Chapter 15

The Syntactic Leibniz Hierarchy: Basement II

15.1 Syntactic Narrow Family Monotonicity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

Recall that $\mathcal{I}$ is **roughly/narrowly family monotone** if, for all $T, T' \in \mathrm{ThFam}^\nmid(\mathcal{I})$,

$$T \leq T' \quad \text{implies} \quad \Omega(T) \leq \Omega(T').$$

In this section we introduce and study a syntactic analog of this concept.

First, we relativize family reflexivity, family symmetry, family transitivity, family compatibility and family modus ponens to $\mathrm{ThFam}^\nmid(\mathcal{I})$.

Let, as above, $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Moreover, suppose that $I^\flat \subseteq N^\flat$ is a collection of natural transformations in $N^\flat$, with two distinguished arguments.

- $I^\flat$ is **roughly family reflexive** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,
$$I^\flat_\Sigma[\phi, \phi] \leq \widetilde{T};$$
- $I^\flat$ is **narrowly family reflexive** if, for all $T \in \mathrm{ThFam}^\nmid(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,
$$I^\flat_\Sigma[\phi, \phi] \leq T.$$

As the following lemma establishes rough and narrow family reflexivity are identical properties.

Lemma 1198 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat \subseteq N^\flat$ a family of natural transformations in $N^\flat$, with two distinguished arguments. $I^\flat$ is roughly family reflexive if and only if it is narrowly family reflexive.*

Proof: Suppose, first, that $I^\flat$ is roughly family reflexive and consider $T \in \mathrm{ThFam}^\nmid(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Since $T \in \mathrm{ThFam}^\nmid(\mathcal{I})$, we have $\widetilde{T} = T$, whence, by rough family reflexivity, $I^\flat_\Sigma[\phi, \phi] \leq \widetilde{T} = T$. Thus, $I^\flat$ is narrowly family reflexive.

Suppose, conversely, that $I^\flat$ is narrowly family reflexive and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Since $\widetilde{T} \in \mathrm{ThFam}^\nmid(\mathcal{I})$, we get, by narrow family reflexivity, $I^\flat_\Sigma[\phi, \phi] \leq \widetilde{T}$. Thus, $I^\flat$ is roughly family reflexive. ■

Let, again, $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat \subseteq N^\flat$ a collection of natural transformations in $N^\flat$, with two distinguished arguments.

- $I^\flat$ is **roughly family symmetric** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$I^\flat_\Sigma[\phi,\psi] \leq \widetilde{T} \quad \text{implies} \quad I^\flat_\Sigma[\psi,\phi] \leq \widetilde{T};$$
- $I^\flat$ is **narrowly family symmetric** if, for all $T \in \mathrm{ThFam}^{\not\sim}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$I^\flat_\Sigma[\phi,\psi] \leq T \quad \text{implies} \quad I^\flat_\Sigma[\psi,\phi] \leq T.$$

Similarly to rough and narrow family reflexivity, rough and narrow family symmetry coincide.

Lemma 1199 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat \subseteq N^\flat$ a family of natural transformations in $N^\flat$, with two distinguished arguments. $I^\flat$ is roughly family symmetric if and only if it is narrowly family symmetric.*

Proof: Suppose, first, that $I^\flat$ is roughly family symmetric and consider $T \in \mathrm{ThFam}^{\not\sim}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $I^\flat_\Sigma[\phi,\psi] \leq T$. Since $T \in \mathrm{ThFam}^{\not\sim}(\mathcal{I})$, we have $\widetilde{T} = T$, whence, by hypothesis, $I^\flat_\Sigma[\phi,\psi] \leq \widetilde{T}$. Applying rough family symmetry, we get $I^\flat_\Sigma[\psi,\phi] \leq \widetilde{T} = T$. Thus, $I^\flat$ is narrowly family symmetric.

Suppose, conversely, that $I^\flat$ is narrowly family symmetric and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $I^\flat_\Sigma[\phi,\psi] \leq \widetilde{T}$. Since $\widetilde{T} \in \mathrm{ThFam}^{\not\sim}(\mathcal{I})$, we get, by narrow family symmetry, $I^\flat_\Sigma[\psi,\phi] \leq \widetilde{T}$. Thus, $I^\flat$ is roughly family symmetric. ∎

Let, once more, $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat \subseteq N^\flat$ a collection of natural transformations in $N^\flat$, with two distinguished arguments.

- $I^\flat$ is **roughly family transitive** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$,
$$I^\flat_\Sigma[\phi,\psi] \cup I^\flat_\Sigma[\psi,\chi] \leq \widetilde{T} \quad \text{implies} \quad I^\flat_\Sigma[\phi,\chi] \leq \widetilde{T};$$
- $I^\flat$ is **narrowly family transitive** if, for all $T \in \mathrm{ThFam}^{\not\sim}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$,
$$I^\flat_\Sigma[\phi,\psi] \cup I^\flat_\Sigma[\psi,\chi] \leq T \quad \text{implies} \quad I^\flat_\Sigma[\phi,\chi] \leq T.$$

Rough and narrow family transitivity also coincide.

Lemma 1200 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat \subseteq N^\flat$ a family of natural transformations in $N^\flat$, with two distinguished arguments. $I^\flat$ is roughly family transitive if and only if it is narrowly family transitive.*

Proof: Similar to the proof of Lemma 1199. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat \subseteq N^\flat$ a collection of natural transformations in $N^\flat$, with two distinguished arguments.

- $I^\flat$ is **roughly family compatible** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\sigma^\flat \in N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,
$$\bigcup_{i<k} \overleftrightarrow{I^\flat}_\Sigma[\phi_i, \psi_i] \leq \widetilde{T} \quad \text{implies} \quad I^\flat_\Sigma[\sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi})] \leq \widetilde{T};$$
- $I^\flat$ is **narrowly family compatible** if, for all $T \in \mathrm{ThFam}^\nmid(\mathcal{I})$, all $\sigma^\flat \in N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,
$$\bigcup_{i<k} \overleftrightarrow{I^\flat}_\Sigma[\phi_i, \psi_i] \leq T \quad \text{implies} \quad I^\flat_\Sigma[\sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi})] \leq T.$$

Rough and narrow family transitivity also coincide.

Lemma 1201 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat \subseteq N^\flat$ a family of natural transformations in $N^\flat$, with two distinguished arguments. $I^\flat$ is roughly family compatible if and only if it is narrowly family compatible.*

Proof: Similar to the proof of Lemma 1199. ∎

Finally, we define the property of possessing the rough and the narrow family modus ponens. Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat \subseteq N^\flat$ a collection of natural transformations in $N^\flat$, with two distinguished arguments.

- $I^\flat$ has the **rough family MP** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi \in \widetilde{T}_\Sigma \quad \text{and} \quad I^\flat_\Sigma[\phi, \psi] \leq \widetilde{T} \quad \text{imply} \quad \psi \in \widetilde{T}_\Sigma;$$
- $I^\flat$ has the **narrow family MP** if, for all $T \in \mathrm{ThFam}^\nmid(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi \in T_\Sigma \quad \text{and} \quad I^\flat_\Sigma[\phi, \psi] \leq T \quad \text{imply} \quad \psi \in T_\Sigma.$$

As with all preceding properties, the rough and narrow family MP turn out to be identical properties.

Lemma 1202 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat \subseteq N^\flat$ a family of natural transformations in $N^\flat$, with two distinguished arguments. $I^\flat$ has the rough family MP if and only if it has the narrow family MP.*

Proof: The proof again follows the lines of the proof of Lemma 1199, but we describe it also in detail.

Suppose, first, that $I^\flat$ has the rough family MP and let $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $I^\flat_\Sigma[\phi,\psi] \le T$. Again, by hypothesis, $\widetilde{T} = T$, whence, we get $\phi \in \widetilde{T}_\Sigma$ and $I^\flat_\Sigma[\phi,\psi] \le \widetilde{T}$. Thus, by rough family MP, we get that $\psi \in \widetilde{T}_\Sigma$, i.e., $\psi \in T_\Sigma$. Thus, $I^\flat$ has the narrow family MP.

Assume, conversely, that $I^\flat$ has the narrow family MP and consider $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \widetilde{T}_\Sigma$ and $I^\flat_\Sigma[\phi,\psi] \le \widetilde{T}$. Since $\widetilde{T} \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, we may apply narrow family MP to conclude that $\psi \in \widetilde{T}_\Sigma$. This proves that $I^\flat$ has the rough family MP. ∎

We say that $\mathcal{I}$ is **syntactically roughly/narrowly family monotone** if there exists $I^\flat \subseteq N^\flat$, with two distinguished arguments, such that $I^\flat$ satisfies:

- narrow family reflexivity;
- narrow family transitivity;
- narrow family compatibility; and
- narrow family MP.

In that case, we call $I^\flat$ a **set of witnessing natural transformations**, or, more simply, **witnessing transformations** (of the syntactic rough/narrow family monotonicity of $\mathcal{I}$).

It turns out that, if $\mathcal{I}$ is a syntactically narrowly family monotone π-institution, with witnessing transformations $I^\flat$, then $\overleftrightarrow{I^\flat}(T)$ is a congruence system on $\mathbf{F}$ compatible with T, for all $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$. This forms a "narrow" analog of Proposition 790.

Proposition 1203 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically narrowly family monotone, with witnessing transformations* $I^\flat$*, then, for all* $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$*,* $\overleftrightarrow{I^\flat}(T)$ *is a congruence system on* $\mathbf{F}$ *compatible with* T*.*

Proof: The proof follows along the lines of the proof of Proposition 790. So we give an outline. Let $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$. The narrow family reflexivity of $I^\flat$ ensures that $\langle \phi, \phi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$. The fact that $\overleftrightarrow{I^\flat}$ is the symmetrization of $I^\flat$ ensures that $\langle \phi, \psi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$ implies that $\langle \psi, \phi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$. The narrow family transitivity of $I^\flat$ guarantees that $\langle \phi, \psi \rangle, \langle \psi, \chi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$ imply $\langle \phi, \chi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$.

Suppose, next, that $\sigma^\flat \in N^\flat$, $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$. Then, the narrow family compatibility of $I^\flat$ ensures that, if, for all $i < k$, $\langle \phi_i, \psi_i \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$, then

$\langle\sigma^\flat_\Sigma(\vec{\phi}),\sigma^\flat_\Sigma(\vec{\psi})\rangle\in I^\flat_\Sigma(T)$. Thus, $\overleftrightarrow{I}^\flat(T)$ is a congruence family on $\mathbf{F}$. However, by Lemma 93, $\overleftrightarrow{I}^\flat(T)$ is a relation system on $\mathbf{F}$. Hence, $\overleftrightarrow{I}^\flat(T)$ is a congruence system on $\mathbf{F}$.

