Chapter 8

The Semantic Leibniz Hierarchy: Over the Top I

8.1 Introduction

The prototypical example of an algebraizable deductive system, classical propositional calculus, has the additional distinctive feature of being *1-algebraizable* or *regularly algebraizable* (see, e.g., Chapter 5 of [64], Chapter 3 (p. 66) of [52] and Section 3.4 of [86]). This means that any two theorems are equivalent or, more generally, that any two sentences belonging to a theory T are equivalent relative to T. In this chapter, we undertake the study of regularity, a property that, when added to algebraizability, yields regular algebraizability, featured among the topmost classes in the entire semantic hierarchy discussed in this monograph.

In Section 8.2, we introduce and study the basic *regularity properties*, which form the basis for developing the regular algebraizability classes in Sections 8.4-8.7. A π-institution $\mathcal{I}$ is *family regular* if, for every theory family T of $\mathcal{I}$, all signatures Σ and all Σ-sentences ϕ and ψ, $\phi,\psi \in T_\Sigma$ implies $\langle\phi,\psi\rangle \in \Omega_\Sigma(T)$. It is *left regular* if it satisfies the same condition, but with T in the hypothesis replaced by $\overleftarrow{T}$, and *right regular* if T is replaced by $\overleftarrow{T}$ in the conclusion instead. Finally, $\mathcal{I}$ is *system regular* if, in the implication defining family regularity, T is restricted to range only over theory systems, instead of being allowed to range over arbitrary theory families. These four properties form a linear hierarchy, with family regularity being the strongest, followed by right regularity, then by left regularity, with the system version being the weakest of the four. Stability causes the collapse of this hierarchy into two levels, since, under stability, system regularity implies left regularity and right regularity implies family regularity. More transparently, systemicity causes a total collapse of the hierarchy into a single class. The family, left and system versions have characterizations involving the Suszko operator and one of its variants. For a sneak preview, $\mathcal{I}$ is system regular if and only if for every signature Σ and all Σ-sentences ϕ and ψ, $\langle\phi,\psi\rangle \in \widehat{\Omega}^{\mathcal{I}}_\Sigma(\overrightarrow{C}(\phi,\psi))$, where $\overrightarrow{C}(\phi,\psi)$ is the least theory system of $\mathcal{I}$ containing ϕ and ψ and $\widehat{\Omega}^{\mathcal{I}}$: $\mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}(\mathcal{I})$ gives, for a given theory system T of $\mathcal{I}$, the largest congruence system $\widehat{\Omega}^{\mathcal{I}}(T)$ compatible with every theory system including T. All four regularity properties transfer, e.g., looking at right regularity, it holds for a π-institution $\mathcal{I}$ if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, all $\mathcal{I}$-filter families T of $\mathcal{A}$, all signatures Σ of $\mathcal{A}$ and all Σ-sentences ϕ, ψ, $\phi,\psi \in T_\Sigma$ implies $\langle\phi,\psi\rangle \in \Omega^{\mathcal{A}}_\Sigma(\overleftarrow{T})$. Finally, the family and system versions have natural characterizations in terms of the form of the filter families/systems, respectively, of the reduced matric families/systems of $\mathcal{I}$. The condition here is that, if $\langle\mathcal{A},T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$, then T is at most a singleton, i.e., each of its components T_Σ has at most one element.

In Section 8.3, we look at *assertionality*, which is the property ensuing from regularity when existence of theorems is also postulated. Thus, a π-institution $\mathcal{I}$ is *family*, *right*, *left* or *system assertional* if it is family, right, left or system regular, respectively, and has theorems. The hierarchy of regularity properties established in Section 8.2 immediately yields an a priori linear assertionality hierarchy with four classes, the family version implying the right, which, in turn, implies the left version, with the system being the weakest of the four versions. However, it turns out that right assertionality is strong enough to imply systemicity and, as a consequence, the family and right versions are equivalent. Thus, the hierarchy consists of only three distinct classes. The weakest property, system assertionality, coupled with systemicity, is equivalent to the strongest, family assertionality. It is straightforward by the definitions that each asertionality property implies its regularity counterpart. More interestingly, each assertionality property implies the corresponding complete reflectivity (c-reflectivity) property (see Section 3.8). All three versions of assertionality transfer. This follows from the fact that both regularity and existence of theorems transfer. Additionally, based on the characterizations of family and system regularity in terms of reduced matrix families/systems, one may obtain similar characterizations of family/system assertionality. Again, for the sake of preview, the condition characterizing family assertionality is that, for every reduced $\mathcal{I}$-matrix family $\langle \mathcal{A}, T \rangle$, T is a singleton, i.e., $|T_\Sigma| = 1$, for all signatures Σ of $\mathcal{A}$.

Having discussed, to some extent, the foundations in Sections 8.2 and 8.3, we embark, in Section 8.4, on the study of algebraizability properties, starting with *regular weak prealgebraizability*. The three classes defined here reflect the type of asssertionality combined with prealgebraicity. Accordingly, a π-institution $\mathcal{I}$ is *regularly weakly family* (RWF) *prealgebraizable* if it is prealgebraic and family assertional. It is *regularly weakly left* (RWL) *prealgebraizable* if it is prealgebraic and left assertional, and it is *regularly weakly system* (RWS) *prealgebraizable* if it is prealgebraic and system assertional. The hierarchy of assertionality properties of Section 8.3 yields that RWF prealgebraizability implies RWL prealgebraizability, which, in turn, implies RWS prealgebraizability. By definition, RWF/L/S prealgebraizability implies, respectively, family/left/system assertionality. More noteworthy, however, is the fact that, since each version of assertionality implies the corresponding c-reflectivity version, RWF/L/S prealgebraizability implies, respectively, WF/L/SC prealgebraizability (see Section 4.2). All three regular weak prealgebraizability properties transfer. This property stems from the transferability of both prealgebraicity and assertionality. It is possible to formulate characterizations of the regular weak prealgebraizability properties in terms of the Leibniz operator viewed as a mapping between ordered sets. E.g., a π-institution $\mathcal{I}$ is RWF prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism, such that, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T/\Omega^{\mathcal{A}}(T)$ is a singleton. The other two characterizations assume similar forms.

In Section 8.5, we switch from regular weak prealgebraizability to *regular weak algebraizability* properties. The former involve prealgebraicity, which, when strengthened to protoalgebraicity, yield the latter. In accordance, a π-institution $\mathcal{I}$ is *regularly weakly family* (RWF) *algebraizable* if it is protoalgebraic and family assertional. It is *regularly weakly left* (RWL) *algebraizable* if it is protoalgebraic and left assertional, and it is *regularly weakly system* (RWS) *algebraizable* if it is protoalgebraic and system assertional. The strengthening of prealgebraicity to protoalgebraicity results in the identification of the left and system versions. Thus, the regular weak algebraizability hierarchy consists of only two distinct classes, that of regularly weakly family algebraizable π-institutions and its proper superclass of regularly weakly system algebraizable π-institutions. Further, in comparing regular weak algebraizability with regular weak prealgebraizability properties, it is revealed that the strongest versions of each, i.e., RWF algebraizability and RWF prealgebraizability, are actually equivalent. In contrast, RWS algebraizability strictly implies RWL prealgebraizability. Again, based on the fact that assertionality implies c-reflectivity, one infers that each regular weak algebraizability property implies the corresponding weak algebraizability property (see Section 4.3). Both regular weak algebraizability properties transfer and both can be characterized in terms of the Leibniz operator seen as a mapping between ordered sets. Clearly, since RWF algebraizability coincides with RWF prealgebraizability, the characterization, given previously, regarding the latter applies to the former as well.

In Section 8.6, we turn to *regular prealgebraizability* properties, which are obtained from the regular weak prealgebraizability properties of Section 8.4, not by strengthening prealgebraicity to protoalgebraicity, as was done in Section 8.5, but, by adding, instead, system extensionality, i.e., by replacing prealgebraicity by preequivalentiality. Consequently, a π-institution $\mathcal{I}$ is *regularly family* (RF) *prealgebraizable* if it is preequivalential (prealgebraic and system extensional) and family assertional. It is *regularly left* (RL) *prealgebraizable* if it is preequivalential and left assertional, and it is *regularly system* (RS) *prealgebraizable* if it is preequivalential and system assertional. RF prealgebraizability implies RL prealgebraizability, which implies RS prealgebraizability, based on the asertionality hierarchy of Section 8.3. Since preequivalentiality implies prealgebraicity, each of the three regular prealgebraizability properties implies the corresponding regular weak prealgebraizability property. Furthermore, since assertionality implies c-reflectivity, each of the regular prealgebraizability properties implies its prealgebraizability counterpart (see Section 5.5). All three regular prealgebraizability properties transfer. In addition, each can be characterized via the use of the Leibniz operator perceived as a mapping between ordered sets. Roughly speaking, these characterizations mimic the ones used in Section 8.4 for regular weak prealgebraizability properties, while adding some form of commutativity with inverse logical extensions, which, by Theorem 327, captures extensionality.

In Section 8.7, the last section of the chapter, we look at *regular algebraizability* properties, which are obtained from the regular prealgebraizability properties of Section 8.6 by strengthening preequivalentiality to equivalentiality or, alternatively, from the regular weak algebraizability properties of Section 8.5 by strengthening protoalgebraicity to equivalentiality. Either point of view leads to defining a π-institution $\mathcal{I}$ being *regularly family* (RF) *algebraizable* if it is equivalential and family assertional, *regularly left* (RL) *algebraizable* if it is equivalential and left assertional, and *regularly system* (RS) *algebraizable* if it is equivalential and system assertional. As transpired with regular weak algebraizability in Section 8.5, the left and system versions are equivalent, and this results in a two-class hierarchy, with RF algebraizability at the top, dominating RS algebraizability. The reasoning naturally leading to the establishment of these classes, permits us to conclude, on the one hand, that each regular algebraizability property implies the corresponding regular prealgebraizability property and, on the other, that each regular algebraizability property implies its regular weak counterpart. But, in addition, in establishing the relations between regular algebraizability and regular prealgebraizability properties, it is seen that the two family versions coincide. A final comparison is made between regular algebraizability and algebraizability (see Section 5.6). Since assertionality implies c-reflectivity, one obtains that each of the two distinct regular algebraizability versions implies the corresponding algebraizability version. Both regular algebraizability properties transfer. Finally, each possesses a characterization via the Leibniz operator, viewed as a mapping between ordered sets, satisfying some additional properties.

8.2 Semantic Regularity

In this chapter, we deal with π-institutions that have theorems and that, in addition, satisfy some form of the *semantic regularity property*, which is detailed in the following

Definition 575 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is* **family regular** *if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$, *all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi, \psi \in T_\Sigma \quad \textit{implies} \quad \langle \phi, \psi \rangle \in \Omega_\Sigma(T);$$

- $\mathcal{I}$ *is* **left regular** *if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$, *all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi, \psi \in \overleftarrow{T}_\Sigma \quad \textit{implies} \quad \langle \phi, \psi \rangle \in \Omega_\Sigma(T);$$

- $\mathcal{I}$ *is* **right regular** *if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\phi, \psi \in T_\Sigma \quad \textit{implies} \quad \langle \phi, \psi \rangle \in \Omega_\Sigma(\overleftarrow{T});$$

- $\mathcal{I}$ *is* **system regular** *if, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\phi, \psi \in T_\Sigma \quad \textit{implies} \quad \langle \phi, \psi \rangle \in \Omega_\Sigma(T).$$

We establish a hierarchy of regularity properties by looking at the relationships that hold between the properties introduced in Definition 575.

Proposition 576 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is family regular, then it is left regular;*

(b) If $\mathcal{I}$ *is family regular, then it is right regular;*

(c) If $\mathcal{I}$ *is left regular, then it is system regular;*

(d) If $\mathcal{I}$ *is right regular, then it is system regular.*

Proof:

(a) Suppose that $\mathcal{I}$ is family regular and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in \overleftarrow{T}_\Sigma$. Then, by Proposition 42, $\phi, \psi \in T_\Sigma$. Thus, by family regularity, $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. Therefore, $\mathcal{I}$ is left regular.

(b) Suppose that $\mathcal{I}$ is family regular and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Then, by family regularity, $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. Therefore, by Proposition 20, $\langle \phi, \psi \rangle \in \Omega_\Sigma(\overleftarrow{T})$. Thus, $\mathcal{I}$ is right regular.

(c) Suppose that $\mathcal{I}$ is left regular and let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Since T is a theory system, $\overleftarrow{T} = T$, whence, $\phi, \psi \in \overleftarrow{T}_\Sigma$. Hence, by left regularity, $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. Therefore, $\mathcal{I}$ is system regular.

(d) Suppose that $\mathcal{I}$ is right regular and let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Then, by right regularity, $\langle \phi, \psi \rangle \in \Omega_\Sigma(\overleftarrow{T})$. But T is a theory system, i.e., $\overleftarrow{T} = T$, whence $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. Thus, $\mathcal{I}$ is system regular.

■

We now show that, in fact, right regularity implies left regularity. This is a more challenging result that requires a technical lemma.

Lemma 577 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is right regular and not systemic, then, for all* $T \in \mathrm{ThFam}(\mathcal{I})\backslash\mathrm{ThSys}(\mathcal{I})$ *and all* $\Sigma \in |\mathbf{Sign}^\flat|$, *such that* $\overleftarrow{T}_\Sigma \subsetneq T_\Sigma$, $\overleftarrow{T}_\Sigma = \varnothing$.

Proof: Suppose that $\mathcal{I}$ is right regular and not systemic and consider $T \in \mathrm{ThFam}(\mathcal{I})\backslash\mathrm{ThSys}(\mathcal{I})$ and $\Sigma \in |\mathbf{Sign}^\flat|$, such that $\overleftarrow{T}_\Sigma \subsetneq T_\Sigma$ and $\overleftarrow{T}_\Sigma \neq \varnothing$. Then, on the one hand, there exists $\phi \in T_\Sigma$, such that $\phi \notin \overleftarrow{T}_\Sigma$ and, on the other, there exists $\psi \in \overleftarrow{T}_\Sigma$. Thus, by the compatibility of $\Omega(\overleftarrow{T})$ with $\overleftarrow{T}$, we get that $\langle \phi, \psi\rangle \notin \Omega_\Sigma(\overleftarrow{T})$, whereas, since $\overleftarrow{T} \leq T$, $\phi, \psi \in T_\Sigma$. Therefore, $\mathcal{I}$ is not right regular, a contradiction. We conclude that $\overleftarrow{T}_\Sigma = \varnothing$. ■

Theorem 578 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is right regular, then it is left regular.*

Proof: Suppose $\mathcal{I}$ is right regular. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in \overleftarrow{T}_\Sigma$. Then, also, $\phi, \psi \in T_\Sigma$.

- If $\mathcal{I}$ is systemic, then, by right regularity, $\langle \phi, \psi\rangle \in \Omega_\Sigma(\overleftarrow{T}) = \Omega_\Sigma(T)$, whence $\mathcal{I}$ is left regular.
- Suppose, now, that $\mathcal{I}$ is not systemic, whence Lemma 577 applies. Since $\phi, \psi \in \overleftarrow{T}_\Sigma$, by Lemma 577, we must have $\overleftarrow{T}_\Sigma = T_\Sigma$. But then, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ such that $\mathbf{Sign}^\flat(\Sigma, \Sigma') \neq \varnothing$, we get $\overleftarrow{T}_{\Sigma'} \neq \varnothing$, whence $\overleftarrow{T}_{\Sigma'} = T_{\Sigma'}$. Thus, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$, the condition

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \in \overleftarrow{T}_{\Sigma'} \quad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}) \in \overleftarrow{T}_{\Sigma'}$$

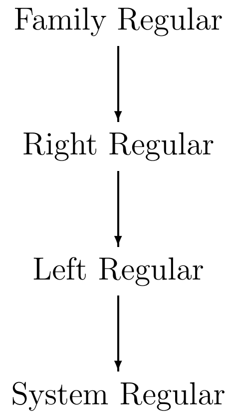
  is equivalent to the condition

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \in T_{\Sigma'} \quad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}) \in T_{\Sigma'}.$$

  Hence, $\langle \phi, \psi\rangle \in \Omega_\Sigma(\overleftarrow{T}) = \Omega_\Sigma(T)$.

We conclude that $\mathcal{I}$ is left regular. ■

Proposition 576 and Theorem 578 establish the hierarchy depicted in the diagram.



We show, next, that, adding stability to system regularity and to right regularity takes us, respectively, into the classes of left regular and family regular π-institutions.

Proposition 579 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is system regular and stable, then it is left regular;*

(b) If $\mathcal{I}$ *is right regular and stable, then it is family regular.*

Proof:

(a) Suppose $\mathcal{I}$ is system regular and stable. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in \overleftarrow{T}_\Sigma$. Since, by Proposition 42, $\overleftarrow{T} \in \mathrm{ThSys}(\mathcal{I})$, we may apply system regularity to conclude that $\langle \phi, \psi \rangle \in \Omega_\Sigma(\overleftarrow{T})$. Therefore, by stability, $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. Thus, $\mathcal{I}$ is left regular.

(b) Suppose $\mathcal{I}$ is right regular and stable. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. By right regularity, we get that $\langle \phi, \psi \rangle \in \Omega_\Sigma(\overleftarrow{T})$. Therefore, by stability, $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. Thus, $\mathcal{I}$ is family regular.

■

Of course, if systemicity is assumed, then all four classes in the regularity hierarchy collapse into a single class.

Proposition 580 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is system regular and systemic, then it is family regular.*

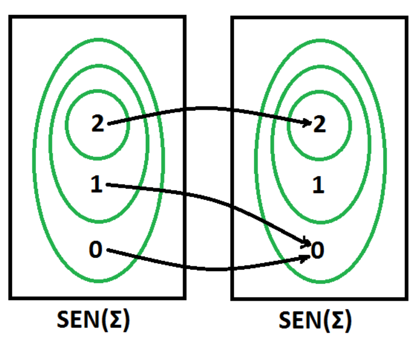
Proof: Under systemicity, all theory families are also theory systems. Hence the conditions defining family and system regularity are identical. ∎

To show that all four classes in the hierarchy above are different, we must present some examples that separate them. The first example provides an unstable π-institution which is left regular but not right regular. This accomplishes two goals:

- It shows that the class of right regular π-institutions is a proper subclass of the class of left regular ones;
- It shows that the converse of Part (a) of Proposition 579 does not hold in general, as the π-institution constructed is left regular but fails to be stable.

Example 581 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial category of natural transformations.*

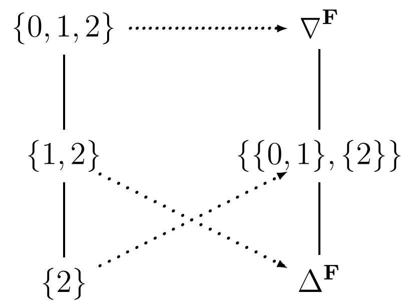


Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $C_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}$*.*

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$*:*

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1, 2\}$	$\{2\}$
$\{0, 1, 2\}$	$\{0, 1, 2\}$

The lattice of theory families and the corresponding Leibniz congruence systems are depicted below:



Since

$$\Omega(\overleftarrow{\{\{1,2\}\}}) = \Omega(\{\{2\}\}) = \{\{\{0,1\},\{2\}\}\} \neq \Delta^{\mathbf{F}} = \Omega(\{\{1,2\}\}),$$

$\mathcal{I}$ is not stable.

