are the facts that, under system extensionality, stability in inherited and, under family (system, respectively) extensionality, prealgebraicity (protoalgebraicity, respectively) is also inherited. Both versions of extensionality transfer. A seemingly weaker version of extensionality is *2-extensionality*. Roughly speaking, 2-extensionality is extensionality restricted to universes generated by two sentences over the same signature. More precisely, a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is *family 2-extensional* if, for every signature Σ, all Σ-sentences ϕ, ψ and every theory family T, $\langle \phi, \psi\rangle \in \Omega_\Sigma(T)$ if and only if $\langle \phi, \psi\rangle \in \Omega_\Sigma^{\langle \phi,\psi\rangle}(T \cap \langle \phi, \psi\rangle)$. If T is quantified over theory systems, *system 2-extensionality* is obtained instead. Despite its apparent weakness in comparison to extensionality, it turns out that a π-institution is family/system extensional if and only if it is family/system 2-extensional, respectively. Extensionality is one manifestation of the property that is used as an analog of commutativity with inverse endomorphisms, employed in the sentential framework to define equivalentiality. An alternative formalization, closer in spirit to commutativity, is introduced in Section 5.3.

In Section 5.3, we study *Leibniz commutativity* or, simply, *commutativity*, a property closer in spirit to the original property used in the sentential context to characterize equivalentiality. Let $\mathbf{F}$ be an algebraic system and X a sentence family of $\mathbf{F}$. Recall the subsystem $\langle X\rangle$ of $\mathbf{F}$ generated by X. A morphism of the form $\langle I, \alpha\rangle : \langle X\rangle \to \mathbf{F}$, where I is the identity functor on signatures, is called an *extension*. Recall also that, if $\mathbf{F}$ happens to be the base algebraic system of a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$, then X induces a π-subinstitution $\mathcal{I}^{\langle X\rangle} = \langle \langle X\rangle, C^{\langle X\rangle}\rangle$ of $\mathcal{I}$ based on $\langle X\rangle$, whose closure system is essentially C restricted on $\langle X\rangle$ and whose theory families are obtained by the theory families of $\mathcal{I}$ via restriction on $\langle X\rangle$. In this enriched context, an extension $\langle I, \alpha\rangle : \langle X\rangle \to \mathbf{F}$ is called *logical*, denoted by $\langle I, \alpha\rangle : \mathcal{I}^{\langle X\rangle} \to \mathcal{I}$, if it preserves the closure structure in the sense that, for all signatures Σ and all $\Phi \subseteq \langle X\rangle_\Sigma$, $\alpha_\Sigma(C_\Sigma^{\langle X\rangle}(\Phi)) \subseteq C_\Sigma(\alpha_\Sigma(\Phi))$. This condition is tantamount to preservation of theory families under α^{-1}, i.e., to $\alpha^{-1}(T)$ being a theory family of $\mathcal{I}^{\langle X\rangle}$, for every theory family T of $\mathcal{I}$. Logical extensions lay the groundwork for building the notion of (*Leibniz*) *commutativity*. We say that a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is *family commuting* if, for all sentence families X of $\mathbf{F}$, all logical extensions $\langle I, \alpha\rangle : \mathcal{I}^{\langle X\rangle} \to \mathcal{I}$ and all theory families T' of $\mathcal{I}^{\langle X\rangle}$, $\alpha(\Omega^{\langle X\rangle}(T')) \leq \Omega(C(\alpha(T')))$. *System commutativity* applies the same condition on theory systems only. A similar, but not identical in general, property is *inverse* (*Leibniz*) *commutativity*. $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is *family inverse commuting* if, for every sentence family X, all logical extensions $\langle I, \alpha \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$ and all theory families T of $\mathcal{I}$, $\alpha^{-1}(\Omega(T)) = \Omega^{\langle X \rangle}(\alpha^{-1}(T))$. *System inverse commutativity* results by quantifying T over theory systems instead. It is elementary to check, based on the definition of $\mathcal{I}^{\langle X \rangle}$, that injection morphisms $\langle I, j \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$ qualify as logical extensions. This permits establishing that family (system, repectively) inverse commutativity implies family (system, respectively) extensionality. Also, since theory systems form a subclass of theory families, it is obvious that family inverse commutativity is stronger than the system version. In addition, it is shown that the system version, coupled with stability, implies the family version. The last results of Section 5.3 are critical for our further investigations.

Since commutativity and inverse commutativity are used mainly in conjunction with monotonicity properties to obtain equivalentiality, it is important that, under system (family) monotonicity (i.e., pre- and protoalgebraicity, respectively), system (family, respectively) commutativity and system (family, respectively) inverse commutativity coincide. Further, in a result that allows us to switch between commutativity properties and the extensionality properties of Section 5.2, and which strengthens a previously mentioned implication, it is shown that system (family) inverse commutativity is equivalent to system (family, respectively) extensionality. Based on these equivalences and a transfer theorem from Section 5.2, it is also shown that both versions of inverse commutativity transfer. Summarizing, the corresponding (system or family) versions of extensionality and 2-extensionality and of inverse commutativity are equivalent without proviso. On the other hand, for these three to be equivalent to the corresponding commutativity version, a sufficient condition is that the corresponding version of monotonicity holds.

In Section 5.4, we define versions of *equivalentiality*, resulting by combining monotonicity and extensionality properties. Since both come in two flavors, we get, a priori, four potentially different equivalentiality classes. A π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is (*family*) *equivalential* if it is protoalgebraic and family extensional. Weakening protoalgebraicity to prealgebraicity we get *family preequivalentiality*. On the other hand, weakening family to system extensionality, we get *system equivalentiality*. Finally, if both properties are weakened in tandem, we get (*system*) *preequivalentiality*. Equivalentiality, as opposed to preequivalentiality, incorporates protoalgebraicity, which implies stability. But, under stability, the two versions of extensionality coincide. This reasoning shows that family and system equivalentiality are identical properties. So when referring to this property, we use the term *equivalentiality*, without qualification. It turns out to be equivalent to preequivalentiality plus stability. All three distinct versions transfer. We also obtain characterizations of both equivalentiality and preequivalentiality in terms of the Leibniz operator seeing as a mapping between lattices of filter families (systems) and congruence systems over arbitrary algebraic systems.

In Section 5.5, we explore the hierarchy of *prealgebraizable π-institutions*. Prealgebraizability results from weak prealgebraizability when prealgebraicity is strengthened to either family or system preequivalentiality, i.e., when either family or system extensionality is added into the mix. Accordingly, two parallel hierarchies mimicking that of weakly prealgebraizable π-institutions, detailed in Chapter 4, are formed depending on the version of preequivalentiality used. If family preequivalentiality is postulated, we get the five classes of *XF prealgebraizable π-institutions*, whereas, if (system) preequivalentiality is dictated, we get five corresponding *X prealgebraizability* classes, where X is a string reflecting which injectivity, reflectivity or complete reflectivity condition is coupled with preequivalentiality, i.e., X can be one of:

- LC for left complete reflectivity;
- LR for left reflectivity;
- FI for family injectivity;
- LI for left injectivity; and
- S for system (injectivity, reflectivity and complete reflectivity all being equivalent under preequivalentiality).

Systemicity leads to a total collapse of the ten classes into a single class. Stability results to FIF and FI prealgebraizable π-institutions being identified and to a collapse of all remaining eight classes into a single class. Thus, it yields a 2-class hierarchy. After showing that all ten prealgebraizability properties transfer, the section is dedicated to obtaining characterization theorems for each of the classes in terms of the Leibniz operator on arbitrary algebraic systems perceived as a mapping between ordered sets. The ten characterizations can be divided into five pairs, each pair addressing XF and X prealgebraizability for the same X in {LC, LR, FI, LI, S}. Making a somewhat arbitrary choice here, we look at the cases of LR and S to provide a flavor of these results. The interested reader is, of course, referred to the main text for further details on all ten properties. A π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is LRF prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is a left order reflecting surjection commuting with inverse logical extensions, which restricts to an order embedding on filter systems. A subtle, but important, change occurs in most pairs in passing from the XF to the X sibling. $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is LR prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is a left order reflecting surjection, which restricts to an order embedding commuting with inverse logical extensions on filter systems. Along similar lines, we get that $\mathcal{I}$ is SF prealgebraizable iff, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ commutes with inverse logical extensions and restricts to an order embedding on filter systems, whereas $\mathcal{I}$ is S prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order embedding commuting with inverse logical extensions.

In Section 5.6, we examine *algebraizability*. This hierarchy results from weak algebraizability when protoalgebraicity is replaced by equivalentiality. Equivalently, it ensues from prealgebraizability when, instead of imposing family or system preequivalentiality, we insist on the stronger condition of equivalentiality. Exactly due to this strengthening, only two classes may be distinguished here, *family algebraizability*, combining equivalentiality with family injectivity, and (*system*) *algebraizability*, coupling equivalentiality with system injectivity. The family version is equivalent to the system version augmented by systemicity. Both flavors transfer. Finally, a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is family algebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism commuting with inverse logical extensions, whereas it is system algebraizable if and only if it is stable and, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism commuting with inverse logical extensions.

5.2 Extensionality

The first two important ingredients in classifying π-institutions according to their algebraic character were:

- the monotonicity properties of the Leibniz operator, which gave rise to the classes of prealgebraic and protoalgebraic π-institutions, as well as the various classes defined using versions of complete monotonicity;
- the various properties involving injectivity and reflectivity, varying from the weakest, system injectivity, to the strongest, family complete reflectivity.

Two additional important properties are the extensionality of the Leibniz operator and the commutativity of the Leibniz operator, which we now introduce and study. The variants studied here will give rise to classes in the equivalential hierarchy of π-institutions and, based on these, in the semantic hierarchy of algebraizable π-institutions (as opposed to *weak* (pre)algebraizability, studied in Chapter 4).

We first define two versions of the extensionality property and two corresponding versions of 2-extensionality, which is an apparently relaxed version of extensionality, but will be shown to be equivalent to extensionality.

Recall from Section 2.12 that, given an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat} \rangle$ and a sentence family $X \in \mathrm{SenFam}(\mathbf{F})$, we denote by $\langle X \rangle = \{\langle X \rangle_{\Sigma}\}_{\Sigma \in |\mathbf{Sign}^{\flat}|}$ the universe of $\mathbf{F}$ generated by X, i.e., $\langle X \rangle = \nu(\overrightarrow{X})$.

Definition 304 (Extensionality) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is called* **family extensional** *if, for all* $X \in \mathrm{SenFam}(\mathcal{I})$ *and all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*
$$\Omega(T) \cap \langle X \rangle^2 = \Omega^{\langle X \rangle}(T \cap \langle X \rangle);$$
- $\mathcal{I}$ *is called* **system extensional** *if, for all* $X \in \mathrm{SenFam}(\mathcal{I})$ *and all* $T \in \mathrm{ThSys}(\mathcal{I})$*,*
$$\Omega(T) \cap \langle X \rangle^2 = \Omega^{\langle X \rangle}(T \cap \langle X \rangle).$$

Taking into account Proposition 89, one obtains the following equivalent formulations.

Lemma 305 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is family (system) extensional if and only if, for all* $X \in \mathrm{SenFam}(\mathcal{I})$ *and all* $T \in \mathrm{ThFam}(\mathcal{I})$ *(*$T \in \mathrm{ThSys}(\mathcal{I})$*, respectively),*
$$\Omega^{\langle X \rangle}(T \cap \langle X \rangle) \leq \Omega(T) \cap \langle X \rangle^2.$$

Proof: Since, by Proposition 89, for all $X \in \mathrm{SenFam}(\mathcal{I})$ and $T \in \mathrm{ThFam}(\mathcal{I})$, the inclusion
$$\Omega(T) \cap \langle X \rangle^2 \leq \Omega^{\langle X \rangle}(T \cap \langle X \rangle)$$
always holds, we get the statement using the definition. ■

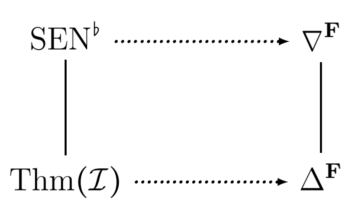
Here is a simple example of a family extensional π-institution.

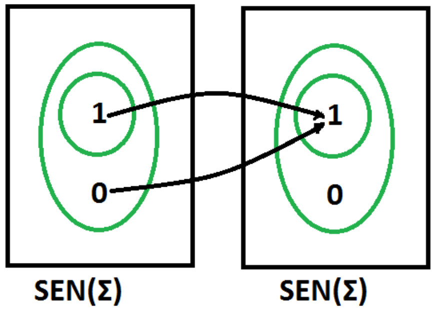
Example 306 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 1$, $\mathrm{SEN}^\flat(f)(1) = 1$*;*
- $N^\flat$ *is the trivial category of natural transformations, consisting of the projections only.*

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution, defined by setting* $\mathcal{C}_\Sigma = \{\{1\}, \{0, 1\}\}$*.*

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.





Note that the only universes are $\mathrm{Thm}(\mathcal{I}) = \{\{1\}\}$ *and* $\mathrm{SEN}^\flat$*. For the second one,* $\Omega(T) \cap \langle X \rangle^2 = \Omega^{\langle X \rangle}(T \cap \langle X \rangle)$ *holds trivially for all* $T \in \mathrm{ThFam}(\mathcal{I})$*, since both sides boil down to* $\Omega(T)$*. For the first, we have*

$$\Omega^{\mathrm{Thm}(\mathcal{I})}(\mathrm{Thm}(\mathcal{I}) \cap \mathrm{Thm}(\mathcal{I})) = \mathrm{Thm}(\mathcal{I})^2 = \Omega(\mathrm{Thm}(\mathcal{I})) \cap \mathrm{Thm}(\mathcal{I})^2;$$
$$\Omega^{\mathrm{Thm}(\mathcal{I})}(\mathrm{SEN}^\flat \cap \mathrm{Thm}(\mathcal{I})) = \mathrm{Thm}(\mathcal{I})^2 = \Omega(\mathrm{SEN}^\flat) \cap \mathrm{Thm}(\mathcal{I})^2.$$

So $\mathcal{I}$ *is family extensional, that is, for all* $X \in \mathrm{SenFam}(\mathcal{I})$ *and all* $T \in \mathrm{ThFam}(\mathcal{I})$*,* $\Omega(T) \cap \langle X \rangle^2 = \Omega^{\langle X \rangle}(T \cap \langle X \rangle)$*.*

We present, now, two examples of π-institutions that are not system extensional.

