Chapter 12

The Syntactic Leibniz Hierarchy: Edifice

12.1 Translations

In this section we discuss translations, interpretations and equivalence that will be used later in the context of algebraizable π-institutions. In the context of algebraizability, the algebraic counterparts of π-institutions may consist of algebraic closure families that lack the property of structurality, i.e., they are not closure systems, as introduced previously. Since these closure families are not structural in general, the corresponding algebraic structures do not constitute π-institutions. To accommodate these, we deal with more general structures that include all π-institutions, but also pairs of algebraic systems and closure families that are non-structural. We call these π-*structures*.

Definition 888 *A* π-**structure** $\mathcal{K} = \langle \mathbf{K}, D\rangle$ *is a pair consisting of:*

- *an algebraic system* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*;*
- *a* $|\mathbf{Sign}|$*-indexed family* $D = \{D_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ *of closure operators* $D_\Sigma : \mathcal{P}\mathrm{SEN}(\Sigma) \to \mathcal{P}\mathrm{SEN}(\Sigma)$*,* $\Sigma \in |\mathbf{Sign}|$*.*

Such a family D *is called a* **closure family** *on* $\mathbf{K}$*.*

Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ and $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ be two algebraic systems. A **translation** $\alpha : \mathbf{K} \to \mathbf{K}'$ is a collection

$$\alpha = \{\alpha_\Sigma\}_{\Sigma\in|\mathbf{Sign}|},$$

where, for all $\Sigma \in |\mathbf{Sign}|$,

$$\alpha_\Sigma : \mathrm{SEN}(\Sigma) \to \mathrm{SenFam}(\mathbf{K}')$$

assigns to each Σ-sentence ϕ of $\mathbf{K}$ a sentence family

$$\alpha_\Sigma[\phi] = \{\alpha_{\Sigma,\Sigma'}[\phi]\}_{\Sigma'\in|\mathbf{Sign}'|}.$$

For $\Sigma \in |\mathbf{Sign}|$, $\Phi \subseteq \mathrm{SEN}(\Sigma)$, we set

$$\alpha_\Sigma[\Phi] = \bigcup\{\alpha_\Sigma[\phi] : \phi \in \Phi\},$$

where the union is, as usual, taken signature-wise and, hence, $\alpha_\Sigma[\Phi] \in \mathrm{SenFam}(\mathbf{K}')$. More generally, for $T \in \mathrm{SenFam}(\mathbf{K})$, we set

$$\alpha[T] = \bigcup\{\alpha_\Sigma[T_\Sigma] : \Sigma \in |\mathbf{Sign}|\}.$$

Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ be algebraic systems and $\mathcal{K} = \langle \mathbf{K}, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}', D'\rangle$ be π-structures based on $\mathbf{K}, \mathbf{K}'$, respectively. An

interpretation $\alpha : \mathcal{K} \to \mathcal{K}'$ is a translation $\alpha : \mathbf{K} \to \mathbf{K}'$, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\phi \in D_\Sigma(\Phi) \quad \text{iff} \quad \alpha_\Sigma[\phi] \leq D'(\alpha_\Sigma[\Phi]).$$

If such an interpretation exists, then $\mathcal{K}$ is said to be **interpretable in** $\mathcal{K}'$.

Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ be algebraic systems and $\mathcal{K} = \langle \mathbf{K}, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}', D'\rangle$ be π-structures based on $\mathbf{K}$, $\mathbf{K}'$, respectively. Let, also,

$$\alpha : \mathcal{K} \to \mathcal{K}' \quad \text{and} \quad \beta : \mathcal{K}' \to \mathcal{K}$$

be interpretations from $\mathcal{K}$ to $\mathcal{K}'$ and from $\mathcal{K}'$ to $\mathcal{K}$, respectively. α and β are said to be **inverses** of each other and the pair $(\alpha, \beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$ is referred to as a **conjugate pair** if:

- for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,
$$D(\phi) = D(\beta[\alpha_\Sigma[\phi]]);$$
- for all $\Sigma' \in |\mathbf{Sign}'|$ and all $\psi \in \mathrm{SEN}'(\Sigma')$,
$$D'(\psi) = D'(\alpha[\beta_{\Sigma'}[\psi]]).$$

The π-structures $\mathcal{K}$ and $\mathcal{K}'$ are called **equivalent** if there exists a conjugate pair $\mathcal{K} \overset{(\alpha,\beta)}{\rightleftarrows} \mathcal{K}'$.

Lemma 889 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *be algebraic systems,* $\mathcal{K} = \langle \mathbf{K}, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}', D'\rangle$ *be π-structures based on* $\mathbf{K}$, $\mathbf{K}'$, *respectively, and* $\alpha : \mathbf{K} \to \mathbf{K}'$, $\beta : \mathbf{K}' \to \mathbf{K}$ *translations. The following are equivalent:*

(i) $\alpha : \mathcal{K} \to \mathcal{K}'$ *is an interpretation and, for all* $\Sigma' \in |\mathbf{Sign}'|$, $\psi \in \mathrm{SEN}'(\Sigma')$, $D'(\psi) = D'(\alpha[\beta_{\Sigma'}[\psi]])$;

(ii) $\beta : \mathcal{K}' \to \mathcal{K}$ *is an interpretation and, for all* $\Sigma \in |\mathbf{Sign}|$, $\phi \in \mathrm{SEN}(\Sigma)$, $D(\phi) = D(\beta[\alpha_\Sigma[\phi]])$.

Proof: By symmetry, it suffices to show (i)⇒(ii).

Suppose, first, that $\Sigma' \in |\mathbf{Sign}'|$ and $\Psi \cup \{\psi\} \subseteq \mathrm{SEN}'(\Sigma')$. Then, we have

$$\begin{array}{lll} \psi \in D'_{\Sigma'}(\Psi) & \text{iff} & D'(\psi) \leq D'(\Psi) \\ & \text{iff} & D'(\alpha[\beta_{\Sigma'}[\psi]]) \leq D'(\alpha[\beta_{\Sigma'}[\Psi]]) \\ & \text{iff} & \alpha[\beta_{\Sigma'}[\psi]] \leq D'(\alpha[\beta_{\Sigma'}[\Psi]]) \\ & \text{iff} & \beta_{\Sigma'}[\psi] \leq D(\beta_{\Sigma'}[\Psi]). \end{array}$$

So $\beta : \mathcal{K}' \to \mathcal{K}$ is an interpretation.

Assume, next, that $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. Then, by the hypothesis applied to $\alpha_\Sigma[\phi] \in \mathrm{SenFam}(\mathbf{K}')$, we have

$$D'(\alpha[\beta[\alpha_\Sigma[\phi]]]) = D'(\alpha_\Sigma[\phi]).$$

Hence, we get that

$$\alpha_\Sigma[\phi] \leq D'(\alpha[\beta[\alpha_\Sigma[\phi]]]) \quad \text{and} \quad \alpha[\beta[\alpha_\Sigma[\phi]]] \leq D'(\alpha_\Sigma[\phi]).$$

Therefore, by the fact that α is an interpretation,

$$\phi \in D_\Sigma(\beta[\alpha_\Sigma[\phi]]) \quad \text{and} \quad \beta[\alpha_\Sigma[\phi]] \leq D(\phi).$$

So we conclude that $D(\phi) = D(\beta[\alpha_\Sigma[\phi]])$. ■

Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ be algebraic systems and $\alpha : \mathbf{K} \to \mathbf{K}'$ a translation. Define the **residual** α^* **of the translation** α,

$$\alpha^* : \mathrm{SenFam}(\mathbf{K}') \to \mathrm{SenFam}(\mathbf{K})$$

by letting, for all $T' \in \mathrm{SenFam}(\mathbf{K}')$,

$$\alpha^*(T') = \{\alpha^*_\Sigma(T')\}_{\Sigma \in |\mathbf{Sign}|}$$

be given, for all $\Sigma \in |\mathbf{Sign}|$, by

$$\alpha^*_\Sigma(T') = \{\phi \in \mathrm{SEN}(\Sigma) : \alpha_\Sigma[\phi] \leq T'\}.$$

The following proposition shows that, when applied to interpretations between π-structures the star operator restricts to mappings from theory families to theory families.

Proposition 890 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *be algebraic systems,* $\mathcal{K} = \langle \mathbf{K}, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}', D'\rangle$ *be* π*-structures based on* $\mathbf{K}$, $\mathbf{K}'$*, respectively, and* $\alpha : \mathcal{K} \to \mathcal{K}'$ *an interpretation. Then, for all* $T' \in \mathrm{ThFam}(\mathcal{K}')$, $\alpha^*(T') \in \mathrm{ThFam}(\mathcal{K})$.

Proof: Suppose $T' \in \mathrm{ThFam}(\mathcal{K}')$ and let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in D_\Sigma(\alpha^*_\Sigma(T'))$. Then, since $\alpha : \mathcal{K} \to \mathcal{K}'$ is an interpretation, we have

$$\alpha_\Sigma[\phi] \leq D'(\alpha[\alpha^*_\Sigma(T')]) \leq D'(T') = T'.$$

Hence $\phi \in \alpha^*_\Sigma(T')$. Since $\Sigma \in |\mathbf{Sign}|$ was arbitrary, we conclude that $\alpha^*_\Sigma(T') \in \mathrm{ThFam}(\mathcal{K})$. ■

In addition, we show that, when $(\alpha, \beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$ form a conjugate pair, then $\beta^* : \mathrm{ThFam}(\mathcal{K}) \to \mathrm{ThFam}(\mathcal{K}')$ and $\alpha^* : \mathrm{ThFam}(\mathcal{K}') \to \mathrm{ThFam}(\mathcal{K})$ are inverse mappings.

Lemma 891 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *be algebraic systems,* $\mathcal{K} = \langle \mathbf{K}, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}', D'\rangle$ *be* π*-structures based on* $\mathbf{K}$, $\mathbf{K}'$, *respectively, and* $(\alpha, \beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *a conjugate pair. Then, for all* $T \in \mathrm{ThFam}(\mathcal{K})$,

$$\alpha^*(\beta^*(T)) = T.$$

Proof: Suppose $(\alpha, \beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$ is a conjugate pair, $T \in \mathrm{ThFam}(\mathcal{K})$ and let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. Then we have

$$\begin{array}{lll} \phi \in \alpha^*_\Sigma(\beta^*(T)) & \text{iff} & \alpha_\Sigma[\phi] \le \beta^*(T) \\ & \text{iff} & \beta[\alpha_\Sigma[\phi]] \le T \\ & \text{iff} & D(\beta[\alpha_\Sigma[\phi]]) \le T \\ & \text{iff} & D_\Sigma(\phi) \le T_\Sigma \\ & \text{iff} & \phi \in T_\Sigma. \end{array}$$

Thus, we conclude that $\alpha^*(\beta^*(T)) = T$. ■

Based on Lemma 891, we can show that $\beta^* : \mathrm{ThFam}(\mathcal{K}) \to \mathrm{ThFam}(\mathcal{K}')$ and $\alpha^* : \mathrm{ThFam}(\mathcal{K}') \to \mathrm{ThFam}(\mathcal{K})$ are bijections.

Lemma 892 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *be algebraic systems,* $\mathcal{K} = \langle \mathbf{K}, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}', D'\rangle$ *be* π*-structures based on* $\mathbf{K}$, $\mathbf{K}'$, *respectively, and* $(\alpha, \beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *a conjugate pair. Then* $\alpha^* : \mathrm{ThFam}(\mathcal{K}') \to \mathrm{ThFam}(\mathcal{K})$ *is a bijection.*

Proof: Let $(\alpha, \beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$ be a conjugate pair. First, by Proposition 890, $\alpha^* : \mathrm{ThFam}(\mathcal{K}') \to \mathrm{ThFam}(\mathcal{K})$ is well-defined. To see that it is surjective, let $T \in \mathrm{ThFam}(\mathcal{K})$. Then, by Proposition 890, $\beta^*(T) \in \mathrm{ThFam}(\mathcal{K}')$ and, by Lemma 891, $\alpha^*(\beta^*(T)) = T$. Thus, α^* is indeed surjective. For injectivity, assume $S', T' \in \mathrm{ThFam}(\mathcal{K}')$, such that $\alpha^*(S') = \alpha^*(T')$. Then, by surjectivity, there exist $S, T \in \mathrm{ThFam}(\mathcal{K})$, such that $\beta^*(S) = S'$ and $\beta^*(T) = T'$. Therefore, we get

$$S = \alpha^*(\beta^*(S)) = \alpha^*(S') = \alpha^*(T') = \alpha^*(\beta^*(T)) = T.$$

But then we get $S' = \beta^*(S) = \beta^*(T) = T'$. we conclude that α^* is also injective and, hence, it is a bijection. ■

In the main theorem of this section, it is shown that if $\mathcal{K}$ and $\mathcal{K}'$ are equivalent π-structures via a conjugate pair $(\alpha, \beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$, then $\beta^* : \mathbf{ThFam}(\mathcal{K}) \to \mathbf{ThFam}(\mathcal{K}')$ and $\alpha^* : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ form a pair of mutually inverse order isomorphisms between the complete lattices of the corresponding theory families.

Recall that, given a π-institution $\mathcal{I}$, we denote by

$$\mathbf{ThFam}(\mathcal{I}) = \langle \mathrm{ThFam}(\mathcal{I}), \le\rangle$$

the complete lattice of theory families of $\mathcal{I}$ ordered by signature-wise inclusion. We extend the notation to the collections of theory families of π-structures. Thus, given a π-structure $\mathcal{K} = \langle \mathbf{K}, D\rangle$, we define

$$\mathbf{ThFam}(\mathcal{K}) = \langle \mathrm{ThFam}(\mathcal{K}), \leq\rangle.$$

Theorem 893 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *be algebraic systems,* $\mathcal{K} = \langle \mathbf{K}, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}', D'\rangle$ *be* π*-structures based on* $\mathbf{K}$, $\mathbf{K}'$*, respectively, and* $(\alpha, \beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *a conjugate pair. Then*

$$\beta^* : \mathbf{ThFam}(\mathcal{K}) \to \mathbf{ThFam}(\mathcal{K}') \quad \textit{and} \quad \alpha^* : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$$

are mutually inverse order isomorphisms.

Proof: We know, by Lemma 892, that β^* and α^* are mutually inverse bijections. Moreover, by definition, they are both order preserving. Thus, each is also order-reflecting, since, e.g., for all $S', T' \in \mathrm{ThFam}(\mathcal{K}')$,

$$\begin{array}{lll}\alpha^*(S') \leq \alpha^*(T') & \text{implies} & \beta^*(\alpha^*(S')) \leq \beta^*(\alpha^*(T')) \\ & \text{implies} & S' \leq T',\end{array}$$

the latter implication following by Lemma 891. ∎

Conversely, it is true that given mutually inverse order isomorphisms between the complete lattices of two π-structures, one may define a conjugate pair between the two that establishes this order-isomorphism via the process that was described above. We provide, next, more details on this inverse process.

Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ be algebraic systems, $\mathcal{K} = \langle \mathbf{K}, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}', D'\rangle$ be π-structures based on $\mathbf{K}$, $\mathbf{K}'$, respectively, and

$$h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$$

an order isomorphism between the corresponding complete lattices of theory families.

