Chapter 9

The Semantic Leibniz Hierarchy: Over the Top II

9.1 Introduction

At the apex of the Leibniz hierarchy of sentential logics one finds the class of algebraizable logics [35, 43] (see, also, Chapter 4 of [64] and Sections 3.2 and 6.5 of [86]). The concept was first introduced by Blok and Pigozzi in [35] for finitary logics. It was later generalized to arbitrary sentential logics by Herrmann [43]. Roughly speaking, a sentential logic is *algebraizable* when there exist a class K of algebras, termed the *equivalent algebraic semantics*, and two translations $\delta \approx \varepsilon$ from formulas to equations, called *defining equations*, and Δ from equations to formulas, called *equivalence formulas*, which are interpretations, i.e., preserve and reflect the logical and the equational closures and vice-versa and, in addition, are inverses of one another in a specific sense. For a detailed study of this framework, apart from the original monograph by Blok and Pigozzi [35] and Herrmann's Dissertation [43], one may consult Chapter 4 of [64] and Sections 3.2 and 6.5 of [86]. Partly due to the historical progression, but also due to the intrinsic importance and ubiquity of finitarity, its key role in studies of classical logical systems and a host of advantageous properties associated with it, the finitary aspects of algebraizability have been extensively studied and tight relations between them have been established. A very illuminative and beautifully written summary of these results, as pertaining to algebraizability, appears in Section 3.4 of [86], which constitutes the inspiration and starting point of the investigations presented here.

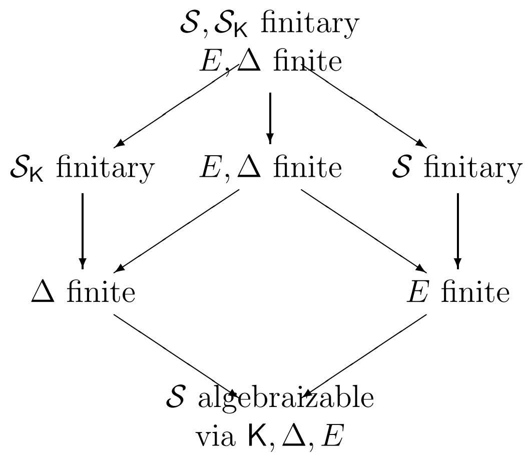
We first give a quick overview of the aforementioned work pertaining to algebraizable sentential logics. We fix an algebraizable sentential logic $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$, with equivalent algebraic semantics the generalized quasivariety K, as witnessed by a set $\delta \approx \varepsilon$ of defining equations and a set Δ of equivalence formulas. Lemma 3.36 of [86] asserts that, in case $\mathcal{S}$ is finitary and has a finite set of equivalence formulas, then every set of equivalence formulas contains a finite subset that also serves as a set of equivalence formulas. Dually, if the equational logic $\mathcal{S}_{\mathsf{K}} = \langle \mathcal{L}, \vDash_{\mathsf{K}} \rangle$ induced by the class K is finitary, i.e., if the class K happens to be a quasivariety, and $\mathcal{S}$ has a finite set of defining equations, then every set of defining equations has a finite subset that also serves in the same capacity. Besides these conditional "finitarization" results, Theorem 3.37 of [86] details some important relationships between the following four conditions: $\mathcal{S}$ finitary; $\mathcal{S}_{\mathsf{K}}$ finitary; $\delta \approx \varepsilon$ finite; and Δ finite. On the one hand, if $\mathcal{S}$ is finitary, $\delta \approx \varepsilon$ may be taken finite, and, dually, if $\mathcal{S}_{\mathsf{K}}$ is finitary, then Δ may be taken to be finite. Moreover, if $\mathcal{S}$ is finitary and has a finite set Δ of equivalence formulas, then $\mathcal{S}_{\mathsf{K}}$ is finitary also, and, dually, if $\mathcal{S}_{\mathsf{K}}$ is finitary and $\mathcal{S}$ has a finite set $\delta \approx \varepsilon$ of defining equations, then $\mathcal{S}$ is finitary also. These implications lead to Corollary 3.38 of [86], which asserts the following three conditional equivalences:

1. If $\delta \approx \varepsilon$ and Δ are both finite, then $\mathcal{S}$ is finitary iff $\mathcal{S}_{\mathsf{K}}$ is finitary.

2. If $\mathcal{S}$ is finitary, then $\mathcal{S}_{\mathsf{K}}$ is finitary iff Δ may be taken finite.

3. If $\mathcal{S}_{\mathsf{K}}$ is finitary, then $\mathcal{S}$ is finitary iff $\delta \approx \varepsilon$ may be taken finite.

These lead to the hierarchy depicted in the diagram shown in Figure 3, p. 137 of [86] and duplicated below.



After outlining the results that lead to the depicted hierarchy, Font gathers pointers to three examples of sentential logics that serve to separate the various classes in the hierarchy. Example 3.41 of [86] revisits Łukasiewicz's infinite valued logic $Ł_\infty$, which is not finitary, has a non-finitary equivalent algebraic semantics, but is algebraized via finite sets of defining equations and equivalence formulas. As a result, it serves to separate classes related by the vertical arrows in the diagram. In Example 3.42 of [86], the so-called Logic of Last Judgement $\mathcal{L}J$, introduced by Herrmann [53], is presented. This is a finitary logic, algebraized by a non-finitary equational consequence, via a single defining equation, but a necessarily infinite set of equivalence formulas. So $\mathcal{L}J$ serves in separating the logics related by the southeast arrows in the diagram. Additional examples that can serve the same purpose were presented by Dellunde [48] and by Lewin, Mikenberg and Schwartze [55]. Finally, in Example 3.43 of [86], Font presents a logic due to Raftery [82]. Raftery's work was motivated by a question posed by Czelakowski in Note 4.5.2 (4) of [64], which was also implicit in Problem 3.18 of [43]. Raftery's logic is not finitary, but is algebraizable, with a finitary equivalent algebraic semantics which is actually a variety, via an infinite set of defining equations and a singleton set of equivalence formulas. It serves in separating the classes of sentential logics connected via the southwest arrows of the diagram. In Section 9.5, we revisit Łukasiewicz's logic and the logics of Dellunde and Raftery in much more detail.

Our own goal in this chapter is to provide analogs of the classes in the finitarity hierarchy of algebraizable sentential logics for logics formalized as π-institutions. The finitarity conditions pertaining to π-institutions remain roughly unchanged. However, keeping in the spirit of dealing with semantically defined classes (i.e., relying on properties of the Leibniz operator) in this part of the monograph, the finitarity conditions regarding $\delta \approx \varepsilon$ and Δ are modified. They are recast as continuity properties of the Leibniz operator and of its inverse. Subject to these modifications, the results obtained for semantically defined finitarity properties pertaining to weakly family algebraizable π-institutions reflect those outlined above for algebraizable sentential logics.

In Section 9.2, we introduce the concept of a π-*structure*, which abstracts that of a π-institution by eliminating the requirement of structurality. That is, a π-structure $\mathcal{I}$ consists of an algebraic system together with a collection of closure operators, one on each of its sentence components, which are not required to satisfy structurality. The *finitary companion* of a π-structure $\mathcal{I}$ is the π-structure obtained by considering the closure family induced by all finite consequences of $\mathcal{I}$. As a consequence, it constitutes the largest finitary π-structure included in $\mathcal{I}$. In addition, it can be shown that it is structural when the given π-structure satisfies structurality, i.e., when it is a π-institution. Finitary companions may also be characterized via their theory families. Namely, a sentence family of a π-institution $\mathcal{I}$ is a theory family of its finitary companion if and only if it is the union of a directed collection of locally finitely generated theory families of $\mathcal{I}$.

In Section 9.3, we focus on some of the fundamental properties that determine the classes in the semantic Leibniz hierarchy and investigate whether they are transferred from a π-institution to its finitary companion and vice-versa, and, if yes, under which conditions. In this vein, protoalgebraicity is shown to hold for a π-institution $\mathcal{I}$ if its finitary companion is protoalgebraic. A similar property holds for family reflectivity. As a consequence, a π-institution is weakly family algebraizable if its finitary companion has the same property. In the opposite direction, for properties of $\mathcal{I}$ to be inherited by its finitary companion, additional provisos are needed. We say that the Leibniz operator of a π-institution is *continuous* if, for every directed collection $\{T^i\}_{i\in I}$ of theory families, such that $\bigcup_{i\in I} T^i$ is also a theory family, $\Omega(\bigcup_{i\in I} T^i) = \bigcup_{i\in I} \Omega(T^i)$. Continuity of the Leibniz operator is a stronger property than, i.e., implies, protoalgebraicity. It turns out that it is also sufficient for the finitary companion of $\mathcal{I}$ to be protoalgebraic, subject to the category of signatures being finite. Along dual lines, we say that the inverse Leibniz operator of a weakly family algebraizable π-institution $\mathcal{I}$ is *continuous* if, for all directed collections $\{\theta^i\}_{i\in I}$ of congruence systems in $\mathrm{ConSys}^*(\mathcal{I})$, such that $\bigcup_{i\in I} \theta^i$ is also a congruence system in $\mathrm{ConSys}^*(\mathcal{I})$, $\Omega^{-1}(\bigcup_{i\in I} \theta^i) = \bigcup_{i\in I} \Omega^{-1}(\theta^i)$. This condition, when supplementing continuity of the Leibniz operator, ensures that weak family algberaizability of a π-institution $\mathcal{I}$, with a finite category of signatures, is inherited by its finitary companion.

Section 9.4 is the main section of the chapter. Here, we establish the semantic finitarity hierarchy of weakly family algebraizable π-institutions, which parallels the hierarchy of sentential logics studied in Section 3.4 of [86] and summarized both in Subsection 1.3.8 and at the beginning of this Introduction. Throughout, the object of study is a weakly family algebraizable π-institution $\mathcal{I}$. Moreover, we denote by $\mathsf{K} := \mathrm{AlgSys}(\mathcal{I})$ and by $\mathcal{Q}^{\mathsf{K}}$ the equational π-structure induced by the class K. Note that $\mathcal{I}$ being weakly family algebraizable ensures that the Leibniz operator $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is an isomorphism, whence the inverse Ω^{-1} is well-defined. It is shown, first, that the finitarity of $\mathcal{I}$ ensures the continuity of the inverse Leibniz operator on $\mathrm{ConSys}^*(\mathcal{I})$ and that, dually, the finitarity of $\mathcal{Q}^{\mathsf{K}}$ guarantees that the Leibniz operator itself is continuous on $\mathrm{ThFam}(\mathcal{I})$. Further, if to the finitarity of $\mathcal{I}$ is added the continuity of the Leibniz operator, then the finitarity of $\mathcal{Q}^{\mathsf{K}}$ follows. Dually, if to the finitarity of $\mathcal{Q}^{\mathsf{K}}$ is added the continuity of the inverse Leibniz operator, then $\mathcal{I}$ is also finitary. These results are summarized in three statements, which parallel those governing sentential logics, stated in Corollary 3.38 of [86]. Namely, under continuity of both the Leibniz operator and its inverse, finitarity of $\mathcal{I}$ is equivalent to finitarity of $\mathcal{Q}^{\mathsf{K}}$. Under finitarity of $\mathcal{I}$, finitarity of $\mathcal{Q}^{\mathsf{K}}$ is equivalent to continuity of the Leibniz operator and, dually, under finitarity of $\mathcal{Q}^{\mathsf{K}}$, finitarity of $\mathcal{I}$ is tantamount to continuity of the inverse Leibniz operator.

