Chapter 10

Elements of Syntax

10.1 Natural Transformations and Parameters

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. Consider a set

$$I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$$

of natural transformations in $N^\flat$. Of course, by definition, each $\sigma^\flat \in I^\flat \subseteq N^\flat$ is finitary, but the arities in the collection may be unbounded, whence the notation becomes handy.

Recall that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^\omega$,

$$I^\flat_\Sigma(\vec{\phi}) = \{\sigma^\flat_\Sigma(\phi_0, \dots, \phi_{k-1}) : \sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat \in I^\flat\}.$$

Moreover, we may view the first n of the arguments in the input sequence as **distinguished** and the remaining as **parameters** or **parametric arguments**. In that case, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} = \langle \phi_0, \dots, \phi_{n-1} \rangle \in \mathrm{SEN}^\flat(\Sigma)$, we define

$$I^\flat_\Sigma[\vec{\phi}] = \{I^\flat_{\Sigma,\Sigma'}[\vec{\phi}]\}_{\Sigma' \in |\mathbf{Sign}^\flat|},$$

where, for all $\Sigma' \in |\mathbf{Sign}^\flat|$,

$$I^\flat_{\Sigma,\Sigma'}[\vec{\phi}] = \bigcup \{I^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}), \vec{\chi}) : f \in \mathbf{Sign}^\flat(\Sigma, \Sigma'), \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')\}.$$

The following diagram illustrates where the various sentences and components sit as we move from inputs to outputs in this construct.

$$\begin{array}{ccc}
\mathrm{SEN}^\flat(\Sigma) & \xrightarrow{\mathrm{SEN}^\flat(f)} & \mathrm{SEN}^\flat(\Sigma') \\
\vec{\phi} & \longmapsto & \mathrm{SEN}^\flat(f)(\vec{\phi}), \vec{\chi} \\
 & & \big\downarrow \\
 & & I^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}), \vec{\chi})
\end{array}$$

Suppose that in $I^\flat$ we take $n = 2$, i.e., we consider only the first two arguments as distinguished and the remaining as parameters. Then, we define for $\sigma : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat \in N^\flat$, the natural transformation $\overline{\sigma} : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$, by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\overline{\sigma}_\Sigma(\phi, \psi, \vec{\chi}) = \sigma_\Sigma(\psi, \phi, \vec{\chi}).$$

Further, we set

$$\overline{I^\flat} = \{\overline{\sigma} : \sigma \in I^\flat\}$$

and

$$\overleftrightarrow{I}^\flat = I^\flat \cup \overline{I^\flat}.$$

It is not difficult to see that, given $I^\flat \subseteq N^\flat$, the collections $\overline{I^\flat}$ and $\overleftrightarrow{I}^\flat$ both consist of natural transformations in $N^\flat$.

Lemma 681 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and let* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *be a collection of natural transformations in* $N^\flat$*. Then* $\overline{I^\flat}, \overleftrightarrow{I^\flat} \subseteq N^\flat$.

Proof: The inclusion $\overline{I^\flat} \subseteq N^\flat$ follows from Proposition 11. Then the second inclusion follows directly from the definition of $\overleftrightarrow{I^\flat}$. ■

Finally, recall that, given an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$, a collection $I^\flat \subseteq N^\flat$, with two distinguished arguments, and $T \in \mathrm{SenFam}(\mathbf{F})$, we define $I^\flat(T) = \{I^\flat_\Sigma(T)\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in I^\flat_\Sigma(T) \quad \text{iff} \quad I^\flat_\Sigma[\phi, \psi] \le T.$$

It was shown in Lemma 93 that $I^\flat(T)$ is a relation system on $\mathbf{F}$, i.e., invariant under signature morphisms.

In what follows we explore some properties that collections of natural transformations may or may not satisfy in π-institutions based on the algebraic systems on which they are defined.

10.2 Reflexivity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. Taking into account Proposition 103, we say that $I^\flat$ is **reflexive in** $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma(\phi, \phi, \vec{\chi}) \subseteq \mathrm{Thm}_\Sigma(\mathcal{I}) \coloneqq C_\Sigma(\varnothing).$$

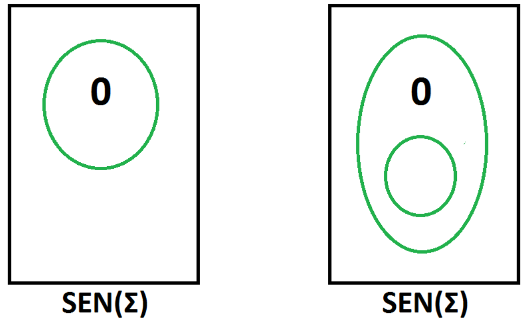
Example 682 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0\}$*;*
- $N^\flat$ *is the trivial category of natural transformations consisting only of the projections.*

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{0\}\}$ *and* $\mathcal{I}' = \langle \mathbf{F}, C' \rangle$ *be the* π*-institution determined by* $\mathcal{C}'_\Sigma = \{\varnothing, \{0\}\}$*.*

Consider the set $I^\flat = \{p^{2,0}\}$*, with* $p^{2,0} : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *be the 2-argument projection function projecting onto the first argument.*

In this case, it is easy to verify that $I^\flat$ *is reflexive in* $\mathcal{I}$ *but* $I^\flat$ *is not reflexive in* $\mathcal{I}'$*.*



As was the case with the various properties of the Leibniz operator that gave rise to the various classes of the semantic hierarchy of π-institutions, the surjectivity of the morphism components in interpreted algebraic systems affords transferring the properties that give rise to the syntactic hierarchy studied in the present chapter from the theory families of a π-institution to the filter families over arbitrary algebraic systems. The key in proving these tranfer properties is Lemma 95, which will be used repeatedly in the proofs throughout the chapter.

The first of this type of transfer properties is the transfer property for reflexivity. In formulating the property it is convenient to adopt the following terminology. We consider, as is usual in this context, a base algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ and a set $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ of natural transformations in $N^\flat$. Given a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, based on $\mathbf{F}$, and an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, we say that I is **reflexive in $\mathcal{A}$** if the collection $I : \mathrm{SEN}^\omega \to \mathrm{SEN}$ of natural transformations in N, that are images of those in $I^\flat$, is reflexive in the π-institution $\langle \mathbf{A}, C^{\mathcal{I},\mathcal{A}} \rangle$, $C^{\mathcal{I},\mathcal{A}}$ being the closure (operator) system whose closed set families are the $\mathcal{I}$-filter families on $\mathcal{A}$.

We use similar terminology for all other properties that we study in this chapter, pertaining to subsets $I^\flat$ of $N^\flat$. In particular, such terminology will be used in all transfer results for these properties.

Proposition 683 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $I^\flat \subseteq N^\flat$ a collection of natural transformations $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $I^\flat$ is reflexive in $\mathcal{I}$ if and only if, for every algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, I is reflexive in $\mathcal{A}$.*

Proof: First, note that if reflexivity of I in $\mathcal{A}$ is assumed, for all $\mathcal{A}$, then it holds, in particular, for $\mathcal{A} = \mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. Moreover $\langle \mathbf{F}, C^{\mathcal{I},\mathcal{F}} \rangle = \mathcal{I}$. Thus, we conclude that $I^\flat$ is reflexive in $\mathcal{I}$.

Suppose, conversely, that $I^\flat$ is reflexive in $\mathcal{I}$. By the surjectivity of $\langle F, \alpha \rangle$, it suffices to show that, for all $\sigma : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat \in I^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi \in \mathrm{SEN}^\flat(\Sigma)$ and all $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\sigma_{F(\Sigma')}(\mathrm{SEN}(F(f))(\alpha_\Sigma(\phi)), \mathrm{SEN}(F(f))(\alpha_\Sigma(\phi)), \alpha_{\Sigma'}(\vec{\chi})) \in C^{\mathcal{I},\mathcal{A}}_{F(\Sigma')}(\varnothing).$$

To this end, let $\sigma : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat \in I^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$ and $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$. Since $C^{\mathcal{I},\mathcal{A}}(\varnothing)$ is, by definition, an $\mathcal{I}$-filter family on $\mathcal{A}$, by Lemma 51, $\alpha^{-1}(C^{\mathcal{I},\mathcal{A}}(\varnothing)) \in \mathrm{ThFam}(\mathcal{I})$. Hence, since $I^\flat$ is reflexive in $\mathcal{I}$, we get

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}) \in \alpha^{-1}_{\Sigma'}(C^{\mathcal{I},\mathcal{A}}_{F(\Sigma')}(\varnothing)).$$

This is equivalent to

$$\alpha_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\mathrm{SEN}^\flat(f)(\phi),\vec{\chi})) \in C^{\mathcal{I},\mathcal{A}}_{F(\Sigma')}(\varnothing),$$

which is, in turn, equivalent to

$$\sigma_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)),\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)),\alpha_{\Sigma'}(\vec{\chi})) \in C^{\mathcal{I},\mathcal{A}}_{F(\Sigma')}(\varnothing).$$

Finally, by the naturality of α, we get the conclusion. Therefore, I is indeed reflexive in $\mathcal{A}$. ■

10.3 Symmetry

We look now at various versions of the symmetry property, taking into account both the duality between local versus global membership and the difference between considering all theory families versus restricting only to theory systems.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. We say that:

- $I^\flat$ has the **local family symmetry in** $\mathcal{I}$ if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma(\phi,\psi,\vec{\xi}) \subseteq T_\Sigma$, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$, implies that $I^\flat_\Sigma(\psi,\phi,\vec{\xi}) \subseteq T_\Sigma$, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$;
- $I^\flat$ has the **local system symmetry in** $\mathcal{I}$ if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma(\phi,\psi,\vec{\xi}) \subseteq T_\Sigma$, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$, implies that $I^\flat_\Sigma(\psi,\phi,\vec{\xi}) \subseteq T_\Sigma$, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$;
- $I^\flat$ has the **global family symmetry in** $\mathcal{I}$ if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma[\phi,\psi] \le T$ implies $I^\flat_\Sigma[\psi,\phi] \le T$;
- $I^\flat$ has the **global system symmetry in** $\mathcal{I}$ if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma[\phi,\psi] \le T$ implies $I^\flat_\Sigma[\psi,\phi] \le T$.

The following proposition establishes a hierarchy of symmetry properties.

Proposition 684 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments.*

(a) *If $I^\flat$ has the local family symmetry, then it has the local system symmetry in $\mathcal{I}$;*

(b) *If $I^\flat$ has the local system symmetry, then it has the global family symmetry in $\mathcal{I}$;*

(c) *$I^\flat$ has the global family symmetry if and only if it has the global system symmetry in $\mathcal{I}$.*

Proof: Parts (a) and one of the implications in Part (c) follow directly from the fact that every theory system of $\mathcal{I}$ is also a theory family of $\mathcal{I}$.

For Part (b), suppose that $I^\flat$ has the local system symmetry in $\mathcal{I}$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $I^\flat_\Sigma[\phi,\psi] \leq T$. Then by Lemma 93, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$,

$$I^\flat_{\Sigma'}[\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi)] \leq T.$$

This implies, by Lemma 99, that, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$I^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), \vec{\xi}) \subseteq \overleftarrow{T}_{\Sigma'}.$$

Since $I^\flat$ has the local system symmetry and, by Proposition 42, $\overleftarrow{T} \in \mathrm{ThSys}(\mathcal{I})$, we get that $I^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \mathrm{SEN}^\flat(f)(\phi), \vec{\xi}) \subseteq \overleftarrow{T}_{\Sigma'} \subseteq T_{\Sigma'}$. Since this holds for all $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$, we conclude that $I^\flat_\Sigma[\psi,\phi] \leq T$. Therefore $I^\flat$ has the global family symmetry in $\mathcal{I}$.

Suppose, finally, that $I^\flat$ has the global system symmetry in $\mathcal{I}$ and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $I^\flat_\Sigma[\phi,\psi] \leq T$. By Lemma 99, we get that $I^\flat_\Sigma[\phi,\psi] \leq \overleftarrow{T}$. Since $I^\flat$ has the global system symmetry and, by Proposition 42, $\overleftarrow{T} \in \mathrm{ThSys}(\mathcal{I})$, we get that $I^\flat_\Sigma[\psi,\phi] \leq \overleftarrow{T}$. Using again Lemma 99, we conclude that $I^\flat_\Sigma[\psi,\phi] \leq T$. Therefore, $I^\flat$ has the global family symmetry in $\mathcal{I}$. ■

Proposition 684 has established the following hierarchy of symmetry properties:

Local Family Symmetry

↓

Local System Symmetry

↓

Global Symmetry

We look, next, at some natural sufficient conditions under which some of these three symmetry properties coincide.

Proposition 685 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$*, with two distinguished arguments.*

(a) If $\mathcal{I}$ *is systemic, then the local family and the local system symmetry coincide;*

(b) If $I^\flat$ *has only two arguments (i.e., is parameter free), then the local system symmetry and the global symmetry coincide.*

Proof: If $\mathcal{I}$ is systemic, then all theory families are theory systems and, hence, the local family and local system symmetries coincide.

Suppose, next that $I^\flat$ is parameter free and has the global system symmetry in $\mathcal{I}$. Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $I^\flat_\Sigma(\phi, \psi) \subseteq T_\Sigma$. Then, by Proposition 99, $I^\flat[\phi, \psi] \leq T$. Thus, by the global system property, $I^\flat_\Sigma[\psi, \phi] \leq T$, which implies that $I^\flat_\Sigma(\psi, \phi) \subseteq T_\Sigma$. Therefore, $I^\flat$ has the local system symmetry in $\mathcal{I}$. ∎

So in the case of a systemic π-institution $\mathcal{I}$, we have the hierarchy pictured on the left, whereas in the case of a parameter-free set of natural transformations we have the hierarchy on the right.

Local Symmetry	Local Family Symmetry
↓	↓
Global Symmetry	Local System/Global Symmetry

Finally, for a systemic π-institution with a parameter-free set of natural transformations all four symmetry properties collapse to a single one.

We provide some examples to show that the implications of Proposition 684 are not equivalences in general, i.e., in the 3-class hierarchy all inclusions of classes of π-institutions with a set of natural transformations satisfying the corresponding symmetry properties are proper inclusions.

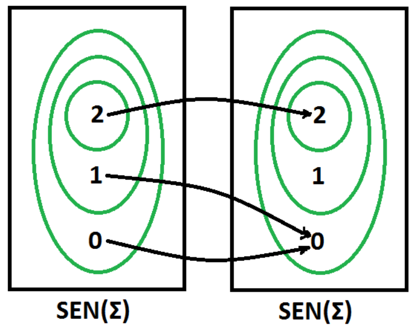
We first present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that have the local system symmetry but not the local family symmetry in $\mathcal{I}$.

Example 686 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single objects* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0, 1, 2\} \to \{0, 1, 2\}$ *given by* $0 \mapsto 0$*,* $1 \mapsto 0$ *and* $2 \mapsto 2$*;*

- *$N^\flat$ is the category of natural transformations generated by the single binary natural transformation $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ defined by letting $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ be given by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 1, & \text{if } (x,y) = (0,1) \\ 0, & \text{otherwise} \end{cases}$$



Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be the π-institution determined by

$$\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}.$$

Note that there are three theory families, but only $\mathrm{Thm}(\mathcal{I})$ and $\mathrm{SEN}^\flat$ are theory systems.

Consider the set $I^\flat = \{\sigma^\flat\}$, with $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ having two distinguished arguments. We show that $I^\flat$ has the local system symmetry in $\mathcal{I}$, but it does not have the local family symmetry in $\mathcal{I}$.

For the local system symmetry note that, if $T = \mathrm{SEN}^\flat$, then the defining implication is trivially true, whereas, if $T = \mathrm{Thm}(\mathcal{I})$, then, since, for all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, $\sigma^\flat_\Sigma(\phi,\psi) \neq 2$, the defining implication is vacuously true. So $I^\flat$ has the local system symmetry in $\mathcal{I}$.

On the other hand, for $T = \{\{1,2\}\} \in \mathrm{ThFam}(\mathcal{I})$, we have $\sigma^\flat_\Sigma(0,1) = 1 \in T_\Sigma$, but $\sigma^\flat_\Sigma(1,0) = 0 \notin T_\Sigma$. Therefore, the implication defining local family symmetry fails for $T = \{\{1,2\}\}$. So $I^\flat$ is not locally family symmetric in $\mathcal{I}$.

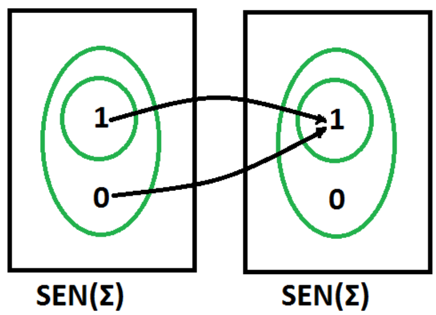
Next, we present an example to show that there is π-institution $\mathcal{I}$ with a set of natural transformations that have the global family symmetry but not the local system symmetry in $\mathcal{I}$.

Example 687 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be the algebraic system determined as follows:*

- *$\mathbf{Sign}^\flat$ is the category with a single objects Σ and a single (non-identity) morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1\} \to \{0,1\}$ *given by* $0 \mapsto 1$ *and* $1 \mapsto 1$*;*
- $N^\flat$ *is the category of natural transformations generated by the single ternary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1\}^3 \to \{0,1\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y,z) = \begin{cases} 0, & \text{if } (x,z) = (1,0) \\ 1, & \text{otherwise} \end{cases}$$



Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\}$*. Note that both theory families,* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, are also theory systems.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the global family symmetry in* $\mathcal{I}$*, but it does not have the local system symmetry in* $\mathcal{I}$*.*

For the global family symmetry note that, if $T = \mathrm{SEN}^\flat$*, then the defining implication is trivially true, whereas, if* $T = \mathrm{Thm}(\mathcal{I})$*, then, since, for all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\sigma^\flat_\Sigma(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), 0) = \sigma^\flat_\Sigma(1,1,0) = 0,$$

the defining implication is vacuously true. So $I^\flat$ *has the global family symmetry in* $\mathcal{I}$*.*

On the other hand, we have $\sigma^\flat_\Sigma(0,1,\xi) = 1$*, for all* $\xi \in \mathrm{SEN}^\flat(\Sigma)$*, but* $\sigma^\flat_\Sigma(1,0,0) = 0 \notin \{1\}$*. Therefore, the implication defining local system symmetry fails for* $\mathrm{Thm}(\mathcal{I})$*. So* $I^\flat$ *is not locally system symmetric in* $\mathcal{I}$*.*

To close the study of symmetry properties, we prove that all three symmetry properties transfer from π-institutions to their models.

Proposition 688 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $I^\flat$ *a collection of natural transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, with two distinguished arguments, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $I^\flat$ *has a symmetry property in* $\mathcal{I}$ *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*,* I *has the corresponding symmetry property in* $\mathcal{A}$*.*

Proof: If I has a symmetry property in $\mathcal{A}$, for all $\mathcal{A}$, then it has the same symmetry in $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. Since $\langle \mathbf{F}, C^{\mathcal{I},\mathcal{F}} \rangle = \mathcal{I}$, we conclude that $I^\flat$ has the corresponding symmetry in $\mathcal{I}$.

Suppose, conversely, that $I^\flat$ has a symmetry in $\mathcal{I}$. We look at each of the three properties in turn.

(a) Suppose $I^\flat$ has the local family symmetry in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$I_{F(\Sigma)}(\alpha_\Sigma(\phi), \alpha_\Sigma(\psi), \alpha_\Sigma(\vec{\xi})) \subseteq T_{F(\Sigma)},$$

for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$. Since this is equivalent to $\alpha_\Sigma(I^\flat_\Sigma(\phi, \psi, \vec{\xi})) \subseteq T_{F(\Sigma)}$, we get that $I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq \alpha^{-1}_\Sigma(T_{F(\Sigma)})$, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$. But, by hypothesis, $I^\flat$ has the local family symmetry in $\mathcal{I}$ and, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$. Therefore, we get that $I^\flat_\Sigma(\psi, \phi, \vec{\xi}) \subseteq \alpha^{-1}_\Sigma(T_{F(\Sigma)})$. This now gives $\alpha_\Sigma(I^\flat_\Sigma(\psi, \phi, \vec{\xi})) \subseteq T_{F(\Sigma)}$, or, equivalently,

$$I_{F(\Sigma)}(\alpha_\Sigma(\psi), \alpha_\Sigma(\phi), \alpha_\Sigma(\vec{\xi})) \subseteq T_{F(\Sigma)}.$$

We conclude that I has the local family symmetry in $\mathcal{A}$.

(b) The case of the local system symmetry can be proven similarly, taking into account that, if $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, then $\alpha^{-1}(T) \in \mathrm{ThSys}(\mathcal{I})$.

(c) Suppose that $I^\flat$ has the global (family) symmetry in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$I_{F(\Sigma)}[\alpha_\Sigma(\phi), \alpha_\Sigma(\psi)] \leq T.$$

Then, we have, by Lemma 95, $I^\flat_\Sigma[\phi, \psi] \leq \alpha^{-1}(T)$. Now, since, by hypothesis, $I^\flat$ has the global family symmetry in $\mathcal{I}$ and, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, we get that $I^\flat_\Sigma[\psi, \phi] \leq \alpha^{-1}(T)$, or, equivalently, by Lemma 95, $I_{F(\Sigma)}[\alpha_\Sigma(\psi), \alpha_\Sigma(\phi)] \leq T$. Thus, I has the global family symmetry in $\mathcal{A}$.

■

10.4 Transitivity

We study next various versions of the transitivity property, taking into account, again, both the duality between local versus global membership and the difference between considering all theory families versus restricting only to theory systems.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. We say that:

- $I^\flat$ has the **local family transitivity in** $\mathcal{I}$ if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq T_\Sigma$ and $I^\flat_\Sigma(\psi, \chi, \vec{\xi}) \subseteq T_\Sigma$, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$, imply that $I^\flat_\Sigma(\phi, \chi, \vec{\xi}) \subseteq T_\Sigma$, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$;
- $I^\flat$ has the **local system transitivity in** $\mathcal{I}$ if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq T_\Sigma$ and $I^\flat_\Sigma(\psi, \chi, \vec{\xi}) \subseteq T_\Sigma$, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$, imply that $I^\flat_\Sigma(\phi, \chi, \vec{\xi}) \subseteq T_\Sigma$, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$;
- $I^\flat$ has the **global family transitivity in** $\mathcal{I}$ if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma[\phi, \psi] \leq T$ and $I^\flat_\Sigma[\psi, \chi] \leq T$ imply $I^\flat_\Sigma[\phi, \chi] \leq T$;
- $I^\flat$ has the **global system transitivity in** $\mathcal{I}$ if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma[\phi, \psi] \leq T$ and $I^\flat_\Sigma[\psi, \chi] \leq T$ imply $I^\flat_\Sigma[\phi, \chi] \leq T$.

The following proposition establishes the hierarchy of transitivity properties.

Proposition 689 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$*, with two distinguished arguments.*

(a) If $I^\flat$ *has the local family transitivity, then it has the local system transitivity in* $\mathcal{I}$*;*

(b) If $I^\flat$ *has the local system transitivity, then it has the global family transitivity in* $\mathcal{I}$*;*

(c) $I^\flat$ *has the global family transitivity if and only if it has the global system transitivity in* $\mathcal{I}$*.*

Proof: The statement in Part (a) as well as one of the two implications of Part (c) follow from the fact that every theory system is also a theory family of $\mathcal{I}$.

For Part (b), suppose that $I^\flat$ has the local system transitivity and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, such that $I^\flat_\Sigma[\phi, \psi] \leq T$ and $I^\flat_\Sigma[\psi, \chi] \leq T$. By Lemma 93, we get that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$I^\flat_{\Sigma'}[\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi)] \leq T, \;\; I^\flat_{\Sigma'}[\mathrm{SEN}^\flat(f)(\psi), \mathrm{SEN}^\flat(f)(\chi)] \leq T.$$

So, by Proposition 99, we get, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\begin{array}{l} I^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), \vec{\xi}) \subseteq \overleftarrow{T}_{\Sigma'}, \\ I^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi}) \subseteq \overleftarrow{T}_{\Sigma'}. \end{array}$$

By local system transitivity, we obtain $I^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi}) \subseteq \overleftarrow{T}_{\Sigma'}$, Since this holds for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$, we conclude that $I^\flat_{\Sigma'}[\phi, \chi] \leq T$, Therefore, $I^\flat$ has the global family transitivity in $\mathcal{I}$.

Finally, suppose that $I^\flat$ has the global system transitivity in $\mathcal{I}$ and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, such that $I^\flat_\Sigma[\phi, \psi] \leq T$ and $I^\flat_\Sigma[\psi, \chi] \leq T$. By Proposition 99, we get $I^\flat_\Sigma[\phi, \psi] \leq \overleftarrow{T}$ and $I^\flat_\Sigma[\psi, \chi] \leq \overleftarrow{T}$. Hence, by global system transitivity, $I^\flat_\Sigma[\phi, \chi] \leq \overleftarrow{T}$. Now, using Proposition 99 again, we conclude that $I^\flat_\Sigma[\phi, \chi] \leq T$. Therefore, $I^\flat$ has the global family transitivity in $\mathcal{I}$. ■

Proposition 689 has established the following hierarchy of transitivity properties:

Local Family Transitivity

$\downarrow$

Local System Transitivity

$\downarrow$

Global Transitivity

We also have the following result regarding natural sufficient conditions under which some of these three transitivity properties coincide.

Proposition 690 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$*, with two distinguished arguments.*

(a) If $\mathcal{I}$ *is systemic, then the local family and the local system transitivity coincide;*

(b) If $I^\flat$ *has only two arguments (i.e., is parameter free), then the local system transitivity and the global transitivity properties coincide.*

Proof: If $\mathcal{I}$ is systemic, then all theory families are theory systems and the local family and local system transitivity properties collapse.

Suppose that $I^\flat$ is parameter-free and that $I^\flat$ has the global (family) transitivity in $\mathcal{I}$. Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, such that $I^\flat_\Sigma(\phi, \psi) \subseteq T_\Sigma$ and $I^\flat_\Sigma(\psi, \chi) \subseteq T_\Sigma$. By Proposition 99, $I^\flat_\Sigma[\phi, \psi] \leq T$ and $I^\flat_\Sigma[\psi, \chi] \leq T$. Thus, by the global family transitivity property, $I^\flat_\Sigma[\phi, \chi] \leq T$, which implies that $I^\flat_\Sigma(\phi, \chi) \subseteq T_\Sigma$. We conclude that $I^\flat$ has the local system transitivity in $\mathcal{I}$. ■

So in the case of a systemic π-institution $\mathcal{I}$, we have the hierarchy pictured on the left, whereas in the case of a parameter-free set of natural transformations we have the hierarchy on the right.

$$\begin{array}{ccc} \text{Local Transitivity} & \qquad & \text{Local Family Transitivity} \\ \downarrow & & \downarrow \\ \text{Global Transitivity} & & \text{Local System/Global Transitivity} \end{array}$$

Finally, for a systemic π-institution with a parameter-free set of natural transformations all four transitivity properties collapse to a single one.

We provide some examples to show that the implications of Proposition 689 are not equivalences in general, i.e., in the 3-class transitivity hierarchy all inclusions of classes of π-institutions with a set of natural transformations satisfying the corresponding transitivity properties are proper inclusions.

First, we present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that has the local system transitivity but not the local family transitivity in $\mathcal{I}$.

Example 691 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single objects* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0, 1, 2\} \to \{0, 1, 2\}$ *given by* $0 \mapsto 0$, $1 \mapsto 0$ *and* $2 \mapsto 2$;
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0, 1, 2\}^2 \to \{0, 1, 2\}$ *be given by*

$$\sigma^\flat_\Sigma(x, y) = \begin{cases} 1, & \textit{if } (x, y) = (0, 1) \textit{ or } (1, 2) \\ 0, & \textit{otherwise} \end{cases}$$

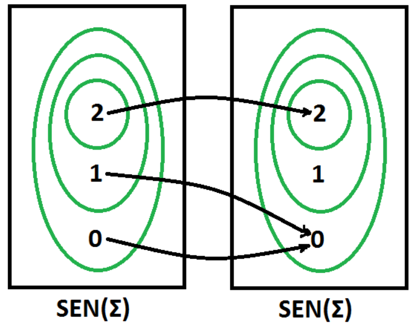
Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

Note that there are three theory families, but only $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$ *are theory systems.*

Consider the set $I^\flat = \{\sigma^\flat\}$, *with* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the local system transitivity in* $\mathcal{I}$, *but it does not have the local family transitivity in* $\mathcal{I}$.

For the local system transitivity note that, if $T = \mathrm{SEN}^\flat$, *then the defining implication is trivially true, whereas, if* $T = \mathrm{Thm}(\mathcal{I})$, *then, since, for all*



$\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $\sigma^\flat_\Sigma(\phi, \psi) \neq 2$, the defining implication is vacuously true. So $I^\flat$ has the local system transitivity in $\mathcal{I}$.

On the other hand, for $T = \{\{1, 2\}\} \in \mathrm{ThFam}(\mathcal{I})$, we have $\sigma^\flat_\Sigma(0, 1) = \sigma^\flat_\Sigma(1, 2) = 1 \in T_\Sigma$, but $\sigma^\flat_\Sigma(0, 2) = 0 \notin T_\Sigma$. Therefore, the implication defining local family transitivity fails for $T = \{\{1, 2\}\}$. So $I^\flat$ is not locally family transitive in $\mathcal{I}$.

We now present an example to show that there is π-institution $\mathcal{I}$, with a set of natural transformations that has the global family transitivity but not the local system transitivity in $\mathcal{I}$.

Example 692 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system determined as follows:*

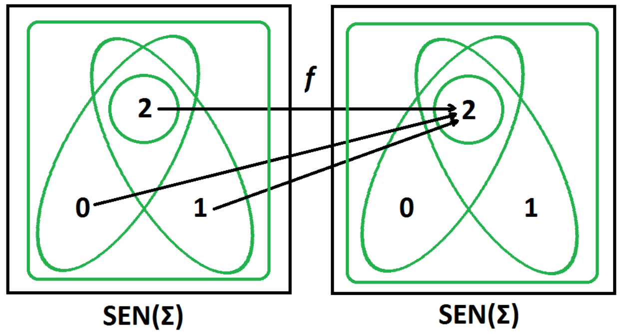
- *$\mathbf{Sign}^\flat$ is the category with a single objects Σ and a single (non-identity) morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- *$\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ is specified by $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ and $\mathrm{SEN}^\flat(f) : \{0, 1, 2\} \to \{0, 1, 2\}$ given by $0 \mapsto 2$, $1 \mapsto 2$ and $2 \mapsto 2$;*
- *$N^\flat$ is the category of natural transformations generated by the single ternary natural transformation $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ defined by letting $\sigma^\flat_\Sigma : \{0, 1, 2\}^3 \to \{0, 1, 2\}$ be given by*

$$\sigma^\flat_\Sigma(x, y, z) = \begin{cases} 0, & \text{if } x = y = 0 \text{ or } x = y = 1 \\ 2, & \text{if } \{x, y\} = \{0, 1\} \text{ or } x = y = z = 2 \\ z, & \text{otherwise} \end{cases}$$

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be the π-institution determined by

$$\mathcal{C}_\Sigma = \{\{2\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

Note that all four theory families, $\mathrm{Thm}(\mathcal{I})$, $T = \{\{0, 2\}\}$, $T' = \{\{1, 2\}\}$ and $\mathrm{SEN}^\flat$, are also theory systems.



Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the global family transitivity in* $\mathcal{I}$*, but it does not have the local system transitivity in* $\mathcal{I}$*.*

For the global family transitivity note that, because, for all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $\sigma^\flat_\Sigma(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), 0) = 0$ *and* $\sigma^\flat_\Sigma(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), 1) = 1$*, the implication of the defining condition is vacuously true for* $\mathrm{Thm}(\mathcal{I})$*,* T *and* T' *and trivially true for* $\mathrm{SEN}^\flat$*. Therefore, we get that* $I^\flat$ *has the global family transitivity in* $\mathcal{I}$*.*

On the other hand, we have $\sigma^\flat_\Sigma(0,1,\xi) = \sigma^\flat_\Sigma(1,0,\xi) = 2$*, for all* $\xi \in \mathrm{SEN}^\flat(\Sigma)$*, but* $\sigma^\flat_\Sigma(0,0,0) = 0 \notin \{2\}$*. Therefore, the implication defining local system transitivity fails for* $\mathrm{Thm}(\mathcal{I})$*. So* $I^\flat$ *does not have the local system transitivity in* $\mathcal{I}$*.*

We close the study of transitivity by providing, again, a transfer property for transitivity.

Proposition 693 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $I^\flat$ *a collection of natural transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, with two distinguished arguments, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $I^\flat$ *has a transitivity property in* $\mathcal{I}$ *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* I *has the corresponding transitivity property in* $\mathcal{A}$*.*

Proof: If I has a transitivity property in $\mathcal{A}$, for all $\mathcal{A}$, then it has the same transitivity in $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. Since $\langle \mathbf{F}, C^{\mathcal{I},\mathcal{F}} \rangle = \mathcal{I}$, we conclude that $I^\flat$ has the corresponding transitivity in $\mathcal{I}$.

Suppose, conversely, that $I^\flat$ has a transitivity property in $\mathcal{I}$. We look at each of the three properties in turn.

(a) Suppose $I^\flat$ has the local family transitivity in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, such that, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$I_{F(\Sigma)}(\alpha_\Sigma(\phi), \alpha_\Sigma(\psi), \alpha_\Sigma(\vec{\xi})) \subseteq T_{F(\Sigma)},$$
$$I_{F(\Sigma)}(\alpha_\Sigma(\psi), \alpha_\Sigma(\chi), \alpha_\Sigma(\vec{\xi})) \subseteq T_{F(\Sigma)}.$$

These are equivalent, respectively, to

$$\alpha_\Sigma(I^\flat_\Sigma(\phi,\psi,\vec{\xi})) \subseteq T_{F(\Sigma)},\ \alpha_\Sigma(I^\flat_\Sigma(\psi,\chi,\vec{\xi})) \subseteq T_{F(\Sigma)},$$

i.e., to $I^\flat_\Sigma(\phi,\psi,\vec{\xi}) \subseteq \alpha^{-1}_\Sigma(T_{F(\Sigma)})$ and $I^\flat_\Sigma(\psi,\chi,\vec{\xi}) \subseteq \alpha^{-1}_\Sigma(T_{F(\Sigma)})$, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$. But, by hypothesis, $I^\flat$ has the local family transitivity in $\mathcal{I}$ and, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$. Therefore, we get that $I^\flat_\Sigma(\phi,\chi,\vec{\xi}) \subseteq \alpha^{-1}_\Sigma(T_{F(\Sigma)})$, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$. Thus, $\alpha_\Sigma(I^\flat_\Sigma(\phi,\chi,\vec{\xi})) \subseteq T_{F(\Sigma)}$ and, hence, $I_{F(\Sigma)}(\alpha_\Sigma(\phi),\alpha_\Sigma(\chi),\alpha_\Sigma(\vec{\xi})) \subseteq T_{F(\Sigma)}$. This, combined with the surjectivity of $\langle F,\alpha\rangle$, proves that I has the local family transitivity in $\mathcal{A}$.

(b) The case of the local system transitivity may be proven similarly, taking into account that, if $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, then $\alpha^{-1}(T) \in \mathrm{ThSys}(\mathcal{I})$.

(c) Suppose that $I^\flat$ has the global (family) transitivity in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A},\langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign},\mathrm{SEN},N\rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi,\chi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$I_{F(\Sigma)}[\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)] \leq T \quad \text{and} \quad I_{F(\Sigma)}[\alpha_\Sigma(\psi),\alpha_\Sigma(\chi)] \leq T.$$

Then, we have, by Lemma 95, $I^\flat_\Sigma[\phi,\psi] \leq \alpha^{-1}(T)$ and $I^\flat_\Sigma[\psi,\chi] \leq \alpha^{-1}(T)$. Since, by hypothesis, $I^\flat$ has the global family transitivity in $\mathcal{I}$ and, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, we get that $I^\flat_\Sigma[\phi,\chi] \leq \alpha^{-1}(T)$, or, equivalently, using again Lemma 95, $I_{F(\Sigma)}[\alpha_\Sigma(\phi),\alpha_\Sigma(\chi)] \leq T$. Thus, I has the global family transitivity in $\mathcal{A}$.

■

10.5 Equivalence

We look now at sets of natural transformations $I^\flat$, with two distinguished arguments, that define (modulo theory families) equivalence relation families on the underlying algebraic system of a π-institution $\mathcal{I}$. We assume that $I^\flat$ has the reflexivity property and study combinations of possible symmetry and transitivity properties that the set of connectives may or may not possess.

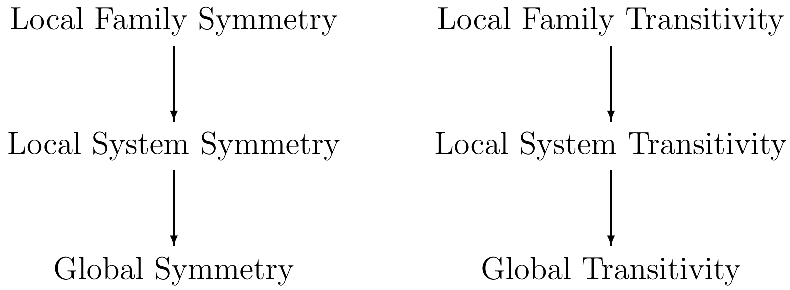
Let $\mathbf{F} = \langle \mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F},C\rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. Let $X,Y \in \{\mathrm{LF},\mathrm{LS},\mathrm{GB}\}$, where LF stands for "Local Family", LS stands for "Local System" and GB stands for "GloBal". We say that $I^\flat$ has the XY**-equivalence property in** $\mathcal{I}$ if it has

(a) reflexivity in $\mathcal{I}$;

(b) X symmetry in $\mathcal{I}$ and

(c) Y transitivity in $\mathcal{I}$.

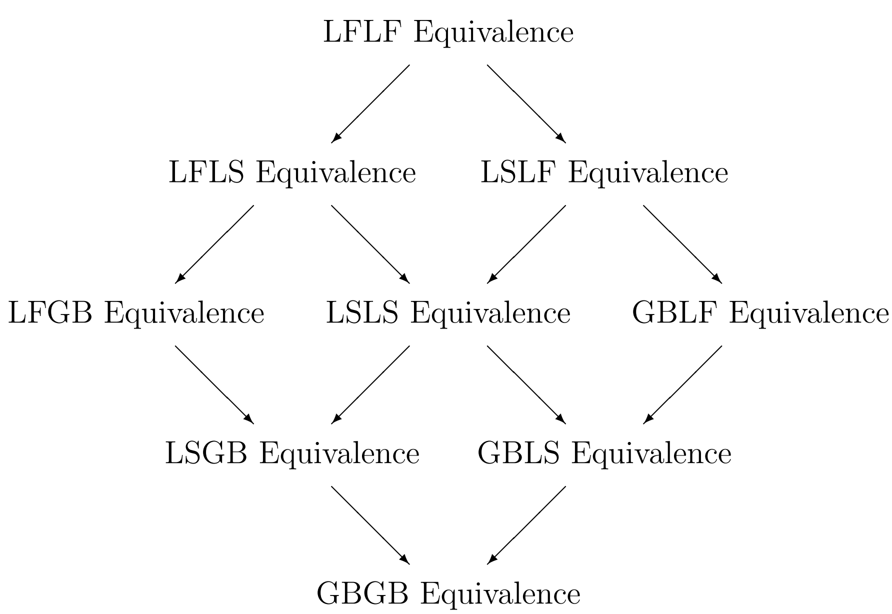
Recall the following hierarchies of symmetry and transitivity properties that we established previously:



From these, we can infer the following hierarchy of equivalence properties:

Corollary 694 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$*, with two distinguished arguments. The nine equivalence properties constitute the hierarchy depicted in the accompanying diagram.*

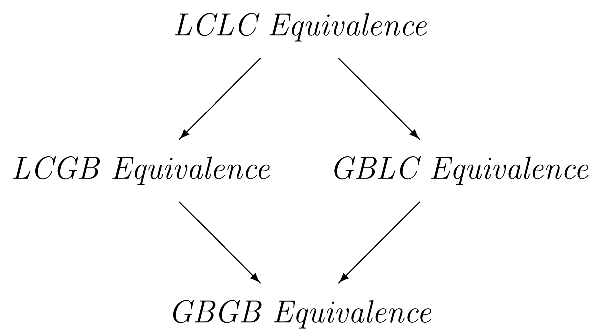
Proof: The statement is a direct consequence of Propositions 684 and 689. ■



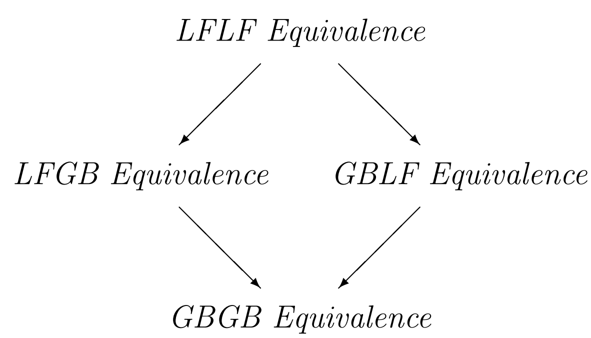
Based on the analysis performed on symmetry and transitivity, we have the following result regarding natural sufficient conditions under which some of the nine equivalence properties above coincide. We let LC stand for "LoCal" to summarize the case when the local family and the local system version of a property coincide.

Corollary 695 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$*, with two distinguished arguments.*

(a) If $\mathcal{I}$ *is systemic, then the equivalence hierarchy collapses to the one depicted below;*



(b) If $I^\flat$ *has only two arguments (i.e., is parameter free), then the equivalence hierarchy collapses to the one depicted below, where the local system versions coincide with (and, hence, are incorporated into) the global versions.*



Proof: The statement follows directly from Propositions 685 and 690. ■

For a systemic π-institution with a parameter-free set of natural transformations, there is only one equivalence property, since all versions of symmetry and all versions of transitivity collapse to a single property.

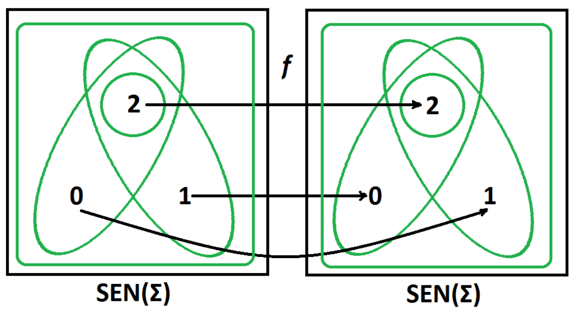
We provide some examples to show that the implications of Proposition 694 are not equivalences in general, i.e., that the nine classes of the equivalence hierarchy are all distinct.

First, we present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that has the LSLF equivalence, but not the LFGB equivalence in $\mathcal{I}$.

Example 696 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = i_\Sigma$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 1$, $1 \mapsto 0$ *and* $2 \mapsto 2$;
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given by*

$\sigma^\flat_\Sigma$	0	1	2
0	2	1	1
1	0	2	0
2	0	1	2



Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by*

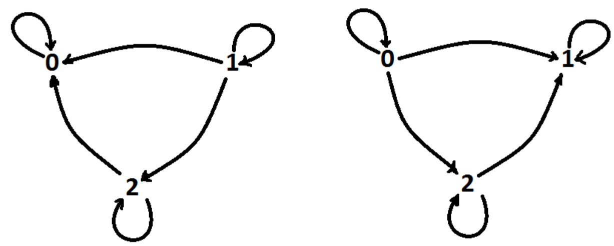
$$\mathcal{C}_\Sigma = \{\{2\},\{0,2\},\{1,2\},\{0,1,2\}\}.$$

Note that there are four theory families, but only $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$ *are theory systems.*

Consider the set $I^\flat = \{\sigma^\flat\}$, *with* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the LSLF equivalence in* $\mathcal{I}$, *but it does not have the LFGB equivalence in* $\mathcal{I}$.

Note, first, that reflexivity is obvious, since, by definition $\sigma^\flat_\Sigma(x,x) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$, *for all* $x \in \mathrm{SEN}^\flat(\Sigma)$. *Local system symmetry is also obvious, since the only theory systems in* $\mathcal{I}$ *are* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$. *Local family transitivity is a little more challenging to verify, but it suffices to observe that the pairs that are related modulo* $T = \{\{0,2\}\}$ *are as shown on the left below and the pairs that are related modulo* $T' = \{\{1,2\}\}$ *are as on the right below. We conclude that* $I^\flat$ *has the LSLF equivalence in* $\mathcal{I}$.

On the other hand, for $T = \{\{0,2\}\} \in \mathrm{ThFam}(\mathcal{I})$, *we have* $\sigma^\flat_\Sigma(1,0) = 0 \in T_\Sigma$, *but* $\sigma^\flat_\Sigma(0,1) = 1 \notin T_\Sigma$. *Therefore, the implication defining local family symmetry fails for* $T = \{\{0,2\}\}$. *So* $I^\flat$ *is not locally family symmetric, and, hence, a fortiori, does not have the LFGB equivalence property in* $\mathcal{I}$.

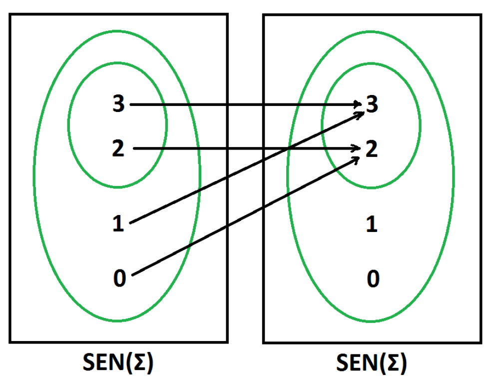


We now present an example to show that there is π-institution $\mathcal{I}$ with a set of natural transformations that has the GBLF equivalence but not the LSGB equivalence in $\mathcal{I}$.

Example 697 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, with* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2,3\} \to \{0,1,2,3\}$ *given by* $0 \mapsto 2$*,* $1 \mapsto 3$*,* $2 \mapsto 2$ *and* $3 \mapsto 3$*;*
- $N^\flat$ *is the category of natural transformations generated by the single ternary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *defined by letting:* $\sigma^\flat_\Sigma : \{0,1,2,3\}^3 \to \{0,1,2,3\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y,z) = \begin{cases} 2, & \text{if } x = y \text{ or } (x,y) = (0,1) \text{ or } z = 2 \text{ or } z = 3 \\ 0, & \text{otherwise} \end{cases}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2,3\}, \{0,1,2,3\}\}.$$

$\mathcal{I}$ *has two theory families,* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, both of which are also theory systems. So it is a systemic* π*-institution.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the GBLF equivalence in* $\mathcal{I}$*, but it does not have the LSGB equivalence in* $\mathcal{I}$*.*

First, note that $\sigma^\flat_\Sigma(\phi,\phi,\psi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, for all* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$*. Thus,* $I^\flat$ *is reflexive in* $\mathcal{I}$*. For global symmetry, the case of* $T = \mathrm{SEN}^\flat$ *is trivial, whereas, for* $T = \mathrm{Thm}(\mathcal{I})$*, observe that, for no* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$*, with* $\phi \neq \psi$*, is it the case that* $\sigma^\flat_\Sigma[\phi,\psi] \leq T$*. Thus, the defining condition holds trivially for* $\mathrm{Thm}(\mathcal{I})$*. So* $I^\flat$ *has the global symmetry in* $\mathcal{I}$*. For local family transitivity, the case of* $T = \mathrm{SEN}^\flat$ *is also trivial and for* $T = \mathrm{Thm}(\mathcal{I})$*, the only pair* $\langle\phi,\psi\rangle \in \mathrm{SEN}^\flat(\Sigma)$*, with* $\phi \neq \psi$*, for which* $\sigma^\flat_\Sigma(\phi,\psi,\xi) \subseteq \mathrm{Thm}_\Sigma(\mathcal{I})$*, for all* $\xi \in \mathrm{SEN}^\flat(\Sigma)$*, is the pair* $(\phi,\psi) = (0,1)$*. So the defining condition holds for* $\mathrm{Thm}(\mathcal{I})$ *also. Thus* $I^\flat$ *has the local family transitivity. We conclude that* $I^\flat$ *has the GBLF equivalence in* $\mathcal{I}$*.*

On the other hand, we have $\sigma^\flat_\Sigma(0,1,\xi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, for all* $\xi \in \mathrm{SEN}^\flat(\Sigma)$*, but* $\sigma^\flat_\Sigma(1,0,0) = 0 \notin \mathrm{Thm}_\Sigma(\mathcal{I})$*. So the implication defining local system symmetry fails for* $\mathrm{Thm}(\mathcal{I})$*. Therefore,* $I^\flat$ *does not have the local system symmetry in* $\mathcal{I}$*.*

Next, we present an example to show that there is π-institution $\mathcal{I}$ with a set of natural transformations that has the LFLS equivalence but not the GBLF equivalence in $\mathcal{I}$.

Example 698 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

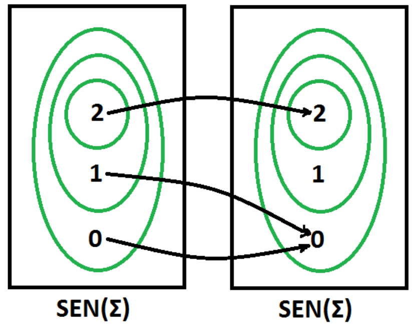
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 0$*,* $1 \mapsto 0$ *and* $2 \mapsto 2$*;*
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given by the following table:*

$\sigma^\flat_\Sigma$	0	1	2
0	2	2	0
1	2	2	1
2	0	1	2

Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}.$$

Note that $\mathcal{I}$ *has three theory families, but only* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$ *are theory systems. So* $\mathcal{I}$ *is not systemic.*



Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the LFLS equivalence in* $\mathcal{I}$*, but it does not have the GBLF equivalence in* $\mathcal{I}$*.*

First, since $\sigma^\flat_\Sigma(\phi,\phi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, for all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,* $I^\flat$ *is reflexive in* $\mathcal{I}$*. Next, observe from the table that, for all* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$*,* $\sigma^\flat_\Sigma(\phi,\psi) = \sigma^\flat_\Sigma(\psi,\phi)$*. Therefore, a fortiori, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*, and all* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$*, if* $\sigma^\flat_\Sigma(\phi,\psi) \in T_\Sigma$*, then* $\sigma^\flat_\Sigma(\psi,\phi) \in T_\Sigma$*, showing that* $I^\flat$ *has the local family symmetry in* $\mathcal{I}$*. For the local system transitivity, the defining implication is trivial in the case of* $\mathrm{SEN}^\flat$*, whereas in the case of* $\mathrm{Thm}(\mathcal{I})$*, it is straightforward to check based on the table defining* $\sigma^\flat_\Sigma$*. Thus,* $I^\flat$ *has indeed the LFLS equivalence in* $\mathcal{I}$*.*

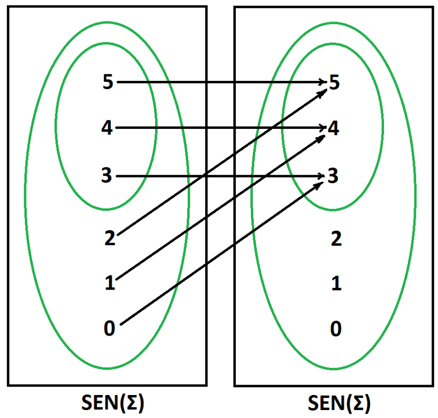
On the other hand, consider the theory family $T = \{\{1,2\}\}$*. We have* $\sigma^\flat_\Sigma(0,1) = 2$ *and* $\sigma^\flat_\Sigma(1,2) = 1$*, i.e.,* $\sigma^\flat_\Sigma(0,1), \sigma^\flat_\Sigma(1,2) \in T_\Sigma$*, whereas* $\sigma^\flat_\Sigma(0,2) = 0 \notin T_\Sigma$*. Therefore,* $I^\flat$ *does not have the local family transitivity and, hence, a fortiori, does not satisfy the GBLF equivalence property in* $\mathcal{I}$*.*

Finally, we present an example to show that there is π-institution $\mathcal{I}$ with a set of natural transformations that has the LFGB equivalence but not the GBLS equivalence in $\mathcal{I}$.

Example 699 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, with* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3,4,5\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2,3,4,5\} \to \{0,1,2,3,4,5\}$ *given by* $0 \mapsto 3$*,* $1 \mapsto 4$*,* $2 \mapsto 5$*,* $3 \mapsto 3$*,* $4 \mapsto 4$ *and* $5 \mapsto 5$*;*
- $N^\flat$ *is the category of natural transformations generated by the single ternary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *defined by letting:* $\sigma^\flat_\Sigma : \{0,1,2,3,4,5\}^3 \to \{0,1,2,3,4,5\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y,z) = \begin{cases} 3, & \textit{if } x = y \textit{ or } \{x,y\} = \{0,1\} \textit{ or } \{x,y\} = \{1,2\} \\ & \quad \textit{or } z = 3 \textit{ or } z = 4 \textit{ or } z = 5 \\ 0, & \textit{otherwise} \end{cases}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be the π-institution determined by

$$\mathcal{C}_\Sigma = \{\{3,4,5\},\{0,1,2,3,4,5\}\}.$$

$\mathcal{I}$ has two theory families, $\mathrm{Thm}(\mathcal{I})$ and $\mathrm{SEN}^\flat$, both of which are also theory systems. So it is a systemic π-institution.

Consider the set $I^\flat = \{\sigma^\flat\}$, with $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ having two distinguished arguments. We show that $I^\flat$ has the LFGB equivalence in $\mathcal{I}$, but it does not have the GBLS equivalence in $\mathcal{I}$.

First, note that $\sigma^\flat_\Sigma(\phi,\phi,\psi) = 3 \in \mathrm{Thm}_\Sigma(\mathcal{I})$, for all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$. Thus, $I^\flat$ is reflexive in $\mathcal{I}$. For local family symmetry, the case of $T = \mathrm{SEN}^\flat$ is trivial, whereas, for $T = \mathrm{Thm}(\mathcal{I})$, observe that, if $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, with $\phi \neq \psi$, are such that $\sigma^\flat_\Sigma(\phi,\psi,\xi) \in \mathrm{Thm}_\Sigma(\mathcal{I})$, for all $\xi \in \mathrm{SEN}^\flat(\Sigma)$, then $\{x,y\} = \{0,1\}$ or $\{x,y\} = \{1,2\}$. Thus, $I^\flat$ is local family symmetric. For global transitivity, the case of $T = \mathrm{SEN}^\flat$ is also trivial and for $T = \mathrm{Thm}(\mathcal{I})$, there is no pair $\langle\phi,\psi\rangle \in \mathrm{SEN}^\flat(\Sigma)$, with $\phi \neq \psi$, for which $\sigma^\flat_\Sigma[\phi,\psi] \subseteq \mathrm{Thm}(\mathcal{I})$. So the defining condition holds trivially for $\mathrm{Thm}(\mathcal{I})$ also. Thus $I^\flat$ has the global transitivity. We conclude that $I^\flat$ has the LFGB equivalence in $\mathcal{I}$.

On the other hand, we have $\sigma^\flat_\Sigma(0,1,\xi) = 3 \in \mathrm{Thm}_\Sigma(\mathcal{I})$ and $\sigma^\flat_\Sigma(1,2,\xi) = 3 \in \mathrm{Thm}_\Sigma(\mathcal{I})$, for all $\xi \in \mathrm{SEN}^\flat(\Sigma)$, but $\sigma^\flat_\Sigma(0,2,0) = 0 \notin \mathrm{Thm}_\Sigma(\mathcal{I})$. So the implication defining local system transitivity fails for $\mathrm{Thm}(\mathcal{I})$. Therefore, $I^\flat$ does not have the local system transitivity in $\mathcal{I}$.

We close the study of equivalence by providing, again, a transfer property.

Corollary 700 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $I^\flat$ a collection of natural transformations $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with two distinguished arguments, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. $I^\flat$ has a transitivity property in $\mathcal{I}$ if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, I has the corresponding transitivity property in $\mathcal{A}$.*

Proof: This follows directly from Propositions 688 and 693. ■

10.6 Antisymmetry

We look next at the antisymmetry property.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. We say that:

- $I^\flat$ has the **local antisymmetry in** $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq \mathrm{Thm}_\Sigma(\mathcal{I})$ and $I^\flat_\Sigma(\psi, \phi, \vec{\xi}) \subseteq \mathrm{Thm}_\Sigma(\mathcal{I})$, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$, imply $\phi = \psi$;

- $I^\flat$ has the **global antisymmetry in** $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma[\phi, \psi] \leq \mathrm{Thm}(\mathcal{I})$ and $I^\flat_\Sigma[\psi, \phi] \leq \mathrm{Thm}(\mathcal{I})$ imply $\phi = \psi$.

The antisymmetry properties stratify in the following hierarchy.

Proposition 701 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. If $I^\flat$ has the local antisymmetry in $\mathcal{I}$, then it has the global antisymmetry in $\mathcal{I}$.*

Proof: Suppose that $I^\flat$ has the local antisymmetry and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $I^\flat_\Sigma[\phi, \psi] \leq \mathrm{Thm}(\mathcal{I})$ and $I^\flat_\Sigma[\psi, \phi] \leq \mathrm{Thm}(\mathcal{I})$. Then we get, in particular, that, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq \mathrm{Thm}_\Sigma(\mathcal{I})$ and $I^\flat_\Sigma(\psi, \phi, \xi) \subseteq \mathrm{Thm}_\Sigma(\mathcal{I})$. Thus, by local antisymmetry, we obtain $\phi = \psi$. We conclude that $I^\flat$ has the global antisymmetry in $\mathcal{I}$. ■

Proposition 701 has established the following hierarchy of antisymmetry properties:

Local Antisymmetry

↓

Global Antisymmetry

We look, next, at a natural sufficient condition under which the antisymmetry properties coincide.

Proposition 702 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. If $I^\flat$ has only two arguments (i.e., is parameter free), then the local antisymmetry and the global antisymmetry properties coincide.*

Proof: Suppose that $I^\flat$ is parameter free and has the global antisymmetry in $\mathcal{I}$. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $I^\flat_\Sigma(\phi,\psi) \subseteq \mathrm{Thm}_\Sigma(\mathcal{I})$ and $I^\flat_\Sigma(\psi,\phi) \subseteq \mathrm{Thm}_\Sigma(\mathcal{I})$. Then, by Proposition 99, $I^\flat_\Sigma[\phi,\psi] \leq \mathrm{Thm}(\mathcal{I})$ and $I^\flat_\Sigma[\psi,\phi] \leq \mathrm{Thm}(\mathcal{I})$. Thus, by global antisymmetry, $\phi = \psi$. Therefore, $I^\flat$ has the local antisymmetry in $\mathcal{I}$. ∎

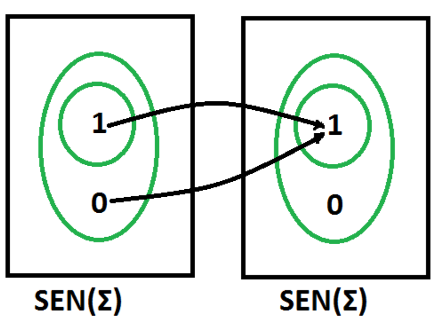
So in the case of a parameter-free set of natural transformations we have a single antisymmetry property.

We provide an example to show that the implication of Proposition 701 is not an equivalence in general. That is, we provide an example of a π-institution $\mathcal{I}$ with a set $I^\flat$ of natural transformations, with two distinguished arguments, that has the global antisymmetry but not the local antisymmetry in $\mathcal{I}$.

Example 703 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single objects* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1\} \to \{0,1\}$ *given by* $0 \mapsto 1$ *and* $1 \mapsto 1$*;*
- $N^\flat$ *is the category of natural transformations generated by the single ternary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1\}^3 \to \{0,1\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y,z) = \begin{cases} 0, & \textit{if } (x,y,z) = (1,1,0) \\ 1, & \textit{otherwise} \end{cases}$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\}$*. Note that there are two theory families,* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, both of which are theory systems.*

Consider the set $I^\flat = \{\sigma^\flat\}$, *with* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the global antisymmetry in* $\mathcal{I}$, *but it does not have the local antisymmetry in* $\mathcal{I}$.

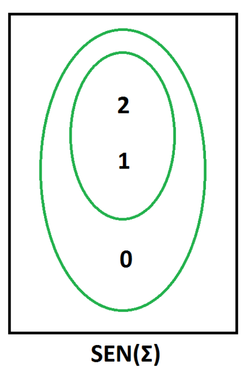
To see that $I^\flat$ *has the global antisymmetry in* $\mathcal{I}$, *it suffices to notice that, for no* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$ *is it the case that* $I^\flat_\Sigma[\phi, \psi] \leq \mathrm{Thm}(\Sigma)$. *Therefore, the defining condition holds vacuously, for all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$.

On the other hand, for $0 \neq 1$, *we have* $\sigma^\flat_\Sigma(0, 1, \xi) = \sigma^\flat_\Sigma(1, 0, \xi) = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$, *for all* $\xi \in \{0, 1\}$. *So* $I^\flat$ *is not locally antisymmetric in* $\mathcal{I}$.

To close the study of antisymmetry properties, we show that they do not transfer from π-institutions to their models. This is to be expected, since the inverse image $\alpha^{-1}(T)$ of the minimum $\mathcal{I}$ filter family of a π-institution $\mathcal{I}$ on an algebraic system $\mathcal{A}$ may not coincide with the theorem system of $\mathcal{I}$.

Example 704 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with a single object* Σ;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$;
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting:* $\sigma^\flat_\Sigma : \{0, 1, 2\}^2 \to \{0, 1, 2\}$ *be given by* $\sigma^\flat_\Sigma(x, y) = 0$, *for all* $x, y \in \mathrm{SEN}^\flat(\Sigma)$.



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{1, 2\}, \{0, 1, 2\}\}.$$

Consider the set $I^\flat = \{\sigma^\flat\}$, *with* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the local antisymmetry in* $\mathcal{I}$, *but that there exists an* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, *with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, *such that* I *does not have the local antisymmetry in* $\mathcal{A}$.