Define $\overrightarrow{h} = \{\overrightarrow{h}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ by letting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\overrightarrow{h}_\Sigma : \mathrm{SEN}(\Sigma) \to \mathrm{SenFam}(\mathbf{K}')$$

be given, for all $\phi \in \mathrm{SEN}(\Sigma)$, by

$$\overrightarrow{h}_\Sigma[\phi] = h^{-1}(D(\phi)).$$

Further, define $\overleftarrow{h} = \{\overleftarrow{h}_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}'|}$ by letting, for all $\Sigma' \in |\mathbf{Sign}'|$,

$$\overleftarrow{h}_{\Sigma'} : \mathrm{SEN}'(\Sigma') \to \mathrm{SenFam}(\mathbf{K})$$

be given, for all $\psi \in \mathrm{SEN}'(\Sigma')$, by

$$\overleftarrow{h}_{\Sigma'}[\psi] = h(D'(\psi)).$$

We show that, the two translations $\overrightarrow{h} : \mathbf{K} \to \mathbf{K}'$ and $\overleftarrow{h} : \mathbf{K}' \to \mathbf{K}$, defined above, constitute interpretations between the corresponding π-structures.

Lemma 894 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *be algebraic systems,* $\mathcal{K} = \langle \mathbf{K}, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}', D'\rangle$ *be* π*-structures based on* $\mathbf{K}$, $\mathbf{K}'$*, respectively, and* $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ *an order isomorphism. Then* $\overleftarrow{h} : \mathcal{K}' \to \mathcal{K}$ *is an interpretation.*

Proof: Suppose $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ is an order isomorphism and let $\Sigma' \in |\mathbf{Sign}'|$ and $\Psi \cup \{\psi\} \subseteq \mathrm{SEN}'(\Sigma')$. Then we have

$$\begin{array}{lll}
\psi \in D'_{\Sigma'}(\Psi) & \text{iff} & D'(\psi) \leq D'(\Psi)\\
& \text{iff} & h(D'(\psi)) \leq h(D'(\Psi))\\
& \text{iff} & h(D'(\psi)) \leq h(\bigvee\{D'(\chi) : \chi \in \Psi\})\\
& \text{iff} & h(D'(\psi)) \leq \bigvee\{h(D'(\chi)) : \chi \in \Psi\}\\
& \text{iff} & \overleftarrow{h}_{\Sigma'}[\psi] \leq \bigvee\{\overleftarrow{h}_{\Sigma'}[\chi] : \chi \in \Psi\}\\
& \text{iff} & \overleftarrow{h}_{\Sigma'}[\psi] \leq D(\overleftarrow{h}_{\Sigma'}[\Psi]).
\end{array}$$

Thus, $\overleftarrow{h} : \mathcal{K}' \to \mathcal{K}$ is indeed an interpretation. ∎

We now know (by symmetry, based on Lemma 894) that $\overrightarrow{h} : \mathcal{K} \to \mathcal{K}'$ and $\overleftarrow{h} : \mathcal{K}' \to \mathcal{K}$ are interpretations. It is, in fact, the case that $(\overrightarrow{h}, \overleftarrow{h}) : \mathcal{K} \rightleftarrows \mathcal{K}'$ form a conjugate pair, as is shown next.

Lemma 895 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *be algebraic systems,* $\mathcal{K} = \langle \mathbf{K}, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}', D'\rangle$ *be* π*-structures based on* $\mathbf{K}$, $\mathbf{K}'$*, respectively, and* $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ *an order isomorphism. Then* $(\overrightarrow{h}, \overleftarrow{h}) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *is a conjugate pair.*

Proof: By Lemma 889, it suffices to show that $\overleftarrow{h} : \mathcal{K}' \to \mathcal{K}$ is an interpretation and that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $D(\phi) = D(\overleftarrow{h}[\overrightarrow{h}_\Sigma[\phi]])$. The former has been shown in Lemma 894. So it suffices to show the latter. To this end, let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. Then we have

$$\begin{array}{lll}
D(\overleftarrow{h}[\overrightarrow{h}_\Sigma[\phi]]) & = & D(\overleftarrow{h}[h^{-1}(D(\phi))])\\
& = & D(\bigcup\{\overleftarrow{h}[\chi] : \chi \in h^{-1}(D(\phi))\})\\
& = & \bigvee\{h(D'(\chi)) : \chi \in h^{-1}(D(\phi))\}\\
& = & h(\bigvee\{D'(\chi) : \chi \in h^{-1}(D(\phi))\})\\
& = & h(h^{-1}(D(\phi)))\\
& = & D(\phi).
\end{array}$$

We conclude that $(\overrightarrow{h}, \overleftarrow{h}) : \mathcal{K} \rightleftarrows \mathcal{K}'$ is a conjugate pair. ∎

Based on Lemma 895, we can now formulate one of the main theorems of this section to the effect that every order isomorphism between the complete lattices of theory families of two π-structures gives rise to a conjugate pair of interpretations that induce the isomorphism via the star construction.

Theorem 896 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ *be algebraic systems,* $\mathcal{K} = \langle \mathbf{K}, D \rangle$, $\mathcal{K}' = \langle \mathbf{K}', D' \rangle$ *be* π*-structures based on* $\mathbf{K}$, $\mathbf{K}'$, *respectively, and* $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ *an order isomorphism. Then* $(\overrightarrow{h}, \overleftarrow{h}) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *is a conjugate pair, such that* $\overrightarrow{h}^* = h$ *and* $\overleftarrow{h}^* = h^{-1}$.

Proof: By Lemma 895, we know that $(\overrightarrow{h}, \overleftarrow{h}) : \mathcal{K} \rightleftarrows \mathcal{K}'$ form a conjugate pair. We show that $\overrightarrow{h}^* = h$. The equality $\overleftarrow{h}^* = h^{-1}$ may be proved similarly. To this end, let $T' \in \mathrm{ThFam}(\mathcal{K}')$. Then we have

$$\begin{array}{rcl}
\overrightarrow{h}^*_\Sigma(T') & = & \{\phi \in \mathrm{SEN}(\Sigma) : \overrightarrow{h}_\Sigma[\phi] \leq T'\} \\
& = & \{\phi \in \mathrm{SEN}(\Sigma) : h^{-1}(D(\phi)) \leq T'\} \\
& = & D_\Sigma(\{\phi \in \mathrm{SEN}(\Sigma) : h^{-1}(D(\phi)) \leq T'\}) \\
& = & D_\Sigma(\{\phi \in \mathrm{SEN}(\Sigma) : D(\phi) \leq h(T')\}) \\
& = & D_\Sigma(\{\phi \in \mathrm{SEN}(\Sigma) : \phi \in h_\Sigma(T')\}) \\
& = & D_\Sigma(h_\Sigma(T')) \\
& = & h_\Sigma(T').
\end{array}$$

Similarly, $\overleftarrow{h}^* = h^{-1}$. ∎

12.2 Transformations

Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an algebraic system and $k \geq 1$ be an integer. Then a **power algebraic system**

$$\mathbf{K}^k = \langle \mathbf{Sign}, \mathrm{SEN}^k, N^k \rangle$$

is the algebraic system whose sentence functor $\mathrm{SEN}^k : \mathbf{Sign} \to \mathbf{Set}$ is the k-th direct power of SEN and whose category N^k of natural transformations consists of k-tuples of natural transformations having the same arity in N.

Let $k, \ell \geq 1$ be integers. A translation $\alpha : \mathbf{K}^k \to \mathbf{K}^\ell$ is called a **transformation** if there exists a set

$$\tau : \mathrm{SEN}^\omega \to \mathrm{SEN}^\ell,$$

in N, with k distinguished arguments, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$,

$$\alpha_\Sigma[\vec{\phi}] = \tau_\Sigma[\vec{\phi}].$$

Moreover a translation $\alpha : \mathbf{K}^k \to \mathbf{K}^\ell$ is called a **natural transformation** if it is a parameter-free transformation, i.e., if there exists $\tau : \mathrm{SEN}^k \to \mathrm{SEN}^\ell$ in N, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$,

$$\alpha_\Sigma[\vec{\phi}] = \tau_\Sigma[\vec{\phi}].$$

Based on the results obtained in Section 12.1, we may formulate some propositions concerning interpretability and equivalence based on transformations.

Proposition 897 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system and* $\mathcal{K} = \langle \mathbf{F}^k, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D'\rangle$ *be two* π*-structures.* $\mathcal{K}$ *is interpretable in* $\mathcal{K}'$ *via a transformation if and only if there exists a set* $\tau : \mathrm{SEN}^\omega \to \mathrm{SEN}^\ell$, *with* k *distinguished arguments, such that, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\Phi \cup \{\vec{\phi}\} \subseteq \mathrm{SEN}(\Sigma)^k$,

$$\vec{\phi} \in D_\Sigma(\Phi) \quad \textit{iff} \quad \tau_\Sigma[\vec{\phi}] \leq D'(\tau_\Sigma[\Phi]).$$

If $\mathcal{K}$ *is interpretable in* $\mathcal{K}'$ *as above, then it is equivalent to* $\mathcal{K}'$ *via a conjugate pair* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *of transformations if and only if, for all* $\Sigma \in |\mathbf{Sign}|$, *all* $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$ *and all* $\Psi \cup \{\vec{\psi}\} \subseteq \mathrm{SEN}(\Sigma)^\ell$,

- $\vec{\psi} \in D'_\Sigma(\Psi)$ *iff* $I_\Sigma[\vec{\psi}] \leq D(I_\Sigma[\Psi])$;
- $D'(\vec{\psi}) = D'(\tau[I_\Sigma[\vec{\psi}]])$;
- $D(\vec{\phi}) = D(I[\tau_\Sigma[\vec{\phi}]])$.

Proof: This is a restatement of the definition of interpretability under the additional hypothesis that the corresponding interpretations are transformations. ■

Proposition 898 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system and* $\mathcal{K} = \langle \mathbf{K}^k, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D'\rangle$ *be two* π*-structures.* $\mathcal{K}$ *is equivalent to* $\mathcal{K}'$ *via a conjugate pair* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *of transformations if and only if one of the following equivalent conditions hold:*

(a) $\tau : \mathcal{K} \to \mathcal{K}'$ *is an interpretation and, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\vec{\psi} \in \mathrm{SEN}(\Sigma)^\ell$, $D'(\vec{\psi}) = D'(\tau[I_\Sigma[\vec{\psi}]])$;

(b) $I : \mathcal{K}' \to \mathcal{K}$ *is an interpretation and, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$, $D(\vec{\phi}) = D(I[\tau_\Sigma[\vec{\phi}]])$.

Proof: Directly by Lemma 889. ■

Taking the point of view of order isomorphisms between lattices of theory families, we would like to have a concept ensuring that such an isomorphism is induced not merely by a conjugate pair of translations, as is asserted by

Theorem 896, but, more emphatically, by a conjugate pair of transformations. We focus on this task next.

Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system and $\mathcal{K} = \langle \mathbf{K}^k, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D'\rangle$ be two π-structures based on $\mathbf{K}^k$, $\mathbf{K}^\ell$, respectively. An order isomorphism $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ is called **transformational** if there exist sets

- $\tau : \mathrm{SEN}^\omega \to \mathrm{SEN}^\ell$ in N, with k distinguished arguments;
- $I : \mathrm{SEN}^\omega \to \mathrm{SEN}^k$ in N, with ℓ distinguished arguments,

such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$ and all $\vec{\psi} \in \mathrm{SEN}(\Sigma)^\ell$,

$$\overrightarrow{h}_\Sigma[\vec{\phi}] = D'(\tau_\Sigma[\vec{\phi}]) \quad \text{and} \quad \overleftarrow{h}_\Sigma[\vec{\psi}] = D(I_\Sigma[\vec{\psi}]).$$

These conditions are, by definition, equivalent, respectively, to the conditions

$$h^{-1}(D(\vec{\phi})) = D'(\tau_\Sigma[\vec{\phi}]) \quad \text{and} \quad h(D'(\vec{\psi})) = D(I_\Sigma[\vec{\psi}]).$$

In this case, we say that h is **induced by** $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$. (Note that, since we will be able to show that (τ, I) is a conjugate pair of transformations, this notation makes sense.)

In fact, the defining conditions yield some crucial relations between theory families, as in shown in the following lemma.

Lemma 899 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system,* $\mathcal{K} = \langle \mathbf{K}^k, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D'\rangle$ *be two* π*-structures and* $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ *a transformational order isomorphism induced by* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$. *Then, for all* $\Sigma \in |\mathbf{Sign}|$, *all* $\Phi \subseteq \mathrm{SEN}(\Sigma)^k$ *and all* $\Psi \subseteq \mathrm{SEN}(\Sigma)^\ell$,

$$h^{-1}(D(\Phi)) = D'(\tau_\Sigma[\Phi]) \quad \textit{and} \quad h(D'(\Psi)) = D(I_\Sigma[\Psi]).$$

Proof: By symmetry, it suffices to show the first equation. We have, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)^k$,

$$\begin{array}{rcl} h^{-1}(D(\Phi)) & = & h^{-1}(\bigvee_{\phi\in\Phi} D(\phi)) \quad (\text{join in } \mathbf{ThFam}(\mathcal{K}))\\ & = & \bigvee_{\phi\in\Phi} h^{-1}(D(\phi)) \quad (h^{-1} \text{ order isomorphism})\\ & = & \bigvee_{\phi\in\Phi} D'(\tau_\Sigma[\phi]) \quad (h^{-1}(D(\phi)) = \overrightarrow{h}_\Sigma[\phi])\\ & = & D'(\bigcup_{\phi\in\Phi} \tau_\Sigma[\phi]) \quad (\text{join in } \mathbf{ThFam}(\mathcal{K}'))\\ & = & D'(\tau_\Sigma[\Phi]). \quad (\text{by definition}) \end{array}$$

The second equation now follows by symmetry. ■

Now we are in a position to show that a transformational order isomorphism between the lattices of theory families of two π-structures is induced by a conjugate pair of transformations between the two π-structures and, as a consequence, gives rise to an equivalence $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ via a conjugate pair of transformations.