It only remains to show that $\overleftrightarrow{I}^\flat(T)$ is compatible with T. Assume that $\phi\in T_\Sigma$ and $\langle\phi,\psi\rangle\in\overleftrightarrow{I}^\flat{}_\Sigma(T)$. Since $I^\flat\subseteq\overleftrightarrow{I}^\flat$, we get, by the narrow family MP of $I^\flat$, that $\psi\in T_\Sigma$. Thus, $\overleftrightarrow{I}^\flat(T)$ is also compatible with T. ■

Proposition 1203 shows that $\overleftrightarrow{I}^\flat$ defines Leibniz congruence systems of theory families in $\mathrm{ThFam}^\sharp(\mathcal{I})$. Following similar terminology adopted in Chapter 14, we say that $I^\flat$ **roughly** or **narrowly defines Leibniz congruence systems** of theory families in $\mathcal{I}$ if, for all $T\in\mathrm{ThFam}^\sharp(\mathcal{I})$,

$$\overleftrightarrow{I}^\flat(T)=\Omega(T).$$

Then, in what is an analog of Corollary 791, we obtain

Corollary 1204 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is syntactically narrowly family monotone, with witnessing transformations* $I^\flat$*, then* $I^\flat$ *narrowly defines Leibniz congruence systems of theory families in* $\mathcal{I}$.

Proof: By Proposition 1203 and Corollary 98. ■

Corollary 1204 allows establishing the fact that syntactic narrow family monotonicity implies (semantic) narrow family monotonicity. This forms an analog of Theorem 792.

Theorem 1205 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is syntactically narrowly family monotone, then it is narrowly family monotone.*

Proof: Suppose that $\mathcal{I}$ is syntactically narrowly family monotone with witnessing transformations $I^\flat$. Let $T,T'\in\mathrm{ThFam}^\sharp(\mathcal{I})$, such that $T\leq T'$. Then

$$\begin{array}{rcll}\Omega(T)&=&\overleftrightarrow{I}^\flat(T)&\text{(by Corollary 1204)}\\&\leq&\overleftrightarrow{I}^\flat(T')&\text{(by Lemma 94)}\\&=&\Omega(T').&\text{(by Corollary 1204)}\end{array}$$

Thus, $\mathcal{I}$ is narrowly family monotone. ■

We now introduce the notion of the rough/narrow reflexive core of a π-institution $\mathcal{I}$ in a way analogous to the reflexive core, which was introduced in Chapter 11. Its introduction will enable us to provide a characterization of the syntactical narrow family monotonicity property and to establish a relationship between this property and its semantic counterpart.

Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and $\mathcal{I}=\langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$.

- The **rough reflexive core** of $\mathcal{I}$ is the collection

$$\begin{array}{rcl}\widetilde{R}^{\mathcal{I}} & = & \{\rho^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}(\mathcal{I}))(\forall \Sigma \in |\mathbf{Sign}^\flat|) \\ & & (\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\rho^\flat_\Sigma[\phi,\phi] \leq \widetilde{T})\};\end{array}$$

- The **narrow reflexive core** of $\mathcal{I}$ is the collection

$$\begin{array}{rcl}R^{\mathcal{I}\natural} & = & \{\rho^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}^{\natural}(\mathcal{I}))(\forall \Sigma \in |\mathbf{Sign}^\flat|) \\ & & (\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\rho^\flat_\Sigma[\phi,\phi] \leq T)\}.\end{array}$$

These two notions are identical, as shown in the following proposition, and this justifies the usage of the terms rough and narrow reflexive core interchangeably in this context.

Proposition 1206 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then* $\widetilde{R}^{\mathcal{I}} = R^{\mathcal{I}\natural}$*.*

Proof: On the one hand, if $\rho^\flat \in \widetilde{R}^{\mathcal{I}}$, $T \in \mathrm{ThFam}^{\natural}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, then, by the definition of the rough reflexive core, $\rho^\flat_\Sigma[\phi,\phi] \leq \widetilde{T} = T$, where the equality follows from the assumption that $T \in \mathrm{ThFam}^{\natural}(\mathcal{I})$. This shows that $\rho^\flat \in R^{\mathcal{I}\natural}$. On the other hand, if $\rho^\flat \in R^{\mathcal{I}\natural}$, $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, then, since $\widetilde{T} \in \mathrm{ThFam}^{\natural}(\mathcal{I})$, we get by the definition of $R^{\mathcal{I}\natural}$, $\rho^\flat_\Sigma[\phi,\phi] \leq \widetilde{T}$. This shows that $\rho^\flat \in \widetilde{R}^{\mathcal{I}}$. ■

Given any theory family in $\mathrm{ThFam}^{\natural}(\mathcal{I})$, the relation system $R^{\mathcal{I}\natural}(T)$ is a reflexive relation system on $\mathbf{F}$. This forms an analog of Lemma 773.

Lemma 1207 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then, for all* $T \in \mathrm{ThFam}^{\natural}(\mathcal{I})$*,* $R^{\mathcal{I}\natural}(T)$ *is a reflexive relation system on* $\mathbf{F}$*.*

Proof: Let $T \in \mathrm{ThFam}^{\natural}(\mathcal{I})$. By Lemma 93, $R^{\mathcal{I}\natural}(T)$ is a relation system on $\mathbf{F}$. For reflexivity, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. By the definition of the narrow reflexive core, $R^{\mathcal{I}\natural}_\Sigma[\phi,\phi] \leq T$. Thus, $\langle\phi,\phi\rangle \in R^{\mathcal{I}\natural}_\Sigma(T)$ and, therefore, $R^{\mathcal{I}\natural}(T)$ is reflexive. ■

As in Lemma 775, it may also be established that $R^{\mathcal{I}\natural}(T)$ is a symmetric relation system on $\mathbf{F}$, for all $T \in \mathrm{ThFam}^{\natural}(\mathcal{I})$.

Lemma 1208 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then, for all* $T \in \mathrm{ThFam}^{\natural}(\mathcal{I})$*,* $R^{\mathcal{I}\natural}(T)$ *is a symmetric relation system on* $\mathbf{F}$*.*

Proof: Let $T \in \mathrm{ThFam}^{\natural}(\mathcal{I})$. Again, Lemma 93 shows that $R^{\mathcal{I}\natural}(T)$ is a relation system. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle\phi,\psi\rangle \in R^{\mathcal{I}\natural}_\Sigma(T)$. Equivalently, $R^{\mathcal{I}\natural}_\Sigma[\phi,\psi] \leq T$. Consider any $\rho^\flat \in R^{\mathcal{I}\natural}$. By the definition of

$R^{\mathcal{I}\natural}$, we get that $\overline{\rho^\flat} \in R^{\mathcal{I}\natural}$. Therefore, by the hypothesis, $\overline{\rho^\flat}_\Sigma[\phi,\psi] \leq T$. But this gives $\rho^\flat_\Sigma[\psi,\phi] \leq T$. Since this holds for all $\rho^\flat \in R^{\mathcal{I}\natural}$, we conclude that $R^{\mathcal{I}\natural}_\Sigma[\psi,\phi] \leq T$. Hence, $\langle\psi,\phi\rangle \in R^{\mathcal{I}\natural}_\Sigma(T)$. Therefore, $R^{\mathcal{I}\natural}(T)$ is a symmetric relation system on $\mathbf{F}$. ∎

Continuing the study of sequence of properties of $R^{\mathcal{I}\natural}(T)$, we show that, for all theory families $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, $R^{\mathcal{I}\natural}(T)$ has the compatibility property in $\mathbf{F}$.

Lemma 1209 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Then, for all $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, $R^{\mathcal{I}\natural}(T)$ has the compatibility property in $\mathbf{F}$.*

Proof: Let $T \in \mathrm{ThFam}^\natural(\mathcal{I})$. We rely on Corollary 12. Let $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ is in $N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle\phi,\psi\rangle \in R^{\mathcal{I}\natural}_\Sigma(T)$ or, equivalently, $R^{\mathcal{I}\natural}_\Sigma[\phi,\psi] \leq T$. Let $\rho^\flat : (\mathrm{SEN}^\flat)^n \to \mathrm{SEN}^\flat$ be arbitrary in $R^{\mathcal{I}\natural}$. We consider the natural transformation $\rho'^\flat : (\mathrm{SEN}^\flat)^{n+k} \to \mathrm{SEN}^\flat$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\zeta,\eta,\vec{\chi},\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$, by

$$\rho'^\flat_\Sigma(\zeta,\eta,\vec{\chi},\vec{\xi}) = \rho^\flat_\Sigma(\sigma^\flat_\Sigma(\zeta,\vec{\chi}),\sigma^\flat_\Sigma(\eta,\vec{\chi}),\vec{\xi}).$$

Note that, since $\sigma^\flat \in N^\flat$, $\rho^\flat \in N^\flat$ and

$$\begin{aligned}\rho'^\flat \;=\; & \rho^\flat \circ \langle \sigma^\flat \circ \langle p^{n+k,0}, p^{n+k,2},\ldots,p^{n+k,k}\rangle, \sigma^\flat \circ \langle p^{n+k,1}, p^{n+k,2},\ldots,p^{n+k,k}\rangle,\\ & p^{n+k,k+1},\ldots,p^{n+k,n+k-1}\rangle,\end{aligned}$$

we get that $\rho'^\flat \in N^\flat$. Moreover, for all $T' \in \mathrm{ThFam}^\natural(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\zeta,\vec{\chi},\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{aligned}\rho'^\flat_\Sigma(\zeta,\zeta,\vec{\chi},\vec{\xi}) &= \rho^\flat_\Sigma(\sigma^\flat_\Sigma(\zeta,\vec{\chi}),\sigma^\flat_\Sigma(\zeta,\vec{\chi}),\vec{\xi}) \quad \text{(by definition of } \rho'^\flat)\\ &\in T'_\Sigma. \quad (\text{since } \rho^\flat \in R^{\mathcal{I}\natural}).\end{aligned}$$

Thus, by the definition of the narrow reflexive core, we get that $\rho'^\flat \in R^{\mathcal{I}\natural}$.

Now since $\rho'^\flat \in R^{\mathcal{I}\natural}$ and, by hypothesis, $R^{\mathcal{I}\natural}_\Sigma[\phi,\psi] \leq T$, we get, in particular, that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\chi},\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}),\vec{\xi}) \in T_{\Sigma'}.$$

Hence, a fortiori, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\sigma^\flat_\Sigma(\phi,\vec{\chi})),\mathrm{SEN}^\flat(f)(\sigma^\flat_\Sigma(\psi,\vec{\chi})),\vec{\xi}) \in T_{\Sigma'}.$$

This proves that

$$\rho^\flat_\Sigma[\sigma^\flat_\Sigma(\phi,\vec{\chi}),\sigma^\flat_\Sigma(\psi,\vec{\chi})] \leq T.$$

Since this holds for all $\rho^\flat \in R^{\mathcal{I}\natural}$, we get that $R^{\mathcal{I}\natural}_\Sigma[\sigma^\flat_\Sigma(\phi,\vec{\chi}),\sigma^\flat_\Sigma(\psi,\vec{\chi})] \leq T$ or, equivalently, $\langle\sigma^\flat_\Sigma(\phi,\vec{\chi}),\sigma^\flat_\Sigma(\psi,\vec{\chi})\rangle \in R^{\mathcal{I}\natural}_\Sigma(T)$. Therefore, $R^{\mathcal{I}\natural}(T)$ has the congruence compatibility property in $\mathbf{F}$. ∎

We now show, in an analog of Theorem 799, that possession of the narrow family modus ponens by the narrow reflexive core intrinsically characterizes syntactic narrow family monotonicity. We start by showing that possession of the narrow family MP by the narrow reflexive core is necessary for syntactic narrow family monotonicity. This forms an analog of Theorem 796.

