Chapter 6

The Semantic Leibniz Hierarchy: Under the Bottom I

6.1 Introduction

The study of some of the lowest classes in the Leibniz hierarchy of abstract algebraic logic presupposes in a certain sense that the logics studied have theorems. This occurs because the defining conditions of those classes do not hold for nontrivial logics without theorems. Realizing this shortcoming, Moraschini, in Chapter 3 of his Doctoral Dissertation [87] (see, also, [89]) introduced and studied weaker versions accommodating logics without theorems. Our investigations in this chapter have their origins in Moraschini's work, but are suitably adapted to cover logics formalized as π-institutions. At the π-institution level, an injective, and, hence, a fortiori, a reflective or completely reflective, π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ must have theorems. Otherwise, both $\mathrm{SEN}^\flat$ and $\overline{\varnothing}$ are theory families, with $\Omega(\mathrm{SEN}^\flat) = \nabla^{\mathbf{F}} = \Omega(\overline{\varnothing})$ and this contradicts injectivity. So, if one wishes to allow, in a context where injectivity is enforced, π-institutions without theorems, the condition of injectivity must be weakened to either exclude, or bypass in some other way, theory families with empty components. In this chapter we present two such attempts. The first is based on the notion of rough equivalence, under which two theory families are identified if, at those signatures Σ where they differ, one has an empty and the other a $\mathrm{SEN}^\flat(\Sigma)$ component. The second, more straightforward, approach disregards all theory families with at least one empty component. The collection of theory families all of whose components are nonempty is denoted by $\mathrm{ThFam}^{\nsharp}(\mathcal{I})$ and, similarly, $\mathrm{ThSys}^{\nsharp}(\mathcal{I})$ stands for the collection of all theory systems all of whose components are nonempty.

In Section 6.2, we introduce the notion of rough equivalence between theory families of a π-institution. Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Given a theory family T of $\mathcal{I}$, we define its *rough companion* or *associate* $\widetilde{T}$ to be the theory family resulting from T by replacing each empty Σ-component by $\mathrm{SEN}^\flat(\Sigma)$. Then we say that two theory families T, T' are *roughly equivalent*, written $T \sim T'$, if they have the same rough companion, i.e., if $\widetilde{T} = \widetilde{T'}$. Rough equivalence is an equivalence relation on theory families. The equivalence class of T is denoted by $\widetilde{[T]}$ and the collection of all rough equivalence classes by $\widetilde{\mathrm{ThFam}}(\mathcal{I})$. When restricted to theory systems, it is still an equivalence relation and the equivalence class of a theory system T is denoted $\underset{\sim}{\widetilde{[T]}}$, whereas the corresponding collection of rough equivalence classes by $\widetilde{\mathrm{ThSys}}(\mathcal{I})$. The key observation making rough equivalence appropriate as a vehicle for defining classes at the bottom of the Leibniz hierarchy is that, if two theory families are roughly equivalent, then they have identical associated Leibniz congruence systems. That is, the Leibniz operator is constant on rough equivalence classes and, hence, may be viewed as an operator on $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ or on $\widetilde{\mathrm{ThSys}}(\mathcal{I})$, depending on the context. The remainder of Section 6.2 deals with several technical issues concerning rough equivalence. First, by definition, $\widetilde{T}$ is the largest theory family in the class $\widetilde{[T]}$. On the other hand, even if T is a theory system, $\widetilde{T}$ may not be one. Nevertheless, $\widetilde{[T]}$ still has a largest element, which, in that case, is clearly different from $\widetilde{T}$. Another drawback is that, even when T and T' are roughly equivalent, it may not be the case that $\overleftarrow{T}$ and $\overleftarrow{T'}$ are roughly equivalent. This introduces some unexpected complications when studying the Leibniz hierarchies based on roughness and narrowness. On the positive side, given an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and an $\mathcal{I}$-filter family $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we have $\widetilde{\alpha^{-1}(T)} = \alpha^{-1}(\widetilde{T})$. Form this it follows that, for all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, if T and T' are roughly equivalent, then so are $\alpha^{-1}(T)$ and $\alpha^{-1}(T')$.

In Section 6.3, we introduce some weakened versions of systemicity adapted to the study of roughness and narrowness. A π-institution is *roughly systemic* if, for every theory family T, $\overleftarrow{T} \sim T$. It is called *narrowly systemic* if, for every theory family T, with all components nonempty, i.e., such that $T \in \mathrm{ThFam}^{\nparallel}(\mathcal{I})$, $\overleftarrow{T} = T$. Finally, it is called *exclusively systemic* if, for all $T \in \mathrm{ThFam}^{\nparallel}(\mathcal{I})$, such that $\overleftarrow{T} \in \mathrm{ThSys}^{\nparallel}(\mathcal{I})$, we have $\overleftarrow{T} = T$. Systemicity implies both rough and narrow systemicity, and each of these two implies exclusive systemicity. On the other hand, for π-institutions having theorems all four properties become identical.

In Section 6.4, we turn to the study of rough injectivity properties. These are obtained by combining injectivity with rough equivalence. A π-institution $\mathcal{I}$ is *roughly family injective* if, for all theory families T and T', $\Omega(T) = \Omega(T')$ implies $T \sim T'$. $\mathcal{I}$ is *roughly left injective* if the same condition holds, but in the conclusion T, T' are replaced by $\overleftarrow{T}$, $\overleftarrow{T'}$, respectively. It is *roughly right injective* if, similarly, the same condition holds, with T, T' in the hypothesis replaced by $\overleftarrow{T}$, $\overleftarrow{T'}$, respectively. Finally, $\mathcal{I}$ is *roughly system injective* if the implication defining rough family injectivity holds, but with T, T' allowed to range over theory systems only, instead of over arbitrary theory families. Rough right injectivity is strong enough to imply rough systemicity. It also implies rough family injectivity, which implies rough system injectivity. Rough left injectivity also implies rough system injectivity. Moreover, rough right injectivity is equivalent to rough system injectivity and rough systemicity, whereas rough system injectivity, coupled with stability, implies rough left injectivity. All four rough injectivity properties are equivalent to the corresponding injectivity properties under availability of theorems. In addition, all four rough injectivity properties transfer. Section 6.4 concludes with characterizations of the family and system versions in terms of the Leibniz operator Ω, viewed as a mapping from $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ and $\widetilde{\mathrm{ThSys}}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

In Section 6.5, we study narrow injectivity properties. These are defined like the injectivity properties of Section 3.6, but only theory families with all components nonempty are taken into account. Accordingly, a π-institution

$\mathcal{I}$ is *narrowly family injective* if, for all theory families $T, T' \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $\Omega(T) = \Omega(T')$ implies $T = T'$. In the *left version* T and T' are replaced in the conclusion by $\overleftarrow{T}$ and $\overleftarrow{T'}$, respectively. In the *right version* the same replacement occurs in the hypothesis, whereas the *system version* results by imposing the same condition as in the family version, but T, T' are allowed to range only over $\mathrm{ThSys}^{\nmid}(\mathcal{I})$. Narrow right injectivity implies exclusive systemicity, but does not imply either rough or narrow systemicity. The narrow injectivity hierarchy recovers the linearity of the injectivity hierarchy, which was established in Section 3.6. Narrow right injectivity implies the family version, which implies the left version, which, in turn, implies the system version. The latter, coupled with narrow systemicity, implies narrow right injectivity. A comparison is made between corresponding narrow and rough injectivity properties. The family versions are identical. The left versions are incomparable. For both right and system versions, the rough properties imply the corresponding narrow properties. Each of the narrow injectivity properties is identical to the corresponding injectivity property in the presence of theorems. In addition, all four of them transfer. The section concludes by formulating characterization theorems for the family and system versions in terms of the Leibniz operator seen as a mapping from $\mathrm{ThFam}^{\nmid}(\mathcal{I})$ and $\mathrm{ThSys}^{\nmid}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

In Sections 6.6 and 6.7, we undertake the study of rough and narrow reflectivity properties, respectively, following the format of the study of rough and narrow injectivity from Sections 6.4 and 6.5. Subsequently, Sections 6.8 and 6.9, still following the same paradigm, present an analogous study of rough and narrow complete reflectivity properties. A π-institution $\mathcal{I}$ is *roughly family reflective* if, for all theory families T, T', $\Omega(T) \leq \Omega(T')$ implies $\widetilde{T} \leq \widetilde{T}'$. *Rough left* and *rough right reflectivity* result by replacing T and T' in the conclusion and in the hypothesis, respectively, by $\overleftarrow{T}$ and $\overleftarrow{T'}$. *Rough system reflectivity* imposes the same condition as the family version, but applies it only to theory systems. Rough right reflectivity implies rough systemicity. Moreover, it implies rough family reflectivity, which implies rough system reflectivity. The left version also implies the system version. Rough right reflectivity is equivalent to the system version plus rough systemicity and, furthermore, the system version, augmented by stability, implies rough left reflectivity. Comparing with previously studied properties, it is fairly obvious that each version of rough reflectivity implies the corresponding rough injectivity version. In addition, each rough reflectivity version is equivalent to the corresponding reflectivity version under the existence of theorems. All four rough reflectivity properties transfer. Finally, characterizations are provided of rough family and rough system reflectivity in terms of the Leibniz operator, perceived as a mapping from $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ and $\widetilde{\mathrm{ThSys}}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

In Section 6.7, we turn to narrow reflectivity. A π-institution $\mathcal{I}$ is *nar-*

rowly family reflective if, for all theory families T, T', with all components nonempty, $\Omega(T) \leq \Omega(T')$ implies $T \leq T'$. As before, in the *left version* T and T' are replaced in the conclusion by $\overleftarrow{T}$ and $\overleftarrow{T'}$, respectively, and, in the *right version* the same change is applied in the hypothesis instead. *Narrow system reflectivity* stipulates that the same condition as in the family version hold, but applied only to theory systems with all components nonempty. Narrow family reflectivity implies exclusive systemicity. In terms of the narrow reflectivity hierarchy, the right version is the strongest, followed by the family, then the left and, finally, the system version. Narrow system reflectivity and narrow systemicity imply narrow right reflectivity. Comparisons between the rough reflectivity and the narrow reflectivity classes lead to conclusions similar to those obtained in the injectivity case. The two family versions are equivalent, the left versions are incomparable, whereas rough right and rough system reflectivity imply, respectively, narrow right and narrow system reflectivity. Each narrow reflectivity property implies in a straightforward way the corresponding narrow injectivity property and, moreover, gets identified with the corresponding reflectivity property in the presence of theorems. All four narrow reflectivity properties transfer. Finally, the family and system versions may be characterized in terms of the Leibniz operator, viewed as a mapping from $\mathrm{ThFam}^{\nleq}(\mathcal{I})$ and $\mathrm{ThSys}^{\nleq}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

Section 6.8 starts the study of complete reflectivity with the rough versions, continued in Section 6.9 with the narrow versions. A π-institution $\mathcal{I}$ is *roughly family c-reflective* if, for every collection $\mathcal{T} \cup \{T'\}$ of theory families, $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$ implies $\bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T'}$. In the *left version* all theory families in the conclusion appear in their arrow versions and, in the *right version* the same happens in the hypothesis instead. Finally, the *system version* stipulates that the same condition as in the family version hold, by $\mathcal{T} \cup \{T'\}$ ranges over collections of theory systems only. The hierarchy established here mimics the one of rough reflectivity properties. Rough right c-reflectivity implies the family version, which implies the system version, which is also implied by rough left c-reflectivity. Rough system c-reflectivity and rough systemicity together are equivalent to rough right c-reflectivity. Moreover, rough system c-reflectivity, coupled with stability, implies the left version. It is clear that each rough c-reflectivity property implies the corresponding rough reflectivity property and, further, each rough c-reflectivity property is equivalent to the corresponding c-reflectivity property in the presence of theorems. All four rough c-reflectivity properties transfer and, as previously, one may formulate characterizations of rough family and rough system c-reflectivity in terms of Ω, seen as a mapping from $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ and $\widetilde{\mathrm{ThSys}}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

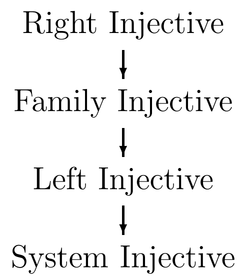
Section 6.9 continues the study of complete reflectivity by looking at narrow c-reflectivity properties. A π-institution $\mathcal{I}$ is *narrowly family c-reflective* if, for every collection $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nleq}(\mathcal{I})$, $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$ implies

$\bigcap \mathcal{T} \le T'$. The *left* and *right versions* are obtained as before by replacing all theory families in the conclusion and in the hypothesis, respectively, by their arrow versions, whereas the *system version* imposes the condition above for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^{\nmid}(\mathcal{I})$. Narrow family c-reflectivity implies exclusive systemicity. The narrow c-reflectivity hierarchy reflects the structure of the narrow reflectivity hierarchy. The right version is the strongest, followed by the family version, then by the left version, while the system version is the weakest of the four. Narrow system c-reflectivity and narrow systemicity imply narrow right c-reflectivity. Comparisons between the rough and narrow versions also follow along lines similar to those between rough and narrow reflectivity properties. The family versions are equivalent, the left versions are incomparable, whereas for both the right and the system versions, rough c-reflectivity implies the corresponding narrow c-reflectivity version. Clearly, each of the four narrow c-reflectivity properties implies the corresponding narrow reflectivity property. As was the case in the rough setting, each narrow c-reflectivity property is identified with the corresponding c-reflectivity property in the presence of theorems. The section concludes with transfer theorems and with characterizations of narrow family and narrow system c-reflectivity via Ω, perceived as a mapping from $\mathrm{ThFam}^{\nmid}(\mathcal{I})$ and $\mathrm{ThSys}^{\nmid}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

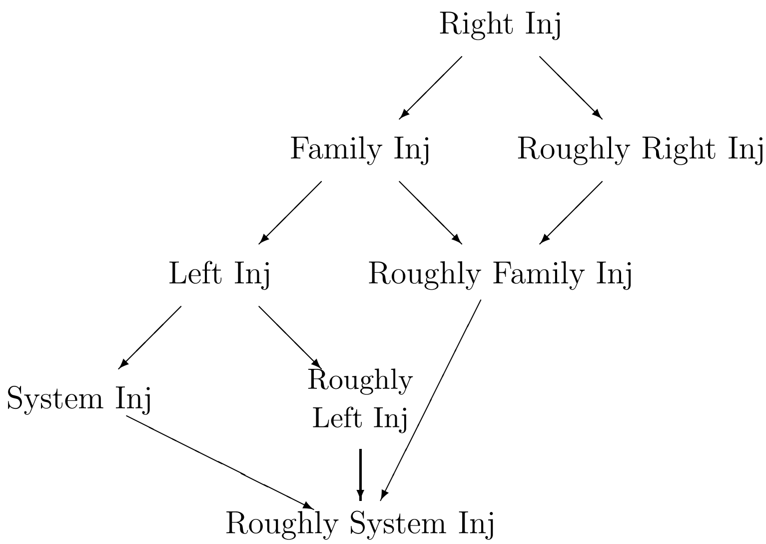
As is clear from all features described, if one considers π-institutions with theorems, the rough and narrow properties become identical to the corresponding properties studied in Chapter 3. Consequently, presence or absence of theorems is a critical characteristic underlying the considerations and hierarchies established in Sections 6.2-6.9. In Section 6.10, the concluding section of the chapter, we turn to some conditions characterizing the existence of theorems via the Frege equivalence family and the Lindenbaum equivalence family operators, introduced in Section 2.11. More precisely, we show that a π-institution $\mathcal{I}$ has theorems if and only if the Frege operator $\lambda : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{EqvFam}(\mathbf{F})$ is injective. Other equivalent conditions to the availability of theorems are the injectivity of the Lindenbaum operator $\widetilde{\lambda}^{\mathcal{I}} : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{EqvFam}(\mathbf{F})$ or, alternatively, its complete reflectivity. Finally, a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ has theorems if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, the induced π-institution $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ has theorems. This constitutes a sort of transfer theorem for the property of possessing theorems.

6.2 Rough Equivalence

Recall from Chapter 3 the injectivity hierarchy, depicted in the following diagram, lying close to the bottom of the semantic Leibniz hierarchy.



Our goal in this section is to add new classes to the semantic Leibniz hierarchy that lie below those injectivity classes. We will eventually build the following hierarchy:



Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. Given a theory family $T \in \mathrm{ThFam}(\mathcal{I})$, the **rough companion** or **rough associate** or **rough representative of** T, denoted $\widetilde{T}$, is the theory family of $\mathcal{I}$ that results from T after replacing every empty Σ-component by the corresponding universe $\mathrm{SEN}^\flat(\Sigma)$. More formally, we set

$$\widetilde{T} = \{\widetilde{T}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|},$$

where, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\widetilde{T}_\Sigma = \begin{cases} T_\Sigma, & \text{if } T_\Sigma \neq \varnothing \\ \mathrm{SEN}^\flat(\Sigma), & \text{if } T_\Sigma = \varnothing \end{cases}.$$

The operator $\widetilde{\ }: \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$ is clearly idempotent:

Lemma 368 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. Then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\widetilde{T}} = \widetilde{T}$.

Proof: We have, by construction, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\widetilde{T}_\Sigma \neq \varnothing$, whence, we get, by definition, $\widetilde{\widetilde{T}}_\Sigma = \widetilde{T}_\Sigma$. ■

Define on $\mathrm{ThFam}(\mathcal{I})$ the relation $\sim \subseteq \mathrm{ThFam}(\mathcal{I})^2$ of **rough equivalence** by setting, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$T \sim T' \quad \text{iff} \quad \widetilde{T} = \widetilde{T'}.$$

It is not difficult to see that rough equivalence is indeed an equivalence relation on the collection of theory families of $\mathcal{I}$, since it is the relational kernel of the mapping $\widetilde{\ } : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$. We call two theories $T, T' \in \mathrm{ThFam}(\mathcal{I})$ **roughly equivalent** if $T \sim T'$. We denote by $\widetilde{[T]}$ the rough equivalence class of a theory family T and let $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ be the collection of all rough equivalence classes of theory families of $\mathcal{I}$.

Since the collection of theory systems of $\mathcal{I}$ is a subcollection of the collection of theory families of $\mathcal{I}$, the rough equivalence relation restricts to an equivalence relation, which we also term **rough equivalence**, on the collection $\mathrm{ThSys}(\mathcal{I})$. We denote by $\widetilde{[T]}$ the rough equivalence class of a theory system T and let $\widetilde{\mathrm{ThSys}}(\mathcal{I})$ be the collection of all rough equivalence classes of theory systems of $\mathcal{I}$.

We now introduce a notation that will prove very handy in subsequent considerations, especially in contexts where the π-institutions under scrutiny may not have theorems. Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We define:

- $\mathrm{ThFam}^{\lightning}(\mathcal{I})$ to be the collection of all theory families of $\mathcal{I}$ with all components nonempty.

  Note that

  $$\begin{array}{rcl} \mathrm{ThFam}^{\lightning}(\mathcal{I}) & = & \{T \in \mathrm{ThFam}(\mathcal{I}) : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(T_\Sigma \neq \varnothing)\} \\ & = & \{T \in \mathrm{ThFam}(\mathcal{I}) : \widetilde{T} = T\}. \end{array}$$

  Note, also, that, in case $\mathcal{I}$ has theorems, $\mathrm{ThFam}^{\lightning}(\mathcal{I}) = \mathrm{ThFam}(\mathcal{I})$.

- $\mathrm{ThSys}^{\lightning}(\mathcal{I})$ to be the collection of all theory systems of $\mathcal{I}$ with all components nonempty.

  Note that

  $$\begin{array}{rcl} \mathrm{ThSys}^{\lightning}(\mathcal{I}) & = & \{T \in \mathrm{ThSys}(\mathcal{I}) : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(T_\Sigma \neq \varnothing)\} \\ & = & \{T \in \mathrm{ThSys}(\mathcal{I}) : \widetilde{T} = T\}. \end{array}$$

  Note, again, that, in case $\mathcal{I}$ has theorems, $\mathrm{ThSys}^{\lightning}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$.

A key result in our use of the rough equivalence relation to define the semantic hierarchy classes "down under" is the realization that two roughly equivalent theory families have the same Leibniz congruence family and, as a result, the Leibniz operator may be unambiguously applied on rough equivalence classes of theory families. This follows from the fact that, for every theory family T, T and $\widetilde{T}$ share the same Leibniz congruence system.

Proposition 369 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *Then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\Omega(T) = \Omega(\widetilde{T}).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$ and $\phi \in \widetilde{T}_\Sigma$.

- If $T_\Sigma = \varnothing$, then $\widetilde{T}_\Sigma = \mathrm{SEN}^\flat(\Sigma)$ and, hence, $\psi \in \widetilde{T}_\Sigma$;
- If $T_\Sigma \neq \varnothing$, then $\widetilde{T}_\Sigma = T_\Sigma$ and, hence, by the compatibility of $\Omega(T)$ with T, we get $\psi \in T_\Sigma = \widetilde{T}_\Sigma$.

We conclude that $\Omega(T)$ is compatible with $\widetilde{T}$ and, hence, $\Omega(T) \leq \Omega(\widetilde{T})$.

Suppose, conversely, that $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in \Omega_\Sigma(\widetilde{T})$ and $\phi \in T_\Sigma$. Then $T_\Sigma \neq \varnothing$, whence $\widetilde{T}_\Sigma = T_\Sigma$. Thus, $\phi \in \widetilde{T}_\Sigma$ and, by the compatibility of $\Omega(\widetilde{T})$ with $\widetilde{T}$, we get that $\psi \in \widetilde{T}_\Sigma = T_\Sigma$. Thus, $\Omega(\widetilde{T})$ is compatible with T and we get $\Omega(\widetilde{T}) \leq \Omega(T)$.

We conclude that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(\widetilde{T}) = \Omega(T)$. ■

Theorem 370 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *Then, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$T \sim T' \quad \textit{implies} \quad \Omega(T) = \Omega(T').$$

Proof: Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \sim T'$. Then, by definition, $\widetilde{T} = \widetilde{T'}$. Thus, we get, by Proposition 369,

$$\Omega(T) = \Omega(\widetilde{T}) = \Omega(\widetilde{T'}) = \Omega(T')$$

and T and T' have, indeed, the same Leibniz congruence system. ■

We define, next, an ordering relation on the rough equivalence classes of theory families of a π-institution $\mathcal{I}$. But we start by looking at maximal elements.

Proposition 371 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *For every* $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\widetilde{T} = \max \widetilde{[T]}.$$

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$ and consider $T' \in \widetilde{[T]}$. Then, clearly, $T' \leq \widetilde{T'} = \widetilde{T}$. Therefore, $\widetilde{T}$ is a maximum element in $\widetilde{[T]}$. ∎

What is, perhaps, more surprising is that each rough equivalence class in $\mathrm{ThSys}(\mathcal{I})$ also has a maximum element. First, we show that it has maximal elements and then prove that there cannot exist two different maximal elements and, hence, that it has a maximum element.

Proposition 372 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,* $\widetilde{[T]}$ *has a maximal element.*

Proof: Let $T \in \mathrm{ThSys}(\mathcal{I})$. We show that every chain in $\widetilde{[T]}$ has an upper bound in $\widetilde{[T]}$. Then the conclusion follows by applying Zorn's Lemma. Assume that $\{T^i : i \in I\}$ is a chain in $\widetilde{[T]}$. We consider $\bigcup_{i \in I} T^i$.

- $\bigcup_{i \in I} T^i \in \mathrm{ThFam}(\mathcal{I})$: Let $\Sigma \in |\mathbf{Sign}^\flat|$. If, for some $j \in I$, $T^j_\Sigma \neq \varnothing$ and $T^j_\Sigma \neq \mathrm{SEN}^\flat(\Sigma)$, then, since all members of $\{T^i : i \in I\}$ are roughly equivalent, we have $\bigcup_{i \in I} T^i_\Sigma = T^j_\Sigma$ is a Σ-theory. If, on the other hand, $T^i_\Sigma = \varnothing$, for all $i \in I$, then $\bigcup_{i \in I} T^i_\Sigma = \varnothing = T^i_\Sigma$, which is again a Σ-theory. Finally, if, for some $i \in I$, $T^i = \mathrm{SEN}^\flat(\Sigma)$, then $\bigcup_{i \in I} T^i_\Sigma = \mathrm{SEN}^\flat(\Sigma)$, which is again a Σ-theory. Therefore, we conclude that $\bigcup_{i \in I} T^i \in \mathrm{ThFam}(\mathcal{I})$.

- $\bigcup_{i \in I} T^i \in \mathrm{ThSys}(\mathcal{I})$: Let $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \bigcup_{i \in I} T^i_\Sigma$. Then, for some $i \in I$, $\phi \in T^i_\Sigma$. Since $T^i \in \mathrm{ThSys}(\mathcal{I})$, we have $\mathrm{SEN}^\flat(f)(\phi) \in T^i_{\Sigma'}$ and, therefore, $\mathrm{SEN}^\flat(f)(\phi) \in \bigcup_{i \in I} T^i_{\Sigma'}$. We conclude that $\bigcup_{i \in I} T^i \in \mathrm{ThSys}(\mathcal{I})$.

- $\bigcup_{i \in I} T^i \sim T$: Let $\Sigma \in |\mathbf{Sign}^\flat|$. If $\bigcup_{i \in I} T^i_\Sigma = \varnothing$, then $T^i_\Sigma = \varnothing$, for all $i \in I$, and hence, $\widetilde{T^i}_\Sigma = \mathrm{SEN}^\flat(\Sigma)$. Therefore, $\widetilde{\bigcup_{i \in I} T^i}_\Sigma = \mathrm{SEN}^\flat(\Sigma) = \widetilde{T^i}_\Sigma$.

  Suppose, next, that $\bigcup_{i \in I} T^i_\Sigma \neq \varnothing$. Thus, there exists $j \in I$, such that $T^j_\Sigma \neq \varnothing$. If there exists $i \in I$, such that $T^i_\Sigma = \mathrm{SEN}^\flat(\Sigma)$, then $\bigcup_{i \in I} T^i_\Sigma = \mathrm{SEN}^\flat(\Sigma)$, whence

  $$(\widetilde{\bigcup_{i \in I} T^i})_\Sigma = \mathrm{SEN}^\flat(\Sigma) = \widetilde{T^i}_\Sigma.$$

  So assume that $T^i_\Sigma \neq \mathrm{SEN}^\flat(\Sigma)$, for all $i \in I$. Then, we conclude that $T^j_\Sigma \neq \varnothing, \mathrm{SEN}^\flat(\Sigma)$ and, therefore, since all T^i's are roughly equivalent, $T^i_\Sigma = T^j_\Sigma$, for all $i \in I$. Then $\bigcup_{i \in I} T^i_\Sigma = T^j_\Sigma$ and, therefore,

  $$(\widetilde{\bigcup_{i \in I} T^i})_\Sigma = T^j_\Sigma = \widetilde{T^j}_\Sigma.$$

  Thus, $\widetilde{\bigcup_{i \in I} T^i} = \widetilde{T}$ and we conclude that $\bigcup_{i \in I} T^i \in \widetilde{[T]}$.

Therefore, $\bigcup_{i\in I} T^i$ is clearly an upper bound of $\{T^i : i \in I\}$ in $\widetilde{[T]}$. By Zorn's Lemma, we conclude that $\widetilde{[T]}$ has a maximal element. ■

Now we show that in a rough equivalence class in $\mathrm{ThSys}(\mathcal{I})$, there cannot exist two different maximal elements and, therefore, that every rough equivalence class in $\mathrm{ThSys}(\mathcal{I})$ has a maximum element.

Theorem 373 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,* $\widetilde{[T]}$ *has a maximum element.*

Proof: The proof is quite similar to the proof of Proposition 372. The key is to show that, if $T', T'' \in \mathrm{ThSys}(\mathcal{I})$, such that $T' \sim T''$, then $T' \cup T'' \in \mathrm{ThSys}(\mathcal{I})$, such that $T' \cup T'' \sim T$. So, unless $T' = T''$, not both can be maximal in $\widetilde{[T]}$.

- $T' \cup T'' \in \mathrm{ThFam}(\mathcal{I})$: Let $\Sigma \in |\mathbf{Sign}^\flat|$. If $T'_\Sigma \neq \emptyset, \mathrm{SEN}^\flat(\Sigma)$ and $T''_\Sigma \neq \emptyset, \mathrm{SEN}^\flat(\Sigma)$, then, since $T' \sim T''$, $T'_\Sigma = T''_\Sigma$. Thus, $T'_\Sigma \cup T''_\Sigma = T'_\Sigma$ and, hence, it is a Σ-theory. If $T'_\Sigma = T''_\Sigma = \emptyset$, then $T'_\Sigma \cup T''_\Sigma = \emptyset$, which is again a Σ-theory. Otherwise, $T'_\Sigma \cup T''_\Sigma = \mathrm{SEN}^\flat(\Sigma)$, which is also a Σ-theory.

- $T' \cup T'' \in \mathrm{ThSys}(\mathcal{I})$: Let $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T'_\Sigma \cup T''_\Sigma$. Then $\phi \in T'_\Sigma$ or $\phi \in T''_\Sigma$. In the first case, $\mathrm{SEN}^\flat(f)(\phi) \in T'_{\Sigma'}$ and, in the second, $\mathrm{SEN}^\flat(f)(\phi) \in T''_{\Sigma'}$. In either case, $\mathrm{SEN}^\flat(f)(\phi) \in T'_{\Sigma'} \cup T''_{\Sigma'}$. Thus, $T' \cup T'' \in \mathrm{ThSys}(\mathcal{I})$.

- $T' \cup T'' \sim T$: Let $\Sigma \in |\mathbf{Sign}^\flat|$. If $T'_\Sigma \cup T''_\Sigma = \emptyset$, then $T'_\Sigma = T''_\Sigma = \emptyset$. So $\widetilde{T' \cup T''}_\Sigma = \mathrm{SEN}^\flat(\Sigma) = \widetilde{T'}_\Sigma = \widetilde{T}_\Sigma$.

  If $T'_\Sigma \cup T''_\Sigma \neq \emptyset$, then $T'_\Sigma \neq \emptyset$ or $T''_\Sigma \neq \emptyset$, say, without loss of generality, $T'_\Sigma \neq \emptyset$. If $T'_\Sigma \neq \mathrm{SEN}^\flat(\Sigma)$, then, since $T' \sim T''$, $T'_\Sigma = T''_\Sigma$ and, hence $T'_\Sigma \cup T''_\Sigma = T'_\Sigma$ and we have $\widetilde{T' \cup T''}_\Sigma = T'_\Sigma = \widetilde{T'}_\Sigma = \widetilde{T}_\Sigma$. If, on the other hand, $T'_\Sigma = \mathrm{SEN}^\flat(\Sigma)$, then $T'_\Sigma \cup T''_\Sigma = \mathrm{SEN}^\flat(\Sigma)$, whence $\widetilde{T' \cup T''}_\Sigma = \mathrm{SEN}^\flat(\Sigma) = \widetilde{T'}_\Sigma = \widetilde{T}_\Sigma$.

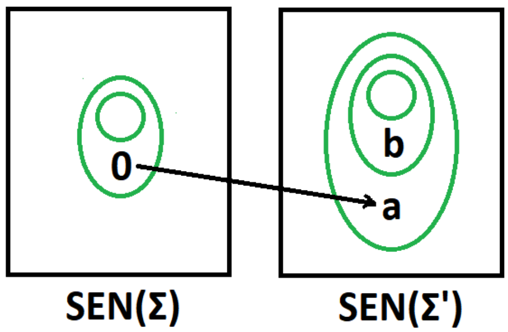
Therefore $T' \cup T'' \in \mathrm{ThSys}(\mathcal{I})$ and $T' \cup T'' \sim T$. We conclude that all maximal elements in $\widetilde{[T]}$ must be equal, i.e., $\widetilde{[T]}$ has a maximum element. ■

It is worth noting, however, that the maximum element of a class $\widetilde{[T]}$ may not be $\widetilde{T} = \max \widetilde{[T]}$, since this theory family may not be a theory system.

Example 374 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$;
- $N^\flat$ *is the trivial clone.*



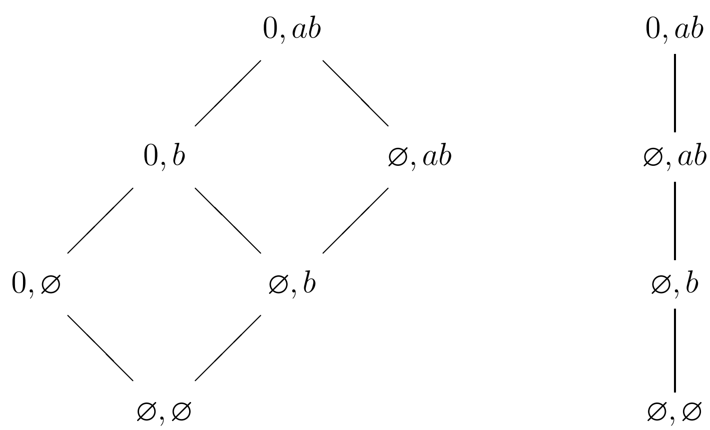
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{0\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a,b\}\}.$$

There are six theory families, but only four theory systems. The action of $\overleftarrow{\ }$ *on theory families is given in the table below.*

T	$\overleftarrow{T}$
$\varnothing, \varnothing$	$\varnothing, \varnothing$
$0, \varnothing$	$\varnothing, \varnothing$
$\varnothing, b$	$\varnothing, b$
$0, b$	$\varnothing, b$
$\varnothing, ab$	$\varnothing, ab$
$0, ab$	$0, ab$

The complete lattice of theory families is shown on the left:



That of the theory systems is shown on the right. Now note that

$$\max\lfloor\widetilde{\{\varnothing,\{b\}\}}\rfloor = \{\varnothing,\{b\}\},$$

whereas $\widetilde{\{\varnothing,\{b\}\}} = \{\{0\},\{b\}\} \notin \mathrm{ThSys}(\mathcal{I})$.

For what follows, we also need to point out the fact that, roughly speaking, the $\sim$ operator does not interact smoothly with the $\overleftarrow{\ }$ operator. More precisely, for arbitrary π-institutions, and theory families T, T',

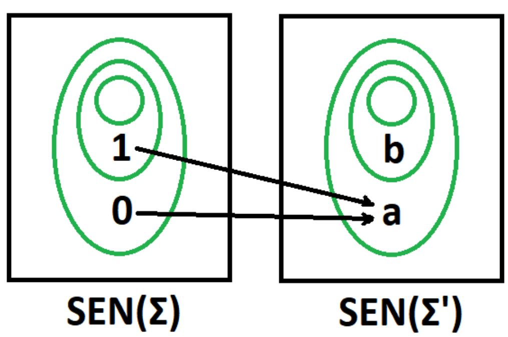
$$\text{the relation } T \sim T' \text{ does not imply, in general, that } \overleftarrow{T} \sim \overleftarrow{T'}.$$

We showcase the potential failure by giving a *counterexample to the statement*, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$T \sim T' \quad \text{implies} \quad \overleftarrow{T} \sim \overleftarrow{T'}.$$

Example 375 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = a$*;*
- $N^\flat$ *is the trivial clone.*



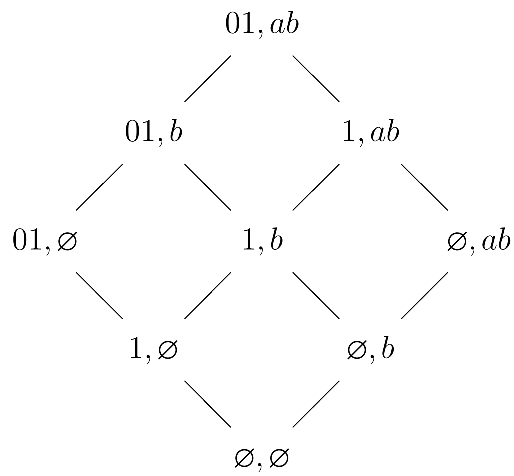
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a,b\}\}.$$

There are nine theory families, but only five theory systems. The action of $\overleftarrow{\ }$ *on theory families is given in the table below.*

T	$\overleftarrow{T}$	T	$\overleftarrow{T}$
$\varnothing, \varnothing$	$\varnothing, \varnothing$	$\varnothing, ab$	$\varnothing, ab$
$1, \varnothing$	$\varnothing, \varnothing$	$01, b$	$\varnothing, b$
$\varnothing, b$	$\varnothing, b$	$1, ab$	$1, ab$
$01, \varnothing$	$\varnothing, \varnothing$	$01, ab$	$01, ab$
$1, b$	$\varnothing, b$		

The lattice of theory families of $\mathcal{I}$ is shown in the diagram.



Consider $T = \{\{1\},\{a,b\}\}$ and $T' = \{\{1\},\varnothing\}$. We clearly have $\widetilde{T} = \widetilde{T'} = T$, whence $T \sim T'$. On the other hand,

$$\overleftarrow{T} = T \nsim \{\varnothing,\varnothing\} = \overleftarrow{T'}.$$

Therefore, even though $T \sim T'$, it is not the case that $\overleftarrow{T} \sim \overleftarrow{T'}$.

To establish some transfer theorems for the classes to be introduced shortly, we need a few results pertaining to the interaction of rough equivalence with inverse images. Key to these considerations is the following technical lemma to the effect that a filter family has an empty component if and only if its inverse theory family has a corresponding empty component. This is a relatively easy consequence of the surjectivity of interpretation morphisms.

Lemma 376 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$,*

$$T_{F(\Sigma)} = \varnothing \quad \textit{iff} \quad \alpha_\Sigma^{-1}(T_{F(\Sigma)}) = \varnothing.$$

Proof: Let $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $\Sigma \in |\mathbf{Sign}^\flat|$. If $T_{F(\Sigma)} = \varnothing$, then, obviously, $\alpha^{-1}(T_{F(\Sigma)}) = \varnothing$. If, conversely, $\alpha_\Sigma^{-1}(T_{F(\Sigma)}) = \varnothing$, then, by surjectivity of $\langle F,\alpha\rangle$, $T_{F(\Sigma)} = \alpha_\Sigma(\alpha_\Sigma^{-1}(T_{F(\Sigma)})) = \alpha_\Sigma(\varnothing) = \varnothing$. ■

We can now show that the maximum $\widetilde{\alpha^{-1}(T)}$ of the rough equivalence class of the theory family $\alpha^{-1}(T)$ in the π-institution $\mathcal{I}$ coincides with the inverse image $\alpha^{-1}(\widetilde{T})$ of the maximum $\widetilde{T}$ of the rough equivalence class of the $\mathcal{I}$-filter family T of $\mathcal{A}$ in $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

Theorem 377 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. Then*

$$\widetilde{\alpha^{-1}(T)} = \alpha^{-1}(\widetilde{T}).$$

Proof: Let $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $\Sigma \in |\mathbf{Sign}^\flat|$. We separate two cases depending on whether or not $\alpha_\Sigma^{-1}(T_{F(\Sigma)}) = \varnothing$.