Theorem 900 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *be an algebraic system,* $\mathcal{K} = \langle \mathbf{K}^k, D \rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D' \rangle$ *be two* π*-structures and* $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ *a transformational order isomorphism induced by* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$. *Then* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *is a conjugate pair of transformations.*

Proof: We use Proposition 898. Let $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\vec{\phi}\} \subseteq \mathrm{SEN}(\Sigma)^k$ and $\vec{\psi} \in \mathrm{SEN}(\Sigma)^\ell$. We then have:

$$\begin{array}{rll} \vec{\phi} \in D_\Sigma(\Phi) & \text{iff} & D_\Sigma(\vec{\phi}) \le D_\Sigma(\Phi) \\ & \text{iff} & h^{-1}(D(\vec{\phi})) \le h^{-1}(D(\Phi)) \quad (h \text{ order isomorphism}) \\ & \text{iff} & D'(\tau_\Sigma[\vec{\phi}]) \le D'(\tau_\Sigma[\Phi]) \quad (\text{Lemma 899}) \\ & \text{iff} & \tau_\Sigma[\vec{\phi}] \le D'(\tau_\Sigma[\Phi]). \end{array}$$

Thus, $\tau : \mathcal{K} \to \mathcal{K}'$ is an interpretation. Moreover, we have:

$$\begin{array}{rcll} D'(\vec{\psi}) & = & h^{-1}(h(D'(\vec{\psi}))) & (h \text{ order isomorphism}) \\ & = & h^{-1}(D(I_\Sigma[\vec{\psi}])) & (h \text{ transformational}) \\ & = & D'(\tau[I_\Sigma[\vec{\psi}]]). & (\text{Lemma 899}) \end{array}$$

We conclude by Proposition 898, that $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ is a conjugate pair of transformations. ■

As a consequence, we have the following

Theorem 901 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *be an algebraic system,* $\mathcal{K} = \langle \mathbf{K}^k, D \rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D' \rangle$ *be two* π*-structures and* $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ *a transformational order isomorphism induced by* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$. *Then the* π*-structures* $\mathcal{K}$ *and* $\mathcal{K}'$ *are equivalent via the conjugate pair* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *of transformations.*

Proof: This follows directly by Theorem 900. ■

Similarly, for interpretability and equivalence based on natural transformations, we have the following corresponding propositions.

Proposition 902 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *be an algebraic system and* $\mathcal{K} = \langle \mathbf{K}^k, D \rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D' \rangle$ *be two* π*-structures.* $\mathcal{K}$ *is interpretable in* $\mathcal{K}'$ *via a natural transformation if and only if there exists a set* $\tau : \mathrm{SEN}^k \to \mathrm{SEN}^\ell$ *in* N*, such that, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\Phi \cup \{\vec{\phi}\} \subseteq \mathrm{SEN}(\Sigma)^k$,

$$\vec{\phi} \in D_\Sigma(\Phi) \quad \textit{iff} \quad \tau_\Sigma[\vec{\phi}] \le D'(\tau_\Sigma[\Phi]).$$

If $\mathcal{K}$ *is interpretable in* $\mathcal{K}'$ *as above, then it is equivalent to* $\mathcal{K}'$ *via a conjugate pair* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *of natural transformations if and only if, for all* $\Sigma \in |\mathbf{Sign}|$*, all* $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$ *and all* $\Psi \cup \{\vec{\psi}\} \subseteq \mathrm{SEN}(\Sigma)^\ell$,

- $\vec{\psi} \in D'_\Sigma(\Psi)$ *iff* $I_\Sigma[\vec{\psi}] \le D(I_\Sigma[\Psi])$;

- $D'(\vec{\psi}) = D'(\tau[I_\Sigma[\vec{\psi}]])$;
- $D(\vec{\phi}) = D(I[\tau_\Sigma[\vec{\phi}]])$.

Proof: This is a restatement of the definition of interpretability under the additional hypothesis that the corresponding interpretations are natural transformations. ∎

Proposition 903 *Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system and $\mathcal{K} = \langle \mathbf{K}^k, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D'\rangle$ be two π-structures. $\mathcal{K}$ is equivalent to $\mathcal{K}'$ via a conjugate pair $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ of natural transformations if and only if one of the following equivalent conditions hold:*

(a) $\tau : \mathcal{K} \to \mathcal{K}'$ is an interpretation and, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\psi} \in \mathrm{SEN}(\Sigma)^\ell$, $D'(\vec{\psi}) = D'(\tau[I_\Sigma[\vec{\psi}]])$;

(b) $I : \mathcal{K}' \to \mathcal{K}$ is an interpretation and, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$, $D(\vec{\phi}) = D(I[\tau_\Sigma[\vec{\phi}]])$.

Proof: Directly by Lemma 889. ∎

In terms of order isomorphisms between lattices of theory families, we have analogs of preceding results that allow us to work with isomorphisms that are induced by conjugate pairs of natural transformations.

Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system and $\mathcal{K} = \langle \mathbf{K}^k, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D'\rangle$ be two π-structures based on $\mathbf{K}^k$, $\mathbf{K}^\ell$, respectively. An order isomorphism $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ is called **natural** if there exist sets

- $\tau : \mathrm{SEN}^k \to \mathrm{SEN}^\ell$ in N;
- $I : \mathrm{SEN}^\ell \to \mathrm{SEN}^k$ in N,

such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$ and $\vec{\psi} \in \mathrm{SEN}(\Sigma)^\ell$,

$$\overrightarrow{h}_\Sigma[\vec{\phi}] = D'(\tau_\Sigma[\vec{\phi}]) \quad \text{and} \quad \overleftarrow{h}_\Sigma[\vec{\psi}] = D(I_\Sigma[\vec{\psi}]).$$

In this case, we say that h is **induced by** the pair of natural transformations $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$.

Similarly, with the case of a transformational isomorphism, we can show that a natural order isomorphism between the lattices of theory families of two π-structures is induced by a conjugate pair of natural transformations between the two π-structures.

Theorem 904 *Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system, $\mathcal{K} = \langle \mathbf{K}^k, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D'\rangle$ be two π-structures and $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ a natural order isomorphism induced by $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$. Then $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ is a conjugate pair of natural transformations.*

Proof: This follows from Theorem 900. ∎

As a consequence, we have the following analog of Theorem 901.

Theorem 905 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system,* $\mathcal{K} = \langle \mathbf{K}^k, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D'\rangle$ *be two* π*-structures and* $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ *a natural order isomorphism induced by* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$*. Then the* π*-structures* $\mathcal{K}$ *and* $\mathcal{K}'$ *are equivalent via the conjugate pair* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *of natural transformations.*

Proof: This follows directly by Theorem 904. ∎

We now revert to the case of a base algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ and a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ based on $\mathbf{F}$. Our focus, in this standard context, will be on $\mathbf{F}$ itself, on the one hand, and on $\mathbf{F}^2$, on the other. In the context of $\mathbf{F}^2$, given $\Sigma \in |\mathbf{Sign}^\flat|$, we sometimes denote a pair $\langle \phi, \psi\rangle \in \mathrm{SEN}^\flat(\Sigma)^2$ in the equational form

$$\phi \approx \psi.$$

Given a π-structure $\mathcal{Q} = \langle \mathbf{F}^2, D\rangle$, we say that $\mathcal{Q}$ is **equational** if the following five axioms hold:

(R) $\phi \approx \phi \in D_\Sigma(\varnothing)$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$;

(S) $\psi \approx \phi \in D_\Sigma(\phi \approx \psi)$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$;

(T) $\phi \approx \chi \in D_\Sigma(\phi \approx \psi, \psi \approx \chi)$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$;

(C) $\sigma^\flat_\Sigma(\vec{\phi}) \approx \sigma^\flat_\Sigma(\vec{\psi}) \in D_\Sigma(\{\phi_i \approx \psi_i : i < k\})$, for all $\sigma^\flat \in N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi_i, \psi_i \in \mathrm{SEN}^\flat(\Sigma)$, $i < k$;

(I) $\mathrm{SEN}^\flat(f)(\phi) \approx \mathrm{SEN}^\flat(f)(\psi) \in D_{\Sigma'}(\phi \approx \psi)$, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$.

Note that according to the relevant definitions introduced in Chapter 2, the meaning of (I) is that the Σ'-component of the least theory family including $\phi \approx \psi$ in its Σ-component includes $\mathrm{SEN}^\flat(f)(\phi) \approx \mathrm{SEN}^\flat(f)(\psi)$.

These properties are termed **reflexivity**, **symmetry**, **transitivity**, **compatibility** and **invariance**, respectively. The first three ensure that, for all $E \in \mathrm{SenFam}(\mathbf{F}^2)$, $D(E)$ is an equivalence family. The fourth one ensures that $D(E)$ is a congruence family and the last that it is a congruence system, i.e., invariant under the action of signature morphisms. In fact, the following characterization theorem holds, showing that a π-structure is equational if and only if it is structural and all its closure families are congruence systems on $\mathbf{F}$ if and only if it is the equational π-structure relative to a class K of $\mathbf{F}$-algebraic systems according to the definition given in Section 2.17.

Theorem 906 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{Q} = \langle \mathbf{F}^2, D\rangle$ *a* π*-structure. The following statements are equivalent:*

(i) $\mathcal{Q}$ *is equational;*

(ii) *For all* $\theta \in \mathrm{SenFam}(\mathcal{Q})$, $D(\theta) \in \mathrm{ConSys}(\mathbf{F})$*;*

(iii) *For some class* K *of* $\mathbf{F}$*-algebraic systems,* $D = D^{\mathsf{K}}$*.*

Proof:

(i)⇒(ii) Suppose $\mathcal{Q}$ is equational and let $\theta \in \mathrm{SenFam}(\mathcal{Q})$. We must show that $D(\theta) = \{D_\Sigma(\theta)\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ is a congruence system on $\mathbf{F}$. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$. Since $\mathcal{Q}$ is equational, we have $\phi \approx \phi \in D_\Sigma(\varnothing) \subseteq D_\Sigma(\theta)$. So $D_\Sigma(\theta)$ is reflexive. Suppose, next, that $\phi \approx \psi \in D_\Sigma(\theta)$. Since $\mathcal{Q}$ is equational, we get $\psi \approx \phi \in D_\Sigma(\phi \approx \psi) \subseteq D_\Sigma(\theta)$. Hence, $D_\Sigma(\theta)$ is also symmetric. Further, if $\phi \approx \psi, \psi \approx \chi \in D_\Sigma(\theta)$, then, since $\mathcal{Q}$ is equational, we get $\phi \approx \chi \in D_\Sigma(\phi \approx \psi, \psi \approx \chi) \subseteq D_\Sigma(\theta)$. Thus, $D_\Sigma(\theta)$ is also transitive and, hence, an equivalence relation on $\mathrm{SEN}^\flat(\Sigma)$.

Suppose, now, that $\sigma^\flat \in N^\flat$, $\phi_i, \psi_i \in \mathrm{SEN}^\flat(\Sigma)$, for $i < k$, such that $\phi_i \approx \psi_i \in D_\Sigma(\theta)$, for all $i < k$. Since $\mathcal{Q}$ is equational, we get $\sigma^\flat_\Sigma(\vec{\phi}) \approx \sigma^\flat_\Sigma(\vec{\psi}) \in D_\Sigma(\{\phi_i \approx \psi_i : i < k\}) \subseteq D_\Sigma(\theta)$. Hence, $D_\Sigma(\theta)$ is a congruence family on $\mathbf{F}$. Finally, if $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \approx \psi \in D_\Sigma(\theta)$, then, again based on the fact that $\mathcal{Q}$ is equational, we obtain $\mathrm{SEN}^\flat(f)(\phi) \approx \mathrm{SEN}^\flat(f)(\psi) \in D_{\Sigma'}(\phi \approx \psi) \subseteq D_{\Sigma'}(\theta)$, whence $D(\theta)$ is a congruence system on $\mathbf{F}$, as was to be shown.

(ii)⇒(iii) Suppose D satisfies (ii). We construct a class K of $\mathbf{F}$-algebraic systems as follows. For $\theta \in \mathrm{SenFam}(\mathcal{Q})$, define

$$\mathcal{F}^\theta = \langle \mathbf{F}^\theta, \langle I, \pi^\theta \rangle \rangle \coloneqq \langle \mathbf{F}/D(\theta), \langle I, \pi^{D(\theta)} \rangle \rangle$$

and set

$$\mathsf{K} = \{\mathcal{F}^\theta : \theta \in \mathrm{SenFam}(\mathcal{Q})\}.$$

Note that the definition of $\mathcal{F}^\theta$ makes sense, since, by hypothesis, $D(\theta) \in \mathrm{ConSys}(\mathbf{F})$, for all $\theta \in \mathrm{SenFam}(\mathcal{Q})$. Our task now is to show that $D = D^{\mathsf{K}}$. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$, $\theta \cup \{\phi \approx \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)^2$.

Suppose, first, that $\phi \approx \psi \in D_\Sigma(\theta)$ and let $\theta' \in \mathrm{SenFam}(\mathcal{Q})$, such that $\pi^{\theta'}_\Sigma(\theta) \subseteq \Delta^{\mathbf{F}/D(\theta')}_\Sigma$. This is equivalent to $\theta_\Sigma \subseteq D_\Sigma(\theta'_\Sigma)$. Hence, we obtain $\phi \approx \psi \in D_\Sigma(\theta) \subseteq D_\Sigma(\theta')$. Thus, $\pi^{\theta'}_\Sigma(\phi) = \pi^{\theta'}_\Sigma(\psi)$. We conclude that $\phi \approx \psi \in D^{\mathsf{K}}_\Sigma(\theta)$. Hence, $D \leq D^{\mathsf{K}}$.

Assume, conversely, that $\phi \approx \psi \notin D_\Sigma(\theta)$. Then, clearly, for $\mathcal{F}^\theta \in \mathsf{K}$, we get $\pi^\theta_\Sigma(D_\Sigma(\theta)) \subseteq \Delta^{\mathbf{F}/D(\theta)}_\Sigma$, but $\pi^\theta_\Sigma(\phi) \neq \pi^\theta_\Sigma(\psi)$. Hence, $\phi \approx \psi \notin D^{\mathsf{K}}_\Sigma(\theta)$. Therefore, $D^{\mathsf{K}} \leq D$ and, hence, $D = D^{\mathsf{K}}$.

(iii)⇒(i) This implication was shown in Proposition 115, which was proven by appealing to the implication (iii)⇒(ii), which was, in turn, the content of Proposition 30.

■

We have the following useful technical lemma, where, for $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$, we use the abbreviation

$$\vec{\phi} \approx \vec{\psi} = \{\phi_i \approx \psi_i : i < k\}.$$

Lemma 907 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{Q} = \langle \mathbf{F}^2, D \rangle$ *an equational* π*-structure. Then, for all* $\delta^\flat, \epsilon^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\delta^\flat_\Sigma(\vec{\psi}) \approx \epsilon^\flat_\Sigma(\vec{\psi}) \in D_\Sigma(\vec{\phi} \approx \vec{\psi}, \delta^\flat_\Sigma(\vec{\phi}) \approx \epsilon^\flat_\Sigma(\vec{\phi})).$$

Proof: We have, for all $\delta^\flat, \epsilon^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rcl} \delta^\flat_\Sigma(\vec{\psi}) \approx \epsilon^\flat_\Sigma(\vec{\psi}) & \in & D_\Sigma(\delta^\flat_\Sigma(\vec{\psi}) \approx \delta^\flat_\Sigma(\vec{\phi}), \delta^\flat_\Sigma(\vec{\phi}) \approx \epsilon^\flat_\Sigma(\vec{\phi}), \epsilon^\flat_\Sigma(\vec{\phi}) \approx \epsilon^\flat_\Sigma(\vec{\psi})) \\ & & \text{(by transitivity)} \\ & \subseteq & D_\Sigma(\vec{\phi} \approx \vec{\psi}, \delta^\flat_\Sigma(\vec{\phi}) \approx \epsilon^\flat_\Sigma(\vec{\phi})). \\ & & \text{(by symmetry and compatibility)} \end{array}$$

This proves the lemma. ■

Lemma 907 has the following corollary:

Corollary 908 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{Q} = \langle \mathbf{F}^2, D \rangle$ *an equational* π*-structure. Then, for all* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*, with* k *distinguished arguments, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\tau^\flat_\Sigma[\vec{\psi}] \leq D(\vec{\phi} \approx \vec{\psi}, \tau^\flat_\Sigma[\vec{\phi}]).$$

Proof: This follows from Lemma 907, using the reflexivity and the invariance of the closure family D. ■

We next show that, if a π-institution $\mathcal{I}$, based on an algebraic system $\mathbf{F}$, happens to be equivalent to an equational π-structure $\mathcal{Q}$, based on $\mathbf{F}^2$, via a conjugate pair $(\tau, I) : \mathcal{I} \rightleftarrows \mathcal{Q}$ of transformations, then $\mathcal{I}$ is syntactically protoalgebraic with set of witnessing transformations I.