In Section 9.5, we take a brief detour to present in detail three examples of sentential logics, which serve to separate the classes in the finitarity hierarchy of algebraizable sentential logics, presented in Section 3.4 of [86]. Even though our focus here is not on sentential logics, we showed in Section 1.1 how a sentential logic gives rise to a π-institution in a rather straightforward way. Accordingly, the purpose of presenting these three sentential logics is to construct, based on them, corresponding π-institutions that will serve to separate the classes in the finitarity hierarchy of weakly family algebraizable π-institutions, studied in Section 9.4. The constructions of the π-institutions, based on the sentential logics introduced here, and the separation properties they help establish will be described in some detail in Section 9.6.

The first example is Łukasiewicz's infinite valued logic (see, e.g., Example 3.41 of [86]). It is a logic over a language with three binary connectives $\wedge$, $\vee$, $\to$ and one unary connective $\neg$. It is semantically defined via a logical matrix. It is shown that it is not finitary, but that it is algebraizable via a singleton set of defining equations $E(x) = \{x \approx \top\}$, where $\top := x \to x$, and the doubleton set of equivalence formulas $\Delta(x,y) = \{x \to y, y \to x\}$. This logic serves to separate the classes of sentential logics related by vertical arrows in the finitarity hierarchy of algebraizable sentential logics, depicted in the preceding diagram. The second example is a logic introduced by Dellunde in [48]. It is a logic defined over a language with one binary connective $\leftrightarrow$ and one unary connective $\Box$. It is defined via a Hilbert calculus and, as a result, it is finitary. It is shown that it is regularly algebraizable via the infinite set of equivalence formulas $\Delta(x,y) = \{\Box^n x \leftrightarrow \Box^n y : n \in \omega\}$. According to the general theory, regular algebraizability implies that the set of defining equations is the singleton $E(x) = \{x \approx \top\}$, where $\top := x \leftrightarrow x$ defines the unique element in the filter of any reduced matrix of the logic. What is key for our purposes is that there does not exist a finite subset $\Delta_0 \subseteq \Delta$ that also serves as a set of equivalence formulas for this logic. Consequently, this example serves in separating those classes in the hierarchy of sentential logics connected via southeast arrows in the diagram. The last example presented in Section 9.5 is a logic introduced by Raftery [82]. It is a logic defined over a language with one binary connective $\leftrightarrow$ and three unary connectives π_1, π_2 and $\Diamond$. It is semantically defined as a weakening of another logic, which, in turn, is defined using a logical matrix. The weakening, roughly speaking, is obtained by considering an entire variety of algebras to which the underlying algebra of this logical matrix belongs. Raftery shows that neither logic is finitary and that, in addition, the weaker logic, corresponding to the variety, is algebraizable via an infinite set of defining equations and a singleton set of equivalence formulas. As a result, Raftery's logic serves as an example separating the classes related by the southwest arrows in the finitarity hierarchy depicted in the preceding diagram.

In Section 9.6, we use the framework outlined in Section 1.1 to formalize the three sentential logics of Section 9.5 as π-institutions. The resulting examples enable us to separate the classes in the semantic finitarity hierarchy of weakly family algebraizable π-institutions, studied in Section 9.4, in a way that parallels the separation of the classes in the hierarchy of algebraizable sentential logics.

9.2 The Finitary Companion

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. A **π-structure** $\mathcal{I} = \langle \mathbf{F}, D \rangle$, **based on F**, is like a π-institution except that D is a **closure family** on $\mathbf{F}$ instead of a closure system, i.e., the only requirement is that

$$D_\Sigma : \mathcal{P}\mathrm{SEN}^\flat(\Sigma) \to \mathcal{P}\mathrm{SEN}^\flat(\Sigma)$$

be a closure operator on $\mathrm{SEN}^\flat(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}^\flat|$. On the other hand, D is not required to be structural. A heavier use of π-structures will be encountered in Chapter 12, where the concept will be defined anew and more details given. D is called the **closure family of the π-structure** $\mathcal{I}$. Note that π-structures generalize π-institutions.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, D \rangle$ be a π-structure based on $\mathbf{F}$. Define the family

$$D^f = \{D^f_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$$

by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$D^f_\Sigma : \mathcal{P}\mathrm{SEN}^\flat(\Sigma) \to \mathcal{P}\mathrm{SEN}^\flat(\Sigma)$$

be given, for all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$, by

$$D^f_\Sigma(\Phi) = \bigcup\{D_\Sigma(\Phi') : \Phi' \subseteq_f \Phi\},$$

where $\subseteq_f$ denotes the finite subset relation.

It is not hard to show that D^f is a finitary closure family on $\mathbf{F}$ and that, moreover, it is a closure system (i.e., structural) in case D itself happens to be structural.

Lemma 651 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, D\rangle$ *a* π*-structure based on* $\mathbf{F}$*. Then* D^f *is a finitary closure family on* $\mathbf{F}$*. Further, if* D *is structural, i.e., if* $\mathcal{I}$ *is a* π*-institution, then* D^f *is also structural.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \Phi$. Then $\phi \in D_\Sigma(\phi) \subseteq D^f_\Sigma(\Phi)$. Thus, D^f is inflationary.

Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \Psi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\Phi \subseteq \Psi$. If $\phi \in D^f_\Sigma(\Phi)$, then there exists $\Phi' \subseteq_f \Phi$, such that $\phi \in D_\Sigma(\Phi')$. But $\Phi' \subseteq_f \Phi \subseteq \Psi$, whence, $\phi \in D^f_\Sigma(\Psi)$. Thus, $D^f_\Sigma(\Phi) \subseteq D^f_\Sigma(\Psi)$ and D^f is also monotone.

Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in D^f_\Sigma(D^f_\Sigma(\Phi))$. Then, there exists $\Phi' \subseteq_f D^f_\Sigma(\Phi)$, such that $\phi \in D_\Sigma(\Phi')$. Since $\Phi' \subseteq D^f_\Sigma(\Phi)$, for all $\phi' \in \Phi'$, there exists $\Phi'^{\phi'} \subseteq_f \Phi$, such that $\phi' \in D_\Sigma(\Phi'^{\phi'})$. Hence, we get

$$\phi \in D_\Sigma(\Phi') \subseteq D_\Sigma(\bigcup_{\phi' \in \Phi'} \Phi'^{\phi'}).$$

Since $\bigcup_{\phi' \in \Phi'} \Phi'^{\phi'} \subseteq_f \Phi$, we get, by definition, $\phi \in D^f_\Sigma(\Phi)$. Thus, D^f is also idempotent.

Finally, to show finitarity, let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in D^f_\Sigma(\Phi)$. Thus, there exists $\Phi' \subseteq_f \Phi$, such that $\phi \in D_\Sigma(\Phi')$. Then, by definition, $\phi \in D^f_\Sigma(\Phi')$. Thus, $D^f_\Sigma(\Phi) = \bigcup_{\Phi' \subseteq_f \Phi} D^f_\Sigma(\Phi')$ and, hence, D^f is a finitary closure family on $\mathbf{F}$.

To prove the last statement concerning structurality, assume that D is structural. Let $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in D^f_\Sigma(\Phi)$. Then, there exists $\Phi' \subseteq_f \Phi$, such that $\phi \in D_\Sigma(\Phi')$. Since D is assumed structural, we get $\mathrm{SEN}^\flat(f)(\phi) \in D_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi'))$. But Φ' being a finite subset of Φ, $\mathrm{SEN}^\flat(f)(\Phi')$ is a finite subset of $\mathrm{SEN}^\flat(f)(\Phi)$, whence, $\mathrm{SEN}^\flat(f)(\phi) \in D^f_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))$. This shows that D^f is also structural. ∎

The following proposition provides a characterization of D^f.

Proposition 652 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, D\rangle$ *a* π*-structure based on* $\mathbf{F}$*. Then* D^f *is the largest finitary closure family on* $\mathbf{F}$ *lying below* D *in the* $\leq$ *ordering.*

Proof: By Lemma 651, D^f is a finitary closure family. By its definition and the monotonicity of D, it is clear that $D^f \leq D$. To complete the proof, suppose D' is a finitary closure family on $\mathbf{F}$, such that $D' \leq D$. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in D'_\Sigma(\Phi)$. Since, by hypothesis, D' is finitary, there exists $\Phi' \subseteq_f \Phi$, such that $\phi \in D'_\Sigma(\Phi')$. Since, also by hypothesis, $D' \leq D$, we get $\phi \in D_\Sigma(\Phi')$. Thus, since $\Phi' \subseteq_f \Phi$, we get, by definition of D^f, $\phi \in D^f_\Sigma(\Phi)$. Thus, $D' \leq D^f$ and, therefore, D^f is the largest finitary closure family below D. ■

Corollary 653 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *Then* C^f *is the largest finitary closure system on* $\mathbf{F}$ *lying below* C *in the* $\leq$ *ordering.*

Proof: By Proposition 652, C^f is the largest finitary closure family lying below C. But, by Lemma 651, it is a closure system on $\mathbf{F}$. Hence, it is the largest finitary closure system lying below C. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, D \rangle$ be a π-structure based on $\mathbf{F}$. We call D^f the **finitary companion of** D. Moreover, we set $\mathcal{I}^f = \langle \mathbf{F}, D^f \rangle$ and call it the **finitary companion of** $\mathcal{I}$. Of course, these terms apply, in particular, to the case of π-institutions.

We would like to provide an alternative characterization of the finitary companion that is also very useful in various applications of the notion. With an eye towards this goal, we make the following definitions.

Definition 654 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\mathcal{T} \cup \{T\} \subseteq \mathrm{ThFam}(\mathcal{I})$.

- T *is called* **locally finitely generated** *if, for all* $\Sigma \in |\mathbf{Sign}^\flat|$, *there exists* $\Phi_\Sigma \subseteq_f T_\Sigma$, *such that* $T_\Sigma = C_\Sigma(\Phi_\Sigma)$.
- $\mathcal{T}$ *is* **locally finitely generated** *if all its theory families are locally finitely generated.*

The following proposition provides a characterization of those sentence families of a π-institution $\mathcal{I}$ that are theory families of its finitary companion.

Proposition 655 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $T \in \mathrm{SenFam}(\mathbf{F})$. *Then* $T \in \mathrm{ThFam}(\mathcal{I}^f)$ *if and only if, there exists a directed locally finitely generated collection* $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$, *such that*

$$T = \bigcup_{i \in I} T^i.$$

Proof: Let $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$ be a directed locally finitely generated collection. Since it is locally finitely generated, we have, by definition, $T^i \in \mathrm{ThFam}(\mathcal{I}^f)$, for all $i \in I$. But, by Lemma 651, $\mathcal{I}^f$ is finitary. Thus, by Proposition 112, it is continuous. Hence $\bigcup_{i\in I} T^i \in \mathrm{ThFam}(\mathcal{I}^f)$.

Suppose, conversely, that $T \in \mathrm{ThFam}(\mathcal{I}^f)$. Set

$$\mathcal{T} = \{C(X) : X \leq_{lf} T\},$$

where $\leq_{lf}$ denotes the locally finite subfamily relation. It is clear, by its definition, that $\mathcal{T}$ is locally finitely generated. Suppose $C(X), C(Y) \in \mathcal{T}$. Then $C(X \cup Y) \in \mathcal{T}$ and, moreover, $C(X), C(Y) \leq C(X \cup Y)$. Hence, $\mathcal{T}$ is also directed. Finally, it is not difficult to see that $T = \bigcup \mathcal{T}$. Thus, the declared characterization holds. ■

9.3 π-Institutions & Companions: Hierarchy

In this section, we study how some of the properties that have been used to build hierarchies of π-institutions are inherited by the finitary companion of a π-institution from the π-institution itself and vice-versa. In some instances the inheritance is immediate, but, in others, additional conditions need to be imposed. We focus on the property of weak family algebraizability. That is the reason of selecting the few properties studied here versus some of the remaining properties introduced previously.