- Suppose $\alpha_\Sigma^{-1}(T_{F(\Sigma)}) = \varnothing$. Then, by Lemma 376, we get $T_{F(\Sigma)} = \varnothing$. Thus, we get

$$\widetilde{\alpha^{-1}(T)}_\Sigma = \mathrm{SEN}^\flat(\Sigma) = \alpha_\Sigma^{-1}(\mathrm{SEN}(F(\Sigma))) = \alpha_\Sigma^{-1}(\widetilde{T}_{F(\Sigma)}).$$

- Suppose $\alpha_\Sigma^{-1}(T_{F(\Sigma)}) \neq \varnothing$. Then, by Lemma 376, we get $T_{F(\Sigma)} \neq \varnothing$. Thus, we get

$$\widetilde{\alpha^{-1}(T)}_\Sigma = \alpha_\Sigma^{-1}(T_{F(\Sigma)}) = \alpha_\Sigma^{-1}(\widetilde{T}_{F(\Sigma)}).$$

In either case, we have $\widetilde{\alpha^{-1}(T)}_\Sigma = \alpha_\Sigma^{-1}(\widetilde{T}_{F(\Sigma)})$. Therefore, we get $\widetilde{\alpha^{-1}(T)} = \alpha^{-1}(\widetilde{T})$. ∎

This implies that rough equivalence interacts smoothly with inverse images. More precisely, given two $\mathcal{I}$-filter families $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T \sim T'$ in $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ if and only if $\alpha^{-1}(T) \sim \alpha^{-1}(T')$ in $\mathrm{ThFam}(\mathcal{I})$. Contrast this with the rather rocky interaction between rough equivalence and the $\overleftarrow{\ }$ operator, as detailed before (and in) Example 375.

Corollary 378 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. Then*

$$T \sim T' \quad \textit{iff} \quad \alpha^{-1}(T) \sim \alpha^{-1}(T').$$

Proof: Let $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. We get

$$\begin{array}{lll} \alpha^{-1}(T) \sim \alpha^{-1}(T') & \text{iff} & \widetilde{\alpha^{-1}(T)} = \widetilde{\alpha^{-1}(T')} \quad \text{(Definition of } \sim) \\ & \text{iff} & \alpha^{-1}(\widetilde{T}) = \alpha^{-1}(\widetilde{T'}) \quad \text{(Theorem 377)} \\ & \text{iff} & \widetilde{T} = \widetilde{T'} \quad \text{(Surjectivity of } \langle F, \alpha \rangle) \\ & \text{iff} & T \sim T'. \quad \text{(Definition of } \sim) \end{array}$$

This establishes the conclusion. ∎

6.3 Roughness and Systemicity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. Recall that $\mathcal{I}$ was called **systemic** if all its theory families are theory systems. This can be expressed in symbols by writing $\mathrm{ThFam}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$ or, alternatively, by the assertion that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} = T$.

We now introduce three other *systemicity* properties that are inspired by the original, but avoid in some way the consideration of theory families with empty components or take into account the rough equivalence relation between theory families.

- We say that $\mathcal{I}$ is **roughly systemic** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} \sim T$;
- We say that $\mathcal{I}$ is **narrowly systemic** if, for all $T \in \mathrm{ThFam}^{\not\flat}(\mathcal{I})$, $\overleftarrow{T} = T$;
- We say that $\mathcal{I}$ is **exclusively systemic** if, for all $T \in \mathrm{ThFam}^{\not\flat}(\mathcal{I})$, such that $\overleftarrow{T} \in \mathrm{ThSys}^{\not\flat}(\mathcal{I})$, $\overleftarrow{T} = T$.

The inclusions between these four classes are straightforward and recounted in the following

Proposition 379 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$.*

(a) If $\mathcal{I}$ is systemic, then it is both roughly and narrowly systemic;

(b) If $\mathcal{I}$ is roughly systemic, then it is exclusively systemic;

(c) If $\mathcal{I}$ is narrowly systemic, then it is exclusively systemic.

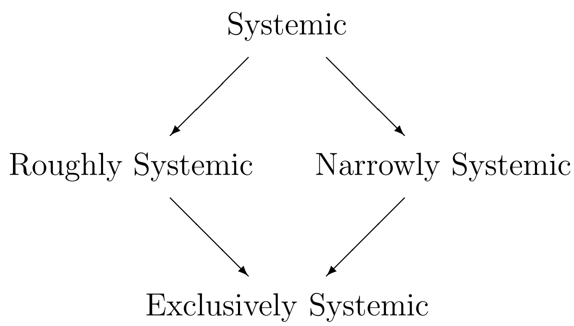
Proof:

(a) Suppose that $\mathcal{I}$ is systemic. If $T \in \mathrm{ThFam}(\mathcal{I})$, then $T = \overleftarrow{T}$. Thus, $\widetilde{T} = \widetilde{\overleftarrow{T}}$, i.e., $T \sim \overleftarrow{T}$ and, hence, $\mathcal{I}$ is roughly systemic. On the other hand, if $T \in \mathrm{ThFam}^{\not\flat}(\mathcal{I})$, then, since $\mathrm{ThFam}^{\not\flat}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I})$, we get, by hypothesis, $\overleftarrow{T} = T$. Thus, $\mathcal{I}$ is also narrowly systemic.

(b) Suppose that $\mathcal{I}$ is roughly systemic. Let $T \in \mathrm{ThFam}^{\not\flat}(\mathcal{I})$, such that $\overleftarrow{T} \in \mathrm{ThSys}^{\not\flat}(\mathcal{I})$. Since $\mathrm{ThFam}^{\not\flat}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I})$, we get, by hypothesis, $\overleftarrow{T} \sim T$, i.e., $\widetilde{\overleftarrow{T}} = \widetilde{T}$. However, since $T \in \mathrm{ThFam}^{\not\flat}(\mathcal{I})$ and $\overleftarrow{T} \in \mathrm{ThSys}^{\not\flat}(\mathcal{I})$, we conclude that $\overleftarrow{T} = \widetilde{\overleftarrow{T}} = \widetilde{T} = T$. Thus, $\mathcal{I}$ is exclusively systemic.

(c) Suppose $\mathcal{I}$ is narrowly systemic. Then, by hypothesis, for all $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$ and, therefore, a fortiori, for all such T, such that $\overleftarrow{T} \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$, we get that $\overleftarrow{T} = T$. Hence, $\mathcal{I}$ is exclusively systemic. ∎

Proposition 379 establishes the following *hierarchy of roughness and systemicity properties*:



A related result, which partially explains the introduction of the roughness and systemicity classes and which, in fact, forms the undercurrent of much of the ideas underlying developments in the entire chapter, assures that all three bottom properties actually coincide with systemicity itself, in case the π-institution under consideration has theorems.

Proposition 380 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is exclusively systemic and has theorems, then it is systemic.*

Proof: Suppose $\mathcal{I}$ has theorems. Then $\mathrm{ThFam}^{\lightning}(\mathcal{I}) = \mathrm{ThFam}(\mathcal{I})$, and $\mathrm{ThSys}^{\lightning}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$. Moreover, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} \in \mathrm{ThSys}(\mathcal{I}) = \mathrm{ThSys}^{\lightning}(\mathcal{I})$. Therefore, the defining condition of exclusive systemicity is equivalent to asserting that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} = T$, i.e., it is equivalent to systemicity. ∎

We present two examples that will show that all four classes in the roughness and systemicity hierarchy depicted above are indeed different. The first example shows that the southwest arrows represent proper inclusions, i.e.,

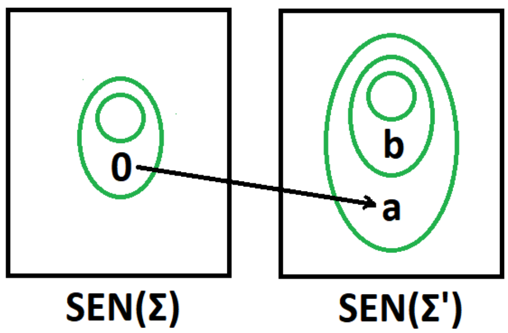
- Systemic π-institutions form a proper subclass of roughly systemic π-institutions;

- Exclusively systemic π-institutions form a proper subclass of narrowly systemic π-institutions.

This is accomplished by presenting a π-institution which is roughly systemic but fails to be narrowly systemic.

Example 381 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$;
- $N^\flat$ *is the trivial clone.*



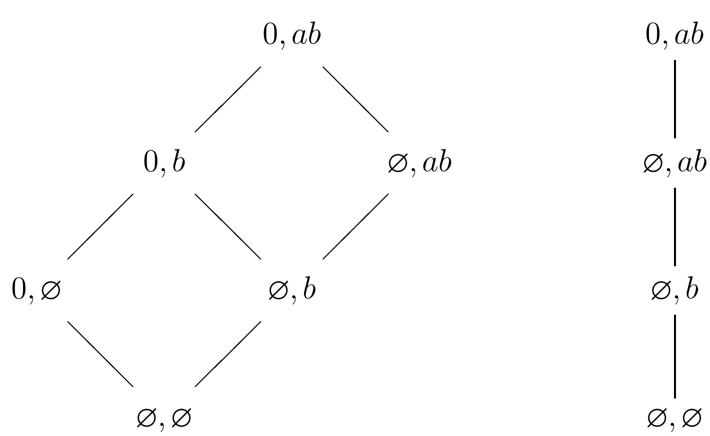
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{0\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a, b\}\}.$$

There are six theory families, but only four theory systems. The action of $\overleftarrow{\ }$ *on theory families is given in the table below.*

T	$\overleftarrow{T}$
$\varnothing, \varnothing$	$\varnothing, \varnothing$
$0, \varnothing$	$\varnothing, \varnothing$
$\varnothing, b$	$\varnothing, b$
$0, b$	$\varnothing, b$
$\varnothing, ab$	$\varnothing, ab$
$0, ab$	$0, ab$

The complete lattice of theory families is shown on the left, whereas that of the theory systems is shown on the right.



To check that this π-institution is roughly systemic, it is only necessary to focus on theory families T for which $\overleftarrow{T} \neq T$. There are two such, namely $T = \{\{0\}, \varnothing\}$ and $T = \{\{0\}, \{b\}\}$. We have (using obvious shorthand):

$$\overleftarrow{\widetilde{0, \varnothing}} = \widetilde{\varnothing, \varnothing} = 0, ab = \widetilde{0, \varnothing};$$
$$\overleftarrow{\widetilde{0, b}} = \widetilde{\varnothing, b} = 0, b = \widetilde{0, b}.$$

Thus, $\mathcal{I}$ is indeed roughly systemic. On the other hand, for the theory $T = \{\{0\}, \{b\}\}$ above, we have $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$ and, moreover, $\overleftarrow{T} = \{\varnothing, \{b\}\} \neq T$. Hence, $\mathcal{I}$ fails to be narrowly systemic.

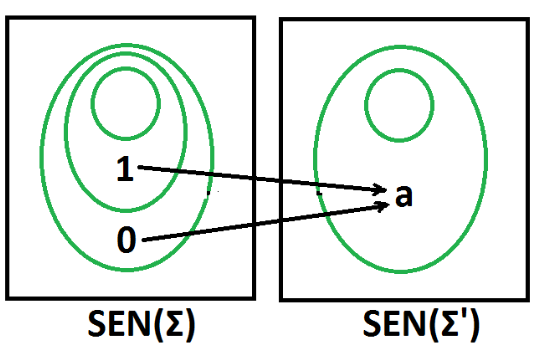
The second example shows that the southeast arrows represent proper inclusions, i.e.,

- The class of systemic π-institutions is a proper subclass of that of narrowly systemic π-institutions;
- The class of roughly systemic π-institutions forms a proper subclass of that of exclusively systemic π-institutions.

It exhibits a π-institution which is narrowly systemic but not roughly systemic.

Example 382 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

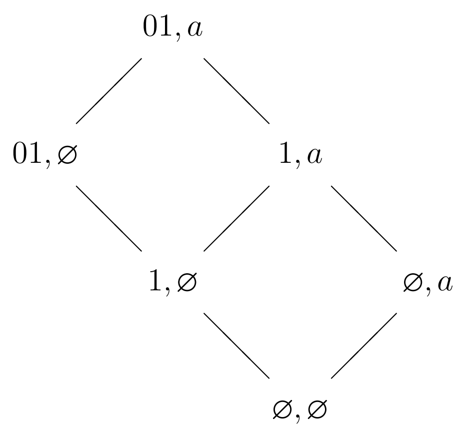
- *$\mathbf{Sign}^\flat$ is the category with objects Σ and Σ' and a unique (non-identity) morphism $f : \Sigma \to \Sigma'$;*
- *$\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ is defined by $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a\}$ and $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = a$;*
- *$N^\flat$ is the trivial clone.*



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{a\}\}.$$

Clearly, there are six theory families in $\mathrm{ThFam}(\mathcal{I})$*, only four of which are theory systems, and only two of which are in* $\mathrm{ThFam}^{\natural}(\mathcal{I})$*. The lattice of theory families is shown in the diagram:*



Since $\mathrm{ThFam}^{\natural}(\mathcal{I}) = \{\{1,a\},\{01,a\}\}$ *and* $\overleftarrow{1,a} = 1,a$ *and* $\overleftarrow{01,a} = 01,a$*, we get that* $\mathcal{I}$ *is narrowly systemic. On the other hand, consider* $T = \{\{1\},\varnothing\}$*. We have*

$$\widetilde{\overleftarrow{1,\varnothing}} = \widetilde{\varnothing,\varnothing} = 01,a \neq 1,a = \widetilde{1,\varnothing},$$

whence, $\overleftarrow{1,\varnothing} \nsim 1,\varnothing$ *and, hence,* $\mathcal{I}$ *is not roughly systemic.*

Finally, it is not difficult to show that rough systemicity implies stability.

Lemma 383 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a π-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly systemic, then it is stable.*

Proof: Suppose $\mathcal{I}$ is roughly systemic and let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, by rough systemicity, $\overleftarrow{T} \sim T$, i.e., $\widetilde{\overleftarrow{T}} = \widetilde{T}$. Therefore, using Proposition 369, we get $\Omega(\overleftarrow{T}) = \Omega(\widetilde{\overleftarrow{T}}) = \Omega(\widetilde{T}) = \Omega(T)$. This shows that $\mathcal{I}$ is stable. ■

6.4 Rough Injectivity

In this section we study classes of π-institutions defined using injectivity properties of the Leibniz operator applied on rough equivalence classes.

Definition 384 (Rough Injectivity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a π-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is called* **roughly family injective** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$\Omega(T) = \Omega(T') \quad \textit{implies} \quad T \sim T';$$

- $\mathcal{I}$ *is called* **roughly left injective** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$\Omega(T) = \Omega(T') \quad \textit{implies} \quad \overleftarrow{T} \sim \overleftarrow{T'}.$$

- $\mathcal{I}$ *is called* **roughly right injective** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'}) \quad \textit{implies} \quad T \sim T'.$$

- $\mathcal{I}$ *is called* **roughly system injective** *if, for all* $T, T' \in \mathrm{ThSys}(\mathcal{I})$,

$$\Omega(T) = \Omega(T') \quad \textit{implies} \quad T \sim T'.$$

Recall that, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, based on an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$, we say that $\mathcal{I}$ is *roughly systemic* if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} \sim T$. In an analog of Lemma 207, we show that rough right injectivity implies rough systemicity and, hence, by Theorem 370, stability.

Lemma 385 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly right injective, then it is roughly systemic.*

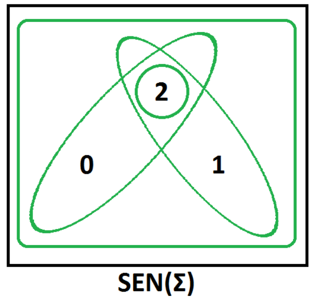
Proof: Suppose that $\mathcal{I}$ is roughly right injective and let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, we have, by Proposition 42, that $\overleftarrow{\overleftarrow{T}} = \overleftarrow{T}$. Therefore, we get $\Omega(\overleftarrow{\overleftarrow{T}}) = \Omega(\overleftarrow{T})$. Hence, by rough right injectivity, we get that $\overleftarrow{T} \sim T$. Hence $\mathcal{I}$ is roughly systemic. ■

We give another example to show that the converse of Lemma 385 does not hold in general. That is, that there exists a roughly systemic π-institution that is not roughly right injective.

Example 386 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with the single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the clone generated by the three unary natural transformations* $\sigma^\flat, \tau^\flat, \rho^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *given by the following table:*

x	$\sigma^\flat_\Sigma(x)$	$\tau^\flat_\Sigma(x)$	$\rho^\flat_\Sigma(x)$
0	0	0	0
1	2	1	0
2	2	1	2

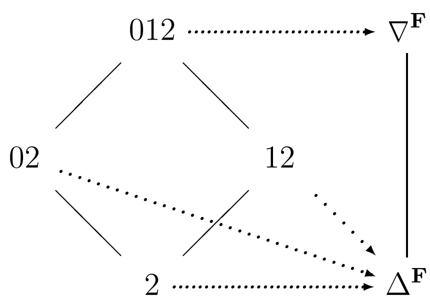


Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}.$$

$\mathcal{I}$ has theorems and, therefore, rough equivalence coincides with the identity relation on $\mathrm{ThFam}(\mathcal{I})$. Moreover, since $\mathbf{Sign}^\flat$ is trivial, $\mathcal{I}$ is systemic. These observations imply that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T \sim \overleftarrow{T}$ and, hence, $\mathcal{I}$ is roughly systemic.

On the other hand, the lattice of theory families of $\mathcal{I}$ and the corresponding Leibniz congruence systems are given in the diagram.



Since

$$\Omega(\overleftarrow{\{\{0,2\}\}}) = \Omega(\{\{0,2\}\}) = \Delta^{\mathbf{F}} = \Omega(\{\{1,2\}\}) = \Omega(\overleftarrow{\{\{1,2\}\}}),$$

whereas $\{\{0,2\}\} \nsim \{\{1,2\}\}$, we get that $\mathcal{I}$ is not roughly right injective.

Next we look into establishing the *rough injectivity hierarchy* of π-institutions. The following relationships can be established between the four rough injectivity classes.

Proposition 387 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) If $\mathcal{I}$ is roughly right injective, then it is roughly family injective;

(b) If $\mathcal{I}$ is roughly family injective, then it is roughly system injective;

(c) If $\mathcal{I}$ is roughly left injective, then it is roughly system injective.

Proof:

(a) Suppose that $\mathcal{I}$ is roughly right injective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. By Lemma 385, $\mathcal{I}$ is roughly systemic, whence $\overleftarrow{T} \sim T$ and $\overleftarrow{T'} \sim T'$. Thus, by Theorem 370, we get

$$\Omega(\overleftarrow{T}) = \Omega(T) = \Omega(T') = \Omega(\overleftarrow{T'}).$$

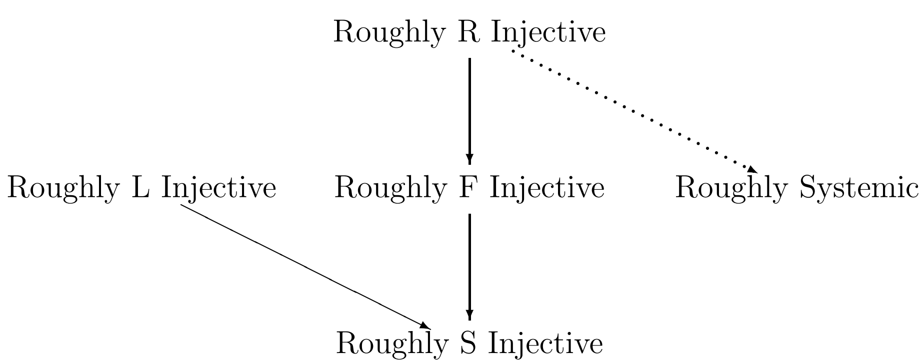
Now applying rough right injectivity gives $T \sim T'$. Hence, $\mathcal{I}$ is roughly family injective.

(b) Suppose that $\mathcal{I}$ is roughly family injective and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, by rough family injectivity, we get $T \sim T'$. Therefore, $\mathcal{I}$ is roughly system injective.

(c) Suppose that $\mathcal{I}$ is roughly left injective and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. By rough left injectivity, we conclude that $\overleftarrow{T} \sim \overleftarrow{T'}$. However, since T, T' are theory systems, we have $\overleftarrow{T} = T$ and $\overleftarrow{T'} = T'$. Hence we get $T \sim T'$ and $\mathcal{I}$ is roughly system injective.

■

We have now established the following **rough injectivity hierarchy** of π-institutions.



We formulate, next, two additional properties concerning the relationships between rough injectivity classes. First, it turns out that the separating property between rough right injectivity and rough system injectivity is exactly rough systemicity.

Proposition 388 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is roughly right injective if and only if it is roughly system injective and roughly systemic.*

Proof: Suppose, first, that $\mathcal{I}$ is roughly right injective. Then, by Lemma 385, it is roughly systemic and by Proposition 387 it is roughly system injective.

Suppose conversely, that $\mathcal{I}$ is roughly system injective and roughly systemic and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$. By rough system injectivity and Proposition 42, we get $\overleftarrow{T} \sim \overleftarrow{T'}$. Hence, by rough systemicity, $T \sim \overleftarrow{T} \sim \overleftarrow{T'} \sim T'$. Thus, $\mathcal{I}$ is roughly right injective. ■

Second, we show that rough system injectivity together with stability imply rough left injectivity.

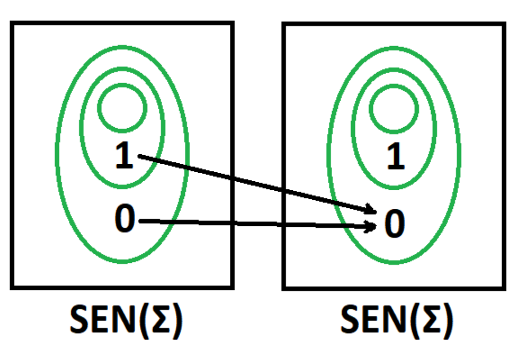
Proposition 389 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is roughly system injective and stable, then it is roughly left injective.*

Proof: Suppose that $\mathcal{I}$ is roughly system injective and stable and consider $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, by stability, $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$. Hence, since $\overleftarrow{T}, \overleftarrow{T'} \in \mathrm{ThSys}(\mathcal{I})$, by rough system injectivity, $\overleftarrow{T} \sim \overleftarrow{T'}$. This shows that $\mathcal{I}$ is roughly left injective. ■

Even though rough left injectivity does imply rough system injectivity, as was shown in Proposition 387, rough left injectivity does not imply stability in general, as is shown in the following example, and, hence, the converse of Proposition 389 fails in general.

Example 390 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

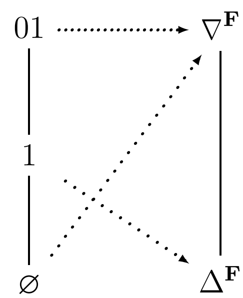
- *$\mathbf{Sign}^\flat$ is the category with the single object Σ and a single (non-identity) morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- *$\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ is defined by $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$ and $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$;*
- *$N^\flat$ is the trivial clone, consisting of the projections only.*



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\}.$$

$\mathcal{I}$ has three theory families $\{\varnothing\}$, $\{\{1\}\}$ and $\{\{0,1\}\}$, but only two theory systems, $\{\varnothing\}$ and $\{\{0,1\}\}$. The lattice of theory families of $\mathcal{I}$ and the corresponding Leibniz congruence systems are given in the diagram.



The only theory families for which $\Omega(T) = \Omega(T')$ are $T = \{\{0,1\}\}$ and $T' = \{\varnothing\}$. For those, we get $\overleftarrow{T} = T \sim T' = \overleftarrow{T'}$. Therefore, $\mathcal{I}$ is roughly left injective. On the other hand, we get $\Omega(\overleftarrow{\{\{1\}\}}) = \Omega(\{\varnothing\}) = \nabla^{\mathbf{F}} \neq \Delta^{\mathbf{F}} = \Omega(\{\{1\}\})$. Therefore, $\mathcal{I}$ is not a stable π-institution.

We now present three examples to show that all inclusions established between rough injectivity classes and depicted in the diagram above are proper inclusions. The first example will show that the class of roughly right injective π-institutions is a proper subclass of the class of roughly family injective π-institutions.

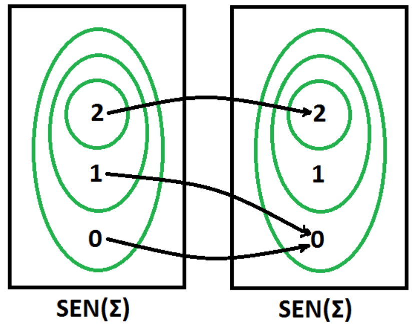
Example 391 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object Σ and a single non-identity morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$;
- *$N^\flat$ is the trivial clone.*

Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by setting $\mathcal{C}_\Sigma = \{\{2\}, \{1,2\}, \{0,1,2\}\}$. Since $\mathcal{I}$ has theorems, rough equivalence on $\mathrm{ThFam}(\mathcal{I})$ *coincides with the identity relation.*

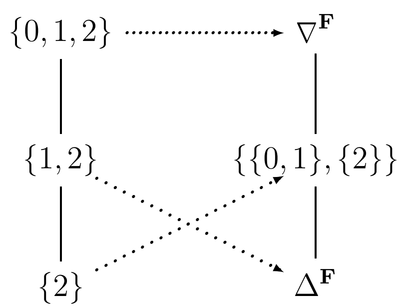
The following table gives the theory families and the theory systems of the π-institution $\mathcal{I}$:

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$



Since $\{\{1,2\}\}$ is a theory family that is not a theory system, $\mathcal{I}$ is not systemic. Thus, rough equivalence being the identity, $\mathcal{I}$ is not roughly systemic and, hence, by Lemma 385, $\mathcal{I}$ is not roughly right injective.

The lattice of theory families and the corresponding Leibniz congruence systems are depicted below:



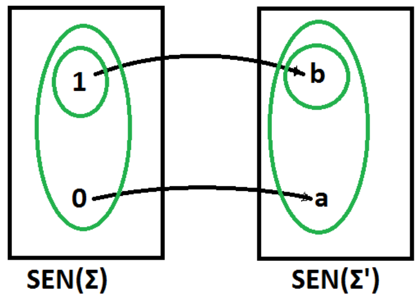
It is obvious from the diagram that $\mathcal{I}$ is family injective, and therefore, rough equivalence being the identity, it is also roughly family injective.

Returning more explicitly to right rough injectivity, note that for $T = \{\{2\}\}$ and $T' = \{\{1,2\}\}$, we have $\overleftarrow{T} = T = \overleftarrow{T'}$, whence $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$, whereas, obviously, $T \neq T'$ and, hence, $T \not\approx T'$.

The second example shows that there exists a roughly left injective π-institution that is not roughly family injective.

Example 392 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- *$\mathbf{Sign}^\flat$ is the category with objects Σ and Σ' and a unique (non-identity) morphism $f : \Sigma \to \Sigma'$;*
- *$\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ is defined by $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ and $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$;*
- *$N^\flat$ is the trivial clone.*



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ by stipulating that

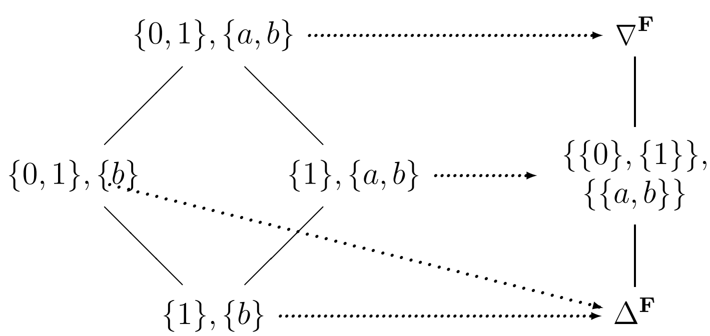
$$\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\},\{a,b\}\}.$$

Again, since $\mathcal{I}$ has theorems, rough equivalence coincides with the identity relation on $\mathrm{ThFam}(\mathcal{I})$*.*

The following table shows the action of $\overleftarrow{\ }$ on theory families, where rows correspond to T_Σ and columns to $T_{\Sigma'}$ and each entry is written as $\overleftarrow{T}_\Sigma, \overleftarrow{T}_{\Sigma'}$.

$\leftarrow$	$\{b\}$	$\{a,b\}$
$\{1\}$	$\{1\},\{b\}$	$\{1\},\{a,b\}$
$\{0,1\}$	$\{1\},\{b\}$	$\{0,1\},\{a,b\}$

The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right:



Since the only two theory families that have the same Leibniz congruence system are $\{\{0,1\},\{b\}\}$ and $\{\{1\},\{b\}\}$ and it holds that

$$\overleftarrow{\{\{0,1\},\{b\}\}} = \overleftarrow{\{\{1\},\{b\}\}} = \{\{1\},\{b\}\},$$

we conclude that $\mathcal{I}$ is left injective. Moreover, since rough equivalence coincides with the identity, $\mathcal{I}$ is also roughly left injective.

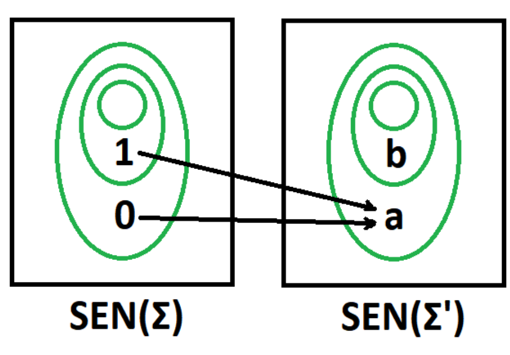
From the diagram, it is also clear that $\mathcal{I}$ is not family injective, since the two theory families $\{\{0,1\},\{b\}\}$ and $\{\{1\},\{b\}\}$ have the same Leibniz *congruence system. The same counterexample, keeping in mind the fact that rough equivalence coincides with the identity, showcases that $\mathcal{I}$ is not roughly family injective either.*

The third example shows that there exists a roughly family injective π-institution that is not roughly left injective. Combined with the preceding example, it has the effect of establishing the following facts:

- The classes of roughly family injective and roughly left injective π-institutions are incomparable. Contrast this with the case of injectivity, where family injectivity implies left injectivity.
- The class of roughly family injective π-institutions is properly contained in the class of roughly system injective π-institutions.
- Similarly, the class of roughly left injective π-institutions is a proper subclass of the class of roughly system injective π-institutions.

Example 393 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects Σ and Σ' and a unique (non-identity) morphism $f : \Sigma \to \Sigma'$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = a$*;*
- $N^\flat$ *is the trivial clone.*



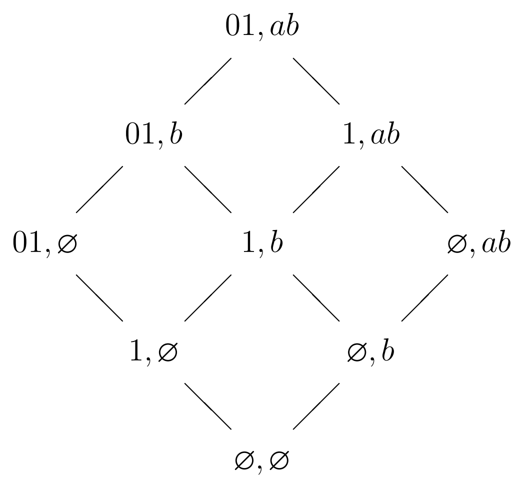
Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a,b\}\}.$$

There are nine theory families, but only five theory systems. The action of $\leftarrow$ *on theory families is given in the table below.*

T	$\overleftarrow{T}$	T	$\overleftarrow{T}$
$\varnothing,\varnothing$	$\varnothing,\varnothing$	$\varnothing,ab$	$\varnothing,ab$
$1,\varnothing$	$\varnothing,\varnothing$	$01,b$	$\varnothing,b$
$\varnothing,b$	$\varnothing,b$	$1,ab$	$1,ab$
$01,\varnothing$	$\varnothing,\varnothing$	$01,ab$	$01,ab$
$1,b$	$\varnothing,b$		

The lattice of theory families of $\mathcal{I}$ *is shown in the diagram.*



We show that $\mathcal{I}$ *is roughly family injective. The following table summarizes the theory families together with their associated Leibniz congruence systems.*

T	$\Omega(T)$
$\{\varnothing,\varnothing\},\{01,\varnothing\},\{\varnothing,ab\},\{01,ab\}$	$\nabla^{\mathbf{F}}$
$\{\varnothing,b\},\{01,b\}$	$\{\nabla^{\mathbf{F}}_{\Sigma},\Delta^{\mathbf{F}}_{\Sigma'}\}$
$\{1,\varnothing\},\{1,ab\}$	$\{\Delta^{\mathbf{F}}_{\Sigma},\nabla^{\mathbf{F}}_{\Sigma'}\}$
$\{1,b\}$	$\Delta^{\mathbf{F}}$

Since, every row on the left column of this table contains roughly equivalent theory families, we conclude that $\mathcal{I}$ *is roughly family injective.*

On the other hand, consider $T = \{1,ab\}$ *and* $T' = \{1,\varnothing\}$*. We have*

$$\overleftarrow{T} = \{1,ab\} \not\sim \{\varnothing,\varnothing\} = \overleftarrow{T'},$$

but

$$\Omega(T) = \Omega(\{1,ab\}) = \{\Delta^{\mathbf{F}}_{\Sigma},\nabla^{\mathbf{F}}_{\Sigma'}\} = \Omega(\{1,\varnothing\}) = \Omega(T').$$

We conclude that $\mathcal{I}$ *is not roughly left injective.*

We now clarify the connections between rough injectivity and injectivity classes. It turns out that membership in an injectivity class implies membership in the corresponding rough injectivity class and, also, possession of theorems. Conversely, membership in a rough injectivity class plus possession of theorems entails membership in the corresponding injectivity class.

Theorem 394 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is right injective if and only if it is roughly right injective and has theorems;*

(b) $\mathcal{I}$ *is family injective if and only if it is roughly family injective and has theorems;*

(c) $\mathcal{I}$ *is left injective if and only if it is roughly left injective and has theorems;*

(d) $\mathcal{I}$ *is system injective if and only if it is roughly system injective and has theorems.*

Proof:

(a) Suppose that $\mathcal{I}$ is right injective. First, note that $\Omega(\overleftarrow{\mathrm{SEN}^\flat}) = \Omega(\mathrm{SEN}^\flat) = \nabla^{\mathbf{F}} = \Omega(\varnothing) = \Omega(\overleftarrow{\varnothing})$. Thus, if $\mathcal{I}$ does not have theorems, $\mathrm{SEN}^\flat = \varnothing$, a contradiction. Therefore, $\mathcal{I}$ has theorems. Second, if $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$, then, by right injectivity, $T = T'$ and, hence, $T \sim T'$. Thus, $\mathcal{I}$ is roughly right injective.

Assume, conversely, that $\mathcal{I}$ is roughly right injective and has theorems. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$. Then, by rough right injectivity, we get $T \sim T'$. On the other hand, since $\mathcal{I}$ has theorems, rough equivalence collapses to the identity relation, whence $T = T'$. Therefore, $\mathcal{I}$ is right injective.

(b) Suppose that $\mathcal{I}$ is family injective. First, note that $\Omega(\mathrm{SEN}^\flat) = \nabla^{\mathbf{F}} = \Omega(\varnothing)$. Thus, if $\mathcal{I}$ does not have theorems, $\mathrm{SEN}^\flat = \varnothing$, a contradiction. Therefore, $\mathcal{I}$ has theorems. Second, if $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$, then, by family injectivity, $T = T'$ and, hence, $T \sim T'$. Thus, $\mathcal{I}$ is roughly family injective.

Assume, conversely, that $\mathcal{I}$ is roughly family injective and has theorems. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, by rough family injectivity, we get $T \sim T'$. On the other hand, since $\mathcal{I}$ has theorems, rough equivalence collapses to the identity relation, whence $T = T'$. Therefore, $\mathcal{I}$ is family injective.

(c) Suppose that $\mathcal{I}$ is left injective. First, note that $\Omega(\overleftarrow{\mathrm{SEN}^\flat}) = \nabla^{\mathbf{F}} = \Omega(\varnothing)$. Thus, if $\mathcal{I}$ does not have theorems, $\mathrm{SEN}^\flat = \overleftarrow{\mathrm{SEN}^\flat} = \overleftarrow{\varnothing} = \varnothing$, a contradiction. Therefore, $\mathcal{I}$ has theorems. Second, if $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$, then, by left injectivity, $\overleftarrow{T} = \overleftarrow{T'}$ and, hence, $\overleftarrow{T} \sim \overleftarrow{T'}$. Thus, $\mathcal{I}$ is roughly left injective.

