Chapter 18

Properties of Selected Classes

18.1 Protoalgebraic π-Institutions

18.1.1 The Correspondence Theorem

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

Recall that $\mathcal{I}$ is **protoalgebraic** if the Leibniz operator is monotone on theory families, i.e., if, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$T \leq T' \quad \text{implies} \quad \Omega(T) \leq \Omega(T').$$

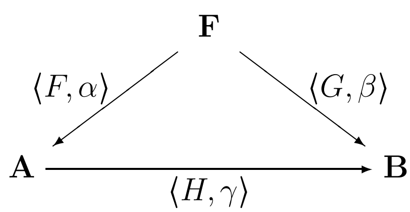
Recall, also, that, by Theorem 175, every protoalgebraic π-institution is stable and that, moreover, by Theorem 179, $\mathcal{I}$ is protoalgebraic if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$T \leq T' \quad \text{implies} \quad \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T').$$

The π-institution $\mathcal{I}$ has the **compatibility property** if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, with $T \leq T'$, and all $\theta \in \mathrm{ConSys}(\mathbf{A})$,

$$\theta \text{ compatible with } T \quad \text{implies} \quad \theta \text{ compatible with } T'.$$

The π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ has the **filter correspondence property** if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$, and surjective morphism $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$, with $H : \mathbf{Sign} \to \mathbf{Sign}'$ an isomorphism,



and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$,

$$\gamma^{-1}(\widehat{\gamma}(T) \vee T') = T \vee \gamma^{-1}(T'),$$

where $\widehat{\gamma}(T) = C^{\mathcal{I},\mathcal{B}}(\gamma(T))$ is the least $\mathcal{I}$-filter family on $\mathcal{B}$ that includes $\gamma(T)$.

Our goal is to show that both the compatibility property and the filter correspondence property characterize protoalgebraic π-institutions. We start with a lemma to the effect that, for every $\mathcal{I}$-filter family T of $\mathcal{A}$, if the kernel of $\langle H, \gamma \rangle$ happens to be compatible with T, then $\gamma(T)$ is already an $\mathcal{I}$-filter family of $\mathcal{B}$ and, therefore, $\widehat{\gamma}(T) = \gamma(T)$.

Lemma 1333 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ $\mathbf{F}$-algebraic systems, and $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism, with $H : \mathbf{Sign} \to \mathbf{Sign}'$ an isomorphism. If $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\mathrm{Ker}(\langle H, \gamma \rangle)$ is compatible with T, then $\widehat{\gamma}(T) = \gamma(T)$.*

Proof: By definition, $\gamma(T) \leq \widehat{\gamma}(T)$ always holds. To show the reverse inequality, it suffices to show that $\gamma(T)$, under the hypothesis of the compatibility of $\mathrm{Ker}(\langle H,\gamma\rangle)$ with T, is an $\mathcal{I}$-filter family of $\mathcal{B}$. So assume $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$, and let $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$, such that

$$\beta_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq \gamma_{F(\Sigma')}(T_{F(\Sigma')}).$$

This gives

$$\gamma_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))) \subseteq \gamma_{F(\Sigma')}(T_{F(\Sigma')}).$$

By the postulated compatibility of $\mathrm{Ker}(\langle H,\gamma\rangle)$ with T, we obtain

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq T_{F(\Sigma')}.$$

Since $\phi \in C_\Sigma(\Phi)$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get that

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \in T_{F(\Sigma')}.$$

Thus, $\gamma_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi))) \in \gamma_{F(\Sigma')}(T_{F(\Sigma')})$, and, therefore,

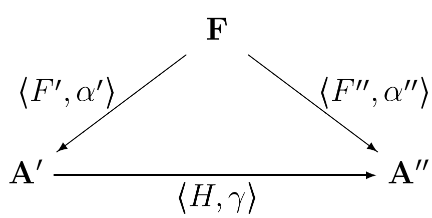
$$\beta_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \in \gamma_{F(\Sigma')}(T_{F(\Sigma')}).$$

This shows that $\gamma(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$ and, hence, $\widehat{\gamma}(T) = \gamma(T)$. ■

Next, we give an equivalent formulation of the Filter Correspondence Property.

