Chapter 17

The Syntactic Leibniz Hierarchy: Attic II

17.1 Finitary Companions Revisited

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall from Chapter 9 the construction of the *finitary companion* $\mathcal{I}^f = \langle \mathbf{F}, C^f \rangle$ of $\mathcal{I}$. It is defined, by setting, for all $\Sigma in |\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$C^f_\Sigma(\Phi) = \bigcup\{C_\Sigma(\Phi') : \Phi' \subseteq_f \Phi\},$$

where $\subseteq_f$ denotes the finite subset relation. It was shown in Corollary 653 that $\mathcal{I}^f$ is the largest finitary π-institution based on $\mathbf{F}$ that lies below $\mathcal{I}$ is the $\leq$ ordering. Furthermore, even though it is obvious, based on $\mathcal{I}^f \leq \mathcal{I}$, that $\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$, Proposition 655 provided a characterization of those sentence families of $\mathbf{F}$ that are theory families of $\mathcal{I}^f$. More concretely, it asserted that $T \in \mathrm{ThFam}(\mathcal{I}^f)$ if and only if it is the union of a directed locally finitely generated collection of theory families of $\mathcal{I}$.

Turning now to the Leibniz hierarchy, some of the semantic aspects of which, in relation to finitarity, were studied in some detail in Chapter 9, it was proven in Lemma 656 that protoalgebraicity is inherited by $\mathcal{I}$ from $\mathcal{I}^f$, i.e., if $\mathcal{I}^f$ is protoalgebraic, then so is $\mathcal{I}$ itself. This is a rather simple consequence of the fact that $\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$.

Recall from Chapter 11 the definition of the *reflexive core* $R^{\mathcal{I}}$ of a π-institution $\mathcal{I}$. It consists of all natural transformations $\rho^\flat$ in $N^\flat$, with two distinguished arguments, having the property that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\rho^\flat_\Sigma[\phi, \phi] \leq \mathrm{Thm}(\mathcal{I}).$$

It is not very difficult to show that the reflexive core of the finitary companion $\mathcal{I}^f$ of a π-institution $\mathcal{I}$ is included in that of $\mathcal{I}$.

Lemma 1304 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *Then*

$$R^{\mathcal{I}^f} \subseteq R^{\mathcal{I}}.$$

Proof: Suppose $\rho^\flat \in R^{\mathcal{I}^f}$ and consider $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. We have

$$\begin{array}{lll} \rho^\flat_\Sigma[\phi, \phi] & \leq & \mathrm{Thm}(\mathcal{I}^f) \quad (\rho^\flat \in R^{\mathcal{I}^f}) \\ & \leq & \mathrm{Thm}(\mathcal{I}). \quad (\mathrm{Thm}(\mathcal{I}) \in \mathrm{ThFam}(\mathcal{I}^f)) \end{array}$$

Thus, by definition, $\rho^\flat \in R^{\mathcal{I}}$. It follows that $R^{\mathcal{I}^f} \subseteq R^{\mathcal{I}}$. ■

Recall that the reflexive core $R^{\mathcal{I}}$ is said to be *Leibniz* if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi])).$$

From the fact that $R^{\mathcal{I}^f} \subseteq R^{\mathcal{I}}$ it follows at once that, if $\mathcal{I}^f$ is protoalgebraic and $R^{\mathcal{I}^f}$ is Leibniz in $\mathcal{I}^f$, then $R^{\mathcal{I}}$ is Leibniz in $\mathcal{I}$.

Proposition 1305 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}^f$ *is protoalgebraic and* $R^{\mathcal{I}^f}$ *is Leibniz in* $\mathcal{I}^f$, *then so is* $R^{\mathcal{I}}$ *in* $\mathcal{I}$.

Proof: Suppose that $\mathcal{I}^f$ is protoalgebraic and $R^{\mathcal{I}^f}$ is Leibniz in $\mathcal{I}^f$. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. We then have

$$\begin{array}{lll}\langle \phi, \psi \rangle & \in & \Omega_\Sigma(C^f(R^{\mathcal{I}^f}_\Sigma[\phi, \psi])) \quad (R^{\mathcal{I}^f} \text{ Leibniz in } \mathcal{I}^f) \\ & \subseteq & \Omega_\Sigma(C^f(R^{\mathcal{I}}_\Sigma[\phi, \psi])) \quad (\text{Lemma 1304 and hypothesis}) \\ & \subseteq & \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi])). \quad (\text{Corollary 653 and hypothesis})\end{array}$$

Therefore, $R^{\mathcal{I}}$ is Leibniz in $\mathcal{I}$. ■

We can now show that syntactic protoalgebraicity is inherited by a π-institution $\mathcal{I}$ from its finitary companion $\mathcal{I}^f$. This forms an analog in the syntactic context of Lemma 656.

Theorem 1306 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}^f$ *is syntactically protoalgebraic, then so is* $\mathcal{I}$.

Proof: Suppose $\mathcal{I}^f$ is syntactically protoalgebraic. By Theorem 805, it is protoalgebraic and its reflexive core $R^{\mathcal{I}^f}$ is Leibniz in $\mathcal{I}^f$. Therefore, by Lemma 656, $\mathcal{I}$ is protoalgebraic and, by Proposition 1305, $R^{\mathcal{I}}$ is Leibniz in $\mathcal{I}$. Therefore, again by Theorem 805, $\mathcal{I}$ is syntactically protoalgebraic. ■

Recalling Theorem 799, which characterizes syntactic protoalgebraicity in terms of the global family modus ponens property of the reflexive core, we derive the following

Corollary 1307 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $R^{\mathcal{I}^f}$ *has the global family MP in* $\mathcal{I}^f$, *then* $R^{\mathcal{I}}$ *has the global family MP in* $\mathcal{I}$.