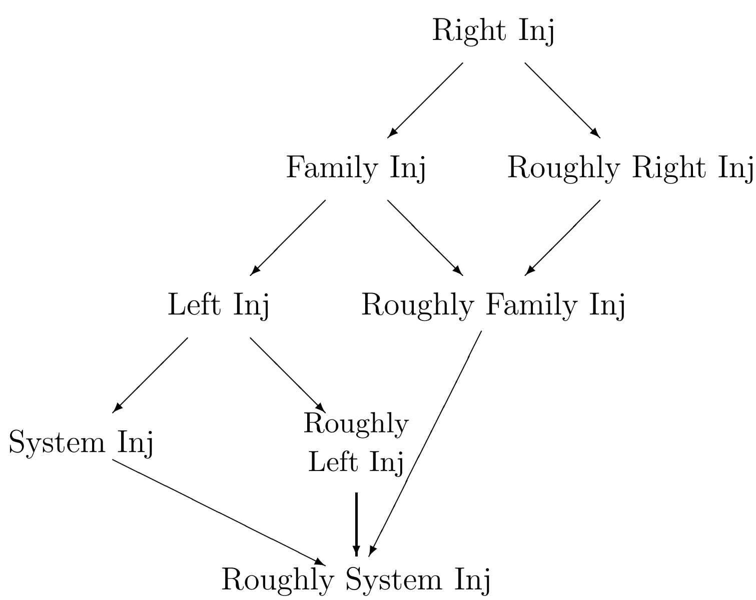
Assume, conversely, that $\mathcal{I}$ is roughly left injective and has theorems. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, by rough left injectivity, we get $\overleftarrow{T} \sim \overleftarrow{T'}$. On the other hand, since $\mathcal{I}$ has theorems, rough equivalence collapses to the identity relation, whence $\overleftarrow{T} = \overleftarrow{T'}$. Therefore, $\mathcal{I}$ is left injective.

(d) Suppose that $\mathcal{I}$ is system injective. First, note that $\Omega(\mathrm{SEN}^\flat) = \nabla^{\mathbf{F}} = \Omega(\varnothing)$. Thus, if $\mathcal{I}$ does not have theorems, $\mathrm{SEN}^\flat = \varnothing$, a contradiction. Therefore, $\mathcal{I}$ has theorems. Second, if $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$, then, by system injectivity, $T = T'$ and, hence, $T \sim T'$. Thus, $\mathcal{I}$ is roughly system injective.

Assume, conversely, that $\mathcal{I}$ is roughly system injective and has theorems. Let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, by rough system injectivity, we get $T \sim T'$. On the other hand, since $\mathcal{I}$ has theorems, rough equivalence collapses to the identity relation, whence $T = T'$. Therefore, $\mathcal{I}$ is system injective.

■

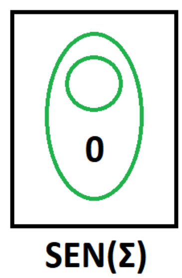
The work in Section 3.6, together with the work done in the present section and Theorem 394, reveal the following hierarchy of injectivity and rough injectivity classes, which was previewed at the beginning of Section 6.2.



To complete the demonstration that all classes in the depicted hierarchy are distinct we provide an example of a π-institution which belongs to all steps in the rough injectivity hierarchy but possesses none of the four (gentle) injectivity properties.

Example 395 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

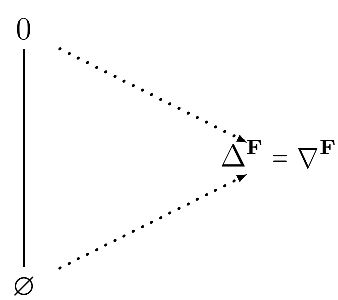
- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0\}$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{0\}\}.$$

$\mathcal{I}$ *is systemic and its lattice of theory families and corresponding Leibniz congruence systems are shown in the diagram.*



$\mathcal{I}$ *is roughly system injective, since* $\Omega(T) = \Omega(T')$ *implies* $T \sim T'$*. Since* $\mathcal{I}$ *is also systemic, it is, a fortiori, roughly systemic and stable. Now, by either direct calculation or based on Propositions 388 and 389, we get that* $\mathcal{I}$ *is also roughly right injective (and, hence, roughly family injective) and roughly left injective, respectively.*

On the other hand, since $\varnothing \neq \{0\}$ *but* $\Omega(\varnothing) = \nabla^{\mathbf{F}} = \Omega(\{0\})$*,* $\mathcal{I}$ *is not system injective and, hence, a fortiori,* $\mathcal{I}$ *has none of the four injectivity properties.*

The rough injectivity properties transfer from the theory families/systems of a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ to all $\mathcal{I}$-filter families/systems on arbitrary $\mathbf{F}$-algebraic systems.

Theorem 396 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is roughly right injective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $\Omega^{\mathcal{A}}(\overleftarrow{T}) = \Omega^{\mathcal{A}}(\overleftarrow{T'})$ *implies* $T \sim T'$*;*

(b) $\mathcal{I}$ *is roughly family injective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$ *implies* $T \sim T'$*;*

(c) $\mathcal{I}$ *is roughly left injective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$ *implies* $\overleftarrow{T} \sim \overleftarrow{T'}$*;*

(d) $\mathcal{I}$ *is roughly system injective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$*,* $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$ *implies* $T \sim T'$*.*

Proof:

(a) The "if" follows by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account that, by Lemma 51, $\mathrm{ThFam}(\mathcal{I}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F})$.

For the "only if", suppose that $\mathcal{I}$ is roughly right injective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(\overleftarrow{T}) = \Omega^{\mathcal{A}}(\overleftarrow{T'})$. Then $\alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T})) = \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'}))$. So, by Proposition 24, $\Omega(\alpha^{-1}(\overleftarrow{T})) = \Omega(\alpha^{-1}(\overleftarrow{T'}))$. Hence, by Lemma 6, $\Omega(\overleftarrow{\alpha^{-1}(T)}) = \Omega(\overleftarrow{\alpha^{-1}(T')})$. Since, by Lemma 51, $\alpha^{-1}(T)$, $\alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$, we get, by applying rough right injectivity, $\alpha^{-1}(T) \sim \alpha^{-1}(T')$. Thus, by Corollary 378, $T \sim T'$.

(b) The "if" follows as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is roughly family injective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$. Then $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) = \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. So, by Proposition 24, $\Omega(\alpha^{-1}(T)) = \Omega(\alpha^{-1}(T'))$. Since, by Lemma 51, $\alpha^{-1}(T)$, $\alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$, we get, by applying rough family injectivity, $\alpha^{-1}(T) \sim \alpha^{-1}(T')$. Thus, by Corollary 378, $T \sim T'$.

(c) The "if" follows as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is roughly left injective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$. Then $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) = \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. So, by Proposition 24, $\Omega(\alpha^{-1}(T)) = \Omega(\alpha^{-1}(T'))$. Since, by Lemma 51, $\alpha^{-1}(T), \alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$, we get, by applying rough left injectivity, $\overleftarrow{\alpha^{-1}(T)} \sim \overleftarrow{\alpha^{-1}(T')}$. Thus, by Lemma 6, $\alpha^{-1}(\overleftarrow{T}) \sim \alpha^{-1}(\overleftarrow{T'})$. Hence, by Corollary 378, $\overleftarrow{T} \sim \overleftarrow{T'}$.

(d) Similar to Part (b).

■

Finally, we may recast the rough injectivity classes in terms of the injectivity of mappings from posets of classes of theory or filter families/systems into posets of congruence systems.

Proposition 397 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The following statements are equivalent:*

(a) $\mathcal{I}$ is roughly family injective;

(b) $\Omega : \widetilde{\mathrm{ThFam}}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is injective;

(c) $\Omega^{\mathcal{A}} : \widetilde{\mathrm{FiFam}}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is injective, for every $\mathbf{F}$-algebraic system $\mathcal{A}$.

Similarly, for system injectivity, we have

Proposition 398 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The following statements are equivalent:*

(a) $\mathcal{I}$ is roughly system injective;

(b) $\Omega : \widetilde{\mathrm{ThSys}}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is injective;

(c) $\Omega^{\mathcal{A}} : \widetilde{\mathrm{FiSys}}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is injective, for every $\mathbf{F}$-algebraic system $\mathcal{A}$.

6.5 Narrow Injectivity

In this section we study classes of π-institutions defined using injectivity properties of the Leibniz operator restricted to $\mathrm{ThFam}^{\natural}(\mathcal{I})$. We call those *narrow injectivity* properties in analogy with the terminology adopted in Section 6.3, differentiating rough systemicity and narrow systemicity, the two strongest properties combining systemicity with rough equivalence.

Definition 399 (Narrow Injectivity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is called* **narrowly family injective** *if, for all* $T, T' \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$*,*
$$\Omega(T) = \Omega(T') \quad \textit{implies} \quad T = T';$$
- $\mathcal{I}$ *is called* **narrowly left injective** *if, for all* $T, T' \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$*,*
$$\Omega(T) = \Omega(T') \quad \textit{implies} \quad \overleftarrow{T} = \overleftarrow{T'}.$$
- $\mathcal{I}$ *is called* **narrowly right injective** *if, for all* $T, T' \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$*,*
$$\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'}) \quad \textit{implies} \quad T = T'.$$
- $\mathcal{I}$ *is called* **narrowly system injective** *if, for all* $T, T' \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$*,*
$$\Omega(T) = \Omega(T') \quad \textit{implies} \quad T = T'.$$

These narrow injectivity properties have the following useful characterizations.

Proposition 400 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is narrowly family injective if and only if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$*,* $\Omega(T) = \Omega(T')$ *implies* $T \sim T'$*;*

(b) $\mathcal{I}$ *is narrowly left injective if and only if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$*,* $\Omega(T) = \Omega(T')$ *implies* $\overleftarrow{\widetilde{T}} = \overleftarrow{\widetilde{T'}}$*;*

(c) $\mathcal{I}$ *is narrowly right injective if and only if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$*,* $\Omega(\overleftarrow{\widetilde{T}}) = \Omega(\overleftarrow{\widetilde{T'}})$ *implies* $T \sim T'$*;*

(d) $\mathcal{I}$ *is narrowly system injective if and only if, for all* $T, T' \in \mathrm{ThSys}(\mathcal{I})$*, such that* $\widetilde{T}, \widetilde{T'} \in \mathrm{ThSys}(\mathcal{I})$*,* $\Omega(T) = \Omega(T')$ *implies* $T \sim T'$*.*

Proof:

(a) Suppose that $\mathcal{I}$ is narrowly family injective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Consider $\widetilde{T}, \widetilde{T'} \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$. By Proposition 369, $\Omega(\widetilde{T}) = \Omega(T) = \Omega(T') = \Omega(\widetilde{T'})$. Thus, by hypothesis, $\widetilde{T} = \widetilde{T'}$, i.e., $T \sim T'$. Therefore, the asserted condition holds.

Assume, conversely, that the asserted condition holds and let $T, T' \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, since $\mathrm{ThFam}^{\lightning}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I})$, we get, by hypothesis, $T \sim T'$, i.e., $\widetilde{T} = \widetilde{T'}$. Since, however, $T, T' \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, we get $T = \widetilde{T} = \widetilde{T'} = T'$. Thus, $\mathcal{I}$ is narrowly family injective.

(b) Suppose that $\mathcal{I}$ is narrowly left injective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then $\widetilde{T}, \widetilde{T'} \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$ and, by Proposition 369, $\Omega(\widetilde{T}) = \Omega(\widetilde{T'})$. Thus, by hypothesis, $\overleftarrow{\widetilde{T}} = \overleftarrow{\widetilde{T'}}$.

Assume, conversely, that the asserted condition holds and let $T, T' \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, by hypothesis, $\overleftarrow{\widetilde{T}} = \overleftarrow{\widetilde{T'}}$. Since, however, $T, T' \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, we get $\overleftarrow{T} = \overleftarrow{\widetilde{T}} = \overleftarrow{\widetilde{T'}} = \overleftarrow{T'}$. Therefore, $\mathcal{I}$ is narrowly left injective.

(c) Suppose that $\mathcal{I}$ is narrowly right injective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(\overleftarrow{\widetilde{T}}) = \Omega(\overleftarrow{\widetilde{T'}})$. Since $\widetilde{T}, \widetilde{T'} \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, we get, by hypothesis, $\widetilde{T} = \widetilde{T'}$, i.e., $T \sim T'$.

Assume, conversely, that the asserted condition holds and let $T, T' \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$. Then, since $T, T' \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, we get $\Omega(\overleftarrow{\widetilde{T}}) = \Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'}) = \Omega(\overleftarrow{\widetilde{T'}})$. Now, by hypothesis, $T \sim T'$, i.e., $\widetilde{T} = \widetilde{T'}$ and, therefore, $T = T'$. We conclude that $\mathcal{I}$ is narrowly right injective.

(d) Suppose $\mathcal{I}$ is narrowly system injective and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T}, \widetilde{T'} \in \mathrm{ThSys}(\mathcal{I})$ and $\Omega(T) = \Omega(T')$. Then $\widetilde{T}, \widetilde{T'} \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$ and, by Proposition 369, $\Omega(\widetilde{T}) = \Omega(T) = \Omega(T') = \Omega(\widetilde{T'})$. Thus, by hypothesis, $\widetilde{T} = \widetilde{T'}$, i.e., $T \sim T'$.

Assume, conversely, that the asserted condition holds and let $T, T' \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, since $T, T' \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$, we get $\widetilde{T} = T, \widetilde{T'} = T' \in \mathrm{ThSys}(\mathcal{I})$ and, therefore, by hypothesis, $T \sim T'$, i.e., $\widetilde{T} = \widetilde{T'}$. But this gives $T = \widetilde{T} = \widetilde{T'} = T'$. Thus, $\mathcal{I}$ is narrowly system injective.

■

It will be shown, next, in an analog of Lemma 385, that narrow right injectivity implies exclusive systemicity. Recall that, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, based on an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$, we say that $\mathcal{I}$ is *exclusively systemic* if, for all $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, such that $\overleftarrow{T} \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$, $\overleftarrow{T} = T$.

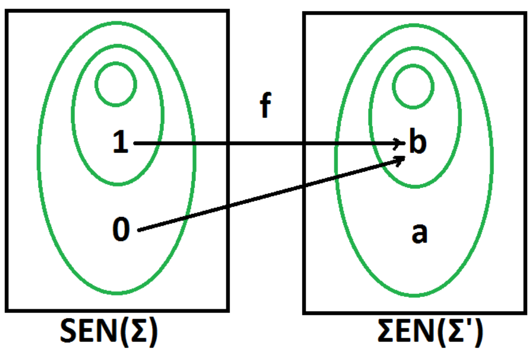
Lemma 401 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is narrowly right injective, then it is exclusively systemic.*

Proof: Assume $\mathcal{I}$ is narrowly right injective and let $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, such that $\overleftarrow{T} \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$. Then, since, by Proposition 2, $\overleftarrow{\overleftarrow{T}} = \overleftarrow{T}$, we have $\Omega(\overleftarrow{\overleftarrow{T}}) = \Omega(\overleftarrow{T})$ and, hence, by narrow right injectivity, $\overleftarrow{T} = T$. Therefore, $\mathcal{I}$ is exclusively systemic. ■

However, as opposed to rough right injectivity, as the next examples demonstrate, narrow right injectivity implies neither rough nor narrow systemicity, in general. The first example showcases a π-institution which is narrowly right injective, but fails to be roughly systemic.

Example 402 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*,* $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = b$*,* $\mathrm{SEN}^\flat(f)(1) = b$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a,b\}\}.$$

Clearly, there are only four theory families in $\mathrm{ThFam}^\natural(\mathcal{I})$*, all of which are theory systems. Their lattice together with the associated Leibniz congruence systems are shown in the diagram:*

$$01, ab \qquad\qquad \nabla^{\mathbf{F}}$$

$$01, b \qquad 1, ab \qquad \nabla^{\mathbf{F}}_\Sigma, \Delta^{\mathbf{F}}_{\Sigma'} \qquad \Delta^{\mathbf{F}}_\Sigma, \nabla^{\mathbf{F}}_{\Sigma'}$$

$$1, b \qquad\qquad \Delta^{\mathbf{F}}$$

From this diagram and the fact that all theory families depicted are theory systems, we can see that, for all $T, T' \in \mathrm{ThFam}^\natural(\mathcal{I})$*,*

$$\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'}) \quad \text{implies} \quad T = T'.$$

Therefore, $\mathcal{I}$ is indeed narrowly right injective.

On the other hand, consider $T = \{\{1\}, \varnothing\}$. Then we have

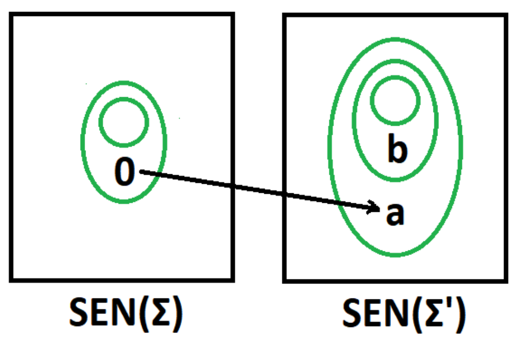
$$\widetilde{\overleftarrow{T}} = \widetilde{\widetilde{\varnothing}} = \mathrm{SEN}^\flat \neq \{\{1\},\{a,b\}\} = \widetilde{T}.$$

This shows that $\overleftarrow{T} \not\approx T$ and, therefore, $\mathcal{I}$ is not roughly systemic.

The next example exhibits a π-institution which is also narrowly right injective, but fails to be narrowly systemic.

Example 403 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- *$\mathbf{Sign}^\flat$ is the category with objects Σ and Σ' and a unique morphism $f : \Sigma \to \Sigma'$;*
- *$\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ is defined by $\mathrm{SEN}^\flat(\Sigma) = \{0\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ and $\mathrm{SEN}^\flat(f)(0) = a$;*
- *$N^\flat$ is the trivial clone.*



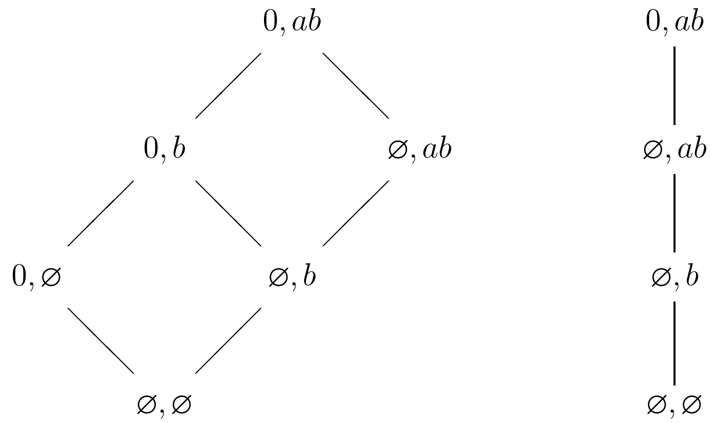
Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\varnothing, \{0\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a,b\}\}.$$

There are six theory families, but only four theory systems. The action of $\overleftarrow{\ }$ on theory families is given in the table below.

T	$\overleftarrow{T}$
$\varnothing, \varnothing$	$\varnothing, \varnothing$
$0, \varnothing$	$\varnothing, \varnothing$
$\varnothing, b$	$\varnothing, b$
$0, b$	$\varnothing, b$
$\varnothing, ab$	$\varnothing, ab$
$0, ab$	$0, ab$

The complete lattice of theory families is shown on the left.



That of the theory systems is shown on the right.

The only theory families in $\mathrm{ThFam}^{\nmid}(\mathcal{I})$ *are* $T = \{\{0\}, \{b\}\}$ *and* $\mathrm{SEN}^\flat$. *Since*

$$\Omega(\overleftarrow{T}) = \Omega(\varnothing, b) = \nabla^{\mathbf{F}}_\Sigma, \Delta^{\mathbf{F}}_\Sigma \neq \nabla^{\mathbf{F}} = \Omega(\mathrm{SEN}^\flat) = \Omega(\overleftarrow{\mathrm{SEN}^\flat}),$$

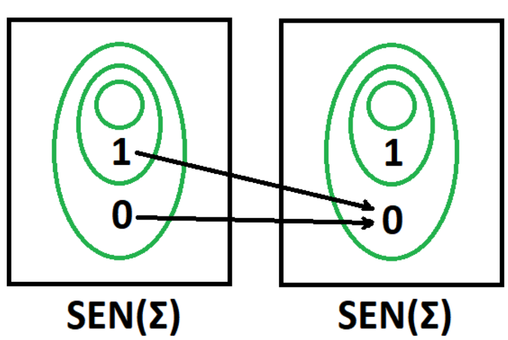
we conclude that $\mathcal{I}$ *is narrowly right injective.*

On the other hand, since $\overleftarrow{T} = \{\varnothing, \{b\}\} \neq T$, $\mathcal{I}$ *is not narrowly systemic.*

The converse of Lemma 401 fails in general. That is, there exists a π-institution which is exclusively systemic but is not narrowly right injective.

Example 404 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

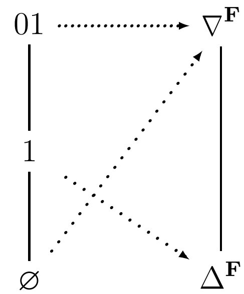
- $\mathbf{Sign}^\flat$ *is the category with the single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$;
- $N^\flat$ *is the trivial clone, consisting of the projections only.*



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\}.$$

$\mathcal{I}$ has three theory families, $\varnothing$, $\{\{1\}\}$ and $\{\{0,1\}\}$, but only two theory systems, $\varnothing$ and $\{\{0,1\}\}$. The lattice of theory families of $\mathcal{I}$ and the corresponding Leibniz congruence systems are given in the diagram.



Since there exists only one theory family T in $\mathrm{ThFam}^{\natural}(\mathcal{I})$, such that $\overleftarrow{T} \in \mathrm{ThSys}^{\natural}(\mathcal{I})$, namely $T = \mathrm{SEN}^\flat$, and $\overleftarrow{\mathrm{SEN}^\flat} = \mathrm{SEN}^\flat$, $\mathcal{I}$ is exclusively systemic. On the other hand, $\{\{1\}\}, \mathrm{SEN}^\flat \in \mathrm{ThFam}^{\natural}(\mathcal{I})$ and

$$\Omega(\overleftarrow{1}) = \Omega(\varnothing) = \nabla^{\mathbf{F}} = \Omega(\mathrm{SEN}^\flat) = \Omega(\overleftarrow{\mathrm{SEN}^\flat}),$$

but $\{\{1\}\} \neq \mathrm{SEN}^\flat$. Therefore, $\mathcal{I}$ fails to be narrowly right injective.

Following a similar vein, we establish a weakened analog of Lemma 207 for narrow right injectivity. This will play a key role in some of the classifications obtained in this and in subsequent sections.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that $\mathcal{I}$ is **narrowly stable** if, for all $T \in \mathrm{ThFam}^{\natural}(\mathcal{I})$, $\Omega(\overleftarrow{T}) = \Omega(T)$. We return to this notion and study it in more detail in Section 7.2.

Lemma 405 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is narrowly right injective, then it is narrowly stable.*

Proof: Suppose that $\mathcal{I}$ is narrowly right injective. Let $T \in \mathrm{ThFam}^{\natural}(\mathcal{I})$. If $T \neq \overleftarrow{T} \in \mathrm{ThFam}^{\natural}(\mathcal{I})$, then, since $\overleftarrow{\overleftarrow{T}} = \overleftarrow{T}$, we would get $\Omega(\overleftarrow{\overleftarrow{T}}) = \Omega(\overleftarrow{T})$ and, hence, by narrow right injectivity, $\overleftarrow{T} = T$, a contradiction. Thus, we get that, for all $T \in \mathrm{ThFam}^{\natural}(\mathcal{I})$,

$$\overleftarrow{T} = T \quad \text{or} \quad \overleftarrow{T} \notin \mathrm{ThFam}^{\natural}(\mathcal{I}).$$

If $T \in \mathrm{ThFam}^{\not\vdash}(\mathcal{I})$ is such that $\overleftarrow{T} = \overline{\varnothing}$, then we would have $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{\mathrm{SEN}^\flat}) = \nabla^{\mathbf{F}}$, whence, by narrow right injectivity, $T = \mathrm{SEN}^\flat$, giving $\overline{\varnothing} = \overleftarrow{T} = \overleftarrow{\mathrm{SEN}^\flat} = \mathrm{SEN}^\flat$, a contradiction. Thus, $\overleftarrow{T} \neq \overline{\varnothing}$. So, there exists $P \in |\mathbf{Sign}^\flat|$, such that $\overleftarrow{T}_P \neq \varnothing$. If for such P, $\overleftarrow{T}_P \neq T_P$, then, setting $T' = \{T'_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, with $T'_\Sigma = \begin{cases} T_\Sigma, & \text{if } \Sigma \neq P \\ \overleftarrow{T}_P, & \text{if } \Sigma = P \end{cases}$, we would get $T, T' \in \mathrm{ThFam}^{\not\vdash}(\mathcal{I})$, with $\overleftarrow{T'} = \overleftarrow{T}$ and $T' \neq T$, contradicting narrow right injectivity. Therefore, for all $T \in \mathrm{ThFam}^{\not\vdash}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\overleftarrow{T}_\Sigma = T_\Sigma \quad \text{or} \quad \overleftarrow{T}_\Sigma = \varnothing.$$

Based on this fact, given $T \in \mathrm{ThFam}^{\not\vdash}(\mathcal{I})$, we partition the signatures into Part I, consisting of $\Sigma \in |\mathbf{Sign}^\flat|$, such that $\overleftarrow{T}_\Sigma = T_\Sigma$, and Part II, consisting of $\Sigma \in |\mathbf{Sign}^\flat|$, such that $\overleftarrow{T}_\Sigma = \varnothing$. Note that no morphism can have a domain of Type I and a codomain of Type II. Thus, letting $T' = \{T'_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, with

$$T'_\Sigma = \begin{cases} T_\Sigma, & \text{if } \Sigma \text{ is of Type I} \\ \mathrm{SEN}^\flat(\Sigma), & \text{if } \Sigma \text{ is of Type II} \end{cases},$$

we get $\overleftarrow{T'}_\Sigma = \overleftarrow{T}_\Sigma = T_\Sigma$, if Σ is of Type I, and, by the displayed condition above, $\overleftarrow{T'}_\Sigma = \varnothing$ or $\mathrm{SEN}^\flat(\Sigma)$, if Σ is of Type II. In either case, it follows by Theorem 370 that $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$, whence, by narrow right injectivity, $T = T'$. We finally conclude that

$$\begin{array}{rcll} \Omega(\overleftarrow{T}) & = & \Omega(\overleftarrow{T'}) & (T = T') \\ & = & \Omega(T') & (\text{Theorem 370}) \\ & = & \Omega(T). & (T = T') \end{array}$$

Therefore, $\mathcal{I}$ is narrowly stable. ■

We establish, next, the narrow injectivity hierarchy. The following proposition forms an analog of Proposition 387, which established the rough injectivity hierarchy.

Proposition 406 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is narrowly right injective, then it is narrowly family injective;*

(b) If $\mathcal{I}$ *is narrowly family injective, then it is narrowly left injective;*

(c) If $\mathcal{I}$ *is narrowly left injective, then it is narrowly system injective.*

Proof:

(a) Suppose that $\mathcal{I}$ is narrowly right injective and let $T,T' \in \mathrm{ThFam}^{\natural}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. By hypothesis and Lemma 405, $\Omega(\overleftarrow{T}) = \Omega(T) = \Omega(T') = \Omega(\overleftarrow{T'})$. By narrow right injectivity, we conclude that $T = T'$. Hence, $\mathcal{I}$ is narrowly family injective.

(b) Suppose that $\mathcal{I}$ is narrowly family injective and let $T,T' \in \mathrm{ThFam}^{\natural}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, by hypothesis, $T = T'$, whence, $\overleftarrow{T} = \overleftarrow{T'}$. Thus, $\mathcal{I}$ is narrowly left injective.

(c) Suppose that $\mathcal{I}$ is narrowly left injective and let $T,T' \in \mathrm{ThSys}^{\natural}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, by hypothesis, we get $\overleftarrow{T} = \overleftarrow{T'}$. Therefore, since T,T' are theory systems, $T = T'$ and, hence, $\mathcal{I}$ is narrowly system injective.

■

We have now established the following **narrow injectivity hierarchy** of π-institutions.

Narrowly R Injective

↓

Narrowly F Injective

↓

Narrowly L Injective

↓

Narrowly S Injective

We give some additional relations governing the hierarchy of narrow injectivity. The following proposition may be viewed as an analog of Propositions 388 and 389.

Proposition 407 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is narrowly system injective and narrowly systemic, then it is narrowly right injective.*

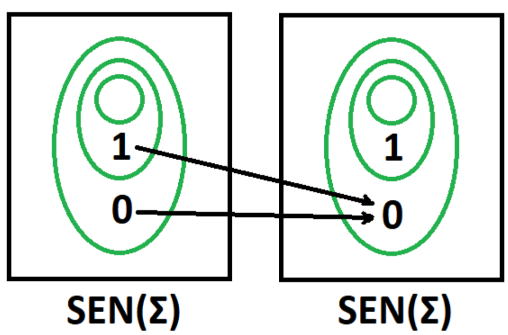
Proof: Suppose $\mathcal{I}$ is narrowly system injective and narrowly systemic. Let $T,T' \in \mathrm{ThFam}^{\natural}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$. By narrow systemicity, $T = \overleftarrow{T}$ and $T' = \overleftarrow{T'}$. Hence, on the one hand, $\Omega(T) = \Omega(T')$ and, on the other, $T,T' \in \mathrm{ThSys}^{\natural}(\mathcal{I})$. Thus, by narrow system injectivity, $T = T'$. Thus, $\mathcal{I}$ is narrowly right injective. ■

It was shown in Example 403 that narrow right injectivity does not imply, in general, narrow systemicity. Thus, the converse of Proposition 407 does not hold in general.

We present three examples to show that all inclusions established between the narrow injectivity classes and shown in the preceding diagram are indeed proper inclusions. The first example depicts a π-institution which is narrowly family injective but not narrowly right injective. This shows that the class of narrowly right injective π-institutions constitutes a proper subclass of the class of narrowly family injective π-institutions.

Example 408 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

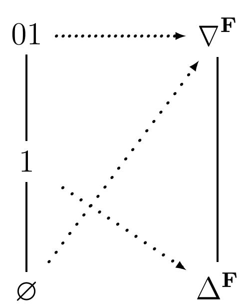
- $\mathbf{Sign}^\flat$ *is the category with the single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$*;*
- $N^\flat$ *is the trivial clone, consisting of the projections only.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0, 1\}\}.$$

$\mathcal{I}$ *has three theory families,* $\varnothing$*,* $\{\{1\}\}$ *and* $\{\{0, 1\}\}$*, but only two theory systems,* $\varnothing$ *and* $\{\{0, 1\}\}$*. The lattice of theory families of* $\mathcal{I}$ *and the corresponding Leibniz congruence systems are given in the diagram.*



Since there exists only two theory families in $\mathrm{ThFam}^{\nmid}(\mathcal{I})$, $\{\{0,1\}\}$ *and* $\{\{1\}\}$, *and* $\Omega(\{\{0,1\}\}) \neq \Omega(\{\{1\}\})$, $\mathcal{I}$ *is trivially narrowly family injective. On the other hand,* $\{\{1\}\}, \{\{0,1\}\} \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$ *and*

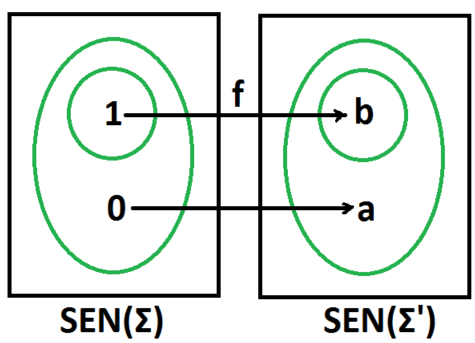
$$\Omega(\overleftarrow{\{\{1\}\}}) = \Omega(\{\varnothing\}) = \nabla^{\mathbf{F}} = \Omega(\{\{0,1\}\}) = \Omega(\overleftarrow{\{\{0,1\}\}}),$$

but $\{\{1\}\} \neq \{\{0,1\}\}$. *Therefore,* $\mathcal{I}$ *fails to be narrowly right injective.*

The next example depicts a π-institution which is narrowly left injective but not narrowly family injective. This shows that the class of narrowly family injective π-institutions constitutes a proper subclass of the class of narrowly left injective π-institutions.

Example 409 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$;
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

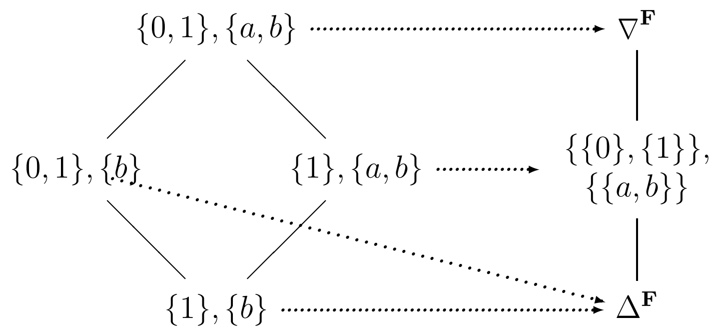
$$\mathcal{C}_\Sigma = \{\{1\}, \{0,1\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\}, \{a,b\}\}.$$

Since $\mathcal{I}$ *has theorems, we get* $\mathrm{ThFam}^{\nmid}(\mathcal{I}) = \mathrm{ThFam}(\mathcal{I})$. *Hence, both narrow family injectivity and narrow left injectivity coincide with family injectivity and left injectivity, respectively.*

The following table shows the action of $\overleftarrow{\ }$ *on theory families, where rows correspond to* T_Σ *and columns to* $T_{\Sigma'}$ *and each entry is written as* $\overleftarrow{T}_\Sigma, \overleftarrow{T}_{\Sigma'}$.

$\leftarrow$	$\{b\}$	$\{a,b\}$
$\{1\}$	$\{1\}, \{b\}$	$\{1\}, \{a,b\}$
$\{0,1\}$	$\{1\}, \{b\}$	$\{0,1\}, \{a,b\}$

The diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right.



Since the only two theory families that have the same Leibniz congruence system are $\{\{0,1\},\{b\}\}$ *and* $\{\{1\},\{b\}\}$ *and it holds that*

$$\overleftarrow{\{\{0,1\},\{b\}\}} = \overleftarrow{\{\{1\},\{b\}\}} = \{\{1\},\{b\}\},$$

we conclude that $\mathcal{I}$ *is left injective. Therefore, taking into account the remark above, we get that* $\mathcal{I}$ *is also narrowly left injective.*

From the diagram, it is also clear that $\mathcal{I}$ *is not family injective, since the two theory families* $\{\{0,1\},\{b\}\}$ *and* $\{\{1\},\{b\}\}$ *have the same Leibniz congruence system. The same counterexample shows that* $\mathcal{I}$ *is not narrowly family injective either.*

We finish the sequence of examples by presenting a narrowly system injective π-institution which, however, fails to be narrowly left injective. This example shows that narrowly left injective π-institutions form a proper subclass of the class of narrowly system injective π-institutions.

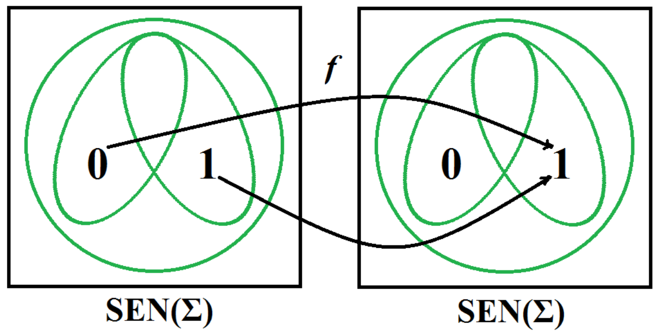
Example 410 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 1$ *and* $\mathrm{SEN}^\flat(f)(1) = 1$*;*
- $N^\flat$ *is the trivial clone.*

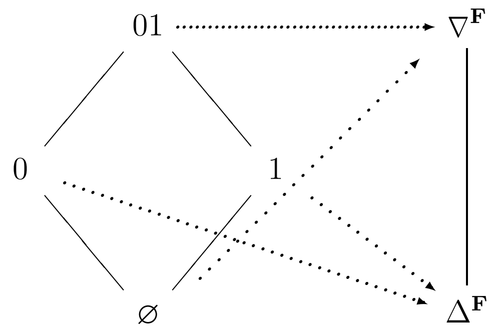
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\varnothing, \{0\}, \{1\}, \{0,1\}\}$*.*

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$*.*

T	$\overleftarrow{T}$
$\varnothing$	$\varnothing$
$\{0\}$	$\varnothing$
$\{1\}$	$\{1\}$
$\{0,1\}$	$\{0,1\}$



The lattice of theory families and the corresponding Leibniz congruence systems are depicted below.



It is obvious from the diagram that for no $T,T' \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, *such that* $T \neq T'$ *is it the case that* $\Omega(T) = \Omega(T')$. *Therefore,* $\mathcal{I}$ *is trivially narrowly system injective. On the other hand, for* $T = \{\{0\}\}$, $T' = \{\{1\}\}$, *both members of* $\mathrm{ThFam}^{\nmid}(\mathcal{I})$, *we have* $\Omega(T) = \Omega(T') = \Delta^{\mathbf{F}}$, *whereas* $\overleftarrow{T} = \{\varnothing\} \neq \{\{1\}\} = \overleftarrow{T'}$. *Therefore,* $\mathcal{I}$ *fails to be narrowly left injective.*

We turn now to the relationships between corresponding classes of the rough injectivity and the narrow injectivity hierarchies.

First, it is easy to see, using the characterization in Part (a) of Proposition 400 that the two types of family injectivity involved coincide.

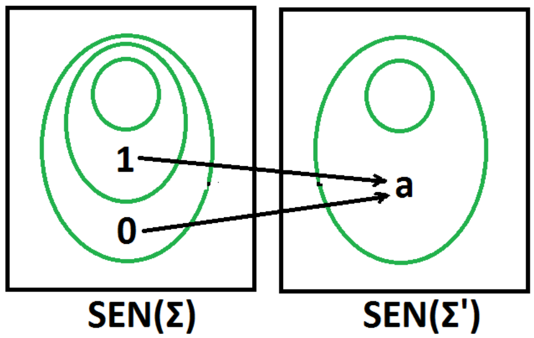
Corollary 411 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is roughly family injective if and only if it is narrowly family injective.*

Proof: Part (a) of Proposition 400. ■

Unfortunately, the relationship between the remaining classes are not so straightforward, due to the necessity of investigating the mode of interaction between rough equivalence and the $\leftarrow$ operator. We look, next, at the two classes of left injective π-institutions. We start by showing that the class of narrow left injective π-institutions is not included in the class of roughly left injective π-institutions. The next example exhibits a π-institution which is narrowly left injective but not roughly left injective.