Proposition 1334 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *has the filter correspondence property iff*

> *for all* $\mathcal{I}$*-matrix families* $\mathfrak{A} = \langle \mathcal{A}', T'\rangle$, $\mathfrak{A}'' = \langle \mathcal{A}'', T''\rangle$ *and strict surjective matrix morphism* $\langle H,\gamma\rangle : \mathfrak{A}' \to \mathfrak{A}''$, *with* $H : \mathbf{Sign}' \to \mathbf{Sign}''$ *an isomorphism,*



$$T = \gamma^{-1}(\gamma(T)), \quad \textit{for all } T' \leq T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}').$$

Proof: Suppose, first, that $\mathcal{I}$ has the Filter Correspondence Property and consider $\mathcal{I}$-matrix families $\mathfrak{A} = \langle \mathcal{A}', T'\rangle$, $\mathfrak{A}'' = \langle \mathcal{A}'', T''\rangle$, a strict surjective matrix morphism $\langle H,\gamma\rangle : \mathfrak{A}' \to \mathfrak{A}''$, with $H : \mathbf{Sign}' \to \mathbf{Sign}''$ an isomorphism, and $T' \le T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$. Then we have

$$\begin{array}{rcl}\gamma^{-1}(\gamma(T)) & \le & \gamma^{-1}(\widehat{\gamma}(T) \vee T'') \quad (\gamma(T) \le \widehat{\gamma}(T))\\ & = & T \vee \gamma^{-1}(T'') \quad \text{(Filter Correspondence)}\\ & = & T \vee T' \quad (\langle H,\gamma\rangle \text{ strict})\\ & = & T. \quad (T' \le T \text{ by hypothesis})\end{array}$$

Thus, the displayed property holds. Assume, conversely, that the displayed property holds. We must show that $\mathcal{I}$ has the Filter Correspondence Property. So suppose that $\mathcal{A} = \langle \mathbf{A}', \langle F', \alpha'\rangle\rangle$, $\mathcal{A}'' = \langle \mathbf{A}'', \langle F'', \alpha''\rangle\rangle$ are $\mathbf{F}$-algebraic systems, $\langle H,\gamma\rangle : \mathcal{A}' \to \mathcal{A}''$ a surjective morphism, with $H : \mathbf{Sign}' \to \mathbf{Sign}''$ an isomorphism, and let $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$ and $T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}'')$. Our goal is to show that

$$\gamma^{-1}(\widehat{\gamma}(T') \vee T'') = T' \vee \gamma^{-1}(T'').$$

Notice that $\langle H,\gamma\rangle : \langle \mathcal{A}', \gamma^{-1}(T'')\rangle \to \langle \mathcal{A}'', T''\rangle$ is a strict surjective morphism, with H an isomorphism, and $\gamma^{-1}(T'') \le T' \vee \gamma^{-1}(T'')$. Thus, we fit the setup of the hypothesis, which allows us to conclude that

$$\gamma^{-1}(\gamma(T' \vee \gamma^{-1}(T''))) = T' \vee \gamma^{-1}(T'').$$

So, it suffices, in turn, to show that $\widehat{\gamma}(T') \vee T'' = \gamma(T' \vee \gamma^{-1}(T''))$ and, since, $\mathrm{Ker}(\langle H,\gamma\rangle)$ is compatible with T' (having $\gamma^{-1}(\gamma(T')) = T'$, by hypothesis), it suffices, by Lemma 1333, to show that

$$\gamma(T') \vee T'' = \gamma(T' \vee \gamma^{-1}(T'')).$$

The left to right inclusion is obvious, since $\gamma(T'), T'' \le \gamma(T' \vee \gamma^{-1}(T''))$. Conversely, note that, taking into account the hypothesis, $T', \gamma^{-1}(T'') \le \gamma^{-1}(\gamma(T') \vee T'')$. Therefore, $T' \vee \gamma^{-1}(T'') \le \gamma^{-1}(\gamma(T') \vee T'')$ and, therefore, $\gamma(T' \vee \gamma^{-1}(T'')) \le \gamma(T') \vee T''$ and, hence, the right to left inclusion also holds. Thus, the Filter Correspondence Property holds. ■

Now we proceed with the formulation and proof of the main theorem.

