Chapter 16

The Syntactic Leibniz Hierarchy: Attic I

16.1 Introduction

In this chapter our goal is to develop a hierarchy analogous to the one developed in Chapter 8, but on the syntactic side. The key on the semantic side, developed in Chapter 8, was the property of regularity of a π-institution. The family version of the property asserts that, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, based on an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$, $\mathcal{I}$ is *family regular* if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi, \psi \in T_\Sigma \quad \text{implies} \quad \langle \phi, \psi \rangle \in \Omega_\Sigma(T).$$

By combining this property with pre- or proto-algebraicity, on the one hand, and with the existence of theorems, on the other, which subsumes complete reflectivity, one obtains various classes in the regular (weak) (pre)algebraizability hierarchy, which were studied in some detail in Chapter 8.

In this chapter, as our interest shifts to the syntactic side, the role played by of pre- and proto-algebraicity is assumed by syntactic pre- and proto-algebraicity, respectively, and the existence of theorems is replaced by the existence of natural theorems. By adding these features to regularity, one obtains the classes of the syntactically regularly (weakly) (pre)algebraizable π-institutions, which dominate, in general, the corresponding semantic classes. Roughly speaking, the hierarchy that we are aiming for here has the general shape depicted in the accompanying diagram. Of course various classes are present at each level, since the properties shown have various flavors, or versions, that may be used at each of the combinations depicted.

$$\begin{tikzcd}
 & & & \text{Synt Reg Algble} \arrow[dl] \arrow[dr] & & & \\
 & & \text{Synt Reg Equiv} \arrow[dl] \arrow[dr] & & \text{Synt Reg Weak Algble} \arrow[dl] \arrow[dr] & & \\
 & \text{Synt Equiv} \arrow[dl] \arrow[dr] & & \text{Synt Reg Proto} \arrow[dl] \arrow[dr] & & \text{Synt Assrt} \arrow[dl] \arrow[dr] & \\
\text{Ext} & & \text{Synt Proto} & & \text{Regular} & & \text{Nat Thms}
\end{tikzcd}$$

16.2 Regularity of Transformations

To prepare us for the main developments, we start by looking closely at the various versions of the regularity property of a family of natural transformations in a given π-institution.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Moreover, let $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ be a collection of natural transformations in $N^\flat$, having two distinguished arguments. We define the following properties:

- $I^\flat$ has the **family regularity in** $\mathcal{I}$, or is **family regular in** $\mathcal{I}$, if, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi, \psi \in T_\Sigma \quad \text{implies} \quad I^\flat_\Sigma[\phi, \psi] \le T;$$
- $I^\flat$ has the **left regularity in** $\mathcal{I}$, or is **left regular in** $\mathcal{I}$, if, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi, \psi \in \overleftarrow{T}_\Sigma \quad \text{implies} \quad I^\flat_\Sigma[\phi, \psi] \le T;$$
- $I^\flat$ has the **right regularity in** $\mathcal{I}$, or is **right regular in** $\mathcal{I}$, if, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi, \psi \in T_\Sigma \quad \text{implies} \quad I^\flat_\Sigma[\phi, \psi] \le \overleftarrow{T};$$
- $I^\flat$ has the **system regularity in** $\mathcal{I}$, or is **system regular in** $\mathcal{I}$, if, for all $T \in \mathrm{ThSys}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi, \psi \in T_\Sigma \quad \text{implies} \quad I^\flat_\Sigma[\phi, \psi] \le T.$$

Recalling that, by Proposition 99, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, we have
$$I^\flat_\Sigma[\phi, \psi] \le T \quad \text{iff} \quad I^\flat_\Sigma[\phi, \psi] \le \overleftarrow{T},$$
it is easy to see that the four properties defined above collapse in pairs and, therefore, there are only two distinct ones. This is detailed in the following:

Proposition 1246 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a collection of natural transformations in* $N^\flat$*, with two distinguished arguments.*

(a) $I^\flat$ *is family regular in* $\mathcal{I}$ *if and only if it is right regular in* $\mathcal{I}$*;*

(b) $I^\flat$ *is system regular in* $\mathcal{I}$ *if and only if it is left regular in* $\mathcal{I}$*.*

Proof: By Proposition 99, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, we have
$$I^\flat_\Sigma[\phi, \psi] \le T \quad \text{iff} \quad I^\flat_\Sigma[\phi, \psi] \le \overleftarrow{T}.$$

Thus, taking into account the definitions of family and right regularity, the equivalence of Part (a) becomes clear. We turn now to Part (b).

Assume, first, that $I^\flat$ is left regular in $\mathcal{I}$ and let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Since $T \in \mathrm{ThSys}(\mathcal{I})$, $\overleftarrow{T} = T$, whence, by hypothesis, $\phi, \psi \in \overleftarrow{T}_\Sigma$. Thus, by left regularity, $I^\flat_\Sigma[\phi, \psi] \le T$. This shows that $I^\flat$ has the system regularity in $\mathcal{I}$.

Suppose, conversely, that $I^\flat$ is system regular in $\mathcal{I}$ and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in \overleftarrow{T}_\Sigma$. Since $\overleftarrow{T} \in \mathrm{ThSys}(\mathcal{I})$, we get, by system regularity, $I^\flat_\Sigma[\phi, \psi] \le \overleftarrow{T}$. Therefore, by Proposition 99, $I^\flat_\Sigma[\phi, \psi] \le T$, showing that $I^\flat$ has the left regularity in $\mathcal{I}$. ■

Based on Proposition 1246, we use the term **family regular** to refer to family/right regularity and the term **system regular** for system/left regularity. As far as the relation between these two distinct properties, it is straightforward to see that, as is typical with almost all properties studied in the monograph, system regularity is weaker than family regularity.

Proposition 1247 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a collection of natural transformations in* $N^\flat$*, having two distinguished arguments. If* $I^\flat$ *is family regular in* $\mathcal{I}$*, then it is system regular in* $\mathcal{I}$*.*

Proof: This is clear from the definitions, since the condition defining system regularity is a specialization of that defining family regularity, where T is allowed to range over theory systems only. ■

Thus, the following hierarchy of regularity properties emerges.

$$\begin{tikzcd} I^\flat \text{ Family/Right Regular} \arrow[d] \\ I^\flat \text{ System/Left Regular} \end{tikzcd}$$

It is also easy to see that, in case $\mathcal{I}$ is systemic, the two properties of being family and system regular are identified.

Proposition 1248 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a collection of natural transformations in* $N^\flat$*. If* $\mathcal{I}$ *is systemic, then* $I^\flat$ *is system regular if and only if it is family regular in* $\mathcal{I}$*.*

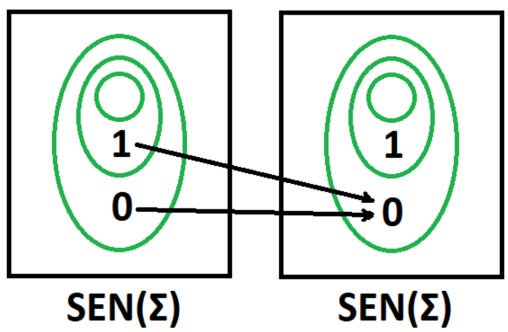
Proof: If $\mathcal{I}$ is systemic, then $\mathrm{ThFam}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$, whence the two conditions defining family and system regularity are identical. ■

And it is not difficult to show that this hierarchy does not collapse, in general.

Example 1249 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with the single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$;
- $N^\flat$ *is the clone of natural transformations generated by the binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$, *specified by*

$$\sigma^\flat_\Sigma(x,y) = 0, \quad \textit{for all } x,y \in \mathrm{SEN}^\flat(\Sigma).$$



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\}.$$

$\mathcal{I}$ *has three theory families* $\overline{\varnothing}$, $\{\{1\}\}$ *and* $\mathrm{SEN}^\flat$, *but only two theory systems,* $\overline{\varnothing}$ *and* $\mathrm{SEN}^\flat$. *Consider* $I^\flat = \{\sigma^\flat\}$. *Since, the only theory systems are* $\overline{\varnothing}$ *and* $\mathrm{SEN}^\flat$, $I^\flat$ *is trivially system regular. On the other hand, for* $T = \{\{1\}\}$, *we get,* $1 \in T_\Sigma$, *but* $I^\flat_\Sigma[1,1] = \{\{0\}\} \nleq \{\{1\}\}$, *whence* $I^\flat$ *is not family regular in* $\mathcal{I}$.

We close the section by showing that the two versions of regularity transfer from the family $I^\flat$ to $I^\mathcal{A}$, for all $\mathbf{F}$-algebraic systems $\mathcal{A}$.

Proposition 1250 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a collection of natural transformations in* $N^\flat$, *with two distinguished arguments.*

(a) $I^\flat$ *is family regular in* $\mathcal{I}$ *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $I^\mathcal{A}$ *is family regular in* $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$;

(b) $I^\flat$ *is system regular in* $\mathcal{I}$ *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $I^\mathcal{A}$ *is system regular in* $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$.

Proof:

(a) The "if" follows easily by considering the **F**-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and recalling from Lemma 51 that $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}) = \mathrm{ThFam}(\mathcal{I})$.

Assume, conversely, that $I^\flat$ is family regular in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an **F**-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \in T_{F(\Sigma)}$. Then, $\phi, \psi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})$. By Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, whence, by the postulated family regularity of $I^\flat$ in $\mathcal{I}$, we get that $I_\Sigma^\flat[\phi, \psi] \leq \alpha^{-1}(T)$. Thus, by Lemma 95, we get $I_{F(\Sigma)}^{\mathcal{A}}[\alpha_\Sigma(\phi), \alpha_\Sigma(\psi)] \leq T$. Taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, if $\phi, \psi \in T_\Sigma$, then $I_\Sigma^{\mathcal{A}}[\phi, \psi] \leq T$. Therefore, $I^{\mathcal{A}}$ is family regular in $\langle \mathcal{A}, C^{\mathcal{I}, \mathcal{A}} \rangle$.

(b) This follows along very similar lines.

■

16.3 Syntactic Regular PreAlgebraicity

In the next result, we connect the property of regularity of a collection of natural transformations with the property of regularity of a π-institution $\mathcal{I}$, studied in Chapter 8. More specifically, we show that, in case the π-institution under consideration is syntactically pre- (proto-)algebraic with $I^\flat$ a collection of witnessing transformations, then family (system) regularity of $I^\flat$ is equivalent to $\mathcal{I}$ being family (system) regular. Since the combination of syntactic pre- and proto-algebraicity with regularity turns out to be an important property in its own right, we give it a name, partly inspired by the results that follow. Recall that there are two kinds of syntactic monotonicity, namely syntactic prealgebraicity and syntactic protoalgebraicity, and two kinds of regularity properties of collections of natural transformations, namely family regularity and system regularity. Thus, by combining syntactic monotonicity properties with regularity properties, we obtain, a priori, four versions.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

- $\mathcal{I}$ is said to be **syntactically family regularly protoalgebraic** if it is syntactically protoalgebraic, with a witnessing collection $I^\flat$ of transformations, which is family regular in $\mathcal{I}$;

- $\mathcal{I}$ is said to be **syntactically system regularly protoalgebraic** if it is syntactically protoalgebraic, with a witnessing collection $I^\flat$ of transformations, which is system regular in $\mathcal{I}$;

- $\mathcal{I}$ is said to be **syntactically family regularly prealgebraic** if it is syntactically prealgebraic, with a witnessing collection $I^\flat$ of transformations, which is family regular in $\mathcal{I}$;
- $\mathcal{I}$ is said to be **syntactically system regularly prealgebraic** if it is syntactically prealgebraic, with a witnessing collection $I^\flat$ of transformations, which is system regular in $\mathcal{I}$.