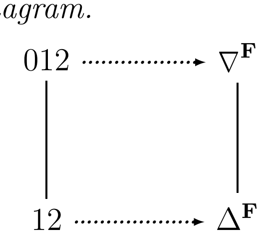
Example 307 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

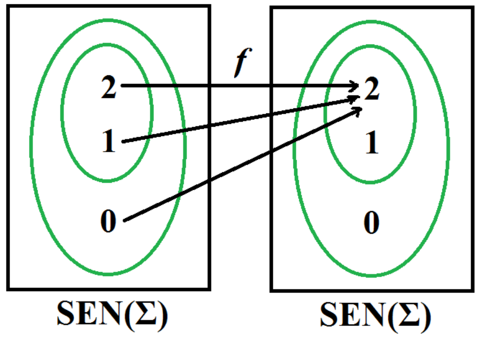
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f)(x) = 2$*, for all* $x \in \{0, 1, 2\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by the following table:*

$\sigma^\flat_\Sigma$	0	1	2
0	0	0	2
1	0	1	1
2	2	1	2

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution defined by setting* $\mathcal{C}_\Sigma = \{\{1, 2\}, \{0, 1, 2\}\}$*.*

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.





For the universe $\mathbf{X} = \{\{1,2\}\}$ *and the theory system* $T = \{\{1,2\}\}$, *we get*

$$\Omega(T) \cap \mathbf{X}^2 = \{\{1\},\{2\}\} \subsetneq \{\{1,2\}\} = \Omega^{\mathbf{X}}(T \cap \mathbf{X}).$$

Therefore, $\mathcal{I}$ *is not system extensional.*

And here is a second example of a non-system extensional π-institution.

Example 308 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with the single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, a, b, 1\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the two binary natural transformations* $\wedge : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *and* $\vee : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by the following tables:*

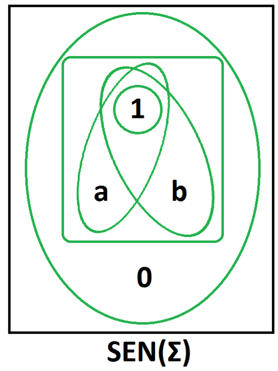
$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

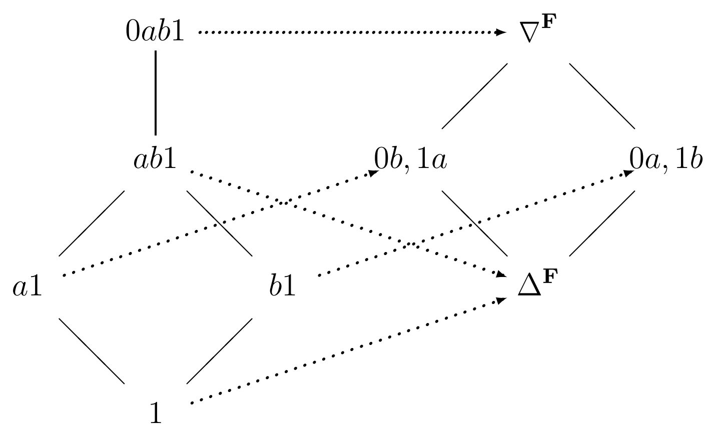
Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution defined by setting*

$$\mathcal{C}_\Sigma = \{\{1\}, \{a,1\}, \{b,1\}, \{a,b,1\}, \{0,a,b,1\}\}.$$

The lattice of theory families and the corresponding Leibniz congruence



systems are shown in the diagram.



For the universe $\mathbf{X} = \{\{0, a, 1\}\}$ *and the theory system* $T = \{\{a, b, 1\}\}$, *we get*

$$\Omega(T) \cap \mathbf{X}^2 = \{\{0\}, \{a\}, \{1\}\} \subsetneq \{\{0\}, \{a, 1\}\} = \Omega^{\mathbf{X}}(T \cap \mathbf{X}).$$

Therefore, $\mathcal{I}$ *is not system extensional and, a fortiori, not family extensional either.*

The following clarifies the relation between family and system extensionality.

Proposition 309 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is family extensional, then it is system extensional.*

(b) If $\mathcal{I}$ *is system extensional and stable, then it is family extensional.*

Proof:

(a) Since all theory systems are also theory families, it follows that every family extensional π-institution is also system extensional.

(b) Suppose that $\mathcal{I}$ is system extensional and stable. Let $X \in \mathrm{SenFam}(\mathcal{I})$ and $T \in \mathrm{ThFam}(\mathcal{I})$. Then we have

$$\begin{array}{rcll} \Omega^{\langle X\rangle}(T \cap \langle X\rangle) & \leq & \Omega^{\langle X\rangle}(\overleftarrow{T \cap \langle X\rangle}) & \text{(by Proposition 20)} \\ & = & \Omega^{\langle X\rangle}(\overleftarrow{T} \cap \langle X\rangle) & \text{(by Lemma 3)} \\ & = & \Omega(\overleftarrow{T}) \cap \langle X\rangle^2 & \text{(by system extensionality)} \\ & = & \Omega(T) \cap \langle X\rangle^2. & \text{(by stability)} \end{array}$$

By Lemma 305, $\mathcal{I}$ is family extensional.

■

According to Proposition 309 we have the following **extensionality hierarchy**:

Family Extensional

↓

System Extensional

The reverse, however, does not hold in general, as the following example, exhibiting a π-institution which is system but not family extensional, shows.

Example 310 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

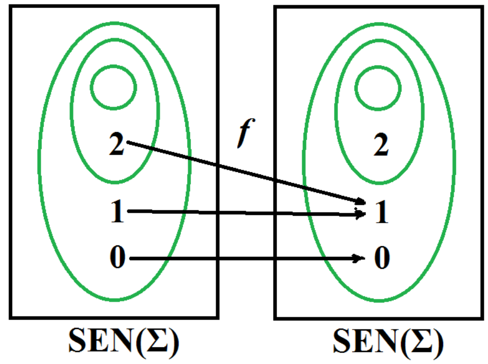
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 1$ *and* $\mathrm{SEN}^\flat(f)(2) = 1$*;*
- $N^\flat$ *is the category of natural transformations generated by the binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by the following table:*

$\sigma^\flat_\Sigma$	0	1	2
0	1	1	2
1	1	1	1
2	2	1	2

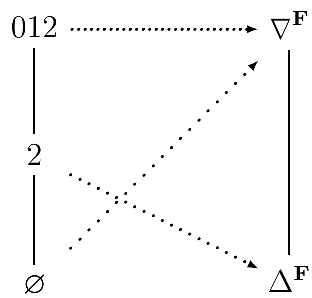
Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution defined by setting*

$$\mathcal{C}_\Sigma = \{\varnothing, \{2\}, \{0,1,2\}\}.$$

$\mathcal{I}$ *has three theory families,* $\overline{\varnothing}$*,* $T = \{\{2\}\}$ *and* $\mathrm{SEN}^\flat$*, but only* $\overline{\varnothing}$ *and* $\mathrm{SEN}^\flat$ *are theory systems. The lattice of theory families and the corresponding*



Leibniz congruence systems are shown in the diagram.



Moreover, $\mathbf{F}$ *has five universes* $\{\{0\}\}$, $\{\{1\}\}$, $\{\{0,1\}\}$, $\{\{1,2\}\}$ *and* $\{\{0,1,2\}\}$. *Since the only theory systems of* $\mathcal{I}$ *are* $\overline{\varnothing}$ *and* $\mathrm{SEN}^\flat$, *it is trivial to check that* $\mathcal{I}$ *is system extensional.*

For the universe $\mathbf{X} = \{\{0,1\}\}$ *and the theory family* $T = \{\{2\}\}$, *we get*

$$\Omega(T) \cap \mathbf{X}^2 = \{\{0\},\{1\}\} \subsetneq \{\{0,1\}\} = \Omega^{\mathbf{X}}(T \cap \mathbf{X}).$$

Therefore, $\mathcal{I}$ *is not family extensional.*

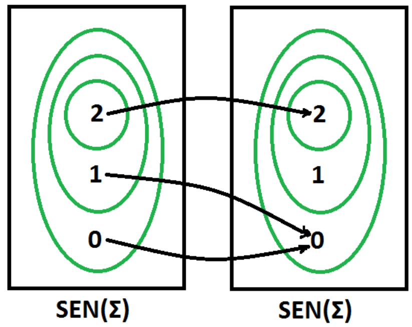
Moreover, as the following example shows, the converse of Part (b) of Proposition 309 does not hold in general, i.e., stability is not necessary for family extensionality.

Example 311 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$;

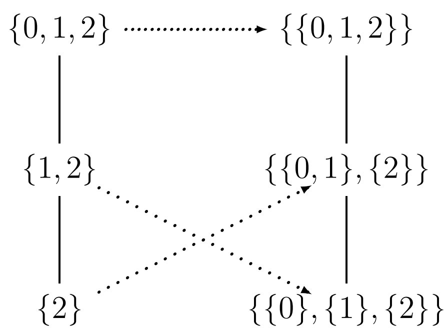
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}$.

$\mathcal{I}$ *has three theory families,* $\mathrm{Thm}(\mathcal{I})$, $T = \{\{1,2\}\}$ *and* $\mathrm{SEN}^\flat$, *but only* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$ *are theory systems.*

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



It is not difficult to check that $\mathcal{I}$ *is family extensional, that is, for all* $X \in \mathrm{SenFam}(\mathcal{I})$ *and all* $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\Omega(T) \cap \langle X \rangle^2 = \Omega^{\langle X \rangle}(T \cap \langle X \rangle).$$

In fact, $\mathbf{F}$ *has five universes* $\{\{0\}\}$, $\{\{2\}\}$, $\{\{0,1\}\}$, $\{\{0,2\}\}$ *and* $\{\{0,1,2\}\}$, *only two of which are proper and non-singletons.* $\mathcal{I}$ *has three theory families, two of which are different from* $\mathrm{SEN}^\flat$. *Thus, there are only four cases to check, shown below, adopting, for brevity, an obvious shorthand notation.*

$$\begin{aligned}
&\Omega(2) \cap (01)^2 = \{\{0,1\}\} = \Omega^{01}(2 \cap 01),\\
&\Omega(12) \cap (01)^2 = \{\{0\},\{1\}\} = \Omega^{01}(12 \cap 01),\\
&\Omega(2) \cap (02)^2 = \{\{0\},\{2\}\} = \Omega^{02}(2 \cap 02),\\
&\Omega(12) \cap (02)^2 = \{\{0\},\{2\}\} = \Omega^{02}(12 \cap 02).
\end{aligned}$$

Clearly, since for $T \in \mathrm{ThFam}(\mathcal{I}) \backslash \mathrm{ThSys}(\mathcal{I})$,

$$\Omega(T) = \Delta^{\mathbf{F}} \neq \{\{0,1\},\{2\}\} = \Omega(\mathrm{Thm}(\mathcal{I})) = \Omega(\overleftarrow{T}),$$

$\mathcal{I}$ *is not stable.*

A related result is that, under system extensionality, stability is inherited by π-subinstitutions of a given π-institution.

Proposition 312 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a system extensional* π*-institution based on* $\mathbf{F}$ *and* $\mathbf{F}' = \langle \mathbf{Sign}^\flat, \mathrm{SEN}'^\flat, N'^\flat \rangle \leq \mathbf{F}$ *an algebraic subsystem of* $\mathbf{F}$*. If* $\mathcal{I}$ *is stable, then* $\mathcal{I}' = \langle \mathbf{F}', C' \rangle$ *is also stable.*

Proof: Suppose that $\mathcal{I}$ is system extensional and stable. Let $T \in \mathrm{ThFam}(\mathcal{I})$. Then we have

$$\begin{array}{rcll} \Omega^{\mathbf{F}'}(\overleftarrow{T \cap \mathrm{SEN}'^\flat}) & = & \Omega^{\mathbf{F}'}(\overleftarrow{T} \cap \mathrm{SEN}'^\flat) & \text{(by Lemma 3)} \\ & = & \Omega^{\mathbf{F}}(\overleftarrow{T}) \cap (\mathrm{SEN}'^\flat)^2 & \text{(by system extensionality)} \\ & = & \Omega^{\mathbf{F}}(T) \cap (\mathrm{SEN}'^\flat)^2 & \text{(by stability)} \\ & \leq & \Omega^{\mathbf{F}'}(T \cap \mathrm{SEN}'^\flat). & \text{(by Proposition 89)} \end{array}$$

Since, by Proposition 20, the reverse inclusion always holds, we conclude that $\mathcal{I}'$ is also stable. ∎

A similar preservation result, under extensionality, may also be proven with regards to pre- and protoalgebraicity. More precisely, we show that if a π-institution is family extensional and protoalgebraic, then all its π-subinstitutions are also protoalgebraic. Analogously, if a π-institution is system extensional and prealgebraic, then prealgebraicity is inherited by all its π-subinstitutions.

Proposition 313 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is family extensional and protoalgebraic, then, for all* $\mathbf{F}' = \langle \mathbf{Sign}^\flat, \mathrm{SEN}'^\flat, N'^\flat \rangle \leq \mathbf{F}$*,* $\mathcal{I}' = \langle \mathbf{F}', C' \rangle$ *is also protoalgebraic;*

(b) If $\mathcal{I}$ *is system extensional and prealgebraic, then, for all* $\mathbf{F}' = \langle \mathbf{Sign}^\flat, \mathrm{SEN}'^\flat, N'^\flat \rangle \leq \mathbf{F}$*,* $\mathcal{I}' = \langle \mathbf{F}', C' \rangle$ *is also prealgebraic.*

Proof: We only prove Part (a). Part (b) may be proven similarly. Suppose that $\mathcal{I}$ is family extensional and protoalgebraic and let $\mathbf{F}' \leq \mathbf{F}$. If $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$, then, by protoalgebraicity, $\Omega^{\mathbf{F}}(T) \leq \Omega^{\mathbf{F}}(T')$. Thus, $\Omega^{\mathbf{F}}(T) \cap (\mathrm{SEN}'^\flat)^2 \leq \Omega^{\mathbf{F}}(T') \cap (\mathrm{SEN}'^\flat)^2$. Therefore, by family extensionality, $\Omega^{\mathbf{F}'}(T \cap \mathrm{SEN}'^\flat) \leq \Omega^{\mathbf{F}'}(T' \cap \mathrm{SEN}'^\flat)$. By Proposition 87, we conclude that $\mathcal{I}'$ is also protoalgebraic. ∎

There are transfer theorems that hold for both system and family extensionality.