We show that $\mathcal{I}$ is left regular, i.e., that, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\phi,\psi \in \mathrm{SEN}(\Sigma)$, if $\phi,\psi \in \overleftarrow{T}_\Sigma$, then $\langle\phi,\psi\rangle \in \Omega_\Sigma(T)$.

- *If $T = \{\{0,1,2\}\}$, then, for all ϕ,ψ, $\langle\phi,\psi\rangle \in \nabla^{\mathbf{F}}_\Sigma = \Omega_\Sigma(\{\{0,1,2\}\})$;*
- *If $T \neq \{\{0,1,2\}\}$, then $\phi,\psi \in \overleftarrow{T}_\Sigma$ implies $\phi = \psi = 2$, whence, $\langle\phi,\psi\rangle \in \Delta^{\mathbf{F}}_\Sigma \subseteq \Omega_\Sigma(T)$.*

On the other hand, for $T = \{\{1,2\}\}$, we have $1,2 \in T_\Sigma$, but

$$\langle 1,2\rangle \notin \{\{\{0,1\},\{2\}\}\} = \Omega_\Sigma(\{\{2\}\}) = \Omega_\Sigma(\overleftarrow{T}).$$

Therefore, $\mathcal{I}$ is not right regular.

The second example presents a π-institution which is right regular, but fails to be family regular.

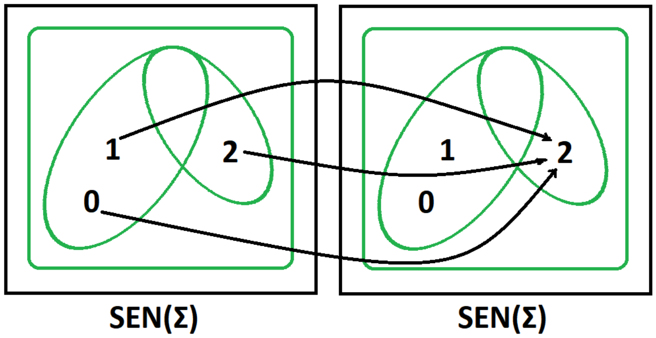
Example 582 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be the algebraic system defined as follows:*

- *$\mathbf{Sign}^\flat$ is the category with a single object Σ and a unique (non-identity) morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- *$\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ is defined by $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ and*

  $$\mathrm{SEN}^\flat(f)(0) = 2,\ \mathrm{SEN}^\flat(f)(1) = 2,\ \mathrm{SEN}^\flat(f)(2) = 2;$$

- *$N^\flat$ is the category of natural transformations generated by the unary natural transformation $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ determined by*

  $$\sigma^\flat_\Sigma(0) = 0,\ \sigma^\flat_\Sigma(1) = 2,\ \sigma^\flat_\Sigma(2) = 2.$$



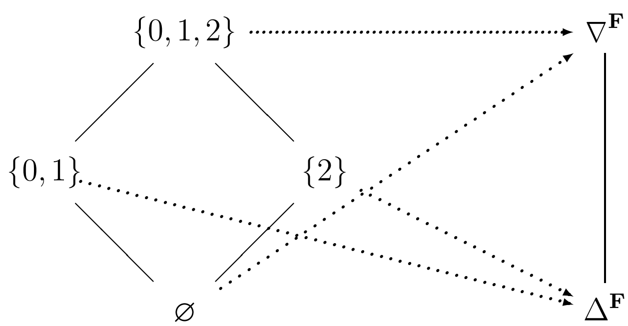
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{0,1\}, \{2\}, \{0,1,2\}\}.$$

The following table shows the action of $\overleftarrow{}$ *on theory families.*

T	$\varnothing$	$\{0,1\}$	$\{2\}$	$\{0,1,2\}$
$\overleftarrow{T}$	$\varnothing$	$\varnothing$	$\{2\}$	$\{0,1,2\}$

The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right:



We show, first, that $\mathcal{I}$ *is right regular, i.e., that it satisfies, for all* $T \in \mathrm{ThFam}(\mathcal{I})$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi, \psi \in T_\Sigma$ *implies* $\langle \phi, \psi \rangle \in \Omega_\Sigma(\overleftarrow{T})$.

- *If* $T = \{\varnothing\}$, *then the conclusion is vacuously true;*
- *If* $T = \{\{0,1\}\}$, *then, since* $\Omega(\overleftarrow{T}) = \Omega(\{\varnothing\}) = \nabla^{\mathbf{F}}$, *the conclusion is trivial;*
- *If* $T = \{\{2\}\}$, *then* $\phi, \psi \in T_\Sigma$ *implies* $\phi = \psi = 2$, *whence* $\langle \phi, \psi \rangle \in \Delta^{\mathbf{F}}_\Sigma \subseteq \Omega_\Sigma(\overleftarrow{T})$;
- *If* $T = \{\{0,1,2\}\}$, *then, since* $\Omega(\overleftarrow{T}) = \Omega(T) = \nabla^{\mathbf{F}}$, *the conclusion is trivial.*

On the other hand, for $T = \{\{0,1\}\}$, *we have* $0,1 \in T_\Sigma$, *whereas* $\langle 0,1\rangle \notin \Delta^{\mathbf{F}}_\Sigma = \Omega_\Sigma(T)$. *We conclude that* $\mathcal{I}$ *is not family regular.*

The last example shows a system regular π-institution which fails to be left regular.

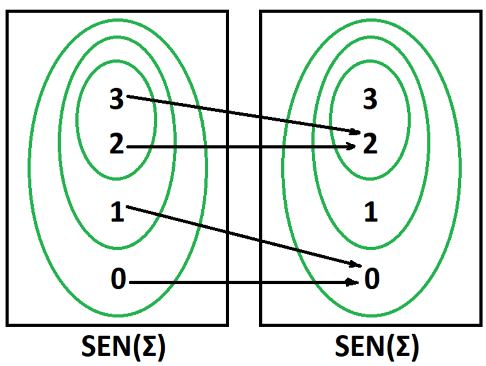
Example 583 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3\}$ *and*

  $$\mathrm{SEN}^\flat(f)(0) = 0,\ \mathrm{SEN}^\flat(f)(1) = 0,\ \mathrm{SEN}^\flat(f)(2) = 2,\ \mathrm{SEN}^\flat(f)(3) = 2;$$

- $N^\flat$ *is the category of natural transformations generated by the unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *determined by*

x	0	1	2	3
$\sigma^\flat_\Sigma(x)$	0	1	0	1



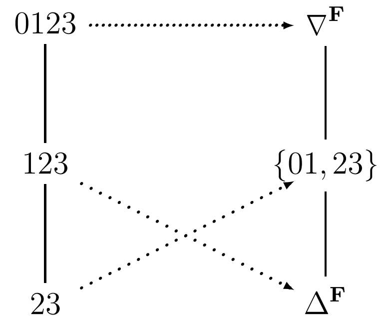
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{2,3\},\{1,2,3\},\{0,1,2,3\}\}.$$

The following table shows the action of $\overleftarrow{\ }$ *on theory families.*

T	$\{2,3\}$	$\{1,2,3\}$	$\{0,1,2,3\}$
$\overleftarrow{T}$	$\{2,3\}$	$\{2,3\}$	$\{0,1,2,3\}$

The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in *terms of blocks) on the right:*



We show, first, that $\mathcal{I}$ is system regular, i.e., that it satisfies, for all $T \in \mathrm{ThSys}(\mathcal{I})$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi, \psi \in T_\Sigma$ implies $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$.

- *If $T = \{\{2,3\}\}$, then, $\phi = \psi$ or $\{\phi, \psi\} = \{2,3\}$. In either case $\langle \phi, \psi \rangle \in \{\{0,1\},\{2,3\}\} = \Omega_\Sigma(T)$;*
- *If $T = \{\{0,1,2,3\}\}$, then, since $\Omega(T) = \nabla^{\mathbf{F}}$, the conclusion is trivial.*

On the other hand, for $T = \{\{1,2,3\}\}$, we have $2, 3 \in \{2,3\} = \overleftarrow{T}_\Sigma$, whereas $\langle 2, 3 \rangle \notin \Delta^{\mathbf{F}}_\Sigma = \Omega_\Sigma(T)$. We conclude that $\mathcal{I}$ is not left regular.

We provide, next, characterizations of three of the four regularity classes in terms of the Suszko operator acting on the theory families of a π-institution.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Given a theory system $T \in \mathrm{ThSys}(\mathcal{I})$, we set

$$\widehat{\Omega}^{\mathcal{I}}(T) = \bigcap \{\Omega(T') : T \leq T' \in \mathrm{ThSys}(\mathcal{I})\},$$

a system version of the Suszko operator on $\mathcal{I}$.

Theorem 584 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is family regular if and only if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \widetilde{\Omega}^{\mathcal{I}}_\Sigma(C(\phi, \psi));$$

(b) $\mathcal{I}$ is left regular if and only if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \widetilde{\Omega}^{\mathcal{I}}_\Sigma(\overrightarrow{C}(\phi, \psi));$$

(c) $\mathcal{I}$ is system regular if and only if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \widehat{\Omega}^{\mathcal{I}}_\Sigma(\overrightarrow{C}(\phi, \psi)).$$

Proof:

(a) Suppose $\mathcal{I}$ is family regular. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then, we have, by family regularity, $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$, for all $T \in \mathrm{ThFam}(\mathcal{I})$, such that $\phi, \psi \in T_\Sigma$. Therefore, by the definition of $\widetilde{\Omega}^{\mathcal{I}}$,

$$\begin{array}{rcl}\langle \phi, \psi \rangle & \in & \bigcap\{\Omega_\Sigma(T) : T \in \mathrm{ThFam}(\mathcal{I}), \phi, \psi \in T_\Sigma\} \\ & = & \widetilde{\Omega}^{\mathcal{I}}_\Sigma(C(\phi, \psi)).\end{array}$$

Assume, conversely, that the displayed condition holds. To show family regularity, let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Then, $C(\phi, \psi) \leq T$, whence, by the hypothesis and the monotonicity of $\widetilde{\Omega}^{\mathcal{I}}$,

$$\langle \phi, \psi \rangle \in \widetilde{\Omega}^{\mathcal{I}}_\Sigma(C(\phi, \psi)) \subseteq \widetilde{\Omega}^{\mathcal{I}}_\Sigma(T) \subseteq \Omega_\Sigma(T).$$

Hence, $\mathcal{I}$ is family regular.

(b) Suppose $\mathcal{I}$ is left regular. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then, we have, by left regularity, $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$, for all $T \in \mathrm{ThFam}(\mathcal{I})$, such that $\phi, \psi \in \overleftarrow{T}_\Sigma$. Therefore, by the definition of $\widetilde{\Omega}^{\mathcal{I}}$,

$$\begin{array}{rcl}\langle \phi, \psi \rangle & \in & \bigcap\{\Omega_\Sigma(T) : T \in \mathrm{ThFam}(\mathcal{I}), \phi, \psi \in \overleftarrow{T}_\Sigma\} \\ & = & \bigcap\{\Omega_\Sigma(T) : T \in \mathrm{ThFam}(\mathcal{I}), \overrightarrow{\{\phi, \psi\}} \leq T\} \\ & = & \widetilde{\Omega}^{\mathcal{I}}_\Sigma(\overrightarrow{C}(\phi, \psi)).\end{array}$$

Assume, conversely, that the displayed condition holds. To show left regularity, let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in \overleftarrow{T}_\Sigma$. Then, $\overrightarrow{\{\phi, \psi\}} \leq T$, whence, by the hypothesis and the monotonicity of $\widetilde{\Omega}^{\mathcal{I}}$,

$$\langle \phi, \psi \rangle \in \widetilde{\Omega}^{\mathcal{I}}_\Sigma(\overrightarrow{C}(\phi, \psi)) \subseteq \widetilde{\Omega}^{\mathcal{I}}_\Sigma(T) \subseteq \Omega_\Sigma(T).$$

Hence, $\mathcal{I}$ is left regular.

(c) Suppose $\mathcal{I}$ is system regular. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then, we have, by system regularity, $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$, for all $T \in \mathrm{ThSys}(\mathcal{I})$, such that $\phi, \psi \in T_\Sigma$. Therefore, by the definition of $\widehat{\Omega}^{\mathcal{I}}$,

$$\begin{array}{rcl}\langle \phi, \psi \rangle & \in & \bigcap\{\Omega_\Sigma(T) : T \in \mathrm{ThSys}(\mathcal{I}), \phi, \psi \in T_\Sigma\} \\ & = & \bigcap\{\Omega_\Sigma(T) : T \in \mathrm{ThSys}(\mathcal{I}), \overrightarrow{\{\phi, \psi\}} \leq T\} \\ & = & \widehat{\Omega}^{\mathcal{I}}_\Sigma(\overrightarrow{C}(\phi, \psi)).\end{array}$$

Assume, conversely, that the displayed condition holds. To show system regularity, let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Then, $\overrightarrow{\{\phi, \psi\}} \leq T$, whence, by the hypothesis and the monotonicity of $\widehat{\Omega}^{\mathcal{I}}$,

$$\langle \phi, \psi \rangle \in \widehat{\Omega}^{\mathcal{I}}_\Sigma(\overrightarrow{C}(\phi, \psi)) \subseteq \widehat{\Omega}^{\mathcal{I}}_\Sigma(T) \subseteq \Omega_\Sigma(T).$$

Hence, $\mathcal{I}$ is system regular.

■

We show, next, that all four regularity properties transfer from theory families/systems to $\mathcal{I}$-filter families/systems over arbitrary **F**-algebraic systems.

Theorem 585 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* **F**.

(a) $\mathcal{I}$ *is family regular if and only if, for every* **F***-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, *all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, *all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\phi, \psi \in T_\Sigma \quad \textit{implies} \quad \langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_\Sigma(T);$$

(b) $\mathcal{I}$ *is right regular if and only if, for every* **F***-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, *all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, *all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\phi, \psi \in T_\Sigma \quad \textit{implies} \quad \langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_\Sigma(\overleftarrow{T});$$

(c) $\mathcal{I}$ *is left regular if and only if, for every* **F***-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, *all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, *all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\phi, \psi \in \overleftarrow{T}_\Sigma \quad \textit{implies} \quad \langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_\Sigma(T);$$

(d) $\mathcal{I}$ *is system regular if and only if, for every* **F***-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, *all* $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, *all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\phi, \psi \in T_\Sigma \quad \textit{implies} \quad \langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_\Sigma(T).$$

Proof:

(a) The "if" follows easily by considering the **F**-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and recalling from Lemma 51 that $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}) = \mathrm{ThFam}(\mathcal{I})$.

Assume, conversely, that $\mathcal{I}$ is family regular and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an **F**-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \in T_{F(\Sigma)}$. Then, we get $\phi, \psi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})$. By Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, whence, by family regularity, we get that $\langle \phi, \psi \rangle \in \Omega_\Sigma(\alpha^{-1}(T))$. Thus, by Proposition 24, we get $\langle \phi, \psi \rangle \in \alpha_\Sigma^{-1}(\Omega^{\mathcal{A}}_{F(\Sigma)}(T))$. Hence, $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \Omega^{\mathcal{A}}_{F(\Sigma)}(T)$. Taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, if $\phi, \psi \in T_\Sigma$, then $\langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_\Sigma(T)$.

(b) The "if" follows as in Part (a).

Assume, conversely, that $\mathcal{I}$ is right regular and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an **F**-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\in T_{F(\Sigma)}$. Then, we get $\phi,\psi\in\alpha_\Sigma^{-1}(T_{F(\Sigma)})$. By Lemma 51, $\alpha^{-1}(T)\in\mathrm{ThFam}(\mathcal{I})$, whence, by right regularity, we get that $\langle\phi,\psi\rangle\in\Omega_\Sigma(\overleftarrow{\alpha^{-1}(T)})$. By Lemma 6, we get $\langle\phi,\psi\rangle\in\Omega_\Sigma(\alpha^{-1}(\overleftarrow{T}))$. Thus, by Proposition 24, we get $\langle\phi,\psi\rangle\in\alpha_\Sigma^{-1}(\Omega^{\mathcal{A}}_{F(\Sigma)}(\overleftarrow{T}))$. Hence, $\langle\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\rangle\in\Omega^{\mathcal{A}}_{F(\Sigma)}(\overleftarrow{T})$. Taking into account the surjectivity of $\langle F,\alpha\rangle$, we conclude that, for all $\Sigma\in|\mathbf{Sign}|$ and all $\phi,\psi\in\mathrm{SEN}(\Sigma)$, if $\phi,\psi\in T_\Sigma$, then $\langle\phi,\psi\rangle\in\Omega^{\mathcal{A}}_\Sigma(\overleftarrow{T})$.

(c) The "if" follows as in Part (a).

Assume, conversely, that $\mathcal{I}$ is left regular and let $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system, $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma\in|\mathbf{Sign}^\flat|$ and $\phi,\psi\in\mathrm{SEN}^\flat(\Sigma)$, such that $\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\in\overleftarrow{T}_{F(\Sigma)}$. Then, we get $\phi,\psi\in\alpha_\Sigma^{-1}(\overleftarrow{T}_{F(\Sigma)})$, i.e., by Lemma 6, $\phi,\psi\in\overleftarrow{\alpha^{-1}(T)}_\Sigma$. By Lemma 51, $\alpha^{-1}(T)\in\mathrm{ThFam}(\mathcal{I})$, whence, by left regularity, we get that $\langle\phi,\psi\rangle\in\Omega_\Sigma(\alpha^{-1}(T))$. Thus, by Proposition 24, we get $\langle\phi,\psi\rangle\in\alpha_\Sigma^{-1}(\Omega^{\mathcal{A}}_{F(\Sigma)}(T))$. Hence, $\langle\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\rangle\in\Omega^{\mathcal{A}}_{F(\Sigma)}(T)$. Taking into account the surjectivity of $\langle F,\alpha\rangle$, we conclude that, for all $\Sigma\in|\mathbf{Sign}|$ and all $\phi,\psi\in\mathrm{SEN}(\Sigma)$, if $\phi,\psi\in\overleftarrow{T}_\Sigma$, then $\langle\phi,\psi\rangle\in\Omega^{\mathcal{A}}_\Sigma(T)$.

(d) Similar to Part (a). ∎

We also have the following characterizations in terms of reduced $\mathcal{I}$-matrix families and $\mathcal{I}$-matrix systems.