First, we show that protoalgebraicity is passed up to $\mathcal{I}$ from its finitary companion $\mathcal{I}^f$.

Lemma 656 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a π-institution based on* $\mathbf{F}$. *If* $\mathcal{I}^f = \langle \mathbf{F}, C^f\rangle$ *is protoalgebraic, then so is* $\mathcal{I}$.

Proof: Suppose $\mathcal{I}^f$ is protoalgebraic. Then, by definition, the Leibniz operator $\Omega : \mathrm{ThFam}(\mathcal{I}^f) \to \mathrm{ConSys}^*(\mathcal{I}^f)$ is monotone. Since $C^f \leq C$, we have $\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$. Therefore, the Leibniz operator $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is also monotone. We conclude that $\mathcal{I}$ is protoalgebraic. ■

Similarly, if $\mathcal{I}^f$ is family reflective, then so is $\mathcal{I}$.

Lemma 657 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a π-institution based on* $\mathbf{F}$. *If* $\mathcal{I}^f = \langle \mathbf{F}, C^f\rangle$ *is family reflective, then so is* $\mathcal{I}$.

Proof: Suppose $\mathcal{I}^f$ is family reflective. Then, by definition, the Leibniz operator $\Omega : \mathrm{ThFam}(\mathcal{I}^f) \to \mathrm{ConSys}^*(\mathcal{I}^f)$ is order reflecting. Since $C^f \leq C$, we have $\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$. Therefore, a fortiori, the Leibniz operator

$\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is also order reflecting. We conclude that $\mathcal{I}$ is family reflective. ■

Combining Lemmas 656 and 657, we get that weak family algebraizability for a π-institution is obtained, provided that its finitary companion has the same property.

Proposition 658 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}^f = \langle \mathbf{F}, C^f \rangle$ *is weakly family algebraizable, then so is* $\mathcal{I}$.

Proof: Suppose $\mathcal{I}^f$ is weakly family algebraizable. Then it is protoalgebraic and family reflective. Thus, by Lemmas 656 and 657, respectively, $\mathcal{I}$ is also protoalgebraic and family reflective. Hence, by definition, $\mathcal{I}$ is weakly family algebraizable. ■

Now we turn to the question of the same properties passing down to $\mathcal{I}^f$ from $\mathcal{I}$. In this direction, additional conditions are needed to ensure inheritance.

Given a directed family $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$ it is not, in general, the case that $\bigcup_{i\in I} T^i$ is a theory family of $\mathcal{I}$. However, as we saw in Proposition 112, this is always the case when $\mathcal{I}$ is a finitary π-institution.

Motivated by this consideration, we define the following property of the Leibniz operator:

Definition 659 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *The Leibniz operator*

$$\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$$

is **continuous** *if, for every directed family* $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$, *such that* $\bigcup_{i\in I} T^i \in \mathrm{ThFam}(\mathcal{I})$,

$$\Omega(\bigcup_{i\in I} T^i) = \bigcup_{i\in I} \Omega(T^i).$$

It is easy to see that continuity implies protoalgebraicity.

Lemma 660 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is continuous, then* $\mathcal{I}$ *is protoalgebraic.*

Proof: Suppose Ω is continuous and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$. Then $T' = T \cup T'$ and we get

$$\Omega(T) \cup \Omega(T') = \Omega(T \cup T') = \Omega(T').$$

Hence, $\Omega(T) \leq \Omega(T')$ and $\mathcal{I}$ is protoalgebraic. ■

We saw in Lemma 656 that protoalgebraicity of the finitary companion $\mathcal{I}^f$ of a π-institution $\mathcal{I}$ ensures that $\mathcal{I}$ is also protoalgebraic. We now see that working over finite signature categories and imposing the stronger property of continuity of the Leibniz operator on $\mathcal{I}$ ensure that $\mathcal{I}^f$ is protoalgebraic.

Lemma 661 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is continuous, then* $\mathcal{I}^f$ *is protoalgebraic.*

Proof: Suppose that Ω is continuous on $\mathrm{ThFam}(\mathcal{I})$. Then, by Lemma 660, $\mathcal{I}$ is protoalgebraic. To show that $\mathcal{I}^f$ is protoalgebraic, let $T, T' \in \mathrm{ThFam}(\mathcal{I}^f)$, such that $T \le T'$. By Proposition 655, there exist directed locally finitely generated collections $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$ and $\{T'^j : j \in J\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that

$$T = \bigcup_{i \in I} T^i \quad \text{and} \quad T' = \bigcup_{j \in J} T'^j.$$

Since, by hypothesis, $T \le T'$, we get, for all $i \in I$, $T^i \le \bigcup_{j \in J} T'^j$. Since $\mathbf{Sign}^\flat$ is finite, T^i is locally finitely generated and $\{T'^j : j \in J\}$ is directed, there exists $j_i \in J$, such that $T^i \le T'^{j_i}$, for all $i \in I$. Now we get

$$\begin{array}{rcll}
\Omega(T) & = & \Omega(\bigcup_{i \in I} T^i) & (T = \bigcup_{i \in I} T^i) \\
& = & \bigcup_{i \in I} \Omega(T^i) & (\Omega \text{ continuous}) \\
& \le & \bigcup_{i \in I} \Omega(T'^{j_i}) & (T^i \le T'^{j_i} \text{ and protoalgebraicity}) \\
& \le & \bigcup_{j \in J} \Omega(T'^j) & (\text{Set Theory}) \\
& = & \Omega(\bigcup_{j \in J} T'^j) & (\Omega \text{ continuous}) \\
& = & \Omega(T'). & (T' = \bigcup_{j \in J} T'^j)
\end{array}$$

Thus, $\mathcal{I}^f$ is protoalgebraic. ■

Lemma 661 allows us to prove the following result, giving sufficient conditions for weak family algebraizability to be inherited by the finitary companion $\mathcal{I}^f$ from a π-institution $\mathcal{I}$.

Definition 662 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a weakly family algebraizable* π*-institution based on* $\mathbf{F}$*. The inverse* $\Omega^{-1} : \mathrm{ConSys}^*(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$ *of the Leibniz operator is* **continuous** *if, for every directed family* $\{\theta^i : i \in I\} \subseteq \mathrm{ConSys}^*(\mathcal{I})$*, such that* $\bigcup_{i \in I} \theta^i \in \mathrm{ConSys}^*(\mathcal{I})$*,*

$$\Omega^{-1}(\bigcup_{i \in I} \theta^i) = \bigcup_{i \in I} \Omega^{-1}(\theta^i).$$

Theorem 663 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a weakly family algebraizable* π*-institution based on* $\mathbf{F}$*. If*

$$\mathrm{ThFam}(\mathcal{I}) \underset{\Omega^{-1}}{\overset{\Omega}{\rightleftarrows}} \mathrm{ConSys}^*(\mathcal{I})$$

are continuous, then $\mathcal{I}^f$ *is also weakly family algebraizable.*

Proof: By Lemma 661, $\mathcal{I}^f$ is protoalgebraic. Thus, it suffices to show that $\mathcal{I}^f$ is also family injective. To this end, let $T, T' \in \mathrm{ThFam}(\mathcal{I}^f)$, such that $\Omega(T) = \Omega(T')$. By Proposition 655, there exist directed locally finitely generated collections $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$ and $\{T'^j : j \in J\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that

$$T = \bigcup_{i \in I} T^i \quad \text{and} \quad T' = \bigcup_{j \in J} T'^j.$$

Now we work as follows:

$$\begin{array}{rcl}
T & = & \bigcup_{i\in I} T^i \\
 & = & \bigcup_{i\in I} \Omega^{-1}(\Omega(T^i)) \\
 & = & \Omega^{-1}(\bigcup_{i\in I} \Omega(T^i)) \\
 & & (\bigcup_{i\in I} \Omega(T^i) \in \mathrm{ConSys}^*(\mathcal{I}) \text{ and } \Omega^{-1} \text{ continuous}) \\
 & = & \Omega^{-1}(\Omega(\bigcup_{i\in I} T^i)) \\
 & & (\bigcup_{i\in I} T^i \in \mathrm{ThFam}(\mathcal{I}) \text{ and } \Omega \text{ continuous}) \\
 & = & \Omega^{-1}(\Omega(\bigcup_{j\in J} T'^j)) \quad (\Omega(T) = \Omega(T')) \\
 & = & \Omega^{-1}(\bigcup_{j\in J} \Omega(T'^j)) \\
 & & (\bigcup_{j\in J} T'^j \in \mathrm{ThFam}(\mathcal{I}) \text{ and } \Omega \text{ continuous}) \\
 & = & \bigcup_{j\in J} \Omega^{-1}(\Omega(T'^j)) \\
 & & (\bigcup_{j\in J} \Omega(T'^j) \in \mathrm{ConSys}^*(\mathcal{I}) \text{ and } \Omega^{-1} \text{ continuous}) \\
 & = & \bigcup_{j\in J} T'^j \\
 & = & T'.
\end{array}$$

Hence, $\mathcal{I}^f$ is family injective and, thus, weakly family algebraizable. ∎

9.4 Finitarity and Continuity

In this section, we establish some results pertaining to the finitarity of weakly family algebraizable π-institutions. We stay with semantic notions, using the Leibniz operator, and aim at establishing relations between various aspects of finitarity.

We begin by showing that the finitarity of a weakly family algebraizable π-institution $\mathcal{I}$ entails the continuity of the inverse Leibniz operator on the $\mathcal{I}^*$-congruence systems on $\mathcal{F}$.

Proposition 664 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a weakly family algebraizable* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is finitary, then* $\Omega^{-1} : \mathrm{ConSys}^*(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$ *is continuous.*

Proof: Suppose that $\{\theta^i : i \in I\} \subseteq \mathrm{ConSys}^*(\mathcal{I})$ is directed, such that $\bigcup_{i \in I} \theta^i \in \mathrm{ConSys}^*(\mathcal{I})$. Since $\{\theta^i : i \in I\} \subseteq \mathrm{ConSys}^*(\mathcal{I})$, there exist $T^i \in \mathrm{ThFam}(\mathcal{I})$, such that $\theta^i = \Omega(T^i)$, for all $i \in I$. Note that, since, by Theorem 296, Ω is an order isomorphism, $\{T^i : i \in I\}$ is also directed. Moreover, since $\mathcal{I}$ is finitary, we have, by Proposition 112, $\bigvee_{i\in I} T^i = \bigcup_{i\in I} T^i$. Hence, we get

$$\begin{array}{rcll} \bigcup_{i\in I}\theta^i & = & \bigvee_{i\in I}\theta^i & (\bigcup_{i\in I}\theta^i \in \mathrm{ConSys}^*(\mathcal{I})) \\ & = & \bigvee_{i\in I}\Omega(T^i) & (\theta^i = \Omega(T^i)) \\ & = & \Omega(\bigvee_{i\in I} T^i) & (\Omega \text{ order isomorphism}) \\ & = & \Omega(\bigcup_{i\in I} T^i). & (\mathcal{I} \text{ finitary}) \end{array}$$

From this, we get

$$\Omega^{-1}(\bigcup_{i\in I}\theta^i) = \bigcup_{i\in I} T^i = \bigcup_{i\in I}\Omega^{-1}(\theta^i).$$

Hence Ω^{-1} is indeed continuous. ■

Next, we show that the finitarity of the π-structure $\mathcal{Q}^{\mathsf{K}} = \langle \mathbf{F}, D^{\mathsf{K}}\rangle$, where $\mathsf{K} = \mathrm{AlgSys}(\mathcal{I})$, for a weakly family algebraizable π-institution $\mathcal{I}$, entails the continuity of the Leibniz operator on the collection of theory families of $\mathcal{I}$.