Theorem 314 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is family (system) extensional if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, all* $Y \in \mathrm{SenFam}(\mathcal{A})$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *(*$T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$*, respectively)*

$$\Omega^{\mathcal{A}}(T) \cap \langle Y \rangle^2 = \Omega^{\langle Y \rangle}(T \cap \langle Y \rangle).$$

Proof: We present the proof for theory families. The case of theory systems is similar.

The "if" direction follows by taking $\mathcal{A} = \mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and observing that, in that case, the displayed condition reduces to the definition of family extensionality.

For the "only if", assume that $\mathcal{I}$ is family extensional and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system, $Y \in \mathrm{SenFam}(\mathcal{A})$, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \Omega^{\langle Y \rangle}_{F(\Sigma)}(T \cap \langle Y \rangle).$$

Then we have

$$\begin{array}{lll}
\langle \phi, \psi \rangle & \in & \alpha_\Sigma^{-1}(\Omega^{\langle Y \rangle}_{F(\Sigma)}(T \cap \langle Y \rangle)) \quad \text{(set theory)} \\
& = & \Omega_\Sigma^{\alpha^{-1}(\langle Y \rangle)}(\alpha^{-1}(T) \cap \alpha^{-1}(\langle Y \rangle)) \quad \text{(Corollary 92)} \\
& = & \Omega_\Sigma(\alpha^{-1}(T)) \cap \alpha_\Sigma^{-1}(\langle Y \rangle_{F(\Sigma)})^2 \quad \text{(hypothesis)} \\
& = & \alpha_\Sigma^{-1}(\Omega^{\mathcal{A}}_{F(\Sigma)}(T)) \cap \alpha_\Sigma^{-1}(\langle Y \rangle_{F(\Sigma)})^2 \quad \text{(Proposition 24)} \\
& = & \alpha_\Sigma^{-1}(\Omega^{\mathcal{A}}_{F(\Sigma)}(T) \cap \langle Y \rangle^2_{F(\Sigma)}). \quad \text{(set theory)}
\end{array}$$

Therefore $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \Omega^{\mathcal{A}}_{F(\Sigma)}(T) \cap \langle Y \rangle^2_{F(\Sigma)}$. Since, by Proposition 89, the opposite inclusion always holds, we get, taking into account the surjectivity of $\langle F, \alpha \rangle$, that

$$\Omega^{\mathcal{A}}(T) \cap \langle Y \rangle^2 = \Omega^{\langle Y \rangle}(T \cap \langle Y \rangle).$$

The conclusion now follows. ■

We define, next, the second property, a seemingly relaxed version of extensionality that we call 2-extensionality.

Definition 315 (2-Extensionality) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is called* **family 2-extensional** *if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(T) \quad \textit{iff} \quad \langle \phi, \psi \rangle \in \Omega_\Sigma^{\langle \phi, \psi \rangle}(T \cap \langle \phi, \psi \rangle);$$

- $\mathcal{I}$ *is called* **system 2-extensional** *if, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(T) \quad \textit{iff} \quad \langle \phi, \psi \rangle \in \Omega_\Sigma^{\langle \phi, \psi \rangle}(T \cap \langle \phi, \psi \rangle).$$

Taking into account Proposition 89, one obtains the following equivalent formulations.

Lemma 316 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is family (system) 2-extensional if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$ *(*$T \in \mathrm{ThSys}(\mathcal{I})$*, respectively), all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\langle \phi, \psi \rangle \in \Omega_\Sigma^{\langle \phi, \psi \rangle}(T \cap \langle \phi, \psi \rangle) \quad \textit{implies} \quad \langle \phi, \psi \rangle \in \Omega_\Sigma(T).$$

Proof: By Proposition 89, for all $T \in \mathrm{ThFam}(\mathcal{I})$, the inclusion

$$\Omega(T) \cap \langle \phi, \psi \rangle^2 \le \Omega^{\langle \phi, \psi \rangle}(T \cap \langle \phi, \psi \rangle)$$

always holds. Since $\phi, \psi \in \langle \phi, \psi \rangle_\Sigma$, if $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$, then $\langle \phi, \psi \rangle \in \Omega_\Sigma^{\langle \phi, \psi \rangle}(T \cap \langle \phi, \psi \rangle)$.

Thus, 2-extensionality is, by definition, equivalent to

$$\langle \phi, \psi \rangle \in \Omega_\Sigma^{\langle \phi, \psi \rangle}(T \cap \langle \phi, \psi \rangle) \quad \text{implies} \quad \langle \phi, \psi \rangle \in \Omega_\Sigma(T).$$

■

It turns out that the corresponding versions of extensionality and 2-extensionality are equivalent. That extensionality implies 2-extensionality is fairly straightforward.

Proposition 317 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is family (system) extensional, then it is family (system, respectively) 2-extensional.*

Proof: We present the proof for theory families. The case of theory systems is similar. Suppose $\mathcal{I}$ is family extensional and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in \Omega_\Sigma^{\langle \phi, \psi \rangle}(T \cap \langle \phi, \psi \rangle)$. Then, by family extensionality, we get that $\langle \phi, \psi \rangle \in \Omega_\Sigma(T) \cap \langle \phi, \psi \rangle_\Sigma^2$, which implies that $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. Thus, by Lemma 316, $\mathcal{I}$ is family 2-extensional. ■

The full equivalence is given in the following

Theorem 318 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is family (system) extensional if and only if it is family (system, respectively) 2-extensional.*

Proof: Again we prove only the equivalence of the family versions of the two properties, since the system versions can be proven similarly.

The "only if" was the content of Proposition 317. For the "if", suppose that $\mathcal{I}$ is family 2-extensional and let $X \in \mathrm{SenFam}(\mathcal{I})$, $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \notin \Omega_\Sigma(T) \cap \langle X \rangle_\Sigma^2$.

If $\langle \phi, \psi \rangle \notin \langle X \rangle_\Sigma^2$, then, a fortiori, $\langle \phi, \psi \rangle \notin \Omega_\Sigma^{\langle X \rangle}(T \cap \langle X \rangle)$, and we are done.

If, on the other hand, $\langle\phi,\psi\rangle \in \langle X\rangle^2_\Sigma$, then, we have $\langle\phi,\psi\rangle \notin \Omega_\Sigma(T)$. Thus, by hypothesis, $\langle\phi,\psi\rangle \notin \Omega^{\langle\phi,\psi\rangle}_\Sigma(T \cap \langle\phi,\psi\rangle)$. So, by Theorem 19, there exist $\sigma^\flat \in N^\flat$, $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and $\vec{\chi} \in \langle\phi,\psi\rangle_{\Sigma'}$, such that (without loss of generality)

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}) \in T_{\Sigma'} \cap \langle\phi,\psi\rangle_{\Sigma'} \quad \text{but} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}) \notin T_{\Sigma'} \cap \langle\phi,\psi\rangle_{\Sigma'}.$$

Since $\phi,\psi \in \langle X\rangle_\Sigma$ and $\vec{\chi} \in \langle\phi,\psi\rangle_{\Sigma'}$, we get that

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}) \in T_{\Sigma'} \cap \langle X\rangle_{\Sigma'} \quad \text{but} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}) \notin T_{\Sigma'} \cap \langle X\rangle_{\Sigma'}.$$

Thus, again by Theorem 19, we get $\langle\phi,\psi\rangle \notin \Omega^{\langle X\rangle}_\Sigma(T \cap \langle X\rangle)$. Hence, $\Omega^{\langle X\rangle}(T \cap \langle X\rangle) \leq \Omega(T) \cap \langle X\rangle^2$. We now conclude, using Lemma 305, that $\mathcal{I}$ is family extensional. ■

5.3 Leibniz Commutativity

Another important property is that of commutativity with a special type of logical morphism, which we now introduce and study.

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and $X \in \mathrm{SenFam}(\mathbf{F})$. An algebraic system morphism of the form $\langle I,\alpha\rangle : \langle X\rangle \to \mathbf{F}$, where $I : \mathbf{Sign}^\flat \to \mathbf{Sign}^\flat$ is the identity functor, will be called an **extension**.

Further, given a π-institution $\mathcal{I} = \langle\mathbf{F},C\rangle$ based on $\mathbf{F}$, an extension $\langle I,\alpha\rangle : \langle X\rangle \to \mathbf{F}$ is said to be **logical** if it is a logical morphism $\langle I,\alpha\rangle : \mathcal{I}^{\langle X\rangle} \to \mathcal{I}$, where $\mathcal{I}^{\langle X\rangle} = \langle\langle X\rangle, C^{\langle X\rangle}\rangle$ is the π-subinstitution of $\mathcal{I}$ induced by $\langle X\rangle$. In other words $\langle I,\alpha\rangle : \langle X\rangle \to \mathbf{F}$ is a logical extension if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \langle X\rangle_\Sigma$,

$$\alpha_\Sigma(C^{\langle X\rangle}_\Sigma(\Phi)) \subseteq C_\Sigma(\alpha_\Sigma(\Phi)).$$

This is abbreviated to $\alpha(C^{\langle X\rangle}(\Phi)) \leq C(\alpha(\Phi))$.

Using Lemma 47, we get the following characterization:

Corollary 319 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$, $X \in \mathrm{SenFam}(\mathbf{F})$ and $\langle I,\alpha\rangle : \langle X\rangle \to \mathbf{F}$ an extension. $\langle I,\alpha\rangle : \mathcal{I}^{\langle X\rangle} \to \mathcal{I}$ is a logical extension if and only if*

$$\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I}^{\langle X\rangle}),\ \textit{for all } T \in \mathrm{ThFam}(\mathcal{I}).$$

Proof: Immediate by Lemma 47. ■

We now define the two notions of Leibniz commutativity that we wish to study.

Definition 320 (Leibniz Commutativity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is called* **family (Leibniz) commuting** *if the Leibniz operator on theory families commutes with logical extensions, i.e., if, for every* $X \in \mathrm{SenFam}(\mathcal{I})$*, all logical extensions* $\langle I, \alpha \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$ *and all* $T' \in \mathrm{ThFam}(\mathcal{I}^{\langle X \rangle})$,
$$\alpha(\Omega^{\langle X \rangle}(T')) \leq \Omega(C(\alpha(T')));$$
- $\mathcal{I}$ *is called* **system (Leibniz) commuting** *if the Leibniz operator on theory systems commutes with logical extensions, i.e., if, for every* $X \in \mathrm{SenFam}(\mathcal{I})$*, all logical extensions* $\langle I, \alpha \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$ *and all* $T' \in \mathrm{ThSys}(\mathcal{I}^{\langle X \rangle})$,
$$\alpha(\Omega^{\langle X \rangle}(T')) \leq \Omega(C(\alpha(T'))).$$

We now give a useful characterization of those two properties, in the case of protoalgebraic and of prealgebraic π-institutions, respectively. To do this, however, we need some preliminary work. First, we note that injection morphisms are logical extensions.

Lemma 321 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *For all* $X \in \mathrm{SenFam}(\mathcal{I})$, $\langle I, j \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$ *is a logical extension, where* $\langle I, j \rangle : \langle X \rangle \to \mathbf{F}$ *is the injection morphism.*

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, we have
$$j^{-1}(T) = T \cap \langle X \rangle \in \mathrm{ThFam}(\mathcal{I}^{\langle X \rangle}),$$
where the membership follows by Proposition 87. Therefore, by Corollary 319, $\langle I, j \rangle$ is a logical extension. ■

Next we define two alternative versions of Leibniz commutativity, which we term inverse Leibniz commutativity.

Definition 322 (Inverse (Leibniz) Commutativity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is* **family inverse (Leibniz) commuting** *if, for all* $X \in \mathrm{SenFam}(\mathcal{I})$*, all logical extensions* $\langle I, \alpha \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$ *and all* $T \in \mathrm{ThFam}(\mathcal{I})$,
$$\alpha^{-1}(\Omega(T)) = \Omega^{\langle X \rangle}(\alpha^{-1}(T));.$$
- $\mathcal{I}$ *is* **system inverse (Leibniz) commuting** *if, for all* $X \in \mathrm{SenFam}(\mathcal{I})$*, all logical extensions* $\langle I, \alpha \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$ *and all* $T \in \mathrm{ThSys}(\mathcal{I})$,
$$\alpha^{-1}(\Omega(T)) = \Omega^{\langle X \rangle}(\alpha^{-1}(T)).$$

We now show that inverse commutativity implies extensionality. Naturally enough, we have two versions of this implication.

Proposition 323 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is family (system) inverse commuting, then it is family (system, respectively) extensional.*

Proof: We show the family version. The system version is similar.

Assume that $\mathcal{I}$ is family inverse commuting and let $X \in \mathrm{SenFam}(\mathcal{I})$, $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$\langle \phi, \psi \rangle \in \Omega_\Sigma^{\langle X \rangle}(T \cap \langle X \rangle).$$

Considering the injection morphism $\langle I, j \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$, which is a logical extension by Lemma 321, the hypothesis can be rewritten as $\langle \phi, \psi \rangle \in \Omega_\Sigma^{\langle X \rangle}(j^{-1}(T))$. Thus, by inverse family commutativity, $\langle \phi, \psi \rangle \in j_\Sigma^{-1}(\Omega_\Sigma(T))$. But this is equivalent to $\langle \phi, \psi \rangle \in \Omega_\Sigma(T) \cap \langle X \rangle_\Sigma^2$. We conclude, using Lemma 305, that $\mathcal{I}$ is family extensional. ∎

It is clear that family inverse commutativity implies system inverse commutativity. We show, next, that under stability, the system and the family versions of inverse commutativity coincide.

Proposition 324 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is family inverse commuting, then it is system inverse commuting;*

(b) If $\mathcal{I}$ *is system inverse commuting and stable, then it is also family inverse commuting.*

Proof: Family inverse commutativity always implies system inverse commutativity. Conversely, assume that $\mathcal{I}$ is stable and system inverse commuting and let $X \in \mathrm{SenFam}(\mathcal{I})$, $\langle I, \alpha \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$ a logical extension and $T \in \mathrm{ThFam}(\mathcal{I})$. Then we have

$$\begin{array}{rcll}
\alpha^{-1}(\Omega(T)) & = & \alpha^{-1}(\Omega(\overleftarrow{T})) & \text{(stability)} \\
& = & \Omega^{\langle X \rangle}(\alpha^{-1}(\overleftarrow{T})) & \text{(system inverse commutativity)} \\
& = & \Omega^{\langle X \rangle}(\overleftarrow{\alpha^{-1}(T)}) & \text{(Lemma 6)} \\
& = & \Omega^{\langle X \rangle}(\alpha^{-1}(T)). & \text{(Propositions 323 and 312)}
\end{array}$$

Thus, $\mathcal{I}$ is family inverse commuting. ∎

Finally, the promised characterization that relates family (system) commutativity with family (system) inverse commutativity under the hypothesis of proto(pre)algebraicity. We present the two results separately for the sake of clarity.