Theorem 1335 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(i) $\mathcal{I}$ *is protoalgebraic;*

(ii) $\mathcal{I}$ *has the compatibility property;*

(iii) $\mathcal{I}$ *has the filter correspondence property.*

Proof:

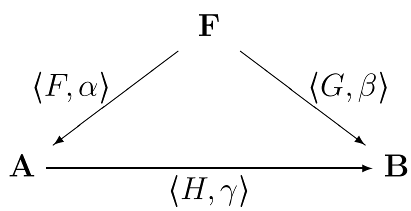
(i)⇒(ii) Suppose $\mathcal{I}$ is protoalgebraic and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system, $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, with $T \le T'$, and $\theta \in \mathrm{ConSys}(\mathbf{A})$, such that θ is compatible with T. Then we have

$$\begin{array}{rcll} \theta & \le & \Omega(T) & \text{(by the compatibility of } \theta \text{ with } T) \\ & \le & \Omega(T'). & \text{(by protoalgebraicity)} \end{array}$$

We conclude that θ is also compatible with T' and, hence, $\mathcal{I}$ has the compatibility property.

(ii)⇒(i) Suppose that $\mathcal{I}$ has the compatibility property and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \le T'$. Then $\Omega(T) \in \mathrm{ConSys}(\mathbf{A})$ and, by the definition of a Leibniz congruence system, it is compatible with T. Now it follows by the compatibility property, that $\Omega(T)$ is also compatible with T'. Hence $\Omega(T) \le \Omega(T')$. We conclude that $\mathcal{I}$ is protoalgebraic.

(ii)⇒(iii) Suppose that $\mathcal{I}$ has the compatibility property and consider $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta\rangle\rangle$, a commutative triangle



with $H : \mathbf{Sign} \to \mathbf{Sign}'$ an isomorphism, and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. Note that it is always the case that

$$T \vee \gamma^{-1}(T') \le \gamma^{-1}(\widehat{\gamma}(T) \vee T').$$

Thus, it suffices to show that, under the hypothesis of compatibility, the reverse inclusion also holds.

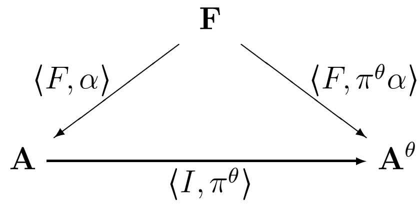
Consider, temporarily, $X \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\gamma^{-1}(T') \le X$. Since $\mathrm{Ker}(\langle H, \gamma\rangle)$ is compatible with $\gamma^{-1}(T')$, by the postulated compatibility property, it is also compatible with X. Thus, by Lemma 1333, $\widehat{\gamma}(X) = \gamma(X)$. Moreover, we have $T' \le \gamma(X) = \widehat{\gamma}(X)$.

Now set $X = T \vee \gamma^{-1}(T')$ and reason as follows:

$$\begin{array}{rcll} \gamma^{-1}(\widehat{\gamma}(T) \vee T') & \le & \gamma^{-1}(\widehat{\gamma}(X) \vee T') & (T \le X) \\ & = & \gamma^{-1}(\widehat{\gamma}(X)) & (T' \le \widehat{\gamma}(X)) \\ & = & \gamma^{-1}(\gamma(X)) & (\widehat{\gamma}(X) = \gamma(X)) \\ & = & X. & (\mathrm{Ker}(\langle H, \gamma\rangle) \text{ compatible with } X) \end{array}$$

So we get $\gamma^{-1}(\widehat{\gamma}(T) \vee T') = T \vee \gamma^{-1}(T')$ and $\mathcal{I}$ has the correspondence property.