Theorem 1210 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically narrowly family monotone, then* $R^{\mathcal{I}\nmid}$ *has the narrow family MP.*

Proof: Suppose that $\mathcal{I}$ is syntactically narrowly family monotone with witnessing transformations $I^\flat$. Since, by definition, $I^\flat$ is narrowly family reflexive, we get, by definition of $R^{\mathcal{I}\nmid}$, $I^\flat \subseteq R^{\mathcal{I}\nmid}$. Thus, since $I^\flat$ has narrow family MP in $\mathcal{I}$, we get that, a fortiori, $R^{\mathcal{I}\nmid}$ also satisfies the narrow family MP. ∎

Possession of narrow family MP by $R^{\mathcal{I}\nmid}$ implies that $R^{\mathcal{I}\nmid}$ has the narrow family transitivity in $\mathcal{I}$. This proposition forms an analog of Proposition 797.

Proposition 1211 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}\nmid}$ *has the narrow family MP, then it also has the narrow family transitivity in* $\mathcal{I}$*.*

Proof: Suppose that $R^{\mathcal{I}\nmid}$ has the narrow family MP and let $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle, \langle \psi, \chi \rangle \in R^{\mathcal{I}\nmid}_\Sigma(T)$. This means that $R^{\mathcal{I}\nmid}_\Sigma[\phi, \psi] \leq T$ and $R^{\mathcal{I}\nmid}_\Sigma[\psi, \chi] \leq T$. Then, by Lemma 1209, we get that, for all $\rho^\flat \in R^{\mathcal{I}\nmid}$, and all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$R^{\mathcal{I}\nmid}_{\Sigma'}[\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), \vec{\xi}), \rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi})] \leq T.$$

But, by hypothesis, $R^{\mathcal{I}\nmid}_\Sigma[\phi, \psi] \leq T$ and $R^{\mathcal{I}\nmid}$ has the narrow family MP. Therefore, for all $\rho^\flat \in R^{\mathcal{I}\nmid}$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi}) \subseteq T_{\Sigma'},$$

i.e., $R^{\mathcal{I}\nmid}_\Sigma[\phi, \chi] \leq T$. This shows $\langle \phi, \chi \rangle \in R^{\mathcal{I}\nmid}_\Sigma(T)$ and, hence, $R^{\mathcal{I}\nmid}$ is narrowly family transitive in $\mathcal{I}$. ∎

We are now ready to show a converse of Theorem 1210, i.e., that possession of the narrow family MP by $R^{\mathcal{I}\nmid}$ suffices to establish the syntactic narrow family monotonicity of $\mathcal{I}$, since, in that case, $R^{\mathcal{I}\nmid}$ serves as a family of witnessing transformations. The following constitutes an analog of Theorem 798.

Theorem 1212 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $R^{\mathcal{I}\natural}$ *has the narrow family MP, then* $\mathcal{I}$ *is syntactically narrowly family monotone, with witnessing transformations* $R^{\mathcal{I}\natural}$.

Proof: By Lemma 1207, $R^{\mathcal{I}\natural}$ is narrowly family reflexive in $\mathcal{I}$. By Lemma 1208, $R^{\mathcal{I}\natural}$ is narrowly family symmetric in $\mathcal{I}$. By hypothesis and Proposition 1211, it is narrowly family transitive in $\mathcal{I}$. By Lemma 1209 it has the narrow family compatibility property in $\mathcal{I}$. Finally, by hypothesis, it has the narrow family MP in $\mathcal{I}$. We conclude that $\mathcal{I}$ is syntactically narrowly family monotone, with witnessing transformations $R^{\mathcal{I}\natural}$. ■

Theorems 1210 and 1212 provide the promised characterization of syntactic narrow family monotonicity in terms of the narrow family MP of the narrow reflexive core.

$$\begin{array}{c}\mathcal{I} \text{ is Syntactically Narrow}\\ \text{Family Monotone}\end{array} \longleftrightarrow \begin{array}{c}R^{\mathcal{I}\natural} \text{ has Narrow Family}\\ \text{Modus Ponens}\end{array}.$$

Theorem 1213 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically narrowly family monotone if and only if* $R^{\mathcal{I}\natural}$ *has the narrow family MP in* $\mathcal{I}$.

Proof: Theorem 1210 gives the "only if" and the "if" is by Theorem 1212. ■

A related alternative characterization asserts that syntactic narrow family monotonicity amounts to the narrow definability of Leibniz congruence systems of theory families by the narrow reflexive core. This result forms an analog of Theorem 801.

$$\begin{array}{c}\mathcal{I} \text{ is Syntactically Narrow}\\ \text{Family Monotone}\end{array} \longleftrightarrow \begin{array}{c}R^{\mathcal{I}\natural} \text{ Defines Leibniz Congruence}\\ \text{Systems of Theory Families}\end{array}.$$

Theorem 1214 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically narrowly family monotone if and only if, for all* $T \in \mathrm{ThFam}^\natural(\mathcal{I})$,

$$\Omega(T) = R^{\mathcal{I}\natural}(T).$$

Proof: If $\mathcal{I}$ is syntactically narrowly family monotone, then, by Theorem 1210, $R^{\mathcal{I}\natural}$ has the narrow family MP in $\mathcal{I}$. Thus, by Theorem 1212, $R^{\mathcal{I}\natural}$ is a family of witnessing transformations for the syntactic narrow family monotonicity of $\mathcal{I}$. Thus, by Corollary 1204, for all $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, $\Omega(T) = R^{\mathcal{I}\natural}(T)$.

Suppose, conversely, that the displayed condition holds. Then $R^{\mathcal{I}\nparallel}$ is narrowly family reflexive, narrowly family transitive and has the narrow family compatibility property and the narrow family MP. Hence, it constitutes a collection of witnessing transformations and, therefore, $\mathcal{I}$ is syntactically narrowly family monotone. ∎

In the case of syntactic protoalgebraicity, in Chapter 11, it was shown that the property that separates syntactic protoalgebraicity from protoalgebraicity is the Leibniz compatibility property with respect to the theory family generated by the reflexive core, i.e., the property that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi])).$$

The task of characterizing those π-institutions that are syntactically narrowly family monotone among those that are narrowly family monotone is more involved. The additional complications arise from the fact that the class of theory families $\mathrm{ThFam}^{\nparallel}(\mathcal{I})$ may not be, in general, closed under (signature-wise) intersections and, hence, may not possess a least element. Therefore, to pinpoint syntactic narrow family monotonicity inside the class of narrow family monotone π-institutions, we need to devise a suitable analog of the Leibniz compatibility property with respect to the theory family generated by the narrow reflexive core.

To introduce this analog and to understand how it comes about and how it extends the Leibniz property, we interject a small discussion. Recall that a π-institution $\mathcal{I}$ is *protoalgebraic* if its Leibniz operator is monotone on theory families. Recall, also, that its reflexive core $R^{\mathcal{I}}$ is said to be *Leibniz* if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi])).$$

If a π-institution is protoalgebraic and has a Leibniz reflexive core, then it satisfies the global family modus ponens. This was shown in Chapter 11 using the following method. Considering $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $R^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T$, we get

- $\langle \phi, \psi \rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi]))$ first, by applying the Leibniz property;
- $\Omega(C(R^{\mathcal{I}}_\Sigma[\phi, \psi])) \leq \Omega(T)$, by applying the hypothesis that $R^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T$ and the postulated protoalgebraicity of $\mathcal{I}$.

However, in case of narrow family monotonicity, the plausibility of $R^{\mathcal{I}\nparallel}_\Sigma[\phi, \psi]$ having some empty components makes it likely that, in the second stage, narrow family monotonicity may not be applicable to ensure the inclusion $\Omega(C(R^{\mathcal{I}\nparallel}_\Sigma[\phi, \psi])) \leq \Omega(T)$.

An obvious remedy is to restrict attention to those π-institutions in which $C(R^{\mathcal{I}\nparallel}_\Sigma[\phi, \psi]) \in \mathrm{ThFam}^{\nparallel}(\mathcal{I})$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, and leave the Leibniz property unaltered. A more relaxed approach is to assume that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, the poset

$$[R^{\mathcal{I}\natural}_\Sigma[\phi,\psi]) \coloneqq \{T \in \mathrm{ThFam}^\natural(\mathcal{I}) : R^{\mathcal{I}\natural}_\Sigma[\phi,\psi] \leq T\}$$

satisfies the descending chain condition and to postulate that every minimal element $T \in [R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$ satisfies $\langle\phi,\psi\rangle \in \Omega_\Sigma(T)$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.

- For $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, define

$$[R^{\mathcal{I}\natural}_\Sigma[\phi,\psi]) \coloneqq \{T \in \mathrm{ThFam}^\natural(\mathcal{I}) : R^{\mathcal{I}\natural}_\Sigma[\phi,\psi] \leq T\};$$

- For $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, $\mathcal{I}$ is called $\langle\Sigma,\phi,\psi\rangle$-**reflexively covered** if, for every theory family $T \in [R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$, there exists minimal $T' \in [R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$, such that $T' \leq T$;
- $\mathcal{I}$ is called **reflexively covered** if it is $\langle\Sigma,\phi,\psi\rangle$-reflexively covered, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$.

Given $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, we write

$$\min[R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$$

for the collection of minimal elements in $[R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. We say that the narrow reflexive core $R^{\mathcal{I}\natural}$ of $\mathcal{I}$ is **Leibniz** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$ and all $T \in \min[R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$,

$$\langle\phi,\psi\rangle \in \Omega_\Sigma(T).$$

We show, in an analog of Proposition 785, that, if $R^{\mathcal{I}\natural}$ has the narrow family MP, then it is Leibniz. In fact, the proof demonstrates that, under the narrow family MP, a stronger property than that of being Leibniz holds; more concretely, that for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$ and all $T \in [R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$,

$$\langle\phi,\psi\rangle \in \Omega_\Sigma(T).$$

Proposition 1215 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $R^{\mathcal{I}\natural}$ *has the narrow family MP, then for all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$ *and all* $T \in [R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$*,* $\langle\phi,\psi\rangle \in \Omega_\Sigma(T)$.

Proof: Suppose $R^{\mathcal{I}\natural}$ has the narrow family MP and let $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $R^{\mathcal{I}\natural}_\Sigma[\phi,\psi] \leq T$. To verify that $\langle\phi,\psi\rangle \in \Omega_\Sigma(T)$, we use Theorem 19. Let $\sigma^\flat \in N^\flat$, $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and

$\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$, such that $\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}) \in T_{\Sigma'}$. Since $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, by Lemma 1209,

$$R^{\mathcal{I}\nleq}_{\Sigma'}[\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi})] \leq T.$$

Thus, since, by hypothesis, $R^{\mathcal{I}\nleq}$ has the narrow family MP, we obtain

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}) \in T_{\Sigma'}.$$

By symmetry, we conclude that, for all $\sigma^\flat \in N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}) \in T_{\Sigma'} \quad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}) \in T_{\Sigma'}.$$

Hence, by Theorem 19, $\langle\phi,\psi\rangle \in \Omega_\Sigma(T)$ and, therefore, $R^{\mathcal{I}\nleq}$ is Leibniz. ■

Corollary 1216 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $R^{\mathcal{I}\nleq}$ has the narrow family MP, then it is Leibniz.*

Proof: Directly by Proposition 1215. ■

In the opposite direction, when dealing with reflexively covered π-institutions, we may show that narrow family monotonicity combined with the Leibniz property of the narrow reflexive core imply that the narrow reflexive core has the narrow family modus ponens in $\mathcal{I}$.