The definitions are partially justified by the following propositions that relate them to the semantical notions of family, right, left and system regularity.

Proposition 1251 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically protoalgebraic* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$*.*

(a) $\mathcal{I}$ *is family regular if and only if* $I^\flat$ *is family regular in* $\mathcal{I}$*;*

(b) $\mathcal{I}$ *is left regular if and only if* $I^\flat$ *is system regular in* $\mathcal{I}$*.*

Proof: Let $\mathcal{I}$ be a syntactically protoalgebraic π-institution, with witnessing transformations $I^\flat$.

(a) This part is easy to see, since, by syntactic protoalgebraicity, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(T) \quad \text{iff} \quad I^\flat_\Sigma[\phi, \psi] \leq T.$$

(b) Suppose, first, that $\mathcal{I}$ is left regular and let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Since $T \in \mathrm{ThSys}(\mathcal{I})$, $\phi, \psi \in \overleftarrow{T}_\Sigma$. By the left regularity of $\mathcal{I}$, we get $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$, whence, by syntactic protoalgebraicity, $I^\flat_\Sigma[\phi, \psi] \leq T$. This shows that $I^\flat$ is system regular in $\mathcal{I}$.

Assume, conversely, that $I^\flat$ is system regular in $\mathcal{I}$ and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in \overleftarrow{T}_\Sigma$. Then, since $\overleftarrow{T} \in \mathrm{ThSys}(\mathcal{I})$, by the system regularity of $I^\flat$, $I^\flat_\Sigma[\phi, \psi] \leq \overleftarrow{T} \leq T$, whence, by syntactic protoalgebraicity, $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. Therefore, $\mathcal{I}$ is left regular.

■

Proposition 1252 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically prealgebraic* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$*.*

(a) $\mathcal{I}$ *is right regular if and only if* $I^\flat$ *is family regular in* $\mathcal{I}$*;*

(b) $\mathcal{I}$ is system regular if and only if $I^\flat$ is system regular in $\mathcal{I}$.

Proof: Let $\mathcal{I}$ be a syntactically prealgebraic π-institution, with witnessing transformations $I^\flat$.

(a) Suppose, first, that $\mathcal{I}$ is right regular and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. By the right regularity of $\mathcal{I}$, we get $\langle \phi, \psi \rangle \in \Omega_\Sigma(\overleftarrow{T})$, whence, since $\overleftarrow{T} \in \mathrm{ThSys}(\mathcal{I})$, we get, by syntactic prealgebraicity, $I^\flat_\Sigma[\phi, \psi] \le \overleftarrow{T} \le T$. This shows that $I^\flat$ is family regular in $\mathcal{I}$.

Assume, conversely, that $I^\flat$ is family regular in $\mathcal{I}$ and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Then, by the family regularity of $I^\flat$, $I^\flat_\Sigma[\phi, \psi] \le T$. Hence, by Proposition 99, we get $I^\flat_\Sigma[\phi, \psi] \le \overleftarrow{T}$. Since $\overleftarrow{T} \in \mathrm{ThSys}(\mathcal{I})$, by syntactic prealgebraicity, $\langle \phi, \psi \rangle \in \Omega_\Sigma(\overleftarrow{T})$. Therefore, $\mathcal{I}$ is right regular.

(b) This part is straightforward, since, by syntactic prealgebraicity, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$ if and only if $I^\flat_\Sigma[\phi, \psi] \le T$.

■

Propositions 1251 and 1252 may be viewed as partial justifications for the definitions of syntactic regular pre- and proto-algebraicity Moreover, recalling the following hierarchies of syntactic pre- and protoalgebraicity, of the regularity properties of $I^\flat$ and of semantic regularity,

$$\begin{tikzcd}
 & & \text{Family Regular} \arrow[d] \\
\text{Synt ProtoAlgic} \arrow[d] & I^\flat \text{ Family Reg} \arrow[d] & \text{Right Regular} \arrow[d] \\
\text{Synt PreAlgic} & I^\flat \text{ System Reg} & \text{Left Regular} \arrow[d] \\
 & & \text{System Regular}
\end{tikzcd}$$

the following hierarchy of syntactic classes of regularly pre- and protoalgebraic π-institutions emerges.

$$\begin{tikzcd}
& \text{Synt Fam Reg ProtoAlgic} \arrow[dl] \arrow[dr] & \\
\text{Synt Fam Reg PreAlgic} \arrow[dr] & & \text{Synt Sys Reg ProtoAlgic} \arrow[dl] \\
& \text{Synt Sys Reg PreAlgic} &
\end{tikzcd}$$

Furthermore, these four classes relate with their immediate subordinate properties on the syntactic side, as shown in the following diagram

$$\begin{tikzcd}
& \text{Synt Fam Reg ProtoAlgic} \arrow[dl] \arrow[dr] & \\
\text{Synt Sys Reg ProtoAlgic} \arrow[d] \arrow[dr] & & \text{Synt Fam Reg PreAlgic} \arrow[dl] \arrow[d] \\
\text{Synt ProtoAlgic} \arrow[d] & \begin{array}{c}\text{Synt Sys}\\ \text{Reg PreAlgic}\end{array} \arrow[dl] \arrow[dr] & I^\flat \text{ Fam Reg} \arrow[d] \\
\text{Synt PreAlgic} & & I^\flat \text{ Sys Reg}
\end{tikzcd}$$

and with the four semantic regularity classes, as revealed by Propositions 1251 and 1252, as shown in the following diagram.

$$\begin{tikzcd}
& \text{Synt Fam Reg ProtoAlgic} \arrow[dl] \arrow[d] \arrow[dr] & \\
\begin{array}{c}\text{Synt Sys}\\ \text{Reg ProtoAlgic}\end{array} \arrow[d] \arrow[ddrr] & \begin{array}{c}\text{Synt Fam}\\ \text{Reg PreAlgic}\end{array} \arrow[dl] \arrow[dr] & \text{Fam Reg} \arrow[d] \\
\text{Synt Sys Reg PreAlgic} \arrow[ddrr] & & \text{Right Reg} \arrow[d] \\
& & \text{Left Reg} \arrow[d] \\
& & \text{Sys Reg}
\end{tikzcd}$$

Theorem 584, which provided a characterization of both family and of system regularity in terms of the Suszko operator and of a system version of the Suszko operator, respectively, gives rise to the following characterizations of family and system regularity of witnessing collections of natural transformations for the proto- and pre-algebraicity, respectively, of a π-institution.

Corollary 1253 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a collection of natural transformations in* $N^\flat$*, with two distinguished arguments.*

(a) If $\mathcal{I}$ *is syntactically protoalgebraic, with witnessing transformations* $I^\flat$*, then* $I^\flat$ *is family regular in* $\mathcal{I}$ *if and only if, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\langle \phi, \psi \rangle \in \widetilde{\Omega}^{\mathcal{I}}_{\Sigma}(C(\phi, \psi));$$

(ba) If $\mathcal{I}$ *is syntactically prealgebraic, with witnessing transformations* $I^\flat$*, then* $I^\flat$ *is system regular in* $\mathcal{I}$ *if and only if, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\langle \phi, \psi \rangle \in \widetilde{\overleftrightarrow{\Omega}}^{\mathcal{I}}_{\Sigma}(\overrightarrow{C}(\phi, \psi)).$$

Proof: Part (a) follows by combining Part (a) of Proposition 1251 with the characterization of family regularity given in Theorem 584. Similarly, Part (b) follows by combing Part (b) of Proposition 1252 with the characterization of system regularity given in Theorem 584. ∎

The next results form transfer theorems, asserting that all four types of syntactic regularity, studied here, transfer from a π-institution to all its generalized matrix families/systems. We start with the two types obtained by strengthening syntactic protoalgebraicity.

Theorem 1254 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically family (system, respectively) regularly protoalgebraic, with witnessing transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$*, if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *(and all* $T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$*, respectively), all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,*

- $\langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_{\Sigma}(T)$ *iff* $I^{\mathcal{A}}_{\Sigma}[\phi, \psi] \le T$*;*
- $\phi, \psi \in T_\Sigma$ *implies* $I^{\mathcal{A}}_{\Sigma}[\phi, \psi] \le T$ *(*$\phi, \psi \in T'_\Sigma$ *implies* $I^{\mathcal{A}}_{\Sigma}[\phi, \psi] \le T'$*, respectively).*

Proof: $\mathcal{I}$ is syntactically regularly protoalgebraic if and only if, by definition, it is syntactically protoalgebraic, with witnessing transformations $I^\flat$, which are family regular, if and only if, by Theorem 810 and Proposition 1250, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ is syntactically protoalgebraic, with witnessing transformations $I^{\mathcal{A}}$, which are family regular in $\langle\mathcal{A}, C^{\mathcal{I},\mathcal{A}}\rangle$, if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, the two conditions asserted in the statement hold.

The case of system regularity may be treated similarly. ■

We close with the two types that only require syntactic prealgebraicity.

Theorem 1255 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically family (system, respectively) regularly prealgebraic with witnessing transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$*, if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle\mathbf{A}, \langle F, \alpha\rangle\rangle$*, with* $\mathbf{A} = \langle\mathbf{Sign}, \mathrm{SEN}, N\rangle$*, all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and* $T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$*, all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,*

- $\langle\phi, \psi\rangle \in \Omega^{\mathcal{A}}_\Sigma(T')$ *iff* $I^{\mathcal{A}}_\Sigma[\phi, \psi] \leq T'$*;*
- $\phi, \psi \in T_\Sigma$ *implies* $I^{\mathcal{A}}_\Sigma[\phi, \psi] \leq T$ *(*$\phi, \psi \in T'_\Sigma$ *implies* $I^{\mathcal{A}}_\Sigma[\phi, \psi] \leq T'$*, respectively).*

Proof: Similar to the proof of Theorem 1254. ■

16.4 Syntactic Regular (Pre-)Equivalentiality

Syntactic regular pre- and proto-algebraicity were defined by combining syntactic pre- and proto-algebraicity, respectively, with versions of regularity. If we upgrade syntactic pre- and proto-algebraicity to syntactic preequivalentiality and equivalentiality, respectively, then we obtain, analogously, versions of syntactic regular preequivalentiality and syntactic regular equivalentiality, respectively.

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.