(iii)⇒(ii) Suppose that $\mathcal{I}$ has the correspondence property and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system, $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, with $T \leq T'$, and $\theta \in \mathrm{ConSys}(\mathbf{A})$, such that θ is compatible with T. We look at the commutative diagram



and calculate

$$\begin{array}{rcl}(\pi^\theta)^{-1}(\widehat{\pi^\theta}(T')) & = & (\pi^\theta)^{-1}(\widehat{\pi^\theta}(T') \vee \widehat{\pi^\theta}(T)) \quad (T \leq T') \\ & = & T' \vee (\pi^\theta)^{-1}(\widehat{\pi^\theta}(T)) \quad \text{(correspondence property)} \\ & \leq & T' \vee T \quad (\theta \text{ compatible with } T) \\ & = & T'. \quad (T \leq T')\end{array}$$

Thus, θ is also compatible with T' and $\mathcal{I}$ has the compatibility property. ■

As a consequence we obtain the *Correspondence Theorem*, which asserts that, under the same hypothesis, $\langle H, \gamma\rangle$ induces an order isomorphism between the principal filter of the lattice $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})$ generated by $\gamma^{-1}(T')$ and the principal filter of the lattice $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{B})$ generated by T'.

Theorem 1336 (Correspondence Theorem) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a protoalgebraic* π*-institution based on* $\mathbf{F}$*. Let, also,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*,* $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta\rangle\rangle$ *be* $\mathbf{F}$*-algebraic systems and* $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism, with* $H : \mathbf{Sign} \to \mathbf{Sign}'$ *an isomorphism, and* $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$*. Then,* $Y \mapsto \gamma^{-1}(Y)$*,* $T' \leq Y \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$*, defines an order isomorphism* $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'} \cong \mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})^{\gamma^{-1}(T')}$*.*

Proof: $\gamma^{-1} : \mathbf{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'} \to \mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})^{\gamma^{-1}(T')}$ is well defined by Corollary 55 and it is clearly monotone. Furthermore, $\widehat{\gamma} : \mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})^{\gamma^{-1}(T')} \to \mathbf{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$ is also well-defined and monotone. So it suffices to show that, for all $T' \leq Y \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, $\widehat{\gamma}(\gamma^{-1}(Y)) = Y$ and that, for all $\gamma^{-1}(T') \leq X \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\gamma^{-1}(\widehat{\gamma}(X)) = X$.

First, for $T' \leq Y \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, since $\langle H, \gamma\rangle$ is surjective, $\gamma(\gamma^{-1}(Y)) = Y$ and, therefore, $\widehat{\gamma}(\gamma^{-1}(Y)) = Y$, since $Y \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. For the other equation, if $\gamma^{-1}(T') \leq X \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we have,

$$\begin{array}{rcl}\gamma^{-1}(\widehat{\gamma}(X)) & = & \gamma^{-1}(\widehat{\gamma}(X) \vee T') \quad (\gamma^{-1}(T') \leq X \Rightarrow T' \leq \widehat{\gamma}(X)) \\ & = & X \vee \gamma^{-1}(T') \quad \text{(correspondence property)} \\ & = & X. \quad (\gamma^{-1}(T') \leq X)\end{array}$$

So $\gamma^{-1} : \mathbf{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'} \cong \mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})^{\gamma^{-1}(T')}$. ■

18.1.2 The Homomorphism Theorem

We show that, in the case of protoalgebraic π-institutions $\mathcal{I}$, every surjective morphism of $\mathcal{I}$-matrix families gives rise to a corresponding surjective morphism between their reductions. This establishes a "reduction" functor and, moreover, gives rise to a version of the Homomorphism Theorem of Universal Algebra.

Recall that, given a base algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ and an $\mathbf{F}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, we denote by $\mathfrak{A}^*$ the reduction of $\mathfrak{A}$, i.e.,

$$\mathfrak{A}^* = \langle \mathcal{A}/\Omega^{\mathcal{A}}(T), T/\Omega^{\mathcal{A}}(T) \rangle.$$