Proposition 1217 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a reflexively covered, narrowly family monotone π-institution based on $\mathbf{F}$. If $R^{\mathcal{I}\nleq}$ is Leibniz, then it has the narrow family MP in $\mathcal{I}$.*

Proof: Let $\mathcal{I}$ be a reflexively covered π-institution. Suppose that $\mathcal{I}$ is narrowly family monotone and that $R^{\mathcal{I}\nleq}$ is Leibniz. Let $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $R^{\mathcal{I}\nleq}_\Sigma[\phi,\psi] \leq T$. Since $\mathcal{I}$ is reflexively covered, there exists $T' \in \min[R^{\mathcal{I}\nleq}_\Sigma[\phi,\psi]]$, such that $T' \leq T$. Now we have

$$\begin{array}{lll}\langle\phi,\psi\rangle & \in & \Omega_\Sigma(T') \quad (\text{since } R^{\mathcal{I}\nleq} \text{ is Leibniz and } T' \in \min[R^{\mathcal{I}\nleq}_\Sigma[\phi,\psi]])\\ & \subseteq & \Omega_\Sigma(T). \quad (\text{since } T' \leq T \text{ and } \mathcal{I} \text{ is narrowly family monotone})\end{array}$$

Therefore, since $\phi \in T_\Sigma$, we get, by the compatibility of $\Omega(T)$ with T, that $\psi \in T_\Sigma$. We conclude that $R^{\mathcal{I}\nleq}$ has the narrow family MP in $\mathcal{I}$. ■

Thus, at least for reflexively covered π-institutions, it is possible to show that the class of syntactically narrowly monotone ones inside the class of the narrowly monotone ones can be characterized exactly by the Leibniz property of the narrow reflexive core. This forms a partial analog of Theorem 805 in the narrow context.

Theorem 1218 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a reflexively covered* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically narrowly family monotone if and only if it is narrowly family monotone and has a Leibniz narrow reflexive core.*

Proof: Let $\mathcal{I}$ be a reflexively covered π-institution.

Suppose, first, that $\mathcal{I}$ is syntactically narrowly family monotone. Then it is narrowly family monotone by Theorem 1205. Moreover, its narrow reflexive core has the narrow family MP by Theorem 1210 and, hence, by Corollary 1216, its narrow reflexive core is Leibniz.

Suppose, conversely, that $\mathcal{I}$ is narrowly family monotone with a Leibniz narrow reflexive core. Then, by Proposition 1217, its narrow reflexive core has the narrow family MP and, therefore, by Theorem 1212, $\mathcal{I}$ is syntactically narrowly family monotone, with witnessing transformations $R^{\mathcal{I}\nmid}$. ■

15.2 Syntactic Narrow System Monotonicity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

Recall that $\mathcal{I}$ is **narrowly system monotone** if, for all $T, T' \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$,

$$T \le T' \quad \text{implies} \quad \Omega(T) \le \Omega(T').$$

In this section, in analogy with Section 15.1, we introduce and study a syntactic analog of this concept.

First, the concepts of narrow family reflexivity, narrow family symmetry, narrow family transitivity, narrow family compatibility and narrow family modus ponens can all be relativized to $\mathrm{ThSys}^{\nmid}(\mathcal{I})$.

Let, as above, $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Moreover, suppose that $I^\flat \subseteq N^\flat$ is a collection of natural transformations in $N^\flat$, with two distinguished arguments.

- $I^\flat$ is **narrowly system reflexive** if, for all $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,
$$I^\flat_\Sigma[\phi, \phi] \le T;$$
- $I^\flat$ is **narrowly system symmetric** if, for all $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$I^\flat_\Sigma[\phi, \psi] \le T \quad \text{implies} \quad I^\flat_\Sigma[\psi, \phi] \le T;$$
- $I^\flat$ is **narrowly system transitive** if, for all $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$,
$$I^\flat_\Sigma[\phi, \psi] \cup I^\flat_\Sigma[\psi, \chi] \le T \quad \text{implies} \quad I^\flat_\Sigma[\phi, \chi] \le T;$$

- $I^\flat$ is **narrowly system compatible** if, for all $T \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$, all $\sigma^\flat \in N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,
$$\bigcup_{i<k} \overleftrightarrow{I^\flat}_\Sigma[\phi_i, \psi_i] \leq T \quad \text{implies} \quad I^\flat_\Sigma[\sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi})] \leq T;$$

- $I^\flat$ has the **narrow system MP** if, for all $T \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi \in T_\Sigma \quad \text{and} \quad I^\flat_\Sigma[\phi, \psi] \leq T \quad \text{imply} \quad \psi \in T_\Sigma.$$

We say that $\mathcal{I}$ is **syntactically narrowly system monotone** if there exists $I^\flat \subseteq N^\flat$, with two distinguished arguments, such that $I^\flat$ satisfies:

- narrow system reflexivity;
- narrow system transitivity;
- narrow system compatibility; and
- narrow system MP.

In that case, we call $I^\flat$ a **set of witnessing natural transformations**, or, more simply, **witnessing transformations** (of the syntactic narrow system monotonicity of $\mathcal{I}$).

It turns out that, if $\mathcal{I}$ is a syntactically narrowly system monotone π-institution, with witnessing transformations $I^\flat$, then $\overleftrightarrow{I^\flat}(T)$ is a congruence system on $\mathbf{F}$ compatible with T, for all $T \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$. This forms a system analog of Proposition 1203.

Proposition 1219 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically narrowly system monotone, with witnessing transformations* $I^\flat$*, then, for all* $T \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$*,* $\overleftrightarrow{I^\flat}(T)$ *is a congruence system on* $\mathbf{F}$ *compatible with* T*.*

Proof: The proof is similar to that of Proposition 1203. Let $T \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$. The narrow system reflexivity of $I^\flat$ ensures that $\langle \phi, \phi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$. The fact that $\overleftrightarrow{I^\flat}$ is the symmetrization of $I^\flat$ ensures that $\langle \phi, \psi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$ implies that $\langle \psi, \phi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$. The narrow system transitivity of $I^\flat$ guarantees that $\langle \phi, \psi \rangle, \langle \psi, \chi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$ imply $\langle \phi, \chi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$.

Suppose, next, that $\sigma^\flat \in N^\flat$, $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$. Then, the narrow system compatibility of $I^\flat$ ensures that, if, for all $i < k$, $\langle \phi_i, \psi_i \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$, then $\langle \sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi}) \rangle \in I^\flat_\Sigma(T)$. Thus, $\overleftrightarrow{I^\flat}(T)$ is a congruence family on $\mathbf{F}$. However, by Lemma 93, $\overleftrightarrow{I^\flat}(T)$ is a relation system on $\mathbf{F}$. Hence, $\overleftrightarrow{I^\flat}(T)$ is a congruence system on $\mathbf{F}$.

It only remains to show that $\overleftrightarrow{I^\flat}(T)$ is compatible with T. Assume that $\phi \in T_\Sigma$ and $\langle\phi,\psi\rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$. Since $I^\flat \subseteq \overleftrightarrow{I^\flat}$, we get, by the narrow system MP of $I^\flat$, that $\psi \in T_\Sigma$. Thus, $\overleftrightarrow{I^\flat}(T)$ is also compatible with T. ■

Proposition 1219 shows that $\overleftrightarrow{I^\flat}$ defines Leibniz congruence systems of theory systems in $\mathrm{ThSys}^{\nleq}(\mathcal{I})$. Again, following terminology adopted in Section 15.1, we say that $I^\flat$ **narrowly defines Leibniz congruence systems** of theory systems in $\mathcal{I}$ if, for all $T \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$,

$$\overleftrightarrow{I^\flat}(T) = \Omega(T).$$

Then, in what is an analog of Corollary 1204, we obtain

Corollary 1220 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically narrowly system monotone, with witnessing transformations* $I^\flat$*, then* $I^\flat$ *narrowly defines Leibniz congruence systems of theory systems in* $\mathcal{I}$*.*

Proof: By Proposition 1219 and Corollary 98. ■

Corollary 1220 shows that syntactic narrow system monotonicity implies (semantic) narrow system monotonicity. This forms an analog of Theorem 1205.

Theorem 1221 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically narrowly system monotone, then it is narrowly system monotone.*

Proof: Suppose that $\mathcal{I}$ is syntactically narrowly system monotone with witnessing transformations $I^\flat$. Let $T, T' \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$, such that $T \leq T'$. Then

$$\begin{array}{rcll}\Omega(T) & = & \overleftrightarrow{I^\flat}(T) & \text{(by Corollary 1220)}\\ & \leq & \overleftrightarrow{I^\flat}(T') & \text{(by Lemma 94)}\\ & = & \Omega(T'). & \text{(by Corollary 1220)}\end{array}$$

Thus, $\mathcal{I}$ is narrowly system monotone. ■

We now introduce the notion of the narrow reflexive system core of a π-institution $\mathcal{I}$ in a way analogous to the narrow reflexive core, which was introduced in Section 15.1. Its introduction will enable us to provide a characterization of the syntactical narrow system monotonicity property and to establish a relationship between this property and its semantic counterpart.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The **narrow reflexive system core** of $\mathcal{I}$ is the collection

$$\begin{aligned} R^{\mathcal{I}s} \;=\; & \{\rho^\flat \in N^\flat : (\forall T \in \mathrm{ThSys}^\nleq(\mathcal{I}))(\forall \Sigma \in |\mathbf{Sign}^\flat|) \\ & \qquad (\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\rho^\flat_\Sigma[\phi, \phi] \le T)\}. \end{aligned}$$

Given any theory system in $\mathrm{ThSys}^\nleq(\mathcal{I})$, the relation system $R^{\mathcal{I}s}(T)$ is a reflexive relation system on $\mathbf{F}$. This forms an analog of Lemma 1207.

Lemma 1222 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *Then, for all* $T \in \mathrm{ThSys}^\nleq(\mathcal{I})$, $R^{\mathcal{I}s}(T)$ *is a reflexive relation system on* $\mathbf{F}$.

Proof: Let $T \in \mathrm{ThSys}^\nleq(\mathcal{I})$. By Lemma 93, $R^{\mathcal{I}s}(T)$ is a relation system on $\mathbf{F}$. For reflexivity, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. By the definition of the narrow reflexive system core, $R^{\mathcal{I}s}_\Sigma[\phi, \phi] \le T$. Thus, $\langle \phi, \phi \rangle \in R^{\mathcal{I}s}_\Sigma(T)$ and, therefore, $R^{\mathcal{I}s}(T)$ is reflexive. ■

As in Lemma 1208, we establish that $R^{\mathcal{I}s}(T)$ is a symmetric relation system on $\mathbf{F}$, for all $T \in \mathrm{ThSys}^\nleq(\mathcal{I})$.

Lemma 1223 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *Then, for all* $T \in \mathrm{ThSys}^\nleq(\mathcal{I})$, $R^{\mathcal{I}s}(T)$ *is a symmetric relation system on* $\mathbf{F}$.