Theorem 586 *Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and $\mathcal{I}=\langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is family regular if and only if, for every $\langle\mathcal{A},T\rangle\in\mathrm{MatFam}^*(\mathcal{I})$, with $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$, $\mathbf{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$, and all $\Sigma\in|\mathbf{Sign}|$, $|T_\Sigma|\le1$;

(b) $\mathcal{I}$ is system regular if and only if, for every $\langle\mathcal{A},T\rangle\in\mathrm{MatSys}^*(\mathcal{I})$, with $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$, $\mathbf{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$, and all $\Sigma\in|\mathbf{Sign}|$, $|T_\Sigma|\le1$.

Proof:

(a) Suppose, first, that $\mathcal{I}$ is family regular. Let $\langle\mathcal{A},T\rangle\in\mathrm{MatFam}^*(\mathcal{I})$, $\Sigma\in|\mathbf{Sign}|$ and $\phi,\psi\in\mathrm{SEN}(\Sigma)$, such that $\phi,\psi\in T_\Sigma$. Then, we have, using Theorem 585 and the fact that $\langle\mathcal{A},T\rangle$ is reduced,

$$\langle\phi,\psi\rangle\in\Omega^{\mathcal{A}}_\Sigma(T)=\Delta^{\mathcal{A}}_\Sigma,$$

whence $\phi=\psi$. Therefore, $|T_\Sigma|\le1$.

Suppose, conversely, that the given condition holds. Let $T\in\mathrm{ThFam}(\mathcal{I})$, $\Sigma\in|\mathbf{Sign}^\flat|$ and $\phi,\psi\in\mathrm{SEN}^\flat(\Sigma)$, such that $\phi,\psi\in T_\Sigma$. Then, $\langle\mathcal{F}/\Omega(T),$

$T/\Omega(T)\rangle$ is reduced and, moreover, $\phi/\Omega_\Sigma(T), \psi/\Omega_\Sigma(T) \in T_\Sigma/\Omega_\Sigma(T)$. Hence, by hypothesis, $\phi/\Omega_\Sigma(T) = \psi/\Omega_\Sigma(T)$, i.e., $\langle\phi,\psi\rangle \in \Omega_\Sigma(T)$. We conclude that $\mathcal{I}$ is family regular.

(b) Suppose, first, that $\mathcal{I}$ is system regular. Let $\langle\mathcal{A},T\rangle \in \mathrm{MatSys}^*(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\phi,\psi \in T_\Sigma$. Then, we have, using Theorem 585 and the fact that $\langle\mathcal{A},T\rangle$ is reduced,

$$\langle\phi,\psi\rangle \in \Omega^{\mathcal{A}}_\Sigma(T) = \Delta^{\mathcal{A}}_\Sigma,$$

whence $\phi = \psi$. Therefore, $|T_\Sigma| \leq 1$.

Suppose, conversely, that the given condition holds. Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi,\psi \in T_\Sigma$. Then, $\langle\mathcal{F}/\Omega(T), T/\Omega(T)\rangle$ is a reduced $\mathcal{I}$-matrix system. Moreover, we have $\phi/\Omega_\Sigma(T)$, $\psi/\Omega_\Sigma(T) \in T_\Sigma/\Omega_\Sigma(T)$. Hence, by hypothesis, $\phi/\Omega_\Sigma(T) = \psi/\Omega_\Sigma(T)$, i.e., $\langle\phi,\psi\rangle \in \Omega_\Sigma(T)$. We conclude that $\mathcal{I}$ is system regular.

■

8.3 Assertionality

In this section, we introduce the *assertionality hierarchy* of π-institutions. The properties defining this hierarchy are obtained simply by adding to the various properties defining the regularity hierarchy the stipulation that $\mathcal{I}$ have theorems.

Definition 587 (Assertionality) *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is* **family assertional** *if it is family regular and has theorems;*
- $\mathcal{I}$ *is* **left assertional** *if it is left regular and has theorems;*
- $\mathcal{I}$ *is* **right assertional** *if it is right regular and has theorems;*
- $\mathcal{I}$ *is* **system assertional** *if it is system regular and has theorems.*

Definition 587 and Proposition 576 allow us to obtain the following a priori assertionality hierarchy of π-institutions.

Family Assertional

$\downarrow$

Right Assertional

$\downarrow$

Left Assertional

$\downarrow$

System Assertional

However, using the characterizing properties included in the following proposition, we shall see that right assertionality implies systemicity and, hence, the classes of family assertional and right assertional π-institutions coincide.

Proposition 588 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is family assertional if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$ *and all* $\Sigma \in |\mathbf{Sign}^\flat|$, $T_\Sigma = t_\Sigma / \Omega_\Sigma(T)$, *for some* $t_\Sigma \in \mathrm{Thm}_\Sigma(\mathcal{I})$;

(b) $\mathcal{I}$ *is right assertional if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$ *and all* $\Sigma \in |\mathbf{Sign}^\flat|$, $T_\Sigma = t_\Sigma / \Omega_\Sigma(\overleftarrow{T})$, *for some* $t_\Sigma \in \mathrm{Thm}_\Sigma(\mathcal{I})$;

(c) $\mathcal{I}$ *is left assertional if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$ *and all* $\Sigma \in |\mathbf{Sign}^\flat|$, $\overleftarrow{T}_\Sigma = t_\Sigma / \Omega_\Sigma(T)$, *for some* $t_\Sigma \in \mathrm{Thm}_\Sigma(\mathcal{I})$;

(d) $\mathcal{I}$ *is system assertional if and only if, for all* $T \in \mathrm{ThSys}(\mathcal{I})$ *and all* $\Sigma \in |\mathbf{Sign}^\flat|$, $T_\Sigma = t_\Sigma / \Omega_\Sigma(T)$, *for some* $t_\Sigma \in \mathrm{Thm}_\Sigma(\mathcal{I})$.

Proof: If, in a certain context, a π-institution $\mathcal{I}$ has theorems, we shall use t_Σ to denote an arbitrary Σ-theorem of $\mathcal{I}$, $\Sigma \in |\mathbf{Sign}^\flat|$.

(a) Suppose that $\mathcal{I}$ is family assertional and let $T \in \mathrm{ThFam}(\mathcal{I})$ and $\Sigma \in |\mathbf{Sign}^\flat|$. If $\phi \in T_\Sigma$, then $\phi, t_\Sigma \in T_\Sigma$, whence, by family regularity, $\langle \phi, t_\Sigma \rangle \in \Omega_\Sigma(T)$, i.e., $\phi \in t_\Sigma / \Omega_\Sigma(T)$. On the other hand, if $\langle \phi, t_\Sigma \rangle \in \Omega_\Sigma(T)$, then, since $t_\Sigma \in T_\Sigma$, we get, by the compatibility of $\Omega(T)$ with T, $\phi \in T_\Sigma$.

Suppose, conversely, that, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$, $T_\Sigma = t_\Sigma / \Omega_\Sigma(T)$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Then, by hypothesis

$$\phi \ \Omega_\Sigma(T) \ t_\Sigma \ \Omega_\Sigma(T) \ \psi,$$

whence $\mathcal{I}$ is family regular.

(b) Suppose that $\mathcal{I}$ is right assertional and let $T \in \mathrm{ThFam}(\mathcal{I})$ and $\Sigma \in |\mathbf{Sign}^\flat|$. If $\phi \in T_\Sigma$, then $\phi, t_\Sigma \in T_\Sigma$, whence, by right regularity, $\langle\phi, t_\Sigma\rangle \in \Omega_\Sigma(\overleftarrow{T})$, i.e., $\phi \in t_\Sigma/\Omega_\Sigma(\overleftarrow{T})$. On the other hand, if $\langle\phi, t_\Sigma\rangle \in \Omega_\Sigma(\overleftarrow{T})$, then, since $t_\Sigma \in \overleftarrow{T}_\Sigma$, we get, by the compatibility of $\Omega(\overleftarrow{T})$ with $\overleftarrow{T}$, $\phi \in \overleftarrow{T}_\Sigma \subseteq T_\Sigma$.

Suppose, conversely, that, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$, $T_\Sigma = t_\Sigma/\Omega_\Sigma(\overleftarrow{T})$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Then, by hypothesis

$$\phi \ \Omega_\Sigma(\overleftarrow{T}) \ t_\Sigma \ \Omega_\Sigma(\overleftarrow{T}) \ \psi,$$

whence $\mathcal{I}$ is right regular.

(c) Suppose that $\mathcal{I}$ is left assertional and let $T \in \mathrm{ThFam}(\mathcal{I})$ and $\Sigma \in |\mathbf{Sign}^\flat|$. If $\phi \in \overleftarrow{T}_\Sigma$, then $\phi, t_\Sigma \in \overleftarrow{T}_\Sigma$, whence, by left regularity, $\langle\phi, t_\Sigma\rangle \in \Omega_\Sigma(T)$, i.e., $\phi \in t_\Sigma/\Omega_\Sigma(T)$. On the other hand, if $\langle\phi, t_\Sigma\rangle \in \Omega_\Sigma(T)$, then, since $\Omega(T) \le \Omega(\overleftarrow{T})$, we get $\langle\phi, t_\Sigma\rangle \in \Omega_\Sigma(\overleftarrow{T})$. But $t_\Sigma \in \overleftarrow{T}_\Sigma$, whence, by the compatibility of $\Omega(\overleftarrow{T})$ with $\overleftarrow{T}$, $\phi \in \overleftarrow{T}_\Sigma$.

Suppose, conversely, that, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$, $\overleftarrow{T}_\Sigma = t_\Sigma/\Omega_\Sigma(T)$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in \overleftarrow{T}_\Sigma$. Then, by hypothesis

$$\phi \ \Omega_\Sigma(T) \ t_\Sigma \ \Omega_\Sigma(T) \ \psi,$$

whence $\mathcal{I}$ is left regular.

(d) Similar to Part (a).

■

Using the characterizations in Proposition 588, we can show that right assertionality implies systemicity.

Proposition 589 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is right assertional, then* $\mathcal{I}$ *is systemic.*

Proof: Suppose that $\mathcal{I}$ is right assertional. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$. Then, by right assertionality and Proposition 588, $\langle\phi, t_\Sigma\rangle \in \Omega_\Sigma(\overleftarrow{T})$, for some $t_\Sigma \in \mathrm{Thm}_\Sigma(\mathcal{I})$. But $t_\Sigma \in \overleftarrow{T}_\Sigma$, whence, by compatibility of $\Omega(\overleftarrow{T})$ with $\overleftarrow{T}$, $\phi \in \overleftarrow{T}_\Sigma$. Therefore, $T \le \overleftarrow{T}$ and, hence, $T \in \mathrm{ThSys}(\mathcal{I})$. Thus, $\mathcal{I}$ is systemic. ■

Proposition 590 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is right assertional if and only if* $\mathcal{I}$ *is family assertional.*

Proof: If $\mathcal{I}$ is family assertional, then, by definition, it is family regular and has theorems, whence, by Proposition 576, it is right regular and has theorems and, therefore, by definition, it is right assertional.

Suppose, conversely, that $\mathcal{I}$ is right assertional. Then, by Proposition 589, it is systemic and, hence, a fortiori, stable. Therefore, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, if $\phi, \psi \in T_\Sigma$, then, by right assertionality, $\langle \phi, \psi \rangle \in \Omega_\Sigma(\overleftarrow{T})$ and, hence, by stability, $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. Therefore, $\mathcal{I}$ is family regular and, hence, family assertional. ■

We can also show easily that, in case $\mathcal{I}$ is systemic, the entire assertionailty hierarchy collapses into a single class.

Proposition 591 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is family assertional if and only if it is system assertional and systemic.*

Proof: If $\mathcal{I}$ is systemic, the conditions defining family assertionality and system assertionality coincide.

On the other hand, if $\mathcal{I}$ is family assertional, then, by definition, it is family regular and has theorems, whence, by Proposition 576, it is right regular and has theorems. Thus, by definition, $\mathcal{I}$ is right assertional and, hence, by Proposition 589, it is systemic. Moreover, using again Proposition 576, we conclude that $\mathcal{I}$ is also system assertional. ■

Thus, we get, regarding the assertionality hierarchy the following

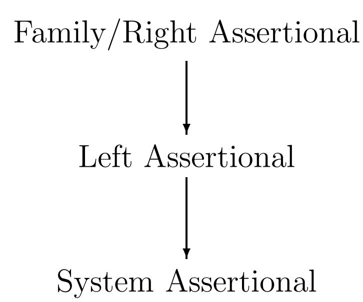
Proposition 592 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) *If* $\mathcal{I}$ *is family/right assertional, then it is left assertional;*

(b) *If* $\mathcal{I}$ *is left assertional, then it is system assertional.*

Proof: By Definition 587, Proposition 576 and Proposition 591. ■

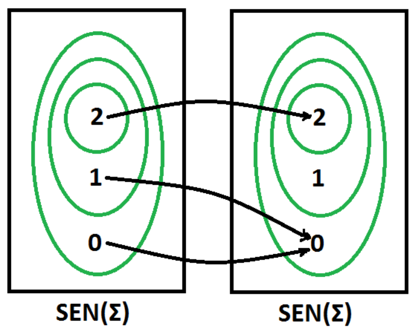
Proposition 592 establishes the **assertionality hierarchy** depicted in the accompanying diagram.



We show, next, that all three classes are different, by constructing two examples to separate them. The first is an example of a left assertional π-institution which fails to satisfy family assertionality.

Example 593 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial category of natural transformations.*



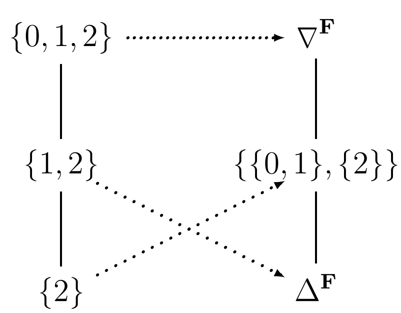
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}$*.*

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$*:*

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1, 2\}$	$\{2\}$
$\{0, 1, 2\}$	$\{0, 1, 2\}$

Since $\mathcal{I}$ *is not systemic, then, by Proposition 591, it fails to be family assertional.*

The lattice of theory families and the corresponding Leibniz congruence systems are depicted below:



Clearly, $\mathcal{I}$ *has theorems. Thus, to show that it is left assertional, it suffices to show, by Proposition 588, that, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,* $\overleftarrow{T}_\Sigma = 2/\Omega_\Sigma(T)$*.*

- $\overleftarrow{\{\{2\}\}}_\Sigma = \{2\} = 2/\Omega_\Sigma(\{\{2\}\})$;
- $\overleftarrow{\{\{1,2\}\}}_\Sigma = \{2\} = 2/\Omega_\Sigma(\{\{1,2\}\})$;
- $\overleftarrow{\{\{0,1,2\}\}}_\Sigma = \{0,1,2\} = 2/\Omega_\Sigma(\{\{0,1,2\}\})$.

The next example showcases a system assertional π-institution which is not left assertional.

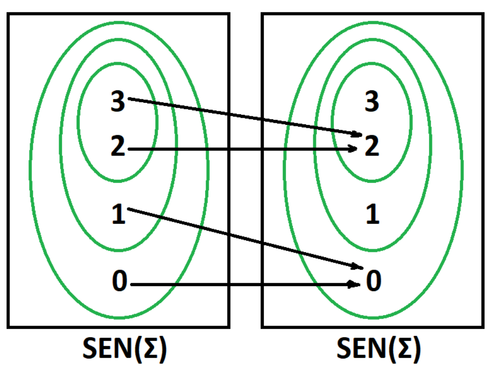
Example 594 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3\}$ *and*

  $$\mathrm{SEN}^\flat(f)(0) = 0,\ \mathrm{SEN}^\flat(f)(1) = 0,\ \mathrm{SEN}^\flat(f)(2) = 2,\ \mathrm{SEN}^\flat(f)(3) = 2;$$

- $N^\flat$ *is the category of natural transformations generated by the unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *determined by*

x	0	1	2	3
$\sigma^\flat_\Sigma(x)$	0	1	0	1



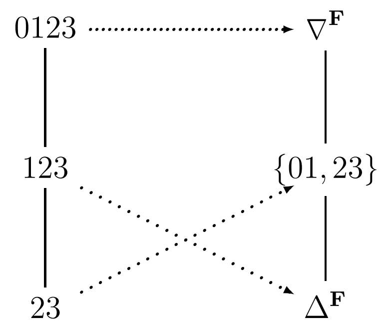
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{2,3\},\{1,2,3\},\{0,1,2,3\}\}.$$

The following table shows the action of $\overleftarrow{\ }$ *on theory families.*

T	$\{2,3\}$	$\{1,2,3\}$	$\{0,1,2,3\}$
$\overleftarrow{T}$	$\{2,3\}$	$\{2,3\}$	$\{0,1,2,3\}$

The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right:



Clearly, $\mathcal{I}$ has theorems. To see that $\mathcal{I}$ is system assertional, it suffices to show, by Proposition 588, that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $T_\Sigma = 2/\Omega_\Sigma(T)$. We do have indeed:

- $\{2,3\} = 2/\Omega_\Sigma(\{\{2,3\}\})$;
- $\{0,1,2,3\} = 2/\Omega_\Sigma(\{\{0,1,2,3\}\})$.

On the other hand, for $T = \{\{1,2,3\}\}$, we have $2, 3 \in \{2,3\} = \overleftarrow{T}_\Sigma$, whereas $\langle 2,3\rangle \notin \Delta^{\mathbf{F}}_\Sigma = \Omega_\Sigma(T)$. We conclude that $\mathcal{I}$ is not left regular and, hence, a fortiori, not left assertional either.

We proceed by exploring the relationships that hold between the various classes of the assertionality hierarchy, introduced in the present section, with the classes of the regularity hierarchy, which were introduced in Section 8.2. We have the following straightforward implications, which follow directly from the definitions involved.

Proposition 595 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.*

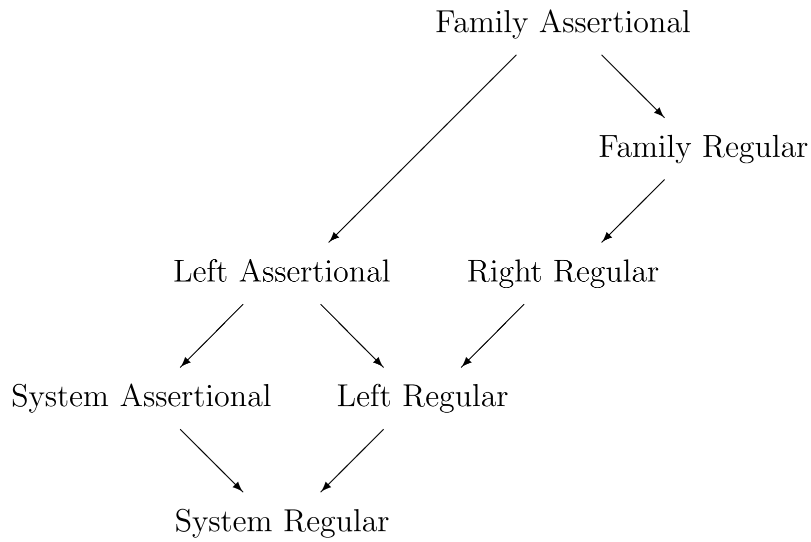
(a) If $\mathcal{I}$ is family assertional, then it is family regular;

(b) If $\mathcal{I}$ is left assertional, then it is left regular;

(c) If $\mathcal{I}$ is system assertional, then it is system regular.