Moreover, extending this notation, given $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, we set

$$\phi^* = \phi/\Omega^{\mathcal{A}}_\Sigma(T) \in \mathrm{SEN}^*(\Sigma).$$

Theorem 1337 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a protoalgebraic π-institution based on $\mathbf{F}$. Further, let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$, be $\mathbf{F}$-algebraic systems, $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ and $\mathfrak{A}' = \langle \mathcal{A}', T' \rangle$ be $\mathcal{I}$-matrix families and $\langle H, \gamma \rangle : \mathfrak{A} \to \mathfrak{A}'$ a surjective morphism. Then, there exists a surjective morphism $\langle H, \gamma^* \rangle : \mathfrak{A}^* \to \mathfrak{A}'^*$, given, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, by*

$$\gamma^*_\Sigma(\phi^*) = \gamma_\Sigma(\phi)^*.$$

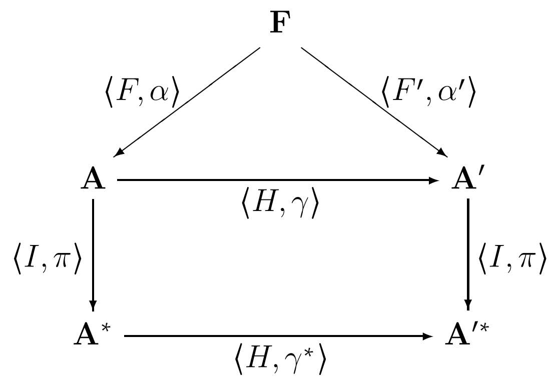
Proof: First, we show that, for all $\Sigma \in |\mathbf{Sign}|$, $\gamma^*_\Sigma : \mathrm{SEN}^*(\Sigma) \to \mathrm{SEN}'^*(H(\Sigma))$ is well-defined. Indeed, suppose $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\phi^* = \psi^*$, i.e., $\langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_\Sigma(T)$. Then, since $T \leq \gamma^{-1}(T')$, we get, by protoalgebraicity, $\langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_\Sigma(\gamma^{-1}(T'))$, whence, by Proposition 24, $\langle \phi, \psi \rangle \in \gamma^{-1}_\Sigma(\Omega^{\mathcal{A}'}_{H(\Sigma)}(T'))$, and, hence, $\langle \gamma_\Sigma(\phi), \gamma_\Sigma(\psi) \rangle \in \Omega^{\mathcal{A}'}_{H(\Sigma)}(T')$, or, equivalently, $\gamma_\Sigma(\phi)^* = \gamma_\Sigma(\psi)^*$.

Next we see that $\gamma^* : \mathrm{SEN}^* \to \mathrm{SEN}'^* \circ H$ is a natural transformation. To this end, let $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\phi \in \mathrm{SEN}(\Sigma)$. Then we have

$$\begin{array}{ccc}
\mathrm{SEN}^*(\Sigma) & \xrightarrow{\ \gamma^*_\Sigma\ } & \mathrm{SEN}'^*(H(\Sigma)) \\
{\scriptstyle \mathrm{SEN}^*(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{SEN}'^*(H(f))} \\
\mathrm{SEN}^*(\Sigma') & \xrightarrow[\ \gamma^*_{\Sigma'}\ ]{} & \mathrm{SEN}'^*(H(\Sigma'))
\end{array}$$

$$\begin{array}{rcl}
\mathrm{SEN}'^*(H(f))(\gamma^*_\Sigma(\phi^*)) & = & \mathrm{SEN}'^*(H(f))(\gamma_\Sigma(\phi)^*) \\
& = & \mathrm{SEN}'(H(f))(\gamma_\Sigma(\phi))^* \\
& = & \gamma_{\Sigma'}(\mathrm{SEN}(f)(\phi))^* \\
& = & \gamma^*_{\Sigma'}(\mathrm{SEN}(f)(\phi)^*) \\
& = & \gamma^*_{\Sigma'}(\mathrm{SEN}^*(f)(\phi^*)).
\end{array}$$

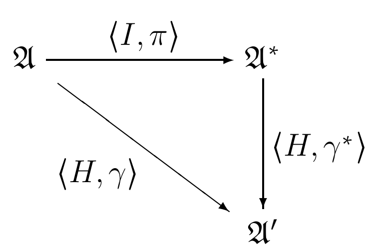
Surjectivity of $\langle H,\gamma^*\rangle : \mathcal{A}^* \to \mathcal{A}'^*$ follows from the fact that $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{A}'$ is surjective. So it suffices to show that $\langle F',\pi\alpha'\rangle = \langle H,\gamma^*\rangle \circ \langle F,\pi,\alpha\rangle$ and that $\langle H,\gamma^*\rangle : \mathfrak{A}^* \to \mathfrak{A}'^*$ is a matrix family morphism. The first equation follows from the fact that the upper triangle of the diagram commutes by hypothesis and the rectangle commutes by the definition of $\langle H,\gamma^*\rangle$.