Theorem 325 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a protoalgebraic* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is family commuting if and only if it is family inverse commuting.*

Proof: Note, first, that, for all $X \in \mathrm{SenFam}(\mathcal{I})$, all logical extensions $\langle F, \alpha \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$ and all $T \in \mathrm{ThFam}(\mathcal{I})$, $\alpha^{-1}(\Omega(T))$ is a congruence system on $\langle X \rangle$ that is compatible with $\alpha^{-1}(T)$. Thus, by the maximality property of the Leibniz congruence system, we have, regardless of commutativity, that, for all $X \in \mathrm{SenFam}(\mathcal{I})$, all $\langle I, \alpha \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$ and all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\alpha^{-1}(\Omega(T)) \leq \Omega^{\langle X \rangle}(\alpha^{-1}(T)).$$

Therefore, it suffices to show that $\mathcal{I}$ is family commuting if and only if, for all $X \in \mathrm{SenFam}(\mathcal{I})$, all $\langle I, \alpha \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$ and all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\Omega^{\langle X \rangle}(\alpha^{-1}(T)) \leq \alpha^{-1}(\Omega(T)).$$

For the "only if" direction, assume that $\mathcal{I}$ is family commuting and let $X \in \mathrm{SenFam}(\mathcal{I})$, $\langle I, \alpha \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$, $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in \Omega^{\langle X \rangle}_\Sigma(\alpha^{-1}(T))$. Then we have

$$\begin{array}{rcll}
\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle & \in & \alpha_\Sigma(\Omega^{\langle X \rangle}_\Sigma(\alpha^{-1}(T))) & \\
& \subseteq & \Omega_\Sigma(C(\alpha(\alpha^{-1}(T)))) & \text{(commutativity)} \\
& \subseteq & \Omega_\Sigma(C(T)) & \text{(protoalgebraicity)} \\
& = & \Omega_\Sigma(T). &
\end{array}$$

We conclude that $\langle \phi, \psi \rangle \in \alpha_\Sigma^{-1}(\Omega_\Sigma(T))$. Therefore, $\mathcal{I}$ is family inverse commuting.

For the "if" direction, assume $\mathcal{I}$ is family inverse commuting and let $X \in \mathrm{SenFam}(\mathcal{I})$, $\langle I, \alpha \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$ and $T' \in \mathrm{ThFam}(\mathcal{I}^{\langle X \rangle})$. Then we have

$$\begin{array}{rcl}
\alpha(\Omega^{\langle X \rangle}(T')) & \leq & \alpha(\Omega^{\langle X \rangle}(\alpha^{-1}(C(\alpha(T'))))) \\
& & \qquad \text{(Propositions 323 and 313)} \\
& = & \alpha(\alpha^{-1}(\Omega(C(\alpha(T'))))) \\
& & \qquad \text{(inverse commutativity)} \\
& \leq & \Omega(C(\alpha(T'))). \quad \text{(set theory)}
\end{array}$$

Thus, $\mathcal{I}$ is family commuting. ■

Similarly, we may obtain the following analog for the system versions of the corresponding properties.

Theorem 326 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a prealgebraic* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is system commuting if and only if it is system inverse commuting.*

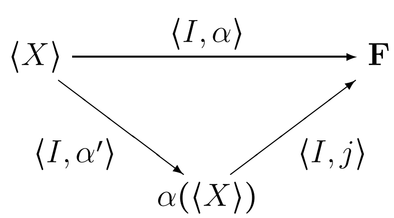
Proof: Along the lines of the proof of Theorem 325. ■

In Proposition 323 we saw that inverse commutativity implies extensionality. We now show that extensionality is in fact equivalent to inverse commutativity.

Theorem 327 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is family (system) inverse commuting if and only if it is family (system, respectively) extensional.*

Proof: By Proposition 323, family inverse commutativity implies family extensionality.

Suppose, conversely, that $\mathcal{I}$ is family extensional and let $X \in \mathrm{SenFam}(\mathcal{I})$, $\langle I, \alpha \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$ be a logical extension and $T \in \mathrm{ThFam}(\mathcal{I})$. We exploit the epi-mono factorization of $\langle I, \alpha \rangle$ provided in Proposition 81:



We have

$$\begin{aligned}
\Omega^{\langle X \rangle}(\alpha^{-1}(T)) &= \Omega^{\langle X \rangle}(\alpha'^{-1}(j^{-1}(T))) \quad (\langle I, \alpha \rangle = \langle I, j \rangle \circ \langle I, \alpha' \rangle) \\
&= \Omega^{\langle X \rangle}(\alpha'^{-1}(T \cap \mathrm{SEN}^{\flat \alpha})) \quad (\text{definition of } \langle I, j \rangle) \\
&= \alpha'^{-1}(\Omega^{\alpha(\langle X \rangle)}(T \cap \mathrm{SEN}^{\flat \alpha})) \quad (\text{Proposition 24}) \\
&= \alpha'^{-1}(\Omega(T) \cap (\mathrm{SEN}^{\flat \alpha})^2) \quad (\text{extensionality}) \\
&= \alpha'^{-1}(j^{-1}(\Omega(T))) \quad (\text{definition of } \langle I, j \rangle) \\
&= \alpha^{-1}(\Omega(T)). \quad (\langle I, \alpha \rangle = \langle I, j \rangle \circ \langle I, \alpha' \rangle)
\end{aligned}$$

Therefore, $\mathcal{I}$ is family inverse commuting.

The system version can be proven analogously. ■

Finally, we have the following transfer theorem for inverse commutativity.

Theorem 328 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is family (system) inverse commuting if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, the* π*-institution* $\langle \mathbf{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ *is family (system, respectively) inverse commuting.*

Proof: This follows by combining Theorem 327 with Theorem 314. ■

5.4 Equivalential π-Institutions

By combining prealgebraicity or protoalgebraicity, on the one hand, with system or family extensionality, on the other, we obtain another hierarchy, the hierarchy of *equivalential π-institutions*. The terminology is built by abiding to the following guidelines:

- The qualification "system" or "family" refers to the version of extensionality employed;

- "preequivalential" or "equivalential" is used depending on whether prealgebraicity or protoalgebraicity is assumed.

According to this nomenclature, we may define four classes of π-institutions as follows:

Definition 329 ((Pre)Equivalentiality) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

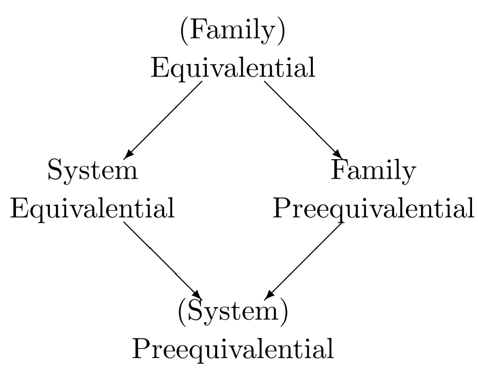
- $\mathcal{I}$ *is* **(family) equivalential** *if it is protoalgebraic and family extensional;*

- $\mathcal{I}$ *is* **system equivalential** *if it is protoalgebraic and system extensional;*

- $\mathcal{I}$ *is* **family preequivalential** *if it is prealgebraic and family extensional;*

- $\mathcal{I}$ *is* **(system) preequivalential** *if it is prealgebraic and system extensional.*

A priori, these four classes form the hierarchy depicted in the diagram.



However, it is easy to show that family and system equivalentiality are equivalent properties.

Proposition 330 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is family equivalential if and only if it is system equivalential.*

Proof: First, if $\mathcal{I}$ is family equivalential, then it is also system equivalential, since family extensionality implies system extensionality.

Suppose, conversely, that $\mathcal{I}$ is system equivalential. Then, by Theorem 175, it is stable and, by definition, it is system extensional, whence, by Proposition 309, it is also family extensional. Since it is protoalgebraic and family extensional, it is family equivalential. ■

Taking into account Proposition 330, we call $\mathcal{I}$ **equivalential** if it is protoalgebraic and (family or system) extensional.

Using this terminology, the hierarchy depicted in the preceding diagram reduces to the following linear **equivalentiality hierarchy**:

Equivalential

↓

Family Preequivalential

↓

(System) Preequivalential

It is easy to see that the separating property of the top level from the bottom level in the hierarchy is exactly stability.

Proposition 331 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is equivalential if and only if it is system preequivalential and stable.*

Proof: If $\mathcal{I}$ is equivalential, then it is trivially system preequivalential. Moreover, it is protoalgebraic and, therefore, by Theorem 175, it is stable.

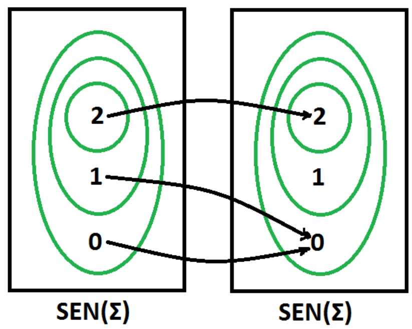
Suppose, conversely, that $\mathcal{I}$ is system prequivalential and stable. Then, by definition, it is system extensional, prealgebraic and stable. Thus, again by Theorem 175, it is system extensional and protoalgebraic and, hence, by Proposition 330, equivalential. ■

Examples are in order to show that the inclusions between the three classes of the equivalentiality hierarchy are proper.

We revisit, first, a familiar example of a π-institution that turns out to be family preequivalential but not equivalential.

Example 332 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

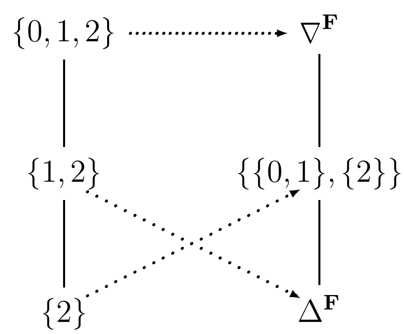
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$;
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}$.

The theory family $\{\{1,2\}\}$ *is not a theory system.*

The structure of the lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



It is clear from the diagram that $\mathcal{I}$ *is prealgebraic but not protoalgebraic. So to see that it is family preequivalential but not equivalential, it suffices to show that* $\mathcal{I}$ *is family extensional. We can easily see that* $\mathbf{F}$ *has two nontrivial proper universes and three theory families:*

Universes	$\mathbf{F}_{01} = \{\{0,1\}\}$	$\mathbf{F}_{02} = \{\{0,2\}\}$	
Theory Families	$\mathrm{Thm}(\mathcal{I})$	$T = \{\{1,2\}\}$	$\mathrm{SEN}^\flat$

For verification we perform the following calculations, since the case of $\mathrm{SEN}^\flat$ *is trivial:*

$$\begin{array}{rcccl}
\Omega^{\mathbf{F}_{01}}(\mathrm{Thm}(\mathcal{I}) \cap \mathbf{F}_{01}) & = & \mathbf{F}^2_{01} & = & \Omega(\mathrm{Thm}(\mathcal{I})) \cap \mathbf{F}^2_{01};\\
\Omega^{\mathbf{F}_{02}}(\mathrm{Thm}(\mathcal{I}) \cap \mathbf{F}_{02}) & = & \Delta^{\mathbf{F}_{02}} & = & \Omega(\mathrm{Thm}(\mathcal{I})) \cap \mathbf{F}^2_{02};\\
\Omega^{\mathbf{F}_{01}}(T \cap \mathbf{F}_{01}) & = & \Delta^{\mathbf{F}_{01}} & = & \Omega(T) \cap \mathbf{F}^2_{01};\\
\Omega^{\mathbf{F}_{02}}(T \cap \mathbf{F}_{02}) & = & \Delta^{\mathbf{F}_{02}} & = & \Omega(T) \cap \mathbf{F}^2_{02}.
\end{array}$$

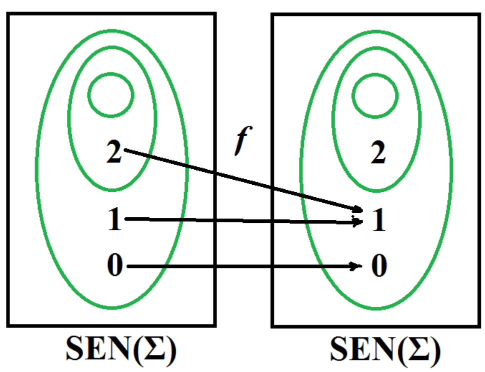
We conclude that $\mathcal{I}$ is indeed family extensional. Thus, $\mathcal{I}$ is an example of a family preequivalential π-institution, which is not equivalential.

Next we give an example of a π-institution that is system preequivalential but not family preequivalential.

Example 333 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- *$\mathbf{Sign}^\flat$ is the category with a single object Σ and a single non-identity morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- *$\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ is the functor specified by $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ and $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 1$ and $\mathrm{SEN}^\flat(f)(2) = 1$;*
- *$N^\flat$ is the category of natural transformations generated by the binary natural transformation $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ defined by the following table:*

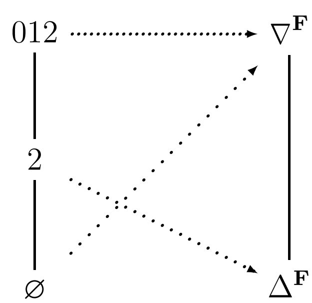
$\sigma^\flat_\Sigma$	0	1	2
0	1	1	2
1	1	1	1
2	2	1	2



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be the π-institution defined by setting

$$\mathcal{C}_\Sigma = \{\varnothing, \{2\}, \{0, 1, 2\}\}.$$

$\mathcal{I}$ has three theory families, $\overline{\varnothing}$, $T = \{\{2\}\}$ and $\mathrm{SEN}^\flat$, but only $\overline{\varnothing}$ and $\mathrm{SEN}^\flat$ are theory systems. The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



From the diagram and the fact that $T \notin \mathrm{ThSys}(\mathcal{I})$ *it follows that* $\mathcal{I}$ *is prealgebraic.*

$\mathbf{F}$ *has five universes* $\{\{0\}\}$, $\{\{1\}\}$, $\{\{0,1\}\}$, $\{\{1,2\}\}$ *and* $\{\{0,1,2\}\}$. *Since the only theory systems of* $\mathcal{I}$ *are* $\overline{\varnothing}$ *and* $\mathrm{SEN}^\flat$, *it is trivial to check that* $\mathcal{I}$ *is system extensional. Hence, being prealgebraic and system extensional,* $\mathcal{I}$ *is preequivalential.*

For the universe $\mathbf{X} = \{\{0,1\}\}$ *and the theory family* $T = \{\{2\}\}$, *we get*

$$\Omega(T) \cap \mathbf{X}^2 = \{\{0\},\{1\}\} \subsetneq \{\{0,1\}\} = \Omega^{\mathbf{X}}(T \cap \mathbf{X}).$$

This shows that $\mathcal{I}$ *is not family extensional and, hence,* $\mathcal{I}$ *is not family preequivalential.*

Theorems 179 and 314 allow us to formulate transfer results for (pre)equivalential π-institutions.