Proof: Let $T \in \mathrm{ThSys}^\nleq(\mathcal{I})$. Again, Lemma 93 shows that $R^{\mathcal{I}s}(T)$ is a relation system. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in R^{\mathcal{I}s}_\Sigma(T)$. Equivalently, $R^{\mathcal{I}s}_\Sigma[\phi, \psi] \le T$. Consider any $\rho^\flat \in R^{\mathcal{I}s}$. By the definition of $R^{\mathcal{I}s}$, we get that $\overline{\rho^\flat} \in R^{\mathcal{I}s}$. Therefore, by the hypothesis, $\overline{\rho^\flat}_\Sigma[\phi, \psi] \le T$. But this gives $\rho^\flat_\Sigma[\psi, \phi] \le T$. Since this holds for all $\rho^\flat \in R^{\mathcal{I}s}$, we conclude that $R^{\mathcal{I}s}_\Sigma[\psi, \phi] \le T$. Hence, $\langle \psi, \phi \rangle \in R^{\mathcal{I}s}_\Sigma(T)$. Therefore, $R^{\mathcal{I}s}(T)$ is a symmetric relation system on $\mathbf{F}$. ■

We now show that, for all theory systems $T \in \mathrm{ThSys}^\nleq(\mathcal{I})$, $R^{\mathcal{I}s}(T)$ has the compatibility property in $\mathbf{F}$. This forms an analog of Lemma 1209.

Lemma 1224 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *Then, for all* $T \in \mathrm{ThSys}^\nleq(\mathcal{I})$, $R^{\mathcal{I}s}(T)$ *has the compatibility property in* $\mathbf{F}$.

Proof: Let $T \in \mathrm{ThSys}^\nleq(\mathcal{I})$. We rely on Corollary 12. Let $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ is in $N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in R^{\mathcal{I}s}_\Sigma(T)$ or, equivalently, $R^{\mathcal{I}s}_\Sigma[\phi, \psi] \le T$. Let $\rho^\flat : (\mathrm{SEN}^\flat)^n \to \mathrm{SEN}^\flat$ be arbitrary in $R^{\mathcal{I}s}$. We consider the natural transformation $\rho'^\flat : (\mathrm{SEN}^\flat)^{n+k} \to \mathrm{SEN}^\flat$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\zeta, \eta, \vec{\chi}, \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$, by

$$\rho'^\flat_\Sigma(\zeta, \eta, \vec{\chi}, \vec{\xi}) = \rho^\flat_\Sigma(\sigma^\flat_\Sigma(\zeta, \vec{\chi}), \sigma^\flat_\Sigma(\eta, \vec{\chi}), \vec{\xi}).$$

Note that, since $\sigma^\flat \in N^\flat$, $\rho^\flat \in N^\flat$ and

$$\begin{array}{rcl}\rho'^\flat & = & \rho^\flat \circ \langle \sigma^\flat \circ \langle p^{n+k,0}, p^{n+k,2}, \dots, p^{n+k,k}\rangle, \sigma^\flat \circ \langle p^{n+k,1}, p^{n+k,2}, \dots, p^{n+k,k}\rangle, \\ & & p^{n+k,k+1}, \dots, p^{n+k,n+k-1}\rangle,\end{array}$$

we get that $\rho'^\flat \in N^\flat$. Moreover, for all $T' \in \mathrm{ThSys}^\nmid(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\zeta, \vec{\chi}, \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rcll}\rho'^\flat_\Sigma(\zeta, \zeta, \vec{\chi}, \vec{\xi}) & = & \rho^\flat_\Sigma(\sigma^\flat_\Sigma(\zeta, \vec{\chi}), \sigma^\flat_\Sigma(\zeta, \vec{\chi}), \vec{\xi}) & \text{(by definition of } \rho'^\flat) \\ & \in & T'_\Sigma. \quad (\text{since } \rho^\flat \in R^{\mathcal{I}s}). &\end{array}$$

Thus, by the definition of the narrow reflexive system core, we get that $\rho'^\flat \in R^{\mathcal{I}s}$.

Now since $\rho'^\flat \in R^{\mathcal{I}s}$ and, by hypothesis, $R^{\mathcal{I}s}_\Sigma[\phi, \psi] \leq T$, we get, in particular, that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi}, \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}), \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}), \vec{\xi}) \in T_{\Sigma'}.$$

Hence, a fortiori, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\sigma^\flat_\Sigma(\phi, \vec{\chi})), \mathrm{SEN}^\flat(f)(\sigma^\flat_\Sigma(\psi, \vec{\chi})), \vec{\xi}) \in T_{\Sigma'}.$$

This proves that

$$\rho^\flat_\Sigma[\sigma^\flat_\Sigma(\phi, \vec{\chi}), \sigma^\flat_\Sigma(\psi, \vec{\chi})] \leq T.$$

Since this holds for all $\rho^\flat \in R^{\mathcal{I}s}$, we get that $R^{\mathcal{I}s}_\Sigma[\sigma^\flat_\Sigma(\phi, \vec{\chi}), \sigma^\flat_\Sigma(\psi, \vec{\chi})] \leq T$ or, equivalently, $\langle \sigma^\flat_\Sigma(\phi, \vec{\chi}), \sigma^\flat_\Sigma(\psi, \vec{\chi})\rangle \in R^{\mathcal{I}s}_\Sigma(T)$. Therefore, $R^{\mathcal{I}s}(T)$ has the congruence compatibility property in $\mathbf{F}$. ∎

We now show, in an analog of Theorem 1213, that possession of the narrow system modus ponens by the narrow reflexive system core intrinsically characterizes syntactic narrow system monotonicity. We start by showing that possession of the narrow system MP by the narrow reflexive core is necessary for syntactic narrow system monotonicity. This forms an analog of Theorem 1210.

Theorem 1225 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically narrowly system monotone, then* $R^{\mathcal{I}s}$ *has the narrow system MP.*

Proof: Suppose that $\mathcal{I}$ is syntactically narrowly system monotone with witnessing transformations $I^\flat$. Since, by definition, $I^\flat$ is narrowly system reflexive, we get, by definition of $R^{\mathcal{I}s}$, $I^\flat \subseteq R^{\mathcal{I}s}$. Thus, since $I^\flat$ has the narrow system MP in $\mathcal{I}$, we get that, a fortiori, $R^{\mathcal{I}s}$ also satisfies the narrow system MP. ∎

If $R^{\mathcal{I}s}$ has the narrow system MP, then it has the narrow system transitivity in $\mathcal{I}$. This proposition forms an analog of Proposition 1211.

Proposition 1226 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $R^{\mathcal{I}s}$ *has the narrow system MP, then it also has the narrow system transitivity in* $\mathcal{I}$.

Proof: Suppose that $R^{\mathcal{I}s}$ has the narrow system MP and let $T \in \mathrm{ThSys}^\nmid(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle, \langle \psi, \chi \rangle \in R^{\mathcal{I}s}_\Sigma(T)$. This means that $R^{\mathcal{I}s}_\Sigma[\phi, \psi] \le T$ and $R^{\mathcal{I}s}_\Sigma[\psi, \chi] \le T$. Then, by Lemma 1224, we get that, for all $\rho^\flat \in R^{\mathcal{I}s}$, and all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\begin{aligned} R^{\mathcal{I}s}_{\Sigma'}[\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), \vec{\xi}),& \\ \rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi})] &\le T. \end{aligned}$$

But, by hypothesis, $R^{\mathcal{I}s}_\Sigma[\phi, \psi] \le T$ and $R^{\mathcal{I}s}$ has the narrow system MP. Therefore, for all $\rho^\flat \in R^{\mathcal{I}s}$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi}) \subseteq T_{\Sigma'},$$

i.e., $R^{\mathcal{I}s}_\Sigma[\phi, \chi] \le T$. This shows $\langle \phi, \chi \rangle \in R^{\mathcal{I}s}_\Sigma(T)$ and, hence, $R^{\mathcal{I}s}$ is narrowly system transitive in $\mathcal{I}$. ∎

We are now ready to show a converse of Theorem 1225, i.e., that possession of the narrow system MP by $R^{\mathcal{I}s}$ suffices to establish the syntactic narrow system monotonicity of $\mathcal{I}$, since, in that case, $R^{\mathcal{I}s}$ serves as a family of witnessing transformations. The following constitutes an analog of Theorem 1212.

Theorem 1227 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $R^{\mathcal{I}s}$ *has the narrow system MP, then* $\mathcal{I}$ *is syntactically narrowly system monotone, with witnessing transformations* $R^{\mathcal{I}s}$.

Proof: By Lemma 1222, $R^{\mathcal{I}s}$ is narrowly system reflexive in $\mathcal{I}$. By Lemma 1223, $R^{\mathcal{I}s}$ is narrowly system symmetric in $\mathcal{I}$. By hypothesis and Proposition 1226, it is narrowly system transitive in $\mathcal{I}$. By Lemma 1224 it has the narrow system compatibility property in $\mathcal{I}$. Finally, by hypothesis, it has the narrow system MP in $\mathcal{I}$. We conclude that $\mathcal{I}$ is syntactically narrowly system monotone, with witnessing transformations $R^{\mathcal{I}s}$. ∎

Theorems 1225 and 1227 provide the promised characterization of syntactic narrow system monotonicity in terms of the narrow system MP of the narrow reflexive system core.

$$\begin{array}{c} \mathcal{I} \text{ is Syntactically Narrow} \\ \text{System Monotone} \end{array} \longleftrightarrow \begin{array}{c} R^{\mathcal{I}s} \text{ has Narrow System} \\ \text{Modus Ponens} \end{array}.$$

Theorem 1228 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically narrowly system monotone if and only if* $R^{\mathcal{I}s}$ *has the narrow system MP in* $\mathcal{I}$.

Proof: Theorem 1225 gives the "only if" and the "if" is by Theorem 1227. ∎

A related alternative characterization asserts that syntactic narrow system monotonicity amounts to the narrow definability of Leibniz congruence systems of theory systems by the narrow reflexive system core. This result forms an analog of Theorem 1214.

$$\begin{array}{c} \mathcal{I} \text{ is Syntactically Narrow} \\ \text{System Monotone} \end{array} \longleftrightarrow \begin{array}{c} R^{\mathcal{I}s} \text{ Defines Leibniz Congruence} \\ \text{Systems of Theory Systems} \end{array}.$$

Theorem 1229 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically narrowly system monotone if and only if, for all* $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$,

$$\Omega(T) = R^{\mathcal{I}s}(T).$$

Proof: If $\mathcal{I}$ is syntactically narrowly system monotone, then, by Theorem 1225, $R^{\mathcal{I}s}$ has the narrow system MP in $\mathcal{I}$. Thus, by Theorem 1227, $R^{\mathcal{I}s}$ is a family of witnessing transformations for the syntactic narrow system monotonicity of $\mathcal{I}$. Thus, by Corollary 1220, for all $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, $\Omega(T) = R^{\mathcal{I}s}(T)$.

Suppose, conversely, that the displayed condition holds. Then $R^{\mathcal{I}s}$ is narrowly system reflexive, narrowly system transitive and has the narrow system compatibility property and the narrow system MP. Hence, it constitutes a collection of witnessing transformations and, therefore, $\mathcal{I}$ is syntactically narrowly system monotone. ∎

To prove an analog of Theorem 1218, which, in a certain restricted sense characterizes syntactic narrow family monotonicity inside the class of narrow family monotone π-institutions, we create a suitable analog of the Leibniz compatibility property with respect to the theory family generated by the narrow reflexive system core. Once more, the difficulty in this case, similarly with that described in some detail in Section 15.1, arises from the fact that $\mathrm{ThSys}^{\nmid}(\mathcal{I})$ may not be, in general, closed under signature-wise intersections.

