Chapter 13

The Syntactic Leibniz Hierarchy: Parameterlessness

13.1 The Binary Reflexive Core

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall that the **reflexive core** of $\mathcal{I}$ is the collection

$$\begin{aligned} R^{\mathcal{I}} &= \{\rho^\flat \in N^\flat : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\rho^\flat_\Sigma[\phi,\phi] \leq \mathrm{Thm}(\mathcal{I}))\} \\ &= \{\rho^\flat \in N^\flat : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \phi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)) \\ &\qquad\qquad (\rho^\flat_\Sigma(\phi,\phi,\vec{\chi}) \subseteq \mathrm{Thm}_\Sigma(\mathcal{I}))\}. \end{aligned}$$

We define the **binary reflexive core** of $\mathcal{I}$ as the collection

$$B^{\mathcal{I}} : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$$

of binary natural transformations in $N^\flat$ given by:

$$\begin{aligned} B^{\mathcal{I}} &= \{\rho^\flat \in N^\flat : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\rho^\flat_\Sigma[\phi,\phi] \leq \mathrm{Thm}(\mathcal{I}))\} \\ &= \{\rho^\flat \in N^\flat : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\rho^\flat_\Sigma(\phi,\phi) \subseteq \mathrm{Thm}_\Sigma(\mathcal{I}))\}. \end{aligned}$$

It turns out that the binary reflexive core of $\mathcal{I}$ coincides with the collection $\ddot{R}^{\mathcal{I}}$.

Proposition 952 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then* $B^{\mathcal{I}} = \ddot{R}^{\mathcal{I}}$*.*

Proof: By Theorem 107. ∎

In view of Proposition 952, we drop the notation $B^{\mathcal{I}}$ and denote the binary reflexive core of $\mathcal{I}$ by the symbol $\ddot{R}^{\mathcal{I}}$, without fear of ambiguity.

13.2 Syntactic PreEquivalentiality

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. Recall that $\mathcal{I}$ is **preequivalential** if it is prealgebraic and system extensional, i.e., if:

- For all $T, T' \in \mathrm{ThSys}(\mathcal{I})$,

$$T \leq T' \quad \text{implies} \quad \Omega(T) \leq \Omega(T');$$

- For all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(T) \quad \text{iff} \quad \langle \phi, \psi \rangle \in \Omega_\Sigma^{\langle \phi,\psi \rangle}(T \cap \langle \phi, \psi \rangle).$$

We say that $\mathcal{I}$ is **syntactically preequivalential** if there exists $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, without parameters, such that $I^\flat$ has:

- reflexivity;

- global system transitivity;
- global system compatibility; and
- global system modus ponens.

Note that because all these conditions are imposed on theory systems and because $I^\flat$ is parameter-free, they are all equivalent to the corresponding local properties. Therefore, an equivalent definition would require reflexivity, local system transitivity, local system compatibility and local system modus ponens. Because of this, we sometimes omit the global/local specification and simply say "system" in qualifying the corresponding property.

In case $\mathcal{I}$ is syntactically preequivalential, we call $I^\flat$ a **set of witnessing natural transformations**, or, more simply, **witnessing transformations** (of/for the syntactic preequivalentiality of $\mathcal{I}$).

An interesting first observation, that will prove handy later, is that syntactic preequivalentiality is inherited by all π-subinstitutions of a syntactically preequivalential π-institution.

Theorem 953 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$ *and* $\mathcal{I}' = \langle \mathbf{F}', C' \rangle$ *a π-subinstitution of* $\mathcal{I}$ *induced by the algebraic subsystem* $\mathbf{F}' = \langle \mathbf{Sign}^\flat, \mathrm{SEN}'^\flat, N'^\flat \rangle \leq \mathbf{F}$. *If* $\mathcal{I}$ *is syntactically preequivalential with witnessing transformations* $I^\flat \subseteq N^\flat$, *then* $\mathcal{I}'$ *is also syntactically preequivalential, with witnessing transformations* $I'^\flat \in N'^\flat$.

Proof: Suppose that $\mathcal{I}$ is syntactically preequivalential with witnessing transformations $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$. To prove the conclusion, it suffices to show that $I'^\flat$ is reflexive, system transitive and has both the system compatibility and the system modus ponens in $\mathcal{I}'$.

Let, first, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}'^\flat(\Sigma)$. Then, clearly, $I'^\flat_\Sigma[\phi,\phi] \leq \mathrm{SEN}'^\flat$ and $I'^\flat_\Sigma[\phi,\phi] = I^\flat_\Sigma[\phi,\phi] \leq \mathrm{Thm}(\mathcal{I})$. So we get that

$$I'^\flat_\Sigma[\phi,\phi] \leq \mathrm{Thm}(\mathcal{I}) \cap \mathrm{SEN}'^\flat = \mathrm{Thm}(\mathcal{I}').$$

It follows that $I'^\flat$ is reflexive in $\mathcal{I}'$.

Suppose, next, that $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi,\chi \in \mathrm{SEN}'^\flat(\Sigma)$, such that

$$I'^\flat_\Sigma[\phi,\psi] \leq T \cap \mathrm{SEN}'^\flat \quad \text{and} \quad I'^\flat_\Sigma[\psi,\chi] \leq T \cap \mathrm{SEN}'^\flat.$$

Then $I^\flat_\Sigma[\phi,\psi] \leq T$ and $I^\flat_\Sigma[\psi,\chi] \leq T$, whence, by the global system transitivity of $I^\flat$ in $\mathcal{I}$, we get that $I^\flat_\Sigma[\phi,\chi] \leq T$. Since, by hypothesis, $\phi,\chi \in \mathrm{SEN}'^\flat(\Sigma)$, we get that $I'^\flat_\Sigma[\phi,\chi] \leq T \cap \mathrm{SEN}'^\flat$. This shows that $I'^\flat$ is globally system transitive in $\mathcal{I}'$.

For system compatibility, let $T \in \mathrm{ThSys}(\mathcal{I})$, $\sigma^\flat \in N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi,\vec{\chi} \in \mathrm{SEN}'^\flat(\Sigma)$, such that $\overleftrightarrow{I'^\flat}_\Sigma[\phi,\psi] \leq T \cap \mathrm{SEN}'^\flat$. Then we get that

$\overleftrightarrow{I}^\flat{}_\Sigma[\phi,\psi] \le T$, whence, we obtain $I^\flat_\Sigma[\sigma^\flat_\Sigma(\phi,\vec{\chi}),\sigma^\flat_\Sigma(\psi,\vec{\chi})] \le T$. Since $\sigma^\flat \in N^\flat$ and $\phi,\psi,\vec{\chi} \in \mathrm{SEN}'^\flat(\Sigma)$, this yields that

$$I'^\flat_\Sigma[\sigma'^\flat_\Sigma(\phi,\vec{\chi}),\sigma'^\flat_\Sigma(\psi,\vec{\chi})] \le T \cap \mathrm{SEN}'^\flat.$$

Therefore, $I'^\flat$ has the system compatibility in $\mathcal{I}'$.

For the system MP, assume that $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}'^\flat(\Sigma)$, such that $\phi \in T_\Sigma \cap \mathrm{SEN}'^\flat(\Sigma)$ and $I'^\flat_\Sigma[\phi,\psi] \le T \cap \mathrm{SEN}'^\flat$. Then $\phi \in T_\Sigma$ and $I^\flat_\Sigma[\phi,\psi] \le T$, whence we get that $\psi \in T_\Sigma$. Since, by hypothesis, $\psi \in \mathrm{SEN}'^\flat(\Sigma)$, we get that $\psi \in T_\Sigma \cap \mathrm{SEN}'^\flat(\Sigma)$. Therefore, $I'^\flat$ has the system MP in $\mathcal{I}'$.

We conclude that $\mathcal{I}'$ is also syntactically preequivalential with witnessing transformations $I'^\flat$. ■

Since $I^\flat$ is, a fortiori, a set of witnessing transformations for the syntactic prealgebraicity of $\mathcal{I}$, we get the following result, based on Corollary 770.

Corollary 954 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically preequivalential, with witnessing transformations* $I^\flat$*, if and only if, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,*

$$\overleftrightarrow{I}^\flat(T) = \Omega(T).$$

Proof: The "only if" is by Corollary 770. The "if" is clear, since the given condition implies that $\overleftrightarrow{I}^\flat$ satisfies reflexivity, global system transitivity, global system compatibility and global system modus ponens. ■

Based on Corollary 954, it is easy to see that syntactic preequivalentiality transfers from a π-institution to all its gmatrix families.

Theorem 955 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically preequivalential, with witnessing transformations* $I^\flat$*, if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$*, the* $\mathcal{I}$*-gmatrix family* $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}}\rangle$ *is syntactically preequivalential.*

Proof: The "if" follows by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I,\iota\rangle\rangle$. For the "only if", assume that $\mathcal{I}$ is syntactically preequivalential, with witnessing transformations $I^\flat$, and let $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$. Then, we have

$$\begin{array}{rll}
\langle \alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\rangle \in \overleftrightarrow{I}^{\mathcal{A}}_{F(\Sigma)}(T) & \text{iff} & \overleftrightarrow{I}^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)] \le T \\
& \text{iff} & \overleftrightarrow{I}^\flat_\Sigma[\phi,\psi] \le \alpha^{-1}(T) \\
& \text{iff} & \langle \phi,\psi\rangle \in \Omega_\Sigma(\alpha^{-1}(T)) \\
& \text{iff} & \langle \phi,\psi\rangle \in \alpha^{-1}_\Sigma(\Omega^{\mathcal{A}}_{F(\Sigma)}(T)) \\
& \text{iff} & \langle \alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\rangle \in \Omega^{\mathcal{A}}_{F(\Sigma)}(T).
\end{array}$$

Taking into account the surjectivity of $\langle F, \alpha\rangle$, it follows, by Corollary 954, that $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}}\rangle$ is also syntactically preequivalential, with witnessing transformations $I^{\mathcal{A}}$. ■

It turns out that syntactic preequivalentiality implies preequivalentiality.

Theorem 956 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically preequivalential, then it is preequivalential.*

Proof: Suppose that $\mathcal{I}$ is syntactically preequivalential, with witnessing transformations $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$. Then, it is a fortiori syntactically prealgebraic and, hence, by Theorem 771, prealgebraic. Thus, the Leibniz operator is monotone on theory systems. It suffices, therefore, to show that $\mathcal{I}$ is system extensional. To this end, let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$\langle \phi, \psi\rangle \in \Omega_\Sigma^{\langle \phi,\psi\rangle}(T \cap \langle \phi, \psi\rangle).$$

Thus, by Theorem 953 and Corollary 954,

$$\overleftrightarrow{I'^\flat}_\Sigma[\phi, \psi] \le T \cap \langle \phi, \psi\rangle.$$

Therefore, $\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi] \le T$, which, again by Corollary 954, implies that $\langle \phi, \psi\rangle \in \Omega_\Sigma(T)$. Since, by Proposition 89, the reverse inclusion always holds, $\mathcal{I}$ is also system extensional and, hence, preequivalential. ■

Apart from the definability of Leibniz congruence systems of theory systems, syntactic preequivalentiality has some additional important consequences. Namely, it implies that the binary reflexive core has the system modus ponens and that it also has the extensionality property. Before we look at those results more closely, we give a key lemma to the effect that in a syntactically preequivalential π-institution, any set of witnessing transformations is included in the binary reflexive core of the π-institution.

Lemma 957 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a syntactically preequivalential* π*-institution, with witnessing transformations* $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$*. Then* $I^\flat \subseteq \ddot{R}^{\mathcal{I}}$*.*

Proof: Since $I^\flat$ is parameter free and reflexive in $\mathcal{I}$, we get, by definition of $B^{\mathcal{I}}$ and Proposition 952, that $I^\flat \subseteq B^{\mathcal{I}} = \ddot{R}^{\mathcal{I}}$. ■

Now we formalize the fact that syntactic preequivalentiality implies the system modus ponens property for the binary reflexive core.

Proposition 958 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically preequivalential, then* $\ddot{R}^{\mathcal{I}}$ *has the system modus ponens in* $\mathcal{I}$*.*

Proof: Suppose that $\mathcal{I}$ is syntactically preequivalential and let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T$. Then $\phi \in T_\Sigma$ and, by Lemma 957, $I^\flat_\Sigma[\phi, \psi] \leq T$. Since $I^\flat$ has the system MP in $\mathcal{I}$, we conclude that $\psi \in T_\Sigma$. Therefore, $\ddot{R}^{\mathcal{I}}$ has the system MP in $\mathcal{I}$. ■

The next property that is implied by syntactic preequivalentiality is the extensionality of the binary reflexive core. Before introducing the concept, we take a look at a technical lemma that will serve to justify its formulation.

Lemma 959 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $X, Y \in \mathrm{SenSys}(\mathcal{I})$*. Then the following conditions are equivalent:*

(a) For all $T \in \mathrm{ThFam}(\mathcal{I})$*,* $X \leq T$ *if and only if* $Y \leq T$*;*

(b) For all $T \in \mathrm{ThSys}(\mathcal{I})$*,* $X \leq T$ *if and only if* $Y \leq T$*.*

Proof: That (a)⇒(b) is obvious, since every theory system of $\mathcal{I}$ is also a theory family. For (b)⇒(a), assume that (b) holds and let $T \in \mathrm{ThFam}(\mathcal{I})$, such that $X \leq T$. Then, by Lemma 1, $\overleftarrow{X} \leq \overleftarrow{T}$. Since $X \in \mathrm{SenSys}(\mathcal{I})$, by Proposition 2, we get that $X \leq \overleftarrow{T}$. Therefore, by hypothesis, since $\overleftarrow{T} \in \mathrm{ThSys}(\mathcal{I})$, we obtain $Y \leq \overleftarrow{T} \leq T$. By symmetry, we conclude that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $X \leq T$ if and only if $Y \leq T$. ■

Due to Lemma 959 and the fact that both the reflexive core and the binary reflexive core yield sentence systems of $\mathcal{I}$ under substitution, we make the following definition (without the need for distinguishing between a family versus system version):

Definition 960 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. The binary reflexive core* $\ddot{R}^{\mathcal{I}}$ *is* **extensional in** $\mathcal{I}$ *if and only if, for all* $T \in \mathrm{ThSys}(\mathcal{I})$ *(or equivalently, by Lemma 959, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*), all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T \quad \textit{if and only if} \quad R^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T.$$

Note that, since, by Lemma 104, $\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi] \leq R^{\mathcal{I}}_\Sigma[\phi, \psi]$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, the right-to-left implication in Definition 960 always holds. Therefore one has, equivalently, that $\ddot{R}^{\mathcal{I}}$ is extensional if and only if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T \quad \text{implies} \quad R^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T.$$

This can be taken to justify the name chosen for this property.

As mentioned previously, and shown in the next proposition, syntactic preequivalentiality implies the extensionality of the binary reflexive core:

Proposition 961 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically preequivalential, then* $\ddot{R}^{\mathcal{I}}$ *is extensional.*

Proof: Suppose $\mathcal{I}$ is a syntactically preequivalential π-institution, with witnessing transformations $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$. Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T$. Then, by Lemma 957, we get that $\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi] \leq T$. Thus, by Corollary 954, we get that $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. Since $\mathcal{I}$ is syntactically preequivalential, it is a fortiori syntactically prealgebraic, whence, by Theorems 781 and 782, we get that $R^{\mathcal{I}}$ is a set of witnessing transformations for the prealgebraicity of $\mathcal{I}$ and, hence, by Theorems 782 and 783, $R^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T$. Since the reverse inclusion always holds, we conclude that $\ddot{R}^{\mathcal{I}}$ is indeed extensional in $\mathcal{I}$. ■

As a result of preceding work we obtain the following

Theorem 962 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically preequivalential, then* $\ddot{R}^{\mathcal{I}}$ *has the system modus ponens and is extensional in* $\mathcal{I}$*.*

Proof: By Propositions 958 and 961. ■

We provide, next, a characterization of syntactic preequivalentiality in terms of the preceding two properties of the binary reflexive core of the π-institution, namely system modus ponens and extensionality. Later, we use this characterization to provide an exact description of those preequivalential π-institutions which are syntactically preequivalential.

In proving the reverse implication of that included in Theorem 962, we now show that, if $\ddot{R}^{\mathcal{I}}$ has the system modus ponens and is extensional in $\mathcal{I}$, then $\mathcal{I}$ is syntactically preequivalential.

Theorem 963 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\ddot{R}^{\mathcal{I}}$ *has the system modus ponens and is extensional in* $\mathcal{I}$*, then* $\mathcal{I}$ *is syntactically preequivalential, with witnessing transformations* $\ddot{R}^{\mathcal{I}}$*.*

Proof: If $\ddot{R}^{\mathcal{I}}$ has the system MP, then, by Lemma 104, $R^{\mathcal{I}}$ has a fortiori the global system MP. Thus, by Theorem 781, $\mathcal{I}$ is syntactically prealgebraic with witnessing transformations $R^{\mathcal{I}}$. Thus, $R^{\mathcal{I}}$ is globally system reflexive, globally system transitive, has the global system compatibility property and the global system MP. Moreover, by the extensionality of $\ddot{R}^{\mathcal{I}}$, all these properties transfer from $R^{\mathcal{I}}$ to $\ddot{R}^{\mathcal{I}}$. We conclude that $\mathcal{I}$ is syntactically preequivalential with witnessing transformations $\ddot{R}^{\mathcal{I}}$. ■

Theorems 962 and 963 provide the promised characterization of syntactic preequivalentiality in terms of the system modus ponens and the extensionality of the binary reflexive core.

$$\begin{array}{rcl}\mathcal{I} \text{ is Syntactically Preequivalential} & \longleftrightarrow & \ddot{R}^{\mathcal{I}} \text{ has System MP} \\ & + & \ddot{R}^{\mathcal{I}} \text{ is Extensional.}\end{array}$$

Theorem 964 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically preequivalential if and only if* $\ddot{R}^{\mathcal{I}}$ *has the system modus ponens and is extensional in* $\mathcal{I}$.

Proof: Theorem 962 gives the "only if" and the "if" is by Theorem 963. ∎

If $\mathcal{I}$ is syntactically preequivalential, then $\ddot{R}^{\mathcal{I}}$ defines Leibniz congruence systems of theory systems in $\mathcal{I}$. This proposition may be viewed as a special case of Corollary 954, since $\ddot{R}^{\mathcal{I}}$ forms a set of witnessing transformations.

Proposition 965 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\ddot{R}^{\mathcal{I}}$ *has system modus ponens and is extensional in* $\mathcal{I}$*, then, for all* $T \in \mathrm{ThSys}(\mathcal{I})$,

$$\Omega(T) = \ddot{R}^{\mathcal{I}}(T).$$

Proof: Let $T \in \mathrm{ThSys}(\mathcal{I})$. If $\ddot{R}^{\mathcal{I}}$ has the system modus ponens and is extensional, then, by Theorem 963, $\mathcal{I}$ is syntactically preequivalential with witnessing transformations $\ddot{R}^{\mathcal{I}}$. Therefore, by Corollary 954, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\Omega(T) = \ddot{R}^{\mathcal{I}}(T)$. ∎

We also get another related characterization of syntactic preequivalentiality.

$$\begin{array}{l}\mathcal{I} \text{ is Syntactically Preequivalential} \\ \qquad\qquad \longleftrightarrow \ddot{R}^{\mathcal{I}} \text{ Defines Leibniz Congruence Systems} \\ \qquad\qquad\qquad \text{of Theory Systems in } \mathcal{I}.\end{array}$$

Theorem 966 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically preequivalential if and only if, for all* $T \in \mathrm{ThSys}(\mathcal{I})$,

$$\Omega(T) = \ddot{R}^{\mathcal{I}}(T).$$

Proof: If $\mathcal{I}$ is syntactically preequivalential, then, by Theorem 962, $\ddot{R}^{\mathcal{I}}$ has the system modus ponens and is extensional in $\mathcal{I}$. Thus, by Proposition 965, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\Omega(T) = \ddot{R}^{\mathcal{I}}(T)$.

Conversely, if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\ddot{R}^{\mathcal{I}}(T) = \Omega(T)$, then, $\ddot{R}^{\mathcal{I}}$ is reflexive, system transitive, has the system compatibility and the system modus ponens. Thus, $\mathcal{I}$ is syntactically preequivalential with witnessing transformations $\ddot{R}^{\mathcal{I}}$. ∎

We finally show that the property that separates preequivalentiality from syntactic preequivalentiality is exactly a sort of a local Leibniz compatibility property with respect to the theory system generated by the binary reflexive core.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall that $R^{\mathcal{I}}$ is **Leibniz** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi])).$$

Similarly, we say that $\ddot{R}^{\mathcal{I}}$ is **Leibniz** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega^{\langle \phi, \psi \rangle}_\Sigma(C(\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi]) \cap \langle \phi, \psi \rangle).$$

We show next that, if $\ddot{R}^{\mathcal{I}}$ has the system modus ponens and is extensional in $\mathcal{I}$, then $\ddot{R}^{\mathcal{I}}$ is Leibniz.

Proposition 967 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\ddot{R}^{\mathcal{I}}$ has the system modus ponens and is extensional in $\mathcal{I}$, then $\ddot{R}^{\mathcal{I}}$ is Leibniz.*

Proof: Suppose that $\ddot{R}^{\mathcal{I}}$ has the system MP and is extensional and let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. By Theorem 963, $\mathcal{I}$ is syntactically preequivalential, with witnessing transformations $\ddot{R}^{\mathcal{I}}$. Hence, it is a fortiori syntactically prealgebraic, with witnessing transformations $\ddot{R}^{\mathcal{I}}$. Thus, by Theorem 788, we get

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi])).$$

Then, by Theorem 89, we get

$$\langle \phi, \psi \rangle \in \Omega^{\langle \phi, \psi \rangle}_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi]) \cap \langle \phi, \psi \rangle).$$

By hypothesis (more precisely the extensionality of $\ddot{R}^{\mathcal{I}}$), we get $C(R^{\mathcal{I}}_\Sigma[\phi, \psi]) = C(\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi])$. Therefore, we conclude that $\langle \phi, \psi \rangle \in \Omega^{\langle \phi, \psi \rangle}_\Sigma(C(\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi]) \cap \langle \phi, \psi \rangle)$. So $\ddot{R}^{\mathcal{I}}$ is Leibniz. ∎

In the opposite direction, in a *preequivalential* π-institution $\mathcal{I}$, if the binary reflexive core is Leibniz, then it has the system modus ponens and is extensional in $\mathcal{I}$.

Proposition 968 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a preequivalential π-institution based on $\mathbf{F}$. If $\ddot{R}^{\mathcal{I}}$ is Leibniz, then $\ddot{R}^{\mathcal{I}}$ has the system modus ponens and is extensional in $\mathcal{I}$.*

Proof: Suppose that $\mathcal{I}$ is preequivalential and that $\ddot{R}^{\mathcal{I}}$ is Leibniz.

For the system MP, suppose that $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$, $\ddot{R}^{\mathcal{I}}_\Sigma[\phi,\psi] \leq T$. Then, since $\ddot{R}^{\mathcal{I}}$ is Leibniz,

$$\langle \phi, \psi \rangle \in \Omega^{\langle \phi,\psi \rangle}_\Sigma (C(\ddot{R}^{\mathcal{I}}_\Sigma[\phi,\psi]) \cap \langle \phi, \psi \rangle).$$

Thus, since $\mathcal{I}$ is system extensional,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(C(\ddot{R}^{\mathcal{I}}_\Sigma[\phi,\psi])).$$

By hypothesis, $C(\ddot{R}^{\mathcal{I}}_\Sigma[\phi,\psi]) \leq T$, whence, by preequivalentiality,

$$\Omega(C(\ddot{R}^{\mathcal{I}}_\Sigma[\phi,\psi])) \leq \Omega(T).$$

We, thus, get that $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. Therefore, by compatibility of $\Omega(T)$ with T, we obtain $\psi \in T_\Sigma$, showing that $\ddot{R}^{\mathcal{I}}$ has the system MP in $\mathcal{I}$.

For extensionality, assume that $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\ddot{R}^{\mathcal{I}}_\Sigma[\phi,\psi] \leq T$. Following the initial argument of the preceding paragraph mutatis mutandis we obtain that $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. But since $\ddot{R}^{\mathcal{I}}$ has the system MP, a fortiori $R^{\mathcal{I}}$ has the global system MP, whence, by Proposition 783, $R^{\mathcal{I}}_\Sigma[\phi,\psi] \leq T$. Since the reverse inclusion always holds, we conclude that $\ddot{R}^{\mathcal{I}}$ is extensional. ∎

We now show that a π-institution is syntactically preequivalential if and only if it is preequivalential and it has a Leibniz binary reflexive core.

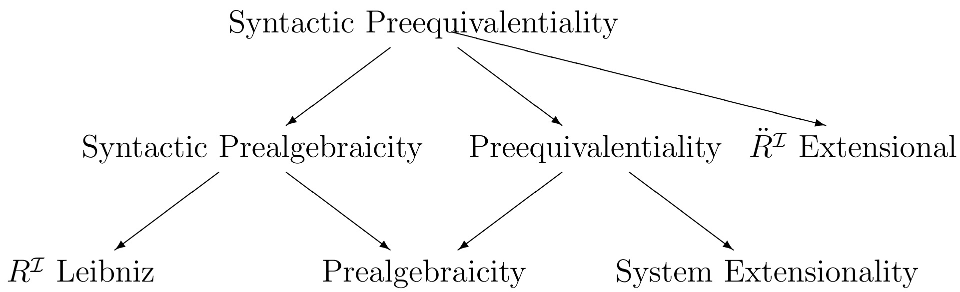
Syntactic Preequivalentiality
= $\ddot{R}^{\mathcal{I}}$ has System MP + $\ddot{R}^{\mathcal{I}}$ is Extensional
= $\ddot{R}^{\mathcal{I}}$ Defines Leibniz Congruence Systems of Theory Systems in $\mathcal{I}$
= Preequivalentiality + $\ddot{R}^{\mathcal{I}}$ is Leibniz

Theorem 969 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically preequivalential if and only if it is preequivalential and has a Leibniz binary reflexive core.*

Proof: Suppose, first, that $\mathcal{I}$ is syntactically preequivalential. Then it is preequivalential by Theorem 956. Moreover, its binary reflexive core has the system modus ponens and is extensional, by Theorem 964, and, hence, by Proposition 967, its binary reflexive core is Leibniz.

Suppose, conversely, that $\mathcal{I}$ is preequivalential with a Leibniz binary reflexive core. Then, by Proposition 968, its binary reflexive core has the system MP and is extensional. Therefore, by Theorem 964, $\mathcal{I}$ is syntactically preequivalential. ∎

We have the following part of a hierarchy:



13.3 Syntactic Equivalentiality

We now define the class of syntactically equivalential π-institutions. The difference between equivalentiality and preequivalentiality is that the system versions of the properties defining the latter are replaced by their corresponding family versions. Otherwise, the developments are exactly parallel.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$.

Recall that $\mathcal{I}$ is **equivalential** if it is protoalgebraic and family extensional, i.e., if:

- For all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$T \leq T' \quad \text{implies} \quad \Omega(T) \leq \Omega(T');$$

- For all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(T) \quad \text{iff} \quad \langle \phi, \psi \rangle \in \Omega_\Sigma^{\langle \phi, \psi \rangle}(T \cap \langle \phi, \psi \rangle).$$

Recall, also, that protoalgebraicity implies stability. If a π-institution is stable, then it is family extensional if and only if it is system extensional. Thus, under protoalgebraicity, system and family extensionality coincide, and, therefore, $\mathcal{I}$ being equivalential is equivalent to $\mathcal{I}$ being protoalgebraic and system extensional.

We say that $\mathcal{I}$ is **syntactically equivalential** if there exists $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, without parameters, such that $I^\flat$ has:

- reflexivity;
- global family transitivity;
- global family compatibility; and
- global family modus ponens.

We emphasize that, in opposition to the case of the corresponding system properties, in this case, the latter three conditions are not equivalent to the corresponding local properties. So one cannot dispense with the qualification "global" in the defining conditions.

In case $\mathcal{I}$ is syntactically equivalential, we call $I^\flat$ a **set of witnessing natural transformations**, or, more simply, **witnessing transformations** (of the syntactic equivalentiality of $\mathcal{I}$).

As was the case with syntactic preequivalentiality, syntactic equivalentiality is inherited by all π-subinstitutions of a syntactically equivalential π-institution.

Theorem 970 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\mathcal{I}' = \langle \mathbf{F}', C'\rangle$ *a* π*-subinstitution of* $\mathcal{I}$ *induced by the algebraic subsystem* $\mathbf{F}' = \langle \mathbf{Sign}^\flat, \mathrm{SEN}'^\flat, N'^\flat\rangle \leq \mathbf{F}$*. If* $\mathcal{I}$ *is syntactically equivalential with witnessing transformations* $I^\flat \subseteq N^\flat$*, then* $\mathcal{I}'$ *is also syntactically equivalential, with witnessing transformations* $I'^\flat \in N'^\flat$*.*

Proof: Suppose that $\mathcal{I}$ is syntactically equivalential with witnessing transformations $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$. To prove the conclusion, it suffices to show that $I'^\flat$ is reflexive, globally family transitive and has both the global family compatibility and the global family modus ponens in $\mathcal{I}'$.

Let, first, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}'^\flat(\Sigma)$. Then, clearly, $I'^\flat_\Sigma[\phi,\phi] \leq \mathrm{SEN}'^\flat$ and $I'^\flat_\Sigma[\phi,\phi] = I^\flat_\Sigma[\phi,\phi] \leq \mathrm{Thm}(\mathcal{I})$. So we get that

$$I'^\flat_\Sigma[\phi,\phi] \leq \mathrm{Thm}(\mathcal{I}) \cap \mathrm{SEN}'^\flat = \mathrm{Thm}(\mathcal{I}').$$

It follows that $I'^\flat$ is reflexive in $\mathcal{I}'$.

Suppose, next, that $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi,\chi \in \mathrm{SEN}'^\flat(\Sigma)$, such that

$$I'^\flat_\Sigma[\phi,\psi] \leq T \cap \mathrm{SEN}'^\flat \quad \text{and} \quad I'^\flat_\Sigma[\psi,\chi] \leq T \cap \mathrm{SEN}'^\flat.$$

Then $I^\flat_\Sigma[\phi,\psi] \leq T$ and $I^\flat_\Sigma[\psi,\chi] \leq T$, whence by the global family transitivity of $I^\flat$ in $\mathcal{I}$, we get that $I^\flat_\Sigma[\phi,\chi] \leq T$. Since, by hypothesis, $\phi,\chi \in \mathrm{SEN}'^\flat(\Sigma)$, we get that $I'^\flat_\Sigma[\phi,\chi] \leq T \cap \mathrm{SEN}'^\flat$. This shows that $I'^\flat$ is globally family transitive in $\mathcal{I}'$.

For global family compatibility, let $T \in \mathrm{ThFam}(\mathcal{I})$, $\sigma^\flat \in N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi,\vec{\chi} \in \mathrm{SEN}'^\flat(\Sigma)$, such that $\overleftrightarrow{I'^\flat}_\Sigma[\phi,\psi] \leq T \cap \mathrm{SEN}'^\flat$. Then we get that $\overleftrightarrow{I^\flat}_\Sigma[\phi,\psi] \leq T$, whence, we obtain $I^\flat_\Sigma[\sigma^\flat_\Sigma(\phi,\vec{\chi}),\sigma^\flat_\Sigma(\psi,\vec{\chi})] \leq T$. Since $\sigma^\flat \in N^\flat$ and $\phi,\psi,\vec{\chi} \in \mathrm{SEN}'^\flat(\Sigma)$, this yields that

$$I'^\flat_\Sigma[\sigma'^\flat_\Sigma(\phi,\vec{\chi}),\sigma'^\flat_\Sigma(\psi,\vec{\chi})] \leq T \cap \mathrm{SEN}'^\flat.$$

Therefore, $I'^\flat$ has the global family compatibility in $\mathcal{I}'$.

For the global family MP, assume that $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}'^\flat(\Sigma)$, such that $\phi \in T_\Sigma \cap \mathrm{SEN}'^\flat(\Sigma)$ and $I'^\flat_\Sigma[\phi, \psi] \leq T \cap \mathrm{SEN}'^\flat$. Then $\phi \in T_\Sigma$ and $I^\flat_\Sigma[\phi, \psi] \leq T$, whence we get that $\psi \in T_\Sigma$. Since, by hypothesis, $\psi \in \mathrm{SEN}'^\flat(\Sigma)$, we get that $\psi \in T_\Sigma \cap \mathrm{SEN}'^\flat(\Sigma)$. Therefore, $I'^\flat$ has the global family MP in $\mathcal{I}'$.

We conclude that $\mathcal{I}'$ is also syntactically equivalential with witnessing transformations $I'^\flat$. ■

Since $I^\flat$ is, a fortiori, a set of witnessing transformations for the syntactic protoalgebraicity of $\mathcal{I}$, we get the following result, based on Corollary 791.

Corollary 971 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically equivalential, with witnessing transformations* $I^\flat$*, if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\overleftrightarrow{I^\flat}(T) = \Omega(T).$$

Proof: The "only if" is by Corollary 791. The "if" is clear, since the given condition implies that $\overleftrightarrow{I^\flat}$ satisfies reflexivity, global family transitivity, global family compatibility and global family modus ponens. ■

Based on Corollary 971, it is easy to see that syntactic equivalentiality also transfers from a π-institution to all its gmatrix families.

Theorem 972 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically equivalential, with witnessing transformations* $I^\flat$*, if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, the* $\mathcal{I}$*-gmatrix family* $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ *is syntactically equivalential.*

Proof: Analogous to the proof of Theorem 955. ■

Syntactic equivalentiality implies equivalentiality.

Theorem 973 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically equivalential, then it is equivalential.*

Proof: Suppose that $\mathcal{I}$ is syntactically equivalential, with witnessing transformations $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$. Then, it is a fortiori syntactically protoalgebraic and, hence, by Theorem 792, protoalgebraic. Thus, the Leibniz operator is monotone on theory families. Since syntactical equivalentiality implies syntactical preequivalentiality, by Theorem 956, we get that $\mathcal{I}$ is also system extensional. ■

In analogy with syntactic preequivalentiality, syntactic equivalentiality implies that the binary reflexive core has the global family modus ponens and that it also has the extensionality property.

We first formalize the fact that syntactic equivalentiality implies the global family modus ponens property of the binary reflexive core.

Proposition 974 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically equivalential, then* $\ddot{R}^{\mathcal{I}}$ *has the global family modus ponens in* $\mathcal{I}$*.*

Proof: Suppose that $\mathcal{I}$ is syntactically equivalential, with witnessing transformations $I^\flat$, and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T$. Then $\phi \in T_\Sigma$ and, by Lemma 957, $I^\flat_\Sigma[\phi, \psi] \leq T$. Since $I^\flat$ has the global family MP in $\mathcal{I}$, we conclude that $\psi \in T_\Sigma$. Therefore, $\ddot{R}^{\mathcal{I}}$ also has the global family MP in $\mathcal{I}$. ■

We now show that syntactic equivalentiality implies the extensionality of the binary reflexive core.

Corollary 975 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically equivalential, then* $\ddot{R}^{\mathcal{I}}$ *is extensional.*

Proof: Since syntactic equivalentiality implies syntactic preequivalentiality, we get the conclusion by applying Proposition 961. ■

We summarize these two important consequences of syntactic equivalentiality in the following

Theorem 976 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically equivalential, then* $\ddot{R}^{\mathcal{I}}$ *has the global family modus ponens and is extensional in* $\mathcal{I}$*.*

Proof: By Propositions 974 and 975. ■

We provide, next, a characterization of syntactic equivalentiality in terms of the preceding two properties of the binary reflexive core of the π-institution, namely global family modus ponens and extensionality. As with preequivalentiality, we use this characterization to provide an exact description of those equivalential π-institutions which are syntactically equivalential.

In proving the reverse implication of that included in Theorem 976, we show that, if $\ddot{R}^{\mathcal{I}}$ has the global family modus ponens and is extensional in $\mathcal{I}$, then $\mathcal{I}$ is syntactically equivalential.

Theorem 977 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\ddot{R}^{\mathcal{I}}$ *has the global family modus ponens and is extensional in* $\mathcal{I}$*, then* $\mathcal{I}$ *is syntactically equivalential, with witnessing transformations* $\ddot{R}^{\mathcal{I}}$*.*

Proof: If $\ddot{R}^{\mathcal{I}}$ has the global family MP, then, by Lemma 104, $R^{\mathcal{I}}$ has a fortiori the global family MP. Thus, by Theorem 798, $\mathcal{I}$ is syntactically protoalgebraic with witnessing transformations $R^{\mathcal{I}}$. Thus, $R^{\mathcal{I}}$ is reflexive, globally family transitive, has the global family compatibility property and the global family

MP. Moreover, by the extensionality of $\ddot{R}^{\mathcal{I}}$, all these properties transfer from $R^{\mathcal{I}}$ to $\ddot{R}^{\mathcal{I}}$. We conclude that $\mathcal{I}$ is syntactically equivalential with witnessing transformations $\ddot{R}^{\mathcal{I}}$. ∎

Theorems 976 and 977 provide the promised characterization of syntactic equivalentiality in terms of the global family modus ponens and the extensionality of the binary reflexive core.

$$\begin{array}{lcl}\mathcal{I} \text{ is Syntactically Equivalential} & \longleftrightarrow & \ddot{R}^{\mathcal{I}} \text{ has Global Family MP} \\ & + & \ddot{R}^{\mathcal{I}} \text{ is Extensional.}\end{array}$$

Theorem 978 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically equivalential if and only if* $\ddot{R}^{\mathcal{I}}$ *has the global family modus ponens and is extensional in* $\mathcal{I}$*.*

Proof: Theorem 976 gives the "only if" and the "if" is by Theorem 977. ∎

If $\mathcal{I}$ is syntactically equivalential, then $\ddot{R}^{\mathcal{I}}$ defines Leibniz congruence systems of theory families in $\mathcal{I}$. This proposition may be viewed as a special case of Corollary 971, since $\ddot{R}^{\mathcal{I}}$ forms a set of witnessing transformations.

Proposition 979 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\ddot{R}^{\mathcal{I}}$ *has the global family modus ponens and is extensional in* $\mathcal{I}$*, then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\Omega(T) = \ddot{R}^{\mathcal{I}}(T).$$

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$. If $\ddot{R}^{\mathcal{I}}$ has the global family modus ponens and is extensional, then, by Theorem 977, $\mathcal{I}$ is syntactically equivalential with witnessing transformations $\ddot{R}^{\mathcal{I}}$. Therefore, by Corollary 971, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) = \ddot{R}^{\mathcal{I}}(T)$. ∎

We also get another related characterization of syntactic equivalentiality.

$$\begin{array}{l}\mathcal{I} \text{ is Syntactically Equivalential} \\ \qquad\qquad \longleftrightarrow \ddot{R}^{\mathcal{I}} \text{ Defines Leibniz Congruence Systems} \\ \qquad\qquad\qquad \text{of Theory Families in } \mathcal{I}.\end{array}$$

Theorem 980 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically equivalential if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\Omega(T) = \ddot{R}^{\mathcal{I}}(T).$$

Proof: If $\mathcal{I}$ is syntactically equivalential, then, by Theorem 976, $\ddot{R}^{\mathcal{I}}$ has the global family modus ponens and is extensional in $\mathcal{I}$. Thus, by Proposition 979, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) = \ddot{R}^{\mathcal{I}}(T)$.

Conversely, if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\ddot{R}^{\mathcal{I}}(T) = \Omega(T)$, then, $\ddot{R}^{\mathcal{I}}$ is reflexive, globally family transitive and has the global family compatibility and the global family modus ponens. Thus, $\mathcal{I}$ is syntactically equivalential with witnessing transformations $\ddot{R}^{\mathcal{I}}$. ■

We finally show that the property that separates equivalentiality from syntactic equivalentiality is the Leibniz property of the binary reflexive core.

We show first that, if $\ddot{R}^{\mathcal{I}}$ has the global family modus ponens and is extensional in $\mathcal{I}$, then $\ddot{R}^{\mathcal{I}}$ is Leibniz.

Corollary 981 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\ddot{R}^{\mathcal{I}}$ *has the global family modus ponens and is extensional in* $\mathcal{I}$*, then* $\ddot{R}^{\mathcal{I}}$ *is Leibniz.*

Proof: Since $\ddot{R}^{\mathcal{I}}$ having the global family MP is stronger than having the system MP, the conclusion follows from Proposition 967. ■

In the opposite direction, in an *equivalential* π-institution $\mathcal{I}$, if the binary reflexive core is Leibniz, then it has the global family modus ponens and is extensional in $\mathcal{I}$.

Proposition 982 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *an equivalential* π*-institution based on* $\mathbf{F}$*. If* $\ddot{R}^{\mathcal{I}}$ *is Leibniz, then* $\ddot{R}^{\mathcal{I}}$ *has the global family modus ponens and is extensional in* $\mathcal{I}$*.*

Proof: Suppose that $\mathcal{I}$ is equivalential and that $\ddot{R}^{\mathcal{I}}$ is Leibniz.

For the global family MP, suppose that $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$, $\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T$. Then, since $\ddot{R}^{\mathcal{I}}$ is Leibniz,

$$\langle \phi, \psi \rangle \in \Omega^{\langle \phi, \psi \rangle}_\Sigma (C(\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi]) \cap \langle \phi, \psi \rangle).$$

Thus, since $\mathcal{I}$ is system extensional,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma (C(\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi])).$$

By hypothesis, $C(\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi]) \leq T$, whence, by equivalentiality,

$$\Omega(C(\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi])) \leq \Omega(T).$$

We, thus, get that $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. Therefore, by compatibility of $\Omega(T)$ with T, we obtain $\psi \in T_\Sigma$, showing that $\ddot{R}^{\mathcal{I}}$ has the global family MP in $\mathcal{I}$.

Since equivalentiality implies preequivalentiality, the extensionality of $\ddot{R}^{\mathcal{I}}$ follows from Proposition 968. ■

We now show that a π-institution is syntactically equivalential if and only if it is equivalential and has a Leibniz binary reflexive core.

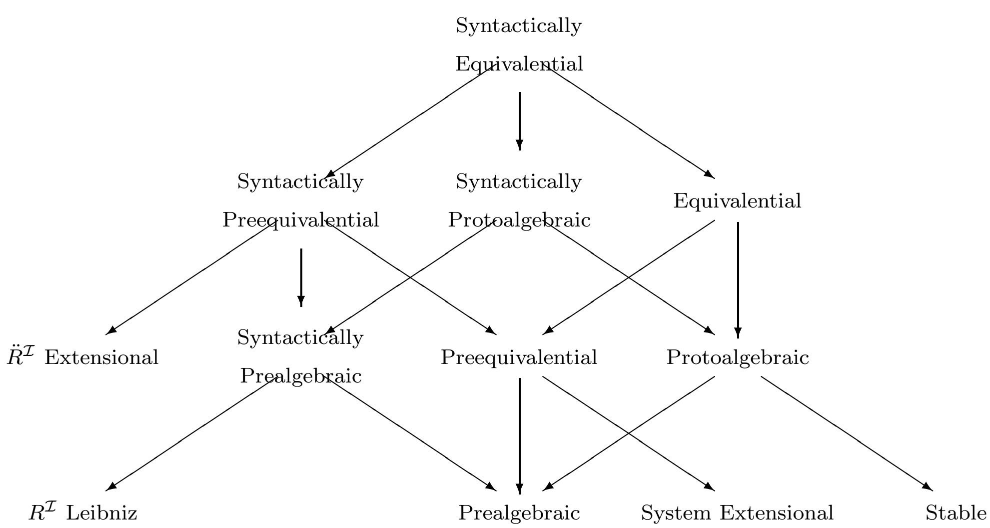
Syntactic Equivalentiality
$= \ddot{R}^{\mathcal{I}}$ has Global Family MP + $\ddot{R}^{\mathcal{I}}$ is Extensional
$= \ddot{R}^{\mathcal{I}}$ Defines Leibniz Congruence Systems of Theory Families in $\mathcal{I}$
= Equivalentiality + $\ddot{R}^{\mathcal{I}}$ is Leibniz

Theorem 983 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is syntactically equivalential if and only if it is equivalential and has a Leibniz binary reflexive core.*

Proof: Suppose, first, that $\mathcal{I}$ is syntactically equivalential. Then it is equivalential by Theorem 973. Moreover, its binary reflexive core has the global family modus ponens and is extensional, by Theorem 978, and, hence, by Corollary 981, its binary reflexive core is Leibniz.

Suppose, conversely, that $\mathcal{I}$ is equivalential with a Leibniz binary reflexive core. Then, by Proposition 982, its binary reflexive core has the global family MP and is extensional. Therefore, by Theorem 978, $\mathcal{I}$ is syntactically equivalential. ■

We have now established the following hierarchy of properties:



13.4 Strong Truth Equationality

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is **strongly (family) truth equational** if there exists a set $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ (with a single distinguished argument), such that, for every $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

In that case, we call $\tau^\flat$ a **set of witnessing equations** (of/for the strong truth equationality of $\mathcal{I}$).

Note that, since $\tau^\flat$ is parameter-free and $\Omega(T)$ is invariant under signature morphisms, strong truth equationality may be defined equivalently by the condition, for every $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma(\phi) \subseteq \Omega_\Sigma(T).$$

Since truth equationality implies systemicity, we get, a fortiori, that strong truth equationality implies systemicity.

We introduce next the unary Suszko core of a π-institution. Analogously with the Suszko core, the unary Suszko core enables one to obtain:

- A characterization of strong truth equationality in terms of the solubility property of the unary Suszko core of the π-institution.
- An exact description of those family c-reflective π-institutions which are strongly truth equational.
- A characterization of those truth equational π-institutions which are strongly truth equational.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. The **unary Suszko core** of $\mathcal{I}$ is the collection

$$\dot{S}^{\mathcal{I}} = \{\sigma^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2 \in N^\flat : (\forall T \in \mathrm{ThFam}(\mathcal{I}))(\sigma^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T)\}.$$

By Lemma 821, this definition is equivalent to setting

$$\begin{array}{rcl}\dot{S}^{\mathcal{I}} & = & \{\sigma^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2 \in N^\flat : (\forall \Sigma \in |\mathbf{Sign}^\flat|) \\ & & \qquad (\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\sigma^\flat_\Sigma(\phi) \in \widetilde{\Omega}^{\mathcal{I}}_\Sigma(C(\phi))\}.\end{array}$$

Note that the unary Suszko core of a π-institution is included in the Suszko core, i.e., we have

Lemma 984 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then* $\dot{S}^{\mathcal{I}} \subseteq S^{\mathcal{I}}$*.*

Proof: Every pair of unary natural transformations in $N^\flat$ that satisfies the membership criterion for $\dot{S}^{\mathcal{I}}$ also satisfies the condition for membership in $S^{\mathcal{I}}$. ■

Lemma 984 yields immediately the following consequence.

Corollary 985 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $T \in \mathrm{ThFam}(\mathcal{I})$*, we have*

$$S^{\mathcal{I}}(\Omega(T)) \leq \dot{S}^{\mathcal{I}}(\Omega(T)).$$

Proof: We have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rcll} \phi \in S^{\mathcal{I}}_\Sigma(\Omega(T)) & \text{iff} & S^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T) & \text{(definition)} \\ & \text{implies} & \dot{S}^{\mathcal{I}}[\phi] \le \Omega(T) & \text{(Lemma 984)} \\ & \text{iff} & \phi \in \dot{S}^{\mathcal{I}}_\Sigma(\Omega(T)). & \text{(definition)} \end{array}$$

Therefore, $S^{\mathcal{I}}(\Omega(T)) \le \dot{S}^{\mathcal{I}}(\Omega(T))$. ∎

Either directly by the definition or using Proposition 832 together with Corollary 985, we get the following

Proposition 986 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For every* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$T \le \dot{S}^{\mathcal{I}}(\Omega(T)).$$

Proof: We have $T \le S^{\mathcal{I}}(\Omega(T)) \le \dot{S}^{\mathcal{I}}(\Omega(T))$, where the first inclusion is by Lemma 832 and the second by Corollary 985. ∎

Similarly with the Suszko core, the unary Suszko core of a π-institution may or may not satisfy the reverse inclusion of Proposition 986, a property that was called *solubility*.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the unary Suszko core of $\mathcal{I}$ is **soluble** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\dot{S}^{\mathcal{I}}(\Omega(T)) \le T.$$

Note that $\dot{S}^{\mathcal{I}}$ is soluble if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\dot{S}^{\mathcal{I}}_\Sigma(\phi) \subseteq \Omega_\Sigma(T) \quad \text{implies} \quad \phi \in T_\Sigma.$$

It turns out that possession of the solubility property by the unary Suszko core intrinsically characterizes strong truth equationality. To show the necessity of solubility observe, first, that, in case a π-institution is strongly truth equational, the witnessing equations form a subset of the unary Suszko core.

Lemma 987 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is strongly truth equational, with witnessing equations* $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2 \subseteq N^\flat$*, then* $\tau^\flat \subseteq \dot{S}^{\mathcal{I}}$*.*

Proof: Suppose that $\mathcal{I}$ is strongly truth equational with witnessing equations $\tau^\flat$. Then, $\mathcal{I}$ is, a fortiori, truth equational, with the same witnessing equations. It follows, by Lemma 835, that $\tau^\flat \subseteq S^{\mathcal{I}}$. Since $\tau^\flat$ consists of unary equations and they satisfy the membership criterion for $S^{\mathcal{I}}$, it follows that they also satisfy the condition for membership in $\dot{S}^{\mathcal{I}}$. Therefore, we get that $\tau^\flat \subseteq \dot{S}^{\mathcal{I}}$. ∎

Now we prove the necessity of the solubility of the unary Suszko core for strong truth equationality.

Theorem 988 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is strongly truth equational, then* $\dot{S}^{\mathcal{I}}$ *is soluble.*

Proof: Suppose that $\mathcal{I}$ is strongly truth equational, with witnessing equations $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll} \dot{S}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) & \text{implies} & \tau^\flat_\Sigma[\phi] \leq \Omega(T) \quad \text{(Lemma 987)} \\ & \text{iff} & \phi \in T_\Sigma. \quad \text{(strong truth equationality)} \end{array}$$

Thus, $\dot{S}^{\mathcal{I}}$ is soluble. ■

The reverse implication also holds and completes the promised characterization of strong truth equationality in terms of the solubility of the unary Suszko core.

Theorem 989 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\dot{S}^{\mathcal{I}}$ *is soluble, then* $\mathcal{I}$ *is strongly truth equational, with witnessing equations* $\dot{S}^{\mathcal{I}}$.

Proof: It suffices to show that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \dot{S}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

The left-to-right implication is given in Proposition 986, whereas the converse is ensured by the postulated solubility of $\dot{S}^{\mathcal{I}}$. ■

Theorems 988 and 989 provide the promised characterization of strong truth equationality in terms of the solubility of the unary Suszko core.

$$\mathcal{I} \text{ is Strongly Truth Equational} \longleftrightarrow \dot{S}^{\mathcal{I}} \text{ is Soluble.}$$

Theorem 990 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is strongly truth equational if and only if* $\dot{S}^{\mathcal{I}}$ *is soluble.*

Proof: Theorem 988 gives the "only if" and the "if" is by Theorem 989. ■

If $\mathcal{I}$ is strongly truth equational, then the unary Suszko core defines theory families in $\mathcal{I}$ in terms of their Leibniz congruence systems. This proposition may be viewed as a special case of Proposition 828, since $\dot{S}^{\mathcal{I}}$ forms a maximal set of witnessing equations of the strong truth equationality of $\mathcal{I}$.

Proposition 991 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\dot{S}^{\mathcal{I}}$ *is soluble, then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$,

$$T = \dot{S}^{\mathcal{I}}(\Omega(T)).$$

Proof: If $\dot{S}^{\mathcal{I}}$ is soluble, then, by Theorem 989, $\dot{S}^{\mathcal{I}}$ forms a set of witnessing equations for the strong truth equationality of $\mathcal{I}$. Therefore, by Proposition 828, we get that, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $T = \dot{S}^{\mathcal{I}}(\Omega(T))$. ∎

This property provides another characterization of strong truth equationality. We say that $\dot{S}^{\mathcal{I}}$ **defines theory families** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T = \dot{S}^{\mathcal{I}}(\Omega(T))$. Then we have:

$$\mathcal{I} \text{ is Strongly Truth Equational} \longleftrightarrow \dot{S}^{\mathcal{I}} \text{ Defines Theory Families.}$$

Theorem 992 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is strongly truth equational if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$,

$$T = \dot{S}^{\mathcal{I}}(\Omega(T)).$$

Proof: If $\mathcal{I}$ is truth equational, then, by Theorem 990, $\dot{S}^{\mathcal{I}}$ is soluble. Thus, by Proposition 991, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T = \dot{S}^{\mathcal{I}}(\Omega(T))$.

Conversely, if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T = \dot{S}^{\mathcal{I}}(\Omega(T))$, then, $\dot{S}^{\mathcal{I}}$ is soluble. Thus, again by Theorem 990, $\dot{S}^{\mathcal{I}}$ is a set of witnessing equations and $\mathcal{I}$ is strongly truth equational. ∎

It turns out that the property that separates family complete reflectivity from strong truth equationality is exactly the adequacy property of the unary Suszko core. Roughly speaking, this property ensures that the unary Suszko core is rich enough to define Suszko congruence systems in terms of the Leibniz congruence systems of theory families that it selects via inclusion.

We have the following relationship connecting the unary Suszko core with both Leibniz and Suszko congruence systems.

Proposition 993 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\bigcap\{\Omega(T) : \dot{S}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \leq \widetilde{\Omega}^{\mathcal{I}}(C(\phi)).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, we have, using Lemma 984,

$$S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) \quad \text{implies} \quad \dot{S}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

Therefore, $\{\Omega(T) : S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \subseteq \{\Omega(T) : \dot{S}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}$. We conclude that

$$\bigcap\{\Omega(T) : \dot{S}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \leq \bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \leq \widetilde{\Omega}^{\mathcal{I}}(C(\phi)),$$

where the last inclusion is based on Proposition 841. ∎

Again it is possible, but not necessary, that the unary Suszko core of a π-institution satisfies, for every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, the reverse inclusion of that given in Proposition 993:

$$\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) \leq \bigcap\{\Omega(T) : \dot{S}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.$$

Intuitively speaking, this means that the unary Suszko core $\dot{S}^{\mathcal{I}}$ is rich enough to allow, for every Σ-sentence ϕ, the determination of those theory families whose Leibniz congruence systems form a covering of the Suszko congruence system of $C(\phi)$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. We say that the unary Suszko core $\dot{S}^{\mathcal{I}}$ of $\mathcal{I}$ is **adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) = \bigcap\{\Omega(T) : \dot{S}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.$$

Based on our preceding work, it is not difficult to see that, if $\dot{S}^{\mathcal{I}}$ is soluble, then it is adequate.

Corollary 994 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $\dot{S}^{\mathcal{I}}$ is soluble, then it is adequate.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rcll} \widetilde{\Omega}^{\mathcal{I}}(C(\phi)) & = & \bigcap\{\Omega(T) : \phi \in T_\Sigma\} & (\text{definition of } \widetilde{\Omega}^{\mathcal{I}}(C(\phi))) \\ & = & \bigcap\{\Omega(T) : \dot{S}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}. & \\ & & (\text{solubility of } \dot{S}^{\mathcal{I}} \text{ and Proposition 991}) & \end{array}$$

We conclude that $\dot{S}^{\mathcal{I}}$ is adequate. ■

In the opposite direction, in a *family c-reflective* π-institution $\mathcal{I}$, if the unary Suszko core is adequate, then it is also soluble.

Proposition 995 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a family c-reflective π-institution based on $\mathbf{F}$. If $\dot{S}^{\mathcal{I}}$ is adequate, then it is soluble.*

Proof: Suppose that $\mathcal{I}$ is family c-reflective and that $\dot{S}^{\mathcal{I}}$ is adequate. We must show that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$

$$\phi \in T_\Sigma \quad \text{iff} \quad \dot{S}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

The implication left-to-right is always satisfied by Proposition 986. For the converse, assume that $\dot{S}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. Then, by the adequacy of $\dot{S}^{\mathcal{I}}$, we get that $\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) \leq \Omega(T)$. Thus, by family c-reflectivity and Lemma 826, we conclude that $C(\phi) \leq T$, which gives $\phi \in T_\Sigma$. ■

We finally show that a π-institution is strongly truth equational if and only if it is family c-reflective and has an adequate unary Suszko core.

$$\begin{array}{lll}\text{Strong Truth Equationality} & = & \dot{S}^{\mathcal{I}} \text{ Soluble}\\ & = & \dot{S}^{\mathcal{I}} \text{ Defines Theory Families}\\ & = & \text{Family c-Reflectivity} + \dot{S}^{\mathcal{I}} \text{ Adequate}\end{array}$$

Theorem 996 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is strongly truth equational if and only if it is family c-reflective and has an adequate unary Suszko core.*

Proof: Suppose, first, that $\mathcal{I}$ is strongly truth equational. Then it is family c-reflective by Theorem 829. Moreover, its unary Suszko core is soluble by Theorem 990 and, hence, by Corollary 994, its unary Suszko core is adequate.

Suppose, conversely, that $\mathcal{I}$ is family c-reflective with an adequate unary Suszko core. Then, by Proposition 995, its unary Suszko core is soluble and, therefore, by Theorem 990, $\mathcal{I}$ is strongly truth equational. ∎

We close the section with a result relating the unary Suszko core with the Suszko core. More precisely, we show that adequacy of the unary Suszko core implies adequacy of the Suszko core.

Proposition 997 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\dot{S}^{\mathcal{I}}$ *is adequate, then* $S^{\mathcal{I}}$ *is adequate.*

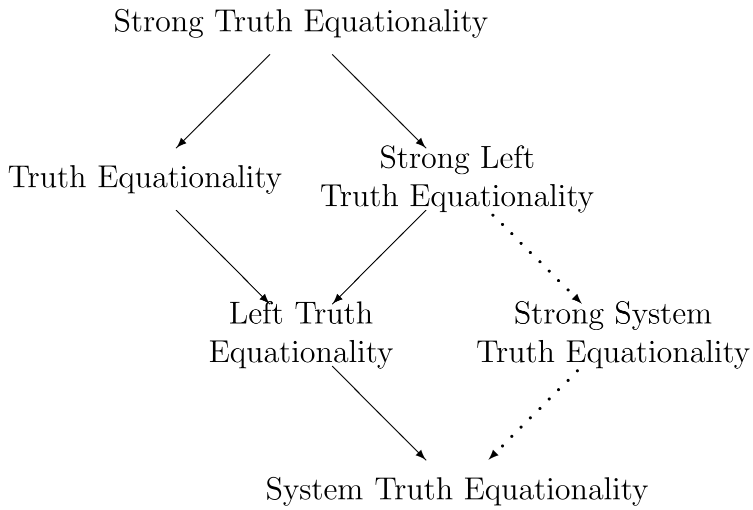
Proof: Suppose that $\dot{S}^{\mathcal{I}}$ is adequate. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{lll}\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) & \leq & \bigcap\{\Omega(T) : \dot{S}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \quad (\dot{S}^{\mathcal{I}} \text{ adequate})\\ & \leq & \bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \quad (\dot{S}^{\mathcal{I}} \subseteq S^{\mathcal{I}})\\ & \leq & \widetilde{\Omega}^{\mathcal{I}}(C(\phi)). \quad \text{(Proposition 841)}\end{array}$$

Hence, $\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) = \bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}$, and $S^{\mathcal{I}}$ is adequate. ∎

13.5 Strong Left Truth Equationality

We now undertake the study of strong left truth equationality. This combines, in a certain sense, the study of left truth equationality with that of strong truth equationality. Strong left truth equationality has the same relation to left truth equationality as strong truth equationality has to (family) truth equationality. After this study, we will have the following hierarchy of truth equationality properties, which will be further augmented in the following section by adjoining strong system truth equationality:



Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is **strongly left truth equational** if there exists a set $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ (with a single distinguished argument), such that, for every $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

In that case, we call $\tau^\flat$ a **set of witnessing equations** (of the strong left truth equationality of $\mathcal{I}$).

Note that, similarly to strong truth equationality, since $\tau^\flat$ is parameter-free and $\Omega(T)$ is invariant under signature morphisms, strong left truth equationality may be defined equivalently by the condition, for every $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma(\phi) \subseteq \Omega_\Sigma(T).$$

Recall that strong truth equationality implies systemicity. Therefore, if a π-institution $\mathcal{I}$ is strongly truth equational, we get, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{iff} \quad \phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T),$$

whence $\mathcal{I}$ is also strongly left truth equational.

We introduce next the unary left Suszko core of a π-institution. Analogously with the left Suszko core and the unary Suszko core, the unary left Suszko core enables one to obtain:

- A characterization of strong left truth equationality in terms of the solubility property of the unary left Suszko core of the π-institution.

- An exact description of those left c-reflective π-institutions which are strongly left truth equational.
- A characterization of those left truth equational π-institutions which are strongly left truth equational.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The **unary left Suszko core** of $\mathcal{I}$ is the collection

$$\dot{L}^{\mathcal{I}} = \{\sigma^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2 \in N^\flat : (\forall T \in \mathrm{ThFam}(\mathcal{I}))(\sigma^\flat[\overleftarrow{T}] \leq \widetilde{\Omega}^{\mathcal{I}}(T)\}.$$

There are a couple of different possible characterizations of the unary left Suszko core that can be given. One is in terms of theory systems in place of theory families and another uses theory systems generated by single sentences.

Proposition 998 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *Then*

$$\dot{L}^{\mathcal{I}} = \{\sigma^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2 \in N^\flat : (\forall T \in \mathrm{ThSys}(\mathcal{I}))(\sigma^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T)\}.$$

Proof: By Proposition 852, we have that

$$L^{\mathcal{I}} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThSys}(\mathcal{I}))(\sigma^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T)\}.$$

Thus, the conclusion follows by applying the $\dot{}$ operator on both sides, i.e., by intersecting both sides with the set of all pairs of unary natural transformations in $N^\flat$. ∎

With Proposition 998 at hand, the second characterization follows from Lemma 822.

Corollary 999 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *Then*

$$\begin{array}{rcl}\dot{L}^{\mathcal{I}} & = & \{\sigma^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2 \in N^\flat : (\forall \Sigma \in |\mathbf{Sign}^\flat|) \\ & & \qquad (\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\sigma^\flat_\Sigma(\phi) \in \widetilde{\Omega}^{\mathcal{I}}_\Sigma(C(\overrightarrow{\phi}))\}.\end{array}$$

Proof: By combining Proposition 998 with Lemma 822. ∎

Note that the unary left Suszko core of a π-institution is included in the left Suszko core, i.e., we have

Lemma 1000 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *Then* $\dot{L}^{\mathcal{I}} \subseteq L^{\mathcal{I}}$.

Proof: Every pair of unary natural transformations in $N^\flat$ that satisfies the membership criterion for $\dot{L}^{\mathcal{I}}$ also satisfies the condition for membership in $L^{\mathcal{I}}$. ■

Lemma 1000 yields immediately the following consequence.

Corollary 1001 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $T \in \mathrm{ThFam}(\mathcal{I})$*, we have*

$$L^{\mathcal{I}}(\Omega(T)) \leq \dot{L}^{\mathcal{I}}(\Omega(T)).$$

Proof: By Theorem 107 and Corollary 105. ■

Either directly by the definition or using Proposition 853 together with Corollary 1001, we get the following

Proposition 1002 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For every* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\overleftarrow{T} \leq \dot{L}^{\mathcal{I}}(\Omega(T)).$$

Proof: We have $\overleftarrow{T} \leq L^{\mathcal{I}}(\Omega(T)) \leq \dot{L}^{\mathcal{I}}(\Omega(T))$, where the first inclusion is by Lemma 853 and the second by Corollary 1001. ■

Similarly with the left Suszko core, the unary left Suszko core of a π-institution may or may not satisfy the reverse inclusion of Proposition 1002, a property that was called left solubility.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the unary left Suszko core of $\mathcal{I}$ is **left soluble** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\dot{L}^{\mathcal{I}}(\Omega(T)) \leq \overleftarrow{T}.$$

Note that $\dot{L}^{\mathcal{I}}$ is left soluble if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\dot{L}^{\mathcal{I}}_\Sigma(\phi) \subseteq \Omega_\Sigma(T) \quad \text{implies} \quad \phi \in \overleftarrow{T}_\Sigma.$$

It turns out that possession of left solubility by the unary left Suszko core intrinsically characterizes strong left truth equationality. To show the necessity of left solubility observe, first, that, in case a π-institution is strongly left truth equational, the witnessing equations form a subset of the unary left Suszko core.

Lemma 1003 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is strongly truth equational, with witnessing equations* $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2 \subseteq N^\flat$*, then* $\tau^\flat \subseteq \dot{L}^{\mathcal{I}}$*.*

Proof: Suppose that $\mathcal{I}$ is strongly left truth equational with witnessing equations $\tau^\flat$. Then, $\mathcal{I}$ is, a fortiori, left truth equational, with the same witnessing equations. It follows, by Lemma 857, that $\tau^\flat \subseteq L^{\mathcal{I}}$. Since $\tau^\flat$ consists of unary equations and they satisfy the membership criterion for $L^{\mathcal{I}}$, it follows that they also satisfy the condition for membership in $\dot{L}^{\mathcal{I}}$. Therefore, we get that $\tau^\flat \subseteq \dot{L}^{\mathcal{I}}$. ■

Now we prove the necessity of the left solubility of the unary left Suszko core for strong left truth equationality.

Theorem 1004 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is strongly left truth equational, then* $\dot{L}^{\mathcal{I}}$ *is left soluble.*

Proof: Suppose that $\mathcal{I}$ is strongly left truth equational, with witnessing equations $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll} \dot{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) & \text{implies} & \tau^\flat_\Sigma[\phi] \leq \Omega(T) \quad \text{(Lemma 1003)} \\ & \text{iff} & \phi \in \overleftarrow{T}_\Sigma. \quad \text{(strong left truth equationality)} \end{array}$$

Thus, $\dot{L}^{\mathcal{I}}$ is left soluble. ■

The reverse implication also holds and provides the characterization of strong left truth equationality in terms of the left solubility of the unary left Suszko core.

Theorem 1005 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\dot{L}^{\mathcal{I}}$ *is left soluble, then* $\mathcal{I}$ *is strongly left truth equational, with witnessing equations* $\dot{L}^{\mathcal{I}}$*.*

Proof: It suffices to show that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{iff} \quad \dot{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

The left-to-right implication is given in Proposition 1002, whereas the converse is ensured by the postulated left solubility of $\dot{L}^{\mathcal{I}}$. ■

Theorems 1004 and 1005 provide the promised characterization of strong left truth equationality in terms of the left solubility of the unary left Suszko core.

$$\mathcal{I} \text{ is Strongly Left Truth Equational} \longleftrightarrow \dot{L}^{\mathcal{I}} \text{ is Left Soluble.}$$

Theorem 1006 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is strongly left truth equational if and only if* $\dot{L}^{\mathcal{I}}$ *is left soluble.*

Proof: Theorem 1004 gives the "only if" and the "if" is by Theorem 1005. ■

If $\mathcal{I}$ is strongly left truth equational, then the unary left Suszko core defines theory families in $\mathcal{I}$, up to arrow, in terms of their Leibniz congruence systems. So, analogously to preceding situations, $\dot{L}^{\mathcal{I}}$ forms a maximal set of witnessing equations of the strong left truth equationality of $\mathcal{I}$.

Proposition 1007 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\dot{L}^{\mathcal{I}}$ *is left soluble, then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\overleftarrow{T} = \dot{L}^{\mathcal{I}}(\Omega(T)).$$

Proof: If $\dot{L}^{\mathcal{I}}$ is left soluble, then, by Theorem 1005, $\dot{L}^{\mathcal{I}}$ forms a set of witnessing equations for the strong left truth equationality of $\mathcal{I}$. Therefore, by Proposition 849, we get that, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} = \dot{L}^{\mathcal{I}}(\Omega(T))$. ■

This property provides another characterization of strong left truth equationality. We say that $\dot{L}^{\mathcal{I}}$ **defines theory families up to arrow** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} = \dot{L}^{\mathcal{I}}(\Omega(T))$. Then we have:

$\mathcal{I}$ is Strongly Left Truth Equational
$\longleftrightarrow$ $\dot{L}^{\mathcal{I}}$ Defines Theory Families Up to Arrow.

Theorem 1008 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is strongly left truth equational if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\overleftarrow{T} = \dot{L}^{\mathcal{I}}(\Omega(T)).$$

Proof: If $\mathcal{I}$ is strongly left truth equational, then, by Theorem 1006, $\dot{L}^{\mathcal{I}}$ is left soluble. Thus, by Proposition 1007, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} = \dot{L}^{\mathcal{I}}(\Omega(T))$.

Conversely, if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} = \dot{L}^{\mathcal{I}}(\Omega(T))$, then, $\dot{L}^{\mathcal{I}}$ is left soluble. Thus, again by Theorem 1006, $\dot{L}^{\mathcal{I}}$ is a set of witnessing equations and $\mathcal{I}$ is strongly left truth equational. ■

In analogy with strong truth equationality and family c-reflectivity, the property that separates left complete reflectivity from strong left truth equationality is exactly the left adequacy of the unary left Suszko core. Roughly speaking, this property ensures that the unary left Suszko core is rich enough to define Suszko congruence systems in terms of the Leibniz congruence systems of theory families that it selects via inclusion.

We have the following relationship connecting the unary left Suszko core with both Leibniz and Suszko congruence systems.

Proposition 1009 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\bigcap \{\Omega(T) : \dot{L}^{\mathcal{I}}_{\Sigma}[\phi] \leq \Omega(T)\} \leq \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, we have, using Lemma 1000,

$$L^{\mathcal{I}}_{\Sigma}[\phi] \leq \Omega(T) \quad \text{implies} \quad \dot{L}^{\mathcal{I}}_{\Sigma}[\phi] \leq \Omega(T).$$

Therefore, $\{\Omega(T) : L^{\mathcal{I}}_{\Sigma}[\phi] \leq \Omega(T)\} \subseteq \{\Omega(T) : \dot{L}^{\mathcal{I}}_{\Sigma}[\phi] \leq \Omega(T)\}$. We conclude that

$$\bigcap \{\Omega(T) : \dot{L}^{\mathcal{I}}_{\Sigma}[\phi] \leq \Omega(T)\} \leq \bigcap \{\Omega(T) : L^{\mathcal{I}}_{\Sigma}[\phi] \leq \Omega(T)\} \leq \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})),$$

where the last inclusion is based on Proposition 863. ∎

Again it is possible, but not necessary, that the unary left Suszko core of a π-institution satisfies, for every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, the reverse inclusion of that given in Proposition 1009:

$$\widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) \leq \bigcap \{\Omega(T) : \dot{L}^{\mathcal{I}}_{\Sigma}[\phi] \leq \Omega(T)\}.$$

Intuitively speaking, this means that the unary left Suszko core $\dot{L}^{\mathcal{I}}$ is rich enough to allow, for every signature Σ and for every Σ-sentence ϕ, the determination of those theory families whose Leibniz congruence systems form a covering of the Suszko congruence system of $C(\overrightarrow{\phi})$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the unary left Suszko core $\dot{L}^{\mathcal{I}}$ of $\mathcal{I}$ is **left adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) = \bigcap \{\Omega(T) : \dot{L}^{\mathcal{I}}_{\Sigma}[\phi] \leq \Omega(T)\}.$$

Based on our preceding work, it is not difficult to see that, if $\dot{L}^{\mathcal{I}}$ is left soluble, then it is left adequate.

Corollary 1010 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\dot{L}^{\mathcal{I}}$ *is left soluble, then it is left adequate.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rcll} \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) & = & \bigcap \{\Omega(T) : \overrightarrow{\phi} \leq T\} & (\text{definition of } \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi}))) \\ & = & \bigcap \{\Omega(T) : \phi \in \overleftarrow{T}_{\Sigma}\} & (\text{definition of } \overleftarrow{T}) \\ & = & \bigcap \{\Omega(T) : \dot{L}^{\mathcal{I}}_{\Sigma}[\phi] \leq \Omega(T)\}. & \\ & & (\text{left solubility of } \dot{L}^{\mathcal{I}} \text{ and Proposition 1007}) & \end{array}$$

We conclude that $\dot{L}^{\mathcal{I}}$ is left adequate. ∎

In the opposite direction, in a *left c-reflective* π-institution $\mathcal{I}$, if the unary left Suszko core is left adequate, then it is also left soluble.

Proposition 1011 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a left c-reflective* π*-institution based on* $\mathbf{F}$. *If* $\dot{L}^{\mathcal{I}}$ *is left adequate, then it is left soluble.*

Proof: Suppose that $\mathcal{I}$ is left c-reflective and that $\dot{L}^{\mathcal{I}}$ is left adequate. We must show that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{iff} \quad \dot{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

The implication left-to-right is always satisfied by Proposition 1002. For the converse, assume that $\dot{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. Then, by the left adequacy of $\dot{L}^{\mathcal{I}}$, we get that $\widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) \leq \Omega(T)$. Thus, by left c-reflectivity and Lemma 868, we conclude that $\overleftarrow{C(\overrightarrow{\phi})} \leq \overleftarrow{T}$. This implies $\phi \in \overleftarrow{T}_\Sigma$. ■

We finally show that a π-institution is strongly left truth equational if and only if it is left c-reflective and has a left adequate unary left Suszko core.

Strong Left Truth Equationality
= $\dot{L}^{\mathcal{I}}$ Left Soluble
= $\dot{L}^{\mathcal{I}}$ Defines Theory Families Up to Arrow
= Left c-Reflectivity + $\dot{L}^{\mathcal{I}}$ Left Adequate

Theorem 1012 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is strongly left truth equational if and only if it is left c-reflective and has a left adequate unary left Suszko core.*

Proof: Suppose, first, that $\mathcal{I}$ is strongly left truth equational. Then it is left c-reflective by Theorem 850. Moreover, its unary left Suszko core is left soluble by Theorem 1006 and, hence, by Corollary 1010, its unary left Suszko core is left adequate.

Suppose, conversely, that $\mathcal{I}$ is family c-reflective with a left adequate unary left Suszko core. Then, by Proposition 1011, its unary left Suszko core is left soluble. Hence, by Theorem 1006, $\mathcal{I}$ is strongly left truth equational. ■

We close the section with a result relating the unary left Suszko core with the left Suszko core. More precisely, we show that left adequacy of the unary left Suszko core implies left adequacy of the left Suszko core.

Proposition 1013 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\dot{L}^{\mathcal{I}}$ *is left adequate, then* $L^{\mathcal{I}}$ *is left adequate.*

Proof: Suppose that $\dot{L}^{\mathcal{I}}$ is left adequate. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rcll}\widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) & \leq & \bigcap\{\Omega(T) : \dot{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} & (\dot{L}^{\mathcal{I}} \text{ left adequate})\\ & \leq & \bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} & (\dot{L}^{\mathcal{I}} \subseteq L^{\mathcal{I}})\\ & \leq & \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})). & (\text{Proposition } 863)\end{array}$$

Hence, $\widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) = \bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}$, and $L^{\mathcal{I}}$ is left adequate. ∎

13.6 Strong System Truth Equationality

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is **strongly system truth equational** if there exists a set $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ (with a single distinguished argument), such that, for every $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

In that case, we call $\tau^\flat$ a **set of witnessing equations** (of/for the strong system truth equationality of $\mathcal{I}$).

Again, since $\tau^\flat$ is parameter-free and $\Omega(T)$ is invariant under signature morphisms, strong system truth equationality may be defined equivalently by the condition, for every $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma(\phi) \subseteq \Omega_\Sigma(T).$$

We introduce next the unary system core of a π-institution. Analogously with the system core, the unary system core enables one to obtain:

- A characterization of strong system truth equationality in terms of the solubility property of the unary system core of the π-institution.
- An exact description of those system c-reflective π-institutions which are strongly system truth equational.
- A characterization of those system truth equational π-institutions which are strongly system truth equational.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Recall that, for $T \in \mathrm{ThSys}(\mathcal{I})$, we have introduced the notation

$$\widehat{\Omega}^{\mathcal{I}}(T) = \bigcap\{\Omega(T') : T \leq T' \in \mathrm{ThSys}(\mathcal{I})\}.$$

This is a variant of the Suszko operator, allowing one to zoom in on the theory system structure of the π-institution under consideration, which forms naturally the focus in the present section.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The **unary system core** of $\mathcal{I}$ is the collection

$$\dot{Z}^{\mathcal{I}} = \{\sigma^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2 \in N^\flat : (\forall T \in \mathrm{ThSys}(\mathcal{I}))(\sigma^\flat[T] \leq \widehat{\Omega}^{\mathcal{I}}(T))\}.$$

Note that the unary system core of a π-institution is included in the system core, i.e., we have

Lemma 1014 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then* $\dot{Z}^{\mathcal{I}} \subseteq Z^{\mathcal{I}}$*.*

Proof: Every pair of unary natural transformations in $N^\flat$ that satisfies the membership criterion for $\dot{Z}^{\mathcal{I}}$ also satisfies the condition for membership in $Z^{\mathcal{I}}$. ■

Moreover, we have the following relationship between the sentence families defined via the Leibniz congruence systems by the system and the unary system core.

Corollary 1015 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $T \in \mathrm{ThFam}(\mathcal{I})$*, we have*

$$Z^{\mathcal{I}}(\Omega(T)) \leq \dot{Z}^{\mathcal{I}}(\Omega(T)).$$

Proof: By Theorem 107 and Corollary 105. ■

The relation between the unary Suszko core, the unary left Suszko core and the unary system core of a π-institution $\mathcal{I}$ is given in the following proposition, forming an analog of Proposition 874, concerning the general (non-unary) analogs of these sets.

Proposition 1016 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\dot{S}^{\mathcal{I}} \subseteq \dot{L}^{\mathcal{I}} \subseteq \dot{Z}^{\mathcal{I}}$*;*

(b) For every relation family θ *on* $\mathbf{F}$*,* $\dot{Z}^{\mathcal{I}}(\theta) \leq \dot{L}^{\mathcal{I}}(\theta) \leq \dot{S}^{\mathcal{I}}(\theta)$*.*

Proof: From Proposition 874, we have that $S^{\mathcal{I}} \subseteq L^{\mathcal{I}} \subseteq Z^{\mathcal{I}}$. Thus, Part (a) follows by applying the $\dot{}$ operator (which is monotone) to this chain of inclusions. Part (b) follows form Part (a) and the relevant definitions. ■

Either directly by the definition or using Proposition 875 together with Corollary 1015, we get the following

Proposition 1017 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For every* $T \in \mathrm{ThSys}(\mathcal{I})$*,*

$$T \leq \dot{Z}^{\mathcal{I}}(\Omega(T)).$$

Proof: We have $T \leq Z^{\mathcal{I}}(\Omega(T)) \leq \dot{Z}^{\mathcal{I}}(\Omega(T))$, where the first inclusion is by Lemma 875 and the second by Corollary 1015. ∎

The unary system core of a π-institution may or may not satisfy the reverse inclusion of Proposition 1017, a property that was called previously, in similar contexts, *solubility*.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the unary system core of $\mathcal{I}$ is **soluble** if, for all $T \in \mathrm{ThSys}(\mathcal{I})$,

$$\dot{Z}^{\mathcal{I}}(\Omega(T)) \leq T.$$

Note that $\dot{Z}^{\mathcal{I}}$ is soluble if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\dot{Z}^{\mathcal{I}}_\Sigma(\phi) \subseteq \Omega_\Sigma(T) \quad \text{implies} \quad \phi \in T_\Sigma.$$

It turns out that possession of the solubility property by the unary system core intrinsically characterizes strong system truth equationality. To show the necessity of solubility, we observe, once again, that, in case a π-institution is strongly system truth equational, the witnessing equations form a subset of the unary system core.

Lemma 1018 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is strongly system truth equational, with witnessing equations $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2 \subseteq N^\flat$, then $\tau^\flat \subseteq \dot{Z}^{\mathcal{I}}$.*

Proof: Suppose that $\mathcal{I}$ is strongly system truth equational with witnessing equations $\tau^\flat$. Then, $\mathcal{I}$ is, a fortiori, system truth equational, with the same witnessing equations. It follows, by Lemma 877, that $\tau^\flat \subseteq Z^{\mathcal{I}}$. Since $\tau^\flat$ consists of unary equations and they satisfy the membership criterion for $Z^{\mathcal{I}}$, it follows that they also satisfy the condition for membership in $\dot{Z}^{\mathcal{I}}$. Therefore, we get that $\tau^\flat \subseteq \dot{Z}^{\mathcal{I}}$. ∎

Now we prove the necessity of the solubility of the unary system core for strong system truth equationality.

Theorem 1019 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is strongly system truth equational, then $\dot{Z}^{\mathcal{I}}$ is soluble.*

Proof: Suppose that $\mathcal{I}$ is strongly system truth equational, with witnessing equations $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$. Then, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{llll}\dot{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) & \text{implies} & \tau^\flat_\Sigma[\phi] \leq \Omega(T) & \text{(Lemma 1018)}\\ & \text{iff} & \phi \in T_\Sigma, & \end{array}$$

where the last equivalence is based on the postulated strong system truth equationality of $\mathcal{I}$. Thus, $\dot{Z}^{\mathcal{I}}$ is soluble. ∎

The reverse implication completes the promised characterization of strong system truth equationality in terms of the solubility of the unary system core.

Theorem 1020 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\dot{Z}^{\mathcal{I}}$ *is soluble, then* $\mathcal{I}$ *is strongly system truth equational, with witnessing equations* $\dot{Z}^{\mathcal{I}}$*.*

Proof: It suffices to show that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \dot{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

The left-to-right implication is given in Proposition 1017, whereas the converse is ensured by the postulated solubility of $\dot{Z}^{\mathcal{I}}$. ∎

Theorems 1019 and 1020 provide the promised characterization of strong system truth equationality in terms of the solubility of the unary system core.

$$\mathcal{I} \text{ is Strongly System Truth Equational} \longleftrightarrow \dot{Z}^{\mathcal{I}} \text{ is Soluble.}$$

Theorem 1021 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is strongly system truth equational if and only if* $\dot{Z}^{\mathcal{I}}$ *is soluble.*

Proof: Theorem 1019 gives the "only if" and the "if" is by Theorem 1020. ∎

If $\mathcal{I}$ is strongly system truth equational, then the unary system core defines theory systems in $\mathcal{I}$ in terms of their Leibniz congruence systems. This proposition may be viewed as a special case of Proposition 871, since $\dot{Z}^{\mathcal{I}}$ forms a maximal set of witnessing equations of the strong system truth equationality of $\mathcal{I}$.

Proposition 1022 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\dot{Z}^{\mathcal{I}}$ *is soluble, then, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,*

$$T = \dot{Z}^{\mathcal{I}}(\Omega(T)).$$

Proof: If $\dot{Z}^{\mathcal{I}}$ is soluble, then, by Theorem 1020, $\dot{Z}^{\mathcal{I}}$ forms a set of witnessing equations for the strong system truth equationality of $\mathcal{I}$. Therefore, by Proposition 871, we get that, for every $T \in \mathrm{ThSys}(\mathcal{I})$, $T = \dot{Z}^{\mathcal{I}}(\Omega(T))$. ∎

This property provides another characterization of strong system truth equationality. We say that $\dot{Z}^{\mathcal{I}}$ **defines theory systems** if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $T = \dot{Z}^{\mathcal{I}}(\Omega(T))$. Then we have:

$$\mathcal{I} \text{ is Strongly System Truth Equational} \longleftrightarrow \dot{Z}^{\mathcal{I}} \text{ Defines Theory Systems.}$$

Theorem 1023 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is strongly system truth equational if and only if, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,*

$$T = \dot{Z}^{\mathcal{I}}(\Omega(T)).$$

Proof: If $\mathcal{I}$ is strongly system truth equational, then, by Theorem 1021, $\dot{Z}^{\mathcal{I}}$ is soluble. Thus, by Proposition 1022, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $T = \dot{Z}^{\mathcal{I}}(\Omega(T))$.

Conversely, if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $T = \dot{Z}^{\mathcal{I}}(\Omega(T))$, then, $\dot{Z}^{\mathcal{I}}$ is soluble. Thus, again by Theorem 1021, $\dot{Z}^{\mathcal{I}}$ is a set of witnessing equations and $\mathcal{I}$ is strongly system truth equational. ∎

It turns out that the property that separates system complete reflectivity from strong system truth equationality is exactly the adequacy property of the unary system core. Roughly speaking, this property ensures that the unary system core is rich enough to define the congruence system $\widehat{\Omega}^{\mathcal{I}}(T)$ of a theory system T in terms of the Leibniz congruence systems of theory systems that it selects via inclusion.

Proposition 1024 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \dot{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \leq \widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $T \in \mathrm{ThSys}(\mathcal{I})$, we have, using Lemma 1014,

$$Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) \quad \text{implies} \quad \dot{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

Therefore,

$$\begin{aligned} &\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \\ &\qquad \subseteq \{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \dot{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}. \end{aligned}$$

We conclude that

$$\begin{aligned} &\bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \dot{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \\ &\qquad \leq \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \leq \widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})), \end{aligned}$$

where the last inclusion is based on Proposition 883. ∎

It is possible, but not necessary, that the unary system core of a π-institution satisfies, for every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, the reverse inclusion of that given in Proposition 1024:

$$\widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) \leq \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \dot{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.$$

Intuitively speaking, this means that the unary system core $\dot{Z}^{\mathcal{I}}$ is rich enough to allow, for every signature Σ and every Σ-sentence ϕ, the determination of those theory systems whose Leibniz congruence systems form a covering of $\widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi}))$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the unary system core $\dot{Z}^{\mathcal{I}}$ of $\mathcal{I}$ is **adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) = \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \dot{Z}^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)\}.$$

Based on our preceding work, it is not difficult to see that, if $\dot{Z}^{\mathcal{I}}$ is soluble, then it is adequate.

Corollary 1025 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\dot{Z}^{\mathcal{I}}$ is soluble, then it is adequate.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rcl} \widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) & = & \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \phi \in T_\Sigma\} \\ & & (\text{definition of } \widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi}))) \\ & = & \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \dot{Z}^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)\}. \\ & & (\text{solubility of } \dot{Z}^{\mathcal{I}} \text{ and Proposition 1022}) \end{array}$$

We conclude that $\dot{Z}^{\mathcal{I}}$ is adequate. ■

In the opposite direction, in a *system c-reflective* π-institution $\mathcal{I}$, if the unary system core is adequate, then it is also soluble.

Proposition 1026 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a system c-reflective π-institution based on $\mathbf{F}$. If $\dot{Z}^{\mathcal{I}}$ is adequate, then it is soluble.*

Proof: Suppose that $\mathcal{I}$ is system c-reflective and that $\dot{Z}^{\mathcal{I}}$ is adequate. We must show that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \dot{Z}^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T).$$

The implication left-to-right is always satisfied by Proposition 1017. For the converse, assume that $\dot{Z}^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)$. Then, by the adequacy of $\dot{Z}^{\mathcal{I}}$, we get that $\widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) \le \Omega(T)$. Thus, by system c-reflectivity and Lemma 885, we conclude that $C(\overrightarrow{\phi}) \le T$, which gives $\phi \in T_\Sigma$. ■

We finally show that a π-institution is strongly system truth equational if and only if it is system c-reflective and has an adequate unary system core.

Strong System Truth Equationality
= $\dot{Z}^{\mathcal{I}}$ Soluble
= $\dot{Z}^{\mathcal{I}}$ Defines Theory Systems
= System c-Reflectivity + $\dot{Z}^{\mathcal{I}}$ Adequate

Theorem 1027 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is strongly system truth equational if and only if it is system c-reflective and has an adequate unary system core.*

Proof: Suppose, first, that $\mathcal{I}$ is strongly system truth equational. Then it is system c-reflective by Theorem 872. Moreover, its unary system core is soluble by Theorem 1021 and, hence, by Corollary 1025, its unary system core is adequate.

Suppose, conversely, that $\mathcal{I}$ is system c-reflective with an adequate unary system core. Then, by Proposition 1026, its unary system core is soluble and, therefore, by Theorem 1021, $\mathcal{I}$ is strongly system truth equational. ∎

We close the section with a result relating the unary system core with the system core. More precisely, we show that adequacy of the unary system core implies adequacy of the system core.

Proposition 1028 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\dot{Z}^{\mathcal{I}}$ *is adequate, then* $Z^{\mathcal{I}}$ *is adequate.*

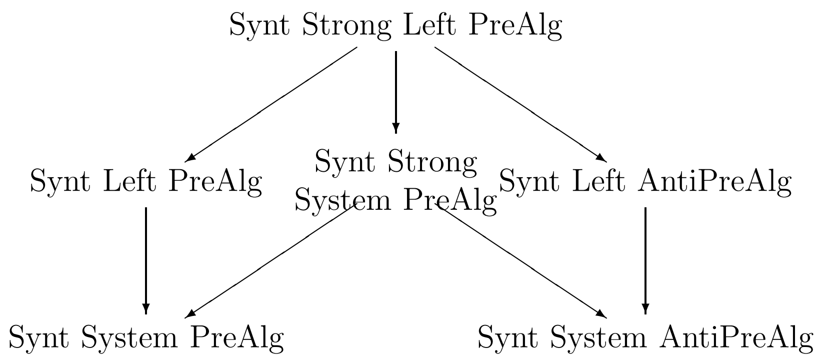
Proof: Suppose that $\dot{Z}^{\mathcal{I}}$ is adequate. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{aligned} \widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) &\leq \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \dot{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \\ &\quad (\dot{Z}^{\mathcal{I}} \text{ adequate}) \\ &\leq \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \\ &\quad (\dot{Z}^{\mathcal{I}} \subseteq Z^{\mathcal{I}}) \\ &\leq \widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})). \quad \text{(Proposition 883)} \end{aligned}$$

Hence, $\widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) = \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}$, and $Z^{\mathcal{I}}$ is adequate. ∎

13.7 Syntactic Left PreAlgebraizability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution. Recall that $\mathcal{I}$ belongs to one of the classes of the *prealgebraizability hierarchy* when its Leibniz operator is monotone on theory systems and it has a certain kind of extensionality and a certain kind of injectivity or reflectivity or complete reflectivity property. We now turn to corresponding properties defined via "syntactic" means. Keeping a level of consistency, we will call *syntactically prealgebraizable* any π-institution whose Leibniz operator on theory systems is definable via a set of binary natural transformations in $N^\flat$, i.e., a parameter free set of natural transformations in $N^\flat$, and, additionally, has a certain kind of definability property of truth, via a, possibly, parameter free set of equations in $N^\flat$. If the situation is reversed and definability of truth is required to be via a parameter free set of equations, but that is not demanded of the definability of Leibniz congruence systems, then we obtain the classes of *anti-prealgebraizable π-institutions*, a term concocted here to convey a kind of chiral symmetry in applying "parameterlessness". The hierarchy we aim for consists of the six classes depicted in the following diagram.



Membership in the classes of the central column imposes parameter free definability of both the Leibniz operator on theory systems and a kind of parameter free definability of truth. Membership in the classes in the left column insists only on parameter free definability of the Leibniz operator, whereas, symmetrically, membership in the classes of the right column postulates only a kind of parameter free definability of truth.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. In agreement with preestablished nomenclature, we say that $\mathcal{I}$ is $\ddot{R}^{\mathcal{I}}\dot{L}^{\mathcal{I}}$-**fortified** if it has

- a Leibniz binary reflexive core and
- a left adequate unary left Suszko core.

We say that $\mathcal{I}$ is **syntactically strongly left prealgebraizable** if it is

- $\ddot{R}^{\mathcal{I}}\dot{L}^{\mathcal{I}}$-fortified;
- preequivalential (i.e., prealgebraic and system extensional);
- left c-reflective.

Our preceding work in this chapter has paved the way for the following important characterization of syntactic strong left prealgebraizability.

Theorem 1029 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is syntactically strongly left prealgebraizable if and only if it is syntactically preequivalential and strongly left truth equational.*

Proof: We have that $\mathcal{I}$ is syntactically strongly left prealgebraizable if and only if, by definition, it is

- preequivalential and has a Leibniz binary reflexive core;
- left c-reflective and has a left adequate unary left Suszko core;

if and only if, by Theorems 969 and 1012, is it syntactically preequivalential and strongly left truth equational. ∎

An alternative characterization along similar lines relates the syntactic with the corresponding semantic notions introduced in the context of prealgebraizability.

Theorem 1030 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically strongly left prealgebraizable if and only if it is LC prealgebraizable and* $\ddot{R}^{\mathcal{I}}\dot{L}^{\mathcal{I}}$*-fortified.*

Proof: We have that $\mathcal{I}$ is syntactically strongly left prealgebraizable if and only if, by definition,

- it is preequivalential and left c-reflective;
- it has a Leibniz binary reflexive core and a left adequate unary left Suszko core;

if and only if, by definition, it is LC prealgebraizable and $\ddot{R}^{\mathcal{I}}\dot{L}^{\mathcal{I}}$-fortified. ∎

This characterization in terms of semantic properties and preceding work on transference of properties from theory families/systems to filter families/systems on arbitrary algebraic systems yield yet another characterization of syntactic strong left prealgebraizability, which may also be viewed as a kind of transfer property for this class in its own right.

Theorem 1031 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically strongly left prealgebraizable if and only if it is* $\ddot{R}^{\mathcal{I}}\dot{L}^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ *is monotone on* $\mathcal{I}$*-filter systems, system extensional and left c-reflective.*

Proof: We have that $\mathcal{I}$ is syntactically strongly left prealgebraizable if and only if, by Theorem 1030, it is $\ddot{R}^{\mathcal{I}}\dot{L}^{\mathcal{I}}$-fortified and LC prealgebraizable if and only if, by Theorem 349, it is $\ddot{R}^{\mathcal{I}}\dot{L}^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ is monotone on $\mathcal{I}$-filter systems, system extensional and left c-reflective. ∎

Turning now to characterizations involving property preserving mappings between posets of filter families and congruence systems, we have the following result:

Theorem 1032 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly left prealgebraizable if and only if it is* $\ddot{R}^{\mathcal{I}}\dot{L}^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is a left completely order reflecting surjection that restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

that commutes with inverse logical extensions.

Proof: We have that $\mathcal{I}$ is syntactically strongly left prealgebraizable if and only if, by Theorem 1030, it is $\ddot{R}^{\mathcal{I}}\dot{L}^{\mathcal{I}}$-fortified and LC prealgebraizable if and only if, by Theorem 355 it is $\ddot{R}^{\mathcal{I}}\dot{L}^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is a left completely order reflecting surjection that restricts to an order embedding $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ that commutes with inverse logical extensions. ∎

Finally, in terms of conjugate pairs of transformations, we get the following analog of Theorem 949.

Theorem 1033 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly left prealgebraizable if and only if it is strongly left truth equational and its systemic skeleton* $\mathcal{K}^{\mathcal{I}}$ *is equivalent to* $\mathcal{Q}^{\mathcal{I}\bullet}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}\bullet}$ *of natural transformations.*

Proof: Suppose, first, that $\mathcal{I}$ is strongly left truth equational and $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair of natural transformations. Then it is strongly left truth equational and, by Theorem 941, it is syntactically preequivalential. Thus, by Theorem 1029, it is syntactically strongly left prealgebraizable.

Suppose, conversely, that $\mathcal{I}$ is syntactically strongly left prealgebraizable. Then, by Theorem 1029, it is strongly left truth equational and syntactically preequivalential. Hence, by Theorem 934, it is syntactically WS prealgebraizable. Now it follows by Theorem 940 that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a pair $(\tau^\flat, \overleftrightarrow{I}^\flat)$, where $I^\flat$ witnesses the syntactic preequivalentiality and $\tau^\flat$ the syntactic strong left truth equationality of $\mathcal{I}$, and, hence, by definition, they constitute a conjugate pair of natural transformations. ∎

Again, the equivalence of the systemic skeleton with some algebraic π-structure via a conjugate pair of natural transformations, coupled with strong left truth equationality, is sufficient to ensure syntactic strong left prealgebraizability.

Theorem 1034 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically strongly left prealgebraizable if and only if it is strongly left truth equational and its systemic skeleton is equivalent to an algebraic* π*-structure via a conjugate pair of natural transformations.*

Proof: If $\mathcal{I}$ is syntactically strongly left prealgebraizable, then, by Theorem 1033, it is strongly left truth equational and its systemic skeleton is equivalent to an algebraic π-structure via a conjugate pair of natural transformations. Suppose, conversely, that $\mathcal{I}$ is strongly left truth equational and its systemic skeleton is equivalent to an algebraic π-structure via a conjugate pair of natural transformations. Then, it is strongly left truth equational and, by Proposition 928, it is syntactically preequivalential. Therefore, by Theorem 1029, it is syntactically strongly left prealgebraizable. ■

Finally, in terms of order isomorphisms between theory family lattices, we have the following alternative characterization of syntactically strongly left prealgebraizable π-institutions:

Theorem 1035 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically strongly left prealgebraizable if and only if it is strongly left truth equational and there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{K}^{\mathcal{I}}) \to \mathbf{ThFam}(\mathcal{Q})$*, induced by a conjugate pair* $(\tau^\flat, I^\flat)$ *of natural transformations, where* $\mathcal{Q}$ *is an algebraic* π*-structure.*

Proof: The "only if" follows by Theorem 1034 and Theorem 893. The "if" is given by Theorem 901 and Theorem 1034. ■

Flanking the class of syntactically strongly left prealgebraizable π-institutions are the classes of *syntactically left prealgebraizable* and *syntactically left antiprealgebraizable* π-institutions. These two classes are defined formally now.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

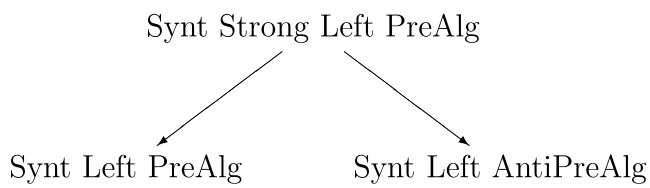
- $\mathcal{I}$ is **syntactically left prealgebraizable** if it is:
  - $\ddot{R}^{\mathcal{I}} L^{\mathcal{I}}$-fortified;
  - preequivalential;
  - left c-reflective;
- $\mathcal{I}$ is **syntactically left antiprealgebraizable** if it is:
  - $R^{\mathcal{I}} \dot{L}^{\mathcal{I}}$-fortified;
  - prealgebraic;

– left c-reflective.

For both of these classes we have analogs of many of the results proven above for syntactic strong left prealgebraizability. Before formulating them, let us observe that, since:

- preequivalentiality is stronger than prealgebraicity;
- under prealgebraicity, $\ddot{R}^{\mathcal{I}}$ Leibniz implies $R^{\mathcal{I}}$ Leibniz; and
- $\dot{L}^{\mathcal{I}}$ left adequate implies $L^{\mathcal{I}}$ left adequate,

we get, immediately from the definitions, the following hierarchical relations between the upper three classes in the echelon formation of the preceding diagram.



We now provide examples to show that the two inclusions are proper. The first is an example of a π-institution which is syntactically left prealgebraizable but not syntactically strongly left prealgebraizable.

Example 1036 EXAMPLE NOT FOUND YET!!

Next, we give an example of a syntactically left antiprealgebraizable π-institution which fails to be syntactically strongly left prealgebraizable.

Example 1037 EXAMPLE NOT FOUND YET!!

The following analog of Theorem 1029 relates these two chiral types of syntactic left prealgebraizability with various classes introduced previously, providing some important characterizations.

Theorem 1038 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is syntactically left prealgebraizable if and only if it is syntactically preequivalential and left truth equational.*

(b) $\mathcal{I}$ *is syntactically left antiprealgebraizable if and only if it is syntactically prealgebraic and strongly left truth equational.*

Proof:

(a) We have that $\mathcal{I}$ is syntactically left prealgebraizable if and only if, by definition, it is preequivalential, with a Leibniz binary reflexive core, and left c-reflective, with a left adequate left Suszko core, if and only if, by Theorems 969 and 870, is it syntactically preequivalential and left truth equational.

(b) Similarly, $\mathcal{I}$ is syntactically left antiprealgebraizable if and only if, by definition, it is prealgebraic, with a Leibniz reflexive core, and left c-reflective, with a left adequate unary left Suszko core, if and only if, by Theorems 788 and 1012, is it syntactically prealgebraic and strongly left truth equational.

■

An alternative characterization, analogous to that of Theorem 1030, relates the syntactic with the corresponding semantic notions.

Theorem 1039 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is syntactically left prealgebraizable if and only if it is LC prealgebraizable and* $\ddot{R}^{\mathcal{I}} L^{\mathcal{I}}$*-fortified.*

(b) $\mathcal{I}$ *is syntactically left antiprealgebraizable if and only if it is weakly LC prealgebraizable and* $R^{\mathcal{I}} \dot{L}^{\mathcal{I}}$*-fortified.*

Proof:

(a) We have that $\mathcal{I}$ is syntactically left prealgebraizable if and only if, by definition, it is preequivalential and left c-reflective and, moreover, it has a Leibniz binary reflexive core and a left adequate left Suszko core. This happens if and only if, by definition, it is LC prealgebraizable and $\ddot{R}^{\mathcal{I}} L^{\mathcal{I}}$-fortified.

(b) Similarly, $\mathcal{I}$ is syntactically left antiprealgebraizable if and only if, by definition, it is prealgebraic and left c-reflective and, moreover, it has a Leibniz reflexive core and a left adequate unary left Suszko core. This happens if and only if, by definition, it is weakly LC prealgebraizable and $R^{\mathcal{I}} \dot{L}^{\mathcal{I}}$-fortified.

■

This characterization in terms of semantic properties and preceding work on transference of properties from theory families/systems to filter families/systems on arbitrary algebraic systems yield a kind of transfer property for syntactic left (anti)prealgebraizability.

Theorem 1040 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) *$\mathcal{I}$ is syntactically left prealgebraizable if and only if it is $\ddot{R}^{\mathcal{I}}L^{\mathcal{I}}$-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ *is monotone on $\mathcal{I}$-filter systems, system extensional and left c-reflective.*

(b) *$\mathcal{I}$ is syntactically left antiprealgebraizable if and only if it is $R^{\mathcal{I}}\dot{L}^{\mathcal{I}}$-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ *is monotone on $\mathcal{I}$-filter systems and left c-reflective.*

Proof: We prove only Part (a), since Part (b) is similar. We have that $\mathcal{I}$ is syntactically left prealgebraizable if and only if, by Theorem 1039, it is $\ddot{R}^{\mathcal{I}}L^{\mathcal{I}}$-fortified and LC prealgebraizable if and only if, by Theorem 349, it is $\ddot{R}^{\mathcal{I}}L^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ is monotone on $\mathcal{I}$-filter systems, system extensional and left c-reflective. ■

Turning now to characterizations involving property preserving mappings between posets of filter families and of congruence systems, we have the following result:

Theorem 1041 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$.

(a) *$\mathcal{I}$ is syntactically left prealgebraizable if and only if it is $\ddot{R}^{\mathcal{I}}L^{\mathcal{I}}$-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is a left completely order reflecting surjection that restricts to an order embedding $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ *that commutes with inverse logical extensions.*

(b) *$\mathcal{I}$ is syntactically left antiprealgebraizable if and only if it is $R^{\mathcal{I}}\dot{L}^{\mathcal{I}}$-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is a left completely order reflecting surjection that restricts to an order embedding $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$.

Proof: Again we show only Part (a). Part (b) is similar. We have that $\mathcal{I}$ is syntactically left prealgebraizable if and only if, by Theorem 1039, it is $\ddot{R}^{\mathcal{I}}L^{\mathcal{I}}$-fortified and LC prealgebraizable if and only if, by Theorem 355 it is $\ddot{R}^{\mathcal{I}}L^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is a left completely order reflecting surjection that restricts to an order embedding $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ that commutes with inverse logical extensions. ■

Finally, in terms of conjugate pairs of transformations, we get the following analog of Theorem 1033.

Theorem 1042 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is syntactically left prealgebraizable if and only if it is left truth equational and its systemic skeleton* $\mathcal{K}^{\mathcal{I}}$ *is equivalent to* $\mathcal{Q}^{\mathcal{I}\bullet}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}\bullet}$ *of transformations, with* $I^\flat$ *natural.*

(b) $\mathcal{I}$ *is syntactically left antiprealgebraizable if and only if it is strongly left truth equational and its systemic skeleton* $\mathcal{K}^{\mathcal{I}}$ *is equivalent to* $\mathcal{Q}^{\mathcal{I}\bullet}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}\bullet}$ *of transformations, with* $\tau^\flat$ *natural.*

Proof:

(a) Suppose, first, that $\mathcal{I}$ is left truth equational and $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $I^\flat$ natural. Then it is left truth equational and, by Theorem 941, it is syntactically preequivalential. Thus, by Theorem 1038, it is syntactically left prealgebraizable.

Suppose, conversely, that $\mathcal{I}$ is syntactically left prealgebraizable. Then, by Theorem 1038, it is left truth equational and syntactically preequivalential. Hence, by Theorem 934, it is syntactically WS prealgebraizable. Now it follows by Theorem 940 that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a pair $(\tau^\flat, \overleftrightarrow{I^\flat})$, where $I^\flat$ witnesses the syntactic preequivalentiality and $\tau^\flat$ the left truth equationality of $\mathcal{I}$, and, hence, by definition, they constitute a conjugate pair of transformations, with $I^\flat$ natural.

(b) Suppose, first, that $\mathcal{I}$ is strongly left truth equational and $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $\tau^\flat$ natural. Then it is strongly left truth equational and, by Theorem 941, it is syntactically prealgebraic. Thus, by Theorem 1038, it is syntactically left antiprealgebraizable.

Suppose, conversely, that $\mathcal{I}$ is syntactically left antiprealgebraizable. Then, by Theorem 1038, it is strongly left truth equational and syntactically prealgebraic. Hence, by Theorem 934, it is syntactically WS prealgebraizable. Now it follows by Theorem 940 that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a pair $(\tau^\flat, \overleftrightarrow{I^\flat})$, where $I^\flat$ witnesses the syntactic prealgebraicity and $\tau^\flat$ the strong left truth equationality of $\mathcal{I}$, and, hence, by definition, they constitute a conjugate pair of transformations, with $\tau^\flat$ natural.

■

The equivalence of the systemic skeleton with some algebraic π-structure via a conjugate pair of transformations, exhibiting the required one-sided naturality condition, coupled with either left truth equationality or strong left truth equationality, depending on the case considered, is sufficient to ensure syntactic left (anti)prealgebraizability.

Theorem 1043 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is syntactically left prealgebraizable if and only if it is left truth equational and its systemic skeleton is equivalent to an algebraic* π*-structure via a conjugate pair* $(\tau^\flat, I^\flat)$ *of transformations, with* $I^\flat$ *natural.*

(b) $\mathcal{I}$ *is syntactically left antiprealgebraizable if and only if it is strongly left truth equational and its systemic skeleton is equivalent to an algebraic* π*-structure via a conjugate pair* $(\tau^\flat, I^\flat)$ *of transformations, with* $\tau^\flat$ *natural.*

Proof:

(a) If $\mathcal{I}$ is syntactically left prealgebraizable, then, by Theorem 1042, it is left truth equational and its systemic skeleton is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations with $I^\flat$ natural. Suppose, conversely, that $\mathcal{I}$ is left truth equational and its systemic skeleton is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $I^\flat$ natural. Then, it is left truth equational and, by Proposition 928, it is syntactically preequivalential. Therefore, by Theorem 1038, it is syntactically left prealgebraizable.

(b) If $\mathcal{I}$ is syntactically left antiprealgebraizable, then, by Theorem 1042, it is strongly left truth equational and its systemic skeleton is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $\tau^\flat$ natural. Suppose, conversely, that $\mathcal{I}$ is strongly left truth equational and its systemic skeleton is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $\tau^\flat$ natural. Then, it is strongly left truth equational and, by Proposition 928, it is syntactically prealgebraic. Therefore, by Theorem 1038, it is syntactically left antiprealgebraizable.

■

Finally, in terms of order isomorphisms between theory family lattices, we have the following alternative characterization of syntactically left (anti)prealgebraizable π-institutions:

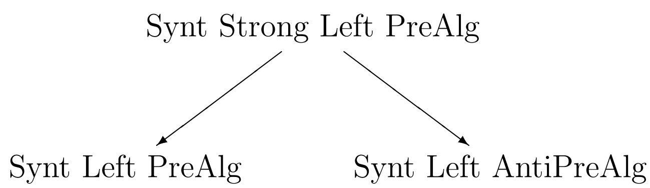
Theorem 1044 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is syntactically left prealgebraizable if and only if it is left truth equational and there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{K}^{\mathcal{I}}) \to \mathbf{ThFam}(\mathcal{Q})$*, induced by a conjugate pair* $(\tau^\flat, I^\flat)$ *of transformations, where* $\mathcal{Q}$ *is an algebraic* π*-structure and* $I^\flat$ *is natural.*

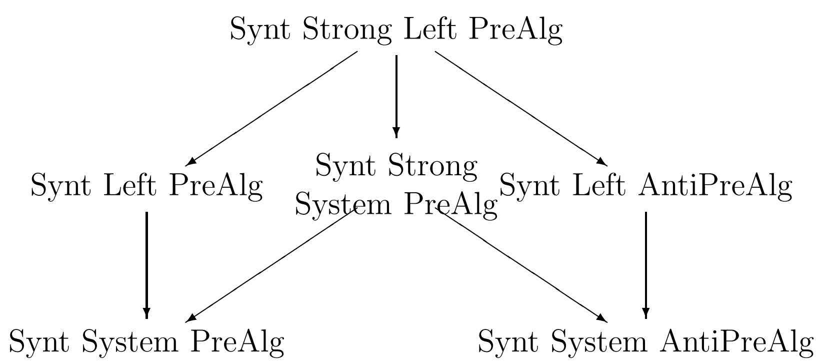
(b) *$\mathcal{I}$ is syntactically left antiprealgebraizable if and only if it is strongly left truth equational and there exists a transformational order isomorphism $h : \mathbf{ThFam}(\mathcal{K}^{\mathcal{I}}) \to \mathbf{ThFam}(\mathcal{Q})$, induced by a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, where $\mathcal{Q}$ is an algebraic π-structure and $\tau^\flat$ is natural.*

Proof: The "only if" follows by Theorem 1043 and Theorem 893. The "if" is given by Theorem 901 and Theorem 1043. ∎

In this section we have introduced the three syntactic left prealgebraizability classes



In the next section, we shall introduce, following a similar path, the remaining three syntactic prealgebraizability classes, namely those of the system prealgebraizable π-institutions, in order to complete the syntactic prealgebraizability hierarchy that was described at the beginning of the section:



13.8 Syntactic System PreAlgebraizability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that $\mathcal{I}$ is **$\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified** if it has

- a Leibniz binary reflexive core; and
- an adequate unary system core.

We say that $\mathcal{I}$ is **syntactically strongly system prealgebraizable** if it is

- $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified;
- preequivalential (i.e., prealgebraic and system extensional);

- system c-reflective.

An analog of Theorem 1029 provides an important characterization of syntactic strong system prealgebraizability in terms of lower classes in the syntactic hierarchy.

Theorem 1045 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically strongly system prealgebraizable if and only if it is syntactically preequivalential and strongly system truth equational.*

Proof: We have that $\mathcal{I}$ is syntactically strongly system prealgebraizable if and only if, by definition, it is

- preequivalential and has a Leibniz binary reflexive core;
- system c-reflective and has an adequate unary system core;

if and only if, by Theorems 969 and 1027, is it syntactically preequivalential and strongly system truth equational. ■

An analog of Theorem 1030 gives an alternative characterization of the syntactic notion in terms of the corresponding semantic notions.

Theorem 1046 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically strongly system prealgebraizable if and only if it is system prealgebraizable and* $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$*-fortified.*

Proof: We have that $\mathcal{I}$ is syntactically strongly system prealgebraizable if and only if, by definition,

- it is preequivalential and system c-reflective;
- it has a Leibniz binary reflexive core and an adequate unary system core;

if and only if, by definition, it is system prealgebraizable and, also, $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified. ■

This characterization in terms of semantic properties and preceding work on transference of properties from theory systems to filter systems on arbitrary algebraic systems yield yet another characterization of syntactic strong system prealgebraizability analogous to that of Theorem 1031, which may also be viewed as a kind of transfer property for this class.

Theorem 1047 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically strongly system prealgebraizable if and only if it is* $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ *is monotone on theory systems, system extensional and system c-reflective.*

Proof: We have that $\mathcal{I}$ is syntactically strongly system prealgebraizable if and only if, by Theorem 1046, it is $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified and system prealgebraizable if and only if, by Theorem 349, it is $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, $\Omega^{\mathcal{A}}$ is monotone on $\mathcal{I}$-filter systems, system extensional and system c-reflective. ∎

Turning now to characterizations involving property preserving mappings between posets of filter families and of congruence systems, we have the following result:

Theorem 1048 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically strongly system prealgebraizable if and only if it is* $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is an order embedding which commutes with inverse logical extensions.

Proof: We have that $\mathcal{I}$ is syntactically strongly system prealgebraizable if and only if, by Theorem 1046, it is $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified and system prealgebraizable if and only if, by Theorem 353, it is $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order embedding which commutes with inverse logical extensions. ∎

Finally, in an analog of Theorem 1033, using conjugate pairs of transformations, we get

Theorem 1049 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically strongly system prealgebraizable if and only if its systemic skeleton* $\mathcal{K}^{\mathcal{I}}$ *is equivalent to* $\mathcal{Q}^{\mathcal{I}\bullet}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}\bullet}$ *of natural transformations.*

Proof: Suppose, first, that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair of natural transformations. Then, by Theorem 942, it is strongly system truth equational and, by Theorem 941, it is syntactically preequivalential. Thus, by Theorem 1045, it is syntactically strongly system prealgebraizable.

Suppose, conversely, that $\mathcal{I}$ is syntactically strongly system prealgebraizable. Then, by Theorem 1045, it is strongly system truth equational and syntactically preequivalential. Hence, by Theorem 934, it is syntactically WS prealgebraizable. Now it follows by Theorem 940 that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a pair $(\tau^\flat, \overleftrightarrow{I}^\flat)$, where $I^\flat$ witnesses the syntactic preequivalentiality and $\tau^\flat$ the syntactic strong system truth equationality of $\mathcal{I}$, and, hence, by definition, they constitute a conjugate pair of natural transformations. ∎

Analogously with Theorem 1034, the equivalence of the systemic skeleton with some algebraic π-structure via a conjugate pair of natural transformations suffices to ensure syntactic strong system prealgebraizability.

Theorem 1050 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly system prealgebraizable if and only if its systemic skeleton is equivalent to an algebraic* π*-structure via a conjugate pair of natural transformations.*

Proof: If $\mathcal{I}$ is syntactically strongly system prealgebraizable, then, by Theorem 1049, its systemic skeleton is equivalent to an algebraic π-structure via a conjugate pair of natural transformations. Suppose, conversely, that the systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to an algebraic π-structure via a conjugate pair of natural transformations. Then, by Proposition 928, it is syntactically preequivalential and, by Theorem 942, it is strongly system truth equational. Therefore, by Theorem 1045, it is syntactically strongly system prealgebraizable. ■

Finally, in terms of order isomorphisms between theory family lattices, we have the following analog of Theorem 1035, providing an alternative characterization of syntactically strongly system prealgebraizable π-institutions.

Theorem 1051 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly system prealgebraizable if and only if there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{K}^{\mathcal{I}}) \to \mathbf{ThFam}(\mathcal{Q})$*, induced by a conjugate pair* $(\tau^\flat, I^\flat)$ *of natural transformations, where* $\mathcal{Q}$ *is an algebraic* π*-structure.*

Proof: The "only if" follows by Theorem 1050 and Theorem 893. The "if" is given by Theorem 901 and Theorem 1050. ■

In the case of syntactic strong left prealgebraizability, studied in the preceding section, below that class sat two wider classes obtained by weakening the naturality requirement either on the side of the witnesses of prealgebraicity or on the side of the witnesses of truth equationality. Similarly here, we get below the class of syntactically strongly system prealgebraizable π-institutions the classes of *syntactically system prealgbebraizable* and *syntactically system antiprealgebraizable* π-institutions. These two classes are defined formally now.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

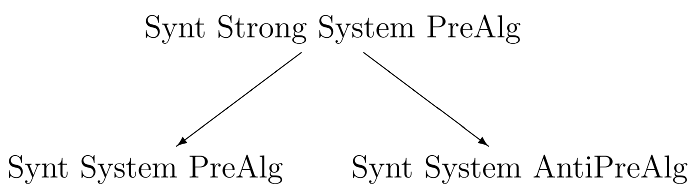
- $\mathcal{I}$ is **syntactically system prealgebraizable** if it is:
  - $\ddot{R}^{\mathcal{I}} Z^{\mathcal{I}}$-fortified;
  - preequivalential;
  - system c-reflective;
- $\mathcal{I}$ is **syntactically system antiprealgebraizable** if it is:

– $R^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified;
– prealgebraic;
– system c-reflective.

For both of these classes we have analogs of many of the results proven above for syntactic strong system prealgebraizability. Again, since:

- preequivalentiality is stronger than prealgebraicity;
- under prealgebraicity, $\ddot{R}^{\mathcal{I}}$ Leibniz implies $R^{\mathcal{I}}$ Leibniz; and
- $\dot{Z}^{\mathcal{I}}$ adequate implies $Z^{\mathcal{I}}$ adequate,

we get, immediately from the definitions the following hierarchical relations between the upper three classes in the echelon formation of the preceding diagram.



We now provide examples to show that the two inclusions are proper. The first is an example of a π-institution which is syntactically system prealgebraizable but not syntactically strongly system prealgebraizable.

Example 1052 EXAMPLE NOT FOUND YET!!

Next, we give an example of a syntactically system antiprealgebraizable π-institution which fails to be syntactically strongly system prealgebraizable.

Example 1053 EXAMPLE NOT FOUND YET!!

The following analog of Theorem 1038 relates these two chiral types of syntactic system prealgebraizability with various classes introduced previously, providing some important characterizations.

Theorem 1054 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is syntactically system prealgebraizable if and only if it is syntactically preequivalential and system truth equational.*

(b) $\mathcal{I}$ *is syntactically system antiprealgebraizable if and only if it is syntactically prealgebraic and strongly system truth equational.*

Proof:

(a) We have that $\mathcal{I}$ is syntactically system prealgebraizable if and only if, by definition, it is preequivalential, with a Leibniz binary reflexive core, and system c-reflective, with an adequate system core, if and only if, by Theorems 969 and 887, is it syntactically preequivalential and system truth equational.

(b) We have that $\mathcal{I}$ is syntactically system antiprealgebraizable if and only if, by definition, it is prealgebraic, with a Leibniz reflexive core, and system c-reflective, with an adequate unary system core, if and only if, by Theorems 788 and 1027, is it syntactically prealgebraic and strongly system truth equational.

■

An alternative characterization, analogous to that of Theorem 1039, relates the syntactic with the corresponding semantic notions.

Theorem 1055 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is syntactically system prealgebraizable if and only if it is system prealgebraizable and* $\ddot{R}^{\mathcal{I}} Z^{\mathcal{I}}$*-fortified.*

(b) $\mathcal{I}$ *is syntactically system antiprealgebraizable if and only if it is weakly system prealgebraizable and* $R^{\mathcal{I}} \dot{Z}^{\mathcal{I}}$*-fortified.*

Proof:

(a) We have that $\mathcal{I}$ is syntactically system prealgebraizable if and only if, by definition, it is preequivalential and system c-reflective and, moreover, it has a Leibniz binary reflexive core and an adequate system core. This happens if and only if, by definition, it is system prealgebraizable and $\ddot{R}^{\mathcal{I}} Z^{\mathcal{I}}$-fortified.

(b) Similarly, $\mathcal{I}$ is syntactically system antiprealgebraizable if and only if, by definition, it is prealgebraic and system c-reflective and, moreover, it has a Leibniz reflexive core and an adequate unary system core. This happens if and only if, by definition, it is weakly system prealgebraizable and $R^{\mathcal{I}} \dot{Z}^{\mathcal{I}}$-fortified.

■

As far as transferring the properties defining syntactic system (anti)prealgebraizability from theory systems to filter systems on arbitrary algebraic systems, we get the following transfer theorem.

Theorem 1056 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) *$\mathcal{I}$ is syntactically system prealgebraizable if and only if it is $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$-fortified and, for every* **F***-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ is monotone on $\mathcal{I}$-filter systems, system extensional and system c-reflective.*

(b) *$\mathcal{I}$ is syntactically system antiprealgebraizable if and only if it is $R^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified and, for every* **F***-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ is monotone on $\mathcal{I}$-filter systems and system c-reflective.*

Proof: We prove only Part (a). The proof of Part (b) follows along similar lines. We have that $\mathcal{I}$ is syntactically system prealgebraizable if and only if, by Theorem 1055, it is $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$-fortified and system prealgebraizable if and only if, by Theorem 349, it is $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$-fortified and, for every **F**-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ is monotone on $\mathcal{I}$-filter systems, system extensional and system c-reflective. ■

As far as characterizations involving property preserving mappings between posets of filter families and of congruence systems, we have the following analog of Theorem 1041.

Theorem 1057 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on* **F***.*

(a) *$\mathcal{I}$ is syntactically system prealgebraizable if and only if it is $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$-fortified and, for every* **F***-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is an order embedding which commutes with inverse logical extensions.

(b) *$\mathcal{I}$ is syntactically system antiprealgebraizable if and only if it is $R^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified and, for every* **F***-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is an order embedding.

Proof: Again we show only Part (a), since Part (b) follows similar reasoning. We have that $\mathcal{I}$ is syntactically system prealgebraizable if and only if, by Theorem 1055, it is $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$-fortified and system prealgebraizable if and only if, by Theorem 353 it is $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$-fortified and, for every **F**-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order embedding which commutes with inverse logical extensions. ■

Finally, in terms of conjugate pairs of transformations, we get the following analog of Theorem 1042.

Theorem 1058 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on* **F***.*

(a) *$\mathcal{I}$ is syntactically system prealgebraizable if and only if its systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}\bullet}$ of transformations, with $I^\flat$ natural.*

(b) *$\mathcal{I}$ is syntactically system antiprealgebraizable if and only if its systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}\bullet}$ of transformations, with $\tau^\flat$ natural.*

Proof:

(a) Suppose, first, that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $I^\flat$ natural. Then, by Theorem 941, it is syntactically preequivalential and, by Theorem 942, it is system truth equational. Thus, by Theorem 1054, it is syntactically system prealgebraizable.

Suppose, conversely, that $\mathcal{I}$ is syntactically system prealgebraizable. Then, by Theorem 1054, it is syntactically preequivalential and system truth equational. Hence, by Theorem 934, it is syntactically WS prealgebraizable. Now it follows by Theorem 940 that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a pair $(\tau^\flat, \overleftrightarrow{I}^\flat)$, where $I^\flat$ witnesses the syntactic preequivalentiality and $\tau^\flat$ the system truth equationality of $\mathcal{I}$, and, hence, by definition, they constitute a conjugate pair of transformations, with $I^\flat$ natural.

(b) Suppose, first, $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $\tau^\flat$ natural. Then, by Theorem 941, it is syntactically prealgebraic and, by Theorem 942, it is strongly system truth equational. Thus, by Theorem 1054, it is syntactically system antiprealgebraizable.

Suppose, conversely, that $\mathcal{I}$ is syntactically system antiprealgebraizable. Then, by Theorem 1054, it is syntactically prealgebraic and strongly system truth equational. Hence, by Theorem 934, it is syntactically WS prealgebraizable. Now it follows by Theorem 940 that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a pair $(\tau^\flat, \overleftrightarrow{I}^\flat)$, where $I^\flat$ witnesses the syntactic prealgebraicity and $\tau^\flat$ the strong system truth equationality of $\mathcal{I}$, and, hence, by definition, they constitute a conjugate pair of transformations, with $\tau^\flat$ natural.

■

The equivalence of the systemic skeleton with some algebraic π-structure via a conjugate pair of transformations, exhibiting the required one-sided naturality condition, is sufficient to ensure syntactic system (anti)prealgebraizability. This constitutes an analog of Theorem 1043.

Theorem 1059 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ *is syntactically system prealgebraizable if and only if its systemic skeleton is equivalent to an algebraic* π*-structure via a conjugate pair* $(\tau^\flat, I^\flat)$ *of transformations, with* $I^\flat$ *natural.*

(b) $\mathcal{I}$ *is syntactically system antiprealgebraizable if and only if its systemic skeleton is equivalent to an algebraic* π*-structure via a conjugate pair* $(\tau^\flat, I^\flat)$ *of transformations, with* $\tau^\flat$ *natural.*

Proof:

(a) If $\mathcal{I}$ is syntactically system prealgebraizable, then, by Theorem 1058, its systemic skeleton is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations with $I^\flat$ natural. Suppose, conversely, that the systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $I^\flat$ natural. Then, by Proposition 928, it is syntactically preequivalential and, by Theorem 942, it is system truth equational. Therefore, by Theorem 1054, it is syntactically system prealgebraizable.

(b) If $\mathcal{I}$ is syntactically system antiprealgebraizable, then, by Theorem 1058, its systemic skeleton is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $\tau^\flat$ natural. Suppose, conversely, that $\mathcal{K}^{\mathcal{I}}$ is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $\tau^\flat$ natural. Then, by Proposition 928, it is syntactically prealgebraic and, by Theorem 942, it is strongly system truth equational. Therefore, by Theorem 1054, it is syntactically system antiprealgebraizable.

■

Finally, in terms of order isomorphisms between theory family lattices, we have the following analog of Theorem 1044, giving an alternative characterization of syntactically system (anti)prealgebraizable π-institutions:

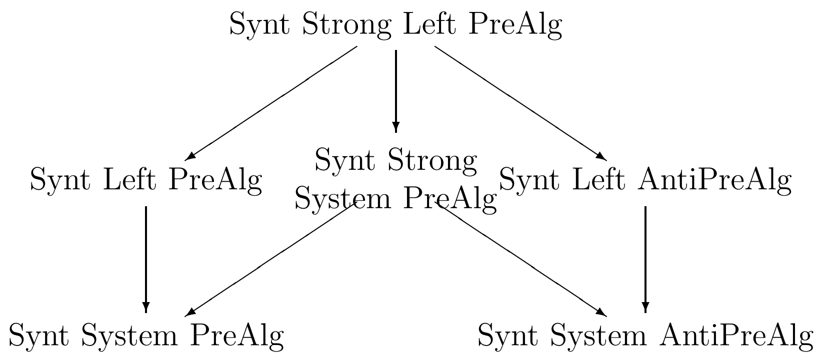
Theorem 1060 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is syntactically system prealgebraizable if and only if there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{K}^{\mathcal{I}}) \to \mathbf{ThFam}(\mathcal{Q})$*, induced by a conjugate pair* $(\tau^\flat, I^\flat)$ *of transformations, where* $\mathcal{Q}$ *is an algebraic* π*-structure and* $I^\flat$ *is natural.*

(b) $\mathcal{I}$ *is syntactically system antiprealgebraizable if and only if there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{K}^{\mathcal{I}}) \to \mathbf{ThFam}(\mathcal{Q})$*, induced by a conjugate pair* $(\tau^\flat, I^\flat)$ *of transformations, where* $\mathcal{Q}$ *is an algebraic* π*-structure and* $\tau^\flat$ *is natural.*

Proof: The "only if" follows by Theorem 1059 and Theorem 893. The "if" is given by Theorem 901 and Theorem 1059. ■

Finally, since we have now described in detail the six classes of the syntactic prealgebraizability hierarchy, it is only appropriate to pause and look for examples that separate the left prealgebraizability from the system prealgebraizability classes, i.e., examples showing that the vertical arrows in the following 6-class diagram

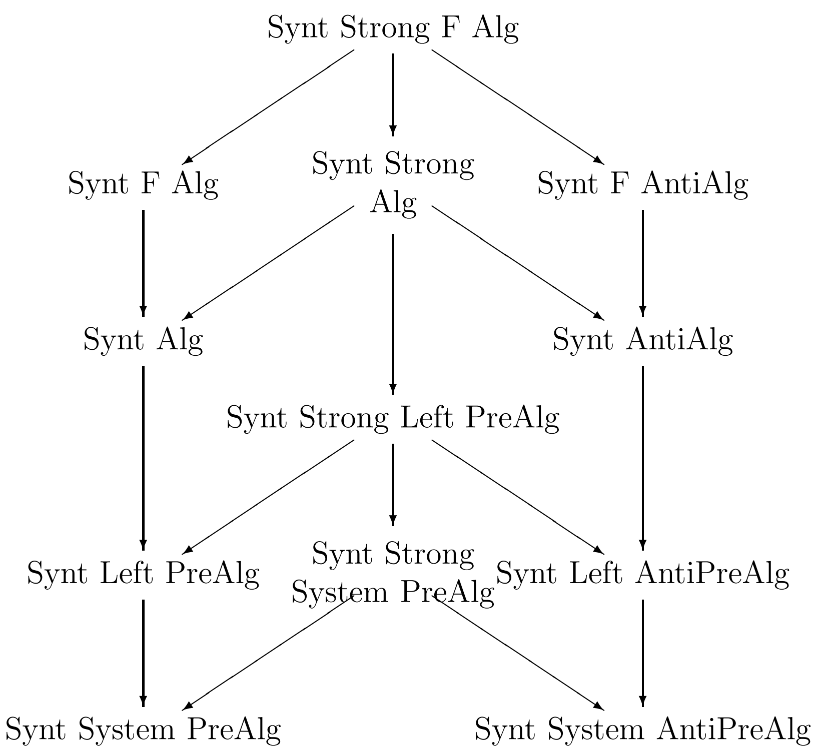


represent, in fact, proper inclusions. We can do this in one swoop by exhibiting an example of a syntactically strong system prealgebraizable π-institution which is neither syntactically left prealgebraizable nor syntactically left antiprealgebraizable.

Example 1061 EXAMPLE NOT FOUND YET!

13.9 Syntactic Family Algebraizability

We now preview the full hierarchy of syntactically prealgebraizable π-institutions that will be established in this section. The bottom six classes are the ones established in the preceding two sections, where prealgebraizability refers to the fact that monotonicity is only applied to theory systems. The top six classes concern syntactic algebraizability, where monotonicity is applied to all theory families. The bottom row of this upper tier consists of those π-institutions, where c-reflectivity is postulated only for theory systems. The very top row above it refers to applying left c-reflectivity to theory families. In the second from top class, system (or, equivalently, left c-reflectivity) is postulated in conjunction with family monotonicity and at the very top row family c-reflectivity is combined with family monotonicity. Finally, as far as columns go, they incorporate meanings similar to the ones described for the cohorts of classes introduced in the preceding sections. The left column applies parameter freeness only to the equivalence natural transformations witnessing syntactic protoalgebraicity. The right column insists on parameter freeness for the defining equations that witness the truth equationality of the π-institution, whereas the middle column is combing those properties and consists of those classes of π-institutions that are syntactically protoalgebraic and syntactic truth equational, with both properties having parameter free witnessing transformations and witnessing equations, respectively. The complete picture that emerges at the end of this and the next section adds to the six-class diagram concluding the previous section six more classes, those positioned at the top two rows.



We start by defining the class at the apex of the diagram. Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that $\mathcal{I}$ is $\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$**-fortified** if it has

- a Leibniz binary reflexive core; and
- an adequate unary Suszko core.

We say that $\mathcal{I}$ is **syntactically strongly family algebraizable** if it is

- $\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$-fortified;
- equivalential (i.e., protoalgebraic and family extensional);
- family c-reflective.

Based on previous work, we can formulate the following important characterization of syntactic strong family algebraizability.

Theorem 1062 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly family algebraizable if and only if it is syntactically equivalential and strongly truth equational.*

Proof: We have that $\mathcal{I}$ is syntactically strongly family algebraizable if and only if, by definition, it is

- equivalential and has a Leibniz binary reflexive core;
- family c-reflective and has an adequate unary Suszko core;

if and only if, by Theorems 983 and 996, is it syntactically preequivalential and strongly left truth equational. ∎

An alternative characterization along similar lines relates the syntactic with the corresponding semantic notions introduced in the context of algebraizability.

Theorem 1063 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly family algebraizable if and only if it is family algebraizable and* $\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$*-fortified.*

Proof: We have that $\mathcal{I}$ is syntactically strongly family algebraizable if and only if, by definition,

- it is equivalential and family c-reflective;
- it has a Leibniz binary reflexive core and an adequate unary Suszko core;

if and only if, by definition, it is family algebraizable (recall that family injectivity and family c-reflectivity coincide under protoalgebraicity) and $\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$-fortified. ∎

This characterization in terms of semantic properties and preceding work on transference of properties from theory families to filter families on arbitrary algebraic systems yields another characterization of syntactic strong family algebraizability, which may also be viewed as a kind of transfer property in its own right.

Theorem 1064 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly family algebraizable if and only if it is* $\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\Omega^{\mathcal{A}}$ *is monotone on* $\mathcal{I}$*-filter families, family extensional and family c-reflective.*

Proof: We have that $\mathcal{I}$ is syntactically strongly family algebraizable if and only if, by Theorem 1063, it is $\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$-fortified and family algebraizable if and only if, by Theorem 364, it is $\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ is monotone and injective (equivalently c-reflective) on $\mathcal{I}$-filter families and family extensional. ■

Turning now to characterizations involving property preserving mappings between posets of filter families and of congruence systems, we have the following result:

Theorem 1065 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly family algebraizable if and only if it is* $\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is an order isomorphism that commutes with inverse logical extensions.

Proof: We have that $\mathcal{I}$ is syntactically strongly family algebraizable if and only if, by Theorem 1063, it is $\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$-fortified and family algebraizable if and only if, by Theorem 366 it is $\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order isomorphism that commutes with inverse logical extensions. ■

Finally, in terms of conjugate pairs of transformations, we get the following analog of Theorem 949.

Theorem 1066 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly family algebraizable if and only if it is equivalent to its associated algebraic* π*-structure* $\mathcal{Q}^{\mathcal{I}*}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$ *of natural transformations.*

Proof: Suppose, first, that $\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair of natural transformations. Then it is syntactically equivalential by Corollary 910 and it is family truth equational by Theorem 911. Thus, by Theorem 1062, it is syntactically strongly family algebraizable.

Suppose, conversely, that $\mathcal{I}$ is syntactically strongly family algebraizable. Then, by Theorem 1062, it is strongly family truth equational and syntactically equivalential. Hence, by Theorem 913, it is syntactically WF algebraizable. Now it follows by Theorem 919 that $\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a pair $(\tau^\flat, \overleftrightarrow{I^\flat})$, where $I^\flat$ witnesses the syntactic equivalentiality and $\tau^\flat$ the syntactic strong truth equationality of $\mathcal{I}$, and, hence, by definition, they constitute a conjugate pair of natural transformations. ■

It turns out, in this case also, that the equivalence of the π-institution with some algebraic π-structure via a conjugate pair of natural transformations is sufficient to ensure syntactic strong family algebraizability.

Theorem 1067 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly family algebraizable if and only if it is equivalent to an algebraic* π*-structure via a conjugate pair of natural transformations.*

Proof: If $\mathcal{I}$ is syntactically strongly family algebraizable, then, by Theorem 1066, it is equivalent to an algebraic π-structure via a conjugate pair of natural transformations. Suppose, conversely, that $\mathcal{I}$ is equivalent to an algebraic π-structure via a conjugate pair of natural transformations. Then, it is syntactically equivalential by Corollary 910 and it is strongly family truth equational by Theorem 911. Therefore, by Theorem 1062, it is syntactically strongly family algebraizable. ∎

Finally, in terms of order isomorphisms between theory family lattices, we have the following alternative characterization of syntactically strongly family algebraizable π-institutions:

Theorem 1068 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly family algebraizable if and only if there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{I}) \to \mathbf{ThFam}(\mathcal{Q})$*, induced by a conjugate pair* $(\tau^\flat, I^\flat)$ *of natural transformations, where* $\mathcal{Q}$ *is an algebraic* π*-structure.*

Proof: The "only if" follows by Theorem 1067 and Theorem 893. The "if" is given by Theorem 901 and Theorem 1067. ∎

Lying just underneath the class of syntactically strongly family algebraizable π-institutions are the classes of *syntactically family algebraizable* and *syntactically family antialgebraizable* π-institutions. These two classes are defined formally now.

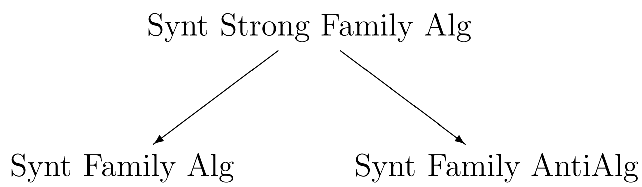
Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

- $\mathcal{I}$ is **syntactically family algebraizable** if it is:
  - $\ddot{R}^{\mathcal{I}} S^{\mathcal{I}}$-fortified;
  - equivalential;
  - family c-reflective;
- $\mathcal{I}$ is **syntactically family antialgebraizable** if it is:
  - $R^{\mathcal{I}} \dot{S}^{\mathcal{I}}$-fortified;
  - protoalgebraic;
  - family c-reflective.

We formulate analogs of many of the results proven previously for the various kinds of syntactic prealgebraizability properties. Observe, first, that, since:

- equivalentiality is stronger than protoalgebraicity;
- under prealgebraicity, $\ddot{R}^{\mathcal{I}}$ Leibniz implies $R^{\mathcal{I}}$ Leibniz; and
- $\dot{S}^{\mathcal{I}}$ adequate implies $S^{\mathcal{I}}$ adequate,

we get, immediately from the definitions the following hierarchical relations between the three topmost classes in the syntactic algebraizability hierarchy.



We now provide examples to show that the two inclusions are proper. The first is an example of a π-institution which is syntactically family algebraizable but not syntactically strongly family algebraizable.

Example 1069 EXAMPLE NOT FOUND YET!!

Next, we give an example of a syntactically family antialgebraizable π-institution which fails to be syntactically strongly family algebraizable.

Example 1070 EXAMPLE NOT FOUND YET!!

The following analog of Theorem 1062 relates these two chiral sorts of syntactic family (anti)algebraizability with various classes introduced previously, providing some important characterizations.

Theorem 1071 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is syntactically family algebraizable if and only if it is syntactically equivalential and family truth equational.*

(b) $\mathcal{I}$ *is syntactically family antialgebraizable if and only if it is syntactically protoalgebraic and strongly family truth equational.*

Proof:

(a) We have that $\mathcal{I}$ is syntactically family algebraizable if and only if, by definition, it is equivalential, with a Leibniz binary reflexive core, and family c-reflective, with an adequate Suszko core, if and only if, by Theorems 983 and 847, is it syntactically equivalential and family truth equational.

(b) We have that $\mathcal{I}$ is syntactically family antialgebraizable if and only if, by definition, it is protoalgebraic, with a Leibniz reflexive core, and family c-reflective, with an adequate unary Suszko core, if and only if, by Theorems 805 and 996, is it syntactically protoalgebraic and strongly family truth equational.

■

An alternative characterization, analogous to that of Theorem 1063, relates the syntactic with the corresponding semantic notions.

Theorem 1072 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is syntactically family algebraizable if and only if it is family algebraizable and* $\ddot{R}^{\mathcal{I}} S^{\mathcal{I}}$*-fortified.*

(b) $\mathcal{I}$ *is syntactically family antialgebraizable if and only if it is family algebraizable and* $R^{\mathcal{I}} \dot{S}^{\mathcal{I}}$*-fortified.*

Proof:

(a) We have that $\mathcal{I}$ is syntactically family algebraizable if and only if, by definition, it is equivalential and family c-reflective and, moreover, it has a Leibniz binary reflexive core and an adequate Suszko core. This happens if and only if, by definition, it is family algebraizable and $\ddot{R}^{\mathcal{I}} S^{\mathcal{I}}$-fortified.

(b) Similarly, $\mathcal{I}$ is syntactically family antialgebraizable if and only if, by definition, it is equivalential and family c-reflective and, moreover, it has a Leibniz reflexive core and an adequate unary Suszko core. This happens if and only if, by definition, it is family algebraizable and $R^{\mathcal{I}} \dot{S}^{\mathcal{I}}$-fortified.

■

The characterization in terms of semantic properties and preceding work on transference of properties from theory families to filter families on arbitrary algebraic systems yield a transfer property for syntactic family (anti)-prealgebraizability.

Theorem 1073 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is syntactically family algebraizable if and only if it is* $\ddot{R}^{\mathcal{I}} S^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ *is monotone on* $\mathcal{I}$*-filter families, family extensional and family c-reflective.*

(b) $\mathcal{I}$ *is syntactically family antialgebraizable if and only if it is* $R^{\mathcal{I}} \dot{S}^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ *is monotone on* $\mathcal{I}$*-filter families and family c-reflective.*

Proof: We prove only Part (a), since Part (b) is similar. We have that $\mathcal{I}$ is syntactically family algebraizable if and only if, by Theorem 1072, it is $\ddot{R}^{\mathcal{I}}S^{\mathcal{I}}$-fortified and family algebraizable if and only if, by Theorem 349, it is $\ddot{R}^{\mathcal{I}}S^{\mathcal{I}}$-fortified and, for every **F**-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ is monotone on $\mathcal{I}$-filter families, family extensional and family c-reflective. ■

Turning now to characterizations involving property preserving mappings between posets of filter families and of congruence systems, we have the following result:

Theorem 1074 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is syntactically family algebraizable if and only if it is* $\ddot{R}^{\mathcal{I}}S^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is an order isomorphism that commutes with inverse logical extensions.

(b) $\mathcal{I}$ *is syntactically family antialgebraizable if and only if it is* $R^{\mathcal{I}}\dot{S}^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is an order isomorphism.

Proof: Again we show only Part (a). Part (b) is similar. We have that $\mathcal{I}$ is syntactically family algebraizable if and only if, by Theorem 1072, it is $\ddot{R}^{\mathcal{I}}S^{\mathcal{I}}$-fortified and family algebraizable if and only if, by Theorem 366 it is $\ddot{R}^{\mathcal{I}}S^{\mathcal{I}}$-fortified and, for every **F**-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order isomorphism that commutes with inverse logical extensions. ■

In terms of conjugate pairs of transformations, we get the following analog of Theorem 1066.

Theorem 1075 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is syntactically family algebraizable if and only if it is equivalent to* $\mathcal{Q}^{\mathcal{I}*}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$ *of transformations, with* $I^\flat$ *natural.*

(b) $\mathcal{I}$ *is syntactically family antialgebraizable if and only if it is equivalent to* $\mathcal{Q}^{\mathcal{I}*}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$ *of transformations, with* $\tau^\flat$ *natural.*

Proof:

(a) Suppose, first, that $\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $I^\flat$ natural. Then, by Corollary 910, it is syntactically equivalential and, by Theorem 911, it is family truth equational. Thus, by Theorem 1071, it is syntactically family algebraizable.

Suppose, conversely, that $\mathcal{I}$ is syntactically family algebraizable. Then, by Theorem 1071, it is left truth equational and syntactically equivalential. Hence, by Theorem 913, it is syntactically WF prealgebraizable. Now it follows by Theorem 919 that $\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a pair $(\tau^\flat, \overleftrightarrow{I^\flat})$, where $I^\flat$ witnesses the syntactic equivalentiality and $\tau^\flat$ the syntactic family truth equationality of $\mathcal{I}$, and, hence, by definition, they constitute a conjugate pair of transformations, with $I^\flat$ natural.

(b) Suppose, first, that $\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $\tau^\flat$ natural. Then, by Theorem 909, it is syntactically protoalgebraic and, by Theorem 911, it is strongly family truth equational. Thus, by Theorem 1071, it is syntactically family antialgebraizable.

Suppose, conversely, that $\mathcal{I}$ is syntactically family antialgebraizable. Then, by Theorem 1071, it is strongly left truth equational and syntactically protoalgebraic. Hence, by Theorem 913, it is syntactically WF prealgebraizable. Now it follows by Theorem 919 that $\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a pair $(\tau^\flat, \overleftrightarrow{I^\flat})$, where $I^\flat$ witnesses the syntactic protoalgebraicity and $\tau^\flat$ the strong family truth equationality of $\mathcal{I}$, and, hence, by definition, they constitute a conjugate pair of transformations, with $\tau^\flat$ natural.

■

The equivalence of the π-institution with some algebraic π-structure via a conjugate pair of transformations exhibiting the required one-sided naturality condition suffices to ensure syntactic family (anti)algebraizability.

Theorem 1076 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is syntactically family algebraizable if and only if is equivalent to an algebraic* π*-structure via a conjugate pair* $(\tau^\flat, I^\flat)$ *of transformations, with* $I^\flat$ *natural.*

(b) $\mathcal{I}$ *is syntactically family antialgebraizable if and only if it is equivalent to an algebraic* π*-structure via a conjugate pair* $(\tau^\flat, I^\flat)$ *of transformations, with* $\tau^\flat$ *natural.*

Proof:

(a) If $\mathcal{I}$ is syntactically family algebraizable, then, by Theorem 1075, it is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations with $I^\flat$ natural. Suppose, conversely, that $\mathcal{I}$ is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $I^\flat$ natural. Then, by Theorem 909, it is syntactically equivalential and, by Theorem 911, it is family truth equational. Therefore, by Theorem 1071, it is syntactically family algebraizable.

(b) If $\mathcal{I}$ is syntactically family antialgebraizable, then, by Theorem 1075, it is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $\tau^\flat$ natural. Suppose, conversely, that $\mathcal{I}$ is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $\tau^\flat$ natural. Then, by Theorem 909, it is syntactically protoalgebraic and by Theorem 911, it is strongly family truth equational. Therefore, by Theorem 1071, it is syntactically family antialgebraizable.

■

Finally, in terms of order isomorphisms between theory family lattices, we have the following alternative characterization of syntactic family (anti)algebraizability:

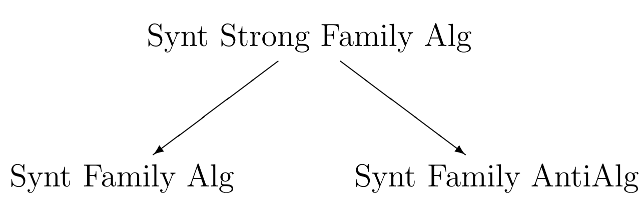
Theorem 1077 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is syntactically family algebraizable if and only if there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{I}) \to \mathbf{ThFam}(\mathcal{Q})$*, induced by a conjugate pair* $(\tau^\flat, I^\flat)$ *of transformations, where* $\mathcal{Q}$ *is an algebraic* π*-structure and* $I^\flat$ *is natural.*

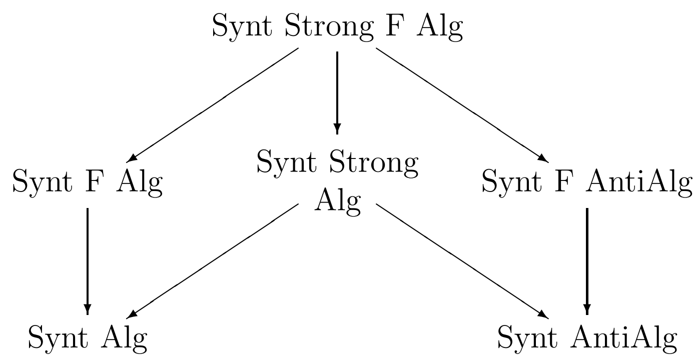
(b) $\mathcal{I}$ *is syntactically family antialgebraizable if and only if there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{I}) \to \mathbf{ThFam}(\mathcal{Q})$*, induced by a conjugate pair* $(\tau^\flat, I^\flat)$ *of transformations, where* $\mathcal{Q}$ *is an algebraic* π*-structure and* $\tau^\flat$ *is natural.*

Proof: The "only if" follows by Theorem 1076 and Theorem 893. The "if" is given by Theorem 901 and Theorem 1076. ■

In this section we have introduced the three syntactic family algebraizability classes



In the next section, we shall introduce the remaining three syntactic algebraizability classes, namely those of the syntactically algebraizable π-institutions, in order to complete the syntactic algebraizability hierarchy that was described at the beginning of the section:



13.10 Syntactic Algebraizability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall that $\mathcal{I}$ is $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-**fortified** if it has a Leibniz binary reflexive core and an adequate unary system core. We say that $\mathcal{I}$ is **syntactically strongly algebraizable** if it is

- $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified;
- equivalential (i.e., protoalgebraic and family extensional);
- system c-reflective.

An analog of Theorem 1062 provides an important characterization of syntactic strong algebraizability in terms of lower classes in the syntactic hierarchy.

Theorem 1078 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is syntactically strongly algebraizable if and only if it is syntactically equivalential and strongly system truth equational.*

Proof: We have that $\mathcal{I}$ is syntactically strongly algebraizable if and only if, by definition, it is

- equivalential and has a Leibniz binary reflexive core;
- system c-reflective and has an adequate unary system core;

if and only if, by Theorems 983 and 1027, is it syntactically preequivalential and strongly system truth equational. ∎

An analog of Theorem 1063 gives an alternative characterization of the syntactic notion in terms of the corresponding semantic notions.

Theorem 1079 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly algebraizable if and only if it is (system) algebraizable and* $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$*-fortified.*

Proof: We have that $\mathcal{I}$ is syntactically strongly algebraizable if and only if, by definition,

- it is equivalential and system c-reflective;
- it has a Leibniz binary reflexive core and an adequate unary system core;

if and only if, by definition, it is algebraizable and $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified. ■

This characterization in terms of semantic properties and preceding work on transference of properties from theory families/systems to filter families/systems on arbitrary algebraic systems yield another characterization of syntactic strong algebraizability analogous to that of Theorem 1064, which may also be viewed as a kind of transfer property for syntactically strongly algebraizable π-institutions.

Theorem 1080 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly algebraizable if and only if it is* $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\Omega^{\mathcal{A}}$ *is monotone on* $\mathcal{I}$*-filter families, family extensional and system c-reflective.*

Proof: We have that $\mathcal{I}$ is syntactically strongly algebraizable if and only if, by Theorem 1079, it is $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified and algebraizable if and only if, by Theorem 349, it is $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ is monotone on $\mathcal{I}$-filter families, family extensional and system c-reflective. ■

Turning now to characterizations involving property preserving mappings between posets of filter families and of congruence systems, we have the following analog of Theorem 1065.

Theorem 1081 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly algebraizable if and only if it is* $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$*-fortified, stable and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is an order isomorphism that commutes with inverse logical extensions.

Proof: We have that $\mathcal{I}$ is syntactically strongly (system) algebraizable if and only if, by Theorem 1079, it is $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified and algebraizable if and only if, by Theorem 365, it is $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified, stable and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order isomorphism that commutes with inverse logical extensions. ■

Finally, in an analog of Theorem 1066, using conjugate pairs of transformations, we get

Theorem 1082 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly algebraizable if and only if it is stable and its systemic skeleton* $\mathcal{K}^{\mathcal{I}}$ *is equivalent to* $\mathcal{Q}^{\mathcal{I}*}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$ *of natural transformations.*

Proof: Suppose, first, that $\mathcal{I}$ is stable and $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair of natural transformations. Then, by Theorem 929, it is syntactically equivalential and, by Theorem 941, it is strongly system truth equational. Thus, by Theorem 1078, it is syntactically strongly algebraizable.

Suppose, conversely, that $\mathcal{I}$ is syntactically strongly algebraizable. Then, by Theorem 1078, it is syntactically equivalential and strongly system truth equational. Hence, by Theorem 923, it is syntactically weakly algebraizable. Now it follows by Theorem 927 that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a pair $(\tau^\flat, \overleftrightarrow{I}^\flat)$, where $I^\flat$ witnesses the syntactic equivalentiality and $\tau^\flat$ the strong system truth equationality of $\mathcal{I}$, and, hence, by definition, they constitute a conjugate pair of natural transformations. ■

Analogously with Theorem 1067, the equivalence of the systemic skeleton with some algebraic π-structure via a conjugate pair of natural transformations, coupled with stability, suffices to ensure syntactic strong algebraizability.

Theorem 1083 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly algebraizable if and only if it is stable and its systemic skeleton is equivalent to an algebraic* π*-structure via a conjugate pair of natural transformations.*

Proof: If $\mathcal{I}$ is syntactically strongly algebraizable, then, by Theorem 1082, it is stable and its systemic skeleton is equivalent to an algebraic π-structure via a conjugate pair of natural transformations. Suppose, conversely, that $\mathcal{I}$ is stable and that its systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to an algebraic π-structure via a conjugate pair of natural transformations. Then, by Theorem 929, it is syntactically equivalential and, by Theorem 930, it is strongly system truth equational. Therefore, by Theorem 1078, it is syntactically strongly algebraizable. ■

Finally, in terms of order isomorphisms between theory family lattices, we have the following analog of Theorem 1065, providing an alternative characterization of syntactic strong algebraizability.

Theorem 1084 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly algebraizable if and only if it is stable and there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{K}^{\mathcal{I}}) \to \mathbf{ThFam}(\mathcal{Q})$*, induced by a conjugate pair* $(\tau^\flat, I^\flat)$ *of natural transformations, where* $\mathcal{Q}$ *is an algebraic* π*-structure.*

Proof: The "only if" follows by Theorem 1083 and Theorem 893. The "if" is given by Theorem 901 and Theorem 1083. ■

As with all other strong (pre)algebraizability classes, studied before, below the class of syntactically strongly algebraizable π-institutions sit two wider classes obtained by weakening the naturality requirement either on the side of the witnesses of prealgebraicity or on the side of the witnesses of truth equationality, namely the classes of *syntactically algebraizable* and *syntactically antialgebraizable* π-institutions.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

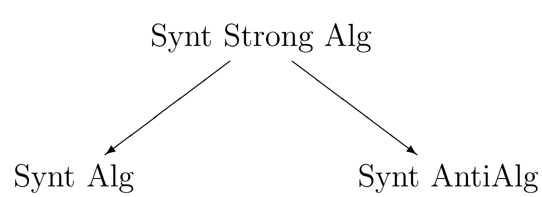
- $\mathcal{I}$ is **syntactically algebraizable** if it is:
  - $\ddot{R}^{\mathcal{I}} Z^{\mathcal{I}}$-fortified;
  - equivalential;
  - system c-reflective;
- $\mathcal{I}$ is **syntactically antialgebraizable** if it is:
  - $R^{\mathcal{I}} \dot{Z}^{\mathcal{I}}$-fortified;
  - protoalgebraic;
  - system c-reflective.

We now conclude the chapter by formulating analogs of many of the results proven above for syntactic strong algebraizability for these two new classes of π-institutions.

Firs, observe, once more, that, since:

- equivalentiality implies protoalgebraicity;
- under protoalgebraicity, $\ddot{R}^{\mathcal{I}}$ Leibniz implies $R^{\mathcal{I}}$ Leibniz; and
- $\dot{Z}^{\mathcal{I}}$ adequate implies $Z^{\mathcal{I}}$ adequate,

we get the following hierarchical relations between the three classes in the second-from-top tier of the syntactic algebraizability hierarchy.



Examples are in order to show that the two inclusions are proper. The first is an example of a π-institution which is syntactically algebraizable but not syntactically strongly algebraizable.

Example 1085 EXAMPLE NOT FOUND YET!!

Next, we give an example of a syntactically antialgebraizable π-institution which fails to be syntactically strongly algebraizable.

Example 1086 EXAMPLE NOT FOUND YET!!

The following analog of Theorem 1078 relates the two chiral types of syntactic algebraizability with various classes introduced previously, providing some important characterizations.

Theorem 1087 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is syntactically algebraizable if and only if it is syntactically equivalential and system truth equational.*

(b) $\mathcal{I}$ *is syntactically antialgebraizable if and only if it is syntactically protoalgebraic and strongly system truth equational.*

Proof:

(a) We have that $\mathcal{I}$ is syntactically algebraizable if and only if, by definition, it is equivalential, with a Leibniz binary reflexive core, and system c-reflective, with an adequate system core, if and only if, by Theorems 983 and 887, is it syntactically equivalential and system truth equational.

(b) We have that $\mathcal{I}$ is syntactically antialgebraizable if and only if, by definition, it is protoalgebraic, with a Leibniz reflexive core, and system c-reflective, with an adequate unary system core, if and only if, by Theorems 805 and 1027, is it syntactically protoalgebraic and strongly system truth equational.

■

An alternative characterization, analogous to that of Theorem 1079, relates the syntactic with the corresponding semantic notions.

Theorem 1088 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is syntactically algebraizable if and only if it is algebraizable and* $\ddot{R}^{\mathcal{I}} Z^{\mathcal{I}}$*-fortified.*

(b) $\mathcal{I}$ *is syntactically antialgebraizable if and only if it is weakly algebraizable and* $R^{\mathcal{I}} \dot{Z}^{\mathcal{I}}$*-fortified.*

Proof:

(a) We have that $\mathcal{I}$ is syntactically algebraizable if and only if, by definition, it is equivalential and system c-reflective and, moreover, it has a Leibniz binary reflexive core and an adequate system core. This happens if and only if, by definition, it is algebraizable and $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$-fortified.

(b) Similarly, $\mathcal{I}$ is syntactically antialgebraizable if and only if, by definition, it is protoalgebraic and system c-reflective and, moreover, it has a Leibniz reflexive core and an adequate unary system core. This happens if and only if, by definition, it is weakly algebraizable and $R^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified.

■

The properties defining syntactic (anti)algebraizability transfer from theory families/systems to filter families/systems on arbitrary algebraic systems. More precisely, we obtain the following analog of Theorem 1080.

Theorem 1089 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is syntactically algebraizable if and only if it is* $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ *is monotone on* $\mathcal{I}$*-filter families, family extensional and system c-reflective.*

(b) $\mathcal{I}$ *is syntactically antialgebraizable if and only if it is* $R^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ *is monotone on* $\mathcal{I}$*-filter families and system c-reflective.*

Proof: We prove only Part (a). Part (b) is similar. We have that $\mathcal{I}$ is syntactically algebraizable if and only if, by Theorem 1088, it is $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$-fortified and algebraizable if and only if, by Theorem 363, it is $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ is monotone on $\mathcal{I}$-filter families, family extensional and system c-reflective. ■

Turning now to characterizations involving property preserving mappings between posets of filter families and of congruence systems, we have the following analog of Theorem 1081.

Theorem 1090 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is syntactically algebraizable if and only if it is* $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$*-fortified, stable and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is an order isomorphism that commutes with inverse logical extensions.

(b) $\mathcal{I}$ is syntactically antialgebraizable if and only if it is $R^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified, stable and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is an order isomorphism.

Proof:

(a) $\mathcal{I}$ is syntactically algebraizable if and only if, by Theorem 1088, it is $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$-fortified and algebraizable if and only if, by Theorem 365 it is $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$-fortified, stable and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order isomorphism that commutes with inverse logical extensions.

(b) $\mathcal{I}$ is syntactically antialgebraizable if and only if, by Theorem 1088, it is $R^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified and weakly algebraizable if and only if, by Theorem 298 it is $R^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified, stable and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order isomorphism. ■

Finally, in terms of conjugate pairs of transformations, we get the following analog of Theorem 1082.

Theorem 1091 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is syntactically algebraizable if and only if it is stable and its systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}}$ via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}}$ of transformations, with $I^\flat$ natural.

(b) $\mathcal{I}$ is syntactically antialgebraizable if and only if it is stable and its systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}}$ via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}}$ of transformations, with $\tau^\flat$ natural.

Proof:

(a) Suppose, first, that $\mathcal{I}$ is stable and that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}}$ via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $I^\flat$ natural. Then, by Theorem 929, it is syntactically equivalential and, by Theorem 930, it is system truth equational. Thus, by Theorem 1087, it is syntactically algebraizable.

Suppose, conversely, that $\mathcal{I}$ is syntactically algebraizable. Then, by Theorem 1087, it is syntactically equivalential and system truth equational. Hence, by Theorem 923, it is syntactically weakly algebraizable. Now it follows by Theorem 927 that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}}$ via a pair $(\tau^\flat, \overleftrightarrow{I}^\flat)$, where $I^\flat$ witnesses the syntactic equivalentiality and $\tau^\flat$ the system truth equationality of $\mathcal{I}$, and, hence, by definition, they constitute a conjugate pair of transformations, with $I^\flat$ natural.

(b) Suppose, first, that $\mathcal{I}$ is stable and that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}}$ via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $\tau^\flat$ natural. Then, by Theorem 929, it is syntactically protoalgebraic and, by Theorem 930, it is strongly system truth equational. Thus, by Theorem 1087, it is syntactically antialgebraizable.

Suppose, conversely, that $\mathcal{I}$ is syntactically antialgebraizable. Then, by Theorem 1087, it is syntactically protoalgebraic and strongly system truth equational. Hence, by Theorem 923, it is syntactically weakly algebraizable. Now it follows by Theorem 927 that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}}$ via a pair $(\tau^\flat, \overleftrightarrow{I}^\flat)$, where $I^\flat$ witnesses the syntactic protoalgebraicity and $\tau^\flat$ the strong system truth equationality of $\mathcal{I}$, and, hence, by definition, they constitute a conjugate pair of transformations, with $\tau^\flat$ natural.

■

The equivalence of the systemic skeleton with some algebraic π-structure via a conjugate pair of transformations, exhibiting the required one-sided naturality condition, is sufficient to ensure syntactic system (anti)algebraizability. This constitutes an analog of Theorem 1083.

Theorem 1092 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is syntactically algebraizable if and only if it is stable and its systemic skeleton is equivalent to an algebraic* π*-structure via a conjugate pair* $(\tau^\flat, I^\flat)$ *of transformations, with* $I^\flat$ *natural.*

(b) $\mathcal{I}$ *is syntactically antialgebraizable if and only if it is stable and its systemic skeleton is equivalent to an algebraic* π*-structure via a conjugate pair* $(\tau^\flat, I^\flat)$ *of transformations, with* $\tau^\flat$ *natural.*

Proof:

(a) If $\mathcal{I}$ is syntactically algebraizable, then, by Theorem 1091, it is stable and its systemic skeleton is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations with $I^\flat$ natural. Suppose, conversely, that $\mathcal{I}$ is stable and that its systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $I^\flat$ natural. Then, by Proposition 929, it is syntactically equivalential and, by Theorem 930, it is system truth equational. Therefore, by Theorem 1087, it is syntactically algebraizable.

(b) If $\mathcal{I}$ is syntactically antialgebraizable, then, by Theorem 1091, it is stable and its systemic skeleton is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $\tau^\flat$ natural. Suppose, conversely, that $\mathcal{I}$ is stable and that $\mathcal{K}^{\mathcal{I}}$ is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with

$\tau^\flat$ natural. Then, by Proposition 929, it is syntactically protoalgebraic and, by Theorem 930, it is strongly system truth equational. Therefore, by Theorem 1087, it is syntactically antialgebraizable. ∎

Finally, in terms of order isomorphisms between theory family lattices, we have the following analog of Theorem 1084, giving an alternative characterization of syntactically (anti)algebraizable π-institutions:

Theorem 1093 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is syntactically algebraizable if and only if it is stable and there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{K}^{\mathcal{I}}) \to \mathbf{ThFam}(\mathcal{Q})$*, induced by a conjugate pair* $(\tau^\flat, I^\flat)$ *of transformations, where* $\mathcal{Q}$ *is an algebraic* π*-structure and* $I^\flat$ *is natural.*

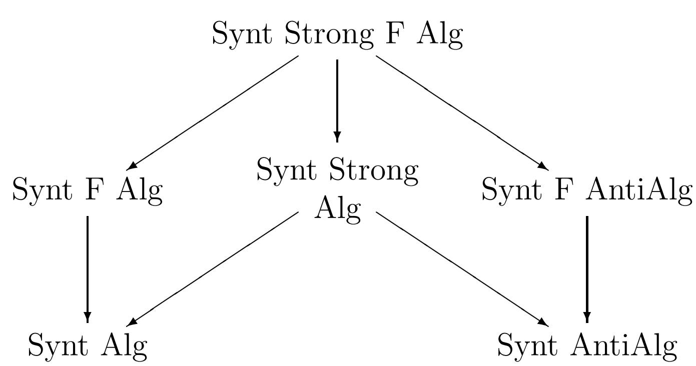
(b) $\mathcal{I}$ *is syntactically antialgebraizable if and only if it is stable and there exists a transformational order isomorphism*

$$h : \mathbf{ThFam}(\mathcal{K}^{\mathcal{I}}) \to \mathbf{ThFam}(\mathcal{Q}),$$

induced by a conjugate pair $(\tau^\flat, I^\flat)$ *of transformations, where* $\mathcal{Q}$ *is an algebraic* π*-structure and* $\tau^\flat$ *is natural.*

Proof: The "only if" follows by Theorem 1092 and Theorem 893. The "if" is given by Theorem 901 and Theorem 1092. ∎

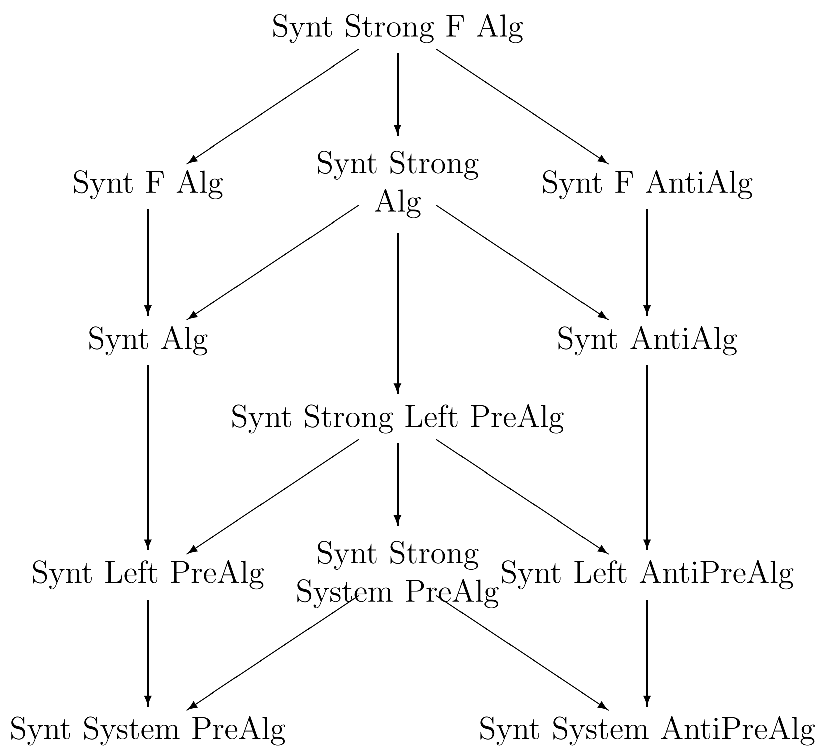
To close the chapter, we have some class separating work to do. First of all, since we have now described in detail the six classes of the syntactic algebraizability hierarchy, it is only appropriate to pause and look for examples that separate the family algebraizability classes, i.e., those in the top-most tier, from the algebraizability classes, that is those immediately below them. In other words, we are looking for examples that show that the vertical arrows in the accompanying diagram



represent, in fact, proper inclusions. We can do this all at once by exhibiting an example of a syntactically strongly algebraizable π-institution which is neither syntactically family algebraizable nor syntactically family antialgebraizable.

Example 1094 EXAMPLE NOT FOUND YET!

Last, since the syntactic algebraizability classes, shown in the bottom row of the preceding diagram, dominate the syntactic left prealgebraizability classes in the 12-class hierarchy, we also need examples to separate syntactically algebraizable from syntactically left prealgebraizable π-institutions, i.e., examples showing that the longish vertical arrows in the diagram



represent proper inclusions. Again, in a single strike, this can be accomplished by providing an example of a syntactically strongly left prealgebraizable π-institution which is neither syntactically algebraizable nor syntactically antialgebraizable.

Example 1095 EXAMPLE NOT FOUND YET!