Proof: If $R^{\mathcal{I}^f}$ has the global family MP in $\mathcal{I}^f$, then, by Theorem 799, $\mathcal{I}^f$ is syntactically protoalgebraic. Thus, by Theorem 1306, $\mathcal{I}$ is syntactically protoalgebraic, whence, again by Theorem 799, applied in the opposite direction, $R^{\mathcal{I}}$ has the global family MP in $\mathcal{I}$. ■

Alternatively, instead of deriving the implication in Corollary 1307 by applying Theorem 1306, we may prove it first and then use Theorem 799 to establish that syntactic protoalgebraicity of $\mathcal{I}^f$ implies the syntactic protoalgebraicity of $\mathcal{I}$. We outline this reasoning also, at the expense of having to repeat Corollary 1307 and Theorem 1306.

Lemma 1308 (Corollary 1307) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $R^{\mathcal{I}^f}$ *has the global family MP in* $\mathcal{I}^f$, *then* $R^{\mathcal{I}}$ *has the global family MP in* $\mathcal{I}$.

Proof: Suppose $R^{\mathcal{I}^f}$ has the global family MP in $\mathcal{I}^f$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$\phi \in T_\Sigma \quad \text{and} \quad R^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T.$$

By Lemma 1304, we get

$$\phi \in T_\Sigma \quad \text{and} \quad R^{\mathcal{I}^f}_\Sigma[\phi, \psi] \leq T.$$

But $\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$ and $R^{\mathcal{I}^f}$ is assumed to have the global family MP in $\mathcal{I}^f$. Thus, $\psi \in T_\Sigma$. This proves that $R^{\mathcal{I}}$ has the global family MP in $\mathcal{I}$. ■

Corollary 1309 (Theorem 1306) *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}^f$ is syntactically protoalgebraic, then so is $\mathcal{I}$.*

Proof: Suppose $\mathcal{I}^f$ is syntactically protoalgebraic. Then, by Theorem 799, $R^{\mathcal{I}^f}$ has the global family MP in $\mathcal{I}^f$. Thus, by Lemma 1308, $R^{\mathcal{I}}$ has the global family MP in $\mathcal{I}$. Hence, again by applying Theorem 799, only now in the reverse direction, $\mathcal{I}$ is syntactically protoalgebraic. ■

A similar work can be undertaken concerning truth equationality, based on an analog of Lemma 657, but referring to family c-reflectivity, which can be proved in a similar fashion as Lemma 657. We now provide the details.

It is straightforward to see, first of all, that family complete reflectivity is also inherited by $\mathcal{I}$ itself by its finitary companion.

Lemma 1310 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}^f$ is family c-reflective, then so is $\mathcal{I}$.*

Proof: If $\mathcal{I}^f$ is family c-reflective, then, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I}^f)$,

$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \quad \text{implies} \quad \bigcap_{T \in \mathcal{T}} T \leq T'.$$

In particular, the condition holds if quantification is restricted over the collection $\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$. Therefore, $\mathcal{I}$ is family c-reflective. ■

It is not very hard either to see that the the Suszko core of the finitary companion $\mathcal{I}^f$ of a π-institution $\mathcal{I}$ is contained in the Suszko core of $\mathcal{I}$ itself, just as was the case with the reflexive core. Recall that the Suszko core $S^{\mathcal{I}}$ of a π-institution $\mathcal{I}$ consists of those natural transformations $\sigma^\flat$ in $N^\flat$, with a single distinguished argument, such that, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\sigma^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T).$$

This means, of course, that, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{implies} \quad \sigma^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T).$$

Lemma 1311 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *Then*

$$S^{\mathcal{I}^f} \subseteq S^{\mathcal{I}}.$$

Proof: Suppose that $\sigma^\flat \in S^{\mathcal{I}^f}$ and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$. Then, since $\sigma^\flat \in S^{\mathcal{I}^f}$ and $T \in \mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$, we get $\sigma^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}^f}(T) \leq \widetilde{\Omega}^{\mathcal{I}}(T)$, where the second inclusion follows from the fact that $\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$. Therefore, we conclude that $\sigma^\flat \in S^{\mathcal{I}}$. Hence, $S^{\mathcal{I}^f} \subseteq S^{\mathcal{I}}$. ∎

With this result available, we can see that, if $\mathcal{I}^f$ is family c-reflective and its Suszko core is adequate, then the Suszko core of $\mathcal{I}$ is also adequate. Recall that adequacy of the Suszko core $S^{\mathcal{I}}$ means that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) = \bigcap\{\Omega(T) : T \in \mathrm{ThFam}(\mathcal{I}) \text{ and } S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.$$

Recall also, that the right-to-left inclusion always holds. So the definition is tantamount to the assertion that the left-to-right inclusion also holds.

Proposition 1312 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}^f$ *is family c-reflective and* $S^{\mathcal{I}^f}$ *is adequate in* $\mathcal{I}^f$, *then so is* $S^{\mathcal{I}}$ *in* $\mathcal{I}$.

Proof: Suppose $\mathcal{I}^f$ is family c-reflective and that $S^{\mathcal{I}^f}$ is adequate. Then, by Theorem 847, $\mathcal{I}^f$ is truth equational, whence, by Theorem 840, for all $T \in \mathrm{ThFam}(\mathcal{I}^f)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad S^{\mathcal{I}^f}_\Sigma[\phi] \leq \Omega(T).$$

Consider $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. We have

$$\begin{array}{rcl}
\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) & = & \bigcap\{\Omega(T) : T \in \mathrm{ThFam}(\mathcal{I}) \text{ and } \phi \in T_\Sigma\} \\
& & (\text{Definition of } \widetilde{\Omega}^{\mathcal{I}}) \\
& = & \bigcap\{\Omega(T) : T \in \mathrm{ThFam}(\mathcal{I}) \text{ and } S^{\mathcal{I}^f}_\Sigma[\phi] \leq \Omega(T)\} \\
& & (\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f) \text{ and displayed equivalence}) \\
& \leq & \bigcap\{\Omega(T) : T \in \mathrm{ThFam}(\mathcal{I}) \text{ and } S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}. \\
& & (\text{Lemma 1311})
\end{array}$$

Thus, by definition, $S^{\mathcal{I}}$ is also adequate in $\mathcal{I}$. ∎

We can now show that truth equationality is inherited by a π-institution $\mathcal{I}$ from its finitary companion $\mathcal{I}^f$. This forms an analog of Lemma 1306.