Theorem 334 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is equivalential if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, $\Omega^{\mathcal{A}}$ *is monotone on* $\mathcal{I}$*-filter families and, for all* $Y \in \mathrm{SenFam}(\mathcal{A})$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$\Omega^{\mathcal{A}}(T) \cap \langle Y\rangle^2 = \Omega^{\langle Y\rangle}(T \cap \langle Y\rangle).$$

Proof: This follows from Theorems 179 and 314. ∎

Similarly, we have the following versions for the preequivalentiality properties:

Theorem 335 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is family (system) preequivalential if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, $\Omega^{\mathcal{A}}$ *is monotone on* $\mathcal{I}$*-filter systems and, for all* $Y \in \mathrm{SenFam}(\mathcal{A})$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *(*$T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, *respectively),*

$$\Omega^{\mathcal{A}}(T) \cap \langle Y\rangle^2 = \Omega^{\langle Y\rangle}(T \cap \langle Y\rangle).$$

Proof: This follows from Theorems 179 and 314. ∎

The definitions of equivalentiality and of system preequivalentiality may be recast in terms of properties of mappings between the lattice of theory families/systems and congruence systems. We have the following

Theorem 336 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is equivalential if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*:*

- *The mapping* $\Omega : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ *is monotone;*

- *The following diagram commutes, for every* $Y \in \mathrm{SenFam}(\mathcal{A})$.

$$\begin{array}{ccc}
\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) & \xrightarrow{\ \Omega\ } & \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}) \\
{\scriptstyle - \cap \langle Y\rangle}\big\downarrow & & \big\downarrow{\scriptstyle - \cap \langle Y\rangle^2} \\
\mathrm{FiFam}^{\mathcal{I}^{\alpha^{-1}(\langle Y\rangle)}}(\langle Y\rangle) & \xrightarrow[\Omega^{\langle Y\rangle}]{} & \mathrm{ConSys}^{\mathcal{I}^{\alpha^{-1}(\langle Y\rangle)}}(\langle Y\rangle)
\end{array}$$

Proof: The "only if" follows from Theorem 334. The "if" follows by considering the **F**-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$. ∎

The version for system preequivalentiality has the following form.

Theorem 337 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* **F**. $\mathcal{I}$ *is preequivalential if and only if, for every* **F***-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*:*

- *The mapping* $\Omega : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ *is monotone;*
- *The following diagram commutes, for every* $Y \in \mathrm{SenFam}(\mathcal{A})$.

$$\begin{array}{ccc}
\mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) & \xrightarrow{\ \Omega\ } & \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}) \\
{\scriptstyle - \cap \langle Y\rangle}\big\downarrow & & \big\downarrow{\scriptstyle - \cap \langle Y\rangle^2} \\
\mathrm{FiSys}^{\mathcal{I}^{\alpha^{-1}(\langle Y\rangle)}}(\langle Y\rangle) & \xrightarrow[\Omega^{\langle Y\rangle}]{} & \mathrm{ConSys}^{\mathcal{I}^{\alpha^{-1}(\langle Y\rangle)}}(\langle Y\rangle)
\end{array}$$

Proof: Along the lines of Theorem 336, using Theorem 335. ∎

We now state formally some straightforward relationships between the classes in the equivalential hierarchy and those in the monotonicity hierarchy.

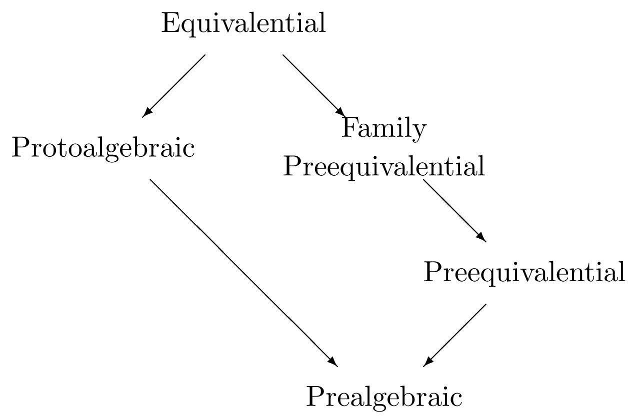
Proposition 338 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* **F**.

(a) If $\mathcal{I}$ *is equivalential, then it is protoalgebraic;*

(b) If $\mathcal{I}$ *is system preequivalential, then it is prealgebraic.*

Proof: Both statements follow directly from the definitions of equivalentiality and system preequivalentiality. ∎

Proposition 338 establishes the following hierarchy:



The next example shows that the two inclusions from the classes in the equivalential hierarchy to the monotonicity hierarchy are proper inclusions. More precisely, a π-institution is constructed that is protoalgebraic but fails to be (system) preequivalential.

Example 339 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, a, b, 1\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the two binary natural transformations* $\wedge, \vee : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by the following tables:*

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

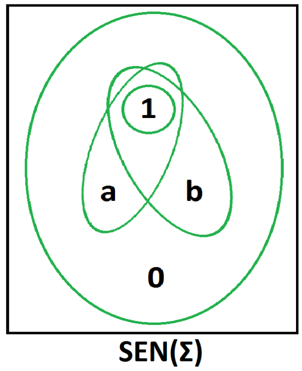
$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

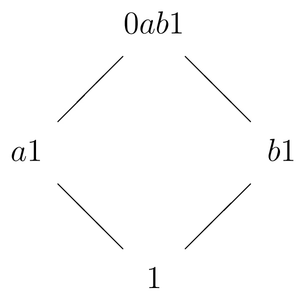
Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution defined by setting*

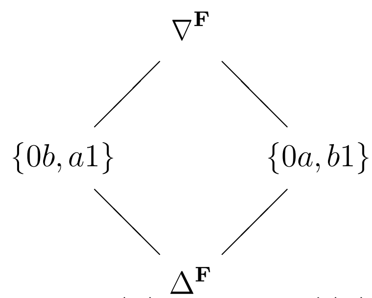
$$\mathcal{C}_\Sigma = \{\{1\}, \{a, 1\}, \{b, 1\}, \{0, a, b, 1\}\}.$$

$\mathcal{I}$ *has four theory families, all of which are also theory systems.*

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.







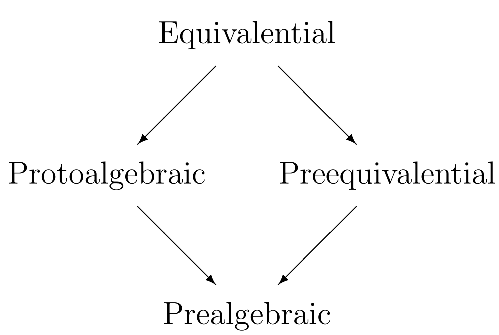
From the diagram, we can see that $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is an order isomorphism, whence,* $\mathcal{I}$ *is, in particular, protoalgebraic.*

On the other hand, for the universe $\mathbf{X} = \{\{0, a, 1\}\}$ *and the theory system* $T = \{\{1\}\}$*, we get*

$$\Omega(T) \cap \mathbf{X}^2 = \{\{0\}, \{a\}, \{1\}\} \subsetneq \{\{0, a\}, \{1\}\} = \Omega^{\mathbf{X}}(T \cap \mathbf{X}).$$

Thus, $\mathcal{I}$ *is not system extensional and, therefore, it fails to be (system) preequivalential.*

In our future work we will deal mostly with equivalential and system preequivalential π-institutions, referring to them as *equivalential* and *preequivalential*, respectively (as has already been suggested). So we focus mostly on the following part of the hierarchy:



Whenever the need to refer to family preequivalential π-institutions arises, the "family" qualification shall not be omitted to avoid confusion.

5.5 PreAlgebraizability

We study now the hierarchy that results by taking the various classes of weakly prealgebraizable π-institutions and adding to them family or system extensionality. Equivalently, we may replace prealgebraicity by either family or system preequivalentiality. Since, for every weak prealgebraizability class, we have two strengthening (or replacement) options, we get a sort of a double (or parallel) hierarchy whose classes are defined formally as follows and which is depicted in the accompanying diagram.

Definition 340 (Family PreAlgebraizability) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is* **left completely reflective family prealgebraizable***, or* **LCF prealgebraizable** *for short, if it is family preequivalential and left completely reflective, i.e., if it is system monotone, family extensional and left completely reflective;*

- $\mathcal{I}$ *is* **left reflective family prealgebraizable***, or* **LRF prealgebraizable** *for short, if it is family preequivalential and left reflective, i.e., if it is system monotone, family extensional and left reflective;*

- $\mathcal{I}$ *is* **family injective family prealgebraizable***, or* **FIF prealgebraizable** *for short, if it is family preequivalential and family injective, i.e., if it is system monotone, family extensional and family injective;*

- $\mathcal{I}$ *is* **left injective family prealgebraizable***, or* **LIF prealgebraizable** *for short, if it is family preequivalential and left injective, i.e., if it is system monotone, family extensional and left injective;*

- $\mathcal{I}$ *is* **system family prealgebraizable***, or* **SF prealgebraizable** *for short, if it is family preequivalential and system injective, i.e., if it is system monotone, family extensional and system injective.*

Definition 341 (PreAlgebraizability) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

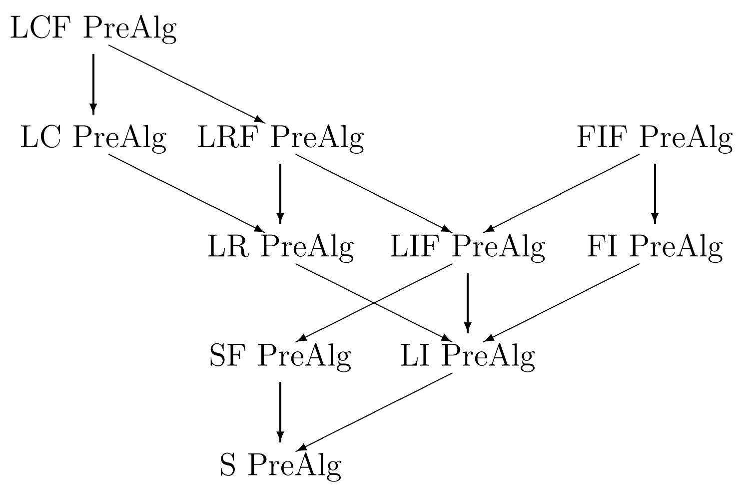
- $\mathcal{I}$ *is* **left completely reflective prealgebraizable***, or* **LC prealgebraizable** *for short, if it is preequivalential and left completely reflective, i.e., if it is system monotone, system extensional and left completely reflective;*

- $\mathcal{I}$ *is* **left reflective prealgebraizable***, or* **LR prealgebraizable** *for short, if it is preequivalential and left reflective, i.e., if it is system monotone, system extensional and left reflective;*

- *$\mathcal{I}$ is* **family injective prealgebraizable**, *or* **FI prealgebraizable** *for short, if it is preequivalential and family injective, i.e., if it is system monotone, system extensional and family injective;*

- *$\mathcal{I}$ is* **left injective prealgebraizable**, *or* **LI prealgebraizable** *for short, if it is preequivalential and left injective, i.e., if it is system monotone, system extensional and left injective;*

- *$\mathcal{I}$ is* **system prealgebraizable**, *or* **S prealgebraizable** *for short, if it is preequivalential and system injective, i.e., if it is system monotone, system extensional and system injective.*



The nomenclature here uses the term "prealgebraizable" to suggest that we are applying prealgebraicity. The first two qualifying capitals reflect the kind of injectivity, reflectivity or c-reflectivity that is applied and, finally, the addition or omission of "F" conveys whether family or system extensionality is applied, i.e., (together with prealgebraicity) whether family or system preequivalentiality is postulated. For instance, a π-institution is *LRF prealgebraizable* if it is

- prealgebraic;
- left reflective;
- family extensional,

i.e., if it is family preequivalential and left reflective or, equivalently, if it is weakly LR prealgebraizable and family extensional.

Directly from corresponding theorems pertaining to weakly prealgebraizable π-institutions, we obtain the following results that clarify the status of this hierarchy under systemicity, on the one hand, and under the weaker condition of stability, on the other.

Theorem 342 *For systemic π-institutions, all ten classes shown in the hierarchical diagram coincide.*

Proof: First, if $\mathcal{I}$ is systemic, then it is, a fortiori, stable. Therefore, by Proposition 309, the properties of family and system extensionality coincide. Thus, the two parallel hierarchies of the diagram collapse into one. Finally, by Theorem 269, all these five classes coincide. Therefore, restricted to systemic π-institutions, the entire hierarchy of the diagram collapses into a single class. ∎

Theorem 343 *For stable π-institutions, the ten-class prealgebraizability hierarchy shown in the diagram collapses to only two different classes, as shown in the diagram below*

$$\begin{array}{c} \textit{F Prealgebraizable} \\ \downarrow \\ \textit{S Prealgebraizable} \end{array}$$

where F Prealgebraizable encompasses the classes of FIF and FI Prealgebraizable π-institutions and S Prealgebraizable encompasses the remaining eight classes in the original hierarchy.

