Chapter 11

The Syntactic Leibniz Hierarchy: Foundations

11.1 Syntactic Prealgebraicity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

Recall that $\mathcal{I}$ is **prealgebraic** if, for all $T, T' \in \mathrm{ThSys}(\mathcal{I})$,

$$T \le T' \quad \text{implies} \quad \Omega(T) \le \Omega(T').$$

We say that $\mathcal{I}$ is **syntactically prealgebraic** if there exists $I^\flat \subseteq N^\flat$, with two distinguished arguments, such that $I^\flat$ has:

- reflexivity;
- global system transitivity;
- global system compatibility; and
- global system modus ponens.

In that case, we call $I^\flat$ a **set of witnessing natural transformations**, or, more simply, **witnessing transformations** (of the syntactic prealgebraicity of $\mathcal{I}$).

It turns out that, if $\mathcal{I}$ is a syntactically prealgebraic π-institution, with witnessing transformations $I^\flat$, then $\overleftrightarrow{I^\flat}(T)$ is a congruence system on $\mathbf{F}$ compatible with T, for all $T \in \mathrm{ThSys}(\mathcal{I})$. As a consequence, using Corollary 98, we may conclude that, for all $T \in \mathrm{ThSys}(\mathcal{I})$,

$$\overleftrightarrow{I^\flat}(T) = \Omega(T).$$

Proposition 769 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is syntactically prealgebraic, with witnessing transformations $I^\flat$, then, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\overleftrightarrow{I^\flat}(T)$ is a congruence system on $\mathbf{F}$ compatible with T.*

Proof: Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$.

Since $I^\flat$ is reflexive in $\mathcal{I}$, we get that $I^\flat_\Sigma[\phi, \phi] \le \mathrm{Thm}(\mathcal{I}) \le T$. Therefore, $\overleftrightarrow{I^\flat}_\Sigma[\phi, \phi] \le T$, which shows that $\langle \phi, \phi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$.

Suppose, next, that $\langle \phi, \psi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$. Thus, $\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi] \le T$. By the definition of $\overleftrightarrow{I^\flat}$, we get $\overleftrightarrow{I^\flat}_\Sigma[\psi, \phi] \le T$ and, hence, $\langle \psi, \phi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$.

Next, assume that $\langle \phi, \psi \rangle, \langle \psi, \chi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$. Then we get $\langle \phi, \psi \rangle$, $\langle \psi, \chi \rangle$, $\langle \psi, \phi \rangle$, $\langle \chi, \psi \rangle \in I^\flat_\Sigma(T)$. Since $I^\flat$ is globally system transitive in $\mathcal{I}$, we conclude that $\langle \phi, \chi \rangle$, $\langle \chi, \phi \rangle \in I^\flat_\Sigma(T)$ and, therefore, $\langle \phi, \chi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$.

To show the congruence property, assume that $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ is a natural transformation in $N^\flat$ and that $\langle \phi_i, \psi_i \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$, for all $i < k$. Thus, since $I^\flat$ has the global system compatibility in $\mathcal{I}$, we get that $\langle \sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi})\rangle \in I^\flat_\Sigma(T)$. By symmetry, we also get $\langle \sigma^\flat_\Sigma(\vec{\psi}), \sigma^\flat_\Sigma(\vec{\phi})\rangle \in I^\flat_\Sigma(T)$ and, hence, that $\langle \sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi})\rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$.

Finally, since by Lemma 93, $\overleftrightarrow{I^\flat}(T)$ is a relation system on $\mathbf{F}$, we conclude that $\overleftrightarrow{I^\flat}(T)$ is a congruence system on $\mathbf{F}$.

To conclude the proof, note that, if $\phi \in T_\Sigma$ and $\langle \phi, \psi\rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$, then $\psi \in T_\Sigma$ by the global system modus ponens of $I^\flat$ in $\mathcal{I}$ and the fact that $I^\flat \subseteq \overleftrightarrow{I^\flat}$. ∎

Based on Proposition 769, we can conclude that $\overleftrightarrow{I^\flat}$ defines the Leibniz congruence systems of the theory systems of $\mathcal{I}$.

Corollary 770 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically prealgebraic, with witnessing transformations* $I^\flat$*, if and only if, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,*

$$\overleftrightarrow{I^\flat}(T) = \Omega(T).$$

Proof: The only if is by Proposition 769 and Corollary 98. The if is obvious, since the displayed equations immediately implies the four properties of $\overleftrightarrow{I^\flat}$ defining syntactic prealgebraicity. ∎

Corollary 770 has as an immediate consequence the important fact that syntactic prealgebraicity implies (semantic) prealgebraicity.

Theorem 771 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically prealgebraic, then it is prealgebraic.*

Proof: Suppose that $\mathcal{I}$ is syntactically prealgebraic with witnessing transformations $I^\flat$. Let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $T \leq T'$. Then

$$\begin{array}{rcll}\Omega(T) & = & \overleftrightarrow{I^\flat}(T) & \text{(by Corollary 770)}\\ & \leq & \overleftrightarrow{I^\flat}(T') & \text{(by Lemma 94)}\\ & = & \Omega(T'). & \text{(by Corollary 770)}\end{array}$$

Thus, $\mathcal{I}$ is prealgebraic. ∎

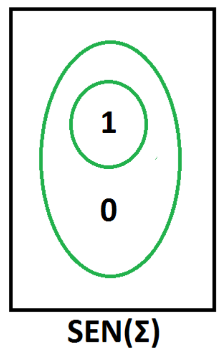
The following example shows that the inclusion of Theorem 771 is proper.

Example 772 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with a single object* Σ*;*

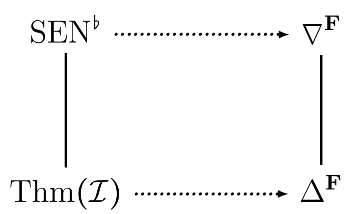
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting:* $\sigma^\flat_\Sigma : \{0,1\}^2 \to \{0,1\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y) = 1, \quad \textit{for all } x,y \in \{0,1\}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\}$*.*

$\mathcal{I}$ *has two theory families,* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, which are also theory systems. Clearly,* $\mathrm{Thm}(\mathcal{I}) \leq \mathrm{SEN}^\flat$*. Moreover,* $\Omega(\mathrm{Thm}(\mathcal{I})) = \Delta^{\mathbf{F}}$ *and* $\Omega(\mathrm{SEN}^\flat) = \nabla^{\mathbf{F}}$*. Since* $\Omega(\mathrm{Thm}(\mathcal{I})) \leq \Omega(\mathrm{SEN}^\flat)$*,* $\mathcal{I}$ *is prealgebraic.*



On the other hand, there does not exist $I^\flat \subseteq N^\flat$*, such that* $I^\flat$ *has the required properties to constitute a witnessing set of transformations in* $\mathcal{I}$*. Any set containing projections cannot satisfy reflexivity and the set consisting only of* $\sigma^\flat$ *does not satisfy the modus ponens property. We conclude that* $\mathcal{I}$ *is not syntactically prealgebraic.*

We provide, next, a characterization of syntactic prealgebraicity in terms of the global system modus ponens property of a subset of natural transformations intrinsically associated with the π-institution. Later, we use this characterization to provide an exact description of those prealgebraic π-institutions which are syntactically prealgebraic.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$. We define the **reflexive core** of $\mathcal{I}$ to be the collection

$$R^{\mathcal{I}} = \{\rho^\flat \in N^\flat : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\rho^\flat_\Sigma[\phi,\phi] \leq \mathrm{Thm}(\mathcal{I}))\}.$$

Note that the defining condition is equivalent to asserting that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\rho^\flat_\Sigma(\phi, \phi, \vec{\chi}) \in \mathrm{Thm}_\Sigma(\mathcal{I}).$$

It is clear that $R^{\mathcal{I}}(T)$ is a reflexive relation system on $\mathbf{F}$, for every theory family T of $\mathcal{I}$.

Lemma 773 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,* $R^{\mathcal{I}}(T)$ *is a reflexive relation system on* $\mathbf{F}$*.*

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$. That $R^{\mathcal{I}}(T)$ is a relation system follows from Lemma 93. For reflexivity, it is required that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\langle \phi, \phi \rangle \in R^{\mathcal{I}}_\Sigma(T)$. But this is equivalent to $R^{\mathcal{I}}_\Sigma[\phi, \phi] \leq T$, which certainly holds, since, by definition of $R^{\mathcal{I}}$, $R^{\mathcal{I}}_\Sigma[\phi, \phi] \leq \mathrm{Thm}(\mathcal{I}) \leq T$. ∎

Now, using Proposition 97, we draw a useful conclusion about the role of the reflexive core in determining the Leibniz congruence system associated with a given theory family.

Proposition 774 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\Omega(T) \leq R^{\mathcal{I}}(T).$$

Proof: By Lemma 773 and Proposition 97. ∎

We next show that, for every theory family T of $\mathcal{I}$, $R^{\mathcal{I}}(T)$ is also a symmetric relation system on $\mathbf{F}$.

Lemma 775 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,* $R^{\mathcal{I}}(T)$ *is a symmetric relation system on* $\mathbf{F}$*.*

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$. That $R^{\mathcal{I}}(T)$ is a relation system follows from Lemma 93. To show that it is symmetric, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in R^{\mathcal{I}}_\Sigma(T)$. Equivalently, $R^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T$. Now consider any $\rho^\flat \in R^{\mathcal{I}}$. By the definition of $R^{\mathcal{I}}$, we get that $\overline{\rho^\flat} \in R^{\mathcal{I}}$. Therefore, by the hypothesis, $\overline{\rho^\flat}_\Sigma[\phi, \psi] \leq T$. But this gives $\rho^\flat_\Sigma[\psi, \phi] \leq T$. Since this holds for all $\rho^\flat \in R^{\mathcal{I}}$, we conclude that $R^{\mathcal{I}}_\Sigma[\psi, \phi] \leq T$. Hence, $\langle \psi, \phi \rangle \in R^{\mathcal{I}}_\Sigma(T)$. Therefore, $R^{\mathcal{I}}(T)$ is a symmetric relation system on $\mathbf{F}$. ∎

We turn, next, to the congruence compatibility property. More precisely, we show that, for all theory families T of $\mathcal{I}$, $R^{\mathcal{I}}(T)$ has the compatibility property in $\mathbf{F}$.

Lemma 776 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,* $R^{\mathcal{I}}(T)$ *has the compatibility property in* $\mathbf{F}$*.*

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$. Note that, because of Corollary 12, it suffices to show that, for all $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$, and all $\phi, \psi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in R^{\mathcal{I}}_\Sigma(T) \quad \text{implies} \quad \langle \sigma^\flat_\Sigma(\phi, \vec{\chi}), \sigma^\flat_\Sigma(\psi, \vec{\chi}) \rangle \in R^{\mathcal{I}}_\Sigma(T).$$

Suppose, $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ is in $N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in R^{\mathcal{I}}_\Sigma(T)$ or, equivalently, $R^{\mathcal{I}}_\Sigma[\phi, \psi] \le T$. Let $\rho^\flat : (\mathrm{SEN}^\flat)^n \to \mathrm{SEN}^\flat$ be arbitrary in $R^{\mathcal{I}}$. We consider the natural transformation $\rho'^\flat : (\mathrm{SEN}^\flat)^{n+k} \to \mathrm{SEN}^\flat$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\zeta, \eta, \vec{\chi}, \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$, by

$$\rho'^\flat_\Sigma(\zeta, \eta, \vec{\chi}, \vec{\xi}) = \rho^\flat_\Sigma(\sigma^\flat_\Sigma(\zeta, \vec{\chi}), \sigma^\flat_\Sigma(\eta, \vec{\chi}), \vec{\xi}).$$

Now note that, since $\sigma^\flat \in N^\flat$, $\rho^\flat \in N^\flat$ and

$$\begin{aligned} \rho'^\flat \;=\; \rho^\flat \circ \langle & \sigma^\flat \circ \langle p^{n+k,0}, p^{n+k,2}, \dots, p^{n+k,k} \rangle, \sigma^\flat \circ \langle p^{n+k,1}, p^{n+k,2}, \dots, p^{n+k,k} \rangle, \\ & p^{n+k,k+1}, \dots, p^{n+k,n+k-1} \rangle, \end{aligned}$$

we get, by the definition of a category of natural transformations, that $\rho'^\flat \in N^\flat$.

Next, note that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\zeta, \vec{\chi}, \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{aligned} \rho'^\flat_\Sigma(\zeta, \zeta, \vec{\chi}, \vec{\xi}) &= \rho^\flat_\Sigma(\sigma^\flat_\Sigma(\zeta, \vec{\chi}), \sigma^\flat_\Sigma(\zeta, \vec{\chi}), \vec{\xi}) \quad \text{(by definition of } \rho'^\flat) \\ &\in \mathrm{Thm}_\Sigma(\mathcal{I}). \quad \text{(since } \rho^\flat \in R^{\mathcal{I}}). \end{aligned}$$

Thus, by the definition of the reflexive core, we get that $\rho'^\flat \in R^{\mathcal{I}}$.

Now since $\rho'^\flat \in R^{\mathcal{I}}$ and, by hypothesis, $R^{\mathcal{I}}_\Sigma[\phi, \psi] \le T$, we get, in particular, that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, and $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi}, \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}), \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}), \vec{\xi}) \in T_{\Sigma'}.$$

Hence, a fortiori, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\sigma^\flat_\Sigma(\phi, \vec{\chi})), \mathrm{SEN}^\flat(f)(\sigma^\flat_\Sigma(\psi, \vec{\chi})), \vec{\xi}) \in T_{\Sigma'}.$$

This proves that

$$\rho^\flat_\Sigma[\sigma^\flat_\Sigma(\phi, \vec{\chi}), \sigma^\flat_\Sigma(\psi, \vec{\chi})] \le T.$$

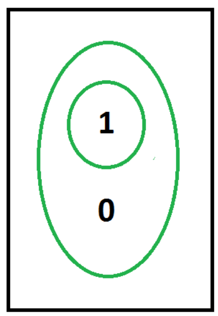
Since this holds for all $\rho^\flat \in R^{\mathcal{I}}$, we get that $R^{\mathcal{I}}_\Sigma[\sigma^\flat_\Sigma(\phi, \vec{\chi}), \sigma^\flat_\Sigma(\psi, \vec{\chi})] \le T$ or, equivalently, $\langle \sigma^\flat_\Sigma(\phi, \vec{\chi}), \sigma^\flat_\Sigma(\psi, \vec{\chi}) \rangle \in R^{\mathcal{I}}_\Sigma(T)$. Therefore, $R^{\mathcal{I}}(T)$ has the congruence compatibility property in $\mathbf{F}$. ■

It is possible, but not necessary, that the reflexive core of a π-institution has the global system modus ponens. To see this, we present two examples. In the first example, we look at a π-institution $\mathcal{I}$ whose reflexive core $R^{\mathcal{I}}$ does have the global system modus ponens in $\mathcal{I}$.

Example 777 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with a single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting:* $\sigma^\flat_\Sigma : \{0,1\}^2 \to \{0,1\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 0, & \text{if } (x,y) = (1,0) \\ 1, & \text{otherwise} \end{cases}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\}$*. The only theory families,* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, are also theory systems.*

Note that $\sigma^\flat \in R^{\mathcal{I}}$*, since, for all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,* $\sigma^\flat_\Sigma(\phi,\phi) = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*. On the other hand, no projection natural transformation can be in the reflexive core.*

To see that $R^{\mathcal{I}}$ *satisfies the global system modus ponens in* $\mathcal{I}$*, note that it does so trivially for the theory system* $\mathrm{SEN}^\flat$*, whereas for* $\mathrm{Thm}(\mathcal{I})$*, it is possible that* $\sigma^\flat_\Sigma(\phi,\psi) = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$ *and* $\phi = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$ *only if* $\psi = 1$*. Thus,* $R^{\mathcal{I}}$ *has the global system MP in* $\mathcal{I}$*, as claimed.*

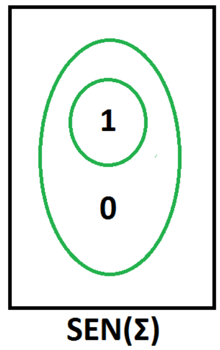
Next, we present an example of a π-institution $\mathcal{I}$ whose reflexive core $R^{\mathcal{I}}$ does not have the global system modus ponens in $\mathcal{I}$.

Example 778 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with a single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*;*

- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting:* $\sigma^\flat_\Sigma : \{0,1\}^2 \to \{0,1\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y) = 1, \quad \textit{for all } x,y \in \{0,1\}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\}$*. So its two theory families,* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, are also theory systems.*

Note that $\sigma^\flat \in R^{\mathcal{I}}$*, since, for all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,* $\sigma^\flat_\Sigma(\phi,\phi) = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*. On the other hand, no projection natural transformation can be in the reflexive core.*

To see that $R^{\mathcal{I}}$ *does not satisfy the global system modus ponens in* $\mathcal{I}$*, note that* $1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$ *and that* $\sigma^\flat_\Sigma(1,0) = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, but* $0 \notin \mathrm{Thm}_\Sigma(\mathcal{I})$*. Thus,* $R^{\mathcal{I}}$ *does not have the global system MP in* $\mathcal{I}$*.*

It turns out that possession of the global system modus ponens by the reflexive core intrinsically characterizes syntactic prealgebraicity. We can show, at the outset, that the reflexive core having the global system modus ponens is necessary for syntactic prealgebraicity.

Theorem 779 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically prealgebraic, then* $R^{\mathcal{I}}$ *has the global system modus ponens.*

Proof: Suppose that $\mathcal{I}$ is syntactically prealgebraic with witnessing transformations $I^\flat$. Thus, $I^\flat$ has reflexivity, global system transitivity, global system compatibility and the global system modus ponens in $\mathcal{I}$. Since $I^\flat$ is reflexive in $\mathcal{I}$, we get, by the definition of the reflexive core, that $I^\flat \subseteq R^{\mathcal{I}}$. But, then, since, by hypothesis, $I^\flat$ has the global system modus ponens, it follows that, a fortiori, $R^{\mathcal{I}}$ has the global system modus ponens in $\mathcal{I}$. ■

In proving the reverse implication, we now show that having the global system modus ponens implies the global system transitivity property of the reflexive core.

Proposition 780 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^\mathcal{I}$ *has the global system modus ponens, then it also has the global system transitivity in* $\mathcal{I}$*.*

Proof: Suppose that $R^\mathcal{I}$ has the global system modus ponens in $\mathcal{I}$ and let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle, \langle \psi, \chi \rangle \in R^\mathcal{I}_\Sigma(T)$. This means that $R^\mathcal{I}_\Sigma[\phi, \psi] \leq T$ and $R^\mathcal{I}_\Sigma[\psi, \chi] \leq T$. Then, by Lemma 776, we get that, for all $\rho^\flat \in R^\mathcal{I}$, and all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\begin{aligned} R^\mathcal{I}_{\Sigma'}[\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), \vec{\xi}), \\ \rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi})] \leq T. \end{aligned}$$

Since $R^\mathcal{I}_\Sigma[\phi, \psi] \leq T$, we get by the global system MP of $R^\mathcal{I}$ that, for all $\rho^\flat \in R^\mathcal{I}$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi}) \subseteq T_{\Sigma'}.$$

Thus, $R^\mathcal{I}_\Sigma[\phi, \chi] \leq T$, whence $\langle \phi, \chi \rangle \in R^\mathcal{I}_\Sigma(T)$. Therefore $R^\mathcal{I}$ has the global system transitivity in $\mathcal{I}$. ■

Proposition 780 closes a line of work that was started with the definition of a reflexive core and with Lemma 773.

Theorem 781 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^\mathcal{I}$ *has the global system modus ponens, then* $\mathcal{I}$ *is syntactically prealgebraic, with witnessing transformations* $R^\mathcal{I}$*.*

Proof: By Lemma 773, $R^\mathcal{I}$ is reflexive in $\mathcal{I}$. By Lemma 775, $\mathcal{I}$ is globally family symmetric in $\mathcal{I}$. By hypothesis and Proposition 780, it is globally system transitive in $\mathcal{I}$. By Lemma 776 it has the global family compatibility property in $\mathcal{I}$. Finally, by hypothesis, it has the global system modus ponens in $\mathcal{I}$. We conclude that $\mathcal{I}$ is syntactically prealgebraic with witnessing transformations $R^\mathcal{I}$. ■

Theorems 779 and 781 provide the promised characterization of syntactic prealgebraicity in terms of the global system modus ponens of the reflexive core.

$$\mathcal{I} \text{ is Syntactically Prealgebraic} \longleftrightarrow R^\mathcal{I} \text{ has Global System MP.}$$

Theorem 782 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically prealgebraic if and only if* $R^\mathcal{I}$ *has the global system modus ponens in* $\mathcal{I}$*.*

Proof: Theorem 779 gives the "only if" and the "if" is by Theorem 781. ■

If $\mathcal{I}$ is syntactically prealgebraic, then $R^{\mathcal{I}}$ defines Leibniz congruence systems of theory systems in $\mathcal{I}$. This proposition may be viewed as a special case of Corollary 770, since $R^{\mathcal{I}}$ forms a set of witnessing transformations that, in addition, has the global family symmetry in $\mathcal{I}$.

Proposition 783 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $R^{\mathcal{I}}$ *has the global system modus ponens, then, for all* $T \in \mathrm{ThSys}(\mathcal{I})$,

$$\Omega(T) = R^{\mathcal{I}}(T).$$

Proof: Let $T \in \mathrm{ThSys}(\mathcal{I})$. If $R^{\mathcal{I}}$ has the global system modus ponens, then, by Lemma 773, Lemma 775, Lemma 776, the hypothesis and Proposition 780, $R^{\mathcal{I}}(T)$ is a congruence system that is compatible with T. Therefore, by Corollary 98, we get that $\Omega(T) = R^{\mathcal{I}}(T)$. ■

We also get another related characterization of syntactic prealgebraicity.

$\mathcal{I}$ is Syntactically Prealgebraic
$\longleftrightarrow$ $R^{\mathcal{I}}$ Defines Leibniz Congruence Systems
of Theory Systems in $\mathcal{I}$.

Theorem 784 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically prealgebraic if and only if, for all* $T \in \mathrm{ThSys}(\mathcal{I})$,

$$\Omega(T) = R^{\mathcal{I}}(T).$$

Proof: If $\mathcal{I}$ is syntactically prealgebraic, then, by Theorem 782, $R^{\mathcal{I}}$ has the global system modus ponens in $\mathcal{I}$. Thus, by Proposition 783, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\Omega(T) = R^{\mathcal{I}}(T)$.

Conversely, if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $R^{\mathcal{I}}(T) = \Omega(T)$, then, $R^{\mathcal{I}}$ is reflexive, globally system transitive, has the global family compatibility and the global system modus ponens. Thus, $R^{\mathcal{I}}$ is a set of witnessing transformations and $\mathcal{I}$ is syntactically prealgebraic. ■

We finally show that the property that separates prealgebraicity from syntactic prealgebraicity is exactly the Leibniz compatibility property with respect to the theory system generated by the reflexive core.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that $R^{\mathcal{I}}$ is **Leibniz** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi])).$$

We show that, if $R^{\mathcal{I}}$ has the global system modus ponens, then it is Leibniz.