To finish the proof, we calculate, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $\phi^* \in T_\Sigma/\Omega^{\mathcal{A}}_\Sigma(T)$ if and only if, by compatibility, $\phi \in T_\Sigma$ implies, by hypothesis, $\phi \in \gamma_\Sigma^{-1}(T'_{H(\Sigma)})$ if and only if $\gamma_\Sigma(\phi) \in T'_{H(\Sigma)}$ if and only if, by compatibility, $\gamma_\Sigma(\phi)^* \in T'_{H(\Sigma)}/\Omega^{\mathcal{A}'}_{H(\Sigma)}(T')$ if and only if, by the definition of γ^*, $\gamma^*_\Sigma(\phi^*) \in T'_{H(\Sigma)}/\Omega^{\mathcal{A}'}_{H(\Sigma)}(T')$ if and only if $\phi^* \in (\gamma^*_\Sigma)^{-1}(T'_{H(\Sigma)}/\Omega^{\mathcal{A}'}_{H(\Sigma)}(T'))$. ■

We also have the following construction.

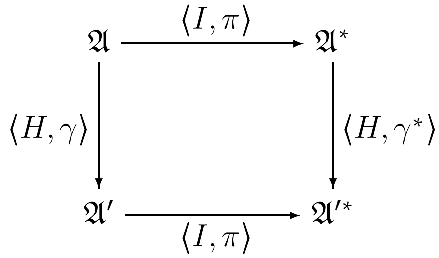
Corollary 1338 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a protoalgebraic* π*-institution based on* $\mathbf{F}$*. Let, also* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, $\mathcal{A}' = \langle \mathbf{A}', \langle F',\alpha'\rangle\rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$*, be* $\mathbf{F}$*-algebraic systems,* $\mathfrak{A} = \langle \mathcal{A}, T\rangle$ *an* $\mathcal{I}$*-matrix family and* $\mathfrak{A}' = \langle \mathcal{A}', T'\rangle$ *a reduced* $\mathcal{I}$*-matrix family and* $\langle H,\gamma\rangle : \mathfrak{A} \to \mathfrak{A}'$ *a surjective morphism.*



There exists a unique surjective morphism $\langle H,\gamma^*\rangle : \mathfrak{A}^* \to \mathfrak{A}'$ *that makes the triangle commute.*

Proof: By Theorem 1337, there exists a surjective matrix morphism $\langle H,\gamma^*\rangle :$

$\mathfrak{A}^* \to \mathfrak{A}'^*$, such that the following rectangle commutes:



But, by hypothesis, $\mathfrak{A}'$ is reduced, whence $\mathfrak{A}'^* = \mathfrak{A}'$ and $\langle I, \pi\rangle = \langle I, \iota\rangle : \mathfrak{A}' \to \mathfrak{A}'^*$ is the identity morphism. We now obtain the triangle depicted in the diagram of the statement. ∎

Let us denote by $\mathbf{MatFam}(\mathcal{I})$ the category of $\mathcal{I}$-matrix families with surjective matrix morphisms between them and, similarly, $\mathbf{MatFam}^*(\mathcal{I})$ the category of reduced $\mathcal{I}$-matrix families with surjective matrix morphisms between them. Then, based on Theorem 1337, we obtain the following functor.