Proof: Indeed, by Proposition 309, the properties of family and system extensionality coincide under stability. Therefore, the five pairs of parallel classes of the original hierarchy coincide, giving a 5-class hierarchy. But, according to Theorem 273, under stability, these five classes reduce to only two, as shown in the diagram of the statement. ∎

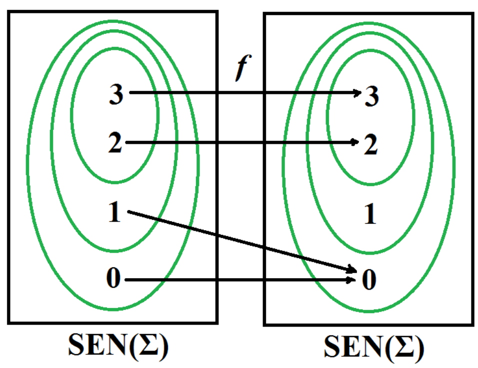
A few examples are now in order to separate the various classes of this prealgebraizability hierarchy. The first example serves in separating each pair of the two parallel hierarchies shown in the diagram. Namely, a π-institution is constructed which is LC prealgebraizable and FI prealgebraizable and, hence, belongs to all five levels of the lower hierarchy, but fails to be SF prealgebraizable and, as a consequence, belongs to none of the five upper levels of the hierarchy.

Example 344 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2, 3\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$, $\mathrm{SEN}^\flat(f)(2) = 2$ *and* $\mathrm{SEN}^\flat(f)(3) = 3$*;*

- $N^\flat$ *is the category of natural transformations generated by the binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by the following table:*

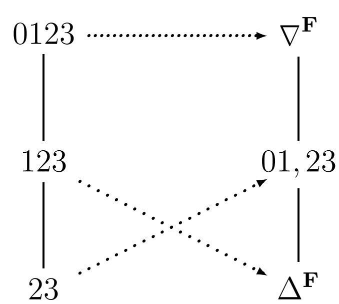
$\sigma^\flat_\Sigma$	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	2	2
3	0	1	2	3



Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution defined by setting*

$$C_\Sigma = \{\{2,3\},\{1,2,3\},\{0,1,2,3\}\}.$$

$\mathcal{I}$ *has three theory families, but only two theory systems. The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.*



From the diagram, we can see that $\mathcal{I}$ *is prealgebraic, i.e., that* Ω *is monotone on* $\mathrm{ThSys}(\mathcal{I})$*, and, also, left c-reflective and family injective.*

To see that $\mathcal{I}$ *is system extensional, note that* $\mathbf{F}$ *has eleven universes,* $\{\{0\}\}$, $\{\{2\}\}$, $\{\{3\}\}$, $\{\{0,1\}\}$, $\{\{0,2\}\}$, $\{\{0,3\}\}$, $\{\{2,3\}\}$, $\{\{0,1,2\}\}$, $\{\{0,1,3\}\}$, $\{\{0,2,3\}\}$ *and* $\{\{0,1,2,3\}\}$*, seven of which are proper and non-singletons. Moreover,* $\mathcal{I}$ *has two theory systems, only one of which is proper. Thus, we* *have seven cases to check, shown below adopting obvious shorthand notation:*

$$\begin{aligned}
&\Omega(23) \cap \{01\}^2 = \{01\} = \Omega^{01}(\varnothing) = \Omega^{01}(23 \cap 01);\\
&\Omega(23) \cap \{02\}^2 = \{0,2\} = \Omega^{02}(2) = \Omega^{01}(23 \cap 02);\\
&\Omega(23) \cap \{03\}^2 = \{0,3\} = \Omega^{03}(3) = \Omega^{01}(23 \cap 03);\\
&\Omega(23) \cap \{23\}^2 = \{23\} = \Omega^{23}(23) = \Omega^{01}(23 \cap 23);\\
&\Omega(23) \cap \{012\}^2 = \{01,2\} = \Omega^{012}(2) = \Omega^{01}(23 \cap 012);\\
&\Omega(23) \cap \{013\}^2 = \{01,3\} = \Omega^{013}(3) = \Omega^{01}(23 \cap 013);\\
&\Omega(23) \cap \{023\}^2 = \{0,23\} = \Omega^{023}(23) = \Omega^{01}(23 \cap 023).
\end{aligned}$$

On the other hand, for the universe $\mathbf{X} = \{\{0,2,3\}\}$ *and the theory system* $T = \{\{1,2,3\}\}$*, we get*

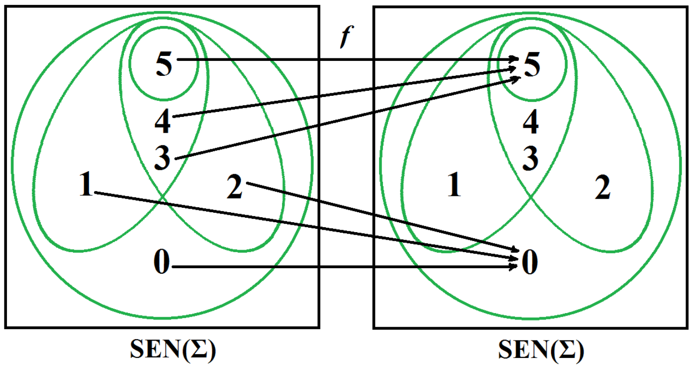
$$\Omega(T) \cap \mathbf{X}^2 = \{\{0\},\{2\},\{3\}\} \lneqq \{\{0\},\{2,3\}\} = \Omega^{\mathbf{X}}(T \cap \mathbf{X}).$$

Thus, $\mathcal{I}$ *is not family extensional and, therefore, it fails to be SF prealgebraizable.*

We now present examples that separate each parallel step from the one immediately below it. The first is an example of an LRF prealgebraizable π-institution that fails to be LC prealgebraizable. This shows that LCF prealgebraizable π-institutions form a proper subclass of the class of LRF prealgebraizable ones and that the class of LC prealgebraizable π-institutions is a proper subclass of the class of LR prealgebraizable π-institutions.

Example 345 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3,4,5\}$ *and*
$$\begin{aligned}
&\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = \mathrm{SEN}^\flat(f)(2) = 0,\\
&\mathrm{SEN}^\flat(f)(3) = \mathrm{SEN}^\flat(f)(4) = \mathrm{SEN}^\flat(f)(5) = 5;
\end{aligned}$$
- $N^\flat$ *is the category of natural transformations generated by the two unary natural transformations* $\sigma^\flat, \tau^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, with*
$$\sigma^\flat_\Sigma, \tau^\flat_\Sigma : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SEN}^\flat(\Sigma)$$
defined by
  - $\sigma^\flat_\Sigma(3) = 1$ *and* $\sigma^\flat_\Sigma(x) = 0$*, for all* $x \in \{0,1,2,4,5\}$*;*
  - $\sigma^\flat_\Sigma(4) = 2$ *and* $\sigma^\flat_\Sigma(x) = 0$*, for all* $x \in \{0,1,2,3,5\}$*.*



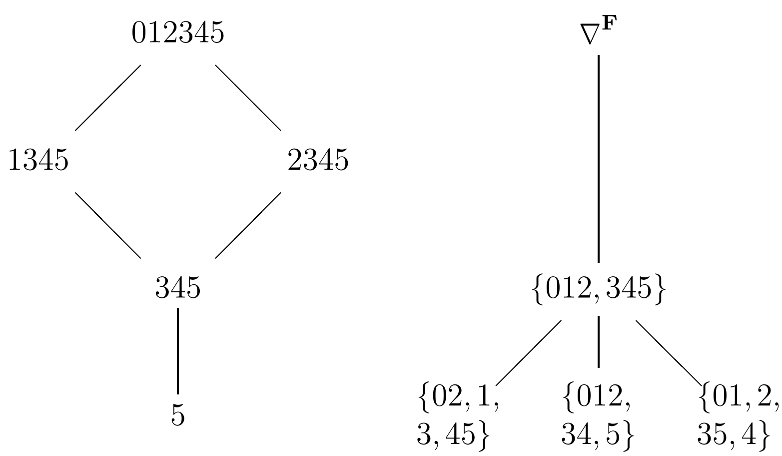
Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by setting

$$\mathcal{C}_\Sigma = \{\{5\}, \{3,4,5\}, \{1,3,4,5\}, \{2,3,4,5\}, \{0,1,2,3,4,5\}\}.$$

$\mathcal{I}$ has five theory families but only three theory systems. The action of $\overleftarrow{\ }$ on theory families is given by the following table.

T	$\overleftarrow{T}$
$\{5\}$	$\{5\}$
$\{3,4,5\}$	$\{3,4,5\}$
$\{1,3,4,5\}$	$\{3,4,5\}$
$\{2,3,4,5\}$	$\{3,4,5\}$
$\{0,1,2,3,4,5\}$	$\{0,1,2,3,4,5\}$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



From the diagram, it is clear that $\mathcal{I}$ is prealgebraic, i.e., that, for all $T, T' \in \mathrm{ThSys}(\mathcal{I})$, $T \le T'$ implies $\Omega(T) \le \Omega(T')$. Moreover, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$, if $\Omega(T) \le \Omega(T')$, then $\overleftarrow{T} \le \overleftarrow{T'}$, i.e., $\mathcal{I}$ is left reflective. On the *other hand, setting,* $T^1 = \{\{1,3,4,5\}\}$, $T^2 = \{\{2,3,4,5\}\}$ *and* $T' = \{\{5\}\}$, *we get*

$$\begin{array}{rcl}\Omega(T^1)\cap\Omega(T^2) & = & \{\{02,1,3,45\}\}\cap\{\{01,2,35,4\}\}\\ & = & \Delta^{\mathbf{F}}\\ & \leq & \{\{012,34,5\}\} = \Omega(T'),\end{array}$$

whereas

$$\overleftarrow{T^1}\cap\overleftarrow{T^2} = \{\{3,4,5\}\}\cap\{\{3,4,5\}\} = \{\{3,4,5\}\} \nleq \{\{5\}\} = \overleftarrow{T'}.$$

Hence, $\mathcal{I}$ *is not left completely reflective. Hence to see that* $\mathcal{I}$ *is LRF prealgebraizable but not LC prealgebraizable, it suffices to show that it is family extensional. The verification is routine, but rather tedious. Note that* $\mathbf{F}$ *has eleven proper and non-trivial universes, namely* $\{\{0,1\}\}$, $\{\{0,2\}\}$, $\{\{0,5\}\}$, $\{\{0,1,2\}\}$, $\{\{0,1,5\}\}$, $\{\{0,2,5\}\}$, $\{\{0,1,2,5\}\}$, $\{\{0,1,3,5\}\}$, $\{\{0,2,4,5\}\}$, $\{\{0,1,2,3,5\}\}$ *and* $\{\{0,1,2,4,5\}\}$. *Moreover, it has four proper theory families,* $T^1 = \{\{5\}\}$, $T^2 = \{\{3,4,5\}\}$, $T^3 = \{\{1,3,4,5\}\}$ *and* $T^4 = \{\{2,3,4,5\}\}$. *So, one has to check forty-four cases in total which are summarized in the following table, where each entry in the column labeled by universe* $\mathbf{F}'$ *and the row labeled by theory family* T *shows the congruence system* $\Omega(T)\cap\mathbf{F}'^2 = \Omega^{\mathbf{F}'}(T\cap\mathbf{F}')$ *in shorthand block notation.*

	01	02	05	012	015	025	0125
5	01	02	0,5	012	01,5	02,5	012,5
345	01	02	0,5	012	01,5	02,5	012,5
1345	0,1	02	0,5	02,1	0,1,5	02,5	02,1,5
2345	01	0,2	0,5	01,2	01,5	0,2,5	01,2,5

	0135	0245	01235	01245
5	01,3,5	02,4,5	012,3,5	012,4,5
345	01,35	02,45	012,35	012,45
1345	0,1,3,5	02,45	02,1,3,5	02,1,45
2345	01,35	0,2,4,5	01,2,35	01,2,4,5

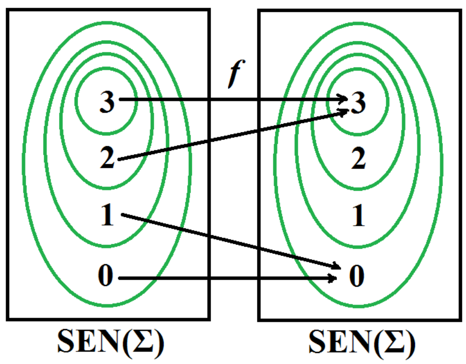
The second example is an example of an LIF prealgebraizable π-institution that fails to be LR prealgebraizable. This shows, on the one hand, that the class of LRF prealgebraizable π-institutions is a proper subclass of the class of LIF prealgebraizable π-institutions and, on the other, that the class of LR prealgebraizable π-institutions is a proper subclass of the class of LI prealgebraizable π-institutions.

Example 346 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f:\Sigma\to\Sigma$, *such that* $f\circ f = f$;

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$*,* $\mathrm{SEN}^\flat(f)(2) = 3$ *and* $\mathrm{SEN}^\flat(f)(3) = 3$*;*
- $N^\flat$ *is the category of natural transformations generated by the unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined by the following table:*

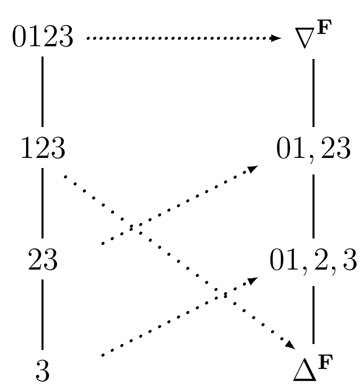
x	0	1	2	3
$\sigma^\flat_\Sigma(x)$	0	1	1	0



Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution defined by setting*

$$\mathcal{C}_\Sigma = \{\{3\},\{2,3\},\{1,2,3\},\{0,1,2,3\}\}.$$

$\mathcal{I}$ *has four theory families, but only three theory systems. The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.*



From the diagram, we can see that $\mathcal{I}$ *is prealgebraic, i.e., that* Ω *is monotone on* $\mathrm{ThSys}(\mathcal{I})$ *and, also left injective. But* $\mathcal{I}$ *is not left reflective, since* $\Omega(\{1,2,3\}) \le \Omega(\{3\})$*, whereas* $\overleftarrow{\{1,2,3\}} = \{2,3\} \nleq \{3\} = \overleftarrow{\{3\}}$*. Therefore, to see that* $\mathcal{I}$ *is LIF prealgebraizable but not LR prealgebraizable, it suffices to show that it is family extensional.*

Note that $\mathbf{F}$ *has three proper and non-singleton universes,* $\{\{0,1\}\}$, $\{\{0,3\}\}$ *and* $\{\{0,1,3\}\}$. *Moreover,* $\mathcal{I}$ *has three proper theory families. Thus, we only have nine cases to check, shown in the following array, which in the row labeled by theory family* T *and the column labeled by universe* $\mathbf{F}'$ *shows the congruence system* $\Omega(T)\cap\mathbf{F}'^2 = \Omega^{\mathbf{F}'}(T\cap\mathbf{F}')$ *in an obvious shorthand notation in terms of blocks.*

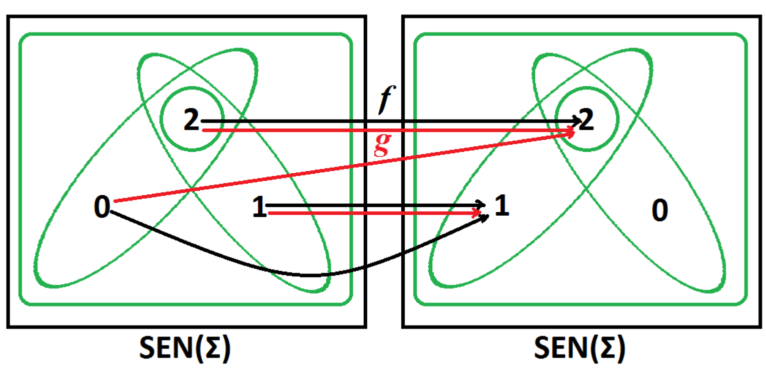
	01	03	013
3	01	0,3	01,3
23	01	0,3	01,3
123	0,1	0,3	0,1,3

We conclude that $\mathcal{I}$ *is LIF prealgebraizable but not LR prealgebraizable.*

The third example is an example of an LIF prealgebraizable π-institution that fails to be FI prealgebraizable. This shows that the class of FIF prealgebraizable π-institutions is a proper subclass of the class of LIF prealgebraizable π-institutions and that the class of FI prealgebraizable π-institutions is a proper subclass of the class of LI prealgebraizable π-institutions.