Proposition 785 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}}$ *has the global system modus ponens, then it is Leibniz.*

Proof: Suppose $R^{\mathcal{I}}$ has the global system modus ponens. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. To show that $\langle \phi, \psi \rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi]))$, we use the criterion for membership given in Theorem 19. To this end, let $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ be in $N^\flat$, $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$, such that

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\xi}) \in C_{\Sigma'}(R^{\mathcal{I}}_\Sigma[\phi, \psi]).$$

By Lemma 776,

$$R^{\mathcal{I}}_{\Sigma'}[\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\xi}), \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\xi})] \le C(R^{\mathcal{I}}_\Sigma[\phi, \psi]).$$

Since, by hypothesis, $R^{\mathcal{I}}$ has the global system modus ponens, we obtain that $\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\xi}) \in C_{\Sigma'}(R^{\mathcal{I}}_\Sigma[\phi, \psi])$. By symmetry, we now have that

$$\begin{array}{l}\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\xi}) \in C_{\Sigma'}(R^{\mathcal{I}}_\Sigma[\phi, \psi]) \\ \qquad\qquad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\xi}) \in C_{\Sigma'}(R^{\mathcal{I}}_\Sigma[\phi, \psi]).\end{array}$$

Therefore, by Theorem 19, we conclude that $\langle \phi, \psi \rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi]))$, and, hence, $R^{\mathcal{I}}$ is Leibniz. ■

Here is an example of a π-institution $\mathcal{I}$, with a Leibniz reflexive core not having the global system modus ponens.

Example 786 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

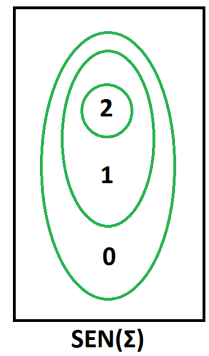
- $\mathbf{Sign}^\flat$ *is the trivial category with single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0, 1, 2\}^2 \to \{0, 1, 2\}$ *be given, for all* $x, y \in \mathrm{SEN}^\flat(\Sigma)$*, by*

$$\sigma^\flat_\Sigma(x, y) = \begin{cases} 2, & \text{if } x = y \text{ or } \{x, y\} = \{0, 1\} \\ 0, & \text{otherwise} \end{cases}.$$

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

$\mathcal{I}$ *has three theory families* $\mathrm{Thm}(\mathcal{I})$*,* $T = \{\{1, 2\}\}$ *and* $\mathrm{SEN}^\flat$*, all of which are theory systems.*



Note that $R^{\mathcal{I}} = \{\sigma^\flat\}$. *We show that* $R^{\mathcal{I}}$ *is Leibniz, but does not have the global system modus ponens.*

To verify the Leibniz property, note that, if $\phi = \psi$ *the conclusion is trivial. If* $\phi \neq \psi$, *then, if* $\{\phi,\psi\} \neq \{0,1\}$, *then* $R^{\mathcal{I}}_\Sigma[\phi,\psi] = \{\{0\}\}$, *whence* $C(R^{\mathcal{I}}_\Sigma[\phi,\psi]) = \mathrm{SEN}^\flat$ *and, therefore,*

$$\Omega(C(R^{\mathcal{I}}_\Sigma[\phi,\psi])) = \nabla^{\mathbf{F}}$$

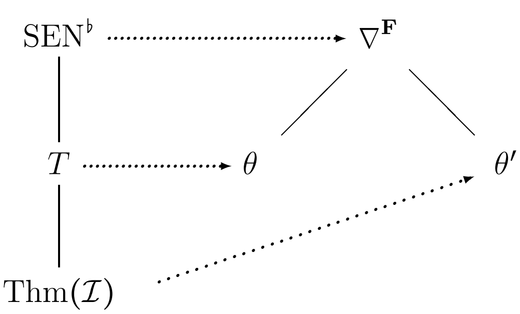
and the conclusion follows. Otherwise, if $\{\phi,\psi\} = \{0,1\}$, *then* $C(R^{\mathcal{I}}_\Sigma[\phi,\psi]) = \mathrm{Thm}(\mathcal{I})$, *whence*

$$\Omega(C(R^{\mathcal{I}}_\Sigma[\phi,\psi])) = \{\{0,1\},\{2\}\}$$

and the conclusion follows. Therefore, $R^{\mathcal{I}}$ *is Leibniz.*

On the other hand, we have $1 \in \{1,2\}$ *and* $R^{\mathcal{I}}_\Sigma[1,0] \leq \{\{1,2\}\}$, *whereas* $0 \notin \{1,2\}$. *Therefore,* $R^{\mathcal{I}}$ *fails to have the global system modus ponens in* $\mathcal{I}$.

We note, with a nod to what is to follow, that $\mathcal{I}$ *is not prealgebraic, since, as is clear by the poset diagrams of theory systems and associated Leibniz congruence systems, the Leibniz operator is not monotonic on theory systems (here* $\theta = \{\{0\},\{1,2\}\}$ *and* $\theta' = \{\{0,1\},\{2\}\}$).



In the opposite direction, and on the positive side, in a *prealgebraic* π-institution $\mathcal{I}$, if the reflexive core is Leibniz, then it does have the global system modus ponens in $\mathcal{I}$.

Proposition 787 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a prealgebraic* π*-institution based on* $\mathbf{F}$. *If* $R^{\mathcal{I}}$ *is Leibniz, then it has the global system modus ponens in* $\mathcal{I}$.

Proof: Suppose that $\mathcal{I}$ is prealgebraic and that $R^{\mathcal{I}}$ is Leibniz. Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $R^{\mathcal{I}}_\Sigma[\phi,\psi] \le T$. Now we have

$$\begin{array}{rcll}\langle\phi,\psi\rangle & \in & \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi,\psi])) & \text{(since } R^{\mathcal{I}} \text{ is Leibniz)}\\ & \subseteq & \Omega_\Sigma(T). & \text{(since } R^{\mathcal{I}}_\Sigma[\phi,\psi] \le T \text{ and } \mathcal{I} \text{ is prealgebraic)}\end{array}$$

Therefore, since $\phi \in T_\Sigma$, we get, by the compatibility of $\Omega(T)$ with T, that $\psi \in T_\Sigma$. We conclude that $R^{\mathcal{I}}$ has the global system modus ponens in $\mathcal{I}$. ■

We now show that a π-institution is syntactically prealgebraic if and only if it is prealgebraic and it has a Leibniz reflexive core.

$$\begin{array}{rcl}\text{Syntactic Prealgebraicity} & = & R^{\mathcal{I}} \text{ has Global System MP}\\ & = & R^{\mathcal{I}} \text{ Defines Leibniz Congruence Systems}\\ & & \quad\text{of Theorem Systems in } \mathcal{I}\\ & = & \text{Prealgebraicity} + R^{\mathcal{I}} \text{ is Leibniz}\end{array}$$

Theorem 788 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is syntactically prealgebraic if and only if it is prealgebraic and has a Leibniz reflexive core.*

Proof: Suppose, first, that $\mathcal{I}$ is syntactically prealgebraic. Then it is prealgebraic by Theorem 771. Moreover, its reflexive core has the global family modus ponens by Theorem 782 and, hence, by Proposition 785, its reflexive core is Leibniz.

Suppose, conversely, that $\mathcal{I}$ is prealgebraic with a Leibniz reflexive core. Then, by Proposition 787, its reflexive core has the global system modus ponens and, therefore, by Theorem 782, $\mathcal{I}$ is syntactically prealgebraic. ■

It is not difficult to see that syntactic prealgebraicity transfers from a π-institution $\mathcal{I}$ to all its generalized matrix families.

Theorem 789 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is syntactically prealgebraic, with witnessing transformations $I^\flat$, if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle$, the generalized matrix family $\langle\mathcal{A},C^{\mathcal{I},\mathcal{A}}\rangle$ is syntactically prealgebraic, with witnessing transformations $I^{\mathcal{A}}$.*

Proof: The "if" follows by considering the algebraic system $\mathcal{F} = \langle\mathbf{F},\langle I,\iota\rangle\rangle$. For the "only if", assume that $\mathcal{I}$ is syntactically prealgebraic, with witnessing transformations $I^\flat$, and let $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle\mathbf{Sign},\mathrm{SEN},N\rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$. We have

$$\begin{array}{rcl}\langle\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\rangle \in T_{F(\Sigma)} & \text{iff} & \langle\phi,\psi\rangle \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})\\ & \text{iff} & I^\flat_\Sigma[\phi,\psi] \le \alpha^{-1}(T)\\ & \text{iff} & I^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)] \le T.\end{array}$$

Taking into account the surjectivity of $\langle F,\alpha\rangle$, we conclude that $\langle\mathcal{A},C^{\mathcal{I},\mathcal{A}}\rangle$ is syntactically prealgebraic, with witnessing transformations $I^{\mathcal{A}}$. ■

11.2 Syntactic Protoalgebraicity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$.

Recall that $\mathcal{I}$ is **protoalgebraic** if, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$T \leq T' \quad \text{implies} \quad \Omega(T) \leq \Omega(T').$$

We say that $\mathcal{I}$ is **syntactically protoalgebraic** if there exists $I^\flat \subseteq N^\flat$, with two distinguished arguments, such that $I^\flat$ has:

- reflexivity;
- global family transitivity;
- global family compatibility; and
- global family modus ponens.

In that case, we call $I^\flat$ a **set of witnessing natural transformations**, or, more simply, **witnessing transformations** (of the syntactic protoalgebraicity of $\mathcal{I}$).

It turns out that, if $\mathcal{I}$ is a syntactically protoalgebraic π-institution, with witnessing transformations $I^\flat$, then $\overleftrightarrow{I^\flat}(T)$ is a congruence system on $\mathbf{F}$ compatible with T, for all $T \in \mathrm{ThFam}(\mathcal{I})$. As a consequence, using Corollary 98, we may conclude that, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\overleftrightarrow{I^\flat}(T) = \Omega(T).$$

Proposition 790 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is syntactically protoalgebraic, with witnessing transformations $I^\flat$, then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftrightarrow{I^\flat}(T)$ is a congruence system on $\mathbf{F}$ compatible with T.*

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$.

Since $I^\flat$ is reflexive in $\mathcal{I}$, we get that $I^\flat_\Sigma[\phi, \phi] \leq \mathrm{Thm}(\mathcal{I}) \leq T$. Therefore, $\overleftrightarrow{I^\flat}_\Sigma[\phi, \phi] \leq T$, which shows that $\langle \phi, \phi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$.

Suppose, next, that $\langle \phi, \psi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$. Thus, $\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi] \leq T$. By the definition of $\overleftrightarrow{I^\flat}$, we then get $\overleftrightarrow{I^\flat}_\Sigma[\psi, \phi] \leq T$ and, hence, $\langle \psi, \phi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$.

Next, assume that $\langle \phi, \psi \rangle, \langle \psi, \chi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$. Then we get $\langle \phi, \psi \rangle$, $\langle \psi, \chi \rangle$, $\langle \psi, \phi \rangle$, $\langle \chi, \psi \rangle \in I^\flat_\Sigma(T)$. Since $I^\flat$ is transitive in $\mathcal{I}$, we conclude that $\langle \phi, \chi \rangle$, $\langle \chi, \phi \rangle \in I^\flat_\Sigma(T)$ and, therefore, $\langle \phi, \chi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$.

To show the congruence property, assume that $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ is a natural transformation in $N^\flat$ and that $\langle \phi_i, \psi_i \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$, for all $i < k$. Thus, since $I^\flat$ has the compatibility property in $\mathcal{I}$, we get that $\langle \sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi})\rangle \in I^\flat_\Sigma(T)$. By symmetry, we also get $\langle \sigma^\flat_\Sigma(\vec{\psi}), \sigma^\flat_\Sigma(\vec{\phi})\rangle \in I^\flat_\Sigma(T)$ and, hence, that $\langle \sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi})\rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$.

Finally, since by Lemma 93, $\overleftrightarrow{I^\flat}(T)$ is a relation system on $\mathbf{F}$, we conclude that $\overleftrightarrow{I^\flat}(T)$ is a congruence system on $\mathbf{F}$.

To conclude the proof, note that, if $\phi \in T_\Sigma$ and $\langle \phi, \psi\rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$, then $\psi \in T_\Sigma$ by the global family modus ponens of $I^\flat$ in $\mathcal{I}$ and the fact that $I^\flat \subseteq \overleftrightarrow{I^\flat}$. ■

Based on Proposition 790, we can conclude that $\overleftrightarrow{I^\flat}$ defines the Leibniz congruence systems of the theory families of $\mathcal{I}$.

Corollary 791 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically protoalgebraic, with witnessing transformations* $I^\flat$*, if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\overleftrightarrow{I^\flat}(T) = \Omega(T).$$

Proof: The "only if" is by Proposition 790 and Corollary 98. The "if" is again obvious, as in Corollary 770. ■

Corollary 791 has as an immediate consequence the important fact that syntactic protoalgebraicity implies (semantic) protoalgebraicity.

Theorem 792 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically protoalgebraic, then it is protoalgebraic.*

Proof: Suppose that $\mathcal{I}$ is syntactically protoalgebraic with witnessing transformations $I^\flat$. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$. Then

$$\begin{array}{rcll}\Omega(T) & = & \overleftrightarrow{I^\flat}(T) & \text{(by Corollary 791)}\\ & \leq & \overleftrightarrow{I^\flat}(T') & \text{(by Lemma 94)}\\ & = & \Omega(T'). & \text{(by Corollary 791)}\end{array}$$

Thus, $\mathcal{I}$ is protoalgebraic. ■

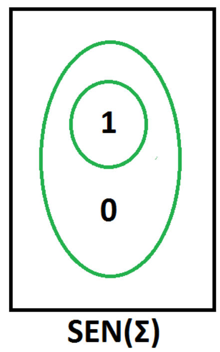
The following example shows that the inclusion of Theorem 792 is proper.

Example 793 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with a single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$*;*

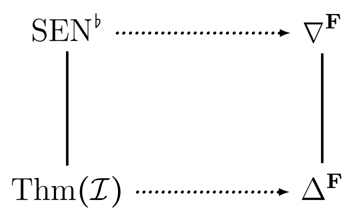
- *$N^\flat$ is the category of natural transformations generated by the single binary natural transformation $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ defined by letting: $\sigma^\flat_\Sigma : \{0,1\}^2 \to \{0,1\}$ be given by*

$$\sigma^\flat_\Sigma(x,y) = 1, \quad \textit{for all } x,y \in \{0,1\}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be the π-institution determined by $\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\}$.

$\mathcal{I}$ has two theory families, $\mathrm{Thm}(\mathcal{I})$ and $\mathrm{SEN}^\flat$, such that $\mathrm{Thm}(\mathcal{I}) \leq \mathrm{SEN}^\flat$. Moreover, $\Omega(\mathrm{Thm}(\mathcal{I})) = \Delta^{\mathbf{F}}$ and $\Omega(\mathrm{SEN}^\flat) = \nabla^{\mathbf{F}}$. Since $\Omega(\mathrm{Thm}(\mathcal{I})) \leq \Omega(\mathrm{SEN}^\flat)$, $\mathcal{I}$ is protoalgebraic.



On the other hand, there does not exist $I^\flat \subseteq N^\flat$, such that $I^\flat$ has the required properties to constitute a witnessing set of transformations in $\mathcal{I}$. Any set containing projections cannot satisfy reflexivity and the set consisting only of $\sigma^\flat$ does nor satisfy the modus ponens property. We conclude that $\mathcal{I}$ is not syntactically protoalgebraic.

We now work towards a dual goal. We first provide a characterization of syntactic protoalgebraicity in terms of the global family modus ponens property of the reflexive core of the π-institution. Then, we use this characterization to provide an exact description of those protoalgebraic π-institutions which are syntactically protoalgebraic.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Recall that the **reflexive core** of $\mathcal{I}$ is the collection

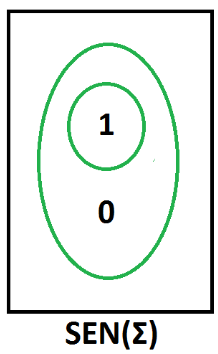
$$R^{\mathcal{I}} = \{\rho^\flat \in N^\flat : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\rho^\flat_\Sigma[\phi,\phi] \leq \mathrm{Thm}(\mathcal{I}))\}.$$

It is possible, but not necessary, that the reflexive core of a π-institution has the global family modus ponens. To see this, we present two examples. In the first example, we look at a π-institution $\mathcal{I}$ whose reflexive core $R^{\mathcal{I}}$ does have the global family modus ponens in $\mathcal{I}$.

Example 794 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with a single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting:* $\sigma^\flat_\Sigma : \{0,1\}^2 \to \{0,1\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 0, & \text{if } (x,y) = (1,0) \\ 1, & \text{otherwise} \end{cases}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\}$.

Note that $\sigma^\flat \in R^{\mathcal{I}}$*, since, for all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,* $\sigma^\flat_\Sigma(\phi,\phi) = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*. On the other hand, no projection natural transformation can be in the reflexive core.*

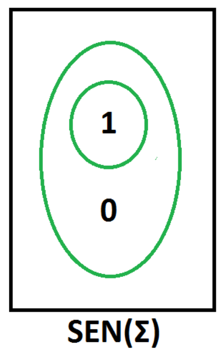
To see that $R^{\mathcal{I}}$ *satisfies the modus ponens in* $\mathcal{I}$*, note that it does so trivially for the theory family* $\mathrm{SEN}^\flat$*, whereas for* $\mathrm{Thm}(\mathcal{I})$*, it is possible that* $\sigma^\flat_\Sigma(\phi,\psi) = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$ *and* $\phi = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$ *only if* $\psi = 1$*. Thus,* $R^{\mathcal{I}}$ *has the global family MP in* $\mathcal{I}$*, as claimed.*

Next, we present an example of a π-institution $\mathcal{I}$ whose reflexive core $R^{\mathcal{I}}$ does not have the global family modus ponens in $\mathcal{I}$.

Example 795 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with a single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting:* $\sigma^\flat_\Sigma : \{0,1\}^2 \to \{0,1\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y) = 1, \quad \textit{for all } x,y \in \{0,1\}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\}$*.*

Note that $\sigma^\flat \in R^\mathcal{I}$*, since, for all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,* $\sigma^\flat_\Sigma(\phi,\phi) = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*. On the other hand, no projection natural transformation can be in the reflexive core.*

To see that $R^\mathcal{I}$ *does not satisfy the modus ponens in* $\mathcal{I}$*, note that* $1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$ *and that* $\sigma^\flat_\Sigma(1,0) = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, but* $0 \notin \mathrm{Thm}_\Sigma(\mathcal{I})$*. Thus,* $R^\mathcal{I}$ *does not have the global family MP in* $\mathcal{I}$*.*

It turns out that possession of the global family modus ponens by the reflexive core intrinsically characterizes syntactic protoalgebraicity. We can show, at the outset, that the reflexive core having the global family modus ponens is necessary for syntactic protoalgebraicity. Thus, there is no point in exploring syntactic protoalgebraicity unless the π-institution $\mathcal{I}$ under scrutiny is such that $R^\mathcal{I}$ has the global family MP.

Theorem 796 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically protoalgebraic, then* $R^\mathcal{I}$ *has the global family modus ponens.*

Proof: Suppose that $\mathcal{I}$ is syntactically protoalgebraic with witnessing transformations $I^\flat$. Thus, $I^\flat$ has reflexivity, global family transitivity, global family compatibility and the global family modus ponens in $\mathcal{I}$. Since $I^\flat$ is reflexive in $\mathcal{I}$, we get, by the definition of the reflexive core, that $I^\flat \subseteq R^\mathcal{I}$. But, then, since, by hypothesis, $I^\flat$ has the global family modus ponens, it follows that, a fortiori, $R^\mathcal{I}$ has the global family modus ponens in $\mathcal{I}$. ∎

To prove the reverse implication, we show, first, that having the global family modus ponens implies the global family transitivity property of the reflexive core.

Proposition 797 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^\mathcal{I}$ *has the global family modus ponens, then it also has the global family transitivity in* $\mathcal{I}$*.*

Proof: Suppose that $R^\mathcal{I}$ has the global family modus ponens in $\mathcal{I}$ and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle, \langle \psi, \chi \rangle \in R^\mathcal{I}_\Sigma(T)$. This means that $R^\mathcal{I}_\Sigma[\phi, \psi] \leq T$ and $R^\mathcal{I}_\Sigma[\psi, \chi] \leq T$. Then, by Lemma 776, we get that, for all $\rho^\flat \in R^\mathcal{I}$, and all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$R^\mathcal{I}_{\Sigma'}[\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), \vec{\xi}), \\ \rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi})] \leq T.$$

Since $R^\mathcal{I}_\Sigma[\phi, \psi] \leq T$, we get by the global family MP of $R^\mathcal{I}$ that, for all $\rho^\flat \in R^\mathcal{I}$, and all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi}) \subseteq T_{\Sigma'}.$$

Thus, $R^\mathcal{I}_\Sigma[\phi, \chi] \leq T$, whence $\langle \phi, \chi \rangle \in R^\mathcal{I}_\Sigma(T)$. Therefore $R^\mathcal{I}$ is globally family transitive in $\mathcal{I}$. ∎

Proposition 797 closes a line of work that was started with the definition of a reflexive core and goes back to Lemma 773.

Theorem 798 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^\mathcal{I}$ *has the global family modus ponens, then* $\mathcal{I}$ *is syntactically protoalgebraic, with witnessing transformations* $R^\mathcal{I}$*.*

Proof: By Lemma 773, $R^\mathcal{I}$ is reflexive in $\mathcal{I}$. By Lemma 775, $\mathcal{I}$ is globally family symmetric in $\mathcal{I}$. By hypothesis and Proposition 797, it is globally family transitive in $\mathcal{I}$. By Lemma 776 it has the global family compatibility property in $\mathcal{I}$. Finally, by hypothesis, it has the global family modus ponens in $\mathcal{I}$. We conclude that $\mathcal{I}$ is syntactically protoalgebraic with witnessing transformations $R^\mathcal{I}$. ∎

Theorems 796 and 798 provide the promised characterization of syntactic protoalgebraicity in terms of the global family modus ponens of the reflexive core.

$$\mathcal{I} \text{ is Syntactically Protoalgebraic} \longleftrightarrow R^\mathcal{I} \text{ has Global Family MP.}$$

Theorem 799 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically protoalgebraic if and only if* $R^\mathcal{I}$ *has the global family modus ponens in* $\mathcal{I}$*.*

Proof: Theorem 796 gives the "only if" and the "if" is by Theorem 798. ■

If $\mathcal{I}$ is syntactically protoalgebraic, then $R^\mathcal{I}$ defines Leibniz congruence systems in $\mathcal{I}$. This proposition may be viewed as a special case of Corollary 791, since $R^\mathcal{I}$ forms a set of witnessing transformations that, in addition, has the global family symmetry in $\mathcal{I}$.

Proposition 800 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^\mathcal{I}$ *has the global family modus ponens, then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\Omega(T) = R^\mathcal{I}(T).$$

Proof: If $R^\mathcal{I}$ has the global family modus ponens, then, by Lemma 773, Lemma 775, Lemma 776, the hypothesis and Proposition 797, $R^\mathcal{I}(T)$ is a congruence system that is compatible with T. Therefore, by Corollary 98, we get that $\Omega(T) = R^\mathcal{I}(T)$. ■

We also get (almost) for free another related characterization of syntactic protoalgebraicity.