Theorem 1313 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}^f$ *is truth equational, then so is* $\mathcal{I}$.

Proof: Suppose $\mathcal{I}^f$ is truth equational. By Theorem 847, it is family c-reflective and its Suszko core $S^{\mathcal{I}^f}$ is adequate in $\mathcal{I}^f$. Therefore, by Lemma 1310, $\mathcal{I}$ is family c-reflective and, by Proposition 1312, $S^{\mathcal{I}}$ is adequate in $\mathcal{I}$. Therefore, again by Theorem 847, $\mathcal{I}$ is truth equational. ∎

Theorem 840 characterized truth equationality in terms of the solubility property of the Suszko core. In fact, the solubility of the Suszko core is the condition asserting that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) \quad \text{implies} \quad \phi \in T_\Sigma.$$

Since the reverse implication always holds, the condition is equivalent to the assertion that, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\phi \in T_\Sigma \quad \text{iff} \quad S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

Corollary 1314 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $S^{\mathcal{I}^f}$ *is soluble in* $\mathcal{I}^f$*, then* $S^{\mathcal{I}}$ *is soluble in* $\mathcal{I}$.

Proof: If $S^{\mathcal{I}^f}$ is soluble in $\mathcal{I}^f$, then, by Theorem 838, $\mathcal{I}^f$ is truth equational. Thus, by Theorem 1313, $\mathcal{I}$ is also truth equational, whence, again by Theorem 838, applied in the opposite direction, $S^{\mathcal{I}}$ is soluble in $\mathcal{I}$. ∎

Once more, as was the case with syntactic protoalgebraicity, instead of deriving the implication in Corollary 1314 by applying Theorem 1313, we may prove it first and then use Theorem 838 to establish that truth equationality of $\mathcal{I}^f$ implies truth equationality of $\mathcal{I}$.

Lemma 1315 (Corollary 1314) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $S^{\mathcal{I}^f}$ *is soluble in* $\mathcal{I}^f$*, then* $S^{\mathcal{I}}$ *is soluble in* $\mathcal{I}$.

Proof: Suppose $S^{\mathcal{I}^f}$ is soluble in $\mathcal{I}^f$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. hence, by Lemma 1311, we get $S^{\mathcal{I}^f}_\Sigma[\phi] \leq \Omega(T)$. But $\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$ and $S^{\mathcal{I}^f}$ is assumed to be soluble in $\mathcal{I}^f$. Thus, $\phi \in T_\Sigma$. This proves that $S^{\mathcal{I}}$ is soluble in $\mathcal{I}$. ∎

Corollary 1316 (Theorem 1313) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}^f$ *is truth equational, then so is* $\mathcal{I}$.

Proof: Suppose $\mathcal{I}^f$ is truth equational. Then, by Theorem 838, $S^{\mathcal{I}^f}$ is soluble in $\mathcal{I}^f$. Thus, by Lemma 1315, $S^{\mathcal{I}}$ is soluble in $\mathcal{I}$. Hence, again by applying Theorem 838, only now in the reverse direction, $\mathcal{I}$ is truth equational. ∎

We conclude the section by synthesizing Theorems 1306 and 1313. Recall that a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is syntactically weakly family algebraizable if it is

- protoalgebraic;
- family c-reflective;
- $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified, i.e., has a Leibniz reflexive core and an adequate Suszko core.

By Theorem 913, $\mathcal{I}$ is syntactically weakly family algebraizable if and only if it is syntactically protoalgebraic and family truth equational. Thus, we get

Theorem 1317 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}^f$ *is syntactically weakly family algebraizable, then so is* $\mathcal{I}$*.*

Proof: If $\mathcal{I}^f$ is syntactically weakly family algebraizable, then, by Theorem 913, it is syntactically protoalgebraic and family truth equational. Hence, by Theorems 1306 and 1313, $\mathcal{I}$ possesses the same properties. Therefore, applying again Theorem 913 in the reverse direction, we conclude that $\mathcal{I}$ is also syntactically weakly family algebraizable. ■

In Section 9.4, we saw that the continuity of the Leibniz operator is one of the key properties when studying finitarity conditions. Lemma 660 showed that, if $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is continuous, then $\mathcal{I}$ is protoalgebraic. That is asserting the continuity of the Leibniz operator strengthens protoalgebraicity. Additionally, it was proven in Lemma 661 that, if $\mathbf{Sign}^\flat$ is finite, then continuity of Ω also ensures that the finitary companion $\mathcal{I}^f$ of $\mathcal{I}$ is also protoalgebraic.

We begin, here, our parallel treatment on the syntactic side by showing that, maintaining the finiteness of $\mathbf{Sign}^\flat$, the condition that $\mathcal{I}$ be syntactically protoalgebraic, with a finite collection of parameter-free witnessing transformations $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$, constitutes an additional strengthening on protoalgebraicity, on top of the continuity of Ω.