Example 347 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

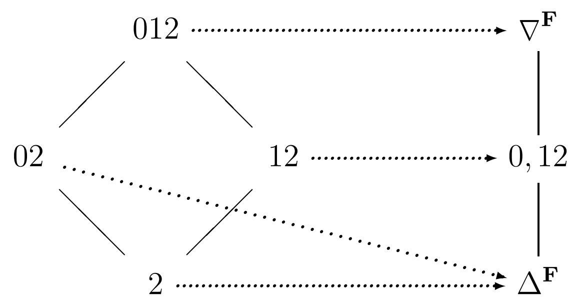
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and two (non-identity) morphisms* $f,g:\Sigma\to\Sigma$, *such that* $f\circ f = f$, $f\circ g = g$, $g\circ f = f$ *and* $g\circ g = g$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and, on morphisms* $\mathrm{SEN}^\flat(f)(0) = 1$, $\mathrm{SEN}^\flat(f)(1) = 1$, $\mathrm{SEN}^\flat(f)(2) = 2$ *and* $\mathrm{SEN}^\flat(g)(0) = 2$, $\mathrm{SEN}^\flat(g)(1) = 1$ *and* $\mathrm{SEN}^\flat(g)(2) = 2$;
- $N^\flat$ *is the trivial category of natural transformations.*



Let $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *be the* π*-institution defined by setting*

$$\mathcal{C}_\Sigma = \{\{2\},\{0,2\},\{1,2\},\{0,1,2\}\}.$$

$\mathcal{I}$ has four theory families, but only three theory systems. The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



From the diagram, we can see that $\mathcal{I}$ is prealgebraic and left injective. But $\mathcal{I}$ is clearly not family injective, since the theory families $\{\{2\}\}$ and $\{\{0,2\}\}$ map to the same congruence system. Therefore, to see that $\mathcal{I}$ is LIF prealgebraizable but not FI prealgebraizable, it suffices to show that it is family extensional.

Note that $\mathbf{F}$ has only one proper and non-singleton universe, $\{\{1,2\}\}$, and three proper theory families $\{\{2\}\}$, $\{\{0,2\}\}$ and $\{\{1,2\}\}$. Thus, we only have three cases to check, shown below in a shorthand notation.

$$\begin{array}{l}\Omega(2)\cap(12)^2=\{1,2\}=\Omega^{12}(2)=\Omega^{12}(2\cap 12);\\ \Omega(02)\cap(12)^2=\{1,2\}=\Omega^{12}(2)=\Omega^{12}(02\cap 12);\\ \Omega(12)\cap(12)^2=\{12\}=\Omega^{12}(12)=\Omega^{12}(12\cap 12).\end{array}$$

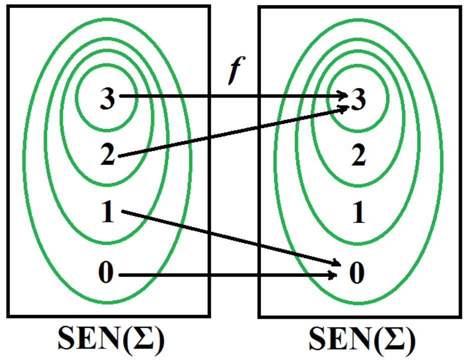
We conclude that $\mathcal{I}$ is LIF prealgebraizable but not FI prealgebraizable.

The last example in this series is an example of an SF prealgebraizable π-institution that fails to be LI prealgebraizable. This shows that LIF prealgebraizable π-institutions form a proper subclass of the class of SF prealgebraizable ones and that the class of LI prealgebraizable π-institutions is a proper subclass of the class of S prealgebraizable ones.

Example 348 *Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object Σ and a single (non-identity) morphism $f:\Sigma\to\Sigma$, such that $f\circ f=f$;*
- $\mathrm{SEN}^\flat:\mathbf{Sign}^\flat\to\mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma)=\{0,1,2,3\}$ *and* $\mathrm{SEN}^\flat(f)(0)=0$, $\mathrm{SEN}^\flat(f)(1)=0$, $\mathrm{SEN}^\flat(f)(2)=3$ *and* $\mathrm{SEN}^\flat(f)(3)=3$;
- $N^\flat$ *is the category of natural transformations generated by the unary natural transformation $\sigma^\flat:\mathrm{SEN}^\flat\to\mathrm{SEN}^\flat$ defined by the following table:*

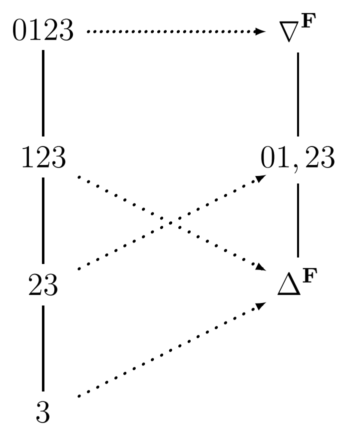
x	0	1	2	3
$\sigma^\flat_\Sigma(x)$	3	2	1	0



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be the π-institution defined by setting

$$\mathcal{C}_\Sigma = \{\{3\},\{2,3\},\{1,2,3\},\{0,1,2,3\}\}.$$

$\mathcal{I}$ has four theory families, but only three theory systems. The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



From the diagram, we can see that $\mathcal{I}$ is prealgebraic, i.e., that Ω is monotone on $\mathrm{ThSys}(\mathcal{I})$ *and, also system injective, i.e., Ω is injective on theory systems. But $\mathcal{I}$ is not left injective, since $\Omega(\{1,2,3\}) = \Omega(\{3\})$, whereas $\overleftarrow{\{1,2,3\}} = \{2,3\} \neq \{3\} = \overleftarrow{\{3\}}$. Therefore, to see that $\mathcal{I}$ is SF prealgebraizable but not LI prealgebraizable, it suffices to show that it is family extensional.*

Note that $\mathbf{F}$ has only one proper and non-singleton universe, $\{\{0,3\}\}$. Moreover, $\mathcal{I}$ has three proper theory families. Thus, we have only three cases to check, shown below in shorthand notation:

$$\begin{array}{l}\Omega(3) \cap \{03\}^2 = \{0,3\} = \Omega^{03}(3) = \Omega^{03}(3 \cap 03);\\ \Omega(23) \cap \{03\}^2 = \{0,3\} = \Omega^{03}(3) = \Omega^{03}(23 \cap 03);\\ \Omega(123) \cap \{03\}^2 = \{0,3\} = \Omega^{03}(3) = \Omega^{03}(123 \cap 03).\end{array}$$

We conclude that $\mathcal{I}$ is SF prealgebraizable but not LI prealgebraizable.

We now turn to establishing transfer properties for the π-institutions belonging to the various classes of the preceding hierarchy. We do this by formulating a comprehensive result encompassing the transference of all ten properties of the above hierarchy. It is hoped that, despite its all-encompassing character, the formulation will be sufficiently clear.

Theorem 349 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *belongs to one of the ten prealgebraizability classes in the prealgebraizability hierarchy if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, the Leibniz operator on* $\mathcal{A}$*, relative to* $\mathcal{I}$*, satisfies the properties defining the corresponding class.*

For example, $\mathcal{I}$ is FIF prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, the Leibniz operator on $\mathcal{A}$ is monotone on $\mathcal{I}$-filter systems, injective on $\mathcal{I}$-filter families and family extensional, i.e.,

- for all $T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, $T \leq T'$ implies $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$;
- for all $T, T' \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$ implies $T = T'$;
- for all $Y \in \mathrm{SenFam}(\mathcal{A})$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$\Omega^{\mathcal{A}}(T) \cap \langle Y \rangle^2 = \Omega^{\langle Y \rangle}(T \cap \langle Y \rangle).$$

Proof: First, observe that the "if" is trivially satisfied, since, if the postulated conditions hold for every $\mathbf{F}$-algebraic system, then they hold, in particular, for $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and this ensures that, by definition, $\mathcal{I}$ belongs to the corresponding prealgebraizability class.

So we turn to the "only if". First, in all cases $\mathcal{I}$ is prealgebraic, i.e., system monotone, and this property transfers to all $\mathbf{F}$-algebraic systems and $\mathcal{I}$-filter systems by Theorem 179. Then, depending on whether $\mathcal{I}$ belongs to one of the classes in the upper or the lower hierarchy of the two parallel hierarchies, it is family or system extensional, respectively. But, by Theorem 314, both of these properties transfer. Finally, depending on the class $\mathcal{I}$ is postulated to belong to, it satisfies one of the properties of system injectivity, family injectivity, left injectivity, left reflectivity or left c-reflectivity. The first three properties transfer by Theorem 214, the fourth transfers by Theorem 225 and the last transfers by Theorem 240. Therefore, the conclusion holds for each of the ten prealgebraizability classes in the prealgebraizability hierarchy. ■

Finally, we turn to characterizations of the classes in the hierarchy in the form of isomorphism theorems between lattices of theory families/systems and lattices of congruence systems. We start, first with FIF prealgebraizability.

Theorem 350 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is FIF prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a bijection which commutes with inverse logical extensions and which restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: The proof is based on Theorem 267, characterizing weak FI prealgebraizbility. We have that $\mathcal{I}$ is FIF prealgebraizable if and only if, by definition, it is weakly FI prealgebraizable and family extensional if and only if, by Theorem 267 and Theorem 327, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is a bijection which commutes with inverse logical extensions and which restricts to an order embedding $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. ■

FI prealgebraizability is characterized in a similar way, the difference being that commutativity with inverse logical extensions is restricted to the application of the Leibniz operator on $\mathcal{I}$-filter systems only, rather than being valid for its operation on the entire collection of $\mathcal{I}$-filter families.

Theorem 351 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is FI prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a bijection which restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

that commutes with inverse logical extensions.

Proof: Similar to the proof of Theorem 350. ■

We turn now to a similar characterization of SF prealgebraizability.

Theorem 352 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is SF prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *commutes with inverse logical extensions and restricts to an order embedding*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: The proof is based on Theorem 256, characterizing weak system prealgebraizability. We have that $\mathcal{I}$ is SF prealgebraizable if and only if, by definition, it is weakly system prealgebraizable and family extensional if and only if, by Theorem 256 and Theorem 327, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$ commutes with inverse logical extensions and restricts to an order embedding $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$. ∎

S prealgebraizability is characterized in a similar way, the difference being that commutativity with inverse logical extensions is restricted to the application of the Leibniz operator on $\mathcal{I}$-filter systems only, rather than to its operation on the entire collection of $\mathcal{I}$-filter families.

Theorem 353 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is S prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$$

is an order embedding which commutes with inverse logical extensions.

Proof: Similar to the proof of Theorem 352. ∎

We continue with LCF prealgebraizability.

Theorem 354 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is LCF prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$$

is a left completely order reflecting surjection, which commutes with inverse logical extensions and which restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A}).$$

Proof: The proof is based on Theorem 276, characterizing weak LC prealgebraizbility. We have that $\mathcal{I}$ is LCF prealgebraizable if and only if, by definition, it is weakly LC prealgebraizable and family extensional, if and only if, by Theorem 276 and Theorem 327, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$ is a left completely order reflecting surjection, which commutes with inverse logical extensions and which restricts to an order embedding $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$. ∎

LC prealgebraizability is characterized in a similar way, the difference being that commutativity with inverse logical extensions is restricted to the application of the Leibniz operator on $\mathcal{I}$-filter systems only, rather than to its operation on the entire collection of $\mathcal{I}$-filter families.

Theorem 355 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is LC prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left completely order reflecting surjection that restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

that commutes with inverse logical extensions.

Proof: Similar to the proof of Theorem 354. ■

A characterization of LRF prealgebraizability in the same spirit follows.

Theorem 356 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is LRF prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left order reflecting surjection which commutes with inverse logical extensions and which restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: The proof is based on Theorem 279, characterizing weak LR prealgebraizability. We have that $\mathcal{I}$ is LRF prealgebraizable if and only if, by definition, it is weakly LR prealgebraizable and family extensional, if and only if, by Theorem 279 and Theorem 327, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is a left order reflecting surjection which commutes with inverse logical extensions and which restricts to an order embedding $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. ■

LR prealgebraizability is characterized in a similar way, the difference being that commutativity with inverse logical extensions is restricted to the application of the Leibniz operator on $\mathcal{I}$-filter systems only, rather than to its operation on the entire collection of $\mathcal{I}$-filter families.

Theorem 357 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is LR prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left order reflecting surjection that restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

that commutes with inverse logical extensions.

Proof: Similar to the proof of Theorem 356. ■

Finally, along the same lines we obtain a characterization of LIF prealgebraizability.