$\mathcal{I}$ is Syntactically Protoalgebraic
$\longleftrightarrow$ $R^\mathcal{I}$ Defines Leibniz Congruence Systems.

Theorem 801 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically protoalgebraic if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\Omega(T) = R^\mathcal{I}(T).$$

Proof: If $\mathcal{I}$ is syntactically protoalgebraic, then, by Theorem 799, $R^\mathcal{I}$ has the family modus ponens in $\mathcal{I}$. Thus, by Proposition 800, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) = R^\mathcal{I}(T)$.

Conversely, if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $R^\mathcal{I}(T) = \Omega(T)$, then, $R^\mathcal{I}$ is reflexive, globally family transitive, has the global family compatibility and the global family modus ponens. Thus, $R^\mathcal{I}$ is a set of witnessing transformations and $\mathcal{I}$ is syntactically protoalgebraic. ■

We finally show that the property that separates protoalgebraicity from syntactic protoalgebraicity is exactly the Leibniz compatibility property with respect to the theory family generated by the reflexive core.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall that $R^\mathcal{I}$ is **Leibniz** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(C(R^\mathcal{I}_\Sigma[\phi, \psi])).$$

We have shown in Proposition 785 that, if $R^\mathcal{I}$ has the global system modus ponens, then it is Leibniz.

Corollary 802 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $R^\mathcal{I}$ has the global family modus ponens, then it is Leibniz.*

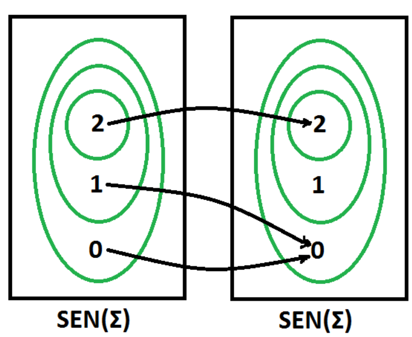
Proof: Directly from Proposition 785 ■

Here is an example of a π-institution $\mathcal{I}$, with a Leibniz reflexive core not having the global family modus ponens.

Example 803 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object Σ and a single (non-identity) morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 0$, $1 \mapsto 0$ *and* $2 \mapsto 2$;
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ defined by letting $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ be given, for all $a, b \in \mathrm{SEN}^\flat(\Sigma)$, by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 2, & \text{if } x = y \text{ or } \{x,y\} = \{0,1\} \\ 0, & \text{otherwise} \end{cases}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be the π-institution determined by

$$\mathcal{C}_\Sigma = \{\{2\}, \{1,2\}, \{0,1,2\}\}.$$

$\mathcal{I}$ has three theory families $\mathrm{Thm}(\mathcal{I})$*,* $T = \{\{1,2\}\}$ *and* $\mathrm{SEN}^\flat$*, but only two theory systems* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*.*

Note that $R^{\mathcal{I}} = \{\sigma^\flat\}$*. We show that* $R^{\mathcal{I}}$ *is Leibniz, but does not have the global family modus ponens.*

To verify the Leibniz property, note that, if $\phi = \psi$ *the conclusion is trivial and, if* $\{\phi,\psi\} \neq \{0,1\}$*, then* $C_\Sigma(R^{\mathcal{I}}_\Sigma[\phi,\psi]) = \mathrm{SEN}^\flat(\Sigma)$*, whence*

$$\Omega(C(R^{\mathcal{I}}_\Sigma[\phi,\psi])) = \nabla^{\mathbf{F}}$$

and the conclusion follows. Finally, if $\{\phi,\psi\} = \{0,1\}$*, then* $C_\Sigma(R^{\mathcal{I}}_\Sigma[\phi,\psi]) = \{2\}$*, whence* $\Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi,\psi])) = \{\{0,1\},\{2\}\}$ *and, therefore,*

$$\langle\phi,\psi\rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi,\psi])),$$

as required. We conclude that $R^{\mathcal{I}}$ *is Leibniz.*

On the other hand, we have $1 \in \{1,2\}$ *and* $R^{\mathcal{I}}_\Sigma[1,0] \leq \{\{1,2\}\}$*, whereas* $0 \notin \{1,2\}$*. Therefore,* $R^{\mathcal{I}}$ *fails to have the global family modus ponens in* $\mathcal{I}$*.*

In the opposite direction, and on the positive side, in a *protoalgebraic* π-institution $\mathcal{I}$, if the reflexive core is Leibniz, then it has the global family modus ponens in $\mathcal{I}$.

Proposition 804 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a protoalgebraic* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}}$ *is Leibniz, then it has the global family modus ponens in* $\mathcal{I}$*.*

Proof: Suppose that $\mathcal{I}$ is protoalgebraic and that $R^{\mathcal{I}}$ is Leibniz. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $R^{\mathcal{I}}_\Sigma[\phi,\psi] \leq T$. Now we have

$$\begin{array}{rcll} \langle\phi,\psi\rangle & \in & \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi,\psi])) & \text{(since } R^{\mathcal{I}} \text{ is Leibniz)} \\ & \subseteq & \Omega_\Sigma(T). & \text{(since } R^{\mathcal{I}}_\Sigma[\phi,\psi] \leq T \text{ and } \mathcal{I} \text{ is protoalgebraic)} \end{array}$$

Therefore, since $\phi \in T_\Sigma$, we get, by the compatibility of $\Omega(T)$ with T, that $\psi \in T_\Sigma$. We conclude that $R^{\mathcal{I}}$ has the global family modus ponens in $\mathcal{I}$. ■

We now show that a π-institution is syntactically protoalgebraic if and only if it is protoalgebraic and has a Leibniz reflexive core.

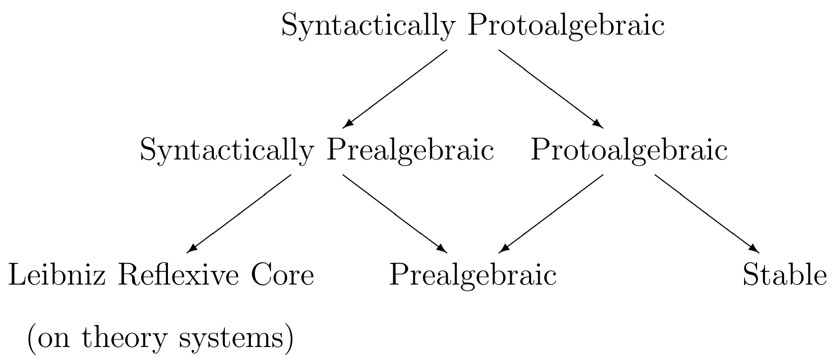
Syntactic Protoalgebraicity	=	$R^{\mathcal{I}}$ has Global Family MP
	=	$R^{\mathcal{I}}$ Defines Leibniz Congruence Systems
	=	Protoalgebraicity + $R^{\mathcal{I}}$ is Leibniz

Theorem 805 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically protoalgebraic if and only if it is protoalgebraic and has a Leibniz reflexive core.*

Proof: Suppose, first, that $\mathcal{I}$ is syntactically protoalgebraic. Then it is protoalgebraic by Theorem 792. Moreover, its reflexive core has the global family modus ponens by Theorem 799 and, hence, by Corollary 802, its reflexive core is Leibniz.

Suppose, conversely, that $\mathcal{I}$ is protoalgebraic with a Leibniz reflexive core. Then, by Proposition 804, its reflexive core has the global family modus ponens and, therefore, by Theorem 799, $\mathcal{I}$ is syntactically protoalgebraic. ■

We have now established the following hierarchy of properties:



In fact, it turns out that many of the given characterizations of syntactic protoalgebraicity can be recast in terms of the corresponding ones concerning syntactic prealgebraicity by adding stability. The main result that allows this connection is the one corresponding to Theorem 175, but concerning syntactic protoalgebraicity and syntactic prealgebraicity rather than their respective semantic versions.

Theorem 806 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically protoalgebraic if and only if it is syntactically prealgebraic and stable.*

Proof: Suppose, first, that $\mathcal{I}$ is syntactically protoalgebraic. Then, it is, a fortiori, syntactically prealgebraic. Moreover, by Theorem 792, it is protoalgebraic. Therefore, by Theorem 175, it is stable.

Suppose, conversely, that $\mathcal{I}$ is syntactically prealgebraic and stable. Consider $T \in \mathrm{ThFam}(\mathcal{I})$. Then we have

$$\begin{array}{rcll} \Omega(T) & = & \Omega(\overleftarrow{T}) & \text{(stability)} \\ & = & R^{\mathcal{I}}(\overleftarrow{T}) & \text{(syntactic prealgebraicity and Theorem 784)} \\ & = & R^{\mathcal{I}}(T). & \text{(Proposition 99)} \end{array}$$

By Theorem 801, we conclude that $\mathcal{I}$ is syntactically protoalgebraic. ■

Now we obtain, almost for free, the following corollaries, which contain the promised characterizations involving stability.

Corollary 807 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically protoalgebraic if and only if it is stable and* $R^\mathcal{I}$ *has the global system modus ponens.*

Proof: We have that $\mathcal{I}$ is syntactically protoalgebraic if and only if, by Theorem 806, it is syntactically prealgebraic and stable, if and only if, by Theorem 782, it is stable and $R^\mathcal{I}$ has the global system modus ponens. ∎

Corollary 808 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically protoalgebraic if and only if it is stable and* $R^\mathcal{I}$ *defines Leibniz congruence systems of theory systems of* $\mathcal{I}$*, i.e., for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,* $\Omega(T) = R^\mathcal{I}(T)$*.*

Proof: We have that $\mathcal{I}$ is syntactically protoalgebraic if and only if, by Theorem 806, it is syntactically prealgebraic and stable, if and only if, by Theorem 784, it is stable and $R^\mathcal{I}$ defines Leibniz congruence systems of theory systems in $\mathcal{I}$. ∎

Corollary 809 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically protoalgebraic if and only if it is prealgebraic, stable and* $R^\mathcal{I}$ *is Leibniz.*

Proof: We have that $\mathcal{I}$ is syntactically protoalgebraic if and only if, by Theorem 806, it is syntactically prealgebraic and stable, if and only if, by Theorem 788, it is prealgebraic, stable and $R^\mathcal{I}$ is Leibniz. ∎

Finally, it is not difficult to see, in this case as well, that syntactic protoalgebraicity transfers from a π-institution $\mathcal{I}$ to all its generalized matrix families.

Theorem 810 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically protoalgebraic, with witnessing transformations* $I^\flat$*, if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, the generalized matrix family* $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ *is syntactically protoalgebraic, with witnessing transformations* $I^\mathcal{A}$*.*

Proof: The proof mimics the proof of Theorem 789. ∎

11.3 Matrix Semantics

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall that an $\mathcal{I}$-**matrix family** is a pair $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, where $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ is an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^\mathcal{I}(\mathcal{A})$ is an $\mathcal{I}$-filter family on $\mathcal{A}$. The class of all $\mathcal{I}$-matrix families is denoted by MatFam($\mathcal{I}$). $\mathcal{I}$-matrix families on $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle\rangle$, i.e., pairs of the form $\langle \mathcal{F}, T\rangle$, where $T \in$ ThFam($\mathcal{I}$), are called **Lindenbaum $\mathcal{I}$-matrix families**. The collection of all Lindenbaum $\mathcal{I}$-matrix families is denoted by LMatFam($\mathcal{I}$).

Four subclasses of MatFam($\mathcal{I}$) are distinguished and will be of particular interest to us in the upcoming sections. These are:

- The class LMatFam*($\mathcal{I}$) of all **reduced Lindenbaum $\mathcal{I}$-matrix families**:

$$\mathrm{LMatFam}^*(\mathcal{I}) = \{\langle \mathcal{F}/\Omega(T), T/\Omega(T)\rangle : T \in \mathrm{ThFam}(\mathcal{I})\};$$

- The class LMatFam^{Su}($\mathcal{I}$) of all **Suszko reduced Lindenbaum $\mathcal{I}$-matrix families**:

$$\mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I}) = \{\langle \mathcal{F}/\widetilde{\Omega}^{\mathcal{I}}(T), T/\widetilde{\Omega}^{\mathcal{I}}(T)\rangle : T \in \mathrm{ThFam}(\mathcal{I})\};$$

- The class MatFam*($\mathcal{I}$) of all **reduced $\mathcal{I}$-matrix families**:

$$\mathrm{LMatFam}^*(\mathcal{I}) = \{\langle \mathcal{A}, T\rangle \in \mathrm{MatFam}(\mathcal{I}) : \Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}\};$$

- The class MatFam^{Su}($\mathcal{I}$) of all **Suszko reduced $\mathcal{I}$-matrix families**:

$$\mathrm{MatFam}^{Su}(\mathcal{I}) = \{\langle \mathcal{A}, T\rangle \in \mathrm{MatFam}(\mathcal{I}) : \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Delta^{\mathcal{A}}\}.$$

The following characterizations of the last two classes are well-known and very useful in practice.

Proposition 811 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a π-institution based on* $\mathbf{F}$. *Then the following equalities hold (where the classes are perceived as being closed under isomorphism):*

(a) $\mathrm{MatFam}^*(\mathcal{I}) = \{\langle \mathcal{A}/\Omega^{\mathcal{A}}(T), T/\Omega^{\mathcal{A}}(T)\rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}$;

(b) $\mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I}) = \{\langle \mathcal{F}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T), T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)\rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}$.

Proof: We prove Part (a). Part (b) can be proven similarly. First, if $\langle \mathcal{A}, T\rangle \in$ MatFam*($\mathcal{I}$), then, since, by definition, $\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}$, we get that $\langle \mathcal{A}, T\rangle \cong \langle \mathcal{A}/\Omega^{\mathcal{A}}(T), T/\Omega^{\mathcal{A}}(T)\rangle$. For the reverse inclusion, it suffices to observe that, given an $\mathbf{F}$-algebraic system $\mathcal{A}$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we have, essentially due to the definition of the Leibniz congruence system, that

$$\Omega^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(T/\Omega^{\mathcal{A}}(T)) = \Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}.$$

Therefore, $\langle \mathcal{A}/\Omega^{\mathcal{A}}(T), T/\Omega^{\mathcal{A}}(T)\rangle \in \mathrm{MatFam}^*(\mathcal{I})$. ∎

It turns out that all four classes of $\mathcal{I}$-matrix families defined above form matrix family semantics for the π-institution $\mathcal{I}$. More precisely, given an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ and a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, a class M of $\mathcal{I}$-matrix families is called a **matrix (family) semantics for** $\mathcal{I}$ if $C = C^{\mathrm{M}}$, i.e., for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, $\phi \in C_\Sigma(\Phi)$ if and only if, for all $\langle \mathcal{A}, T \rangle \in \mathrm{M}$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq T_{F(\Sigma')} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \in T_{F(\Sigma')}.$$

Proposition 812 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathcal{I}$*. The four classes*

$$\mathrm{LMatFam}^*(\mathcal{I}), \mathrm{LMatFam}^{\mathsf{Su}}(\mathcal{I}), \mathrm{MatFam}^*(\mathcal{I}) \textit{ and } \mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})$$

are all matrix semantics for $\mathcal{I}$*.*

Proof: Let M be any of these four matrix family classes. Since M consists of $\mathcal{I}$-matrix families, we have that $C \le C^{\mathrm{M}}$.

For the converse, note that the following inclusions hold:

$$\mathrm{LMatFam}^{\mathsf{Su}}(\mathcal{I}) \longrightarrow \mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})$$

$$\mathrm{LMatFam}^*(\mathcal{I}) \longrightarrow \mathrm{MatFam}^*(\mathcal{I}) \longrightarrow \mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})$$

Therefore, we have, by definition, the inclusions

$$C^{\mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})} \longrightarrow C^{\mathrm{LMatFam}^{\mathsf{Su}}(\mathcal{I})}$$

$$C^{\mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})} \longrightarrow C^{\mathrm{MatFam}^*(\mathcal{I})} \longrightarrow C^{\mathrm{LMatFam}^*(\mathcal{I})}$$

It follows that it suffices to show that the two reduced Lindenbaum matrix family classes satisfy $C^{\mathrm{LMatFam}^{\mathsf{Su}}(\mathcal{I})} \le C$ and $C^{\mathrm{LMatFam}^*(\mathcal{I})} \le C$. We show the first inclusion, since the second can be proven similarly.

Suppose $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that

$$\phi \in C_\Sigma^{\mathrm{LMatFam}^{\mathsf{Su}}(\mathcal{I})}(\Phi).$$

Let $T \in \mathrm{ThFam}(\mathcal{I})$, such that $\Phi \subseteq T_\Sigma$. Then, $\Phi/\widetilde{\Omega}^{\mathcal{I}}_\Sigma(T) \subseteq T_\Sigma/\widetilde{\Omega}^{\mathcal{I}}_\Sigma(T)$. Since $\langle \mathcal{F}/\widetilde{\Omega}^{\mathcal{I}}(T), T/\widetilde{\Omega}^{\mathcal{I}}(T) \rangle \in \mathrm{LMatFam}^{\mathsf{Su}}(\mathcal{I})$, we get, by hypothesis, $\phi/\widetilde{\Omega}^{\mathcal{I}}_\Sigma(T) \in T_\Sigma/\widetilde{\Omega}^{\mathcal{I}}_\Sigma(T)$. Thus, using the compatibility of $\widetilde{\Omega}^{\mathcal{I}}(T)$ with T, we get that $\phi \in T_\Sigma$. Since $T \in \mathrm{ThFam}(\mathcal{I})$ was arbitrary, we conclude that $\phi \in C_\Sigma(\Phi)$. ■

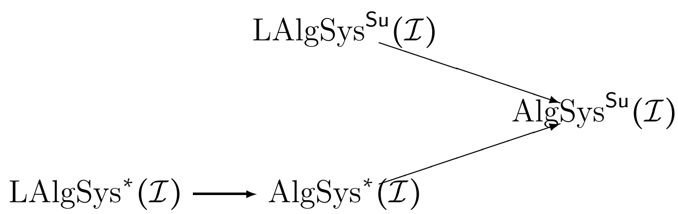
We denote the classes of the underlying **F**-algebraic systems of the matrix families in $\mathrm{LMatFam}^*(\mathcal{I})$, $\mathrm{LMatFam}^{\mathsf{Su}}(\mathcal{I})$, $\mathrm{MatFam}^*(\mathcal{I})$ and $\mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})$, respectively, by

$$\mathrm{LAlgSys}^*(\mathcal{I}), \mathrm{LAlgSys}^{\mathsf{Su}}(\mathcal{I}), \mathrm{AlgSys}^*(\mathcal{I}) \text{ and } \mathrm{AlgSys}^{\mathsf{Su}}(\mathcal{I}).$$

So we have

$$\begin{array}{rcl}\mathrm{LAlgSys}^*(\mathcal{I}) & = & \{\mathcal{F}/\Omega(T) : T \in \mathrm{ThFam}(\mathcal{I})\};\\ \mathrm{LAlgSys}^{\mathsf{Su}}(\mathcal{I}) & = & \{\mathcal{F}/\widetilde{\Omega}^{\mathcal{I}}(T) : T \in \mathrm{ThFam}(\mathcal{I})\};\\ \mathrm{AlgSys}^*(\mathcal{I}) & = & \{\mathcal{A} : (\exists T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}})\};\\ \mathrm{AlgSys}^{\mathsf{Su}}(\mathcal{I}) & = & \{\mathcal{A} : (\exists T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Delta^{\mathcal{A}})\}.\end{array}$$

We clearly have the following inclusion relationships between those four classes of **F**-algebraic systems:



11.4 Algebraic Semantics

In the study of logical systems formalized as π-institutions and, more specifically, as related to their algebraic properties, the notions of an algebraic semantics and that of equational definability of truth are paramount. We introduce and study these two notions in this section.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Consider a class K of **F**-algebraic systems. We define the closure system $C^{\mathrm{K}} : \mathcal{P}(\mathrm{SEN}^\flat)^2 \to \mathcal{P}(\mathrm{SEN}^\flat)^2$, by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $C^{\mathrm{K}}_\Sigma : \mathcal{P}(\mathrm{SEN}^\flat(\Sigma)^2) \to \mathcal{P}(\mathrm{SEN}^\flat(\Sigma)^2)$ be given, for all $E \cup \{\phi \approx \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)^2$, by

$$\begin{array}{lll}\phi \approx \psi \in C^{\mathrm{K}}_\Sigma(E) & \text{iff} & \text{for all } \mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle \in \mathrm{K},\\ & & \alpha_\Sigma(E) \subseteq \Delta^{\mathcal{A}}_{F(\Sigma)} \text{ implies } \alpha_\Sigma(\phi) = \alpha_\Sigma(\psi).\end{array}$$

Given a set $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ of natural transformations in $N^\flat$, with a single distinguished argument, we say that the class K of **F**-algebraic systems is a $\tau^\flat$-**algebraic semantics for** $\mathcal{I}$, or, more simply, a $\tau^\flat$-**semantics for** $\mathcal{I}$, if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in C_\Sigma(\Phi) \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq C^{\mathrm{K}}(\tau^\flat_\Sigma[\Phi]).$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution, based on $\mathbf{F}$, and M a class of $\mathcal{I}$-matrix families. Given a set $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ of natural transformations in $N^\flat$, we say that

truth is $\tau^\flat$-equationally definable in M, or, more simply, that **truth is $\tau^\flat$-definable in** M if, for all $\langle\mathcal{A},T\rangle\in$ M, with $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$ and $\mathbf{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$, all $\Sigma\in|\mathbf{Sign}|$ and all $\phi\in\mathrm{SEN}(\Sigma)$,

$$\phi\in T_\Sigma \quad\text{iff}\quad \tau^{\mathcal{A}}_\Sigma[\phi]\leq\Delta^{\mathcal{A}}.$$

It turns out that classes of algebraic systems forming $\tau^\flat$-semantics for a π-institution and classes of matrix families in which truth is $\tau^\flat$-definable are closely interrelated. To express this connection, we first formulate a technical lemma.