Proposition 1318 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically protoalgebraic, with a finite parameter-free collection* $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *of witnessing transformations, then* $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is continuous.*

Proof: Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\mathbf{Sign}^\flat$ finite, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a syntactically protoalgebraic π-institution based on $\mathbf{F}$, with a finite parameter-free collection $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ of witnessing transformations. Suppose $\{T^i : i \in I\}$ is a directed collection of theory families of $\mathcal{I}$, such that $\bigcup_{i \in I} T^i \in \mathrm{ThFam}(\mathcal{I})$. Then, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}\langle \phi, \psi \rangle \in \Omega_\Sigma(\bigcup_{i \in I} T^i) & \text{iff} & I^\flat_\Sigma[\phi, \psi] \leq \bigcup_{i \in I} T^i \\ & \text{iff} & I^\flat_\Sigma[\phi, \psi] \leq T^i, \text{ some } i \in I, \\ & \text{iff} & \langle \phi, \psi \rangle \in \Omega_\Sigma(T^i), \text{ some } i \in I, \\ & \text{iff} & \langle \phi, \psi \rangle \in \bigcup_{i \in I} \Omega_\Sigma(T^i).\end{array}$$

Note that the second equivalence employs both the fact that $\mathbf{Sign}^\flat$ is finite and the fact that $I^\flat$ is finite and parameter-free. Thus, $\Omega(\bigcup_{i\in I}T^i) = \bigcup_{i\in I}\Omega(T^i)$ and, hence, Ω is indeed continuous. ■

We next see that this stronger condition than the continuity of the Leibniz operator suffices to ensure that $\mathcal{I}^f$ is also syntactically protoalgebraic, with the same collection of witnessing transformations. Thus, the following proposition may be viewed as a syntactic analog of Lemma 661.

Proposition 1319 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically protoalgebraic, with a finite and parameter-free collection* $I^\flat$ *of witnessing transformations, then* $\mathcal{I}^f$ *is also syntactically protoalgebraic, with the same collection of witnessing transformations.*

Proof: Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\mathbf{Sign}^\flat$ finite, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a syntactically protoalgebraic π-institution, with a finite and parameter-free collection $I^\flat$ of witnessing transformations. Let $T \in \mathrm{ThFam}(\mathcal{I}^f)$. Then, by Proposition 655, there exists a directed locally finitely generated collection $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $T = \bigcup_{i\in I}T^i$. Now we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rll}
\langle\phi,\psi\rangle \in \Omega_\Sigma(T) & \text{iff} & \langle\phi,\psi\rangle \in \Omega_\Sigma(\bigcup_{i\in I}T^i)\\
& \text{iff} & \langle\phi,\psi\rangle \in \bigcup_{i\in I}\Omega_\Sigma(T^i) \quad \text{(Proposition 1318)}\\
& \text{iff} & \langle\phi,\psi\rangle \in \Omega_\Sigma(T^i),\ \text{some } i \in I,\\
& \text{iff} & \overleftrightarrow{I^\flat_\Sigma}[\phi,\psi] \leq T^i,\ \text{some } i \in I,\\
& \text{iff} & \overleftrightarrow{I^\flat_\Sigma}[\phi,\psi] \leq \bigcup_{i\in I}T^i\\
& \text{iff} & \overleftrightarrow{I^\flat_\Sigma}[\phi,\psi] \leq T.
\end{array}$$

Again, note that the one-before-the-last equivalence employs both the fact that $\mathbf{Sign}^\flat$ is finite and the fact that $I^\flat$ is finite and parameter-free. Therefore, by Corollary 791, $\mathcal{I}^f$ is also syntactically protoalgebraic, with the same collection $I^\flat$ of witnessing transformations. ■

Suppose, now, that $\mathbf{Sign}^\flat$ is finite and $\mathcal{I}$ is weakly family algebraizable, so that $\Omega^{-1} : \mathrm{ConSys}^*(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$ be defined. An analog of Proposition 1318 asserts that, if $\mathcal{I}$ is truth equational, with a finite and parameter-free witnessing family $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ of equations, then the inverse Leibniz operator Ω^{-1} is continuous. Thus, under these hypotheses, the truth equationality of $\mathcal{I}$ via a finite, parameter-free collection of witnessing equations is stronger than the continuity of Ω^{-1}.

Proposition 1320 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a weakly family algebraizable* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is truth equational, with a finite parameter-free collection* $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *of witnessing equations, then* $\Omega^{-1} : \mathrm{ConSys}^*(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$ *is continuous.*

Proof: Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\mathbf{Sign}^\flat$ finite, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a weakly family algebraizable π-institution, which is, in addition, truth equational, with a finite parameter-free collection $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ of witnessing equations. Let $\{\theta^i : i \in I\}$ be a directed collection of $\mathcal{I}^*$-congruence systems, such that $\bigcup_{i \in I} \theta^i \in \mathrm{ConSys}^*(\mathcal{I})$. Now we get, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll} \phi \in \Omega_\Sigma^{-1}(\bigcup_{i \in I} \theta^i) & \text{iff} & \tau_\Sigma^\flat[\phi] \leq \bigcup_{i \in I} \theta^i \\ & \text{iff} & \tau_\Sigma^\flat[\phi] \leq \theta^i, \text{ some } i \in I, \\ & \text{iff} & \phi \in \Omega_\Sigma^{-1}(\theta^i), \text{ some } i \in I, \\ & \text{iff} & \phi \in \bigcup_{i \in I} \Omega_\Sigma^{-1}(\theta^i). \end{array}$$

Thus, Ω^{-1} is indeed continuous. ■

Recall from Theorem 663 that given a weakly family algebraizable π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, based on an algebraic system $\mathbf{F}$ over a finite category of signatures, the continuity of both $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ and $\Omega^{-1} : \mathrm{ConSys}^*(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$ are sufficient to ensure that $\mathcal{I}^f$ is also weakly family algebraizable. In Propositions 1318 and 1320, by comparison, it was shown that the continuities of Ω and Ω^{-1} are strengthened by assuming, respectively, that $\mathcal{I}$ is syntactically protoalgebraic, with a finite, parameter-free witnessing family of transformations, and that $\mathcal{I}$ is family truth equational, with a finite, parameter-free witnessing family of equations. We show, next, in an analog of Theorem 663, that imposing these two stronger conditions on $\mathcal{I}$ suffices to ensure that syntactic strong algebraizability transfers from $\mathcal{I}$ to its finitary companion $\mathcal{I}^f$.