- $\mathcal{I}$ is said to be **syntactically family regularly equivalential** if it is syntactically equivalential, with a witnessing collection $I^\flat$ of transformations, which is family regular in $\mathcal{I}$;
- $\mathcal{I}$ is said to be **syntactically system regularly equivalential** if it is syntactically equivalential, with a witnessing collection $I^\flat$ of transformations, which is system regular in $\mathcal{I}$;
- $\mathcal{I}$ is said to be **syntactically family regularly preequivalential** if it is syntactically preequivalential, with a witnessing collection $I^\flat$ of transformations, which is family regular in $\mathcal{I}$;
- $\mathcal{I}$ is said to be **syntactically system regularly preequivalential** if it is syntactically preequivalential, with a witnessing collection $I^\flat$ of transformations, which is system regular in $\mathcal{I}$.

Analogs of Propositions 1251 and 1252 may be proven. They follow the same lines of proof, the only difference being that the witnessing collections of transformations we are dealing with in this case, as opposed to the cases of syntactic pre- and proto-algebraicity, are parameter free.

Corollary 1256 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically equivalential* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$*.*

(a) $\mathcal{I}$ *is family regular if and only if* $I^\flat$ *is family regular in* $\mathcal{I}$*;*

(b) $\mathcal{I}$ *is left regular if and only if* $I^\flat$ *is system regular in* $\mathcal{I}$*.*

Proof: By Proposition 1251, taking into account the fact that syntactic equivalentiality is equivalent to syntactic protoalgebraicity via a parameter free collection of transformations. ∎

Corollary 1257 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically preequivalential* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$*.*

(a) $\mathcal{I}$ *is right regular if and only if* $I^\flat$ *is family regular in* $\mathcal{I}$*;*

(b) $\mathcal{I}$ *is system regular if and only if* $I^\flat$ *is system regular in* $\mathcal{I}$*.*

Proof: By Proposition 1252, taking into account the fact that syntactic preequivalentiality is equivalent to syntactic prealgebraicity via a parameter free collection of transformations. ∎

Recalling the following hierarchies of syntactic (pre)equivalentiality, of the regularity properties of $I^\flat$ and of semantic regularity,

$$\begin{tikzcd}
 & & \text{Family Regular} \arrow[d] \\
\text{Synt Equiv} \arrow[d] & I^\flat \text{ Family Reg} \arrow[d] & \text{Right Regular} \arrow[d] \\
\text{Synt PreEquiv} & I^\flat \text{ System Reg} & \text{Left Regular} \arrow[d] \\
 & & \text{System Regular}
\end{tikzcd}$$

the following hierarchy of syntactic classes of regularly (pre)equivalential π-institutions arises.

$$
\begin{tikzcd}
 & \text{Synt Fam Reg Equiv} \arrow[dl] \arrow[dr] & \\
\text{Synt Fam Reg PreEquiv} \arrow[dr] & & \text{Synt Sys Reg Equiv} \arrow[dl] \\
 & \text{Synt Sys Reg PreEquiv} &
\end{tikzcd}
$$

Furthermore, these four classes relate to their immediate subordinate properties on the syntactic side, as shown in the following diagram

$$
\begin{tikzcd}
 & \text{Synt Fam Reg Equiv} \arrow[dl] \arrow[dr] & \\
\text{Synt Sys Reg Equiv} \arrow[d] \arrow[dr] & & \text{Synt Fam Reg PreEquiv} \arrow[dl] \arrow[d] \\
\text{Synt Equiv} \arrow[d] & \begin{array}{c}\text{Synt Sys}\\ \text{Reg PreEquiv}\end{array} \arrow[dl] \arrow[dr] & I^\flat \text{ Fam Reg} \arrow[d] \\
\text{Synt PreEquiv} & & I^\flat \text{ Sys Reg}
\end{tikzcd}
$$

Moreover, from the fact that syntactic equivalentiality is equivalent to syntactic protoalgebraicity, with a parameter free witnessing collection of transformations, and, similarly for preequivalentiality and prealgebraicity, we get, immediately from the definitions. the following hierarchy of classes of π-institutions involving syntactic regular pre- and proto-algebraicity and syntactic regular (pre)equivalentiality.

$$
\begin{tikzcd}
 & \text{Synt Fam Reg Equiv} \arrow[dl] \arrow[d] \arrow[dr] & \\
\begin{array}{c}\text{Synt Fam}\\ \text{Reg PreEquiv}\end{array} \arrow[d] \arrow[dr] & \begin{array}{c}\text{Synt Fam}\\ \text{Reg ProtoAlgic}\end{array} \arrow[dl] \arrow[dr] & \begin{array}{c}\text{Synt Sys}\\ \text{Reg Equiv}\end{array} \arrow[dl] \arrow[d] \\
\begin{array}{c}\text{Synt Fam}\\ \text{Reg PreAlgic}\end{array} \arrow[dr] & \begin{array}{c}\text{Synt Sys}\\ \text{Reg PreEquiv}\end{array} \arrow[d] & \begin{array}{c}\text{Synt Sys}\\ \text{Reg ProtoAlgic}\end{array} \arrow[dl] \\
 & \text{Synt Sys Reg PreAlgic} &
\end{tikzcd}
$$

An analog to Corollary 1253 adjusts its contents to address the special case in which the collection $I^\flat$ witnessing syntactic pre- or proto-algebraicity is parameter free, thus giving rise, instead, to syntactic preequivalentiality or equivalentiality, respectively.

Corollary 1258 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *a collection of natural transformations in* $N^\flat$ *(with both arguments distinguished).*

(a) *If* $\mathcal{I}$ *is syntactically equivalential, with witnessing transformations* $I^\flat$*, then* $I^\flat$ *is family regular in* $\mathcal{I}$ *if and only if, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*
$$\langle \phi, \psi \rangle \in \widetilde{\Omega}^{\mathcal{I}}_\Sigma(C(\phi, \psi));$$

(b) *If* $\mathcal{I}$ *is syntactically preequivalential, with witnessing transformations* $I^\flat$*, then* $I^\flat$ *is system regular in* $\mathcal{I}$ *if and only if, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*
$$\langle \phi, \psi \rangle \in \widehat{\Omega}^{\mathcal{I}}_\Sigma(\overrightarrow{C}(\phi, \psi)).$$

Proof: Each part is a consequence of the corresponding part of Corollary 1253 and the fact that $I^\flat$ is assumed to be parameter free. ■

Finally, the transfer theorems for syntactic regular pre- and proto-algebraicity, Theorems 1254 and 1255, may also be easily adapted to provide analogous transfer theorems for syntactic regular equivalentiality and preequivalentiality, respectively.

Corollary 1259 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically family (system, respectively) regularly equivalential, with witnessing transformations* $I^\flat : (\mathrm{SEN}^\flat)^2 \to$

$\mathrm{SEN}^\flat$, *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, *with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, *all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *(and all* $T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$*, respectively), all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

- $\langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_\Sigma(T)$ *iff* $I^{\mathcal{A}}_\Sigma[\phi, \psi] \leq T$;
- $\phi, \psi \in T_\Sigma$ *implies* $I^{\mathcal{A}}_\Sigma[\phi, \psi] \leq T$ $(\phi, \psi \in T'_\Sigma$ *implies* $I^{\mathcal{A}}_\Sigma[\phi, \psi] \leq T'$*, respectively).*

Proof: Directly from Theorem 1254. ∎

We close with the two types that only require syntactic preequivalentiality.

Corollary 1260 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically family (system, respectively) regularly pre-equivalential with witnessing transformations* $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$, *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, *with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, *all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and* $T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, *all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

- $\langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_\Sigma(T')$ *iff* $I^{\mathcal{A}}_\Sigma[\phi, \psi] \leq T'$;
- $\phi, \psi \in T_\Sigma$ *implies* $I^{\mathcal{A}}_\Sigma[\phi, \psi] \leq T$ $(\phi, \psi \in T'_\Sigma$ *implies* $I^{\mathcal{A}}_\Sigma[\phi, \psi] \leq T'$*, respectively).*

Proof: Follows from Theorem 1255. ∎

16.5 Syntactic Assertionality

In this section, we study some of the consequences of adding to the various versions of semantic regularity, studied in detail in Section 8.2, the property of having natural theorems.

Recall, first, from Section 8.2, that there are four distinct types of semantic regularity, namely, family, right, left and system, which form the hierarchy depicted in the left diagram (obtained in Section 8.2).

$$\begin{tikzcd}
\text{Family Regular} \arrow[d] & \text{Family/Right Assertional} \arrow[dd] \\
\text{Right Regular} \arrow[d] & \\
\text{Left Regular} \arrow[d] & \text{Left Assertional} \arrow[d] \\
\text{System Regular} & \text{System Assertional}
\end{tikzcd}$$

If to the various regularity conditions, one adds the existence of theorems, then one obtains the semantic assertionality classes, which were studied in detail in Section 8.3, where it was shown that they form the hierarchy depicted in the diagram on the right.

Additionally, it was shown in Section 8.3 that these three assertionality classes dominate, respectively, the three corresponding complete reflectivity classes. This is shown in the third diagram, reproduced here from Section 8.3.

$$\begin{tikzcd}
& \text{Family Assertional} \arrow[dl] \arrow[dr] & & \\
\text{Family c-Reflective} \arrow[dr] & & \text{Left Assertional} \arrow[dl] \arrow[dr] & \\
& \text{Left c-Reflective} \arrow[dr] & & \text{System Assertional} \arrow[dl] \\
& & \text{System c-Reflective} &
\end{tikzcd}$$

In this section, we study the classes arising by adding to the various flavors of semantic regularity the property of possessing natural theorems. Since the property of possessing natural theorems is strictly stronger that having theorems, there are, in accordance with the results recalled from Section 8.3 above, only three potentially different classes of π-institutions arising. These, of course, dominate the corresponding assertionality classes. The π-institution members of these classes are termed *syntactically assertional*. A strong motivation for introducing these three classes lies in the fact that lifting the possession of theorems to that of the existence of natural theorems, in tandem with semantic regularity, is enough to allow passing from the semantic classes of completely reflective π-institutions to the corresponding syntactic classes of truth equational π-institutions.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

- $\mathcal{I}$ is **syntactically family assertional** if it is family regular and has natural theorems;
- $\mathcal{I}$ is **syntactically left assertional** if it is left regular and has natural theorems;
- $\mathcal{I}$ is **syntactically right assertional** if it is right regular and has natural theorems;

- $\mathcal{I}$ is **syntactically system assertional** if it is system regular and has natural theorems.

First, it is easy to see that syntactic family and syntactic right assertionality coincide.

Proposition 1261 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically family assertional if and only if it is syntactically right assertional.*

Proof: $\mathcal{I}$ is syntactically family assertional iff, by definition, it is family assertional and has natural theorems iff, by Proposition 591, it is right assertional and has natural theorems iff, by definition, it is syntactically right assertional. ■

Given Proposition 1261, asserting that syntactic family and syntactic right assertionality coincide, we may establish the hierarchy of syntactic assertionality classes.