Lemma 813 *Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and let $\tau^\flat:(\mathrm{SEN}^\flat)^\omega\to(\mathrm{SEN}^\flat)^2$ be a set of natural transformations in $N^\flat$. Suppose $\mathfrak{A}=\langle\mathcal{A},T\rangle$, with $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$ and $\mathbf{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$, is an $\mathbf{F}$-matrix family in which truth is $\tau^\flat$-definable. Then, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\Phi\cup\{\phi\}\subseteq\mathrm{SEN}^\flat(\Sigma)$,*

$$\phi\in C^{\mathfrak{A}}_\Sigma(\Phi)\quad\textit{iff}\quad\tau^\flat_\Sigma[\phi]\leq C^{\mathcal{A}}(\tau^\flat_\Sigma[\Phi]).$$

Proof: Let $\Sigma\in|\mathbf{Sign}^\flat|$ and $\Phi\cup\{\phi\}\subseteq\mathrm{SEN}^\flat(\Sigma)$. Then we have the following sequence of equivalent statements:

$\phi\in C^{\mathfrak{A}}_\Sigma(\Phi)$ iff, for all $\Sigma'\in|\mathbf{Sign}^\flat|, f\in\mathbf{Sign}^\flat(\Sigma,\Sigma')$,
$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))\subseteq T_{F(\Sigma')}$
implies $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi))\in T_{F(\Sigma')}$
iff, for all $\Sigma'\in|\mathbf{Sign}^\flat|, f\in\mathbf{Sign}^\flat(\Sigma,\Sigma')$,
$\tau^{\mathcal{A}}_{F(\Sigma')}[\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))]\leq\Delta^{\mathcal{A}}$
implies $\tau^{\mathcal{A}}_{F(\Sigma')}[\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi))]\leq\Delta^{\mathcal{A}}$
iff, for all $\Sigma'\in|\mathbf{Sign}^\flat|, f\in\mathbf{Sign}^\flat(\Sigma,\Sigma')$,
$\tau^{\mathcal{A}}_{F(\Sigma')}[\mathrm{SEN}(F(f))(\alpha_\Sigma(\Phi))]\leq\Delta^{\mathcal{A}}$
implies $\tau^{\mathcal{A}}_{F(\Sigma')}[\mathrm{SEN}(F(f))(\alpha_\Sigma(\phi))]\leq\Delta^{\mathcal{A}}$
iff, by Lemma 93,
$\tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\Phi)]\leq\Delta^{\mathcal{A}}$ implies $\tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi)]\leq\Delta^{\mathcal{A}}$
iff, by surjectivity of $\langle F,\alpha\rangle$,
$\alpha(\tau^\flat_\Sigma[\Phi])\leq\Delta^{\mathcal{A}}$ implies $\alpha(\tau^\flat_\Sigma[\phi])\leq\Delta^{\mathcal{A}}$
iff $\tau^\flat_\Sigma[\phi]\leq C^{\mathcal{A}}(\tau^\flat_\Sigma[\Phi])$.

■

Now we establish the promised relationship between algebraic semantics and matrix semantics.

Theorem 814 *Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, $\mathcal{I}=\langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$ and $\tau^\flat:(\mathrm{SEN}^\flat)^\omega\to(\mathrm{SEN}^\flat)^2$ a set of natural transformation in $N^\flat$. A class* K *of $\mathbf{F}$-algebraic systems is a $\tau^\flat$-semantics for $\mathcal{I}$ if and only if it is the class of underlying algebraic systems of some matrix semantics* M *for $\mathcal{I}$ in which truth is $\tau^\flat$-definable.*

Proof: Suppose, first, that M is a matrix semantics for $\mathcal{I}$ in which truth is $\tau^\flat$-definable and let K be the class of its underlying algebraic systems. Then we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}\phi \in C_\Sigma(\Phi) & \text{iff} & \phi \in C^{\mathrm{M}}_\Sigma(\Phi) \quad \text{(M a matrix semantics)}\\ & \text{iff} & (\forall \mathfrak{A} \in \mathrm{M})(\phi \in C^{\mathfrak{A}}_\Sigma(\Phi)) \quad \text{(by definition)}\\ & \text{iff} & (\forall \mathcal{A} \in \mathrm{K})(\tau^\flat_\Sigma[\phi] \leq C^{\mathcal{A}}(\tau^\flat_\Sigma[\Phi])) \quad \text{(by Lemma 813)}\\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq C^{\mathrm{K}}(\tau^\flat_\Sigma[\Phi]). \quad \text{(by definition)}\end{array}$$

Thus, K is a $\tau^\flat$-semantics for $\mathcal{I}$.

Suppose, conversely, that K is a $\tau^\flat$-semantics for $\mathcal{I}$. Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle \in$ K, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$. Define, for all $\Sigma \in |\mathbf{Sign}|$,

$$T^{\mathcal{A},\tau}_\Sigma = \{\phi \in \mathrm{SEN}(\Sigma) : \tau^{\mathcal{A}}_\Sigma[\phi] \leq \Delta^{\mathcal{A}}\},$$

and set $T^{\mathcal{A},\tau} = \{T^{\mathcal{A},\tau}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$. Then, let

$$\mathrm{M} = \{\langle \mathcal{A}, T^{\mathcal{A},\tau}\rangle : \mathcal{A} \in \mathrm{K}\}.$$

Note that K is the class of all underlying algebraic systems of the matrix systems in M and, also, that, for all $\mathcal{A} \in \mathrm{K}$, truth is $\tau^\flat$-definable in $\langle \mathcal{A}, T^{\mathcal{A},\tau}\rangle$ by the definition of $T^{\mathcal{A},\tau}$. Thus, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}\phi \in C_\Sigma(\Phi) & \text{iff} & \tau^\flat_\Sigma[\phi] \leq C^{\mathrm{K}}(\tau^\flat_\Sigma[\Phi]) \quad \text{(K a } \tau^\flat\text{-semantics)}\\ & \text{iff} & (\forall \mathcal{A} \in \mathrm{K})(\tau^\flat_\Sigma[\phi] \leq C^{\mathcal{A}}(\tau^\flat_\Sigma[\Phi])) \quad \text{(by definition)}\\ & \text{iff} & (\forall \mathfrak{A} \in \mathrm{M})(\phi \in C^{\mathfrak{A}}_\Sigma(\Phi)) \quad \text{(by Lemma 813)}\\ & \text{iff} & \phi \in C^{\mathrm{M}}_\Sigma(\Phi). \quad \text{(by definition)}\end{array}$$

We conclude that M is a matrix semantics for $\mathcal{I}$. ∎

Corollary 815 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution, based on* $\mathbf{F}$ *and* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *a set of natural transformation in* $N^\flat$*. If truth is* $\tau^\flat$*-definable in any of the classes*

$$\mathrm{LMatFam}^*(\mathcal{I}), \mathrm{LMatFam}^{\mathsf{Su}}(\mathcal{I}), \mathrm{MatFam}^*(\mathcal{I}) \textit{ or } \mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I}),$$

then, the corresponding class

$$\mathrm{LAlgSys}^*(\mathcal{I}), \mathrm{LAlgSys}^{\mathsf{Su}}(\mathcal{I}), \mathrm{AlgSys}^*(\mathcal{I}) \textit{ or } \mathrm{AlgSys}^{\mathsf{Su}}(\mathcal{I})$$

is a $\tau^\flat$*-semantics for* $\mathcal{I}$*.*

Proof: This follows from Theorem 814, since the four displayed classes of $\mathcal{I}$-matrix families are matrix semantics for $\mathcal{I}$ and the four displayed classes of algebraic systems are the respective classes of their underlying algebraic systems. ∎

11.5 Truth Equationality

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that $\mathcal{I}$ is:

- **Leibniz truth equational** if there exists $\tau^\flat \subseteq N^\flat$, with a single distinguished argument, such that truth is $\tau^\flat$-definable in $\mathrm{LMatFam}^*(\mathcal{I})$, i.e., such that, for all $\langle \mathcal{F}/\Omega(T), T/\Omega(T) \rangle \in \mathrm{LMatFam}^*(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi/\Omega_\Sigma(T) \in T_\Sigma/\Omega_\Sigma(T) \quad \text{iff} \quad \tau_\Sigma^{\mathcal{F}/\Omega(T)}[\phi/\Omega_\Sigma(T)] \leq \Delta^{\mathcal{F}/\Omega(T)};$$

- **Universally Leibniz truth equational** if there exists $\tau^\flat \subseteq N^\flat$, with a single distinguished argument, such that truth is $\tau^\flat$-definable in the class $\mathrm{MatFam}^*(\mathcal{I})$, i.e., such that, for all $\langle \mathcal{A}, T \rangle \in \mathrm{MatFam}^*(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,
$$\phi \in T_\Sigma \quad \text{iff} \quad \tau_\Sigma^{\mathcal{A}}[\phi] \leq \Delta^{\mathcal{A}};$$

- **Suszko truth equational** if there exists $\tau^\flat \subseteq N^\flat$, with a single distinguished argument, such that truth is $\tau^\flat$-definable in $\mathrm{LMatFam}^{\mathsf{Su}}(\mathcal{I})$, i.e., such that, for all $\langle \mathcal{F}/\widetilde{\Omega}^{\mathcal{I}}(T), T/\widetilde{\Omega}^{\mathcal{I}}(T) \rangle \in \mathrm{LMatFam}^{\mathsf{Su}}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi/\widetilde{\Omega}^{\mathcal{I}}_\Sigma(T) \in T_\Sigma/\widetilde{\Omega}^{\mathcal{I}}_\Sigma(T) \quad \text{iff} \quad \tau_\Sigma^{\mathcal{F}/\widetilde{\Omega}^{\mathcal{I}}(T)}[\phi/\widetilde{\Omega}^{\mathcal{I}}_\Sigma(T)] \leq \Delta^{\mathcal{F}/\widetilde{\Omega}^{\mathcal{I}}(T)};$$

- **Universally Suszko truth equational** if there exists $\tau^\flat \subseteq N^\flat$, with a single distinguished argument, such that truth is $\tau^\flat$-definable in the class $\mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})$, i.e., such that, for all $\langle \mathcal{A}, T \rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,
$$\phi \in T_\Sigma \quad \text{iff} \quad \tau_\Sigma^{\mathcal{A}}[\phi] \leq \Delta^{\mathcal{A}}.$$

The set $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ will be called a set of **witnessing equations** (of/for the corresponding truth equationality property).

The following proposition provides alternative conditions for testing whether a given π-institution is truth equational with respect to any of the four classes of matrix families considered above.

Proposition 816 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\tau^\flat \subseteq N^\flat$ *having a single distinguished argument.*

(a) $\mathcal{I}$ *is Leibniz truth equational with witnessing equations* $\tau^\flat$ *iff, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*
$$\phi \in T_\Sigma \quad \textit{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T);$$

(b) *$\mathcal{I}$ is universally Leibniz truth equational if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, all* $T \in \mathrm{FiFam}^{\mathcal{A}}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi \in \mathrm{SEN}(\Sigma)$*,*
$$\phi \in T_\Sigma \quad \textit{iff} \quad \tau^{\mathcal{A}}_\Sigma[\phi] \le \Omega^{\mathcal{A}}(T);$$

(c) *$\mathcal{I}$ is Suszko truth equational with witnessing equations* $\tau^\flat$ *iff, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*
$$\phi \in T_\Sigma \quad \textit{iff} \quad \tau^\flat_\Sigma[\phi] \le \widetilde{\Omega}^{\mathcal{I}}(T);$$

(d) *$\mathcal{I}$ is universally Suszko truth equational if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, all* $T \in \mathrm{FiFam}^{\mathcal{A}}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi \in \mathrm{SEN}(\Sigma)$*,*
$$\phi \in T_\Sigma \quad \textit{iff} \quad \tau^{\mathcal{A}}_\Sigma[\phi] \le \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T).$$

Proof:

(a) Suppose, first, that $\mathcal{I}$ is Leibniz truth equational and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then
$$\begin{array}{lll} \phi \in T_\Sigma & \text{iff} & \phi/\Omega_\Sigma(T) \in T_\Sigma/\Omega_\Sigma(T) \quad \text{(by compatibility)} \\ & \text{iff} & \tau^{\mathcal{F}/\Omega(T)}[\phi/\Omega_\Sigma(T)] \le \Delta^{\mathcal{F}/\Omega(T)} \quad \text{(by hypothesis)} \\ & \text{iff} & \tau^\flat_\Sigma[\phi]/\Omega(T) \le \Delta^{\mathcal{F}/\Omega(T)} \quad \text{(by definition)} \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \le \Omega(T). \end{array}$$

Assume, conversely, that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in T_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \le \Omega(T)$. Let $\langle \mathcal{F}/\Omega(T), T/\Omega(T)\rangle \in \mathrm{LMatFam}^*(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then
$$\begin{array}{lll} \phi/\Omega_\Sigma(T) \in T_\Sigma/\Omega_\Sigma(T) & \text{iff} & \phi \in T_\Sigma \quad \text{(by compatibility)} \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \le \Omega(T) \quad \text{(by hypothesis)} \\ & \text{iff} & \tau^\flat_\Sigma[\phi]/\Omega(T) \le \Delta^{\mathcal{F}/\Omega(T)} \quad \text{(by definition)} \\ & \text{iff} & \tau^{\mathcal{F}/\Omega(T)}_\Sigma[\phi/\Omega_\Sigma(T)] \le \Delta^{\mathcal{F}/\Omega(T)}. \end{array}$$

(b) Suppose, first, that $\mathcal{I}$ is universally Leibniz truth equational and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. Then
$$\begin{array}{lll} \phi \in T_\Sigma & \text{iff} & \phi/\Omega^{\mathcal{A}}_\Sigma(T) \in T_\Sigma/\Omega^{\mathcal{A}}_\Sigma(T) \quad \text{(by compatibility)} \\ & \text{iff} & \tau^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}[\phi/\Omega^{\mathcal{A}}_\Sigma(T)] \le \Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)} \quad \text{(by hypothesis)} \\ & \text{iff} & \tau^{\mathcal{A}}_\Sigma[\phi]/\Omega^{\mathcal{A}}(T) \le \Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)} \quad \text{(by definition)} \\ & \text{iff} & \tau^{\mathcal{A}}_\Sigma[\phi] \le \Omega^{\mathcal{A}}(T). \end{array}$$

Assume, conversely, that, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $\phi \in T_\Sigma$ if and only if $\tau^{\mathcal{A}}_\Sigma[\phi] \le \Omega^{\mathcal{A}}(T)$. Let $\langle \mathcal{A}, T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. Then
$$\begin{array}{lll} \phi \in T_\Sigma & \text{iff} & \tau^{\mathcal{A}}_\Sigma[\phi] \le \Omega^{\mathcal{A}}(T) \quad \text{(by hypothesis)} \\ & \text{iff} & \tau^{\mathcal{A}}_\Sigma[\phi] \le \Delta^{\mathcal{A}}. \quad (\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}) \end{array}$$

Parts (c) and (d) follow along similar lines. ∎

We investigate next the relationships between the various types of truth equationality. Our first result is that Leibniz truth equationality and universal Leibniz truth equationality coincide.

Proposition 817 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is Leibniz truth equational if and only if it is universally Leibniz truth equational.*

Proof: First, note that, since $\mathrm{LMatFam}^*(\mathcal{I}) \subseteq \mathrm{MatFam}^*(\mathcal{I})$, universal Leibniz truth equationality trivially implies Leibniz truth equationality. Suppose, conversely, that $\mathcal{I}$ is Leibniz truth equational, with witnessing equations $\tau^\flat \subseteq N^\flat$. Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, we have

$$\begin{array}{lll}\alpha_\Sigma(\phi) \in T_{F(\Sigma)} & \text{iff} & \phi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)}) \quad \text{(set theory)}\\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(\alpha^{-1}(T)) \quad \text{(Proposition 816)}\\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T)) \quad \text{(Proposition 24)}\\ & \text{iff} & \alpha(\tau^\flat_\Sigma[\phi]) \leq \Omega^{\mathcal{A}}(T) \quad \text{(set theory)}\\ & \text{iff} & \tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi)] \leq \Omega^{\mathcal{A}}(T). \quad \text{(Lemma 96)}\end{array}$$

Taking into account the surjectivity of $\langle F, \alpha\rangle$ and Proposition 816, we conclude that $\mathcal{I}$ is universally Leibniz truth equational. ∎

In the next proposition, we show that (universal) Leibniz truth equationality and universal Suszko truth equationality are also identical properties.

Theorem 818 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is universally Leibniz truth equational if and only if it is universally Suszko truth equational.*

Proof: Since $\mathrm{MatFam}^*(\mathcal{I}) \subseteq \mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})$, it follows that universal Suszko truth equationality implies universal Leibniz truth equationality. Suppose, conversely, that $\mathcal{I}$ is universally Leibniz truth equational, with witnessing equations $\tau^\flat$. Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. By Proposition 816, it suffices to show that

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^{\mathcal{A}}_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T).$$

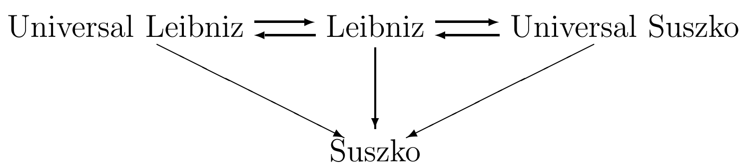
If $\phi \in T_\Sigma$, then $\phi \in T'_\Sigma$, for all $T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Thus, by hypothesis, $\tau^{\mathcal{A}}_\Sigma[\phi] \leq \Omega^{\mathcal{A}}(T')$. But then we have

$$\tau^{\mathcal{A}}_\Sigma[\phi] \leq \bigcap_{T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})} \Omega^{\mathcal{A}}(T') = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T).$$

Suppose, conversely, that $\tau^{\mathcal{A}}_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. Then, since $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T)$, we get that $\tau^{\mathcal{A}}[\phi] \leq \Omega^{\mathcal{A}}(T)$, whence, by hypothesis, $\phi \in T_\Sigma$.

Using Proposition 816, we conclude that $\mathcal{I}$ is universally Suszko truth equational. ■

Since, for any π-institution $\mathcal{I}$, $\mathrm{LMatFam}^{\mathsf{Su}}(\mathcal{I}) \subseteq \mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})$, we have, trivially, that universal Suszko truth equationality implies Suszko truth equationality. Therefore, we get the following picture involving implications between the various truth equationality properties:



Next, we present an example showing that the top-to-bottom implication is not an equivalence in general. I.e., we construct an example of a π-institution, which is Suszko truth equational but not Leibniz truth equational.

Example 819 EXAMPLE NOT FOUND YET!

We call a π-institution that is (universally) Leibniz truth equational, or equivalently, universally Suszko truth equational, a **family truth-equational π-institution**, or more simply, a **truth equational π-institution**.

Combining these results with Corollary 815, we get the following

Corollary 820 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a π-institution and* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$. *If* $\mathcal{I}$ *is truth equational with witnessing equations* $\tau^\flat$, *then the three classes* $\mathrm{LAlgSys}^*(\mathcal{I})$, $\mathrm{AlgSys}^*(\mathcal{I})$ *and* $\mathrm{AlgSys}^{\mathsf{Su}}(\mathcal{I})$ *are* $\tau^\flat$*-semantics for* $\mathcal{I}$.

Proof: By the definition of truth equationality and Corollary 815. ■

11.6 More on Truth Equationality

We start this section by looking closely at a property similar to the one defining truth equationality.

Lemma 821 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$ *and* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *a set of natural transformations in* $N^\flat$. *The following statements are equivalent:*

(a) *For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\tau^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(C(\phi))$;

(b) *For all* $T \in \mathrm{ThFam}(\mathcal{I})$, $\tau^\flat[T] \le \widetilde{\Omega}^{\mathcal{I}}(T)$.

Proof: For (a)⇒(b), assume that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\tau^\flat_\Sigma[\phi] \le \widetilde{\Omega}^{\mathcal{I}}(C(\phi))$, and let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in T_\Sigma$,

$$\begin{array}{lll}\tau^\flat_\Sigma[\phi] & \le & \widetilde{\Omega}^{\mathcal{I}}(C(\phi)) \quad \text{(hypothesis)}\\ & \le & \widetilde{\Omega}^{\mathcal{I}}(T). \quad \text{(monotonicity of } \widetilde{\Omega}^{\mathcal{I}})\end{array}$$

Therefore, $\tau^\flat[T] \le \widetilde{\Omega}^{\mathcal{I}}(T)$.

For (b)⇒(a), assume that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\tau^\flat[T] \le \widetilde{\Omega}^{\mathcal{I}}(T)$, and let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, by hypothesis,

$$\tau^\flat_\Sigma[\phi] \le \tau^\flat[C(\phi)] \le \widetilde{\Omega}^{\mathcal{I}}(C(\phi)).$$

■

A very similar property holds replacing theory families by theory systems and using the arrow operators.

Lemma 822 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *a set of natural transformations in* $N^\flat$*. The following statements are equivalent:*

(a) *For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\tau^\flat_\Sigma[\phi] \le \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi}))$;

(b) *For all* $T \in \mathrm{ThSys}(\mathcal{I})$, $\tau^\flat[T] \le \widetilde{\Omega}^{\mathcal{I}}(T)$.

Proof: For (a)⇒(b), assume that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\tau^\flat_\Sigma[\phi] \le \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi}))$, and let $T \in \mathrm{ThSys}(\mathcal{I})$. Then, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in T_\Sigma$, $C(\overrightarrow{\phi}) \le T$ and, hence,

$$\begin{array}{lll}\tau^\flat_\Sigma[\phi] & \le & \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) \quad \text{(hypothesis)}\\ & \le & \widetilde{\Omega}^{\mathcal{I}}(T). \quad \text{(monotonicity of } \widetilde{\Omega}^{\mathcal{I}})\end{array}$$

Therefore, $\tau^\flat[T] \le \widetilde{\Omega}^{\mathcal{I}}(T)$.

For (b)⇒(a), assume that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\tau^\flat[T] \le \widetilde{\Omega}^{\mathcal{I}}(T)$, and let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, by hypothesis,

$$\tau^\flat_\Sigma[\phi] \le \tau^\flat[C(\overrightarrow{\phi})] \le \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})).$$

■

The property studied in Lemma 821 is one that is satisfied by every π-institution possessing a $\tau^\flat$-semantics.

Proposition 823 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *a set of natural transformations in* $N^\flat$*. If* $\mathcal{I}$ *has a* $\tau^\flat$*-semantics, then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\tau^\flat_\Sigma[\phi] \le \widetilde{\Omega}^{\mathcal{I}}(C(\phi)).$$

Proof: Suppose that K is a $\tau^\flat$-semantics for $\mathcal{I}$ and let $\delta^\flat \approx \epsilon^\flat$ be an arbitrary equation in $\tau^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Our goal is to show that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\langle \delta^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}), \epsilon^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi})\rangle \in \widetilde{\Omega}^{\mathcal{I}}_{\Sigma'}(C(\phi)).$$

By the characterization theorem for membership in the Suszko congruence system, and using symmetry, it suffices to show that, for all $\sigma^\flat \in N^\flat$, all $\Sigma'' \in |\mathbf{Sign}^\flat|$, all $g \in \mathbf{Sign}^\flat(\Sigma',\Sigma'')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma'')$,

$$\begin{array}{ccccc}
\Sigma & \xrightarrow{\quad f \quad} & \Sigma' & \xrightarrow{\quad g \quad} & \Sigma'' \\
\phi & \mapsto & \mathrm{SEN}^\flat(\phi) & \mapsto & \mathrm{SEN}^\flat(gf)(\phi) \\
 & & \vec{\chi} & \mapsto & \mathrm{SEN}^\flat(g)(\vec{\chi}) \\
 & & & & \vec{\xi}
\end{array}$$

$$\begin{array}{l}\sigma^\flat_{\Sigma''}(\mathrm{SEN}^\flat(g)(\epsilon^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi})),\vec{\xi}) \\ \qquad \in C_{\Sigma''}(\phi,\sigma^\flat_{\Sigma''}(\mathrm{SEN}^\flat(g)(\delta^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi})),\vec{\xi})).\end{array}$$

This is equivalent to showing

$$\begin{array}{l}\sigma^\flat_{\Sigma''}(\epsilon^\flat_{\Sigma''}(\mathrm{SEN}^\flat(gf)(\phi),\mathrm{SEN}^\flat(g)(\vec{\chi})),\vec{\xi}) \\ \qquad \in C_{\Sigma''}(\phi,\sigma^\flat_{\Sigma''}(\delta^\flat_{\Sigma''}(\mathrm{SEN}^\flat(gf)(\phi),\mathrm{SEN}^\flat(g)(\vec{\chi})),\vec{\xi})).\end{array}$$

To show this, however, it suffices to show that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\chi},\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\sigma^\flat_{\Sigma'}(\epsilon^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\vec{\xi}) \in C_{\Sigma'}(\phi,\sigma^\flat_{\Sigma'}(\delta^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\vec{\xi})).$$

This, now, follows from the fact that K is a $\tau^\flat$-semantics for $\mathcal{I}$ and that, obviously,

$$\begin{array}{l}\tau^\flat_{\Sigma'}[\sigma^\flat_{\Sigma'}(\epsilon^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\vec{\xi})] \\ \qquad \le C^{\mathrm{K}}(\tau^\flat_\Sigma[\phi],\tau^\flat_{\Sigma'}[\sigma^\flat_{\Sigma'}(\delta^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\vec{\xi})]).\end{array}$$

■

Now we obtain the following consequence:

Corollary 824 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *a set of natural transformations in* $N^\flat$*. If* $\mathcal{I}$ *has a* $\tau^\flat$*-semantics, then. for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\tau^\flat[T] \le \widetilde{\Omega}^{\mathcal{I}}(T).$$

Proof: By Proposition 823 and Lemma 821. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$. We say that the Suszko operator:

- is **universally family injective** if, for every **F**-algebraic system $\mathcal{A}$, and all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T') \quad \text{implies} \quad T = T';$$

- has the **universal family minimality** property if, for every **F**-algebraic system $\mathcal{A}$, and every $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$ is the least theory family of $\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$.