Example 412 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

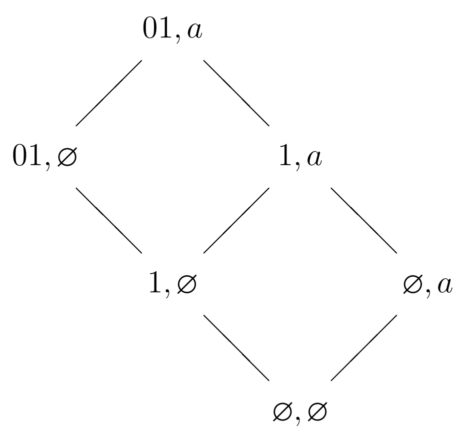
- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a\}$ *and* $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = a$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{a\}\}.$$

Clearly, there are six theory families in $\mathrm{ThFam}(\mathcal{I})$*, only four of which are theory systems, and only two of which are in* $\mathrm{ThFam}^{\natural}(\mathcal{I})$*. The lattice of theory families is shown in the diagram:*



Since $\mathrm{ThFam}^{\natural}(\mathcal{I}) = \{\{1,a\},\{01,a\}\}$ *and* $\Omega(1,a) = \{\Delta^{\mathbf{F}}_\Sigma, \nabla^{\mathbf{F}}_{\Sigma'}\} \neq \nabla^{\mathbf{F}} = \Omega(01,a)$*, it follows that* $\mathcal{I}$ *is trivially narrowly left injective.*

On the other hand, consider $T = \{1,\varnothing\}$ *and* $T' = \{1,a\}$*. We have* $\Omega(1,\varnothing) = \{\Delta^{\mathbf{F}}_\Sigma, \nabla^{\mathbf{F}}_{\Sigma'}\} = \Omega(1,a)$*, but*

$$\overleftarrow{\widetilde{1,\varnothing}} = \widetilde{\varnothing,\varnothing} = 01,a \neq 1,a = \widetilde{1,a} = \overleftarrow{\widetilde{1,a}}.$$

This proves that $\mathcal{I}$ *is not roughly left injective.*

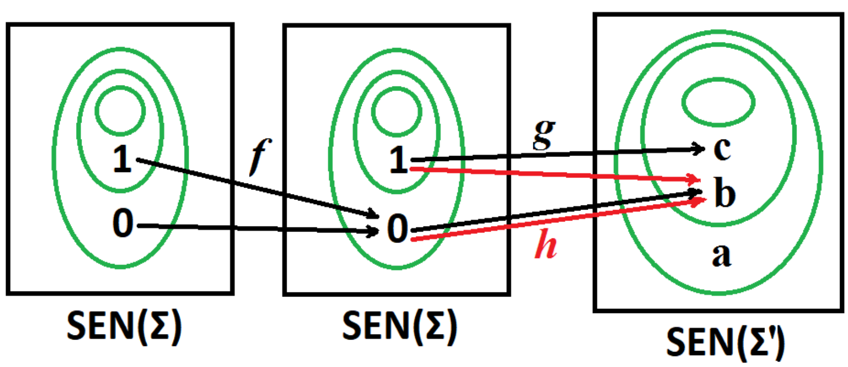
We now look at a π-institution that is roughly left injective, while it fails to be narrowly left injective. Combined with Example 412, this will show that the two left injectivity classes, rough and narrow, are incomparable from the point of view of inclusion.

Example 413 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and three nonidentity morphisms* $f : \Sigma \to \Sigma$ *and* $g, h : \Sigma \to \Sigma'$*, such that* $f \circ f = f$*,* $g \circ f = h$ *and* $h \circ f = h$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$*,* $\mathrm{SEN}^\flat(\Sigma') = \{a, b, c\}$*,* $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = 0$*,* $\mathrm{SEN}^\flat(g)(0) = b$*,* $\mathrm{SEN}^\flat(g)(1) = c$ *and* $\mathrm{SEN}^\flat(h)(0) = \mathrm{SEN}^\flat(h)(1) = b$*;*
- $N^\flat$ *is the clone generated by a single binarry natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$*, whose components are defined by the following tables:*

$\sigma^\flat_\Sigma$	0	1
0	0	1
1	1	1

$\sigma^\flat_{\Sigma'}$	a	b	c
a	a	a	c
b	a	b	c
c	c	c	c



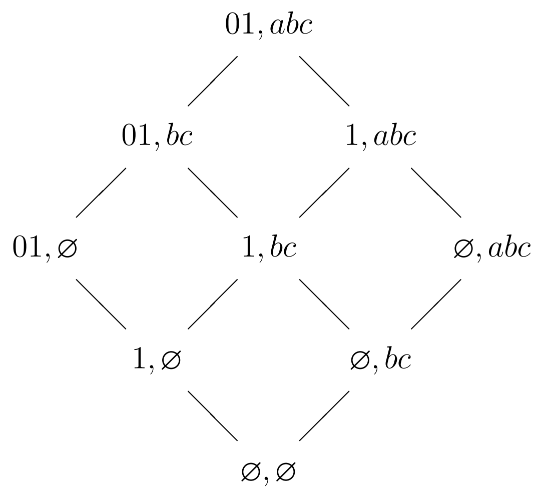
It is not difficult, albeit slightly tedious, to check that this is a well-defined natural transformation. We summarize the checking in the accompanying table.

(x, y)	$f(\sigma^\flat_\Sigma(x, y)) = \sigma^\flat_\Sigma(f(x), f(y))$	$g(\sigma^\flat_\Sigma(x, y)) = \sigma^\flat_{\Sigma'}(g(x), g(y))$	$h(\sigma^\flat_\Sigma(x, y)) = \sigma^\flat_{\Sigma'}(h(x), h(y))$
$(0, 0)$	$0 = 0$	$b = b$	$b = b$
$(0, 1)$	$0 = 0$	$c = c$	$b = b$
$(1, 0)$	$0 = 0$	$c = c$	$b = b$
$(1, 1)$	$0 = 0$	$c = c$	$b = b$

Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b,c\}, \{a,b,c\}\}.$$

Clearly, there are nine theory families in $\mathrm{ThFam}(\mathcal{I})$*, five of which are theory systems, and four of which are in* $\mathrm{ThFam}^{\sharp}(\mathcal{I})$*. The lattice of theory families is shown in the diagram:*



The action of $\overleftarrow{\ }$ *on theory families is given in the following table.*

T	$\overleftarrow{T}$	T	$\overleftarrow{T}$
$01, abc$	$01, abc$	$\varnothing, abc$	$\varnothing, abc$
$01, bc$	$01, bc$	$1, \varnothing$	$\varnothing, \varnothing$
$1, abc$	$\varnothing, abc$	$\varnothing, bc$	$\varnothing, bc$
$01, \varnothing$	$\varnothing, \varnothing$	$\varnothing, \varnothing$	$\varnothing, \varnothing$
$1, bc$	$\varnothing, bc$		

The table below provides the Leibniz congruence systems associated with the theory families of $\mathcal{I}$*.*

T	$\Omega(T)$
$\{01, abc\}, \{01, \varnothing\}, \{\varnothing, abc\}, \{\varnothing, \varnothing\}$	$\nabla^{\mathbf{F}}$
$\{1, abc\}, \{1, \varnothing\}$	$\{\Delta^{\mathbf{F}}_\Sigma, \nabla^{\mathbf{F}}_{\Sigma'}\}$
$\{01, bc\}, \{1, bc\}, \{\varnothing, bc\}$	$\Delta^{\mathbf{F}}$

To see that $\mathcal{I}$ *is roughly left injective, note that all elements in a single row of the table have associated theory systems that are roughly equivalent.*

$$\overleftarrow{\widetilde{\{01, abc\}}} = \overleftarrow{\widetilde{\{01, \varnothing\}}} = \overleftarrow{\widetilde{\{\varnothing, abc\}}} = \overleftarrow{\widetilde{\{\varnothing, \varnothing\}}} = \{01, abc\};$$
$$\overleftarrow{\widetilde{\{1, abc\}}} = \overleftarrow{\widetilde{\{1, \varnothing\}}} = \{01, abc\};$$
$$\overleftarrow{\widetilde{\{01, bc\}}} = \overleftarrow{\widetilde{\{1, bc\}}} = \overleftarrow{\widetilde{\{\varnothing, bc\}}} = \{01, bc\}.$$

But $\mathcal{I}$ *is not narrowly left injective. In fact, setting* $T = \{1, bc\}$ *and* $T' = \{01, bc\}$*, we get* $\Omega(T) = \Omega(T') = \Delta^{\mathbf{F}}$*, whereas* $\overleftarrow{T} = \{\varnothing, bc\} \neq \{01, bc\} = \overleftarrow{T'}$*.*

We turn, next to the relationship between the two kinds of right injectivity. We show, first, that rough right injectivity implies narrow right injectivity.

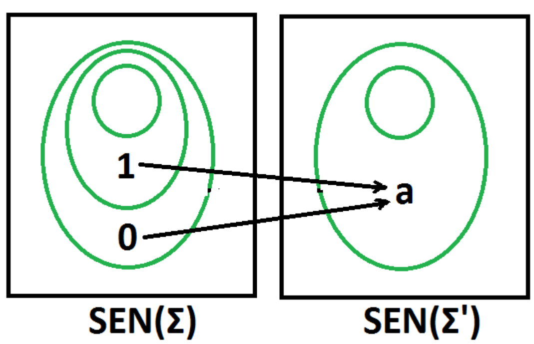
Proposition 414 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly right injective, then it is narrowly right injective.*

Proof: Suppose $\mathcal{I}$ is roughly right injective and let $T, T' \in \mathrm{ThFam}^{\not\sim}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$. By rough right injectivity, we get that $T \sim T'$, i.e., that $\widetilde{T} = \widetilde{T'}$. Since, however, $T, T' \in \mathrm{ThFam}^{\not\sim}(\mathcal{I})$, we get $T = \widetilde{T} = \widetilde{T'} = T'$. Therefore, $\mathcal{I}$ is narrowly right injective. ∎

The converse, on the other hand, does not hold in general, as the following example demonstrates.

Example 415 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

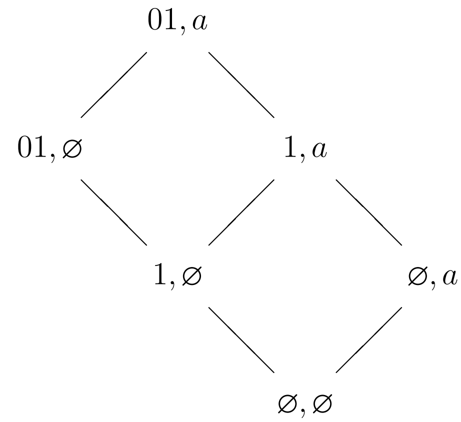
- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$*,* $\mathrm{SEN}^\flat(\Sigma') = \{a\}$ *and* $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = a$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$C_\Sigma = \{\varnothing, \{1\}, \{0, 1\}\} \quad \textit{and} \quad C_{\Sigma'} = \{\varnothing, \{a\}\}.$$

Clearly, there are six theory families in $\mathrm{ThFam}(\mathcal{I})$*, only four of which are theory systems and only two of which are in* $\mathrm{ThFam}^{\not\sim}(\mathcal{I})$*. The lattice of* *theory families is shown in the diagram.*



The only theory families in $\mathrm{ThFam}^{\nmid}(\mathcal{I})$ *are* $\{1,a\}$ *and* $\{01,a\}$*. Moreover,*

$$\Omega(\overleftarrow{\{1,a\}}) = \Omega(\{1,a\}) = \{\Delta^{\mathbf{F}}_{\Sigma}, \nabla^{\mathbf{F}}_{\Sigma}\} \neq \nabla^{\mathbf{F}} = \Omega(\{01,a\}) = \Omega(\overleftarrow{\{01,a\}}).$$

Thus, $\mathcal{I}$ *is trivially narrowly right injective.*

On the other hand, letting $T = \{1,\varnothing\}$ *and* $T' = \{01,\varnothing\}$*, we get*

$$\Omega(\overleftarrow{T}) = \Omega(\{\varnothing,\varnothing\}) = \nabla^{\mathbf{F}} = \Omega(\{\varnothing,\varnothing\}) = \Omega(\overleftarrow{T'}),$$

but, clearly, $\widetilde{T} = \{1,a\} \neq \{01,a\} = \widetilde{T'}$*, i.e.,* $T \nsim T'$*. Therefore,* $\mathcal{I}$ *is not roughly right injective.*

Finally, we look at system injectivity. Again, it turns out that rough system injectivity implies narrow system injectivity. However, the converse fails in general.

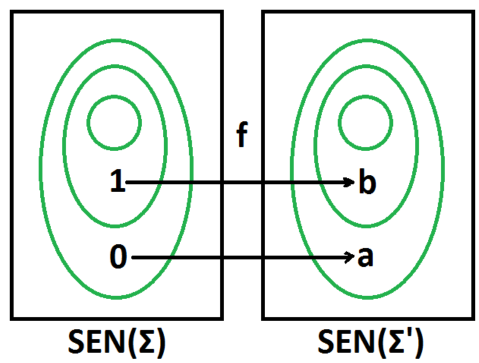
Proposition 416 *Let* $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat} \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly system injective, then it is narrowly system injective.*

Proof: Suppose $\mathcal{I}$ is roughly system injective and let $T, T' \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, by rough system injectivity, $T \sim T'$, i.e., $\widetilde{T} = \widetilde{T'}$. However, since $T, T' \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, we get $T = \widetilde{T} = \widetilde{T'} = T'$. Therefore, $\mathcal{I}$ is narrowly system injective. ■

And now we present an example of a π-institution that is narrowly system injective but not roughly system injective. This, combined with Proposition 416, shows that the class of narrowly system injective π-institutions properly contains the class of roughly system injective π-institutions.

Example 417 *Let* $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat} \rangle$ *be the algebraic system defined as follows:*

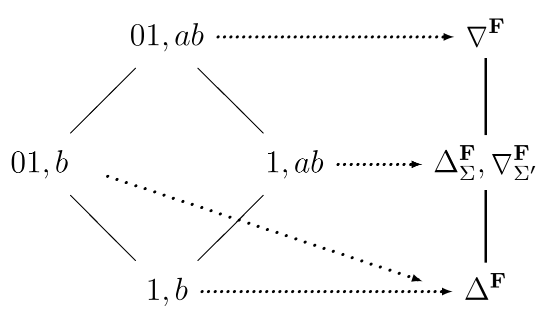
- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique morphism* $f : \Sigma \to \Sigma'$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$;
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a,b\}\}.$$

There are only four theory families in $\mathrm{ThFam}^\natural(\mathcal{I})$, *all of which except for* $\{01, b\}$ *are theory systems. Their lattice together with the associated Leibniz congruence systems are shown in the diagram:*

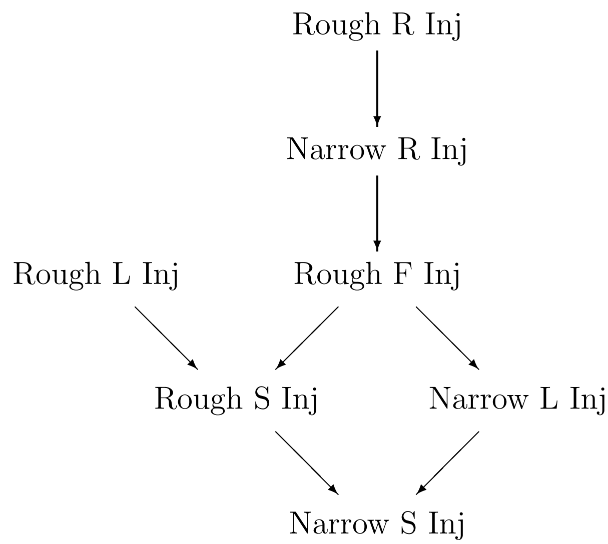


From this diagram we see that for no $T, T' \in \mathrm{ThSys}^\natural(\mathcal{I})$, *with* $T \neq T'$ *is it the case that* $\Omega(T) = \Omega(T')$. *Therefore,* $\mathcal{I}$ *is trivially narrowly system injective.*

On the other hand, consider $T = \{1, b\}$, $T' = \{\varnothing, b\} \in \mathrm{ThSys}(\mathcal{I})$. *Even though* $T \nleq T'$, *we have* $\Omega(T) = \Delta^{\mathbf{F}} = \Omega(T')$. *Hence,* $\mathcal{I}$ *is not roughly system injective.*

The results obtained and the counterexamples presented, thus far, reveal the following mixed hierarchy of rough and narrow injectivity classes of π-

institutions.



A theorem, analogous to Theorem 394 asserts that ordinary injectivity is equivalent to narrow injectivity in the presence of theorems. This holds for all four injectivity classes.

Theorem 418 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is right injective if and only if it is narrowly right injective and has theorems;*

(b) $\mathcal{I}$ *is family injective if and only if it is narrowly family injective and has theorems;*

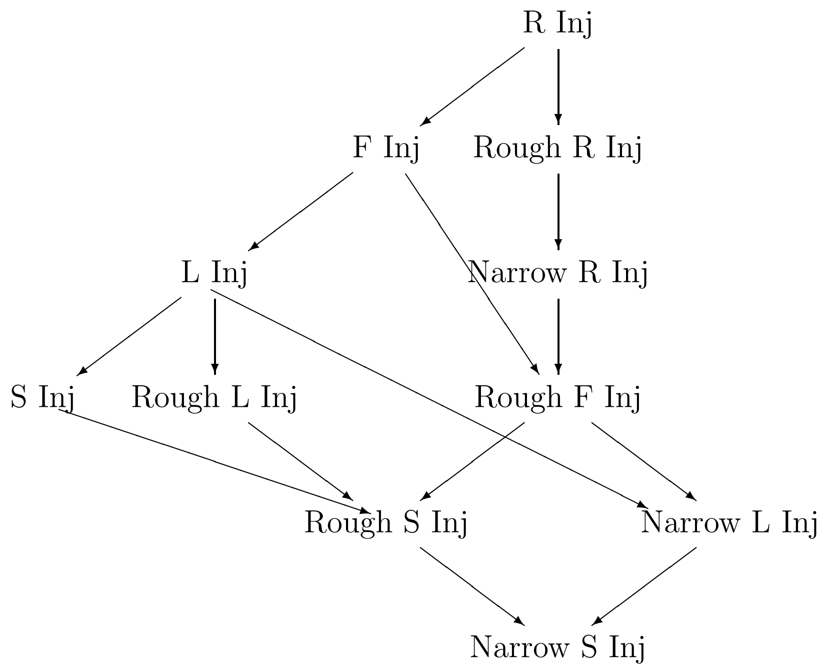
(c) $\mathcal{I}$ *is left injective if and only if it is narrowly left injective and has theorems;*

(d) $\mathcal{I}$ *is system injective if and only if it is narrowly system injective and has theorems.*

Proof: By Theorem 394, if $\mathcal{I}$ has one of the four injectivity properties, then it has theorems. Moreover, by the same theorem, an injectivity property implies the corresponding rough injectivity property and, by Corollary 411, Proposition 414 and Proposition 416, each implies the corresponding narrow injectivity property except in the case of left injectivity, where (as actually in all other cases, as well) one can easily see directly, that left injectivity implies narrow left injectivity, since the defining condition of the latter is a specialization of that of the former.

All converses are also easily verified, since, in the presence of theorems, $\mathrm{ThFam}^{\nleq}(\mathcal{I}) = \mathrm{ThFam}(\mathcal{I})$ and $\mathrm{ThSys}^{\nleq}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$, which makes the four defining conditions for the narrow classes identical with the corresponding conditions for the ordinary injectivity classes. ∎

We now have the following hierarchy.



The narrow injectivity properties transfer from the theory families/systems of a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ to all $\mathcal{I}$-filter families/systems on arbitrary $\mathbf{F}$-algebraic systems. This result forms an analog of Theorem 396, which applied to rough injectivity classes.

Theorem 419 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is narrowly right injective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}\natural}(\mathcal{A})$*,* $\Omega^{\mathcal{A}}(\overleftarrow{T}) = \Omega^{\mathcal{A}}(\overleftarrow{T'})$ *implies* $T = T'$*;*

(b) $\mathcal{I}$ *is narrowly family injective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}\natural}(\mathcal{A})$*,* $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$ *implies* $T = T'$*;*

(c) $\mathcal{I}$ *is narrowly left injective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}\natural}(\mathcal{A})$*,* $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$ *implies* $\overleftarrow{T} = \overleftarrow{T'}$*;*

(d) $\mathcal{I}$ *is narrowly system injective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiSys}^{\mathcal{I}\natural}(\mathcal{A})$*,* $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$ *implies* $T = T'$*.*

Proof: The proof follows the steps of the proofs of the various parts of Theorem 214, but, in addition, it takes into account Lemma 376. We do Part (a) in detail to give a flavor of what is involved.

The "if" follows by considering the **F**-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account that $\mathrm{ThFam}^{\nmid}(\mathcal{I}) = \mathrm{FiFam}^{\mathcal{I}\nmid}(\mathcal{F})$, by Lemmas 51 and 376.

For the "only if", suppose that $\mathcal{I}$ is narrowly right injective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an **F**-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}\nmid}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(\overleftarrow{T}) = \Omega^{\mathcal{A}}(\overleftarrow{T'})$. Then $\alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T})) = \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'}))$. So, by Proposition 24, $\Omega(\alpha^{-1}(\overleftarrow{T})) = \Omega(\alpha^{-1}(\overleftarrow{T'}))$. Hence, by Lemma 6, $\Omega(\overleftarrow{\alpha^{-1}(T)}) = \Omega(\overleftarrow{\alpha^{-1}(T')})$. Since, by Lemmas 51 and 376, $\alpha^{-1}(T), \alpha^{-1}(T') \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, we get, by applying narrow right injectivity, $\alpha^{-1}(T) = \alpha^{-1}(T')$. This yields, taking into account the surjectivity of $\langle F, \alpha \rangle$, $T = T'$. ■

We finally recast narrow injectivity in terms of the injectivity of mappings from posets of theory or filter families/systems into posets of congruence systems. The following results form, roughly, analogs of Propositions 397 and 398, respectively, except that special attention must be paid to the fact that neither $\mathrm{ThFam}^{\nmid}(\mathcal{I})$ nor $\mathrm{FiFam}^{\mathcal{I}\nmid}(\mathcal{A})$ is necessarily a lattice.

Proposition 420 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* **F**. *The following statements are equivalent:*

(a) $\mathcal{I}$ *is narrowly family injective;*

(b) $\Omega : \mathrm{ThFam}^{\nmid}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is injective;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}\nmid}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is injective, for every* **F***-algebraic system* $\mathcal{A}$.

Similarly, for system injectivity, we have

Proposition 421 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* **F**. *The following statements are equivalent:*

(a) $\mathcal{I}$ *is narrowly system injective;*

(b) $\Omega : \mathrm{ThSys}^{\nmid}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is injective;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}\nmid}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is injective, for every* **F***-algebraic system* $\mathcal{A}$.

6.6 Rough Reflectivity

In this section we study classes of π-institutions defined using reflectivity properties of the Leibniz operator applied on rough equivalence classes.

Definition 422 (Rough Reflectivity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is called* **roughly family reflective** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,
$$\Omega(T) \le \Omega(T') \quad \textit{implies} \quad \widetilde{T} \le \widetilde{T'}.$$
- $\mathcal{I}$ *is called* **roughly left reflective** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,
$$\Omega(T) \le \Omega(T') \quad \textit{implies} \quad \widetilde{\overleftarrow{T}} \le \widetilde{\overleftarrow{T'}}.$$
- $\mathcal{I}$ *is called* **roughly right reflective** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,
$$\Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'}) \quad \textit{implies} \quad \widetilde{T} \le \widetilde{T'}.$$
- $\mathcal{I}$ *is called* **roughly system reflective** *if, for all* $T, T' \in \mathrm{ThSys}(\mathcal{I})$,
$$\Omega(T) \le \Omega(T') \quad \textit{implies} \quad \widetilde{T} \le \widetilde{T'}.$$

In a partial analog of Lemma 218, we show that rough right reflectivity implies rough systemicity and, hence, by Theorem 370, stability.

Lemma 423 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly right reflective, then it is roughly systemic.*

Proof: Suppose that $\mathcal{I}$ is roughly right reflective and let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, we have, by Proposition 42, $\overleftarrow{\overleftarrow{T}} = \overleftarrow{T}$. Therefore, we get $\Omega(\overleftarrow{\overleftarrow{T}}) = \Omega(\overleftarrow{T})$. Hence, by rough right reflectivity, we get that $\widetilde{\overleftarrow{T}} = \widetilde{T}$, i.e., $\overleftarrow{T} \sim T$. Hence $\mathcal{I}$ is roughly systemic. ■

Next we look into establishing the *rough reflectivity hierarchy* of π-institutions. The following relationships can be established between the four rough reflectivity classes.

Proposition 424 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is roughly right reflective, then it is roughly family reflective;*

(b) If $\mathcal{I}$ *is roughly family reflective, then it is roughly system reflective;*

(c) If $\mathcal{I}$ *is roughly left reflective, then it is roughly system reflective.*

Proof:

(a) Suppose that $\mathcal{I}$ is roughly right reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \le \Omega(T')$. By Lemma 423, $\mathcal{I}$ is roughly systemic, whence $\overleftarrow{T} \sim T$ and $\overleftarrow{T'} \sim T'$. Thus, by Theorem 370, we get

$$\Omega(\overleftarrow{T}) = \Omega(T) \le \Omega(T') = \Omega(\overleftarrow{T'}).$$

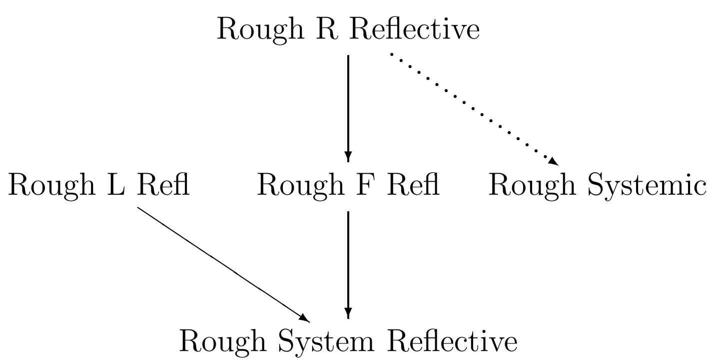
Now applying rough right reflectivity, we get $\widetilde{T} \le \widetilde{T'}$. This proves that $\mathcal{I}$ is roughly family reflective.

(b) Suppose that $\mathcal{I}$ is roughly family reflective and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\Omega(T) \le \Omega(T')$. Then, by rough family reflectivity, we get $\widetilde{T} \le \widetilde{T'}$, whence, $\mathcal{I}$ is roughly system reflective.

(c) Suppose that $\mathcal{I}$ is roughly left reflective and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\Omega(T) \le \Omega(T')$. By rough left reflectivity, we conclude that $\widetilde{\overleftarrow{T}} \le \widetilde{\overleftarrow{T'}}$. However, since T, T' are theory systems, we have $\overleftarrow{T} = T$ and $\overleftarrow{T'} = T'$. Hence we get $\widetilde{T} \le \widetilde{T'}$ and $\mathcal{I}$ is roughly system reflective.

■

We have now established the following **rough reflectivity hierarchy** of π-institutions.



We formulate two additional properties concerning the relationships between rough reflectivity classes. First, rough right reflectivity turns out to be equivalent to rough system reflectivity combined with rough systemicity.

Proposition 425 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *$\mathcal{I}$ is roughly right reflective if and only if it is roughly system reflective and roughly systemic.*

Proof: Suppose, first, that $\mathcal{I}$ is roughly right reflective. Then, by Lemma 423, it is roughly systemic and by Proposition 424 it is roughly system reflective.

Suppose, conversely, that $\mathcal{I}$ is roughly system reflective and roughly systemic and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'})$. By rough system reflectivity and Proposition 42, we get $\widetilde{\overleftarrow{T}} \le \widetilde{\overleftarrow{T'}}$. Hence, by rough systemicity, $\widetilde{T} = \widetilde{\overleftarrow{T}} \le \widetilde{\overleftarrow{T'}} = \widetilde{T'}$. Thus, $\mathcal{I}$ is roughly right reflective. ■

Second, we show that rough system reflectivity together with stability imply rough left reflectivity.

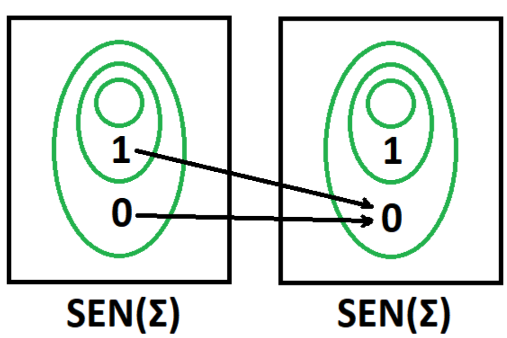
Proposition 426 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly system reflective and stable, then it is roughly left reflective.*

Proof: Suppose that $\mathcal{I}$ is roughly system reflective and stable and consider $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \le \Omega(T')$. Then, by stability $\Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'})$. Hence, since $\overleftarrow{T}, \overleftarrow{T'} \in \mathrm{ThSys}(\mathcal{I})$, by rough system reflectivity, $\widetilde{\overleftarrow{T}} \le \widetilde{\overleftarrow{T'}}$. This shows that $\mathcal{I}$ is roughly left reflective. ■

We present three examples to show that all inclusions established between rough reflectivity classes and depicted in the diagram above are proper inclusions. The first example will show that the class of roughly right reflective π-institutions is a proper subclass of the class of roughly family reflective π-institutions.

Example 427 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

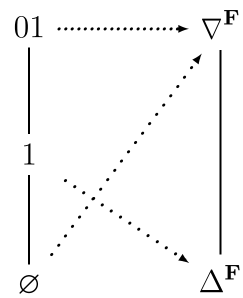
- $\mathbf{Sign}^\flat$ *is the category with the single object* Σ *and a single (no-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0, 1\}\}.$$

$\mathcal{I}$ has three theory families $\{\varnothing\}$, $\{\{1\}\}$ and $\{\{0,1\}\}$, but only two theory systems, $\{\varnothing\}$ and $\{\{0,1\}\}$. The lattice of theory families of $\mathcal{I}$ and the corresponding Leibniz congruence systems are given in the diagram.



It is easy to see that $\mathcal{I}$ is roughly family reflective. Suppose that for $T,T' \in$ $\mathrm{ThFam}(\mathcal{I})$, $\Omega(T) \leq \Omega(T')$.

- *If $\Omega(T') = \Delta^{\mathbf{F}}$, then $\Omega(T) = \Delta^{\mathbf{F}}$, whence $T' = T = \{\{1\}\}$. Thus, $\widetilde{T} \leq \widetilde{T'}$.*
- *If $\Omega(T') = \nabla^{\mathbf{F}}$, then $T' = \{\varnothing\}$ or $T' = \{\{0,1\}\}$. In either case, $\widetilde{T} \leq \{\{0,1\}\} = \widetilde{T'}$.*

On the other hand, for $T = \{\{1\}\}$, we get $\widetilde{T} = \{\{1\}\} \neq \{\{0,1\}\} = \widetilde{\{\varnothing\}} = \widetilde{\overleftarrow{T}}$, whence $T \not\approx \overleftarrow{T}$ and, hence, $\mathcal{I}$ is not roughly systemic. Therefore, by Lemma 423, $\mathcal{I}$ is not roughly right reflective.

The second example shows that there exists a roughly left reflective π-institution that is not roughly family reflective.

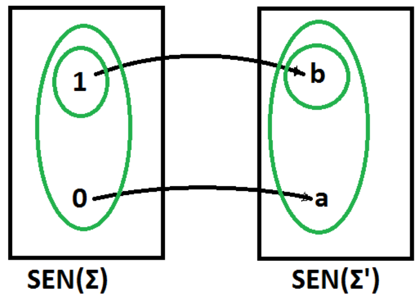
Example 428 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- *$\mathbf{Sign}^\flat$ is the category with objects Σ and Σ' and a unique (non-identity) morphism $f : \Sigma \to \Sigma'$;*
- *$\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ is defined by $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ and $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$;*
- *$N^\flat$ is the trivial clone.*

Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\},\{a,b\}\}.$$

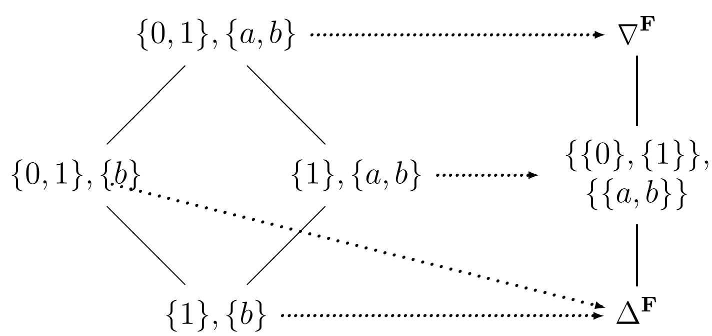
Again, since $\mathcal{I}$ has theorems, rough equivalence coincides with the identity relation on $\mathrm{ThFam}(\mathcal{I})$.



The following table shows the action of $\leftarrow$ *on theory families, where rows correspond to* T_Σ *and columns to* $T_{\Sigma'}$ *and each entry is written as* $\overleftarrow{T}_\Sigma, \overleftarrow{T}_{\Sigma'}$.

$\leftarrow$	$\{b\}$	$\{a,b\}$
$\{1\}$	$\{1\},\{b\}$	$\{1\},\{a,b\}$
$\{0,1\}$	$\{1\},\{b\}$	$\{0,1\},\{a,b\}$

The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right:



To see that $\mathcal{I}$ *is roughly left reflective, suppose* $T, T' \in \mathrm{ThFam}(\mathcal{I})$, *such that* $\Omega(T) \leq \Omega(T')$.

- *If* $\Omega(T') = \nabla^{\mathbf{F}}$, *then* $T' = \{\{0,1\},\{a,b\}\}$, *whence* $\overleftarrow{T} \leq \{\{0,1\},\{a,b\}\} = \overleftarrow{T'}$ *and, hence,* $\overleftarrow{\overleftarrow{T}} \leq \overleftarrow{\overleftarrow{T'}}$.
- *If* $\Omega(T') = \{\{\{0\},\{1\}\},\{\{a,b\}\}\}$, *then* $T' = \{\{1\},\{a,b\}\}$ *and* $T = \{\{0,1\},\{b\}\}$ *or* $T = \{\{1\},\{b\}\}$. *In either case* $\overleftarrow{T} = \{\{1\},\{b\}\} \leq T' = \overleftarrow{T'}$ *and, hence,* $\overleftarrow{\overleftarrow{T}} \leq \overleftarrow{\overleftarrow{T'}}$.
- *If* $\Omega(T') = \Delta^{\mathbf{F}}$, *then both* T *and* T' *have to be either* $\{\{0,1\},\{b\}\}$ *or* $\{\{1\},\{b\}\}$. *Thus, we get* $\overleftarrow{T} = \{\{1\},\{b\}\} = \overleftarrow{T'}$ *and, hence,* $\overleftarrow{\overleftarrow{T}} \leq \overleftarrow{\overleftarrow{T'}}$.

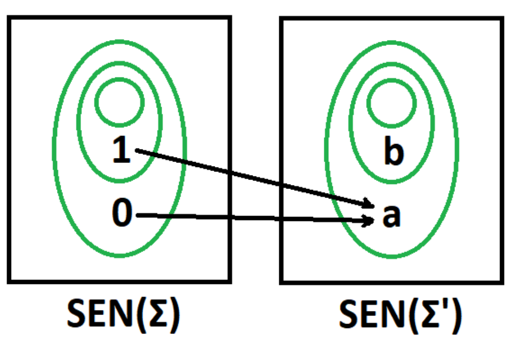
On the other hand, we have $\Omega(\{\{0,1\},\{b\}\}) \le \Omega(\{\{1\},\{b\}\})$, *but, clearly,* $\{\{0,1\},\{b\}\} \nleq \{\{1\},\{b\}\}$. *Thus, since rough equivalence is the identity on* $\mathrm{ThFam}(\mathcal{I})$, *we conclude that* $\mathcal{I}$ *is not roughly family reflective.*

The third example shows that there exists a roughly family reflective π-institution that is not roughly left reflective. Combined with the preceding example, it has the effect of establishing the following facts:

- The classes of roughly family reflective and roughly left reflective π-institutions are incomparable.
- The class of roughly family reflective π-institutions is properly contained in the class of roughly system reflective π-institutions.
- Similarly, the class of roughly left reflective π-institutions is a proper subclass of the class of roughly system reflective π-institutions.

Example 429 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = a$;
- $N^\flat$ *is the trivial clone.*



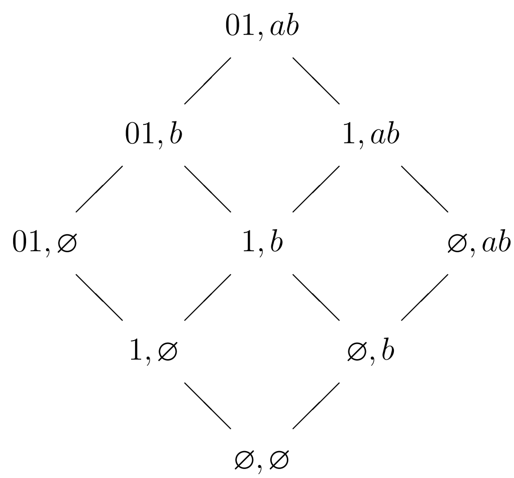
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a,b\}\}.$$

There are nine theory families, but only five theory systems. The action of $\overleftarrow{\ }$ *on theory families is given in the table below.*

T	$\overleftarrow{T}$	T	$\overleftarrow{T}$
$\varnothing,\varnothing$	$\varnothing,\varnothing$	$\varnothing,ab$	$\varnothing,ab$
$1,\varnothing$	$\varnothing,\varnothing$	$01,b$	$\varnothing,b$
$\varnothing,b$	$\varnothing,b$	$1,ab$	$1,ab$
$01,\varnothing$	$\varnothing,\varnothing$	$01,ab$	$01,ab$
$1,b$	$\varnothing,b$		

The lattice of theory families of $\mathcal{I}$ *is shown in the diagram.*



We show that $\mathcal{I}$ *is roughly family reflective. The following table summarizes the theory families together with their associated Leibniz congruence systems.*

T	$\Omega(T)$
$\{\varnothing,\varnothing\},\{01,\varnothing\},\{\varnothing,ab\},\{01,ab\}$	$\nabla^{\mathbf{F}}$
$\{\varnothing,b\},\{01,b\}$	$\{\nabla^{\mathbf{F}}_{\Sigma},\Delta^{\mathbf{F}}_{\Sigma'}\}$
$\{1,\varnothing\},\{1,ab\}$	$\{\Delta^{\mathbf{F}}_{\Sigma},\nabla^{\mathbf{F}}_{\Sigma'}\}$
$\{1,b\}$	$\Delta^{\mathbf{F}}$

Let $T,T' \in \mathrm{ThFam}(\mathcal{I})$*, such that* $\Omega(T) \leq \Omega(T')$*.*

- *If* $\Omega(T') = \nabla^{\mathbf{F}}$*, then* $\widetilde{T} \leq \{01,ab\} = \widetilde{T}'$*.*
- *If* $\Omega(T') = \{\Delta^{\mathbf{F}}_{\Sigma},\nabla^{\mathbf{F}}_{\Sigma'}\}$*, then* $\widetilde{T} = \{1,ab\}$ *or* $\widetilde{T} = \{1,b\}$ *and, hence,* $\widetilde{T} \leq \{1,ab\} = \widetilde{T}'$*.*
- *If* $\Omega(T') = \{\nabla^{\mathbf{F}}_{\Sigma},\Delta^{\mathbf{F}}_{\Sigma'}\}$*, then* $\widetilde{T} = \{01,b\}$ *or* $\widetilde{T} = \{1,b\}$ *and, hence,* $\widetilde{T} \leq \{01,b\} = \widetilde{T}'$*.*
- *If* $\Omega(T') = \Delta^{\mathbf{F}}$*, then* $\widetilde{T} = \{1,b\} = \widetilde{T}'$*.*

On the other hand, consider $T = \{1, \varnothing\}$ *and* $T' = \{1, ab\}$. *We have*

$$\Omega(T) = \Omega(\{1,\varnothing\}) = \{\Delta^{\mathbf{F}}_{\Sigma}, \nabla^{\mathbf{F}}_{\Sigma'}\} = \Omega(\{1,ab\}) = \Omega(T'),$$

whereas

$$\widetilde{\overleftarrow{T}} = \widetilde{\{\varnothing,\varnothing\}} = \{01, ab\} \nleq T' = \widetilde{T'} = \widetilde{\overleftarrow{T'}}.$$

Hence, $\mathcal{I}$ *is not roughly left reflective.*

We look, next, at the connections between rough reflectivity and rough injectivity classes. It turns out that membership in a rough reflectivity class implies membership in the corresponding rough injectivity class. We have the following straightforward inclusions.

Theorem 430 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is roughly right reflective, then it is roughly right injective;*

(b) If $\mathcal{I}$ *is roughly family reflective, then it is roughly family injective;*

(c) If $\mathcal{I}$ *is roughly left reflective, then it is roughly left injective;*

(d) If $\mathcal{I}$ *is roughly system reflective, then it is roughly system injective.*

Proof:

(a) Suppose that $\mathcal{I}$ is roughly right reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$. Then, by hypothesis, $\widetilde{T} \leq \widetilde{T'}$ and $\widetilde{T'} \leq \widetilde{T}$, whence $\widetilde{T} = \widetilde{T'}$, i.e., $T \sim T'$. Therefore, $\mathcal{I}$ is roughly right injective.

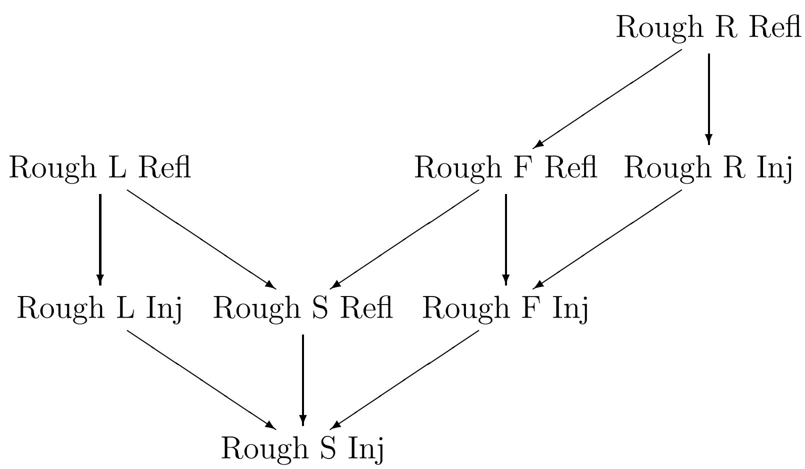
(b) Suppose that $\mathcal{I}$ is roughly family reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, by hypothesis, $\widetilde{T} \leq \widetilde{T'}$ and $\widetilde{T'} \leq \widetilde{T}$, whence $\widetilde{T} = \widetilde{T'}$, i.e., $T \sim T'$. Therefore, $\mathcal{I}$ is roughly family injective.

(c) Suppose that $\mathcal{I}$ is roughly left reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, by hypothesis, $\widetilde{\overleftarrow{T}} \leq \widetilde{\overleftarrow{T'}}$ and $\widetilde{\overleftarrow{T'}} \leq \widetilde{\overleftarrow{T}}$, whence $\widetilde{\overleftarrow{T}} = \widetilde{\overleftarrow{T'}}$, i.e., $\overleftarrow{T} \sim \overleftarrow{T'}$. Therefore, $\mathcal{I}$ is roughly left injective.

(d) Similar to Part (b).

■

Theorem 430 establishes the mixed rough hierarchy depicted in the diagram.

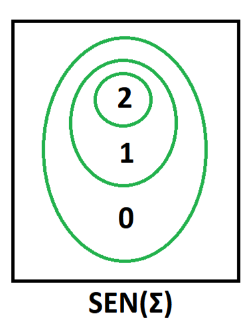


To see that all classes in the hierarchy are different, we give an example of a π-institution satisfying all four rough injectivity properties, which is not, however, roughly system reflective and, therefore, a fortiori, belongs to none of the four rough reflectivity classes.

Example 431 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with the single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the clone generated by the single unary operation* $\sigma : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *determined by the following table:*

x	0	1	2
$\sigma^\flat_\Sigma(x)$	0	0	2

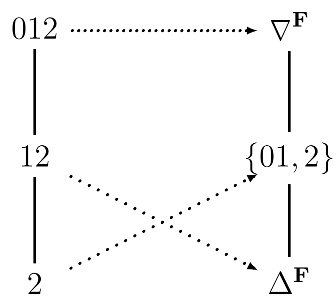


Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{1\}, \{0, 1\}, \{0, 1, 2\}\}.$$

$\mathcal{I}$ has three theory families $\{\{2\}\}$ and $\{\{1,2\}\}$ and $\{\{0,1,2\}\}$, all of which are theory systems. Moreover, $\mathcal{I}$ has theorems. It follows that the action of $\overleftarrow{\ }$ is trivial and that rough equivalence in $\mathcal{I}$ coincides with the identity relation on $\mathrm{ThFam}(\mathcal{I})$.

The lattice of theory families of $\mathcal{I}$ and the corresponding Leibniz congruence systems are given in the diagram.



We show that $\mathcal{I}$ is both roughly right and roughly left injective and, hence, belongs to all four classes in the rough injectivity hierarchy.

- *Suppose $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$. Then $\Omega(T) = \Omega(T')$ and, hence, $T = T'$, i.e., $T \sim T'$. Thus, $\mathcal{I}$ is rough right injective.*
- *Suppose $\Omega(T) = \Omega(T')$. Then $T = T'$. This gives $\overleftarrow{T} = \overleftarrow{T'}$, which, in turn, implies $\overleftarrow{T} \sim \overleftarrow{T'}$. Thus, $\mathcal{I}$ is roughly left injective.*

On the other hand, we have $\Omega(\{\{1,2\}\}) = \Delta^{\mathbf{F}} \leq \{\{0,1\},\{2\}\} = \Omega(\{2\})$, but $\{\{1,2\}\} \nleq \{\{2\}\}$, whence $\mathcal{I}$ is not roughly system reflective.

We now clarify the connections between rough reflectivity and reflectivity classes. It turns out that membership in a reflectivity class implies membership in the corresponding rough reflectivity class and, also, possession of theorems and, conversely, that membership in a rough reflectivity class plus possession of theorems entails membership in the corresponding reflectivity class.

Theorem 432 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is right/family reflective if and only if it is roughly right reflective and has theorems;

(b) $\mathcal{I}$ is right/family reflective if and only if it is roughly family reflective and has theorems;

(c) $\mathcal{I}$ is left reflective if and only if it is roughly left reflective and has theorems;

(d) $\mathcal{I}$ is system reflective if and only if it is roughly system reflective and has theorems.

Proof:

(a) Suppose that $\mathcal{I}$ is right reflective. Then, by Proposition 228, it is right injective. Hence, by Theorem 394, it has theorems. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Then, by right reflectivity, $T \leq T'$. Since $\mathcal{I}$ has theorems, $\widetilde{T} = T$ and $\widetilde{T'} = T'$. Therefore, $\widetilde{T} \leq \widetilde{T'}$ and $\mathcal{I}$ is roughly right reflective.

Assume, conversely, that $\mathcal{I}$ is roughly right reflective and has theorems. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Then, by rough right reflectivity, we get $\widetilde{T} \leq \widetilde{T'}$. On the other hand, since $\mathcal{I}$ has theorems, $\widetilde{T} = T$ and $\widetilde{T'} = T'$. Therefore, $T \leq T'$ and $\mathcal{I}$ is right reflective.

(b) Suppose that $\mathcal{I}$ is family reflective. Then, by Proposition 228, it is family injective. Hence, by Theorem 394, it has theorems. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then, by family reflectivity, $T \leq T'$. Since $\mathcal{I}$ has theorems, $\widetilde{T} = T$ and $\widetilde{T'} = T'$. Therefore, $\widetilde{T} \leq \widetilde{T'}$ and $\mathcal{I}$ is roughly family reflective.

Assume, conversely, that $\mathcal{I}$ is roughly family reflective and has theorems. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then, by rough family reflectivity, we get $\widetilde{T} \leq \widetilde{T'}$. On the other hand, since $\mathcal{I}$ has theorems, $\widetilde{T} = T$ and $\widetilde{T'} = T'$. Therefore, $T \leq T'$ and $\mathcal{I}$ is family reflective.

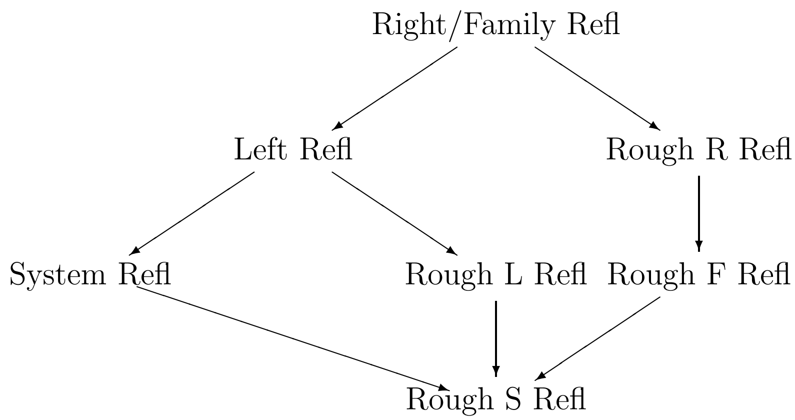
(c) Suppose that $\mathcal{I}$ is left reflective. Then, by Proposition 228, it is left injective. Hence, by Theorem 394, it has theorems. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then, by left reflectivity, $\overleftarrow{T} \leq \overleftarrow{T'}$. Since $\mathcal{I}$ has theorems, $\widetilde{\overleftarrow{T}} = \overleftarrow{T}$ and $\widetilde{\overleftarrow{T'}} = \overleftarrow{T'}$. Therefore, $\widetilde{\overleftarrow{T}} \leq \widetilde{\overleftarrow{T'}}$ and $\mathcal{I}$ is roughly left reflective.

Assume, conversely, that $\mathcal{I}$ is roughly left reflective and has theorems. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then, by rough left reflectivity, we get $\widetilde{\overleftarrow{T}} \leq \widetilde{\overleftarrow{T'}}$. On the other hand, since $\mathcal{I}$ has theorems, $\widetilde{\overleftarrow{T}} = \overleftarrow{T}$ and $\widetilde{\overleftarrow{T'}} = \overleftarrow{T'}$. Therefore, $\overleftarrow{T} \leq \overleftarrow{T'}$ and $\mathcal{I}$ is left reflective.

(d) Similar to Part (b).

■

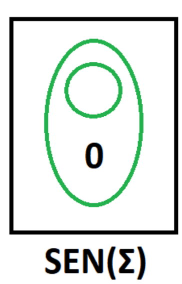
The work in Chapter 3, together with the work done in the present section and Theorem 432, reveal a hierarchy of reflectivity and rough reflectivity classes shown in the accompanying diagram.



To complete the demonstration that all classes in the depicted hierarchy are distinct we provide an example of a π-institution which belongs to all steps in the rough reflectivity hierarchy but possesses none of the four (gentle) reflectivity properties.

Example 433 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

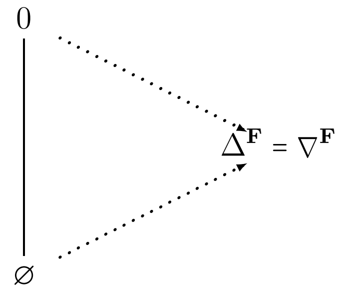
- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0\}$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{0\}\}.$$

$\mathcal{I}$ *is systemic and its lattice of theory families and corresponding Leibniz* *congruence systems are shown in the diagram.*



Note that $\widetilde{\{\{0\}\}} = \widetilde{\{\varnothing\}} = \{\{0\}\}$, *whence, trivially,* $\mathcal{I}$ *is both roughly right and roughly left reflective.*

On the other hand, since $\Omega(\{\{0\}\}) = \nabla^{\mathbf{F}} = \Omega(\{\varnothing\})$, *whereas* $\{\{0\}\} \nleq \{\varnothing\}$, $\mathcal{I}$ *is not system reflective and, hence, a fortiori,* $\mathcal{I}$ *has none of the four reflectivity properties.*

The rough reflectivity properties transfer from the theory families/systems of a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ to all $\mathcal{I}$-filter families/systems on arbitrary $\mathbf{F}$-algebraic systems.

Theorem 434 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$.

- *(a)* $\mathcal{I}$ *is roughly right reflective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(\overleftarrow{T}) \leq \Omega^{\mathcal{A}}(\overleftarrow{T'})$ *implies* $\widetilde{T} \leq \widetilde{T'}$;
- *(b)* $\mathcal{I}$ *is roughly family reflective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ *implies* $\widetilde{T} \leq \widetilde{T'}$;
- *(c)* $\mathcal{I}$ *is roughly left reflective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ *implies* $\widetilde{\overleftarrow{T}} \leq \widetilde{\overleftarrow{T'}}$;
- *(d)* $\mathcal{I}$ *is roughly system reflective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ *implies* $\widetilde{T} \leq \widetilde{T'}$.

Proof:

(a) The "if" follows by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$ and taking into account that, by Lemma 51, $\mathrm{ThFam}(\mathcal{I}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F})$.

For the "only if", suppose that $\mathcal{I}$ is roughly right reflective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that

$\Omega^{\mathcal{A}}(\overleftarrow{T}) \le \Omega^{\mathcal{A}}(\overleftarrow{T'})$. Then $\alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T})) \le \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'}))$. So, by Proposition 24, $\Omega(\alpha^{-1}(\overleftarrow{T})) \le \Omega(\alpha^{-1}(\overleftarrow{T'}))$. Hence, by Lemma 6, $\Omega(\overleftarrow{\alpha^{-1}(T)}) \le \Omega(\overleftarrow{\alpha^{-1}(T')})$. Since, by Lemma 51, $\alpha^{-1}(T), \alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$, we get, by applying rough right reflectivity, $\widetilde{\alpha^{-1}(T)} \le \widetilde{\alpha^{-1}(T')}$. Thus, by Theorem 377, $\alpha^{-1}(\widetilde{T}) \le \alpha^{-1}(\widetilde{T'})$. Therefore, taking into account the surjectivity of $\langle F, \alpha\rangle$, we conclude that $\widetilde{T} \le \widetilde{T'}$.

(b) The "if" follows as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is roughly family reflective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')$. Then $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \le \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. So, by Proposition 24, $\Omega(\alpha^{-1}(T)) \le \Omega(\alpha^{-1}(T'))$. Since, by Lemma 51, $\alpha^{-1}(T), \alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$, we get, by applying rough family reflectivity, $\widetilde{\alpha^{-1}(T)} \le \widetilde{\alpha^{-1}(T')}$. Thus, by Theorem 377, $\alpha^{-1}(\widetilde{T}) \le \alpha^{-1}(\widetilde{T'})$. Therefore, taking into account the surjectivity of $\langle F, \alpha\rangle$, we conclude that $\widetilde{T} \le \widetilde{T'}$.

(c) The "if" follows as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is roughly left reflective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')$. Then $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \le \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. So, by Proposition 24, $\Omega(\alpha^{-1}(T)) \le \Omega(\alpha^{-1}(T'))$. Since, by Lemma 51, $\alpha^{-1}(T), \alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$, we get, by applying rough left reflectivity, $\widetilde{\overleftarrow{\alpha^{-1}(T)}} \le \widetilde{\overleftarrow{\alpha^{-1}(T')}}$. Thus, by Lemma 6, $\widetilde{\alpha^{-1}(\overleftarrow{T})} \le \widetilde{\alpha^{-1}(\overleftarrow{T'})}$. Hence, by Theorem 377, $\alpha^{-1}(\widetilde{\overleftarrow{T}}) \le \alpha^{-1}(\widetilde{\overleftarrow{T'}})$. Therefore, taking into account the surjectivity of $\langle F, \alpha\rangle$, we conclude that $\widetilde{\overleftarrow{T}} \le \widetilde{\overleftarrow{T'}}$.

(d) Similar to Part (b).

■

Finally, we may recast the rough reflectivity classes in terms of the order reflectivity of mappings from posets of classes of theory or filter families/systems into posets of congruence systems.

Note for the following, that the collections $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ and $\widetilde{\mathrm{ThSys}}(\mathcal{I})$ may be ordered by setting, respectively, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$\widetilde{[T]} \le \widetilde{[T']} \quad \text{iff} \quad \widetilde{T} \le \widetilde{T'}$$

and, for all $T, T' \in \mathrm{ThSys}(\mathcal{I})$

$$\widetilde{\lfloor T \rfloor} \le \widetilde{\lfloor T' \rfloor} \quad \text{iff} \quad \widetilde{T} \le \widetilde{T'}.$$

We denote by $\widetilde{\mathbf{ThFam}}(\mathcal{I}) = \langle \widetilde{\mathrm{ThFam}}(\mathcal{I}), \le\rangle$ and $\widetilde{\mathbf{ThSys}}(\mathcal{I}) = \langle \widetilde{\mathrm{ThSys}}(\mathcal{I}), \le\rangle$, respectively, the corresponding ordered sets.

Proposition 435 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is roughly family reflective;*

(b) $\Omega : \widetilde{\mathbf{ThFam}}(\mathcal{I}) \to \mathbf{ConSys}^*(\mathcal{I})$ *is order reflecting;*

(c) $\Omega^{\mathcal{A}} : \widetilde{\mathbf{FiFam}}^{\mathcal{I}}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is order reflecting, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

Similarly, for system reflectivity, we have

Proposition 436 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is roughly system reflective;*

(b) $\Omega : \widetilde{\mathbf{ThSys}}(\mathcal{I}) \to \mathbf{ConSys}^*(\mathcal{I})$ *is order reflecting;*

(c) $\Omega^{\mathcal{A}} : \widetilde{\mathbf{FiSys}}^{\mathcal{I}}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is order reflecting, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

6.7 Narrow Reflectivity

In this section we study classes of π-institutions defined using reflectivity properties of the Leibniz operator restricted to $\mathrm{ThFam}^{\nleq}(\mathcal{I})$. We call those *narrow reflectivity* properties in analogy with the terminology adopted when differentiating rough injectivity and narrow injectivity classes.

Definition 437 (Narrow Reflectivity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is called* **narrowly family reflective** *if, for all* $T, T' \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$,

$$\Omega(T) \le \Omega(T') \quad \textit{implies} \quad T \le T';$$

- $\mathcal{I}$ *is called* **narrowly left reflective** *if, for all* $T, T' \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$,

$$\Omega(T) \le \Omega(T') \quad \textit{implies} \quad \overleftarrow{T} \le \overleftarrow{T'}.$$

- $\mathcal{I}$ *is called* **narrowly right reflective** *if, for all* $T, T' \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$,

$$\Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'}) \quad \textit{implies} \quad T \le T'.$$

- $\mathcal{I}$ *is called* **narrowly system reflective** *if, for all* $T, T' \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$,

$$\Omega(T) \le \Omega(T') \quad \textit{implies} \quad T \le T'.$$

The narrow reflectivity properties have the following characterizations, paralleling those given for the narrow injectivity classes in Proposition 400.

Proposition 438 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is narrowly family reflective if and only if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) \le \Omega(T')$ *implies* $\widetilde{T} \le \widetilde{T'}$*;*

(b) $\mathcal{I}$ *is narrowly left reflective if and only if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) \le \Omega(T')$ *implies* $\overleftarrow{\widetilde{T}} \le \overleftarrow{\widetilde{T'}}$*;*

(c) $\mathcal{I}$ *is narrowly right reflective if and only if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(\overleftarrow{\widetilde{T}}) \le \Omega(\overleftarrow{\widetilde{T'}})$ *implies* $\widetilde{T} \le \widetilde{T'}$*;*

(d) $\mathcal{I}$ *is narrowly system reflective if and only if, for all* $T, T' \in \mathrm{ThSys}(\mathcal{I})$, *such that* $\widetilde{T}, \widetilde{T'} \in \mathrm{ThSys}(\mathcal{I})$, $\Omega(T) \le \Omega(T')$ *implies* $\widetilde{T} \le \widetilde{T'}$*.*

Proof:

(a) Suppose that $\mathcal{I}$ is narrowly family reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \le \Omega(T')$. Consider $\widetilde{T}, \widetilde{T'} \in \mathrm{ThFam}^\natural(\mathcal{I})$. By Proposition 369, $\Omega(\widetilde{T}) = \Omega(T) \le \Omega(T') = \Omega(\widetilde{T'})$. Thus, by hypothesis, $\widetilde{T} \le \widetilde{T'}$. Therefore, the asserted condition holds.

Assume, conversely, that the asserted condition holds and let $T, T' \in \mathrm{ThFam}^\natural(\mathcal{I})$, such that $\Omega(T) \le \Omega(T')$. Then, since $\mathrm{ThFam}^\natural(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I})$, we get, by hypothesis, $\widetilde{T} \le \widetilde{T'}$. Since, however, $T, T' \in \mathrm{ThFam}^\natural(\mathcal{I})$, we get $T = \widetilde{T} \le \widetilde{T'} = T'$. Thus, $\mathcal{I}$ is narrowly family reflective.

(b) Suppose that $\mathcal{I}$ is narrowly left reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \le \Omega(T')$. Then $\widetilde{T}, \widetilde{T'} \in \mathrm{ThFam}^\natural(\mathcal{I})$ and, by Proposition 369, $\Omega(\widetilde{T}) \le \Omega(\widetilde{T'})$. Thus, by hypothesis, $\overleftarrow{\widetilde{T}} \le \overleftarrow{\widetilde{T'}}$.

Assume, conversely, that the asserted condition holds and let $T, T' \in \mathrm{ThFam}^\natural(\mathcal{I})$, such that $\Omega(T) \le \Omega(T')$. Then, by hypothesis, $\overleftarrow{\widetilde{T}} \le \overleftarrow{\widetilde{T'}}$. Since, however, $T, T' \in \mathrm{ThFam}^\natural(\mathcal{I})$, we get $\overleftarrow{T} = \overleftarrow{\widetilde{T}} \le \overleftarrow{\widetilde{T'}} = \overleftarrow{T'}$. Therefore, $\mathcal{I}$ is narrowly left reflective.

(c) Suppose that $\mathcal{I}$ is narrowly right reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(\overleftarrow{\widetilde{T}}) \le \Omega(\overleftarrow{\widetilde{T'}})$. Since $\widetilde{T}, \widetilde{T'} \in \mathrm{ThFam}^\natural(\mathcal{I})$, we get, by hypothesis, $\widetilde{T} \le \widetilde{T'}$.

Assume, conversely, that the asserted condition holds and let $T, T' \in \mathrm{ThFam}^\natural(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'})$. Then, since $T, T' \in \mathrm{ThFam}^\natural(\mathcal{I})$, we get $\Omega(\overleftarrow{\widetilde{T}}) = \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'}) = \Omega(\overleftarrow{\widetilde{T'}})$. Now, by hypothesis, $\widetilde{T} \leq \widetilde{T'}$ and, therefore, $T \leq T'$. We conclude that $\mathcal{I}$ is narrowly right reflective.

(d) Suppose $\mathcal{I}$ is narrowly system reflective and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T}, \widetilde{T'} \in \mathrm{ThSys}(\mathcal{I})$ and $\Omega(T) \leq \Omega(T')$. Then $\widetilde{T}, \widetilde{T'} \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$ and, by Proposition 369, $\Omega(\widetilde{T}) = \Omega(T) \leq \Omega(T') = \Omega(\widetilde{T'})$. Thus, by hypothesis, $\widetilde{T} \leq \widetilde{T'}$.

Assume, conversely, that the asserted condition holds and let $T, T' \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then, since $T, T' \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, we get $\widetilde{T} = T, \widetilde{T'} = T' \in \mathrm{ThSys}(\mathcal{I})$ and, therefore, by hypothesis, $\widetilde{T} \leq \widetilde{T'}$. But this gives $T = \widetilde{T} \leq \widetilde{T'} = T'$. Thus, $\mathcal{I}$ is narrowly system reflective. ∎

As was shown in Lemma 401, narrow right injectivity implies exclusive systemicity. In the next lemma, we show that narrow family reflectivity also implies exclusive systemicity.

Lemma 439 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is narrowly family reflective, then it is exclusively systemic.*

Proof: Assume $\mathcal{I}$ is narrowly family reflective and let $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, such that $\overleftarrow{T} \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$. By Proposition 20, $\Omega(T) \leq \Omega(\overleftarrow{T})$. Thus, by hypothesis, $T \leq \overleftarrow{T}$. Since, by Proposition 2, the reverse inclusion always holds, we get $\overleftarrow{T} = T$. Thus, $\mathcal{I}$ is exclusively systemic. ∎

Lemma 405 also has the following direct consequence.

Corollary 440 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is narrowly right reflective, then it is narrowly stable.*

Proof: Since narrow right reflectivity implies narrow right injectivity, this follows directly from Lemma 405. ∎

We establish, next the *narrow reflectivity hierarchy*. The following proposition forms an analog of Proposition 406, which established the narrow injectivity hierarchy.

Proposition 441 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is narrowly right reflective, then it is narrowly family reflective;*

(b) If $\mathcal{I}$ *is narrowly family reflective, then it is narrowly left reflective;*

(c) If $\mathcal{I}$ *is narrowly left reflective, then it is narrowly system reflective.*

Proof:

(a) Suppose that $\mathcal{I}$ is narrowly right reflective and let $T, T' \in \mathrm{ThFam}^{\natural}(\mathcal{I})$, such that $\Omega(T) \le \Omega(T')$. By Corollary 440, $\mathcal{I}$ is narrowly stable. Now we obtain $\Omega(\overleftarrow{T}) = \Omega(T) \le \Omega(T') = \Omega(\overleftarrow{T'})$. Hence, by narrow right reflectivity, $T \le T'$. Hence, $\mathcal{I}$ is narrowly family reflective.

(b) Suppose that $\mathcal{I}$ is narrowly family reflective and let $T, T' \in \mathrm{ThFam}^{\natural}(\mathcal{I})$, such that $\Omega(T) \le \Omega(T')$. Then, by hypothesis, $T \le T'$, whence, by Lemma 1, $\overleftarrow{T} \le \overleftarrow{T'}$. Thus, $\mathcal{I}$ is narrowly left reflective.

(c) Suppose that $\mathcal{I}$ is narrowly left reflective and let $T, T' \in \mathrm{ThSys}^{\natural}(\mathcal{I})$, such that $\Omega(T) \le \Omega(T')$. Then, by hypothesis, we get $\overleftarrow{T} \le \overleftarrow{T'}$. Therefore, since T, T' are theory systems, $T \le T'$ and, hence, $\mathcal{I}$ is narrowly system reflective.

■

We have now established the following **narrow reflectivity hierarchy** of π-institutions.

Narrowly R Reflective

$\downarrow$

Narrowly F Reflective

$\downarrow$

Narrowly L Reflective

$\downarrow$

Narrowly S Reflective

We give an additional result pertaining to the hierarchy of narrow reflectivity properties depicted in the diagram. The following proposition may be viewed as an analog of Proposition 407.

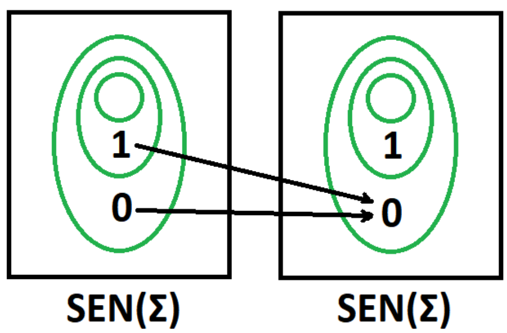
Proposition 442 *Let* $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat} \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is narrowly system reflective and narrowly systemic, then it is narrowly right reflective.*

Proof: Suppose $\mathcal{I}$ is narrowly system reflective and narrowly systemic. Let $T, T' \in \mathrm{ThFam}^{\natural}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'})$. By narrow systemicity, $T = \overleftarrow{T}$ and $T' = \overleftarrow{T'}$. Hence, on the one hand $\Omega(T) \le \Omega(T')$ and, on the other, $T, T' \in \mathrm{ThSys}^{\natural}(\mathcal{I})$. Thus, by narrow system reflectivity, $T \le T'$. Thus, $\mathcal{I}$ is narrowly right reflective. ■

We present three examples to show that all inclusions established between the narrow reflectivity classes and shown in the preceding diagram are indeed proper inclusions. The first example depicts a π-institution which is narrowly family reflective but not narrowly right reflective. This shows that the class of narrowly right reflective π-institutions constitutes a proper subclass of the class of narrowly family reflective π-institutions.

Example 443 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

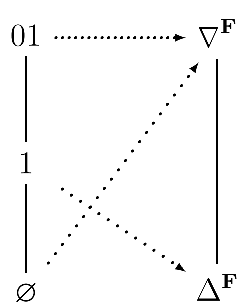
- $\mathbf{Sign}^\flat$ *is the category with the single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$*;*
- $N^\flat$ *is the trivial clone, consisting of the projections only.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\}.$$

$\mathcal{I}$ *has three theory families,* $\varnothing$, $\{\{1\}\}$ *and* $\{\{0,1\}\}$*, but only two theory systems,* $\varnothing$ *and* $\{\{0,1\}\}$*. The lattice of theory families of* $\mathcal{I}$ *and the corresponding Leibniz congruence systems are given in the diagram.*



Since the Leibniz operator is an isomorphism on $\mathrm{ThFam}^\natural(\mathcal{I})$*,* $\mathcal{I}$ *is narrowly family reflective. On the other hand,* $\{\{1\}\}, \{\{0,1\}\} \in \mathrm{ThFam}^\natural(\mathcal{I})$ *and*

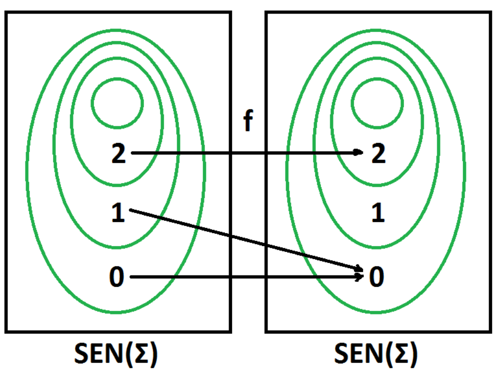
$$\Omega(\overleftarrow{\{\{1\}\}}) = \Omega(\{\varnothing\}) = \nabla^{\mathbf{F}} = \Omega(\{\{0,1\}\}) = \Omega(\overleftarrow{\{\{0,1\}\}}),$$

but $\{\{1\}\} \neq \{\{0,1\}\}$. *Therefore,* $\mathcal{I}$ *is not narrowly right injective and, a fortiori, it fails to be narrowly right reflective.*

The next example depicts a π-institution which is narrowly left reflective but not narrowly family reflective. This shows that the class of narrowly family reflective π-institutions is a proper subclass of the class of narrowly left reflective π-institutions.

Example 444 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

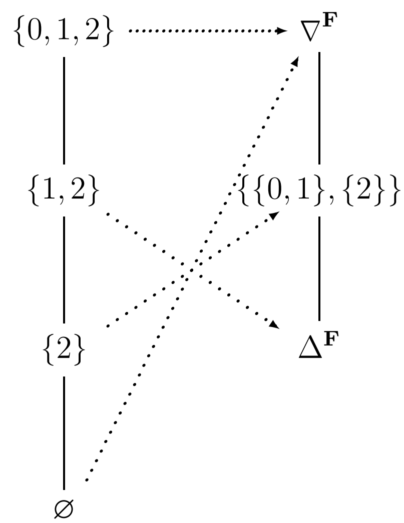
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$, $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$;
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{2\}, \{1,2\}, \{0,1,2\}\}.$$

$\mathcal{I}$ *has four theory families, but only three theory systems, namely* $\varnothing$, $\{2\}$ *and* $\{0,1,2\}$. *Moreover, clearly,* $\mathrm{ThFam}^{\nparallel}(\mathcal{I}) = \{\{2\}, \{1,2\}, \{0,1,2\}\}$. *The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right:*



There are three pairs (T,T'), *with* $T,T' \in \mathrm{ThFam}^\natural(\mathcal{I})$ *and* $T \neq T'$, *such that* $\Omega(T) \leq \Omega(T')$, *namely,*

$$(\{1,2\},\{0,1,2\}), \quad (\{2\},\{0,1,2\}), \quad (\{1,2\},\{2\}).$$

For all three, we get $\overleftarrow{T} \leq \overleftarrow{T'}$. *Thus,* $\mathcal{I}$ *is narrowly left reflective. On the other hand, for* $T = \{1,2\}$ *and* $T' = \{2\}$, *even though* $\Omega(T) \leq \Omega(T')$, *we get* $T \nleq T'$, *whence* $\mathcal{I}$ *fails to be narrowly family reflective.*

We finish the sequence of examples by presenting a narrowly system reflective π-institution which fails to be narrowly left reflective. This example shows that narrowly left reflective π-institutions form a proper subclass of the class of narrowly system reflective π-institutions.

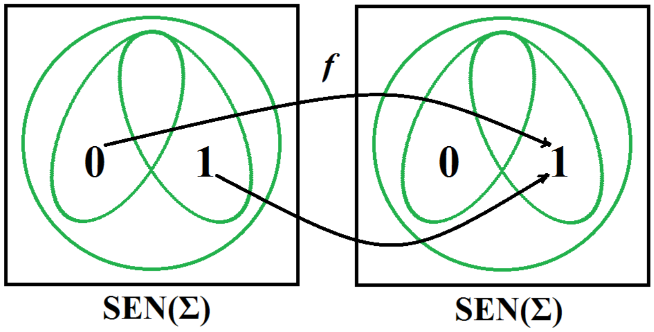
Example 445 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 1$ *and* $\mathrm{SEN}^\flat(f)(1) = 1$;
- $N^\flat$ *is the trivial clone.*

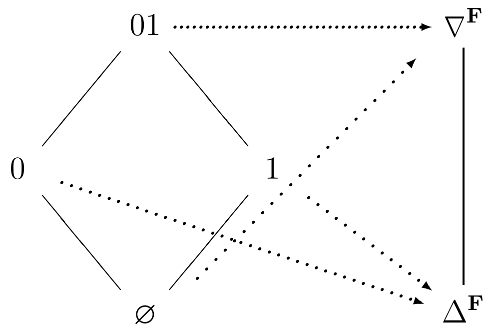
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\varnothing, \{0\}, \{1\}, \{0,1\}\}$.

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$.

T	$\overleftarrow{T}$
$\varnothing$	$\varnothing$
$\{0\}$	$\varnothing$
$\{1\}$	$\{1\}$
$\{0,1\}$	$\{0,1\}$



The lattice of theory families and the corresponding Leibniz congruence systems are depicted below.



It is obvious from the diagram that the Leibniz operator is an isomorphism on $\mathrm{ThSys}^{\nmid}(\mathcal{I})$*. Therefore,* $\mathcal{I}$ *is narrowly system reflective. On the other hand, for* $T = \{\{0\}\}$*,* $T' = \{\{1\}\}$*, both members of* $\mathrm{ThFam}^{\nmid}(\mathcal{I})$*, we have* $\Omega(T) = \Omega(T') = \Delta^{\mathbf{F}}$*, whereas* $\overleftarrow{T} = \{\varnothing\} \neq \{\{1\}\} = \overleftarrow{T'}$*. Therefore,* $\mathcal{I}$ *is not narrowly left injective and, a fortiori, it fails to be narrowly left reflective.*

We turn now to the relationships between corresponding classes of the rough reflectivity and the narrow reflectivity hierarchies. These parallel the ones already established between the rough injectivity and narrow injectivity classes in Section 6.5.

Using the characterization in Part (a) of Proposition 438, we can immediately see that the two types of family reflectivity coincide.

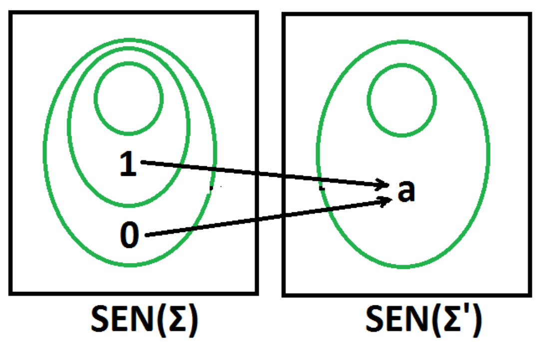
Corollary 446 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is roughly family reflective if and only if it is narrowly family reflective.*

Proof: Part (a) of Proposition 438. ■

As was the case with rough and narrow injectivity properties, the relationships between the remaining classes are not so straightforward, due to the necessity of investigating the mode of interaction between rough equivalence and the $\leftarrow$ operator. Starting with the two left reflectivity classes, we show that the class of narrow left reflective π-institutions is not included in the class of roughly left reflective π-institutions. This is accomplished by constructing a π-institution which is narrowly left reflective but not roughly left reflective.

Example 447 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

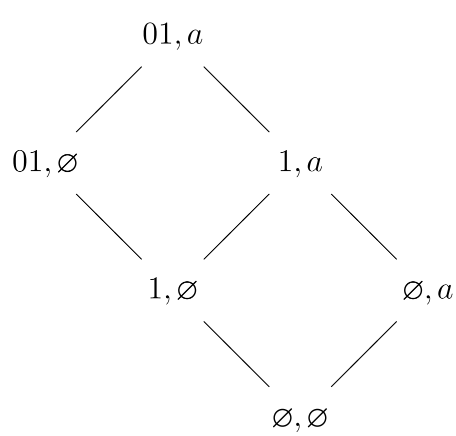
- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$*,* $\mathrm{SEN}^\flat(\Sigma') = \{a\}$ *and* $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = a$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$C_\Sigma = \{\varnothing, \{1\}, \{0, 1\}\} \quad \textit{and} \quad C_{\Sigma'} = \{\varnothing, \{a\}\}.$$

Clearly, there are six theory families in $\mathrm{ThFam}(\mathcal{I})$*, only four of which are theory systems, and only two of which are in* $\mathrm{ThFam}^{\lightning}(\mathcal{I})$*. The lattice of theory families is shown in the diagram:*



The only pair (T,T'), *with* $T,T' \in \mathrm{ThFam}^{\not\leq}(\mathcal{I})$, $T \neq T'$ *and* $\Omega(T) \leq \Omega(T')$ *is* $(\{1,a\},\ \{01,a\})$, *Since,* $\overleftarrow{\{1,a\}} = \{1,a\} \leq \{01,a\} = \overleftarrow{\{01,a\}}$, *it follows that* $\mathcal{I}$ *is narrowly left reflective.*

On the other hand, consider $T = \{1,\varnothing\}$ *and* $T' = \{1,a\}$. *We have* $\Omega(1,\varnothing) = \{\Delta^{\mathbf{F}}_{\Sigma}, \nabla^{\mathbf{F}}_{\Sigma'}\} = \Omega(1,a)$, *but*

$$\overleftarrow{\widetilde{1,\varnothing}} = \widetilde{\varnothing,\varnothing} = \{01,a\} \nleq \{1,a\} = \widetilde{1,a} = \overleftarrow{\widetilde{1,a}}.$$

This proves that $\mathcal{I}$ *is not roughly left reflective.*

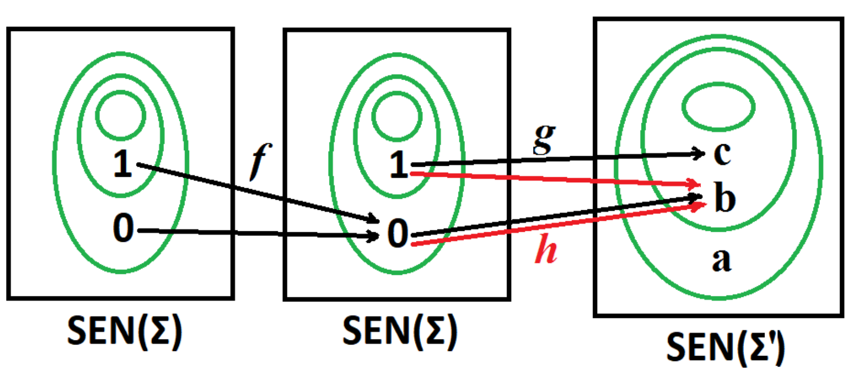
We exhibit, next a π-institution that is roughly left reflective, while it fails to be narrowly left reflective. Combined with Example 447, this will show that the two left reflectivity classes, rough and narrow, are incomparable from the point of view of inclusion.

Example 448 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and three nonidentity morphisms* $f : \Sigma \to \Sigma$ *and* $g,h : \Sigma \to \Sigma'$, *such that* $f \circ f = f$, $g \circ f = h$ *and* $h \circ f = h$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b,c\}$, $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = 0$, $\mathrm{SEN}^\flat(g)(0) = b$, $\mathrm{SEN}^\flat(g)(1) = c$ *and* $\mathrm{SEN}^\flat(h)(0) = \mathrm{SEN}^\flat(h)(1) = b$;
- $N^\flat$ *is the clone generated by a single binarry natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$, *whose components are defined by the following tables:*

$\sigma^\flat_\Sigma$	0	1
0	0	1
1	1	1

$\sigma^\flat_{\Sigma'}$	a	b	c
a	a	a	c
b	a	b	c
c	c	c	c



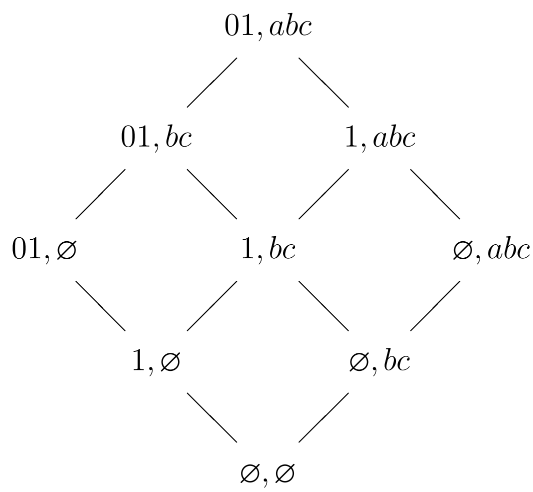
It is not difficult, albeit slightly tedious, to check that this is a well-defined natural transformation. We summarize the checking in the accompanying table.

(x,y)	$f(\sigma^\flat_\Sigma(x,y))$ $= \sigma^\flat_\Sigma(f(x),f(y))$	$g(\sigma^\flat_\Sigma(x,y))$ $= \sigma^\flat_{\Sigma'}(g(x),g(y))$	$h(\sigma^\flat_\Sigma(x,y))$ $= \sigma^\flat_{\Sigma'}(h(x),h(y))$
$(0,0)$	$0 = 0$	$b = b$	$b = b$
$(0,1)$	$0 = 0$	$c = c$	$b = b$
$(1,0)$	$0 = 0$	$c = c$	$b = b$
$(1,1)$	$0 = 0$	$c = c$	$b = b$

Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\varnothing,\{1\},\{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing,\{b,c\},\{a,b,c\}\}.$$

Clearly, there are nine theory families in $\mathrm{ThFam}(\mathcal{I})$*, five of which are theory systems, and four of which are in* $\mathrm{ThFam}^{\nleq}(\mathcal{I})$*. The lattice of theory families is shown in the diagram:*



The action of $^{\leftarrow}$ *on theory families is given in the following table.*

T	$\overleftarrow{T}$	T	$\overleftarrow{T}$
$01, abc$	$01, abc$	$\varnothing, abc$	$\varnothing, abc$
$01, bc$	$01, bc$	$1, \varnothing$	$\varnothing, \varnothing$
$1, abc$	$\varnothing, abc$	$\varnothing, bc$	$\varnothing, bc$
$01, \varnothing$	$\varnothing, \varnothing$	$\varnothing, \varnothing$	$\varnothing, \varnothing$
$1, bc$	$\varnothing, bc$		

The table below provides the Leibniz congruence systems associated with the theory families of $\mathcal{I}$.

T	$\Omega(T)$
$\{01,abc\},\{01,\varnothing\},\{\varnothing,abc\},\{\varnothing,\varnothing\}$	$\nabla^{\mathbf{F}}$
$\{1,abc\},\{1,\varnothing\}$	$\{\Delta^{\mathbf{F}}_\Sigma,\nabla^{\mathbf{F}}_{\Sigma'}\}$
$\{01,bc\},\{1,bc\},\{\varnothing,bc\}$	$\Delta^{\mathbf{F}}$

To see that $\mathcal{I}$ is roughly left reflective, suppose that $\Omega(T) \leq \Omega(T')$. We separate cases depending on $\Omega(T')$.

- *If $\Omega(T') = \Delta^{\mathbf{F}}$, then $T, T' \in \{\{01, bc\}, \{1, bc\}, \{\varnothing, b\}\}$, whence $\widetilde{\overleftarrow{T}} = \{01, bc\} = \widetilde{\overleftarrow{T'}}$;*

- *If $\Omega(T') = \{\Delta^{\mathbf{F}}_\Sigma, \nabla^{\mathbf{F}}_{\Sigma'}\}$ or $\Omega(T') = \nabla^{\mathbf{F}}$, then $\widetilde{\overleftarrow{T}} \leq \{01, abc\} = \widetilde{\overleftarrow{T'}}$.*

On the other hand, for $T = \{01, bc\}$ and $T' = \{1, bc\}$, we get $\Omega(T) = \Delta^{\mathbf{F}} = \Omega(T')$, whereas $\overleftarrow{T} = \{01, bc\} \nleq \{\varnothing, bc\} = \overleftarrow{T'}$. Therefore, $\mathcal{I}$ is not narrowly left reflective.

We turn, next to the relationship between the two kinds of right reflectivity. We show, first, that rough right reflectivity implies narrow right reflectivity.

Proposition 449 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is roughly right reflective, then it is narrowly right reflective.*

Proof: Suppose $\mathcal{I}$ is roughly right reflective and let $T, T' \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. By rough right reflectivity, we get that $\widetilde{T} \leq \widetilde{T'}$. Since, however, $T, T' \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, we get $T = \widetilde{T} \leq \widetilde{T'} = T'$. Therefore, $\mathcal{I}$ is narrowly right reflective. ■

The converse, on the other hand, does not hold in general, as the following example demonstrates.

Example 450 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

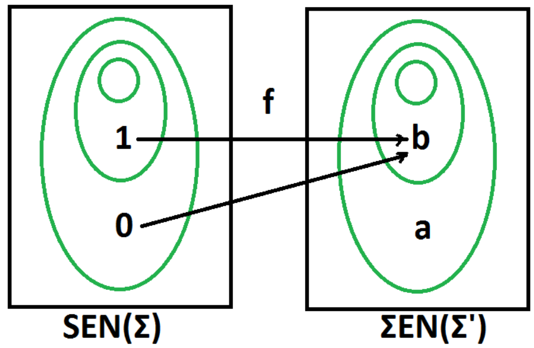
- $\mathbf{Sign}^\flat$ *is the category with objects Σ and Σ' and a unique morphism $f : \Sigma \to \Sigma'$;*

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = b$, $\mathrm{SEN}^\flat(f)(1) = b$*;*

- *$N^\flat$ is the trivial clone.*

Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0, 1\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a, b\}\}.$$



Clearly, there are only four theory families in $\mathrm{ThFam}^{\natural}(\mathcal{I})$*, all of which are theory systems. Their lattice together with the associated Leibniz congruence systems are shown in the diagram:*

$$\begin{array}{ccc} & 01, ab & \\ 01, b & & 1, ab \\ & 1, b & \end{array} \qquad \begin{array}{ccc} & \nabla^{\mathbf{F}} & \\ \nabla^{\mathbf{F}}_{\Sigma}, \Delta^{\mathbf{F}}_{\Sigma'} & & \Delta^{\mathbf{F}}_{\Sigma}, \nabla^{\mathbf{F}}_{\Sigma'} \\ & \Delta^{\mathbf{F}} & \end{array}$$

From this diagram and the fact that all theory families in $\mathrm{ThFam}^{\natural}(\mathcal{I})$ *are theory systems, we see that, for all* $T, T' \in \mathrm{ThFam}^{\natural}(\mathcal{I})$*,*

$$\Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'}) \quad \text{iff} \quad \Omega(T) \le \Omega(T') \quad \text{iff} \quad T \le T'.$$

Therefore, $\mathcal{I}$ *is indeed narrowly right reflective.*

On the other hand, consider $T = \{01, ab\}$ *and* $T' = \{1, \varnothing\}$*. Then we have*

$$\Omega(\overleftarrow{T}) = \Omega(01, ab) = \nabla^{\mathbf{F}} = \Omega(\overline{\varnothing}) = \Omega(\overleftarrow{T'}),$$

whereas $\widetilde{01, ab} = \{01, ab\} \nleq \{1, ab\} = \widetilde{1, \varnothing}$*. This shows that* $\mathcal{I}$ *is not roughly right reflective.*

Finally, we look at system reflectivity. We show that rough system reflectivity implies narrow system reflectivity, but that the converse implication fails in general.

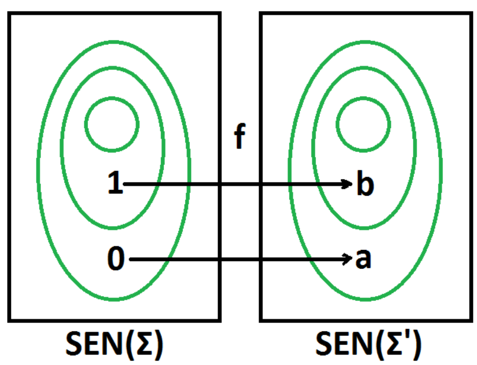
Proposition 451 *Let* $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat} \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly system reflective, then it is narrowly system reflective.*

Proof: Suppose $\mathcal{I}$ is roughly system reflective and let $T,T' \in \mathrm{ThSys}^{\natural}(\mathcal{I})$, such that $\Omega(T) \le \Omega(T')$. Then, by rough system reflectivity, $\widetilde{T} \le \widetilde{T'}$. However, since $T,T' \in \mathrm{ThSys}^{\natural}(\mathcal{I})$, we get $T = \widetilde{T} \le \widetilde{T'} = T'$. Therefore, $\mathcal{I}$ is narrowly system reflective. ∎

And now we present an example of a π-institution that is narrowly system reflective but not roughly system reflective. This, combined with Proposition 451, shows that the class of narrowly system reflective π-institutions properly contains the class of roughly system reflective π-institutions.

Example 452 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

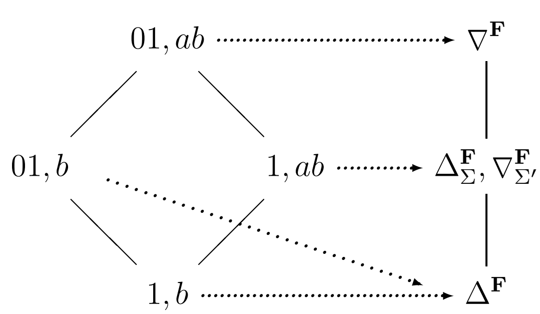
- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a,b\}\}.$$

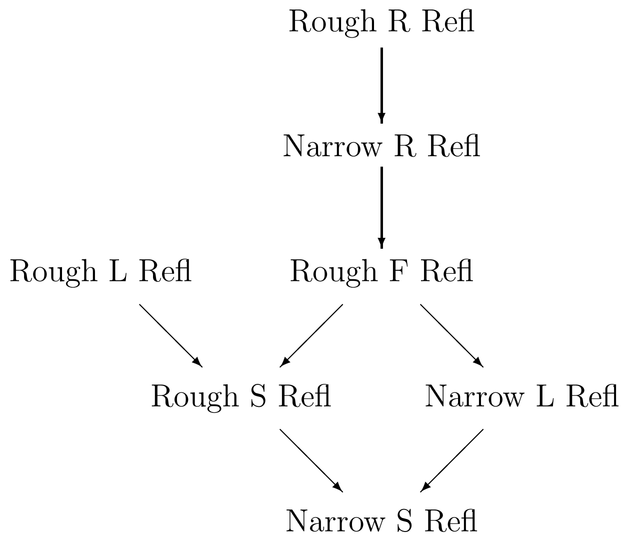
There are only four theory families in $\mathrm{ThFam}^{\natural}(\mathcal{I})$*, all of which except for* $\{01, b\}$ *are theory systems. Their lattice together with the associated Leibniz congruence systems are shown in the diagram:*



From this diagram we see that for all $T, T' \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$*, we get* $\Omega(T) \leq \Omega(T')$ *if and only if* $T \leq T'$*. Therefore,* $\mathcal{I}$ *is narrowly system reflective.*

On the other hand, consider $T = \{\varnothing, b\}$*,* $T' = \{1, b\} \in \mathrm{ThSys}(\mathcal{I})$*. Even though* $\widetilde{T} = \{01, b\} \nleq \{1, b\} = \widetilde{T'}$*, we have* $\Omega(T) = \Delta^{\mathbf{F}} = \Omega(T')$*. Hence,* $\mathcal{I}$ *is not roughly system reflective.*

The results obtained and the counterexamples presented, thus far, reveal the following mixed hierarchy of rough and narrow reflectivity classes of π-institutions, paralleling the one presented for rough and narrow injectivity properties.



As far as narrow injectivity versus narrow reflectivity properties, it is easy to show that a narrow reflectivity property implies the corresponding narrow injectivity property. (In fact, this observation, formalized in Proposition 453, has already been used before, e.g., in the proof of Part (a) of Proposition 441.)

Proposition 453 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is narrowly family reflective, then it is narrowly family injective;*

(b) If $\mathcal{I}$ *is narrowly left reflective, then it is narrowly left injective;*

(c) If $\mathcal{I}$ *is narrowly right reflective, then it is narrowly right injective;*

(d) If $\mathcal{I}$ *is narrowly system reflective, then it is narrowly system injective.*

Proof: We only deal with the family case, since the other three implications are equally straightforward to prove.

Assume that $\mathcal{I}$ is narrowly family reflective and let $T, T' \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Since this implies that $\Omega(T) \leq \Omega(T')$ and that

$\Omega(T') \le \Omega(T)$, we get, by applying narrow family reflectivity, that $T \le T'$ and $T' \le T$. Therefore, $T = T'$ and, hence, $\mathcal{I}$ is narrowly family injective. ■

Turning to the relationships between narrow reflectivity classes and corresponding reflectivity classes, we prove a theorem, analogous to Theorem 418, asserting that ordinary reflectivity is equivalent to narrow reflectivity in the presence of theorems.

Theorem 454 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is family reflective if and only if it is narrowly family reflective and has theorems;*

(b) $\mathcal{I}$ *is left reflective if and only if it is narrowly left reflective and has theorems;*

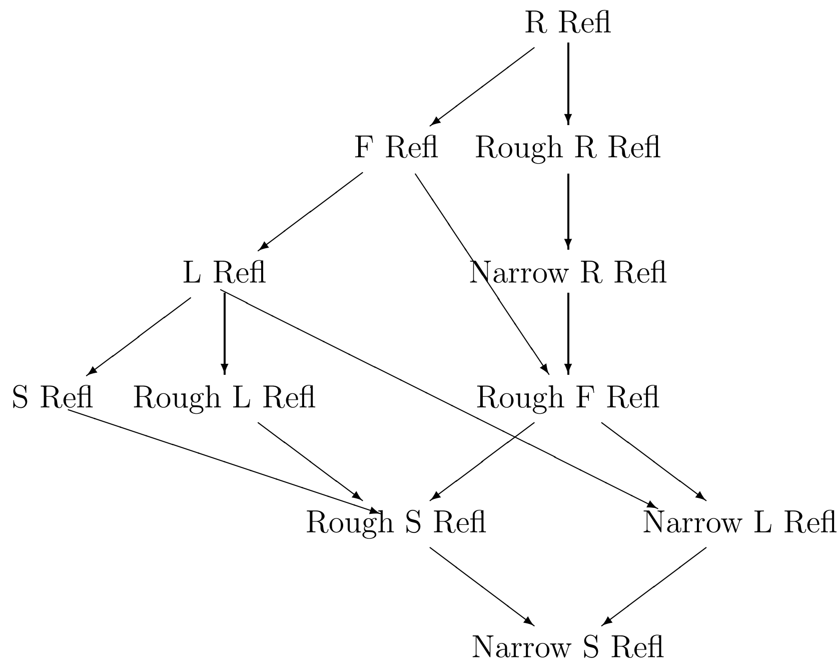
(c) $\mathcal{I}$ *is right reflective if and only if it is narrowly right reflective and has theorems;*

(d) $\mathcal{I}$ *is system reflective if and only if it is narrowly system reflective and has theorems.*

Proof: By Theorem 432, if $\mathcal{I}$ has one of the four reflectivity properties, then it has theorems. Moreover, by the same theorem, a reflectivity property implies the corresponding rough reflectivity property and, by Corollary 446, Proposition 449 and Proposition 451, each implies the corresponding narrow reflectivity property except in the case of left reflectivity, where (as actually in all other cases, as well) one can easily see directly that left reflectivity implies narrow left reflectivity, since the defining condition of the latter is a special case of that of the former.

All converses are also easily verified, since, in the presence of theorems, $\mathrm{ThFam}^{\natural}(\mathcal{I}) = \mathrm{ThFam}(\mathcal{I})$ and $\mathrm{ThSys}^{\natural}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$, which makes the four defining conditions for the narrow classes identical with the corresponding conditions for the ordinary reflectivity classes. ■

We now have the following hierarchy.



The narrow reflectivity properties transfer from the theory families/systems of a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ to all $\mathcal{I}$-filter families/systems on arbitrary $\mathbf{F}$-algebraic systems. This result forms an analog of Theorem 419, which applied to narrow injectivity classes.

Theorem 455 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is narrowly right reflective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}^\natural}(\mathcal{A})$*,* $\Omega^{\mathcal{A}}(\overleftarrow{T}) \leq \Omega^{\mathcal{A}}(\overleftarrow{T'})$ *implies* $T \leq T'$*;*

(b) $\mathcal{I}$ *is narrowly family reflective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}^\natural}(\mathcal{A})$*,* $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ *implies* $T \leq T'$*;*

(c) $\mathcal{I}$ *is narrowly left reflective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}^\natural}(\mathcal{A})$*,* $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ *implies* $\overleftarrow{T} \leq \overleftarrow{T'}$*;*

(d) $\mathcal{I}$ *is narrowly system reflective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiSys}^{\mathcal{I}^\natural}(\mathcal{A})$*,* $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ *implies* $T \leq T'$*.*

Proof: The proof follows the steps of the proofs of the various parts of Theorem 419. We do Part (a) in detail to give a flavor of what is involved.

The "if" follows by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account that $\mathrm{ThFam}^{\lightning}(\mathcal{I}) = \mathrm{FiFam}^{\mathcal{I}\lightning}(\mathcal{F})$, by Lemmas 51 and 376.

For the "only if", suppose that $\mathcal{I}$ is narrowly right reflective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}\lightning}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(\overleftarrow{T}) \leq \Omega^{\mathcal{A}}(\overleftarrow{T'})$. Then $\alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T})) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'}))$. So, by Proposition 24, $\Omega(\alpha^{-1}(\overleftarrow{T})) \leq \Omega(\alpha^{-1}(\overleftarrow{T'}))$. Hence, by Lemma 6, $\Omega(\overleftarrow{\alpha^{-1}(T)}) \leq \Omega(\overleftarrow{\alpha^{-1}(T')})$. Since, by Lemmas 51 and 376, $\alpha^{-1}(T), \alpha^{-1}(T') \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, we get, by applying narrow right reflectivity, $\alpha^{-1}(T) \leq \alpha^{-1}(T')$. This yields, taking into account the surjectivity of $\langle F, \alpha \rangle$, $T \leq T'$. ■

We finally recast narrow reflectivity in terms of the order reflectivity of mappings from posets of theory or filter families/systems into posets of congruence systems. The following results form analogs of Propositions 420 and 421, respectively, addressing reflectivity instead of injectivity properties.

Proposition 456 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is narrowly family reflective;*

(b) $\Omega : \mathrm{ThFam}^{\lightning}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is order reflecting;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}\lightning}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is order reflecting, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

Similarly, for system reflectivity, we have

Proposition 457 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is narrowly system reflective;*

(b) $\Omega : \mathrm{ThSys}^{\lightning}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is order reflecting;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}\lightning}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is order reflecting, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

6.8 Rough Complete Reflectivity

In this section we study classes of π-institutions defined using complete reflectivity properties of the Leibniz operator applied on rough equivalence classes.

Definition 458 (Rough c-Reflectivity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is called* **roughly family completely reflective**, *or* **roughly family c-reflective** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,

$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \quad \textit{implies} \quad \bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T'}.$$

- $\mathcal{I}$ *is called* **roughly left completely reflective**, *or* **roughly left c-reflective** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,

$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \quad \textit{implies} \quad \bigcap_{T \in \mathcal{T}} \overleftarrow{\widetilde{T}} \leq \overleftarrow{\widetilde{T'}}.$$

- $\mathcal{I}$ *is called* **roughly right completely reflective**, *or* **roughly right c-reflective** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,

$$\bigcap_{T \in \mathcal{T}} \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'}) \quad \textit{implies} \quad \bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T'}.$$

- $\mathcal{I}$ *is called* **roughly system completely reflective**, *or* **roughly system c-reflective** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$,

$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \quad \textit{implies} \quad \bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T'}.$$

As was shown to be the case with rough right reflectivity in Lemma 423, we show that rough right c-reflectivity implies rough systemicity and, hence, by Theorem 370, stability.

Lemma 459 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly right completely reflective, then it is roughly systemic.*

Proof: This is a consequence of Lemma 423, since rough right c-reflectivity implies trivially rough right reflectivity. ■

Next we establish the *rough c-reflectivity hierarchy* of π-institutions.

Proposition 460 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is roughly right c-reflective, then it is roughly family c-reflective;*

(b) If $\mathcal{I}$ *is roughly family c-reflective, then it is roughly system c-reflective;*

(c) If $\mathcal{I}$ *is roughly left c-reflective, then it is roughly system c-reflective.*

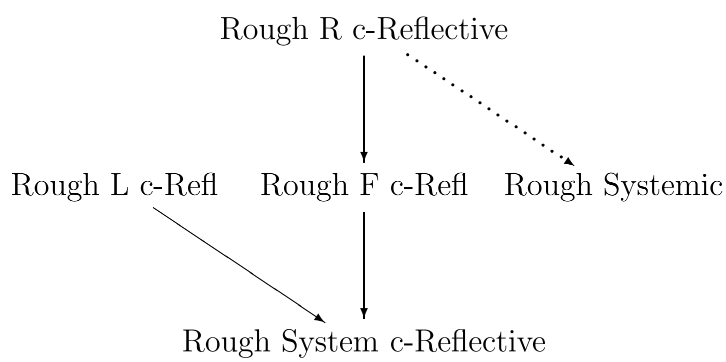
Proof:

(a) Suppose $\mathcal{I}$ is roughly right c-reflective and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. By Lemma 459, $\mathcal{I}$ is roughly systemic, whence $\overleftarrow{T} \sim T$, for all $T \in \mathcal{T}$, and $\overleftarrow{T'} \sim T'$. Thus, by Theorem 370, we get
$$\bigcap_{T\in\mathcal{T}} \Omega(\overleftarrow{T}) = \bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T') = \Omega(\overleftarrow{T'}).$$
Now applying rough right c-reflectivity, we get $\bigcap_{T\in\mathcal{T}} \widetilde{T} \leq \widetilde{T'}$. This proves that $\mathcal{I}$ is roughly family c-reflective.

(b) Suppose $\mathcal{I}$ is roughly family c-reflective and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. Then, by rough family c-reflectivity, we get $\bigcap_{T\in\mathcal{T}} \widetilde{T} \leq \widetilde{T'}$, whence, $\mathcal{I}$ is roughly system c-reflective.

(c) Suppose $\mathcal{I}$ is roughly left c-reflective and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. By rough left c-reflectivity, we conclude that $\bigcap_{T\in\mathcal{T}} \overleftarrow{\widetilde{T}} \leq \overleftarrow{\widetilde{T'}}$. However, since $\mathcal{T} \cup \{T'\}$ consists of theory systems, we have $\overleftarrow{T} = T$, for all $T \in \mathcal{T}$, and $\overleftarrow{T'} = T'$. Hence we get $\bigcap_{T\in\mathcal{T}} \widetilde{T} \leq \widetilde{T'}$ and, hence, $\mathcal{I}$ is roughly system reflective. ■

We have now established the following **rough complete reflectivity hierarchy** of π-institutions.



We formulate two additional properties concerning the relationships between rough c-reflectivity classes. First, rough right c-reflectivity turns out to be equivalent to rough system c-reflectivity combined with rough systemicity.

Proposition 461 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is roughly right c-reflective if and only if it is roughly system c-reflective and roughly systemic.*

Proof: Suppose, first, that $\mathcal{I}$ is roughly right c-reflective. Then, by Lemma 459, it is roughly systemic and by Proposition 460 it is roughly system c-reflective.

Suppose, conversely, that $\mathcal{I}$ is roughly system c-reflective and roughly systemic and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. By rough system c-reflectivity and Proposition 42, we get $\bigcap_{T\in\mathcal{T}}\widetilde{\overleftarrow{T}} \leq \widetilde{\overleftarrow{T'}}$. Hence, by rough systemicity, $\bigcap_{T\in\mathcal{T}}\widetilde{T} = \bigcap_{T\in\mathcal{T}}\overleftarrow{\widetilde{T}} \leq \overleftarrow{\widetilde{T'}} = \widetilde{T'}$. Thus, $\mathcal{I}$ is roughly right c-reflective. ■

Second, we show that rough system c-reflectivity together with stability imply rough left c-reflectivity.

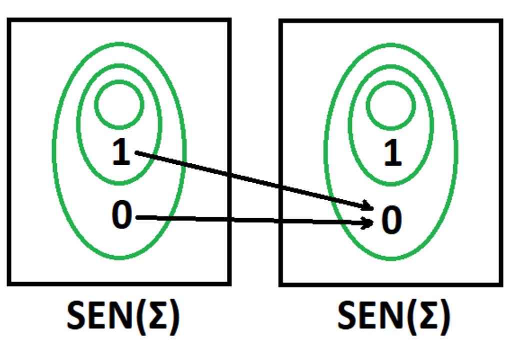
Proposition 462 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is roughly system c-reflective and stable, then it is roughly left c-reflective.*

Proof: Suppose that $\mathcal{I}$ is roughly system c-reflective and stable and consider $\mathcal{T}\cup\{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}}\Omega(T) \leq \Omega(T')$. Then, by stability $\bigcap_{T\in\mathcal{T}}\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Hence, since $\{\overleftarrow{T} : T \in \mathcal{T}\}\cup\{\overleftarrow{T'}\} \subseteq \mathrm{ThSys}(\mathcal{I})$, by rough system c-reflectivity, $\bigcap_{T\in\mathcal{T}}\widetilde{\overleftarrow{T}} \leq \widetilde{\overleftarrow{T'}}$. This shows that $\mathcal{I}$ is roughly left c-reflective. ■

We present three examples to show that all inclusions established between rough c-reflectivity classes and depicted in the diagram above are proper inclusions. The first example will show that the class of roughly right c-reflective π-institutions is a proper subclass of the class of roughly family c-reflective π-institutions.

Example 463 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be the algebraic system defined as follows:*

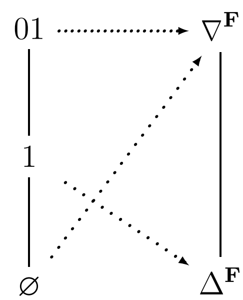
- $\mathbf{Sign}^\flat$ *is the category with the single object Σ and a single (non-identity) morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$;
- $N^\flat$ *is the trivial clone.*



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\}.$$

$\mathcal{I}$ has three theory families $\{\varnothing\}$, $\{\{1\}\}$ and $\{\{0,1\}\}$, but only two theory systems, $\{\varnothing\}$ and $\{\{0,1\}\}$. The lattice of theory families of $\mathcal{I}$ and the corresponding Leibniz congruence systems are given in the diagram.



It is easy to see that $\mathcal{I}$ is roughly family c-reflective. Suppose that for $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$.

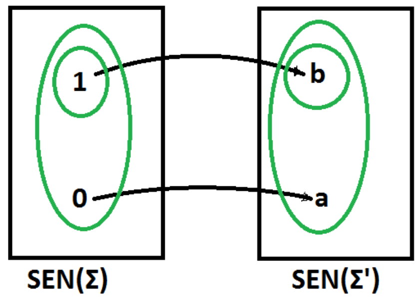
- *If $\Omega(T') = \Delta^{\mathbf{F}}$, then $\bigcap_{T \in \mathcal{T}} \Omega(T) = \Delta^{\mathbf{F}}$, whence $T' = \{\{1\}\}$ and $\{\{1\}\} \in \mathcal{T}$. Thus, $\bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \{\{1\}\} = \widetilde{T'}$.*
- *If $\Omega(T') = \nabla^{\mathbf{F}}$, then $T' = \{\varnothing\}$ or $T' = \{\{0,1\}\}$. In either case, $\bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \{\{0,1\}\} = \widetilde{T'}$.*

On the other hand, for $T = \{\{1\}\}$, we get $\widetilde{T} = \{\{1\}\} \neq \{\{0,1\}\} = \widetilde{\{\varnothing\}} = \widetilde{\overleftarrow{T}}$, whence $T \nsim \overleftarrow{T}$ and, hence, $\mathcal{I}$ is not roughly systemic. Therefore, by Lemma 459, $\mathcal{I}$ is not roughly right c-reflective.

The second example shows that there exists a roughly left c-reflective π-institution that is not roughly family c-reflective.

Example 464 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- *$\mathbf{Sign}^\flat$ is the category with objects Σ and Σ' and a unique (non-identity) morphism $f : \Sigma \to \Sigma'$;*
- *$\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ is defined by $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ and $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$;*
- *$N^\flat$ is the trivial clone.*



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ by stipulating that

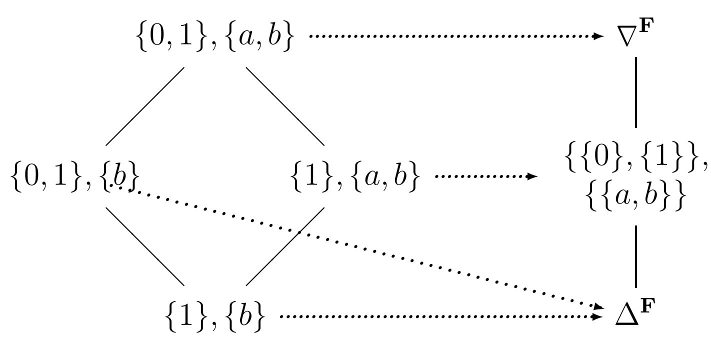
$$\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\},\{a,b\}\}.$$

Again, since $\mathcal{I}$ has theorems, rough equivalence coincides with the identity relation on $\mathrm{ThFam}(\mathcal{I})$.

The following table shows the action of $\overleftarrow{\ }$ *on theory families.*

T	$\{1,b\}$	$\{01,b\}$	$\{1,ab\}$	$\{01,ab\}$
$\overleftarrow{T}$	$\{1,b\}$	$\{1,b\}$	$\{1,ab\}$	$\{01,ab\}$

The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right:



We show, first, that $\mathcal{I}$ is roughly left c-reflective. Suppose $\mathcal{T}\cup\{T'\} \subseteq$ $\mathrm{ThFam}(\mathcal{I})$, *such that $\bigcap_{T\in\mathcal{T}}\Omega(T) \le \Omega(T')$.*

- *If $\Omega(T') = \nabla^{\mathbf{F}}$, then $T' = \{\{0,1\},\{a,b\}\}$, whence*

$$\bigcap_{T\in\mathcal{T}}\overleftarrow{T} \le \{\{0,1\},\{a,b\}\} = \overleftarrow{T'}$$

and, hence, $\bigcap_{T\in\mathcal{T}}\overleftarrow{\widetilde{T}} \le \overleftarrow{\widetilde{T'}}$.

- *If* $\Omega(T') = \{\{\{0\},\{1\}\},\{\{a,b\}\}\}$, *then* $T' = \{\{1\},\{a,b\}\}$ *and one of* $\{\{0,1\},\{b\}\}$ *or* $\{\{1\},\{a,b\}\}$ *or* $\{\{1\},\{b\}\}$ *must belong to* $\mathcal{T}$. *In either case*
$$\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \le \{\{1\},\{a,b\}\} = \overleftarrow{T'}$$
and, hence, $\bigcap_{T\in\mathcal{T}} \overleftarrow{\overleftarrow{T}} \le \overleftarrow{\overleftarrow{T'}}$.

- *If* $\Omega(T') = \Delta^{\mathbf{F}}$, *then* T' *must be either* $\{\{0,1\},\{b\}\}$ *or* $\{\{1\},\{b\}\}$ *and, moreover,* $\{\{0,1\},\{b\}\}$ *or* $\{\{1\},\{b\}\}$ *is in* $\mathcal{T}$. *Thus, we get*
$$\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \le \{\{1\},\{b\}\} = \overleftarrow{T'}$$
and, hence, $\bigcap_{T\in\mathcal{T}} \overleftarrow{\overleftarrow{T}} \le \overleftarrow{\overleftarrow{T'}}$.

On the other hand, we have $\Omega(\{\{0,1\},\{b\}\}) \le \Omega(\{\{1\},\{b\}\})$, *but, clearly,* $\{\{0,1\},\{b\}\} \not\le \{\{1\},\{b\}\}$. *Thus, since rough equivalence is the identity on* $\mathrm{ThFam}(\mathcal{I})$, *we conclude that* $\mathcal{I}$ *is not roughly family c-reflective.*

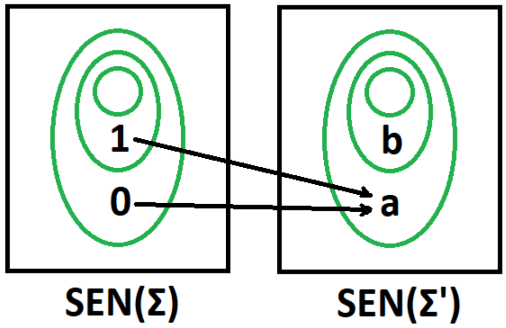
The third example shows that there exists a roughly family c-reflective π-institution that is not roughly left c-reflective. Combined with the preceding example, it has the effect of establishing the following facts:

- The classes of roughly family c-reflective and roughly left c-reflective π-institutions are incomparable.
- The class of roughly family c-reflective π-institutions is properly contained in the class of roughly system c-reflective π-institutions.
- Similarly, the class of roughly left c-reflective π-institutions is a proper subclass of the class of roughly system c-reflective π-institutions.