Proof: Directly from Definition 587. ∎

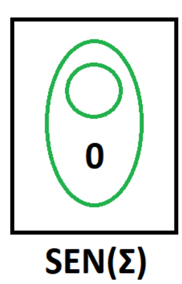
Thus, taking into account Propositions 576 and 592, we have the following mixed assertionality and regularity hierarchy.



An easy example shows that the three southeast arrows from the assertionality classes to the corresponding regularity classes correspond to proper inclusions.

Example 596 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

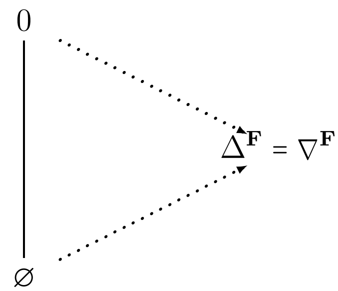
- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0\}$*;*
- $N^\flat$ *is the trivial category of natural transformations.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{0\}\}.$$

$\mathcal{I}$ is systemic and its lattice of theory families and corresponding Leibniz congruence systems are shown in the diagram.



$\mathcal{I}$ is family regular, since, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\langle 0,0\rangle \in \nabla^{\mathbf{F}}_{\Sigma} = \Omega_{\Sigma}(T)$.

On the other hand, since $\mathcal{I}$ does not have theorems, $\mathcal{I}$ does not belong to any of the steps in the assertionality hierarchy.

We examine next, the relationships between the classes in the assertionality hierarchy and those in the complete reflectivity hierarchy.

Theorem 597 *Let $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat}\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.*

(a) If $\mathcal{I}$ is family/right assertional, then it is family/right completely reflective;

(b) If $\mathcal{I}$ is left assertional, then it is left completely reflective;

(c) If $\mathcal{I}$ is system assertional, then it is system completely reflective.

Proof:

(a) Suppose that $\mathcal{I}$ is family assertional. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$, $\Sigma \in |\mathbf{Sign}^{\flat}|$ and $\phi \in \bigcap_{T\in\mathcal{T}} T_{\Sigma}$. By assertionality, there exists $t_{\Sigma} \in \mathrm{Thm}_{\Sigma}(\mathcal{I})$, whence, $\phi, t_{\Sigma} \in T_{\Sigma}$, for all $T \in \mathcal{T}$. Thus, by family regularity, $\langle \phi, t_{\Sigma}\rangle \in \Omega_{\Sigma}(T)$, for all $T \in \mathcal{T}$, i.e., $\langle \phi, t_{\Sigma}\rangle \in \bigcap_{T\in\mathcal{T}} \Omega_{\Sigma}(T)$. By hypothesis, $\langle \phi, t_{\Sigma}\rangle \in \Omega_{\Sigma}(T')$. Therefore, since $t_{\Sigma} \in T'_{\Sigma}$, we get, by compatibility of $\Omega(T')$ with T', $\phi \in T'_{\Sigma}$. We conclude that $\bigcap_{T\in\mathcal{T}} T \leq T'$ and, hence, that $\mathcal{I}$ is family c-reflective.

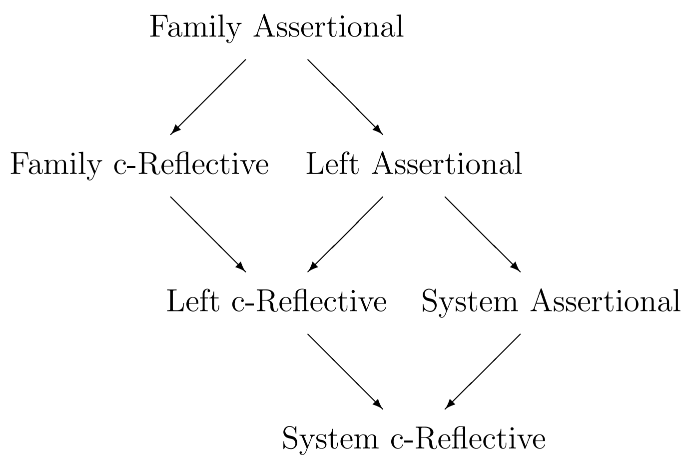
(b) Suppose that $\mathcal{I}$ is left assertional. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$, $\Sigma \in |\mathbf{Sign}^{\flat}|$ and $\phi \in \bigcap_{T\in\mathcal{T}} \overleftarrow{T}_{\Sigma}$. By assertionality, there exists $t_{\Sigma} \in \mathrm{Thm}_{\Sigma}(\mathcal{I})$, whence, $\phi, t_{\Sigma} \in \overleftarrow{T}_{\Sigma}$, for all $T \in \mathcal{T}$. Thus, by left regularity, $\langle \phi, t_{\Sigma}\rangle \in \Omega_{\Sigma}(T)$, for all $T \in \mathcal{T}$, i.e., $\langle \phi, t_{\Sigma}\rangle \in \bigcap_{T\in\mathcal{T}} \Omega_{\Sigma}(T)$. By hypothesis, $\langle \phi, t_{\Sigma}\rangle \in \Omega_{\Sigma}(T') \subseteq \Omega_{\Sigma}(\overleftarrow{T'})$. Therefore, since $t_{\Sigma} \in \overleftarrow{T'}_{\Sigma}$, we get, by compatibility of $\Omega(\overleftarrow{T'})$ with $\overleftarrow{T'}$, $\phi \in \overleftarrow{T'}_{\Sigma}$. We conclude that $\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$ and, hence, that $\mathcal{I}$ is left c-reflective.

(c) Similar to Part (a).

■

Alternatively, Theorem 597 may be proven by employing the characterizations provided in Proposition 588.

Based on the complete reflectivity hierarchy, which was established in Section 3.8, on the assertionality hierarchy established in Proposition 592 and on Theorem 597, we get the hierarchy relating assertionality with complete reflectivity classes shown in the diagram.



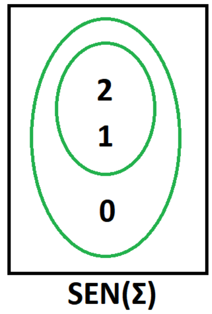
To show that all southwest inclusion arrows, connecting the various assertionality classes with the corresponding c-reflectivity classes, represent proper inclusions we construct an example of a family completely reflective π-institution which fails to be system assertional. Note that, since family c-reflectivity implies family injectivity, any π-institution fulfilling these requirements must have theorems. Therefore, the failure of assertionality must be due to failure of family regularity rather than the absence of theorems.

Example 598 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

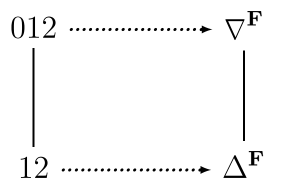
- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, specified by* $\sigma^\flat_\Sigma(0) = 0$*,* $\sigma^\flat_\Sigma(1) = 1$ *and* $\sigma^\flat_\Sigma(2) = 0$*.*

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{1,2\},\{0,1,2\}\}.$$



$\mathcal{I}$ is systemic and its lattice of theory families and corresponding Leibniz congruence systems are shown in the diagram.



Since the lattice of theory families of $\mathcal{I}$ is order isomorphic with the lattice of $\mathrm{AlgSys}^*(\mathcal{I})$*-congruence systems, $\mathcal{I}$ is family completely reflective.*

On the other hand, for $T = \{\{2,3\}\}$, we have $2,3 \in T_\Sigma$, but $\langle 2,3\rangle \notin \Delta^{\mathbf{F}}_\Sigma = \Omega_\Sigma(T)$, whence $\mathcal{I}$ is not system regular and, hence, a fortiori, belongs to none of the three classes in the assertionality hierarchy.

We show, next, that all assertionality properties transfer from theory families/systems to $\mathcal{I}$-filter families/systems over arbitrary $\mathbf{F}$-algebraic systems. This is a consequence of the facts that, by Theorem 585, all regularity properties transfer and, also, that the property of having theorems carries from the collection of all theory families to the collections of all filter systems over arbitrary algebraic systems, as seen in Lemma 376.

Theorem 599 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is family (respectively, left, system) assertional if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, $\langle \mathbf{A}, C^{\mathcal{I},\mathcal{A}}\rangle$ is family (respectively, left, system) assertional.*

Proof: Directly from Lemma 376 and Theorem 585. ■

We also have the following characterizations in terms of reduced $\mathcal{I}$-matrix families and $\mathcal{I}$-matrix systems.

Theorem 600 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is family assertional if and only if, for every $\langle \mathcal{A}, T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$, with $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, and all $\Sigma \in |\mathbf{Sign}|$, $|T_\Sigma| = 1$;

(b) *$\mathcal{I}$ is system assertional if and only if, for every $\langle \mathcal{A}, T\rangle \in \mathrm{MatSys}^*(\mathcal{I})$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, and all $\Sigma \in |\mathbf{Sign}|$, $|T_\Sigma| = 1$.*

Proof:

(a) Suppose, first, that $\mathcal{I}$ is family assertional. Then, by definition, it is family regular. Thus, by Theorem 586, for all $\langle \mathcal{A}, T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$, $|T_\Sigma| \le 1$. However, by family assertionality, $\mathcal{I}$ has theorems, whence, by Lemma 376, $|T_\Sigma| = 1$.

Suppose, conversely, that the given condition holds. Then $\mathcal{I}$ has theorems and, by Lemma 586, it is family regular. Therefore, $\mathcal{I}$ is family assertional.

(b) Similar to Part (a).

■

8.4 Regular Weak Prealgebraizability

We look, next, at those classes of π-institutions that are formed by adding prealgebraicity to the various levels of assertionality.

Definition 601 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.*

- *$\mathcal{I}$ is* **regularly weakly family prealgebraizable**, *or* **RWF prealgebraizable** *for short, if it is prealgebraic and family assertional;*
- *$\mathcal{I}$ is* **regularly weakly left prealgebraizable**, *or* **RWL prealgebraizable** *for short, if it is prealgebraic and left assertional;*
- *$\mathcal{I}$ is* **regularly weakly system prealgebraizable**, *or* **RWS prealgebraizable** *for short, if it is prealgebraic and system assertional.*

Based on the assertionality hierarchy established in Proposition 592, we have the following

Proposition 602 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.*

(a) *If $\mathcal{I}$ is regularly weakly family prealgebraizable, then it is regularly weakly left prealgebraizable;*

(b) *If $\mathcal{I}$ is regularly weakly left prealgebraizable, then it is regularly weakly system prealgebraizable.*

Proof: Straightforward by combining Definition 601 and Proposition 592. ■

Proposition 602 establishes the **regular weak prealgebraizability hierarchy** depicted in the following diagram.

Regular Weak Family Prealgebraizable

↓

Regular Weak Left Prealgebraizable

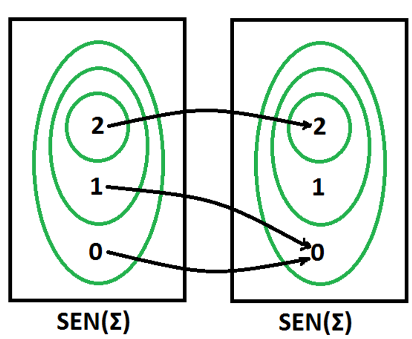
↓

Regular Weak System Prealgebraizable

We reuse two examples to show that all classes in this hierarchy are different, i.e., that the arrows in the diagram represent proper inclusions. The first describes a π-institution that is regularly weakly left prealgebraizable but fails to be regularly weakly family prealgebraizable, thus showing that the family class is properly included in the left class.

Example 603 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$;
- $N^\flat$ *is the trivial category of natural transformations.*



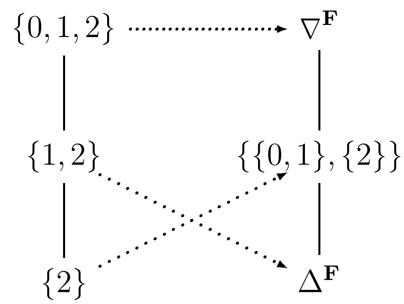
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}$.

The following table gives the theory families and the theory systems of the π-institution $\mathcal{I}$:

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

Since $\mathcal{I}$ is not systemic, by Proposition 591, it fails to be family assertional and, hence, it is not regularly weakly family prealgebraizable.

The lattice of theory families and the corresponding Leibniz congruence systems are depicted below:



Since the only theory systems of $\mathcal{I}$ are $\{\{2\}\}$ and $\{\{0,1,2\}\}$, it is clear that Ω is monotone on theory systems and, hence, $\mathcal{I}$ is prealgebraic. Clearly, $\mathcal{I}$ has theorems. Thus, to complete the proof that it is regularly weakly left prealgebraizable, it suffices to show that it is left assertional, i.e., by Proposition 588, that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T}_\Sigma = 2/\Omega_\Sigma(T)$.

- $\overleftarrow{\{\{2\}\}}_\Sigma = \{2\} = 2/\Omega_\Sigma(\{\{2\}\})$;
- $\overleftarrow{\{\{1,2\}\}}_\Sigma = \{2\} = 2/\Omega_\Sigma(\{\{1,2\}\})$;
- $\overleftarrow{\{\{0,1,2\}\}}_\Sigma = \{0,1,2\} = 2/\Omega_\Sigma(\{\{0,1,2\}\})$.

The second example presents a regularly weakly system prealgebraizable π-institution that is not regularly weakly left prealgebraizable.

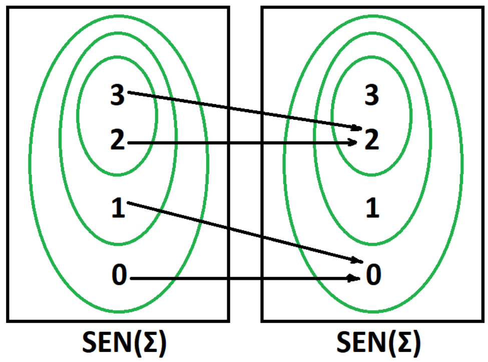
Example 604 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object Σ and a unique (non-identity) morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3\}$ *and*

  $$\mathrm{SEN}^\flat(f)(0) = 0,\ \mathrm{SEN}^\flat(f)(1) = 0,\ \mathrm{SEN}^\flat(f)(2) = 2,\ \mathrm{SEN}^\flat(f)(3) = 2;$$

- *$N^\flat$ is the category of natural transformations generated by the unary natural transformation $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ determined by*

x	0	1	2	3
$\sigma^\flat_\Sigma(x)$	0	1	0	1



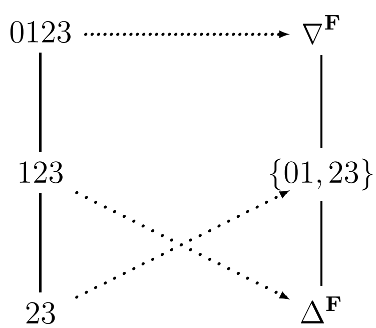
Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\{2,3\},\{1,2,3\},\{0,1,2,3\}\}.$$

The following table shows the action of $\overleftarrow{}$ on theory families.

T	$\{2,3\}$	$\{1,2,3\}$	$\{0,1,2,3\}$
$\overleftarrow{T}$	$\{2,3\}$	$\{2,3\}$	$\{0,1,2,3\}$

The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right:



Since the only theory systems of $\mathcal{I}$ are $\{\{2,3\}\}$ and $\{\{0,1,2,3\}\}$, it is obvious that Ω is monotone on theory systems and, hence, that $\mathcal{I}$ is prealgebraic. Clearly, $\mathcal{I}$ has theorems. To see that $\mathcal{I}$ is regularly weakly system prealgebraizable it suffices to show that it is system assertional, i.e., by Proposition 588, that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $T_\Sigma = 2/\Omega_\Sigma(T)$. We do have indeed:

- $\{2,3\} = 2/\Omega_\Sigma(\{\{2,3\}\})$;
- $\{0,1,2,3\} = 2/\Omega_\Sigma(\{\{0,1,2,3\}\})$.

On the other hand, for $T = \{\{1,2,3\}\}$*, we have* $2,3 \in \{2,3\} = \overleftarrow{T}_\Sigma$*, whereas* $\langle 2,3\rangle \notin \Delta^{\mathbf{F}}_\Sigma = \Omega_\Sigma(T)$*. We conclude that* $\mathcal{I}$ *is not left regular and, hence, a fortiori, it is not regularly weakly left prealgebraizable.*

We investigate, next, the relationships that hold between the various regular weak prealgebraizability classes, introduced in the present section, and the corresponding assertional classes, that were introduced in Section 8.3.

Directly from the definitions involved, we get the following

Proposition 605 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

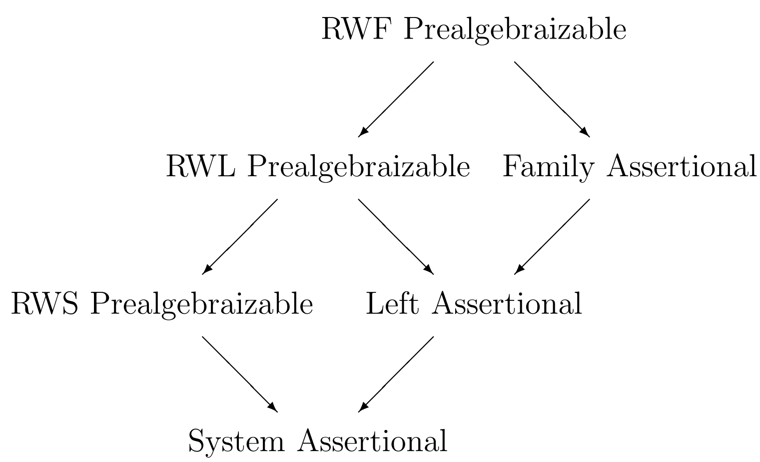
(a) If $\mathcal{I}$ *is regularly weakly family prealgebraizable, then it is family assertional;*

(b) If $\mathcal{I}$ *is regularly weakly left prealgebraizable, then it is left assertional;*

(c) If $\mathcal{I}$ *is regularly weakly system prealgebraizable, then it is system assertional.*

Proof: Directly from Definition 601. ∎

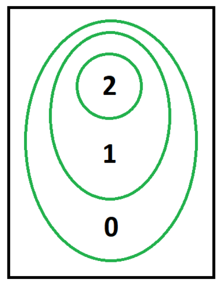
Therefore, we get the mixed regular weak prealgebraizability and assertionality hierarchy depicted in the diagram.



To show that all classes in this hierarchy are different, we provide an example of a π-institution that is family assertional, and, thus, belongs to all three assertionality classes, but fails to be regularly weakly system prealgebraizable, whence it belongs to none of three steps in the regular weak prealgebraizability hierarchy. This example shows that all three southeast arrows represent proper inclusions.