Universal family injectivity and universal family minimality of the Suszko operator turn out to be equivalent properties.

Theorem 825 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *The Suszko operator is universally family injective if and only if it has the universal family minimality property.*

Proof: Suppose, first, that the Suszko operator has the universal family minimality property. Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an **F**-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T')$. Then both $T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$ and $T'/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T')$ are $\mathcal{I}$-filter families on the **F**-algebraic system $\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T')$. Thus, by the universal family minimality property, $T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = T'/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T')$. Since $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T')$, we get that $T = T'$. So $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ is universally family injective.

Suppose, conversely, that the Suszko operator is universally family injective. Consider an **F**-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Let T' be the least $\mathcal{I}$-filter family on $\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. Since we have $T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))$, we get, by minimality, that $T' \le T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. But then, by the monotonicity of the Suszko operator, we get that

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(T') \le \widetilde{\Omega}^{\mathcal{I},\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)) = \Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}$$

and, therefore,

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(T') = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))(= \Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}).$$

Hence, by universal family injectivity, $T' = T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$, which proves that the Suszko operator has the universal family minimality property. ■

Finally, recall that a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is called **family c-reflective** if, for every $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,

$$\bigcap_{T \in \mathcal{T}} \Omega(T) \le \Omega(T') \quad \text{implies} \quad \bigcap \mathcal{T} \le T'.$$

Also recall that, by the Transfer Theorem **??**, $\mathcal{I}$ is family c-reflective if and only if, for every **F**-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T') \quad \text{implies} \quad \bigcap \mathcal{T} \le T'.$$

We may call this latter property **universal family complete reflectivity** or **universal family c-reflectivity**.

Our goal, in closing this section is to show that the family injectivity of the Suszko operator (and, hence, by Theorem 825, its universal family minimality) is equivalent to the (universal) family c-reflecitvity of $\mathcal{I}$.

We provide, first, an alternative characterization of universal family c-reflectivity involving both the Suszko and the Leibniz congruence systems.

Lemma 826 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is (universally) family c-reflective if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T') \quad \textit{implies} \quad T \leq T'.$$

Proof: Assume, first, that $\mathcal{I}$ is universally family c-reflective and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. Thus, by the definition of the Suszko operator,

$$\bigcap\{\Omega^{\mathcal{A}}(T'') : T \leq T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \leq \Omega^{\mathcal{A}}(T').$$

Using universal family c-reflectivity, we get that

$$\bigcap\{T'' : T \leq T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \leq T'.$$

Hence, $T \leq T'$, as required.

Suppose, conversely, that, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ implies $T \leq T'$. Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. Then we have

$$\begin{array}{rcll}
\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(\bigcap_{T \in \mathcal{T}} T) & \leq & \bigcap_{T \in \mathcal{T}} \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) & (\text{monotonicity of } \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}) \\
& \leq & \bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) & (\text{since } \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T)) \\
& \leq & \Omega^{\mathcal{A}}(T'). & (\text{by hypothesis})
\end{array}$$

Using the hypothesis, we conclude that $\bigcap \mathcal{T} \leq T'$. Therefore, $\mathcal{I}$ is family c-reflective. ■

Finally, we show that family c-reflectivity is identical with the universal family injectivity of the Suszko operator.

Theorem 827 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is (universally) family c-reflective if and only if the Suszko operator is universally family injective.*

Proof: Suppose, first, that the Suszko operator is universally family injective. To show that $\mathcal{I}$ is family c-reflective, we use Lemma 826. Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$.

This implies that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$ is compatible with T'. We consider the natural transformation

$$\langle I,\gamma\rangle : \mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \to \mathcal{A}/\Omega^{\mathcal{A}}(T').$$

Since $T'/\Omega^{\mathcal{A}}(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T'))$, we get

$$\gamma^{-1}(T'/\Omega^{\mathcal{A}}(T')) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)),$$

i.e., $T'/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))$. By universal family injectivity of the Suszko operator and Theorem 825, we get that $T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \le T'/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. Taking into account the compatibility of $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$ with T', pointed out above, we get that $T \le T'$.

Assume, conversely, that $\mathcal{I}$ is (universally) family c-reflective. Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T,T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T')$. Then, we have $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')$ and $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T') \le \Omega^{\mathcal{A}}(T)$, whence, by hypothesis and Lemma 826, $T \le T'$ and $T' \le T$, showing that $T = T'$. Thus, the Suszko operator is universally family injective. ■

In a nutshell we have the following three equivalent properties, given in Theorems 825 and 827.

Suszko is Universally Family Injective

$\downarrow\uparrow$

Suszko has Universal Family Minimality

$\downarrow\uparrow$

$\mathcal{I}$ is (Universally) Family c-Reflective

11.7 Truth Equationality and c-Reflectivity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$. Recall that $\mathcal{I}$ was called *family c-reflective* if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,

$$\bigcap_{T\in\mathcal{T}} \Omega(T) \le \Omega(T') \quad \text{implies} \quad \bigcap_{T\in\mathcal{T}} T \le T'.$$

Family c-reflectivity implies family reflectivity, i.e., the property that, for all $T,T' \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) \le \Omega(T')$ implies $T \le T'$. Finally, family c-reflectivity is a property strong enough to imply systemicity. Therefore, a π-institution is family c-reflective if and only if it is system c-reflective and systemic.

Recall, also, that $\mathcal{I}$ was called (*family*) *truth equational* if there exists $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with a single distinguished argument, such that, for every $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \le \Omega(T).$$

In that case, $\tau^\flat$ is termed a **set of witnessing equations** (of/for the truth equationality of $\mathcal{I}$).

Note again that truth equationality implies systemicity. In fact, if $\mathcal{I}$ is truth equational with witnessing equations $\tau^\flat$, then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, we get

$$\begin{array}{rcl} \tau^\flat_\Sigma[\phi] \leq \Omega(\overleftarrow{T}) & \text{iff} & \phi \in \overleftarrow{T}_\Sigma \\ & \text{implies} & \phi \in T_\Sigma \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(T) \\ & \text{implies} & \tau^\flat_\Sigma[\phi] \leq \Omega(\overleftarrow{T}). \end{array}$$

So all statements above are equivalent showing that $\overleftarrow{T} = T$. Thus, $\mathcal{I}$ is systemic.

It turns out that, if $\mathcal{I}$ is a truth equational π-institution, with witnessing equations $\tau^\flat$, then $\tau^\flat(\Omega(T))$ is exactly equal to T, i.e., that the witnessing equations reflect theory families.

Proposition 828 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is truth equational, with witnessing equations* $\tau^\flat$*, then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\tau^\flat(\Omega(T)) = T.$$

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rcll} \phi \in \tau^\flat_\Sigma(\Omega(T)) & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(T) & \text{(definition)} \\ & \text{iff} & \phi \in T_\Sigma. & \text{(truth equationality)} \end{array}$$

■

Proposition 828 has as an immediate consequence the important fact that truth equationality implies family c-reflectivity.

Theorem 829 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is truth equational, then it is family c-reflective.*

Proof: Suppose that $\mathcal{I}$ is truth equational with witnessing equations $\tau^\flat$. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. Then

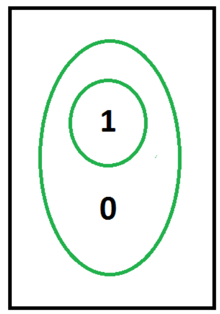
$$\begin{array}{rcll} \bigcap_{T\in\mathcal{T}} T & = & \bigcap_{T\in\mathcal{T}} \tau^\flat(\Omega(T)) & \text{(Proposition 828)} \\ & = & \tau^\flat(\bigcap_{T\in\mathcal{T}} \Omega(T)) & \text{(set theory)} \\ & \leq & \tau^\flat(\Omega(T')) & \text{(hypothesis and Lemma 94)} \\ & = & T'. & \text{(Proposition 828)} \end{array}$$

Thus, $\mathcal{I}$ is family c-reflective. ■

The following example shows that the inclusion of Theorem 829 is proper.

Example 830 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*;*
- $N^\flat$ *is the trivial category of natural transformations consisting of the projections only.*



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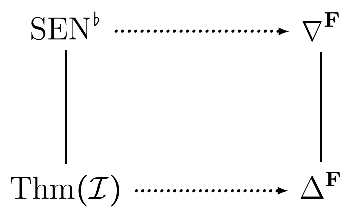
Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\}$*.*

$\mathcal{I}$ *has two theory families,* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, which are also theory systems. Clearly,* $\mathrm{Thm}(\mathcal{I}) \leq \mathrm{SEN}^\flat$*. Moreover,* $\Omega(\mathrm{Thm}(\mathcal{I})) = \Delta^{\mathbf{F}}$ *and* $\Omega(\mathrm{SEN}^\flat) = \nabla^{\mathbf{F}}$*.* $\mathcal{I}$ *is clearly family c-reflective.*



On the other hand, there does not exist $\tau^\flat \subseteq N^\flat$*, such that* $I^\flat$ *has the required properties to constitute a witnessing set of equations for the truth equationality in* $\mathcal{I}$*. Any set consisting of projections only cannot satisfy the required condition since* $\tau^\flat(\Omega(T))$ *can only be* $\mathrm{SEN}^\flat$ *or* $\overline{\varnothing}$*.*

We now work towards a dual goal. We first provide a characterization of truth equationality in terms of the solubility property of the Suszko core of the π-institution. Then, we provide an exact description of those family c-reflective π-institutions which are truth equational.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. We define the **Suszko core** $S^{\mathcal{I}}$ of $\mathcal{I}$ to be the collection

$$S^{\mathcal{I}} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}(\mathcal{I}))(\sigma^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T))\}.$$

By Lemma 821, this definition is equivalent to setting

$$S^{\mathcal{I}} = \{\sigma^\flat \in N^\flat : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\sigma^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(C(\phi)))\}.$$

The Suszko core has a list of interesting properties:

Proposition 831 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$.

(a) $\iota \approx \iota \in S^{\mathcal{I}}$*, where* $\iota : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *denotes the identity;*

(b) If $\delta^\flat \approx \epsilon^\flat \in S^{\mathcal{I}}$*, then* $\epsilon^\flat \approx \delta^\flat \in S^{\mathcal{I}}$*;*

(c) If $\delta^\flat \approx \epsilon^\flat, \epsilon^\flat \approx \zeta^\flat \in S^{\mathcal{I}}$*, then* $\delta^\flat \approx \zeta^\flat \in S^{\mathcal{I}}$*;*

(d) If $\delta^\flat \approx \epsilon^\flat \in S^{\mathcal{I}}$*, then, for all* $\sigma^\flat \in N^\flat$*,*

$$\sigma^\flat \circ \langle \delta^\flat(\vec{p}), \vec{q} \rangle \approx \sigma^\flat \circ \langle \epsilon^\flat(\vec{p}), \vec{q} \rangle \in S^{\mathcal{I}},$$

where $\vec{p}, \vec{q}$ *denote vectors of projections*

$$\vec{p} = \langle p^{k+n-1,0}, \dots, p^{k+n-1,k-1} \rangle, \vec{q} = \langle p^{k+n-1,k}, \dots, p^{k+n-1,k+n-2} \rangle,$$

with k *the maximum arity between* $\delta^\flat$ *and* $\epsilon^\flat$*, and* n *the arity of* $\sigma^\flat$*.*

Proof:

(a) Since, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in T_\Sigma$,

$$(\iota \approx \iota)_\Sigma[\phi] \leq \Delta^{\mathbf{F}} \leq \widetilde{\Omega}^{\mathcal{I}}(T),$$

we get, by definition, $\iota \approx \iota \in S^{\mathcal{I}}$.

(b) Suppose that $\delta^\flat \approx \epsilon^\flat \in S^{\mathcal{I}}$. Then, by definition, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in T_\Sigma$, $(\delta^\flat \approx \epsilon^\flat)_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T)$. By the symmetry property of the Suszko congruence system $\widetilde{\Omega}^{\mathcal{I}}(T)$, we conclude that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in T_\Sigma$, $(\epsilon^\flat \approx \delta^\flat)_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T)$. Therefore, $\epsilon^\flat \approx \delta^\flat \in S^{\mathcal{I}}$.

(c) This follows along the lines of Part (b), using the transitivity of the Suszko congruence system $\widetilde{\Omega}^{\mathcal{I}}(T)$ instead of its symmetry.

(d) Suppose that $\delta^\flat \approx \epsilon^\flat \in S^{\mathcal{I}}$ and $\sigma^\flat \in N^\flat$. Then, by definition of $S^{\mathcal{I}}$, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in T_\Sigma$, $(\delta^\flat \approx \epsilon^\flat)_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T)$. Thus, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\langle \delta^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}), \epsilon^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \rangle \in \widetilde{\Omega}^{\mathcal{I}}_{\Sigma'}(T).$$

But, then, by the congruence compatibility property of $\widetilde{\Omega}^{\mathcal{I}}(T)$, we get that, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\langle \sigma^\flat_{\Sigma'}(\delta^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\vec{\xi}), \sigma^\flat_{\Sigma'}(\epsilon^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\vec{\xi})\rangle \in \widetilde{\Omega}^{\mathcal{I}}_{\Sigma'}(T).$$

Since $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and $\vec{\chi},\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$ were arbitrary, we get

$$(\sigma^\flat \circ \langle \delta^\flat(\vec{p}),\vec{q}\rangle \approx \sigma^\flat \circ \langle \epsilon^\flat(\vec{p}),\vec{q}\rangle)_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T).$$

Finally, since $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in T_\Sigma$ were arbitrary, we conclude that

$$\sigma^\flat \circ \langle \delta^\flat(\vec{p}),\vec{q}\rangle \approx \sigma^\flat \circ \langle \epsilon^\flat(\vec{p}),\vec{q}\rangle \in S^{\mathcal{I}}.$$

■

It is clear, by the definitions involved, that the Suszko core of a π-institution satisfies the following property:

Proposition 832 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. For every* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$T \leq S^{\mathcal{I}}(\Omega(T)).$$

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{llll} \phi \in T_\Sigma & \text{implies} & S^{\mathcal{I}}_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T) & (\text{definition of } S^{\mathcal{I}}) \\ & \text{implies} & S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T). & (\widetilde{\Omega}^{\mathcal{I}}(T) \leq \Omega(T)) \end{array}$$

Thus, we get that $T \leq S^{\mathcal{I}}(\Omega(T))$. ■

It is possible, but not necessary, that the Suszko core of a π-institution satisfies the reverse inclusion. We call this property solubility.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. We say that the Suszko core of $\mathcal{I}$ is **soluble** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$S^{\mathcal{I}}(\Omega(T)) \leq T.$$

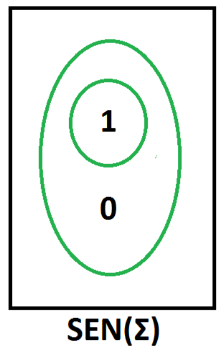
In other words, $S^{\mathcal{I}}$ is soluble if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) \quad \text{implies} \quad \phi \in T_\Sigma.$$

We present two examples to showcase the possibilities. In the first example, we look at a π-institution $\mathcal{I}$ whose Suszko core $S^{\mathcal{I}}$ is soluble.

Example 833 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

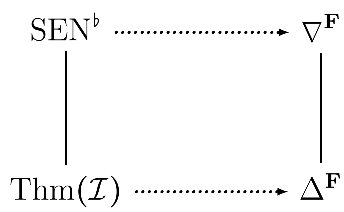
- $\mathbf{Sign}^\flat$ *is the trivial category with single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*;*



- $N^\flat$ *is the category of natural transformations generated by the single unary natural transformations* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$, *specified by setting* $\sigma^\flat_\Sigma(x) = 1$, *for all* $x \in \mathrm{SEN}^\flat(\Sigma)$.

Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\}, \{0,1\}\}$.

$\mathcal{I}$ *has two theory families,* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$, *which are also theory systems. Moreover, the structure of its posets of theory families and of their associated Leibniz congruence systems is given below.*



One can see that the Suszko core of $\mathcal{I}$ *is given by*

$$S^{\mathcal{I}} = \{\iota \approx \iota, \iota \approx \sigma^\flat, \sigma^\flat \approx \iota, \sigma^\flat \approx \sigma^\flat\}.$$

Since the implication

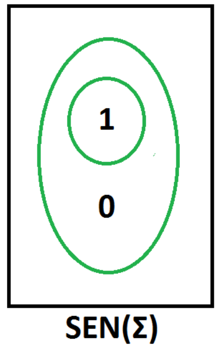
$$S^{\mathcal{I}}_\Sigma[\phi] \le T \quad \textit{implies} \quad \phi \in T_\Sigma$$

holds universally, we conclude that the Suszko core of $\mathcal{I}$ *is soluble.*

Next, we present an example of a π-institution $\mathcal{I}$ whose Suszko core $S^{\mathcal{I}}$ is not soluble.

Example 834 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with single object* Σ;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$;



- $N^\flat$ *is the trivial category of natural transformations consisting of the projections only.*

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\}$.

$\mathcal{I}$ *has two theory families,* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, which are also theory systems. Moreover, the structure of its posets of theory families and of their associated Leibniz congruence systems is given below.*

$$\begin{array}{ccc} \mathrm{SEN}^\flat & \cdots\cdots\rightarrow & \nabla^{\mathbf{F}} \\ | & & | \\ \mathrm{Thm}(\mathcal{I}) & \cdots\cdots\rightarrow & \Delta^{\mathbf{F}} \end{array}$$

One can see that the Suszko core of $\mathcal{I}$ *is given by*

$$S^{\mathcal{I}} = \{\iota \approx \iota\}.$$

We, thus, have that

$$S^{\mathcal{I}}_\Sigma[0] \leq \Delta^{\mathbf{F}} = \Omega(\mathrm{Thm}(\mathcal{I})) \quad \textit{but} \quad 0 \notin \mathrm{Thm}_\Sigma(\mathcal{I}).$$

Therefore $S^{\mathcal{I}}$ *is not soluble.*

It turns out that possession of the solubility property by the Suszko core intrinsically characterizes truth equationality. We can show, at the outset, that the Suszko core being soluble is necessary for truth equationality. Thus, there is no point in trying to discover witnessing equations unless the Suszko core of the π-institution $\mathcal{I}$ under scrutiny is soluble.

To show this, observe, first, that, in case a π-institution is truth equational, the witnessing equations form a subset of the Suszko core.

Lemma 835 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is truth equational, with witnessing equations* $\tau^\flat \subseteq N^\flat$*, then* $\tau^\flat \subseteq S^{\mathcal{I}}$*.*

Proof: By truth equationality, for all $T \in \text{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \text{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

Thus, for all $T \in \text{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \text{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll} \phi \in T_\Sigma & \text{iff} & (\forall T \leq T' \in \text{ThFam}(\mathcal{I}))(\phi \in T'_\Sigma) \\ & \text{iff} & (\forall T \leq T' \in \text{ThFam}(\mathcal{I}))(\tau^\flat_\Sigma[\phi] \leq \Omega(T')) \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \bigcap\{\Omega(T') : T \leq T' \in \text{ThFam}(\mathcal{I})\} \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T). \end{array}$$

We conclude, by the definition of $S^{\mathcal{I}}$, that $\tau^\flat \subseteq S^{\mathcal{I}}$. ∎

Now we prove the necessity of solubility for truth equationality.

Theorem 836 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is truth equational, then* $S^{\mathcal{I}}$ *is soluble.*

Proof: Suppose that $\mathcal{I}$ is truth equational, with witnessing equations $\tau^\flat$. Then, for all $T \in \text{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \text{SEN}^\flat(\Sigma)$,

$$\begin{array}{llll} S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) & \text{implies} & \tau^\flat_\Sigma[\phi] \leq \Omega(T) & \text{(Lemma 835)} \\ & \text{iff} & \phi \in T_\Sigma. & \text{(truth equationality)} \end{array}$$

Thus, $S^{\mathcal{I}}$ is soluble. ∎

The reverse implication, which also holds and completes the promised characterization of truth equationality in terms of the Suszko core, is presented in the following result.

Theorem 837 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $S^{\mathcal{I}}$ *is soluble, then* $\mathcal{I}$ *is truth equational, with witnessing equations* $S^{\mathcal{I}}$*.*

Proof: It suffices to show that, for all $T \in \text{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \text{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

The left-to-right implication is given in Proposition 832, whereas the converse is ensured by the postulated solubility of $S^{\mathcal{I}}$. ∎

Theorems 836 and 837 provide the promised characterization of truth equationality in terms of the solubility of the Suszko core.

$$\mathcal{I} \text{ is Truth Equational} \longleftrightarrow S^{\mathcal{I}} \text{ is Soluble.}$$

Theorem 838 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is truth equational if and only if* $S^{\mathcal{I}}$ *is soluble.*

Proof: Theorem 836 gives the "only if" and the "if" is by Theorem 837. ■

If $\mathcal{I}$ is truth equational, then the Suszko core defines theory families in $\mathcal{I}$ in terms of their Leibniz congruence systems. This proposition may be viewed as a special case of Proposition 828, since $S^{\mathcal{I}}$ forms a maximal set of witnessing equations for the truth equationality of $\mathcal{I}$.