Example 465 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = a$;
- $N^\flat$ *is the trivial clone.*

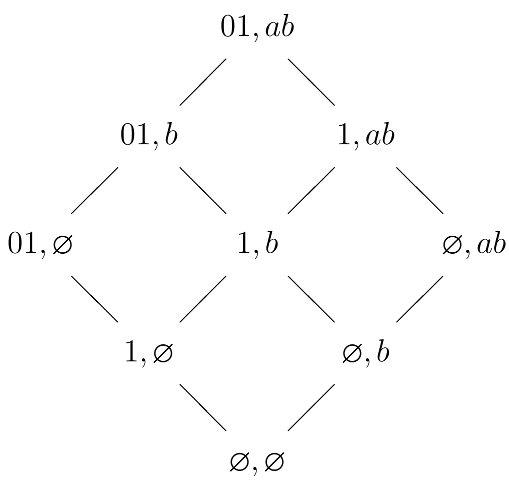
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by stipulating that*
$$\mathcal{C}_\Sigma = \{\varnothing,\{1\},\{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing,\{b\},\{a,b\}\}.$$



There are nine theory families, but only five theory systems. The action of $\leftarrow$ *on theory families is given in the table below.*

T	$\overleftarrow{T}$	T	$\overleftarrow{T}$
$\varnothing, \varnothing$	$\varnothing, \varnothing$	$\varnothing, ab$	$\varnothing, ab$
$1, \varnothing$	$\varnothing, \varnothing$	$01, b$	$\varnothing, b$
$\varnothing, b$	$\varnothing, b$	$1, ab$	$1, ab$
$01, \varnothing$	$\varnothing, \varnothing$	$01, ab$	$01, ab$
$1, b$	$\varnothing, b$		

The lattice of theory families of $\mathcal{I}$ *is shown in the diagram.*



We show that $\mathcal{I}$ *is roughly family c-reflective. The following table summarizes the theory families together with their associated Leibniz congruence systems.*

T	$\Omega(T)$
$\{\varnothing, \varnothing\}, \{01, \varnothing\}, \{\varnothing, ab\}, \{01, ab\}$	$\nabla^{\mathbf{F}}$
$\{\varnothing, b\}, \{01, b\}$	$\{\nabla^{\mathbf{F}}_{\Sigma}, \Delta^{\mathbf{F}}_{\Sigma'}\}$
$\{1, \varnothing\}, \{1, ab\}$	$\{\Delta^{\mathbf{F}}_{\Sigma}, \nabla^{\mathbf{F}}_{\Sigma'}\}$
$\{1, b\}$	$\Delta^{\mathbf{F}}$

Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$*, such that* $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$*.*

- *If* $\Omega(T') = \nabla^{\mathbf{F}}$*, then* $\bigcap_{T\in\mathcal{T}}\widetilde{T} \leq \{01, ab\} = \widetilde{T'}$.
- *If* $\Omega(T') = \{\Delta^{\mathbf{F}}_{\Sigma}, \nabla^{\mathbf{F}}_{\Sigma'}\}$*, then* $\mathcal{T}$ *must include one of the theory families* $\{1,\varnothing\}$*,* $\{1,ab\}$*,* $\{1,b\}$*. Hence,* $\bigcap_{T\in\mathcal{T}}\widetilde{T} \leq \{1, ab\} = \widetilde{T'}$.
- *If* $\Omega(T') = \{\nabla^{\mathbf{F}}_{\Sigma}, \Delta^{\mathbf{F}}_{\Sigma'}\}$*, then* $\mathcal{T}$ *must include one of the theory families* $\{\varnothing,b\}$*,* $\{01,b\}$*,* $\{1,b\}$*. Hence,* $\bigcap_{T\in\mathcal{T}}\widetilde{T} \leq \{01, b\} = \widetilde{T'}$.
- *If* $\Omega(T') = \Delta^{\mathbf{F}}$*, then* $\bigcap_{T\in\mathcal{T}}\Omega(T) = \Delta^{\mathbf{F}}$ *and* $\widetilde{T'} = \widetilde{\{1,b\}} = \{1,b\}$.
  - *If* $\{1,b\} \in \mathcal{T}$*, then* $\bigcap_{T\in\mathcal{T}}\widetilde{T} \leq \{1,b\} = \widetilde{T'}$*;*
  - *If* $\{1,b\} \notin \mathcal{T}$*, then* $\mathcal{T}$ *must include at least one member of each of the pairs*
  $$\{\varnothing,b\},\ \{01,b\} \quad \text{and} \quad \{1,\varnothing\},\ \{1,ab\}.$$
  Thus, $\bigcap_{T\in\mathcal{T}}\widetilde{T} \leq \{01,b\} \cap \{1,ab\} = \{1,b\} = \widetilde{T'}$.

On the other hand, consider $T = \{1,\varnothing\}$ *and* $T' = \{1,ab\}$*. We have*

$$\Omega(T) = \Omega(\{1,\varnothing\}) = \{\Delta^{\mathbf{F}}_{\Sigma}, \nabla^{\mathbf{F}}_{\Sigma'}\} = \Omega(\{1,ab\}) = \Omega(T'),$$

whereas

$$\overleftarrow{\widetilde{T}} = \widetilde{\{\varnothing,\varnothing\}} = \{01,ab\} \nleq T' = \widetilde{T'} = \overleftarrow{\widetilde{T'}}.$$

hence, $\mathcal{I}$ *is not roughly left c-reflective.*

We look, next, at the connections between rough c-reflectivity and rough reflectivity classes. Membership in a rough c-reflectivity class implies, in a straightforward way, membership in the corresponding rough reflectivity class.

Theorem 466 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is roughly right c-reflective, then it is roughly right reflective;*

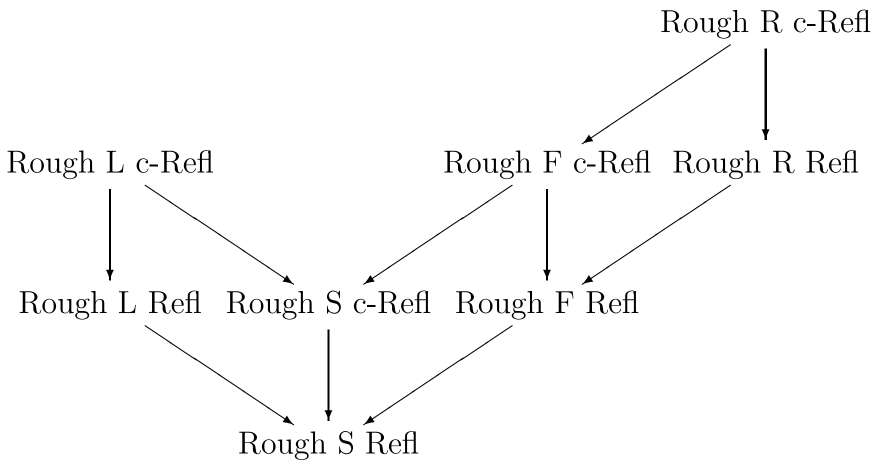
(b) If $\mathcal{I}$ *is roughly family c-reflective, then it is roughly family reflective;*

(c) If $\mathcal{I}$ *is roughly left c-reflective, then it is roughly left reflective;*

(d) If $\mathcal{I}$ *is roughly system c-reflective, then it is roughly system reflective.*

Proof: The property of being, e.g., roughly right reflective is a specialization of the property of being roughly right c-reflective, where one replaces the class $\mathcal{T}$ of theory families by the singleton $\{T\}$. The same holds for the remaining three types of rough reflectivity and rough c-reflectivity, respectively. ■

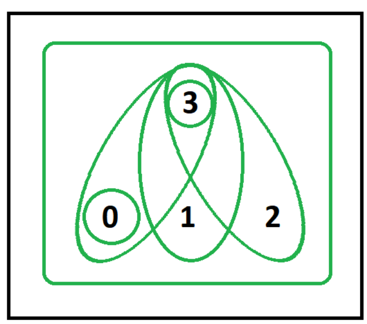
Theorem 466 establishes the mixed rough hierarchy depicted in the diagram.



To see that all classes in the hierarchy are different, we give an example of a π-institution satisfying all four rough reflectivity properties, which is not, however, roughly system c-reflective and, therefore, a fortiori, belongs to none of the four rough c-reflectivity classes.

Example 467 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with the single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2, 3\}$*;*
- $N^\flat$ *is the trivial clone.*



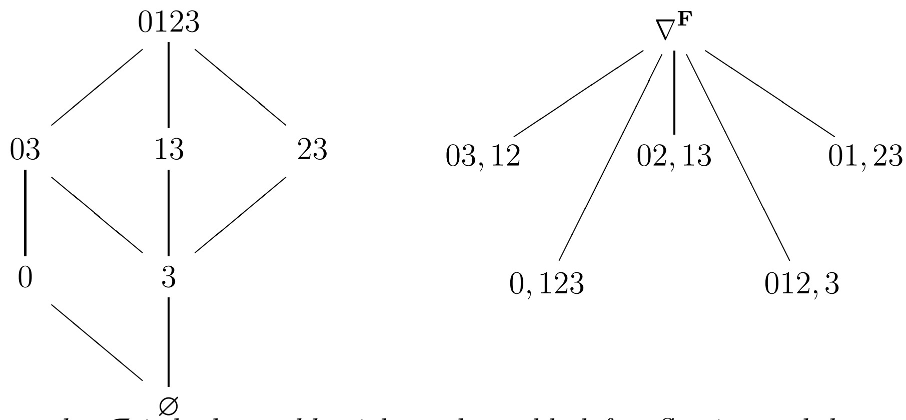
SEN(Σ)

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{0\}, \{3\}, \{0,3\}, \{1,3\}, \{2,3\}, \{0,1,2,3\}\}.$$

$\mathcal{I}$ *has seven theory families all of which are theory systems. It follows that the action of* $\overleftarrow{\ }$ *is trivial. Moreover, the only non-singleton rough equivalence class is the one consisting of* $\{\varnothing\}$ *and* $\{\{0,1,2,3\}\}$*.*

The lattice of theory families of $\mathcal{I}$ and the corresponding Leibniz congruence systems are given in the diagram.



We show that $\mathcal{I}$ is both roughly right and roughly left reflective and, hence, belongs to all four classes in the rough reflectivity hierarchy. Note that, since $\mathcal{I}$ is systemic, both rough reflectivity properties boil down to showing that, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$\Omega(T) \leq \Omega(T') \quad \textit{implies} \quad \widetilde{T} \leq \widetilde{T'}.$$

- *If $\Omega(T') = \nabla^{\mathbf{F}}$, then $T' = \{\varnothing\}$ or $T' = \{\{0,1,2,3\}\}$. Therefore, $\widetilde{T} \leq \{\{0,1,2,3\}\} = \widetilde{T'}$;*
- *If $\Omega(T') \neq \nabla^{\mathbf{F}}$, then, since $\Omega(T) \leq \Omega(T')$, we must have $T = T'$ and, hence, $\widetilde{T} \leq \widetilde{T'}$.*

On the other hand, we have

$$\Omega(\{03\}) \cap \Omega(\{3\}) = \{03,12\} \cap \{012,3\} = \{0,12,3\} \leq \{0,123\} = \Omega(\{\{0\}\}),$$

whereas

$$\widetilde{\{03\}} \cap \widetilde{\{3\}} = \{03\} \cap \{3\} = \{3\} \nleq \{0\} = \widetilde{\{0\}}.$$

Hence, $\mathcal{I}$ is not roughly system c-reflective and, therefore, it belongs to none of the four rough c-reflectivity classes.

We explore, next, the connections between rough c-reflectivity and c-reflectivity classes. By analogy with the case of reflectivity and rough reflectivity (Theorem 432), we get that membership in a c-reflectivity class implies membership in the corresponding rough c-reflectivity class and, also, possession of theorems. Conversely, membership in a rough c-reflectivity class plus possession of theorems entails membership in the corresponding c-reflectivity class.

Theorem 468 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) *$\mathcal{I}$ is right/family c-reflective if and only if it is roughly right c-reflective and has theorems;*

(b) *$\mathcal{I}$ is right/family c-reflective if and only if it is roughly family c-reflective and has theorems;*

(c) *$\mathcal{I}$ is left c-reflective if and only if it is roughly left c-reflective and has theorems;*

(d) *$\mathcal{I}$ is system c-reflective if and only if it is roughly system c-reflective and has theorems.*

Proof:

(a) Suppose that $\mathcal{I}$ is right c-reflective. Then, by Proposition 243, it is right reflective. Hence, by Theorem 432, it has theorems. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Then, by right c-reflectivity, $\bigcap_{T\in\mathcal{T}} T \leq T'$. Since $\mathcal{I}$ has theorems, $\widetilde{T} = T$, for all $T \in \mathcal{T}$, and $\widetilde{T'} = T'$. Therefore, $\bigcap_{T\in\mathcal{T}} \widetilde{T} \leq \widetilde{T'}$ and $\mathcal{I}$ is roughly right c-reflective.

Assume, conversely, that $\mathcal{I}$ is roughly right c-reflective and has theorems. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Then, by rough right c-reflectivity, we get $\bigcap_{T\in\mathcal{T}} \widetilde{T} \leq \widetilde{T'}$. On the other hand, since $\mathcal{I}$ has theorems, $\widetilde{T} = T$, for all $T \in \mathcal{T}$, and $\widetilde{T'} = T'$. Therefore, $\bigcap_{T\in\mathcal{T}} T \leq T'$ and $\mathcal{I}$ is right c-reflective.

(b) Suppose that $\mathcal{I}$ is family c-reflective. Then, by Proposition 243, it is family reflective. Hence, by Theorem 432, it has theorems. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. Then, by family c-reflectivity, $\bigcap_{T\in\mathcal{T}} T \leq T'$. Since $\mathcal{I}$ has theorems, $\widetilde{T} = T$, for all $T \in \mathcal{T}$, and $\widetilde{T'} = T'$. Therefore, $\bigcap_{T\in\mathcal{T}} \widetilde{T} \leq \widetilde{T'}$ and $\mathcal{I}$ is roughly family c-reflective.

Assume, conversely, that $\mathcal{I}$ is roughly family c-reflective and has theorems. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. Then, by rough family c-reflectivity, we get $\bigcap_{T\in\mathcal{T}} \widetilde{T} \leq \widetilde{T'}$. On the other hand, since $\mathcal{I}$ has theorems, $\widetilde{T} = T$, for all $T \in \mathcal{T}$, and $\widetilde{T'} = T'$. Therefore, $\bigcap_{T\in\mathcal{T}} T \leq T'$ and $\mathcal{I}$ is family c-reflective.

(c) Suppose that $\mathcal{I}$ is left c-reflective. Then, by Proposition 243, it is left reflective. Hence, by Theorem 432, it has theorems. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. Then, by left reflectivity, $\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$. Since $\mathcal{I}$ has theorems, $\widetilde{\overleftarrow{T}} = \overleftarrow{T}$, for all $T \in \mathcal{T}$, and $\widetilde{\overleftarrow{T'}} = \overleftarrow{T'}$. Therefore, $\bigcap_{T\in\mathcal{T}} \widetilde{\overleftarrow{T}} \leq \widetilde{\overleftarrow{T'}}$ and $\mathcal{I}$ is roughly left c-reflective.

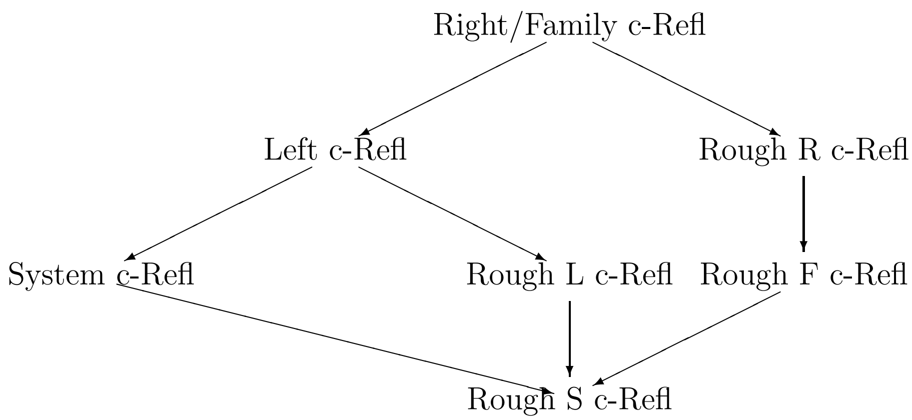
Assume, conversely, that $\mathcal{I}$ is roughly left c-reflective and has theorems. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. Then, by rough left c-reflectivity, we get $\bigcap_{T\in\mathcal{T}} \widetilde{\overleftarrow{T}} \leq \widetilde{\overleftarrow{T'}}$. On the other hand, since

$\mathcal{I}$ has theorems, $\overset{\sim}{\overleftarrow{T}} = \overleftarrow{T}$, for all $T \in \mathcal{T}$, and $\overset{\sim}{\overleftarrow{T'}} = \overleftarrow{T'}$. Therefore, $\bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$ and $\mathcal{I}$ is left c-reflective.

(d) Similar to Part (b).

■

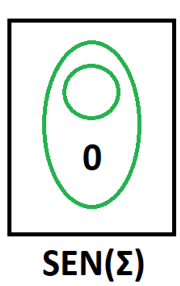
The work in Chapter 3, together with the work done in the present section and Theorem 468, reveal a hierarchy of c-reflectivity and rough c-reflectivity classes shown in the accompanying diagram.



To complete the demonstration that all classes in the depicted hierarchy are distinct we provide an example of a π-institution which belongs to all steps in the rough c-reflectivity hierarchy but possesses none of the four (gentle) c-reflectivity properties.

Example 469 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

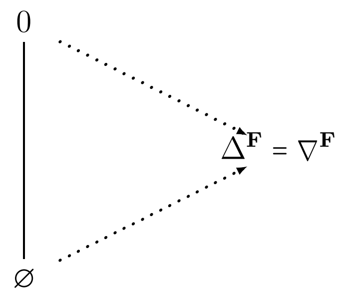
- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0\}$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{0\}\}.$$

$\mathcal{I}$ *is systemic and its lattice of theory families and corresponding Leibniz congruence systems are shown in the diagram.*



Note that $\widetilde{\{\{0\}\}} = \widetilde{\{\varnothing\}} = \{\{0\}\}$*, whence, trivially,* $\mathcal{I}$ *is both roughly right and roughly left c-reflective.*

On the other hand, since $\Omega(\{\{0\}\}) = \nabla^{\mathbf{F}} = \Omega(\{\varnothing\})$*, whereas* $\{\{0\}\} \not\leq \{\varnothing\}$*,* $\mathcal{I}$ *is not system c-reflective and, hence, a fortiori,* $\mathcal{I}$ *has none of the four c-reflectivity properties.*

As was shown to be the case with the rough reflectivity properties in Theorem 434, the rough c-reflectivity properties transfer from the theory families/systems of a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ to all $\mathcal{I}$-filter families/systems on arbitrary $\mathbf{F}$-algebraic systems.

Theorem 470 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is roughly right c-reflective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(\overleftarrow{T}) \leq \Omega^{\mathcal{A}}(\overleftarrow{T'})$ *implies* $\bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T'}$*;*

(b) $\mathcal{I}$ *is roughly family c-reflective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ *implies* $\bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T'}$*;*

(c) $\mathcal{I}$ *is roughly left c-reflective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ *implies* $\bigcap_{T \in \mathcal{T}} \overleftarrow{\widetilde{T}} \leq \overleftarrow{\widetilde{T'}}$*;*

(d) $\mathcal{I}$ *is roughly system c-reflective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $\mathcal{T} \cup \{T'\} \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$*,* $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ *implies* $\bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T'}$*.*

Proof:

(a) The "if" follows by considering the **F**-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account that, by Lemma 51, $\mathrm{ThFam}(\mathcal{I}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F})$.

For the "only if", suppose that $\mathcal{I}$ is roughly right c-reflective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an **F**-algebraic system and $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(\overleftarrow{T}) \leq \Omega^{\mathcal{A}}(\overleftarrow{T'})$. Then $\alpha^{-1}(\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(\overleftarrow{T})) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'}))$, whence, $\bigcap_{T \in \mathcal{T}} \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T})) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'}))$. So, by Proposition 24, $\bigcap_{T \in \mathcal{T}} \Omega(\alpha^{-1}(\overleftarrow{T})) \leq \Omega(\alpha^{-1}(\overleftarrow{T'}))$. By Lemma 6,

$$\bigcap_{T \in \mathcal{T}} \Omega(\overleftarrow{\alpha^{-1}(T)}) \leq \Omega(\overleftarrow{\alpha^{-1}(T')}).$$

Since, by Lemma 51, $\{\alpha^{-1}(T) : T \in \mathcal{T}\} \cup \{\alpha^{-1}(T')\} \subseteq \mathrm{ThFam}(\mathcal{I})$, we get, by applying rough right c-reflectivity,

$$\bigcap_{T \in \mathcal{T}} \widetilde{\alpha^{-1}(T)} \leq \widetilde{\alpha^{-1}(T')}.$$

Thus, by Theorem 377, $\bigcap_{T \in \mathcal{T}} \alpha^{-1}(\widetilde{T}) \leq \alpha^{-1}(\widetilde{T'})$, i.e., $\alpha^{-1}(\bigcap_{T \in \mathcal{T}} \widetilde{T}) \leq \alpha^{-1}(\widetilde{T'})$. Therefore, taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that $\bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T'}$.

(b) The "if" follows as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is roughly family c-reflective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an **F**-algebraic system and $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. Then we get $\alpha^{-1}(\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$, whence, $\bigcap_{T \in \mathcal{T}} \alpha^{-1}(\Omega^{\mathcal{A}}(T)) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. So, by Proposition 24, $\bigcap_{T \in \mathcal{T}} \Omega(\alpha^{-1}(T)) \leq \Omega(\alpha^{-1}(T'))$. Since, by Lemma 51,

$$\{\alpha^{-1}(T) : T \in \mathcal{T}\} \cup \{\alpha^{-1}(T')\} \subseteq \mathrm{ThFam}(\mathcal{I}),$$

we get, by applying rough family c-reflectivity, $\bigcap_{T \in \mathcal{T}} \widetilde{\alpha^{-1}(T)} \leq \widetilde{\alpha^{-1}(T')}$. Thus, by Theorem 377, $\bigcap_{T \in \mathcal{T}} \alpha^{-1}(\widetilde{T}) \leq \alpha^{-1}(\widetilde{T'})$, i.e., $\alpha^{-1}(\bigcap_{T \in \mathcal{T}} \widetilde{T}) \leq \alpha^{-1}(\widetilde{T'})$. Therefore, taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that $\bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T'}$.

(c) The "if" follows as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is roughly left c-reflective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an **F**-algebraic system and $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. Then $\alpha^{-1}(\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$, whence, $\bigcap_{T \in \mathcal{T}} \alpha^{-1}(\Omega^{\mathcal{A}}(T)) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. So, by Proposition 24, $\bigcap_{T \in \mathcal{T}} \Omega(\alpha^{-1}(T)) \leq \Omega(\alpha^{-1}(T'))$. Since, by Lemma 51,

$$\{\alpha^{-1}(T) : T \in \mathcal{T}\} \cup \{\alpha^{-1}(T')\} \subseteq \mathrm{ThFam}(\mathcal{I}),$$

we get, by applying rough left c-reflectivity, $\bigcap_{T \in \mathcal{T}} \overleftarrow{\widetilde{\alpha^{-1}(T)}} \leq \overleftarrow{\widetilde{\alpha^{-1}(T')}}$. Thus, by Lemma 6, $\bigcap_{T \in \mathcal{T}} \widetilde{\alpha^{-1}(\overleftarrow{T})} \leq \widetilde{\alpha^{-1}(\overleftarrow{T'})}$. Hence, by Theorem 377,

$\bigcap_{T\in\mathcal{T}}\alpha^{-1}(\overleftarrow{\widetilde{T}}) \le \alpha^{-1}(\overleftarrow{\widetilde{T'}})$, i.e., $\alpha^{-1}(\bigcap_{T\in\mathcal{T}}\overleftarrow{\widetilde{T}}) \le \alpha^{-1}(\overleftarrow{\widetilde{T'}})$. Therefore, taking into account the surjectivity of $\langle F,\alpha\rangle$, we conclude that $\bigcap_{T\in\mathcal{T}}\overleftarrow{\widetilde{T}} \le \overleftarrow{\widetilde{T'}}$.

(d) Similar to Part (b).

■

Finally, we may recast the rough c-reflectivity classes in terms of complete order reflectivity of mappings from posets of classes of theory or filter families/systems into posets of congruence systems.

Recall that the collections $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ and $\widetilde{\mathrm{ThSys}}(\mathcal{I})$ may be ordered by setting, respectively, for all $T,T' \in \mathrm{ThFam}(\mathcal{I})$,

$$\widetilde{\lceil T\rceil} \le \widetilde{\lceil T'\rceil} \quad \text{iff} \quad \widetilde{T} \le \widetilde{T'}$$

and, for all $T,T' \in \mathrm{ThSys}(\mathcal{I})$,

$$\widetilde{\lfloor T\rfloor} \le \widetilde{\lfloor T'\rfloor} \quad \text{iff} \quad \widetilde{T} \le \widetilde{T'},$$

and that the corresponding ordered sets are denoted by $\widetilde{\mathbf{ThFam}}(\mathcal{I})$ and $\widetilde{\mathbf{ThSys}}(\mathcal{I})$, respectively.

Proposition 471 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *The following statements are equivalent:*

(a) $\mathcal{I}$ *is roughly family c-reflective;*

(b) $\Omega : \widetilde{\mathbf{ThFam}}(\mathcal{I}) \to \mathbf{ConSys}^*(\mathcal{I})$ *is completely order reflecting;*

(c) $\Omega^{\mathcal{A}} : \widetilde{\mathbf{FiFam}}^{\mathcal{I}}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is completely order reflecting, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$.

Similarly, for system c-reflectivity, we have

Proposition 472 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *The following statements are equivalent:*

(a) $\mathcal{I}$ *is roughly system c-reflective;*

(b) $\Omega : \widetilde{\mathbf{ThSys}}(\mathcal{I}) \to \mathbf{ConSys}^*(\mathcal{I})$ *is completely order reflecting;*

(c) $\Omega^{\mathcal{A}} : \widetilde{\mathbf{FiSys}}^{\mathcal{I}}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is completely order reflecting, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$.

6.9 Narrow Complete Reflectivity

In this section we study classes of π-institutions defined using complete reflectivity properties of the Leibniz operator restricted to $\mathrm{ThFam}^{\nleq}(\mathcal{I})$. We call those *narrow complete reflectivity* properties in analogy with the terminology adopted when differentiating rough reflectivity and narrow reflectivity classes.

Definition 473 (Narrow c-Reflectivity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is called* **narrowly family completely reflective**, *or* **narrowly family c-reflective** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nleq}(\mathcal{I})$,
$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \quad \textit{implies} \quad \bigcap_{T \in \mathcal{T}} T \leq T';$$
- $\mathcal{I}$ *is called* **narrowly left completely reflective**, *or* **narrowly left c-reflective** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nleq}(\mathcal{I})$,
$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \quad \textit{implies} \quad \bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'};$$
- $\mathcal{I}$ *is called* **narrowly right completely reflective**, *or* **narrowly right c-reflective** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nleq}(\mathcal{I})$,
$$\bigcap_{T \in \mathcal{T}} \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'}) \quad \textit{implies} \quad \bigcap_{T \in \mathcal{T}} T \leq T';$$
- $\mathcal{I}$ *is called* **narrowly system completely reflective**, *or* **narrowly system c-reflective** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^{\nleq}(\mathcal{I})$,
$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \quad \textit{implies} \quad \bigcap_{T \in \mathcal{T}} T \leq T'.$$

The narrow complete reflectivity properties have the following characterizations, paralleling those given for the narrow reflectivity classes, given in Proposition 438.

Proposition 474 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is narrowly family c-reflective if and only if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$ *implies* $\bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T'}$;

(b) $\mathcal{I}$ *is narrowly left c-reflective if and only if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$ *implies* $\bigcap_{T \in \mathcal{T}} \overleftarrow{\widetilde{T}} \leq \overleftarrow{\widetilde{T'}}$;

(c) *$\mathcal{I}$ is narrowly right c-reflective if and only if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\bigcap_{T\in\mathcal{T}} \Omega(\overleftarrow{\widetilde{T}}) \le \Omega(\overleftarrow{\widetilde{T'}})$ implies $\bigcap_{T\in\mathcal{T}} \widetilde{T} \le \widetilde{T'}$;*

(d) *$\mathcal{I}$ is narrowly system c-reflective if and only if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $\{\widetilde{T} : T \in \mathcal{T}\} \cup \{\widetilde{T'}\} \subseteq \mathrm{ThSys}(\mathcal{I})$, $\bigcap_{T\in\mathcal{T}} \Omega(T) \le \Omega(T')$ implies $\bigcap_{T\in\mathcal{T}} \widetilde{T} \le \widetilde{T'}$.*

Proof: The proofs of the various parts mimic those of the corresponding parts for the narrow reflectivity properties, presented in detail in Proposition 438. Thus, we only do Part (b) in detail here, trusting that the reader may easily reproduce the other proofs.

Suppose that $\mathcal{I}$ is narrowly left c-reflective and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \le \Omega(T')$. Then $\{\widetilde{T} : T \in \mathcal{T}\} \cup \{\widetilde{T'}\} \subseteq \mathrm{ThFam}^{\lightning}(\mathcal{I})$ and, by Proposition 369, $\bigcap_{T\in\mathcal{T}} \Omega(\widetilde{T}) \le \Omega(\widetilde{T'})$. Thus, by hypothesis, $\bigcap_{T\in\mathcal{T}} \overleftarrow{\widetilde{\widetilde{T}}} \le \overleftarrow{\widetilde{\widetilde{T'}}}$.

Assume, conversely, that the asserted condition holds and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\lightning}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \le \Omega(T')$. Then, by hypothesis, $\bigcap_{T\in\mathcal{T}} \overleftarrow{\widetilde{\widetilde{T}}} \le \overleftarrow{\widetilde{\widetilde{T'}}}$. Since, however, $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\lightning}(\mathcal{I})$, we get

$$\bigcap_{T\in\mathcal{T}} \overleftarrow{T} = \bigcap_{T\in\mathcal{T}} \overleftarrow{\widetilde{\widetilde{T}}} \le \overleftarrow{\widetilde{\widetilde{T'}}} = \overleftarrow{T'}.$$

Therefore, $\mathcal{I}$ is narrowly left c-reflective. ■

As was shown in Lemma 439, narrow family reflectivity implies exclusive systemicity. Since narrow family c-reflectivity implies narrow family reflectivity, it follows that it also implies exclusive systemicity.

Corollary 475 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is narrowly family c-reflective, then it is exclusively systemic.*

Proof: If $\mathcal{I}$ is narrowly family c-reflective, then it is, a fortiori, narrow family reflective, whence, by Lemma, 439, it is exclusively systemic. ■

Similarly, the fact that narrow right c-reflectivity strengthens narrow right reflectivity, implies immediately the following

Corollary 476 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is narrowly right c-reflective, then it is narrowly stable.*

Proof: Since narrow right c-reflectivity implies narrow right reflectivity, this follows from Corollary 440. ■

We establish, next the *narrow c-reflectivity hierarchy*. The following proposition forms an analog of Proposition 441, which dealt with the narrow reflectivity hierarchy.

Proposition 477 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(c) If $\mathcal{I}$ *is narrowly right c-reflective, then it is narrowly family c-reflective;*

(b) If $\mathcal{I}$ *is narrowly family c-reflective, then it is narrowly left c-reflective;*

(c) If $\mathcal{I}$ *is narrowly left c-reflective, then it is narrowly system c-reflective.*

Proof:

(a) Suppose that $\mathcal{I}$ is narrowly right c-reflective and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nsim}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. By Corollary 476, $\mathcal{I}$ is narrowly stable. Now we obtain $\bigcap_{T \in \mathcal{T}} \Omega(\overleftarrow{T}) = \bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') = \Omega(\overleftarrow{T'})$. Hence, by narrow right c-reflectivity, $\bigcap \mathcal{T} \leq T'$. Hence, $\mathcal{I}$ is narrowly family c-reflective.

(b) Suppose that $\mathcal{I}$ is narrowly family c-reflective and consider $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nsim}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. Then, by hypothesis, $\bigcap_{T \in \mathcal{T}} T \leq T'$, whence, by Lemma 1, $\overleftarrow{\bigcap_{T \in \mathcal{T}} T} \leq \overleftarrow{T'}$. Thus, by Lemma 3, $\bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$. Thus, $\mathcal{I}$ is narrowly left c-reflective.

(c) Suppose that $\mathcal{I}$ is narrowly left c-reflective and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^{\nsim}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. Then, by hypothesis, we get $\bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$. Therefore, since $\mathcal{T} \cup \{T'\}$ is a collection of theory systems, $\bigcap_{T \in \mathcal{T}} T \leq T'$ and, hence, $\mathcal{I}$ is narrowly system c-reflective.

■

We have now established the following **narrow complete reflectivity hierarchy** of π-institutions.

Narrowly R c-Reflective
↓
Narrowly F c-Reflective
↓
Narrowly L c-Reflective
↓
Narrowly S c-Reflective

We give an additional result pertaining to the hierarchy of narrow complete reflectivity properties depicted in the diagram. It forms an analog of Proposition 442, establishing a similar result for the narrow reflectivity hierarchy.

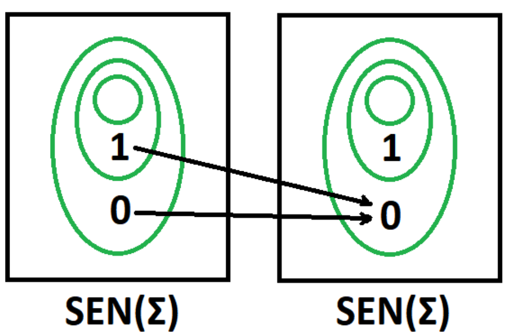
Proposition 478 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is narrowly system c-reflective and narrowly systemic, then it is narrowly right c-reflective.*

Proof: Suppose $\mathcal{I}$ is narrowly system c-reflective and narrowly systemic. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nparallel}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. By narrow systemicity, $T = \overleftarrow{T}$, for all $T \in \mathcal{T}$, and $T' = \overleftarrow{T'}$. Hence, on the one hand, $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$ and, on the other, $\{T : T \in \mathcal{T}\} \cup \{T'\} \subseteq \mathrm{ThSys}^{\nparallel}(\mathcal{I})$. Thus, by narrow system c-reflectivity, $\bigcap_{T \in \mathcal{T}} T \leq T'$. Thus, $\mathcal{I}$ is narrowly right c-reflective. ■

We present three examples to show that all inclusions established between the narrow c-reflectivity classes and shown in the preceding diagram are indeed proper inclusions. The first example depicts a π-institution which is narrowly family c-reflective but not narrowly right c-reflective. This shows that the class of narrowly right c-reflective π-institutions constitutes a proper subclass of the class of narrowly family c-reflective π-institutions.

Example 479 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

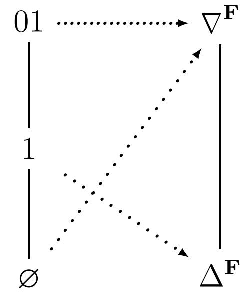
- $\mathbf{Sign}^\flat$ *is the category with the single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$;
- $N^\flat$ *is the trivial clone, consisting of the projections only.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0, 1\}\}.$$

$\mathcal{I}$ *has three theory families,* $\varnothing$, $\{\{1\}\}$ *and* $\{\{0, 1\}\}$, *but only two theory systems,* $\varnothing$ *and* $\{\{0, 1\}\}$. *The lattice of theory families of* $\mathcal{I}$ *and the corresponding Leibniz congruence systems are given in the diagram.*



Since the Leibniz operator is an isomorphism on $\mathrm{ThFam}^{\lightning}(\mathcal{I})$, $\mathcal{I}$ *is narrowly family c-reflective. On the other hand,* $\{\{1\}\}, \{\{0,1\}\} \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$ *and*

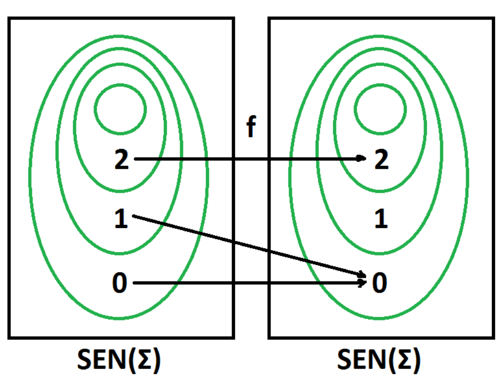
$$\Omega(\overleftarrow{\{\{1\}\}}) = \Omega(\{\varnothing\}) = \nabla^{\mathbf{F}} = \Omega(\{\{0,1\}\}) = \Omega(\overleftarrow{\{\{0,1\}\}}),$$

but $\{\{1\}\} \neq \{\{0,1\}\}$. *Therefore,* $\mathcal{I}$ *is not narrowly right injective and, a fortiori, it fails to be narrowly right c-reflective.*

The next example depicts a π-institution which is narrowly left c-reflective but not narrowly family c-reflective. This shows that the class of narrowly family c-reflective π-institutions is a proper subclass of the class of narrowly left c-reflective π-institutions.