Proposition 665 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a weakly family algebraizable* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{Q}^{\mathsf{K}}$ *is finitary, for* $\mathsf{K} = \mathrm{AlgSys}(\mathcal{I})$*, then* $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is continuous.*

Proof: Suppose that $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$ is directed, such that $\bigcup_{i\in I} T^i \in \mathrm{ThFam}(\mathcal{I})$. Since $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\Omega(T^i) \in \mathrm{ThFam}(\mathcal{Q}^{\mathsf{K}})$, where $\mathsf{K} = \mathrm{AlgSys}(\mathcal{I})$. Moreover, since, by Theorem 296, Ω is an order isomorphism, $\{\Omega(T^i) : i \in I\}$ is also directed. Hence, since $\mathcal{Q}^{\mathsf{K}}$ is finitary, by Proposition 112, we have $\bigvee_{i\in I}\Omega(T^i) = \bigcup_{i\in I}\Omega(T^i)$. Hence, we get

$$\begin{array}{rcll} \Omega(\bigcup_{i\in I} T^i) & = & \Omega(\bigvee_{i\in I} T^i) & (\bigcup_{i\in I} T^i \in \mathrm{ThFam}(\mathcal{I})) \\ & = & \bigvee_{i\in I}\Omega(T^i) & (\Omega \text{ order isomorphism}) \\ & = & \bigcup_{i\in I}\Omega(T^i). & (\mathcal{Q}^{\mathsf{K}} \text{ finitary}) \end{array}$$

Hence, Ω is continuous. ■

We saw in Proposition 664 that finitarity of a weakly family algebraizable π-institution $\mathcal{I}$ entails the continuity of the inverse Leibniz operator. If, to the finitarity of $\mathcal{I}$, we add continuity of the Leibniz operator Ω, then finitarity of $\mathcal{Q}^{\mathsf{K}}$ is ensured, where $\mathsf{K} = \mathrm{AlgSys}(\mathcal{I})$.

Proposition 666 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a weakly family algebraizable* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is finitary, and* $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is continuous, then* $\mathcal{Q}^{\mathsf{K}}$*, where* $\mathsf{K} = \mathrm{AlgSys}(\mathcal{I})$*, is also finitary.*

Proof: Assume that $\mathcal{I}$ is a finitary, weakly family algebraizable π-institution and that Ω is continuous. Let $\{\theta^i : i \in I\} \subseteq \mathrm{ConSys}^*(\mathcal{I})$ be a directed family of congruence systems. Then, there exist $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that

$\Omega(T^i) = \theta^i$, for all $i \in I$. Moreover, since $\{\theta^i : i \in I\}$ is directed and Ω is, by Theorem 296, an order isomorphism, $\{T^i : i \in I\}$ is also directed. Hence, since $\mathcal{I}$ is finitary, by Proposition 112, $\bigvee_{i\in I} T^i = \bigcup_{i\in I} T^i$. Now we have

$$\begin{array}{rcll} \bigvee_{i\in I}\theta^i & = & \bigvee_{i\in I}\Omega(T^i) & (\Omega(T^i) = \theta^i)\\ & = & \Omega(\bigvee_{i\in I} T^i) & (\Omega \text{ order isomorphism})\\ & = & \Omega(\bigcup_{i\in I} T^i) & (\mathcal{I} \text{ finitary})\\ & = & \bigcup_{i\in I}\Omega(T^i) & (\Omega \text{ continuous})\\ & = & \bigcup_{i\in I}\theta^i. & (\Omega(T^i) = \theta^i) \end{array}$$

Thus, $\mathrm{ConSys}^*(\mathcal{I})$ is closed under directed unions and, therefore, by Proposition 112, $\mathcal{Q}^{\mathsf{K}}$ is finitary. ∎

Dually, we have seen in Proposition 665 that if $\mathcal{Q}^{\mathsf{K}}$ is finitary, where $\mathsf{K} = \mathrm{AlgSys}(\mathcal{I})$, then Ω is continuous. If, to the finitarity of $\mathcal{Q}^{\mathsf{K}}$, we add the continuity of Ω^{-1}, then, finitarity of $\mathcal{I}$ is ensured.

Proposition 667 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a weakly family algebraizable* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{Q}^{\mathsf{K}}$ *is finitary, where* $\mathsf{K} = \mathrm{AlgSys}^*(\mathcal{I})$*, and* $\Omega^{-1} : \mathrm{ConSys}^*(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$ *is continuous, then* $\mathcal{I}$ *is also finitary.*

Proof: Assume that $\mathcal{I}$ is a weakly family algebraizable π-institution, such that $\mathcal{Q}^{\mathsf{K}}$, $\mathsf{K} = \mathrm{AlgSys}(\mathcal{I})$, is finitary, and that Ω^{-1} is continuous. Let $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$ be a directed collection of theory families. Then, since, by Theorem 296, Ω is an order isomorphism, $\{\Omega(T^i) : i \in I\}$ is a directed family of congruence systems. Since $\mathcal{Q}^{\mathsf{K}}$ is finitary, by Proposition 112, $\bigvee_{i\in I}\Omega(T^i) = \bigcup_{i\in I}\Omega(T^i)$. Now we have

$$\begin{array}{rcll} \bigvee_{i\in I} T^i & = & \Omega^{-1}(\Omega(\bigvee_{i\in I} T^i)) & (\Omega \text{ isomorphism})\\ & = & \Omega^{-1}(\bigvee_{i\in I}\Omega(T^i)) & (\Omega \text{ order isomorphism})\\ & = & \Omega^{-1}(\bigcup_{i\in I}\Omega(T^i)) & (\mathcal{Q}^{\mathsf{K}} \text{ finitary})\\ & = & \bigcup_{i\in I}\Omega^{-1}(\Omega(T^i)) & (\Omega^{-1} \text{ continuous})\\ & = & \bigcup_{i\in I} T^i. & (\Omega \text{ isomorphism}) \end{array}$$

Thus, $\mathrm{ThFam}(\mathcal{I})$ is closed under directed unions and, hence, by Proposition 112, $\mathcal{I}$ is finitary. ∎

Gathering together all conclusions drawn during the studies undertaken in this section, we get the following

Corollary 668 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a weakly family algebraizable* π*-institution based on* $\mathbf{F}$ *and set* $\mathsf{K} = \mathrm{AlgSys}(\mathcal{I})$*.*

(a) If both Ω *and* Ω^{-1} *are continuous, then* $\mathcal{I}$ *is finitary if and only if* $\mathcal{Q}^{\mathsf{K}}$ *is finitary.*

(b) If $\mathcal{I}$ is finitary, then $\mathcal{Q}^{\mathsf{K}}$ is finitary if and only if $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is continuous.*

(c) If $\mathcal{Q}^{\mathsf{K}}$ is finitary, then $\mathcal{I}$ is finitary if and only if $\Omega^{-1} : \mathrm{ConSys}^*(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$ *is continuous.*

In each of (a)-(c), if the two equivalent conditions hold, then all four finitarity conditions hold.

Proof:

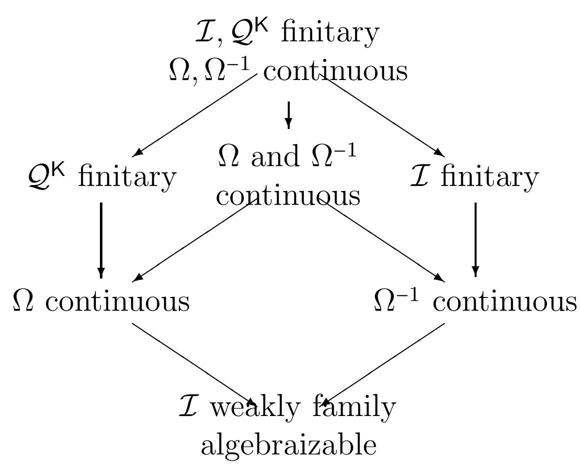
(a) Suppose Ω and Ω^{-1} are continuous. Then, if $\mathcal{I}$ is finitary, $\mathcal{Q}^{\mathsf{K}}$ is finitary, by Proposition 666, and if $\mathcal{Q}^{\mathsf{K}}$ is finitary, then $\mathcal{I}$ is finitary, by Proposition 667.

(b) Suppose that $\mathcal{I}$ is finitary. Then, by Proposition 664, Ω^{-1} is continuous. If $\mathcal{Q}^{\mathsf{K}}$ is finitary, then Ω is continuous, by Proposition 665. On the other hand, if Ω is continuous, then $\mathcal{Q}^{\mathsf{K}}$ is finitary, by Proposition 666.

(c) Suppose that $\mathcal{Q}^{\mathsf{K}}$ is finitary. Then, by Proposition 665, Ω is continuous. If $\mathcal{I}$ is finitary, then Ω^{-1} is continuous, by Proposition 664. On the other hand, if Ω^{-1} is continuous, then $\mathcal{I}$ is finitary, by Proposition 667.

We turn to the last statement. For Part (a), assume that $\mathcal{I}$ and $\mathcal{Q}^{\mathsf{K}}$ are finitary. Then, we get, by Propositions 664 and 665, that Ω and Ω^{-1} are continuous. For Part (b), if $\mathcal{I}$ is finitary, $\mathcal{Q}^{\mathsf{K}}$ is finitary and $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is continuous, then, by Proposition 664, Ω^{-1} is continuous. A similar reasoning applies to Part (c). ■

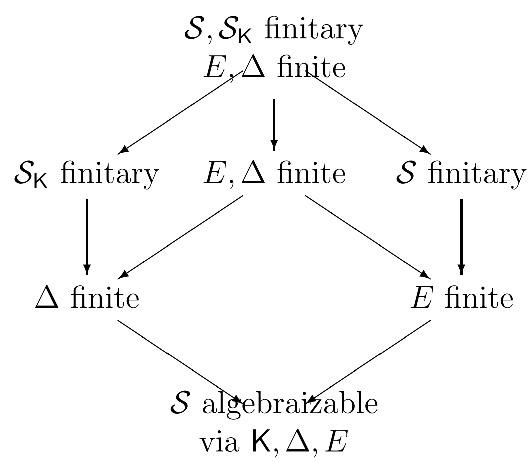
We summarize our conclusions in the accompanying diagram. At the bottom is situated the underlying assumption (holding at all levels) that $\mathcal{I}$ is weakly family algebraizable. The top consists of the situation in which all four finitarity conditions hold, i.e., both $\mathcal{I}$ and $\mathcal{Q}^{\mathsf{K}}$, for $\mathsf{K} = \mathrm{AlgSys}(\mathcal{I})$, are finitary and both Ω and Ω^{-1} are continuous. The intermediate classes constitute the various different intermediate possibilities that were detailed previously.



This diagram is a modified version of the one shown in Figure 3 of Section 3.4 of [86]. Instead of dealing with sentential logics, it concerns the more general case of π-institutions and, instead of being expressed in terms of syntactic constructs (which in the case of sentential logics turn out to be equivalent), it relies entirely on corresponding properties of the Leibniz operator and its inverse. These analogies and similarities will be exploited in the remainder of the chapter to obtain examples that separate the various classes involved in this hierarchy by adapting appropriate examples that serve an analogous purpose in the framework of sentential logics.

9.5 The Case of Sentential Logics

In the Introduction and, briefly, in concluding Section 9.4, we pointed out that the semantic finitarity hierarchy of π-institutions reflects the finitarity hierarchy presented in Section 3.4 of [86], which is duplicated below.



In [86], Font presents examples of sentential logics to separate the classes in this hierarchy. In this section, we revisit some of them in detail and, then, rely on them in Section 9.6 to separate the classes of π-institutions shown in the hierarchy of Section 9.4. We provide, now, a brief overview of the examples chosen and what each accomplishes, before describing them in full detail.

The first example we present is Łukasiewicz's infinite valued logic $Ł_\infty$. This logic is introduced in Example 1.12 of Section 1.2 of [86]. In Example 1.15, in the same section, in conjunction with Exercise 1.26 of [86], it is shown that it is not finitary. On the other hand, in Example 3.41 of Section 3.4 of [86], it is shown that it is finitely algebraizable, with defining equations $E(x) = \{x \approx \top\}$ and equivalence formulas $\Delta(x,y) = \{x \to y, y \to x\}$, both finite. Thus, $Ł_\infty$ serves in showing that the three vertical arrows in the preceding diagram represent proper inclusions. As Font points out in [86], more information about $Ł_\infty$ and similar logics may be found in specialized references, such as [61, 65, 56] and Chapters 1, 2 and 6 of [83].

The second example that Font presents in Section 3.4 of [86] is Hermann's Last Judgement Logic $\mathcal{L}J$ [53]. This is a finitary logic which is syntactically defined and which is algebraizable with a single defining equation $E(x) = \{\neg x \approx \neg(x \to x)\}$ and an infinite set of equivalence formulas $\Delta(x,y) = \{\Box^n(x \to y) \approx \Box^n(y \to x) : n \geq 0\}$. So this logic serves to separate all classes related by southeast arrows in the diagram. The same separations may be attained by a logic introduced by Dellunde in [48]. Dellunde's logic is actually the logic we opt to present as our second example. Here, we shall name it Dellunde's Logic $\mathcal{D}$.

The last example, presented in Section 3.4 of [86] is a logic introduced by Raftery in [82]. This is a semantically defined logic which is not finitary but is algebraizable via a finitary equational consequence, with an infinite set of defining equations and a single equivalence formula. So this logic, which we shall refer to as Raftery's Logic $\mathcal{R}$, shows that all southwest arrows in the diagram represent proper inclusions. This will be the third, and last example, presented in detail in this section.

9.5.1 Łukasiewicz's Infinite Valued Logic

We begin with Łukasiewicz's infinite valued logic $Ł_\infty$. Define an algebra $\mathbf{A} = \langle A, \wedge, \vee, \to, \neg \rangle$ as follows:

- The universe A is the unit interval $A = [0,1]$.
- The operations are defined, for all $a, b \in A$, by:
  - $a \wedge b = \min\{a, b\}$;
  - $a \vee b = \max\{a, b\}$;
  - $a \to b = \min\{1, 1 - a + b\} = \begin{cases} 1, & \text{if } a \leq b \\ 1 - a + b, & \text{if } a > b \end{cases}$;
  - $\neg a = 1 - a$.

Łukasiewicz's infinite valued logic $Ł_\infty = \langle \mathcal{L}, \vdash_\infty \rangle$ is the sentential logic over the language $\mathcal{L} = \{\wedge, \vee, \to, \neg\}$ defined, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, by

$$\Gamma \vdash_\infty \varphi \quad \text{iff} \quad \begin{array}{l} \text{for every homomorphism } h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \\ h(\Gamma) \subseteq \{1\} \quad \text{implies} \quad h(\varphi) = 1. \end{array}$$

Over the same language $\mathcal{L}$, we define the derived binary connective $\oplus$ by setting

$$x \oplus y := \neg x \to y.$$

This operation satisfies some key properties.

Lemma 669 *For all $a, b \in A$,*

$$a \oplus b = \begin{cases} 1, & \text{if } a + b \geq 1, \\ a + b, & \text{if } a + b < 1 \end{cases}.$$

Proof: Let $a, b \in A$. We perform a straightforward calculation using the definitions of the operations in $\mathbf{A}$.

$$\begin{aligned} a \oplus b &= \neg a \to b = (1 - a) \to b \\ &= \left\{ \begin{array}{ll} 1, & \text{if } 1 - a \leq b \\ 1 - (1 - a) + b, & \text{if } 1 - a > b \end{array} \right\} = \left\{ \begin{array}{ll} 1, & \text{if } a + b \geq 1 \\ a + b, & \text{if } a + b < 1 \end{array} \right. . \end{aligned}$$

■

Lemma 669 helps us establish the following

Lemma 670 *For all $a \in A$ and all $n \geq 2$,*

$$\underbrace{a \oplus \cdots \oplus a}_{n} = \begin{cases} 1, & \text{if } a \geq \frac{1}{n} \\ na, & \text{if } a < \frac{1}{n} \end{cases}.$$

Proof: We use induction on n. Lemma 669 guarantees that the formula holds for $n = 2$. Assume that the formula holds for some $n \geq 2$. Then, we get

$$\begin{aligned} \underbrace{a \oplus \cdots \oplus a}_{n+1} &= \underbrace{(a \oplus \cdots \oplus a)}_{n} \oplus a = \left\{ \begin{array}{ll} 1 \oplus a, & \text{if } a \geq \frac{1}{n} \\ (na) \oplus a, & \text{if } a < \frac{1}{n} \end{array} \right. \\ &= \left\{ \begin{array}{ll} 1, & \text{if } a \geq \frac{1}{n} \\ 1, & \text{if } (n+1)a \geq 1 \\ (n+1)a, & \text{if } (n+1)a < 1 \end{array} \right\} = \left\{ \begin{array}{ll} 1, & \text{if } a \geq \frac{1}{n+1} \\ (n+1)a, & \text{if } a < \frac{1}{n+1} \end{array} \right. . \end{aligned}$$

■

We now show that $Ł_\infty$ is not finitary.

Theorem 671 *Łukasiewicz's infinite valued logic $Ł_\infty$ is not finitary.*

Proof: We set

$$\Phi = \{\underbrace{(x \oplus \cdots \oplus x)}_{n} \to y : n \geq 2\} \cup \{\neg x \to y\}.$$

We show that $\Phi \vdash_\infty y$, but $\Phi_0 \nvdash_\infty y$, for any finite $\Phi_0 \subseteq \Phi$.

Suppose $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$ is such that

$$h(\neg y \to x) = 1 \quad \text{and} \quad h(\underbrace{(x \oplus \cdots \oplus x)}_{n} \to y) = 1, \text{ for all } n \geq 2.$$

By definition, these imply that $h(x) + h(y) \geq 1$ and $h(\underbrace{x \oplus \cdots \oplus x}_{n}) \leq h(y)$, for all $n \geq 2$.

- If $h(x) = 0$, then, by the first inequality, $h(y) = 1$.
- If $h(x) \neq 0$, then $h(x) \geq \frac{1}{n}$, for some $n > 0$. In this case, $h(\underbrace{x \oplus \cdots \oplus x}_{n}) = 1$, whence, by the second inequality, $h(y) = 1$.

Since, in either case, $h(y) = 1$, we get that $\Phi \vdash_\infty y$.

To refute finitarity, assume, towards obtaining a contradiction, that, for some finite $\Phi_0 \subseteq \Phi$, $\Phi_0 \vdash_\infty y$. Then, there exists $k \geq 2$, such that

$$\{(\underbrace{x \oplus \cdots \oplus x}_{n}) \to y : 2 \leq n \leq k\} \cup \{\neg x \to y\} \vdash_\infty y.$$

Consider a homomorphism $h : \mathbf{Fm}_\mathcal{L}(V) \to \mathbf{A}$, such that

$$h(x) = \frac{1}{k+1} \quad \text{and} \quad h(y) = \frac{k}{k+1}.$$

Then, we have

$$\begin{aligned} &h(\neg x \to y) = (1 - h(x)) \to y = \tfrac{k}{k+1} \to \tfrac{k}{k+1} = 1; \\ &h((\underbrace{x \oplus \cdots \oplus x}_{n}) \to y) = h((\underbrace{x \oplus \cdots \oplus x}_{n})) \to h(y) \overset{n<k+1}{=} \tfrac{n}{k+1} \to \tfrac{k}{k+1} \overset{n \leq k}{=} 1. \end{aligned}$$

On the other hand, $h(y) = \frac{k}{k+1} \neq 1$. Therefore, $\Phi_0 \nvdash_\infty y$, contrary to hypothesis. We conclude that Ł$_\infty$ is not a finitary sentential logic. ■

Our final result pertaining to this logic is that it is algebraizable, via the class $\{\mathbf{A}\}$, with defining equation $E(x) = \{x \approx \top\}$, where $\top := x \to x$ (interpreted as 1), and equivalence formulas $\Delta(x, y) = \{x \to y, y \to x\}$. In the proof, we will rely on the general theory of algebraizable logics (see, e.g., Sections 3.2 and 6.5 of [86] or Section 4.5 of [64]).

Theorem 672 *Lukasiewicz's infinite value logic Ł$_\infty$ is algebraizable via the class $\{\mathbf{A}\}$, with defining equations $E(x) = \{x \approx \top\}$ and equivalence formulas $\Delta(x, y) = \{x \to y, y \to x\}$.*

Proof: According to the general theory of algebraizability, it suffices to show that, for all $\Gamma \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}_\mathcal{L}(V)$,

$$\begin{gathered} \Gamma \vdash_\infty \varphi \quad \text{iff} \quad \{\gamma \approx \top : \gamma \in \Gamma\} \vDash_\mathcal{A} \varphi \approx \top, \\ \varphi \approx \psi \dashv\vDash_\mathbf{A} \{\varphi \to \psi \approx \top, \psi \to \varphi \approx \top\}. \end{gathered}$$

For the first, note that

$$\begin{aligned} \Gamma \vdash_\infty \varphi \quad &\text{iff} \quad (\forall h : \mathbf{Fm}_\mathcal{L}(V) \to \mathbf{A})(h(\Gamma) \subseteq \{1\} \text{ implies } h(\varphi) = 1) \\ &\text{iff} \quad (\forall h : \mathbf{Fm}_\mathcal{L}(V) \to \mathbf{A})((\forall \gamma \in \Gamma)(h(\gamma) = 1) \text{ implies } h(\varphi) = 1) \\ &\text{iff} \quad \{\gamma \approx \top : \gamma \in \Gamma\} \vDash_\mathbf{A} \varphi \approx \top. \end{aligned}$$

Finally, noting that, for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$, we have $h(\varphi \to \psi) = 1$ iff $h(\varphi) \leq h(\psi)$, we get

$$h(\varphi) = h(\psi) \quad \text{iff} \quad h(\varphi \to \psi) = h(\psi \to \varphi) = 1,$$

whence $\varphi \approx \psi \dashv\vDash_{\mathbf{A}} \{\varphi \to \psi \approx \top, \psi \to \varphi \approx \top\}$. The conclusion now follows. ■

9.5.2 Dellunde's Logic

We switch to the second example of the section, a logic due to Dellunde [48].

Let $\mathcal{L} = \{\leftrightarrow, \Box\}$ be the algebraic language consisting of a binary operation $\leftrightarrow$ and a unary operation $\Box$. **Dellunde's logic** $\mathcal{D} = \langle \mathcal{L}, \vdash_D \rangle$ is the logic over the language $\mathcal{L}$ defined by the following Hilbert style calculus, where x, y and x_1, y_1, x_2, y_2 denote distinct variables:

(1) $\vdash_D x \leftrightarrow x$;

(2) $x, x \leftrightarrow y \vdash_D y$;

(3) $x, y \vdash_D \Box^n x \leftrightarrow \Box^n y$, for all $n \in \omega$;

(4) $x_1 \leftrightarrow y_1, x_2 \leftrightarrow y_2 \vdash_D \Box^n(x_1 \leftrightarrow x_2) \leftrightarrow \Box^n(y_1 \leftrightarrow y_2)$, for all $n \in \omega$.

Since $\mathcal{D}$ is defined via a Hilbert calculus, it is finitary. We further define

$$\Delta(x, y) = \{\Box^n x \leftrightarrow \Box^n y : n \in \omega\}.$$

Dellunde shows that $\mathcal{D}$ is 1-equivalential, which implies that it is regularly algebraizable [53].

Theorem 673 *Dellunde's logic $\mathcal{D} = \langle \mathcal{L}, \vdash_D \rangle$ is regularly algebraizable.*

Proof: It suffices to show that, for distinct variables x, y, x_1, y_1, x_2, y_2, the following hold:

(R) $\vdash_D \Delta(x, x)$;

(MP) $x, \Delta(x, y) \vdash_D y$;

(RP) $\Delta(x, y) \vdash_D \Delta(\Box x, \Box y)$ and

$\Delta(x_1, y_1), \Delta(x_2, y_2) \vdash_D \Delta(x_1 \leftrightarrow x_2, y_1 \leftrightarrow y_2)$;

(RG) $x, y \vdash_D \Delta(x, y)$.

By (1), we have $\vdash_D x \leftrightarrow x$. By structurality, $\vdash_D \Box^n x \leftrightarrow \Box^n x$, for all $n \in \omega$. That is, $\vdash_D \Delta(x,x)$. So (R) holds.

Rule (2) assures that $x, x \leftrightarrow y \vdash_D y$. Now, note that $x \leftrightarrow y \in \Delta(x,y)$ and apply monotonicity of entailment to get $x, \Delta(x,y) \vdash_D y$. That is, (MP) holds.

The first rule in (RP) is a consequence of monotonicity, since

$$\begin{array}{rcl}\Delta(\Box x, \Box y) & = & \{\Box^n \Box x \leftrightarrow \Box^n \Box y : n \in \omega\} \\ & = & \{\Box^n x \leftrightarrow \Box^n y : n \geq 1\} \\ & \subseteq & \{\Box^n x \leftrightarrow \Box^n y : n \in \omega\} \\ & = & \Delta(x,y).\end{array}$$

For the second rule in (RP), note that (4) gives $x_1 \leftrightarrow y_1, x_2 \leftrightarrow y_2 \vdash_D \Delta(x_1 \leftrightarrow x_2, y_1 \leftrightarrow y_2)$. On the other hand, $x_1 \leftrightarrow y_1 \in \Delta(x_1,y_1)$ and $x_2 \leftrightarrow y_2 \in \Delta(x_2,y_2)$. Therefore, we conclude that $\Delta(x_1,y_1), \Delta(x_2,y_2) \vdash_D \Delta(x_1 \leftrightarrow x_2, y_1 \leftrightarrow y_2)$, whence (RP) holds.

Finally, note that, by (3), (RG) holds.

We conclude that $\mathcal{D}$ is regularly algebraizable, with a singleton set of defining equations $E(x) = \{x \approx \top\}$, where $\top := x \leftrightarrow x$ is a unary term interpreted as the unique element of the $\mathcal{D}$-filter of any reduced $\mathcal{D}$-matrix, and set of equivalence formulas $\Delta(x,y)$. ■

Finally, Dellunde shows that $\mathcal{D}$ is not finitely equivalential, i.e., that there does not exist a finite subset $\Delta_0 \subseteq \Delta$ that can also serve as a set of equivalence formulas. In relation to this, see Lemma 3.36 in Section 3.4 of [86].

Theorem 674 *Dellunde's logic $\mathcal{D} = \langle \mathcal{L}, \vdash_D \rangle$ is not finitely equivalential, i.e., there exists no finite $\Delta_0 \subseteq \Delta$ which is also a set of equivalence formulas for $\mathcal{D}$.*

Proof: Assume, towards a contradiction, that there exists finite $\Delta_0 \subseteq \Delta$, which serves as a set of equivalence formulas for $\mathcal{D}$. Then, there exists a maximum $m \in \omega$, such that $\Box^m x \leftrightarrow \Box^m y \in \Delta_0$. To obtain a contradiction, we construct a $\mathcal{D}$-matrix $\mathfrak{A} = \langle \mathbf{A}, F \rangle$ and choose elements $c, d \in A$, such that

$$\Delta_0^{\mathbf{A}}(c,d) \subseteq F \quad \text{but} \quad \langle c,d \rangle \notin \Omega_{\mathbf{A}}(F).$$

As a preparatory step in defining the $\mathcal{L}$-algebra $\mathbf{A}$, we define on $\omega \times \omega$ the following equivalence relation:

$$R = \mathrm{Id}_{\omega\times\omega} \cup \{\langle \langle i,j \rangle, \langle k,\ell \rangle \rangle : i = k, i < j, k < \ell\}.$$

The algebra $\mathbf{A} = \langle A, \leftrightarrow^{\mathbf{A}}, \Box^{\mathbf{A}} \rangle$ is defined as follows:

- $A = \omega \times \omega$;
- The operations are defined, for all $i, j \in \omega$ and all $a, b \in \omega \times \omega$,

- $\Box^{\mathbf{A}}(\langle i,j\rangle) = \langle i+1,j\rangle$;
- $\leftrightarrow^{\mathbf{A}} (a,b) = \begin{cases} \langle 1,0\rangle, & \text{if } \langle a,b\rangle \in R \\ \langle 0,0\rangle, & \text{if } \langle a,b\rangle \notin R \end{cases}$.

The filter $F = \{\langle 1,0\rangle\}$ and the elements $c,d \in A$ are chosen as $c = \langle 0,m+1\rangle$ and $d = \langle 0,m+2\rangle$, where, recall that, $m = \max\{k : \Box^k x \leftrightarrow \Box^k y \in \Delta_0\}$. It suffices, now, to show the following:

(a) $\mathfrak{A} = \langle \mathbf{A}, F\rangle$ is a $\mathcal{D}$-matrix, i.e., F is closed under all $\mathcal{D}$-rules;

(b) $\Delta_0^{\mathbf{A}}(c,d) \subseteq F$;

(c) $\langle c,d\rangle \notin \Omega_{\mathbf{A}}(F)$.

For (a), let $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$ be arbitrary. Then:

- For all $n \in \omega$, $h(\Box^n x \leftrightarrow \Box^n x) = h(\Box^n x) \leftrightarrow^{\mathbf{A}} h(\Box^n x) = \langle 1,0\rangle \in F$.
- Suppose $h(x) = \langle 1,0\rangle$ and $h(x \leftrightarrow y) = \langle 1,0\rangle$. So $h(x) = \langle 1,0\rangle$ and $h(x) \leftrightarrow^{\mathbf{A}} h(y) = \langle 1,0\rangle$. Since $h(x) = \langle 1,0\rangle$ and $1 \nleq 0$, we get $h(x) = h(y)$. So, $h(y) = \langle 1,0\rangle \in F$.
- If $h(x) = h(y) = \langle 1,0\rangle$, then $h(\Box^n x) = \Box^{\mathbf{A}^n} h(x) = \Box^{\mathbf{A}^n} h(y) = h(\Box^n y)$, whence, $h(\Box^n x \leftrightarrow \Box^n y) = h(\Box^n x) \leftrightarrow^{\mathbf{A}} h(\Box^n y) = \langle 1,0\rangle \in F$.
- If $h(x_1 \leftrightarrow y_1) = h(x_2 \leftrightarrow y_2) = \langle 1,0\rangle$, then, since R is an equivalence relation, it follows from

$$\begin{array}{ccc} h(x_1) & \text{—}R\text{—} & h(y_1) \\ \vdots & & \vdots \\ R & & R \\ \vdots & & \vdots \\ h(x_2) & \text{—}R\text{—} & h(y_2) \end{array}$$

that $\langle h(x_1),h(x_2)\rangle \in R$ iff $\langle h(y_1),h(y_2)\rangle \in R$, i.e., that

$$h(x_1 \leftrightarrow x_2) = h(y_1 \leftrightarrow y_2) = \begin{cases} \langle 1,0\rangle, & \text{if } \langle h(x_1),h(x_2)\rangle \in R \\ \langle 0,0\rangle, & \text{if } \langle h(x_1),h(x_2)\rangle \notin R \end{cases}.$$

Then, we obtain, for all $n \in \omega$, $h(\Box^n(x_1 \leftrightarrow x_2)) = h(\Box^n(y_1 \leftrightarrow y_2))$, which yields that $h(\Box^n(x_1 \leftrightarrow x_2) \leftrightarrow \Box^n(y_1 \leftrightarrow y_2)) = \langle 1,0\rangle$.

Thus, $\mathfrak{A}$ is indeed a $\mathcal{D}$-matrix.

For (b), suppose $\Box^k x \leftrightarrow \Box^k y \in \Delta_0$, i.e., $k \leq m$. Then, we have

$$\begin{array}{rcl} \Box^{\mathbf{A}^k} c \leftrightarrow^{\mathbf{A}} \Box^{\mathbf{A}^k} d & = & \Box^{\mathbf{A}^k}\langle 0,m+1\rangle \leftrightarrow^{\mathbf{A}} \Box^{\mathbf{A}^k}\langle 0,m+2\rangle \\ & = & \langle k,m+1\rangle \leftrightarrow^{\mathbf{A}} \langle k,m+2\rangle \\ & = & \langle 1,0\rangle. \end{array}$$

So, $\Delta_0^{\mathbf{A}}(c,d) \subseteq F$.

Finally, for (c), observe that

$$\begin{array}{rcl}\Box^{\mathbf{A}^{m+1}}c \leftrightarrow^{\mathbf{A}} \Box^{\mathbf{A}^{m+1}}d & = & \Box^{\mathbf{A}^{m+1}}\langle 0,m+1\rangle \leftrightarrow^{\mathbf{A}} \Box^{\mathbf{A}^{m+1}}\langle 0,m+2\rangle \\ & = & \langle m+1,m+1\rangle \leftrightarrow^{\mathbf{A}} \langle m+1,m+2\rangle \\ & = & \langle 0,0\rangle \notin F.\end{array}$$

Therefore, by Theorem 673 and the general theory of algebraizability, $\langle c,d\rangle \notin \Omega_{\mathbf{A}}(F)$.

The conjunction of assertions (a), (b) and (c) shows that Δ_0 is not a set of equivalence formulas for $\mathcal{D}$ and, consequently, taking into account the finitarity of $\mathcal{D}$ and Lemma 3.36 of Section 3.4 of [86], $\mathcal{D}$ does not possess a finite set of equivalence formulas. ■

9.5.3 Raftery's Logic

Finally, we turn to a detailed description of Raftery's logic [82].

The construction unfolds in several stages. It starts with the set

$$B = \{0,1\} \cup (\{0,1\} \times \{0,1\})^\omega$$

consisting of the bits 0 and 1 and of infinite sequences of pairs of bits. On this set B, three unary operations π_1, π_2 and $\Diamond$ are defined by setting, for all $b \in \{0,1\}$ and all $\langle\langle b_0,b_0'\rangle,\langle b_1,b_1'\rangle,\ldots\rangle \in (\{0,1\}\times\{0,1\})^\omega$,

$$\begin{array}{rl}\pi_1(b) = b, & \pi_1(\langle\langle b_0,b_0'\rangle,\langle b_1,b_1'\rangle,\ldots\rangle) = b_0;\\ \pi_2(b) = b, & \pi_2(\langle\langle b_0,b_0'\rangle,\langle b_1,b_1'\rangle,\ldots\rangle) = b_0';\\ \Diamond b = b, & \Diamond(\langle\langle b_0,b_0'\rangle,\langle b_1,b_1'\rangle,\ldots\rangle) = \langle\langle b_1,b_1'\rangle,\langle b_2,b_2'\rangle,\ldots\rangle.\end{array}$$

In the next stage, Raftery constructs the universe A of the algebra $\mathbf{A}$ that forms the algebraic reduct of the logical matrix used to define Raftery's logic. This is accomplished by closing under the formation of ordered pairs.

$$\begin{array}{rcl}B^{[1]} & = & B;\\ B^{[n]} & = & (\bigcup_{0<m<n} B^{[m]}) \times (\bigcup_{0<m<n} B^{[m]}), \quad n > 1;\end{array}$$

and, finally,

$$A = \bigcup_{0<n\in\omega} B^{[n]}.$$

First, observe that no element of B is an ordered pair and that every element of $A - B$ is an ordered pair.

To define the algebra $\mathbf{A}$, the operations introduced previously on B are extended on A. We set, for all $\langle a,a'\rangle \in A - B$,

$$\pi_1(\langle a,a'\rangle) = a, \qquad \pi_2(\langle a,a'\rangle) = a', \qquad \Diamond\langle a,a'\rangle = \langle a,a'\rangle.$$

To complete the specification of $\mathbf{A}$, we add a "pair forming" binary operation $\leftrightarrow$, defined, for all $a, a' \in A$, by

$$a \leftrightarrow a' = \langle a, a' \rangle.$$

So the algebra used to specify Raftery's logic is $\mathbf{A} = \langle A, \leftrightarrow, \pi_1, \pi_2, \Diamond \rangle$. It is easy to check that $\mathbf{A}$ satisfies the equations

$$\begin{array}{rcl} \pi_1(x \leftrightarrow y) & \approx & x, \\ \pi_2(x \leftrightarrow y) & \approx & y, \\ \Diamond(x \leftrightarrow y) & \approx & x \leftrightarrow y. \end{array}$$

We now define two logical systems semantically. The first is defined via a logical matrix with underlying algebra $\mathbf{A}$. We define the matrix $\mathfrak{A} = \langle \mathbf{A}, D \rangle$, where D is the set of so-called "diagonal elements" of A, i.e., the elements

- 0 and 1;
- $\langle \langle b_0, b_0 \rangle, \langle b_1, b_1 \rangle, \ldots \rangle$, for $b_0, b_1, \ldots \in \{0, 1\}$;
- $\langle a, a \rangle$, for $a \in A$.

This matrix $\mathfrak{A}$ specifies the logic $\mathcal{S}_{\mathfrak{A}} = \langle \mathcal{L}, \vdash_{\mathfrak{A}} \rangle$ in the standard way, i.e., for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\begin{array}{lll} \Gamma \vdash_{\mathfrak{A}} \varphi & \text{iff} & \text{for every } h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \\ & & h(\Gamma) \subseteq D \quad \text{implies} \quad h(\varphi) \in D. \end{array}$$

The second logical system is defined using a variety V of $\mathcal{L}$-algebras, for $\mathcal{L} = \{\leftrightarrow, \pi_1, \pi_2, \Diamond\}$, namely the variety axiomatized by the three equations

$$\begin{array}{rcl} \pi_1(x \leftrightarrow y) & \approx & x, \\ \pi_2(x \leftrightarrow y) & \approx & y, \\ \Diamond(x \leftrightarrow y) & \approx & x \leftrightarrow y. \end{array}$$

We set $\delta_i(x) = \pi_1(\Diamond^i x)$ and $\varepsilon_i(x) = \pi_2(\Diamond^i x)$ and define **Raftery's logic** $\mathcal{R} = \langle \mathcal{L}, \vdash_R \rangle$ by setting, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Gamma \vdash_{\mathcal{R}} \varphi \quad \text{iff} \quad (\delta \approx \varepsilon)(\Gamma) \vDash_V (\delta \approx \varepsilon)(\phi),$$

i.e., $\Gamma \vdash_{\mathcal{R}} \varphi$ iff, for every $\mathbf{A} \in V$, all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$ and all $j \in \omega$,

$$\begin{array}{l} \delta_i^{\mathbf{A}}(h(\gamma)) = \varepsilon_i^{\mathbf{A}}(h(\gamma)), \text{ for all } i \in \omega, \gamma \in \Gamma, \\ \qquad\qquad\qquad\qquad \text{implies} \quad \delta_j^{\mathbf{A}}(h(\varphi)) = \varepsilon_j^{\mathbf{A}}(h(\varphi)). \end{array}$$

The first result relating the logics $\mathcal{S}_{\mathfrak{A}}$ and $\mathcal{R}$ asserts that the latter is a weakening of the former.

Lemma 675 *Raftery's logic* $\mathcal{R} = \langle \mathcal{L}, \vdash_R \rangle$ *is a weakening of* $\mathcal{S}_{\mathfrak{A}} = \langle \mathcal{L}, \vdash_{\mathfrak{A}} \rangle$.

Proof: Since, as remarked previously, $\mathbf{A}$ satisfies the three equations axiomatizing the variety V, we get that $\mathbf{A} \in V$. Consequently, it suffices to show that for all $a \in A$,

$$a \in D \quad \text{iff} \quad \delta_i^{\mathbf{A}}(a) = \epsilon_i^{\mathbf{A}}(a), \text{ for all } i \in \omega.$$

Suppose, first, that $a \in D$.

- If $a \in \{0,1\}$, then $\delta_i^{\mathbf{A}}(a) = \pi_1^{\mathbf{A}}(a)(\Diamond^{\mathbf{A}^i} a) = \pi_1^{\mathbf{A}}(a) = a = \pi_2^{\mathbf{A}}(a) = \pi_2^{\mathbf{A}}(a)(\Diamond^{\mathbf{A}^i} a) = \varepsilon_i^{\mathbf{A}}(a)$.
- If $a = \langle\langle b_0, b_0\rangle, \langle b_1, b_1\rangle, \ldots\rangle$, then
$$\begin{aligned}\delta_i^{\mathbf{A}}(a) &= \pi_1^{\mathbf{A}}(\Diamond^{\mathbf{A}^i} a) = \pi_1^{\mathbf{A}}(\langle\langle b_i, b_i\rangle, \langle b_{i+1}, b_{i+1}\rangle, \ldots\rangle) = b_i \\ &= \pi_2^{\mathbf{A}}(\langle\langle b_i, b_i\rangle, \langle b_{i+1}, b_{i+1}\rangle, \ldots\rangle) = \pi_2^{\mathbf{A}}(\Diamond^{\mathbf{A}^i} a) = \varepsilon_i^{\mathbf{A}}(a).\end{aligned}$$
- If $a = \langle a', a'\rangle$, with $a' \in A$, then
$$\begin{aligned}\delta_i^{\mathbf{A}}(\langle a', a'\rangle) &= \pi_1^{\mathbf{A}}(\Diamond^{\mathbf{A}^i} \langle a', a'\rangle) = \pi_1^{\mathbf{A}}(\langle a', a'\rangle) = a' \\ &= \pi_2^{\mathbf{A}}(\langle a', a'\rangle) = \pi_2^{\mathbf{A}}(\Diamond^{\mathbf{A}^i} \langle a', a'\rangle) = \varepsilon_i^{\mathbf{A}}(\langle a', a'\rangle).\end{aligned}$$

Assume, conversely, that $\delta_i^{\mathbf{A}}(a) = \varepsilon_i^{\mathbf{A}}(a)$, for all $i \in \omega$. This means $\pi_1^{\mathbf{A}}(\Diamond^{\mathbf{A}^i} a) = \pi_2^{\mathbf{A}}(\Diamond^{\mathbf{A}^i} a)$, for all $i \in \omega$. if $a = 0$ or $a = 1$, there is nothing to prove. If $a = \langle\langle b_0, b_0'\rangle, \langle b_1, b_1'\rangle, \ldots\rangle$, then the i-th equation gives $b_i = b_i'$. So we conclude that $a \in D$. Finally, if $a = \langle a', a''\rangle$, for some $a', a'' \in A$, then the equations ensure that $a' = a''$ and, therefore, $a = \langle a', a''\rangle \in D$. ∎

To verify that Raftery's logic accomplishes its mission, one has to establish that it is not finitary, but that it is algebraizable with the variety V as its equivalent algebraic semantics. Then, by Theorem 3.37 and Corollary 3.38 of Section 3.4 of [86], it becomes clear that the algebraization of $\mathcal{R}$ is carried out by a necessarily infinite set of defining equations and a set of equivalence formulas that may be taken to be finite. We formalize the second statement first.

Theorem 676 (Fact 9 of [82]) *Raftery's logic* $\mathcal{R} = \langle \mathcal{L}, \vdash_R\rangle$ *is algebraizable with equivalent algebraic semantics* V *via the set of defining equations* $\delta(x) \approx \varepsilon(x) = \{\delta_i(x) \approx \varepsilon_i(x) : i \in \omega\}$ *and the equivalence formula* $\Delta(x,y) = \{x \leftrightarrow y\}$.

Proof: By the definition of $\vdash_R$, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Gamma \vdash_R \varphi \quad \text{iff} \quad (\delta \approx \varepsilon)(\Gamma) \vDash_V (\delta \approx \varepsilon)(\varphi).$$

Moreover, for all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\begin{array}{rcl}(\delta \approx \varepsilon)(\varphi \leftrightarrow \psi) & = & \pi_1(\Diamond^i(\varphi \leftrightarrow \psi)) \approx \pi_2(\Diamond^i(\varphi \leftrightarrow \psi)), i \in I \\ & \dashv\vDash_V & \pi_1(\varphi \leftrightarrow \psi) \approx \pi_2(\varphi \leftrightarrow \psi) \\ & & (\text{since } V \vDash \Diamond(x \leftrightarrow y) \approx x \leftrightarrow y) \\ & \dashv\vDash_V & \varphi \approx \psi \\ & & (\text{since } V \vDash \pi_1(x \leftrightarrow y) \approx x \\ & & \text{and } V \vDash \pi_2(x \leftrightarrow y) \approx y).\end{array}$$

By the general theory of algebraizable logics, these two conditions suffice to guarantee the conclusion. ■

And, finally, we show that $\mathcal{R}$ is not finitary.

Theorem 677 (Fact 10 of [82]) *The logics* $\mathcal{S}_{\mathfrak{A}} = \langle \mathcal{L}, \vdash_{\mathfrak{A}} \rangle$ *and* $\mathcal{R} = \langle \mathcal{L}, \vdash_R \rangle$ *are not finitary.*

Proof: Note that the two conditions established in the proof of Theorem 676, which suffice to establish algebraizability, imply, by the general theory of algebraizability (see, e.g., Exercise 39 of Section 3.2 of [86]), that

$$\{\delta_i(x) \leftrightarrow \varepsilon_i(x) : i \in \omega\} \vdash_R x$$

also holds. In addition, since, by Lemma 675, $\mathcal{R} \leq \mathcal{S}_{\mathfrak{A}}$,

$$\{\delta_i(x) \leftrightarrow \varepsilon_i(x) : i \in \omega\} \vdash_{\mathfrak{A}} x.$$

So to prove that $\mathcal{S}_{\mathfrak{A}}$ and $\mathcal{R}$ are not finitary, it suffices to show that, for no finite $K \subseteq \omega$ is it the case that $\{\delta_k(x) \leftrightarrow \varepsilon_k(x) : k \in K\} \vdash_{\mathfrak{A}} x$.