Theorem 909 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\mathcal{Q} = \langle \mathbf{F}^2, D \rangle$ *an equational* π*-structure. If* $\mathcal{I}$ *is equivalent to* $\mathcal{Q}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}$ *of transformations, then* $\mathcal{I}$ *is syntactically protoalgebraic with witnessing transformations* $I^\flat$*.*

Proof: By definition, it suffices to show that $I^\flat : \mathrm{SEN}^\omega \to \mathrm{SEN}$, with two distinguished arguments, is reflexive, globally family transitive and has the global family compatibility and the global family modus ponens in $\mathcal{I}$. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have, in turn:

- By reflexivity of $\mathcal{Q}$, $\phi \approx \phi \in D_\Sigma(\varnothing)$. Hence, by interpretability, we get $I^\flat_\Sigma[\phi, \phi] \leq C(\varnothing)$. Therefore, $I^\flat$ is reflexive in $\mathcal{I}$;
- By transitivity of $\mathcal{Q}$, $\phi \approx \chi \in D_\Sigma(\phi \approx \psi, \psi \approx \chi)$. Hence, by interpretability, we get $I^\flat_\Sigma[\phi, \chi] \leq C(I^\flat_\Sigma[\phi, \psi], I^\flat_\Sigma[\psi, \chi])$. Therefore, $I^\flat$ is globally family transitive in $\mathcal{I}$;
- By the reflexivity and compatibility of $\mathcal{Q}$, we have, for all $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in N and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, that $\sigma^\flat_\Sigma(\phi, \vec{\chi}) \approx \sigma^\flat_\Sigma(\psi, \vec{\chi}) \in D_\Sigma(\phi \approx \psi)$. Hence, by interpretability,
$$I^\flat_\Sigma[\sigma^\flat_\Sigma(\phi, \vec{\chi}), \sigma^\flat_\Sigma(\psi, \vec{\chi})] \leq C(I^\flat_\Sigma[\phi, \psi]).$$
Therefore, $I^\flat$ has the global family compatibility in $\mathcal{I}$;
- Finally, for global family MP, we have
$$\begin{array}{rcl} C(\psi) & = & C(I^\flat[\tau^\flat_\Sigma[\psi]]) \quad \text{(by equivalence)} \\ & \leq & C(I^\flat_\Sigma[\phi, \psi], I^\flat[\tau^\flat_\Sigma[\phi]]) \\ & & \text{(by Lemma 907 and interpretability)} \\ & = & C(I^\flat_\Sigma[\phi, \psi], \phi). \quad \text{(by equivalence)} \end{array}$$
Thus, for all $T \in \mathrm{ThFam}(\mathcal{I})$, if $\phi \in T_\Sigma$ and $I^\flat_\Sigma[\phi, \psi] \leq T$, then $\psi \in T_\Sigma$, i.e., $I^\flat$ has the global family modus ponens in $\mathcal{I}$.

We conclude that $\mathcal{I}$ is syntactically protoalgebraic with witnessing transformations $I^\flat$. ∎

As a consequence of Theorem 909, we obtain

Corollary 910 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\mathcal{Q} = \langle \mathbf{F}^2, D \rangle$ *an equational* π*-structure. If* $\mathcal{I}$ *is equivalent to* $\mathcal{Q}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}$ *of natural transformations, then* $\mathcal{I}$ *is syntactically equivalential with witnessing transformations* $I^\flat$.

Proof: By Theorem 909, $\mathcal{I}$ is syntactically protoalgebraic with witnessing transformations $I^\flat$. Since $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ is parameter free, we conclude that $\mathcal{I}$ is syntactically equivalential with witnessing transformations $I^\flat$. ∎

Using Theorem 909, we can also show that, if a π-institution $\mathcal{I}$, based on an algebraic system $\mathbf{F}$, happens to be equivalent to an equational π-structure $\mathcal{Q}$, based on $\mathbf{F}^2$, via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}$ of transformations, then $\mathcal{I}$ is family truth equational, with witnessing equations $\tau^\flat$.

Theorem 911 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\mathcal{Q} = \langle \mathbf{F}^2, D \rangle$ *an equational* π*-structure. If* $\mathcal{I}$ *is equivalent to* $\mathcal{Q}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}$ *of transformations, then* $\mathcal{I}$ *is family truth equational, with witnessing equations* $\tau^\flat$*.*

Proof: By definition, it suffices to show that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

We, indeed, have, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll} \phi \in T_\Sigma & \text{iff} \quad I^\flat[\tau_\Sigma[\phi]] \leq T & ((\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q} \text{ an equivalence}) \\ & \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T). & \text{(by Theorem 909 and Corollary 791)} \end{array}$$

Therefore, $\mathcal{I}$ is family truth equational, with witnessing equations $\tau^\flat$. ■

We close the section by showing that equivalence between a given π-institution and an equational π-structure established via conjugate pairs of transformations is essentially unique in the sense that both the closure family on $\mathbf{F}^2$ must be unique and the closures of the translations used must be identical. More precisely, we have the following

Theorem 912 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Suppose that* $\mathcal{Q}^1 = \langle \mathbf{F}^2, D^1 \rangle$ *and* $\mathcal{Q}^2 = \langle \mathbf{F}^2, D^2 \rangle$ *are equational* π*-structures that are equivalent to* $\mathcal{I}$ *via the conjugate pairs* $\langle \tau^1, I^1 \rangle : \mathcal{I} \rightleftarrows \mathcal{Q}^1$ *and* $\langle \tau^2, I^2 \rangle : \mathcal{I} \rightleftarrows \mathcal{Q}^2$*, respectively, of transformations. Then, we have:*

(a) $D^1 = D^2$ $(=: D)$ *and, hence,* $\mathcal{Q}^1 = \mathcal{Q}^2$ $(=: \mathcal{Q})$*;*

(b) For all $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,* $C(I^1_\Sigma[\phi, \psi]) = C(I^2_\Sigma[\phi, \psi])$*;*

(c) For all $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,* $D(\tau^1_\Sigma[\phi]) = D(\tau^2_\Sigma[\phi])$*.*

Proof: By Theorem 909, we know that both I^1 and I^2 are witnessing the syntactic protoalgebraicity of $\mathcal{I}$. Thus, by Corollary 791, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^1_\Sigma[\phi, \psi] \leq T \quad \text{iff} \quad \langle \phi, \psi \rangle \in \Omega_\Sigma(T) \quad \text{iff} \quad I^2_\Sigma[\phi, \psi] \leq T.$$

We conclude that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $C(I^1_\Sigma[\phi, \psi]) = C(I^2_\Sigma[\phi, \psi])$, which proves Part (b).

For Part (a), suppose that $\Sigma \in |\mathbf{Sign}^\flat|$ and $E \cup \{\phi \approx \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)^2$. Then, we have

$$\begin{array}{lll} \phi \approx \psi \in D^1_\Sigma(E) & \text{iff} \quad I^1_\Sigma[\phi, \psi] \leq C(I^1_\Sigma[E]) & \text{(interpretability)} \\ & \text{iff} \quad C(I^1_\Sigma[\phi, \psi]) \leq C(I^1_\Sigma[E]) & \\ & \text{iff} \quad C(I^2_\Sigma[\phi, \psi]) \leq C(I^2_\Sigma[E]) & \text{(Part (b))} \\ & \text{iff} \quad I^2_\Sigma[\phi, \psi] \leq C(I^2_\Sigma[E]) & \\ & \text{iff} \quad \phi \approx \psi \in D^2_\Sigma(E). & \text{(interpretability)} \end{array}$$

Therefore, we get that $D^1 = D^2$. This justifies using $D := D^1 = D^2$ and since the π-structures $\mathcal{Q}^1$ and $\mathcal{Q}^2$, which are both based on $\mathbf{F}^2$, have the same closure families, we obtain $\mathcal{Q} := \mathcal{Q}^1 = \mathcal{Q}^2$.

Finally, for Part (c), suppose that $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, we have

$$\begin{aligned} D(\tau^1_\Sigma[\phi]) \leq D(\tau^2_\Sigma[\phi]) \quad & \text{iff} \quad \tau^1_\Sigma[\phi] \leq D(\tau^2_\Sigma[\phi]) \\ & \text{iff} \quad I^2[\tau^1_\Sigma[\phi]] \leq C(I^2[\tau^2_\Sigma[\phi]]) \quad \text{(interpretability)} \\ & \text{iff} \quad I^2[\tau^1_\Sigma[\phi]] \leq C(\phi) \quad \text{(equivalence)} \\ & \text{iff} \quad I^1[\tau^1_\Sigma[\phi]] \leq C(\phi) \quad \text{(Part (b))} \\ & \text{iff} \quad \phi \in C_\Sigma(\phi). \quad \text{(equivalence)} \end{aligned}$$

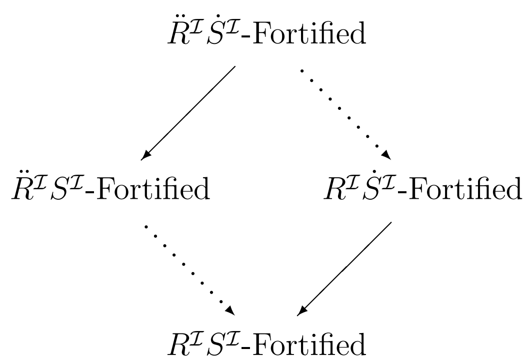
By symmetry, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $D(\tau^1_\Sigma[\phi]) = D(\tau^2_\Sigma[\phi])$. This proves Part (c) and concludes the proof of the theorem. ■

12.3 Syntactic Weak Family Algebraizability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. We say that:

- $\mathcal{I}$ is $R^{\mathcal{I}}S^{\mathcal{I}}$-(**syntactically**) **fortified** if $R^{\mathcal{I}}$ is Leibniz and $S^{\mathcal{I}}$ is adequate;
- $\mathcal{I}$ is $R^{\mathcal{I}}\dot{S}^{\mathcal{I}}$-(**syntactically**) **fortified** if $R^{\mathcal{I}}$ is Leibniz and $\dot{S}^{\mathcal{I}}$ is adequate;
- $\mathcal{I}$ is $\ddot{R}^{\mathcal{I}}S^{\mathcal{I}}$-(**syntactically**) **fortified** if $\ddot{R}^{\mathcal{I}}$ is Leibniz and $S^{\mathcal{I}}$ is adequate;
- $\mathcal{I}$ is $\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$-(**syntactically**) **fortified** if $\ddot{R}^{\mathcal{I}}$ is Leibniz and $\dot{S}^{\mathcal{I}}$ is adequate.

Recall that, by Proposition 997, if $\dot{S}^{\mathcal{I}}$ is adequate, then $S^{\mathcal{I}}$ is adequate. Moreover, since, by Proposition 952, $\ddot{R}^{\mathcal{I}} \subseteq R^{\mathcal{I}}$, it follows that, under the assumption of prealgebraicity, if $\ddot{R}^{\mathcal{I}}$ is Leibniz, then $R^{\mathcal{I}}$ is Leibniz. Thus, we have the following **syntactic fortification hierarchy** (in which the dotted arrows hold under prealgebraicity):



$\mathcal{I}$ is **syntactically weakly family algebraizable** (abbreviated to **syntactically WF algebraizable**) if:

- $\mathcal{I}$ is $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified;
- $\mathcal{I}$ is protoalgebraic;
- $\mathcal{I}$ is family injective.

By Theorem 288, under protoalgebraicity, the properties of family injectivity, family reflectivity and family c-reflectivity coincide. This enables us to formulate the following alternative characterization of syntactic WF algebraizability.

Theorem 913 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically WF algebraizable if and only if it is syntactically protoalgebraic and family truth equational.*

Proof: Assume that $\mathcal{I}$ is syntactically WF algebraizable. Then, on the one hand, it is protoalgebraic and has a Leibniz reflexive core. Thus, by Theorem 805, it is syntactically protoalgebraic. On the other, it is, by Theorem 288, family c-reflective and has an adequate Suszko core. Therefore, by Theorem 847, it is family truth equational.

Assume, conversely, that $\mathcal{I}$ is syntactically protoalgebraic and family truth equational. Then, by Theorem 805, it is protoalgebraic and has a Leibniz reflexive core, and, by Theorem 847, it is family c-reflective and has an adequate Suszko core. Therefore, $\mathcal{I}$ is syntactically WF algebraizable. ∎

Directly from the definitions, we may derive the following relationship between the semantic and syntactic WF algebraizability classes of π-institutions.

Theorem 914 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically WF algebraizable if and only if* $\mathcal{I}$ *is WF algebraizable and* $R^{\mathcal{I}}S^{\mathcal{I}}$*-fortified.*

Proof: $\mathcal{I}$ is syntactically WF algebraizable if and only if, by definition, it is $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified, protoalgebraic and family injective, i.e., iff it is, by definition, $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified and WF algebraizable. ∎

Previous results, put together, also allow us to provide an alternative characterization of syntactic weak family algebraizability in terms of isomorphisms between complete lattices of theory families.

Theorem 915 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically WF algebraizable if and only if it is* $R^{\mathcal{I}}S^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism.

Proof: We have that $\mathcal{I}$ is syntacticaly WF algebraizable if and only if, by Theorem 914, it is $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified and WF algebraizable, if and only if, by Theorem 296, it it $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism. ∎

Next, we show that syntactic WF algebraizability may also be characterized by the existence of an equivalence between the π-institution and its algebraic π-structure counterpart via a pair of conjugate transformations.

We embark on the path by defining first the algebraic π-structure $\mathcal{Q}^{\mathcal{I}*}$ associated with a given π-institution $\mathcal{I}$. We recall some concepts that we have already introduced previously which culminate in the definition of $\mathcal{Q}^{\mathcal{I}*}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Recall the definition of the class $\mathrm{AlgSys}^*(\mathcal{I})$ of all reduced $\mathbf{F}$-algebraic systems:

$$\mathrm{AlgSys}^*(\mathcal{I}) = \{\mathcal{A} : (\exists T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}})\}.$$

Given an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, we define the class of $\mathcal{I}^*$-**congruence systems on** $\mathcal{A}$ by

$$\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}) = \{\theta \in \mathrm{ConSys}(\mathbf{A}) : \mathcal{A}/\theta \in \mathrm{AlgSys}^*(\mathcal{I})\}.$$

It turns out that congruence systems in $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ have a straightforward characterization.