Proposition 1321 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\mathbf{Sign}^\flat$ finite, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a syntactically protoalgebraic π-institution, with a finite parameter-free collection $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ of witnessing transformations. If $\mathcal{I}$ is family truth equational, with a finite and parameter-free collection $\tau^\flat$ of witnessing equations, then $\mathcal{I}^f$ is also family truth equational, with the same collection of witnessing equations.*

Proof: Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\mathbf{Sign}^\flat$ finite, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a weakly family algebraizable π-institution, which is family truth equational, with a finite and parameter-free collection $I^\flat$ of witnessing equations. Let $T \in \mathrm{ThFam}(\mathcal{I}^f)$. Then, by Proposition 655, there exists a directed locally finitely generated collection $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $T = \bigcup_{i \in I} T^i$. Now we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll} \phi \in T_\Sigma & \text{iff} & \phi \in \bigcup_{i \in I} T_\Sigma^i \\ & \text{iff} & \phi \in T_\Sigma^i, \text{ some } i \in I, \\ & \text{iff} & \tau_\Sigma^\flat[\phi] \leq \Omega(T^i), \text{ some } i \in I, \\ & \text{iff} & \tau_\Sigma^\flat[\phi] \leq \bigcup_{i \in I} \Omega(T^i) \\ & \text{iff} & \tau_\Sigma^\flat[\phi] \leq \Omega(\bigcup_{i \in I} T^i) \quad \text{(Proposition 1318)} \\ & \text{iff} & \tau_\Sigma^\flat[\phi] \leq \Omega(T). \end{array}$$

Again, note that the fourth equivalence employs both the fact that $\mathbf{Sign}^\flat$ is finite and the fact that $\tau^\flat$ is finite and parameter-free. We conclude that $\mathcal{I}^f$ is also family truth equational, with the same collection $\tau^\flat$ of witnessing equations. ■

Putting together Propositions we finally obtain the promised analog of Theorem 663.

Theorem 1322 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically strongly family algebraizable* π*-institution, via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}^\mathsf{K}$ *consisting of finite and parameter-free collections of transformations. Then* $\mathcal{I}^f$ *is also syntactically strongly family algebraizable, via the same conjugate pair of transformations.*

Proof: We simply put together Propositions 1319 and 1321. ■

17.2 Natural Finitarity

This section deals with concepts analogous to those studied in Section 9.4, but in the syntactic, rather than in the semantic, context. In the semantic context, the four key ingredients of our study were the finitarity of the π-institutions involved as well as the continuity of the Leibniz operator and its inverse. Recall that for the inverse to be defined in the context under consideration, the general underlying hypothesis that the π-institution $\mathcal{I}$ be weakly family algebraizable was adhered to. In the present, syntactic, context, we assume that $\mathcal{I}$ is syntactically strongly family algebraizable, that is, syntactically family algebraizable via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}^\mathsf{K}$, where both $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ and $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ are parameter-free witnessing collections of equations and of transformations, respectively. The four notions involved are the properties of $\mathcal{I}$ and $\mathcal{Q}^\mathsf{K}$ being naturally finitary, a strengthening of finitarity, and those of $\tau^\flat$ and $I^\flat$ being finite, also strengthening the continuity of the Leibniz operator and its inverse operator. But let us embark on the developments so as to clarify these introductory remarks and to make the concepts and the details involved precise.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall that $\mathcal{I}$ is *finitary* if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$, there exists $\Phi' \subseteq_f \Phi$, such that $\phi \in C_\Sigma(\Phi')$. Equivalently, $\mathcal{I}$ is finitary if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$C_\Sigma(\Phi) = \bigcup \{C_\Sigma(\Phi') : \Phi' \subseteq_f \Phi\}.$$

We say that $\mathcal{I}$ is **naturally finitary** if it is finitary and, in addition, the following condition holds:

(NATFIN) If, for some collections $\mu,\nu:(\mathrm{SEN}^\flat)^\omega\to\mathrm{SEN}^\flat$ of natural transformations in $N^\flat$, such that $|\mu|<\infty$, it holds that, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\vec{\phi}\in\mathrm{SEN}^\flat(\Sigma)$,

$$\mu_\Sigma[\vec{\phi}]\leq C(\nu_\Sigma[\vec{\phi}]),$$

then, there exists a finite subset $\nu'\subseteq\nu$, such that, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\vec{\phi}\in\mathrm{SEN}^\flat(\Sigma)$,

$$\mu_\Sigma[\vec{\phi}]\leq C(\nu'_\Sigma[\vec{\phi}]).$$

It is not difficult to see that, if $\mathcal{I}$ is naturally finitary, the implication resulting from (NATFIN) by replacing the two inclusions by equalities of closure families also holds.

Lemma 1323 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is naturally finitary, then, for all* $\mu,\nu:(\mathrm{SEN}^\flat)^\omega\to\mathrm{SEN}^\flat$ *in* $N^\flat$*, with* $|\mu|<\infty$*, such that, for all* $\Sigma\in|\mathbf{Sign}^\flat|$ *and all* $\vec{\phi}\in\mathrm{SEN}^\flat(\Sigma)$*,* $C(\mu_\Sigma[\vec{\phi}])=C(\nu_\Sigma[\vec{\phi}])$*, there exists a finite* $\nu'\subseteq\nu$*, such that, for all* $\Sigma\in|\mathbf{Sign}^\flat|$ *and all* $\vec{\phi}\in\mathrm{SEN}^\flat(\Sigma)$*,* $C(\mu_\Sigma[\vec{\phi}])=C(\nu'_\Sigma[\vec{\phi}])$*.*