To introduce this analog and to understand how it comes about and how it extends the Leibniz property, we elaborate further on the relevant discussion initiated in Section 15.1. Recall that a π-institution $\mathcal{I}$ is *prealgebraic* if its Leibniz operator is monotone on theory systems. Recall, also, once more, that its reflexive core $R^{\mathcal{I}}$ is said to be *Leibniz* if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi])).$$

If a π-institution is prealgebraic and has a Leibniz reflexive core, then it satisfies the global system modus ponens. This was shown in Chapter 11 using the following method. Considering $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $R^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T$, we get

- $\langle \phi, \psi \rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi]))$ first, by applying the Leibniz property;
- $\Omega(C(R^{\mathcal{I}}_\Sigma[\phi, \psi])) \leq \Omega(T)$, by applying the hypothesis that $R^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T$ and the postulated prealgebraicity of $\mathcal{I}$ and observing at the same time that $C(R^{\mathcal{I}}_\Sigma[\phi, \psi]) \in \mathrm{ThSys}(\mathcal{I})$, since $R^{\mathcal{I}}_\Sigma[\phi, \psi]$ is a sentence system.

However, in case of narrow system monotonicity, the plausibility of $R^{\mathcal{I}s}_\Sigma[\phi, \psi]$ having some empty components makes it likely that, in the second stage, narrow system monotonicity may not be applicable to ensure the inclusion $\Omega(C(R^{\mathcal{I}s}_\Sigma[\phi, \psi])) \leq \Omega(T)$. To deal with this plausibility, we assume, in a similar way as before, that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, the poset

$$[R^{\mathcal{I}s}_\Sigma[\phi, \psi]) \coloneqq \{T \in \mathrm{ThSys}^{\notin}(\mathcal{I}) : R^{\mathcal{I}s}_\Sigma[\phi, \psi] \leq T\}$$

satisfies the descending chain condition and to postulate that every minimal element $T \in [R^{\mathcal{I}s}_\Sigma[\phi, \psi])$ satisfies $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

- For $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, define
$$[R^{\mathcal{I}s}_\Sigma[\phi, \psi]) \coloneqq \{T \in \mathrm{ThSys}^{\notin}(\mathcal{I}) : R^{\mathcal{I}s}_\Sigma[\phi, \psi] \leq T\};$$
- For $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $\mathcal{I}$ is called **$\langle \Sigma, \phi, \psi \rangle$-reflexively system covered** if, for every theory system $T \in [R^{\mathcal{I}s}_\Sigma[\phi, \psi])$, there exists minimal $T' \in [R^{\mathcal{I}s}_\Sigma[\phi, \psi])$, such that $T' \leq T$;
- $\mathcal{I}$ is called **reflexively system covered** if it is $\langle \Sigma, \phi, \psi \rangle$-reflexively system covered, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$.

Given $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, we write

$$\min [R^{\mathcal{I}s}_\Sigma[\phi, \psi])$$

for the collection of minimal elements in $[R^{\mathcal{I}s}_\Sigma[\phi, \psi])$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the narrow reflexive system core $R^{\mathcal{I}s}$ of $\mathcal{I}$ is **Leibniz** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$ and all $T \in \min [R^{\mathcal{I}s}_\Sigma[\phi, \psi])$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(T).$$

We show, in an analog of Proposition 1215, that, if $R^{\mathcal{I}s}$ has the narrow system MP, then it is Leibniz. In fact, the proof demonstrates that, under the narrow system MP, a stronger property than that of being Leibniz holds; more concretely, that for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$ and all $T \in [R^{\mathcal{I}s}_\Sigma[\phi,\psi])$ (and not only for $T \in \min[R^{\mathcal{I}s}_\Sigma[\phi,\psi]))$,

$$\langle \phi, \psi\rangle \in \Omega_\Sigma(T).$$

Proposition 1230 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}s}$ *has the narrow system MP, then for all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$ *and all* $T \in [R^{\mathcal{I}s}_\Sigma[\phi,\psi])$*,* $\langle \phi, \psi\rangle \in \Omega_\Sigma(T)$*.*

Proof: Suppose $R^{\mathcal{I}s}$ has the narrow system MP and let $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $R^{\mathcal{I}s}_\Sigma[\phi,\psi] \leq T$. To verify that $\langle \phi, \psi\rangle \in \Omega_\Sigma(T)$, we use Theorem 19. Let $\sigma^\flat \in N^\flat$, $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$, such that $\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \in T_{\Sigma'}$. Since $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, by Lemma 1224,

$$R^{\mathcal{I}s}_{\Sigma'}[\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}), \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi})] \leq T.$$

Thus, since, by hypothesis, $R^{\mathcal{I}s}$ has the narrow system MP, we obtain

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}) \in T_{\Sigma'}.$$

By symmetry, we conclude that, for all $\sigma^\flat \in N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \in T_{\Sigma'} \quad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}) \in T_{\Sigma'}.$$

Hence, by Theorem 19, $\langle \phi, \psi\rangle \in \Omega_\Sigma(T)$. ■

Corollary 1231 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}s}$ *has the narrow system MP, then it is Leibniz.*

Proof: Directly by Proposition 1230. ■

In the opposite direction, when dealing with reflexively system covered π-institutions, we may show that narrow system monotonicity combined with the Leibniz property of the narrow reflexive system core imply that the narrow reflexive system core has the narrow system modus ponens in $\mathcal{I}$. The following proposition forms an analog of Proposition 1217 in the system context.

Proposition 1232 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a reflexively system covered, narrowly system monotone* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}s}$ *is Leibniz, then it has the narrow system MP in* $\mathcal{I}$*.*

Proof: Let $\mathcal{I}$ be a reflexively system covered π-institution. Suppose that $\mathcal{I}$ is narrowly system monotone and that $R^{\mathcal{I}s}$ is Leibniz. Let $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $R^{\mathcal{I}s}_\Sigma[\phi,\psi] \leq T$. Since $\mathcal{I}$ is reflexively system covered, there exists $T' \in \min[R^{\mathcal{I}s}_\Sigma[\phi,\psi])$, such that $T' \leq T$. Now we have

$$\begin{array}{lll}\langle\phi,\psi\rangle & \in & \Omega_\Sigma(T') \quad (\text{since } R^{\mathcal{I}s} \text{ is Leibniz and } T' \in \min[R^{\mathcal{I}s}_\Sigma[\phi,\psi])) \\ & \subseteq & \Omega_\Sigma(T). \quad (\text{since } T' \leq T \text{ and } \mathcal{I} \text{ is narrowly system monotone})\end{array}$$

Therefore, since $\phi \in T_\Sigma$, we get, by the compatibility of $\Omega(T)$ with T, that $\psi \in T_\Sigma$. We conclude that $R^{\mathcal{I}s}$ has the narrow system MP in $\mathcal{I}$. ■

Thus, at least for reflexively system covered π-institutions, it is possible to show that the class of syntactically narrowly system monotone ones inside the class of the narrowly system monotone ones can be characterized exactly by the Leibniz property of the narrow reflexive system core. This forms a partial analog of Theorem 1218.

Theorem 1233 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a reflexively system covered* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically narrowly system monotone if and only if it is narrowly system monotone and has a Leibniz narrow reflexive system core.*

Proof: Let $\mathcal{I}$ be a reflexively system covered π-institution.

Suppose, first, that $\mathcal{I}$ is syntactically narrowly system monotone. Then it is narrowly system monotone by Theorem 1221. Moreover, its narrow reflexive system core has the narrow system MP by Theorem 1225 and, hence, by Corollary 1231, its narrow reflexive system core is Leibniz.

Suppose, conversely, that $\mathcal{I}$ is narrowly system monotone with a Leibniz narrow reflexive system core. Then, by Proposition 1232, its narrow reflexive system core has the narrow system MP and, therefore, by Theorem 1227, $\mathcal{I}$ is syntactically narrowly system monotone, with witnessing transformations $R^{\mathcal{I}s}$. ■

15.3 Syntactic Narrow Right Monotonicity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $I^\flat \subseteq N^\flat$ a collection of natural transformations in $N^\flat$, with two distinguished arguments. Recall from Proposition 99, that, for all $T \in \mathrm{SenFam}(\mathbf{F})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma[\phi,\psi] \leq T \quad \text{iff} \quad I^\flat_\Sigma[\phi,\psi] \leq \overleftarrow{T}. \tag{15.1}$$

Let, now, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$. We may attempt to define "syntactic narrow left monotonicity" as the existence of a collection

$I^\flat \subseteq N^\flat$, with two distinguished arguments, such that, for all $T \in \mathrm{ThFam}^{\nsharp}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma[\phi,\psi] \leq \overleftarrow{T} \quad \text{iff} \quad \langle \phi,\psi\rangle \in \Omega_\Sigma(T).$$

Because of the the preceding remark, however, this condition would amount exactly to defining syntactic narrow family monotonicity. On the other hand, syntactic narrow system monotonicity is equivalent, again based on the remark above, to asserting the existence of $I^\flat \subseteq N^\flat$, with two distinguished arguments, such that, for all $T \in \mathrm{ThFam}^{\nsharp}(\mathcal{I})$, with $\overleftarrow{T} \in \mathrm{ThSys}^{\nsharp}(\mathcal{I})$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma[\phi,\psi] \leq T \quad \text{iff} \quad \langle \phi,\psi\rangle \in \Omega_\Sigma(\overleftarrow{T}).$$

If we drop the restriction that $\overleftarrow{T}$ be in $\mathrm{ThSys}^{\nsharp}(\mathcal{I})$, thus allowing the condition above to be imposed on the wider class of all $T \in \mathrm{ThFam}^{\nsharp}(\mathcal{I})$, we obtain a concept slightly more general that syntactic narrow system monotonicity, which we term *syntactic narrow right monotonicity*. We study this notion in more detail in this section, following the study of syntactic narrow family (and system) monotonicity, carried out in the preceding sections of the chapter.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Recall that $\mathcal{I}$ is **narrowly right monotone** if, for all $T, T' \in \mathrm{ThFam}^{\nsharp}(\mathcal{I})$,

$$T \leq T' \quad \text{implies} \quad \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'}).$$

In this section, following the work on syntactic narrow family monotonicity of Section 15.1, we introduce and study a syntactic analog of narrow right monotonicity.

First, the concepts of narrow system reflexivity, narrow system symmetry, narrow system transitivity, narrow system compatibility and narrow system modus ponens are recast to accommodate theory systems that arise by applying the arrow operator $\overleftarrow{\ }$ on theory families in $\mathrm{ThFam}^{\nsharp}(\mathcal{I})$. Note that such theory systems include, of course, all theory systems in $\mathrm{ThSys}^{\nsharp}(\mathcal{I})$, since these arise by applying the arrow operator on themselves.

Let, as above, $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Moreover, suppose that $I^\flat \subseteq N^\flat$ is a collection of natural transformations in $N^\flat$, with two distinguished arguments.