Proposition 916 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$,

$$\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}) = \{\theta \in \mathrm{ConSys}(\mathbf{A}) : (\exists T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(\Omega^{\mathcal{A}}(T) = \theta)\}.$$

Proof: Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system.

Suppose, first, that $\theta \in \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. By definition, $\mathcal{A}/\theta \in \mathrm{AlgSys}^*(\mathcal{I})$. Thus, there exists $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)$, such that

$$\Omega^{\mathcal{A}/\theta}(T') = \Delta^{\mathcal{A}/\theta}.$$

By applying the inverse of the quotient morphism $\langle I, \pi^\theta\rangle : \mathcal{A} \to \mathcal{A}/\theta$, we get

$$(\pi^\theta)^{-1}(\Omega^{\mathcal{A}/\theta}(T')) = (\pi^\theta)^{-1}(\Delta^{\mathcal{A}/\theta}).$$

Since $\langle I, \pi^\theta\rangle$ is surjective, we get by Proposition 24 and by Corollary 55, that $(\pi^\theta)^{-1}(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and

$$\Omega^{\mathcal{A}}((\pi^\theta)^{-1}(T')) = \theta.$$

Therefore, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \theta$.

Suppose, conversely, that $\theta \in \mathrm{ConSys}(\mathbf{A})$, with $\Omega^{\mathcal{A}}(T) = \theta$, for some $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, we have $\Omega^{\mathcal{A}/\theta}(T/\theta) = \Delta^{\mathcal{A}/\theta}$ and, therefore, by definition, $\mathcal{A}/\theta \in \mathrm{AlgSys}^*(\mathcal{I})$, implying that $\theta \in \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. ∎

In general, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ and an $\mathbf{F}$-algebraic system $\mathcal{A}$, the family $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ of $\mathcal{I}^*$-congruence systems on $\mathcal{A}$ need not be closed under signature-wise intersections, i.e., may not form a closure family on $\mathbf{A}^2$. However, we can show that, if $\mathcal{I}$ is protoalgebraic, this is always the case.

Proposition 917 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a protoalgebraic* π*-institution based on* $\mathbf{F}$*. Then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is closed under arbitrary intersections and, therefore, forms a closure family on* $\mathbf{A}^2$.

Proof: First, note that $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ has a top element $\nabla^{\mathcal{A}}$. To see this, observe that $\mathcal{A}/\nabla^{\mathcal{A}}$ is a trivial algebraic system, which is always a member of $\mathrm{AlgSys}^*(\mathcal{I})$.

It suffices now to show that $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is closed under arbitrary intersections. To this end, suppose $\theta^i \in \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$, for $i \in I$. By Proposition 916, for all $i \in I$, there exists $T^i \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T^i) = \theta^i$. But, by Lemma 23 and protoalgebraicity, we get that

$$\Omega^{\mathcal{A}}(\bigcap_{i\in I} T^i) = \bigcap_{i\in I}\Omega^{\mathcal{A}}(T^i) = \bigcap_{i\in I}\theta^i.$$

Now, again by Proposition 916, we conclude that $\bigcap_{i\in I}\theta^i \in \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. ∎

Applying Proposition 917 to the algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$, where $\langle I, \iota\rangle : \mathbf{F} \to \mathbf{F}$ is the identity morphism, we get the following

Corollary 918 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a protoalgebraic* π*-institution based on* $\mathbf{F}$*. Then,* $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{F})$ *is closed under arbitrary intersections and, therefore, forms a closure family on* $\mathbf{F}^2$.

Proof: This is a special case of Proposition 917. ∎

Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a protoalgebraic π-institution. We define, in accordance with Corollary 918, the **algebraic π-structure $\mathcal{Q}^{\mathcal{I}*}$ associated with $\mathcal{I}$** to be the π-structure

$$\mathcal{Q}^{\mathcal{I}*} = \langle \mathbf{F}^2, D^{\mathcal{I}*}\rangle,$$

where $D^{\mathcal{I}*}$ is the closure (operator) family corresponding to the closure family $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{F})$.

Our first result in connecting syntactic WF algebraizability with the associated algebraic π-structure shows that, if a π-institution is syntactically WF algebraizable, then it is equivalent to its associated algebraic π-structure via a conjugate pair of transformations.

Theorem 919 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically WF algebraizable* π*-institution based on* $\mathbf{F}$*. Then* $\mathcal{I}$ *is equivalent to* $\mathcal{Q}^{\mathcal{I}*}$ *via a conjugate pair* $(\tau^\flat, \overleftrightarrow{I^\flat}) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$ *of transformations. More precisely:*

- $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, with two distinguished arguments, is a set of witnessing transformations of the syntactic protoalgebraicity of* $\mathcal{I}$*;*
- $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$*, with a single distinguished argument, is a set of witnessing equations for the family truth equationality of* $\mathcal{I}$*.*

Proof: Suppose that $\mathcal{I}$ is syntactically WF algebraizable. Then, by definition, $\mathcal{I}$ is syntactically protoalgebraic and family truth equational. Therefore, there exist a set $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ of natural transformations in $N^\flat$, with two distinguished arguments, witnessing the syntactic protoalgebraicity of $\mathcal{I}$, and a set $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ of natural transformations in $N^\flat$, with a single distinguished argument, witnessing family truth equationality. To verify the conclusion, observe, first, that $\tau^\flat_\Sigma : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SenFam}(\mathbf{F}^2)$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, as the sentence family $\tau^\flat_\Sigma[\phi]$ and $\overleftrightarrow{I^\flat}_\Sigma : \mathrm{SEN}^\flat(\Sigma)^2 \to \mathrm{SenFam}(\mathbf{F})$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, as the sentence family $\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi]$ are as required. Therefore, by Proposition 898, it suffices to show that:

(a) For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,
$$\phi \in C_\Sigma(\Phi) \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq D^{\mathcal{I}*}(\tau^\flat_\Sigma[\Phi]);$$

(b) For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$D^{\mathcal{I}*}(\phi \approx \psi) = D^{\mathcal{I}*}(\tau^\flat[\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi]]).$$

For (a), let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. Note that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, we have, by family truth equationality,
$$\begin{array}{rcl}\Phi \subseteq T_\Sigma & \text{iff} & \tau^\flat_\Sigma[\Phi] \leq \Omega(T);\\ \phi \in T_\Sigma & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(T).\end{array}$$
Therefore, $\phi \in C_\Sigma(\Phi)$ if and only if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Phi \subseteq T_\Sigma$ implies $\phi \in T_\Sigma$, if and only if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\tau^\flat_\Sigma[\Phi] \leq \Omega(T)$ implies $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$, if and only if, by Proposition 916, $\tau^\flat_\Sigma[\phi] \leq D^{\mathcal{I}*}(\tau^\flat_\Sigma[\Phi])$.

For (b), let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have, for all $T \in \mathrm{ThFam}(\mathcal{I})$,
$$\begin{array}{rcll}\phi \approx \psi \in \Omega_\Sigma(T) & \text{iff} & \overleftrightarrow{I^\flat}_\Sigma[\phi, \psi] \leq T & \text{(Corollary 791)}\\ & \text{iff} & \tau^\flat[\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi]] \leq \Omega(T). & \text{(truth equationality)}\end{array}$$

Using again Proposition 916, we conclude that

$$D^{\mathcal{I}*}(\phi \approx \psi) = D^{\mathcal{I}*}(\tau^\flat[\overset{\leftrightarrow}{I^\flat}_\Sigma[\phi,\psi]]).$$

Therefore $\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via $(\tau^\flat, \overset{\leftrightarrow}{I^\flat}) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$. ■

Putting together Theorems 909, 911 and 919, we get the following fundamental result to the effect that syntactic WF algebraizability boils down to the equivalence of a π-institution with its associated algebraic π-structure via a conjugate pair of transformations.

Theorem 920 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically WF algebraizable if and only if it is equivalent to* $\mathcal{Q}^{\mathcal{I}*}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$ *of transformations.*

Proof: If $\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair of transformations, then, by Theorem 909, it is syntactically protoalgebraic and, by Theorem 911, it is family truth equational. Therefore, by definition, it is syntactically WF algebraizable.

If, conversely, $\mathcal{I}$ is syntactically WF algebraizable, then, by Theorem 919, it is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair of transformations. ■

We close the section by slightly generalizing the preceding characterization. Namely, we show that existence of an equivalence with an algebraic π-structure induced by conjugate transformations is sufficient to yield syntactic WF algebraizability.

Theorem 921 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically WF algebraizable if and only if it is equivalent to an algebraic* π*-structure via a conjugate pair of transformations.*

Proof: If $\mathcal{I}$ is syntactically WF algebraizable, then the conclusion follows from Theorem 920. Conversely, if $\mathcal{I}$ is equivalent to an algebraic π-structure via a conjugate pair of transformations, then it is syntactically protoalgebraic by Theorem 909 and family truth equational by Theorem 911, whence it is syntactically WF algebraizable. ■

Taking into account Theorem 901, we have the following alternative characterization of syntactically WF algebraizable π-institutions:

Theorem 922 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically WF algebraizable if and only if there is a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{I}) \to \mathbf{ThFam}(\mathcal{Q})$*, where* $\mathcal{Q}$ *is an algebraic* π*-structure.*

Proof: The "only if" follows by Theorem 921 and Theorem 893. The "if" is given by Theorem 901 and Theorem 921. ■

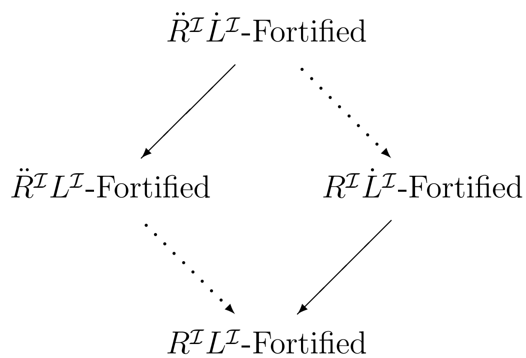
12.4 Syntactic Weak Algebraizability

Syntactic WF algebraizability determines one of the highest levels of the main algebraic hierarchy of π-institutions. Since every syntactically WF algebraizable π-institution is, in particular, family reflective, it follows that every syntactically WF algebraizable π-institution is systemic. To avoid systemicity, one has to weaken the hypothesis of family reflectivity. In this section we follow this line of thought by keeping the assumption of syntactic protoalgebraicity, but insisting only that the π-institution is system truth equational, rather than family truth equational.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that:

- $\mathcal{I}$ is $R^\mathcal{I} L^\mathcal{I}$-(**syntactically**) **fortified** if $R^\mathcal{I}$ is Leibniz and $L^\mathcal{I}$ is left adequate;
- $\mathcal{I}$ is $R^\mathcal{I} \dot{L}^\mathcal{I}$-(**syntactically**) **fortified** if $R^\mathcal{I}$ is Leibniz and $\dot{L}^\mathcal{I}$ is left adequate;
- $\mathcal{I}$ is $\ddot{R}^\mathcal{I} L^\mathcal{I}$-(**syntactically**) **fortified** if $\ddot{R}^\mathcal{I}$ is Leibniz and $L^\mathcal{I}$ is left adequate;
- $\mathcal{I}$ is $\ddot{R}^\mathcal{I} \dot{L}^\mathcal{I}$-(**syntactically**) **fortified** if $\ddot{R}^\mathcal{I}$ is Leibniz and $\dot{L}^\mathcal{I}$ is left adequate.

Similarly with the Suszko core, it can be seen that, if $\dot{L}^\mathcal{I}$ is left adequate, then $L^\mathcal{I}$ is left adequate. Moreover, since, by Proposition 952, $\ddot{R}^\mathcal{I} \subseteq R^\mathcal{I}$, it follows that, under the assumption of prealgebraicity, if $\ddot{R}^\mathcal{I}$ is Leibniz, then $R^\mathcal{I}$ is Leibniz. Thus, we have the following **syntactic left fortification hierarchy** (in which the dotted arrows hold under prealgebraicity):



$\mathcal{I}$ is **syntactically weakly algebraizable** (abbreviated to **syntactically W algebraizable**) if:

- $\mathcal{I}$ is $R^\mathcal{I} L^\mathcal{I}$-fortified;
- $\mathcal{I}$ is protoalgebraic;

- $\mathcal{I}$ is system injective.

By Corollary 300, under protoalgebraicity, the six properties of system injectivity, left injectivity, system reflectivity, left reflectivity, system complete reflectivity and left complete reflectivity coincide. This enables us to formulate the following alternative characterization of syntactic weak algebraizability.

Theorem 923 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically weakly algebraizable if and only if it is syntactically protoalgebraic and system (or, equivalently, left) truth equational.*

Proof: Assume that $\mathcal{I}$ is syntactically weakly algebraizable. Then, on the one hand, it is protoalgebraic and has a Leibniz reflexive core. Thus, by Theorem 805, it is syntactically protoalgebraic. On the other, it is, by Theorem 300, left c-reflective and has a left adequate left Suszko core. Therefore, by Theorem **??**, it is left truth equational.

Assume, conversely, that $\mathcal{I}$ is syntactically protoalgebraic and left truth equational. Then, by Theorem 805, it is protoalgebraic and has a Leibniz reflexive core, and, by Theorem 870, it is left c-reflective and has a left adequate left Suszko core. Therefore, by definition, $\mathcal{I}$ is syntactically weakly algebraizable. ■

Directly from the definitions, we may derive the following relationship between the semantic and syntactic weak algebraizability classes of π-institutions.

Theorem 924 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically weakly algebraizable if and only if* $\mathcal{I}$ *is weakly algebraizable and* $R^{\mathcal{I}} L^{\mathcal{I}}$*-fortified.*

Proof: $\mathcal{I}$ is syntactically weakly algebraizable if and only if, by definition, it is $R^{\mathcal{I}} L^{\mathcal{I}}$-fortified, protoalgebraic and system injective, i.e., iff it is, by definition, $R^{\mathcal{I}} L^{\mathcal{I}}$-fortified and weakly algebraizable. ■

Previous results, put together, also allow us to provide an alternative characterization of syntactic weak algebraizability in terms of isomorphisms between complete lattices of theory systems.

Theorem 925 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically weakly algebraizable if and only if it is* $R^{\mathcal{I}} L^{\mathcal{I}}$*-fortified, stable and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism.

Proof: We have that $\mathcal{I}$ is syntactically weakly algebraizable if and only if, by Theorem 924, it is $R^{\mathcal{I}}L^{\mathcal{I}}$-fortified and weakly algebraizable, if and only if, by Theorem 298, it it $R^{\mathcal{I}}L^{\mathcal{I}}$-fortified, stable and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism. ■

Next, we show that syntactic weak algebraizability may also be characterized by stability in conjunction with the existence of an equivalence between the systemic skeleton of a π-institution and its algebraic π-structure counterpart via a pair of conjugate transformations. To start, we define the *systemic skeleton* of a given π-institution.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall that $\mathrm{ThSys}(\mathcal{I})$ forms a complete lattice $\mathbf{ThSys}(\mathcal{I}) = \langle \mathrm{ThSys}(\mathcal{I}), \leq \rangle$ under signature wise inclusion. Therefore, we are justified in defining the π-structure

$$\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}} \rangle$$

of $\mathcal{I}$ by stipulating that $K^{\mathcal{I}} : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}$ is the closure family on $\mathbf{F}$ corresponding to the closed set family $\mathrm{ThSys}(\mathcal{I})$. We call $\mathcal{K}^{\mathcal{I}}$ the **systemic skeleton** of $\mathcal{I}$.