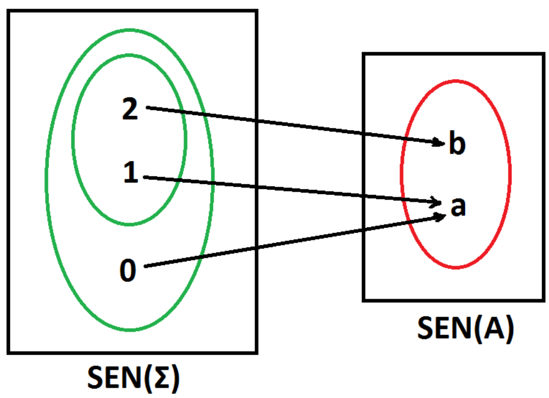
Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *be the algebraic system determined as follows:*

- **Sign** *is the trivial category with a single object* A*;*
- $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}(A) = \{a, b\}$*;*
- N *is the category of natural transformations generated by the single binary natural transformation* $\sigma : \mathrm{SEN}^2 \to \mathrm{SEN}$ *defined by letting:* $\sigma_A : \{a,b\}^2 \to \{a,b\}$ *be given by* $\sigma_A(x,y) = a$*, for all* $x, y \in \mathrm{SEN}(A)$*.*

$\mathbf{A}$ *is an* $N^\flat$*-algebraic system, as can be seen by sending* $\sigma^\flat \mapsto \sigma$ *and extending to categories by composition.*

Now let $\langle F, \alpha\rangle : \mathbf{F} \to \mathbf{A}$ *be the morphism defined as follows:*

- $F : \mathbf{Sign}^\flat \to \mathbf{Sign}$ *is the obvious functor between trivial categories;*
- $\alpha : \mathrm{SEN}^\flat \to \mathrm{SEN} \circ F$ *is defined by setting* $\alpha_\Sigma(0) = a$*,* $\alpha_\Sigma(1) = a$ *and* $\alpha_\Sigma(2) = b$*.*



Our goal is to show that $I^\flat$ *has the local antisymmetry in* $\mathcal{I}$ *but that* I *does not have the local antisymmetry in* $\mathcal{A}$*. We have, for all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,* $\sigma^\flat_\Sigma(\phi,\psi) \notin \mathrm{Thm}_\Sigma(\mathcal{I})$ *and* $\sigma^\flat_\Sigma(\psi,\phi) \notin \mathrm{Thm}_\Sigma(\mathcal{I})$*, whence the defining condition of local antisymmetry for* $I^\flat$ *is vacuously true. So* $I^\flat$ *is locally antisymmetric in* $\mathcal{I}$*.*

On the other hand, note that the least $\mathcal{I}$*-filter system on* $\mathcal{A}$ *is* SEN*. Moreover, we have* $\sigma_A(a,b) = \sigma_A(b,a) = a \in \mathrm{SEN}(A)$*, with* $a \neq b$*. Thus* $I = \{\sigma\}$ *does not have local antisymmetry in* $\mathcal{A}$*.*

10.7 Order

We look next at sets of natural transformations $I^\flat$, with two distinguished arguments, that define (modulo theory families) partial order families on the underlying algebraic system of a π-institution $\mathcal{I}$. We assume that $I^\flat$ has the reflexivity property and study combinations of possible antisymmetry and transitivity properties that the set of connectives may or may not possess.

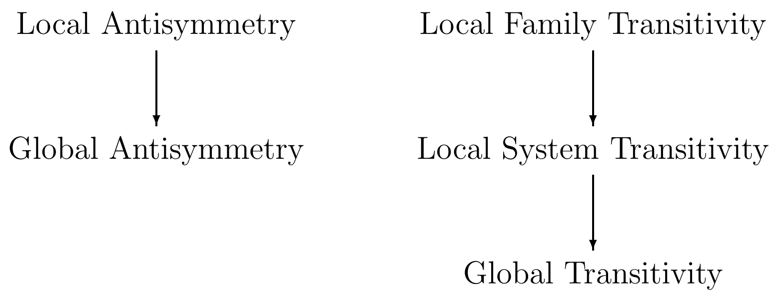
Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. Let $X \in \{LC, GB\}$, where LC and GB stand for "LoCal" and "GloBal", respectively, and let $Y \in \{LF, LS, GB\}$, where LF stands for "Local Family" and LS for "Local System". We say that $I^\flat$ has the XY **poset property in** $\mathcal{I}$ if it has

(a) reflexivity in $\mathcal{I}$;

(b) X antisymmetry in $\mathcal{I}$ and

(c) Y transitivity in $\mathcal{I}$.

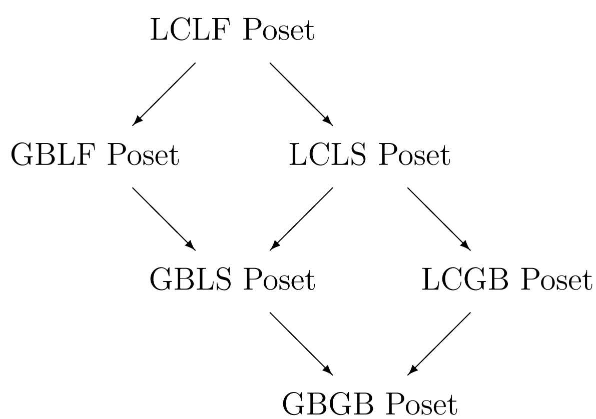
Recall, again, the following hierarchies of antisymmetry and of transitivity properties:



From these, we can infer the following hierarchy of equivalence properties:

Corollary 705 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. The six poset properties of $I^\flat$ satisfy the hierarchy depicted in the accompanying diagram.*

Proof: The statement is a direct consequence of Propositions 701 and 689. ■



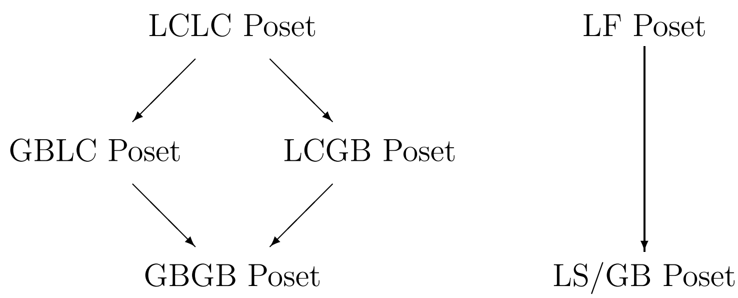
Based on the analyses performed on antisymmetry and transitivity, we have the following result regarding natural sufficient conditions under which some of these poset properties coincide.

Corollary 706 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$*, with two distinguished arguments.*

(a) If $\mathcal{I}$ *is systemic, then the equivalence hierarchy collapses to the one depicted on the left of the accompanying diagram;*

(b) If $I^\flat$ *has only two arguments (i.e., is parameter free), then the equivalence hierarchy collapses to the one depicted on the right of the diagram, where, since there is only one antisymmetry property, the qualifications refer to the type of transitivity that holds.*

Proof: The statement follows directly from Propositions 690 and 702. ■



For a systemic π-institution with a parameter-free set of natural transformations, there is only one poset property, since the two versions of antisymmetry and all three versions of transitivity collapse, respectively, to a single property.

We provide some examples to show that the implications of Corollary 705 are not equivalences, i.e., the six classes of the poset hierarchy are all distinct in general.

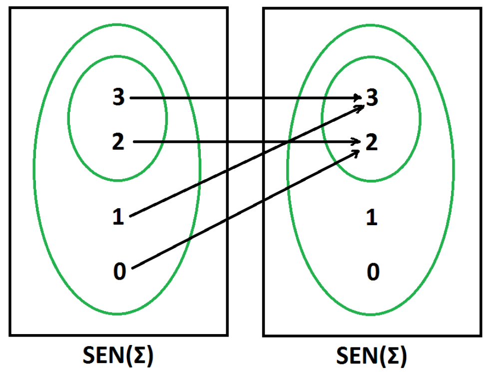
First, we present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that has the GBLF poset property, but not the LCGB poset property in $\mathcal{I}$.

Example 707 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2,3\} \to \{0,1,2,3\}$ *given by* $0 \mapsto 2$*,* $1 \mapsto 3$*,* $2 \mapsto 2$ *and* $3 \mapsto 3$*;*
- $N^\flat$ *is the category of natural transformations generated by the single ternary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *defined by letting*

$\sigma^\flat_\Sigma : \{0,1,2,3\}^3 \to \{0,1,2,3\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y,z) = \begin{cases} 2, & \text{if } x = y \text{ or } (x,y) = (0,1) \text{ or } (x,y) = (1,0) \\ & \quad \text{or } z = 2 \text{ or } z = 3 \\ 0, & \text{otherwise} \end{cases}$$



Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2,3\},\{0,1,2,3\}\}.$$

Note that both theory families, $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, are also theory systems. So* $\mathcal{I}$ *is a systemic* π*-institution.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the GBLF poset property in* $\mathcal{I}$*, but it does not have the LCGB poset property in* $\mathcal{I}$*.*

Note, first, that reflexivity is obvious, since, by definition, for all $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,* $\sigma^\flat_\Sigma(\phi,\phi,\xi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, for all* $\xi \in \mathrm{SEN}^\flat(\Sigma)$*. For global antisymmetry, note that if, for some* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$*,* $\sigma^\flat_\Sigma[\phi,\psi] \leq \mathrm{Thm}(\mathcal{I})$ *and* $\sigma^\flat_\Sigma[\psi,\phi] \leq \mathrm{Thm}(\mathcal{I})$*, then we must have* $\phi = \psi$*. Finally, for local family transitivity, the defining equation holds trivially for* $T = \mathrm{SEN}^\flat$*, whereas, if for some* $\phi,\psi,\chi \in \mathrm{SEN}^\flat(\Sigma)$*, with* $\phi \neq \chi$*,* $\sigma^\flat_\Sigma(\phi,\psi,\xi) \in \{2,3\}$ *and* $\sigma^\flat_\Sigma(\psi,\chi,\xi) \in \{2,3\}$*, for all* $\xi \in \mathrm{SEN}^\flat(\Sigma)$*, we must have* $\phi = \psi$ *or* $\psi = \chi$*, whence the condition is satisfied in this case as well. Thus,* $I^\flat$ *is also locally family transitive in* $\mathcal{I}$ *and, therefore, has the GBLF poset property in* $\mathcal{I}$*.*

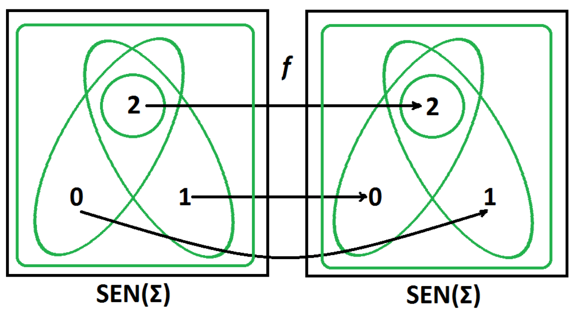
On the other hand, since $\sigma^\flat_\Sigma(0,1,\xi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$ *and* $\sigma^\flat_\Sigma(1,0,\xi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, for all* $\xi \in \mathrm{SEN}^\flat(\Sigma)$*,* $I^\flat$ *does not have the local antisymmetry property. A fortiori,* $I^\flat$ *does not have the LCGB poset property in* $\mathcal{I}$*.*

Next, we present an example to show that there is a π-institution $\mathcal{I}$ with a set of natural transformations that has the LCLS poset property but not the GBLF poset property in $\mathcal{I}$.

Example 708 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$, *with* $f \circ f = i_\Sigma$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 1$, $1 \mapsto 0$ *and* $2 \mapsto 2$;
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting:* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y,z) = \begin{cases} 0, & \text{if } (x,y) = (0,1) \text{ or } (x,y) = (1,2) \\ 1, & \text{if } (x,y) = (1,0) \text{ or } (x,y) = (0,2) \\ 2, & \text{otherwise} \end{cases}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by*

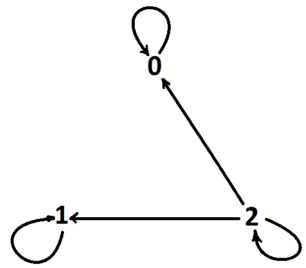
$$\mathcal{C}_\Sigma = \{\{2\},\{0,2\},\{1,2\},\{0,1,2\}\}.$$

$\mathcal{I}$ *has four theory families, but only* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$ *are theory systems. So* $\mathcal{I}$ *is not systemic.*

Consider the set $I^\flat = \{\sigma^\flat\}$, *with* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the LCLS poset property but it does not have the GBLF poset property in* $\mathcal{I}$.

First, note that $\sigma^\flat_\Sigma(\phi,\phi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$, *for all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$. *Thus,* $I^\flat$ *is reflexive in* $\mathcal{I}$. *For the local antisymmetry, note that for no* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, *with* $\phi \neq \psi$ *is it the case that both* $\sigma^\flat_\Sigma(\phi,\psi) = \sigma^\flat_\Sigma(\psi,\phi) = 2$. *Finally, for the local system transitivity, the defining condition is trivially satisfied for* $\mathrm{SEN}^\flat$, *whereas the pairs related modulo* $\mathrm{Thm}(\mathcal{I})$ *are as in the following diagram, an examination of which verifies transitivity. Therefore* $I^\flat$ *is also locally system transitive in* $\mathcal{I}$ *and, hence has the LCLS poset property in* $\mathcal{I}$.

On the other hand, for $T = \{\{0,2\}\} \in \mathrm{ThFam}(\mathcal{I})$, *we have* $\sigma^\flat_\Sigma(0,1) = \sigma^\flat_\Sigma(1,2) = 0 \in T_\Sigma$, *whereas* $\sigma^\flat_\Sigma(0,2) = 1 \notin T_\Sigma$. *So the implication defining local family transitivity fails for* T. *Therefore,* $I^\flat$ *does not have the local family transitivity and, a fortiori, does not have the GBLF poset property in* $\mathcal{I}$.

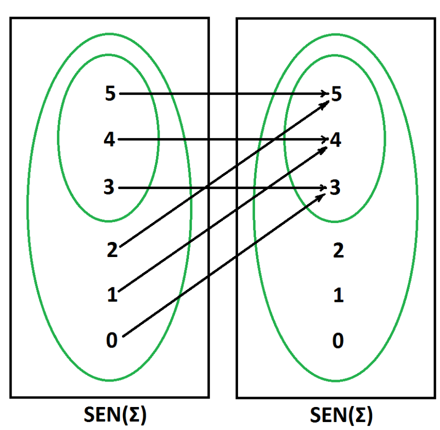


Finally, we look at an example that shows that there is π-institution $\mathcal{I}$ with a set of natural transformations that has the LCGB poset property but not the GBLS poset property in $\mathcal{I}$.

Example 709 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$, *with* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3,4,5\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2,3,4,5\} \to \{0,1,2,3,4,5\}$ *given by* $0 \mapsto 3$, $1 \mapsto 4$, $2 \mapsto 5$, $3 \mapsto 3$, $4 \mapsto 4$ *and* $5 \mapsto 5$;
- $N^\flat$ *is the category of natural transformations generated by the single ternary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *defined by letting:* $\sigma^\flat_\Sigma : \{0,1,2,3,4,5\}^3 \to \{0,1,2,3,4,5\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y,z) = \begin{cases} 3, & \text{if } x = y \text{ or } (x,y) = (0,1) \text{ or } (x,y) = (1,2) \\ & \quad \text{or } z = 3 \text{ or } z = 4 \text{ or } z = 5 \\ 0, & \text{otherwise} \end{cases}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{3,4,5\}, \{0,1,2,3,4,5\}\}.$$

$\mathcal{I}$ has two theory families, $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, both of which are also theory systems. So it is a systemic π-institution.*

Consider the set $I^\flat = \{\sigma^\flat\}$, with $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ having two distinguished arguments. We show that $I^\flat$ has the LCGB poset property in $\mathcal{I}$, but not the GBLS poset property in $\mathcal{I}$.

First, note that, for all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\sigma^\flat_\Sigma(\phi,\phi,\xi) = 3 \in \mathrm{Thm}_\Sigma(\mathcal{I})$, for all $\xi \in \mathrm{SEN}^\flat(\Sigma)$. Thus, $I^\flat$ is reflexive in $\mathcal{I}$. For local antisymmetry, it suffices to observe that, for no $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, with $\phi \neq \psi$, is it the case that $\sigma^\flat_\Sigma(\phi,\psi,\xi) \in \mathrm{Thm}_\Sigma(\mathcal{I})$ and $\sigma^\flat_\Sigma(\psi,\phi,\xi) \in \mathrm{Thm}_\Sigma(\mathcal{I})$, for all $\xi \in \mathrm{SEN}^\flat(\Sigma)$. For global transitivity, the defining condition holds trivially for $T = \mathrm{SEN}^\flat$, whereas for $T = \mathrm{Thm}(\mathcal{I})$, it suffices to note that, for no $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, with $\phi \neq \psi$, is it the case that $\sigma^\flat_\Sigma[\phi,\psi] \leq \mathrm{Thm}(\mathcal{I})$. We conclude that $I^\flat$ has the LCGB poset property in $\mathcal{I}$.

On the other hand, we have $\sigma^\flat_\Sigma(0,1,\xi) = 3 \in \mathrm{Thm}_\Sigma(\mathcal{I})$ and $\sigma^\flat_\Sigma(1,2,\xi) = 3 \in \mathrm{Thm}_\Sigma(\mathcal{I})$, for all $\xi \in \mathrm{SEN}^\flat(\Sigma)$, but $\sigma^\flat_\Sigma(0,2,0) = 0 \notin \mathrm{Thm}_\Sigma(\mathcal{I})$. So the implication defining local system transitivity fails for $\mathrm{Thm}(\mathcal{I})$*. Therefore, $I^\flat$ does not have the local system transitivity in $\mathcal{I}$ and, hence, a fortiori, it does not have the GBLS poset property in $\mathcal{I}$.*

Because of the non-transference of antisymmetry, which was shown in Example 704, it is to be expected that none of the poset properties transfers from a π-institution to its models. We provide an example that showcases a π-institution $\mathcal{I}$, with a set $I^\flat$ of natural transformations having two distinguished arguments, that has the LCLF poset property, but one of whose models does not have the GBGB poset property.

Example 710 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be the algebraic system determined as follows:*

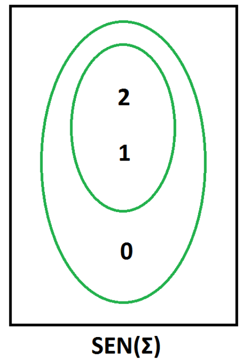
- $\mathbf{Sign}^\flat$ *is the trivial category with a single object Σ;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$*;*
- *$N^\flat$ is the category of natural transformations generated by the single binary natural transformation $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ defined by letting: $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ be given, for all $x,y \in \mathrm{SEN}^\flat(\Sigma)$, by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}.$$

Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be the π-institution determined by $\mathcal{C}_\Sigma = \{\{1,2\},\{0,1,2\}\}$.

Consider the set $I^\flat = \{\sigma^\flat\}$, with $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ having two distinguished arguments. We show that $I^\flat$ has the LCLF poset property in $\mathcal{I}$, but that there exists an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, such that I does not have the GBGB poset property in $\mathcal{A}$.

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be the algebraic system determined as follows:



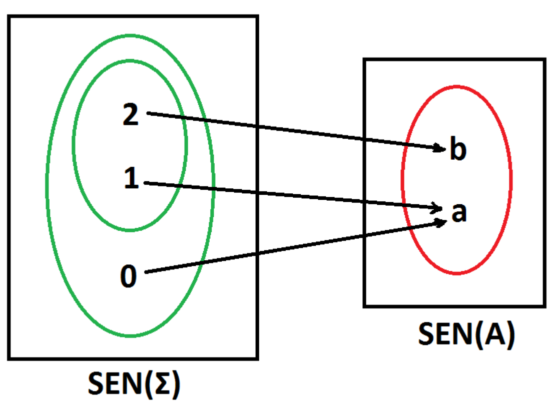
- **Sign** *is the trivial category with a single object* A*;*
- $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}(A) = \{a,b\}$*;*
- N *is the category of natural transformations generated by the single binary natural transformation* $\sigma : \mathrm{SEN}^2 \to \mathrm{SEN}$ *defined by letting:* $\sigma_A : \{a,b\}^2 \to \{a,b\}$ *be given, for all* $x,y \in \mathrm{SEN}(A)$*, by*

$$\sigma_A(x,y) = a.$$

$\mathbf{A}$ *is an* $N^\flat$*-algebraic system, as can be seen by sending* $\sigma^\flat \mapsto \sigma$ *and extending to categories by composition.*

Now let $\langle F,\alpha\rangle : \mathbf{F} \to \mathbf{A}$ *be the morphism defined as follows:*

- $F : \mathbf{Sign}^\flat \to \mathbf{Sign}$ *is the obvious functor between trivial categories;*
- $\alpha : \mathrm{SEN}^\flat \to \mathrm{SEN} \circ F$ *is defined by setting* $\alpha_\Sigma(0) = a$*,* $\alpha_\Sigma(1) = a$ *and* $\alpha_\Sigma(2) = b$*.*



We show that $I^\flat$ *has the LCLF poset property in* $\mathcal{I}$ *but that* I *does not have the GBGB poset property in* $\mathcal{A}$*.*

First, since $\sigma^\flat_\Sigma(\phi,\phi) = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$, *we have that* $I^\flat$ *is reflexive in* $\mathcal{I}$. *Second, if* $\sigma^\flat_\Sigma(\phi,\psi) = 1 = \sigma^\flat_\Sigma(\psi,\phi)$, *then* $\phi = \psi$. *So* $I^\flat$ *has the local antisymmetry in* $\mathcal{I}$. *Finally, local family transitivity is obvious, since for no* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, *with* $\phi \neq \psi$ *is it the case that* $\sigma^\flat_\Sigma(\phi,\psi) = 1$. *We conclude that* $I^\flat$ *has the LCLF poset property in* $\mathcal{I}$.

On the other hand, note that the least $\mathcal{I}$*-filter system on* $\mathcal{A}$ *is* SEN *and since* $\sigma_A(a,b) = \sigma_A(b,a) = a \in \mathrm{SEN}(A)$, *with* $a \neq b$, $I = \{\sigma\}$ *does not have the global antisymmetry in* $\mathcal{A}$. *So, a fortiori, it does not have the GBGB poset property in* $\mathcal{A}$.

10.8 Compatibility

We look next at various versions of the compatibility property, taking again into account both the duality between local versus global membership and the difference between considering all theory families versus restricting only to theory systems.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. We say that:

- $I^\flat$ has the **local family compatibility in** $\mathcal{I}$ if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, $\overleftrightarrow{I^\flat}_\Sigma(\phi,\psi,\vec{\xi}) \subseteq T_\Sigma$, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$, implies
$$I^\flat_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}),\vec{\xi}) \subseteq T_{\Sigma'},$$
for all $\sigma^\flat \in N^\flat$, $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and $\vec{\chi},\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$;

- $I^\flat$ has the **local system compatibility in** $\mathcal{I}$ if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, $\overleftrightarrow{I^\flat}_\Sigma(\phi,\psi,\vec{\xi}) \subseteq T_\Sigma$, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$, implies
$$I^\flat_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}),\vec{\xi}) \subseteq T_{\Sigma'},$$
for all $\sigma^\flat \in N^\flat$, $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and $\vec{\chi},\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$;

- $I^\flat$ has the **global family compatibility in** $\mathcal{I}$ if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, $\overleftrightarrow{I^\flat}_\Sigma[\phi,\psi] \leq T$ implies
$$I^\flat_{\Sigma'}[\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi})] \leq T,$$
for all $\sigma^\flat \in N^\flat$, $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$;

- $I^\flat$ has the **global system compatibility in** $\mathcal{I}$ if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi] \leq T$ implies

$$I^\flat_{\Sigma'}[\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}), \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi})] \leq T,$$

for all $\sigma^\flat \in N^\flat$, $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$.

The following proposition establishes a hierarchy of compatibility properties.

Proposition 711 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$*, with two distinguished arguments.*

(a) If $I^\flat$ *has the local family compatibility, then it has the local system compatibility in* $\mathcal{I}$*;*

(b) If $I^\flat$ *has the local system compatibility, then it has the global family compatibility in* $\mathcal{I}$*;*

(c) $I^\flat$ *has the global family compatibility in* $\mathcal{I}$ *if and only if it has the global system compatibility in* $\mathcal{I}$*.*

Proof: Part (a) and one of the implications in Part (c) follow directly from the fact that every theory system of $\mathcal{I}$ is also a theory family of $\mathcal{I}$.

For Part (b), suppose that $I^\flat$ has the local system compatibility in $\mathcal{I}$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi] \leq T$. Then by Lemma 93, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\overleftrightarrow{I^\flat}_{\Sigma'}[\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi)] \leq T.$$

This implies, by Lemma 99, that, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\overleftrightarrow{I^\flat}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), \vec{\xi}) \subseteq \overleftarrow{T}_{\Sigma'}.$$

Since $I^\flat$ has the local system compatibility and, by Proposition 42, $\overleftarrow{T} \in \mathrm{ThSys}(\mathcal{I})$, we get that, for all $\sigma^\flat \in N^\flat$, all $\Sigma'' \in |\mathbf{Sign}^\flat|$, all $g \in \mathbf{Sign}^\flat(\Sigma', \Sigma'')$

$$\Sigma \xrightarrow{\ f\ } \Sigma' \xrightarrow{\ g\ } \Sigma''$$

and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$ and $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma'')$,

$$\begin{aligned} I^\flat_{\Sigma''}(&\sigma^\flat_{\Sigma''}(\mathrm{SEN}^\flat(gf)(\phi), \mathrm{SEN}^\flat(g)(\vec{\chi})), \\ &\sigma^\flat_{\Sigma''}(\mathrm{SEN}^\flat(gf)(\psi), \mathrm{SEN}^\flat(g)(\vec{\chi})), \vec{\xi}) \subseteq T_{\Sigma''}, \end{aligned}$$

or, equivalently,

$$I^\flat_{\Sigma''}(\mathrm{SEN}^\flat(g)(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi})), \mathrm{SEN}^\flat(g)(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi})),\vec{\xi}) \subseteq T_{\Sigma''}.$$

Since this holds for all $\Sigma'' \in |\mathbf{Sign}^\flat|$, $g \in \mathbf{Sign}^\flat(\Sigma',\Sigma'')$ and $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma'')$, we conclude that $I^\flat_{\Sigma'}[\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi})] \leq T$. Therefore $I^\flat$ has the global family compatibility in $\mathcal{I}$.

Suppose, finally, that $I^\flat$ has the global system compatibility in $\mathcal{I}$ and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\overleftrightarrow{I^\flat}_\Sigma[\phi,\psi] \leq T$. By Lemma 99, we get that $\overleftrightarrow{I^\flat}_\Sigma[\phi,\psi] \leq \overleftarrow{T}$. Since $I^\flat$ has the global system compatibility and, by Proposition 42, $\overleftarrow{T} \in \mathrm{ThSys}(\mathcal{I})$, we get that, for all $\sigma^\flat \in N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$, $I^\flat_{\Sigma'}[\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi})] \leq \overleftarrow{T}$. Using again Lemma 99, we conclude that $I^\flat_{\Sigma'}[\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi})] \leq T$. Therefore, $I^\flat$ has the global family compatibility in $\mathcal{I}$. ∎

Proposition 711 has established the following hierarchy of compatibility properties:

Local Family Compatibility

↓

Local System Compatibility

↓

Global Compatibility

We look, next, at some natural sufficient conditions under which some of these three compatibility properties coincide.

Proposition 712 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$*, with two distinguished arguments.*

(a) If $\mathcal{I}$ *is systemic, then the local family and the local system compatibility coincide;*

(b) If $I^\flat$ *has only two arguments (i.e., is parameter free), then the local system compatibility and the global compatibility coincide.*

Proof: If $\mathcal{I}$ is systemic, then all theory families are theory systems and, hence, the local family and local system compatibility properties coincide.

Suppose, next that $I^\flat$ is parameter free and has the global system compatibility in $\mathcal{I}$. Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$\overleftrightarrow{I}^\flat{}_\Sigma(\phi,\psi) \subseteq T_\Sigma$. Then, by Proposition 99, $\overleftrightarrow{I}^\flat{}_\Sigma[\phi,\psi] \leq T$. Thus, by the global system compatibility, for all $\sigma^\flat \in N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$, $I^\flat_{\Sigma'}[\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi})] \leq T$, which implies that, for all $\sigma^\flat \in N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$, $I^\flat_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi})) \subseteq T_{\Sigma'}$. Therefore, $I^\flat$ has the local system compatibility in $\mathcal{I}$. ■

So in the case of a systemic π-institution $\mathcal{I}$, we have the hierarchy pictured on the left, whereas in the case of a parameter-free set of natural transformations we have the hierarchy on the right.

Local Compatibility	Local Family Compatibility
↓	↓
Global Compatibility	Local System/Global Compatibility

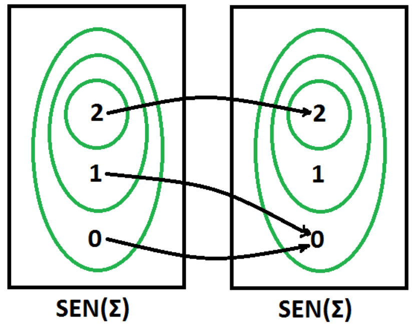
Of course, for a systemic π-institution with a parameter-free set of natural transformations all four compatibility properties coincide.

We provide some examples to show that the implications of Proposition 711 are not equivalences in general, i.e., in the 3-class hierarchy all inclusions of classes of π-institutions with a set of natural transformations satisfying the corresponding compatibility properties are proper inclusions.

We first present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that has the local system compatibility but not the local family compatibility in $\mathcal{I}$.

Example 713 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single objects* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 0$*,* $1 \mapsto 0$ *and* $2 \mapsto 2$*;*
- $N^\flat$ *is the category of natural transformations generated by two binary natural transformations:*
  - $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given by*
  $$\sigma^\flat_\Sigma(x,y) = \begin{cases} 2, & \text{if } x = 2 \text{ or } y = 2 \\ 1, & \text{if } \{x,y\} = \{0,1\} \\ 0, & \text{otherwise} \end{cases}$$
  - $\lambda^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\lambda^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given by*
  $$\lambda^\flat_\Sigma(x,y) = \begin{cases} 2, & \text{if } x = 2 \text{ or } y = 2 \\ 0, & \text{otherwise} \end{cases}$$



Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be the π-institution determined by

$$\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}.$$

Note that there are three theory families, but only $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$ *are theory systems.*

Consider the set $I^\flat = \{\sigma^\flat\}$, with $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ having two distinguished arguments. We show that $I^\flat$ has the local system compatibility in $\mathcal{I}$, but it does not have the local family compatibility in $\mathcal{I}$.

For the local system compatibility note that, if $T = \mathrm{SEN}^\flat$, then the defining implication is trivially true. If, on the other hand, $T = \mathrm{Thm}(\mathcal{I})$, then $\sigma^\flat_\Sigma(\phi,\psi) = 2$ if and only if $\phi = 2$ or $\psi = 2$. But then we get, for all $\chi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\sigma^\flat_\Sigma(\sigma^\flat_\Sigma(\phi,\chi),\sigma^\flat_\Sigma(\psi,\chi)) = 2,$$
$$\sigma^\flat_\Sigma(\lambda^\flat_\Sigma(\phi,\chi),\lambda^\flat_\Sigma(\psi,\chi)) = 2.$$

So $I^\flat$ has the local system compatibility in $\mathcal{I}$.