Proposition 839 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $S^{\mathcal{I}}$ *is soluble, then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$,

$$T = S^{\mathcal{I}}(\Omega(T)).$$

Proof: If $S^{\mathcal{I}}$ is soluble, then, by Theorem 837, $S^{\mathcal{I}}$ forms a set of witnessing equations for the truth equationality of $\mathcal{I}$. Therefore, by Proposition 828, we get that, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $T = S^{\mathcal{I}}(\Omega(T))$. ■

In fact, this property may also be restated as another characterization of truth equationality. Let us say that $S^{\mathcal{I}}$ **defines theory families** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T = S^{\mathcal{I}}(\Omega(T))$. Then we have:

$$\mathcal{I} \text{ is Truth Equational} \longleftrightarrow S^{\mathcal{I}} \text{ Defines Theory Families.}$$

Theorem 840 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is truth equational if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$,

$$T = S^{\mathcal{I}}(\Omega(T)).$$

Proof: If $\mathcal{I}$ is truth equational, then, by Theorem 838, $S^{\mathcal{I}}$ is soluble. Thus, by Proposition 839, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T = S^{\mathcal{I}}(\Omega(T))$.

Conversely, if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T = S^{\mathcal{I}}(\Omega(T))$, then $S^{\mathcal{I}}$ is soluble. Thus, again by Theorem 838, $S^{\mathcal{I}}$ is a set of witnessing equations and $\mathcal{I}$ is truth equational. ■

We finally show that the property that separates family complete reflectivity from truth equationality is exactly the adequacy property of the Suszko core. Roughly speaking, this property ensures that the Suszko core is rich enough to define Suszko congruence systems in terms of the Leibniz congruence systems of theory families that it selects via inclusion.

We have the following relationship connecting the Suszko core with both Leibniz and Suszko congruence systems.

Proposition 841 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \leq \widetilde{\Omega}^{\mathcal{I}}(C(\phi)).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\begin{array}{lll} \phi \in T_\Sigma & \text{implies} & S^{\mathcal{I}}_\Sigma[\phi] \le \widetilde{\Omega}^{\mathcal{I}}(T) \quad \text{(definition of the Suszko core)} \\ & \text{implies} & S^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T). \quad (\widetilde{\Omega}^{\mathcal{I}}(T) \le \Omega(T)) \end{array}$$

Therefore, we have

$$\begin{array}{rcl} \bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)\} & \le & \bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[\phi] \le \widetilde{\Omega}^{\mathcal{I}}(T)\} \\ & \le & \bigcap\{\Omega(T) : \phi \in T_\Sigma\} \\ & = & \widetilde{\Omega}^{\mathcal{I}}(C(\phi)). \end{array}$$

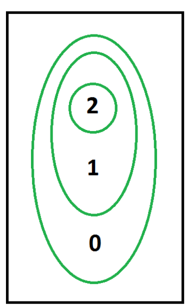
■

We provide an example, next, that shows that the inclusion proven in Proposition 841 is proper, in general. I.e., there exist π-institutions $\mathcal{I}$ in which, for some signature Σ and some Σ-sentence ϕ,

$$\bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)\} \lneqq \widetilde{\Omega}^{\mathcal{I}}(C(\phi)).$$

Example 842 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the trivial category of natural transformations, consisting of the projections only.*



SEN(Σ)

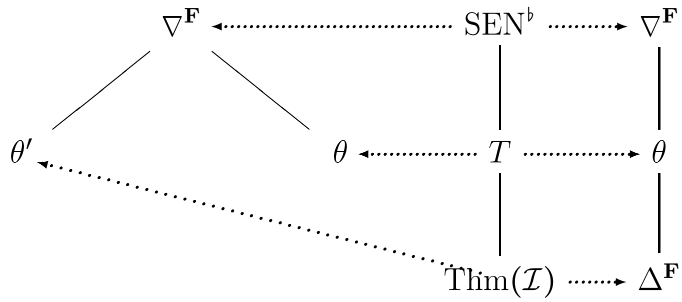
Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

$\mathcal{I}$ *has three theory families* $\mathrm{Thm}(\mathcal{I})$, $T = \{\{1, 2\}\}$ *and* $\mathrm{SEN}^\flat$*, all of which are theory systems.*

Note that $S^{\mathcal{I}} = \{\iota \approx \iota\}$*. Note, also, the structure of the posets of Leibniz congruence systems and of Suszko congruence systems, that are provided in the left and right sides, respectively, of the following diagram, where*

$$T = \{\{1, 2\}\}, \quad \theta = \{\{0\}, \{1, 2\}\}, \quad \theta' = \{\{0, 1\}, \{2\}\}.$$



Taking this into account, it is not difficult to see that

$$\bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[1] \le \Omega(T)\} = \Delta^{\mathbf{F}} \lneqq \theta = \widetilde{\Omega}^{\mathcal{I}}(C(1)).$$

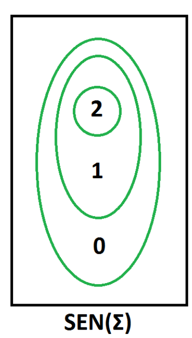
We also give an example of a π-institution $\mathcal{I}$ whose Suszko core $S^{\mathcal{I}}$ is such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) = \bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)\}.$$

Example 843 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the single unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0, 1, 2\} \to \{0, 1, 2\}$ *be given, for all* $x \in \mathrm{SEN}^\flat(\Sigma)$*, by*

$$\sigma^\flat_\Sigma(x) = 2.$$



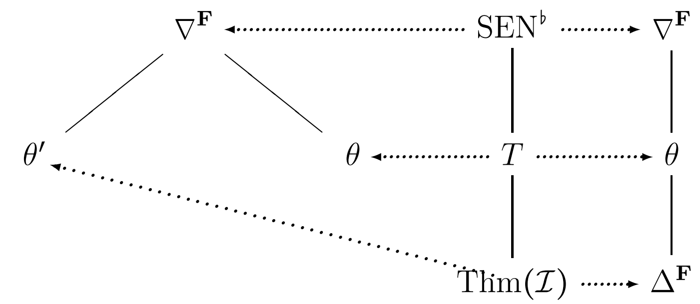
Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

$\mathcal{I}$ *has three theory families* $\mathrm{Thm}(\mathcal{I})$*,* $T = \{\{1, 2\}\}$ *and* $\mathrm{SEN}^\flat$*, all of which are theory systems.*

Note that $S^{\mathcal{I}} = \{\iota \approx \iota, \iota \approx \sigma^\flat, \sigma^\flat \approx \iota, \sigma^\flat \approx \sigma^\flat\}$. *Note, also, the structure of the posets of Leibniz congruence systems and of Suszko congruence systems, that are provided in the left and right sides, respectively, of the following diagram, where*

$$T = \{\{1,2\}\}, \quad \theta = \{\{0\},\{1,2\}\}, \quad \theta' = \{\{0,1\},\{2\}\}.$$



Now we can check:

$$\begin{aligned}
\widetilde{\Omega}^{\mathcal{I}}(C(0)) &= \nabla^{\mathbf{F}} = \Omega(\mathrm{SEN}^\flat) \\
&= \bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[0] \leq \Omega(T)\}; \\
\widetilde{\Omega}^{\mathcal{I}}(C(1)) &= \theta = \Omega(\mathrm{SEN}^\flat) \cap \Omega(T) \\
&= \bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[1] \leq \Omega(T)\}; \\
\widetilde{\Omega}^{\mathcal{I}}(C(2)) &= \Delta^{\mathbf{F}} = \Omega(\mathrm{SEN}^\flat) \cap \Omega(T) \cap \Omega(\mathrm{Thm}(\mathcal{I})) \\
&= \bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[2] \leq \Omega(T)\}.
\end{aligned}$$

We have seen, therefore, through examples, that it is possible, but not necessary, that the Suszko core of a π-institution satisfies, for every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, the reverse inclusion of that given in Proposition 841:

$$\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) \leq \bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.$$

Intuitively speaking, this means that the Suszko core $S^{\mathcal{I}}$ is rich enough to allow, for every signature Σ and every Σ-sentence ϕ, the determination of those theory families whose Leibniz congruence systems form a covering of the Suszko congruence system of $C(\phi)$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$. We say that the Suszko core $S^{\mathcal{I}}$ of $\mathcal{I}$ is **adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) = \bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.$$

Based on our preceding work, it is not difficult to see that, if $S^{\mathcal{I}}$ is soluble, then it is adequate.

Corollary 844 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $S^{\mathcal{I}}$ *is soluble, then it is adequate.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

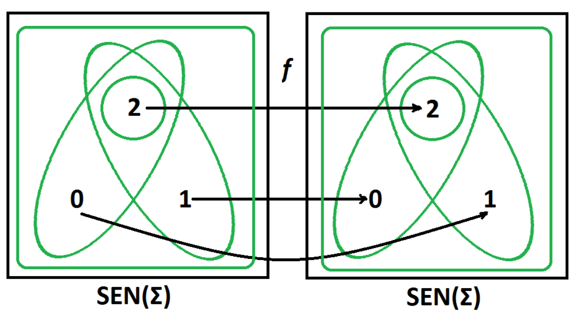
$$\begin{array}{rcll}\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) & = & \bigcap\{\Omega(T) : \phi \in T_\Sigma\} & (\text{definition of } \widetilde{\Omega}^{\mathcal{I}}(C(\phi)))\\ & = & \bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}. & \\ & & (\text{solubility of } S^{\mathcal{I}} \text{ and Proposition 839}) &\end{array}$$

We conclude that $S^{\mathcal{I}}$ is adequate. ■

Here is an example of a π-institution $\mathcal{I}$, with an adequate but not soluble Suszko core.

Example 845 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = i_\Sigma$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 1$, $1 \mapsto 0$ *and* $2 \mapsto 2$*;*
- $N^\flat$ *is the trivial category of natural transformations (consisting of the projections only).*

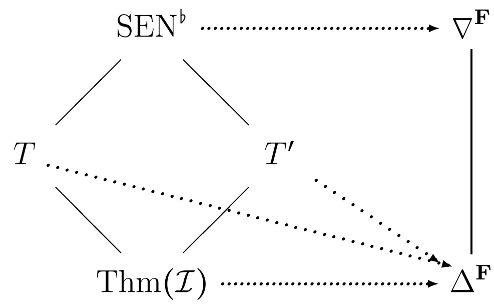


Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\},\{0,2\},\{1,2\},\{0,1,2\}\}.$$

$\mathcal{I}$ *has four theory families* $\mathrm{Thm}(\mathcal{I})$, $T = \{\{0,2\}\}$, $T' = \{\{1,2\}\}$ *and* $\mathrm{SEN}^\flat$, *but only two theory systems* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$. *Therefore, being non-systemic, it can be neither family c-reflective nor truth-equational. The fact that it is not truth equational, together with Theorem 838, reveal that the Suszko core* $S^{\mathcal{I}}$ *is not soluble.*

To verify that $S^{\mathcal{I}}$ *is adequate, we look at the posets of theory families (left), Leibniz congruence systems (right) and Suszko congruence systems* *(right, identical with the Leibniz congruence systems, since the π-institution is protoalgebraic).*



Since $S^{\mathcal{I}} = \{\iota \approx \iota\}$, we verify adequacy of $S^{\mathcal{I}}$ by the following calculation, holding for all $\phi \in \mathrm{SEN}^\flat(\Sigma)$:

$$\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) = \Delta^{\mathbf{F}} = \bigcap_{T \in \mathrm{ThFam}(\mathcal{I})} \Omega(T) = \bigcap \{\Omega(T) : S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.$$

Thus, $S^{\mathcal{I}}$ is in fact adequate but not soluble.

In the opposite direction, and on the positive side, in a *family c-reflective* π-institution $\mathcal{I}$, if the Suszko core is adequate, then it is also soluble.

Proposition 846 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a family c-reflective π-institution based on $\mathbf{F}$. If $S^{\mathcal{I}}$ is adequate, then it is soluble.*

Proof: Suppose that $\mathcal{I}$ is family c-reflective and that $S^{\mathcal{I}}$ is adequate. We must show that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

The implication left-to-right is always satisfied by Proposition 832. For the converse, assume that $S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. Then, by the adequacy of $S^{\mathcal{I}}$, we get that $\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) \leq \Omega(T)$. Thus, by family c-reflectivity and Lemma 826, we conclude that $C(\phi) \leq T$, which gives $\phi \in T_\Sigma$. ■

We finally show that a π-institution is truth equational if and only if it is family c-reflective and has an adequate Suszko core.

Truth Equationality	=	$S^{\mathcal{I}}$ Soluble
	=	$S^{\mathcal{I}}$ Defines Theory Families
	=	Family c-Reflectivity + $S^{\mathcal{I}}$ Adequate

Theorem 847 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is truth equational if and only if it is family c-reflective and has an adequate Suszko core.*

Proof: Suppose, first, that $\mathcal{I}$ is truth equational. Then it is family c-reflective by Theorem 829. Moreover, its Suszko core is soluble by Theorem 838 and, hence, by Corollary 844, its Suszko core is adequate.

Suppose, conversely, that $\mathcal{I}$ is family c-reflective with an adequate Suszko core. Then, by Proposition 846, its Suszko core is soluble and, therefore, by Theorem 838, $\mathcal{I}$ is truth equational. ∎

Finally, it is not difficult to see that, in some sense, truth equationality transfers from a π-institution to all $\mathcal{I}$-matrix families.

Theorem 848 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is truth equational, with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*, if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $T = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$*.*

Proof: Suppose $\mathcal{I}$ is truth equational, with witnessing transformations $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, whence, by hypothesis, $\alpha^{-1}(T) = \tau^\flat(\Omega(\alpha^{-1}(T)))$. Hence, by Proposition 24, $\alpha^{-1}(T) = \tau^\flat(\alpha^{-1}(\Omega^{\mathcal{A}}(T)))$. Therefore, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, we get

$$\begin{array}{lll} \alpha_\Sigma(\phi) \in T_{F(\Sigma)} & \text{iff} & \phi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)}) \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T)) \\ & \text{iff} & \alpha(\tau^\flat_\Sigma[\phi]) \leq \Omega^{\mathcal{A}}(T) \\ & \text{iff} & \tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi)] \leq \Omega^{\mathcal{A}}(T). \quad (\langle F, \alpha \rangle \text{ surjective}) \end{array}$$

Taking again into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $\phi \in T_\Sigma$ if and only if $\tau^{\mathcal{A}}_\Sigma[\phi] \leq \Omega^{\mathcal{A}}(T)$, i.e., $T = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$. ∎

11.8 Left Truth Equationality

In this section, we look at versions of truth equationality and c-reflectivity that can still be applied to general theory families but do not force the π-institutions to be systemic. In the next section we will also look at *system truth equationality*, i.e., truth equationality applied only to theory systems, and at *system c-reflectivity*. In this section we take the "leftist" approach, "left" having the meaning attributed to it in Chapter 3.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$.

Recall that $\mathcal{I}$ is **left c-reflective** if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,

$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \quad \text{implies} \quad \bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}.$$

Left c-reflectivity is not strong enough to imply systemicity. Moreover, left c-reflectivity is a property intermediate between family c-reflectivity and system c-reflectivity.

We say that the π-institution $\mathcal{I}$ is **left truth equational** if there exists $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with a single distinguished argument, such that, for every $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

In that case, we call $\tau^\flat$ a **set of witnessing equations** (of/for the left truth equationality of $\mathcal{I}$).

If $\mathcal{I}$ is a left truth equational π-institution, with witnessing equations $\tau^\flat$, then $\tau^\flat(\Omega(T))$ is exactly equal to $\overleftarrow{T}$, i.e., the witnessing equations reflect theory families only "up to arrow".

Proposition 849 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is left truth equational, with witnessing equations $\tau^\flat$, then, for all $T \in \mathrm{ThFam}(\mathcal{I})$,*

$$\tau^\flat(\Omega(T)) = \overleftarrow{T}.$$

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{llll} \phi \in \tau^\flat_\Sigma(\Omega(T)) & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(T) & \text{(definition)} \\ & \text{iff} & \phi \in \overleftarrow{T}_\Sigma. & \text{(left truth equationality)} \end{array}$$

■

Proposition 849 has as an immediate consequence the important fact that left truth equationality implies left c-reflectivity.

Theorem 850 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is left truth equational, then it is left c-reflective.*

Proof: Suppose that $\mathcal{I}$ is left truth equational with witnessing equations $\tau^\flat$. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. Then

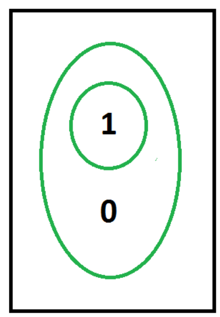
$$\begin{array}{llll} \bigcap_{T \in \mathcal{T}} \overleftarrow{T} & = & \bigcap_{T \in \mathcal{T}} \tau^\flat(\Omega(T)) & \text{(Proposition 849)} \\ & = & \tau^\flat(\bigcap_{T \in \mathcal{T}} \Omega(T)) & \text{(set theory)} \\ & \leq & \tau^\flat(\Omega(T')) & \text{(hypothesis and Lemma 94)} \\ & = & \overleftarrow{T'}. & \text{(Proposition 849)} \end{array}$$

Thus, $\mathcal{I}$ is left c-reflective. ■

The following example shows that the inclusion of Theorem 850 is proper.

Example 851 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*;*
- $N^\flat$ *is the trivial category of natural transformations consisting of the projections only.*



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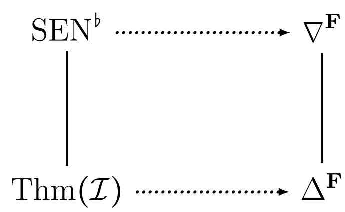
Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\}$.

$\mathcal{I}$ *has two theory families,* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, which are also theory systems. In other words,* $\overleftarrow{\mathrm{Thm}(\mathcal{I})} = \mathrm{Thm}(\mathcal{I})$ *and* $\overleftarrow{\mathrm{SEN}^\flat} = \mathrm{SEN}^\flat$*. Clearly,* $\mathrm{Thm}(\mathcal{I}) \leq \mathrm{SEN}^\flat$*. Moreover,* $\Omega(\mathrm{Thm}(\mathcal{I})) = \Delta^{\mathbf{F}}$ *and* $\Omega(\mathrm{SEN}^\flat) = \nabla^{\mathbf{F}}$*.* $\mathcal{I}$ *is clearly left c-reflective.*



On the other hand, there does not exist $\tau^\flat \subseteq N^\flat$*, such that* $I^\flat$ *has the required properties to constitute a witnessing set of equations for the left truth equationality in* $\mathcal{I}$*. Any set consisting of projections only cannot satisfy the required condition since* $\tau^\flat(\Omega(T))$ *can only be* $\mathrm{SEN}^\flat$ *or* $\overline{\varnothing}$*.*

We provide, next, a characterization of left truth equationality in terms of the left solubility property of the left Suszko core of the π-institution. Then, we provide an exact description of those left c-reflective π-institutions which are left truth equational.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The **left Suszko core** of $\mathcal{I}$ is the collection

$$L^{\mathcal{I}} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}(\mathcal{I}))(\sigma^\flat[\overleftarrow{T}] \leq \widetilde{\Omega}^{\mathcal{I}}(T))\}.$$

There is an alternative way to define the left Suszko core of a π-institution, which may be also viewed as justifying the alternative terminology *system Suszko core* for it, which we state in the form of a property.

Proposition 852 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *Then*

$$L^{\mathcal{I}} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThSys}(\mathcal{I}))(\sigma^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T)\}.$$

Proof: Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution and set

$$M^{\mathcal{I}} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThSys}(\mathcal{I}))(\sigma^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T)\}.$$

Our goal is to show that $M^{\mathcal{I}} = L^{\mathcal{I}}$.

Suppose, first, that $\sigma^\flat \in L^{\mathcal{I}}$ and let $T \in \mathrm{ThSys}(\mathcal{I})$. Then, we have

$$\begin{array}{rcll} \sigma^\flat[T] & = & \sigma^\flat[\overleftarrow{T}] & (T \in \mathrm{ThSys}(\mathcal{I})) \\ & \leq & \widetilde{\Omega}^{\mathcal{I}}(T). & (\sigma^\flat \in L^{\mathcal{I}}) \end{array}$$

Therefore $\sigma^\flat \in M^{\mathcal{I}}$.

Suppose, conversely, that $\sigma^\flat \in M^{\mathcal{I}}$ and let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, we have

$$\begin{array}{rcll} \sigma^\flat[\overleftarrow{T}] & \leq & \widetilde{\Omega}^{\mathcal{I}}(\overleftarrow{T}) & (\sigma^\flat \in M^{\mathcal{I}} \text{ and } \overleftarrow{T} \in \mathrm{ThSys}(\mathcal{I})) \\ & \leq & \widetilde{\Omega}^{\mathcal{I}}(T). & (\overleftarrow{T} \leq T \text{ and monotonicity of } \widetilde{\Omega}^{\mathcal{I}}) \end{array}$$

We conclude that $\sigma^\flat \in L^{\mathcal{I}}$ and, therefore, $M^{\mathcal{I}} = L^{\mathcal{I}}$. ■

Note that, since, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} \leq T$, we get that $S^{\mathcal{I}} \subseteq L^{\mathcal{I}}$, which implies that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $L^{\mathcal{I}}(\Omega(T)) \leq S^{\mathcal{I}}(\Omega(T))$.

Note, also, that for systemic π-institutions the left Suszko core and the Suszko core are identical.

The left Suszko core of a π-institution satisfies the following property relating to the arrow operator:

Proposition 853 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *For every* $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\overleftarrow{T} \leq L^{\mathcal{I}}(\Omega(T)).$$

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{rcll} \phi \in \overleftarrow{T}_\Sigma & \text{implies} & L^{\mathcal{I}}_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T) & (\text{by definition of } L^{\mathcal{I}}) \\ & \text{implies} & L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) & (\widetilde{\Omega}^{\mathcal{I}}(T) \leq \Omega(T)) \\ & \text{iff} & \phi \in L^{\mathcal{I}}(\Omega(T)). & (\text{by definition}) \end{array}$$

■

It is possible, but not necessary, that the left Suszko core of a π-institution satisfies the reverse inclusion. We call this property left solubility.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the left Suszko core of $\mathcal{I}$ is **left soluble** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$L^{\mathcal{I}}(\Omega(T)) \leq \overleftarrow{T}.$$

In other words, $L^{\mathcal{I}}$ is left soluble if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) \quad \text{implies} \quad \phi \in \overleftarrow{T}_\Sigma.$$

We show that this property has an alternative characterization in terms of theory systems.

Proposition 854 *Let $\mathcal{I} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. The left Suszko core $L^{\mathcal{I}}$ of $\mathcal{I}$ is left soluble if and only if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $L^{\mathcal{I}}(\Omega(T)) = T$.*

Proof: For the "only if", assume that $L^{\mathcal{I}}$ is left soluble and let $T \in \mathrm{ThSys}(\mathcal{I})$. Then

$$\begin{array}{rcll} L^{\mathcal{I}}(\Omega(T)) & = & \overleftarrow{T} & \text{(Left Solubility of } L^{\mathcal{I}}) \\ & = & T. & (T \in \mathrm{ThSys}(\mathcal{I})) \end{array}$$

Conversely, assume that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $L^{\mathcal{I}}(\Omega(T)) = T$ and let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, we have

$$\begin{array}{rcll} L^{\mathcal{I}}(\Omega(T)) & \leq & L^{\mathcal{I}}(\Omega(\overleftarrow{T})) & \text{(Proposition 20 and Lemma 94)} \\ & = & \overleftarrow{T}. & \text{(by hypothesis)} \end{array}$$

Thus, $L^{\mathcal{I}}$ is left soluble. ∎

Note that for systemic π-institutions, since the left Suszko core coincides with the Suszko core, left solubility of the left Suszko core coincides with the solubility of the Suszko core. These two properties are, however, different in general and, as the following proposition and example show, solubility of the Suszko core is a stronger property than left solubility of the left Suszko core.