We give an example to show that, in general, $K^{\mathcal{I}}$ is not a π-institution, since $K^{\mathcal{I}} : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}$ may not satisfy structurality.

Example 926 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be defined as follows:*

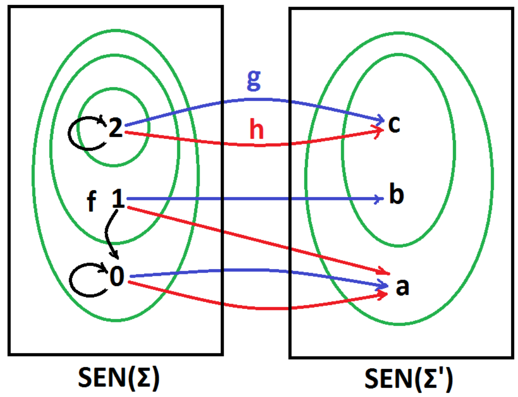
- $\mathbf{Sign}^\flat$ *is the category with objects* Σ, Σ' *and, except the identities, a morphism* $f : \Sigma \to \Sigma$ *and two morphisms* $g, h : \Sigma \to \Sigma'$*, satisfying the following composition rules:*

$$f \circ f = f, \quad gf = h, \quad hf = h.$$

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by setting* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*,* $\mathrm{SEN}^\flat(\Sigma') = \{a, b, c\}$ *and*

$x \in \mathrm{SEN}^\flat(\Sigma)$	$\mathrm{SEN}^\flat(f)(x)$	$\mathrm{SEN}^\flat(g)(x)$	$\mathrm{SEN}^\flat(h)(x)$
0	0	a	a
1	0	b	a
2	2	c	c

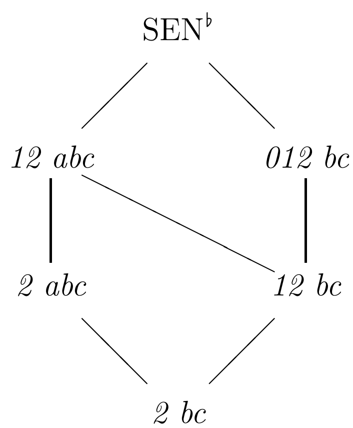
- *Finally,* $N^\flat$ *is the trivial category of natural transformations (consisting of the projections only).*

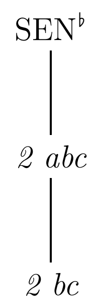


Next define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by setting

$$\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{b,c\},\{a,b,c\}\}.$$

This π-institution has six theory families, having the lattice structure shown on the left below. It has, however, only three theory systems, whose lattice structure is given on the right.





The theory systems of $\mathcal{I}$ are the theory families of the systemic skeleton $\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}} \rangle$. We can see that $\mathcal{K}^{\mathcal{I}}$ is not a π-institution by considering $\Phi = \{1\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. We have

$$\begin{array}{rcl} \mathrm{SEN}^\flat(g)(K_\Sigma^{\mathcal{I}}(\{1\})) & = & \mathrm{SEN}^\flat(g)(\bigcap\{T_\Sigma : \{\{1\},\varnothing\} \leq T \in \mathrm{ThSys}(\mathcal{I})\}) \\ & = & \mathrm{SEN}^\flat(g)(\{0,1,2\}) \\ & = & \{a,b,c\}. \end{array}$$

On the other hand,

$$\begin{array}{rcl} K_{\Sigma'}^{\mathcal{I}}(\mathrm{SEN}^\flat(g)(\{1\})) & = & K_{\Sigma'}^{\mathcal{I}}(\{b\}) \\ & = & \bigcap\{T_{\Sigma'} : \{\varnothing,\{b\}\} \leq T \in \mathrm{ThSys}(\mathcal{I})\} \\ & = & \{b,c\}. \end{array}$$

Therefore

$$\mathrm{SEN}^\flat(g)(K^{\mathcal{I}}_\Sigma(\{1\})) \notin K^{\mathcal{I}}_{\Sigma'}(\mathrm{SEN}^\flat(g)(\{1\}))$$

showing that $K^{\mathcal{I}}$ is not structural and, hence, $\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}}\rangle$ is a π-structure, but not a π-institution.

We now resume our work on the characterization of syntactic weak algebraizability. We will again make use of the algebraic π-structure $\mathcal{Q}^{\mathcal{I}*} = \langle \mathbf{F}^2, D^{\mathcal{I}*}\rangle$ associated with a protoalgebraic π-institution $\mathcal{I}$. Recall that this is the π-structure whose closure family is the one corresponding to the closure set family $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{F})$.

Our first result connecting syntactic weak algebraizability of a π-institution with the associated algebraic π-structure shows that, if a π-institution is syntactically weakly algebraizable, then its systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to its associated algebraic π-structure $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair of transformations.

Theorem 927 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a syntactically weakly algebraizable π-institution based on $\mathbf{F}$. Then $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair $(\tau^\flat, \overleftrightarrow{I}^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$ of transformations. More precisely:*

- *$I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with two distinguished arguments, is a set of witnessing transformations of the syntactic protoalgebraicity of $\mathcal{I}$;*
- *$\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$, with a single distinguished argument, is a set of witnessing equations for the left truth equationality of $\mathcal{I}$.*

Proof: Suppose that $\mathcal{I}$ is syntactically weakly algebraizable. Then, by definition, $\mathcal{I}$ is syntactically protoalgebraic and left truth equational. Therefore, there exist a set $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ of natural transformations in $N^\flat$, with two distinguished arguments, witnessing the syntactic protoalgebraicity of $\mathcal{I}$, and a set $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ of natural transformations in $N^\flat$, with a single distinguished argument, witnessing left truth equationality. To verify the conclusion, observe, first, that $\tau^\flat_\Sigma : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SenFam}(\mathbf{F}^2)$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, as the sentence family $\tau^\flat_\Sigma[\phi]$ and $\overleftrightarrow{I}^\flat_\Sigma : \mathrm{SEN}^\flat(\Sigma)^2 \to \mathrm{SenFam}(\mathbf{F})$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, as the sentence family $\overleftrightarrow{I}^\flat_\Sigma[\phi, \psi]$ are as required. Therefore, by Proposition 898, it suffices to show that:

(a) For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in K^{\mathcal{I}}_\Sigma(\Phi) \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq D^{\mathcal{I}*}(\tau^\flat_\Sigma[\Phi]);$$

(b) For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$D^{\mathcal{I}*}(\phi \approx \psi) = D^{\mathcal{I}*}(\tau^\flat[\overleftrightarrow{I^\flat}_\Sigma[\phi,\psi]]).$$

For (a), let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. Note that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, we have

$$\begin{array}{lll}\Phi \subseteq T_\Sigma & \text{iff} & \Phi \subseteq \overleftarrow{T}_\Sigma \quad (T \in \mathrm{ThSys}(\mathcal{I}))\\ & \text{iff} & \tau^\flat_\Sigma[\Phi] \leq \Omega(T) \quad \text{(left truth equationality)}\end{array}$$

and, similarly,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

Therefore, $\phi \in K^{\mathcal{I}}_\Sigma(\Phi)$ if and only if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\Phi \subseteq T_\Sigma$ implies $\phi \in T_\Sigma$, if and only if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\tau^\flat_\Sigma[\Phi] \leq \Omega(T)$ implies $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$, if and only if, by stability, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\tau^\flat_\Sigma[\Phi] \leq \Omega(T)$ implies $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$, if and only if, by Proposition 916, $\tau^\flat_\Sigma[\phi] \leq D^{\mathcal{I}*}(\tau^\flat_\Sigma[\Phi])$.

For (b), let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have, for all $T \in \mathrm{ThSys}(\mathcal{I})$,

$$\begin{array}{lll}\phi \approx \psi \in \Omega_\Sigma(T) & \text{iff} & \overleftrightarrow{I^\flat}_\Sigma[\phi,\psi] \leq T \quad \text{(Corollary 791)}\\ & \text{iff} & \overleftrightarrow{I^\flat}_\Sigma[\phi,\psi] \leq \overleftarrow{T} \quad (T \in \mathrm{ThSys}(\mathcal{I}))\\ & \text{iff} & \tau^\flat[\overleftrightarrow{I^\flat}_\Sigma[\phi,\psi]] \leq \Omega(T). \quad \text{(left truth equationality)}\end{array}$$

Using again Proposition 916 and stability, we conclude that

$$D^{\mathcal{I}*}(\phi \approx \psi) = D^{\mathcal{I}*}(\tau^\flat[\overleftrightarrow{I^\flat}_\Sigma[\phi,\psi]]).$$

Therefore $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via $(\tau^\flat,\overleftrightarrow{I^\flat}) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$. ■

Towards the converse, we show, first, that, if a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is such that there exists an equivalence $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}$, via a conjugate pair of transformations, between its systemic skeleton and an algebraic π-structure $\mathcal{Q}$, then $I^\flat$ defines Leibniz congruence systems of theory systems of $\mathcal{I}$.

Recall that for a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$, based on an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$, and a set $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ of natural transformations in $N^\flat$, with two distinguished arguments, we define, for all $T \in \mathrm{SenFam}(\mathcal{I})$, $I^\flat(T) = \{I^\flat_\Sigma(T)\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$I^\flat_\Sigma(T) = \{\langle \phi,\psi\rangle \in \mathrm{SEN}^\flat(\Sigma)^2 : I^\flat_\Sigma[\phi,\psi] \leq T\}.$$

Proposition 928 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a π-institution based on* $\mathbf{F}$. *If* $\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}}\rangle$ *is equivalent to an algebraic π-structure* $\mathcal{Q}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}$ *of transformations, then, for all* $T \in \mathrm{ThSys}(\mathcal{I})$, $\Omega(T) = I^\flat(T)$.

Proof: Let $T \in \mathrm{ThSys}(\mathcal{I})$. It suffices to show, by Corollary 98, that $I^\flat(T)$ is a congruence system on $\mathbf{F}$ compatible with T. We know by Lemma 93 that it is a relation system on $\mathbf{F}$.

Suppose $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Since $\mathcal{Q} = \langle \mathbf{F}^2, D\rangle$ is algebraic, we have $\phi \approx \phi \in D_\Sigma(\varnothing)$. Therefore, by interpretability, $I^\flat_\Sigma[\phi,\phi] \leq K^{\mathcal{I}}(\varnothing) = C(\varnothing) \leq T$. Hence, $\langle \phi,\phi\rangle \in I^\flat_\Sigma(T)$ and $I^\flat(T)$ is reflexive.

Suppose, now, that $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$. Since $\mathcal{Q}$ is algebraic, we have that $\psi \approx \phi \in D_\Sigma(\phi \approx \psi)$. Therefore, by interpretability, $I^\flat_\Sigma[\psi,\phi] \leq K^{\mathcal{I}}(I^\flat_\Sigma[\phi,\psi])$. Since $T \in \mathrm{ThSys}(\mathcal{I})$, this implies that, if $I^\flat_\Sigma[\phi,\psi] \leq T$, then $I^\flat_\Sigma[\psi,\phi] \leq T$. In other words $\langle \phi,\psi\rangle \in I^\flat_\Sigma(T)$ implies $\langle \psi,\phi\rangle \in I^\flat_\Sigma(T)$. Therefore, $I^\flat(T)$ is also symmetric.

Suppose, next, that $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi,\chi \in \mathrm{SEN}^\flat(\Sigma)$. Since $\mathcal{Q}$ is algebraic, we have that $\phi \approx \chi \in D_\Sigma(\phi \approx \psi, \psi \approx \chi)$. Therefore, by interpretability, $I^\flat_\Sigma[\phi,\chi] \leq K^{\mathcal{I}}(I^\flat_\Sigma[\phi,\psi], I^\flat_\Sigma[\psi,\chi])$. Since $T \in \mathrm{ThSys}(\mathcal{I})$, this implies that, if $I^\flat_\Sigma[\phi,\psi], I^\flat_\Sigma[\psi,\chi] \leq T$, then $I^\flat_\Sigma[\phi,\chi] \leq T$. In other words, $\langle \phi,\psi\rangle, \langle \psi,\chi\rangle \in I^\flat_\Sigma(T)$ imply $\langle \phi,\chi\rangle \in I^\flat_\Sigma(T)$. Therefore, $I^\flat(T)$ is transitive.

We have now shown that $I^\flat(T)$ is an equivalence system on $\mathbf{F}$. It remains to show that it satisfies the congruence property and that it is compatible with T.

Suppose that $\sigma^\flat \in N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi},\vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$. Since $\mathcal{Q}$ is algebraic, we have that $\sigma^\flat_\Sigma(\vec{\phi}) \approx \sigma^\flat_\Sigma(\vec{\psi}) \in D_\Sigma(\vec{\phi} \approx \vec{\psi})$ (recall that $\vec{\phi} \approx \vec{\psi}$ means $\{\phi_i \approx \psi_i : i < k\}$). Therefore, by interpretability,

$$I^\flat_\Sigma[\sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi})] \leq K^{\mathcal{I}}(\bigcup\{I^\flat_\Sigma[\phi_i,\psi_i] : i < k\}).$$

Since $T \in \mathrm{ThSys}(\mathcal{I})$, this implies that, if, for all $i < k$, $I^\flat_\Sigma[\phi_i,\psi_i] \leq T$, then $I^\flat_\Sigma[\sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi})] \leq T$. In other words $\langle \phi_i,\psi_i\rangle \in I^\flat_\Sigma(T)$, for all $i < k$, imply $\langle \sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi})\rangle \in I^\flat_\Sigma(T)$. Therefore, $I^\flat(T)$ satisfies the congruence property.

Finally, to see that $I^\flat(T)$ is compatible with T, suppose that $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$. Since $\mathcal{Q}$ is algebraic and $\tau^\flat \in N^\flat$, we have, by Lemma 907,

$$\tau^\flat_\Sigma[\psi] \leq D(\tau^\flat_\Sigma[\phi], \phi \approx \psi).$$

By interpretability, this yields

$$I^\flat[\tau^\flat_\Sigma[\psi]] \leq K^{\mathcal{I}}(I^\flat[\tau^\flat_\Sigma[\phi]], I^\flat_\Sigma[\phi,\psi]).$$

Since $(\tau^\flat, I^\flat)$ is a conjugate pair, the latter is equivalent to

$$\psi \in K^{\mathcal{I}}_\Sigma(\phi, I^\flat_\Sigma[\phi,\psi]).$$

In other words, for all $T \in \mathrm{ThSys}(\mathcal{I})$,

$$\phi \in T_\Sigma \quad \text{and} \quad \langle \phi,\psi\rangle \in I^\flat_\Sigma(T) \quad \text{imply} \quad \psi \in T_\Sigma.$$

Hence $I^\flat(T)$ is compatible with T. ∎

Using Proposition 928, we can show that stability and the existence of an equivalence between the systemic skeleton and an algebraic π-structure ensure syntactic protoalgebraicity.