To see that $I^\flat$ does not have the local family compatibility in $\mathcal{I}$, consider the theory family $T = \{\{1,2\}\}$. We have $\sigma^\flat_\Sigma(0,1) = \sigma^\flat_\Sigma(1,0) = 1 \in T_\Sigma$, but

$$\sigma^\flat_\Sigma(\lambda^\flat_\Sigma(1,0),\lambda^\flat_\Sigma(0,0)) = \sigma^\flat_\Sigma(0,0) = 0 \notin T_\Sigma.$$

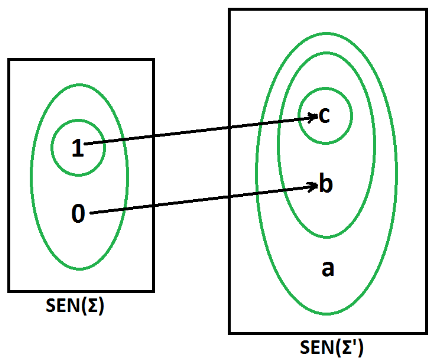
Therefore, the implication defining local family compatibility fails for $T = \{\{1,2\}\}$. So $I^\flat$ does not have locally family compatibility in $\mathcal{I}$.

Next, we present an example to show that there is π-institution $\mathcal{I}$ with a set of natural transformations that has the global (family) compatibility but not the local system compatibility in $\mathcal{I}$.

Example 714 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with two objects Σ, Σ' and a single (non-identity) morphism $f : \Sigma \to \Sigma'$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b,c\}$ and $\mathrm{SEN}^\flat(f) : \{0,1\} \to \{a,b,c\}$ given by $0 \mapsto b$, $1 \mapsto c$;*

- $N^\flat$ *is the category of natural transformations generated by one ternary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$, *defined as follows:*
  - $\sigma^\flat_\Sigma : \{0,1\}^3 \to \{0,1\}$ *be given by*
  $$\sigma^\flat_\Sigma(x,y,z) = 0, \text{ for all } x,y,z \in \{0,1\};$$
  - $\sigma^\flat_{\Sigma'} : \{a,b,c\}^3 \to \{a,b,c\}$ *be given by*
  $$\sigma^\flat_{\Sigma'}(x,y,z) = \begin{cases} a, & \text{if } a \in \{x,y,z\} \\ b, & \text{otherise} \end{cases}$$



Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\}, \quad \mathcal{C}_{\Sigma'} = \{\{c\},\{b,c\},\{a,b,c\}\}.$$

$\mathrm{Thm}(\mathcal{I})$ *has six theory families all of which, except* $\{\{0,1\},\{c\}\}$, *are theory systems.*

Consider the set $I^\flat = \{\sigma^\flat\}$, *with* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the global family compatibility in* $\mathcal{I}$, *but it does not have the local system compatibility in* $\mathcal{I}$.

For the global compatibility note that, if, for some $x,y \in \mathrm{SEN}^\flat(\Sigma)$, *we have* $\sigma^\flat_\Sigma[x,y] \leq T$, *then* $T = \mathrm{SEN}^\flat$. *Similarly, if, for some* $x,y \in \mathrm{SEN}^\flat(\Sigma')$, $\sigma^\flat_{\Sigma'}[x,y] \leq T$, *then* $T_{\Sigma'} = \{a,b,c\}$. *In both cases, the conclusion of the defining implication is trivially true. So* $I^\flat$ *has the global compatibility in* $\mathcal{I}$.

On the other hand, consider the theory system $T = \{\{0,1\},\{b,c\}\}$. *Let* $\phi = 0$ *and* $\psi = 1$. *Then, we have, for all* $z \in \{0,1\}$,

$$\sigma^\flat_\Sigma(0,1,z) = \sigma^\flat_\Sigma(1,0,z) = 0 \in T_\Sigma.$$

On the contrary,

$$\sigma^\flat_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(0),c,c), \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(1),c,c),a) = a \notin T_{\Sigma'}.$$

Therefore, the implication defining local system compatibility fails for T. *So* $I^\flat$ *does not have the local system compatibility in* $\mathcal{I}$.

We close by proving that all three compatibility properties transfer from π-institutions to their models.

Proposition 715 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $I^\flat$ *a collection of natural transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, with two distinguished arguments, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $I^\flat$ *has a compatibility property in* $\mathcal{I}$ *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*,* I *has the corresponding compatibility property in* $\mathcal{A}$*.*

Proof: If I has a compatibility property in $\mathcal{A}$, for all $\mathcal{A}$, then it has the same compatibility property in $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$. Since $\langle \mathbf{F}, C^{\mathcal{I},\mathcal{F}}\rangle = \mathcal{I}$, we conclude that $I^\flat$ has the corresponding compatibility in $\mathcal{I}$.

Suppose, conversely, that $I^\flat$ has a compatibility property in $\mathcal{I}$. We look at each of the three properties in turn.

(a) Suppose $I^\flat$ has the local family compatibility in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, be an $\mathbf{F}$-algebraic system, let $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$\overleftrightarrow{I}_{F(\Sigma)}(\alpha_\Sigma(\phi), \alpha_\Sigma(\psi), \alpha_\Sigma(\vec{\xi})) \subseteq T_{F(\Sigma)},$$

for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$. Since this is equivalent to $\alpha_\Sigma(\overleftrightarrow{I^\flat}_\Sigma(\phi, \psi, \vec{\xi})) \subseteq T_{F(\Sigma)}$, we get that $\overleftrightarrow{I^\flat}_\Sigma(\phi, \psi, \vec{\xi}) \subseteq \alpha_\Sigma^{-1}(T_{F(\Sigma)})$, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$. But, by hypothesis, $I^\flat$ has the local family compatibility in $\mathcal{I}$ and, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$. Therefore, we get that, for all $\lambda^\flat \in N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi}, \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$I^\flat_{\Sigma'}(\lambda^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}), \lambda^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}), \vec{\xi}) \subseteq \alpha_{\Sigma'}^{-1}(T_{F(\Sigma')}).$$

This now gives

$$\alpha_{\Sigma'}(I^\flat_{\Sigma'}(\lambda^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}), \lambda^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}), \vec{\xi})) \subseteq T_{F(\Sigma')},$$

or, equivalently,

$$\begin{aligned} I_{F(\Sigma')}(&\lambda_{F(\Sigma')}(\mathrm{SEN}(F(f))(\alpha_\Sigma(\phi)), \alpha_{\Sigma'}(\vec{\chi})),\\ &\lambda_{F(\Sigma')}(\mathrm{SEN}(F(f))(\alpha_\Sigma(\psi)), \alpha_{\Sigma'}(\vec{\chi})), \alpha_{\Sigma'}(\vec{\xi})) \subseteq T_{F(\Sigma')}.\end{aligned}$$

Taking into account the surjectivity of $\langle F, \alpha\rangle$, we conclude that I has the local family compatibility in $\mathcal{A}$.

(b) The case of the local system compatibility can be proven similarly, taking into account that, if $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, then $\alpha^{-1}(T) \in \mathrm{ThSys}(\mathcal{I})$.

(c) Suppose that $I^\flat$ has the global (family) compatibility in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$\overleftrightarrow{I}_{F(\Sigma)}[\alpha_\Sigma(\phi), \alpha_\Sigma(\psi)] \leq T.$$

Then, we have, by Lemma 95, $\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi] \leq \alpha^{-1}(T)$. Now, since, by hypothesis, $I^\flat$ has the global family compatibility in $\mathcal{I}$ and, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, we get that, for all $\lambda^\flat \in N^\flat$, $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$I^\flat_{\Sigma'}[\lambda^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}), \lambda^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi})] \leq \alpha^{-1}(T),$$

or, equivalently, by Lemma 95,

$$I_{F(\Sigma')}[\alpha_{\Sigma'}(\lambda^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi})), \alpha_{\Sigma'}(\lambda^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}))] \leq T.$$

But this amounts to

$$\begin{aligned} I_{F(\Sigma')}[&\lambda_{F(\Sigma')}(\mathrm{SEN}(F(f))(\alpha_\Sigma(\phi)), \alpha_{\Sigma'}(\vec{\chi})), \\ &\lambda_{F(\Sigma')}(\mathrm{SEN}(F(f))(\alpha_\Sigma(\psi)), \alpha_{\Sigma'}(\vec{\chi}))] \leq T. \end{aligned}$$

Thus, I has the global family compatibility in $\mathcal{A}$.

■

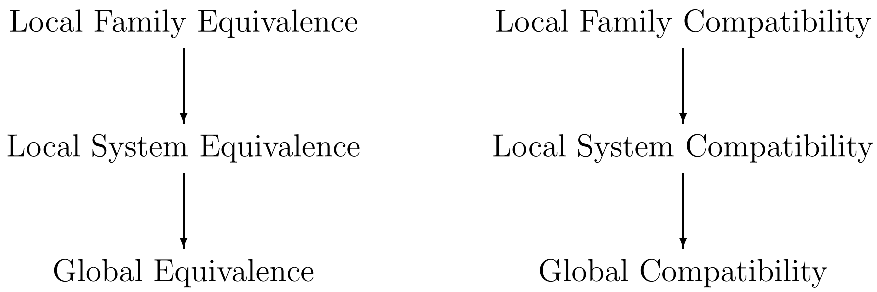
10.9 Congruence

In this section we focus on the three **uniform equivalence properties**, i.e., on LFLF equivalence, LSLS equivalence and GBGB equivalence, and we add to those versions of the compatibility property to obtain several versions of the congruence property.

To fix some terminology, we say that a set $I^\flat$ of natural transformations in a π-institution $\mathcal{I}$ has:

- the **local family equivalence in** $\mathcal{I}$ if it has the LFLF equivalence in $\mathcal{I}$;
- the **local system equivalence in** $\mathcal{I}$ if it has the LSLS equivalence in $\mathcal{I}$;
- the **global equivalence in** $\mathcal{I}$ if it has the GBGB equivalence in $\mathcal{I}$.

By previous work, we know that these three uniform equivalence properties are stratified in the linear hierarchy shown on the left below.



Moreover, by our study of the compatibility properties, we know that they also fall into a similar linear hierarchy, as shown on the right of the diagram.

By combining equivalence with compatibility properties, we obtain nine congruence properties as follows. Let $X, Y \in \{\mathrm{LF}, \mathrm{LS}, \mathrm{GB}\}$, where, as before, LF stands for "Local Family", LS stands for "Local System" and GB stands for "GloBal".

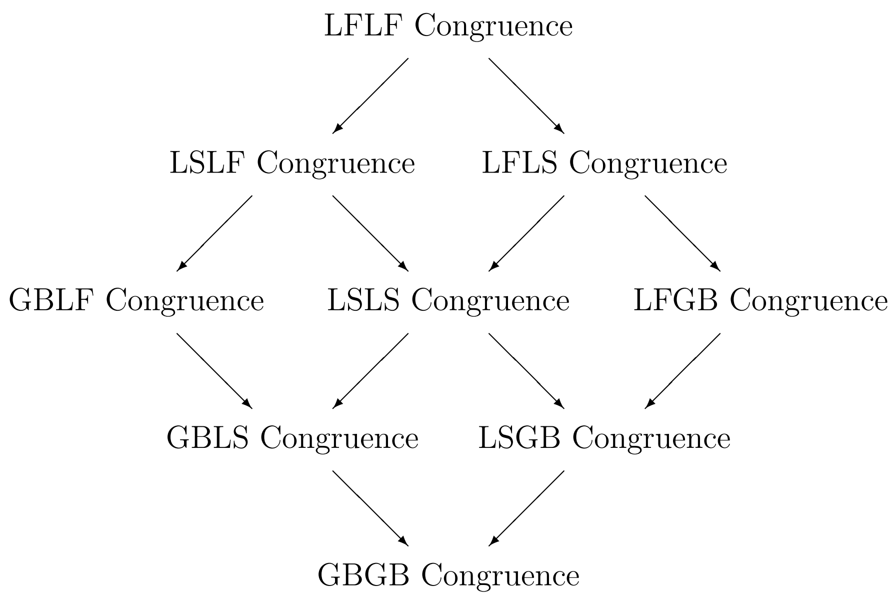
Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. We say that $I^\flat$ has the **XY-congruence in** $\mathcal{I}$ if it has

- the X (uniform) equivalence in $\mathcal{I}$;
- the Y compatibility in $\mathcal{I}$.

Based on the hierarchies of the equivalence and compatibility properties, we obtain the following hierarchical structure for the various flavors of the congruence property.

Corollary 716 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$*, with two distinguished arguments. The nine congruence properties form the hierarchy shown on the accompanying diagram.*

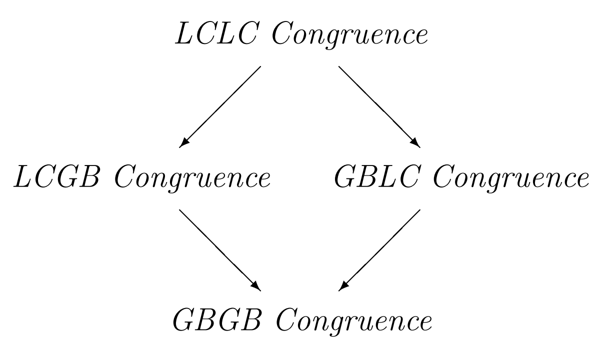
Proof: This follows directly from Corollary 694 and Proposition 711. ∎



Based on the analysis performed on symmetry and transitivity, we have the following result regarding natural sufficient conditions under which some of the nine congruence properties above coincide.

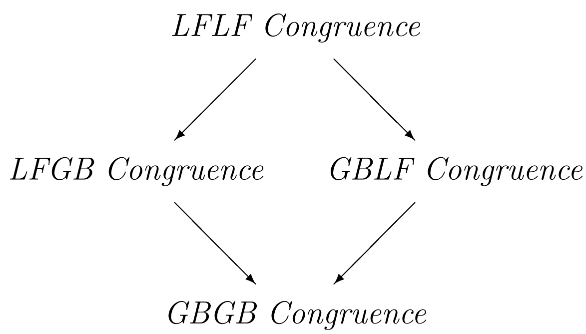
Corollary 717 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$*, with two distinguished arguments.*

(a) If $\mathcal{I}$ *is systemic, then the congruence hierarchy collapses to the one depicted below, where LC (for LoCal) is used to incorporate the LF and LS properties, which coincide;*



(b) If $I^\flat$ *has only two arguments (i.e., is parameter free), then the congruence hierarchy collapses to the one depicted below, where the Local System versions coincide with (and, thus, are incorporated into) the*

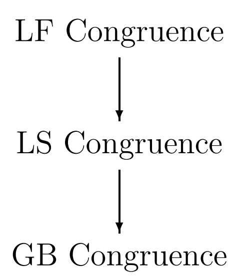
GloBal versions.



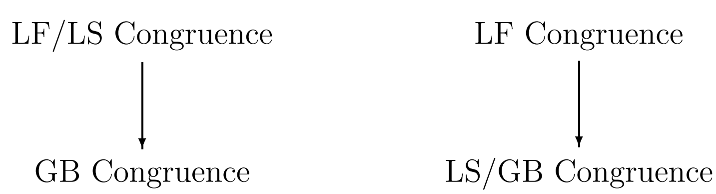
Proof: The statement follows from Corollary 695 and Proposition 712. ■

For a systemic π-institution with a parameter-free set of natural transformations, there is only one congruence property, since all versions of equivalence and all versions of compatibility collapse to a single property.

Instead of studying this entire hierarchy in detail, we refocus, once again, to the uniformly defined classes. So we define **LF congruence**, **LS congruence** and **GB congruence** to mean, respectively, LFLF congruence, LSLS congruence and GBGB congruence. These are the three diagonal classes in the original diagram that form, according to Corollary 716, the subhierarchy depicted below.



And, of course, according to Corollary 717, this reduces to the hierarchy depicted on the left below for a systemic π-institution and to the one depicted on the right below for a parameter free set of natural transformations.



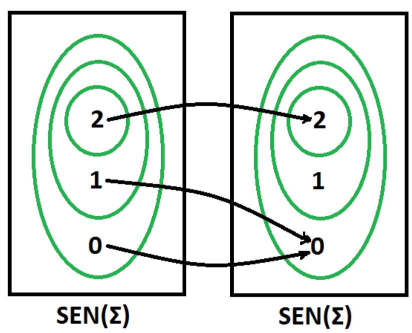
We provide examples to show that the three uniform classes of the congruence hierarchy are distinct in general.

First, we present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that has the LS congruence, but not the LF congruence property in $\mathcal{I}$.

Example 718 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 0$*,* $1 \mapsto 0$ *and* $2 \mapsto 2$*;*
- $N^\flat$ *is the category of natural transformations generated by a single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 2, & \text{if } x = y \text{ or } \{x,y\} = \{0,1\} \\ 1, & \text{if } (x,y) = (1,2) \\ 0, & \text{otherwise} \end{cases}$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\}, \{1,2\}, \{0,1,2\}\}.$$

Note that there are three theory families, but only $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$ *are theory systems.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the local system congruence in* $\mathcal{I}$*, but it does not have the local family congruence in* $\mathcal{I}$*.*

First note that $\sigma^\flat_\Sigma(\phi,\phi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, for all* $\phi \in \{0,1,2\}$*, whence* $I^\flat$ *is reflexive in* $\mathcal{I}$*. Next note that the condition defining local system symmetry holds trivially for* $\mathrm{SEN}^\flat$*, whereas for* $\mathrm{Thm}(\mathcal{I})$*, if* $\sigma^\flat_\Sigma(\phi,\psi) \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, for some* $\phi \neq \psi$*, then* $\{\phi,\psi\} = \{0,1\}$*, whence* $\sigma^\flat_\Sigma(\psi,\phi) \in \mathrm{Thm}_\Sigma(\mathcal{I})$*. So* $I^\flat$ *is locally system symmetric in* $\mathcal{I}$*. For local system transitivity, the defining condition holds, again, trivially for* $\mathrm{SEN}^\flat$*, whereas for* $\mathrm{Thm}(\mathcal{I})$*, it holds due to the fact that* $\sigma^\flat_\Sigma(\phi,\psi) \in \mathrm{Thm}_\Sigma(\mathcal{I})$ *for no* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$*, with* $\phi \neq \psi$*, other than* $(\phi,\psi) = (0,1)$ *or* $(1,0)$*. Thus,* $I^\flat$ *is also locally system transitive in*

$\mathcal{I}$. Finally, note that the condition defining local system compatibility is also trivial for $\mathrm{SEN}^\flat$*, whereas for* $\mathrm{Thm}(\mathcal{I})$*, if $\sigma^\flat_\Sigma(\phi,\psi) = 2$ and $\sigma^\flat_\Sigma(\psi,\phi) = 2$, with $\phi \neq \psi$, then $\{\phi,\psi\} = \{0,1\}$ and, in that case,*

$$\sigma^\flat_\Sigma(\sigma^\flat_\Sigma(\mathrm{SEN}^\flat(h)(\phi),\chi),\sigma^\flat_\Sigma(\mathrm{SEN}^\flat(h)(\psi),\chi)) = 2$$

and

$$\sigma^\flat_\Sigma(\sigma^\flat_\Sigma(\chi,\mathrm{SEN}^\flat(h)(\phi)),\sigma^\flat_\Sigma(\chi,\mathrm{SEN}^\flat(h)(\psi))) = 2,$$

for all $h \in \mathbf{Sign}^\flat(\Sigma,\Sigma)$ and all $\chi \in \{0,1,2\}$. Thus, $I^\flat$ has the local system compatibility in $\mathcal{I}$ and, therefore, has the local system congruence in $\mathcal{I}$.

On the other hand, note that $\sigma^\flat_\Sigma(1,2) = 1 \in \{1,2\}$, but $\sigma^\flat_\Sigma(2,1) = 0 \notin \{1,2\}$. Thus, the local family symmetry condition fails for $T = \{\{1,2\}\} \in \mathrm{ThFam}(\mathcal{I})$. Hence, $I^\flat$ is not locally family symmetric in $\mathcal{I}$ and, therefore, a fortiori, it fails to satisfy the local family congruence in $\mathcal{I}$.

Finally, we present an example to show that there is a π-institution $\mathcal{I}$ with a set of natural transformations that has the GB congruence but not the LS congruence in $\mathcal{I}$.

Example 719 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object Σ and a single (non-identity) morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2,3\} \to \{0,1,2,3\}$ *given by $0 \mapsto 2$, $1 \mapsto 3$, $2 \mapsto 2$ and $3 \mapsto 3$;*
- $N^\flat$ *is the category of natural transformations generated by the single ternary natural transformation $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ defined by letting $\sigma^\flat_\Sigma : \{0,1,2,3\}^3 \to \{0,1,2,3\}$ be given by*

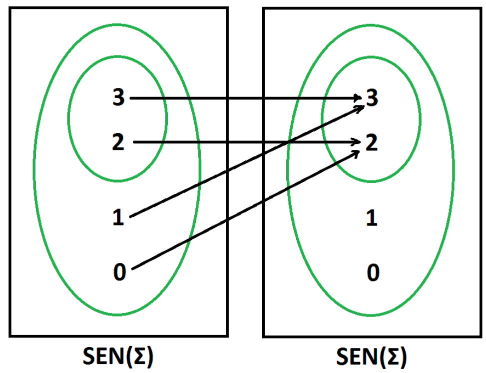
$$\sigma^\flat_\Sigma(x,y,z) = \begin{cases} 2, & \text{if } x = y \text{ or } (x,y) = (0,1) \text{ or } z = 2 \text{ or } z = 3 \\ 0, & \text{otherwise} \end{cases}.$$

Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be the π-institution determined by

$$\mathcal{C}_\Sigma = \{\{2,3\},\{0,1,2,3\}\}.$$

Note that both theory families, $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, are also theory systems. So $\mathcal{I}$ is a systemic π-institution.*

Consider the set $I^\flat = \{\sigma^\flat\}$, with $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ having two distinguished arguments. We show that $I^\flat$ has the GB congruence in $\mathcal{I}$, but it does not have the LS congruence in $\mathcal{I}$.



Note, first, that reflexivity is obvious, since, by definition, for all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\sigma^\flat_\Sigma(\phi,\phi,\xi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$, *for all* $\xi \in \mathrm{SEN}^\flat(\Sigma)$. *For global symmetry, note that if, for some* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, $\sigma^\flat_\Sigma[\phi,\psi] \leq \mathrm{Thm}(\mathcal{I})$, *then we must have* $\phi = \psi$, *whence* $\sigma^\flat_\Sigma[\psi,\phi] \leq \mathrm{Thm}(\mathcal{I})$ *holds. For global transitivity, note again that for no* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, *with* $\phi \neq \psi$, *is it the case that* $\sigma^\flat_\Sigma[\phi,\psi] \leq \mathrm{Thm}(\mathcal{I})$, *whence the condition is satisfied in this case as well. Finally, the same observation leads to the conclusion that* $I^\flat$ *satisfies the global compatibility property in* $\mathcal{I}$. *We conclude that* $I^\flat$ *has the GB congruence in* $\mathcal{I}$.

On the other hand, since $\sigma^\flat_\Sigma(0,1,\xi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$, *for all* $\xi \in \mathrm{SEN}^\flat(\Sigma)$, *but* $\sigma^\flat_\Sigma(1,0,0) = 0 \notin \mathrm{Thm}_\Sigma(\mathcal{I})$, $I^\flat$ *does not have the local system symmetry. A fortiori,* $I^\flat$ *does not have the LS congruence in* $\mathcal{I}$.

And here is a transfer property for the congruence properties that we have focused on in this section.

Corollary 720 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $I^\flat$ *a collection of natural transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$, *with two distinguished arguments, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $I^\flat$ *has a congruence property in* $\mathcal{I}$ *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, *with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, I *has the corresponding congrunece property in* $\mathcal{A}$.

Proof: This follows directly from Corollary 700 and Proposition 715. ■

10.10 Modus Ponens

We turn now to the study of various versions of the modus ponens property, taking again into account both the duality between local versus global membership and the difference between considering all theory families versus restricting only to theory systems.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. We say that:

- $I^\flat$ has the **local family modus ponens** (**local family MP**) **in** $\mathcal{I}$ if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \text{ and, for all } \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma),\ I^\flat_\Sigma(\phi, \psi, \vec{\chi}) \subseteq T_\Sigma \text{ imply } \psi \in T_\Sigma;$$

- $I^\flat$ has the **local system modus ponens** (**local system MP**) **in** $\mathcal{I}$ if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \text{ and, for all } \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma),\ I^\flat_\Sigma(\phi, \psi, \vec{\chi}) \subseteq T_\Sigma \text{ imply } \psi \in T_\Sigma;$$

- $I^\flat$ has the **global family modus ponens** (**global family MP**) **in** $\mathcal{I}$ if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \text{ and } I^\flat_\Sigma[\phi, \psi] \leq T \text{ imply } \psi \in T_\Sigma;$$

- $I^\flat$ has the **global system modus ponens** (**global system MP**) **in** $\mathcal{I}$ if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \text{ and } I^\flat_\Sigma[\phi, \psi] \leq T \text{ imply } \psi \in T_\Sigma.$$

The following proposition establishes the hierarchy of modus ponens rules.

Proposition 721 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments.*

(a) If $I^\flat$ *has the local family MP, then it has both the global family MP in* $\mathcal{I}$ *and the local system MP in* $\mathcal{I}$*;*

(b) If $I^\flat$ *has the global family MP, then it has the global system MP in* $\mathcal{I}$*;*

(c) If $I^\flat$ *has the local system MP, then it has the global system MP in* $\mathcal{I}$*.*

Proof:

(a) Suppose that $I^\flat$ has the local family MP in $\mathcal{I}$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $I^\flat_\Sigma[\phi, \psi] \leq T$. Then, we have, in particular, that, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma(\phi, \psi, \vec{\chi}) \subseteq T_\Sigma$. But then, since $\phi \in T_\Sigma$ and, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma(\phi, \psi, \vec{\chi}) \subseteq T_\Sigma$, we get by the local family MP that $\psi \in T_\Sigma$. We conclude that $I^\flat$ has the global family MP in $\mathcal{I}$.

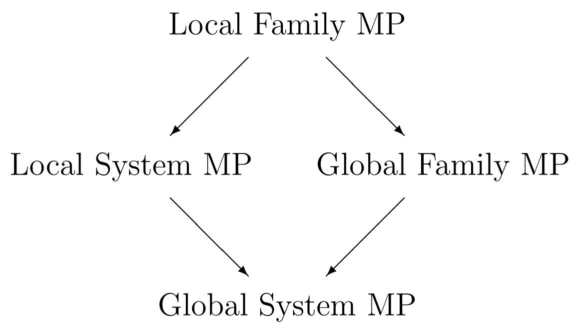
If $I^\flat$ has the local family MP in $\mathcal{I}$, then it has a fortiori the local system MP in $\mathcal{I}$ due to the fact that every theory system of $\mathcal{I}$ is also a theory family.

(b) This follows, similarly to the second part of (a), from the fact that every theory system of $\mathcal{I}$ is also a theory family.

(c) We repeat the argument used in the proof of the first part of (a) except reasoning exclusively in terms of theory systems rather than using arbitrary theory families.

■

Proposition 721 has established the following hierarchy of modus ponens properties, where the southwest arrows are based on the family-system duality whereas the southeast arrows on the local-global duality.



We also note the following regarding natural sufficient conditions under which some of these four classes coincide.

Proposition 722 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments.*

(a) If $\mathcal{I}$ *is systemic, then the local (global) family and the local (global) system MP coincide;*

(b) If $I^\flat$ *has only two arguments (i.e., is parameter free), then the local system MP and the global system MP coincide;*

(c) If $\mathcal{I}$ *is systemic and* $I^\flat$ *is parameter-free, then the local family MP and the global family MP also coincide.*

Proof:

(a) If $\mathcal{I}$ is systemic, then all theory families are theory systems and the family and system properties collapse.

(b) Suppose that $I^\flat$ is parameter-free and that $I^\flat$ has the global system MP in $\mathcal{I}$. Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $I^\flat_\Sigma(\phi, \psi) \subseteq T_\Sigma$. Since T is a theory system, we have, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$I^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi)) = \mathrm{SEN}^\flat(f)(I^\flat_\Sigma(\phi, \psi)) \subseteq T_{\Sigma'}.$$

Equivalently, $I^\flat_\Sigma[\phi, \psi] \leq T$. Thus, by the global system MP, we get that $\psi \in T_\Sigma$. Thus, $I^\flat$ has the local system MP.

(c) This follows from Parts (a) and (b). ∎

So in the case of a systemic π-institution $\mathcal{I}$, we have the hierarchy pictured on the left, whereas in the case of a parameter-free set of natural transformations we have the hierarchy on the right.

$$\begin{array}{ccc} \text{Local MP} & & \text{Local Family MP} \\ \downarrow & & \downarrow \\ \text{Global MP} & & \text{Global Family MP} \\ & & \downarrow \\ & & \text{System MP} \end{array}$$

Finally, for a systemic π-institution with a parameter-free set of natural transformations all four MP properties collapse to a single one.

We provide some examples to show that the implications of Proposition 721 are not equivalences in general, i.e., in the hierarchy shown above all inclusions of classes of π-institutions with a set of natural transformations satisfying the corresponding modus ponens properties are proper inclusions.

We first present an example to show that there is π-institution $\mathcal{I}$ with a set of natural transformations that have the global family MP but not the local system MP in $\mathcal{I}$.

Example 723 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

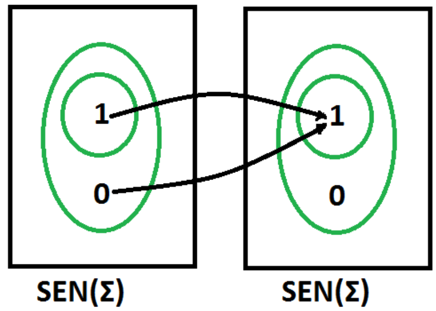
- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1\} \to \{0,1\}$ *given by* $0 \mapsto 1$ *and* $1 \mapsto 1$*;*
- $N^\flat$ *is the category of natural transformations generated by the single ternary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1\}^3 \to \{0,1\}$ *be given by*

$$\sigma^\flat_\Sigma(a,b,c) = \begin{cases} 0, & \text{if } (a,b,c) = (1,1,0) \\ 1, & \text{otherwise} \end{cases}$$

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\}$*.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *having two distinguished arguments.*

Clearly, both $\mathrm{Thm}(\mathcal{I})$ *and* SEN*, which are the only theory families, are also theory systems. We show that* $I^\flat$ *has the global family MP in* $\mathcal{I}$*, but it does not have the local system MP in* $\mathcal{I}$*.*



For the global family MP notice that we only need to check the case with $T = \mathrm{Thm}(\mathcal{I})$, $\phi = 1$ *and* $\psi = 0$. *Since* $\sigma^\flat_\Sigma(\mathrm{SEN}^\flat(f)(1), \mathrm{SEN}^\flat(f)(0), 0) = \sigma^\flat_\Sigma(1,1,0) = 0$, *we have that* $I^\flat_\Sigma[\phi,\psi] \nleq T$, *whence the condition is vacuously satisfied. Therefore, we get that* $I^\flat$ *has the global family MP in* $\mathcal{I}$.

On the other hand, we have $\sigma^\flat_\Sigma(1,0,0) = \sigma^\flat_\Sigma(1,0,1) = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$ *and* $1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$, *but* $0 \notin \mathrm{Thm}_\Sigma(\mathcal{I})$, *which shows that* $I^\flat$ *does not have the local system MP in* $\mathcal{I}$.

Next we present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that have the local system MP but not the global family MP in $\mathcal{I}$.