- $I^\flat$ is **narrowly right reflexive** if, for all $T \in \mathrm{ThFam}^{\nsharp}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma[\phi,\phi] \leq \overleftarrow{T};$$

- $I^\flat$ is **narrowly right symmetric** if, for all $T \in \mathrm{ThFam}^\nleq(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma[\phi,\psi] \leq \overleftarrow{T} \quad \text{implies} \quad I^\flat_\Sigma[\psi,\phi] \leq \overleftarrow{T};$$

- $I^\flat$ is **narrowly right transitive** if, for all $T \in \mathrm{ThFam}^\nleq(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma[\phi,\psi] \cup I^\flat_\Sigma[\psi,\chi] \leq \overleftarrow{T} \quad \text{implies} \quad I^\flat_\Sigma[\phi,\chi] \leq \overleftarrow{T};$$

- $I^\flat$ is **narrowly right compatible** if, for all $T \in \mathrm{ThFam}^\nleq(\mathcal{I})$, all $\sigma^\flat \in N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\bigcup_{i<k} \overleftrightarrow{I^\flat}_\Sigma[\phi_i,\psi_i] \leq \overleftarrow{T} \quad \text{implies} \quad I^\flat_\Sigma[\sigma^\flat_\Sigma(\vec{\phi}),\sigma^\flat_\Sigma(\vec{\psi})] \leq \overleftarrow{T};$$

- $I^\flat$ has the **narrow right MP** if, for all $T \in \mathrm{ThFam}^\nleq(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{and} \quad I^\flat_\Sigma[\phi,\psi] \leq \overleftarrow{T} \quad \text{imply} \quad \psi \in \overleftarrow{T}_\Sigma.$$

Note that, because of Equivalence (15.1), narrow right reflexivity, narrow right symmetry, narrow right transitivity and narrow right compatibility are equivalent, respectively, to narrow family reflexivity, narrow family symmetry, narrow family transitivity and narrow family compatibility. They are simply recast involving the arrow operator, but the change is inessential. On the other hand, narrow right modus ponens is an essentially different property than narrow family modus ponens and it is the critical property that differentiates syntactic narrow right monotonicity from syntactic narrow family monotonicity.

Note, also, that, based on Equivalence (15.1), for all $T \in \mathrm{ThFam}^\nleq(\mathcal{I})$,

$$I^\flat(T) = I^\flat(\overleftarrow{T}).$$

We say that $\mathcal{I}$ is **syntactically narrowly right monotone** if there exists $I^\flat \subseteq N^\flat$, with two distinguished arguments, such that $I^\flat$ satisfies:

- narrow right reflexivity;
- narrow right transitivity;
- narrow right compatibility; and
- narrow right MP.

In that case, we call $I^\flat$ a **set of witnessing natural transformations**, or, more simply, **witnessing transformations** (of the syntactic narrow right monotonicity of $\mathcal{I}$).

It turns out that, if $\mathcal{I}$ is a syntactically narrowly right monotone π-institution, with witnessing transformations $I^\flat$, then $\overleftrightarrow{I^\flat}(T)$ $(:= \overleftrightarrow{I^\flat}(\overleftarrow{T}))$ is a congruence system on $\mathbf{F}$ compatible with $\overleftarrow{T}$, for all $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$. This forms a system analog of Proposition 1203.

Proposition 1234 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically narrowly right monotone, with witnessing transformations* $I^\flat$*, then, for all* $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$*,* $\overleftrightarrow{I^\flat}(T)$ *is a congruence system on* $\mathbf{F}$ *compatible with* $\overleftarrow{T}$*.*

Proof: The proof is similar to that of Proposition 1203. Let $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$. The narrow right reflexivity of $I^\flat$ ensures that $\langle \phi, \phi\rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$. The fact that $\overleftrightarrow{I^\flat}$ is the symmetrization of $I^\flat$ ensures that $\langle \phi, \psi\rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$ implies that $\langle \psi, \phi\rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$. The narrow right transitivity of $I^\flat$ guarantees that $\langle \phi, \psi\rangle, \langle \psi, \chi\rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$ imply $\langle \phi, \chi\rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$.

Suppose, next, that $\sigma^\flat \in N^\flat$, $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$. Then, the narrow right compatibility of $I^\flat$ ensures that, if, for all $i < k$, $\langle \phi_i, \psi_i\rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$, then $\langle \sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi})\rangle \in I^\flat_\Sigma(T)$. Thus, $\overleftrightarrow{I^\flat}(T)$ is a congruence family on $\mathbf{F}$. However, by Lemma 93, $\overleftrightarrow{I^\flat}(T)$ is a relation system on $\mathbf{F}$. Hence, $\overleftrightarrow{I^\flat}(T)$ is a congruence system on $\mathbf{F}$.

It only remains to show that $\overleftrightarrow{I^\flat}(T)$ is compatible with $\overleftarrow{T}$. Assume that $\phi \in \overleftarrow{T}_\Sigma$ and $\langle \phi, \psi\rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$. Since $I^\flat \subseteq \overleftrightarrow{I^\flat}$, we get, by the narrow right MP of $I^\flat$, that $\psi \in \overleftarrow{T}_\Sigma$. Thus, $\overleftrightarrow{I^\flat}(T)$ is also compatible with $\overleftarrow{T}$. ■

Proposition 1234 shows that $\overleftrightarrow{I^\flat}$ defines Leibniz congruence systems of those theory systems of the form $\overleftarrow{T}$, for $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$. We say that $I^\flat$ **narrowly defines Leibniz congruence systems** of theory families in $\mathcal{I}$ **up to arrow** if, for all $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$,

$$\overleftrightarrow{I^\flat}(T) = \Omega(\overleftarrow{T}).$$

Then, in what is an analog of Corollary 1204, we obtain

Corollary 1235 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically narrowly right monotone, with witnessing transformations* $I^\flat$*, then* $I^\flat$ *narrowly defines Leibniz congruence systems of theory families in* $\mathcal{I}$ *up to arrow.*

Proof: By Proposition 1219 and Corollary 98. ∎

This corollary has as immediate consequence the fact that syntactic narrow right monotonicity implies (semantic) narrow right monotonicity. This forms an analog of Theorem 1205.

Theorem 1236 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is syntactically narrowly right monotone, then it is narrowly right monotone.*

Proof: Suppose that $\mathcal{I}$ is syntactically narrowly right monotone with witnessing transformations $I^\flat$. Let $T, T' \in \mathrm{ThFam}^{\nsharp}(\mathcal{I})$, such that $T \leq T'$. Then

$$\begin{array}{rcll} \Omega(\overleftarrow{T}) & = & \overleftrightarrow{I^\flat}(T) & \text{(by Corollary 1235)} \\ & \leq & \overleftrightarrow{I^\flat}(T') & \text{(by Lemma 94)} \\ & = & \Omega(\overleftarrow{T'}). & \text{(by Corollary 1235)} \end{array}$$

Thus, $\mathcal{I}$ is narrowly right monotone. ∎

We now introduce the notion of the narrow reflexive system core of a π-institution $\mathcal{I}$ in a way analogous to the narrow reflexive core, which was introduced in Section 15.1. Its introduction will enable us to provide a characterization of the syntactical narrow system monotonicity property and to establish a relationship between this property and its semantic counterpart.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall from Section 15.1 that the **narrow reflexive core** of $\mathcal{I}$ is the collection

$$\begin{array}{rcl} R^{\mathcal{I}\nsharp} & = & \{\rho^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}^{\nsharp}(\mathcal{I}))(\forall \Sigma \in |\mathbf{Sign}^\flat|) \\ & & \qquad (\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\rho^\flat_\Sigma[\phi, \phi] \leq T)\}. \end{array}$$

Recall, also, from Lemmas 1207, 1208 and 1209, that, given any theory family in $\mathrm{ThFam}^{\nsharp}(\mathcal{I})$, the relation system $R^{\mathcal{I}\nsharp}(T)$ is a reflexive and symmetric relation system on $\mathbf{F}$ that has the congruence compatibility property in $\mathbf{F}$.

We now show, in an analog of Theorem 1213, that possession of the narrow right modus ponens by the narrow reflexive core intrinsically characterizes syntactic narrow right monotonicity. We start by showing that possession of the narrow right MP by the narrow reflexive core is necessary for syntactic narrow right monotonicity. This forms an analog of Theorem 1210.

Theorem 1237 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is syntactically narrowly right monotone, then* $R^{\mathcal{I}\nsharp}$ *has the narrow right MP.*

Proof: Suppose that $\mathcal{I}$ is syntactically narrowly right monotone with witnessing transformations $I^\flat$. Since, by definition, $I^\flat$ is narrowly right reflexive, which is equivalent to being narrowly family reflexive, we get, by definition of $R^{\mathcal{I}\nleftarrow}$, $I^\flat \subseteq R^{\mathcal{I}\nleftarrow}$. Thus, since $I^\flat$ has the narrow right MP in $\mathcal{I}$, we get that, a fortiori, $R^{\mathcal{I}\nleftarrow}$ also satisfies the narrow right MP. ■

If $R^{\mathcal{I}\nleftarrow}$ has the narrow right MP, then it has the narrow right transitivity in $\mathcal{I}$. This proposition forms an analog of Proposition 1211.

Proposition 1238 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $R^{\mathcal{I}\nleftarrow}$ *has the narrow right MP, then it also has the narrow right transitivity in* $\mathcal{I}$.

Proof: Suppose that $R^{\mathcal{I}\nleftarrow}$ has the narrow right MP and let $T \in \mathrm{ThFam}^{\nleftarrow}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle, \langle \psi, \chi \rangle \in R^{\mathcal{I}\nleftarrow}_\Sigma(T)$. This means that $R^{\mathcal{I}\nleftarrow}_\Sigma[\phi, \psi] \leq T$ and $R^{\mathcal{I}\nleftarrow}_\Sigma[\psi, \chi] \leq T$. Then, by Lemma 1224, we get that, for all $\rho^\flat \in R^{\mathcal{I}\nleftarrow}$, and all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$R^{\mathcal{I}\nleftarrow}_{\Sigma'}[\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), \vec{\xi}), \rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi})] \leq T.$$

But, by hypothesis, $R^{\mathcal{I}\nleftarrow}_\Sigma[\phi, \psi] \leq T$ and $R^{\mathcal{I}\nleftarrow}$ has the narrow right MP. Therefore, for all $\rho^\flat \in R^{\mathcal{I}\nleftarrow}$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi}) \subseteq T_{\Sigma'},$$

i.e., $R^{\mathcal{I}\nleftarrow}_\Sigma[\phi, \chi] \leq T$. This shows $\langle \phi, \chi \rangle \in R^{\mathcal{I}\nleftarrow}_\Sigma(T)$ and, hence, $R^{\mathcal{I}\nleftarrow}$ is narrowly right transitive in $\mathcal{I}$. ■

We are now ready to show a converse of Theorem 1237, i.e., that possession of the narrow right MP by $R^{\mathcal{I}\nleftarrow}$ suffices to establish the syntactic narrow right monotonicity of $\mathcal{I}$, since, in that case, $R^{\mathcal{I}\nleftarrow}$ serves as a family of witnessing transformations. The following constitutes an analog of Theorem 1212.

Theorem 1239 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $R^{\mathcal{I}\nleftarrow}$ *has the narrow right MP, then* $\mathcal{I}$ *is syntactically narrowly right monotone, with witnessing transformations* $R^{\mathcal{I}\nleftarrow}$.