Theorem 929 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is stable and its systemic skeleton* $\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}} \rangle$ *is equivalent to an algebraic* π*-structure* $\mathcal{Q}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}$ *of transformations, then* $\mathcal{I}$ *is syntactically protoalgebraic, with witnessing transformations* $I^\flat$*.*

Proof: Suppose that $\mathcal{I}$ is stable and its systemic skeleton $\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}} \rangle$ is equivalent to an algebraic π-structure $\mathcal{Q}$ via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}$ of transformations. Then, we have, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\begin{array}{rcll} \Omega(T) & = & \Omega(\overleftarrow{T}) & \text{(by stability)} \\ & = & I^\flat(\overleftarrow{T}) & \text{(by Proposition 928)} \\ & = & I^\flat(T). & \text{(by Proposition 99)} \end{array}$$

Therefore, $\mathcal{I}$ is syntactically protoalgebraic with witnessing transformations $I^\flat$. ■

Finally, before the main theorem, we show that stability and the existence of a transformational equivalence between the systemic skeleton and an algebraic π-structure ensure left truth equationality.

Theorem 930 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is stable and its systemic skeleton* $\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}} \rangle$ *is equivalent to an algebraic* π*-structure* $\mathcal{Q}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}$ *of transformations, then* $\mathcal{I}$ *is left truth equational, with witnessing equations* $\tau^\flat$*.*

Proof: We have, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rll} \phi \in \overleftarrow{T}_\Sigma & \text{iff} & I^\flat[\tau^\flat_\Sigma[\phi]] \leq \overleftarrow{T} \quad ((\tau^\flat, I^\flat) \text{ an equivalence}) \\ & \text{iff} & I^\flat[\tau^\flat_\Sigma[\phi]] \leq T \quad \text{(by Proposition 99)} \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(T). \quad \text{(by Theorem 929)} \end{array}$$

Therefore, $\mathcal{I}$ is left truth equational, with witnessing equations $\tau^\flat$. ■

Putting together Theorems 929, 930 and 927, we get the following fundamental result to the effect that syntactic weak algebraizability boils down to stability, together with the equivalence of the systemic skeleton of a π-institution with its associated algebraic π-structure via a conjugate pair of transformations.

Theorem 931 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically weakly algebraizable if and only if it is stable and its systemic skeleton* $\mathcal{K}^{\mathcal{I}}$ *is equivalent to* $\mathcal{Q}^{\mathcal{I}*}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$ *of transformations.*

Proof: Suppose, first, that $\mathcal{I}$ is stable and that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair of transformations. Then, by Theorem 929, it is syntactically protoalgebraic and, by Theorem 930, it is left truth equational. Therefore, by definition, it is syntactically weakly algebraizable.

If, conversely, $\mathcal{I}$ is syntactically weakly algebraizable, then, on the one hand, it is protoalgebraic and, therefore, stable, and, on the other, by Theorem 927, it is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair of transformations. ■

Generalizing again, we show that stability together with the existence of an equivalence of the systemic skeleton with an algebraic π-structure, induced by conjugate transformations, is sufficient to yield syntactic weak algebraizability.

Theorem 932 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically weakly algebraizable if and only if it is stable and its systemic skeleton is equivalent to an algebraic* π*-structure via a conjugate pair of transformations.*

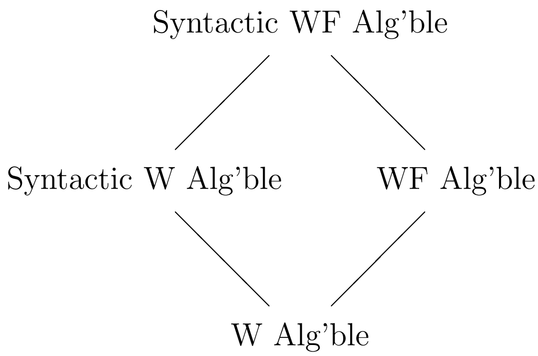
Proof: If $\mathcal{I}$ is syntactically weakly algebraizable, then the conclusion follows from Theorem 931. Conversely, if $\mathcal{K}^{\mathcal{I}}$ is equivalent to an algebraic π-structure via a conjugate pair of transformations, then $\mathcal{I}$ is syntactically protoalgebraic by Theorem 929 and left truth equational by Theorem 930, whence it is syntactically weakly algebraizable. ■

Finally, in terms of order isomorphisms between theory family lattices, we have the following alternative characterization of syntactically weakly algebraizable π-institutions:

Theorem 933 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically weakly algebraizable if and only if it is stable and there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{K}^{\mathcal{I}}) \to \mathbf{ThFam}(\mathcal{Q})$*, where* $\mathcal{Q}$ *is an algebraic* π*-structure.*

Proof: The "only if" follows by Theorem 932 and Theorem 893. The "if" is given by Theorem 901 and Theorem 932. ■

Let us give, in closing the section, the picture of the **weak algebraizability hierarchy** that we have established, consisting of both semantic and syntactic classes of π-institutions.



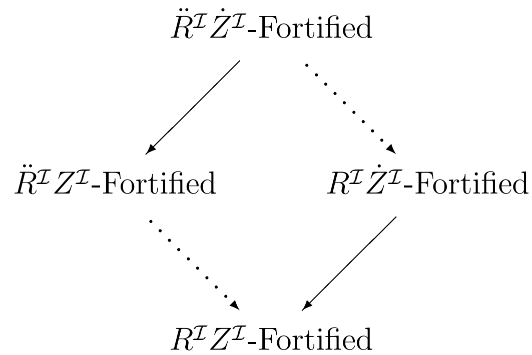
12.5 Syntactic WS PreAlgebraizability

Syntactic WS prealgebraizability, requires, like syntactic WF algebraizability, the monotonicity of the Leibniz operator on theory systems and the injectivity of the Leibniz operator on theory systems but, unlike WF algebraizability, it requires these two properties only on theory systems and not on the entire complete lattice of theory families. As a consequence of this weakened requirement, syntactic WS prealgebraizability implies neither systemicity (as does syntactic WF algebraizability) nor the even weaker condition of stability (as do both kinds of syntactic algebraizability). Thus, as other conditions that were under our scrutiny previously, it allows us to consider for membership π-institutions that are not necessarily stable.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that:

- $\mathcal{I}$ is $R^{\mathcal{I}} Z^{\mathcal{I}}$-(**syntactically**) **fortified** if $R^{\mathcal{I}}$ is Leibniz and $Z^{\mathcal{I}}$ is adequate;
- $\mathcal{I}$ is $R^{\mathcal{I}} \dot{Z}^{\mathcal{I}}$-(**syntactically**) **fortified** if $R^{\mathcal{I}}$ is Leibniz and $\dot{Z}^{\mathcal{I}}$ is adequate;
- $\mathcal{I}$ is $\ddot{R}^{\mathcal{I}} Z^{\mathcal{I}}$-(**syntactically**) **fortified** if $\ddot{R}^{\mathcal{I}}$ is Leibniz and $Z^{\mathcal{I}}$ is adequate;
- $\mathcal{I}$ is $\ddot{R}^{\mathcal{I}} \dot{Z}^{\mathcal{I}}$-(**syntactically**) **fortified** if $\ddot{R}^{\mathcal{I}}$ is Leibniz and $\dot{Z}^{\mathcal{I}}$ is adequate.

Similarly with the Suszko core, it can be seen that, if $\dot{Z}^{\mathcal{I}}$ is adequate, then $Z^{\mathcal{I}}$ is adequate. Moreover, since, by Proposition 952, $\ddot{R}^{\mathcal{I}} \subseteq R^{\mathcal{I}}$, it follows that, under the assumption of prealgebraicity, if $\ddot{R}^{\mathcal{I}}$ is Leibniz, then $R^{\mathcal{I}}$ is Leibniz. Thus, we have the following **syntactic system fortification hierarchy** (in which the dotted arrows hold under prealgebraicity):



$\mathcal{I}$ is **syntactically weakly system prealgebraizable** (abbreviated to **syntactically WS prealgebraizable**) if:

- $\mathcal{I}$ is $R^{\mathcal{I}}Z^{\mathcal{I}}$-fortified;
- $\mathcal{I}$ is prealgebraic;
- $\mathcal{I}$ is system injective.

By Theorem 248, under prealgebraicity, the properties of system injectivity, system reflectivity and system complete reflectivity coincide. As a result, we have the following alternative characterization of syntactic weak system prealgebraizability.

Theorem 934 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically weakly system prealgebraizable if and only if it is syntactically prealgebraic and system truth equational.*

Proof: Assume that $\mathcal{I}$ is syntactically weakly system prealgebraizable. Then, on the one hand, it is prealgebraic and has a Leibniz reflexive core. Thus, by Theorem 788, it is syntactically prealgebraic. On the other, it is, by Theorem 248, system c-reflective and has an adequate system core. Therefore, by Theorem 887, it is system truth equational.

Assume, conversely, that $\mathcal{I}$ is syntactically prealgebraic and system truth equational. Then, by Theorem 788, it is prealgebraic and has a Leibniz reflexive core, and, by Theorem 887, it is system c-reflective and has an adequate system core. Therefore, by definition, $\mathcal{I}$ is syntactically weakly system prealgebraizable. ■

Directly from the definitions, we may derive the following relationship between the semantic and syntactic weak system prealgebraizability classes of π-institutions.

Theorem 935 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically weakly system prealgebraizable if and only if* $\mathcal{I}$ *is weakly system prealgebraizable and* $R^{\mathcal{I}} Z^{\mathcal{I}}$*-fortified.*

Proof: $\mathcal{I}$ is syntactically weakly system prealgebraizable if and only if, by definition, it is $R^{\mathcal{I}} Z^{\mathcal{I}}$-fortified, prealgebraic and system injective, i.e., iff it is, by definition, $R^{\mathcal{I}} Z^{\mathcal{I}}$-fortified and weakly system prealgebraizable. ∎

Previous results, put together, also allow us to provide an alternative characterization of syntactic weak system prealgebraizability in terms of morphisms between complete lattices of theory systems.

Theorem 936 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically weakly system prealgebraizable if and only if it is* $R^{\mathcal{I}} Z^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order embedding.

Proof: We have that $\mathcal{I}$ is syntactically weakly system prealgebraizable if and only if, by Theorem 935, it is $R^{\mathcal{I}} Z^{\mathcal{I}}$-fortified and weakly system prealgebraizable, if and only if, by Theorem 256, it it $R^{\mathcal{I}} Z^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order embedding ∎

Next, we show that syntactic weak system prealgebraizability may also be characterized by the existence of an equivalence between the systemic skeleton of a π-institution and an algebraic π-structure associated with the π-institution (different, in general, than $\mathcal{Q}^{\mathcal{I}*}$) via a pair of conjugate transformations.

We embark on the path by defining first the algebraic π-structure $\mathcal{Q}^{\mathcal{I}\bullet}$ associated with a given π-institution $\mathcal{I}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall the definition of the class $\mathrm{AlgSys}^\bullet(\mathcal{I})$ of all $\mathbf{F}$-algebraic systems reduced with respect to $\mathcal{I}$-filter systems:

$$\mathrm{AlgSys}^\bullet(\mathcal{I}) = \{\mathcal{A} : (\exists T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}))(\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}})\}.$$

Given an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, we define the class of $\mathcal{I}^\bullet$-**congruence systems on** $\mathcal{A}$ by

$$\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A}) = \{\theta \in \mathrm{ConSys}(\mathbf{A}) : \mathcal{A}/\theta \in \mathrm{AlgSys}^\bullet(\mathcal{I})\}.$$

It turns out that congruence systems in $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$ have a straightforward characterization.

Proposition 937 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. Then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A}) = \{\theta \in \mathrm{ConSys}(\mathbf{A}) : (\exists T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}))(\Omega^{\mathcal{A}}(T) = \theta)\}.$$

Proof: Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system.

Suppose, first, that $\theta \in \mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$. By definition, $\mathcal{A}/\theta \in \mathrm{AlgSys}^\bullet(\mathcal{I})$. Thus, there exists $T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}/\theta)$, such that

$$\Omega^{\mathcal{A}/\theta}(T') = \Delta^{\mathcal{A}/\theta}.$$

By applying the inverse of the quotient morphism $\langle I, \pi^\theta \rangle : \mathcal{A} \to \mathcal{A}/\theta$, we get

$$(\pi^\theta)^{-1}(\Omega^{\mathcal{A}/\theta}(T')) = (\pi^\theta)^{-1}(\Delta^{\mathcal{A}/\theta}).$$

Since $\langle I, \pi^\theta \rangle$ is surjective, we get by Proposition 24 and Corollary 55, that $(\pi^\theta)^{-1}(T') \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ and

$$\Omega^{\mathcal{A}}((\pi^\theta)^{-1}(T')) = \theta.$$

Therefore, there exists $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \theta$.

Suppose, conversely, that $\theta \in \mathrm{ConSys}(\mathbf{A})$, with $\Omega^{\mathcal{A}}(T) = \theta$, for some $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$. Then, we have $\Omega^{\mathcal{A}/\theta}(T/\theta) = \Delta^{\mathcal{A}/\theta}$ and, therefore, by definition, $\mathcal{A}/\theta \in \mathrm{AlgSys}^\bullet(\mathcal{I})$, implying that $\theta \in \mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$. ■

In general, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ and an $\mathbf{F}$-algebraic system $\mathcal{A}$, the family $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$ of systemic $\mathcal{I}$-congruence systems on $\mathcal{A}$ need not be closed under signature-wise intersections, i.e., may not form a closure family on $\mathbf{A}^2$. However, we can show that, if $\mathcal{I}$ is prealgebraic, this is always the case.

Proposition 938 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a prealgebraic* π*-institution based on* $\mathbf{F}$*. Then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$ *is closed under arbitrary intersections and, therefore, forms a closure family on* $\mathbf{A}^2$*.*

Proof: First, note that $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$ has a top element $\nabla^{\mathcal{A}}$. To see this, observe that $\mathcal{A}/\nabla^{\mathcal{A}}$ is a trivial algebraic system, which is always a member of $\mathrm{AlgSys}^\bullet(\mathcal{I})$.