Example 724 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

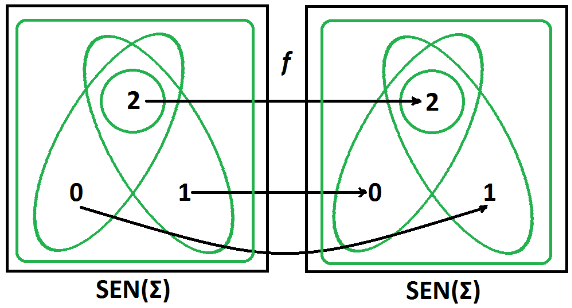
- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = i_\Sigma$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 1$, $1 \mapsto 0$ *and* $2 \mapsto 2$;
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given, for all* $a, b \in \mathrm{SEN}^\flat(\Sigma)$, *by*

$$\sigma^\flat_\Sigma(a,b) = \begin{cases} 1, & \text{if } (a,b) = (2,0) \\ 0, & \text{if } (a,b) = (2,1) \\ 2, & \text{otherwise} \end{cases}.$$

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}.$$

Consider the set $I^\flat = \{\sigma^\flat\}$.



$\mathcal{I}$ has four theory families $\mathrm{Thm}(\mathcal{I})$, $T = \{\{0,2\}\}$, $T' = \{\{1,2\}\}$ *and* $\mathrm{SEN}^\flat$, *but only two theory systems* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$. *We show that $I^\flat$ has the local system MP in $\mathcal{I}$, but it does not have the global family MP in $\mathcal{I}$.*

For the local system MP notice that we only need to check the case for $\mathrm{Thm}(\mathcal{I})$, $\phi = 2$ *and* $\psi = 0$ *or* $\psi = 1$. *Since* $\sigma^\flat_\Sigma(2,0) = 1 \notin \mathrm{Thm}_\Sigma(\mathcal{I})$ *and* $\sigma^\flat_\Sigma(2,1) = 0 \notin \mathrm{Thm}_\Sigma(\mathcal{I})$, *we conclude that $I^\flat$ has the local system MP in $\mathcal{I}$.*

On the other hand, for the theory family T and for $\phi = 0$ and $\psi = 1$, we get that $\phi = 0 \in T_\Sigma$ and $\sigma^\flat_\Sigma(\phi,\psi) = \sigma^\flat_\Sigma(0,1) = 2$ and

$$\sigma^\flat_\Sigma(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi)) = \sigma^\flat_\Sigma(1,0) = 2,$$

whence $\sigma^\flat_\Sigma[0,1] \leq T$. But clearly $1 \notin T_\Sigma$. Therefore $I^\flat$ does not have the global family MP in $\mathcal{I}$.

We prove next a transfer property for modus ponens.

Proposition 725 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $I^\flat$ a collection of natural transformations $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with two distinguished arguments, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $I^\flat$ has a modus ponens property in $\mathcal{I}$ if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, I has the corresponding modus ponens property in $\mathcal{A}$.*

Proof: If I has a modus ponens property in $\mathcal{A}$, for all $\mathcal{A}$, then it has the same modus ponens in $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. Since $\langle \mathbf{F}, C^{\mathcal{I},\mathcal{F}} \rangle = \mathcal{I}$, we conclude that $I^\flat$ has the corresponding modus ponens in $\mathcal{I}$.

Suppose, conversely, that $I^\flat$ has a modus ponens in $\mathcal{I}$. We look at each of the four properties in turn.

(a) Suppose $I^\flat$ has the local family MP in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$ and

$$I_{F(\Sigma)}(\alpha_\Sigma(\phi), \alpha_\Sigma(\psi), \alpha_\Sigma(\vec{\chi})) \subseteq T_{F(\Sigma)},$$

for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$. Since the latter is equivalent to $\alpha_\Sigma(I^\flat_\Sigma(\phi,\psi,\vec{\chi})) \subseteq T_{F(\Sigma)}$, we get that $\phi \in \alpha^{-1}_\Sigma(T_{F(\Sigma)})$ and $I^\flat_\Sigma(\phi,\psi,\vec{\chi}) \subseteq \alpha^{-1}_\Sigma(T_{F(\Sigma)})$, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$. But, by hypothesis, $I^\flat$ has the local family MP in $\mathcal{I}$ and, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$. Therefore, we get that $\psi \in \alpha^{-1}_\Sigma(T_{F(\Sigma)})$, or, equivalently, $\alpha_\Sigma(\psi) \in T_{F(\Sigma)}$. This proves that I has the local family MP in $\mathcal{A}$.

(b) The case of the local system MP can be proven similarly, taking into account that, if $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, then $\alpha^{-1}(T) \in \mathrm{ThSys}(\mathcal{I})$.

(c) Suppose that $I^\flat$ has the global family MP in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$\alpha_\Sigma(\phi) \quad \text{and} \quad I_{F(\Sigma)}[\alpha_\Sigma(\phi), \alpha_\Sigma(\psi)] \le T.$$

Then, we have $\phi \in \alpha^{-1}_\Sigma(T_{F(\Sigma)})$ and, by Lemma 95, $I^\flat_\Sigma[\phi,\psi] \le \alpha^{-1}(T)$. Now, since, by hypothesis, $I^\flat$ has the global family MP in $\mathcal{I}$ and, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, we get that $\psi \in \alpha^{-1}_\Sigma(T_{F(\Sigma)})$, or, equivalently, $\alpha_\Sigma(\psi) \in T_{F(\Sigma)}$. Thus, I has the global family MP in $\mathcal{A}$.

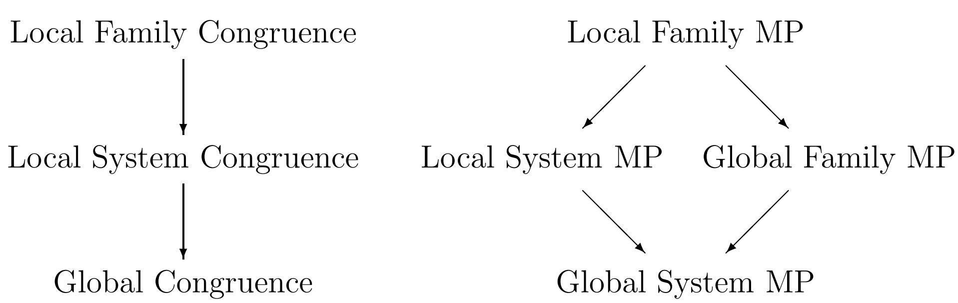
(d) Similar to (c).

■

10.11 Syntactic Protoalgebraicity

In this section we focus on the three **uniform congruence properties**, i.e., on LF congruence, LS congruence and GB congruence, and we add to those versions of the modus ponens property to obtain several versions of the syntactic protoalgebraicity property.

By previous work, we know that the three uniform congruence properties are stratified in the linear hierarchy shown on the left below.



Moreover, by our study of the modus ponens, we know that they fall into the hierarchy shown on the right of the diagram.

By combining equivalence with compatibility properties, we obtain twelve syntactic protoalgebraicity properties as follows. Let $X \in \{\mathrm{LF}, \mathrm{LS}, \mathrm{GB}\}$ and

$Y \in \{\mathrm{LF}, \mathrm{LS}, \mathrm{GF}, \mathrm{GS}\}$, where, as before, LF stands for "Local Family", LS stands for "Local System", GF stands for "Global Family", GS stands for "Global System" and GB stands for "GloBal".

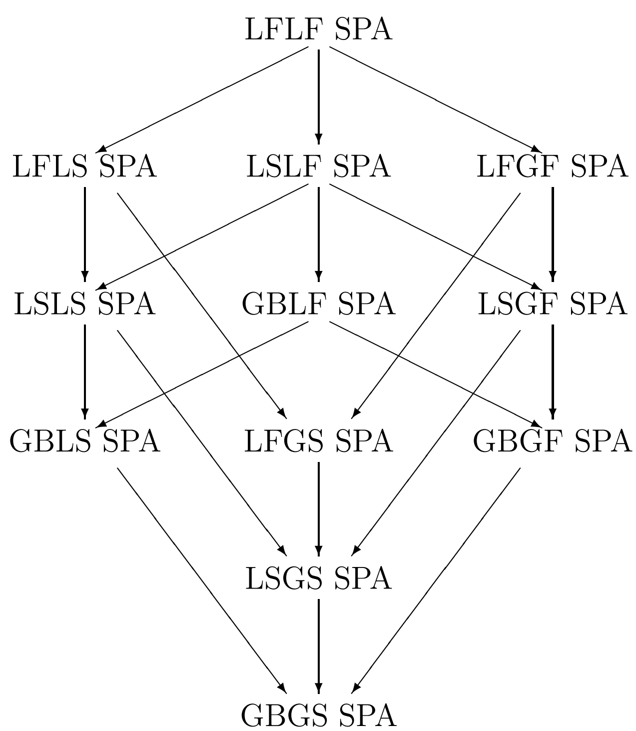
Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. We say that $I^\flat$ has the **XY syntactic protoalgebraicity in $\mathcal{I}$** (**XY SPA in $\mathcal{I}$**) if it has

- the X congruence in $\mathcal{I}$;
- the Y modus ponens in $\mathcal{I}$.

Based on the hierarchies of the congruence and MP properties, we obtain the following hierarchical structure for the various flavors of the syntactic protoalgebraicity property.

Corollary 726 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$ having two distinguished arguments. The twelve syntactic protoalgebraicity properties form the hierarchy shown on the accompanying diagram.*

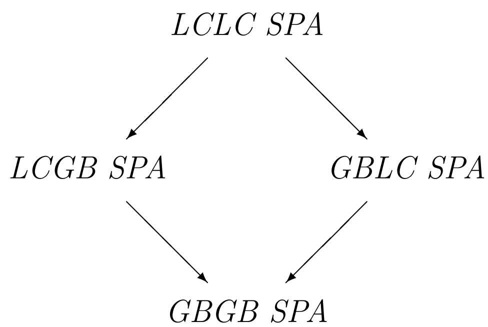
Proof: This follows directly from Corollary 716 and Proposition 721. ■



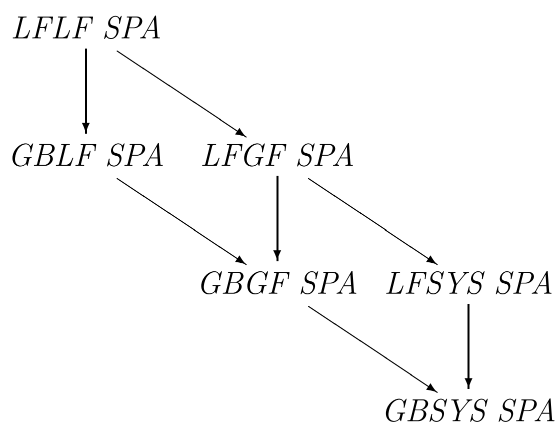
Based on the analysis performed on congruence and modus ponens, we have the following result regarding natural sufficient conditions under which some of the twelve syntactic protoalgebraicity properties above coincide.

Corollary 727 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments.*

(a) If $\mathcal{I}$ *is systemic, then the syntactic protoalgebraicity hierarchy collapses to the one depicted below;*



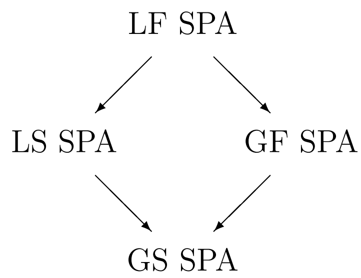
(b) If $I^\flat$ *has only two arguments (i.e., is parameter free), then the syntactic protoalgebraicity hierarchy collapses to the one depicted below, where the Local System versions of congruence coincide with (and are incorporated into) the global versions and the Local and Global System versions of MP also coincide and are denoted by SYS.*



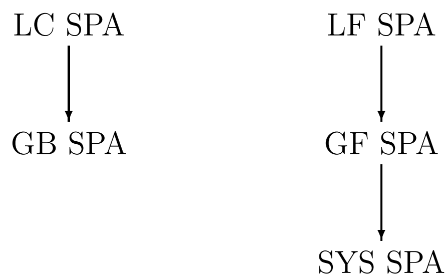
Proof: The statement follows from Corollary 717 and Proposition 722. ∎

For a systemic π-institution with a parameter-free set of natural transformations, there is only one syntactic protoalgebraicity property, since all versions of congruence and all versions of modus ponens collapse to a single property.

Instead of studying this entire hierarchy in detail, we refocus, once again, on the uniformly defined classes. So we define **LF SPA**, **LS SPA**, **GF SPA** and **GS SPA** to mean, respectively, LFLF syntactic, LSLS syntactic, GFGF syntactic and GSGS syntactic protoalgebraicity. These classes form, according to the diagram above, based on Corollary 726, the sub hierarchy depicted below.



Moreover, according to Corollary 727, this reduces to the hierarchy depicted on the left below for a systemic π-institution and to the one depicted on the right below for a parameter free set of natural transformations.



We provide examples to show that the four uniform classes of the syntactic protoalgebraicity hierarchy are distinct in general.

First, we present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that has the LS syntactic protoalgebraicity, but not the GF syntactic protoalgebraicity in $\mathcal{I}$.

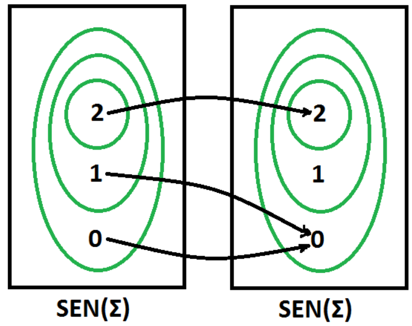
Example 728 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0, 1, 2\} \to \{0, 1, 2\}$ *given by* $0 \mapsto 0$*,* $1 \mapsto 0$ *and* $2 \mapsto 2$*;*
- $N^\flat$ *is the category of natural transformations generated by a single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0, 1, 2\}^2 \to \{0, 1, 2\}$ *be given by*

$$\sigma^\flat_\Sigma(x, y) = \begin{cases} 2, & \text{if } x = y \text{ or } \{x, y\} = \{0, 1\} \\ 1, & \text{if } \{x, y\} = \{1, 2\} \\ 0, & \text{otherwise} \end{cases}$$

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}.$$



Note that there are three theory families, but only $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$ *are theory systems.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the local system syntactic protoalgebraicity in* $\mathcal{I}$*, but it does not have the global family syntactic protoalgebraicity in* $\mathcal{I}$*.*

First note that $\sigma^\flat_\Sigma(\phi,\phi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, for all* $\phi \in \{0,1,2\}$*, whence* $I^\flat$ *is reflexive in* $\mathcal{I}$*.*

Next, note that the condition defining local system symmetry holds trivially for $\mathrm{SEN}^\flat$*, whereas for* $\mathrm{Thm}(\mathcal{I})$*, if* $\sigma^\flat_\Sigma(\phi,\psi) \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, for some* $\phi \neq \psi$*, then* $\{\phi,\psi\} = \{0,1\}$*, whence* $\sigma^\flat_\Sigma(\psi,\phi) \in \mathrm{Thm}_\Sigma(\mathcal{I})$*. So* $I^\flat$ *is local system symmetric in* $\mathcal{I}$*.*

For local system transitivity, the defining condition holds, again, trivially for $\mathrm{SEN}^\flat$*, whereas for* $\mathrm{Thm}(\mathcal{I})$*, it holds due to the fact that* $\sigma^\flat_\Sigma(\phi,\psi) \in \mathrm{Thm}_\Sigma(\mathcal{I})$ *for no* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$*, with* $\phi \neq \psi$*, other than* $(\phi,\psi) = (0,1)$ *or* $(1,0)$*. Thus,* $I^\flat$ *is also local system transitive in* $\mathcal{I}$*.*

Next, note that the condition defining local system compatibility is also trivial for $\mathrm{SEN}^\flat$*, whereas for* $\mathrm{Thm}(\mathcal{I})$*, if* $\sigma^\flat_\Sigma(\phi,\psi) = 2$ *and* $\sigma^\flat_\Sigma(\psi,\phi) = 2$*, with* $\phi \neq \psi$*, then* $\{\phi,\psi\} = \{0,1\}$ *and, in that case,*

$$\sigma^\flat_\Sigma(\sigma^\flat_\Sigma(\mathrm{SEN}^\flat(h)(\phi),\chi),\sigma^\flat_\Sigma(\mathrm{SEN}^\flat(h)(\psi),\chi)) = 2$$

and

$$\sigma^\flat_\Sigma(\sigma^\flat_\Sigma(\chi,\mathrm{SEN}^\flat(h)(\phi)),\sigma^\flat_\Sigma(\chi,\mathrm{SEN}^\flat(h)(\psi))) = 2,$$

for all $h \in \mathbf{Sign}^\flat(\Sigma,\Sigma)$ *and all* $\chi \in \{0,1,2\}$*. Thus,* $I^\flat$ *has the local system compatibility in* $\mathcal{I}$ *and, therefore, has the local system congruence in* $\mathcal{I}$*.*

To finish up, note that, since the only pairs (ϕ,ψ)*, with* $\phi \neq \psi$*, such that* $\sigma^\flat_\Sigma(\phi,\psi) \in \mathrm{Thm}_\Sigma(\mathcal{I})$ *are* $(0,1)$ *and* $(1,0)$ *and for neither of these is* $\phi \in \mathrm{Thm}_\Sigma(\mathcal{I})$*,* $I^\flat$ *has the local system modus ponens in* $\mathcal{I}$ *and, therefore, it has the local system syntactic protoalgebraicity in* $\mathcal{I}$ *as well.*

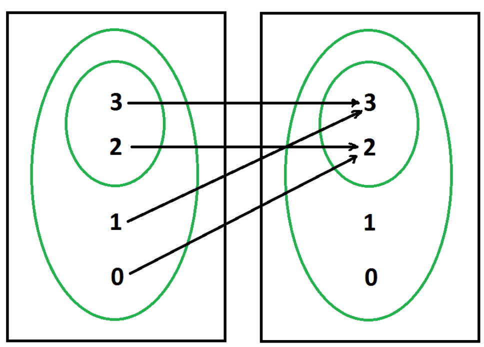
On the other hand, $1 \in \{1,2\}$ *and* $\sigma^\flat_\Sigma[1,0] \leq \{\{1,2\}\}$*, but* $0 \notin \{1,2\}$*. Therefore,* $I^\flat$ *does not have the global family modus ponens in* $\mathcal{I}$ *and, hence, a fortiori, it does not have the global family syntactic protoalgebraicity in* $\mathcal{I}$*.*

Next, we present an example to show that there is π-institution $\mathcal{I}$ with a set of natural transformations that has the GF syntactic protoalgebraicity but not the LS syntactic protoalgebraicity in $\mathcal{I}$.

Example 729 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2,3\} \to \{0,1,2,3\}$ *given by* $0 \mapsto 2$*,* $1 \mapsto 3$*,* $2 \mapsto 2$ *and* $3 \mapsto 3$*;*
- $N^\flat$ *is the category of natural transformations generated by the single ternary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2,3\}^3 \to \{0,1,2,3\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y,z) = \begin{cases} 2, & \text{if } x = y \text{ or } (x,y) = (0,1) \text{ or } z = 2 \text{ or } z = 3 \\ 0, & \text{otherwise} \end{cases}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$C_\Sigma = \{\{2,3\},\{0,1,2,3\}\}.$$

Note that both theory families, $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, are also theory systems. So* $\mathcal{I}$ *is a systemic* π*-institution.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the global family syntactic protoalgebraicity in* $\mathcal{I}$*, but it does not have the local system syntactic protoalgebraicity in* $\mathcal{I}$*.*

Note, first, that reflexivity is obvious, since, by definition, for all $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,* $\sigma^\flat_\Sigma(\phi,\phi,\xi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, for all* $\xi \in \mathrm{SEN}^\flat(\Sigma)$*. For global symmetry, note that if, for some* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$*,* $\sigma^\flat_\Sigma[\phi,\psi] \leq \mathrm{Thm}(\mathcal{I})$*, then we* *must have* $\phi = \psi$, *whence* $\sigma^\flat_\Sigma[\psi,\phi] \le \mathrm{Thm}(\mathcal{I})$ *holds. For global transitivity, note again that for no* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, *with* $\phi \ne \psi$, *is it the case that* $\sigma^\flat_\Sigma[\phi,\psi] \le \mathrm{Thm}(\mathcal{I})$, *whence the condition is satisfied in this case as well. Finally, the same observation leads to the conclusion that* $I^\flat$ *satisfies both the global compatibility property in* $\mathcal{I}$ *and the global modus ponens. We conclude that* $I^\flat$ *has the global family syntactic protoalgebtaicity in* $\mathcal{I}$.

On the other hand, since $\sigma^\flat_\Sigma(0,1,\xi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$, *for all* $\xi \in \mathrm{SEN}^\flat(\Sigma)$, *but* $\sigma^\flat_\Sigma(1,0,0) = 0 \notin \mathrm{Thm}_\Sigma(\mathcal{I})$, $I^\flat$ *does not have the local system symmetry. A fortiori,* $I^\flat$ *does not have the local system congruence and, hence, does not have the local system syntactic protoalgebraicity in* $\mathcal{I}$ *either.*

And here is a transfer property for the syntactic protoalgebraicity properties that we have focused on in this section.

Corollary 730 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $I^\flat$ *a collection of natural transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$, *with two distinguished arguments, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $I^\flat$ *has a (uniform) syntactic protoalgebraicity property in* $\mathcal{I}$ *if and only if, for every algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, *with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, I *has the corresponding syntactic protoalgebraicity property in* $\mathcal{A}$.

Proof: This follows directly from Corollary 720 and Proposition 1440. ■

10.12 Invertibility

We study, next, various versions of the invertibility property, once again based on the local versus global and the theory family versus theory system dualities.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$ having two distinguished arguments. We say that:

- $I^\flat$ has the **local family invertibility in** $\mathcal{I}$ if there exists a set $\tau : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ of natural transformations in $N^\flat$, such that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \text{ iff } \overleftrightarrow{I^\flat}_\Sigma(\tau_\Sigma(\phi),\vec{\xi}) \subseteq T_\Sigma, \text{ for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma);$$

- $I^\flat$ has the **local system invertibility in** $\mathcal{I}$ if there exists a set $\tau : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ of natural transformations in $N^\flat$, such that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \text{ iff } \overleftrightarrow{I^\flat}_\Sigma(\tau_\Sigma(\phi),\vec{\xi}) \subseteq T_\Sigma, \text{ for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma);$$

- $I^\flat$ has the **global family invertibility in** $\mathcal{I}$ if there exists a set $\tau : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ of natural transformations in $N^\flat$, such that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \overleftrightarrow{I^\flat}_\Sigma[\tau_\Sigma(\phi)] \leq T;$$

- $I^\flat$ has the **global system invertibility in** $\mathcal{I}$ if there exists a set $\tau : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ of natural transformations in $N^\flat$, such that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \overleftrightarrow{I^\flat}_\Sigma[\tau_\Sigma(\phi)] \leq T.$$

We look at the hierarchy of invertibility properties.

Proposition 731 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments.*

(a) If $I^\flat$ *has the local (global) family invertibility, then it has the local (global) system invertibility in* $\mathcal{I}$.

(b) If $I^\flat$ *has the local system invertibility, then it has the global system invertibility in* $\mathcal{I}$.

Proof: Since every theory system of $\mathcal{I}$ is a theory family, if $I^\flat$ has the local (global) family invertibility in $\mathcal{I}$, then it has, a fortiori, the local (global) system invertibility in $\mathcal{I}$, with the same witnessing set τ of natural transformations in $N^\flat$.

Suppose, next, that $I^\flat$ has the local system invertibility in $\mathcal{I}$, with witnessing set of natural transformations τ, and let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$.

- If $\phi \in T_\Sigma$, then, since $T \in \mathrm{ThSys}(\mathcal{I})$, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, $\mathrm{SEN}^\flat(f)(\phi) \in T_{\Sigma'}$. Thus, by the local family invertibility,

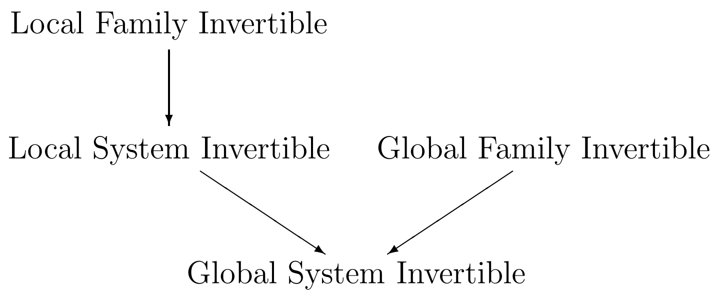
$$I^\flat_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)), \vec{\xi}) \subseteq T_{\Sigma'}, \text{ for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma').$$

This is equivalent to $I^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\tau_\Sigma(\phi)), \vec{\xi}) \subseteq T_{\Sigma'}$. Since $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$ were arbitrary, we conclude that $I^\flat_\Sigma[\tau_\Sigma(\phi)] \leq T$.

- Suppose, conversely, that $I^\flat_\Sigma[\tau_\Sigma(\phi)] \leq T$. This implies $I^\flat_\Sigma(\tau_\Sigma(\phi), \vec{\xi}) \subseteq T_\Sigma$, for all $\xi \in \mathrm{SEN}^\flat(\Sigma)$. Thus, by the local family invertibility, $\phi \in T_\Sigma$.

We conclude that $\phi \in T_\Sigma$ if and only if $I^\flat_\Sigma[\tau_\Sigma(\phi)] \le T$, whence $I^\flat$ has the global system invertibility in $\mathcal{I}$. ∎

Proposition 731 has established the following hierarchy of invertibility properties:



The following holds regarding natural sufficient conditions under which some of these properties coincide.

Proposition 732 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments.*

(a) If $\mathcal{I}$ *is systemic, then the local (global) family invertibility and the local (global) system invertibility properties coincide;*

(b) If $I^\flat$ *is parameter-free, then the local system invertibility and the global system invertibility properties coincide.*

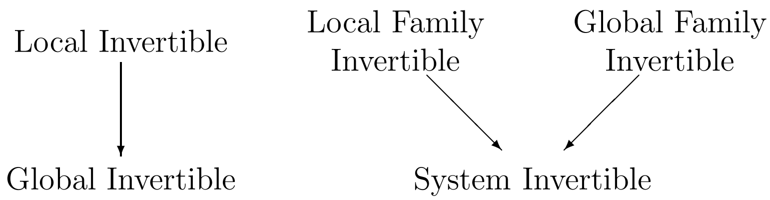
Proof: If $\mathcal{I}$ is systemic, then the local (global) system invertibility property coincides with the local (global) family invertibility property because of the fact that every theory family in $\mathcal{I}$ is also a theory system.

Suppose, next, that $I^\flat$ is parameter-free and that $I^\flat$ has the global system invertibility with witnessing set of natural transformations $\tau : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$. Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$.

- If $\phi \in T_\Sigma$, then, by the global system invertibility, $I^\flat_\Sigma[\tau_\Sigma(\phi)] \le T$. In particular, $I^\flat_\Sigma(\tau_\Sigma(\phi)) \subseteq T_\Sigma$.
- If, conversely, $I^\flat_\Sigma(\tau_\Sigma(\phi)) \subseteq T_\Sigma$, then, since $T \in \mathrm{ThSys}(\mathcal{I})$, we get that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, $I^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\tau_\Sigma(\phi))) \subseteq T_{\Sigma'}$. Hence, $I^\flat_\Sigma[\tau_\Sigma(\phi)] \le T$. Using the global system invertibility, we now conclude that $\phi \in T_\Sigma$.

Thus, the global system invertibility implies the local system invertibility property and, therefore that, provided $I^\flat$ is parameter-free, the local and global system invertibility properties coincide. ∎

So, in the case of a systemic π-institution $\mathcal{I}$, the hierarchy of invertibility properties reduces to the one depicted on the left below, whereas in the case of a parameter-free set of natural transformations $I^\flat$, we get the hierarchy depicted on the right.



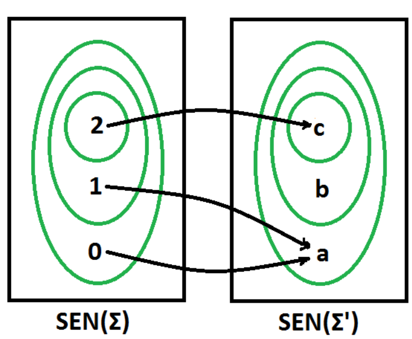
Finally, for a systemic π-institution and a parameter-free set of natural transformations, all four invertibility properties coincide.

We provide some examples to show that the implications of Proposition 731 are not equivalences in general, i.e., in the hierarchy shown above all inclusions of classes of π-institutions with a set of natural transformations satisfying the corresponding invertibility properties are proper inclusions.

We first present an example to show that there is π-institution $\mathcal{I}$ with a set of natural transformations that has the local family invertibility but not the global family invertibility in $\mathcal{I}$.

Example 733 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with two objects* Σ *and* Σ' *and a single (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b,c\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{a,b,c\}$ *given by* $0 \mapsto a$, $1 \mapsto a$ *and* $2 \mapsto c$*;*
- $N^\flat$ *is the trivial category of natural transformations consisting of the projections only.*



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{c\},\{b,c\},\{a,b,c\}\}.$$

Consider the set $I^\flat = \{p^{2,0}\}$*, with* $p^{2,0} : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *being the projection binary natural transformation (onto the first coordinate), viewed as having two distinguished arguments.*

$\mathcal{I}$ *has nine theory families, but only five of those are theory systems. So it is not a systemic* π*-institution. We show that* $I^\flat$ *has the local family invertibility in* $\mathcal{I}$*, but it does not have the global family invertibility in* $\mathcal{I}$*.*

For the local family invertibility, let $\tau \equiv \{\iota \approx \iota\}$*, where* $\iota : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *is the identity (or unary first coordinate projection) natural transformation. Then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\overleftrightarrow{I^\flat}_\Sigma(\iota_\Sigma(\phi), \iota_\Sigma(\phi)) \in T_\Sigma \quad \textit{iff} \quad \phi \in T_\Sigma.$$

Thus, $I^\flat$ *has the local family invertibility in* $\mathcal{I}$*.*

On the other hand, for $T = \{\{1,2\},\{b,c\}\} \in \mathrm{ThFam}(\mathcal{I})$*, we have* $1 \in T_\Sigma$*, but*

$$p^{2,0}_{\Sigma'}(\mathrm{SEN}^\flat(f)(1), \mathrm{SEN}^\flat(f)(1)) = p^{2,0}_{\Sigma'}(a,a) = a \notin T_{\Sigma'}.$$

Therefore $I^\flat$ *does not have the global family invertibility in* $\mathcal{I}$*.*

Next we present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that has the global family invertibility but not the local system invertibility in $\mathcal{I}$.

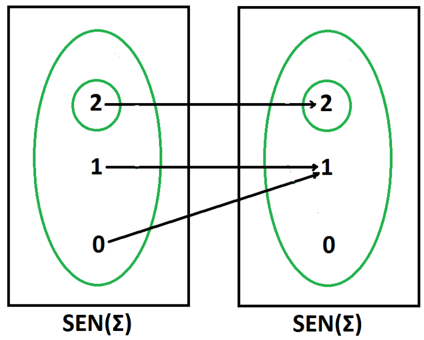
Example 734 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 1$*,* $1 \mapsto 1$ *and* $2 \mapsto 2$*;*
- $N^\flat$ *is the category of natural transformations generated by the single ternary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^3 \to \{0,1,2\}$ *be given, for all* $x,y,z \in \mathrm{SEN}^\flat(\Sigma)$*, by*

$$\sigma^\flat_\Sigma(x,y,z) = \begin{cases} 1, & \textit{if } (x,y,z) = (1,1,2) \\ 2, & \textit{otherwise} \end{cases}.$$

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{2\},\{0,1,2\}\}$*. Consider the set* $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat$ *having two distinguished arguments.*

$\mathcal{I}$ *has two theory families* $\mathrm{Thm}(\mathcal{I})$*,* $\mathrm{SEN}^\flat$ *both of which are also theory systems. So* $\mathcal{I}$ *is systemic. We show that* $I^\flat$ *has the global family invertibility in* $\mathcal{I}$*, but it does not have the local system invertibility in* $\mathcal{I}$*.*



For the global family invertibility, consider $\tau = \{\iota \approx \iota\}$, *where* $\iota : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *is the identity natural transformation. The case of* $\mathrm{SEN}^\flat$ *is trivial, whereas for* $\mathrm{Thm}(\mathcal{I})$, *we have, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and* $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi = 2 \quad \textit{iff} \quad \overleftrightarrow{I}^\flat{}_\Sigma[\phi, \phi] \leq \{\{2\}\},$$

which holds, for all $\phi \in \{0, 1, 2\}$, *as can be checked on a case-by-case basis.*

On the other hand, for the local system invertibility, note that $0 \notin \{2\}$, *but* $\sigma^\flat_\Sigma(\tau_\Sigma(0), \psi) = 2 \in \{2\}$, *for every set of unary natural transformations* $\tau : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$. *We conclude that* $I^\flat$ *does not have the local system invertibility in* $\mathcal{I}$.

Finally, we present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that has the local system invertibility but not the local family invertibility in $\mathcal{I}$.

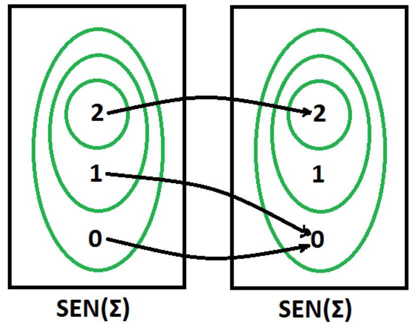
Example 735 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0, 1, 2\} \to \{0, 1, 2\}$ *given by* $0 \mapsto 0$, $1 \mapsto 0$ *and* $2 \mapsto 2$;
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0, 1, 2\}^2 \to \{0, 1, 2\}$ *be given, for all* $x, y \in \mathrm{SEN}^\flat(\Sigma)$, *by*

$$\sigma^\flat_\Sigma(x, y) = \begin{cases} 2, & \textit{if } (x, y) = (2, 2) \\ 0, & \textit{otherwise} \end{cases}.$$

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}.$$



Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat$ *having two distinguished arguments.*

$\mathcal{I}$ *has three theory families, but only* $\mathrm{Thm}(\mathcal{I})$*,* $\mathrm{SEN}^\flat$ *are theory systems. So* $\mathcal{I}$ *is not systemic. We show that* $I^\flat$ *has the local system invertibility in* $\mathcal{I}$*, but it does not have the local family invertibility in* $\mathcal{I}$*.*

For the local system invertibility, consider $\tau = \{\iota \approx \iota\}$*, where* $\iota : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *is the identity natural transformation. The case of* $\mathrm{SEN}^\flat$ *is trivial, whereas for* $\mathrm{Thm}(\mathcal{I})$*, we have to verify that, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\phi = 2 \quad \textit{iff} \quad \overleftrightarrow{I^\flat}_\Sigma(\phi,\phi) \subseteq \{\{2\}\}.$$

But this obviously holds, by the definition of $I^\flat$*.*

On the other hand, for the local family invertibility, note that $1 \in \{1,2\}$*, but*

$$\sigma^\flat_\Sigma(\tau_\Sigma(1)) = 0 \notin \{1,2\},$$

for every set of unary natural transformations $\tau : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*. We conclude that* $I^\flat$ *does not have the local family invertibility in* $\mathcal{I}$*.*

We now prove a transfer property for invertibility.

Proposition 736 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $I^\flat$ *a collection of natural transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, with two distinguished arguments, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $I^\flat$ *has an invertibility property in* $\mathcal{I}$ *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*,* I *has the corresponding invertibility property in* $\mathcal{A}$*.*

Proof: If I has an invertibility property in $\mathcal{A}$, for all $\mathcal{A}$, then it has the same invertibility property in $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$. Since $\langle \mathbf{F}, C^{\mathcal{I},\mathcal{F}}\rangle = \mathcal{I}$, we conclude that $I^\flat$ has the corresponding invertibility property in $\mathcal{I}$.

Suppose, conversely, that $I^\flat$ has an invertibility property in $\mathcal{I}$, with witnessing set of natural transformations $\tau : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$. We look at each of the four properties in turn.

(a) Suppose $I^\flat$ has the global family invertibility in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. We then have

$$\begin{array}{rll} \alpha_\Sigma(\phi) \in T_{F(\Sigma)} & \text{iff} & \phi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)}) \\ & \text{iff} & \overleftrightarrow{I^\flat}_\Sigma[\tau_\Sigma(\phi)] \leq \alpha^{-1}(T) \\ & \text{iff} & \overleftrightarrow{I}_{F(\Sigma)}[\alpha_\Sigma(\tau_\Sigma(\phi))] \leq T \\ & \text{iff} & \overleftrightarrow{I}_{F(\Sigma)}[\tau_{F(\Sigma)}(\alpha_\Sigma(\phi))] \leq T. \end{array}$$

Taking into account the surjetivity of $\langle F, \alpha \rangle$, we conclude that I has the global family invertibility in $\mathcal{A}$.

(b) The global system invertibility follows analogously, taking into account the fact that if $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, then $\alpha^{-1}(T) \in \mathrm{ThSys}(\mathcal{I})$.

(c) Suppose $I^\flat$ has the local family invertibility in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rll} \alpha_\Sigma(\phi) \in T_{F(\Sigma)} & \text{iff} & \phi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)}) \\ & \text{iff} & \overleftrightarrow{I^\flat}_\Sigma(\tau_\Sigma(\phi), \vec{\xi}) \subseteq \alpha_\Sigma^{-1}(T_{F(\Sigma)}), \\ & & \qquad \text{for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma), \\ & \text{iff} & \alpha_\Sigma(\overleftrightarrow{I^\flat}_\Sigma(\tau_\Sigma(\phi), \vec{\xi})) \subseteq T_{F(\Sigma)}, \\ & & \qquad \text{for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma), \\ & \text{iff} & \overleftrightarrow{I}_{F(\Sigma)}(\tau_{F(\Sigma)}(\alpha_\Sigma(\phi)), \alpha_\Sigma(\vec{\xi})) \subseteq T_{F(\Sigma)}, \\ & & \qquad \text{for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma), \end{array}$$

Taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that I has the local family invertibility in $\mathcal{A}$.

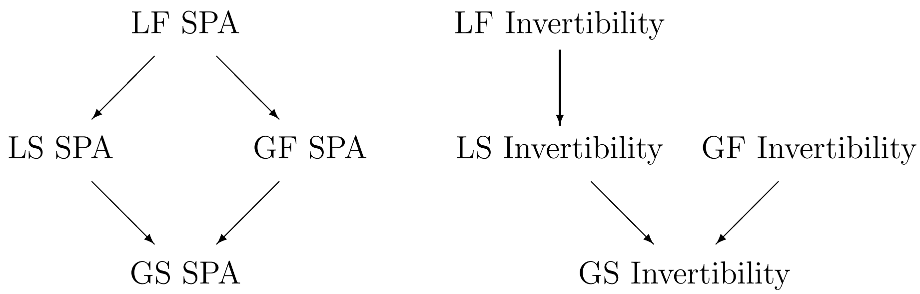
(d) The local system invertibility follows along the same lines, taking again into account the fact that if $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, then $\alpha^{-1}(T) \in \mathrm{ThSys}(\mathcal{I})$.

■

10.13 Syntactic Algebraizability

In this section we focus on the four **uniform syntactic protoalgebraicity properties**, i.e., on LF SPA, LS SPA and GF SPA and GS SPA, and we add to those versions of the invertibility property to obtain several versions of the syntactic algebraizability property.

By previous work, we know that the four uniform SPA properties constitute the hierarchy shown on the left below.



Moreover, by our study of invertibility, we know that the various versions of invertibility fall into the hierarchy shown on the right of the diagram.

By combining syntactic protoalgebraicity with invertibility properties, we obtain sixteen syntactic algebraizability properties as follows. Let $X, Y \in \{\mathrm{LF}, \mathrm{LS}, \mathrm{GF}, \mathrm{GS}\}$, where LF stands for "Local Family", LS stands for "Local System", GF stands for "Global Family" and GS stands for "Global System".

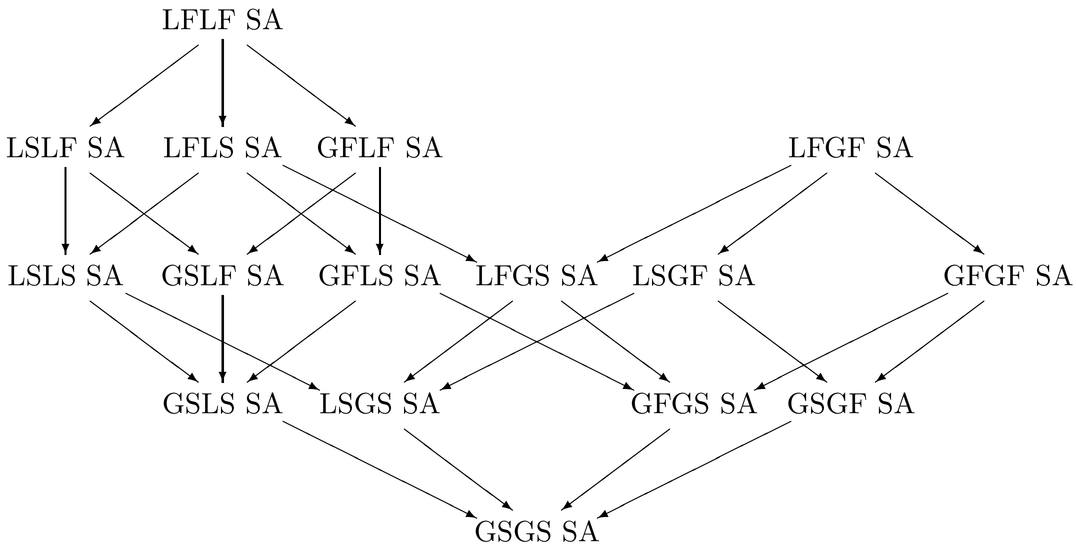
Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. We say that $I^\flat$ has the **XY syntactic algebraizability in $\mathcal{I}$ (XY SA in $\mathcal{I}$)** if it has

- the X syntactic protoalgebraicity in $\mathcal{I}$;
- the Y invertibility in $\mathcal{I}$.

Based on the hierarchies of the syntactic protoalgebraicity and invertibility properties, we obtain the following hierarchical structure for the various flavors of the syntactic algebraizability property.

Corollary 737 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$ having two distinguished arguments. The sixteen syntactic algebraizability properties form the hierarchy shown on the accompanying diagram.*

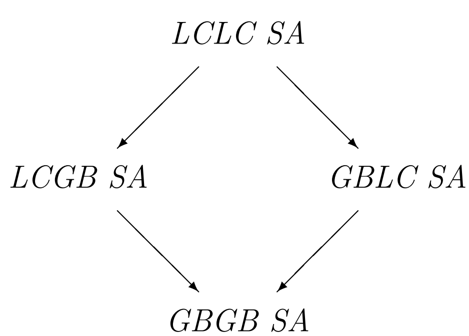
Proof: This follows directly from Corollary 726 and Proposition 731. ∎



Based on the analysis performed on SPA and Invertibility, we have the following result regarding sufficient conditions under which some of the sixteen syntactic algebraizability properties coincide.

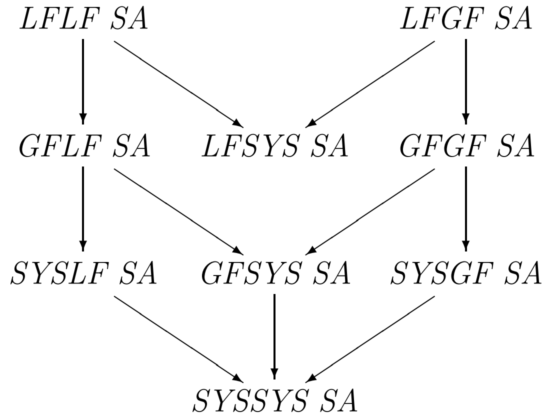
Corollary 738 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments.*

(a) If $\mathcal{I}$ *is systemic, then the syntactic algebraizability hierarchy collapses to the one depicted below;*



(b) If $I^\flat$ *has only two arguments (i.e., is parameter free), then the syntactic algebraizability hierarchy collapses to the one depicted below, where the system versions of both the SPA and the invertibility properties are*

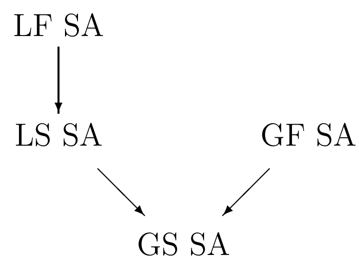
grouped together under the label SYS.



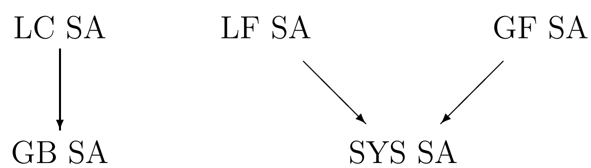
Proof: The statement follows from Corollary 727 and Proposition 732. ■

For a systemic π-institution with a parameter-free set of natural transformations, there is only one syntactic protoalgebraicity property, since all versions of syntactic protoalgebraicity and all versions of invertibility collapse to a single property.

Instead of studying this entire hierarchy in detail, we refocus, once again, on the uniformly defined classes. So we define **LF SA**, **LS SA**, **GF SA** and **GS SA** to mean, respectively, LFLF syntactic, LSLS syntactic, GFGF syntactic and GSGS syntactic algebraizability. These classes form the subhierarchy depicted below.



Moreover, according to Corollary 738, this reduces to the hierarchy depicted on the left below for a systemic π-institution and to the one depicted on the right below for a parameter free set of natural transformations.



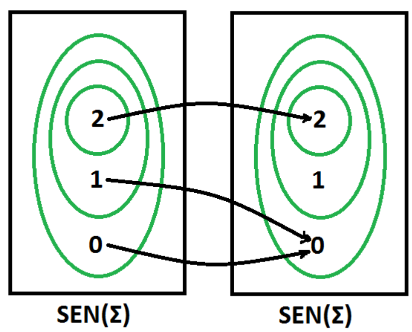
We provide examples to show that the inclusions between the four uniform classes of the syntactic algebraizability hierarchy are proper in general.

First, we present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that has the LS syntactic algebraizability, but not the LF syntactic algebraizability in $\mathcal{I}$.

Example 739 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 0$, $1 \mapsto 0$ *and* $2 \mapsto 2$;
- $N^\flat$ *is the category of natural transformations generated by*
  - *a unary natural transformation* $\lambda^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined by letting* $\lambda^\flat_\Sigma : \{0,1,2\} \to \{0,1,2\}$ *be given by* $\lambda^\flat_\Sigma(x) = 2$, *for all* $x \in \{0,1,2\}$;
  - *a binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 2, & \text{if } x = y \text{ or } \{x,y\} = \{0,1\} \\ 1, & \text{if } \{x,y\} = \{1,2\} \\ 0, & \text{otherwise} \end{cases}$$



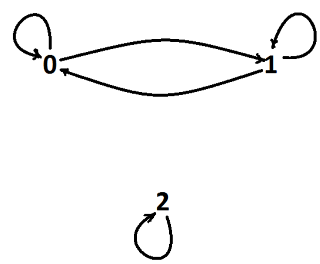
Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\}, \{1,2\}, \{0,1,2\}\}.$$

Note that there are three theory families, but only $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$ *are theory systems. So* $\mathcal{I}$ *is not systemic.*

Consider the set $I^\flat = \{\sigma^\flat\}$, *with* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the local system syntactic algebraizability in* $\mathcal{I}$, *but it does not have the local family syntactic algebraizability in* $\mathcal{I}$.

First, we look at local system equivalence. The defining conditions for reflexivity, symmetry and transitivity of $I^\flat$ *in* $\mathcal{I}$ *are all trivially satisfied for* $\mathrm{SEN}^\flat$. *For the theory system* $\mathrm{Thm}(\mathcal{I})$, *it suffices to observe that the elements*



of $\mathrm{SEN}^\flat(\Sigma)$ *are related in local system equivalence modulo* $\mathrm{Thm}(\mathcal{I})$ *as shown in the diagram. Therefore,* $I^\flat$ *has the local system equivalence in* $\mathcal{I}$.

Next, observe that, for all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, *the pairs* $(\sigma^\flat_\Sigma(\phi,0),\sigma^\flat_\Sigma(\phi,1))$, $(\sigma^\flat_\Sigma(0,\phi),\sigma^\flat_\Sigma(1,\phi))$ *and* $(\lambda^\flat_\Sigma(0),\lambda^\flat_\Sigma(1))$ *are related via* $I^\flat$ *modulo* $\mathrm{Thm}(\mathcal{I})$. *Thus,* $I^\flat$ *has the local system congruence in* $\mathcal{I}$.

Next, note that, since the only pairs (ϕ,ψ), *with* $\phi \neq \psi$, *such that* $\sigma^\flat_\Sigma(\phi,\psi) \in \mathrm{Thm}_\Sigma(\mathcal{I})$ *are* $(0,1)$ *and* $(1,0)$ *and for neither of these is* $\phi \in \mathrm{Thm}_\Sigma(\mathcal{I})$, $I^\flat$ *has the local system modus ponens in* $\mathcal{I}$.

Finally, consider the set $\tau : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *of natural transformations in* $N^\flat$, *given by* $\tau = \{\iota \approx \lambda^\flat\}$, *where* $\iota : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *is the identity natural transformation. Since, for every* $\phi \in \mathrm{SEN}^\flat(\Sigma)$, *we have*

$$\phi \in \mathrm{Thm}_\Sigma(\mathcal{I}) \quad \text{iff} \quad \overleftrightarrow{I}_\Sigma(\phi,\lambda^\flat_\Sigma(\phi)) \subseteq \mathrm{Thm}_\Sigma(\mathcal{I}),$$

we also get that $I^\flat$ *has the local system invertibility in* $\mathcal{I}$ *and, therefore, we conclude that* $I^\flat$ *has the local system algebraizability in* $\mathcal{I}$.

On the other hand, $1 \in \{1,2\}$ *and* $\sigma^\flat_\Sigma(1,0) = 2 \in \{1,2\}$, *but* $0 \notin \{1,2\}$. *Therefore,* $I^\flat$ *does not have the local family modus ponens in* $\mathcal{I}$ *and, hence, a fortiori, it does not have the local family syntactic algebraizability in* $\mathcal{I}$.

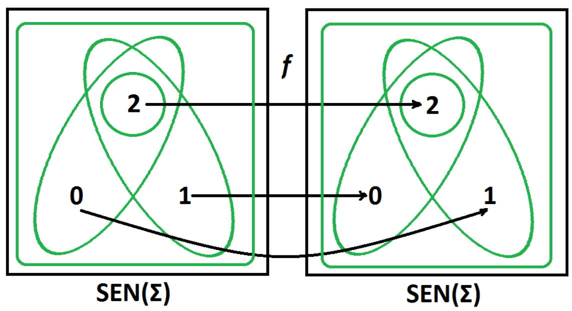
Next, we present an example to show that there is π-institution $\mathcal{I}$ with a set of natural transformations that has the GS syntactic algebraizability but not the GF syntactic algebraizability in $\mathcal{I}$.

Example 740 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = i_\Sigma$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 1$, $1 \mapsto 0$ *and* $2 \mapsto 2$;
- $N^\flat$ *is the category of natural transformations generated by*
  - *a unary natural transformation* $\lambda^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined by letting* $\lambda^\flat_\Sigma : \{0,1,2\} \to \{0,1,2\}$ *be given by* $\lambda^\flat_\Sigma(x) = 2$, *for all* $x \in \{0,1,2\}$;

– *a single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y,z) = \begin{cases} 2, & \text{if } x = y \text{ or } \{x,y\} = \{0,1\} \\ 1, & \text{if } (x,y) = (0,2) \text{ or } (x,y) = (2,0) \\ 0, & \text{if } (x,y) = (1,2) \text{ or } (x,y) = (2,1) \end{cases}.$$



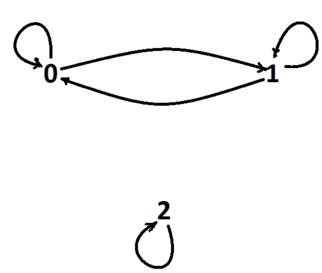
Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\},\{0,2\},\{1,2\},\{0,1,2\}\}.$$

$\mathcal{I}$ *has four theory families* $\mathrm{Thm}(\mathcal{I})$, $T = \{\{0,2\}\}$, $T' = \{\{1,2\}\}$ *and* $\mathrm{SEN}^\flat$, *but only two theory systems* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$. *In particular,* $\mathcal{I}$ *is not systemic.*

Consider the set $I^\flat = \{\sigma^\flat\}$, *with* $\sigma^\flat$ *having both arguments distinguished. We show that* $I^\flat$ *has the global system syntactic algebraizability in* $\mathcal{I}$, *but it does not have the global family syntactic algebraizability in* $\mathcal{I}$.

First, we look at global system equivalence. The defining conditions for reflexivity, symmetry and transitivity of $I^\flat$ *in* $\mathcal{I}$ *are all trivially satisfied for* $\mathrm{SEN}^\flat$. *For the theory system* $\mathrm{Thm}(\mathcal{I})$, *it suffices to observe that the elements of* $\mathrm{SEN}^\flat(\Sigma)$ *are related in global system equivalence modulo* $\mathrm{Thm}(\mathcal{I})$ *as shown in the diagram. Therefore,* $I^\flat$ *has the global system equivalence in* $\mathcal{I}$.



Next, observe that, for all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, *the pairs* $(\sigma^\flat_\Sigma(\phi,0), \sigma^\flat_\Sigma(\phi,1))$, $(\sigma^\flat_\Sigma(0,\phi), \sigma^\flat_\Sigma(1,\phi))$ *and* $(\lambda^\flat_\Sigma(0), \lambda^\flat_\Sigma(1))$ *are related via* $I^\flat$ *modulo* $\mathrm{Thm}(\mathcal{I})$. *Thus,* $I^\flat$ *has the global system congruence in* $\mathcal{I}$.

Next, note that, since the only pairs (ϕ,ψ)*, with* $\phi \neq \psi$*, such that* $\sigma^\flat_\Sigma[\phi,\psi] \leq \mathrm{Thm}(\mathcal{I})$ *are* $(0,1)$ *and* $(1,0)$ *and for neither of these is* $\phi \in \mathrm{Thm}_\Sigma(\mathcal{I})$*,* $I^\flat$ *has the global system modus ponens in* $\mathcal{I}$*.*

Finally, consider the set $\tau : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *of natural transformations in* $N^\flat$*, given by* $\tau = \{\iota \approx \lambda^\flat\}$*, where* $\iota : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *is the identity natural transformation. Since, for every* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*, we have*

$$\phi \in \mathrm{Thm}_\Sigma(\mathcal{I}) \quad \textit{iff} \quad \overleftrightarrow{I}_\Sigma[\phi, \lambda^\flat_\Sigma(\phi)] \leq \mathrm{Thm}(\mathcal{I}),$$

we also get that $I^\flat$ *has the global system invertibility in* $\mathcal{I}$ *and, therefore, we conclude that* $I^\flat$ *has the global system algebraizability in* $\mathcal{I}$*.*

On the other hand, $1 \in \{1,2\}$ *and* $\sigma^\flat_\Sigma[1,0] \leq \{\{1,2\}\}$*, but* $0 \notin \{1,2\}$*. Therefore,* $I^\flat$ *does not have the global family modus ponens in* $\mathcal{I}$ *and, hence, a fortiori, it does not have the global family syntactic algebraizability in* $\mathcal{I}$*.*

Note that the preceding example also shows that there is π-institution $\mathcal{I}$ with a set of natural transformations that has the GS syntactic algebraizability but not the LF syntactic algebraizability in $\mathcal{I}$. We present also an additional example depicting a π-institution $\mathcal{I}$ with a set of natural transformations $I^\flat$, with two distinguished arguments, that has the GS syntactic algebraizability but not the LS syntactic algebraizability in $\mathcal{I}$.

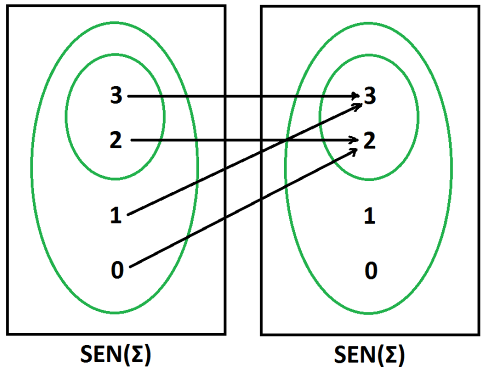
Example 741 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2,3\} \to \{0,1,2,3\}$ *given by* $0 \mapsto 2$*,* $1 \mapsto 3$*,* $2 \mapsto 2$ *and* $3 \mapsto 3$*;*
- $N^\flat$ *is the category of natural transformations generated by*
  - *a unary natural transformation* $\lambda^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined by letting* $\lambda^\flat_\Sigma : \{0,1,2,3\} \to \{0,1,2,3\}$ *be given by* $0 \mapsto 2$*,* $1 \mapsto 3$*,* $2 \mapsto 2$ *and* $3 \mapsto 3$*;*
  - *a ternary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2,3\}^3 \to \{0,1,2,3\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y,z) = \begin{cases} 2, & \text{if } x = y \text{ or } (x,y) = (0,1) \text{ or } z = 2 \text{ or } z = 3 \\ 0, & \text{otherwise} \end{cases}.$$

Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2,3\},\{0,1,2,3\}\}.$$



Note that both theory families, $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, are also theory systems. So* $\mathcal{I}$ *is a systemic* π*-institution.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the global (family or system) syntactic algebraizability in* $\mathcal{I}$*, but it does not have the local (family or system) syntactic algebraizability in* $\mathcal{I}$*.*

Concerning global equivalence, the defining conditions for reflexivity, symmetry and transitivity of $I^\flat$ *in* $\mathcal{I}$ *are all trivially satisfied for* $\mathrm{SEN}^\flat$*. For the theory system* $\mathrm{Thm}(\mathcal{I})$*, it suffices to observe that the relation of global equivalence modulo* $\mathrm{Thm}(\mathcal{I})$ *is the identity relation. Therefore,* $I^\flat$ *has the global system equivalence in* $\mathcal{I}$*. Because of that, the global compatibility and the global modus ponens are trivially satisfied.*

Finally, consider the set $\tau : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *of natural transformations in* $N^\flat$*, given by* $\tau = \{\iota \approx \lambda^\flat\}$*, where* $\iota : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *is the identity natural transformation. Since, for every* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*, we have*

$$\phi \in \mathrm{Thm}_\Sigma(\mathcal{I}) \quad \textit{iff} \quad \overleftrightarrow{I}_\Sigma[\phi, \lambda^\flat_\Sigma(\phi)] \leq \mathrm{Thm}(\mathcal{I}),$$

we also get that $I^\flat$ *has the global system invertibility in* $\mathcal{I}$ *and, therefore, we conclude that* $I^\flat$ *has the global system algebraizability in* $\mathcal{I}$*.*

On the other hand, $\sigma^\flat_\Sigma(0,1,\xi) \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, for all* $\xi \in \mathrm{SEN}^\flat(\Sigma)$*, but* $\sigma^\flat_\Sigma(1,0,0) = 0 \notin \mathrm{Thm}_\Sigma(\mathcal{I})$*, whence* $I^\flat$ *does not have the local symmetry in* $\mathcal{I}$ *and, therefore, a fortiori, fails to satisfy the local syntactic algebraizability in* $\mathcal{I}$*.*

We close with a transfer property for the syntactic algebraizability properties that we have focused on in this section.

Corollary 742 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $I^\flat$ *a collection of natural transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, with two distinguished arguments, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $I^\flat$ *has a (uniform) syntactic algebraizability property in* $\mathcal{I}$ *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*,* I *has the corresponding syntactic algebraizability property in* $\mathcal{A}$*.*

Proof: This follows directly from Corollary 730 and Proposition 736. ∎

10.14 Regularity

We turn now to the study of various versions of the regularity property, based on the local versus global and the theory family versus theory system dualities.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. We say that:

- $I^\flat$ has the **local family regularity in** $\mathcal{I}$ if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi, \psi \in T_\Sigma \text{ imply } I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq T_\Sigma, \text{ for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma);$$
- $I^\flat$ has the **local system regularity in** $\mathcal{I}$ if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi, \psi \in T_\Sigma \text{ imply } I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq T_\Sigma, \text{ for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma);$$
- $I^\flat$ has the **global family regularity in** $\mathcal{I}$ if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi, \psi \in T_\Sigma \text{ imply } I^\flat_\Sigma[\phi, \psi] \leq T;$$
- $I^\flat$ has the **global system regularity in** $\mathcal{I}$ if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi, \psi \in T_\Sigma \text{ imply } I^\flat_\Sigma[\phi, \psi] \leq T.$$

We give now the hierarchy of regularity properties.

Proposition 743 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$ having two distinguished arguments.*

(a) If $I^\flat$ has the global family regularity, then it has the local family regularity in $\mathcal{I}$;

(b) If $I^\flat$ has the local family regularity, then it has the local system regularity in $\mathcal{I}$;

(c) $I^\flat$ has the global system regularity if and only if it has the local system regularity in $\mathcal{I}$.

Proof:

(a) Suppose that $I^\flat$ has the global family regularity in $\mathcal{I}$. Consider $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Then, we have, by hypothesis, $I^\flat_\Sigma[\phi, \psi] \leq T$. But this implies that $I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq T_\Sigma$, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$. Thus, $I^\flat$ has the local family regularity in $\mathcal{I}$.

(b) The conclusion follows directly from the fact that every theory system is a theory family of $\mathcal{I}$.

(c) For the "only if" direction, we repeat the argument used in the proof of Part (a) except reasoning exclusively in terms of theory systems rather than using arbitrary theory families.

Suppose, conversely, that $I^\flat$ has the local system regularity in $\mathcal{I}$. Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Since $T \in \mathrm{ThSys}(\mathcal{I})$, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi) \in T_{\Sigma'}.$$

Thus, by the local system regularity, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$I^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), \vec{\xi}) \subseteq T_{\Sigma'}.$$

Since this holds for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$, we get that $I^\flat_\Sigma[\phi, \psi] \leq T$, Therefore, $I^\flat$ has the global system regularity in $\mathcal{I}$.

■

Proposition 743 has established the following hierarchy of regularity properties:

$$\begin{array}{c}\text{Global Family Regular}\\ \downarrow\\ \text{Local Family Regular}\\ \downarrow\\ \text{System Regular}\end{array}$$

We also note the following regarding natural sufficient conditions under which some of these properties coincide.

Proposition 744 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. If $\mathcal{I}$ is systemic, then all three regularity properties coincide.*

Proof: If $\mathcal{I}$ is systemic, then the (global) system regularity property coincides with the family regularity property and this causes the collapsing of the hierarchy. ■

So in the case of a systemic π-institution $\mathcal{I}$, there is only one possible regularity property.

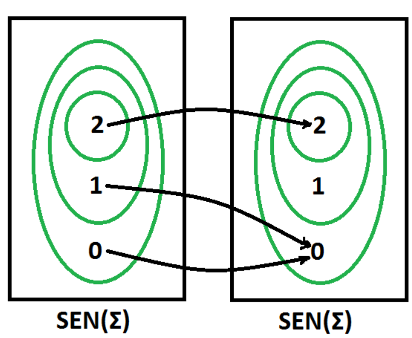
We provide some examples to show that the implications of Proposition 743 are not equivalences in general, i.e., in the hierarchy shown above all inclusions of classes of π-institutions with a set of natural transformations satisfying the corresponding regularity properties are proper inclusions.

We first present an example to show that there is π-institution $\mathcal{I}$ with a set of natural transformations that has the local family regularity but not the global family regularity in $\mathcal{I}$.

Example 745 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 0$*,* $1 \mapsto 0$ *and* $2 \mapsto 2$*;*
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1\}^2 \to \{0,1\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 2, & \text{if } x = 2 \text{ or } y = 2 \\ 1, & \text{if } (x,y) = (1,1) \\ 0, & \text{otherwise} \end{cases} .$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}$*.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *having two distinguished arguments.*

$\mathcal{I}$ has three theory families, but only $\mathrm{Thm}(\mathcal{I})$ *and* SEN *are theory systems. We show that $I^\flat$ has the local family regularity in $\mathcal{I}$, but it does not have the global family regularity in $\mathcal{I}$.*

For the local family regularity note, first, that $\sigma^\flat_\Sigma(2,2) = 2$, which takes care of $\mathrm{Thm}(\mathcal{I})$ *and that the case of* $\mathrm{SEN}^\flat$ *is trivial. So we only need to check the case with $T = \{\{1,2\}\}$. Since $\sigma^\flat_\Sigma(2,2) = \sigma^\flat_\Sigma(1,2) = \sigma^\flat_\Sigma(2,1) = 2$ and $\sigma^\flat_\Sigma(1,1) = 1$, the defining condition for local family regularity is also satisfied for $T = \{\{1,2\}\}$. Therefore, $I^\flat$ has the local family regularity in $\mathcal{I}$.*

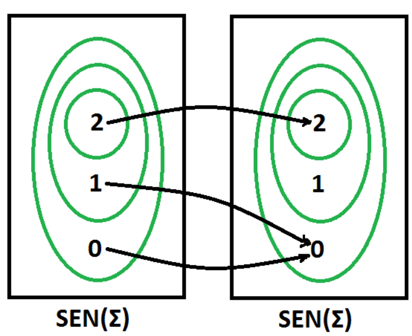
On the other hand, we have $1 \in \{1,2\}$ but $\sigma^\flat_\Sigma(\mathrm{SEN}^\flat(f)(1), \mathrm{SEN}^\flat(f)(1)) = \sigma^\flat_\Sigma(0,0) = 0 \notin \{1,2\}$. Thus, $1 \in T$, but $\sigma^\flat_\Sigma[1,1] \nleq T$, which shows that $I^\flat$ does not have the global family regularity in $\mathcal{I}$.

Next we present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that has the system regularity but not the local family regularity in $\mathcal{I}$.

Example 746 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object Σ and a single (non-identity) morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by $0 \mapsto 0$, $1 \mapsto 0$ and $2 \mapsto 2$;*
- *$N^\flat$ is the category of natural transformations generated by the single binary natural transformation $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ defined by letting $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ be given, for all $a, b \in \mathrm{SEN}^\flat(\Sigma)$, by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 2, & \text{if } x = 2 \text{ or } y = 2 \\ 0, & \text{otherwise} \end{cases}$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be the π-institution determined by $\mathcal{C}_\Sigma = \{\{2\}, \{1,2\}, \{0,1,2\}\}$. Consider the set $I^\flat = \{\sigma^\flat\}$, with $\sigma^\flat$ having two distinguished arguments.

$\mathcal{I}$ has three theory families $\mathrm{Thm}(\mathcal{I})$, $T = \{\{1,2\}\}$ *and* $\mathrm{SEN}^\flat$*, but only two theory systems* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*. We show that $I^\flat$ has the (local) system regularity in $\mathcal{I}$, but it does not have the local family regularity in $\mathcal{I}$.*

For the local system regularity note that $\sigma^\flat_\Sigma(2,2) = 2$, which takes care of $\mathrm{Thm}(\mathcal{I})$*, and that the case of* $\mathrm{SEN}^\flat$ *is trivial.*

On the other hand, for the local family regularity, note that $1 \in T_\Sigma = \{1,2\}$, but $\sigma^\flat_\Sigma(1,1) = 0 \notin T_\Sigma$. Therefore $I^\flat$ does not have the local family regularity in $\mathcal{I}$.

We now prove a transfer property for regularity.

Proposition 747 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $I^\flat$ a collection of natural transformations $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with two distinguished arguments, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. $I^\flat$ has a regularity property in $\mathcal{I}$ if and only if, for every algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, I has the corresponding regularity property in $\mathcal{A}$.*

Proof: If I has a regularity property in $\mathcal{A}$, for all $\mathcal{A}$, then it has the same regularity property in $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$. Since $\langle \mathbf{F}, C^{\mathcal{F}}\rangle = \mathcal{I}$, we conclude that $I^\flat$ has the corresponding regularity property in $\mathcal{I}$.

Suppose, conversely, that $I^\flat$ has a regularity property in $\mathcal{I}$. We look at each of the three properties in turn.

(a) Suppose $I^\flat$ has the global family regularity in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$ and $\alpha_\Sigma(\psi) \in T_{F(\Sigma)}$. Then $\phi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})$ and $\psi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})$. Since, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, we get by global family regularity, $I^\flat_\Sigma[\phi, \psi] \leq \alpha^{-1}(T)$. Thus, by Lemma 95, $I_{F(\Sigma)}[\alpha_\Sigma(\phi), \alpha_\Sigma(\psi)] \leq T$. We conclude that I has the global family regularity in $\mathcal{A}$.

(b) Suppose $I^\flat$ has the local family regularity in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$ and $\alpha_\Sigma(\psi) \in T_{F(\Sigma)}$. Then $\phi \in \alpha^{-1}(T_{F(\Sigma)})$ and $\psi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})$. Since $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, we get by local family regularity, that

$$I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq \alpha_\Sigma^{-1}(T_{F(\Sigma)}), \text{ for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma).$$

Thus, $\alpha_\Sigma(I^\flat_\Sigma(\phi, \psi, \vec{\xi})) \subseteq T_{F(\Sigma)}$ or, equivalently,

$$I_{F(\Sigma)}(\alpha_\Sigma(\phi), \alpha_\Sigma(\psi), \alpha_\Sigma(\vec{\xi})) \subseteq T_{F(\Sigma)}, \text{ for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma).$$

It follows, taking into account the surjectivity of $\langle F, \alpha\rangle$, that I has the local family regularity in $\mathcal{A}$.

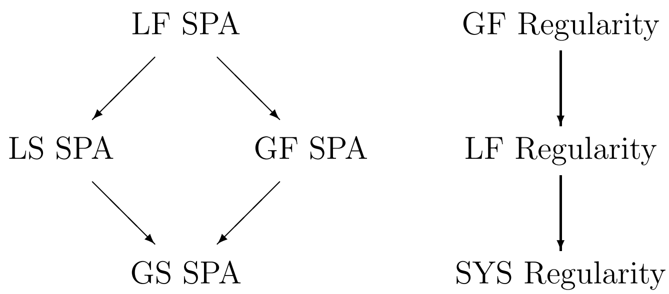
(c) The system regularity follows analogously, taking into account the fact that if $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, then $\alpha^{-1}(T) \in \mathrm{ThSys}(\mathcal{I})$.

■

10.15 Syntactic Regularity

In this section we focus on the four uniform syntactic protoalgebraicity properties, LF SPA, LS SPA, GF SPA and GS SPA, and we add to those versions of the regularity property to obtain several versions of the syntactic regularity property.

By previous work, we know that the four uniform SPA properties constitute the hierarchy shown on the left below.



Moreover, by our study of regularity, we know that the various versions of regularity fall into the linear hierarchy shown on the right of the diagram.

By combining syntactic protoalgebraicity with regularity properties, we obtain twelve syntactic regularity properties as follows. Let $X \in \{\mathrm{LF}, \mathrm{LS}, \mathrm{GF}, \mathrm{GS}\}$ and $Y \in \{\mathrm{LF}, \mathrm{GF}, \mathrm{SYS}\}$, where LF stands for "Local Family", LS stands for "Local System", GF stands for "Global Family", GS stands for "Global System" and SYS stands for "SYStem", abbreviating both the local and the global system properties, in case they are identical.

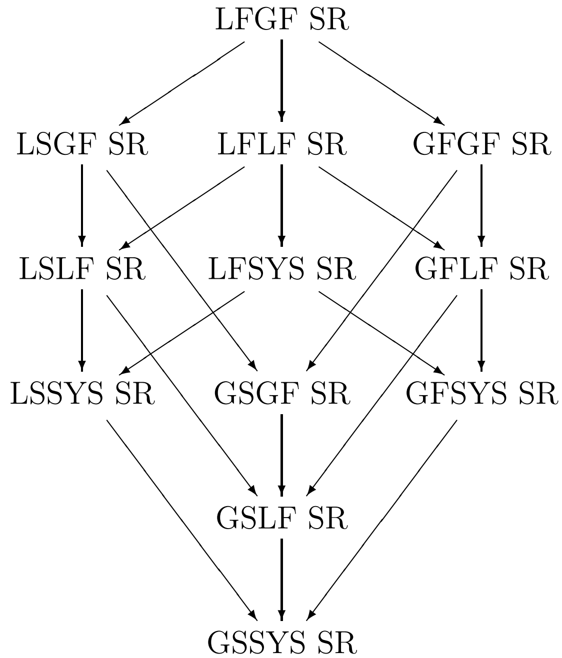
Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$ having two distinguished arguments. We say that $I^\flat$ has the **XY syntactic regularity in $\mathcal{I}$ (XY SR in $\mathcal{I}$)** if it has

- the X syntactic protoalgebraicity in $\mathcal{I}$;
- the Y regularity in $\mathcal{I}$.

Based on the hierarchies of the syntactic protoalgebraicity and regularity properties, we obtain the following hierarchical structure for the various flavors of the syntactic regularity property.

Corollary 748 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$ having two distinguished arguments. The twelve syntactic regularity properties form the hierarchy shown on the accompanying diagram.*

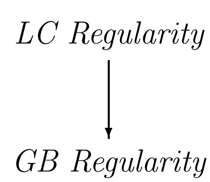
Proof: This follows directly from Corollary 726 and Proposition 743. ∎



Based on the analysis performed on SPA and regularity, we have the following result regarding sufficient conditions under which some of the twelve syntactic regularity properties coincide.

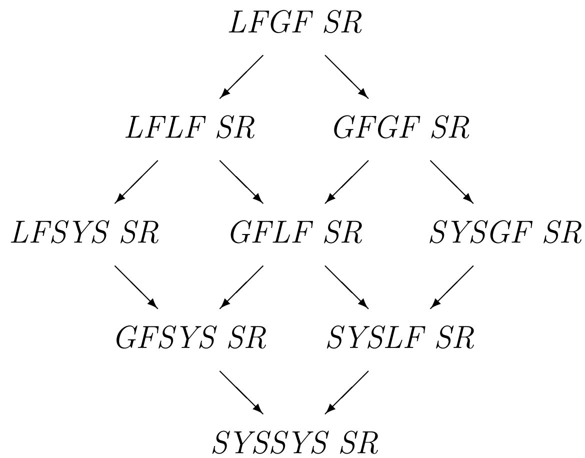
Corollary 749 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$*, with two distinguished arguments.*

(a) If $\mathcal{I}$ *is systemic, then the syntactic regularity hierarchy collapses to the one depicted below;*



(b) If $I^\flat$ *has only two arguments (i.e., is parameter free), then the syntactic regularity hierarchy collapses to the one depicted below, where the System versions of both the SPA and the invertibility properties are*

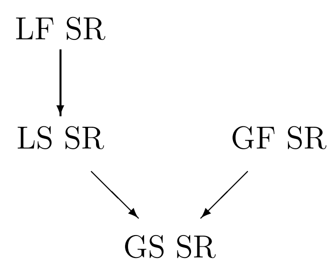
grouped together under the label SYS.



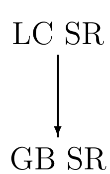
Proof: The statement follows from Corollary 727 and Proposition 743. ■

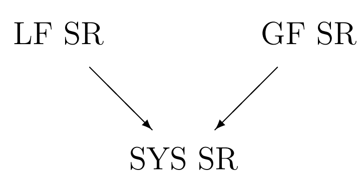
For a systemic π-institution with a parameter-free set of natural transformations, there is only one syntactic regularity property, since all versions of syntactic protoalgebraicity and all versions of regularity collapse to a single property.

Instead of studying this entire hierarchy in detail, we concentrate again on the uniformly defined classes. So we define **LF SR**, **LS SR**, **GF SR** and **GS SR** to mean, respectively, LFLF syntactic, LSLS syntactic, GFGF syntactic and GSGS syntactic regularity. These classes form, according to Corollary 748, the sub hierarchy depicted below.



Moreover, according to Corollary 749, this reduces to the hierarchy depicted on the left below for a systemic π-institution and to the one depicted on the right below for a parameter free set of natural transformations.





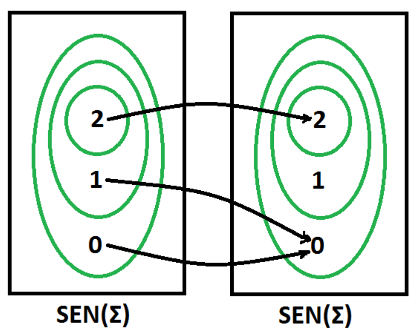
We provide examples to show that the inclusions between these four uniform classes of the syntactic regularity hierarchy are proper in general.

First, we present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that has the LS syntactic regularity, but not the LF syntactic regularity in $\mathcal{I}$.

Example 750 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 0$*,* $1 \mapsto 0$ *and* $2 \mapsto 2$*;*
- $N^\flat$ *is the category of natural transformations generated by a binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 2, & \text{if } x = y \text{ or } \{x,y\} = \{0,1\} \\ 1, & \text{if } \{x,y\} = \{1,2\} \\ 0, & \text{otherwise} \end{cases}$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

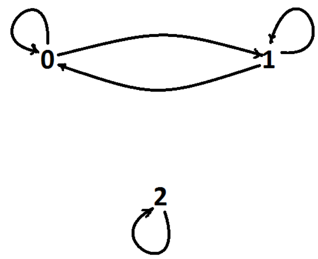
$$\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}.$$

Note that there are three theory families, but only $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$ *are theory systems. So* $\mathcal{I}$ *is not systemic.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the local system syntactic regularity in* $\mathcal{I}$*, but it does not have the local family syntactic regularity in* $\mathcal{I}$*.*

First, we look at local system equivalence. The defining conditions for reflexivity, symmetry and transitivity of $I^\flat$ *in* $\mathcal{I}$ *are all trivially satisfied for* $\mathrm{SEN}^\flat$*. For the theory system* $\mathrm{Thm}(\mathcal{I})$*, it suffices to observe that the elements of* $\mathrm{SEN}^\flat(\Sigma)$ *are related in local system equivalence modulo* $\mathrm{Thm}(\mathcal{I})$ *as shown in the diagram. Therefore,* $I^\flat$ *has the local system equivalence in* $\mathcal{I}$*.*

Next, observe that, for all $\phi \in \mathrm{SEN}^\flat(\Sigma)$*, the pairs* $(\sigma^\flat_\Sigma(\phi,0), \sigma^\flat_\Sigma(\phi,1))$ *and* $(\sigma^\flat_\Sigma(0,\phi), \sigma^\flat_\Sigma(1,\phi))$ *are related via* $I^\flat$ *modulo* $\mathrm{Thm}(\mathcal{I})$*. Thus,* $I^\flat$ *has the local system congruence in* $\mathcal{I}$*.*



Next, note that, since the only pairs (ϕ,ψ), with $\phi \neq \psi$, such that $\sigma^\flat_\Sigma(\phi,\psi) \in \mathrm{Thm}_\Sigma(\mathcal{I})$ are $(0,1)$ and $(1,0)$ and for neither of these is $\phi \in \mathrm{Thm}_\Sigma(\mathcal{I})$, $I^\flat$ has the local system modus ponens in $\mathcal{I}$.

Finally, for local system regularity, note that the defining condition is trivially satisfied for $\mathrm{SEN}^\flat$, whereas, for $\mathrm{Thm}(\mathcal{I})$, we clearly have that, if $\phi,\psi \in \mathrm{Thm}_\Sigma(\mathcal{I})$, then $\phi = \psi = 2$, whence $\sigma^\flat_\Sigma(\phi,\psi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$. Therefore $I^\flat$ has the local system regularity in $\mathcal{I}$ and, therefore, we conclude that $I^\flat$ has the local system syntactic regularity in $\mathcal{I}$.

On the other hand, $1 \in \{1,2\}$ and $\sigma^\flat_\Sigma(1,0) = 2 \in \{1,2\}$, but $0 \notin \{1,2\}$. Therefore, $I^\flat$ does not have the local family modus ponens in $\mathcal{I}$ and, hence, a fortiori, it does not have the local family syntactic regularity in $\mathcal{I}$.

Next, we present an example to show that there is π-institution $\mathcal{I}$ with a set of natural transformations that has the GS syntactic regularity but not the GF syntactic regularity in $\mathcal{I}$.

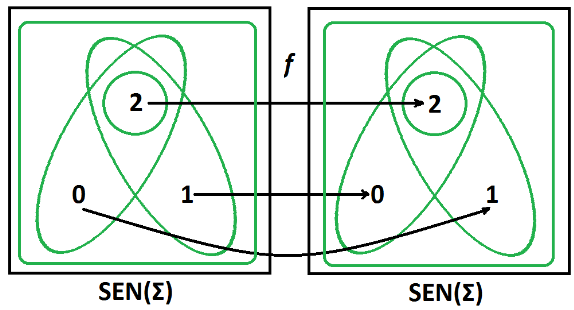
Example 751 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object Σ and a single (non-identity) morphism $f : \Sigma \to \Sigma$, such that $f \circ f = i_\Sigma$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 1$, $1 \mapsto 0$ *and* $2 \mapsto 2$*;*
- *$N^\flat$ is the category of natural transformations generated by a single binary natural transformation $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ defined by letting $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ be given by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 2, & \text{if } x = y \text{ or } \{x,y\} = \{0,1\} \\ 1, & \text{if } \{x,y\} = \{0,2\} \\ 0, & \text{if } \{x,y\} = \{1,2\} \end{cases}$$

Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be the π-institution determined by

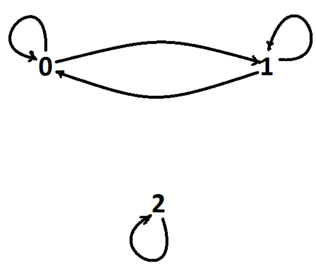
$$\mathcal{C}_\Sigma = \{\{2\},\{0,2\},\{1,2\},\{0,1,2\}\}.$$



$\mathcal{I}$ has four theory families $\mathrm{Thm}(\mathcal{I})$, $T = \{\{0,2\}\}$, $T' = \{\{1,2\}\}$ *and* $\mathrm{SEN}^\flat$, *but only* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$ *are theory systems. In particular, $\mathcal{I}$ is not systemic.*

Consider the set $I^\flat = \{\sigma^\flat\}$, with $\sigma^\flat$ having both arguments distinguished. We show that $I^\flat$ has the global system syntactic regularity in $\mathcal{I}$, but it does not have the global family syntactic regularity in $\mathcal{I}$.

First, we look at global system equivalence. The defining conditions for reflexivity, symmetry and transitivity of $I^\flat$ in $\mathcal{I}$ are all trivially satisfied for $\mathrm{SEN}^\flat$. *For the theory system* $\mathrm{Thm}(\mathcal{I})$, *it suffices to observe that the elements of* $\mathrm{SEN}^\flat(\Sigma)$ *are related in global system equivalence modulo* $\mathrm{Thm}(\mathcal{I})$ *as shown in the diagram. Therefore, $I^\flat$ has the global system equivalence in $\mathcal{I}$.*



Next, observe that, for all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, the pairs $(\sigma^\flat_\Sigma(\phi,0),\sigma^\flat_\Sigma(\phi,1))$ and $(\sigma^\flat_\Sigma(0,\phi),\sigma^\flat_\Sigma(1,\phi))$ are related via $I^\flat$ modulo $\mathrm{Thm}(\mathcal{I})$. *Thus, $I^\flat$ has the global system congruence in $\mathcal{I}$.*

Now note that, since the only pairs (ϕ,ψ), with $\phi \neq \psi$, such that $\sigma^\flat_\Sigma[\phi,\psi] \leq$ $\mathrm{Thm}(\mathcal{I})$ *are $(0,1)$ and $(1,0)$ and for neither of these is $\phi \in \mathrm{Thm}_\Sigma(\mathcal{I})$, $I^\flat$ has the global system modus ponens in $\mathcal{I}$.*

For the global system regularity, note that the defining condition is satisfied trivially for $\mathrm{SEN}^\flat$, *whereas for* $\mathrm{Thm}(\mathcal{I})$, *if $\phi,\psi \in \mathrm{Thm}_\Sigma(\mathcal{I})$, then $\phi,\psi = 2$, whence we get $\sigma^\flat_\Sigma[\phi,\phi] \leq \mathrm{Thm}(\mathcal{I})$. Therefore, we conclude that $I^\flat$ has the global system syntactic regularity in $\mathcal{I}$.*

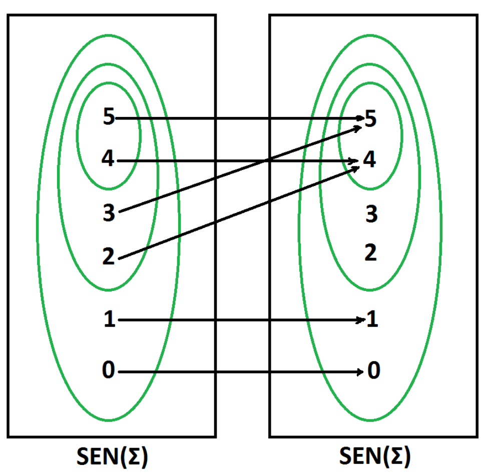
On the other hand, $1 \in \{1,2\}$ and $\sigma^\flat_\Sigma[1,0] \leq \{\{1,2\}\}$, but $0 \notin \{1,2\}$. Therefore, $I^\flat$ does not have the global family modus ponens in $\mathcal{I}$ and, hence, a fortiori, it does not have the global family syntactic regularity in $\mathcal{I}$.

Finally, we present an example of a π-institution $\mathcal{I}$ with a set of natural transformations $I^\flat$, with two distinguished arguments, that has the GS syntactic regularity but not the LS syntactic regularity in $\mathcal{I}$.

Example 752 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3,4,5\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2,3,4,5\} \to \{0,1,2,3,4,5\}$ *given by* $0 \mapsto 0$*,* $1 \mapsto 1$*,* $2 \mapsto 4$*,* $3 \mapsto 5$*,* $4 \mapsto 4$ *and* $5 \mapsto 5$*;*
- $N^\flat$ *is the category of natural transformations generated by a ternary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2,3,4,5\}^3 \to \{0,1,2,3,4,5\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y,z) = \begin{cases} 4, & \text{if } x = y \text{ or } \{x,y\} = \{4,5\} \\ & \quad \text{or } ((x,y) = (1,4) \text{ and } z = 0,1,4,5) \\ & \quad \text{or } ((x,y) = (1,5) \text{ and } z = 0,1,4,5) \\ 2, & \text{else if } \{x,y\} \subseteq \{2,3,4,5\} \\ & \qquad \text{or } (x,y) = (1,2) \text{ or } (x,y) = (1,3) \\ 0, & \text{otherwise} \end{cases}.$$



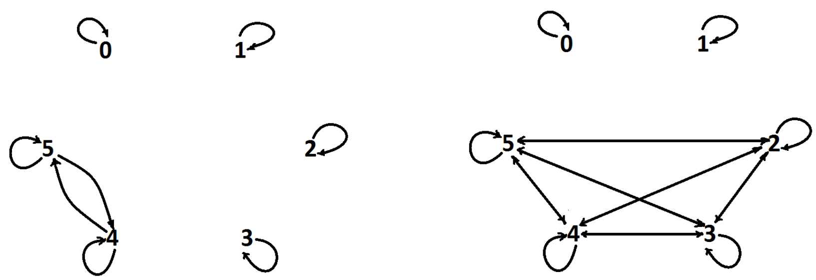
Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{4,5\},\{2,3,4,5\},\{0,1,2,3,4,5\}\}.$$

Note that all three theory families, $\mathrm{Thm}(\mathcal{I})$*,* $T = \{\{2,3,4,5\}\}$ *and* $\mathrm{SEN}^\flat$*, are also theory systems. So* $\mathcal{I}$ *is a systemic* π*-institution.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the global (family or system) syntactic regularity in* $\mathcal{I}$*, but it does not have the local (family or system) syntactic regularity in* $\mathcal{I}$*.*

Concerning global equivalence, the defining conditions for reflexivity, symmetry and transitivity of $I^\flat$ *in* $\mathcal{I}$ *are all trivially satisfied for* $\mathrm{SEN}^\flat$*. For the theory system* $\mathrm{Thm}(\mathcal{I})$*, it suffices to observe that the relation of global equivalence modulo* $\mathrm{Thm}(\mathcal{I})$ *is the binary relation on* $\mathrm{SEN}^\flat(\Sigma)$ *depicted on the left graph in the figure. Moreover, the relation of global equivalence modulo*



T *is the binary relation on* $\mathrm{SEN}^\flat(\Sigma)$ *depicted on the right graph in the figure. Therefore,* $I^\flat$ *has the global system equivalence in* $\mathcal{I}$*.*

Looking at these two graphs and taking into account the definition of $\sigma^\flat$*, we can see that the defining conditions of the global compatibility and the global modus ponens are also satisfied for all three theory systems.*

For global regularity, note again that the defining condition is trivially satisfied for $\mathrm{SEN}^\flat$*, that* $\sigma^\flat_\Sigma[\phi,\psi] \leq \mathrm{Thm}(\mathcal{I})$*, if* $\phi,\psi \in \{4,5\}$*, and that* $\sigma^\flat_\Sigma[\phi,\psi] \leq T$*, if* $\phi,\psi \in \{2,3,4,5\}$*. Thus, we conclude that* $I^\flat$ *has the global regularity in* $\mathcal{I}$ *and, therefore,* $I^\flat$ *has the global system syntactic regularity in* $\mathcal{I}$*.*

On the other hand, $\sigma^\flat_\Sigma(1,2,\xi) = 2 \in T_\Sigma$*, for all* $\xi \in \mathrm{SEN}^\flat(\Sigma)$*, whereas* $\sigma^\flat_\Sigma(2,1,0) = 0 \notin T_\Sigma$*, whence* $I^\flat$ *does not have the local system symmetry in* $\mathcal{I}$ *and, therefore, a fortiori, fails to satisfy the local system syntactic regularity in* $\mathcal{I}$*.*

We close with a transfer property for the (uniform) syntactic regularity properties that we have studied in this section.

Corollary 753 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $I^\flat$ *a collection of natural transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, with two distinguished arguments, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $I^\flat$ *has a (uniform) syntactic regularity property in* $\mathcal{I}$ *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*,* I *has the corresponding syntactic regularity property in* $\mathcal{A}$*.*

Proof: This follows directly from Corollary 730 and Proposition 747. ∎

10.16 Modus Fortis

We conclude with the study of versions of the modus fortis (also known as the Wójcicki or the Rasiowa) property. In the next setion, we call Rasiowa property the combination of syntactic protoalgebraicity with the modus fortis.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$ having two distinguished arguments. We say that:

- $I^\flat$ has the **local family modus fortis** (**local family MF**) **in** $\mathcal{I}$ if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\psi \in T_\Sigma \text{ implies } I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq T_\Sigma, \text{ for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma);$$
- $I^\flat$ has the **local system modus fortis** (**local system MF**) **in** $\mathcal{I}$ if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\psi \in T_\Sigma \text{ implies } I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq T_\Sigma, \text{ for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma);$$
- $I^\flat$ has the **global family modus fortis** (**global family MF**) **in** $\mathcal{I}$ if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\psi \in T_\Sigma \text{ implies } I^\flat_\Sigma[\phi, \psi] \le T;$$
- $I^\flat$ has the **global system modus fortis** (**global system MF**) **in** $\mathcal{I}$ if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\psi \in T_\Sigma \text{ implies } I^\flat_\Sigma[\phi, \psi] \le T.$$

We give now the hierarchy of modus fortis properties.

Proposition 754 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments.*

(a) If $I^\flat$ *has the global family MF, then it has the local family MF in* $\mathcal{I}$*;*

(b) If $I^\flat$ *has the local family MF, then it has the local system MF in* $\mathcal{I}$*;*

(c) $I^\flat$ *has the global system MF if and only if it has the local system MF in* $\mathcal{I}$*.*

Proof:

(a) Suppose that $I^\flat$ has the global family MF in $\mathcal{I}$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\psi \in T_\Sigma$. Then, we have, by hypothesis, $I^\flat_\Sigma[\phi, \psi] \leq T$. This implies that $I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq T_\Sigma$, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$. Thus, $I^\flat$ has the local family MF in $\mathcal{I}$.

(b) The conclusion follows directly from the fact that every theory system is a theory family of $\mathcal{I}$.

(c) For the "only if" direction, we repeat the argument used in the proof of Part (a) except reasoning exclusively in terms of theory systems rather than using arbitrary theory families.

Suppose, conversely, that $I^\flat$ has the local system MF in $\mathcal{I}$. Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\psi \in T_\Sigma$. Since $T \in \mathrm{ThSys}(\mathcal{I})$, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, $\mathrm{SEN}^\flat(f)(\psi) \in T_{\Sigma'}$. Thus, by the local system MF, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$I^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), \vec{\xi}) \subseteq T_{\Sigma'}.$$

Since this holds, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$, we get that $I^\flat_\Sigma[\phi, \psi] \leq T$, Therefore, $I^\flat$ has the global system MF in $\mathcal{I}$.

■

Proposition 754 has established the following hierarchy of Modus Fortis properties:

Global Family MF

↓

Local Family MF

↓

System MF

We also note the following regarding natural sufficient conditions under which some of these properties coincide.

Proposition 755 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments. If* $\mathcal{I}$ *is systemic, then all three modus fortis properties coincide.*

Proof: If $\mathcal{I}$ is systemic, then the (global) system MF coincides with the family MF property and this causes the collapsing of the hierarchy. ■

So in the case of a systemic π-institution $\mathcal{I}$, there is only one possible modus fortis property.

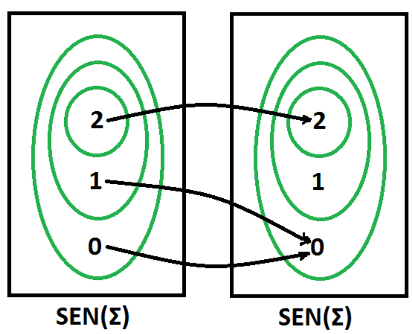
We provide some examples to show that the implications of Proposition 754 are not equivalences in general, i.e., in the hierarchy shown above all inclusions of classes of π-institutions with a set of natural transformations satisfying the corresponding modus fortis properties are proper inclusions.