Proof: By Lemma 1207, $R^{\mathcal{I}\nleftarrow}$ is narrowly right reflexive in $\mathcal{I}$. By Lemma 1208, $R^{\mathcal{I}\nleftarrow}$ is narrowly right symmetric in $\mathcal{I}$. By hypothesis and Proposition 1238, it is narrowly right transitive in $\mathcal{I}$. By Lemma 1209 it has the narrow right compatibility property in $\mathcal{I}$. Finally, by hypothesis, it has the narrow right MP in $\mathcal{I}$. We conclude that $\mathcal{I}$ is syntactically narrowly right monotone, with witnessing transformations $R^{\mathcal{I}\nleftarrow}$. ■

Theorems 1237 and 1239 provide the promised characterization of syntactic narrow right monotonicity in terms of the narrow right MP of the narrow reflexive core.

$$\begin{array}{c}\mathcal{I} \text{ is Syntactically Narrow} \\ \text{Right Monotone}\end{array} \longleftrightarrow \begin{array}{c}R^{\mathcal{I}\natural} \text{ has Narrow Right} \\ \text{Modus Ponens}\end{array}.$$

Theorem 1240 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically narrowly right monotone if and only if* $R^{\mathcal{I}\natural}$ *has the narrow right MP in* $\mathcal{I}$*.*

Proof: Theorem 1237 gives the "only if" and the "if" is by Theorem 1239. ■

A related alternative characterization asserts that syntactic narrow right monotonicity amounts to the narrow definability of Leibniz congruence systems of theory families up to arrow by the narrow reflexive core. This result forms an analog of Theorem 1214.

$$\begin{array}{c}\mathcal{I} \text{ is Syntactically Narrow} \\ \text{Right Monotone}\end{array} \longleftrightarrow \begin{array}{c}R^{\mathcal{I}\natural} \text{ Defines Leibniz Congruence Systems} \\ \text{of Theory Families up to Arrow}\end{array}$$

Theorem 1241 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically narrowly right monotone if and only if, for all* $T \in \mathrm{ThFam}^{\natural}(\mathcal{I})$*,*

$$\Omega(\overleftarrow{T}) = R^{\mathcal{I}\natural}(T).$$

Proof: If $\mathcal{I}$ is syntactically narrowly right monotone, then, by Theorem 1237, $R^{\mathcal{I}\natural}$ has the narrow right MP in $\mathcal{I}$. Thus, by Theorem 1239, $R^{\mathcal{I}\natural}$ is a family of witnessing transformations for the syntactic narrow right monotonicity of $\mathcal{I}$. Thus, by Corollary 1235, for all $T \in \mathrm{ThFam}^{\natural}(\mathcal{I})$, $\Omega(\overleftarrow{T}) = R^{\mathcal{I}\natural}(T)$.

Suppose, conversely, that the displayed condition holds. Then $R^{\mathcal{I}\natural}$ is narrowly right reflexive, narrowly right transitive and has the narrow right compatibility property and the narrow right MP. Hence, it constitutes a collection of witnessing transformations and, therefore, $\mathcal{I}$ is syntactically narrowly right monotone. ■

To prove an analog of Theorem 1218, which, in a sense analogous to that seen for syntactic narrow family monotonicity, characterizes syntactic narrow right monotonicity inside the class of narrow right monotone π-institutions, we create a suitable analog of the Leibniz compatibility property with respect to the theory family generated by the narrow reflexive core. Once more, the difficulty in this case, similarly with that described in some detail in Section 15.1, arises from the fact that $\mathrm{ThFam}^{\natural}(\mathcal{I})$ may not be, in general, closed under signature-wise intersections.

To introduce this analog and to understand how it comes about and how it extends the Leibniz property, we reembark, once more, on a discussion initiated in Section 15.1 and revisit some of the points with relevance in treating the "right" case.

Recall, again, the definition of prealgebraicity and the Leibniz property of the reflexive core of a π-institution. Also recall the method employed to show that, if a π-institution is prealgebraic and has a Leibniz reflexive core, then it satisfies the global system modus ponens, which is done by first applying the Leibniz property and then prealgebraicity. However, in case of narrow right monotonicity, the plausibility of $R^{\mathcal{I}\nleq}_\Sigma[\phi,\psi]$ having some empty components makes it likely that, when one attempts to apply narrow right monotonicity in place of prealgebraicity in the second stage of the argument outlined above, its application in order to derive the inclusion $\Omega(C(R^{\mathcal{I}\nleq}_\Sigma[\phi,\psi])) \leq \Omega(\overleftarrow{T})$ may not be possible. To deal with this plausibility, we assume, in a similar way as before, that the π-institution under consideration is reflexively covered and postulate that every minimal element $T \in [R^{\mathcal{I}\nleq}_\Sigma[\phi,\psi])$ satisfies $\langle\phi,\psi\rangle \in \Omega_\Sigma(T)$, for every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$.

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. For $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, recall the notation

$$[R^{\mathcal{I}\nleq}_\Sigma[\phi,\psi]) \coloneqq \{T \in \mathrm{ThFam}^{\nleq}(\mathcal{I}) : R^{\mathcal{I}\nleq}_\Sigma[\phi,\psi] \leq T\}.$$

Recall, also that $\mathcal{I}$ is said to be *reflexively covered* if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, it is $\langle\Sigma,\phi,\psi\rangle$-reflexively covered, i.e., for every theory family $T \in [R^{\mathcal{I}\nleq}_\Sigma[\phi,\psi])$, there exists minimal $T' \in [R^{\mathcal{I}\nleq}_\Sigma[\phi,\psi])$, such that $T' \leq T$. Recall, furthermore, that, given $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, we write $\min[R^{\mathcal{I}\nleq}_\Sigma[\phi,\psi])$ for the collection of minimal elements in $[R^{\mathcal{I}\nleq}_\Sigma[\phi,\psi])$.

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. We say that the narrow reflexive core $R^{\mathcal{I}\nleq}$ of $\mathcal{I}$ is **right Leibniz** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$ and all $T \in \min[R^{\mathcal{I}\nleq}_\Sigma[\phi,\psi])$,

$$\langle\phi,\psi\rangle \in \Omega_\Sigma(\overleftarrow{T}).$$

We show, in an analog of Proposition 1215, that, if $R^{\mathcal{I}\nleq}$ has the narrow right MP, then it is right Leibniz. In fact, the proof demonstrates that, under the narrow right MP, a stronger property than that of being right Leibniz holds; more concretely, that for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$ and all $T \in [R^{\mathcal{I}\nleq}_\Sigma[\phi,\psi])$ (not only for $T \in \min[R^{\mathcal{I}\nleq}_\Sigma[\phi,\psi])$), $\langle\phi,\psi\rangle \in \Omega_\Sigma(\overleftarrow{T})$.

Proposition 1242 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $R^{\mathcal{I}\nleq}$ has the narrow right MP, then for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$ and all $T \in [R^{\mathcal{I}\nleq}_\Sigma[\phi,\psi])$, $\langle\phi,\psi\rangle \in \Omega_\Sigma(\overleftarrow{T})$.*

Proof: Suppose $R^{\mathcal{I}\nleftrightarrow}$ has the narrow right MP and let $T \in \mathrm{ThFam}^{\nleftrightarrow}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $R^{\mathcal{I}\nleftrightarrow}_\Sigma[\phi, \psi] \leq T$. To verify that $\langle \phi, \psi \rangle \in \Omega_\Sigma(\overleftarrow{T})$, we use Theorem 19. Let $\sigma^\flat \in N^\flat$, $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$, such that $\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \in \overleftarrow{T}_{\Sigma'}$. Since $T \in \mathrm{ThFam}^{\nleftrightarrow}(\mathcal{I})$, by Lemma 1209,

$$R^{\mathcal{I}\nleftrightarrow}_{\Sigma'}[\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}), \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi})] \leq T.$$

Thus, since, by hypothesis, $R^{\mathcal{I}\nleftrightarrow}$ has the narrow right MP, we obtain

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}) \in \overleftarrow{T}_{\Sigma'}.$$

By symmetry, we conclude that, for all $\sigma^\flat \in N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \in \overleftarrow{T}_{\Sigma'} \quad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}) \in \overleftarrow{T}_{\Sigma'}.$$

Hence, by Theorem 19, $\langle \phi, \psi \rangle \in \Omega_\Sigma(\overleftarrow{T})$. ■

Corollary 1243 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}\nleftrightarrow}$ *has the narrow right MP, then it is right Leibniz.*

Proof: Directly by Proposition 1242. ■

To prove a converse, we restrict attention to reflexively covered π-institutions. Inside this class, we may show that narrow right monotonicity combined with the right Leibniz property of the narrow reflexive core imply that the narrow reflexive core has the narrow right modus ponens in $\mathcal{I}$. The following proposition forms an analog of Propositions 1217 and 1232 in the "right" context.

Proposition 1244 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a reflexively covered, narrowly right monotone* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}\nleftrightarrow}$ *is right Leibniz, then it has the narrow right MP in* $\mathcal{I}$*.*

Proof: Let $\mathcal{I}$ be a reflexively covered π-institution. Suppose that $\mathcal{I}$ is narrowly right monotone and that $R^{\mathcal{I}\nleftrightarrow}$ is right Leibniz. Let $T \in \mathrm{ThFam}^{\nleftrightarrow}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \overleftarrow{T}_\Sigma$ and $R^{\mathcal{I}\nleftrightarrow}_\Sigma[\phi, \psi] \leq T$. Since $\mathcal{I}$ is reflexively covered, there exists $T' \in \min[R^{\mathcal{I}\nleftrightarrow}_\Sigma[\phi, \psi]]$, such that $T' \leq T$. Now we have

$$\begin{array}{lll} \langle \phi, \psi \rangle & \in & \Omega_\Sigma(\overleftarrow{T'}) \quad (\text{since } R^{\mathcal{I}\nleftrightarrow} \text{ is right Leibniz and } T' \in \min[R^{\mathcal{I}\nleftrightarrow}_\Sigma[\phi, \psi]]) \\ & \subseteq & \Omega_\Sigma(\overleftarrow{T}). \quad (\text{since } T' \leq T \text{ and } \mathcal{I} \text{ is narrowly right monotone}) \end{array}$$

Therefore, since $\phi \in \overleftarrow{T}_\Sigma$, we get, by the compatibility of $\Omega(\overleftarrow{T})$ with $\overleftarrow{T}$, that $\psi \in \overleftarrow{T}_\Sigma$. We conclude that $R^{\mathcal{I}\natural}$ has the narrow right MP in $\mathcal{I}$. ■

Thus, at least for reflexively covered π-institutions, it is possible to show that the class of syntactically narrowly right monotone ones inside the class of the narrowly right monotone ones can be characterized exactly by the right Leibniz property of the narrow reflexive core. This forms a partial analog of Theorems 1218 and 1233.

Theorem 1245 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a reflexively covered* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically narrowly right monotone if and only if it is narrowly right monotone and has a right Leibniz narrow reflexive core.*

Proof: Let $\mathcal{I}$ be a reflexively covered π-institution.

Suppose, first, that $\mathcal{I}$ is syntactically narrowly right monotone. Then it is narrowly right monotone by Theorem 1236. Moreover, its narrow reflexive core has the narrow right MP by Theorem 1237 and, hence, by Corollary 1243, its narrow reflexive core is right Leibniz.

Suppose, conversely, that $\mathcal{I}$ is narrowly right monotone with a right Leibniz narrow reflexive core. Then, by Proposition 1244, its narrow reflexive core has the narrow right MP and, therefore, by Theorem 1239, $\mathcal{I}$ is syntactically narrowly right monotone, with witnessing transformations $R^{\mathcal{I}\natural}$. ■