It suffices now to show that $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$ is closed under arbitrary intersections. To this end, suppose $\theta^i \in \mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$, for $i \in I$. By Proposition 937, for all $i \in I$, there exists $T^i \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}/\theta^i)$, such that $\Omega^{\mathcal{A}}(T^i) = \theta^i$. But, by Lemma 23 and prealgebraicity, we get that

$$\Omega^{\mathcal{A}}(\bigcap_{i \in I} T^i) = \bigcap_{i \in I} \Omega^{\mathcal{A}}(T^i) = \bigcap_{i \in I} \theta^i.$$

Now, again by Proposition 937, we conclude that $\bigcap_{i\in I}\theta^i \in \mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$. ■

Applying Proposition 938 to the algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$, where $\langle I, \iota\rangle : \mathbf{F} \to \mathbf{F}$ is the identity morphism, we get the following

Corollary 939 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a protoalgebraic* π*-institution based on* $\mathbf{F}$*. Then,* $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{F})$ *is closed under arbitrary intersections and, therefore, forms a closure family on* $\mathbf{F}^2$*.*

Proof: This is a special case of Proposition 938. ■

Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a prealgebraic π-institution. We define, in accordance with Corollary 939, the **systemic algebraic π-structure $\mathcal{Q}^{\mathcal{I}\bullet}$ associated with $\mathcal{I}$** to be the π-structure $\mathcal{Q}^{\mathcal{I}\bullet} = \langle \mathbf{F}^2, D^{\mathcal{I}\bullet}\rangle$, where $D^{\mathcal{I}\bullet}$ is the closure (operator) family corresponding to the closure family $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{F})$.

We recall, also, the defining of the systemic skeleton of $\mathcal{I}$, i.e., of the π-structure

$$\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}}\rangle$$

of $\mathcal{I}$, where $K^{\mathcal{I}} : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}$ is the closure family on $\mathbf{F}$ corresponding to the closet set family $\mathrm{ThSys}(\mathcal{I})$.

Now we have the components needed to resume work on the characterization of syntactic weak system prealgebraizability. Our first result connecting syntactic weak system prealgebraizability of a π-institution with the associated systemic algebraic π-structure shows that, if a π-institution is syntactically weakly system prealgebraizable, then its systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to its associated systemic algebraic π-structure $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair of transformations.

Theorem 940 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a syntactically weakly system prealgebraizable* π*-institution based on* $\mathbf{F}$*. Then* $\mathcal{K}^{\mathcal{I}}$ *is equivalent to* $\mathcal{Q}^{\mathcal{I}\bullet}$ *via a conjugate pair* $(\tau^\flat, \overset{\leftrightarrow}{I^\flat}) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}\bullet}$ *of transformations. More precisely:*

- $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, with two distinguished arguments, is a set of witnessing transformations of the syntactic prealgebraicity of* $\mathcal{I}$*;*
- $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$*, with a single distinguished argument, is a set of witnessing equations for the system truth equationality of* $\mathcal{I}$*.*

Proof: Suppose that $\mathcal{I}$ is syntactically weakly system prealgebraizable. Then, by definition, $\mathcal{I}$ is syntactically prealgebraic and system truth equational. Therefore, there exist a set $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ of natural transformations in $N^\flat$, with two distinguished arguments, witnessing the syntactic prealgebraicity of $\mathcal{I}$, and a set $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ of natural transformations in $N^\flat$, with a single distinguished argument, witnessing system truth equationality. To verify the conclusion, observe, first, that

$\tau^\flat_\Sigma : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SenFam}(\mathbf{F}^2)$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, as the sentence family $\tau^\flat_\Sigma[\phi]$, and $\overleftrightarrow{I}^\flat{}_\Sigma : \mathrm{SEN}^\flat(\Sigma)^2 \to \mathrm{SenFam}(\mathbf{F})$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, as the sentence family $\overleftrightarrow{I}^\flat{}_\Sigma[\phi,\psi]$, are as required. Therefore, by Proposition 898, it suffices to show that:

(a) For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in K^{\mathcal{I}}_\Sigma(\Phi) \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq D^{\mathcal{I}\bullet}(\tau^\flat_\Sigma[\Phi]);$$

(b) For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$D^{\mathcal{I}\bullet}(\phi \approx \psi) = D^{\mathcal{I}\bullet}(\tau^\flat[\overleftrightarrow{I}^\flat{}_\Sigma[\phi,\psi]]).$$

For (a), let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. Note that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, we have, by system truth equationality,

$$\begin{array}{rcl} \Phi \subseteq T_\Sigma & \text{iff} & \tau^\flat_\Sigma[\Phi] \leq \Omega(T) \\ \phi \in T_\Sigma & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(T). \end{array}$$

Therefore, $\phi \in K^{\mathcal{I}}_\Sigma(\Phi)$ if and only if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\Phi \subseteq T_\Sigma$ implies $\phi \in T_\Sigma$, if and only if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\tau^\flat_\Sigma[\Phi] \leq \Omega(T)$ implies $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$, if and only if, by Proposition 937, $\tau^\flat_\Sigma[\phi] \leq D^{\mathcal{I}\bullet}(\tau^\flat_\Sigma[\Phi])$.

For (b), let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have, for all $T \in \mathrm{ThSys}(\mathcal{I})$,

$$\begin{array}{rcll} \phi \approx \psi \in \Omega_\Sigma(T) & \text{iff} & \overleftrightarrow{I}^\flat{}_\Sigma[\phi,\psi] \leq T & \text{(Corollary 770)} \\ & \text{iff} & \tau^\flat[\overleftrightarrow{I}^\flat{}_\Sigma[\phi,\psi]] \leq \Omega(T). & \\ & & \text{(system truth equationality)} & \end{array}$$

Using again Proposition 937, we conclude that

$$D^{\mathcal{I}\bullet}(\phi \approx \psi) = D^{\mathcal{I}\bullet}(\tau^\flat[\overleftrightarrow{I}^\flat{}_\Sigma[\phi,\psi]]).$$

Therefore $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via $(\tau^\flat, \overleftrightarrow{I}^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}\bullet}$. ■

We show, next that the existence of an equivalence between the systemic skeleton of a given π-institution and an algebraic π-structure ensures syntactic prealgebraicity.

Theorem 941 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If the systemic skeleton $\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}} \rangle$ of $\mathcal{I}$ is equivalent to an algebraic π-structure $\mathcal{Q}$ via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}$ of transformations, then $\mathcal{I}$ is syntactically prealgebraic, with witnessing transformations $I^\flat$.*

Proof: Suppose that $\mathcal{K}^\mathcal{I} = \langle \mathbf{F}, K^\mathcal{I}\rangle$ is equivalent to an algebraic π-structure $\mathcal{Q}$ via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{K}^\mathcal{I} \rightleftarrows \mathcal{Q}$ of transformations. Then, we have, by Proposition 928, that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\Omega(T) = I^\flat(T)$. Therefore, $\mathcal{I}$ is syntactically prealgebraic with witnessing transformations $I^\flat$. ∎

Finally, as a last step before the main theorem, we show that the existence of a transformational equivalence between the systemic skeleton of a given π-institution and an algebraic π-structure ensures system truth equationality.

Theorem 942 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *If the systemic skeleton* $\mathcal{K}^\mathcal{I} = \langle \mathbf{F}, K^\mathcal{I}\rangle$ *of* $\mathcal{I}$ *is equivalent to an algebraic* π*-structure* $\mathcal{Q}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^\mathcal{I} \rightleftarrows \mathcal{Q}$ *of transformations, then* $\mathcal{I}$ *is system truth equational, with witnessing equations* $\tau^\flat$.

Proof: We have, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}\phi \in T_\Sigma & \text{iff} & I^\flat[\tau^\flat_\Sigma[\phi]] \leq T \quad ((\tau^\flat, I^\flat) \text{ an equivalence})\\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(T). \quad (\text{by Theorem 941})\end{array}$$

Therefore, $\mathcal{I}$ is system truth equational, with witnessing equations $\tau^\flat$. ∎

Putting together Theorems 941, 942 and 940, we get the following fundamental result to the effect that syntactic weak system prealgebraizability boils down to the existence of an equivalence of the systemic skeleton of a π-institution with its associated systemic algebraic π-structure via a conjugate pair of transformations.

Theorem 943 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically weakly system prealgebraizable if and only if its systemic skeleton* $\mathcal{K}^\mathcal{I}$ *is equivalent to* $\mathcal{Q}^{\mathcal{I}\bullet}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^\mathcal{I} \rightleftarrows \mathcal{Q}^{\mathcal{I}\bullet}$ *of transformations.*

Proof: Suppose, first, that $\mathcal{K}^\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair of transformations. Then, by Theorem 941, it is syntactically prealgebraic and, by Theorem 942, it is system truth equational. Therefore, by definition, it is syntactically weakly system prealgebraizable. If, conversely, $\mathcal{I}$ is syntactically weakly system prealgebraizable, then, by Theorem 940, it is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair of transformations. ∎

It turns out that the existence of an equivalence of the systemic skeleton with an algebraic π-structure, induced by conjugate transformations, is sufficient to yield syntactic weak system prealgebraizability.

Theorem 944 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically weakly system prealgebraizable if and only if its systemic skeleton is equivalent to an algebraic* π*-structure via a conjugate pair of transformations.*

Proof: If $\mathcal{I}$ is syntactically weakly system prealgebraizable, then the conclusion follows from Theorem 943. Conversely, if $\mathcal{K}^{\mathcal{I}}$ is equivalent to an algebraic π-structure via a conjugate pair of transformations, then $\mathcal{I}$ is syntactically prealgebraic by Theorem 941 and system truth equational by Theorem 942, whence it is syntactically weakly system prealgebraizable. ∎

Finally, in terms of order isomorphisms between theory family lattices, we have the following alternative characterization of syntactically weakly system prealgebraizable π-institutions:

Theorem 945 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically weakly system prealgebraizable if and only if there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{K}^{\mathcal{I}}) \to \mathbf{ThFam}(\mathcal{Q})$*, where* $\mathcal{Q}$ *is an algebraic* π*-structure.*

Proof: The "only if" follows by Theorem 944 and Theorem 893. The "if" is given by Theorem 901 and Theorem 944. ∎

12.6 Syntactic WLC PreAlgebraizability

Between syntactic WS prealgebraizability and syntactic weak algebraizability we find the class of syntactic weakly left c-reflective prealgebraizability. This strengthens WS prealgebraizability by replacing system c-reflectivity by the stronger condition of left c-reflectivity. Alternatively, it weakens syntactic weak algebraizability by replacing ptotoalgebraicity by prealgebraicity.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is **syntactically weakly left c-reflectively prealgebraizable** (abbreviated to **syntactically WLC prealgebraizable**) if:

- $\mathcal{I}$ is $R^{\mathcal{I}} L^{\mathcal{I}}$-fortified;
- $\mathcal{I}$ is prealgebraic;
- $\mathcal{I}$ is left c-reflective.

We have the following alternative characterization of syntactic WLC prealgebraizability.

Theorem 946 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically WLC prealgebraizable if and only if it is syntactically prealgebraic and left truth equational.*

Proof: Assume that $\mathcal{I}$ is syntactically WLC prealgebraizable. Then, on the one hand, it is prealgebraic and has a Leibniz reflexive core. Thus, by

Theorem 788, it is syntactically prealgebraic. On the other, it is left c-reflective and has a left adequate left Suszko core. Therefore, by Theorem 870, it is left truth equational.

Assume, conversely, that $\mathcal{I}$ is syntactically prealgebraic and left truth equational. Then, by Theorem 788, it is prealgebraic and has a Leibniz reflexive core, and, by Theorem 870, it is left c-reflective and has a left adequate left Suszko core. Therefore, by definition, $\mathcal{I}$ is syntactically WLC prealgebraizable. ■

Directly from the definitions, we may derive the following relationship between the semantic and syntactic WLC prealgebraizability classes of π-institutions.

Theorem 947 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically WLC prealgebraizable if and only if* $\mathcal{I}$ *is WLC prealgebraizable and* $R^{\mathcal{I}} L^{\mathcal{I}}$*-fortified.*

Proof: $\mathcal{I}$ is syntactically WLC prealgebraizable if and only if, by definition, it is $R^{\mathcal{I}} L^{\mathcal{I}}$-fortified, prealgebraic and left c-reflective, i.e., iff it is, by definition, $R^{\mathcal{I}} L^{\mathcal{I}}$-fortified and WLC prealgebraizable. ■

For an alternative characterization of syntactic WLC prealgebraizability, we take advantage of the corresponding characterization of WLC prealgebraizability in terms of morphisms between complete lattices of theory systems.

Theorem 948 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically WLC prealgebraizable if and only if it is* $R^{\mathcal{I}} L^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left completely order reflecting surjection that restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: We have that $\mathcal{I}$ is syntactically WLC prealgebraizable if and only if, by Theorem 947, it is $R^{\mathcal{I}} L^{\mathcal{I}}$-fortified and WLC prealgebraizable, if and only if, by Theorem 276, it it $R^{\mathcal{I}} L^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left completely order reflecting surjection that restricts to an order embedding $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. ■

Recall that syntactic weak system prealgebraizability was characterized by the existence of an equivalence between the systemic skeleton $K^{\mathcal{I}}$ of a

π-institution $\mathcal{I}$ and the systemic algebraic π-structure $\mathcal{Q}^{\mathcal{I}\bullet}$ associated with the π-institution, via a pair of conjugate transformations. To adapt this characterization to capture syntactic WLC prealgebraizability, we need to postulate alongside this equivalence the property of left truth equationality of the π-institution.

Theorem 949 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically WLC prealgebraizable if and only if it is left truth equational and its systemic skeleton* $\mathcal{K}^{\mathcal{I}}$ *is equivalent to* $\mathcal{Q}^{\mathcal{I}\bullet}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}\bullet}$ *of transformations.*

Proof: Suppose, first, that $\mathcal{I}$ is left truth equational and $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair of transformations. Then, $\mathcal{I}$ is left truth equational and, by Theorem 941, it is syntactically prealgebraic. Therefore, by definition, it is syntactically WLC prealgebraizable. If, conversely, $\mathcal{I}$ is syntactically WLC prealgebraizable, then, by Theorem 946, it is left truth equational and it is weakly system prealgebraizable. Thus, by Theorem 940, it is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair of transformations. ■

Because of Theorem 944, left truth equationality and the existence of an equivalence of the systemic skeleton with an algebraic π-structure, induced by conjugate transformations, is sufficient to yield syntactic WLC prealgebraizability.

Theorem 950 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically WLC prealgebraizable if and only if it is left truth equational and its systemic skeleton is equivalent to an algebraic* π*-structure via a conjugate pair of transformations.*

Proof: If $\mathcal{I}$ is syntactically WLC prealgebraizable, then the conclusion follows from Theorem 949. Conversely, if $\mathcal{K}^{\mathcal{I}}$ is equivalent to an algebraic π-structure via a conjugate pair of transformations, then $\mathcal{I}$ is syntactically prealgebraic by Theorem 941. Since, by hypothesis, it is also left truth equational, it is syntactically WLC prealgebraizable. ■

Finally, in terms of order isomorphisms between theory family lattices, we have the following alternative characterization of syntactically WLC prealgebraizable π-institutions:

Theorem 951 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically WLC prealgebraizable if and only if it is left truth equational and there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{K}^{\mathcal{I}}) \to \mathbf{ThFam}(\mathcal{Q})$*, where* $\mathcal{Q}$ *is an algebraic* π*-structure.*

Proof: The "only if" follows by Theorem 950 and Theorem 893. The "if" is given by Theorem 901 and Theorem 950. ■

Let us give, in closing the section, the picture of the **weak prealgebraizability hierarchy** that we have established, consisting of both semantic and syntactic classes of π-institutions.