We first present an example to show that there is π-institution $\mathcal{I}$ with a set of natural transformations that has the local family MF but not the global family MF in $\mathcal{I}$.

Example 756 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 0$*,* $1 \mapsto 0$ *and* $2 \mapsto 2$*;*
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1\}^2 \to \{0,1\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 2, & \text{if } x = 2 \text{ or } y = 2 \\ 1, & \text{if } x \neq 2 \text{ and } y = 1 \\ 0, & \text{otherwise} \end{cases}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$C_\Sigma = \{\{2\}, \{1,2\}, \{0,1,2\}\}.$$

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *having two distinguished arguments.*

$\mathcal{I}$ *has three theory families, but only* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$ *are theory systems. We show that* $I^\flat$ *has the local family modus fortis, but it does not have the global family modus fortis in* $\mathcal{I}$*.*

For the local family MF note, first, that, for all $x \in \mathrm{SEN}^\flat(\Sigma)$, $\sigma^\flat_\Sigma(x,2) = 2$, *which takes care of* $\mathrm{Thm}(\mathcal{I})$, *and that the case of* $\mathrm{SEN}^\flat$ *is trivial. So we only need to check the case with* $T = \{\{1,2\}\}$. *Since, for all* $x \in \mathrm{SEN}^\flat(\Sigma)$, $\sigma^\flat_\Sigma(x,2) = 2$ *and, also,* $\sigma^\flat_\Sigma(0,1) = \sigma^\flat_\Sigma(1,1) = 1$ *and* $\sigma^\flat_\Sigma(2,1) = 2$, *the defining condition for local family MF is also satisfied for* $T = \{\{1,2\}\}$. *Therefore,* $I^\flat$ *has the local family MF in* $\mathcal{I}$.

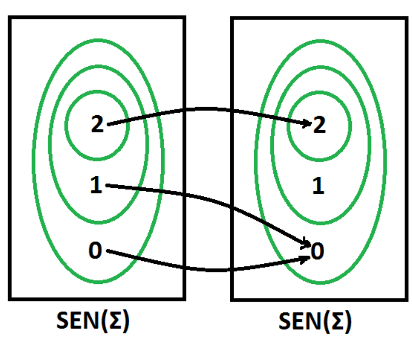
On the other hand, we have $1 \in \{1,2\}$ *but* $\sigma^\flat_\Sigma(\mathrm{SEN}^\flat(f)(0), \mathrm{SEN}^\flat(f)(1)) = \sigma^\flat_\Sigma(0,0) = 0 \notin \{1,2\}$. *Thus,* $1 \in T$, *but* $\sigma^\flat_\Sigma[0,1] \nleq T$, *which shows that* $I^\flat$ *does not have the global family MF in* $\mathcal{I}$.

Next we present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that has the system modus fortis but not the local family modus fortis in $\mathcal{I}$.

Example 757 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 0$, $1 \mapsto 0$ *and* $2 \mapsto 2$;
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given, for all* $a, b \in \mathrm{SEN}^\flat(\Sigma)$, *by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 2, & \text{if } x = 2 \text{ or } y = 2 \\ 0, & \text{otherwise} \end{cases}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\}, \{1,2\}, \{0,1,2\}\}.$$

Consider the set $I^\flat = \{\sigma^\flat\}$, *with* $\sigma^\flat$ *having two distinguished arguments.*

$\mathcal{I}$ has three theory families $\mathrm{Thm}(\mathcal{I})$, $T = \{\{1,2\}\}$ *and* $\mathrm{SEN}^\flat$, *but only two theory systems* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$. *We show that $I^\flat$ has the (local) system MF in $\mathcal{I}$, but it does not have the local family MF in $\mathcal{I}$.*

For the local system MF note that, for all $x \in \mathrm{SEN}^\flat(\Sigma)$, $\sigma^\flat_\Sigma(x,2) = 2$, which takes care of $\mathrm{Thm}(\mathcal{I})$, *and that the case of* $\mathrm{SEN}^\flat$ *is trivial.*

On the other hand, for the local family MF, note that $1 \in T_\Sigma = \{1,2\}$, but $\sigma^\flat_\Sigma(0,1) = 0 \notin T_\Sigma$. Therefore, $I^\flat$ does not have the local family MF in $\mathcal{I}$.

We finally prove a transfer property for modus fortis.

Proposition 758 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $I^\flat$ a collection of natural transformations $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with two distinguished arguments, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. $I^\flat$ has an MF property in $\mathcal{I}$ if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, I has the corresponding MF property in $\mathcal{A}$.*

Proof: If I has an MF property in $\mathcal{A}$, for all $\mathcal{A}$, then it has the same property in $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$. Since $\langle \mathbf{F}, C^{\mathcal{I},\mathcal{F}}\rangle = \mathcal{I}$, we conclude that $I^\flat$ has the corresponding MF property in $\mathcal{I}$.

Suppose, conversely, that $I^\flat$ has an MF property in $\mathcal{I}$. We look at each of the three properties in turn.

(a) Suppose $I^\flat$ has the global family MF in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\alpha_\Sigma(\psi) \in T_{F(\Sigma)}$. Then $\psi \in \alpha^{-1}_\Sigma(T_{F(\Sigma)})$. Since, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, we get by global family MF, $I^\flat_\Sigma[\phi,\psi] \leq \alpha^{-1}(T)$. Thus, by Lemma 95,
$$I_{F(\Sigma)}[\alpha_\Sigma(\phi), \alpha_\Sigma(\psi)] \leq T.$$
We conclude that I has the global family MF in $\mathcal{A}$.

(b) Suppose $I^\flat$ has the local family MF in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\alpha_\Sigma(\psi) \in T_{F(\Sigma)}$. Then $\psi \in \alpha^{-1}_\Sigma(T_{F(\Sigma)})$. Since $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, we get by local family MF, that
$$I^\flat_\Sigma(\phi,\psi,\vec{\xi}) \subseteq \alpha^{-1}_\Sigma(T_{F(\Sigma)}), \text{ for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma).$$
Thus, $\alpha_\Sigma(I^\flat_\Sigma(\phi,\psi,\vec{\xi})) \subseteq T_{F(\Sigma)}$ or, equivalently,
$$I_{F(\Sigma)}(\alpha_\Sigma(\phi), \alpha_\Sigma(\psi), \alpha_\Sigma(\vec{\xi})) \subseteq T_{F(\Sigma)}, \text{ for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma).$$
It follows, taking into account the surjectivity of $\langle F,\alpha\rangle$, that I has the local family MF in $\mathcal{A}$.

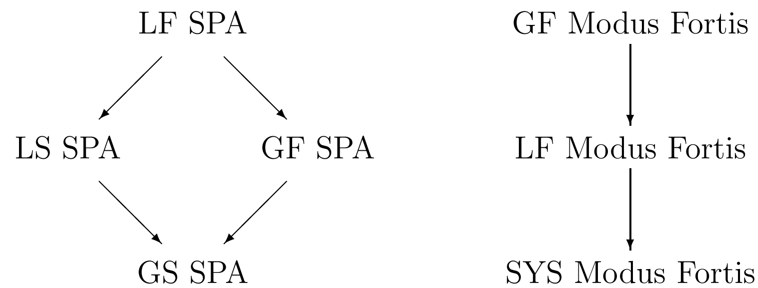
(c) The system MF follows analogously, taking into account the fact that if $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, then $\alpha^{-1}(T) \in \mathrm{ThSys}(\mathcal{I})$.

■

10.17 The Rasiowa Property

In this section we focus again on the four uniform syntactic protoalgebraicity properties, LF SPA, LS SPA, GF SPA and GS SPA, and we add to those versions of the modus fortis property to obtain several versions of the Rasiowa property.

By previous work, we know that the four uniform SPA properties constitute the hierarchy shown on the left below.



Moreover, by our study of modus fortis, we know that the various versions of modus fortis (MF) fall into the linear hierarchy shown on the right of the diagram.

By combining syntactic protoalgebraicity with MF properties, we obtain twelve Rasiowa properties as follows. Let $X \in \{\mathrm{LF}, \mathrm{LS}, \mathrm{GF}, \mathrm{GS}\}$ and $Y \in \{\mathrm{LF}, \mathrm{GF}, \mathrm{SYS}\}$, where LF stands for "Local Family", LS stands for "Local System", GF stands for "Global Family", GS stands for "Global System" and SYS stands for "SYStem", abbreviating both the local and the global system properties, when they are identical.

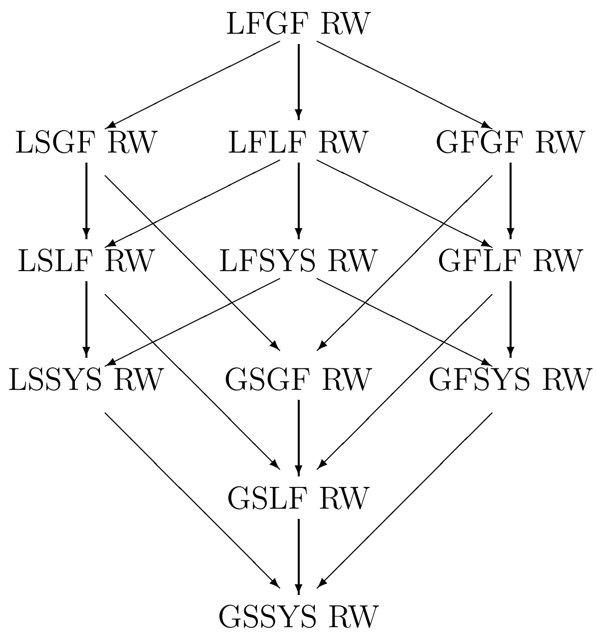
Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$ having two distinguished arguments. We say that $I^\flat$ has the **XY Rasiowa property in $\mathcal{I}$** (**XY RW in $\mathcal{I}$**), or that $I^\flat$ is **XY Rasiowan in $\mathcal{I}$**, if it has

- the X syntactic protoalgebraicity in $\mathcal{I}$;
- the Y modus fortis in $\mathcal{I}$.

Based on the hierarchies of the syntactic protoalgebraicity and MF properties, we obtain the following a priori hierarchical structure for the various flavors of the Rasiowa property.

Corollary 759 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$ having two distinguished arguments. The twelve Rasiowa properties form the hierarchy shown on the accompanying diagram.*

Proof: This follows directly from Corollary 726 and Proposition 754. ∎



It turns out that all these classes collapse to a single class! Indeed, as we show next, the only π-institutions, with a set of natural transformations having two distinguished arguments, satisfying the global system syntactic protoalgebraicity and the system modus fortis are the inconsistent ones. As a consequence, they also satisfy the local family syntactic protoalgebraicity and the global family modus fortis.

Proposition 760 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments. If* $I^\flat$ *has the GSSYS Rasiowa property in* $\mathcal{I}$*, then* $\mathcal{I}$ *is inconsistent.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$. Since $I^\flat$ is reflexive, $I^\flat_\Sigma(\phi, \phi, \vec{\xi}) \subseteq \mathrm{Thm}_\Sigma(\mathcal{I})$, for all $\phi, \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$. Thus, $\mathrm{Thm}_\Sigma(\mathcal{I}) \neq \varnothing$. Fix $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$. Then, for all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, we get, using the SYS Rasiowa property, $I^\flat_\Sigma[\phi, t] \leq \mathrm{Thm}(\mathcal{I})$. Then, by GS symmetry, $I^\flat_\Sigma[t, \phi] \leq \mathrm{Thm}(\mathcal{I})$. Thus, by GS modus ponens, we get $\phi \in \mathrm{Thm}_\Sigma(\mathcal{I})$. Since this holds for all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, we conclude that $\mathrm{Thm}(\mathcal{I}) = \mathrm{SEN}^\flat$ and, therefore, $\mathcal{I}$ is inconsistent. ∎

So the hierarchy of Corollary 759 consists actually of a single property, which we call the **Rasiowa property**, and the only π-institutions satisfying that property are the inconsistent ones.

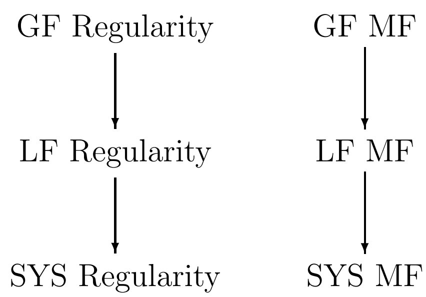
The Rasiowa property also transfers.

Corollary 761 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $I^\flat$ *a collection of natural transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, with two* *distinguished arguments, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $I^\flat$ *has the Rasiowa property in* $\mathcal{I}$ *if and only if, for every algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*,* I *has the Rasiowa property in* $\mathcal{A}$*.*

Proof: This follows directly from Corollary 730 and Proposition 758. ■

10.18 Modus Fortis and Regularity

Recall the hierarchies of the regularity and modus fortis properties that we have introduced previously. These are depicted again below.



The various versions of these three properties are not independent. In fact the modus fortis properties imply the corresponding regularity properties. We prove these straightforward dependencies in the following two propositions.

Proposition 762 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments.*

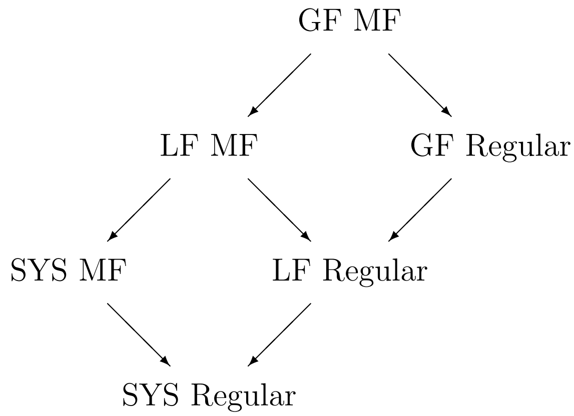
(a) If $I^\flat$ *has the global family MF, then it has the global family regularity in* $\mathcal{I}$*;*

(b) If $I^\flat$ *has the local family MF, then it has the local family regularity in* $\mathcal{I}$*;*

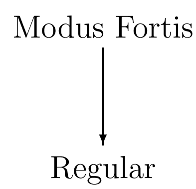
(c) If $I^\flat$ *has the system MF, then it has the system regularity in* $\mathcal{I}$*.*

Proof: We only provide a proof for Part (a), since Parts (b) and (c) can be proved in essentially the same way. So suppose that $I^\flat$ has the global family modus fortis in $\mathcal{I}$ and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Since $\psi \in T_\Sigma$ and $I^\flat$ has the global family modus fortis in $\mathcal{I}$, we get that $I^\flat_\Sigma[\phi, \psi] \leq T$. This show that $I^\flat$ has the global family regularity in $\mathcal{I}$. ■

Proposition 762 together with the previously established hierarchies of regularity and modus fortis properties, establish the following combined hierarchy of these properties.



Recall, now that, if $\mathcal{I}$ is systemic, all three versions of regularity and modus fortis are identified. Therefore, in the case of a systemic π-institution $\mathcal{I}$ with a set $I^\flat$ of natural transformations having two distinguished arguments, the hierarchy above reduces to simply



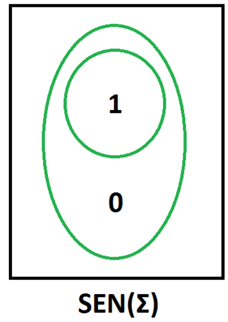
On the other hand, since the property of being parameter-free does not affect either the regularity or the Modus Fortis hierarchies, it has no effect on the mixed hierarchy either.

We present an example of a π-institution $\mathcal{I}$, with a set $I^\flat$ of natural transformations, having two distinguished variables, that has the global family regularity property in $\mathcal{I}$, but does not have the system modus fortis property in $\mathcal{I}$.

Example 763 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1\}^2 \to \{0,1\}$ *be given, for all* $x, y \in \mathrm{SEN}^\flat(\Sigma)$*, by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be the π-institution determined by $\mathcal{C}_\Sigma = \{\{1\}, \{0,1\}\}$. Consider the set $I^\flat = \{\sigma^\flat\}$, with $\sigma^\flat$ having two distinguished arguments.

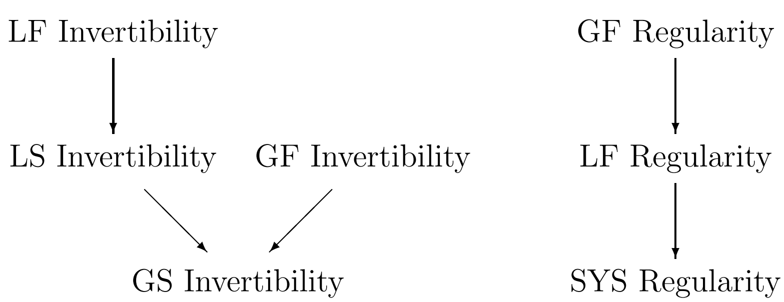
$\mathcal{I}$ has two theory families $\mathrm{Thm}(\mathcal{I})$ and $\mathrm{SEN}^\flat$, both of which are theory systems. So it is a systemic π-institution. We show that $I^\flat$ has the global family regularity in $\mathcal{I}$, but it does not have the system modus fortis in $\mathcal{I}$.

For the global family regularity, note that the condition is trivial when $T = \mathrm{SEN}^\flat$, whereas for $T = \mathrm{Thm}(\mathcal{I})$, if $\phi = \psi = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$, we have $\sigma^\flat_\Sigma(1,1) = 1$, which gives $\sigma^\flat_\Sigma[1,1] \leq \mathrm{Thm}(\mathcal{I})$. Thus, $I^\flat$ is indeed global family regular in $\mathcal{I}$.

On the other hand, note that $1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$, but $\sigma^\flat_\Sigma(0,1) = 0 \notin \mathrm{Thm}_\Sigma(\mathcal{I})$. Therefore, $I^\flat$ does not have the system MF in $\mathcal{I}$.

10.19 Regularity and Invertibility

Recall the hierarchies that we have introduced previously based on invertibility and regularity. These are depicted again below.



Connecting the regularity with the invertibility conditions requires additional hypotheses. Namely, we will suppose that the π-institution under consideration has natural theorems and satisfies some form of the modus ponens property.

Proposition 764 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$, having natural theorems, and $I^\flat$:*

$(\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments.*

(a) If $I^\flat$ *has the global family modus ponens and the global family regularity, then it has the global family invertibility in* $\mathcal{I}$*;*

(b) If $I^\flat$ *has the local family modus ponens and the local family regularity, then it has the local family invertibility in* $\mathcal{I}$*;*

(c) If $I^\flat$ *has the local system modus ponens and the system regularity, then it has the local system invertibility in* $\mathcal{I}$*;*

(d) If $I^\flat$ *has the global system modus ponens and the system regularity, then it has the global system invertibility in* $\mathcal{I}$*.*

Proof: We only provide a proof for Part (a), since Parts (b)-(d) can be proved in essentially the same way. Let $\top^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ be a natural theorem and suppose that $I^\flat$ has the global family modus ponens and the global family regularity in $\mathcal{I}$. Consider the singleton $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ of natural transformations in $N^\flat$, given by

$$\tau^\flat = \{\top^\flat \approx \iota\},$$

where $\iota : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ is the identity natural transformation. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$.

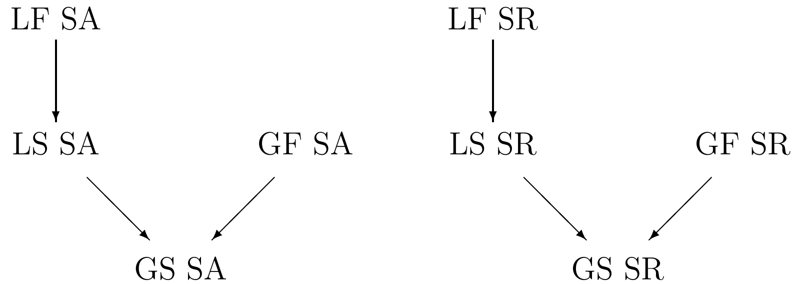
- If $\phi \in T_\Sigma$, then, since $\top^\flat_\Sigma(\phi) \in \mathrm{Thm}_\Sigma(\mathcal{I}) \subseteq T_\Sigma$, we get, by global family regularity, $I^\flat_\Sigma[\top^\flat_\Sigma(\phi), \phi] \leq T$, i.e., $I^\flat_\Sigma[\tau^\flat_\Sigma(\phi)] \leq T$.

- Suppose, conversely, that $I^\flat_\Sigma[\tau^\flat_\Sigma(\phi)] \leq T$. Then $I^\flat_\Sigma[\top^\flat_\Sigma(\phi), \phi] \leq T$. Since $\top^\flat_\Sigma(\phi) \in T_\Sigma$, we get, by global family modus ponens, $\phi \in T_\Sigma$.

We conclude that $\phi \in T_\Sigma$ if and only if $I^\flat_\Sigma[\tau^\flat_\Sigma(\phi)] \leq T$. Thus, $I^\flat$ has the global family invertibility in $\mathcal{I}$, with witnessing set of natural transformations $\tau^\flat$. ■

10.20 The Algebraic Hierarchy

Recall the three hierarchies that we have introduced previously based on uniform combinations of the syntactic protoalgebraizability properties and the invertibility, regularity and modus fortis properties. These formed the hierarchies of syntactically algebraizable (SA), syntactically regular (SR) and Rasiowa properties, respectively. The first two are depicted again below, whereas the last consists of a single property, which, as we saw in Proposition 760, is characteristic of inconsistent π-institutions.



The various versions of these three properties are not independent. Since, as was shown in Proposition 762, the modus fortis properties imply the corresponding regularity properties and, as was shown in Proposition 764, regularity properties, fortified with some form of the modus ponens, imply the corresponding invertibility properties, we obtain ensuing relationships between the Rasiowa, syntactic regularity and syntactic algebraizability properties.

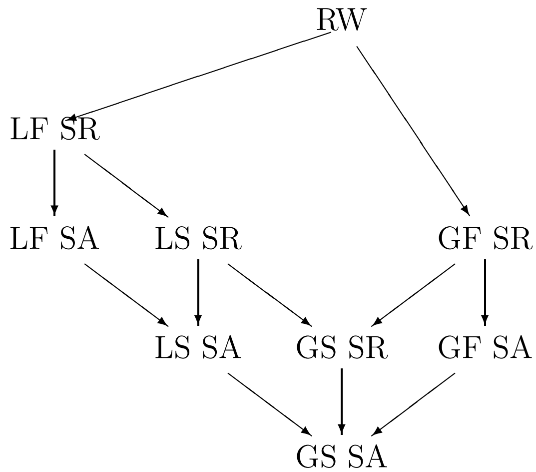
Corollary 765 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments. If* $I^\flat$ *has the Rasiowa property, then it has all four syntactic regularity properties.*

Proof: Directly from the definitions and Proposition 762. ■

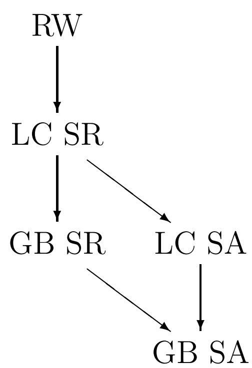
Corollary 766 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments. If* $I^\flat$ *has a syntactic regularity property, then it has the corresponding syntactic algebraizability property in* $\mathcal{I}$.

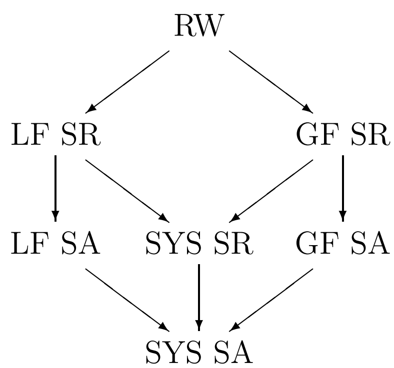
Proof: We present in detail the reasoning for the global family versions. Suppose that $I^\flat$ is a set of natural transformations, with two distinguished arguments, having the global family syntactic regularity in $\mathcal{I}$. Note that, by global family syntactic protoalgebraicity, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma[\phi, \phi] \leq \mathrm{Thm}(\mathcal{I})$. Thus, $\mathcal{I}$ has natural theorems. Moreover, by the definition of global family syntactic regularity, $I^\flat$ has both the global family modus ponens and the global family regularity in $\mathcal{I}$. It follows now, by Proposition 764, that $I^\flat$ has the global family invertibility in $\mathcal{I}$. Thus, it also has the global family syntactic algebraizability in $\mathcal{I}$. ■

Corollaries 765 and 766 together with the previously established hierarchies of syntactic algebraizability, syntactic regularity and Rasiowa properties, establish the following combined hierarchy of these properties.



Recall, now that, if $\mathcal{I}$ is systemic, then the two local versions and the two global versions of syntactic algebraizability become identified and that the same holds for syntactic regularity. Therefore, in the case of a systemic π-institution $\mathcal{I}$ with a set $I^\flat$ of natural transformations having two distinguished arguments, the hierarchy above reduces to the simpler hierarchy shown on the left below.





Furthermore, the property of being parameter-free has the effect of collapsing the two versions of system syntactic algebraizability and the two versions of system syntactic regularity properties. Thus, the hierarchy of the three properties for parameter-free sets of natural transformations $I^\flat$ in $\mathcal{I}$ is given by the diagram shown on the right above.

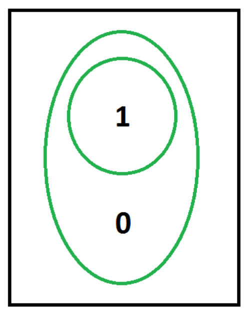
We present some examples to show that all inclusions in the diagram of the hierarchy of Rasiowa, syntactic regularity and syntactic algebraizablity properties are proper in general.

We first present an example of a π-institution $\mathcal{I}$, with a set $I^\flat$ of natural transformations, with two distinguished arguments, that has the local and global family syntactic regularity properties in $\mathcal{I}$, but does not have the Rasiowa property in $\mathcal{I}$.

Example 767 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1\}^2 \to \{0,1\}$ *be given, for all* $x, y \in \mathrm{SEN}^\flat(\Sigma)$*, by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}.$$



SEN(Σ)

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\}, \{0,1\}\}$*.* $\mathcal{I}$ *has two theory families* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, both of which are theory systems. So it is a systemic* π*-institution.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has (all kinds of) the syntactic regularity in* $\mathcal{I}$*, but it does not have the Rasiowa property in* $\mathcal{I}$*.*

First, we look at the equivalence property. The defining conditions for reflexivity, symmetry and transitivity of $I^\flat$ *in* $\mathcal{I}$ *are all trivially satisfied for* $\mathrm{SEN}^\flat$*. For the theory system* $\mathrm{Thm}(\mathcal{I})$*, it suffices to observe that equivalence modulo* $\mathrm{Thm}(\mathcal{I})$ *coincides with the identity relation on* $\mathrm{SEN}^\flat(\Sigma)$*. Therefore,* $I^\flat$ *has the local system equivalence in* $\mathcal{I}$*.*

The fact that equivalence modulo $\mathrm{Thm}(\mathcal{I})$ *is the identity relation immediately implies that* $I^\flat$ *also has the compatibility property and the modus ponens in* $\mathcal{I}$*.*

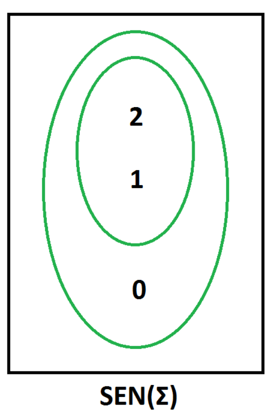
Finally, for regularity, note that the defining condition is trivially satisfied for $\mathrm{SEN}^\flat$*, whereas, for* $\mathrm{Thm}(\mathcal{I})$*, we clearly have that, if* $\phi, \psi \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, then* $\phi = \psi = 1$*, whence* $\sigma^\flat_\Sigma(\phi,\psi) = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*. Therefore* $I^\flat$ *has the regularity property in* $\mathcal{I}$ *and, therefore, we conclude that* $I^\flat$ *has the syntactic regularity in* $\mathcal{I}$*.*

On the other hand, since $\mathcal{I}$ is not an inconsistent π-institution, $I^\flat$ does not have the Rasiowa property in $\mathcal{I}$.

Next, we look at an example of a π-institution $\mathcal{I}$, with a set $I^\flat$ of natural transformations, with two distinguished arguments, that has the local and global family syntactic algebraizability properties in $\mathcal{I}$, but does not possess the global system syntactic regularity in $\mathcal{I}$.

Example 768 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$*;*
- $N^\flat$ *is the category of natural transformations generated by*
  - *a unary natural transformation* $\lambda^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined by letting* $\lambda^\flat_\Sigma : \{0,1,2\} \to \{0,1,2\}$ *be given, for all* $x \in \mathrm{SEN}^\flat(\Sigma)$*, by*
  $$\lambda^\flat_\Sigma(x) = \begin{cases} 2, & \text{if } x = 2 \\ 1, & \text{otherwise} \end{cases};$$
  - *a binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given, for all* $x, y \in \mathrm{SEN}^\flat(\Sigma)$*, by*
  $$\sigma^\flat_\Sigma(x,y) = \begin{cases} 2, & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1,2\},\{0,1,2\}\}$*.* $\mathcal{I}$ *has two theory families* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, both of which are theory systems. So it is a systemic* π*-institution.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has (all kinds of) the syntactic algebraizability in* $\mathcal{I}$*, but it does not have (any kind of) the syntactic regularity in* $\mathcal{I}$*.*

First, we look at the equivalence property. The defining conditions for reflexivity, symmetry and transitivity of $I^\flat$ *in* $\mathcal{I}$ *are all trivially satisfied for* $\mathrm{SEN}^\flat$*. For the theory system* $\mathrm{Thm}(\mathcal{I})$*, it suffices to observe that equivalence modulo* $\mathrm{Thm}(\mathcal{I})$ *coincides with the identity relation on* $\mathrm{SEN}^\flat(\Sigma)$*. Therefore,* $I^\flat$ *has the local system equivalence in* $\mathcal{I}$*.*

The fact that equivalence modulo $\mathrm{Thm}(\mathcal{I})$ *is the identity relation immediately implies that* $I^\flat$ *also has the compatibility property and the modus ponens in* $\mathcal{I}$*.*

Finally, for invertibility, consider the set $\tau : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *of natural transformations in* $N^\flat$*, defined by* $\tau = \{\iota \approx \lambda^\flat\}$*. Note that the defining condition is trivially satisfied for* $\mathrm{SEN}^\flat$*, whereas, for* $\mathrm{Thm}(\mathcal{I})$*, we clearly have that,*

$$\phi \in \mathrm{Thm}_\Sigma(\mathcal{I}) \quad \textit{iff} \quad \sigma^\flat_\Sigma(\phi, \lambda^\flat_\Sigma(\phi)) \in \mathrm{Thm}_\Sigma(\mathcal{I}).$$

Therefore, $I^\flat$ *has the invertibility and, hence, the syntactic algebraizability property in* $\mathcal{I}$*.*

On the other hand, we have $1, 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, but* $\sigma^\flat_\Sigma(1,2) = 0 \notin \mathrm{Thm}_\Sigma(\mathcal{I})$*. Therefore,* $I^\flat$ *fails to have the regularity property and, hence, a fortiori, does not have the syntactic regularity property in* $\mathcal{I}$*.*