Proposition 855 *Let $\mathcal{I} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If the Suszko core $S^{\mathcal{I}}$ of $\mathcal{I}$ is soluble, then the left Suszko core $L^{\mathcal{I}}$ of $\mathcal{I}$ is left soluble.*

Proof: Suppose that $S^{\mathcal{I}}$ is soluble, i.e., for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) \quad \text{implies} \quad \phi \in T_\Sigma.$$

Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. Then, since $S^{\mathcal{I}} \subseteq L^{\mathcal{I}}$, we get that $S^{\mathcal{I}}[\phi] \leq \Omega(T)$. Moreover, since $\Omega(T) \leq$

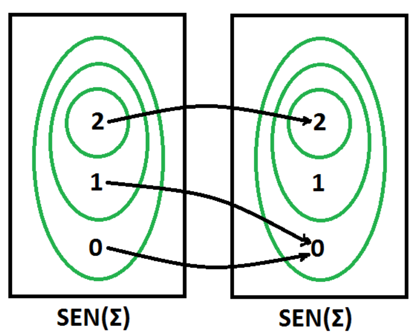
$\Omega(\overleftarrow{T})$, we get that $S^{\mathcal{I}}[\phi] \leq \Omega(\overleftarrow{T})$. Thus, by the solubility of $S^{\mathcal{I}}$, we get that $\phi \in \overleftarrow{T}_\Sigma$. We conclude that $L^{\mathcal{I}}$ is left soluble. ∎

The implication of Proposition 855 is proper in general.

Example 856 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 0$, $1 \mapsto 0$ *and* $2 \mapsto 2$*;*
- $N^\flat$ *is the category of natural transformations generated by the single unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\} \to \{0,1,2\}$ *be given, for all* $x \in \mathrm{SEN}^\flat(\Sigma)$*, by*

$$\sigma^\flat_\Sigma(x) = 2.$$

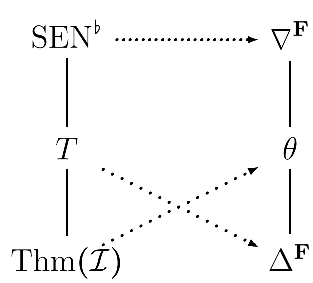


Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}.$$

$\mathcal{I}$ *has three theory families* $\mathrm{Thm}(\mathcal{I})$, $T = \{\{1,2\}\}$ *and* $\mathrm{SEN}^\flat$*, but only two theory systems* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*. So it is not a systemic* π*-institution.*

The posets of theory families and associated Leibniz congruence systems are shown in the following figure (where $T \in \{\{1,2\}\}$ *and* $\theta = \{\{0,1\},\{2\}\}$*):*



Note that

$$S^{\mathcal{I}} = \{\iota \approx \iota, \sigma^\flat \approx \sigma^\flat\},$$

whereas

$$L^{\mathcal{I}} = \{\iota \approx \iota, \iota \approx \sigma^\flat, \sigma^\flat \approx \iota, \sigma^\flat \approx \sigma^\flat\}.$$

We show that $L^{\mathcal{I}}$ is left soluble, but that $S^{\mathcal{I}}$ is not soluble.

The left solubility of $L^{\mathcal{I}}$ can be seen by looking at the defining implication on a case-by-case basis. The case of the theory family $\mathrm{SEN}^\flat$ is trivial as is the case for $\phi = 2$. For $\phi = 0$ or 1 and for the theory families T or $\mathrm{ThFam}(\mathcal{I})$, we have:

- $L^{\mathcal{I}}_\Sigma[0] \leq \Omega(T)$ *is false;*
- $L^{\mathcal{I}}_\Sigma[1] \leq \Omega(T)$ *is false;*
- $L^{\mathcal{I}}_\Sigma[0] \leq \Omega(\mathrm{Thm}(\mathcal{I}))$ *is false;*
- $L^{\mathcal{I}}_\Sigma[1] \leq \Omega(\mathrm{Thm}(\mathcal{I}))$ *is false.*

So in every other case the defining condition is vacuously satisfied.

On the other hand, $S^{\mathcal{I}}_\Sigma[0] \leq \Omega(T)$, but $0 \notin T_\Sigma$, which shows that $S^{\mathcal{I}}$ is not soluble.

It turns out that possession of left solubility by the left Suszko core intrinsically characterizes left truth equationality. We show, first, that the left Suszko core being left soluble is necessary for left truth equationality. To demonstrate this, observe, first, that, in case a π-institution is left truth equational, the witnessing equations form a subset of the left Suszko core.

Lemma 857 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is left truth equational, with witnessing equations $\tau^\flat \subseteq N^\flat$, then $\tau^\flat \subseteq L^{\mathcal{I}}$.*

Proof: By left truth equationality, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

Thus, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}
\phi \in \overleftarrow{T}_\Sigma & \text{iff} & (\forall T \leq T' \in \mathrm{ThFam}(\mathcal{I}))(\phi \in \overleftarrow{T'}_\Sigma) \\
& & \text{(by Lemma 1)} \\
& \text{iff} & (\forall T \leq T' \in \mathrm{ThFam}(\mathcal{I}))(\tau^\flat_\Sigma[\phi] \leq \Omega(T')) \\
& & \text{(left truth equationality; displayed formula above)} \\
& \text{iff} & \tau^\flat_\Sigma[\phi] \leq \bigcap\{\Omega(T') : T \leq T' \in \mathrm{ThFam}(\mathcal{I})\} \\
& & \text{(set theoretically)} \\
& \text{iff} & \tau^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T). \\
& & \text{(by definition of } \widetilde{\Omega}^{\mathcal{I}})
\end{array}$$

We conclude, by the definition of $L^{\mathcal{I}}$, that $\tau^\flat \subseteq L^{\mathcal{I}}$. ■

Now we prove the necessity of left solubility of the left Suszko core for left truth equationality.

Theorem 858 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is left truth equational, then* $L^{\mathcal{I}}$ *is left soluble.*

Proof: Suppose that $\mathcal{I}$ is left truth equational, with witnessing equations $\tau^\flat$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll} L^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T) & \text{implies} & \tau^\flat_\Sigma[\phi] \le \Omega(T) \quad \text{(Lemma 857)} \\ & \text{iff} & \phi \in \overleftarrow{T}_\Sigma. \quad \text{(left truth equationality)} \end{array}$$

Thus, $L^{\mathcal{I}}$ is left soluble. ■

The reverse implication also holds and completes the promised characterization of left truth equationality in terms of the left Suszko core.

Theorem 859 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $L^{\mathcal{I}}$ *is left soluble, then* $\mathcal{I}$ *is left truth equational, with witnessing equations* $L^{\mathcal{I}}$*.*

Proof: It suffices to show that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{iff} \quad L^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T).$$

The left-to-right implication is given in Proposition 853, whereas the converse is ensured by the postulated left solubility of $L^{\mathcal{I}}$. ■

Theorems 858 and 859 provide the promised characterization of left truth equationality in terms of the left solubility of the left Suszko core.

$$\mathcal{I} \text{ is Left Truth Equational} \longleftrightarrow L^{\mathcal{I}} \text{ is Left Soluble.}$$

Theorem 860 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is left truth equational if and only if* $L^{\mathcal{I}}$ *is left soluble.*

Proof: Theorem 858 gives the "only if" and the "if" is by Theorem 859. ■

If $\mathcal{I}$ is left truth equational, then the left Suszko core defines theory families in $\mathcal{I}$ "up to arrow" in terms of their Leibniz congruence systems. This proposition may be viewed as a special case of Proposition 849, since $L^{\mathcal{I}}$ forms a maximal set of witnessing equations of the left truth equationality of $\mathcal{I}$.

Proposition 861 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $L^{\mathcal{I}}$ *is left soluble, then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\overleftarrow{T} = L^{\mathcal{I}}(\Omega(T)).$$

Proof: If $L^{\mathcal{I}}$ is left soluble, then, by Theorem 859, $L^{\mathcal{I}}$ forms a set of witnessing equations for the left truth equationality of $\mathcal{I}$. Therefore, by Proposition 849, we get that, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} = L^{\mathcal{I}}(\Omega(T))$. ∎

This property may be restated as another characterization of left truth equationality. We say that $L^{\mathcal{I}}$ **defines theory families up to arrow** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} = L^{\mathcal{I}}(\Omega(T))$. Then we have:

$\mathcal{I}$ is Left Truth Equational
$\longleftrightarrow$ $L^{\mathcal{I}}$ Defines Theory Families Up to Arrow.

Theorem 862 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is left truth equational if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\overleftarrow{T} = L^{\mathcal{I}}(\Omega(T)).$$

Proof: If $\mathcal{I}$ is left truth equational, then, by Theorem 860, $L^{\mathcal{I}}$ is left soluble. Thus, by Proposition 861, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} = L^{\mathcal{I}}(\Omega(T))$.

Conversely, if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} = L^{\mathcal{I}}(\Omega(T))$, then, $L^{\mathcal{I}}$ is left soluble. Thus, again by Theorem 860, $L^{\mathcal{I}}$ is a set of witnessing equations and $\mathcal{I}$ is left truth equational. ∎

We finally show that the property that separates left complete reflectivity from left truth equationality is a property of the left Suszko core, analogous to the adequacy property introduced previously for the Suszko core, that we call *left adequacy*. Similarly to adequacy, informally speaking, this property ensures that the left Suszko core is rich enough to define Suszko congruence systems in terms of the Leibniz congruence systems of theory families that it selects via inclusion.

The following relationship connects the left Suszko core with both Leibniz and Suszko congruence systems.

Recall that given a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, based on an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$, and a sentence family $T \in \mathrm{SenFam}(\mathcal{I})$, we denote by $\overrightarrow{T}$ the least sentence system of $\mathcal{I}$ that includes T. Because of the structurality of C, it is not difficult to see that $C(\overrightarrow{T}) = \overrightarrow{C(T)}$, for any sentence family T of $\mathcal{I}$.

Proposition 863 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \leq \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\begin{array}{lll} \phi \in \overleftarrow{T}_\Sigma & \text{implies} & L^{\mathcal{I}}_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T) \quad \text{(definition of the left Suszko core)} \\ & \text{implies} & L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T). \quad (\widetilde{\Omega}^{\mathcal{I}}(T) \leq \Omega(T)) \end{array}$$

Therefore, we have

$$\begin{array}{rcl} \bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} & \leq & \bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T)\} \\ & \leq & \bigcap\{\Omega(T) : \phi \in \overleftarrow{T}_\Sigma\} \\ & = & \bigcap\{\Omega(T) : \overrightarrow{\phi} \leq T\} \\ & = & \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})). \end{array}$$

■

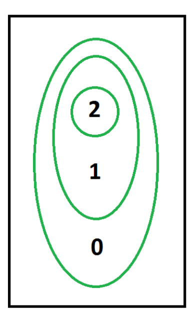
We provide an example, next, that shows that the inclusion proven in Proposition 863 is proper, in general. I.e., there exist π-institutions $\mathcal{I}$ in which, for some signature Σ and some Σ-sentence ϕ,

$$\bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \lneq \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})).$$

Of course, it is convenient that in a systemic π-institution $L^{\mathcal{I}} = S^{\mathcal{I}}$ and, moreover, for every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $C(\overrightarrow{\phi}) = \overrightarrow{C(\phi)} = C(\phi)$, whence Example 842, used to prove proper inclusion following Proposition 841, may be reused.

Example 864 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$*;*
- $N^\flat$ *is the trivial category of natural transformations, consisting of the projections only.*



SEN(Σ)

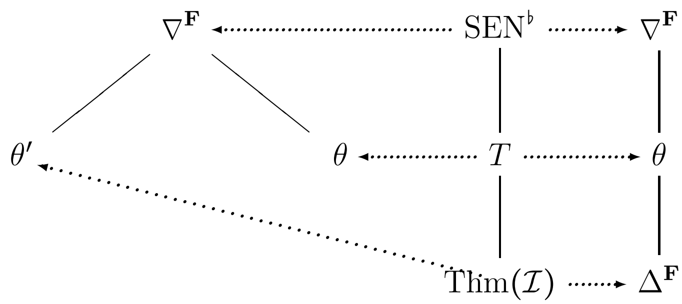
Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}.$$

$\mathcal{I}$ has three theory families $\mathrm{Thm}(\mathcal{I})$, $T = \{\{1,2\}\}$ *and* $\mathrm{SEN}^\flat$, *all of which are theory systems. So $\mathcal{I}$ is systemic.*

We have $L^{\mathcal{I}} = S^{\mathcal{I}} = \{\iota \approx \iota\}$. Furthermore, the structure of the posets of Leibniz congruence systems and of Suszko congruence systems are provided in the left and right sides, respectively, of the following diagram, where

$$T = \{\{1,2\}\}, \quad \theta = \{\{0\},\{1,2\}\}, \quad \theta' = \{\{0,1\},\{2\}\}.$$



Taking this into account, it is not difficult to see that

$$\bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[1] \le \Omega(T)\} = \Delta^{\mathbf{F}} \lneq \theta = \widetilde{\Omega}^{\mathcal{I}}(C(1)) = \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{1})).$$

We also give an example of a π-institution $\mathcal{I}$ whose left Suszko core $L^{\mathcal{I}}$ is such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) = \bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)\}.$$

This again takes after Example 843, since the π-institution used there was systemic.

Example 865 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

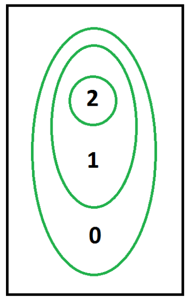
- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$;
- $N^\flat$ *is the category of natural transformations generated by the single unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\} \to \{0,1,2\}$ *be given, for all* $x \in \mathrm{SEN}^\flat(\Sigma)$, *by*

$$\sigma^\flat_\Sigma(x) = 2.$$

Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the π-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}.$$

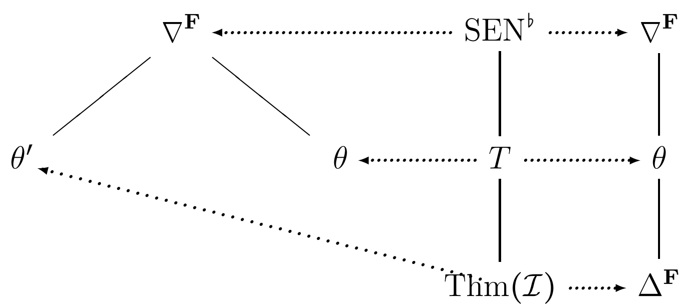
$\mathcal{I}$ has three theory families $\mathrm{Thm}(\mathcal{I})$, $T = \{\{1,2\}\}$ *and* $\mathrm{SEN}^\flat$, *all of which are theory systems. So $\mathcal{I}$ is systemic.*



SEN(Σ)

Note that $L^{\mathcal{I}} = S^{\mathcal{I}} = \{\iota \approx \iota, \iota \approx \sigma^\flat, \sigma^\flat \approx \iota, \sigma^\flat \approx \sigma^\flat\}$. *Note, also, the structure of the posets of Leibniz congruence systems and of Suszko congruence systems, that are provided in the left and right sides, respectively, of the following diagram, where*

$$T = \{\{1,2\}\}, \quad \theta = \{\{0\},\{1,2\}\}, \quad \theta' = \{\{0,1\},\{2\}\}.$$



Now we can check:

$$\begin{array}{rcl}
\widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{0})) = \widetilde{\Omega}^{\mathcal{I}}(C(0)) & = & \nabla^{\mathbf{F}} = \Omega(\mathrm{SEN}^\flat) \\
 & = & \bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[0] \le \Omega(T)\}; \\
\widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{1})) = \widetilde{\Omega}^{\mathcal{I}}(C(1)) & = & \theta = \Omega(\mathrm{SEN}^\flat) \cap \Omega(T) \\
 & = & \bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[1] \le \Omega(T)\}; \\
\widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{2})) =^{\mathcal{I}} \widetilde{\Omega}(C(2)) & = & \Delta^{\mathbf{F}} = \Omega(\mathrm{SEN}^\flat) \cap \Omega(T) \cap \Omega(\mathrm{Thm}(\mathcal{I})) \\
 & = & \bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[2] \le \Omega(T)\}.
\end{array}$$

We have seen, therefore, through examples, that it is possible, but not necessary, that the left Suszko core of a π-institution satisfies, for every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, the reverse inclusion of that given in Proposition 863:

$$\widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) \le \bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)\}.$$

Intuitively speaking, this means that the left Suszko core $L^{\mathcal{I}}$ is rich enough to allow, for every signature Σ and every Σ-sentence ϕ, the determination of those theory families whose Leibniz congruence systems form a covering of the Suszko congruence system of $C(\overrightarrow{\phi})$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the left Suszko core $L^{\mathcal{I}}$ of $\mathcal{I}$ is **left adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) = \bigcap \{\Omega(T) : L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.$$

Based on our preceding work, it is not difficult to see that, if $L^{\mathcal{I}}$ is left soluble, then it is left adequate.

Corollary 866 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $L^{\mathcal{I}}$ *is left soluble, then it is left adequate.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rcll}
\widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) & = & \bigcap\{\Omega(T) : \overrightarrow{\phi} \leq T\} & (\text{definition of } \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi}))) \\
& = & \bigcap\{\Omega(T) : \phi \in \overleftarrow{T}_\Sigma\} & (\text{definition of } \overrightarrow{\phi} \text{ and } \overleftarrow{T}) \\
& = & \bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}. & \\
& & (\text{left solubility of } L^{\mathcal{I}} \text{ and Proposition 861}) &
\end{array}$$

We conclude that $L^{\mathcal{I}}$ is left adequate. ■

Here is an example of a π-institution $\mathcal{I}$, with a left adequate but not left soluble left Suszko core.

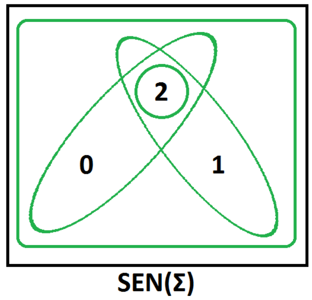
Example 867 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$*;*
- $N^\flat$ *is the category of natural transformations generated by two unary natural transformations:*
  - $\rho^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined by letting* $\rho^\flat_\Sigma : \{0,1,2\} \to \{0,1,2\}$ *be given, for all* $x \in \mathrm{SEN}^\flat(\Sigma)$*, by*

$$\rho^\flat_\Sigma(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x = 1 \\ 2, & \text{if } x = 2 \end{cases}.$$

  - $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\} \to \{0,1,2\}$ *be given, for all* $x \in \mathrm{SEN}^\flat(\Sigma)$*, by*

$$\sigma^\flat_\Sigma(x) = \begin{cases} 2, & \text{if } x = 0 \\ 1, & \text{if } x = 1 \\ 2, & \text{if } x = 2 \end{cases}.$$

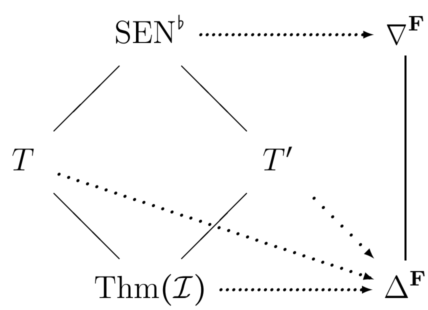


Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be the π-institution determined by

$$\mathcal{C}_\Sigma = \{\{2\},\{0,2\},\{1,2\},\{0,1,2\}\}.$$

$\mathcal{I}$ has four theory families $\mathrm{Thm}(\mathcal{I})$, $T = \{\{0,2\}\}$, $T' = \{\{1,2\}\}$ and $\mathrm{SEN}^\flat$, all of which are theory systems. So it is a systemic π-institution.

The posets of theory families (center), associated Leibniz congruence systems (right) and associate Suszko congruence systems (right, identical with Leibniz congruence systems, since $\mathcal{I}$ is protoalgebraic) are shown in the following figure:



Note that $L^{\mathcal{I}} = \{\iota \approx \iota\}$. We show that $L^{\mathcal{I}}$ is left adequate, but not left soluble. We are omitting arrows from the notation is the following verifications since, as $\mathcal{I}$ is based on $\mathbf{F}$ with trivial $\mathbf{Sign}^\flat$, they play no role in this context.

For left adequacy, we have

$$\begin{aligned}
\widetilde{\Omega}^{\mathcal{I}}(C(0)) &= \widetilde{\Omega}^{\mathcal{I}}(T) = \Delta^{\mathbf{F}} = \bigcap\{\Omega(T''): T'' \in \mathrm{ThFam}(\mathcal{I})\};\\
\widetilde{\Omega}^{\mathcal{I}}(C(1)) &= \widetilde{\Omega}^{\mathcal{I}}(T') = \Delta^{\mathbf{F}} = \bigcap\{\Omega(T''): T'' \in \mathrm{ThFam}(\mathcal{I})\};\\
\widetilde{\Omega}^{\mathcal{I}}(C(2)) &= \widetilde{\Omega}^{\mathcal{I}}(\mathrm{Thm}(\mathcal{I})) = \Delta^{\mathbf{F}} = \bigcap\{\Omega(T''): T'' \in \mathrm{ThFam}(\mathcal{I})\}.
\end{aligned}$$

As for left solubility, note that $L^{\mathcal{I}}_\Sigma[0] \le \Omega(T')$, but that $0 \notin T'_\Sigma$. Thus, $L^{\mathcal{I}}$ is not left soluble.

In the opposite direction, and on the positive side, in a *left c-reflective* π-institution $\mathcal{I}$, if the left Suszko core is left adequate, then it is also left soluble.

First, we note that the following variant of Lemma 826, giving an alternative characterization of left c-reflectivity in terms of both the Suszko and the Leibniz operators, holds.

Lemma 868 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is left c-reflective if and only if, for every* $T, T' \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\widetilde{\Omega}^{\mathcal{I}}(T) \leq \Omega(T') \quad \textit{implies} \quad \overleftarrow{T} \leq \overleftarrow{T'}.$$

Proof: Assume, first, that $\mathcal{I}$ is left c-reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{\Omega}^{\mathcal{I}}(T) \leq \Omega(T')$. By the definition of the Suszko operator,

$$\bigcap\{\Omega(T'') : T \leq T'' \in \mathrm{ThFam}(\mathcal{I})\} \leq \Omega(T').$$

Using left c-reflectivity, we get that

$$\bigcap\{\overleftarrow{T''} : T \leq T'' \in \mathrm{ThFam}(\mathcal{I})\} \leq \overleftarrow{T'}.$$

Hence, using Lemma 1, $\overleftarrow{T} \leq \overleftarrow{T'}$, as required.