Proposition 1262 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) *If* $\mathcal{I}$ *is syntactically family/right assertional, then it is syntactically left assertional;*

(b) *If* $\mathcal{I}$ *is syntactically left assertional, then it is syntactically system assertional.*

Proof: If $\mathcal{I}$ is syntactically family assertional, then it is, by definition, family assertional and has natural theorems, whence, by Proposition 592, it is left assertional and has natural theorems, i.e., it is syntactically left assertional. Similarly, if $\mathcal{I}$ is syntactically left assertional, then it is, by definition, left assertional and has natural theorems, whence, by Proposition 592, it is system assertional and has natural theorems, i.e., it is syntactically system assertional. ■

Proposition 1262 establishes the following hierarchy of **syntactic assertionality** classes, paralleling the corresponding semantic hierarchy established in Section 8.3.

$$\begin{tikzcd}
\text{Syntactically Family/Right Assertional} \arrow[d] \\
\text{Syntactically Left Assertional} \arrow[d] \\
\text{Syntactically System Assertional}
\end{tikzcd}$$

It is not difficult to see that the bottom classes of the hierarchy collapse, if restricted to stable π-institutions, and that the entire hierarchy collapses when considering only systemic π-institutions.

Proposition 1263 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is stable and syntactically system assertional, then it is syntactically left assertional;*

(b) If $\mathcal{I}$ *is systemic and syntactically system assertional, then it is syntactically family assertional.*

Proof: The first statement follows directly from Proposition 579, whereas the second implication is a consequence of Proposition 580. ■

We formalize, next, a result, which is straightforward, establishing the close interrelationships between the syntactic and semantic assertionality classes.

Proposition 1264 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically family (respectively left, system) assertional if and only if it is family (respectively left, system) assertional and has natural theorems.*

Proof: These equivalences follow by the definitions involved, since existence of natural theorems implies having theorems, as was shown in Lemma 1188. ■

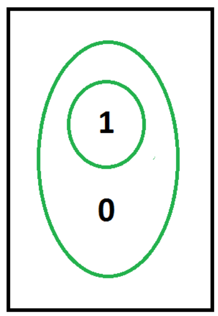
Thus, Proposition 1264 establishes the following relationships between the semantic assertionality and the corresponding syntactic assertionality classes.

$$\begin{tikzcd}
 & & \text{Synt Family Assrt} \arrow[dl] \arrow[dr] & \\
 & \text{Synt Left Assrt} \arrow[dl] \arrow[dr] & & \text{Family Assrt} \arrow[dl] \\
\text{Synt System Assrt} \arrow[dr] & & \text{Left Assrt} \arrow[dl] & \\
 & \text{System Assrt} & &
\end{tikzcd}$$

It is not difficult to show, by providing an example, that the syntactic classes are properly included in the semantic ones. More precisely, we provide an example of a π-institution which is family assertional but fails to be syntactically system assertional. Thus, it belongs to all three semantic assertionality classes but in none of the three syntactic assertionality steps of the hierarchy.

Example 1265 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with a single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$*;*
- $N^\flat$ *is the trivial category of natural transformations.*



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Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\}, \{0, 1\}\}$*.*

$\mathcal{I}$ *has two theory families,* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, which are also theory systems. Moreover,* $\Omega(\mathrm{Thm}(\mathcal{I})) = \Delta^{\mathbf{F}}$ *and* $\Omega(\mathrm{SEN}^\flat) = \nabla^{\mathbf{F}}$*.*

$$\begin{tikzcd}
\mathrm{SEN}^\flat \arrow[r, dotted] \arrow[d, dash] & \nabla^{\mathbf{F}} \arrow[d, dash] \\
\mathrm{Thm}(\mathcal{I}) \arrow[r, dotted] & \Delta^{\mathbf{F}}
\end{tikzcd}$$

Clearly, $\mathcal{I}$ *is family regular, i.e., for all* $T \in \mathrm{ThFam}(\mathcal{I})$*, and all* $x, y \in \{0, 1\}$*, if* $x, y \in T_\Sigma$*, then* $\langle x, y \rangle \in \Omega_\Sigma(T)$*. Further, obviously,* $\mathcal{I}$ *has theorems. Finally, since there are no nontrivial natural transformations in* $N^\flat$*,* $\mathcal{I}$ *does not have natural theorems. Therefore,* $\mathcal{I}$ *is family assertional but it fails to be syntactically system assertional.*

A corollary of the connections established in Proposition 1264 and the characterizations of semantic assertionality classes, given in Proposition 588, provides similar characterizations of the three syntactic assertionality classes.

Corollary 1266 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution having natural theorems and* $\tau : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *a natural theorem.*

(a) *$\mathcal{I}$ is syntactically family assertional if and only if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T = \tau/\Omega(T)$;*

(b) *$\mathcal{I}$ is syntactically left assertional if and only if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} = \tau/\Omega(T)$;*

(c) *$\mathcal{I}$ is syntactically system assertional if and only if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $T = \tau/\Omega(T)$.*

Proof: By combining Propositions 1264 and 588. ■

We conclude the section by establishing the relationships between the three syntactic assertionality classes and the three truth equationality classes, introduced and studied in detail in Chapter 11.

Theorem 1267 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) *If $\mathcal{I}$ is syntactically family assertional, with a natural theorem $\tau : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$, then it is family truth equational, with witnessing equation $\iota \approx \tau$;*

(b) *If $\mathcal{I}$ is syntactically left assertional, with a natural theorem $\tau : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$, then it is left truth equational, with witnessing equation $\iota \approx \tau$;*

(c) *If $\mathcal{I}$ is syntactically system assertional, with a natural theorem $\tau : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$, then it is system truth equational, with witnessing equation $\iota \approx \tau$.*

Proof: We prove Part (a). The other parts can be proven similarly. Suppose that $\mathcal{I}$ is syntactically family assertional, with $\tau : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ a natural theorem. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. To show that $\mathcal{I}$ is family truth equational, with witnessing equation $\iota \approx \tau$, we must establish the equivalence

$$\phi \in T_\Sigma \quad \text{iff} \quad (\iota \approx \tau)_\Sigma[\phi] \leq \Omega(T).$$

Suppose, first, that $\phi \in T_\Sigma$. Since τ is a natural theorem, we also have $\tau_\Sigma(\phi) \in T_\Sigma$. Thus, by family regularity (part of syntactic family assertionality), $\langle \phi, \tau_\Sigma(\phi) \rangle \in \Omega_\Sigma(T)$. But $\Omega(T)$ is a congruence system on $\mathbf{F}$, whence, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\langle \mathrm{SEN}^\flat(f)(\phi), \tau_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \rangle \in \Omega_{\Sigma'}(T),$$

i.e., $(\iota \approx \tau)_\Sigma[\phi] \leq \Omega(T)$.

Assume, conversely, that $(\iota \approx \tau)_\Sigma[\phi] \leq \Omega(T)$. In particular, $\langle \phi, \tau_\Sigma(\phi) \rangle \in \Omega_\Sigma(T)$. However, since τ is a natural theorem, $\tau_\Sigma(\phi) \in T_\Sigma$. Therefore, by the compatibility of $\Omega(T)$ with T, we get that $\phi \in T_\Sigma$. ■

Theorem 1267 establishes the following mixed hierarchy of syntactic assertionality and truth equationality classes.

$$
\begin{tikzcd}
 & & \text{Synt Family Assrt} \arrow[dl] \arrow[dr] & \\
 & \text{Synt Left Assrt} \arrow[dl] \arrow[dr] & & \text{Family TEq} \arrow[dl] \\
\text{Synt System Assrt} \arrow[dr] & & \text{Left TEq} \arrow[dl] & \\
 & \text{System TEq} & &
\end{tikzcd}
$$

It is not difficult to see that the syntactic assertionality classes are properly included in the corresponding truth equationality classes. This is accomplished by exhibiting a π-institution which is family truth equational but fails to be syntactically system assertional.

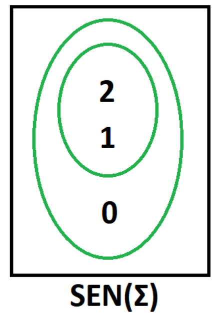
Example 1268 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the clone generated by the unary natural transformations* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$, *specified by*
$$\sigma^\flat_\Sigma(0) = 0, \quad \sigma^\flat_\Sigma(1) = 1, \quad \sigma^\flat_\Sigma(2) = 0,$$
and $\tau^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$, *given by*
$$\tau^\flat_\Sigma(0) = 2, \quad \tau^\flat_\Sigma(1) = 1, \quad \tau^\flat_\Sigma(2) = 2.$$

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*
$$\mathcal{C}_\Sigma = \{\{1, 2\}, \{0, 1, 2\}\}.$$

$\mathcal{I}$ *is systemic and its lattice of theory families and corresponding Leibniz congruence systems are shown in the diagram.*

$$
\begin{tikzcd}
012 \arrow[r, dotted] \arrow[d, dash] & \nabla^{\mathbf{F}} \arrow[d, dash] \\
12 \arrow[r, dotted] & \Delta^{\mathbf{F}}
\end{tikzcd}
$$

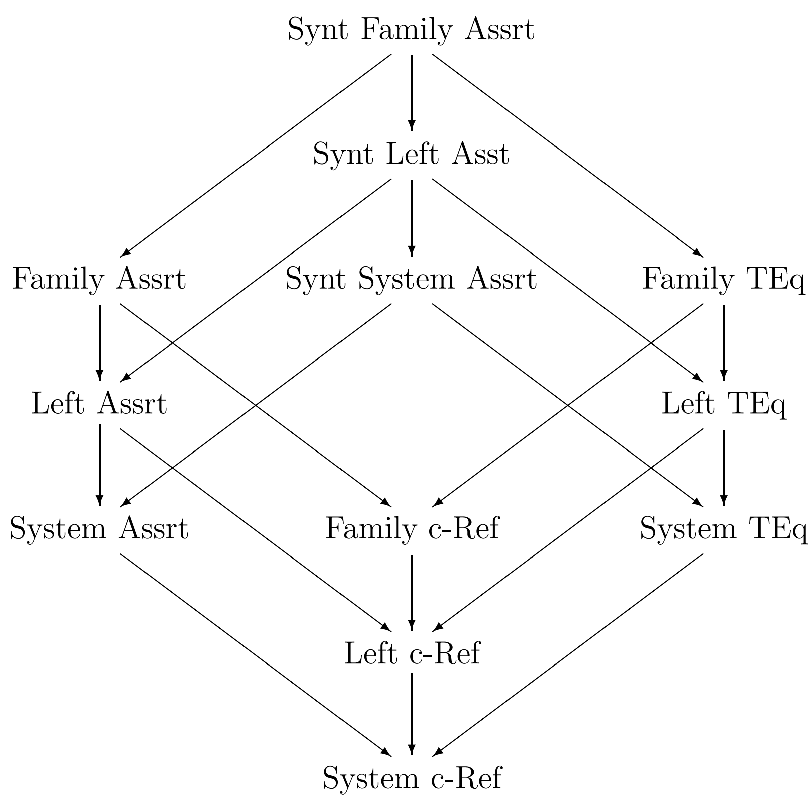


It is not difficult to check that $\mathcal{I}$ *is family truth equational, with witnessing equation* $\iota \approx \tau^\flat$.

On the other hand, $\mathcal{I}$ *is not system regular, since, for* $T = \{\{1,2\}\}$, *we have* $1,2 \in T_\Sigma$, *but* $\langle 1,2\rangle \notin \Delta^{\mathbf{F}}_\Sigma = \Omega_\Sigma(T)$.