Let $j \in \omega - K$ and consider $a = \langle \langle b_0, b_0' \rangle, \langle b_1, b_1' \rangle, \dots \rangle \in B - \{0,1\}$, such that $b_k = b_k'$, for all $k \in K$, but $b_j \neq b_j'$. Now, we compute

$$\delta_i^{\mathbf{A}}(a) \leftrightarrow^{\mathbf{A}} \varepsilon_i^{\mathbf{A}}(a) = \pi_1^{\mathbf{A}}(\Diamond^{\mathbf{A}^i} a) \leftrightarrow^{\mathbf{A}} \pi_2^{\mathbf{A}}(\Diamond^{\mathbf{A}^i} a) = b_i \leftrightarrow^{\mathbf{A}} b_i',$$

whence, $(\delta_k \leftrightarrow \varepsilon_k)^{\mathbf{A}}(a) \in D$, for all $k \in K$, whereas, since $(\delta_j \leftrightarrow \varepsilon_j)^{\mathbf{A}}(a) \notin D$, by what was proven in Lemma 675, $a \notin D$. This shows that $\{\delta_k(x) \leftrightarrow \varepsilon_k(x) : k \in K\} \nvdash_{\mathfrak{A}} x$. Hence $\mathcal{S}_{\mathfrak{A}}$ and, a fortiori, $\mathcal{R}$ are not finitary. ■

So, the logic $\mathcal{R}$ does indeed attain the goal of discovering a non-finitary logic that is elementarily algebraizable (i.e., has a finitary equivalent algebraic semantics).

9.6 Separating Classes of π-Institutions

Using the framework detailed in Section 1.1, we recast the three sentential logics introduced in Section 9.5 as π-institutions and show that they provide examples that serve to separate the classes of π-institutions appearing in the steps of the finitarity hierarchy studied in Section 9.4.

In the first example, we recast Łukasiewicz's infinite valued logic as a π-institution.

Example 678 *Consider the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \mathrm{Fm}_{\mathcal{L}}(V)$, *where* $\mathcal{L} = \{\wedge, \vee, \to, \neg\}$ *is the language of Łukasiewicz's infinite valued logic;*

- $N^\flat$ *is the category of natural transformations generated by the binary operations* $\wedge, \vee, \to : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *and the unary operation* $\neg : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$, *defined as usual on the absolutely free algebra of formulas.*

Now define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$, *where, for all* $\Gamma \cup \{\varphi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\varphi \in C_\Sigma(\Gamma) \quad \textit{iff} \quad \Gamma \vdash_\infty \varphi.$$

By Theorem 671, $\mathcal{I}$ *is not finitary. By Theorem 672 and the general theory of algebraizable logics, for all* $T \in \mathrm{ThFam}(\mathcal{I})$ *and* $\theta = \mathrm{ConSys}^*(\mathcal{I})$,

$$\begin{array}{rcl}\Omega_\Sigma(T) & = & \{\langle \varphi, \psi\rangle \in \mathrm{Fm}^2_{\mathcal{L}}(V) : \varphi \to \psi, \psi \to \varphi \in T_\Sigma\};\\ \Omega^{-1}_\Sigma(\theta) & = & \{\varphi \in \mathrm{Fm}_{\mathcal{L}}(V) : \langle \varphi, \top\rangle \in \theta_\Sigma\}.\end{array}$$

We show that the Leibniz operator $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *and its inverse* $\Omega^{-1} : \mathrm{ConSys}^*(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$ *are continuous. Suppose* $\{T^i\}_{i\in I} \subseteq \mathrm{ThFam}(\mathcal{I})$ *is directed and that* $\bigcup_{i\in I} T^i \in \mathrm{ThFam}(\mathcal{I})$. *Then we get, for all* $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\begin{array}{rll}\langle \varphi, \psi\rangle \in \Omega_\Sigma(\bigcup_{i\in I} T^i) & \textit{iff} & \varphi \to \psi, \psi \to \varphi \in \bigcup_{i\in I} T^i_\Sigma\\ & \textit{iff} & \varphi \to \psi \in T^i_\Sigma, \psi \to \varphi \in T^j_\Sigma, \textit{ for some } i, j \in I,\\ & \textit{iff} & \varphi \to \psi, \psi \to \varphi \in T^k_\Sigma, \textit{ for some } k \in I,\\ & \textit{iff} & \langle \varphi, \psi\rangle \in \Omega_\Sigma(T^k), \textit{ for some } k \in I,\\ & \textit{iff} & \langle \varphi, \psi\rangle \in \bigcup_{i\in I} \Omega_\Sigma(T^i).\end{array}$$

The proof for Ω^{-1} *is similar.*

This π*-institution serves to separate the classes connected by the three vertical arrows in the diagram concluding Section 9.4.*

The second example revisits Dellunde's logic in a similar way.

Example 679 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by setting* $\mathrm{SEN}^\flat(\Sigma) = \mathrm{Fm}_{\mathcal{L}}(V)$, *where* $\mathcal{L} = \{\leftrightarrow, \Box\}$ *is the language of Dellunde's logic;*

- $N^\flat$ *is the category of natural transformations generated by the binary operation* $\leftrightarrow : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *and the unary operation* $\Box : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined as usual on the absolutely free algebra of formulas.*

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$*, by setting, for all* $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$*,*

$$\varphi \in C_\Sigma(\Gamma) \quad \textit{iff} \quad \Gamma \vdash_D \varphi.$$

Since, as remarked in Section 9.5, Dellunde's logic $\mathcal{D}$ *is finitary, so is the* π*-institution* $\mathcal{I}$*. Moreover, by Theorem 673 and the general theory of algebraizable logics, for all* $T \in \mathrm{ThFam}(\mathcal{I})$ *and all* $\theta \in \mathrm{ConSys}^*(\mathcal{I})$*, we have*

$$\begin{array}{rcl}\Omega_\Sigma(T) & = & \{\langle \varphi, \psi \rangle \in \mathrm{Fm}^2_{\mathcal{L}}(V) : \Box^n \varphi \leftrightarrow \Box^n \psi \in T_\Sigma, \text{ for all } n \in \omega\};\\ \Omega^{-1}_\Sigma(\theta) & = & \{\varphi \in \mathrm{Fm}_{\mathcal{L}}(V) : \langle \varphi, \top \rangle \in \theta_\Sigma\}.\end{array}$$

We show that $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is not continuous. Assume to the contrary, and define, for all* $i \in \omega$*,* $T^i = \{T^i_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ *by setting*

$$T^i_\Sigma = C_\Sigma(\{\Box^k x \leftrightarrow \Box^k y : k \leq i\}).$$

Note the following:

(1) $\{T^i\}^\infty_{i=0}$ *is directed;*

(2) $\bigcup^\infty_{i=0} T^i \in \mathrm{ThFam}(\mathcal{I})$*, since* $\mathcal{I}$ *is finitary;*

(3) $\langle x, y \rangle \in \Omega_\Sigma(\bigcup^\infty_{i=0} T^i)$*, since* $\Box^n x \leftrightarrow \Box^n y \in \bigcup^\infty_{i=0} T^i_\Sigma$*, for all* $n \in \omega$*.*

By the hypothesized continuity of Ω*, since* $\langle x, y \rangle \in \bigcup^\infty_{i=0} \Omega_\Sigma(T^i)$*, there exists* $m \in \omega$*, such that* $\langle x, y \rangle \in \Omega_\Sigma(T^m)$*. But this implies that, for all* $n > m$*,*

$$\Box^n x \leftrightarrow \Box^n y \in C_\Sigma(\{\Box^k x \leftrightarrow \Box^k y : k \leq m\}),$$

which contradicts what was shown in Theorem 674.

The π*-institution* $\mathcal{I}$*, constructed here, serves to separate the classes connected by the southeast arrows in the finitarity hierarchy of* π*-institutions, shown at the end of Section 9.4.*

Finally, we formulate an example that employs Raftery's logic $\mathcal{R}$.

Example 680 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \mathrm{Fm}_{\mathcal{L}}(V)$*, where* $\mathcal{L} = \{\leftrightarrow, \pi_1, \pi_2, \Diamond\}$ *is the language of Raftery's logic;*
- $N^\flat$ *is the category of natural transformations generated by the binary operation* $\leftrightarrow : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *and the unary operations* $\pi_1, \pi_2, \Diamond : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined as usual on the absolutely free algebra of formulas.*

Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ by setting, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\varphi \in C_\Sigma(\Gamma) \quad \textit{iff} \quad \Gamma \vdash_R \varphi.$$

By Theorem 677, $\mathcal{I}$ is not finitary. By Theorem 676 and the general theory of algberaizable logics, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\theta \in \mathrm{ConSys}^*(\mathcal{I})$,

$$\begin{array}{rcl}\Omega_\Sigma(T) & = & \{\langle\varphi,\psi\rangle \in \mathrm{Fm}^2_{\mathcal{L}}(V) : \varphi \leftrightarrow \psi \in T_\Sigma\};\\ \Omega^{-1}_\Sigma(\theta) & = & \{\varphi \in \mathrm{Fm}_{\mathcal{L}}(V) : \langle \pi_1(\Box^i\varphi), \pi_2(\Box^i\varphi)\rangle \in \theta_\Sigma, \textit{ for all } i \in \omega\}.\end{array}$$

We may now show that $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is continuous, but $\Omega^{-1} : \mathrm{ConSys}^*(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$ is not continuous.

To show continuity of Ω, assume $\{T^i\}_{i\in I} \subseteq \mathrm{ThFam}(\mathcal{I})$ is directed, such that $\bigcup_{i\in I} T^i \in \mathrm{ThFam}(\mathcal{I})$. Let $\langle\varphi,\psi\rangle \in \Omega_\Sigma(\bigcup_{i\in I} T^i)$. This holds iff $\varphi \leftrightarrow \psi \in \bigcup_{i\in I} T^i_\Sigma$, i.e., iff, for some $i \in I$, $\varphi \leftrightarrow \psi \in T^i$. This is equivalent to $\langle\varphi,\psi\rangle \in \Omega_\Sigma(T^i)$, for some $i \in I$, showing that $\Omega(\bigcup_{i\in I} T^i) = \bigcup_{i\in I} \Omega(T^i)$.

To show that Ω^{-1} is not continuous, let, for all $i \in \omega$, $\theta^i = \{\theta^i_\Sigma\}_{\Sigma\in|\mathbf{Sign}^\flat|} \in \mathrm{ConSys}^*(\mathcal{I})$ be defined by

$$\theta^i_\Sigma = \{\langle\varphi,\psi\rangle \in \mathrm{Fm}^2_{\mathcal{L}}(V) : \{\delta_k(x) \approx \varepsilon_k(x) : k \le i\} \vDash_V \varphi \approx \psi\},$$

where, as before, for all $i \in \omega$,

$$\delta_i(x) = \pi_1(\Diamond^i x) \quad \textit{and} \quad \varepsilon_i(x) = \pi_2(\Diamond^i x).$$

Note that

(1) $\{\theta^i\}_{i=0}^\infty$ is directed;

(2) $\bigcup_{i=0}^\infty \theta^i \in \mathrm{ConSys}^*(\mathcal{I})$, since $\vDash_V$ is finitary;

(3) $x \in \Omega^{-1}_\Sigma(\bigcup_{i=0}^\infty \theta^i)$, since $\delta(x) \approx \varepsilon(x) \subseteq \bigcup_{i=0}^\infty \theta^i_\Sigma$.

If Ω^{-1} were continuous, there would exist $m \in \omega$, such that $x \in \Omega^{-1}_\Sigma(\theta^m)$. But, this would imply that

$$\{\delta_k(x) \approx \varepsilon_k(x) : k \le m\} \vDash_V \delta(x) \approx \varepsilon(x),$$

which yields $\{\delta_k(x) \leftrightarrow \varepsilon_k(x) : k \le m\} \vdash_R x$, contradicting Theorem 677.

The π-institution $\mathcal{I}$, constructed in this example, separates the classes of π-institutions related by the southwest arrows in the finitarity hierarchy shown at the end of Section 9.4.