Example 606 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the trivial category of natural transformations.*

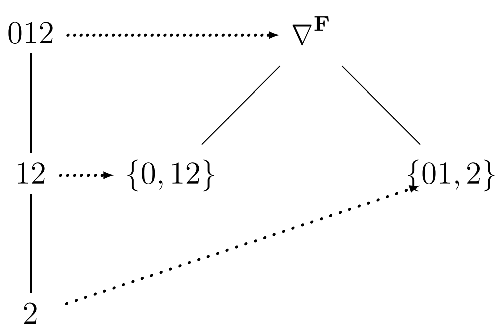


SEN(Σ)

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

$\mathcal{I}$ *is systemic and its lattice of theory families and corresponding Leibniz congruence systems are shown in the diagram.*



$\mathcal{I}$ *has theorems, whence to show that it is family assertional, it suffices to show that, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,* $T_\Sigma = 2/\Omega_\Sigma(T)$*. Indeed, we have:*

- *For* $T = \{\{2\}\}$*,* $\{2\} = 2/\Omega_\Sigma(\{\{2\}\})$*;*
- *For* $T = \{\{1, 2\}\}$*,* $\{1, 2\} = 2/\Omega_\Sigma(\{\{1, 2\}\})$*;*
- *For* $T = \{\{0, 1, 2\}\}$*,* $\{0, 1, 2\} = 2/\Omega_\Sigma(\{\{0, 1, 2\}\})$*.*

On the other hand, since $\{\{2\}\} \leq \{\{1, 2\}\}$*, but* $\Omega(\{\{2\}\}) \nleq \Omega(\{\{1, 2\}\})$*,* $\mathcal{I}$ *is not prealgebraic and, hence, fails to be regularly weakly system prealgebraizable.*

Turning now to the relationship between the regular weak prealgebraizability hierarchy and the weak prealgebraizability hierarchy, we get the following

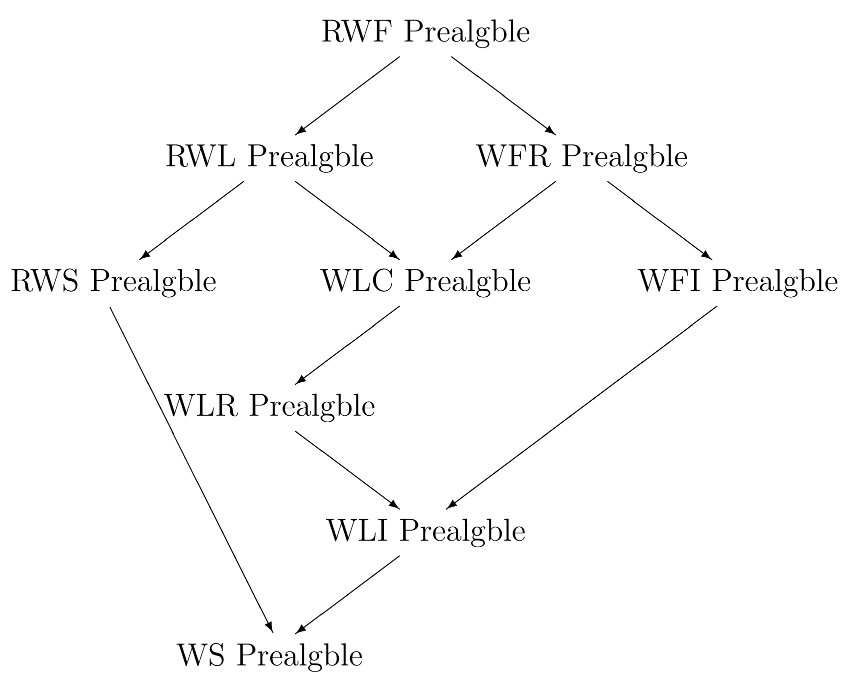
Proposition 607 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- *If* $\mathcal{I}$ *is regularly weakly family prealgebraizable, then it is weakly family (completely) reflective prealgebraizable;*
- *If* $\mathcal{I}$ *is regularly weakly left prealgebraizable, then it is weakly left completely reflective prealgebraizable;*
- *If* $\mathcal{I}$ *is regularly weakly system prealgebraizable, then it is weakly system prealgebraizable.*

Proof: We show Part (a) in detail. The remaining parts can be proved similarly.

Suppose $\mathcal{I}$ is regularly weakly family prealgebraizable. Then, by definition, it is prealgebraic and family assertional. Hence, by Theorem 597, it is prealgebraic and family completely reflective. Thus, by definition, it is weakly family prealgebraizable. ■

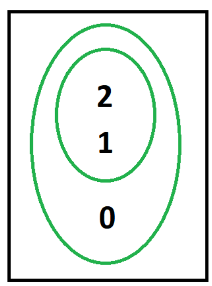
Thus, Proposition 607, together with Proposition 602 and the hierarchy established in Section 4.2, point to the following hierarchy of regularly weakly prealgebraizable and weakly prealgebraizable π-institutions.



Again it is not difficult to see that the classes in the regular weak prealgebraizability hierarchy are different from the classes of weakly prealgebraizable π-institutions. This is accomplished by constructing an example of a π-institution which is weakly family completely reflective prealgebraizable but is not regularly weakly system prealgebraizable.

Example 608 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *specified by* $\sigma^\flat_\Sigma(0) = 0$, $\sigma^\flat_\Sigma(1) = 1$ *and* $\sigma^\flat_\Sigma(2) = 0$.

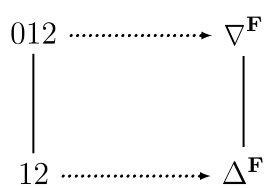


SEN(Σ)

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{1, 2\}, \{0, 1, 2\}\}.$$

$\mathcal{I}$ *is systemic and its lattice of theory families and corresponding Leibniz congruence systems are shown in the diagram.*



Since the lattice of theory families of $\mathcal{I}$ *is order isomorphic with the lattice of* $\mathrm{AlgSys}^*(\mathcal{I})$*-congruence systems,* $\mathcal{I}$ *is weakly family c-reflective prealgebraizable.*

On the other hand, for $T = \{\{2, 3\}\}$*, we have* $2, 3 \in T_\Sigma$*, but* $\langle 2, 3 \rangle \notin \Delta^{\mathbf{F}}_\Sigma = \Omega_\Sigma(T)$*, whence* $\mathcal{I}$ *is not system regular and, hence, a fortiori, it is not regularly weakly system prealgebraizable either.*

Based on existing results, we can show that all three kinds of regular weak prealgebraizability transfer from theory families/systems to filter families/systems over arbitrary $\mathbf{F}$-algebraic systems.

Theorem 609 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is regularly weakly family prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, all* $T', T'' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$*,*

- $T' \leq T''$ *implies* $\Omega^{\mathcal{A}}(T') \leq \Omega^{\mathcal{A}}(T'')$*;*
- $|T_\Sigma / \Omega^{\mathcal{A}}_\Sigma(T)| = 1$*;*

(b) $\mathcal{I}$ *is regularly weakly left prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, all* $T', T'' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$*,*

- $T' \leq T''$ *implies* $\Omega^{\mathcal{A}}(T') \leq \Omega^{\mathcal{A}}(T'')$*;*
- $|\overleftarrow{T}_\Sigma / \Omega^{\mathcal{A}}_\Sigma(T)| = 1$*;*

(c) $\mathcal{I}$ *is regularly weakly system prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, all* $T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$*,*

- $T \leq T'$ *implies* $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$*;*
- $|T_\Sigma / \Omega^{\mathcal{A}}_\Sigma(T)| = 1$*.*

Proof: Combine Theorem 179 with Theorem 599. ■

Finally, we may also adapt previously obtained results characterizing weak prealgebraizability to obtain similar characterizations of regular weak prealgebraizability in terms of mappings between posets of filter families/systems (including theory families/systems) and congruence systems.

Theorem 610 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is regularly weakly family prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism, such that, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$*,* $|T_\Sigma / \Omega^{\mathcal{A}}_\Sigma(T)| = 1$*.*

Proof: Suppose, first, that $\mathcal{I}$ is regularly weakly family prealgebraizable. Then it is, by definition, prealgebraic and, moreover, by definition, Proposition 605 and Theorem 597, it is family c-reflective. Therefore, it is WFR prealgebraizable. Thus, the required isomorphism is given by Theorem 268. The expression for T is obtained by applying Theorem 609.

Assume, conversely, that the postulated condition holds. Then, the hypotheses of Theorem 609, Part (a), are satisfied and, therefore, $\mathcal{I}$ is regularly weakly family prealgebraizable. ∎

Theorem 611 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is regularly weakly left prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order embedding, such that, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$*,* $|\overleftarrow{T}_\Sigma / \Omega^{\mathcal{A}}_\Sigma(T)| = 1$*.*

Proof: Suppose, first, that $\mathcal{I}$ is regularly weakly left prealgebraizable. Then it is, by definition, prealgebraic and, moreover, by definition, Proposition 605 and Theorem 597, it is left c-reflective. Therefore, it is WLC prealgebraizable. Thus, the required embedding is given by Theorem 276. The expression for T is obtained by applying Theorem 609.

Assume, conversely, that the postulated condition holds. Then, the hypotheses of Theorem 609, Part (b), are satisfied and, therefore, $\mathcal{I}$ is regularly weakly left prealgebraizable. ∎

Theorem 612 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is regularly weakly system prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order embedding, such that, for all $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$*,* $|T_\Sigma / \Omega^{\mathcal{A}}_\Sigma(T)| = 1$*.*

Proof: Suppose, first, that $\mathcal{I}$ is regularly weakly system prealgebraizable. Then it is, by definition, prealgebraic and, moreover, by definition, Proposition 605 and Theorem 597, it is system c-reflective. Therefore, it is WS prealgebraizable. Thus, the required embedding is given by Theorem 256. The expression for T is obtained by applying Theorem 609.

Assume, conversely, that the postulated condition holds. Then, the hypotheses of Theorem 609, Part (c), are satisfied and, therefore, $\mathcal{I}$ is regularly weakly system prealgebraizable. ∎

8.5 Regular Weak Algebraizability

We look, next, at those classes of π-institutions that are formed by adding protoalgebraicity to the various levels of assertionality.

Definition 613 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is* **regularly weakly family algebraizable**, *or* **RWF algebraizable** *for short, if it is protoalgebraic and family assertional;*
- $\mathcal{I}$ *is* **regularly weakly left algebraizable**, *or* **RWL algebraizable** *for short, if it is protoalgebraic and left assertional;*
- $\mathcal{I}$ *is* **regularly weakly system algebraizable**, *or* **RWS algebraizable** *for short, if it is protoalgebraic and system assertional.*

Even though there seem to be three classes in the regular weak algebraizability hierarchy, in reality there are only two, since it is easy to see that the classes of regularly weakly left and of regularly weakly system π-institutions coincide.

Proposition 614 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is regularly weakly left algebraizable if and only if it is regularly weakly system algebraizable.*

Proof: The "only if" follows directly by the definition and Proposition 592. For the "if", suppose that $\mathcal{I}$ is regularly weakly system algebraizable. Then it is, a fortiori, protoalgebraic, whence, by Lemma 170, it is stable. Therefore, since $\mathcal{I}$ is system regular and stable, by Proposition 579, it is left regular. We conclude that $\mathcal{I}$ is regularly weakly left algebraizable. ∎

The assertionality hierarchy, established in Proposition 592, and Proposition 614 allow us to establish the following regular weak algebraizability hierarchy.

Proposition 615 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is regularly weakly family algebraizable, then it is regularly weakly system algebraizable.*

Proof: Straightforward by combining Definition 601 and Proposition 592. ∎

The **regular weak algebraizability hierarchy** is depicted in the following diagram.

Regular Weak Family Algebraizable

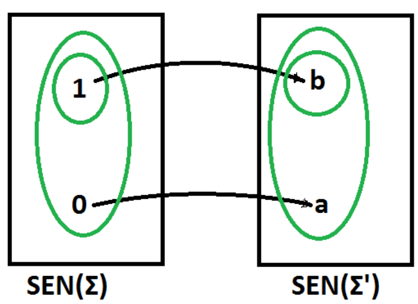
↓

Regular Weak System Algebraizable

We use an example to show that the two classes in this hierarchy are different. Namely, we construct a π-institution that is regularly weakly system algebraizable but fails to be regularly weakly family algebraizable.

Example 616 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$*;*
- $N^\flat$ *is the trivial category of natural transformations.*



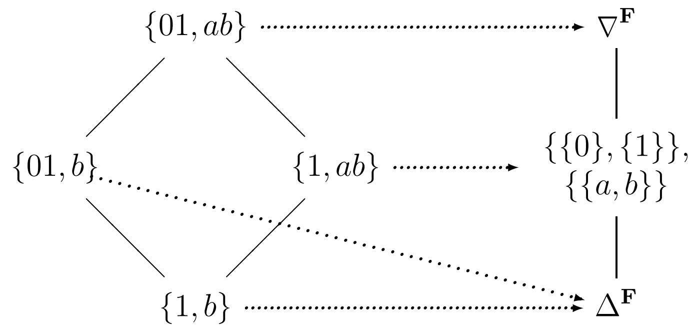
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\},\{a,b\}\}.$$

The following table shows the action of $\overleftarrow{}$ *on theory families, where rows correspond to* T_Σ *and columns to* $T_{\Sigma'}$ *and each entry is written as* $\overleftarrow{T}_\Sigma, \overleftarrow{T}_{\Sigma'}$.

$\leftarrow$	$\{b\}$	$\{a,b\}$
$\{1\}$	$\{1\},\{b\}$	$\{1\},\{a,b\}$
$\{0,1\}$	$\{1\},\{b\}$	$\{0,1\},\{a,b\}$

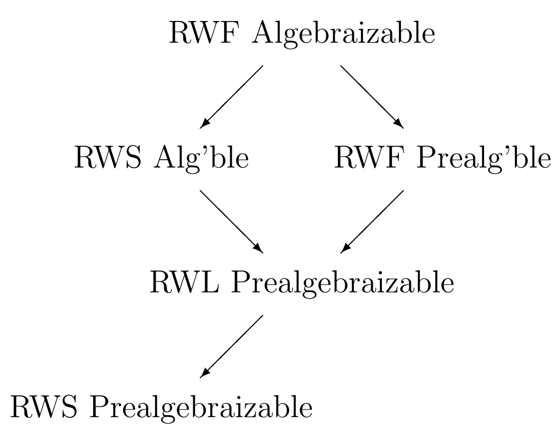
The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in *terms of blocks) on the right:*



The Leibniz operator is monotone on theory families, whence, $\mathcal{I}$ is protoalgebraic. Moreover, $\mathrm{Thm}(\mathcal{I}) = \{\{1\}, \{b\}\}$ *and, for every theory system T, $T_\Sigma = 1/\Omega_\Sigma(T)$ and $T_{\Sigma'} = b/\Omega_{\Sigma'}(T)$. Therefore, $\mathcal{I}$ is system assertional. Thus, $\mathcal{I}$ is regularly weakly system algebraizable.*

On the other hand, for $T = \{\{0,1\}, \{b\}\} \in \mathrm{ThFam}(\mathcal{I})$, we have $0, 1 \in T_\Sigma$, but $\langle 0, 1\rangle \notin \Omega_\Sigma(T)$. Therefore, $\mathcal{I}$ fails to be family regular and, hence, a fortiori, it is not regularly weakly family algebraizable.

We investigate, next, the relationships that hold between the two regular weak algebraizability classes, introduced in the present section, and the three regular weak prealgebraizability classes, that were introduced in Section 8.4. Since, by Theorem 175, protoalgebraicity implies prealgebraicity, we get, a priori, the following mixed hierarchy.



However, we can show that the two top classes of the hierarchies coincide.

Theorem 617 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is regularly weakly family prealgebraizable if and only if it is regularly weakly family algebraizable.*

Proof: The "if" follows from the relevant definitions and the fact that, by Theorem 175, protoalgebraicity implies prealgebraicity. For the "only if", it suffices to show that, under family assertionality, prealgebraicity implies protoalgebraicity. By Theorem 175, it suffices, in turn, to show that family assertionality implies stability and, by Proposition 152, that family assertionality implies systemicity. Indeed, by Theorem 597, family assertionality implies family c-reflectivity and, by Proposition 237, we get that $\mathcal{I}$ is systemic. ∎

Moreover, from the definitions involved, we get the following

Proposition 618 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is regularly weakly system algebraizable, then it is regularly weakly left prealgebraizable.*

Proof: Suppose $\mathcal{I}$ is regularly weakly system algebraizable. Equivalently, by Proposition 614, it is regularly weakly left algebraizable. Then, by definition, it is protoalgebraic and left assertional. Thus, by Theorem 175, it is prealgebraic and left assertional, i.e., by definition, it is regularly weakly left prealgebraizable. ∎

Based on Theorem 617 and Proposition 618, we get the following updated version of the mixed hierarchy shown in the preceding diagram.

RWF (Pre)Algebraizable

↓

RWS/L Algebraizable

↓

RWL Prealgebraizable

↓

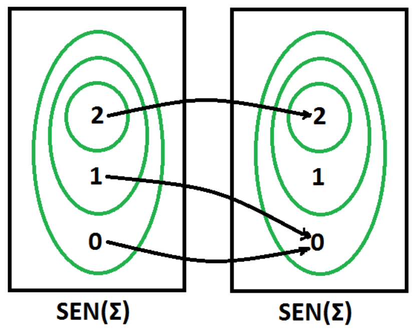
RWS Prealgebraizable

To show that all classes in this hierarchy are different, we provide an example of a π-institution that is regularly weakly left prealgebraizable, but fails to be regularly weakly system algebraizable, i.e., an example that separates the regular weak algebraizability from the regular weak prealgebraizability classes.

Example 619 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$;
- $N^\flat$ *is the trivial category of natural transformations.*

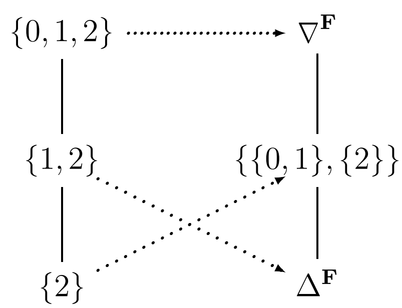


Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}$.

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$.

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

The lattice of theory families and the corresponding Leibniz congruence systems are depicted below.



Since the only theory systems of $\mathcal{I}$ *are* $\{\{2\}\}$ *and* $\{\{0,1,2\}\}$, *it is clear that* Ω *is monotone on theory systems and, hence,* $\mathcal{I}$ *is prealgebraic. Clearly,* $\mathcal{I}$ *has theorems. Thus, to complete the proof that it is regularly weakly left prealgebraizable, it suffices to show that it is left assertional, i.e., by Proposition 588, that, for all* $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T}_\Sigma = 2/\Omega_\Sigma(T)$. *Indeed, we get:*

- $\overleftarrow{\{\{2\}\}}_\Sigma = \{2\} = 2/\Omega_\Sigma(\{\{2\}\})$;
- $\overleftarrow{\{\{1,2\}\}}_\Sigma = \{2\} = 2/\Omega_\Sigma(\{\{1,2\}\})$;

- $\overleftarrow{\{\{0,1,2\}\}}_\Sigma = \{0,1,2\} = 2/\Omega_\Sigma(\{\{0,1,2\}\})$.