Thus, $\mathcal{I}$ *belongs to all three truth equationality classes, but does not satisfy any of the three regularity conditions and, hence, belongs to none of the three syntactic assertionality classes.*

Finally, if we add the corresponding semantic classes of those depicted in the preceding diagram, we get a bigger view of the hierarchy consisting of assertionality (semantic and syntactic) and of complete reflectivity (semantic) and truth equationality (syntactic) classes.



Finally, based on previously established results, we can easily show that the three types of syntactic assertionality transfer from a π-institution to all its generalized matrix families. This constitutes an analog of Theorem 599 in the syntactic context.

Theorem 1269 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically family (respectively, left, system) assertional if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\langle \mathbf{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ *is syntactically family (respectively, left, system) assertional.*

Proof: This follows by putting together Theorem 585, asserting that regularity transfers, and Theorem 1197, asserting that the existence of natural theorems transfers. ■

16.6 Syntactic RW Prealgebraizability

In this section, we deal with three versions of syntactic regular weak prealgebraizability. These arise by combining syntactic prealgebraicity with each of the three versions of syntactic assertionality.

Definition 1270 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is* **syntactically regularly weakly family prealgebraizable***, or* **syntactically RWF prealgebraizable** *for short, if it is syntactically prealgebraic and syntactically family assertional;*
- $\mathcal{I}$ *is* **syntactically regularly weakly left prealgebraizable***, or* **syntactically RWL prealgebraizable** *for short, if it is syntactically prealgebraic and syntactically left assertional;*
- $\mathcal{I}$ *is* **syntactically regularly weakly system prealgebraizable***, or* **syntactically RWS prealgebraizable** *for short, if it is syntactically prealgebraic and syntactically system assertional.*

Based on the syntactic assertionality hierarchy established in Proposition 1262, we have the following

Proposition 1271 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is syntactically regularly weakly family prealgebraizable, then it is syntactically regularly weakly left prealgebraizable;*

(b) *If $\mathcal{I}$ is syntactically regularly weakly left prealgebraizable, then it is syntactically regularly weakly system prealgebraizable.*

Proof: Straightforward by combining Definition 1270 and Proposition 1262. ■

Proposition 1271 establishes the syntactic regular weak prealgebraizability hierarchy depicted in the following diagram.

$$\begin{array}{c}\text{Syntactic Regular Weak Family Prealgebraizable}\\ \downarrow \\ \text{Syntactic Regular Weak Left Prealgebraizable}\\ \downarrow \\ \text{Syntactic Regular Weak System Prealgebraizable}\end{array}$$

Being very close to the apex of the Leibniz hierarchy, just below the other classes that are studied in detail in the remaining sections of the present chapter, it compares favorably (meaning is stronger) to many of the other classes, semantic and syntactic introduced so far.

First, we look at the extant relationships between syntactic regular weak prealgebraizability classes and the four syntactic regular prealgebraicity classes of Section 16.3. It turns out that syntactic regular weak family prealgebraizability implies syntactic family regular protoalgebraicity and that syntactically regular weak system prealgebraizability implies syntactic system regular prealgebraicity. The only implication one can draw from the middle class of syntactically regular weak left prealgebraizability is the trivial one of being syntactically prealgebraic and left regular, which, strictly speaking, lies outside the syntactic hierarchy of Section 16.3.

Proposition 1272 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) *If $\mathcal{I}$ is syntactically RWF prealgebraizable, then it is syntactically family regularly protoalgebraic;*

(b) *If $\mathcal{I}$ is syntactically RWS prealgebraizable, then it is syntactically system regularly prealgebraic.*

Proof:

(a) Suppose that $\mathcal{I}$ is syntactically RWF prealgebraizable. Note that, by definition, $\mathcal{I}$ is syntactically family assertional, i.e., it is family regular and has natural theorems. Thus, by Theorem 1267, it is family truth equational. Thus, by Theorem 829, it is family c-reflective, whence, by Proposition 237, it is systemic. Thus, since, by definition, it is syntactically prealgebraic, it must be syntactically protoalgebraic. This proves that it is syntactically family regularly protoalgebraic.

(b) By definition $\mathcal{I}$ is syntactically system assertional, whence it is system regular. And it is syntactically prealgebraic, also by definition. Thus, it is syntactically system regularly prealgebraic.

∎

Thus, according to Proposition 1272, we get the mixed hierarchy depicted in the diagram.

$$\begin{tikzcd}
 & \text{Synt RWF PreAlgble} \arrow[dl] \arrow[dr] & & \\
\text{Synt Fam Reg ProtoAlgic} \arrow[dr] & & \text{Synt RWL PreAlgble} \arrow[dl] \arrow[dr] & \\
 & \begin{array}{l}\text{Synt PreAlgic}\\ \text{+Left Regular}\end{array} \arrow[dr] & & \text{Synt RWS PreAlgble} \arrow[dl] \\
 & & \text{Synt Sys Reg PreAlgic} &
\end{tikzcd}$$

As far as relationships between the syntactic regular weak prealgebraizability hierarchy and the syntactic assertionality hierarchy are concerned, we have, directly by definition, the following inclusions.

Proposition 1273 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically regularly weakly family (left, system, respectively) prealgebraizable, then it is syntactically family (left, system, respectively) assertional.*

Proof: Directly from Definition 1270. ∎

$$\begin{tikzcd}
 & & \text{Synt RWF PreAlgble} \arrow[dl] \arrow[dr] & \\
 & \text{Synt RWL PreAlgble} \arrow[dl] \arrow[dr] & & \text{Synt Family Assrt} \arrow[dl] \\
\text{Synt RWS PreAlgble} \arrow[dr] & & \text{Synt Left Assrt} \arrow[dl] & \\
 & \text{Synt System Assrt} & &
\end{tikzcd}$$

Finally we look at closer relationships with other classes that are placed relatively high in the Leibniz hierarchy. Still staying with syntactically defined classes, we have the following relationships between the classes in the syntactic regular weak prealgebraizability hierarchy and the classes in the syntactic weak prealgebraizability hierarchy, which were defined in Chapter 12.

Proposition 1274 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is syntactically regularly weakly family prealgebraizable, then it is syntactically weakly family algebraizable;*

(b) If $\mathcal{I}$ *is syntactically regularly weakly left prealgebraizable, then it is syntactically weakly left c-reflective prealgebraizable;*

(c) If $\mathcal{I}$ *is syntactically regularly weakly system prealgebraizable, then it is syntactically weakly system prealgebraizable.*

Proof: We only prove Part (a). Parts (b) and (c) can be proven similarly and are easier. Suppose $\mathcal{I}$ is syntactically regularly weakly family prealgebraizable. Then, it is, by definition syntactically family assertional. Thus, by Theorem 1267, it is family truth equational and, therefore, systemic. Thus, on the one hand, $\mathcal{I}$ is syntactically prealgebraic, and, hence, by systemicity, syntactically protoalgebraic, and, on the other, it is family truth equational. Therefore, it is syntactically weakly family algebraizable. ■

Proposition 1274, establishes the following hierarchy of syntactically regularly weakly prealgebraizable and syntactically weakly prealgebraizable π-institutions.

$$\begin{tikzcd}
 & \text{Synt RWF PreAlgble} \arrow[dl] \arrow[dr] & & \\
\text{Synt WF Algble} \arrow[dr] & & \text{Synt RWL PreAlgble} \arrow[dl] \arrow[dr] & \\
 & \text{Synt WLC PreAlgble} \arrow[dr] & & \text{Synt RWS PreAlgble} \arrow[dl] \\
 & & \text{Synt WS PreAlgble} &
\end{tikzcd}$$

Finally, we reach across to bridge the gap between syntactically and semantically defined prealgebraizability classes. We establish relationaships that govern the syntactic regular weak prealgebraizability classes and the regular weak prealgebraizability classes that were defined in Chapter 8.

Proposition 1275 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is syntactically regularly weakly family prealgebraizable, then it is regularly weakly family algebraizable;*

(b) If $\mathcal{I}$ *is syntactically regularly weakly left prealgebraizable, then it is regularly weakly left prealgebraizable;*

(c) If $\mathcal{I}$ *is syntactically regularly weakly system prealgebraizable, then it is regularly weakly system prealgebraizable.*

Proof: This follows from the facts that, on the one hand, syntactic prealgebraicity implies prealgebraicity and, on the other hand, syntactic family (left, system, respectively) assertionality implies family (left, system, respectively) assertionality. ∎

Proposition 1275 gives rise to the following mixed, semantic and syntactic, hierarchy of regularly weakly prealgebraizable π-institutions.

$$\begin{tikzcd}
 & \text{Synt RWF PreAlgble} \arrow[dl] \arrow[dr] & & \\
\text{RWF Algble} \arrow[dr] & & \text{Synt RWL PreAlgble} \arrow[dl] \arrow[dr] & \\
 & \text{RWL PreAlgble} \arrow[dr] & & \text{Synt RWS PreAlgble} \arrow[dl] \\
 & & \text{RWS PreAlgble} &
\end{tikzcd}$$

Based on existing results, we can show that all three kinds of syntactic regular weak prealgebraizability transfer from theory families/systems to filter families/systems over arbitrary $\mathbf{F}$-algebraic systems. This is the syntactic analog of Theorem 609, which asserted that regular weak prealgebraizability properties transfer from a π-institution to all its generalized matrix families.

Theorem 1276 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically regularly weakly family (left, system, respectively) prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, the* $\mathcal{I}$*-gmatrix family* $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ *is syntactically regularly weakly family (left, system, respectively) prealgebraizable.*

Proof: By Theorem 789, syntactic prealgebraicity transfers. By Theorem 585, the three regularity properties transfer. Finally, by Theorem 1197, the property of possessing natural theorems also transfers. Thus, the properties of being syntactically regularly weakly family, left and system prealgebraizable all transfer from $\mathcal{I}$ to $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$, for all $\mathbf{F}$-algebraic systems $\mathcal{A}$. ■

Finally, we adapt previously obtained results characterizing regular weak prealgebraizability to obtain similar characterizations of syntactic regular weak prealgebraizability in terms of mappings between posets of filter families/systems (including theory families/systems) and congruence systems. Essentially, to the characterizations obtained in Theorems 610, 611 and 612, we add the conditions of having enough natural transformations so that syntactic prealgebraicity is ensured and also the existence of natural theorems so that truth equationality is obtained, rather than having only complete reflectivity.

Theorem 1277 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(i) $\mathcal{I}$ *is syntactically regularly weakly family prealgebraizable;*

(ii) $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is an order isomorphism,* $\mathcal{I}$ *has a Leibniz reflexive core and a natural theorem* $\tau : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, such that, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,* $T = \tau/\Omega(T)$*;*

(iii) *For every* **F***-algebraic system* $\mathcal{A}$*, the clauses of Part (ii) hold for the* π*-institution* $\langle \mathcal{A}.C^{\mathcal{I},\mathcal{A}} \rangle$*.*

Proof: By Theorem 1299, $\mathcal{I}$ is syntactically regularly weakly family prealgebraizable if and only if, for every **F**-algebraic system $\mathcal{A}$, the π-institution $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ is also syntactically regularly weakly family prealgebraizable. Thus, to prove the statement, it suffices to consider the equivalence (i)$\Leftrightarrow$(ii).