Proof: Suppose $\mathcal{I}$ is naturally finitary and let $\mu,\nu:(\mathrm{SEN}^\flat)^\omega\to\mathrm{SEN}^\flat$ in $N^\flat$, with $|\mu|<\infty$, such that, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\vec{\phi}\in\mathrm{SEN}^\flat(\Sigma)$, $C(\mu_\Sigma[\vec{\phi}])=C(\nu_\Sigma[\vec{\phi}])$. Then, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\vec{\phi}\in\mathrm{SEN}^\flat(\Sigma)$, $\mu_\Sigma[\vec{\phi}]\leq C(\nu_\Sigma[\vec{\phi}])$. Thus, by natural finitarity, there exists a finite subset $\nu'\subseteq\nu$, such that, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\vec{\phi}\in\mathrm{SEN}^\flat(\Sigma)$, $\mu_\Sigma[\vec{\phi}]\leq C(\nu'_\Sigma[\vec{\phi}])$. But, then, we obtain, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\vec{\phi}\in\mathrm{SEN}^\flat(\Sigma)$,

$$C(\nu_\Sigma[\vec{\phi}])=C(\mu_\Sigma[\vec{\phi}])\leq C(\nu'_\Sigma[\vec{\phi}])\leq C(\nu_\Sigma[\vec{\phi}]).$$

We conclude that, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\vec{\phi}\in\mathrm{SEN}^\flat(\Sigma)$, $C(\mu_\Sigma[\vec{\phi}])=C(\nu'_\Sigma[\vec{\phi}])$. ■

Starting to take advantage of natural finitarity, we show that it allows to draw the conclusion that, in case of syntactic family algebraizability, the existence of a finite witnessing family of transformations ensures that every witnessing family possesses a finite witnessing subfamily. More precisely, we have

Lemma 1324 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a naturally finitary* π*-institution based on* $\mathbf{F}$*. Suppose* $\mathcal{I}$ *is syntactically family algebraizable, with equivalent guasivariety* K*. If* $\mathcal{I}$ *has a finite witnessing family* $I^\flat:(\mathrm{SEN}^\flat)^2\to\mathrm{SEN}^\flat$ *of transformations, then every witnessing family* $J^\flat:(\mathrm{SEN}^\flat)^2\to\mathrm{SEN}^\flat$ *possesses a finite witnessing subfamily* $J'^\flat$*.*

Proof: Suppose that $\mathcal{I}$ is naturally finitary and syntactically family algebraizable, with equivalent guasivariety K. Let $I^\flat:(\mathrm{SEN}^\flat)^2\to\mathrm{SEN}^\flat$ be a finite set of witnessing transformations and $J^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ a family of witnessing transformations. By Theorem 912, we get that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C(I^\flat_\Sigma[\phi, \psi]) = C(J^\flat_\Sigma[\phi, \psi]).$$

Since $\mathcal{I}$ is naturally finitary and, by hypothesis, $|I^\flat| < \infty$, we get, by Lemma 1323, that there exists finite $J'^\flat \subseteq J^\flat$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C(J'^\flat_\Sigma[\phi, \psi]) = C(I^\flat_\Sigma[\phi, \psi]).$$

Thus, applying Proposition 903, we conclude that $J'^\flat$ is also a witnessing family of transformations. ∎

Dually, we may also prove a corresponding result concerning the witnessing equations for the truth equationality of $\mathcal{I}$.

Lemma 1325 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically strongly family algebraizable* π*-institution based on* $\mathbf{F}$*, with equivalent guasivariety* K*. If* $\mathcal{Q}^\mathsf{K}$ *is naturally finitary and* $\mathcal{I}$ *has a finite witnessing family* $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *of equations, then every witnessing family* $\rho^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *of equations possesses a finite witnessing subfamily* $\rho'^\flat$*.*

Proof: Follows along the lines of the proof of Lemma 1324. Suppose that $\mathcal{I}$ is syntactically strongly family algebraizable, with equivalent guasivariety K, such that $\mathcal{Q}^\mathsf{K}$ is naturally finitary. Let $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ be a finite set of witnessing equations and $\rho^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ a family of witnessing equations. By Theorem 912, we get that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$D^\mathsf{K}(\tau^\flat_\Sigma[\phi]) = D^\mathsf{K}(\rho^\flat_\Sigma[\phi]).$$

Since $\mathcal{Q}^\mathsf{K}$ is naturally finitary and, by hypothesis, $|\tau^\flat| < \infty$, we get, by Lemma 1323, that there exists finite $\rho'^\flat \subseteq \rho^\flat$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$D^\mathsf{K}(\rho'^\flat_\Sigma[\phi]) = D^\mathsf{K}(\tau^\flat_\Sigma[\phi]).$$

Thus, applying Proposition 903, we conclude that $\rho'^\flat$ is also a witnessing family of equations. ∎

We now establish a theorem to the effect that, under natural finitarity and syntactic strong family algebraizability, every witnessing family of equations contains a finite witnessing subfamily. This is the first main result in a series of finitarity results that we aim to prove in the present section, with the ultimate goal of obtaining a hierarchy on the syntactic side, analogous to that obtained on the semantic side at the end of Section 9.4.

Theorem 1326 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a naturally finitary, syntactically strongly family algebraizable* π*-institution, with equivalent guasivariety* K*. Then every witnessing collection* $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *of equations contains a finite subcollection* $\tau'^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$*, which is also a witnessing collection.*

Proof: Suppose $\mathcal{I}$ is naturally finitary and syntactically strongly family algebraizable, with equivalent guasivariety K. Let $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ be a collection of witnessing equations. Then, by definition, there exists a collection $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C(\iota_\Sigma[\phi]) = C(\phi) = C(I^\flat[\tau^\flat_\Sigma[\phi]]).$$

Since $\mathcal{I}$ is naturally finitary, there exist finite $I'^\flat \subseteq I^\flat$ and $\tau'^\flat \subseteq \tau^\flat$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C(\phi) = C(I'^\flat[\tau'^\flat_\Sigma[\phi]]).$$

Thus, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C(\phi) = C(I^\flat[\tau'^\flat_\Sigma[\phi]]).$$

Thus, since, by the properties of $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}^\mathsf{K}$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$,

$$\phi \approx \psi \in D^\mathsf{K}_\Sigma(E) \quad \text{iff} \quad I^\flat_\Sigma[\phi, \psi] \le C(I^\flat_\Sigma[E]),$$

we get, by Proposition 903, that $\tau'^\flat$ is a witnessing family of equations. ■

Dually, we may prove that, under syntactic strong family algebraizability and natural finitarity of the algebraic counterpart $\mathcal{Q}^\mathsf{K}$, every witnessing family of transformations contains a finite witnessing subfamily.