Theorem 358 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is LIF prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left injective surjection which commutes with inverse logical extensions and which restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: The proof is based on Theorem 282, characterizing weak LI prealgebraizability. We have that $\mathcal{I}$ is LIF prealgebraizable if and only if, by definition, it is weakly LI prealgebraizable and family extensional, if and only if, by Theorem 282 and Theorem 327, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is a left injective surjection which commutes with inverse logical extensions and which restricts to an order embedding $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. ■

And, of course, LI prealgebraizability is characterized in a similar way, the difference being that commutativity with inverse logical extensions is restricted to the application of the Leibniz operator on $\mathcal{I}$-filter systems only, rather than to its operation on the entire collection of $\mathcal{I}$-filter families.

Theorem 359 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is LI prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left injective surjection that restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

that commutes with inverse logical extensions.

Proof: Similar to the proof of Theorem 358. ■

5.6 Algebraizability

Since equivalentiality implies protoalgebraicity, the hierarchy of algebraizable π-institutions, which results from the hierarchy of weakly algebraizable π-institutions by replacing protoalgebraicity by equivalentiality, is simpler, reflecting the simplicity of the weak algebraizability hierarchy.

Definition 360 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is* **family algebraizable***, or* **F Algebraizable** *for short, if it is equivalential and family injective, i.e., if it is protoalgebraic, family extensional and family injective;*
- $\mathcal{I}$ *is (***system***)* **algebraizable** *if it is equivalential and system injective, i.e., if it is protoalgebraic, family extensional and system injective.*

These two classes form the following **algebraizability hierarchy**:

Family Algebraizable

↓

(System) Algebraizable

It is clear that these two classes are separated exactly by systemicity, as is shown in the following proposition:

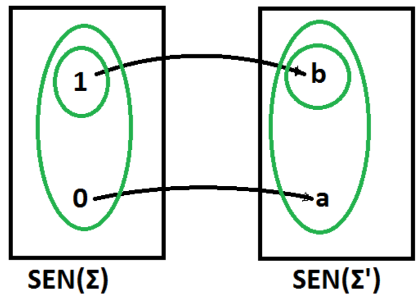
Proposition 361 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I}$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is family algebraizable if and only if it is algebraizable and systemic.*

Proof: We have that $\mathcal{I}$ is family algebraizable if and only if, by definition, it is equivalential and family injective if and only if, by Theorem 291 it is equivalential, systemic and system injective if and only if it is, by definition, algebraizable and systemic. ■

We next present an example to show that these two classes are different. It consists of an algebraizable π-institution, which fails to be family algebraizable.

Example 362 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*



- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$*;*

- $N^\flat$ *is the trivial clone.*

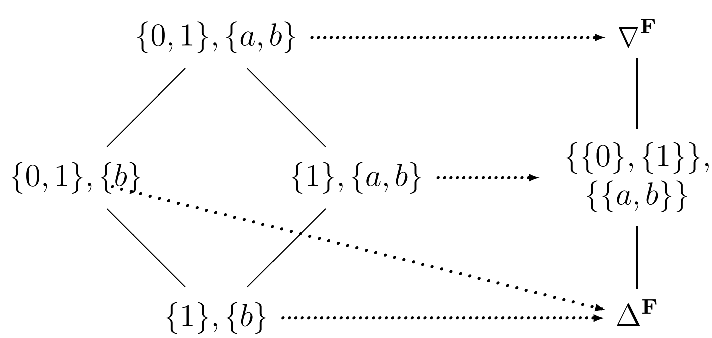
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\},\{a,b\}\}.$$

The table yielding the action of $\overleftarrow{\ }$ *on theory families is shown below.*

$\leftarrow$	$\{b\}$	$\{a,b\}$
$\{1\}$	$\{1\},\{b\}$	$\{1\},\{a,b\}$
$\{0,1\}$	$\{1\},\{b\}$	$\{0,1\},\{a,b\}$

The accompanying diagram gives the structure of the lattice of theory families and the corresponding Leibniz congruence systems.



From the diagram one can check that the Leibniz operator is monotone on theory families and injective on theory systems. Thus, the π*-institution is protoalgebraic and system injective. Moreover, as is shown in the following table, which summarizes the congruence systems of the form* $\Omega(T) \cap \langle X\rangle^2 = \Omega^{\langle X\rangle}(T \cap \langle X\rangle)$ *for the various combinations of nonempty universes and theory* *families, $\mathcal{I}$ is family extensional.*

$\langle X\rangle\backslash T$	$1\ b$	$01\ b$	$1\ ab$	$01\ ab$
$0\ a$	$0\ a$	$0\ a$	$0\ a$	$0\ a$
$0\ ab$	$0\ a,b$	$0\ a,b$	$0\ ab$	$0\ ab$
$1\ b$	$1\ b$	$1\ b$	$1\ b$	$1\ b$
$1\ ab$	$1\ a,b$	$1\ a,b$	$1\ ab$	$1\ ab$
$01\ ab$	$0,1\ a,b$	$0,1\ a,b$	$0,1\ ab$	$01\ ab$

Therefore, $\mathcal{I}$ is clearly equivalential and system injective, i.e., it is algebraizable.

On the other hand, letting $T = \{\{1\},\{b\}\}$ and $T' = \{\{0,1\},\{b\}\}$, we have $\Omega(T) = \Omega(T')$, but $T \neq T'$, whence $\mathcal{I}$ is not family injective and, therefore, it is not family algebraizable.

It is not difficult to show, based on preceding work, that both properties transfer.

Theorem 363 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is algebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, the Leibniz operator on $\mathcal{A}$ is monotone on $\mathcal{I}$-filter families, injective on $\mathcal{I}$-filter systems and family extensional, i.e.,*

- *for all $T,T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T \leq T'$ implies $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$;*
- *for all $T,T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$ implies $T = T'$;*
- *for all $Y \in \mathrm{SenFam}(\mathcal{A})$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,*

$$\Omega^{\mathcal{A}}(T) \cap \langle Y\rangle^2 = \Omega^{\langle Y\rangle}(T \cap \langle Y\rangle).$$

Proof: Suppose, first, that the three conditions hold. Consider the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I,\iota\rangle\rangle$, where $\langle I,\iota\rangle : \mathbf{F} \to \mathbf{F}$ is the identity morphism. By hypothesis, Ω is monotone on theory families and family extensional. Thus, $\mathcal{I}$ is equivalential. Also by hypothesis, Ω is injective on theory systems. Therefore, by definition, $\mathcal{I}$ is algebraizable.

Assume, conversely, that $\mathcal{I}$ is algebraizable. Thus, it is equivalential and system injective, i.e., its Leibniz operator is monotone on theory families, injective on theory systems and family extensional. Now we use Theorems 179, 214 and 314, which guarantee that monotonicity, injectivity and extensionality, respectively, transfer, to conclude that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, the Leibniz operator of $\mathcal{A}$ is monotone on $\mathcal{I}$-filter families, injective on $\mathcal{I}$-filter systems and family extensional. ■

And, similarly, for family algebraizability, we obtain

Theorem 364 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is family algebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, the Leibniz operator on* $\mathcal{A}$ *is monotone and injective on* $\mathcal{I}$*-filter families and family extensional, i.e.,*

- *for all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $T \leq T'$ *implies* $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$*;*
- *for all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$ *implies* $T = T'$*;*
- *for all* $Y \in \mathrm{SenFam}(\mathcal{A})$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*
$$\Omega^{\mathcal{A}}(T) \cap \langle Y \rangle^2 = \Omega^{\langle Y \rangle}(T \cap \langle Y \rangle).$$

Proof: The proof is similar to that given for Theorem 363. It suffices to observe that family injectivity, like system injectivity, also transfers from the theory families of a π-institution $\mathcal{I}$ to all $\mathcal{I}$-filter families on an arbitrary $\mathbf{F}$-algebraic system. ∎

We turn now to characterizations of the classes in the algebraizability hierarchy in terms of order isomorphisms between lattices of filter families/systems and lattices of congruence systems that satisfy additional properties. For algebraizability we have the following characterization.

Theorem 365 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is algebraizable if and only if* $\mathcal{I}$ *is stable and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism that commutes with inverse logical extensions.

Proof: Suppose, first, that $\mathcal{I}$ is algebraizable. Then it is weakly algebraizable and family extensional. Thus, by Theorem 298, $\mathcal{I}$ is stable and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a lattice isomorphism. Commutativity with inverse logical extensions follows by family extensionality and Theorems 327 and 328.

Assume, conversely, that $\mathcal{I}$ is stable and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism that commutes with inverse logical extensions. Then, again by Theorem 298, we get that $\mathcal{I}$ is weakly algebraizable and, by Theorems 328 and 327, that $\mathcal{I}$ is family extensional. It follows, by definition, that $\mathcal{I}$ is algebraizable. ∎

For family algebraizability, we get an analogous characterization.

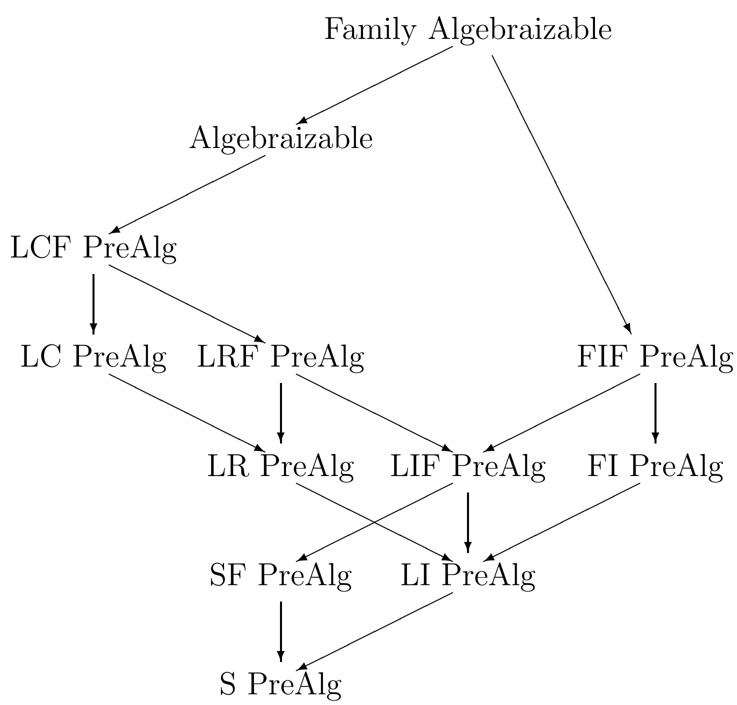
Theorem 366 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is family algebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism that commutes with inverse logical extensions.

Proof: The proof follows along lines similar to the proof of Theorem 365, except references to Theorem 298, characterizing weak algebraizability, must be replaced by referring instead to Theorem 296, which provides a corresponding characterization for weak family algebraizability. ■

Finally, we note that the two classes sit on top of the prealgebraizability hierarchy that was studied in the preceding section. Namely, we have the hierarchy pictured below:

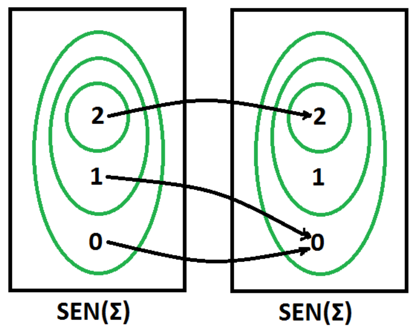


To separate the classes of the algebraizability from those of the prealgebraizability hierarchy, we provide an additional example. It is an example of an LCF and FIF prealgebraizable π-institution which is not algebraizable and, hence, a fortiori, not family algebraizable either.

Example 367 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*

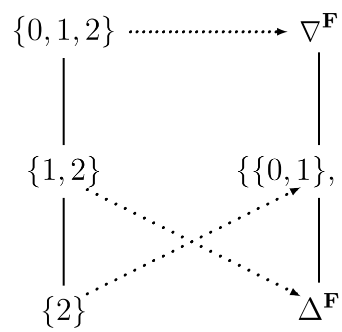
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}$.

The theory family $\{\{1,2\}\}$ *is not a theory system.*

The structure of the lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



Since $\mathcal{I}$ *is not protoalgebraic, it is clear that* $\mathcal{I}$ *is not algebraizable and, a fortiori, it is not family algebraizable either. On the other hand,* $\mathcal{I}$ *is prealgebraic and both left c-reflective and family injective. So, to see that it is both LCF and FIF prealgebraizable, it suffices to show that it is also family extensional. This is done by computing, for all* $T \in \mathrm{ThFam}(\mathcal{I})$ *and all* $X \in \mathrm{SenFam}(\mathcal{I})$ *the congruence systems* $\Omega(T) \cap \langle X\rangle^2$ *and* $\Omega^{\langle X\rangle}(T \cap \langle X\rangle)$ *and verifying that they are identical. This is detailed in the table below:*

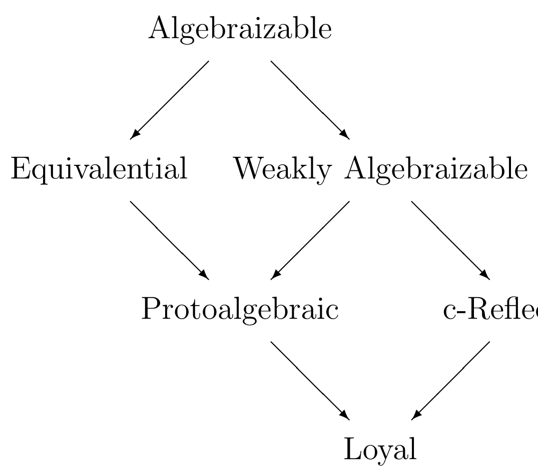
$\langle X\rangle \backslash T$	2	12	012
0	0	0	0
2	2	2	2
01	01	0,1	01
02	0,2	0,2	02
012	01,2	0,1,2	012

We conclude that $\mathcal{I}$ *is family extensional and, therefore, it is, indeed, both LCF and FIF prealgebraizable.*

The last example shows that the hierarchy depicted in the preceding diagram consists of pairwise distinct classes of π-institutions.

5.7 The Semantic Systemic Hierarchy

It is worth stopping momentarily to take a look at the semantic hierarchy that we have studied so far. It has been the case invariably that at each level studied, all classes were identical if restricted to systemic π-institutions. Therefore, considering only systemic π-institutions, one can construct a "skeleton" of the entire hierarchy that is depicted in the accompanying diagram:



It is, therefore, clear that, when restricted to systemic π-institutions, one recovers the fundamental classes and the shape of the well-known Leibniz hierarchy of propositional logics. We view this as a favorable omen that adds credibility to our institutional hierarchical investigations and the hierarchies established through them.