Suppose, first, that $\mathcal{I}$ is syntactically regularly weakly family prealgebraizable. Then it is, by definition, syntactically prealgebraic. Moreover, it is, by definition, syntactically family assertional. Thus, it has a natural theorem τ and it is, by Theorem 1267, family truth equational. Thus, by Theorem 829, it is family c-reflective and, hence, by Proposition 237, systemic. This implies that it is syntactically protoalgebraic and family truth equational. Using Theorem 610, we conclude that Ω is an order isomorphism. By Theorem 788, it has a Leibniz reflexive core and, by Corollary 1266, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T = \tau/\Omega(T)$.

Assume, conversely, that the postulated conditions hold. By Proposition 1275, $\mathcal{I}$ is regularly weakly family prealgebraizable. Hence it is protoalgebraic, which, together with the postulated Leibniz property of the reflexive core, gives, by Corollary 809, that it is syntactically protoalgebraic. Further, by hypothesis and Corollary 1266, it is syntactically family assertional. Thus, by definition, it is syntactically regularly weakly family prealgebraizable. ∎

Analogous characterization theorems may be provided for syntactical regular weak left and system prealgebraizability. The proofs are analogous and are omitted.

Theorem 1278 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* **F***. The following statements are equivalent:*

(i) $\mathcal{I}$ *is syntactically regularly weakly left prealgebraizable;*

(ii) $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is an order embedding,* $\mathcal{I}$ *has a Leibniz reflexive core and a natural theorem* $\tau : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, such that, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,* $\overleftarrow{T} = \tau/\Omega(T)$*;*

(iii) *For every* **F***-algebraic system* $\mathcal{A}$*, the clauses of Part (ii) hold for the* π*-institution* $\langle \mathcal{A}.C^{\mathcal{I},\mathcal{A}} \rangle$*.*

Theorem 1279 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* **F***. The following statements are equivalent:*

(i) $\mathcal{I}$ *is syntactically regularly weakly system prealgebraizable;*

(ii) $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is an order embedding,* $\mathcal{I}$ *has a Leibniz reflexive core and a natural theorem* $\tau : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, such that, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,* $T = \tau/\Omega(T)$*;*

(iii) *For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, the clauses of Part (ii) hold for the* π*-institution* $\langle \mathcal{A}.C^{\mathcal{I},\mathcal{A}} \rangle$*.*

16.7 Syntactic RW Algebraizability

In this section, we deal with three versions of syntactic regular weak algebraizability. These arise by combining syntactic protoalgebraicity with each of the three versions of syntactic assertionality.

Definition 1280 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is* **syntactically regularly weakly family algebraizable***, or* **syntactically RWF algebraizable** *for short, if it is syntactically protoalgebraic and syntactically family assertional;*
- $\mathcal{I}$ *is* **syntactically regularly weakly left algebraizable***, or* **syntactically RWL algebraizable** *for short, if it is syntactically protoalgebraic and syntactically left assertional;*
- $\mathcal{I}$ *is* **syntactically regularly weakly system algebraizable***, or* **syntactically RWS algebraizable** *for short, if it is syntactically protoalgebraic and syntactically system assertional.*

One of the immediate consequences of family assertionality is that the π-institution under consideration must be systemic and, therefore, that pre- and protoalgebraicity coincide. This reasoning has been applied a few times already in the preceding section. It shows that syntactic regular weak family algebraizability coincides with syntactic weak family prealgebraizability.

Proposition 1281 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically regularly weakly family algebraizable if and only if it is syntactically regularly weakly family prealgebraizable.*

Proof: Suppose $\mathcal{I}$ is syntactically regularly weakly family prealgebraizable. Then, by definition, it is family assertional. Thus, by Theorem 1267, it is family truth equational. Hence, by Theorem 829, it is family completely reflective and, hence, by Proposition 237, it is systemic. Since, by definition, it is syntactically prealgebraic, it is, by systemicity, syntactically protoalgebraic. Therefore, being syntactically protoalgebraic and syntactically family assertional, it is syntactically regularly weakly family algebraizable. The reverse implication is trivial. So, equivalence of the two conditions is established. ∎

The second important observation that one can make is that syntactic regular weak left and system algebraizability coincide. This is due to the fact that protoalgebraicity implies stability.

Proposition 1282 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically regularly weakly left algebraizable if and only if it is syntactically regularly weakly system algebraizable.*

Proof: It is easy to see that the left version implies the system version. This follows directly from the fact that syntactic left assertionality implies syntactic system assertionality, established in Proposition 1262. For the converse, assume that $\mathcal{I}$ is syntactically regularly weakly system algebraizable. Then it is, by definition, syntactically protoalgebraic. This implies, by Theorem 805, that it is protoalgebraic. Hence, by Theorem 175, it is stable. Now, also by definition, $\mathcal{I}$ is syntactically system assertional. Thus, by Proposition 1263, it is syntactically left assertional. Being syntactically protoalgebraic and syntactically left assertional, $\mathcal{I}$ is, by definition, syntactically regularly weakly left algebraizable. ∎

Based on the syntactic assertionality hierarchy established in Proposition 1262, we have the following

Corollary 1283 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is syntactically regularly weakly family algebraizable, then it is syntactically regularly weakly system algebraizable.*

Proof: Straightforward by combining Definition 1280 and Proposition 1262. ∎

Proposition 1283 establishes the syntactic regular weak algebraizability hierarchy depicted in the following diagram.

$$\begin{tikzcd}
\text{Syntactic Regular Weak Family Algebraizable} \arrow[d] \\
\text{Syntactic Regular Weak System Algebraizable}
\end{tikzcd}$$

It is easy to see how the two classes introduced in this section fit within a mixed syntactic regular weak (pre)algebraizability hierarchy. Given Proposition 1281, which showed that the top classes in each of the two hierarchies coincide, the picture is completed by the following easy consequence of the definitions involved.

Proposition 1284 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically regularly weakly system algebraizable, then it is syntactically regularly weakly left prealgebraizable.*

Proof: If $\mathcal{I}$ is syntactically regularly weakly system algebraizable, then, by Proposition 1282, it is syntactically regularly weakly left algebraizable, whence, since syntactic protoalgebraicity implies syntactic prealgebraicity, we conclude that $\mathcal{I}$ is syntactically regularly weakly left algebraizable. ■

Thus, the following diagram presents the complete picture consisting of the four syntactic regular weak (pre)algebraizability classes of π-institutions. Compare this with the identical hierarchy revealed on the semantic side in Section 8.5.

$$\begin{array}{c}
\text{Syntactic RWF (Pre)Algebraizable} \\
\downarrow \\
\text{Syntactic RWS Algebraizable} \\
\downarrow \\
\text{Syntactic RWL Prealgebraizable} \\
\downarrow \\
\text{Syntactic RWS Prealgebraizable}
\end{array}$$

To complete the puzzle of the relationships between syntactic regular weak prealgebraizability and syntactic regular pre- and protoalgebraicity classes, it suffices to observe that syntactic regular weak system algebraizability implies, rather trivially, syntactic system regular protoalgebraicity.

Proposition 1285 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically RWS algebraizable, then it is syntactically system regularly protoalgebraic.*

Proof: Suppose that $\mathcal{I}$ is syntactically RWS algebraizable. Note that, by definition, $\mathcal{I}$ is syntactically system assertional, and syntactically protoalgebraic. Hence, it is syntactically system regularly protoalgebraic. ■

Thus, according to both Proposition 1272 and Proposition 1285, we get the following hierarchy, which completes the diagram given in the preceding section after Proposition 1272.

$$\begin{tikzcd}[column sep=tiny]
& \text{Synt RWF Algble} \arrow[dl] \arrow[dr] & & & \\
\text{Synt Fam Reg ProtoAlgic} \arrow[dr] & & \text{Synt RWS Algble} \arrow[dl] \arrow[dr] & & \\
& \text{Synt Sys Reg ProtoAlgic} \arrow[dr] & & \text{Synt RWL PreAlgble} \arrow[dl] \arrow[dr] & \\
& & \begin{array}{l}\text{Synt PreAlgic}\\ \text{+Left Regular}\end{array} \arrow[dr] & & \text{Synt RWS PreAlgble} \arrow[dl] \\
& & & \text{Synt Sys Reg PreAlgic} &
\end{tikzcd}$$

As far as relationships between the syntactic regular weak prealgebraizability hierarchy and the syntactic assertionality hierarchy are concerned, the picture is completed by realizing that syntactic regular weak system algebraizability implies syntactic left assertionality.

Corollary 1286 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically RWS algebraizable, then it is syntactically left assertional.*

Proof: The conclusion follows directly by Proposition 1282. ■

Thus, according to Corollary 1286, and the hierarchy obtained in the preceding section, the interactions with syntactic assertionality properties are as shown in the diagram.

$$
\begin{tikzcd}
 & & & \text{Synt RWF Algble} \arrow[dl] \arrow[dr] & \\
 & & \text{Synt RWS Algble} \arrow[dl] \arrow[dr] & & \text{Synt Family Assrt} \arrow[dl] \\
 & \text{Synt RWL PreAlgble} \arrow[dl] & & \text{Synt Left Assrt} \arrow[ddll] & \\
\text{Synt RWS PreAlgble} \arrow[dr] & & & & \\
 & \text{Synt System Assrt} & & &
\end{tikzcd}
$$

Finally we look at completing the hierarchy diagrams examining the relationships with other classes that are placed relatively high in the Leibniz hierarchy. Staying with syntactically defined classes, we have the following extra relationship between syntactically regularly weakly system algebraizable π-institutions and syntactically weakly (system) algebraizable ones.

Proposition 1287 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically regularly weakly system algebraizable, then it is syntactically weakly (system) algebraizable.*

Proof: Suppose $\mathcal{I}$ is syntactically regularly weakly system algebraizable. Then, it is, by definition syntactically protoalgebraic and system assertional. Thus, by Theorem 1267, it is syntactically protoalgebraic and system truth equational. Therefore, it is, by definition, syntactically weakly (system) algebraizable. ∎

Proposition 1287, in conjunction with Proposition 1274, completes the hierarchy of syntactically regularly weakly (pre)algebraizable and syntactically weakly (pre)algebraizable π-institutions, part of which was shown following Proposition 1274.

$$\begin{tikzcd}
 & \text{Synt RWF Algble} \arrow[dl] \arrow[dr] & & & \\
\text{Synt WF Algble} \arrow[dr] & & \text{Synt RWS Algble} \arrow[dl] \arrow[dr] & & \\
 & \text{Synt W Algble} \arrow[dr] & & \text{Synt RWL PreAlgble} \arrow[dl] \arrow[dr] & \\
 & & \text{Synt WLC PreAlgble} \arrow[dr] & & \text{Synt RWS PreAlgble} \arrow[dl] \\
 & & & \text{Synt WS PreAlgble} &
\end{tikzcd}$$

Finally, we revisit the relationships between syntactically and semantically defined (pre)algebraizability classes. We show that syntactic regular weak system algebraizability implies regular weak system algebraizability. This completes the picture established in Proposition 1275.