On the other hand, since $\{\{2\}\} \leq \{\{1,2\}\}$, *but*

$$\Omega(\{\{2\}\}) = \{\{\{0,1\},\{2\}\}\} \nleq \Delta^{\mathbf{F}} = \Omega(\{\{1,2\}\}),$$

$\mathcal{I}$ *is not protoalgebraic and, hence, it fails to be regularly weakly system algebraizable.*

Turning now to the relationship between regular weak algebraizability and weak algebraizability, we get, by definition

Proposition 620 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

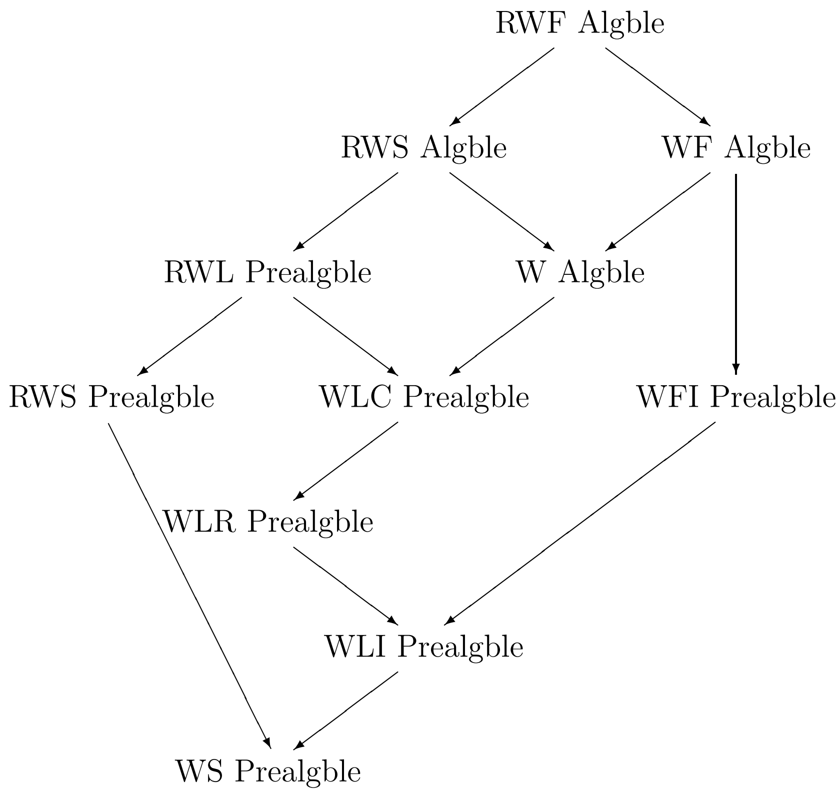
(a) If $\mathcal{I}$ *is regularly weakly family algebraizable, then it is weakly family algebraizable;*

(b) If $\mathcal{I}$ *is regularly weakly system algebraizable, then it is weakly (system/left) algebraizable.*

Proof: For Part (a) note that, by Theorem 617, regular weak family algebraizability coincides with regular weak family prealgebraizability. In turn, by Proposition 607, regular weak family prealgebraizability entails weak family prealgebraizability. But, by Corollary 297, the latter property is identical with weak family algebraizability.

For Part (b), if $\mathcal{I}$ is regularly weakly system algebraizable, then it is, by definition, protoalgebraic and system assertional, whence, by Theorem 597, it is protoalgebraic and system completely reflective. Therefore, it is, by definition, weakly (system or, equivalently, left) algebraizable. ■

Thus, Proposition 620, together with Propositions 607 and 618, point to the following hierarchy of regularly weakly (pre)algebraizable π-institutions and weakly (pre)algebraizable π-institutions.



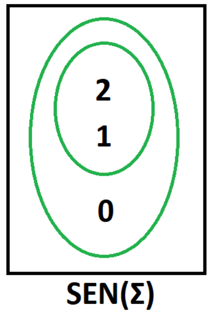
Again it is not difficult to see that the classes in the regular weak algebraizability hierarchy are different from the classes in the weak algebraizability hierarchy. This is accomplished by constructing an example of a π-institution which is weakly family algebraizable but is not regularly weakly system prealgebraizable.

Example 621 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

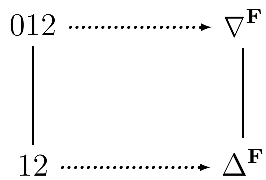
- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *specified by* $\sigma^\flat_\Sigma(0) = 0$*,* $\sigma^\flat_\Sigma(1) = 1$ *and* $\sigma^\flat_\Sigma(2) = 0$*.*

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{1,2\},\{0,1,2\}\}.$$



$\mathcal{I}$ is systemic and its lattice of theory families and corresponding Leibniz congruence systems are shown in the diagram.



Since the lattice of theory families of $\mathcal{I}$ is order isomorphic with the lattice of $\mathrm{AlgSys}^*(\mathcal{I})$*-congruence systems, $\mathcal{I}$ is weakly family algebraizable.*

On the other hand, for $T = \{\{2,3\}\}$, we have $2,3 \in T_\Sigma$, but $\langle 2,3\rangle \notin \Delta^{\mathbf{F}}_\Sigma = \Omega_\Sigma(T)$, whence $\mathcal{I}$ is not system regular. Hence, a fortiori, $\mathcal{I}$ is not regularly weakly system prealgebraizable.

As was the case with regular weak prealgebraizability, we can show that both kinds of regular weak algebraizability transfer from theory families/ systems to filter families/systems over arbitrary **F**-algebraic systems.

Theorem 622 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is regularly weakly family algebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, all $T,T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and all $\Sigma \in |\mathbf{Sign}|$,

- *$T \leq T'$ implies $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$;*
- *$|T_\Sigma/\Omega^{\mathcal{A}}_\Sigma(T)| = 1$;*

(b) $\mathcal{I}$ is regularly weakly system algebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, all $T,T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, all $T'' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ and all $\Sigma \in |\mathbf{Sign}|$,

- *$T \leq T'$ implies $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$;*
- *$|T''_\Sigma/\Omega^{\mathcal{A}}_\Sigma(T)| = 1$.*

Proof: Combine Theorem 179 with Theorems 599 and 600. ∎

Finally, we may also adapt previously obtained results characterizing weak algebraizability to obtain similar characterizations of regular weak algebraizability in terms of mappings between posets of filter families/ systems and congruence systems.

Corollary 623 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is regularly weakly family algebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism, such that, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$*,* $|T_\Sigma / \Omega^{\mathcal{A}}_\Sigma(T)| = 1$*.*

Proof: By Theorems 617 and 610. ∎

Theorem 624 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is regularly weakly system algebraizable if and only if it is stable and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism, such that, for all $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$*,* $|T_\Sigma / \Omega^{\mathcal{A}}_\Sigma(T)| = 1$*.*

Proof: Suppose, first, that $\mathcal{I}$ is regularly weakly system algebraizable. Then it is, by definition, protoalgebraic and, thus, by Theorem 175, stable. Moreover, by Propositions 618 and 605 and Theorem 597, it is system c-reflective. Therefore, it is weakly algebraizable. Thus, the required isomorphism is given by Theorem 268. The expression for T is obtained by applying Theorem 609.

Assume, conversely, that the postulated condition holds. Consider the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. Since Ω on the collection of theory systems is an order isomorphism, it is monotone and, hence, $\mathcal{I}$ is prealgebraic. Thus, by stability and Theorem 175, $\mathcal{I}$ is protoalgebraic. Moreover, by hypothesis and Theorem 609, $\mathcal{I}$ is system assertional. Thus, by definition, $\mathcal{I}$ is regularly weakly system algebraizable. ∎

8.6 Regular Prealgebraizability

We look, next, at those classes of π-institutions that are formed by adding preequivalentiality to the various levels of assertionality.

Definition 625 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is* **regularly family prealgebraizable**, *or* **RF prealgebraizable** *for short, if it is preequivalential and family assertional;*
- $\mathcal{I}$ *is* **regularly left prealgebraizable**, *or* **RL prealgebraizable** *for short, if it is preequivalential and left assertional;*
- $\mathcal{I}$ *is* **regularly system prealgebraizable**, *or* **RS prealgebraizable** *for short, if it is preequivalential and system assertional.*

Based on the assertionality hierarchy established in Proposition 592, we have the following

Proposition 626 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is regularly family prealgebraizable, then it is regularly left prealgebraizable;*

(b) If $\mathcal{I}$ *is regularly left prealgebraizable, then it is regularly system prealgebraizable.*

Proof: Straightforward by combining Definition 625 and Proposition 592. ■

Proposition 626 establishes the **regular prealgebraizability hierarchy** depicted in the following diagram.

Regular Family Prealgebraizable

↓

Regular Left Prealgebraizable

↓

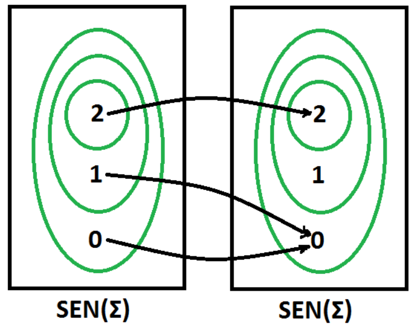
Regular System Prealgebraizable

We give again two examples to show that all classes in this hierarchy are different, i.e., that the arrows in the diagram represent proper inclusions. The first describes a π-institution that is regularly left prealgebraizable but fails to be regularly family prealgebraizable, thus showing that the top arrow stands for a proper inclusion.

Example 627 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial category of natural transformations.*



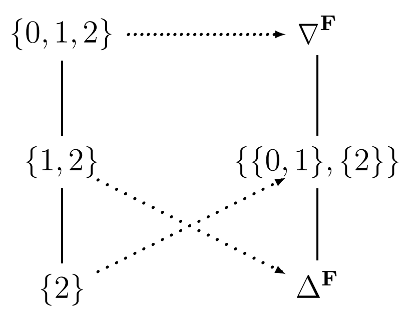
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}$*.*

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$*.*

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

Since $\mathcal{I}$ *is not systemic, by Proposition 591, it fails to be family assertional and, hence, it is not regularly family prealgebraizable.*

The lattice of theory families and the corresponding Leibniz congruence systems are depicted below:



Since the only theory systems of $\mathcal{I}$ *are* $\{\{2\}\}$ *and* $\{\{0,1,2\}\}$*, it is clear that* Ω *is monotone on theory systems and, hence,* $\mathcal{I}$ *is prealgebraic. To see that it is preequivalential, we must also show that it is system extensional. To simplify the process, we note that the only non-trivial proper universes of* $\mathbf{F}$ *are* $\mathbf{X} = \{\{0,1\}\}$ *and* $\mathbf{Y} = \{\{0,2\}\}$*, and the only proper theory system is* $\mathrm{Thm}(\mathcal{I}) = \{\{2\}\}$*. Hence, there are only two cases to check, as shown below (written, as done elsewhere, in shorthand):*

- $\Omega^{\mathbf{X}}(2 \cap \mathbf{X}) = \Omega^{\mathbf{X}}(\varnothing) = \{01\} = \{01,2\} \cap \mathbf{X}^2 = \Omega(2) \cap \mathbf{X}^2$*;*

- $\Omega^{\mathbf{Y}}(2 \cap \mathbf{Y}) = \Omega^{\mathbf{Y}}(2) = \{0,2\} = \{01,2\} \cap \mathbf{Y}^2 = \Omega(2) \cap \mathbf{Y}^2$.

Clearly, $\mathcal{I}$ has theorems. Thus, to complete the proof that it is regularly left prealgebraizable, it suffices to show that it is left assertional, i.e., by Proposition 588, that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T}_\Sigma = 2/\Omega_\Sigma(T)$.

- $\overleftarrow{\{\{2\}\}}_\Sigma = \{2\} = 2/\Omega_\Sigma(\{\{2\}\})$;
- $\overleftarrow{\{\{1,2\}\}}_\Sigma = \{2\} = 2/\Omega_\Sigma(\{\{1,2\}\})$;
- $\overleftarrow{\{\{0,1,2\}\}}_\Sigma = \{0,1,2\} = 2/\Omega_\Sigma(\{\{0,1,2\}\})$.

The second example gives a regularly system prealgebraizable π-institution that is not regularly left prealgebraizable.

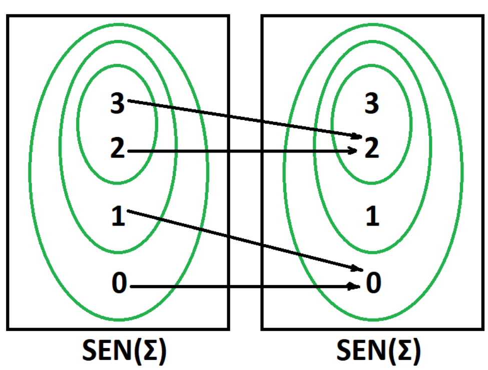
Example 628 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object Σ and a unique (non-identity) morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3\}$ *and*

$$\mathrm{SEN}^\flat(f)(0) = 0,\ \mathrm{SEN}^\flat(f)(1) = 0,\ \mathrm{SEN}^\flat(f)(2) = 2,\ \mathrm{SEN}^\flat(f)(3) = 2;$$

- *$N^\flat$ is the category of natural transformations generated by the unary natural transformation $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ determined by*

x	0	1	2	3
$\sigma^\flat_\Sigma(x)$	0	1	0	1



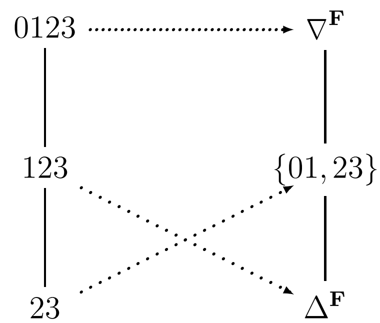
Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\{2,3\},\{1,2,3\},\{0,1,2,3\}\}.$$

The following table shows the action of $\overleftarrow{\ }$ *on theory families.*

T	$\{2,3\}$	$\{1,2,3\}$	$\{0,1,2,3\}$
$\overleftarrow{T}$	$\{2,3\}$	$\{2,3\}$	$\{0,1,2,3\}$

The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right.



Since the only theory systems of $\mathcal{I}$ *are* $\{\{2,3\}\}$ *and* $\{\{0,1,2,3\}\}$, *it is obvious that* Ω *is monotone on theory systems and, hence, that* $\mathcal{I}$ *is prealgebraic. To see that it is preequivalential, it suffices, thus, to show that it is also system extensional. To simplify the process, we note that the only non-trivial proper universes of* $\mathbf{F}$ *are* $\mathbf{X} = \{\{0,1\}\}$, $\mathbf{Y} = \{\{0,2\}\}$ *and* $\mathbf{Z} = \{\{0,1,2\}\}$, *and the only proper theory system is* $\mathrm{Thm}(\mathcal{I}) = \{\{2,3\}\}$. *Hence, there are three cases to check, as shown below (written, as done elsewhere, in shorthand):*

- $\Omega^{\mathbf{X}}(23 \cap \mathbf{X}) = \Omega^{\mathbf{X}}(\varnothing) = \{01\} = \{01,23\} \cap \mathbf{X}^2 = \Omega(23) \cap \mathbf{X}^2$;
- $\Omega^{\mathbf{Y}}(23 \cap \mathbf{Y}) = \Omega^{\mathbf{Y}}(2) = \{0,2\} = \{01,23\} \cap \mathbf{Y}^2 = \Omega(23) \cap \mathbf{Y}^2$;
- $\Omega^{\mathbf{Z}}(23 \cap \mathbf{Z}) = \Omega^{\mathbf{Z}}(2) = \{01,2\} = \{01,23\} \cap \mathbf{Z}^2 = \Omega(23) \cap \mathbf{Z}^2$.

Clearly, $\mathcal{I}$ *has theorems. To see that* $\mathcal{I}$ *is regularly system prealgebraizable it suffices to show that it is system assertional, i.e., by Proposition 588, that, for all* $T \in \mathrm{ThSys}(\mathcal{I})$, $T_\Sigma = 2/\Omega_\Sigma(T)$. *We do have indeed:*

- $\{2,3\} = 2/\Omega_\Sigma(\{\{2,3\}\})$;
- $\{0,1,2,3\} = 2/\Omega_\Sigma(\{\{0,1,2,3\}\})$.

On the other hand, for $T = \{\{1,2,3\}\}$, *we have* $2,3 \in \{2,3\} = \overleftarrow{T}_\Sigma$, *whereas* $\langle 2,3\rangle \notin \Delta^{\mathbf{F}}_\Sigma = \Omega_\Sigma(T)$. *We conclude that* $\mathcal{I}$ *is not left regular and, hence, a fortiori, it is not regularly left prealgebraizable.*

We investigate, next, the relationships that hold between the various regular prealgebraizability classes, introduced in the present section, and the corresponding regular weak prealgebraizability classes, that were introduced in Section 8.4.

Directly from the definitions involved, we get the following

Proposition 629 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

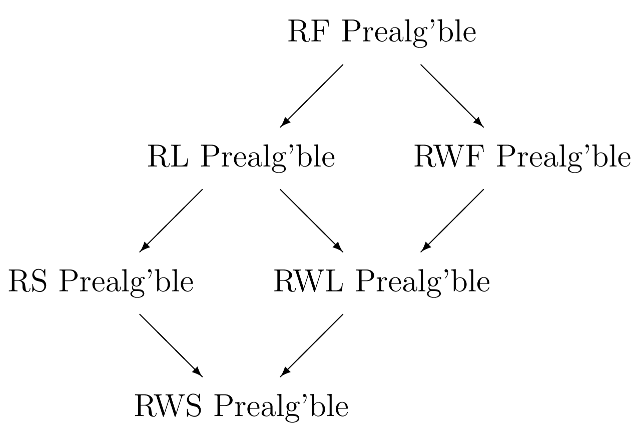
(a) If $\mathcal{I}$ *is regularly family prealgebraizable, then it is regularly weakly family prealgebraizable;*

(b) If $\mathcal{I}$ *is regularly left prealgebraizable, then it is regularly weakly left prealgebraizable;*

(c) If $\mathcal{I}$ *is regularly system prealgebraizable, then it is regularly weakly system prealgebraizable.*

Proof: If $\mathcal{I}$ is regularly family prealgebraizable, then, by definition, it is preequivalential and family assertional. Hence, by Proposition 338, it is prealgebraic and family assertional. Thus, it is, by definition, regularly weakly family prealgebraizable. Parts (b) and (c) can be proven similarly. ■

Therefore, we get the mixed regular prealgebraizability and regular weak prealgebraizability hierarchy depicted in the diagram.