Theorem 1327 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically strongly family algebraizable* π*-institution, with equivalent guasivariety* K*, such that* $\mathcal{Q}^\mathsf{K}$ *is naturally finitary. Then every witnessing collection* $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *of transformations contains a finite subcollection* $I'^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$*, which is also a witnessing collection.*

Proof: Suppose $\mathcal{I}$ is syntactically strongly family algebraizable, with equivalent guasivariety K, such that $\mathcal{Q}^\mathsf{K}$ is naturally finitary. Let $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ be a collection of witnessing transformations. Then, by definition, there exists a collection $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$D^\mathsf{K}(\langle p^{2,0}, p^{2,1} \rangle_\Sigma[\phi, \psi]) = D^\mathsf{K}(\phi \approx \psi) = D^\mathsf{K}(\tau^\flat[I^\flat_\Sigma[\phi, \psi]]).$$

Since $\mathcal{I}$ is naturally finitary, there exist finite $I'^\flat \subseteq I^\flat$ and $\tau'^\flat \subseteq \tau^\flat$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$D^{\mathsf{K}}(\phi \approx \psi) = D^{\mathsf{K}}(\tau'^\flat[I'^\flat_\Sigma[\phi, \psi]]).$$

Thus, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$D^{\mathsf{K}}(\phi \approx \psi) = D^{\mathsf{K}}(\tau^\flat[I'^\flat_\Sigma[\phi, \psi]]).$$

Thus, since, by the properties of $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathsf{K}}$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in C_\Sigma(\Phi) \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq D^{\mathsf{K}}(\tau^\flat_\Sigma[\Phi]),$$

we get, by Proposition 903, that $I'^\flat$ is also a witnessing family of transformations. ■

The following proposition asserts that, under similar hypotheses, but adding finiteness of the signature category, the finitarity of $\mathcal{I}$ and of the witnessing collection $I^\flat$ imply the finitarity of the algebraic counterpart $\mathcal{Q}^{\mathsf{K}}$.

Proposition 1328 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a finitary, syntactically family algebraizable* π*-institution, with equivalent guasivariety* K*. If* $\mathcal{I}$ *has a finite witnessing set* $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *of transformations, then its algebraic counterpart* $\mathcal{Q}^{\mathsf{K}}$ *is also finitary.*

Proof: Suppose $\mathcal{I}$ is a finitary π-institution based on an algebraic system $\mathbf{F}$ over a finite category of signatures. Assume that $\mathcal{I}$ is syntactically family algebraizable, with equivalent guasivariety K and that it has a finite witnessing collection $I^\flat$ of transformations. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$, such that

$$\phi \approx \psi \in D^{\mathsf{K}}_\Sigma(E).$$

Since $I^\flat$ is a witnessing collection of transformations,

$$I^\flat_\Sigma[\phi, \psi] \leq C(I^\flat_\Sigma[E]).$$

Since $\mathbf{Sign}^\flat$ is finite and $I^\flat$ is finite, we get, by the finitarity of $\mathcal{I}$ that there exists finite $E' \subseteq E$, such that $I^\flat_\Sigma[\phi, \psi] \leq C(I^\flat_\Sigma[E'])$. Thus, again by the fact that $I^\flat$ is a set of witnessing transformations, we obtain $\phi \approx \psi \in D^{\mathsf{K}}_\Sigma(E')$. Thus, $\mathcal{Q}^{\mathsf{K}}$ is indeed finitary. ■

A similar result can also be established when focus is shifted from finitarity to natural finitarity.

Proposition 1329 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a naturally finitary, syntactically family algebraizable* π*-institution, with equivalent guasivariety* K*. If* $\mathcal{I}$ *has a finite witnessing set* $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *of transformations, then its algebraic counterpart* $\mathcal{Q}^{\mathsf{K}}$ *is also naturally finitary.*

Proof: Suppose $\mathcal{I}$ is a naturally finitary π-institution based on an algebraic system $\mathbf{F}$ over a finite category of signatures. Assume that $\mathcal{I}$ is syntactically family algebraizable, with equivalent guasivariety K and that it has a finite witnessing collection $I^\flat$ of transformations. By Proposition 1328, we know that $\mathcal{Q}^{\mathsf{K}}$ is finitary. Let $\mu,\nu : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ be collections of natural transformations in $N^\flat$, with $|\mu| < \infty$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\mu_\Sigma[\vec{\phi}] \leq D^{\mathsf{K}}(\nu_\Sigma[\vec{\phi}]).$$

Since $I^\flat$ is a witnessing collection of transformations, we get, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat[\mu_\Sigma[\vec{\phi}]] \leq C(I^\flat[\nu_\Sigma[\vec{\phi}]]).$$

But both μ and $I^\flat$ are finite and, also, $\mathbf{Sign}^\flat$ is assumed to be finite. Hence, since $\mathcal{I}$ is naturally finitary, there exists finite $\nu' \subseteq \nu$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat[\mu_\Sigma[\vec{\phi}]] \leq C(I^\flat[\nu'_\Sigma[\vec{\phi}]]).$$

Therefore, again by the fact that $I^\flat$ is a set of witnessing transformations, we obtain, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\mu_\Sigma[\vec{\phi}] \leq D^{\mathsf{K}}(\nu'_\Sigma[\vec{\phi}]).$$

Thus, $\mathcal{Q}^{\mathsf{K}}$ is indeed naturally finitary. ∎

We turn, next, to results dual to those established in Propositions 1328 and 1329. We start with a dual to Proposition 1328 to the effect that, if $\mathcal{Q}^{\mathsf{K}}$ is finitarty and $\mathcal{I}$ has a finite witnessing collection of equations, then $\mathcal{I}$ is itself finitary.