Suppose, conversely, that, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\Omega}^{\mathcal{I}}(T) \leq \Omega(T')$ implies $\overleftarrow{T} \leq \overleftarrow{T'}$. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. Then we have

$$\begin{array}{rcll} \widetilde{\Omega}^{\mathcal{I}}(\bigcap_{T \in \mathcal{T}} T) & \leq & \bigcap_{T \in \mathcal{T}} \widetilde{\Omega}^{\mathcal{I}}(T) & (\text{monotonicity of } \widetilde{\Omega}^{\mathcal{I}}) \\ & \leq & \bigcap_{T \in \mathcal{T}} \Omega(T) & (\text{since } \widetilde{\Omega}^{\mathcal{I}}(T) \leq \Omega(T)) \\ & \leq & \Omega(T'). & (\text{by hypothesis}) \end{array}$$

Using the hypothesis, we conclude that $\overleftarrow{\bigcap_{T \in \mathcal{T}} T} \leq \overleftarrow{T'}$. Thus, by Lemma 3, $\bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$. Therefore, $\mathcal{I}$ is left c-reflective. ∎

And now for the promised result showing that in a left c-reflective π-institution $\mathcal{I}$, if the left Suszko core is left adequate, then it is also left soluble.

Proposition 869 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a left c-reflective* π*-institution based on* $\mathbf{F}$*. If* $L^{\mathcal{I}}$ *is left adequate, then it is left soluble.*

Proof: Suppose that $\mathcal{I}$ is left c-reflective and that $L^{\mathcal{I}}$ is left adequate. We must show that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{iff} \quad L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

The implication left-to-right is always satisfied by Proposition 853. For the converse, assume that $L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. Then, by the left adequacy of $L^{\mathcal{I}}$, we get that $\widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) \le \Omega(T)$. Thus, by left c-reflectivity and Lemma 868, we conclude that $\overleftarrow{C(\overrightarrow{\phi})} \le \overleftarrow{T}$, which gives $\phi \in \overleftarrow{T}_\Sigma$. ■

We finally show that a π-institution is left truth equational if and only if it is left c-reflective and its left Suszko core is left adequate.

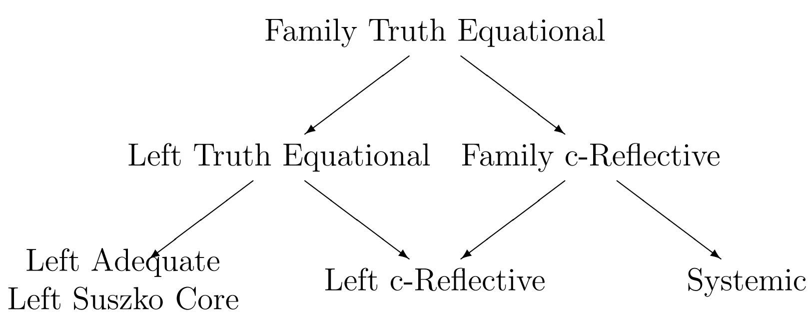
Left Truth Equationality	=	$L^{\mathcal{I}}$ Left Soluble
	=	$L^{\mathcal{I}}$ Defines Theory Families Up to Arrow
	=	Left c-Reflectivity + $L^{\mathcal{I}}$ Left Adequate

Theorem 870 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is left truth equational if and only if it is left c-reflective and has a left adequate left Suszko core.*

Proof: Suppose, first, that $\mathcal{I}$ is left truth equational. Then it is left c-reflective by Theorem 850. Moreover, its left Suszko core is left soluble by Theorem 860 and, hence, by Corollary 866, its left Suszko core is left adequate.

Suppose, conversely, that $\mathcal{I}$ is left c-reflective with a left adequate left Suszko core. Then, by Proposition 869, its left Suszko core is left soluble and, therefore, by Theorem 860, $\mathcal{I}$ is left truth equational. ■

We have now established the following hierarchy of properties:



11.9 System Truth Equationality

In this section, we look at system truth equationality and system c-reflectivity, which can also be applied to a π-institution without forcing it to be systemic. Recall that, by Proposition **??**, system c-reflectivity is a weaker property than left c-reflectivity, i.e., left c-reflectivity, which was used in the characterization of left truth equationality in the preceding section, implies system c-reflectivity.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

Recall that $\mathcal{I}$ is **system c-reflective** if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$,

$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \quad \text{implies} \quad \bigcap_{T \in \mathcal{T}} T \leq T'.$$

Since left c-reflectivity is not strong enough to imply systemicity, system c-reflectivity has, a fortiori, the same property.

We say that the π-institution $\mathcal{I}$ is **system truth equational** if there exists $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ having a single distinguished argument, such that, for every $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

In that case, we call $\tau^\flat$ a **set of witnessing equations** (of/for the system truth equationality of $\mathcal{I}$).

If $\mathcal{I}$ is a system truth equational π-institution, with witnessing equations $\tau^\flat$, then, for $T \in \mathrm{ThSys}(\mathcal{I})$, $\tau^\flat(\Omega(T))$ is exactly equal to T, i.e., the witnessing equations reflect theory systems.

Proposition 871 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *If $\mathcal{I}$ is system truth equational, with witnessing equations $\tau^\flat$, then, for all* $T \in \mathrm{ThSys}(\mathcal{I})$,

$$\tau^\flat(\Omega(T)) = T.$$

Proof: Let $T \in \mathrm{ThSys}(\mathcal{I})$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll} \phi \in \tau^\flat_\Sigma(\Omega(T)) & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(T) \quad \text{(definition)} \\ & \text{iff} & \phi \in T_\Sigma. \quad \text{(system truth equationality)} \end{array}$$

■

Proposition 871 has as an immediate consequence the important fact that system truth equationality implies system c-reflectivity.

Theorem 872 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *If $\mathcal{I}$ is system truth equational, then it is system c-reflective.*

Proof: Suppose that $\mathcal{I}$ is system truth equational with witnessing equations $\tau^\flat$. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. Then

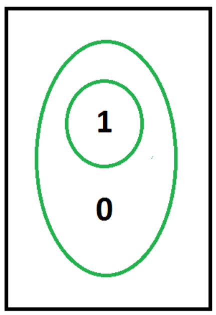
$$\begin{array}{lll} \bigcap_{T \in \mathcal{T}} T & = & \bigcap_{T \in \mathcal{T}} \tau^\flat(\Omega(T)) \quad \text{(Proposition 871)} \\ & = & \tau^\flat(\bigcap_{T \in \mathcal{T}} \Omega(T)) \quad \text{(set theory)} \\ & \leq & \tau^\flat(\Omega(T')) \quad \text{(hypothesis and Lemma 94)} \\ & = & T'. \quad \text{(Proposition 871)} \end{array}$$

Thus, $\mathcal{I}$ is system c-reflective. ■

The following example shows that the inclusion of Theorem 872 is proper.

Example 873 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

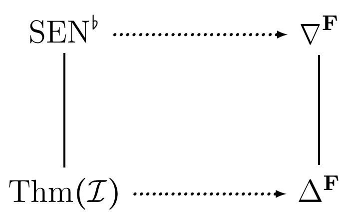
- $\mathbf{Sign}^\flat$ *is the trivial category with single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*;*
- $N^\flat$ *is the trivial category of natural transformations.*



SEN(Σ)

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\}$*.*

$\mathcal{I}$ *has two theory families,* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, which are also theory systems. Clearly,* $\mathrm{Thm}(\mathcal{I}) \le \mathrm{SEN}^\flat$*. Moreover,* $\Omega(\mathrm{Thm}(\mathcal{I})) = \Delta^{\mathbf{F}}$ *and* $\Omega(\mathrm{SEN}^\flat) = \nabla^{\mathbf{F}}$*.* $\mathcal{I}$ *is clearly system c-reflective.*



On the other hand, there does not exist $\tau^\flat \subseteq N^\flat$*, such that* $I^\flat$ *has the required properties to constitute a witnessing set of equations for the system truth equationality in* $\mathcal{I}$*. Any set consisting of projections only cannot satisfy the required condition since* $\tau^\flat(\Omega(T))$ *can only be* $\mathrm{SEN}^\flat$ *or* $\overline{\varnothing}$*.*

We provide, next, a characterization of system truth equationality in terms of the solubility property of the system core of the π-institution. Then, we provide an exact description of those system c-reflective π-institutions which are system truth equational.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. First, for $T \in \mathrm{ThSys}(\mathcal{I})$, we introduce the notation

$$\widehat{\Omega}^{\mathcal{I}}(T) = \bigcap \{\Omega(T') : T \le T' \in \mathrm{ThSys}(\mathcal{I})\}.$$

We now define the **system core** of $\mathcal{I}$ to be the collection

$$Z^{\mathcal{I}} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThSys}(\mathcal{I}))(\sigma^\flat[T] \leq \widehat{\Omega}^{\mathcal{I}}(T)\}.$$

The following proposition clarifies the relation between the Suszko core, the left Suszko core and the system core of a π-institution $\mathcal{I}$.

Proposition 874 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $S^{\mathcal{I}} \subseteq L^{\mathcal{I}} \subseteq Z^{\mathcal{I}}$*;*

(b) For every relation family θ *on* $\mathbf{F}$*,* $Z^{\mathcal{I}}(\theta) \leq L^{\mathcal{I}}(\theta) \leq S^{\mathcal{I}}(\theta)$*.*

Proof: For Part (a), $S^{\mathcal{I}} \subseteq L^{\mathcal{I}}$ has been shown after Proposition 852. For the second inclusion, assume that $\sigma^\flat \in L^{\mathcal{I}}$ and let $T \in \mathrm{ThSys}(\mathcal{I})$. Then we have

$$\begin{array}{rcll} \sigma^\flat[T] & = & \sigma^\flat[\overleftarrow{T}] & (T \in \mathrm{ThSys}(\mathcal{I})) \\ & \leq & \widetilde{\Omega}^{\mathcal{I}}(T) & (\sigma^\flat \in L^{\mathcal{I}}) \\ & \leq & \widehat{\Omega}^{\mathcal{I}}(T). & (\widetilde{\Omega}^{\mathcal{I}}(T) \leq \widehat{\Omega}^{\mathcal{I}}(T)) \end{array}$$

Thus $\sigma^\flat \in Z^\flat$ and $L^{\mathcal{I}} \subseteq Z^{\mathcal{I}}$. Part (b) follows form Part (a) and the relevant definitions. ∎

The system core of a π-institution satisfies the following property related to the Leibniz congruence systems of the theory systems of the π-institution:

Proposition 875 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For every* $T \in \mathrm{ThSys}(\mathcal{I})$*,*

$$T \leq Z^{\mathcal{I}}(\Omega(T)).$$

Proof: Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{rll} \phi \in T_\Sigma & \text{implies} & Z^{\mathcal{I}}_\Sigma[\phi] \leq \widehat{\Omega}^{\mathcal{I}}(T) \quad (\text{by definition of } Z^{\mathcal{I}}) \\ & \text{implies} & Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) \quad (\widehat{\Omega}^{\mathcal{I}}(T) \leq \Omega(T)) \\ & \text{iff} & \phi \in Z^{\mathcal{I}}(\Omega(T)). \quad (\text{by definition}) \end{array}$$

∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the system core $Z^{\mathcal{I}}$ of $\mathcal{I}$ is **soluble** if the converse inclusion to that proven in Proposition 875 holds, i.e., if, for all $T \in \mathrm{ThSys}(\mathcal{I})$

$$Z^{\mathcal{I}}(\Omega(T)) \leq T.$$

Equivalently, $Z^{\mathcal{I}}$ is soluble if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) \quad \text{implies} \quad \phi \in T_\Sigma.$$

We show that left solubility of the left Suszko core implies solubility of the system core of a π-institution.

Proposition 876 *Let $\mathcal{I} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If the left Suszko core $L^{\mathcal{I}}$ of $\mathcal{I}$ is left soluble, then the system core $Z^{\mathcal{I}}$ of $\mathcal{I}$ is soluble.*

Proof: Suppose that $L^{\mathcal{I}}$ is left soluble and let $T \in \mathrm{ThSys}(\mathcal{I})$. Then we have

$$\begin{array}{rcl} Z^{\mathcal{I}}(\Omega(T)) & \leq & L^{\mathcal{I}}(\Omega(T)) \quad \text{(Proposition 874)} \\ & = & T. \quad \text{(hypothesis and Proposition 854)} \end{array}$$

Therefore, $Z^{\mathcal{I}}$ is soluble. ■

It turns out that the property of solubility of the system core intrinsically characterizes system truth equationality. We show, first, that the system core being soluble is necessary for system truth equationality. Observe that, in case a π-institution is system truth equational, the witnessing equations form a subset of the system core.

Lemma 877 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is system truth equational, with witnessing equations $\tau^\flat \subseteq N^\flat$, then $\tau^\flat \subseteq Z^{\mathcal{I}}$.*

Proof: By system truth equationality, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

Thus, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rl} \phi \in T_\Sigma & \text{iff} \quad (\forall T \leq T' \in \mathrm{ThSys}(\mathcal{I}))(\phi \in T'_\Sigma) \\ & \text{iff} \quad (\forall T \leq T' \in \mathrm{ThSys}(\mathcal{I}))(\tau^\flat_\Sigma[\phi] \leq \Omega(T')) \\ & \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \bigcap\{\Omega(T') : T \leq T' \in \mathrm{ThSys}(\mathcal{I})\} \\ & \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \widehat{\Omega}^{\mathcal{I}}(T). \end{array}$$

We conclude, by the definition of $Z^{\mathcal{I}}$, that $\tau^\flat \subseteq Z^{\mathcal{I}}$. ■

Now we prove the necessity of the solubility of the system core for system truth equationality.

Theorem 878 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is system truth equational, then $Z^{\mathcal{I}}$ is soluble.*

Proof: Suppose that $\mathcal{I}$ is system truth equational, with witnessing equations $\tau^\flat$. Then, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rcl} Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) & \text{implies} & \tau^\flat_\Sigma[\phi] \leq \Omega(T) \quad \text{(Lemma 877)} \\ & \text{iff} & \phi \in T_\Sigma. \quad \text{(system truth equationality)} \end{array}$$

Thus, $Z^{\mathcal{I}}$ is soluble. ■

The reverse implication completes the promised characterization of system truth equationality in terms of the system core.

Theorem 879 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $Z^{\mathcal{I}}$ *is soluble, then* $\mathcal{I}$ *is system truth equational, with witnessing equations* $Z^{\mathcal{I}}$*.*

Proof: It suffices to show that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

The left-to-right implication is given in Proposition 875, whereas the converse is ensured by the postulated solubility of $Z^{\mathcal{I}}$. ■

Theorems 878 and 879 provide the promised characterization of system truth equationality in terms of the solubility of the system core.

$$\mathcal{I} \text{ is System Truth Equational} \longleftrightarrow Z^{\mathcal{I}} \text{ is Soluble.}$$

Theorem 880 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is system truth equational if and only if* $Z^{\mathcal{I}}$ *is soluble.*

Proof: Theorem 878 gives the "only if" and the "if" is by Theorem 879. ■

If $\mathcal{I}$ is system truth equational, then the system core defines theory systems in $\mathcal{I}$ in terms of their Leibniz congruence systems. This proposition may be viewed as a special case of Proposition 871, since $Z^{\mathcal{I}}$ forms a maximal set of witnessing equations of the system truth equationality of $\mathcal{I}$.

Proposition 881 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $Z^{\mathcal{I}}$ *is soluble, then, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,*

$$T = Z^{\mathcal{I}}(\Omega(T)).$$

Proof: If $Z^{\mathcal{I}}$ is soluble, then, by Theorem 879, $Z^{\mathcal{I}}$ forms a set of witnessing equations for the system truth equationality of $\mathcal{I}$. Therefore, by Proposition 871, we get that, for every $T \in \mathrm{ThSys}(\mathcal{I})$, $T = Z^{\mathcal{I}}(\Omega(T))$. ■

This property may be restated as another characterization of system truth equationality. We say that $Z^{\mathcal{I}}$ **defines theory systems** if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $T = Z^{\mathcal{I}}(\Omega(T))$. Then we have:

$$\mathcal{I} \text{ is System Truth Equational} \longleftrightarrow Z^{\mathcal{I}} \text{ Defines Theory Systems.}$$

Theorem 882 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is system truth equational if and only if, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,*

$$T = Z^{\mathcal{I}}(\Omega(T)).$$

Proof: If $\mathcal{I}$ is system truth equational, then, by Theorem 990, $Z^{\mathcal{I}}$ is soluble. Thus, by Proposition 881, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $T = Z^{\mathcal{I}}(\Omega(T))$.

Conversely, if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $T = Z^{\mathcal{I}}(\Omega(T))$, then, $Z^{\mathcal{I}}$ is soluble. Thus, again by Theorem 990, $Z^{\mathcal{I}}$ is a set of witnessing equations and $\mathcal{I}$ is system truth equational. ∎

We finally show that the property that separates system complete reflectivity from system truth equationality is a property of the system core that we call adequacy. In analogy to the adequacy of the Suszko core and to the left adequacy of the left Suszko core, this property ensures that the system core is rich enough to define the congruence system $\widehat{\Omega}^{\mathcal{I}}(T)$ of a theory system T in terms of the Leibniz congruence systems of collections of theory systems that it selects via inclusion.

Recall, once more, that given a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$, based on an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$, and a sentence family $T \in \mathrm{SenFam}(\mathcal{I})$, we denote by $\overrightarrow{T}$ the least sentence system of $\mathcal{I}$ that includes T (see Proposition 2).

Proposition 883 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a π-institution based on* $\mathbf{F}$*. For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \leq \widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $T \in \mathrm{ThSys}(\mathcal{I})$,

$$\begin{array}{lll}\phi \in T_\Sigma & \text{implies} & Z^{\mathcal{I}}_\Sigma[\phi] \leq \widehat{\Omega}^{\mathcal{I}}(T) \quad \text{(definition of the system core)}\\ & \text{implies} & Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T). \quad (\widehat{\Omega}^{\mathcal{I}}(T) \leq \Omega(T))\end{array}$$

Therefore, we have

$$\begin{aligned}\bigcap\{\Omega(T) &: T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}\\ &\leq \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } Z^{\mathcal{I}}_\Sigma[\phi] \leq \widehat{\Omega}^{\mathcal{I}}(T)\}\\ &\leq \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \phi \in T_\Sigma\}\\ &= \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \overrightarrow{\phi} \leq T\}\\ &= \widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})).\end{aligned}$$

Therefore, the displayed inclusion always holds. ∎

It is possible, but not necessary, that the system core of a π-institution satisfies, for every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, the reverse inclusion of that given in Proposition 883:

$$\widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) \leq \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.$$

Intuitively speaking, this means that the system core $Z^{\mathcal{I}}$ is rich enough to allow, for every Σ-sentence ϕ, the determination of those theory systems whose Leibniz congruence systems form a covering of the congruence system $\widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi}))$ associated with $C(\overrightarrow{\phi})$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the system core $Z^{\mathcal{I}}$ of $\mathcal{I}$ is **adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) = \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.$$

Based on our preceding work, it is not difficult to see that, if $Z^{\mathcal{I}}$ is soluble, then it is adequate.

Corollary 884 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $Z^{\mathcal{I}}$ *is soluble, then it is adequate.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rcl}
\widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) & = & \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \overrightarrow{\phi} \leq T\} \\
& & (\text{definition of } \widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi}))) \\
& = & \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \phi \in T\} \\
& & (T \in \mathrm{ThSys}(\mathcal{I})) \\
& = & \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}. \\
& & (\text{solubility of } Z^{\mathcal{I}} \text{ and Proposition 881})
\end{array}$$

We conclude that $Z^{\mathcal{I}}$ is adequate. ■

As a partial converse, in a *system c-reflective* π-institution $\mathcal{I}$, if the system core is adequate, then it is also soluble.

First, we prove the following variant of Lemma 826, giving an alternative characterization of system c-reflectivity in terms of both $\widehat{\Omega}^{\mathcal{I}}$ and the Leibniz operator.

Lemma 885 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is system c-reflective if and only if, for every* $T, T' \in \mathrm{ThSys}(\mathcal{I})$*,*

$$\widehat{\Omega}^{\mathcal{I}}(T) \leq \Omega(T') \quad \textit{implies} \quad T \leq T'.$$

Proof: Assume, first, that $\mathcal{I}$ is system c-reflective and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\widehat{\Omega}^{\mathcal{I}}(T) \leq \Omega(T')$. By the definition of the hat operator,

$$\bigcap\{\Omega(T'') : T \leq T'' \in \mathrm{ThSys}(\mathcal{I})\} \leq \Omega(T').$$

Using system c-reflectivity, we get that

$$\bigcap\{T'' : T \leq T'' \in \mathrm{ThSys}(\mathcal{I})\} \leq T'.$$

Hence, we conclude $T \leq T'$, as required.

Suppose, conversely, that, for all $T, T' \in \mathrm{ThSys}(\mathcal{I})$, $\widehat{\Omega}^{\mathcal{I}}(T) \leq \Omega(T')$ implies $T \leq T'$. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. Then we have

$$\begin{array}{rcll}\widehat{\Omega}^{\mathcal{I}}(\bigcap_{T\in\mathcal{T}} T) & \leq & \bigcap_{T\in\mathcal{T}} \widehat{\Omega}^{\mathcal{I}}(T) & \text{(monotonicity of } \widehat{\Omega}^{\mathcal{I}})\\ & \leq & \bigcap_{T\in\mathcal{T}} \Omega(T) & \text{(since } \widehat{\Omega}^{\mathcal{I}}(T) \leq \Omega(T))\\ & \leq & \Omega(T'). & \text{(by hypothesis)}\end{array}$$

Using the hypothesis, we conclude that $\bigcap_{T\in\mathcal{T}} T \leq T'$. Therefore, $\mathcal{I}$ is system c-reflective. ■

And now for the promised result showing that in a system c-reflective π-institution $\mathcal{I}$, if the system core is adequate, then it is also soluble.

Proposition 886 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a system c-reflective* π*-institution based on* $\mathbf{F}$. *If the system core* $Z^{\mathcal{I}}$ *is adequate, then it is soluble.*

Proof: Suppose that $\mathcal{I}$ is system c-reflective and that $Z^{\mathcal{I}}$ is adequate. We must show that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

The implication left-to-right is always satisfied by Proposition 875. For the converse, assume that $Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. Then, by the adequacy of $Z^{\mathcal{I}}$, we get that $\widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) \leq \Omega(T)$. Thus, by system c-reflectivity and Lemma 885, we conclude that $C(\overrightarrow{\phi}) \leq T$, which gives $\phi \in T_\Sigma$. ■

We finally show that a π-institution is system truth equational if and only if it is system c-reflective and its system core is adequate.

System Truth Equationality	=	$Z^{\mathcal{I}}$ Left Soluble
	=	$Z^{\mathcal{I}}$ Defines Theory Systems
	=	System c-Reflectivity + $Z^{\mathcal{I}}$ Adequate

Theorem 887 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is system truth equational if and only if it is system c-reflective and has an adequate system core.*

Proof: Suppose, first, that $\mathcal{I}$ is system truth equational. Then it is system c-reflective by Theorem 872. Moreover, its system core is soluble by Theorem 990 and, hence, by Corollary 884, its system core is adequate.

Suppose, conversely, that $\mathcal{I}$ is system c-reflective with an adequate system core. Then, by Proposition 886, its system core is soluble and, therefore, by Theorem 990, $\mathcal{I}$ is system truth equational. ■

We have now established the following hierarchy of properties: