Chapter 14

The Syntactic Leibniz Hierarchy: Basement I

14.1 Rough/Narrow Truth Equationality

In this section, we study *rough/narrow truth equationality*, the syntactic analog of rough c-reflectivity, which, recalling Corollary 482, coincides with narrow c-reflectivty. It has the same relation to truth equationality as rough c-reflectivity has to c-reflectivity. In other words, it mimics truth equationality, but it is applied only to theory families of a π-institution all of whose components are nonempty.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that $\mathcal{I}$ is **roughly** or **narrowly (family) truth equational** if there exists $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with a single distinguished argument, such that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi/\Omega_\Sigma(T) \in \widetilde{T}_\Sigma/\Omega_\Sigma(T) \quad \text{iff} \quad \tau_\Sigma^{\mathcal{F}/\Omega(T)}[\phi/\Omega_\Sigma(T)] \leq \Delta^{\mathcal{F}/\Omega(T)}.$$

Recall that, by Proposition 369, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(\widetilde{T}) = \Omega(T)$. Thus, $\Omega(T)$ is compatible with $\widetilde{T}$ and, hence, the preceding definition makes sense. The collection $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ is referred to as a set of **witnessing equations** (of/for the rough/narrow truth equationality of $\mathcal{I}$).

Paralleling Proposition 816, we get the following alternative characterization.

Proposition 1096 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ a collection of natural transformations in $N^\flat$, with a single distinguished argument. $\mathcal{I}$ is roughly truth equational, with witnessing equations $\tau^\flat$, if and only if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,*

$$\phi \in \widetilde{T}_\Sigma \quad \textit{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

Proof: Suppose $\mathcal{I}$ is roughly truth equational and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{lll} \phi \in \widetilde{T}_\Sigma & \text{iff} & \phi/\Omega_\Sigma(T) \in \widetilde{T}_\Sigma/\Omega_\Sigma(T) \quad \text{(Proposition 369 and compatibility)} \\ & \text{iff} & \tau_\Sigma^{\mathcal{F}/\Omega(T)}[\phi/\Omega_\Sigma(T)] \leq \Delta^{\mathcal{F}/\Omega(T)} \quad \text{(rough truth equationality)} \\ & \text{iff} & \tau^\flat_\Sigma[\phi]/\Omega(T) \leq \Delta^{\mathcal{F}/\Omega(T)} \quad \text{(by definition)} \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(T). \end{array}$$

Suppose, conversely, that the given condition holds. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{lll} \phi/\Omega_\Sigma(T) \in \widetilde{T}_\Sigma/\Omega_\Sigma(T) & \text{iff} & \phi \in \widetilde{T}_\Sigma \quad \text{(Proposition 369 and compatibility)} \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(T) \quad \text{(by hypothesis)} \\ & \text{iff} & \tau^\flat_\Sigma[\phi]/\Omega(T) \leq \Delta^{\mathcal{F}/\Omega(T)} \\ & \text{iff} & \tau_\Sigma^{\mathcal{F}/\Omega(T)}[\phi/\Omega_\Sigma(T)] \leq \Delta^{\mathcal{F}/\Omega(T)}. \quad \text{(definition)} \end{array}$$

Therefore, $\mathcal{I}$ is roughly truth equational. ■

It is not difficult to see that an alternative way to express rough truth equationality is to assert the same condition that defines truth equationality, excluding, however, those theory families with at least one empty component.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall from Chapter 6 that we denote by $\mathrm{ThFam}^{\nleq}(\mathcal{I})$ the collection of all theory families T of $\mathcal{I}$, such that $T_\Sigma \neq \varnothing$, for all $\Sigma \in |\mathbf{Sign}^\flat|$:

$$\mathrm{ThFam}^{\nleq}(\mathcal{I}) = \{T \in \mathrm{ThFam}(\mathcal{I}) : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(T_\Sigma \neq \varnothing)\}.$$

Recall, also, that, if $\mathcal{I}$ has theorems, then $\mathrm{ThFam}^{\nleq}(\mathcal{I}) = \mathrm{ThFam}(\mathcal{I})$. In particular, this is the case if $\mathcal{I}$ happens to be truth equational.

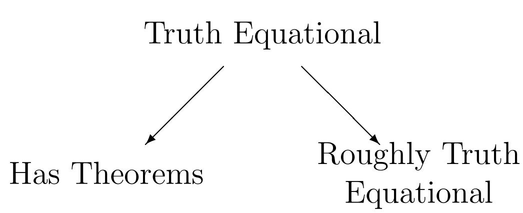
Proposition 1097 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ a collection of natural transformations in $N^\flat$, with a single distinguished argument. $\mathcal{I}$ is roughly truth equational, with witnessing equations $\tau^\flat$, if and only if, for all $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,*

$$\phi \in T_\Sigma \quad \textit{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

Proof: Suppose $\mathcal{I}$ is roughly truth equational, with witnessing equations $\tau^\flat$. Let $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then $\widetilde{T} = T$, whence, by Proposition 1096, $\phi \in T_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$.

Suppose, conversely, that the displayed condition holds. Consider $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, since, by definition of $\widetilde{T}$, we have $\widetilde{T} \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, we get, by hypothesis, $\phi \in \widetilde{T}_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(\widetilde{T})$, whence, using Proposition 369, we conclude that $\phi \in \widetilde{T}_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Therefore, $\mathcal{I}$ is roughly truth equational. ■

As a corollary, we obtain the following key relationship between rough truth equationality and truth equationality.



Corollary 1098 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is truth equational if and only if it is roughly truth equational and has theorems.*

Proof: Suppose, first, that $\mathcal{I}$ is roughly truth equational, with witnessing equations $\tau^\flat$, and that it has theorems. Availability of theorems implies that $\mathrm{ThFam}^{\nleq}(\mathcal{I}) = \mathrm{ThFam}(\mathcal{I})$. Thus, by Proposition 1097, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in T_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Thus, $\mathcal{I}$ is truth equational, with the same witnessing equations $\tau^\flat$.

Assume, conversely, that $\mathcal{I}$ is truth equational, with witnessing equations $\tau^\flat$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, and, hence, a fortiori, for all $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in T_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Hence, again by Proposition 1097, $\mathcal{I}$ is roughly truth equational. Finally, by Theorem 829, $\mathcal{I}$ is family c-reflective and, by Proposition 243, it is family reflective and, hence, family injective. Thus, it must have theorems. ■

Our next goal is to prove an analog of the characterization theorem, Theorem 838, of truth equationality in terms of the solubility of the Suszko core for rough truth equationality.

Rough truth equationality allows the following expression for all theory families with nonempty components, forming an analog of Proposition 828.

Proposition 1099 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *The following conditions are equivalent:*

(i) $\mathcal{I}$ *is roughly truth equational, with witnessing equations* $\tau^\flat$*;*

(ii) *For all* $T \in \mathrm{ThFam}(\mathcal{I})$, $\tau^\flat(\Omega(T)) = \widetilde{T}$*;*

(iii) *For all* $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, $\tau^\flat(\Omega(T)) = T$.

Proof:

(i)⇒(ii) Suppose $\mathcal{I}$ is roughly truth equational, with witnessing equations $\tau^\flat$, and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{llll} \phi \in \tau^\flat_\Sigma(\Omega(T)) & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(T) & \text{(definition)} \\ & \text{iff} & \phi \in \widetilde{T}_\Sigma. & \text{(rough truth equationality)} \end{array}$$

(ii)⇒(iii) Suppose Condition (ii) holds. Let $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Since $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, $T = \widetilde{T}$, whence, by hypothesis, $T = \tau^\flat(\Omega(T))$.

(iii)⇒(i) If, conversely, for all $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, $T = \tau^\flat(\Omega(T))$, then, by Proposition 1097, $\mathcal{I}$ is roughly truth equational, with witnessing equations $\tau^\flat$. ■

Recall from Chapter 6 that, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, $\mathcal{I}$ is called *roughly family c-reflective* if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,

$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \quad \text{implies} \quad \bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T}'.$$

We are now able to show that rough truth equationality implies rough family c-reflectivity. This is an analog of Theorem 829.

Theorem 1100 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly truth equational, then it is roughly family c-reflective.*

Proof: Suppose $\mathcal{I}$ is roughly truth equational, with witnessing equations $\tau^\flat$. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. Then we have

$$\begin{array}{rcll} \bigcap_{T\in\mathcal{T}} \widetilde{T} & = & \bigcap_{T\in\mathcal{T}} \tau^\flat(\Omega(T)) & \text{(Proposition 1099)} \\ & = & \tau^\flat(\bigcap_{T\in\mathcal{T}} \Omega(T)) & \text{(set theory)} \\ & \leq & \tau^\flat(\Omega(T')) & \text{(hypothesis)} \\ & = & \widetilde{T}'. & \text{(Proposition 1099)} \end{array}$$

Thus, $\mathcal{I}$ is roughly family c-reflective. ∎

In the context of rough truth equationality, the notion paralleling the Suszko core is the *rough Suszko core*, a modification of the original which is defined, naturally enough and as, perhaps, was to be expected, by circumventing theory families with empty components.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The **rough Suszko core** $S^{\mathcal{I}\nmid}$ of $\mathcal{I}$ is the collection

$$S^{\mathcal{I}\nmid} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}(\mathcal{I}))(\sigma^\flat[\widetilde{T}] \leq \widetilde{\Omega}^{\mathcal{I}}(\widetilde{T}))\}.$$

As before, an alternative characterization avoids $\widetilde{\ }$ at the expense of restricting quantification over $\mathrm{ThFam}^\nmid(\mathcal{I})$.

Proposition 1101 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then*

$$S^{\mathcal{I}\nmid} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}^\nmid(\mathcal{I}))(\sigma^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T))\}.$$

Proof: Inside this proof we set

$$M^{\mathcal{I}} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}^\nmid(\mathcal{I}))(\sigma^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T))\}.$$

Our goal is to show that $S^{\mathcal{I}\nmid} = M^{\mathcal{I}}$. Suppose, first, that $\sigma^\flat \in S^{\mathcal{I}\nmid}$ and let $T \in \mathrm{ThFam}^\nmid(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$. Since $T \in \mathrm{ThFam}^\nmid(\mathcal{I})$, we get $\widetilde{T} = T$. Hence, by hypothesis, $\phi \in \widetilde{T}_\Sigma$. Thus, since $\sigma^\flat \in S^{\mathcal{I}\nmid}$, we get

$$\sigma^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(\widetilde{T}) = \widetilde{\Omega}^{\mathcal{I}}(T).$$

This proves that $\sigma^\flat \in M^{\mathcal{I}}$. Assume, conversely, that $\sigma^\flat \in M^{\mathcal{I}}$ and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \widetilde{T}_\Sigma$. Since $\widetilde{T} \in \mathrm{ThFam}^\nmid(\mathcal{I})$ and $\sigma^\flat \in M^{\mathcal{I}}$, we get $\sigma^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(\widetilde{T})$, whence, $\sigma^\flat \in S^{\mathcal{I}\nmid}$. This proves that $S^{\mathcal{I}\nmid} = M^{\mathcal{I}}$. ∎

From the definition, it is not difficult to see that any theory family T with all its components nonempty is always included in $S^{\mathcal{I}\nmid}(\Omega(T))$. This forms an analog in the rough context of Proposition 832.

Proposition 1102 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$*,*

$$T \leq S^{\mathcal{I}\nmid}(\Omega(T)).$$

Proof: Suppose $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$, and $\sigma^\flat \in S^{\mathcal{I}\nmid}$. Then, by Proposition 1101, $\sigma^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T) \leq \Omega(T)$. Hence, $S^{\mathcal{I}\nmid}_\Sigma[\phi] \leq \Omega(T)$. By definition, then, $\phi \in S^{\mathcal{I}\nmid}_\Sigma(\Omega(T))$. Since Σ and $\phi \in T_\Sigma$ were arbitrary, we conclude that $T \leq S^{\mathcal{I}\nmid}(\Omega(T))$. ■

The reverse inclusion may or may not hold. If it does, for all $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, we say that the rough Suszko core of $\mathcal{I}$ is *soluble*.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The rough Suszko core $S^{\mathcal{I}\nmid}$ of $\mathcal{I}$ is said to be **soluble** if, for all $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$S^{\mathcal{I}\nmid}_\Sigma[\phi] \leq \Omega(T) \quad \text{implies} \quad \phi \in T_\Sigma.$$

An alternative way to express solubility is to again expand the view to all theory families at the balancing expense of adding rough equivalence representatives.

Lemma 1103 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $S^{\mathcal{I}\nmid}$ *is soluble if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\widetilde{T} = S^{\mathcal{I}\nmid}(\Omega(T)).$$

Proof: $S^{\mathcal{I}\nmid}$ is soluble if and only if, by definition and Proposition 1102, for all $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $T = S^{\mathcal{I}\nmid}(\Omega(T))$, if and only if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{T} = S^{\mathcal{I}\nmid}(\Omega(\widetilde{T}))$, if and only if, by Proposition 369, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{T} = S^{\mathcal{I}\nmid}(\Omega(T))$. ■

As was the case with truth equationality (see Lemma 835), it turns out that, if a given π-institution is roughly truth equational, then any collection of witnessing equations must be included in the rough Suszko core of $\mathcal{I}$. Differently put, in case of rough truth equationality, the rough Suszko core is a candidate for the largest set of witnessing equations.

Lemma 1104 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly truth equational, with witnessing equations* $\tau^\flat$*, then* $\tau^\flat \subseteq S^{\mathcal{I}\nmid}$*.*

Proof: Suppose $\mathcal{I}$ is roughly truth equational, with witnessing equations $\tau^\flat$. Let $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$. Then, for all $T \leq T' \in \mathrm{ThFam}(\mathcal{I})$, we have $T' \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$ and $\phi \in T'_\Sigma$. Thus, by rough truth equationality, and Proposition 1097, $\tau^\flat_\Sigma[\phi] \leq \Omega(T')$. Since T', with the postulated properties was arbitrary,

$$\tau^\flat_\Sigma[\phi] \leq \bigcap\{\Omega(T') : T \leq T'\} = \widetilde{\Omega}^{\mathcal{I}}(T).$$

We conclude, using Proposition 1101, that $\tau^\flat \subseteq S^{\mathcal{I}\nleq}$. ■

We are now ready to prove the equivalence between rough truth equationality and the solubility of the rough Suszko core. In the next theorem, we show that truth equationality implies the solubility of the rough Suszko core. This forms a rough analog of Theorem 836.

Theorem 1105 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly truth equational, then* $S^{\mathcal{I}\nleq}$ *is soluble.*

Proof: Suppose $\mathcal{I}$ is roughly truth equational, with witnessing equations $\tau^\flat$. Let $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $S^{\mathcal{I}\nleq}_\Sigma[\phi] \leq \Omega(T)$. Then, by rough truth equationality and Lemma 1104, $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Again, using rough truth equationality and Proposition 1097, we conclude that $\phi \in T_\Sigma$. This shows that $S^{\mathcal{I}\nleq}$ is soluble. ■

Conversely, in a rough analog of Theorem 837, we show that the solubility of the rough Suszko core of a π-institution implies rough truth equationality.

Theorem 1106 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $S^{\mathcal{I}\nleq}$ *is soluble, then* $\mathcal{I}$ *is roughly truth equational, with witnessing equations* $S^{\mathcal{I}\nleq}$*.*

Proof: Assume $S^{\mathcal{I}\nleq}$ is soluble and let $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. By Proposition 1097, it suffices to show that

$$\phi \in T_\Sigma \quad \text{iff} \quad S^{\mathcal{I}\nleq}_\Sigma[\phi] \leq \Omega(T).$$

If $\phi \in T_\Sigma$, then, by Proposition 1102, $\phi \in S^{\mathcal{I}\nleq}_\Sigma(\Omega(T))$, i.e., $S^{\mathcal{I}\nleq}_\Sigma[\phi] \leq \Omega(T)$. On the other hand, the reverse inclusion is guaranteed by the solubility of $S^{\mathcal{I}\nleq}$. Thus, $\mathcal{I}$ is roughly truth equational, with witnessing equations $S^{\mathcal{I}\nleq}$. ■

Theorems 1105 and 1106 provide the first characterization of rough truth equationality in terms of the solubility of the rough Suszko core. This parallels Theorem 838, which asserted a similar characterization for truth equationality in terms of the solubility of the Suszko core of a π-institution.

$$\mathcal{I} \text{ Roughly Truth Equational} \longleftrightarrow S^{\mathcal{I}\nleq} \text{ Soluble}$$

Theorem 1107 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is roughly truth equational if and only if* $S^{\mathcal{I}\nleq}$ *is soluble.*

Proof: The "if" is by Theorem 1106. The "only if" by Theorem 1105. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the rough Suszko core $S^{\mathcal{I}\natural}$ of $\mathcal{I}$ **roughly defines theory families** if, for al $T \in \mathrm{ThFam}^\natural(\mathcal{I})$,

$$T = S^{\mathcal{I}\natural}(\Omega(T)).$$

Another characterization of rough truth equationality, along the lines of Theorem 840, asserts that it is equivalent to the rough definability of the theory families by the rough Suszko core.

$$\begin{array}{l}\mathcal{I} \text{ Roughly Truth Equational}\\ \qquad\qquad \longleftrightarrow S^{\mathcal{I}\natural} \text{ Roughly Defines Theory Families}\end{array}$$

Theorem 1108 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is roughly truth equational if and only if $S^{\mathcal{I}\natural}$ roughly defines theory families in $\mathcal{I}$.*

Proof: Suppose $\mathcal{I}$ is roughly truth equational. By Theorem 1107, $S^{\mathcal{I}\natural}$ is soluble. Hence, by definition, for all $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, $S^{\mathcal{I}\natural}(\Omega(T)) \leq T$. Since, by Proposition 1102, the reverse always holds, we get, for all $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, $T = S^{\mathcal{I}\natural}(\Omega(T))$. Thus, $S^{\mathcal{I}\natural}$ roughly defines theory families in $\mathcal{I}$. Conversely, if, for all $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, $T = S^{\mathcal{I}\natural}(\Omega(T))$, then $S^{\mathcal{I}\natural}$ is soluble and, therefore, by Theorem 1107, $\mathcal{I}$ is roughly truth equational. ∎

We embark, next, in the process of establishing a connection between rough truth equationality and rough family c-reflectivity by means of the Suszko operator. We start by showing that, in every π-institution $\mathcal{I}$, $T \leq S^{\mathcal{I}\natural}(\Omega(T))$ actually holds for every theory family of $\mathcal{I}$ and not only for those theory families in $\mathrm{ThFam}^\natural(\mathcal{I})$.

Lemma 1109 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. For all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,*

$$\phi \in T_\Sigma \quad \textit{implies} \quad S^{\mathcal{I}\natural}_\Sigma[\phi] \leq \Omega(T).$$

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{rcll}\phi \in T_\Sigma & \text{implies} & \phi \in \widetilde{T}_\Sigma & (T \leq \widetilde{T})\\ & \text{implies} & S^{\mathcal{I}\natural}_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(\widetilde{T}) & (\text{definition of } S^{\mathcal{I}\natural})\\ & \text{implies} & S^{\mathcal{I}\natural}_\Sigma[\phi] \leq \Omega(\widetilde{T}) & (\widetilde{\Omega}^{\mathcal{I}} \leq \Omega)\\ & \text{iff} & S^{\mathcal{I}\natural}_\Sigma[\phi] \leq \Omega(T). & (\text{Proposition 369})\end{array}$$

This establishes the displayed implication. ∎

In the sequel, in dealing with intersections of Leibniz congruence systems, as, e.g., when computing a Suszko congruence system, we shall have the need to switch between arbitrary collections of theory families and collections of theory families having all components nonempty. In all those situations, the following straightforward technical lemma is quite useful.

Lemma 1110 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $X \in \mathrm{SenFam}(\mathbf{F})$ *and* $\theta \in \mathrm{SenFam}(\mathbf{F}^2)$*.*

(a) $\{\Omega(T) : X \leq T \in \mathrm{ThFam}(\mathcal{I})\} = \{\Omega(T) : X \leq T \in \mathrm{ThFam}^{\nplus}(\mathcal{I})\}$*;*

(b) $\{\Omega(T) : X \leq T \in \mathrm{ThFam}(\mathcal{I})$ *and* $\theta \leq \Omega(T)\} = \{\Omega(T) : X \leq T \in \mathrm{ThFam}^{\nplus}(\mathcal{I})$ *and* $\theta \leq \Omega(T)\}$*.*

Proof:

(a) Since $\mathrm{ThFam}^{\nplus}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I})$, it is clear that

$$\{\Omega(T) : X \leq T \in \mathrm{ThFam}^{\nplus}(\mathcal{I})\} \subseteq \{\Omega(T) : X \leq T \in \mathrm{ThFam}(\mathcal{I})\}.$$

To prove the reverse inclusion, let $T \in \mathrm{ThFam}(\mathcal{I})$, such that $X \leq T$. Consider $\widetilde{T} \in \mathrm{ThFam}^{\nplus}(\mathcal{I})$. We get $X \leq T \leq \widetilde{T}$ and, moreover, by Proposition 369, $\Omega(\widetilde{T}) = \Omega(T)$. This proves that $\{\Omega(T) : X \leq T \in \mathrm{ThFam}(\mathcal{I})\} \subseteq \{\Omega(T) : X \leq T \in \mathrm{ThFam}^{\nplus}(\mathcal{I})\}$.

(b) As in Part (a), the right-to-left inclusion is obvious. For the reverse, consider $T \in \mathrm{ThFam}(\mathcal{I})$, such that $X \leq T$ and $\theta \leq \Omega(T)$. Then, again, $\widetilde{T} \in \mathrm{ThFam}^{\nplus}(\mathcal{I})$, such that both $X \leq T \leq \widetilde{T}$ and $\theta \leq \Omega(T) = \Omega(\widetilde{T})$. This shows that the left-to-right inclusion also holds.

■

As a corollary, we obtain, for instance, an alternative expression for the Suszko congruence system associated with a given theory family of a π-institution $\mathcal{I}$.

Corollary 1111 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\widetilde{\Omega}^{\mathcal{I}}(T) = \bigcap\{\Omega(T') : T \leq T' \in \mathrm{ThFam}^{\nplus}(\mathcal{I})\}.$$

Proof: Immediate by the definition of $\widetilde{\Omega}^{\mathcal{I}}$ and Lemma 1110. ■

Based on Lemma 1109, we may show that, for every theory family T, $T \leq S^{\mathcal{I}\nplus}(\widetilde{\Omega}^{\mathcal{I}}(T))$.

Proposition 1112 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $T \in \mathrm{ThFam}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\phi \in T_\Sigma \quad \text{implies} \quad S^{\mathcal{I}\nplus}_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T).$$

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{rll}\phi \in T_\Sigma & \text{implies} & \phi \in T'_\Sigma, \text{ for all } T \le T' \in \mathrm{ThFam}(\mathcal{I})\\ & \text{implies} & S_\Sigma^{\mathcal{I}\natural}[\phi] \le \Omega(T'), \text{ for all } T \le T' \in \mathrm{ThFam}(\mathcal{I})\\ & & \quad \text{(by Lemma 1109)}\\ & \text{iff} & S_\Sigma^{\mathcal{I}\natural}[\phi] \le \widetilde{\Omega}^{\mathcal{I}}(T). \quad (\text{definition of } \widetilde{\Omega}^{\mathcal{I}})\end{array}$$

■

In analogy with the case of rough truth equationality, we may introduce the notion of *adequacy* of the rough Suszko core, which will help in characterizing the relationship between rough truth equationality and rough c-reflectivity. The following proposition, a rough analog of Proposition 841, partially justifies the notion of adequacy that will follow.

Proposition 1113 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\bigcap\{\Omega(T) : S_\Sigma^{\mathcal{I}\natural}[\phi] \le \Omega(T)\} \le \widetilde{\Omega}^{\mathcal{I}}(C(\phi)).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\begin{array}{rll}\phi \in T_\Sigma & \text{implies} & S_\Sigma^{\mathcal{I}\natural}[\phi] \le \widetilde{\Omega}^{\mathcal{I}}(T) \quad (\text{Proposition 1112})\\ & \text{implies} & S_\Sigma^{\mathcal{I}\natural}[\phi] \le \Omega(T). \quad (\widetilde{\Omega}^{\mathcal{I}} \le \Omega)\end{array}$$

Hence,

$$\begin{array}{rcl}\bigcap\{\Omega(T) : S_\Sigma^{\mathcal{I}\natural}[\phi] \le \Omega(T)\} & \le & \bigcap\{\Omega(T) : S_\Sigma^{\mathcal{I}\natural}[\phi] \le \widetilde{\Omega}^{\mathcal{I}}(T)\}\\ & \le & \bigcap\{\Omega(T) : \phi \in T_\Sigma\}\\ & = & \widetilde{\Omega}^{\mathcal{I}}(C(\phi)).\end{array}$$

This is the displayed formula in the statement. ■

If the reverse inclusion of that proven in Proposition 1113 holds, then we say that the rough Suszko core of $\mathcal{I}$ is *adequate*.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. We say that the rough Suszko core $S^{\mathcal{I}\natural}$ of $\mathcal{I}$ is **adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) \le \bigcap\{\Omega(T) : S_\Sigma^{\mathcal{I}\natural}[\phi] \le \Omega(T)\}.$$

We can show right away that solubility of the rough Suszko core implies adequacy.

Corollary 1114 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $S^{\mathcal{I}\natural}$ *is soluble, then it is adequate.*

Proof: Suppose $S^{\mathcal{I}\natural}$ is soluble and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) & = & \bigcap\{\Omega(T) : T \in \mathrm{ThFam}(\mathcal{I}) \text{ and } \phi \in T_\Sigma\} \\ & & \text{(definition of } \widetilde{\Omega}^{\mathcal{I}}) \\ & = & \bigcap\{\Omega(T) : T \in \mathrm{ThFam}^\natural(\mathcal{I}) \text{ and } \phi \in T_\Sigma\} \\ & & \text{(Lemma 1110)} \\ & = & \bigcap\{\Omega(T) : T \in \mathrm{ThFam}^\natural(\mathcal{I}) \text{ and } S^{\mathcal{I}\natural}_\Sigma[\phi] \leq \Omega(T)\} \\ & & \text{(solubility of } S^{\mathcal{I}\natural}) \\ & = & \bigcap\{\Omega(T) : T \in \mathrm{ThFam}(\mathcal{I}) \text{ and } S^{\mathcal{I}\natural}_\Sigma[\phi] \leq \Omega(T)\}. \\ & & \text{(Lemma 1110)}\end{array}$$

Thus, $S^{\mathcal{I}\natural}$ is adequate. ■

We prove, next, the converse of Corollary 1114, under the additional assumption that the π-institution $\mathcal{I}$ under consideration is roughly family c-reflective. This constitutes an analog of Proposition 846.

Proposition 1115 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a roughly family c-reflective* π*-institution based on* $\mathbf{F}$*. If* $S^{\mathcal{I}\natural}$ *is adequate, then it is soluble.*

Proof: Suppose $\mathcal{I}$ is roughly family c-reflective and $S^{\mathcal{I}\natural}$ is adequate. Let $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $S^{\mathcal{I}\natural}_\Sigma[\phi] \leq \Omega(T)$. By the adequacy of $S^{\mathcal{I}\natural}$, we get that $\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) \leq \Omega(T)$. By Lemma 1110,

$$\bigcap\{\Omega(T) : T \in \mathrm{ThFam}^\natural(\mathcal{I}) \text{ and } \phi \in T_\Sigma\} \leq \Omega(T).$$

By rough family c-reflectivity, $\bigcap\{T \in \mathrm{ThFam}^\natural(\mathcal{I}) \text{ and } \phi \in T_\Sigma\} \leq T$. Hence, $\phi \in T_\Sigma$. We conclude that $S^{\mathcal{I}\natural}$ is soluble. ■

We are now in a position to prove the main characterization theorem relating rough truth equationality with rough family c-reflectivity, an analog of Theorem 847, which characterized truth equationality in terms of family c-reflectivity and the adequacy of the Suszko core.

Rough Truth Equationality	=	$S^{\mathcal{I}\natural}$ Soluble
	=	$S^{\mathcal{I}\natural}$ Roughly Defines Theory Families
	=	Rough Family c-Reflectivity
		+ $S^{\mathcal{I}\natural}$ Adequate

Theorem 1116 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is roughly truth equational if and only if it is roughly family c-reflective and has an adequate rough Suszko core.*

Proof: Suppose, first, that $\mathcal{I}$ is roughly truth equational. By Theorem 1100, it is roughly family c-reflective. By Theorem 1105, its rough Suszko core is soluble. Thus, by Corollary 1114, its rough Suszko core is also adequate.

Assume, conversely, that $\mathcal{I}$ is roughly family c-reflective and has an adequate rough Suszko core. Then, by Proposition 1115, its rough Suszko core is also soluble. Hence, by Theorem 1107, $\mathcal{I}$ is roughly truth equational. ■

Even though Theorem 847 formed the inspiration for the formulation of Theorem 1116, we show that it can be obtained as a corollary of the latter. This also exhibits the close connection between the two results which should have been anticipated, given the fact that the work done here is intended to mimic the former, while circumventing potential obstacles due to the absence of theorems.

Corollary 1117 (Theorem 847) *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is truth equational if and only if it is family c-reflective and has an adequate Suszko core.*

Proof: $\mathcal{I}$ is truth equational if and only if, by Corollary 1098, it is roughly truth equational and has theorems if and only if, by Theorem 1116, it is roughly family c-reflective, has theorems and has an adequate rough Suszko core if and only if, by Theorem 468 and the definitions of the Suszko core, the rough Suszko core and their adequacy properties, $\mathcal{I}$ is family c-reflective and its Suszko core is adequate. ■

We close the section by looking at a couple of results that may be perceived either as alternative characterizations of rough truth equationality, involving arbitrary $\mathbf{F}$-algebraic systems, or as transfer theorems.

Theorem 1118 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is roughly truth equational, with witnessing equations $\tau^\flat$, if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,*

$$\phi \in \widetilde{T}_\Sigma \quad \text{iff} \quad \tau^{\mathcal{A}}_\Sigma[\phi] \leq \Omega^{\mathcal{A}}(T).$$

Proof: If the postulated condition holds, then it holds, in particular, for the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. This yields immediately that $\mathcal{I}$ is roughly truth equational.

Suppose, conversely, that $\mathcal{I}$ is roughly truth equational and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rll}
\alpha_\Sigma(\phi) \in \widetilde{T}_{F(\Sigma)} & \text{iff} & \phi \in \widetilde{\alpha^{-1}_\Sigma(\widetilde{T}_{F(\Sigma)})} \\
 & \text{iff} & \phi \in \alpha^{-1}_\Sigma(T_{F(\Sigma)}) \quad \text{(Theorem 377)} \\
 & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(\alpha^{-1}(T)) \quad \text{(hypothesis)} \\
 & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T)) \quad \text{(Proposition 24)} \\
 & \text{iff} & \tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi)] \leq \Omega^{\mathcal{A}}(T). \quad \text{(Lemma 95)}
\end{array}$$

Hence, taking into account the surjectivity of $\langle F,\alpha\rangle$, we conclude that the displayed condition holds. ∎

In analogy with the notation $\mathrm{ThFam}^{\nleq}(\mathcal{I})$, we introduce the following for filter families over arbitrary $\mathbf{F}$-algebraic systems all of whose components are nonempty.

Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, $\mathcal{I}=\langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$ and $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$, with $\mathbf{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$, an $\mathbf{F}$-algebraic system. Define $\mathrm{FiFam}^{\mathcal{I}\nleq}(\mathcal{A})$ to be the collection of all $\mathcal{I}$-filter families T on $\mathcal{A}$, such that $T_\Sigma\neq\varnothing$, for all $\Sigma\in|\mathbf{Sign}|$:

$$\mathrm{FiFam}^{\mathcal{I}\nleq}(\mathcal{A})=\{T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}):(\forall\Sigma\in|\mathbf{Sign}|)(T_\Sigma\neq\varnothing)\}.$$

We now get immediately the following corollary.

Corollary 1119 *Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and $\mathcal{I}=\langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is roughly truth equational, with witnessing equations $\tau^\flat$, if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$, with $\mathbf{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$, all $T\in\mathrm{FiFam}^{\mathcal{I}\nleq}(\mathcal{A})$, all $\Sigma\in|\mathbf{Sign}|$ and all $\phi\in\mathrm{SEN}(\Sigma)$,*

$$\phi\in T_\Sigma\quad\text{iff}\quad\tau^{\mathcal{A}}_\Sigma[\phi]\leq\Omega^{\mathcal{A}}(T).$$

Proof: Suppose that $\mathcal{I}$ is roughly truth equational. Then, if $\mathcal{A}$ is an $\mathbf{F}$-algebraic system and $T\in\mathrm{FiFam}^{\mathcal{I}\nleq}(\mathcal{A})$, we get

$$\begin{array}{rcll}T&=&\widetilde{T}&(T\in\mathrm{FiFam}^{\mathcal{I}\nleq}(\mathcal{A}))\\&=&\tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T)).&(\text{Theorem 1118})\end{array}$$

Suppose, conversely, that the displayed condition holds. Then, if $\mathcal{A}$ is an $\mathbf{F}$-algebraic system and $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get, taking into account that $\widetilde{T}\in\mathrm{FiFam}^{\mathcal{I}\nleq}(\mathcal{A})$,

$$\begin{array}{rcll}\widetilde{T}&=&\tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(\widetilde{T}))&(\text{hypothesis})\\&=&\tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T)).&(\text{Proposition 369})\end{array}$$

This establishes the claimed equivalence. ∎

14.2 Rough Left Truth Equationality

We now turn to *rough left truth equationality*. As the terminology suggests:

- It is in the same relation to rough left c-reflectivity as rough truth equationality is to rough c-reflectivity;
- It is in the same relation to rough truth equationality as left truth equationality is to truth equationality.

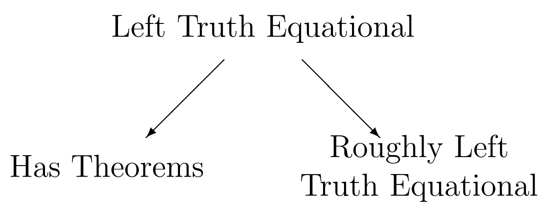
Roughly speaking (in both senses), rough left truth equationality is defined analogously to left truth equationality, but it is applied to rough representatives of theory families so as to avoid theory families with empty components.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that $\mathcal{I}$ is **roughly left truth equational** if there exists $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with a single distinguished argument, such that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in \widetilde{\overleftarrow{T}}_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

The collection $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ is referred to as a set of **witnessing equations** (of/for the rough left truth equationality of $\mathcal{I}$).

The following relationship between rough left truth equationality and left truth equationality, an analog of the relationship between rough truth equationality and truth equationality, presented in Corollary 1098, holds.



Proposition 1120 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is left truth equational if and only if it is roughly left truth equational and has theorems.*

Proof: Suppose, first, that $\mathcal{I}$ is roughly left truth equational, with witnessing equations $\tau^\flat$, and that it has theorems. Availability of theorems implies that $\mathrm{ThFam}^{\lightning}(\mathcal{I}) = \mathrm{ThFam}(\mathcal{I})$. Thus, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in \overleftarrow{T}_\Sigma$ if and only if $\phi \in \widetilde{\overleftarrow{T}}_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Thus, $\mathcal{I}$ is left truth equational, with the same witnessing equations $\tau^\flat$.

Assume, conversely, that $\mathcal{I}$ is left truth equational, with witnessing equations $\tau^\flat$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in \overleftarrow{T}_\Sigma$ iff $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. This clearly implies that $\mathcal{I}$ has theorems, since, otherwise, given that $\Omega(\overline{\varnothing}) = \nabla^{\mathbf{F}} = \Omega(\mathrm{SEN}^\flat)$, we would get $\mathrm{SEN}^\flat = \overleftarrow{\mathrm{SEN}^\flat} = \overleftarrow{\overline{\varnothing}} = \overline{\varnothing}$, a contradiction. Moreover, due to the availability of theorems, we get, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in \widetilde{\overleftarrow{T}}_\Sigma$ if and only if $\phi \in \overleftarrow{T}_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Thus, $\mathcal{I}$ is roughly left truth equational. ∎

Our next goal is to prove an analog of the characterization theorem, Theorem 860, of left truth equationality in terms of the left solubility of the left Suszko core for rough left truth equationality.

Rough left truth equationality allows an expression for $\widetilde{\overleftarrow{T}}$, for all theory families T, in terms of the Leibniz congruence system of T. The following proposition forms an analog of Proposition 1099.

Proposition 1121 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is roughly left truth equational, with witnessing equations* $\tau^\flat$*, if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\overleftarrow{T}} = \tau^\flat(\Omega(T))$.

Proof: $\mathcal{I}$ is roughly left truth equational, with witnessing equations $\tau^\flat$, if and only if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in \widetilde{\overleftarrow{T}}_\Sigma$ iff $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$, if and only if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\overleftarrow{T}} = \tau^\flat(\Omega(T))$. ■

Recall from Chapter 6 that, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, $\mathcal{I}$ is called *roughly left c-reflective* if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,

$$\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T') \quad \text{implies} \quad \bigcap_{T\in\mathcal{T}} \widetilde{\overleftarrow{T}} \leq \widetilde{\overleftarrow{T'}}.$$

We are now able to show that rough left truth equationality implies rough left c-reflectivity. This is an analog of Theorem 1100.

Theorem 1122 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly left truth equational, then it is roughly left c-reflective.*

Proof: Suppose $\mathcal{I}$ is roughly left truth equational, with witnessing equations $\tau^\flat$. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. Then we have

$$\begin{array}{rcll} \bigcap_{T\in\mathcal{T}} \widetilde{\overleftarrow{T}} & = & \bigcap_{T\in\mathcal{T}} \tau^\flat(\Omega(T)) & \text{(Proposition 1121)} \\ & = & \tau^\flat(\bigcap_{T\in\mathcal{T}} \Omega(T)) & \text{(set theory)} \\ & \leq & \tau^\flat(\Omega(T')) & \text{(hypothesis)} \\ & = & \widetilde{\overleftarrow{T'}}. & \text{(Proposition 1121)} \end{array}$$

Thus, $\mathcal{I}$ is roughly left c-reflective. ■

In the context of rough left truth equationality, the notion paralleling the left Suszko core is the *rough left Suszko core*, a modification of the original, which is defined below.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The **rough left Suszko core** $\widetilde{L}^{\mathcal{I}}$ of $\mathcal{I}$ is the collection

$$\begin{array}{rcl} \widetilde{L}^{\mathcal{I}} & = & \{\sigma^\flat \in N^\flat : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \phi \in \mathrm{SEN}^\flat(\Sigma)) \\ & & (\sigma^\flat_\Sigma[\phi] \leq \bigcap\{\Omega(T) : \phi \in \widetilde{\overleftarrow{T}}_\Sigma\})\}. \end{array}$$

From the definition, it is not difficult to see that, for any theory family T, $\widetilde{\overleftarrow{T}}$ is always included in $\widetilde{L}^{\mathcal{I}}(\Omega(T))$. This forms an analog in the rough left context of Proposition 1102.

Proposition 1123 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\widetilde{\overleftarrow{T}} \leq \widetilde{L}^{\mathcal{I}}(\Omega(T)).$$

Proof: Suppose $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \widetilde{\overleftarrow{T}}_\Sigma$, and $\sigma^\flat \in \widetilde{L}^{\mathcal{I}}$. Then, by the definition of $\widetilde{L}^{\mathcal{I}}$, $\sigma^\flat_\Sigma[\phi] \leq \Omega(T)$. Hence, $\widetilde{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. Thus, by definition of $\widetilde{L}^{\mathcal{I}}(\Omega(T))$, $\phi \in \widetilde{L}^{\mathcal{I}}_\Sigma(\Omega(T))$. Since Σ and $\phi \in \widetilde{\overleftarrow{T}}_\Sigma$ were arbitrary, we conclude that $\widetilde{\overleftarrow{T}} \leq \widetilde{L}^{\mathcal{I}}(\Omega(T))$. ■

The reverse inclusion may or may not hold. If it does, for all $T \in \mathrm{ThFam}(\mathcal{I})$, we say that the rough left Suszko core of $\mathcal{I}$ is *left soluble*.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The rough left Suszko core $\widetilde{L}^{\mathcal{I}}$ of $\mathcal{I}$ is said to be **left soluble** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widetilde{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) \quad \text{implies} \quad \phi \in \widetilde{\overleftarrow{T}}_\Sigma.$$

As was the case with rough truth equationality (see Lemma 1104), it turns out that, if a given π-institution is roughly left truth equational, then any collection of witnessing equations must be included in the rough left Suszko core of $\mathcal{I}$. In other words, in case of rough left truth equationality, the rough left Suszko core forms a candidate for the largest collection of witnessing equations.

Lemma 1124 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly left truth equational, with witnessing equations* $\tau^\flat$*, then* $\tau^\flat \subseteq \widetilde{L}^{\mathcal{I}}$*.*

Proof: Suppose $\mathcal{I}$ is roughly left truth equational, with witnessing equations $\tau^\flat$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \widetilde{\overleftarrow{T}}_\Sigma$. Then, by rough left truth equationality, $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Since T was arbitrary,

$$\tau^\flat_\Sigma[\phi] \leq \bigcap\{\Omega(T) : \phi \in \widetilde{\overleftarrow{T}}_\Sigma\}.$$

Hence, since $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$ were arbitrary, we conclude that $\tau^\flat \subseteq \widetilde{L}^{\mathcal{I}}$. ■

We are now ready to prove the equivalence between rough left truth equationality and the left solubility of the rough left Suszko core. In the next theorem, we show that rough left truth equationality implies the left solubility of the rough left Suszko core. This forms an analog of Theorem 1105.

Theorem 1125 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly left truth equational, then* $\widetilde{L}^{\mathcal{I}}$ *is left soluble.*

Proof: Suppose $\mathcal{I}$ is roughly left truth equational, with witnessing equations $\tau^\flat$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\widetilde{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. Then, by rough left truth equationality and Lemma 1124, $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Again, using rough left truth equationality, we conclude that $\phi \in \overleftarrow{\widetilde{T}}_\Sigma$. This shows that $\widetilde{L}^{\mathcal{I}}$ is left soluble. ∎

Conversely, in an analog of Theorem 1106, we show that the left solubility of the rough left Suszko core of a π-institution implies rough left truth equationality.

Theorem 1126 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\widetilde{L}^{\mathcal{I}}$ *is left soluble, then* $\mathcal{I}$ *is roughly left truth equational, with witnessing equations* $\widetilde{L}^{\mathcal{I}}$*.*

Proof: Assume $\widetilde{L}^{\mathcal{I}}$ is left soluble and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. We must show that

$$\phi \in \overleftarrow{\widetilde{T}}_\Sigma \quad \text{iff} \quad \widetilde{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

If $\phi \in \overleftarrow{\widetilde{T}}_\Sigma$, then, by Proposition **??**, $\phi \in \widetilde{L}^{\mathcal{I}}_\Sigma(\Omega(T))$, i.e., $\widetilde{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. On the other hand, the reverse inclusion is guaranteed by the postulated left solubility of $\widetilde{L}^{\mathcal{I}}$. Thus, $\mathcal{I}$ is indeed roughly left truth equational, with witnessing equations $\widetilde{L}^{\mathcal{I}}$. ∎

Theorems 1125 and 1126 provide the first characterization of rough left truth equationality in terms of the left solubility of the rough left Suszko core. This parallels Theorem 1107, which asserted a similar characterization for rough truth equationality in terms of the solubility of the rough Suszko core of a π-institution.

$$\mathcal{I} \text{ Roughly Left Truth Equational} \longleftrightarrow \widetilde{L}^{\mathcal{I}} \text{ Left Soluble}$$

Theorem 1127 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is roughly left truth equational if and only if* $\widetilde{L}^{\mathcal{I}}$ *is left soluble.*

Proof: The "if" is by Theorem 1126. The "only if" by Theorem 1125. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the rough left Suszko core $\widetilde{L}^{\mathcal{I}}$ of $\mathcal{I}$ **roughly defines theory families up to arrow** if, for al $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\overleftarrow{\widetilde{T}} = \widetilde{L}^{\mathcal{I}}(\Omega(T)).$$

Another characterization of rough left truth equationality, along the lines of Theorem 1108, asserts that it is equivalent to the rough definability up to arrow of the theory families by the rough left Suszko core.

$\mathcal{I}$ Roughly Left Truth Equational
$\longleftrightarrow$ $\widetilde{L}^{\mathcal{I}}$ Roughly Defines Theory Families Up to Arrow

Theorem 1128 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is roughly left truth equational if and only if* $\widetilde{L}^{\mathcal{I}}$ *roughly defines theory families in* $\mathcal{I}$ *up to arrow.*

Proof: Suppose $\mathcal{I}$ is roughly left truth equational. By Theorem 1125, $\widetilde{L}^{\mathcal{I}}$ is left soluble. Hence, by definition, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{L}^{\mathcal{I}}(\Omega(T)) \le \overleftarrow{\overleftarrow{T}}$. Since, by Proposition 1123, the reverse always holds, we get, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{\overleftarrow{T}} = \widetilde{L}^{\mathcal{I}}(\Omega(T))$. Thus, $\widetilde{L}^{\mathcal{I}}$ roughly defines theory families in $\mathcal{I}$ up to arrow. Conversely, if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{\overleftarrow{T}} = \widetilde{L}^{\mathcal{I}}(\Omega(T))$, then $\widetilde{L}^{\mathcal{I}}$ is left soluble and, therefore, by Theorem 1126, $\mathcal{I}$ is roughly left truth equational. ∎

We establish, next, a connection between rough left truth equationality and rough left c-reflectivity by means of the rough left Suszko core. To help us is this task, in analogy with the case of rough truth equationality, we introduce the notion of *left adequacy* of the rough left Suszko core. The following proposition, a "left" analog of Proposition 1113, motivates and, in a sense, justifies, the notion of left adequacy that will follow. Its role parallels that of Proposition 1113 in motivating the definition of adequacy of the rough Suszko core.

Proposition 1129 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\bigcap\{\Omega(T) : \widetilde{L}^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)\} \le \bigcap\{\Omega(T) : \phi \in \overleftarrow{\overleftarrow{T}}_\Sigma\}.$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\phi \in \overleftarrow{\overleftarrow{T}}_\Sigma \quad \text{implies} \quad \widetilde{L}^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T). \quad (\text{Definition of } \widetilde{L}^{\mathcal{I}})$$

Hence,

$$\bigcap\{\Omega(T) : \widetilde{L}^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)\} \quad \le \quad \bigcap\{\Omega(T) : \phi \in \overleftarrow{\overleftarrow{T}}_\Sigma\}.$$

This is the displayed formula in the statement. ∎

If the reverse inclusion of that proven in Proposition 1129 holds, then we say that the rough left Suszko core of $\mathcal{I}$ is *left adequate*.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the rough left Suszko core $\widetilde{L}^{\mathcal{I}}$ of $\mathcal{I}$ is **left adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\bigcap\{\Omega(T) : \phi \in \overleftarrow{\overleftarrow{T}}_\Sigma\} \leq \bigcap\{\Omega(T) : \widetilde{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.$$

We can show, in analogy with Corollary 1114, that the left solubility of the rough left Suszko core implies left adequacy.

Corollary 1130 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\widetilde{L}^{\mathcal{I}}$ is left soluble, then it is left adequate.*

Proof: Suppose $\widetilde{L}^{\mathcal{I}}$ is left soluble and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, by left solubility and Proposition 1123, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\phi \in \overleftarrow{\overleftarrow{T}}_\Sigma$ if and only if $\widetilde{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. Therefore,

$$\bigcap\{\Omega(T) : \phi \in \overleftarrow{\overleftarrow{T}}_\Sigma\} \quad = \quad \bigcap\{\Omega(T) : \widetilde{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.$$

Thus, $\widetilde{L}^{\mathcal{I}}$ is left adequate. ■

We prove, next, the converse of Corollary 1130, under the additional assumption that the π-institution $\mathcal{I}$ under consideration is roughly left c-reflective. This constitutes an analog of Proposition 1115.

Proposition 1131 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a roughly left c-reflective π-institution based on $\mathbf{F}$. If $\widetilde{L}^{\mathcal{I}}$ is left adequate, then it is left soluble.*

Proof: Suppose $\mathcal{I}$ is roughly left c-reflective and $\widetilde{L}^{\mathcal{I}}$ is left adequate. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\widetilde{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. By the postulated left adequacy of $\widetilde{L}^{\mathcal{I}}$, we get that $\bigcap\{\Omega(T) : \phi \in \overleftarrow{\overleftarrow{T}}_\Sigma\} \leq \Omega(T)$. By rough left truth equationality, $\bigcap\{\overleftarrow{\overleftarrow{T}} : \phi \in \overleftarrow{\overleftarrow{T}}_\Sigma\} \leq \overleftarrow{\overleftarrow{T}}$. Therefore, $\phi \in \overleftarrow{\overleftarrow{T}}_\Sigma$. We conclude that $\widetilde{L}^{\mathcal{I}}$ is left soluble. ■

We are now in a position to prove the main characterization theorem relating rough left truth equationality with rough left c-reflectivity, an analog of Theorem 1116, which characterized rough truth equationality in terms of rough family c-reflectivity and the adequacy of the rough Suszko core.

Rough Left Truth Equationality	=	$\widetilde{L}^{\mathcal{I}}$ Left Soluble
	=	$\widetilde{L}^{\mathcal{I}}$ Roughly Defines Theory Families Up to Arrow
	=	Rough Left c-Reflectivity + $\widetilde{L}^{\mathcal{I}}$ Left Adequate

Theorem 1132 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is roughly left truth equational if and only if it is roughly left c-reflective and has a left adequate rough left Suszko core.*

Proof: Suppose, first, that $\mathcal{I}$ is roughly left truth equational. By Theorem 1122, it is roughly left c-reflective. By Theorem 1125, its rough left Suszko core is left soluble. Thus, by Corollary 1130, its rough left Suszko core is also left adequate.

Assume, conversely, that $\mathcal{I}$ is roughly left c-reflective and has a left adequate rough left Suszko core. Then, by Proposition 1131, its rough left Suszko core is also left soluble. Hence, by Theorem 1126, $\mathcal{I}$ is roughly left truth equational. ∎

Based on Proposition 1120 and Theorem 468, it is not difficult to show, in an analog of Corollary 1117, that the characterization theorem, Theorem 870, of left truth equationality in terms of left c-reflectivity and the left adequacy of the left Suszko core, can be inferred from Theorem 1132.

Corollary 1133 (Theorem 870) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is left truth equational if and only if it is left c-reflective and has a left adequate left Suszko core.*

Proof: $\mathcal{I}$ is left truth equational if and only if, by Proposition 1120, it is roughly left truth equational and has theorems, if and only if, by Theorem 1132, it is roughly left c-reflective, with a left adequate rough left Suszko core and has theorems, if and only if, by Theorem 468 and the definitions of left Suszko core and rough left Suszko core, it is left c-reflective and has a left adequate left Suszko core. ∎

We close the section by looking at a result, an analog of Theorem 1118, which may be perceived either as an alternative characterization of rough left truth equationality, involving arbitrary $\mathbf{F}$-algebraic systems, or as a transfer theorem.

Theorem 1134 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is roughly left truth equational, with witnessing equations* $\tau^\flat$*, if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi \in \mathrm{SEN}(\Sigma)$*,*

$$\phi \in \overset{\approx}{\overleftarrow{T}}_\Sigma \quad \text{iff} \quad \tau^{\mathcal{A}}_\Sigma[\phi] \leq \Omega^{\mathcal{A}}(T).$$

Proof: If the postulated condition holds, then it holds, in particular, for the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. This yields immediately that $\mathcal{I}$ is roughly left truth equational.

Suppose, conversely, that $\mathcal{I}$ is roughly left truth equational and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rll} \alpha_\Sigma(\phi) \in \overleftarrow{\widetilde{\overleftarrow{T}}}_{F(\Sigma)} & \text{iff} & \phi \in \alpha_\Sigma^{-1}(\underset{\sim}{\overleftarrow{\widetilde{\overleftarrow{T}}}_{F(\Sigma)}}) \\ & \text{iff} & \phi \in \alpha_\Sigma^{-1}(\underset{\sim}{\overleftarrow{T}_{F(\Sigma)}}) \quad \text{(Theorem 377)} \\ & \text{iff} & \phi \in \overleftarrow{\alpha_\Sigma^{-1}(T_{F(\Sigma)})} \quad \text{(Lemma 6)} \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(\alpha^{-1}(T)) \quad \text{(hypothesis)} \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T)) \quad \text{(Proposition 24)} \\ & \text{iff} & \tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi)] \leq \Omega^{\mathcal{A}}(T). \quad \text{(Lemma 95)} \end{array}$$

Hence, taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that the displayed condition holds. ∎

14.3 Narrow Left Truth Equationality

We now turn to *narrow left truth equationality*. As the terminology suggests:

- It is in the same relation to narrow left c-reflectivity as rough left truth equationality is to rough left c-reflectivity;
- It is in the same relation to rough/narrow truth equationality as left truth equationality is to truth equationality.

In a nutshell, narrow left truth equationality is defined analogously to left truth equationality, but care is taken to bypass theory families with empty components.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that $\mathcal{I}$ is **narrowly left truth equational** if there exists $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with a single distinguished argument, such that, for all $T \in \mathrm{ThFam}^{\not\emptyset}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

The collection $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ is referred to as a set of **witnessing equations** (of/for the narrow left truth equationality of $\mathcal{I}$).

An alternative characterization quantifies the relevant condition over all theory families, but it does so at the expense of using the rough operator on one side (and implicitly also on the other).

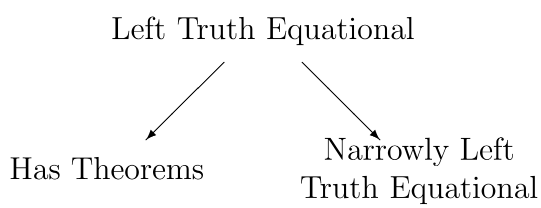
Lemma 1135 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is narrowly left truth equational if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\phi \in \overleftarrow{\widetilde{T}}_\Sigma \quad \textit{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

Proof: Suppose, first, that $\mathcal{I}$ is narrowly left truth equational, with witnessing equations $\tau^\flat$, and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, since $\widetilde{T} \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, we get, by hypothesis, $\phi \in \overleftarrow{\widetilde{T}}_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \le \Omega(\widetilde{T})$. Therefore, by Proposition 369, $\phi \in \overleftarrow{\widetilde{T}}_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \le \Omega(T)$.

Suppose, conversely, that the displayed equivalence holds and let $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$. Then $\widetilde{T} = T$. Thus, by hypothesis, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in \overleftarrow{T}_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \le \Omega(T)$. Therefore, $\mathcal{I}$ is narrowly left truth equational. ∎

The following relationship between rough left truth equationality and left truth equationality, an analog of the relationship between rough truth equationality and truth equationality, presented in Corollary 1098, holds. Note that narrow left truth equationality is in the same relationship to left truth equationality as rough left truth equationality is to left truth equationality, as detailed in Proposition 1120.



Proposition 1136 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is left truth equational if and only if it is narrowly left truth equational and has theorems.*

Proof: Suppose, first, that $\mathcal{I}$ is narrowly left truth equational, with witnessing equations $\tau^\flat$, and that it has theorems. Availability of theorems implies that $\mathrm{ThFam}^{\nmid}(\mathcal{I}) = \mathrm{ThFam}(\mathcal{I})$. Thus, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in \overleftarrow{T}_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \le \Omega(T)$. Thus, $\mathcal{I}$ is left truth equational, with the same witnessing equations $\tau^\flat$.

Assume, conversely, that $\mathcal{I}$ is left truth equational, with witnessing equations $\tau^\flat$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in \overleftarrow{T}_\Sigma$ iff $\tau^\flat_\Sigma[\phi] \le \Omega(T)$. This clearly implies that $\mathcal{I}$ has theorems, since, otherwise, given that $\Omega(\overline{\varnothing}) = \nabla^{\mathbf{F}} = \Omega(\mathrm{SEN}^\flat)$, we would get $\mathrm{SEN}^\flat = \overleftarrow{\mathrm{SEN}^\flat} = \overleftarrow{\overline{\varnothing}} = \overline{\varnothing}$, a contradiction. Moreover, since $\mathrm{ThFam}^{\nmid}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I})$, left truth equationality implies trivially narrow left truth equationality. ∎

Our next goal is to prove an analog of the characterizations, Theorem 860 and Proposition 1121, of left truth equationality and rough left truth equationality, respectively, for narrow left truth equationality.

Narrow left truth equationality allows an expression for $\overleftarrow{T}$, for all theory families T without empty components, or alternatively, for $\overleftarrow{\widetilde{T}}$, for all theory families T, in terms of the Leibniz congruence system of T. The following proposition forms an analog of Proposition 1121.

Proposition 1137 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *Then the following statements are equivalent:*

(i) $\mathcal{I}$ *is narrowly left truth equational, with witnessing equations* $\tau^\flat$*;*

(ii) For all $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $\overleftarrow{T} = \tau^\flat(\Omega(T))$*;*

(iii) For all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{\widetilde{T}} = \tau^\flat(\Omega(T))$.

Proof: Suppose, first, that $\mathcal{I}$ is narrowly left truth equational, with witnessing equations $\tau^\flat$, and let $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{llll} \phi \in \tau^\flat_\Sigma(\Omega(T)) & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(T) & \text{(definition)} \\ & \text{iff} & \phi \in \overleftarrow{T}_\Sigma. & \text{(hypothesis)} \end{array}$$

Suppose, next, that Condition (ii) holds and let $T \in \mathrm{ThFam}(\mathcal{I})$. Then $\widetilde{T} \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, whence, by hypothesis, $\overleftarrow{\widetilde{T}} = \tau^\flat(\Omega(\widetilde{T})) = \tau^\flat(\Omega(T))$, where the last equality holds by Proposition 369. Finally, suppose that Condition (iii) holds and let $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$. Then $\widetilde{T} = T$, whence, we get, by hypothesis, $\overleftarrow{T} = \tau^\flat(\Omega(T))$, showing that $\mathcal{I}$ is narrowly left truth equational. ∎

Recall from Chapter 6 that, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, $\mathcal{I}$ is called *narrowly left c-reflective* if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nmid}(\mathcal{I})$,

$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \quad \text{implies} \quad \bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}.$$

We are now able to show that narrow left truth equationality implies narrow left c-reflectivity. This is an analog of Theorem 1122.

Theorem 1138 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is narrowly left truth equational, then it is narrowly left c-reflective.*

Proof: Suppose $\mathcal{I}$ is narrowly left truth equational, with witnessing equations $\tau^\flat$. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nmid}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. Then we have

$$\begin{array}{lll} \bigcap_{T \in \mathcal{T}} \overleftarrow{T} & = & \bigcap_{T \in \mathcal{T}} \tau^\flat(\Omega(T)) \quad \text{(Proposition 1137)} \\ & = & \tau^\flat(\bigcap_{T \in \mathcal{T}} \Omega(T)) \quad \text{(set theory)} \\ & \leq & \tau^\flat(\Omega(T')) \quad \text{(hypothesis)} \\ & = & \overleftarrow{T'}. \quad \text{(Proposition 1137)} \end{array}$$

Thus, $\mathcal{I}$ is narrowly left c-reflective. ■

In the context of narrow left truth equationality, the notion paralleling the left Suszko core is the *narrow left Suszko core*, a modification of the original, which is defined below.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The **narrow left Suszko core** $L^{\mathcal{I}\lightning}$ of $\mathcal{I}$ is the collection

$$L^{\mathcal{I}\lightning} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}^{\lightning}(\mathcal{I}))(\sigma^\flat[\overleftarrow{T}] \leq \widetilde{\Omega}^{\mathcal{I}}(T))\}.$$

From the definition, it is not difficult to see that, for any theory family T, with all components nonempty, $\overleftarrow{T}$ is always included in $L^{\mathcal{I}\lightning}(\Omega(T))$. This forms an analog in the narrow left context of Propositions 1102 and 1123.

Proposition 1139 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$*,*

$$\overleftarrow{T} \leq L^{\mathcal{I}\lightning}(\Omega(T)).$$

Proof: Suppose $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \overleftarrow{T}_\Sigma$, and $\sigma^\flat \in L^{\mathcal{I}\lightning}$. Then, by the definition of $L^{\mathcal{I}\lightning}$, $\sigma^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T) \leq \Omega(T)$. Hence, $L^{\mathcal{I}\lightning}_\Sigma[\phi] \leq \Omega(T)$. Thus, by definition of $L^{\mathcal{I}\lightning}(\Omega(T))$, $\phi \in L^{\mathcal{I}\lightning}_\Sigma(\Omega(T))$. Since Σ and $\phi \in \overleftarrow{T}_\Sigma$ were arbitrary, we conclude that $\overleftarrow{T} \leq L^{\mathcal{I}\lightning}(\Omega(T))$. ■

The reverse inclusion may or may not hold. If it does, for all $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, we say that the narrow left Suszko core of $\mathcal{I}$ is *left soluble*.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The narrow left Suszko core $L^{\mathcal{I}\lightning}$ of $\mathcal{I}$ is said to be **left soluble** if, for all $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$L^{\mathcal{I}\lightning}_\Sigma[\phi] \leq \Omega(T) \quad \text{implies} \quad \phi \in \overleftarrow{T}_\Sigma.$$

As was the case with rough left truth equationality (see Lemma 1124), it turns out that, if a given π-institution is narrowly left truth equational, then any collection of witnessing equations must be included in the narrow left Suszko core of $\mathcal{I}$; differently put, in case of narrow left truth equationality, the narrow left Suszko core forms a candidate for the largest collection of witnessing equations.

Lemma 1140 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is narrowly left truth equational, with witnessing equations* $\tau^\flat$*, then* $\tau^\flat \subseteq L^{\mathcal{I}\lightning}$*.*

Proof: Suppose $\mathcal{I}$ is narrowly left truth equational, with witnessing equations $\tau^\flat$. Let $T \in \mathrm{ThFam}^\sharp(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \overleftarrow{T}_\Sigma$. Then, for all $T \le T' \in \mathrm{ThFam}(\mathcal{I})$, $\phi \in \overleftarrow{T'}_\Sigma$, whence, by narrow left truth equationality, $\tau^\flat_\Sigma[\phi] \le \Omega(T')$. Since T', with the postulated properties was arbitrary,

$$\tau^\flat_\Sigma[\phi] \le \bigcap\{\Omega(T') : T \le T' \in \mathrm{ThFam}(\mathcal{I})\} = \widetilde{\Omega}^{\mathcal{I}}(T).$$

Hence, $\tau^\flat[\overleftarrow{T}] \le \widetilde{\Omega}^{\mathcal{I}}(T)$. Since $T \in \mathrm{ThFam}^\sharp(\mathcal{I})$ was arbitrary, we conclude that $\tau^\flat \subseteq L^{\mathcal{I}\sharp}$. ∎

We are now ready to prove the equivalence between narrow left truth equationality and the left solubility of the narrow left Suszko core. In the next theorem, we show that narrow left truth equationality implies the left solubility of the narrow left Suszko core. This forms an analog of Theorem 1125.

Theorem 1141 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is narrowly left truth equational, then $L^{\mathcal{I}\sharp}$ is left soluble.*

Proof: Suppose $\mathcal{I}$ is narrowly left truth equational, with witnessing equations $\tau^\flat$. Let $T \in \mathrm{ThFam}^\sharp(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $L^{\mathcal{I}\sharp}_\Sigma[\phi] \le \Omega(T)$. Then, by narrow left truth equationality and Lemma 1140, $\tau^\flat_\Sigma[\phi] \le \Omega(T)$. Again, using narow left truth equationality, we conclude that $\phi \in \overleftarrow{T}_\Sigma$. This shows that $L^{\mathcal{I}\sharp}$ is left soluble. ∎

Conversely, in an analog of Theorem 1126, we show that the left solubility of the narrow left Suszko core of a π-institution implies narrow left truth equationality.

Theorem 1142 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $L^{\mathcal{I}\sharp}$ is left soluble, then $\mathcal{I}$ is narrowly left truth equational, with witnessing equations $L^{\mathcal{I}\sharp}$.*

Proof: Assume $L^{\mathcal{I}\sharp}$ is left soluble and let $T \in \mathrm{ThFam}^\sharp(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. We must show that

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{iff} \quad L^{\mathcal{I}\sharp}_\Sigma[\phi] \le \Omega(T).$$

If $\phi \in \overleftarrow{T}_\Sigma$, then, by Proposition 1139, $\phi \in L^{\mathcal{I}\sharp}_\Sigma(\Omega(T))$, i.e., $L^{\mathcal{I}\sharp}_\Sigma[\phi] \le \Omega(T)$. On the other hand, the reverse inclusion is guaranteed by the postulated left solubility of $L^{\mathcal{I}\sharp}$. Thus, $\mathcal{I}$ is indeed narrowly left truth equational, with witnessing equations $L^{\mathcal{I}\sharp}$. ∎

Theorems 1141 and 1142 provide the first characterization of narrow left truth equationality in terms of the left solubility of the narrow left Suszko core. This parallels Theorem 1127, which asserted a similar characterization for rough left truth equationality in terms of the left solubility of the rough left Suszko core of a π-institution.

$$\mathcal{I} \text{ Narrowly Left Truth Equational } \longleftrightarrow L^{\mathcal{I}\nmid} \text{ Left Soluble}$$

Theorem 1143 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is narrowly left truth equational if and only if* $L^{\mathcal{I}\nmid}$ *is left soluble.*

Proof: The "if" is by Theorem 1142. The "only if" by Theorem 1141. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the narrow left Suszko core $L^{\mathcal{I}\nmid}$ of $\mathcal{I}$ **narrowly defines theory families up to arrow** if, for al $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$,

$$\overleftarrow{T} = L^{\mathcal{I}\nmid}(\Omega(T)).$$

Another characterization of narrow left truth equationality, along the lines of Theorem 1128, asserts that it is equivalent to the narrow definability up to arrow of the theory families by the narrow left Suszko core.

$$\begin{array}{l}\mathcal{I} \text{ Narrowly Left Truth Equational} \\ \qquad \longleftrightarrow L^{\mathcal{I}\nmid} \text{ Narrowly Defines Theory} \\ \qquad\qquad \text{Families Up to Arrow}\end{array}$$

Theorem 1144 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is narrowly left truth equational if and only if* $L^{\mathcal{I}\nmid}$ *narrowly defines theory families in* $\mathcal{I}$ *up to arrow.*

Proof: Suppose $\mathcal{I}$ is narrowly left truth equational. By Theorem 1141, $L^{\mathcal{I}\nmid}$ is left soluble. Hence, by definition, for all $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $L^{\mathcal{I}\nmid}(\Omega(T)) \le \overleftarrow{T}$. Since, by Proposition 1139, the reverse always holds, we get, for all $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $\overleftarrow{T} = L^{\mathcal{I}\nmid}(\Omega(T))$. Thus, $L^{\mathcal{I}\nmid}$ narrowly defines theory families in $\mathcal{I}$ up to arrow. Conversely, if, for all $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $\overleftarrow{T} = L^{\mathcal{I}\nmid}(\Omega(T))$, then $L^{\mathcal{I}\nmid}$ is left soluble and, therefore, by Theorem 1142, $\mathcal{I}$ is narrowly left truth equational. ∎

We would like, next to establish a connection between narrow left truth equationality and narrow left c-reflectivity by means of the narrow left Suszko core. To accomplish this, we introduce an apparently modified version of the Suszko operator, which, however, is identical to the Suszko operator itself. This modified version is convenient for the purpose of handling proofs in a more straightforward and efficient way.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We define the **narrow Suszko operator** $\widetilde{\Omega}^{\mathcal{I}^\sharp}$ by setting, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\widetilde{\Omega}^{\mathcal{I}^\sharp}(T) = \bigcap \{\Omega(T') : T \leq T' \in \mathrm{ThFam}^\sharp(\mathcal{I})\}.$$

By Corollary 1111, we have, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\Omega}^{\mathcal{I}^\sharp}(T) = \widetilde{\Omega}^{\mathcal{I}}(T)$. So this is indeed an apparent and not a substantial change and one can think, without any loss, of $\widetilde{\Omega}^{\mathcal{I}^\sharp}$ as the Suszko operator.

In analogy with the case of rough truth equationality and rough left truth equationality, we may introduce the notion of *left adequacy* of the narrow left Suszko core, which will help in characterizing the relationship between narrow left truth equationality and narrow left c-reflectivity. The following proposition, a "left" analog of Proposition 1113 and an analog of Proposition 1129, justifies the notion of left adequacy that will follow.

Proposition 1145 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,*

$$\bigcap \{\Omega(T) : L_\Sigma^{\mathcal{I}^\sharp}[\phi] \leq \Omega(T)\} \leq \widetilde{\Omega}^{\mathcal{I}^\sharp}(C(\overrightarrow{\phi})).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $T \in \mathrm{ThFam}^\sharp(\mathcal{I})$,

$$\begin{array}{llll} \phi \in \overleftarrow{T}_\Sigma & \text{implies} & L_\Sigma^{\mathcal{I}^\sharp}[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T) & (\text{Definition of } L^{\mathcal{I}^\sharp}) \\ & \text{implies} & L_\Sigma^{\mathcal{I}^\sharp}[\phi] \leq \Omega(T). & (\widetilde{\Omega}^{\mathcal{I}} \leq \Omega) \end{array}$$

Hence,

$$\begin{array}{l} \bigcap \{\Omega(T) : L_\Sigma^{\mathcal{I}^\sharp}[\phi] \leq \Omega(T)\} \\ \quad = \bigcap \{\Omega(T) : T \in \mathrm{ThFam}^\sharp(\mathcal{I}) \text{ and } L_\Sigma^{\mathcal{I}^\sharp}[\phi] \leq \Omega(T)\} \\ \quad \leq \bigcap \{\Omega(T) : T \in \mathrm{ThFam}^\sharp(\mathcal{I}) \text{ and } L_\Sigma^{\mathcal{I}^\sharp}[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T)\} \\ \quad \leq \bigcap \{\Omega(T) : T \in \mathrm{ThFam}^\sharp(\mathcal{I}) \text{ and } \phi \in \overleftarrow{T}_\Sigma\} \\ \quad = \bigcap \{\Omega(T) : T \in \mathrm{ThFam}^\sharp(\mathcal{I}) \text{ and } \overrightarrow{\phi} \leq T\} \\ \quad = \widetilde{\Omega}^{\mathcal{I}^\sharp}(C(\overrightarrow{\phi})). \end{array}$$

This is the displayed formula in the statement. ■

If the reverse inclusion of that proven in Proposition 1145 holds, then we say that the narrow left Suszko core of $\mathcal{I}$ is *left adequate*.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the narrow left Suszko core $L^{\mathcal{I}^\sharp}$ of $\mathcal{I}$ is **left adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widetilde{\Omega}^{\mathcal{I}^\sharp}(C(\overrightarrow{\phi})) \leq \bigcap \{\Omega(T) : L_\Sigma^{\mathcal{I}^\sharp}[\phi] \leq \Omega(T)\}.$$

We can show, in analogy with Corollary 1130, that the left solubility of the narrow left Suszko core implies left adequacy.

Corollary 1146 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $L^{\mathcal{I}\nmid}$ *is left soluble, then it is left adequate.*

Proof: Suppose $L^{\mathcal{I}\nmid}$ is left soluble and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{aligned}
\widetilde{\Omega}^{\mathcal{I}\nmid}(C(\overrightarrow{\phi})) &= \bigcap\{\Omega(T) : T \in \mathrm{ThFam}^{\nmid}(\mathcal{I}) \text{ and } \overrightarrow{\phi} \leq T\} \\
&\quad \text{(definition of } \widetilde{\Omega}^{\mathcal{I}\nmid}) \\
&= \bigcap\{\Omega(T) : T \in \mathrm{ThFam}^{\nmid}(\mathcal{I}) \text{ and } \phi \in \overleftarrow{T}_\Sigma\} \\
&\quad \text{(Definition of } \overrightarrow{\phi} \text{ and } \overleftarrow{T}) \\
&= \bigcap\{\Omega(T) : T \in \mathrm{ThFam}^{\nmid}(\mathcal{I}) \text{ and } L^{\mathcal{I}\nmid}_\Sigma[\phi] \leq \Omega(T)\} \\
&\quad \text{(Left solubility of } L^{\mathcal{I}\nmid}) \\
&= \bigcap\{\Omega(T) : L^{\mathcal{I}\nmid}_\Sigma[\phi] \leq \Omega(T)\}. \quad \text{(Lemma 1110)}
\end{aligned}$$

Thus, $L^{\mathcal{I}\nmid}$ is left adequate. ∎

In order to prove a partial converse of Corollary 1146, we will employ the following characterization of narrow left c-reflectivity.

Lemma 1147 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is narrowly left c-reflective if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$ *and all* $T' \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$*,*

$$\widetilde{\Omega}^{\mathcal{I}\nmid}(T) \leq \Omega(T') \quad \textit{implies} \quad \overleftarrow{T} \leq \overleftarrow{T'}.$$

Proof: Suppose, first, that $\mathcal{I}$ is narrowly left c-reflective and let $T \in \mathrm{ThFam}(\mathcal{I})$ and $T' \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, such that $\widetilde{\Omega}^{\mathcal{I}\nmid}(T) \leq \Omega(T')$. Then, by definition,

$$\bigcap\{\Omega(T'') : T \leq T'' \in \mathrm{ThFam}^{\nmid}(\mathcal{I})\} \leq \Omega(T').$$

Hence, by narrow left c-reflectivity, $\bigcap\{\overleftarrow{T''} : T \leq T'' \in \mathrm{ThFam}^{\nmid}(\mathcal{I})\} \leq \overleftarrow{T'}$. However, $T \leq T''$ implies that $\overleftarrow{T} \leq \overleftarrow{T''}$. Hence, we obtain $\overleftarrow{T} \leq \overleftarrow{T'}$.

Suppose, conversely, that the displayed condition in the statement holds and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nmid}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. Then, since $\mathcal{T} \subseteq \mathrm{ThFam}^{\nmid}(\mathcal{I})$, we get that

$$\bigcap\{\Omega(T'') : \bigcap\mathcal{T} \leq T'' \in \mathrm{ThFam}^{\nmid}(\mathcal{I})\} \leq \Omega(T').$$

By definition, then, $\widetilde{\Omega}^{\mathcal{I}\nmid}(\bigcap\mathcal{T}) \leq \Omega(T')$, whence, by hypothesis, $\overleftarrow{\bigcap\mathcal{T}} \leq \overleftarrow{T'}$. Therefore, $\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$. This shows that $\mathcal{I}$ is narrowly left c-reflective. ∎

We prove, next, the converse of Corollary 1146, under the additional assumption that the π-institution $\mathcal{I}$ under consideration is narrowly left c-reflective. This constitutes an analog of Proposition 1131.

Proposition 1148 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a narrowly left c-reflective* π*-institution based on* $\mathbf{F}$*. If* $L^{\mathcal{I}\natural}$ *is left adequate, then it is left soluble.*

Proof: Suppose $\mathcal{I}$ is narrowly left c-reflective and $L^{\mathcal{I}\natural}$ is left adequate. Let $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $L^{\mathcal{I}\natural}_\Sigma[\phi] \leq \Omega(T)$. By the postulated left adequacy of $L^{\mathcal{I}\natural}$, we get that $\widetilde{\Omega}^{\mathcal{I}\natural}(C(\overrightarrow{\phi})) \leq \Omega(T)$. By narrow left truth equationality and Lemma 1147, $\overleftarrow{C(\overrightarrow{\phi})} \leq \overleftarrow{T}$. Therefore, $\phi \in \overleftarrow{T}_\Sigma$. We conclude that $L^{\mathcal{I}\natural}$ is left soluble. ∎

We are now in a position to prove the main characterization theorem relating narrow left truth equationality with narrow left c-reflectivity, an analog of Theorem 1132, which characterized rough left truth equationality in terms of rough left c-reflectivity and the adequacy of the rough left Suszko core.

$$\begin{array}{rcl}\text{Narrow Left Truth Equationality} & = & L^{\mathcal{I}\natural} \text{ Left Soluble}\\ & = & L^{\mathcal{I}\natural} \text{ Narrowly Defines Theory}\\ & & \quad\text{Families Up to Arrow}\\ & = & \text{Narrow Left c-Reflectivity}\\ & & + \ L^{\mathcal{I}\natural} \text{ Left Adequate}\end{array}$$

Theorem 1149 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is narrowly left truth equational if and only if it is narrowly left c-reflective and has a left adequate narrow left Suszko core.*

Proof: Suppose, first, that $\mathcal{I}$ is narrowly left truth equational. By Theorem 1138, it is narrowly left c-reflective. By Theorem 1141, its narrow left Suszko core is left soluble. Thus, by Corollary 1146, its narrow left Suszko core is also left adequate.

Assume, conversely, that $\mathcal{I}$ is narrowly left c-reflective and has a left adequate narrow left Suszko core. Then, by Proposition 1148, its narrow left Suszko core is also left soluble. Hence, by Theorem 1142, $\mathcal{I}$ is narrowly left truth equational. ∎

Based on Proposition 1136 and Theorem 468, it is not difficult to show, in an analog of Corollary 1133, that the characterization theorem, Theorem 870, of left truth equationality in terms of left c-reflectivity and the left adequacy of the left Suszko core, can be inferred from Theorem 1149.

Corollary 1150 (Theorem 870) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is left truth equational if and only if it is left c-reflective and has a left adequate left Suszko core.*

Proof: $\mathcal{I}$ is left truth equational if and only if, by Proposition 1136, it is narrowly left truth equational and has theorems, if and only if, by Theorem 1149, it is narrowly left c-reflective, with a left adequate narrow left Suszko core and has theorems, if and only if, by Theorem 468 and the definitions of left Suszko core and narrow left Suszko core, it is left c-reflective and has a left adequate left Suszko core. ■

We close the section by looking at a result, an analog of Theorem 1134, which may be perceived either as an alternative characterization of narrow left truth equationality, involving arbitrary **F**-algebraic systems, or as a transfer theorem.

Theorem 1151 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is narrowly left truth equational, with witnessing equations* $\tau^\flat$*, if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, all* $T \in \mathrm{FiFam}^{\mathcal{I}\natural}(\mathcal{A})$*, all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi \in \mathrm{SEN}(\Sigma)$*,*

$$\phi \in \overleftarrow{T}_\Sigma \quad \textit{iff} \quad \tau^{\mathcal{A}}_\Sigma[\phi] \leq \Omega^{\mathcal{A}}(T).$$

Proof: If the postulated condition holds, then it holds, in particular, for the **F**-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. This yields immediately that $\mathcal{I}$ is narrowly left truth equational.

Suppose, conversely, that $\mathcal{I}$ is narrowly left truth equational and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an **F**-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}\natural}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rll} \alpha_\Sigma(\phi) \in \overleftarrow{T}_{F(\Sigma)} & \text{iff} & \phi \in \alpha^{-1}_\Sigma(\overleftarrow{T}_{F(\Sigma)}) \\ & \text{iff} & \phi \in \overleftarrow{\alpha^{-1}_\Sigma(T_{F(\Sigma)})} \quad \text{(Lemma 6)} \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(\alpha^{-1}(T)) \quad \text{(hypothesis)} \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T)) \quad \text{(Proposition 24)} \\ & \text{iff} & \tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi)] \leq \Omega^{\mathcal{A}}(T). \quad \text{(Lemma 95)} \end{array}$$

Hence, taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that the displayed condition holds. ■

14.4 Rough System Truth Equationality

We now turn to *rough system truth equationality*. As the terminology suggests:

- It is in the same relation to rough system c-reflectivity as rough left truth equationality is to rough left c-reflectivity;
- It is in the same relation to rough left truth equationality as system truth equationality is to left truth equationality.

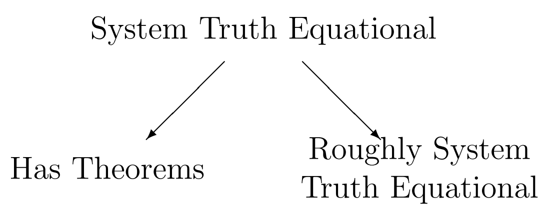
Roughly speaking, rough system truth equationality is defined analogously to system truth equationality, but it is applied to rough representatives of theory systems so as to avoid theory systems with empty components.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that $\mathcal{I}$ is **roughly system truth equational** if there exists $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with a single distinguished argument, such that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in \widetilde{T}_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

The collection $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ is referred to as a set of **witnessing equations** (of/for the rough system truth equationality of $\mathcal{I}$).

The following relationship between rough system truth equationality and system truth equationality, an analog of the relationship between rough truth equationality and truth equationality, presented in Corollary 1098, holds.



Proposition 1152 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is system truth equational if and only if it is roughly system truth equational and has theorems.*

Proof: Suppose, first, that $\mathcal{I}$ is roughly system truth equational, with witnessing equations $\tau^\flat$, and that it has theorems. Availability of theorems implies that $\mathrm{ThSys}^{\lightning}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$. Thus, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in T_\Sigma$ if and only if $\phi \in \widetilde{T}_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Thus, $\mathcal{I}$ is system truth equational, with the same witnessing equations $\tau^\flat$.

Assume, conversely, that $\mathcal{I}$ is system truth equational, with witnessing equations $\tau^\flat$. Then, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in T_\Sigma$ iff $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. This clearly implies that $\mathcal{I}$ has theorems. Moreover, due to the availability of theorems, we get, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in \widetilde{T}_\Sigma$ if and only if $\phi \in T_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Thus, $\mathcal{I}$ is roughly system truth equational. ■

Our next goal is to prove an analog of the characterization theorem, Theorem 1127, of rough left truth equationality in terms of the left solubility of the rough left Suszko core for rough system truth equationality.

Rough system truth equationality allows an expression for $\widetilde{T}$, for all theory systems T, in terms of the Leibniz congruence system of T. The following proposition forms an analog of Proposition 1121.

Proposition 1153 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is roughly system truth equational, with witnessing equations* $\tau^\flat$*, if and only if, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,* $\widetilde{T} = \tau^\flat(\Omega(T))$*.*

Proof: $\mathcal{I}$ is roughly system truth equational, with witnessing equations $\tau^\flat$, if and only if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in \widetilde{T}_\Sigma$ iff $\tau^\flat_\Sigma[\phi] \le \Omega(T)$, if and only if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\widetilde{T} = \tau^\flat(\Omega(T))$. ■

Recall from Chapter 6 that, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$, $\mathcal{I}$ is called *roughly system c-reflective* if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$,

$$\bigcap_{T\in\mathcal{T}} \Omega(T) \le \Omega(T') \quad \text{implies} \quad \bigcap_{T\in\mathcal{T}} \widetilde{T} \le \widetilde{T'}.$$

We are now able to show that rough system truth equationality implies rough system c-reflectivity. This is an analog of Theorem 1122.

Theorem 1154 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly system truth equational, then it is roughly system c-reflective.*

Proof: Suppose $\mathcal{I}$ is roughly system truth equational, with witnessing equations $\tau^\flat$. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \le \Omega(T')$. Then we have

$$\begin{array}{rcll}\bigcap_{T\in\mathcal{T}} \widetilde{T} & = & \bigcap_{T\in\mathcal{T}} \tau^\flat(\Omega(T)) & \text{(Proposition 1153)}\\ & = & \tau^\flat(\bigcap_{T\in\mathcal{T}} \Omega(T)) & \text{(set theory)}\\ & \le & \tau^\flat(\Omega(T')) & \text{(hypothesis)}\\ & = & \widetilde{T'}. & \text{(Proposition 1153)}\end{array}$$

Thus, $\mathcal{I}$ is roughly system c-reflective. ■

In the context of rough system truth equationality, the notion paralleling the rough left Suszko core is the *rough system core*, a modification of the system core, which is defined below.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. The **rough system core** $\widetilde{Z}^{\mathcal{I}}$ of $\mathcal{I}$ is the collection

$$\begin{array}{rcl}\widetilde{Z}^{\mathcal{I}} & = & \{\sigma^\flat \in N^\flat : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \phi \in \mathrm{SEN}^\flat(\Sigma))\\ & & \quad (\sigma^\flat_\Sigma[\phi] \le \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \phi \in \widetilde{T}_\Sigma\})\}.\end{array}$$

From the definition, it is not difficult to see that, for any theory system T, $\widetilde{T}$ is always included in $\widetilde{Z}^{\mathcal{I}}(\Omega(T))$. This forms an analog in the rough left context of Proposition 1123.

Proposition 1155 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $T \in \mathrm{ThSys}(\mathcal{I})$*,*

$$\widetilde{T} \le \widetilde{Z}^{\mathcal{I}}(\Omega(T)).$$

Proof: Suppose $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \widetilde{T}_\Sigma$, and $\sigma^\flat \in \widetilde{Z}^{\mathcal{I}}$. Then, by the definition of $\widetilde{Z}^{\mathcal{I}}$, $\sigma^\flat_\Sigma[\phi] \leq \Omega(T)$. Hence, $\widetilde{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. Thus, by definition of $\widetilde{Z}^{\mathcal{I}}(\Omega(T))$, $\phi \in \widetilde{Z}^{\mathcal{I}}_\Sigma(\Omega(T))$. Since Σ and $\phi \in \widetilde{T}_\Sigma$ were arbitrary, we conclude that $\widetilde{T} \leq \widetilde{Z}^{\mathcal{I}}(\Omega(T))$. ■

The reverse inclusion may or may not hold. If it does, for all $T \in \mathrm{ThSys}(\mathcal{I})$, we say that the rough system core of $\mathcal{I}$ is *soluble*.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The rough system core $\widetilde{Z}^{\mathcal{I}}$ of $\mathcal{I}$ is said to be **soluble** if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widetilde{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) \quad \text{implies} \quad \phi \in \widetilde{T}_\Sigma.$$

As was the case with rough left truth equationality (see Lemma 1124), if a given π-institution is roughly system truth equational, then any collection of witnessing equations must be included in the rough system core of $\mathcal{I}$. In other words, in case of rough system truth equationality, the rough system core forms a candidate for the largest collection of witnessing equations.

Lemma 1156 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is roughly system truth equational, with witnessing equations $\tau^\flat$, then $\tau^\flat \subseteq \widetilde{Z}^{\mathcal{I}}$.*

Proof: Suppose $\mathcal{I}$ is roughly system truth equational, with witnessing equations $\tau^\flat$. Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \widetilde{T}_\Sigma$. Then, by rough system truth equationality, $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Since T was arbitrary,

$$\tau^\flat_\Sigma[\phi] \leq \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \phi \in \widetilde{T}_\Sigma\}.$$

Hence, since $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$ were arbitrary, we conclude that $\tau^\flat \subseteq \widetilde{Z}^{\mathcal{I}}$. ■

We are now ready to prove the equivalence between rough system truth equationality and the solubility of the rough system core. In the next theorem, we show that rough system truth equationality implies the solubility of the rough system core. This forms an analog of Theorem 1125.

Theorem 1157 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is roughly system truth equational, then $\widetilde{Z}^{\mathcal{I}}$ is soluble.*

Proof: Suppose $\mathcal{I}$ is roughly system truth equational, with witnessing equations $\tau^\flat$. Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\widetilde{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. Then, by rough system truth equationality and Lemma 1156, $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Again, using rough system truth equationality, we conclude that $\phi \in \widetilde{T}_\Sigma$. This shows that $\widetilde{Z}^{\mathcal{I}}$ is soluble. ■

Conversely, in an analog of Theorem 1126, we show that the solubility of the rough system core of a π-institution implies rough system truth equationality.

Theorem 1158 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\widetilde{Z}^{\mathcal{I}}$ *is soluble, then* $\mathcal{I}$ *is roughly system truth equational, with witnessing equations* $\widetilde{Z}^{\mathcal{I}}$.

Proof: Assume $\widetilde{Z}^{\mathcal{I}}$ is soluble and let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. We must show that

$$\phi \in \widetilde{T}_\Sigma \quad \text{iff} \quad \widetilde{Z}^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T).$$

If $\phi \in \widetilde{T}_\Sigma$, then, by Proposition 1155, $\phi \in \widetilde{Z}^{\mathcal{I}}_\Sigma(\Omega(T))$, i.e., $\widetilde{Z}^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)$. On the other hand, the reverse inclusion is guaranteed by the postulated solubility of $\widetilde{Z}^{\mathcal{I}}$. Thus, $\mathcal{I}$ is indeed roughly system truth equational, with witnessing equations $\widetilde{Z}^{\mathcal{I}}$. ■

Theorems 1157 and 1158 provide the first characterization of rough system truth equationality in terms of the solubility of the rough system core. This parallels Theorem 1127, which asserted a similar characterization for rough left truth equationality in terms of the left solubility of the rough left Suszko core of a π-institution.

$$\mathcal{I} \text{ Roughly System Truth Equational } \longleftrightarrow \widetilde{Z}^{\mathcal{I}} \text{ Soluble}$$

Theorem 1159 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is roughly system truth equational if and only if* $\widetilde{Z}^{\mathcal{I}}$ *is left soluble.*

Proof: The "if" is by Theorem 1158. The "only if" by Theorem 1157. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the rough system core $\widetilde{Z}^{\mathcal{I}}$ of $\mathcal{I}$ **roughly defines theory systems** if, for al $T \in \mathrm{ThSys}(\mathcal{I})$,

$$\widetilde{T} = \widetilde{Z}^{\mathcal{I}}(\Omega(T)).$$

Another characterization of rough system truth equationality, along the lines of Theorem 1128, asserts that it is equivalent to the rough definability of the theory systems by the rough system core.

$$\mathcal{I} \text{ Roughly System Truth Equational} \\ \longleftrightarrow \widetilde{Z}^{\mathcal{I}} \text{ Roughly Defines Theory Systems}$$

Theorem 1160 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is roughly system truth equational if and only if* $\widetilde{Z}^{\mathcal{I}}$ *roughly defines theory systems in* $\mathcal{I}$.

Proof: Suppose $\mathcal{I}$ is roughly system truth equational. By Theorem 1157, $\widetilde{Z}^{\mathcal{I}}$ is soluble. Hence, by definition, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\widetilde{Z}^{\mathcal{I}}(\Omega(T)) \leq \widetilde{T}$. Since, by Proposition 1155, the reverse inclusion always holds, we get, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\widetilde{T} = \widetilde{Z}^{\mathcal{I}}(\Omega(T))$. Thus, $\widetilde{Z}^{\mathcal{I}}$ roughly defines theory systems in $\mathcal{I}$. Conversely, if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\widetilde{T} = \widetilde{Z}^{\mathcal{I}}(\Omega(T))$, then $\widetilde{Z}^{\mathcal{I}}$ is soluble and, therefore, by Theorem 1158, $\mathcal{I}$ is roughly system truth equational. ■

We establish, next, a connection between rough system truth equationality and rough system c-reflectivity by means of the rough system core. In analogy with the case of rough left truth equationality, we introduce, first, the notion of *adequacy* of the rough system core. The following proposition, a system analog of Proposition 1129, motivates the notion of adequacy that will follow.

Proposition 1161 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{aligned}\bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \textit{ and } \widetilde{Z}^{\mathcal{I}}_{\Sigma}[\phi] \leq \Omega(T)\} \\ \leq \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \textit{ and } \phi \in \widetilde{T}_{\Sigma}\}.\end{aligned}$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $T \in \mathrm{ThSys}(\mathcal{I})$,

$$\phi \in \widetilde{T}_{\Sigma} \quad \text{implies} \quad \widetilde{Z}^{\mathcal{I}}_{\Sigma}[\phi] \leq \Omega(T). \quad (\text{Definition of } \widetilde{Z}^{\mathcal{I}})$$

Hence,

$$\begin{aligned}\bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \widetilde{Z}^{\mathcal{I}}_{\Sigma}[\phi] \leq \Omega(T)\} \\ \leq \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \phi \in \widetilde{T}_{\Sigma}\}.\end{aligned}$$

This is the displayed formula in the statement. ■

If the reverse inclusion of that proven in Proposition 1161 holds, then we say that the rough system core of $\mathcal{I}$ is *adequate*.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the rough system core $\widetilde{Z}^{\mathcal{I}}$ of $\mathcal{I}$ is **adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{aligned}\bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \phi \in \widetilde{T}_{\Sigma}\} \\ \leq \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \widetilde{Z}^{\mathcal{I}}_{\Sigma}[\phi] \leq \Omega(T)\}.\end{aligned}$$

We can show, in analogy with Corollary 1130, that the solubility of the rough system core implies its adequacy.

Corollary 1162 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\widetilde{Z}^{\mathcal{I}}$ *is soluble, then it is adequate.*

Proof: Suppose $\widetilde{Z}^{\mathcal{I}}$ is soluble and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, by solubility and Proposition 1155, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\phi \in \widetilde{T}_\Sigma$ if and only if $\widetilde{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. Therefore,

$$\begin{aligned}&\bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \phi \in \widetilde{T}_\Sigma\}\\ &\qquad = \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \widetilde{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.\end{aligned}$$

Thus, $\widetilde{Z}^{\mathcal{I}}$ is adequate. ∎

We prove, next, the converse of Corollary 1162, under the additional assumption that the π-institution $\mathcal{I}$ under consideration is roughly system c-reflective. This constitutes an analog of Proposition 1131.

Proposition 1163 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a roughly system c-reflective* π*-institution based on* $\mathbf{F}$. *If* $\widetilde{Z}^{\mathcal{I}}$ *is adequate, then it is soluble.*

Proof: Suppose $\mathcal{I}$ is roughly system c-reflective and $\widetilde{Z}^{\mathcal{I}}$ is adequate. Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\widetilde{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. By the postulated adequacy of $\widetilde{Z}^{\mathcal{I}}$, we get that $\bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \phi \in \widetilde{T}_\Sigma\} \leq \Omega(T)$. By rough system truth equationality, $\bigcap\{\widetilde{T} : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \phi \in \widetilde{T}_\Sigma\} \leq \widetilde{T}$. Therefore, $\phi \in \widetilde{T}_\Sigma$. We conclude that $\widetilde{Z}^{\mathcal{I}}$ is soluble. ∎

We are now in a position to prove the main characterization theorem relating rough system truth equationality with rough system c-reflectivity, an analog of Theorem 1132, which characterized rough left truth equationality in terms of rough left c-reflectivity and the left adequacy of the rough left Suszko core.

Rough System Truth Equationality
= $\widetilde{Z}^{\mathcal{I}}$ Soluble
= $\widetilde{Z}^{\mathcal{I}}$ Roughly Defines Theory Systems
= Rough System c-Reflectivity + $\widetilde{Z}^{\mathcal{I}}$ Adequate

Theorem 1164 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is roughly system truth equational if and only if it is roughly system c-reflective and has an adequate rough system core.*

Proof: Suppose, first, that $\mathcal{I}$ is roughly system truth equational. By Theorem 1154, it is roughly system c-reflective. By Theorem 1157, its rough system core is soluble. Thus, by Corollary 1162, its rough system core is also adequate.

Assume, conversely, that $\mathcal{I}$ is roughly system c-reflective and has an adequate rough system core. Then, by Proposition 1163, its rough system core is also soluble. Hence, by Theorem 1158, $\mathcal{I}$ is roughly system truth equational. ∎

Based on Proposition 1152 and Theorem 468, it is not difficult to show, in an analog of Corollary 1133, that the characterization theorem, Theorem 887, of system truth equationality in terms of system c-reflectivity and the adequacy of the system core, can be inferred from Theorem 1164.

Corollary 1165 (Theorem 887) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is system truth equational if and only if it is system c-reflective and has an adequate system core.*

Proof: $\mathcal{I}$ is system truth equational if and only if, by Proposition 1152, it is roughly system truth equational and has theorems, if and only if, by Theorem 1164, it is roughly system c-reflective, with an adequate rough system core and has theorems, if and only if, by Theorem 468 and the definitions of system core and rough system core, it is system c-reflective and has an adequate system core. ∎

We close the section by looking at a result, an analog of Theorem 1134, which may be perceived either as an alternative characterization of rough system truth equationality, involving arbitrary $\mathbf{F}$-algebraic systems, or as a transfer theorem.

Theorem 1166 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is roughly system truth equational, with witnessing equations* $\tau^\flat$*, if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, all* $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$*, all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi \in \mathrm{SEN}(\Sigma)$*,*

$$\phi \in \widetilde{T}_\Sigma \quad \textit{iff} \quad \tau^{\mathcal{A}}_\Sigma[\phi] \leq \Omega^{\mathcal{A}}(T).$$

Proof: If the postulated condition holds, then it holds, in particular, for the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. This yields immediately that $\mathcal{I}$ is roughly system truth equational.

Suppose, conversely, that $\mathcal{I}$ is roughly system truth equational and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rll} \alpha_\Sigma(\phi) \in \widetilde{T}_{F(\Sigma)} & \text{iff} & \phi \in \alpha^{-1}_\Sigma(\widetilde{T}_{F(\Sigma)}) \\ & \text{iff} & \phi \in \widetilde{\alpha^{-1}_\Sigma(T_{F(\Sigma)})} \quad \text{(Theorem 377)} \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(\alpha^{-1}(T)) \quad \text{(hypothesis)} \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T)) \quad \text{(Proposition 24)} \\ & \text{iff} & \tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi)] \leq \Omega^{\mathcal{A}}(T). \quad \text{(Lemma 95)} \end{array}$$

Hence, taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that the displayed condition holds. ∎

14.5 Narrow System Truth Equationality

Finally, we discuss *narrow system truth equationality*, the weakest of all rough/narrow truth equationality conditions. As the terminology suggests:

- It is in the same relation to narrow system c-reflectivity as rough system truth equationality is to rough system c-reflectivity;
- It is in the same relation to narrow truth equationality and narrow left truth equationality as rough system truth equationality is to rough truth equationality and rough left truth equationality, respectively.

In a nutshell, narrow system truth equationality is defined analogously to system truth equationality, but care is taken to bypass theory systems with empty components.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that $\mathcal{I}$ is **narrowly system truth equational** if there exists $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with a single distinguished argument, such that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi/\Omega_\Sigma(T) \in \widetilde{T}_\Sigma/\Omega_\Sigma(T) \quad \text{iff} \quad \tau_\Sigma^{\mathcal{F}/\Omega(T)}[\phi/\Omega_\Sigma(T)] \leq \Delta^{\mathcal{F}/\Omega(T)}.$$

Once more, since, by Proposition 369, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(\widetilde{T}) = \Omega(T)$, $\Omega(T)$ is compatible with $\widetilde{T}$ and, hence, the preceding definition makes sense. The collection $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ is referred to as a set of **witnessing equations** (of/for the narrow system truth equationality of $\mathcal{I}$).

As in Proposition 1096, we get the following alternative characterization.

Proposition 1167 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ a collection of natural transformations in $N^\flat$, with a single distinguished argument. $\mathcal{I}$ is narrowly system truth equational, with witnessing equations $\tau^\flat$, if and only if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,*

$$\phi \in \widetilde{T}_\Sigma \quad \textit{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

Proof: Suppose that $\mathcal{I}$ is narrowly truth equational and let $T \in \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{lll}
\phi \in \widetilde{T}_\Sigma & \text{iff} & \phi/\Omega_\Sigma(T) \in \widetilde{T}_\Sigma/\Omega_\Sigma(T) \quad \text{(Proposition 369 and compatibility)}\\
& \text{iff} & \tau_\Sigma^{\mathcal{F}/\Omega(T)}[\phi/\Omega_\Sigma(T)] \leq \Delta^{\mathcal{F}/\Omega(T)} \quad \text{(by hypothesis)}\\
& \text{iff} & \tau^\flat_\Sigma[\phi]/\Omega(T) \leq \Delta^{\mathcal{F}/\Omega(T)} \quad \text{(by definition)}\\
& \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(T).
\end{array}$$

Suppose, conversely, that the displayed condition holds. Let $T \in \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{llll}\phi/\Omega_\Sigma(T) \in \widetilde{T}_\Sigma/\Omega_\Sigma(T) & \text{iff} & \phi \in \widetilde{T}_\Sigma & \text{(Proposition 369 and compatibility)}\\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(T) & \text{(by hypothesis)}\\ & \text{iff} & \tau^\flat_\Sigma[\phi]/\Omega(T) \leq \Delta^{\mathcal{F}/\Omega(T)} & \\ & \text{iff} & \tau^{\mathcal{F}/\Omega(T)}_\Sigma[\phi/\Omega_\Sigma(T)] \leq \Delta^{\mathcal{F}/\Omega(T)}. & \text{(definition)}\end{array}$$

Therefore, $\mathcal{I}$ is narrowly system truth equational. ■

It is not difficult to see that an alternative way to express narrow system truth equationality is to assert the same condition that defines system truth equationality, excluding, however, those theory systems with at least one empty component.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall from Chapter 6 that we denote by $\mathrm{ThSys}^{\not\emptyset}(\mathcal{I})$ the collection of all theory systems T of $\mathcal{I}$, such that $T_\Sigma \neq \varnothing$, for all $\Sigma \in |\mathbf{Sign}^\flat|$:

$$\mathrm{ThSys}^{\not\emptyset}(\mathcal{I}) = \{T \in \mathrm{ThSys}(\mathcal{I}) : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(T_\Sigma \neq \varnothing)\}.$$

Recall, also, that, if $\mathcal{I}$ has theorems, then $\mathrm{ThSys}^{\not\emptyset}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$. In particular, this is case if $\mathcal{I}$ happens to be system truth equational.

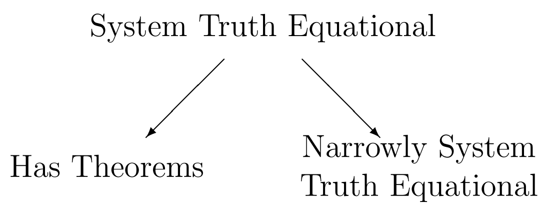
Proposition 1168 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ a collection of natural transformations in $N^\flat$, with a single distinguished argument. $\mathcal{I}$ is narrowly system truth equational, with witnessing equations $\tau^\flat$, if and only if, for all $T \in \mathrm{ThSys}^{\not\emptyset}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,*

$$\phi \in T_\Sigma \quad \textit{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

Proof: Suppose $\mathcal{I}$ is narrowly system truth equational, with witnessing equations $\tau^\flat$. Let $T \in \mathrm{ThSys}^{\not\emptyset}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then $\widetilde{T} = T \in \mathrm{ThSys}(\mathcal{I})$, whence, taking into account Proposition 1167, $\phi \in T_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$.

Suppose, conversely, that the displayed condition holds. Consider $T \in \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, since, by definition of $\widetilde{T}$, we have $\widetilde{T} \in \mathrm{ThSys}^{\not\emptyset}(\mathcal{I})$, we get, by hypothesis, $\phi \in \widetilde{T}_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(\widetilde{T})$, whence, using Proposition 369, we conclude that $\phi \in \widetilde{T}_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Therefore, $\mathcal{I}$ is narrowly system truth equational. ■

As a corollary, we obtain the following key relationship between narrow system truth equationality and system truth equationality, paralleling the one established between system truth equationality and rough system truth equationality in Corollary 1152.



Corollary 1169 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is system truth equational if and only if it is narrowly system truth equational with theorems.*

Proof: Suppose, first, that $\mathcal{I}$ is narrowly system truth equational, with witnessing equations $\tau^\flat$, and that it has theorems. Availability of theorems implies that $\mathrm{ThSys}^{\nmid}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$. Thus, by Proposition 1168, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in T_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \le \Omega(T)$. Thus, $\mathcal{I}$ is system truth equational, with the same witnessing equations $\tau^\flat$.

Assume, conversely, that $\mathcal{I}$ is system truth equational, with witnessing equations $\tau^\flat$. Then, for all $T \in \mathrm{ThSys}(\mathcal{I})$, and, hence, a fortiori, for all $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in T_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \le \Omega(T)$. Hence, again by Proposition 1168, $\mathcal{I}$ is narrowly system truth equational. Finally, by Theorem 872, $\mathcal{I}$ is system c-reflective and, by Proposition 243, it is system reflective and, therefore, system injective. Thus, it must have theorems. ∎

Our next goal is to prove an analog of the characterization theorem, Theorem 1159, of rough system truth equationality in terms of the solubility of the rough system core for the case of narrow system truth equationality.

Narrow system truth equationality allows the following expression for all theory systems with nonempty components, forming an analog of Proposition 1153.

Proposition 1170 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following conditions are equivalent:*

(i) $\mathcal{I}$ *is narrowly system truth equational, with witnessing equations* $\tau^\flat$*;*

(ii) For all $T \in \mathrm{ThSys}(\mathcal{I})$*, such that* $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$*,* $\widetilde{T} = \tau^\flat(\Omega(T))$*;*

(iii) For all $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$*,* $T = \tau^\flat(\Omega(T))$*.*

Proof: Suppose $\mathcal{I}$ is narrowly system truth equational, with witnessing equations $\tau^\flat$, and let $T \in \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{lll} \phi \in \tau^\flat_\Sigma(\Omega(T)) & \text{iff} & \tau^\flat_\Sigma[\phi] \le \Omega(T) \quad \text{(definition)} \\ & \text{iff} & \phi \in \widetilde{T}_\Sigma. \quad \text{(hypothesis and Proposition 1167)} \end{array}$$

This proves Condition (ii). If Condition (ii) holds and $T \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$, then $\widetilde{T} = T \in \mathrm{ThSys}(\mathcal{I})$, whence, by hypothesis, $T = \widetilde{T} = \tau^\flat(\Omega(T))$. Thus, Condition (iii) holds. Finally, assume Condition (iii) holds. Then, by Proposition **??**, $\mathcal{I}$ is narrowly system truth equational, with witnessing equations $\tau^\flat$. ■

Recall from Chapter 6 that, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$, $\mathcal{I}$ is called *narrowly system c-reflective* if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^{\nleq}(\mathcal{I})$,

$$\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T') \quad \text{implies} \quad \bigcap_{T\in\mathcal{T}} T \leq T'.$$

We are now able to show that narrow system truth equationality implies narrow system c-reflectivity. This is an analog of Theorem 1154.

Theorem 1171 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is narrowly system truth equational, then it is narrowly system c-reflective.*

Proof: Suppose $\mathcal{I}$ is narrowly system truth equational, with witnessing equations $\tau^\flat$. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^{\nleq}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. Then we have

$$\begin{array}{rcl}\bigcap_{T\in\mathcal{T}} T & = & \bigcap_{T\in\mathcal{T}} \tau^\flat(\Omega(T)) \quad \text{(Proposition 1170)}\\ & = & \tau^\flat(\bigcap_{T\in\mathcal{T}} \Omega(T)) \quad \text{(set theory)}\\ & \leq & \tau^\flat(\Omega(T')) \quad \text{(hypothesis)}\\ & = & T'. \quad \text{(Proposition 1170)}\end{array}$$

Thus, $\mathcal{I}$ is narrowly system c-reflective. ■

In the context of narrow system truth equationality, the notion paralleling the rough system core, introduced in Section 14.3, is the *narrow system core*, a modification of the original definition of the system core from Chapter 11, which is defined by circumventing theory systems with empty components.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. The **narrow system core** $Z^{\mathcal{I}\nleq}$ of $\mathcal{I}$ is the collection

$$Z^{\mathcal{I}\nleq} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThSys}(\mathcal{I}))(\widetilde{T} \in \mathrm{ThSys}(\mathcal{I}) \Rightarrow \sigma^\flat[\widetilde{T}] \leq \widehat{\Omega}^{\mathcal{I}}(\widetilde{T}))\}.$$

As before, an alternative characterization avoids $\widetilde{\ }$ at the expense of restricting quantification over $\mathrm{ThSys}^{\nleq}(\mathcal{I})$.

Proposition 1172 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then*

$$Z^{\mathcal{I}\nleq} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThSys}^{\nleq}(\mathcal{I}))(\sigma^\flat[T] \leq \widehat{\Omega}^{\mathcal{I}}(T))\}.$$

Proof: Inside this proof we set

$$M^{\mathcal{I}\nleq} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThSys}^{\nleq}(\mathcal{I}))(\sigma^\flat[T] \leq \widehat{\Omega}^{\mathcal{I}}(T))\}.$$

Our goal is to show that $Z^{\mathcal{I}\natural} = M^{\mathcal{I}\natural}$. Suppose, first, that $\sigma^\flat \in Z^{\mathcal{I}\natural}$ and let $T \in \mathrm{ThSys}^\natural(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$. Since $T \in \mathrm{ThSys}^\natural(\mathcal{I})$, we get $\widetilde{T} = T$. Thus, on the one hand, $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$ and, on the other, by hypothesis, $\phi \in \widetilde{T}_\Sigma$. Thus, since $\sigma^\flat \in Z^{\mathcal{I}\natural}$, we get

$$\sigma^\flat_\Sigma[\phi] \leq \widehat{\Omega}^{\mathcal{I}}(\widetilde{T}) = \widehat{\Omega}^{\mathcal{I}}(T).$$

This proves that $\sigma^\flat \in M^{\mathcal{I}\natural}$. Assume, conversely, that $\sigma^\flat \in M^{\mathcal{I}\natural}$ and let $T \in \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \widetilde{T}_\Sigma$. Since $\widetilde{T} \in \mathrm{ThSys}^\natural(\mathcal{I})$ and $\sigma^\flat \in M^{\mathcal{I}\natural}$, we get $\sigma^\flat_\Sigma[\phi] \leq \widehat{\Omega}^{\mathcal{I}}(\widetilde{T})$, whence, $\sigma^\flat \in Z^{\mathcal{I}\natural}$. This proves that $Z^{\mathcal{I}\natural} = M^{\mathcal{I}\natural}$. ■

From the definition, it is not difficult to see that any theory system T with all its components nonempty is always included in $Z^{\mathcal{I}\natural}(\Omega(T))$. This forms an analog in the rough system context of Proposition 1155.

Proposition 1173 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $T \in \mathrm{ThSys}^\natural(\mathcal{I})$,

$$T \leq Z^{\mathcal{I}\natural}(\Omega(T)).$$

Proof: Suppose $T \in \mathrm{ThSys}^\natural(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$, and $\sigma^\flat \in Z^{\mathcal{I}\natural}$. Then, by Proposition **??**, $\sigma^\flat_\Sigma[\phi] \leq \widehat{\Omega}^{\mathcal{I}}(T) \leq \Omega(T)$. Hence, $Z^{\mathcal{I}\natural}_\Sigma[\phi] \leq \Omega(T)$. By definition, then, $\phi \in Z^{\mathcal{I}\natural}_\Sigma(\Omega(T))$. Since Σ and $\phi \in T_\Sigma$ were arbitrary, we conclude that $T \leq Z^{\mathcal{I}\natural}(\Omega(T))$. ■

The reverse inclusion may or may not hold. If it does, for all $T \in \mathrm{ThSys}^\natural(\mathcal{I})$, we say that the narrow system core $Z^{\mathcal{I}\natural}$ of $\mathcal{I}$ is *soluble*.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The narrow system core $Z^{\mathcal{I}\natural}$ of $\mathcal{I}$ is said to be **soluble** if, for all $T \in \mathrm{ThSys}^\natural(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$Z^{\mathcal{I}\natural}_\Sigma[\phi] \leq \Omega(T) \quad \text{implies} \quad \phi \in T_\Sigma.$$

An alternative way to express solubility is to again expand the view to all theory systems, with nonempty components, at the balancing expense of adding rough equivalence representatives. We obtain, thus, an analog of Lemma 1103.

Lemma 1174 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $Z^{\mathcal{I}\natural}$ *is soluble if and only if, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*, such that* $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$,

$$\widetilde{T} = Z^{\mathcal{I}\natural}(\Omega(T)).$$

Proof: $Z^{\mathcal{I}\natural}$ is soluble if and only if, by definition and Proposition 1173, for all $T \in \mathrm{ThSys}^{\natural}(\mathcal{I})$, $T = Z^{\mathcal{I}\natural}(\Omega(T))$. It is easy to see that this holds if and only if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$, $\widetilde{T} = Z^{\mathcal{I}\natural}(\Omega(\widetilde{T}))$. And this is equivalent, by Proposition 369, to the statement that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$, $\widetilde{T} = Z^{\mathcal{I}\natural}(\Omega(T))$. ■

As was the case with rough system truth equationality (see Lemma 1156), it turns out that, if a given π-institution is narrowly system truth equational, then any collection of witnessing equations must be included in the narrow system core of $\mathcal{I}$. That is, in case of narrow system truth equationality, the narrow system core is a candidate for the largest set of witnessing equations.

Lemma 1175 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is narrowly system truth equational, with witnessing equations* $\tau^\flat$*, then* $\tau^\flat \subseteq Z^{\mathcal{I}\natural}$*.*

Proof: Suppose $\mathcal{I}$ is narrowly system truth equational, with witnessing equations $\tau^\flat$. Let $T \in \mathrm{ThSys}^{\natural}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$. Then, for all $T \leq T' \in \mathrm{ThSys}(\mathcal{I})$, we have $T' \in \mathrm{ThSys}^{\natural}(\mathcal{I})$ and $\phi \in T'_\Sigma$. Thus, by narrow system truth equationality and Proposition 1168, $\tau^\flat_\Sigma[\phi] \leq \Omega(T')$. Since T', with the postulated properties was arbitrary,

$$\tau^\flat_\Sigma[\phi] \leq \bigcap\{\Omega(T') : T \leq T' \in \mathrm{ThSys}(\mathcal{I})\} = \widehat{\Omega}^{\mathcal{I}}(T).$$

We conclude, using Proposition 1172, that $\tau^\flat \subseteq Z^{\mathcal{I}\natural}$. ■

We are now ready to prove the equivalence between narrow system truth equationality and the solubility of the narrow system core, an analog of Theorem 1159. First, we show that narrow system truth equationality implies the solubility of the narrow system core. This forms an analog of Theorem 1157.

Theorem 1176 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is narrowly system truth equational, then* $Z^{\mathcal{I}\natural}$ *is soluble.*

Proof: Suppose $\mathcal{I}$ is narrowly system truth equational, with witnessing equations $\tau^\flat$. Let $T \in \mathrm{ThSys}^{\natural}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $Z^{\mathcal{I}\natural}_\Sigma[\phi] \leq \Omega(T)$. Then, by narrow system truth equationality and Lemma 1175, $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Again, using narrow system truth equationality and Proposition 1168, we conclude that $\phi \in T_\Sigma$. This shows that $Z^{\mathcal{I}\natural}$ is soluble. ■

Conversely, the solubility of the narrow system core of a π-institution implies narrow system truth equationality.

Theorem 1177 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $Z^{\mathcal{I}\natural}$ *is soluble, then* $\mathcal{I}$ *is narrowly system truth equational, with witnessing equations* $Z^{\mathcal{I}\natural}$*.*

Proof: Assume $Z^{\mathcal{I}\natural}$ is soluble and let $T \in \mathrm{ThSys}^\natural(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. By Proposition 1168, it suffices to show that

$$\phi \in T_\Sigma \quad \text{iff} \quad Z_\Sigma^{\mathcal{I}\natural}[\phi] \leq \Omega(T).$$

If $\phi \in T_\Sigma$, then, by Proposition 1173, $\phi \in Z_\Sigma^{\mathcal{I}\natural}(\Omega(T))$, i.e., $Z_\Sigma^{\mathcal{I}\natural}[\phi] \leq \Omega(T)$. On the other hand, the reverse inclusion is guaranteed by the postulated solubility of $Z^{\mathcal{I}\natural}$. Thus, $\mathcal{I}$ is indeed narrowly system truth equational, with witnessing equations $Z^{\mathcal{I}\natural}$. ∎

Theorems 1176 and 1177 provide the first characterization of narrow system truth equationality in terms of the solubility of the narrow system core. This parallels Theorem 1159, which asserted a similar characterization for rough system truth equationality in terms of the solubility of the rough system core of a π-institution.

$$\mathcal{I} \text{ Narrowly System Truth Equational} \longleftrightarrow Z^{\mathcal{I}\natural} \text{ Soluble}$$

Theorem 1178 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is narrowly system truth equational if and only if* $Z^{\mathcal{I}\natural}$ *is soluble.*

Proof: The "if" is by Theorem 1177. The "only if" by Theorem 1176. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the narrow system core $Z^{\mathcal{I}\natural}$ of $\mathcal{I}$ **narrowly defines theory systems** if, for al $T \in \mathrm{ThSys}^\natural(\mathcal{I})$,

$$T = Z^{\mathcal{I}\natural}(\Omega(T)).$$

Another characterization of narrow system truth equationality, along the lines of Theorem 1160, asserts that it is equivalent to the narrow definability of the theory systems by the narrow system core.

$$\begin{array}{l}\mathcal{I} \text{ Narrowly System Truth Equational} \\ \qquad\qquad \longleftrightarrow Z^{\mathcal{I}\natural} \text{ Narrowly Defines Theory Systems}\end{array}$$

Theorem 1179 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is narrowly system truth equational if and only if* $Z^{\mathcal{I}\natural}$ *narrowly defines theory systems in* $\mathcal{I}$*.*

Proof: Suppose $\mathcal{I}$ is narrowly system truth equational. By Theorem 1176, $Z^{\mathcal{I}\nleq}$ is soluble. Hence, by definition, for all $T \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$, $Z^{\mathcal{I}\nleq}(\Omega(T)) \leq T$. Since, by Proposition 1173, the reverse inclusion always holds, we get, for all $T \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$, $T = Z^{\mathcal{I}\nleq}(\Omega(T))$. Thus, $Z^{\mathcal{I}\nleq}$ narrowly defines theory systems in $\mathcal{I}$. Conversely, if, for all $T \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$, $T = Z^{\mathcal{I}\nleq}(\Omega(T))$, then $Z^{\mathcal{I}\nleq}$ is soluble and, therefore, by Theorem 1178, $\mathcal{I}$ is narrowly system truth equational. ■

We establish, next, a connection between narrow system truth equationality and narrow system c-reflectivity by means of a variant of the systemic Suszko operator. This variant of the systemic Suszko operator, denoted $\widehat{\Omega}^{\mathcal{I}\nleq}$, is not necessarily identical to the systemic Suszko operator $\widehat{\Omega}^{\mathcal{I}}$ itself, unlike the version of the Suszko operator $\widetilde{\Omega}^{\mathcal{I}\nleq}$, defined in the preceding section, which was introduced only for convenience, but was actually shown to be equivalent to the original version $\widetilde{\Omega}^{\mathcal{I}}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We define the **narrow systemic Suszko operator** $\widehat{\Omega}^{\mathcal{I}\nleq}$ by setting, for all $T \in \mathrm{ThSys}(\mathcal{I})$,

$$\widehat{\Omega}^{\mathcal{I}\nleq}(T) = \bigcap\{\Omega(T') : T \leq T' \in \mathrm{ThSys}^{\nleq}(\mathcal{I})\}.$$

Note that Lemma 1110 and, hence, a hypothetical analog of Corollary 1111, are not applicable in the case of theory systems, since, given $T \in \mathrm{ThSys}(\mathcal{I})$, it may not be the case that $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$. On the other hand, as the following lemma shows, for all $T \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$, $\widehat{\Omega}^{\mathcal{I}\nleq}(T) = \widehat{\Omega}^{\mathcal{I}}(T)$. So, for the case of theory systems, all of whose components are nonempty, the two operators do coincide.

Lemma 1180 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. For all $T \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$,*

$$\widehat{\Omega}^{\mathcal{I}}(T) = \bigcap\{\Omega(T') : T \leq T' \in \mathrm{ThSys}^{\nleq}(\mathcal{I})\}.$$

Proof: Since $\{T' \in \mathrm{ThSys}^{\nleq}(\mathcal{I}) : T \leq T'\} \subseteq \{T' \in \mathrm{ThSys}(\mathcal{I}) : T \leq T'\}$, we get

$$\widehat{\Omega}^{\mathcal{I}}(T) \leq \bigcap\{\Omega(T') : T \leq T' \in \mathrm{ThSys}^{\nleq}(\mathcal{I})\}.$$

But, if $T \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$, then, for all $T' \in \mathrm{ThSys}(\mathcal{I})$, such that $T \leq T'$, we have $T' \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$. Thus, in this particular case, the two collections above are identical and, therefore, equality holds between the two sides in the displayed formula, which proves the lemma. ■

In analogy with the case of rough system truth equationality, we may introduce the notion of *adequacy* of the narrow system core, which will help in characterizing the relationship between narrow system truth equationality and narrow system c-reflectivity. The following proposition, a "narrow" analog of Proposition 1161, justifies the notion of adequacy that will follow.

Proposition 1181 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\bigcap \{\Omega(T) : T \in \mathrm{ThSys}^{\nmid}(\mathcal{I}) \text{ and } Z_\Sigma^{\mathcal{I}\nmid}[\phi] \le \Omega(T)\} \le \widehat{\Omega}^{\mathcal{I}\nmid}(C(\overrightarrow{\phi})).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$,

$$\begin{array}{lll} \phi \in T_\Sigma & \text{implies} & Z_\Sigma^{\mathcal{I}\nmid}[\phi] \le \widehat{\Omega}^{\mathcal{I}}(T) \quad (\text{Definition of } Z^{\mathcal{I}\nmid}) \\ & \text{implies} & Z_\Sigma^{\mathcal{I}\nmid}[\phi] \le \Omega(T). \quad (T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})) \end{array}$$

Hence,

$$\begin{array}{l} \bigcap \{\Omega(T) : T \in \mathrm{ThSys}^{\nmid}(\mathcal{I}) \text{ and } Z_\Sigma^{\mathcal{I}\nmid}[\phi] \le \Omega(T)\} \\ \qquad \le \bigcap \{\Omega(T) : T \in \mathrm{ThSys}^{\nmid}(\mathcal{I}) \text{ and } Z_\Sigma^{\mathcal{I}\nmid}[\phi] \le \widehat{\Omega}^{\mathcal{I}}(T)\} \\ \qquad \le \bigcap \{\Omega(T) : T \in \mathrm{ThSys}^{\nmid}(\mathcal{I}) \text{ and } \phi \in T_\Sigma\} \\ \qquad = \bigcap \{\Omega(T) : T \in \mathrm{ThSys}^{\nmid}(\mathcal{I}) \text{ and } \overrightarrow{\phi} \le T\} \\ \qquad = \widetilde{\Omega}^{\mathcal{I}\nmid}(C(\overrightarrow{\phi})). \end{array}$$

This is the displayed formula in the statement. ■

If the reverse inclusion of that proven in Proposition 1181 holds, then we say that the narrow system core of $\mathcal{I}$ is *adequate*.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the narrow system core $Z^{\mathcal{I}\nmid}$ of $\mathcal{I}$ is **adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widehat{\Omega}^{\mathcal{I}\nmid}(C(\overrightarrow{\phi})) \le \bigcap \{\Omega(T) : T \in \mathrm{ThSys}^{\nmid}(\mathcal{I}) \text{ and } Z_\Sigma^{\mathcal{I}\nmid}[\phi] \le \Omega(T)\}.$$

We can show, in analogy with Corollary 1162, that the solubility of the narrow system core implies its adequacy.

Corollary 1182 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $Z^{\mathcal{I}\nmid}$ *is soluble, then it is adequate.*

Proof: Suppose $Z^{\mathcal{I}\nmid}$ is soluble and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{lll} \widehat{\Omega}^{\mathcal{I}\nmid}(C(\overrightarrow{\phi})) & = & \bigcap \{\Omega(T) : T \in \mathrm{ThSys}^{\nmid}(\mathcal{I}) \text{ and } \overrightarrow{\phi} \le T\} \\ & & (\text{definition of } \widehat{\Omega}^{\mathcal{I}\nmid}) \\ & = & \bigcap \{\Omega(T) : T \in \mathrm{ThSys}^{\nmid}(\mathcal{I}) \text{ and } \phi \in T_\Sigma\} \\ & & (T \in \mathrm{ThSys}(\mathcal{I})) \\ & = & \bigcap \{\Omega(T) : T \in \mathrm{ThSys}^{\nmid}(\mathcal{I}) \text{ and } Z_\Sigma^{\mathcal{I}\nmid}[\phi] \le \Omega(T)\}. \\ & & (\text{Solubility of } Z^{\mathcal{I}\nmid}) \end{array}$$

Thus, $Z^{\mathcal{I}\nmid}$ is adequate. ■

In order to prove a partial converse of Corollary 1182, we will employ the following characterization of narrow system c-reflectivity.

Lemma 1183 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is narrowly system c-reflective if and only if, for all* $T \in \mathrm{ThSys}(\mathcal{I})$ *and all* $T' \in \mathrm{ThSys}^{\nless}(\mathcal{I})$*,*

$$\widehat{\Omega}^{\mathcal{I}^{\nless}}(T) \leq \Omega(T') \quad \textit{implies} \quad T \leq T'.$$

Proof: Suppose, first, that $\mathcal{I}$ is narrowly system c-reflective and let $T \in \mathrm{ThSys}(\mathcal{I})$ and $T' \in \mathrm{ThSys}^{\nless}(\mathcal{I})$, such that $\widehat{\Omega}^{\mathcal{I}^{\nless}}(T) \leq \Omega(T')$. Then, by definition,

$$\bigcap\{\Omega(T'') : T \leq T'' \in \mathrm{ThSys}^{\nless}(\mathcal{I})\} \leq \Omega(T').$$

By narrow system c-reflectivity, $\bigcap\{T'' : T \leq T'' \in \mathrm{ThSys}^{\nless}(\mathcal{I})\} \leq T'$. Thus, $T \leq T'$.

Suppose, conversely, that the displayed condition in the statement holds and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^{\nless}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. Then, since $\mathcal{T} \subseteq \mathrm{ThSys}^{\nless}(\mathcal{I})$, we get that

$$\bigcap\{\Omega(T) : \bigcap \mathcal{T} \leq T'' \in \mathrm{ThSys}^{\nless}(\mathcal{I})\} \leq \Omega(T').$$

By definition, then, $\widehat{\Omega}^{\mathcal{I}^{\nless}}(\bigcap \mathcal{T}) \leq \Omega(T')$, whence, by hypothesis, $\bigcap \mathcal{T} \leq T'$. This shows that $\mathcal{I}$ is narrowly system c-reflective. ∎

We prove, next, a partial converse of Corollary 1182, under the additional assumption that the π-institution $\mathcal{I}$ under consideration is narrowly system c-reflective. This constitutes an analog of Proposition 1163.

Proposition 1184 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a narrowly system c-reflective* π*-institution based on* $\mathbf{F}$*. If* $Z^{\mathcal{I}^{\nless}}$ *is adequate, then it is soluble.*

Proof: Suppose $\mathcal{I}$ is narrowly system c-reflective and $Z^{\mathcal{I}^{\nless}}$ is adequate. Let $T \in \mathrm{ThSys}^{\nless}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $Z^{\mathcal{I}^{\nless}}_\Sigma[\phi] \leq \Omega(T)$. By the postulated adequacy of $Z^{\mathcal{I}^{\nless}}$, we get that $\widehat{\Omega}^{\mathcal{I}^{\nless}}(C(\overrightarrow{\phi})) \leq \Omega(T)$. By narrow system truth equationality and Lemma 1183, $C(\overrightarrow{\phi}) \leq T$. Therefore, $\phi \in T_\Sigma$. We conclude that $Z^{\mathcal{I}^{\nless}}$ is soluble. ∎

We are now in a position to prove the main characterization theorem relating narrow system truth equationality with narrow system c-reflectivity, an analog of Theorem 1164, which characterized rough system truth equationality in terms of rough system c-reflectivity and the adequacy of the rough system core.

Narrow System Truth Equationality
= $Z^{\mathcal{I}^{\nless}}$ Soluble
= $Z^{\mathcal{I}^{\nless}}$ Narrowly Defines Theory Systems
= Narrow System c-Reflectivity
+ $Z^{\mathcal{I}^{\nless}}$ Adequate

Theorem 1185 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is narrowly system truth equational if and only if it is narrowly system c-reflective and has an adequate narrow system core.*

Proof: Suppose, first, that $\mathcal{I}$ is narrowly system truth equational. By Theorem 1171, it is narrowly system c-reflective. By Theorem 1176, its narrow system core is soluble. Thus, by Corollary 1182, its narrow system core is also adequate.

Assume, conversely, that $\mathcal{I}$ is narrowly system c-reflective and has an adequate narrow system core. Then, by Proposition 1184, its narrow system core is soluble. Hence, by Theorem 1177, $\mathcal{I}$ is narrowly system truth equational. ∎

Based on Proposition 1152 and Theorem 468, it is not difficult to show, in an analog of Corollary 1165, that the characterization theorem, Theorem 887, of system truth equationality in terms of system c-reflectivity and the adequacy of the system core, can be inferred from Theorem 1185.

Corollary 1186 (Theorem 887) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is system truth equational if and only if it is system c-reflective and has an adequate system core.*

Proof: $\mathcal{I}$ is system truth equational if and only if, by Proposition 1152, it is narrowly system truth equational and has theorems, if and only if, by Theorem 1185, it is narrowly system c-reflective, with an adequate narrow system core and has theorems, if and only if, by Theorem 468 and the definitions of system core and narrow system core, it is system c-reflective and has an adequate system core. ∎

Finally, we prove an analog of Theorem 1166, which may be perceived either as an alternative characterization of narrow system truth equationality, involving arbitrary $\mathbf{F}$-algebraic systems, or as a transfer theorem.

Theorem 1187 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is narrowly system truth equational, with witnessing equations* $\tau^\flat$*, if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, all* $T \in \mathrm{FiSys}^{\mathcal{I}\nleq}(\mathcal{A})$*, all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi \in \mathrm{SEN}(\Sigma)$*,*

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^{\mathcal{A}}_\Sigma[\phi] \le \Omega^{\mathcal{A}}(T).$$

Proof: If the postulated condition holds, then it holds, in particular, for the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. This yields immediately that $\mathcal{I}$ is narrowly system truth equational.

Suppose, conversely, that $\mathcal{I}$ is narrowly system truth equational and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiSys}^{\mathcal{I}\natural}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rll}\alpha_\Sigma(\phi) \in T_{F(\Sigma)} & \text{iff} & \phi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)}) \\ & \text{iff} & \tau_\Sigma^\flat[\phi] \leq \Omega(\alpha^{-1}(T)) \\ & & \text{(Lemma 6 and hypothesis)} \\ & \text{iff} & \tau_\Sigma^\flat[\phi] \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T)) \quad \text{(Proposition 24)} \\ & \text{iff} & \tau_{F(\Sigma)}^{\mathcal{A}}[\alpha_\Sigma(\phi)] \leq \Omega^{\mathcal{A}}(T). \quad \text{(Lemma 95)}\end{array}$$

Hence, taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that the displayed condition holds. ∎

14.6 Availability of Natural Theorems

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall that, by convention, if $\mathcal{I}$ has theorems, then, for every $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathcal{I}$ has a Σ-theorem, i.e., there exists $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\varnothing)$.

On the other hand, recall from Section 2.6 that we say that a π-institution $\mathcal{I}$ has *natural theorems* if there exists a $\vartheta^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^k$,

$$\vartheta_\Sigma^\flat(\vec{\phi}) \in C_\Sigma(\varnothing).$$

Furthermore, recall that we denote by $\mathrm{NThm}(\mathcal{I})$ the collection of natural theorems of $\mathcal{I}$.

It is straightforward that having natural theorems is a stronger property than having theorems.

Lemma 1188 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ has natural theorems, then it has theorems.*

Proof: Suppose $\vartheta^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$ is a natural theorem. Let $\Sigma \in |\mathbf{Sign}^\flat|$. By convention $\mathrm{SEN}^\flat(\Sigma) \neq \varnothing$. Let $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$. Then, we get $\vartheta_\Sigma^\flat(\vec{\phi}) \in \mathrm{Thm}_\Sigma(\mathcal{I})$. This shows that $\mathcal{I}$ has theorems. ∎

On the other hand, it is easy to find examples of π-institutions with theorems that do not possess natural theorems. For example, every π-institution with at least one non-trivial set of sentences $\mathrm{SEN}^\flat(\Sigma)$, containing both a Σ-theorem and a Σ-non theorem, and with a trivial category of natural transformations, cannot have natural theorems. This follows from the fact that, under these circumstances, no projection natural transformation can be a natural theorem and projection natural transformations are the only ones available because of the triviality of $N^\flat$.

Another useful observation is that every π-institution with natural theorems has at least one at-most-unary natural theorem.

Lemma 1189 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *has natural theorems, then it has at least one at-most-unary natural theorem.*

Proof: Suppose $\mathcal{I}$ has natural theorems and let $\vartheta^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ be a natural theorem in $N^\flat$. If $k = 0$ or 1, then there is nothing to prove. If $k > 1$, then we define $\vartheta'^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\vartheta'^\flat_\Sigma(\phi) = \vartheta^\flat_\Sigma(\underbrace{\phi, \phi, \ldots, \phi}_{k}).$$

Since $\vartheta'^\flat = \vartheta \circ \langle p^{1,0}, p^{1,0}, \ldots, p^{1,0} \rangle$ and $\vartheta^\flat$ is in $N^\flat$, we get that $\vartheta'^\flat$ is in $N^\flat$ also. Moreover, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\vartheta'^\flat_\Sigma(\phi) = \vartheta^\flat_\Sigma(\phi, \ldots, \phi) \in \mathrm{Thm}_\Sigma(\mathcal{I}).$$

Hence $\vartheta'^\flat$ is a unary natural theorem. ∎

We have the following characterization of natural theorems involving the local Frege operator.

Theorem 1190 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, and* $\vartheta^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ *a natural transformation in* $N^\flat$*. Then the following conditions are equivalent:*

(i) $\vartheta^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ *is a natural theorem;*

(ii) For every $T \in \mathrm{ThFam}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\phi \in T_\Sigma \quad \textit{iff} \quad \langle \phi, \vartheta^\flat_\Sigma(\vec{\chi}) \rangle \in \lambda_\Sigma(T);$$

(iii) For every $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\phi \in \mathrm{Thm}_\Sigma(\mathcal{I}) \quad \textit{iff} \quad \langle \phi, \vartheta^\flat_\Sigma(\vec{\chi}) \rangle \in \lambda_\Sigma(\mathrm{Thm}(\mathcal{I})).$$

Proof:

(i)⇒(ii) Assume that $\vartheta^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ is a natural theorem. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$.

- Suppose $\phi \in T_\Sigma$. Then, since $\vartheta^\flat_\Sigma(\vec{\chi}) \in \mathrm{Thm}_\Sigma(\mathcal{I}) \subseteq T_\Sigma$, we get, by definition of $\lambda(T)$, $\langle \phi, \vartheta^\flat_\Sigma(\vec{\chi}) \rangle \in \lambda_\Sigma(T)$.

- On the other hand, assume $\langle\phi,\vartheta^\flat_\Sigma(\vec{\chi})\rangle\in\lambda_\Sigma(T)$. Since $\vartheta^\flat_\Sigma(\vec{\chi})\in\mathrm{Thm}_\Sigma(\mathcal{I})\subseteq T_\Sigma$, we get, by the definition of $\lambda(T)$, $\phi\in T_\Sigma$.

(ii)⇒(iii) Trivial.

(iii)⇒(i) Suppose that (iii) holds. Let $\Sigma\in|\mathbf{Sign}^\flat|$, $\phi\in\mathrm{SEN}^\flat(\Sigma)$ and $\vec{\chi}\in\mathrm{SEN}^\flat(\Sigma)$.

- If $\phi\in\mathrm{Thm}_\Sigma(\mathcal{I})$, then, by hypothesis, $\langle\phi,\vartheta^\flat_\Sigma(\vec{\chi})\rangle\in\lambda_\Sigma(\mathrm{Thm}(\mathcal{I}))$, whence, $\vartheta^\flat_\Sigma(\vec{\chi})\in\mathrm{Thm}_\Sigma(\mathcal{I})$.
- If $\phi\notin\mathrm{Thm}_\Sigma(\mathcal{I})$, then, by hypothesis, $\langle\phi,\vartheta^\flat_\Sigma(\vec{\chi})\rangle\notin\lambda_\Sigma(\mathrm{Thm}(\mathcal{I}))$. Thus, $\theta^\flat_\Sigma(\vec{\chi})\in\mathrm{Thm}_\Sigma(\mathcal{I})$.

We conclude that, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\vec{\chi}\in\mathrm{SEN}^\flat(\Sigma)$, $\vartheta^\flat_\Sigma(\vec{\chi})\in\mathrm{Thm}_\Sigma(\mathcal{I})$. Therefore, $\vartheta^\flat$ is a natural theorem.

■

We provide two additional equivalent conditions in the theorem following the next lemma.

Lemma 1191 *Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, $\mathcal{I}=\langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$, and $\vartheta^\flat:(\mathrm{SEN}^\flat)^k\to\mathrm{SEN}^\flat$ a natural transformation in $N^\flat$. If $\vartheta^\flat$ is a natural theorem, then, for every $\mathbf{F}$-algebraic system $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$, all $\Sigma\in|\mathbf{Sign}|$ and all $\vec{\chi}\in\mathrm{SEN}(\Sigma)$, $\vartheta_\Sigma(\vec{\chi})\in C^{\mathcal{I},\mathcal{A}}_\Sigma(\varnothing)$, i.e., ϑ is a natural theorem of $\langle\mathcal{A},C^{\mathcal{I},\mathcal{A}}\rangle$.*

Proof: Since $\vartheta^\flat$ is a natural theorem, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\vec{\chi}\in\mathrm{SEN}^\flat(\Sigma)$,

$$\vartheta_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi}))=\alpha_\Sigma(\vartheta^\flat_\Sigma(\vec{\chi}))\in\alpha_\Sigma(\mathrm{Thm}_\Sigma(\mathcal{I}))\subseteq C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\varnothing).$$

By the surjectivity of $\langle F,\alpha\rangle$ the conclusion follows. ■

Theorem 1192 *Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, $\mathcal{I}=\langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$, and $\vartheta^\flat:(\mathrm{SEN}^\flat)^k\to\mathrm{SEN}^\flat$ a natural transformation in $N^\flat$. Then the following conditions are equivalent:*

(i) $\vartheta^\flat:(\mathrm{SEN}^\flat)^k\to\mathrm{SEN}^\flat$ *is a natural theorem;*

(ii) For every $\mathbf{F}$-algebraic system $\mathcal{A}$, all $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, all $\Sigma\in|\mathbf{Sign}|$ and all $\phi\in\mathrm{SEN}(\Sigma)$,

$$\phi\in T_\Sigma\quad\text{iff}\quad(\forall\vec{\chi}\in\mathrm{SEN}(\Sigma))(\langle\phi,\vartheta_\Sigma(\vec{\chi})\rangle\in\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T));$$

(iii) For every $\mathbf{F}$-algebraic system $\mathcal{A}$, all $\Sigma\in|\mathbf{Sign}|$ and all $\phi\in\mathrm{SEN}(\Sigma)$,

$$\phi\in C^{\mathcal{I},\mathcal{A}}_\Sigma(\varnothing)\quad\text{iff}\quad(\forall\vec{\chi}\in\mathrm{SEN}(\Sigma))(\langle\phi,\vartheta_\Sigma(\vec{\chi})\rangle\in\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(C^{\mathcal{I},\mathcal{A}}(\varnothing))).$$

Proof:

(i)⇒(ii) Assume that $\vartheta^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ is a natural theorem and let $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. By Lemma 1191, for all $\vec{\chi} \in \mathrm{SEN}(\Sigma)$, $\vartheta^\flat_\Sigma(\vec{\chi}) \in C^{\mathcal{I},\mathcal{A}}_\Sigma(\varnothing)$.

- If $\phi \in T_\Sigma$, then, clearly, for all $\vec{\chi} \in \mathrm{SEN}(\Sigma)$, and all $T \le T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we have $\phi \in T'_\Sigma$ and $\vartheta_\Sigma(\vec{\chi}) \in T'_\Sigma$. Hence, $\langle \phi, \vartheta_\Sigma(\vec{\chi})\rangle \in \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T)$.
- If, for all $\vec{\chi} \in \mathrm{SEN}(\Sigma)$, $\langle \phi, \vartheta_\Sigma(\vec{\chi})\rangle \in \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T)$, then, in particular, for all $\vec{\chi} \in \mathrm{SEN}(\Sigma)$, $\langle \phi, \vartheta_\Sigma(\vec{\chi})\rangle \in \lambda^{\mathcal{A}}_\Sigma(T)$. Since $\vartheta_\Sigma(\vec{\chi}) \in T_\Sigma$, we conclude that $\phi \in T_\Sigma$.

(ii)⇒(iii) Trivial.

(iii)⇒(iv) Suppose that (iii) holds. Consider, first, the trivial algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, with the single signature object $*$ and such that $\mathrm{SEN}(*) = \{0\}$. Then, we have $\langle 0, \vartheta_\Sigma(\vec{0})\rangle = \langle 0, 0\rangle \in \{\langle 0, 0\rangle\} = \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_*(\varnothing)$. If $\varnothing \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, then this would imply, by hypothesis, that $0 \in \varnothing$, a contradiction. Thus, $\varnothing \notin \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A})$. This shows that $\mathcal{I}$ has theorems.

Let, now, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\chi} \in \mathrm{SEN}(\Sigma)$. Take a theorem $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$. Then, by hypothesis, $\langle t, \vartheta^\flat_\Sigma(\vec{\chi})\rangle \in \widetilde{\lambda}^{\mathcal{I}}_\Sigma(\mathrm{Thm}(\mathcal{I})) \subseteq \lambda_\Sigma(\mathrm{Thm}(\mathcal{I}))$. Thus, since $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$, we must have $\vartheta^\flat_\Sigma(\vec{\chi}) \in \mathrm{Thm}_\Sigma(\mathcal{I})$. But $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$ were arbitrary, whence, we conclude that $\vartheta^\flat$ is a natural theorem.

■

We saw that availability of natural theorems is a strictly stronger condition that availability of theorems. We have the following theorem, which follows from preceding results.

Theorem 1193 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an* $\mathbf{F}$*-algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *has natural theorems, then there exists* $\tau : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*, such that, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A}$*, all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi \in \mathrm{SEN}(\Sigma)$*,*

$$\phi \in T_\Sigma \quad \textit{iff}, \quad \textit{for all } \vec{\chi} \in \mathrm{SEN}(\Sigma), \tau_\Sigma(\phi, \vec{\chi}) \subseteq \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T);$$

(b) If the condition in the conclusion of (a) holds, then $\mathcal{I}$ *has theorems.*

Proof:

(a) Suppose $\mathcal{I}$ has natural theorems and let $\vartheta^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ be a natural theorem. Then, we define $\tau^\flat : (\mathrm{SEN}^\flat)^{k+1} \to (\mathrm{SEN}^\flat)^2$, by setting

$$\tau^\flat := \{p^{k+1,0} \approx \vartheta^\flat \circ \langle p^{k+1,1}, \dots, p^{k+1,k}\rangle\}.$$

Then the conclusion follows from Theorem 1192.

(b) Suppose that the conclusion of Part (a) holds. Consider the trivial algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with the single signature object $*$ and such that $\mathrm{SEN}(*) = \{0\}$. If $\mathcal{I}$ does not have theorems, then $\varnothing \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Since $0 \notin \varnothing$, we must have, by hypothesis, $\langle 0, 0 \rangle = \langle 0, \vartheta_\Sigma(\vec{0}) \rangle \notin \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_*(\varnothing) = \{\langle 0, 0 \rangle\}$, a contradiction. Therefore, $\mathcal{I}$ has theorems.

■

We may think of a π-institution that has theorems, but not natural theorems, as having a syntactic deficiency, i.e., not having enough natural transformations in its category of natural transformations to express theoremhood. So in an analogous way with the one used to formulate similar properties through the Leibniz property of the reflexive core and the adequacy of the Suszko core, we make the following definition, taking cue from Theorem 1190.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The **Frege core** $F^{\mathcal{I}}$ of $\mathcal{I}$ is defined by

$$\begin{aligned} F^{\mathcal{I}} \;=\; & \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}(\mathcal{I}))(\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)) \\ & (T_\Sigma \approx \sigma^\flat_\Sigma(\vec{\chi}) \subseteq \widetilde{\lambda}^{\mathcal{I}}_\Sigma(T))\}. \end{aligned}$$

It is not difficult to show that, in case $\mathcal{I}$ has theorems, $F^{\mathcal{I}} = \mathrm{NThm}(\mathcal{I})$.

Proposition 1194 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ has theorems, then $F^{\mathcal{I}} = \mathrm{NThm}(\mathcal{I})$.*

Proof: Suppose that $\mathcal{I}$ has theorems.

Assume $\sigma^\flat \in F^{\mathcal{I}}$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$. Since $\mathcal{I}$ has theorems, there exists $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$. Then, by hypothesis and the definition of $F^{\mathcal{I}}$,

$$\langle t, \sigma^\flat_\Sigma(\vec{\chi}) \rangle \in \widetilde{\lambda}^{\mathcal{I}}_\Sigma(\mathrm{Thm}(\mathcal{I})) \subseteq \lambda_\Sigma(\mathrm{Thm}(\mathcal{I})).$$

Thus, since $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$, $\sigma^\flat_\Sigma(\vec{\chi}) \in \mathrm{Thm}_\Sigma(\mathcal{I})$. Since $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$ were arbitrary, $\sigma^\flat \in \mathrm{NThm}(\mathcal{I})$. Therefore, we get that $F^{\mathcal{I}} \subseteq \mathrm{NThm}(\mathcal{I})$.

Suppose, conversely, that $\sigma^\flat \in \mathrm{NThm}(\mathcal{I})$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $T \leq T' \in \mathrm{ThFam}(\mathcal{I})$. Then, we get $\phi \in T'_\Sigma$ and $\sigma^\flat_\Sigma(\vec{\chi}) \in T'_\Sigma$, whence

$$\phi \in T'_\Sigma \quad \text{iff} \quad \sigma^\flat_\Sigma(\vec{\chi}) \in T'_\Sigma.$$

That is, for all $T \leq T' \in \mathrm{ThFam}(\mathcal{I})$, $\langle \phi, \sigma^\flat_\Sigma(\vec{\chi}) \rangle \in \lambda_\Sigma(T)$. Hence, $\langle \phi, \sigma^\flat_\Sigma(\vec{\chi}) \rangle \in \widetilde{\lambda}^{\mathcal{I}}_\Sigma(T)$. This shows that $\sigma^\flat \in F^{\mathcal{I}}$, whence $\mathrm{NThm}(\mathcal{I}) \subseteq F^{\mathcal{I}}$. ■

In the remainder of the section, we show that a property analogous to adequacy, coupled with possession of theorems, guarantees the existence of natural theorems. The following lemma partly justifies the definition of adequacy.

Proposition 1195 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\bigcap\{\lambda(T) : (\forall \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma))(\phi \approx F^{\mathcal{I}}_\Sigma(\vec{\chi}) \subseteq \lambda_\Sigma(T))\} \leq \widetilde{\lambda}^{\mathcal{I}}(C(\phi)).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. By the definition of the Frege core, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{implies} \quad \phi \approx F^{\mathcal{I}}_\Sigma(\vec{\chi}) \subseteq \widetilde{\lambda}^{\mathcal{I}}_\Sigma(T) \subseteq \lambda_\Sigma(T).$$

Therefore, we get

$$\begin{array}{l}\{T \in \mathrm{ThFam}(\mathcal{I}) : \phi \in T_\Sigma\} \\ \quad \subseteq \{T \in \mathrm{ThFam}(\mathcal{I}) : (\forall \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma))(\phi \approx F^{\mathcal{I}}_\Sigma(\vec{\chi}) \subseteq \lambda_\Sigma(T))\}.\end{array}$$

This, now, yields

$$\bigcap\{\lambda(T) : (\forall \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma))(\phi \approx F^{\mathcal{I}}_\Sigma(\vec{\chi}) \subseteq \lambda_\Sigma(T))\} \leq \widetilde{\lambda}^{\mathcal{I}}(C(\phi)),$$

i.e., the displayed formula in the statement. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the Frege core $F^{\mathcal{I}}$ is **adequate** if the reverse inclusion of the one proved in Proposition 1195 holds, i.e., if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widetilde{\lambda}(C(\phi)) \leq \bigcap\{\lambda(T) : (\forall \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma))(\phi \approx F^{\mathcal{I}}_\Sigma(\vec{\chi}) \subseteq \lambda_\Sigma(T))\}.$$

Theorem 1196 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *has natural theorems if and only if it has theorems and its Frege core is adequate.*

Proof: If $\mathcal{I}$ has natural theorems, then, by Lemma 1188, it has theorems. Moreover, by Proposition 1194, $F^{\mathcal{I}} = \mathrm{NThm}(\mathcal{I})$. Now consider $\tau^\flat \in \mathrm{NThm}(\mathcal{I})$ and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \approx F^{\mathcal{I}}_\Sigma(\vec{\chi}) \subseteq \lambda_\Sigma(T)$. Since $F^{\mathcal{I}} = \mathrm{NThm}(\mathcal{I})$, we get $F^{\mathcal{I}}_\Sigma(\vec{\chi}) \subseteq \mathrm{Thm}_\Sigma(\mathcal{I}) \subseteq T_\Sigma$. Thus, $\phi \in T_\Sigma$. This shows that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{r}\{T \in \mathrm{ThFam}(\mathcal{I}) : (\forall \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma))(\phi \approx F^{\mathcal{I}}_\Sigma(\vec{\chi}) \subseteq \lambda_\Sigma(T))\} \\ \leq \{T \in \mathrm{ThFam}(\mathcal{I}) : \phi \in T_\Sigma\}.\end{array}$$

This proves that $F^{\mathcal{I}}$ is adequate.

Assume, conversely, that $\mathcal{I}$ has theorems and $F^{\mathcal{I}}$ is adequate.

Note that, if $\mathrm{Thm}(\mathcal{I}) = \mathrm{SEN}^\flat$, then $p^{1,0} : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ is a natural theorem. So we may assume that $\overline{\varnothing} \subsetneq \mathrm{Thm}(\mathcal{I}) \subsetneq \mathrm{SEN}^\flat$. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$ and $\phi \in \mathrm{SEN}^\flat(\Sigma) \backslash \mathrm{Thm}_\Sigma(\mathcal{I})$. Then, we get

$$\langle \phi, t \rangle \in \lambda_\Sigma(C(\phi)) \quad \text{but} \quad \langle \phi, t \rangle \notin \lambda_\Sigma(\mathrm{Thm}(\mathcal{I})).$$

Thus, if $F^{\mathcal{I}} = \varnothing$, then

$$\mathrm{Thm}(\mathcal{I}) \in \{T \in \mathrm{ThFam}(\mathcal{I}) : (\forall \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma))(\phi \approx F^{\mathcal{I}}_\Sigma(\vec{\chi}) \subseteq \lambda_\Sigma(T))\}.$$

So $\langle \phi, t\rangle \notin \bigcap\{\lambda_\Sigma(T) : (\forall \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma))(\phi \approx F^{\mathcal{I}}_\Sigma(\vec{\chi}) \subseteq \lambda_\Sigma(T))\}$. Since $\langle \phi, t\rangle \in \bigcap\{\lambda_\Sigma(T) : \phi \in T_\Sigma\}$, we get that

$$\widetilde{\lambda}(C(\phi)) \nleq \bigcap\{\lambda(T) : (\forall \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma))(\phi \approx F^{\mathcal{I}}_\Sigma(\vec{\chi}) \subseteq \lambda_\Sigma(T))\},$$

contrary to the postulated adequacy of $F^{\mathcal{I}}$. ∎

We close the section by showing that having natural theorems is a property that transfers from a π-institution to all $\mathcal{I}$-gmatrix families.

Theorem 1197 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ has natural theorems if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, the $\mathcal{I}$-gmatrix $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}}\rangle$ has natural theorems.*

Proof: The "if" is clear by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$ and taking into account the fact that $C^{\mathcal{I},\mathcal{F}} = C$. The "only if" was proven in Lemma 1191. ∎