Proposition 1330 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a syntactically strongly family algebraizable* π*-institution, with equivalent guasivariety* K*. If* $\mathcal{Q}^{\mathsf{K}}$ *is finitary and* $\mathcal{I}$ *has a finite witnessing set* $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *of equations, then* $\mathcal{I}$ *is also finitary.*

Proof: Suppose $\mathcal{I}$ is a π-institution based on an algebraic system $\mathbf{F}$ over a finite category of signatures. Assume that $\mathcal{I}$ is syntactically strongly family algebraizable, with equivalent guasivariety K, such that $\mathcal{Q}^{\mathsf{K}}$ is finitary, and that it has a finite witnessing collection $\tau^\flat$ of equations. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that

$$\phi \in C_\Sigma(\Phi).$$

Since $\tau^\flat$ is a witnessing collection of equations,

$$\tau^\flat_\Sigma[\phi] \leq D^{\mathsf{K}}(\tau^\flat_\Sigma[\Phi]).$$

Since $\mathbf{Sign}^\flat$ is finite and $\tau^\flat$ is finite, we get, by the finitarity of $\mathcal{Q}^\mathsf{K}$ that there exists finite $\Phi' \subseteq \Phi$, such that $\tau^\flat_\Sigma[\phi] \leq D^\mathsf{K}(\tau^\flat_\Sigma[\Phi'])$. Thus, again by the fact that $\tau^\flat$ is a set of witnessing equations, we obtain $\phi \in C_\Sigma(\Phi')$. Thus, $\mathcal{I}$ is indeed finitary. ■

A dual of Proposition 1329 addresses the case of natural finitarity.

Proposition 1331 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a syntactically strongly family algebraizable* π*-institution, with equivalent guasivariety* K*. If* $\mathcal{Q}^\mathsf{K}$ *is naturally finitary and* $\mathcal{I}$ *has a finite witnessing set* $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *of equations, then* $\mathcal{I}$ *is also naturally finitary.*

Proof: Suppose $\mathcal{I}$ is a π-institution based on an algebraic system $\mathbf{F}$ over a finite category of signatures. Assume that $\mathcal{I}$ is syntactically strongly family algebraizable, with equivalent guasivariety K, such that $\mathcal{Q}^\mathsf{K}$ is naturally finitary, and that it has a finite witnessing collection $\tau^\flat$ of equations. Let $\mu, \nu : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ be collections of natural transformations in $N^\flat$, with $|\mu| < \infty$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\mu_\Sigma[\vec{\phi}] \leq C(\nu_\Sigma[\vec{\phi}]).$$

Since $\tau^\flat$ is a witnessing collection of equations, we get, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\tau^\flat[\mu_\Sigma[\vec{\phi}]] \leq D^\mathsf{K}(\tau^\flat[\nu_\Sigma[\vec{\phi}]]).$$

But both μ and $\tau^\flat$ are finite and, also, $\mathbf{Sign}^\flat$ is assumed to be finite. Hence, since $\mathcal{Q}^\mathsf{K}$ is naturally finitary, there exists finite $\nu' \subseteq \nu$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\tau^\flat[\mu_\Sigma[\vec{\phi}]] \leq D^\mathsf{K}(\tau^\flat[\nu'_\Sigma[\vec{\phi}]]).$$

Therefore, again by the fact that $\tau^\flat$ is a set of witnessing equations, we obtain, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\mu_\Sigma[\vec{\phi}] \leq C(\nu'_\Sigma[\vec{\phi}]).$$

Thus, $\mathcal{I}$ is naturally finitary. ■

Finally, we present a syntactic analog of Corollary 668, which summarizes the conclusions drawn from the study of the various finitarity properties, at the center of the investigations carried out in the present chapter.

Corollary 1332 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a syntactically strongly family algebraizable* π*-institution, via the conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}^\mathsf{K}$*.*

(a) If both $\tau^\flat$ *and* $I^\flat$ *are finite, then* $\mathcal{I}$ *is naturally finitary if and only if* $\mathcal{Q}^\mathsf{K}$ *is naturally finitary;*

(b) *If $\mathcal{I}$ is naturally finitary, then $\mathcal{Q}^{\mathsf{K}}$ is naturally finitary if and only if $I^\flat$ can be taken to be finite;*

(c) *If $\mathcal{Q}^{\mathsf{K}}$ is naturally finitary, then $\mathcal{I}$ is naturally finitary if and only if $\tau^\flat$ can be taken to be finite.*

In each case, if the equivalent alternatives hold, then all four "finitarity" conditions hold.

Proof:

(a) Suppose both $\tau^\flat$ and $I^\flat$ are finite. If $\mathcal{I}$ is naturally finitary, then, by Proposition 1329, $\mathcal{Q}^{\mathsf{K}}$ is also naturally finitary. If, on the other hand, $\mathcal{Q}^{\mathsf{K}}$ is naturally finitary, then, by Proposition 1331, $\mathcal{I}$ is naturally finitary.

(b) Assume that $\mathcal{I}$ is naturally finitary. If $\mathcal{Q}^{\mathsf{K}}$ is naturally finitary, then, by Theorem 1327, $I^\flat$ may be taken to be finite. If, on trhe other hand, $I^\flat$ can be taken to be finite, then, by Proposition 1329, $\mathcal{Q}^{\mathsf{K}}$ is naturally finitary.

(c) Assume $\mathcal{Q}^{\mathsf{K}}$ is naturally finitary. If $\mathcal{I}$ is naturally finitary, then, by Theorem 1326, $\tau^\flat$ may be taken to be finite. If, on the other hand, $\tau^\flat$ may be taken to be finite, then, by Proposition 1331, $\mathcal{I}$ is naturally finitary.

■

In summary, Corollary 1332 establishes the hierarchy depicted below, which parallels in the syntactic context the hierarchy pictured at the end of Chapter 9, concerning the semantic side.