Example 480 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

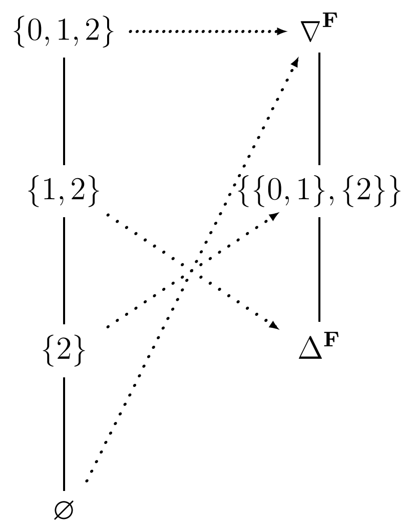
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$, $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$;
- $N^\flat$ *is the trivial clone.*



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\varnothing, \{2\}, \{1,2\}, \{0,1,2\}\}.$$

$\mathcal{I}$ has four theory families, but only three theory systems, namely $\varnothing$, $\{2\}$ and $\{0,1,2\}$. Moreover, clearly, $\mathrm{ThFam}^{\nleq}(\mathcal{I}) = \{\{2\},\{1,2\},\{0,1,2\}\}$. The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right:



To show that $\mathcal{I}$ is narrowly left c-reflective, let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nleq}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. We distinguish the following cases, based on the value of $\Omega(T')$:

- *If $\Omega(T') = \nabla^{\mathbf{F}}$, then $T' = \mathrm{SEN}^\flat$. Hence, we get*

$$\bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \mathrm{SEN}^\flat = \overleftarrow{\mathrm{SEN}^\flat} = \overleftarrow{T'};$$

- *If $\Omega(T') = \{\{0,1\},\{2\}\}$, then $T' = \{2\}$. Then, by hypothesis, $\{2\} \in \mathcal{T}$ or $\{1,2\} \in \mathcal{T}$. Hence, in either case, $\bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \{2\} = T' = \overleftarrow{T'}$;*

- *If $\Omega(T') = \Delta^{\mathbf{F}}$, then $T' = \{1,2\}$ and, by hypothesis, $\{1,2\} \in \mathcal{T}$. Hence, in this case as well, $\bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{\{1,2\}} = \overleftarrow{T'}$.*

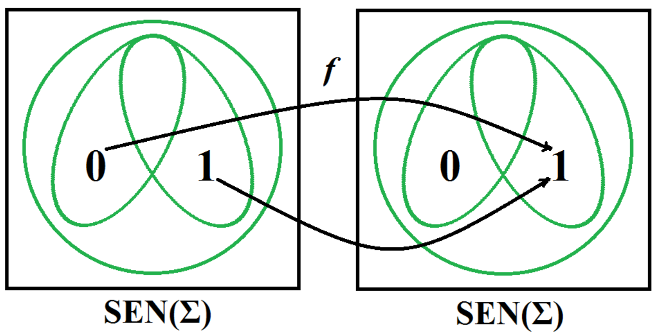
Thus, $\mathcal{I}$ is narrowly left c-reflective.

On the other hand, for $T = \{1,2\}$ and $T' = \{2\}$, even though $\Omega(T) \leq \Omega(T')$, we get $T \nleq T'$, whence $\mathcal{I}$ fails to be narrowly family reflective and, hence, a fortiori, it is not narrow family c-reflective.

We finish the sequence of examples by presenting a narrowly system c-reflective π-institution which fails to be narrowly left c-reflective. This example shows that narrowly left c-reflective π-institutions form a proper subclass of the class of narrowly system c-reflective π-institutions.

Example 481 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 1$ *and* $\mathrm{SEN}^\flat(f)(1) = 1$*;*
- $N^\flat$ *is the trivial clone.*

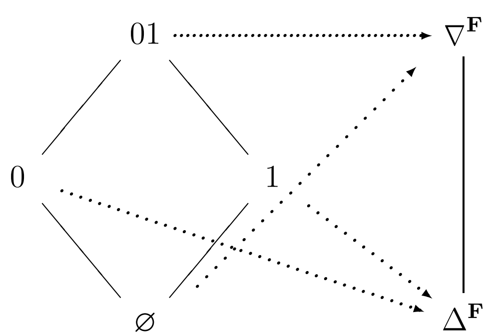


Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\varnothing, \{0\}, \{1\}, \{0,1\}\}$.

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$.

T	$\overleftarrow{T}$
$\varnothing$	$\varnothing$
$\{0\}$	$\varnothing$
$\{1\}$	$\{1\}$
$\{0,1\}$	$\{0,1\}$

The lattice of theory families and the corresponding Leibniz congruence systems are depicted below.



It is obvious from the diagram that the Leibniz operator is an isomorphism on $\mathrm{ThSys}^{\nmid}(\mathcal{I})$. *Therefore,* $\mathcal{I}$ *is narrowly system c-reflective. On the other hand, for* $T = \{\{0\}\}$, $T' = \{\{1\}\}$, *both members of* $\mathrm{ThFam}^{\nmid}(\mathcal{I})$, *we have* $\Omega(T) = \Omega(T') = \Delta^{\mathbf{F}}$, *whereas* $\overleftarrow{T} = \{\varnothing\} \neq \{\{1\}\} = \overleftarrow{T'}$. *Therefore,* $\mathcal{I}$ *is not narrowly left injective and, a fortiori, it fails to be narrowly left c-reflective.*

We turn now to the relationships between corresponding classes of the rough complete reflectivity and the narrow complete reflectivity hierarchies. These parallel the ones already established between the rough reflectivity and narrow reflectivity classes.

Using the characterization in Part (a) of Proposition 474, we can immediately see that the two types of family complete reflectivity coincide.

Corollary 482 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is roughly family c-reflective if and only if it is narrowly family c-reflective.*

Proof: Part (a) of Proposition 474. ■

As was the case with rough and narrow reflectivity properties, the relationships between the remaining classes are more involved. Starting with the two left complete reflectivity classes, we show that the class of narrowly left c-reflective π-institutions is not included in the class of roughly left c-reflective π-institutions. This is accomplished by constructing a π-institution which is narrowly left c-reflective but not roughly left c-reflective.

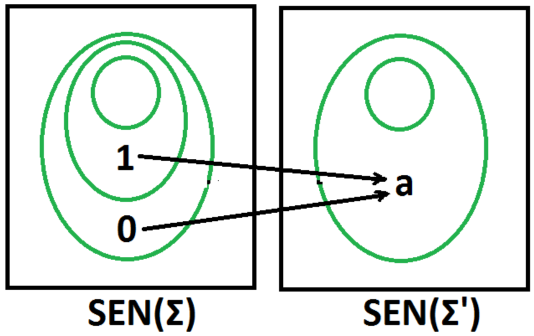
Example 483 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a\}$ *and* $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = a$;
- $N^\flat$ *is the trivial clone.*

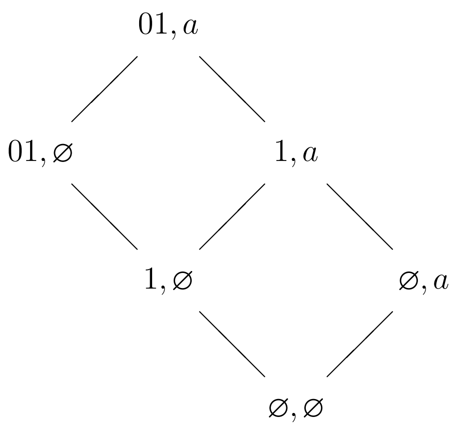
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{a\}\}.$$

Clearly, there are six theory families in $\mathrm{ThFam}(\mathcal{I})$, *only four of which are theory systems, and only two of which are in* $\mathrm{ThFam}^{\nmid}(\mathcal{I})$. *The lattice of*



theory families is shown in the diagram:



To see that $\mathcal{I}$ is narrowly left c-reflective, let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\not\in}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. We reasons by cases, depending on the value of $\Omega(T')$:

- *If $\Omega(T') = \nabla^{\mathbf{F}}$, then $T' = \mathrm{SEN}^\flat$. So we get $\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \leq \mathrm{SEN}^\flat = \overleftarrow{\mathrm{SEN}^\flat} = \overleftarrow{T'}$;*
- *If $\Omega(T') = \Delta^{\mathbf{F}}$, then $T' = \{1, a\}$ and, by hypothesis, we must have $T' \in \mathcal{T}$. Thus, $\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$.*

We conclude that $\mathcal{I}$ is narrowly left c-reflective.

On the other hand, consider $T = \{1, \varnothing\}$ and $T' = \{1, a\}$. We have $\Omega(1, \varnothing) = \{\Delta^{\mathbf{F}}_{\Sigma}, \nabla^{\mathbf{F}}_{\Sigma'}\} = \Omega(1, a)$, but

$$\overleftarrow{\widetilde{1,\varnothing}} = \widetilde{\varnothing,\varnothing} = \{01, a\} \not\leq \{1, a\} = \widetilde{1,a} = \overleftarrow{\widetilde{1,a}}.$$

This proves that $\mathcal{I}$ is not roughly left reflective and, hence, a fortiori, it fails to be roughly left c-reflective.

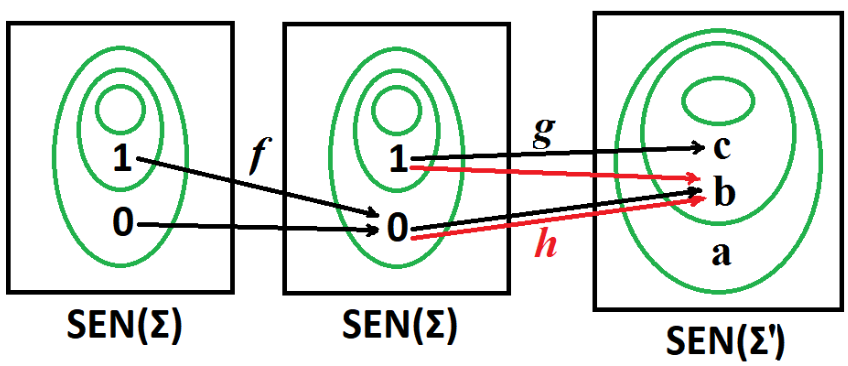
We exhibit, next a π-institution that is roughly left c-reflective, while it fails to be narrowly left c-reflective. Combined with Example 483, this will show that the two left complete reflectivity classes, rough and narrow, are incomparable from the point of view of inclusion.

Example 484 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and three nonidentity morphisms* $f : \Sigma \to \Sigma$ *and* $g, h : \Sigma \to \Sigma'$, *such that* $f \circ f = f$, $g \circ f = h$ *and* $h \circ f = h$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b,c\}$, $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = 0$, $\mathrm{SEN}^\flat(g)(0) = b$, $\mathrm{SEN}^\flat(g)(1) = c$ *and* $\mathrm{SEN}^\flat(h)(0) = \mathrm{SEN}^\flat(h)(1) = b$;
- $N^\flat$ *is the clone generated by a single binarry natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$, *whose components are defined by the following tables:*

$\sigma^\flat_\Sigma$	0	1
0	0	1
1	1	1

$\sigma^\flat_{\Sigma'}$	a	b	c
a	a	a	c
b	a	b	c
c	c	c	c



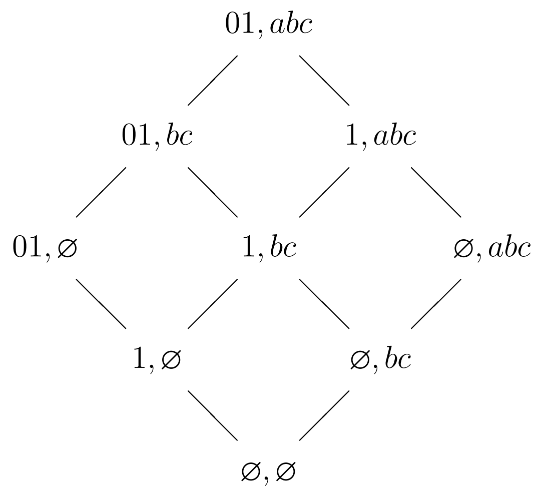
It is not difficult, albeit slightly tedious, to check that this is a well-defined natural transformation. We summarize the checking in the accompanying table.

(x,y)	$f(\sigma^\flat_\Sigma(x,y)) = \sigma^\flat_\Sigma(f(x),f(y))$	$g(\sigma^\flat_\Sigma(x,y)) = \sigma^\flat_{\Sigma'}(g(x),g(y))$	$h(\sigma^\flat_\Sigma(x,y)) = \sigma^\flat_{\Sigma'}(h(x),h(y))$
$(0,0)$	$0=0$	$b=b$	$b=b$
$(0,1)$	$0=0$	$c=c$	$b=b$
$(1,0)$	$0=0$	$c=c$	$b=b$
$(1,1)$	$0=0$	$c=c$	$b=b$

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b,c\}, \{a,b,c\}\}.$$

Clearly, there are nine theory families in $\mathrm{ThFam}(\mathcal{I})$*, five of which are theory systems, and four of which are in* $\mathrm{ThFam}^{\nmid}(\mathcal{I})$*. The lattice of theory families is shown in the diagram:*



The action of $\overleftarrow{\ }$ *on theory families is given in the following table.*

T	$\overleftarrow{T}$	T	$\overleftarrow{T}$
$01, abc$	$01, abc$	$\varnothing, abc$	$\varnothing, abc$
$01, bc$	$01, bc$	$1, \varnothing$	$\varnothing, \varnothing$
$1, abc$	$\varnothing, abc$	$\varnothing, bc$	$\varnothing, bc$
$01, \varnothing$	$\varnothing, \varnothing$	$\varnothing, \varnothing$	$\varnothing, \varnothing$
$1, bc$	$\varnothing, bc$		

The table below provides the Leibniz congruence systems associated with the theory families of $\mathcal{I}$*.*

T	$\Omega(T)$
$\{01, abc\}, \{01, \varnothing\}, \{\varnothing, abc\}, \{\varnothing, \varnothing\}$	$\nabla^{\mathbf{F}}$
$\{1, abc\}, \{1, \varnothing\}$	$\{\Delta^{\mathbf{F}}_{\Sigma}, \nabla^{\mathbf{F}}_{\Sigma'}\}$
$\{01, bc\}, \{1, bc\}, \{\varnothing, bc\}$	$\Delta^{\mathbf{F}}$

To see that $\mathcal{I}$ *is roughly left c-reflective, suppose that* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$*, such that* $\bigcap_{T \in \mathcal{T}} \Omega(T) \le \Omega(T')$*. We separate cases depending on* $\Omega(T')$*.*

- *If* $\Omega(T') = \Delta^{\mathbf{F}}$*, then, by hypothesis, at least one among* $\{01, bc\}$*,* $\{1, bc\}$*,* $\{\varnothing, b\}$ *must be in* $\mathcal{T}$*. But, then, we get* $\bigcap_{T \in \mathcal{T}} \overleftarrow{\overleftarrow{T}} \le \{01, bc\} = \overleftarrow{\overleftarrow{T'}}$*;*
- *If* $\Omega(T') = \{\Delta^{\mathbf{F}}_{\Sigma}, \nabla^{\mathbf{F}}_{\Sigma'}\}$ *or* $\Omega(T') = \nabla^{\mathbf{F}}$*, then* $\bigcap_{T \in \mathcal{T}} \overleftarrow{\overleftarrow{T}} \le \{01, abc\} = \overleftarrow{\overleftarrow{T'}}$*.*

On the other hand, for $T = \{01, bc\}$ *and* $T' = \{1, bc\}$*, we get* $\Omega(T) = \Delta^{\mathbf{F}} = \Omega(T')$*, whereas* $\overleftarrow{T} = \{01, bc\} \nleq \{\varnothing, bc\} = \overleftarrow{T'}$*. Therefore,* $\mathcal{I}$ *is not narrowly left c-reflective.*

We turn, next to the relationship between the two kinds of right c-reflectivity. We show, first, that rough right c-reflectivity implies narrow right c-reflectivity, a direct analog of Proposition 449, which established the corresponding result for the two right reflectivity classes.

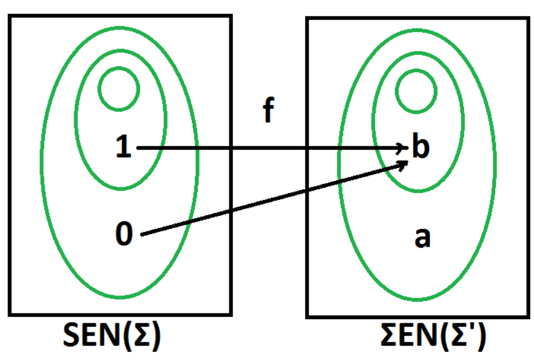
Proposition 485 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is roughly right c-reflective, then it is narrowly right c-reflective.*

Proof: Suppose $\mathcal{I}$ is roughly right reflective and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nmid}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. By rough right reflectivity, we get that $\bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T'}$. Since, however, $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nmid}(\mathcal{I})$, we get $\bigcap_{T \in \mathcal{T}} T = \bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T'} = T'$. Therefore, $\mathcal{I}$ is narrowly right c-reflective. ■

The converse, on the other hand, does not hold in general, as the following example demonstrates.

Example 486 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

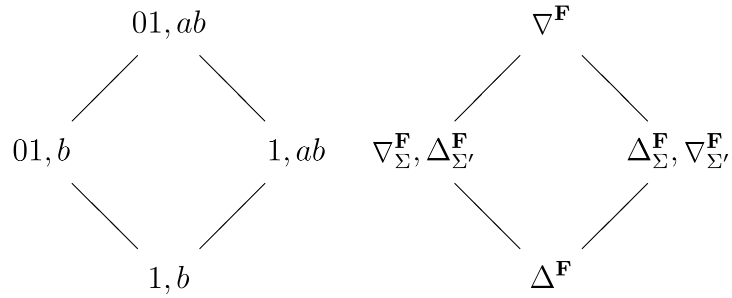
- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique morphism* $f : \Sigma \to \Sigma'$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = b$, $\mathrm{SEN}^\flat(f)(1) = b$;
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0, 1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a, b\}\}.$$

Clearly, there are only four theory families in $\mathrm{ThFam}^{\nmid}(\mathcal{I})$, *all of which are theory systems. Their lattice together with the associated Leibniz congruence* *systems are shown in the diagram:*



From this diagram and the fact that all theory families in $\mathrm{ThFam}^{\nsubseteq}(\mathcal{I})$ *are theory systems, we see that, for all* $T, T' \in \mathrm{ThFam}^{\nsubseteq}(\mathcal{I})$,

$$\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'}) \quad \textit{iff} \quad \Omega(T) \leq \Omega(T') \quad \textit{iff} \quad T \leq T'.$$

Therefore, $\mathcal{I}$ *is indeed narrowly right c-reflective.*

On the other hand, consider $T = \{01, ab\}$ *and* $T' = \{1, \varnothing\}$*. Then we have*

$$\Omega(\overleftarrow{T}) = \Omega(01, ab) = \nabla^{\mathbf{F}} = \Omega(\overline{\varnothing}) = \Omega(\overleftarrow{T'}),$$

whereas $\widetilde{01, ab} = \{01, ab\} \nleq \{1, ab\} = \widetilde{1, \varnothing}$*. This shows that* $\mathcal{I}$ *is not roughly right reflective and, hence, a fortiori, it fails to be roughly right c-reflective.*

Finally, we look at system complete reflectivity. We show that rough system c-reflectivity implies narrow system c-reflectivity, but that the converse implication fails in general.

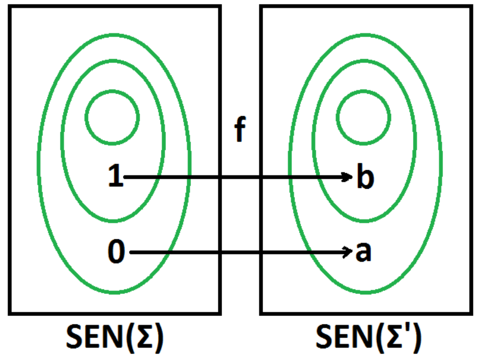
Proposition 487 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly system c-reflective, then it is narrowly system c-reflective.*

Proof: Suppose $\mathcal{I}$ is roughly system c-reflective and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^{\nsubseteq}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. Then, by rough system c-reflectivity, $\bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T'}$. However, since $\mathcal{T} \cup \{T'\} \in \mathrm{ThSys}^{\nsubseteq}(\mathcal{I})$, we get $\bigcap_{T \in \mathcal{T}} T = \bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T'} = T'$. Therefore, $\mathcal{I}$ is narrowly system c-reflective. ■

We present an example of a π-institution that is narrowly system c-reflective but not roughly system c-reflective. This, combined with Proposition 487, shows that the class of narrowly system c-reflective π-institutions properly contains the class of roughly system c-reflective π-institutions.

Example 488 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique morphism* $f : \Sigma \to \Sigma'$;

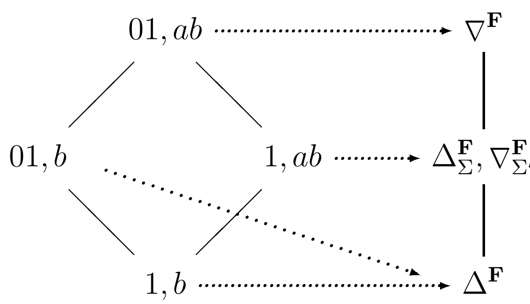


- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$*;*
- $N^\flat$ *is the trivial clone.*

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a,b\}\}.$$

There are only four theory families in $\mathrm{ThFam}^{\not\,}(\mathcal{I})$*, all of which except for* $\{01, b\}$ *are theory systems. Their lattice together with the associated Leibniz congruence systems are shown in the diagram:*

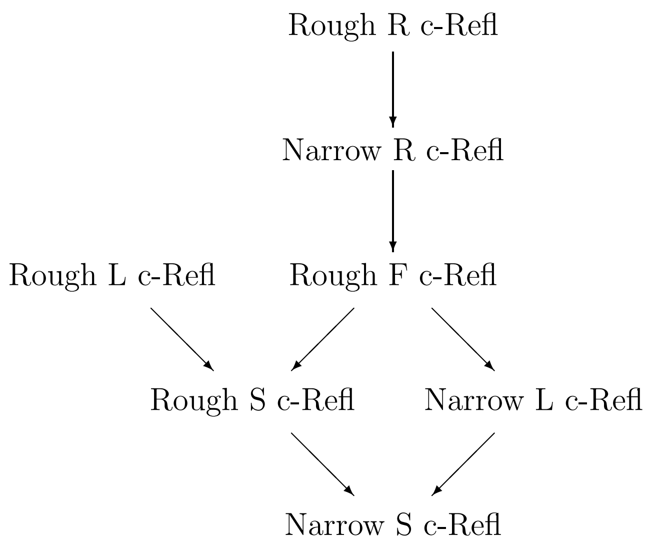


To see that $\mathcal{I}$ *is narrowly system c-reflective, let* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^{\not\,}(\mathcal{I})$*, such that* $\bigcap_{T\in\mathcal{T}} \Omega(T) \le \Omega(T')$*. We distinguish three cases, depending on the value of* $\Omega(T')$*:*

- *If* $\Omega(T') = \nabla^{\mathbf{F}}$*, then* $T' = \mathrm{SEN}^\flat$*. Hence,* $\bigcap_{T\in\mathcal{T}} T \le \mathrm{SEN}^\flat = T'$*;*
- *If* $\Omega(T') = \{\Delta^{\mathbf{F}}_\Sigma, \nabla^{\mathbf{F}}_{\Sigma'}\}$*, then* $T' = \{1, ab\}$*, whence, by hypothesis,* $T' \in \mathcal{T}$ *or* $\{1, b\} \in \mathcal{T}$*. In either case,* $\bigcap_{T\in\mathcal{T}} T \le \{1, ab\} = T'$*;*
- *If* $\Omega(T') = \Delta^{\mathbf{F}}$*, then* $T' = \{1, b\}$ *and, hence, by hypothesis,* $T' \in \mathcal{T}$*, which shows that* $\bigcap_{T\in\mathcal{T}} T \le T'$*.*

On the other hand, consider $T = \{\varnothing, b\}$*,* $T' = \{1, b\} \in \mathrm{ThSys}(\mathcal{I})$*. Even though* $\widetilde{T} = \{01, b\} \nleq \{1, b\} = \widetilde{T'}$*, we have* $\Omega(T) = \Delta^{\mathbf{F}} = \Omega(T')$*. Hence,* $\mathcal{I}$ *is not roughly system reflective and, hence, a fortiori, it is not roughly system c-reflective.*

The results obtained and the counterexamples presented, thus far, reveal the following mixed hierarchy of rough and narrow c-reflectivity classes of π-institutions, paralleling the one presented for rough and narrow reflectivity properties.



We have already used in the context of the preceding examples the fact that a narrow c-reflectivity property implies the corresponding narrow reflectivity property, since the latter is a special case of the former in which $\mathcal{T}$ is taken to be a singleton. These observations are formalized in the following

Proposition 489 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is narrowly family c-reflective, then it is narrowly family reflective;*

(b) If $\mathcal{I}$ *is narrowly left c-reflective, then it is narrowly left reflective;*

(c) If $\mathcal{I}$ *is narrowly right c-reflective, then it is narrowly right reflective;*

(d) If $\mathcal{I}$ *is narrowly system c-reflective, then it is narrowly system reflective.*

Proof: All four reflectivity properties are special cases of the corresponding c-reflectivity properties, in which $\mathcal{T}$ is taken to be a singleton collection of theory families. ■

Turning to the relationships between narrow c-reflectivity classes and corresponding c-reflectivity classes, we prove a theorem, analogous to Theorem 454, asserting that ordinary c-reflectivity is equivalent to narrow c-reflectivity in the presence of theorems.

Theorem 490 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) *$\mathcal{I}$ is family c-reflective if and only if it is narrowly family c-reflective and has theorems;*

(b) *$\mathcal{I}$ is left c-reflective if and only if it is narrowly left c-reflective and has theorems;*

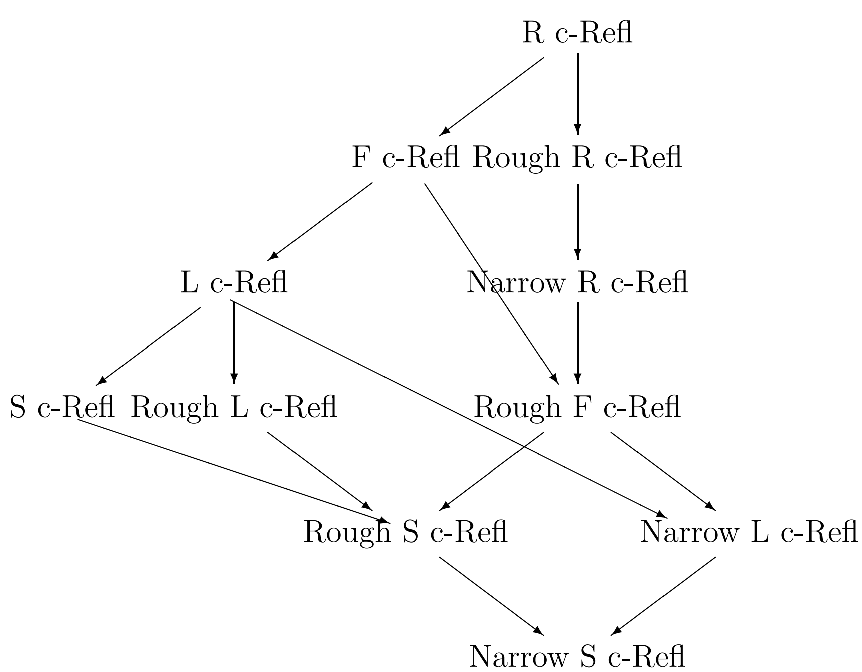
(c) *$\mathcal{I}$ is right c-reflective if and only if it is narrowly right c-reflective and has theorems;*

(d) *$\mathcal{I}$ is system c-reflective if and only if it is narrowly system c-reflective and has theorems.*

Proof: By Theorem 468, if $\mathcal{I}$ has one of the four complete reflectivity properties, then it has theorems. Moreover, by the same theorem, a complete reflectivity property implies the corresponding rough complete reflectivity property and, by Corollary 482, Proposition 485 and Proposition 487, each implies the corresponding narrow complete reflectivity property except in the case of left complete reflectivity, where (as actually in all other cases, as well) one can easily see directly, that left c-reflectivity implies narrow left c-reflectivity, since the defining condition of the latter is a special case of that of the former.

All converses are also easily verified, since, in the presence of theorems, $\mathrm{ThFam}^{\nmid}(\mathcal{I}) = \mathrm{ThFam}(\mathcal{I})$ and $\mathrm{ThSys}^{\nmid}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$, which makes the four defining conditions for the narrow c-reflectivity classes identical with the corresponding conditions for the ordinary c-reflectivity classes. ■

We now have the following hierarchy, paralleling the mixed reflectivity and narrow reflectivity hierarchy, given previously.



The narrow complete reflectivity properties transfer from the theory families/systems of a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ to all $\mathcal{I}$-filter families/systems on arbitrary $\mathbf{F}$-algebraic systems. This result forms an analog of Theorem 455, which applied to narrow reflectivity classes.

Theorem 491 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a π-institution based on* $\mathbf{F}$.

(a) *$\mathcal{I}$ is narrowly right c-reflective if and only if, for all $\mathbf{F}$-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}\natural}(\mathcal{A})$, $\bigcap_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(\overleftarrow{T}) \leq \Omega^{\mathcal{A}}(\overleftarrow{T'})$ *implies* $\bigcap_{T\in\mathcal{T}} T \leq T'$;

(b) *$\mathcal{I}$ is narrowly family c-reflective if and only if, for all $\mathbf{F}$-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}\natural}(\mathcal{A})$, $\bigcap_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ *implies* $\bigcap_{T\in\mathcal{T}} T \leq T'$;

(c) *$\mathcal{I}$ is narrowly left c-reflective if and only if, for all $\mathbf{F}$-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}\natural}(\mathcal{A})$, $\bigcap_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ *implies* $\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$;

(d) *$\mathcal{I}$ is narrowly system c-reflective if and only if, for all $\mathbf{F}$-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiSys}^{\mathcal{I}\natural}(\mathcal{A})$, $\bigcap_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ *implies* $\bigcap_{T\in\mathcal{T}} T \leq T'$.

Proof: The proof follows the steps of the proofs of the various parts of Theorem 455. We do Part (a) in detail to give a flavor of what is involved.

The "if" follows by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$ and taking into account that $\mathrm{ThFam}^\natural(\mathcal{I}) = \mathrm{FiFam}^{\mathcal{I}\natural}(\mathcal{F})$, by Lemmas 51 and 376.

For the "only if", suppose that $\mathcal{I}$ is narrowly right c-reflective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}\natural}(\mathcal{A})$, such that $\bigcap_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(\overleftarrow{T}) \leq \Omega^{\mathcal{A}}(\overleftarrow{T'})$. Then $\alpha^{-1}(\bigcap_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(\overleftarrow{T})) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'}))$. Thus, $\bigcap_{T\in\mathcal{T}} \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T})) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'}))$. So, by Proposition 24, $\bigcap_{T\in\mathcal{T}} \Omega(\alpha^{-1}(\overleftarrow{T})) \leq \Omega(\alpha^{-1}(\overleftarrow{T'}))$. Hence, by Lemma 6, $\bigcap_{T\in\mathcal{T}} \Omega(\overleftarrow{\alpha^{-1}(T)}) \leq \Omega(\overleftarrow{\alpha^{-1}(T')})$. Since, by Lemmas 51 and 376, $\{\alpha^{-1}(T) : T \in \mathcal{T}\} \cup \{\alpha^{-1}(T')\} \subseteq \mathrm{ThFam}^\natural(\mathcal{I})$, we get, by applying narrow right c-reflectivity, $\bigcap_{T\in\mathcal{T}} \alpha^{-1}(T) \leq \alpha^{-1}(T')$ or, equivalently, $\alpha^{-1}(\bigcap_{T\in\mathcal{T}} T) \leq \alpha^{-1}(T')$. This yields, taking into account the surjectivity of $\langle F, \alpha\rangle$, $\bigcap_{T\in\mathcal{T}} T \leq T'$. ■

We finally recast narrow complete reflectivity in terms of the complete order reflectivity of mappings from posets of theory or filter families/systems into posets of congruence systems. The following results form analogs of Propositions 456 and 457, respectively, addressing complete reflectivity instead of reflectivity properties.

Proposition 492 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a π-institution based on* $\mathbf{F}$. *The following statements are equivalent:*

(a) *$\mathcal{I}$ is narrowly family c-reflective;*

(b) $\Omega : \mathrm{ThFam}^{\nleq}(\mathcal{I}) \to \mathrm{ConSys}^{*}(\mathcal{I})$ *is completely order reflecting;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}\nleq}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is completely order reflecting, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$.

Similarly, for system c-reflectivity, we have

Proposition 493 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) *$\mathcal{I}$ is narrowly system c-reflective;*

(b) $\Omega : \mathrm{ThSys}^{\nleq}(\mathcal{I}) \to \mathrm{ConSys}^{*}(\mathcal{I})$ *is completely order reflecting;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}\nleq}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is completely order reflecting, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$.

6.10 Availability of Theorems

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall that, by convention, if $\mathcal{I}$ has theorems, then, for every $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathcal{I}$ has a Σ-theorem, i.e., there exists $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\varnothing)$.

Recall, also, from our work in the present chapter, that all levels of the injectivity, reflectivity and complete reflectivity hierarchies imply the existence of theorems and that, moreover, any rough injectivity, rough reflectivity or rough complete reflectivity property, complemented with the existence of theorems, implies the corresponding (gentle) injectivity, reflectivity or complete reflectivity property, respectively. In other words, insisting on existence of theorems causes all pairs of rough and gentle properties to collapse to a single class.

In this section, due to the importance of the property of "having theorems", we give a few more results characterizing that property.

It turns out that existence of theorems is tantamount to the injectivity of the local Frege operator λ.

Theorem 494 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. $\mathcal{I}$ has theorems if and only if λ is injective.*

Proof: Suppose, first, that $\mathcal{I}$ has theorems. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\lambda(T) = \lambda(T')$. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$. Since $\mathcal{I}$ has theorems, there exists $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$. Then $t \in T_\Sigma$ and, therefore, $\langle \phi, t \rangle \in \lambda_\Sigma(T)$. By hypothesis, $\langle \phi, t \rangle \in \lambda_\Sigma(T')$. But, clearly, $t \in T'_\Sigma$. Hence

$\phi \in T'_\Sigma$. We conclude that $T \le T'$ and, by symmetry, $T = T'$. Thus, λ is injective.

Assume, conversely, that $\mathcal{I}$ does not have theorems. Then, we have $\varnothing, \mathrm{SEN}^\flat \in \mathrm{ThFam}(\mathcal{I})$, with $\varnothing \ne \mathrm{SEN}^\flat$, whereas $\lambda(\varnothing) = \lambda(\mathrm{SEN}^\flat) = \nabla^{\mathbf{F}}$. Therefore, λ is not injective. ∎

It turns out that existence of theorems is also equivalent to both the injectivity and the c-reflectivity of the local Lindenbaum operator $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}$ on all $\mathbf{F}$-algebraic systems.

Theorem 495 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following conditions are equivalent:*

(i) $\mathcal{I}$ *has theorems;*

(ii) $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}$ *is injective, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*;*

(iii) $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}$ *is completely reflective, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

Proof:

(i)⇒(iii) Assume that $\mathcal{I}$ has theorems and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, be an $\mathbf{F}$-algebraic system, $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that

$$\bigcap_{T \in \mathcal{T}} \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \le \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T').$$

Let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in \bigcap_{T \in \mathcal{T}} T_\Sigma$. By hypothesis and the surjectivity of $\langle F, \alpha \rangle$, there exists $t \in C_\Sigma^{\mathcal{I},\mathcal{A}}(\varnothing)$. Then, we have

$$C_\Sigma^{\mathcal{I},\mathcal{A}}(\bigcap_{T \in \mathcal{T}} T_\Sigma, \phi) = C_\Sigma^{\mathcal{I},\mathcal{A}}(\bigcap_{T \in \mathcal{T}} T_\Sigma) = C_\Sigma^{\mathcal{I},\mathcal{A}}(\bigcap_{T \in \mathcal{T}} T_\Sigma, t).$$

Thus, we get

$$\langle \phi, t \rangle \in \widetilde{\lambda}_\Sigma^{\mathcal{I},\mathcal{A}}(\bigcap_{T \in \mathcal{T}} T) \le \bigcap_{T \in \mathcal{T}} \widetilde{\lambda}_\Sigma^{\mathcal{I},\mathcal{A}}(T) \le \widetilde{\lambda}_\Sigma^{\mathcal{I},\mathcal{A}}(T').$$

We conclude that

$$\begin{array}{rcll} \phi & \in & C_\Sigma^{\mathcal{I},\mathcal{A}}(T'_\Sigma, \phi) & \text{(inflationarity)} \\ & = & C_\Sigma^{\mathcal{I},\mathcal{A}}(T'_\Sigma, t) & (\langle \phi, t \rangle \in \widetilde{\lambda}_\Sigma^{\mathcal{I},\mathcal{A}}(T')) \\ & = & T'_\Sigma. & (t \in C_\Sigma^{\mathcal{I},\mathcal{A}}(\varnothing)) \end{array}$$

Therefore, $\bigcap_{T \in \mathcal{T}} T \le T'$ and $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}$ is completely reflective.

(iii)⇒(ii) Complete reflectivity implies injectivity.

(ii)⇒(i) Finally, suppose that $\mathcal{I}$ does not have theorems. We let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be the trivial $\mathbf{F}$-algebraic system, with single signature object $*$ and singleton $\mathrm{SEN}(*) = \{0\}$. Since $\mathcal{I}$ does not have theorems, both $\varnothing$ and SEN are $\mathcal{I}$-filter families of $\mathcal{A}$. Now we have

$$\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\varnothing) = \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\mathrm{SEN}) = \nabla^{\mathcal{A}}.$$

Hence, the Leibniz operator $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}$ is not injective.

■

The property of having theorems clearly transfers from a π-institution to all its gmatrix families.

Theorem 496 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *has theorems if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $T \neq \overline{\varnothing}$*.*

Proof: The right-to-left inclusion follows by considering the algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. For the converse, assume $\mathcal{I}$ has theorems and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Let $\Sigma \in |\mathbf{Sign}^\flat|$. Then, there exists $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$. By definition, $\alpha_\Sigma(t) \in T_{F(\Sigma)}$. Hence, $T \neq \overline{\varnothing}$. ■

Note that an alterrnative way of expressing the assertion of Theorem 496 is to say that $\mathcal{I}$ has theorems if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, the π-institution $\langle \mathbf{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ has theorems.