To show that all classes in this hierarchy are different, we provide an example of a π-institution that is regularly weakly family prealgebraizable, and, thus, belongs to all three regular weak prealgebraizability classes, but fails to be regularly system prealgebraizable, whence it belongs to none of three steps in the regular prealgebraizability hierarchy. This example shows that all three southeast arrows represent proper inclusions.

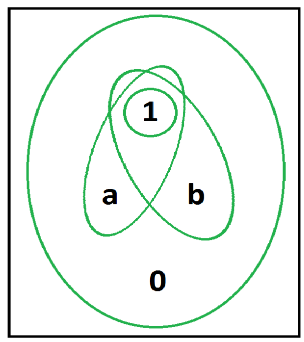
Example 630 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, a, b, 1\}$*;*

- *$N^\flat$ is the category of natural transformations generated by the two binary natural transformations $\wedge, \vee : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ defined by the following tables.*

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1



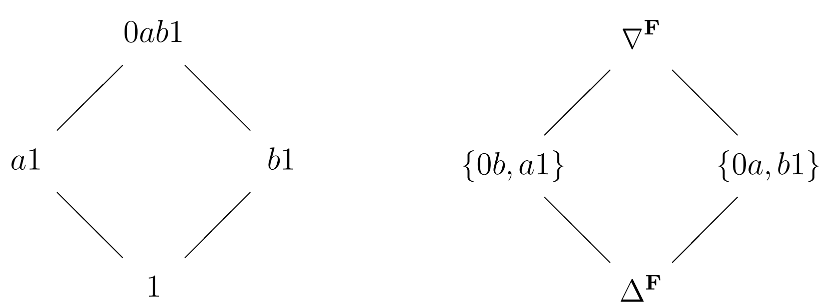
SEN(Σ)

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be the π-institution defined by setting

$$\mathcal{C}_\Sigma = \{\{1\}, \{a, 1\}, \{b, 1\}, \{0, a, b, 1\}\}.$$

$\mathcal{I}$ has four theory families, all of which are also theory systems.

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



From the diagram, we can see that $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is an order isomorphism, whence, $\mathcal{I}$ is weakly family prealgebraizable. Since it has theorems, to show that it is, also, family assertional, it suffices to show that it satisfies, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T_\Sigma = 1/\Omega_\Sigma(T)$. This is easily checked from the diagram above, giving the Leibniz congruence systems corresponding to the various theory families of $\mathcal{I}$.

On the other hand, for the universe $\mathbf{X} = \{\{0,a,1\}\}$ *and the theory system* $T = \{\{1\}\}$, *we get*

$$\Omega(T) \cap \mathbf{X}^2 = \{\{0\},\{a\},\{1\}\} \gneq \{\{0\},\{a,1\}\} = \Omega^{\mathbf{X}}(T \cap \mathbf{X}).$$

Thus, $\mathcal{I}$ *is not system extensional and, therefore, it fails to be (system) preequivalential and, a fortiori, it also fails to be regularly system prealgebraizable.*

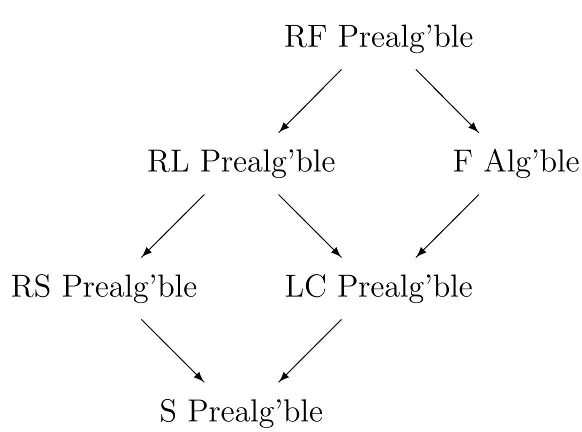
Turning now to the relationship between the regular prealgebraizability hierarchy and the prealgebraizability hierarchy, we get the following

Proposition 631 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

- *If* $\mathcal{I}$ *is regularly family prealgebraizable, then it is family (pre)algebraizable;*
- *If* $\mathcal{I}$ *is regularly left prealgebraizable, then it is left completely reflective prealgebraizable;*
- *If* $\mathcal{I}$ *is regularly system prealgebraizable, then it is system prealgebraizable.*

Proof: We show Part (a) in detail. The remaining parts can be proved similarly. Suppose $\mathcal{I}$ is regularly family prealgebraizable. Then, by definition, it is preequivalential and family assertional. Hence, by Theorem 597, it is preequivalential and family completely reflective. Thus, by definition, it is weakly family (pre)algebraizable. ■

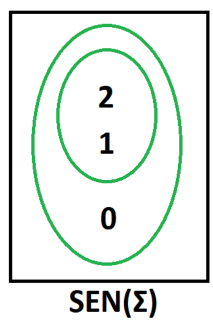
Proposition 631, together with Proposition 626 and the hierarchy established in Section 5.6, point to the following hierarchy of regularly prealgebraizable and (pre)algebraizable π-institutions. Note that the complete hierarchy is larger, but we only show those classes in the (pre)algebraizability hierarchy that are directly related to those in the regular prealgebraizability hierarchy via Theorem 631.



Again it is not difficult to see that the classes in the regular prealgebraizability hierarchy are different from the classes of prealgebraizable π-institutions. This is accomplished by constructing an example of a π-institution which is family completely reflective prealgebraizable (equivalently, family algebraizable), but is not regularly system prealgebraizable.

Example 632 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

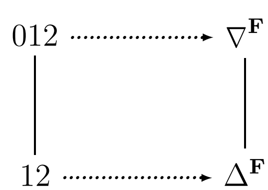
- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *specified by* $\sigma^\flat_\Sigma(0) = 0$*,* $\sigma^\flat_\Sigma(1) = 1$ *and* $\sigma^\flat_\Sigma(2) = 0$*.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{1, 2\}, \{0, 1, 2\}\}.$$

$\mathcal{I}$ *is systemic and its lattice of theory families and corresponding Leibniz congruence systems are shown in the diagram.*



Since the lattice of theory families of $\mathcal{I}$ *is order isomorphic with the lattice of* $\mathrm{AlgSys}^*(\mathcal{I})$*-congruence systems,* $\mathcal{I}$ *is weakly family c-reflective prealgebraizable. To see that it is family c-reflective prealgebraizable, it suffices to show that it is also system extensional. The only nontrivial proper universes of* $\mathbf{F}$ *are* $\mathbf{X} = \{\{0, 1\}\}$ *and* $\mathbf{Y} = \{\{0, 2\}\}$ *and the only proper theory system is* $\{\{1, 2\}\}$*. Thus, we only need to check two cases:*

- $\Omega^{\mathbf{X}}(12 \cap \mathbf{X}) = \Omega^{\mathbf{X}}(1) = \{0,1\} = \Delta^{\mathbf{F}} \cap \mathbf{X}^2 = \Omega(12) \cap \mathbf{X}^2$;
- $\Omega^{\mathbf{Y}}(12 \cap \mathbf{Y}) = \Omega^{\mathbf{Y}}(2) = \{0,2\} = \Delta^{\mathbf{F}} \cap \mathbf{Y}^2 = \Omega(12) \cap \mathbf{Y}^2$.

On the other hand, for $T = \{\{1,2\}\}$, we have $1,2 \in T_\Sigma$, but $\langle 1,2\rangle \notin \Delta^{\mathbf{F}}_\Sigma = \Omega_\Sigma(T)$, whence $\mathcal{I}$ is not system regular and, hence, a fortiori, it is not regularly system prealgebraizable.

Based on existing results, we can show that all three kinds of regular prealgebraizability transfer from theory families/systems to filter families/systems over arbritrary $\mathbf{F}$-algebraic systems.

Theorem 633 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is regularly family prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, every universe $\mathbf{X} \leq \mathbf{A}$, all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, all $T',T'' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ and all $\Sigma \in |\mathbf{Sign}|$,

- *$T' \leq T''$ implies $\Omega^{\mathcal{A}}(T') \leq \Omega^{\mathcal{A}}(T'')$;*
- *$\Omega^{\mathbf{X}}(T' \cap \mathbf{X}) \leq \Omega^{\mathcal{A}}(T') \cap \mathbf{X}^2$;*
- *$|T_\Sigma/\Omega^{\mathcal{A}}_\Sigma(T)| = 1$;*

(b) $\mathcal{I}$ is regularly left prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, every universe $\mathbf{X} \leq \mathbf{A}$, all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, all $T',T'' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ and all $\Sigma \in |\mathbf{Sign}|$,

- *$T' \leq T''$ implies $\Omega^{\mathcal{A}}(T') \leq \Omega^{\mathcal{A}}(T'')$;*
- *$\Omega^{\mathbf{X}}(T' \cap \mathbf{X}) \leq \Omega^{\mathcal{A}}(T') \cap \mathbf{X}^2$;*
- *$|\overleftarrow{T}_\Sigma/\Omega^{\mathcal{A}}_\Sigma(T)| = 1$;*

(c) $\mathcal{I}$ is regularly weakly system prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, every universe $\mathbf{X} \leq \mathbf{A}$, all $T,T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ and all $\Sigma \in |\mathbf{Sign}|$,

- *$T \leq T'$ implies $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$;*
- *$\Omega^{\mathbf{X}}(T \cap \mathbf{X}) \leq \Omega^{\mathcal{A}}(T) \cap \mathbf{X}^2$;*
- *$|T_\Sigma/\Omega^{\mathcal{A}}_\Sigma(T)| = 1$.*

Proof: Combine Theorems 179 and 314 with Theorem 599. ■

Finally, we adapt previously obtained results characterizing prealgebraizability to obtain similar characterizations of regular prealgebraizability in terms of mappings between posets of filter families/ systems (including theory families/systems) and congruence systems.

Theorem 634 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is regularly family prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism that commutes with inverse logical extensions, such that, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$*,* $|T_\Sigma / \Omega^{\mathcal{A}}_\Sigma(T)| = 1$*.*

Proof: Suppose, first, that $\mathcal{I}$ is regularly family prealgebraizable. Then it is, by definition, preequivalential and, moreover, by definition, Proposition 629, Proposition 605 and Theorem 597, it is family c-reflective. Therefore, it is F (pre)algebraizable. Thus, the required isomorphism is given by Theorem 366. The expression for T is obtained by applying Theorem 600.

Assume, conversely, that the postulated condition holds. Consider the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. Since Ω is an order isomorphism, which commutes with inverse logical extensions, $\mathcal{I}$ is preequivalential. Moreover, by hypothesis and Theorem 600, $\mathcal{I}$ is family assertional. Thus, by definition, $\mathcal{I}$ is regularly family prealgebraizable. ∎

Theorem 635 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is regularly left prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order embedding that commutes with inverse logical extensions, such that, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$*,* $|\overleftarrow{T}_\Sigma / \Omega^{\mathcal{A}}_\Sigma(T)| = 1$*.*

Proof: Suppose, first, that $\mathcal{I}$ is regularly left prealgebraizable. Then it is, by definition, preequivalential and, moreover, by definition, Propositions 629 and 605 and Theorem 597, it is left c-reflective. Therefore, it is LC prealgebraizable. Thus, the required embedding is given by Theorem 355. The expression for T is obtained by applying Theorem 600.

Assume, conversely, that the postulated condition holds. Consider the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. Since Ω on the collection of theory systems is an order embedding that commutes with inverse logical extensions, $\mathcal{I}$ is preequivalential. Moreover, by hypothesis and Theorem 600, $\mathcal{I}$ is left assertional. Thus, by definition, $\mathcal{I}$ is regularly left prealgebraizable. ∎

Theorem 636 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is regularly system prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order embedding that commutes with inverse logical extensions, such that, for all $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$*,* $|T_\Sigma / \Omega^{\mathcal{A}}_\Sigma(T)| = 1$*.*

Proof: Suppose, first, that $\mathcal{I}$ is regularly system prealgebraizable. Then it is, by definition, preequivalential and, moreover, by definition, Propositions 629 and 605 and Theorem 597, it is system c-reflective. Therefore, it is system prealgebraizable. Thus, the required embedding is given by Theorem 353. The expression for T is obtained by applying Theorem 600.

Assume, conversely, that the postulated condition holds. Consider the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. Since Ω on the collection of theory systems is an order embedding that commutes with inverse logical extensions, $\mathcal{I}$ is preequivalential. Moreover, by hypothesis and Theorem 600, $\mathcal{I}$ is system assertional. Thus, by definition, $\mathcal{I}$ is regularly system prealgebraizable. ■

8.7 Regular Algebraizability

We look, next, at those classes of π-institutions that are formed by adding equivalentiality to the various levels of assertionality.

Definition 637 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is* **regularly family algebraizable***, or* **RF algebraizable** *for short, if it is equivalential and family assertional;*
- $\mathcal{I}$ *is* **regularly left algebraizable***, or* **RL algebraizable** *for short, if it is equivalential and left assertional;*
- $\mathcal{I}$ *is* **regularly system algebraizable***, or* **RS algebraizable** *for short, if it is equivalential and system assertional.*

Even though, there are apparently three classes in the regular algebraizability hierarchy, in reality there are only two, since, as was the case with regular weak algebraizability, the classes of regularly left and of regularly system π-institutions coincide.

Proposition 638 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is regularly left algebraizable if and only if it is regularly system algebraizable.*

Proof: The "only if" follows directly by the definition and Proposition 592. For the "if", suppose that $\mathcal{I}$ is regularly system algebraizable. Then it is, a fortiori, equivalential and, hence, protoalgebraic. Thus, by Lemma 170, it is stable. Therefore, since $\mathcal{I}$ is system regular and stable, by Proposition 579, it is left regular. We conclude that $\mathcal{I}$ is regularly left algebraizable. ■

The assertionality hierarchy, established in Proposition 592, and Proposition 638 allow us to establish the following regular algebraizability hierarchy.

Proposition 639 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is regularly family algebraizable, then it is regularly system algebraizable.*

Proof: Straightforward by combining Definition 637 and Proposition 592, and taking into account Proposition 638. ∎

The **regular algebraizability hierarchy** is depicted in the following diagram.

Regular Family Algebraizable

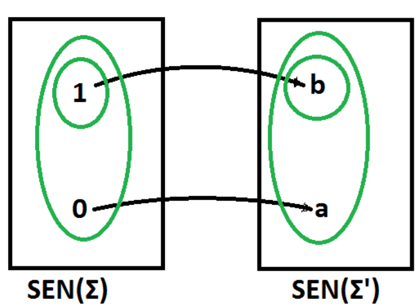
↓

Regular System Algebraizable

We use an example to show that the two classes in this hierarchy are different. Namely, we construct a π-institution that is regularly system algebraizable but fails to be regularly family algebraizable.

Example 640 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*,* $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$*,* $\mathrm{SEN}^\flat(f)(1) = b$*;*
- $N^\flat$ *is the trivial category of natural transformations.*



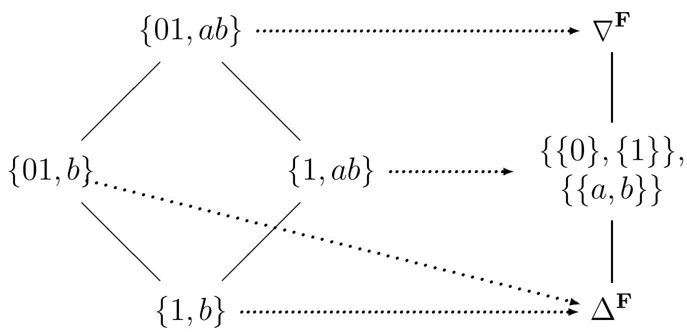
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{1\}, \{0,1\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\}, \{a,b\}\}.$$

The following table shows the action of $^{\leftarrow}$ *on theory families, where rows correspond to* T_Σ *and columns to* $T_{\Sigma'}$ *and each entry is written as* $\overleftarrow{T}_\Sigma, \overleftarrow{T}_{\Sigma'}$.

$\leftarrow$	$\{b\}$	$\{a,b\}$
$\{1\}$	$\{1\},\{b\}$	$\{1\},\{a,b\}$
$\{0,1\}$	$\{1\},\{b\}$	$\{0,1\},\{a,b\}$

The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right:



The Leibniz operator is monotone on theory families, whence, $\mathcal{I}$ *is protoalgebraic. To see that it is equivalential, we must show that it is family extensional. The only non-trivial proper subuniverses of* $\mathbf{F}$ *are* $\mathbf{X} = \{\{0\},\{a,b\}\}$ *and* $\mathbf{Y} = \{\{1\},\{a,b\}\}$*. Moreover, there are only three theory families different from* $\mathrm{SEN}^\flat$*. Thus, we have six cases to examine, accomplished below:*

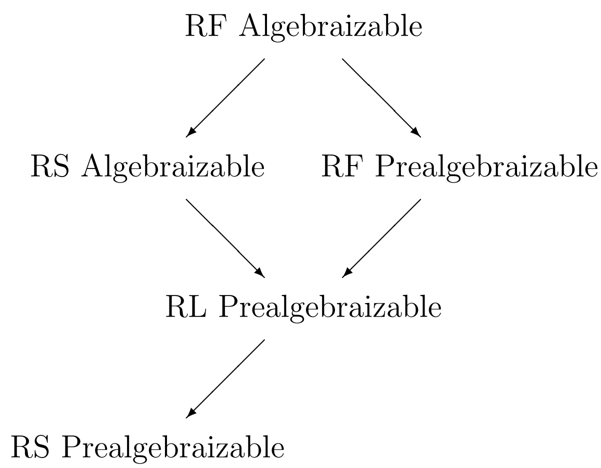
- $\Omega^{\mathbf{X}}(\{1,b\} \cap \mathbf{X}) = \Omega^{\mathbf{X}}(\{\varnothing,b\}) = \{\Delta^{\mathbf{X}}_\Sigma, \nabla^{\mathbf{X}}_{\Sigma'}\} = \Delta^{\mathbf{F}} \cap \mathbf{X}^2 = \Omega(\{1,b\}) \cap \mathbf{X}^2$;
- $\Omega^{\mathbf{X}}(\{01,b\} \cap \mathbf{X}) = \Omega^{\mathbf{X}}(\{0,b\}) = \Delta^{\mathbf{X}} = \Delta^{\mathbf{F}} \cap \mathbf{X}^2 = \Omega(\{01,b\}) \cap \mathbf{X}^2$;
- $\Omega^{\mathbf{X}}(\{1,ab\} \cap \mathbf{X}) = \Omega^{\mathbf{X}}(\{\varnothing,ab\}) = \nabla^{\mathbf{X}} = \{\Delta^{\mathbf{F}}_\Sigma, \nabla^{\mathbf{F}}_{\Sigma'}\} \cap \mathbf{X}^2 = \Omega(\{1,ab\}) \cap \mathbf{X}^2$;
- $\Omega^{\mathbf{Y}}(\{1,b\} \cap \mathbf{Y}) = \Omega^{\mathbf{Y}}(\{1,b\}) = \Delta^{\mathbf{Y}} = \Delta^{\mathbf{F}} \cap \mathbf{Y}^2 = \Omega(\{1,b\}) \cap \mathbf{Y}^2$;
- $\Omega^{\mathbf{Y}}(\{01,b\} \cap \mathbf{Y}) = \Omega^{\mathbf{Y}}(\{1,b\}) = \Delta^{\mathbf{Y}} = \Delta^{\mathbf{F}} \cap \mathbf{Y}^2 = \Omega(\{01,b\}) \cap \mathbf{Y}^2$;
- $\Omega^{\mathbf{Y}}(\{1,ab\} \cap \mathbf{Y}) = \Omega^{\mathbf{Y}}(\{1,ab\}) = \nabla^{\mathbf{Y}} = \{\Delta^{\mathbf{F}}_\Sigma, \nabla^{\mathbf{F}}_{\Sigma'}\} \cap \mathbf{Y}^2 = \Omega(\{1,ab\}) \cap \mathbf{Y}^2$.