Proposition 1288 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically regularly weakly system algebraizable, then it is regularly weakly system algebraizable.*

Proof: This follows from the facts that, on the one hand, by Theorem 805, syntactic protoalgebraicity implies protoalgebraicity and, on the other hand, by Proposition 1264, syntactic system assertionality implies system assertionality. ∎

Propositions 1275 and 1288 give rise to the following mixed, semantic and syntactic, hierarchy of regularly weakly (pre)algebraizable π-institutions, which completes the hierarchy shown after Proposition 1275.

$$
\begin{tikzcd}[column sep=small]
 & \text{Synt RWF Algble} \arrow[dl] \arrow[dr] & & & \\
\text{RWF Algble} \arrow[dr] & & \text{Synt RWS Algble} \arrow[dl] \arrow[dr] & & \\
 & \text{RWS Algble} \arrow[dr] & & \text{Synt RWL PreAlgble} \arrow[dl] \arrow[dr] & \\
 & & \text{RWL PreAlgble} \arrow[dr] & & \text{Synt RWS PreAlgble} \arrow[dl] \\
 & & & \text{RWS PreAlgble} &
\end{tikzcd}
$$

As was the case with the three syntactic regular weak prealgebraizability classes, we may show that syntactic regular weak system algebraizability also transfers from theory families/systems to filter families/systems over arbritrary **F**-algebraic systems. This completes Transfer Theorem 1276.

Theorem 1289 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically regularly weakly system algebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, the* $\mathcal{I}$*-gmatrix family* $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ *is syntactically regularly weakly system algebraizable.*

Proof: By Theorem 810, syntactic protoalgebraicity transfers. By Theorem 585, system regularity transfers. Finally, by Theorem 1197, the property of possessing natural theorems also transfers. Thus, syntactic regular weak system algebraizability transfers from $\mathcal{I}$ to $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$, for all **F**-algebraic systems $\mathcal{A}$. This establishes the theorem. ■

Finally, we adapt previously obtained results characterizing syntactic regular weak prealgebraizability to obtain a similar characterization of syntactic regular weak system algebraizability in terms of mappings between posets of filter families/ systems (including theory families/systems) and congruence systems. This completes Theorem 1277.

Theorem 1290 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(i) $\mathcal{I}$ *is syntactically regularly weakly system algebraizable;*

(ii) *$\mathcal{I}$ is stable, $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is an order isomorphism, $\mathcal{I}$ has a Leibniz reflexive core and a natural theorem $\tau : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$, such that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $T = \tau/\Omega(T)$;*

(iii) *For every $\mathbf{F}$-algebraic system $\mathcal{A}$, the clauses of Part (ii) hold for the π-institution $\langle \mathcal{A}.C^{\mathcal{I},\mathcal{A}} \rangle$.*

Proof: By Theorem 1289, $\mathcal{I}$ is syntactically regularly weakly system algebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, the π-institution $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ is also syntactically regularly weakly system algebraizable. Thus, to prove the statement, it suffices to consider the equivalence (i)$\Leftrightarrow$(ii).

Suppose, first, that $\mathcal{I}$ is syntactically regularly weakly system algebraizable. Then, it is, in particular, by Proposition 1288, regularly weakly system algebraizable, and, by definition, syntactically protoalgebraic and syntactically system assertional. By Theorem 624, $\mathcal{I}$ is stable, $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is an order isomorphism and, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $T = \tau/\Omega(T)$, where τ is a natural theorem, whose existence is guaranteed by syntactic assertionality. Finally, syntactic protoalgebraicity implies, by Theorem 805, that $\mathcal{I}$ has a Leibniz reflexive core.

Assume, conversely, that the postulated conditions hold. By Theorem 624, $\mathcal{I}$ is regularly weakly system algebraizable. Hence it is protoalgebraic, which, together with the postulated Leibniz property of the reflexive core, gives, by Theorem 805, that it is syntactically protoalgebraic. Further, since it is regularly weakly system algebraizable, it is, in particular, system regular and, by hypothesis, has natural theorems. Thus, it is syntactically system assertional. Hence, being syntactically protoalgebraic and syntactically system assertional, it is, by definition, syntactically regularly weakly system algebraizable. ■

16.8 Syntactic Regular (Pre)Algebraizability

In this section, we deal with the four versions of syntactic regular (pre)algebraizability, corresponding to the four versions of syntactic regular weak (pre)algebraizability that were studied in the preceding two sections. These arise by combining syntactic (pre)equivalentiality with each of the three versions of syntactic assertionality. They give rise to a four-element linear hierarchy that parallels that of syntactically regularly weakly (pre)algebraizable π-institutions and lies directly above it. The four classes, introduced and studied in the present section, lie at the very apex of the Leibniz hierarchies that were studied in detail in the monograph, and which form the backbone of the field of categorical abstract algebraic logic.

A priori, one may define six different classes of syntactically regularly (pre)algebraizable π-institutions. Three of these classes use syntactic preequivalentiality.

Definition 1291 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is* **syntactically regularly family prealgebraizable***, or* **syntactically RF prealgebraizable** *for short, if it is syntactically preequivalential and syntactically family assertional;*
- $\mathcal{I}$ *is* **syntactically regularly left prealgebraizable***, or* **syntactically RL prealgebraizable** *for short, if it is syntactically preequivalential and syntactically left assertional;*
- $\mathcal{I}$ *is* **syntactically regularly system prealgebraizable***, or* **syntactically RS prealgebraizable** *for short, if it is syntactically preequivalential and syntactically system assertional.*

Three more classes use syntactic equivalentiality.

Definition 1292 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is* **syntactically regularly family algebraizable***, or* **syntactically RF algebraizable** *for short, if it is syntactically equivalential and syntactically family assertional;*
- $\mathcal{I}$ *is* **syntactically regularly left algebraizable***, or* **syntactically RL algebraizable** *for short, if it is syntactically equivalential and syntactically left assertional;*
- $\mathcal{I}$ *is* **syntactically regularly system algebraizable***, or* **syntactically RS algebraizable** *for short, if it is syntactically equivalential and syntactically system assertional.*

We can show that similar relationships to those holding between the syntactic regular weak (pre)algebraizability classes are valid in this case also, leading to the collapsing of the six-class hierarchy (which, a priori, would look as in the accompanying figure)

$$
\begin{tikzcd}
 & \text{Synt RF Algble} \arrow[dl] \arrow[dr] & & \\
\text{Synt RF PreAlgble} \arrow[dr] & & \text{Synt RL Algble} \arrow[dl] \arrow[dr] & \\
 & \text{Synt RL PreAlgble} \arrow[dr] & & \text{Synt RS Algble} \arrow[dl] \\
 & & \text{Synt RS PreAlgble} &
\end{tikzcd}
$$

to only four classes forming a linear hierarchy.

The top classes of syntactically regularly family prealgebraizable and algebraizable π-institutions coincide. Moreover, in the algebraizability case, syntactic regular left algebraizability turns out to be identical with syntactic regular system algebraizability. These relationships are presented formally in the following proposition.

Proposition 1293 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is syntactically regularly family prealgebraizable if and only if it is syntactically regularly family algebraizable;*

(b) $\mathcal{I}$ *is syntactically regularly left algebraizable if and only if it is syntactically regularly system algebraizable.*

Proof:

(a) Of course the right-to-left implication is trivial, since, by definition (see Section 13.2 and 13.3), syntactic equivalentiality implies syntactic preequivalentiality. On the other hand, by Theorem 1267, syntactic family assertionality implies family truth equationality, which, in turn, implies, by Theorem 829, family c-reflectivity and, hence, by Lemma 233, systemicity. Thus, under the given hypothesis, syntactic preequivalentiality coincides with syntactic equivalentiality.

(b) Again, since it is obvious that syntactical regular left algebraizability implies syntactical system algebraizability, in view of the fact (Proposition 1262) that syntactical left assertionality implies syntactical system assertionality, one must focus on the reverse implication. However, syntactic system algebraizability entails syntactic protoalgebraicity, which implies, by Theorem 792, protoalgebraicity, which, in turn, by Lemma 170, implies stability. And under stability, by Proposition 1263, syntactic left assertionality and syntactic system assertionality coincide. ■

Now the following implications are straightforward and establish the hierarchy obtained from the preceding diagram, if one takes into account the pairwise identification of classes proven in Proposition 1293.

Proposition 1294 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is syntactically regularly family (pre)algebraizable, then it is syntactically regularly system (left) algebraizable;*

(b) If $\mathcal{I}$ *is syntactically regularly system (left) algebraizable, then it is syntactically regularly left prealgebraizable;*

(c) *If $\mathcal{I}$ is syntactically regularly left prealgebraizable, then it is syntactically regularly system prealgebraizable.*

Proof: Part (a) relies on the fact that, by Proposition 1262, syntactic family assertionality is stronger than syntactic system assertionality. Part (b) relies on the fact that syntactic equivalentiality implies syntactic preequivalentiality. Finally, Part (c) is a direct consequence of syntactic system assertionality being dominated by syntactic left assertionality (Proposition 1262). ∎

Proposition 1294, which takes into account the identifications of Proposition 1293, establishes the syntactic regular (pre)algebraizability hierarchy depicted in the following diagram.

$$\begin{array}{c}\text{Syntactic Regular Family Algebraizable}\\ \downarrow\\ \text{Syntactic Regular System Algebraizable}\\ \downarrow\\ \text{Syntactic Regular Left Prealgebraizable}\\ \downarrow\\ \text{Syntactic Regular System Prealgebraizable}\end{array}$$

We look, next, at the relationships between syntactic regular (pre)algebraizability classes and the four syntactic regular (pre)equivalentiality classes of Section 16.4. Syntactic regular family algebraizability implies syntactic family regular equivalentiality, syntactic regular system algebraizability implies syntactic system regular equivalentiality and syntactic regular system prealgebraizability implies syntactic system regular preequivalentiality. However, from syntactic regular left prealgebraizability we can only make the trivial deduction of syntactic preequivalentiality and left regularity. Strictly speaking, the combination of these two properties does not form a class in the syntactic hierarchy of Section 16.4.

Proposition 1295 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

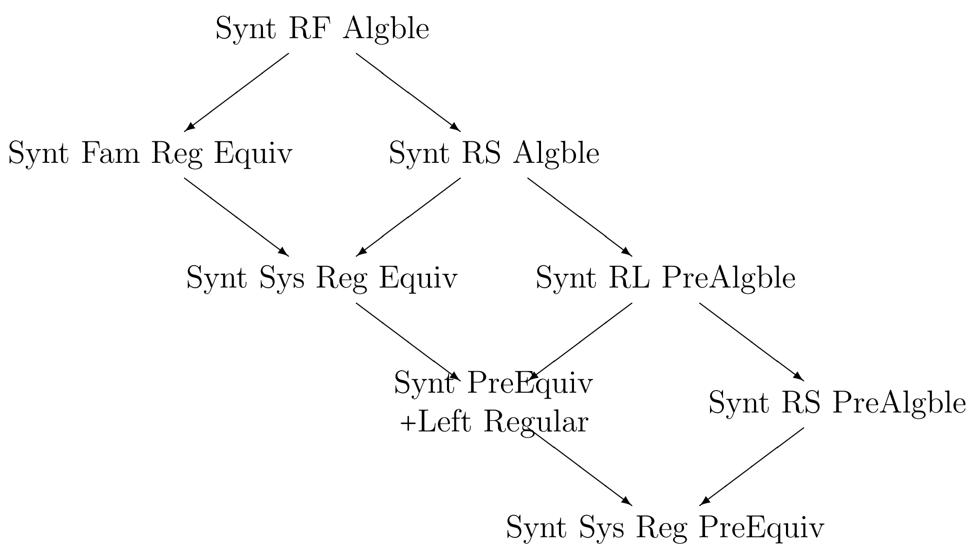
(a) *If $\mathcal{I}$ is syntactically RF algebraizable, then it is syntactically family regularly equivalential;*

(b) *If $\mathcal{I}$ is syntactically RS algebraizable, then it is syntactically system regularly equivalential;*

(c) If $\mathcal{I}$ is syntactically RS prealgebraizable, then it is syntactically system regularly preequivalential.

Proof: For Part (a) observe that, by definition, $\mathcal{I}$ is syntactically equivalential and syntactically family assertional, which implies that it is family regular. Thus, it is syntactically family regularly equivalential. Similarly, for Part (b), $\mathcal{I}$ is, by definition, syntactically equivalential and syntactically system assertional, which implies system regularity. Thus, it is syntactically system regularly equivalential. Finally, in Part (c), $\mathcal{I}$ is, by definition, syntactically preequivalential and syntactically system assertional, whence, once more, it is also syntactically system regular. Hence, it is syntactically system regularly preequivalential. ■

Thus, according to Proposition 1295, we get the mixed hierarchy depicted in the diagram.



We do not dwell on relationships between the syntactic regular (pre)algebraizability classes and the syntactic assertionlity classes, since those are direct consequences of the relationships, already established in the preceding section, between syntactic regular weak (pre)algebraizability classes and the syntactic assertionality classes, once the following, also easily obtainable, relations between syntactic regular (pre)algebraizability classes and syntactic regular weak (pre)algebraizability classes are established.

Proposition 1296 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) If $\mathcal{I}$ is syntactically regularly family (system, respectively) algebraizable, then it is syntactically regularly weakly family (system, respectively) algebraizable;

(b) *If $\mathcal{I}$ is syntactically regularly left (system, respectively) prealgebraizable, then it is syntactically regularly weakly left (system, respectively) prealgebraizable.*

Proof: Directly from the definitions involved. ■

Thus, we get a comprehensive picture of the syntactic regular prealgebraizability hierarchy, including both weak and "strong" (meaning non-weak) classes.

$$\begin{tikzcd}
 & \text{Synt RF Algble} \arrow[dl] \arrow[dr] & & & \\
\text{Synt RWF Algble} \arrow[dr] & & \text{Synt RS Algble} \arrow[dl] \arrow[dr] & & \\
 & \text{Synt RWS Algble} \arrow[dr] & & \text{Synt RL PreAlgble} \arrow[dl] \arrow[dr] & \\
 & & \text{Synt RWL PreAlgble} \arrow[dr] & & \text{Synt RS PreAlgble} \arrow[dl] \\
 & & & \text{Synt RWS PreAlgble} &
\end{tikzcd}$$

Finally we look at the relationships with other classes that are placed just below syntactically regularly (pre)algebraizable π-institutions, namely, the classes in the syntactic (pre)algebraizablity hierarchy and those in the (semantic) regular (pre)algebraizability hierarchy. The former hierarchy was studied in detail in Chapter 12, whereas the latter was studied in Chapter 8. Starting with the relationships between the syntactic regular (pre)algebraizability and the syntactic (pre)algebraizability classes, we get the following

Proposition 1297 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) *If $\mathcal{I}$ is syntactically regularly family (system, respectively) algebraizable, then it is syntactically family (system, respectively) algebraizable;*

(b) *If $\mathcal{I}$ is syntactically regularly left (system, respectively) prealgebraizable, then it is syntactically left (system, respectively) prealgebraizable.*

Proof: Part (a) follows from the fact that syntactic family and system assertionality imply, respectively, family and system truth equationality. Part (b), similarly, follows from the fact that syntactic left and system assertionality imply, respectively, left and system truth equationality. All the aforementioned implications, forming the key to the inclusions in the statement, are the subject of Theorem 1267. ∎

Proposition 1297, establishes the following mixed hierarchy of syntactically regularly (pre)algebraizable and syntactically (pre)algebraizable π-institutions.

$$
\begin{tikzcd}
 & \text{Synt RF Algble} \arrow[dl] \arrow[dr] & & & \\
\text{Synt F Algble} \arrow[dr] & & \text{Synt RS Algble} \arrow[dl] \arrow[dr] & & \\
 & \text{Synt S Algble} \arrow[dr] & & \text{Synt RL PreAlgble} \arrow[dl] \arrow[dr] & \\
 & & \text{Synt L PreAlgble} \arrow[dr] & & \text{Synt RS PreAlgble} \arrow[dl] \\
 & & & \text{Synt S PreAlgble} &
\end{tikzcd}
$$

We close with the relationships between syntactically and semantically defined regular (pre)algebraizability classes.

Proposition 1298 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is syntactically regularly family (system, respectively) algebraizable, then it is regularly family (system, respectively) algebraizable;*

(b) If $\mathcal{I}$ *is syntactically regularly left (system, respectively) prealgebraizable, then it is regularly left (system, respectively) prealgebraizable.*

Proof: This follows from the facts that, on the one hand, syntactic pre- and protoalgebraicity imply respectively pre- and protoalgebraicity, and, on the other hand, syntactic family (left, system, respectively) assertionality implies family (left, system, respectively) assertionality. The former implications are established in Theorems 771 and 792. The latter are by Proposition 1264. ∎

Proposition 1298 gives rise to the following mixed, semantic and syntactic, hierarchy of regularly (pre)algebraizable π-institutions.

$$
\begin{tikzcd}
& \text{Synt RF Algble} \arrow[dl] \arrow[dr] & & & \\
\text{RF Algble} \arrow[dr] & & \text{Synt RS Algble} \arrow[dl] \arrow[dr] & & \\
& \text{RS Algble} \arrow[dr] & & \text{Synt RL PreAlgble} \arrow[dl] \arrow[dr] & \\
& & \text{RL PreAlgble} \arrow[dr] & & \text{Synt RS PreAlgble} \arrow[dl] \\
& & & \text{RS PreAlgble} &
\end{tikzcd}
$$

As was the case with syntactic regular weak (pre)algebraizability, all four flavors of syntactic regular (pre)algebraizability transfer from theory families/systems to filter families/systems over arbitrary $\mathbf{F}$-algebraic systems. This is a "strong" analog of Theorems 1276 and 1289, which asserted that syntactic regular weak (pre)algebraizability properties transfer from a π-institution to all its generalized matrix families.

Theorem 1299 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is syntactically regularly family (system, respectively) algebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, the* $\mathcal{I}$*-gmatrix family* $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ *is syntactically regularly family (system, respectively) algebraizable;*

(b) $\mathcal{I}$ *is syntactically regularly left (system, respectively) prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, the* $\mathcal{I}$*-gmatrix family* $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ *is syntactically regularly left (system, respectively) prealgebraizable.*

Proof: By Theorems 955 and 972, syntactic preequivalentiality and syntactic equivalentiality transfer. By Theorem 585, the three regularity properties transfer. Finally, by Theorem 1197, the property of possessing natural theorems also transfers. Thus, all four syntactic regular (pre)algebraizability properties transfer from $\mathcal{I}$ to $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$, for all $\mathbf{F}$-algebraic systems $\mathcal{A}$. ■

Finally, we obtain characterizations of syntactically regular (pre)algebraizability in terms of mappings between posets of filter families/ systems (including theory families/systems) and congruence systems.

Theorem 1300 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(i) $\mathcal{I}$ *is syntactically regularly family algebraizable;*

(ii) $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is an order isomorphism commuting with inverse logical extensions,* $\mathcal{I}$ *has a Leibniz binary reflexive core and a natural theorem* $\tau : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, such that, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,* $T = \tau/\Omega(T)$*;*

(iii) For every $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, the clauses of Part (ii) hold for the* π*-institution* $\langle \mathcal{A}.C^{\mathcal{I},\mathcal{A}} \rangle$*.*

Proof: By Theorem 1299, $\mathcal{I}$ is syntactically regularly family algebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, the π-institution $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ is also syntactically regularly family algebraizable. Thus, to prove the statement, it suffices to consider the equivalence (i)$\Leftrightarrow$(ii).

Suppose, first, that $\mathcal{I}$ is syntactically regularly family algebraizable. Then it is, by definition, syntactically equivalential and syntactically family assertional. Thus, it has a natural theorem τ, it is family regular and it is, by Theorem 1267, family truth equational. Using Corollary 649, we conclude that Ω is an order isomorphism commuting with inverse logical extensions, by Theorem 983, that it has a Leibniz binary reflexive core and, by Corollary 1266, that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T = \tau/\Omega(T)$.

Assume, conversely, that the postulated conditions hold. By Corollary 649, $\mathcal{I}$ is regularly family algebraizable. Hence it is equivalential, whence, together with the postulated Leibniz property of the binary reflexive core, we obtain, by Corollary 983, that it is syntactically equivalential. Further, by hypothesis and Corollary 1266, it is syntactically family assertional. Thus, by definition, it is syntactically regularly family algebraizable. ■

Theorem 1301 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(i) $\mathcal{I}$ *is syntactically regularly system algebraizable;*

(ii) $\mathcal{I}$ *is stable,* $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is an order isomorphism commuting with inverse logical extensions,* $\mathcal{I}$ *has a Leibniz binary reflexive core and a natural theorem* $\tau : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, such that, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,* $T = \tau/\Omega(T)$*;*

(iii) For every $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, the clauses of Part (ii) hold for the* π*-institution* $\langle \mathcal{A}.C^{\mathcal{I},\mathcal{A}} \rangle$*.*

Proof: Similar to that of Theorem 1300. ■

Analogous characterization theorems may be provided for the syntactic regular prealgebraizability properties. The proofs are also similar and are, therefore, omitted.

Theorem 1302 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(i) $\mathcal{I}$ *is syntactically regularly left prealgebraizable;*

(ii) $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is an order embedding commuting with inverse logical extensions,* $\mathcal{I}$ *has a Leibniz binary reflexive core and a natural theorem* $\tau : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, such that, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,* $\overleftarrow{T} = \tau/\Omega(T)$*;*

(iii) For every $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, the clauses of Part (ii) hold for the* π*-institution* $\langle \mathcal{A}.C^{\mathcal{I},\mathcal{A}} \rangle$*.*

Theorem 1303 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(i) $\mathcal{I}$ *is syntactically regularly system prealgebraizable;*

(ii) $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is an order embedding commuting with inverse logical extensions,* $\mathcal{I}$ *has a Leibniz binary reflexive core and a natural theorem* $\tau : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, such that, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,* $T = \tau/\Omega(T)$*;*

(iii) For every $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, the clauses of Part (ii) hold for the* π*-institution* $\langle \mathcal{A}.C^{\mathcal{I},\mathcal{A}} \rangle$*.*