We showed that $\mathcal{I}$ *is equivalential. We also have,* $\mathrm{Thm}(\mathcal{I}) = \{\{1\},\{b\}\}$ *and, for every theory system* T, $T_\Sigma = 1/\Omega_\Sigma(T)$ *and* $T_{\Sigma'} = b/\Omega_{\Sigma'}(T)$*. Therefore,* $\mathcal{I}$ *is system assertional. Thus,* $\mathcal{I}$ *is regularly system algebraizable.*

On the other hand, for $T = \{\{0,1\},\{b\}\} \in \mathrm{ThFam}(\mathcal{I})$*, we have* $0,1 \in T_\Sigma$*, but* $\langle 0,1\rangle \notin \Omega_\Sigma(T)$*. Therefore,* $\mathcal{I}$ *fails to be family regular and, hence,* $\mathcal{I}$ *is not regularly family algebraizable.*

We investigate, next, the relationships that hold between the two regular algebraizability classes, introduced in the present section, and the three regular prealgebraizability classes, that were introduced in Section 8.6. Since, by

Proposition 331, equivalentiality implies preequivalentiality, we get, a priori, the following mixed hierarchy.



As was the case with the corresponding weak classes, we can show that the top classes of the regular prealgebraizability and the regular algebraizability hierarchies coincide.

Theorem 641 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is regularly family prealgebraizable if and only if it is regularly family algebraizable.*

Proof: The "if" follows from the relevant definitions and the fact that, by Proposition 331, equivalentiality implies preequivalentiality. For the "only if", it suffices to show that, under family assertionality, preequivalentiality implies equivalentiality. By Proposition 331, it suffices, in turn, to show that family assertionality implies stability and, by Proposition 152, that family assertionality implies systemicity. Indeed, by Theorem 597, family assertionality implies family c-reflectivity and, by Proposition 237, we get that $\mathcal{I}$ is systemic. ■

Moreover, from the definitions involved, we get the following

Proposition 642 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is regularly system algebraizable, then it is regularly left prealgebraizable.*

Proof: Suppose $\mathcal{I}$ is regularly system algebraizable. Equivalently, by Proposition 638, it is regularly left algebraizable. Then, by definition, it is equivalential and left assertional. Thus, by Proposition 331, it is preequivalential and left assertional, i.e., by definition, it is regularly left prealgebraizable. ■

Based on Theorem 641 and Proposition 642, we get the following updated version of the mixed hierarchy shown in the preceding diagram.

RF (Pre)Algebraizable

↓

RS Algebraizable

↓

RL Prealgebraizable

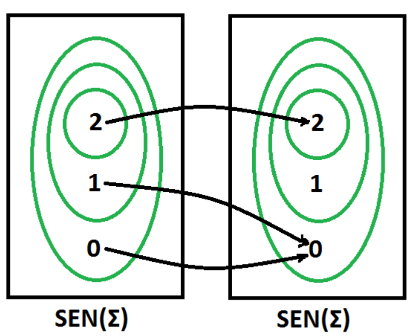
↓

RS Prealgebraizable

To show that all classes in this hierarchy are different, we provide an example of a π-institution that is regularly left prealgebraizable, but fails to be regularly system algebraizable, i.e., an example that separates the regular algebraizability from the regular prealgebraizability classes.

Example 643 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial category of natural transformations.*

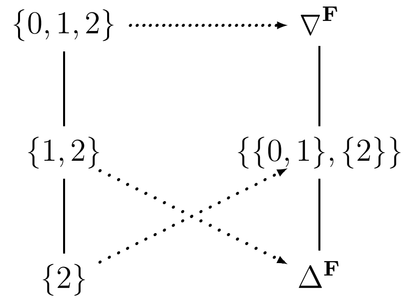


Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $C_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}$.

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$*:*

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

The lattice of theory families and the corresponding Leibniz congruence systems are depicted below:



Since the only theory systems of $\mathcal{I}$ are $\{\{2\}\}$ and $\{\{0,1,2\}\}$, it is clear that Ω is monotone on theory systems and, hence, $\mathcal{I}$ is prealgebraic. To see that it is preequivalential, we must also show that it is system extensional. To simplify the process, we note that the only non-trivial proper universes of $\mathbf{F}$ are $\mathbf{X} = \{\{0,1\}\}$ and $\mathbf{Y} = \{\{0,2\}\}$ and the only proper theory system is $\mathrm{Thm}(\mathcal{I}) = \{\{2\}\}$. Hence, there are two cases to check, as shown below (written, as done elsewhere, in shorthand):

- $\Omega^{\mathbf{X}}(2 \cap \mathbf{X}) = \Omega^{\mathbf{X}}(\varnothing) = \{01\} = \{01,2\} \cap \mathbf{X}^2 = \Omega(2) \cap \mathbf{X}^2$*;*
- $\Omega^{\mathbf{Y}}(2 \cap \mathbf{Y}) = \Omega^{\mathbf{Y}}(2) = \{0,2\} = \{01,2\} \cap \mathbf{Y}^2 = \Omega(2) \cap \mathbf{Y}^2$*.*

Clearly, $\mathcal{I}$ has theorems. Thus, to complete the proof that it is regularly left prealgebraizable, it suffices to show that it is left assertional, i.e., by Proposition 588, that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T}_\Sigma = 2/\Omega_\Sigma(T)$. Indeed, we get:

- $\overleftarrow{\{\{2\}\}}_\Sigma = \{2\} = 2/\Omega_\Sigma(\{\{2\}\})$*;*
- $\overleftarrow{\{\{1,2\}\}}_\Sigma = \{2\} = 2/\Omega_\Sigma(\{\{1,2\}\})$*;*
- $\overleftarrow{\{\{0,1,2\}\}}_\Sigma = \{0,1,2\} = 2/\Omega_\Sigma(\{\{0,1,2\}\})$*.*

On the other hand, since $\{\{2\}\} \leq \{\{1,2\}\}$, but

$$\Omega(\{\{2\}\}) = \{\{\{0,1\},\{2\}\}\} \nleq \Delta^{\mathbf{F}} = \Omega(\{\{1,2\}\}),$$

$\mathcal{I}$ is not protoalgebraic and, hence, a fortioti, it is not equivalential. As a consequence, it fails to be regularly system algebraizable.

Turning now to the relationship between regular (pre)algebraizability and regular weak (pre)algebraizability, we complete the picture given in Section 8.6.

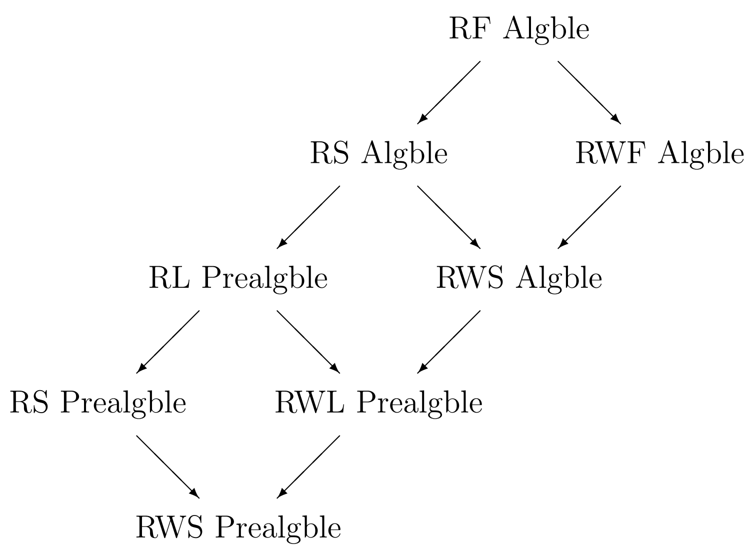
Proposition 644 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) If $\mathcal{I}$ is regularly family algebraizable, then it is regularly weakly family algebraizable;

(b) If $\mathcal{I}$ is regularly system algebraizable, then it is regularly weakly system algebraizable.

Proof: By Definition 329, equivalentiality implies protoalgebraicity. From this fact, and Definitions 637 and 613, both implications follow directly. ■

Thus, Proposition 644, together with Propositions 642 and 629, point to the following hierarchy of regularly (pre)algebraizable π-institutions and regularly weakly (pre)algebraizable π-institutions.



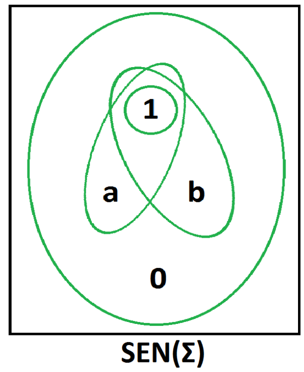
To see that all southeast arrows represent proper inclusions, we give an example of a regularly weakly family algebraizable π-institution which fails to be regularly system prealgebraizable.

Example 645 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, a, b, 1\}$*;*
- $N^\flat$ *is the category of natural transformations, generated by the two binary natural transformations* $\wedge, \vee : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$*, defined by the following tables:*

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

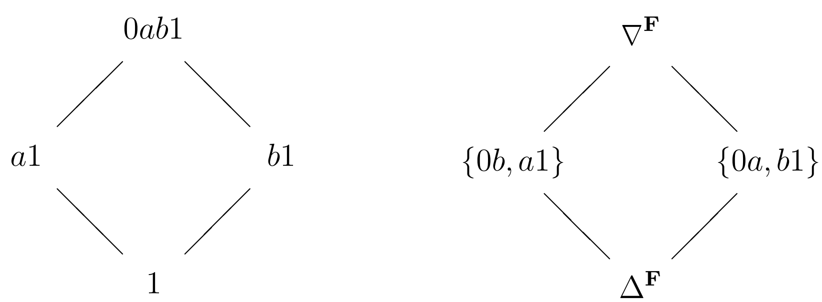


Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be the π-institution, defined by setting

$$\mathcal{C}_\Sigma = \{\{1\},\{a,1\},\{b,1\},\{0,a,b,1\}\}.$$

$\mathcal{I}$ has four theory families, all of which are also theory systems.

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



From the diagram, we can see that $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is an order isomorphism, whence, $\mathcal{I}$ is weakly family algebraizable. Since it has theorems, to show that it is, also, family assertional, it suffices to show that it satisfies, for all $T \in \mathrm{Thfam}(\mathcal{I})$, $T_\Sigma = 1/\Omega_\Sigma(T)$. This is easily checked from the diagram above, giving the Leibniz congruence systems corresponding to the various theory families of $\mathcal{I}$.

On the other hand, for the universe $\mathbf{X} = \{\{0,a,1\}\}$ and the theory system $T = \{\{1\}\}$, we get

$$\Omega(T) \cap \mathbf{X}^2 = \{\{0\},\{a\},\{1\}\} \subsetneq \{\{0\},\{a,1\}\} = \Omega^{\mathbf{X}}(T \cap \mathbf{X}).$$

Thus, $\mathcal{I}$ is not system extensional and, therefore, it fails to be (system) prequivalential and, a fortiori, it also fails to be regularly system prealgebraizable.

Turning now to the relationship between regular (pre)algebraizability and (pre)algebraizability, we get, by definition,

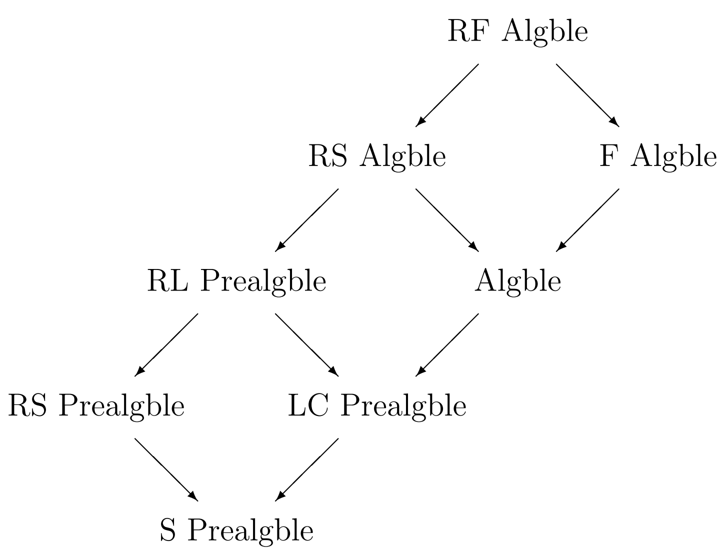
Proposition 646 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.*

(a) If $\mathcal{I}$ is regularly family algebraizable, then it is family algebraizable;

(b) If $\mathcal{I}$ is regularly system algebraizable, then it is (system) algebraizable.

Proof: For Part (a) note that, by definition, $\mathcal{I}$ is regularly family algebraizable if and only if it is equivalential and family assertional. Thus, by Theorem 597, it is equivalential and family completely reflective. Thus, by Definition 360, it is family algebraizable. Part (b) follows along similar lines. ∎

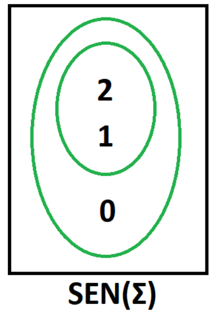
Proposition 646 completes the picture given by Proposition 631 and Proposition 642, establishing the following mixed regular (pre)algebraizability and (pre)algebraizability hierarchies, where, on the prealgebraizability side, only the classes immediately interacting with the regular prealgebraizability classes are shown.



Again it is not difficult to see that the classes in the regular (pre)algebraizability hierarchy are different from the classes in the (pre)algebraizability hierarchy. This is accomplished by constructing an example of a π-institution which is family algebraizable but is not regularly system prealgebraizable.

Example 647 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

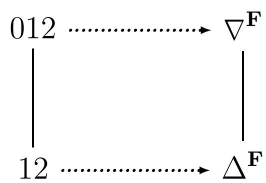
- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *specified by* $\sigma^\flat_\Sigma(0) = 0$, $\sigma^\flat_\Sigma(1) = 1$ *and* $\sigma^\flat_\Sigma(2) = 0$.



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\{1,2\}, \{0,1,2\}\}.$$

$\mathcal{I}$ is systemic and its lattice of theory families and corresponding Leibniz congruence systems are shown in the diagram.



Since the lattice of theory families of $\mathcal{I}$ is order isomorphic with the lattice of $\mathrm{AlgSys}^*(\mathcal{I})$*-congruence systems, $\mathcal{I}$ is weakly family algebraizable. To see that it is family algebraizable, it suffices to show that it is also family extensional. The only nontrivial proper universes of $\mathbf{F}$ are $\mathbf{X} = \{\{0,1\}\}$ and $\mathbf{Y} = \{\{0,2\}\}$ and the only proper theory family is $\{\{1,2\}\}$. Thus, we only need to check two cases:*

- $\Omega^{\mathbf{X}}(12 \cap \mathbf{X}) = \Omega^{\mathbf{X}}(1) = \{0,1\} = \Delta^{\mathbf{F}} \cap \mathbf{X}^2 = \Omega(12) \cap \mathbf{X}^2$;
- $\Omega^{\mathbf{Y}}(12 \cap \mathbf{Y}) = \Omega^{\mathbf{Y}}(2) = \{0,2\} = \Delta^{\mathbf{F}} \cap \mathbf{Y}^2 = \Omega(12) \cap \mathbf{Y}^2$.

On the other hand, for $T = \{\{2,3\}\}$, we have $2,3 \in T_\Sigma$, but $\langle 2,3 \rangle \notin \Delta^{\mathbf{F}}_\Sigma = \Omega_\Sigma(T)$, whence $\mathcal{I}$ is not system regular and, hence, a fortiori, it is not regularly system prealgebraizable.

As was the case with regular weak algebraizability, we can show that both kinds of regular algebraizability transfer from theory families/ systems to filter families/systems over arbitrary $\mathbf{F}$-algebraic systems.

Theorem 648 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is regularly family algebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, all $\mathbf{X} \leq \mathbf{A}$, all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and all $\Sigma \in |\mathbf{Sign}|$,

- $T \leq T'$ *implies* $\Omega(T) \leq \Omega(T')$;
- $\Omega^{\mathbf{X}}(T \cap \mathbf{X}) = \Omega(T) \cap \mathbf{X}^2$;
- $|T_\Sigma / \Omega_\Sigma(T)| = 1$;

(b) *$\mathcal{I}$ is regularly system algebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, all $\mathbf{X} \leq \mathbf{A}$, all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, all $T'' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ and all $\Sigma \in |\mathbf{Sign}|$,*

- $T \leq T'$ *implies* $\Omega(T) \leq \Omega(T')$;
- $\Omega^{\mathbf{X}}(T \cap \mathbf{X}) = \Omega(T) \cap \mathbf{X}^2$;
- $|T''_\Sigma / \Omega_\Sigma(T)| = 1$.

Proof: Combine Theorem 334 with Theorem 599. ■

Finally, we obtain characterizations of regular algebraizability in terms of mappings between posets of filter families/ systems and congruence systems.

Corollary 649 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is regularly family algebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism commuting with inverse logical extensions, such that, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and all $\Sigma \in |\mathbf{Sign}|$, $|T_\Sigma / \Omega^{\mathcal{A}}_\Sigma(T)| = 1$.

Proof: By Theorems 641 and 634. ■

Theorem 650 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is regularly system algebraizable if and only if it is stable and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism commuting with inverse logical extensions, such that, for all $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ and all $\Sigma \in |\mathbf{Sign}|$, $|T_\Sigma / \Omega^{\mathcal{A}}_\Sigma(T)| = 1$.

Proof: Suppose, first, that $\mathcal{I}$ is regularly system algebraizable. Then it is, by definition, equivalential and, thus, by Proposition 331, stable. Moreover, by Proposition 646, it is algebraizable, whence, the required isomorphism is given by Theorem 365. The expression for T is obtained by applying Theorem 600.

Assume, conversely, that the postulated condition holds. Consider the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. Since Ω is an order isomorphism that commutes with inverse logical extensions and $\mathcal{I}$ is stable, $\mathcal{I}$ is, by Theorem 365, algebraizable. Hence, $\mathcal{I}$ is, a fortiori, equivalential. Moreover, by hypothesis and Theorem 600, $\mathcal{I}$ is system assertional. Thus, by definition, $\mathcal{I}$ is regularly system algebraizable. ■