Theorem 1339 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *Then*

$$^* : \mathbf{MatFam}(\mathcal{I}) \to \mathbf{MatFam}^*(\mathcal{I})$$

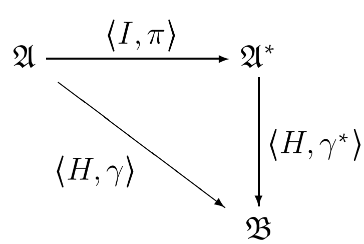
is a functor. The subcategory $\mathbf{MatFam}^*(\mathcal{I})$ *is a reflective subcategory of* $\mathbf{MatFam}(\mathcal{I})$ *with* * *a reflector from* $\mathbf{MatFam}(\mathcal{I})$ *to* $\mathbf{MatFam}^*(\mathcal{I})$.

Proof: Given $\mathcal{A} \in \mathrm{MatFam}(\mathcal{I})$, it is easy to see that $\langle I, \iota^*\rangle : \mathfrak{A}^* \to \mathfrak{A}^*$ is the identity matrix morphism. For the composition property, assume $\mathfrak{A}$, $\mathfrak{A}'$, $\mathfrak{A}'' \in \mathrm{MathFam}(\mathcal{I})$, and $\langle G, \beta\rangle : \mathfrak{A} \to \mathfrak{A}'$, $\langle H, \gamma\rangle : \mathfrak{A}' \to \mathfrak{A}''$ be matrix morphisms. Then, we have, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{rcl}(\gamma_{G(\Sigma)} \circ \beta_\Sigma)^*(\phi^*) & = & \gamma_{G(\Sigma)}(\beta_\Sigma(\phi))^* \\ & = & \gamma^*_{G(\Sigma)}(\beta_\Sigma(\phi)*) \\ & = & \gamma^*_{G(\Sigma)}(\beta^*_\Sigma(\phi^*)).\end{array}$$

Thus, $(\langle H, \gamma\rangle \circ \langle G, \beta\rangle)^* = \langle H, \gamma\rangle^* \circ \langle G, \beta\rangle^*$. Therefore, $^* : \mathbf{MatFam}(\mathcal{I}) \to \mathbf{MatFam}^*(\mathcal{I})$ is a functor.

As far a s reflectivity is concerned, for every $\mathfrak{A} \in \mathrm{MatFam}^*(\mathcal{I})$, we consider the natural quotient morphism $\langle I, \pi\rangle : \mathfrak{A} \to \mathfrak{A}^*$. Given reduced $\mathfrak{B} \in \mathrm{MatFam}^*(\mathcal{I})$ and a surjective morphism $\langle H, \gamma\rangle : \mathfrak{A} \to \mathfrak{B}$, the surjective morphism $\langle H, \gamma^*\rangle : \mathfrak{A}^* \to \mathfrak{B}$ of Corollary 1338 is the unique surjective morphism such that the following diagram commutes.



Thus, $\mathbf{MatFam}^*(\mathcal{I})$ is a reflective subcategory of $\mathbf{MatFam}(\mathcal{I})$ with * a reflector from $\mathbf{MatFam}(\mathcal{I})$ to $\mathbf{MatFam}^*(\mathcal{I})$. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. Given an $\mathcal{I}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, we denote by $\mathbf{FiFam}^{\mathcal{I}}(\mathfrak{A})$ the principal filter of the complete lattice $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})$ generated by the $\mathcal{I}$-filter family T:

$$\mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}) = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : T \leq T'\}.$$

Recall that this set is also denoted by $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$, without explicit reference to the matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$.

The Correspondence Theorem allows us to prove the following.

Theorem 1340 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a protoalgebraic π-institution based on $\mathbf{F}$. For every $\mathcal{I}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,*

$$\mathbf{FiFam}^{\mathcal{I}}(\mathfrak{A}) \cong \mathbf{FiFam}^{\mathcal{I}}(\mathfrak{A}^*).$$

Proof: By The Correspondence Theorem 1336, with $\mathcal{B} = \mathcal{A}/\Omega^{\mathcal{A}}(T)$, $\langle H, \gamma \rangle = \langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T)$ and $T' = T/\Omega^{\mathcal{A}}(T)$, we get

$$\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T))^{T/\Omega^{\mathcal{A}}(T)} \cong \mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})^{\pi^{-1}(T/\Omega^{\mathcal{A}}(T))}.$$

But this amounts to $\mathbf{FiFam}^{\mathcal{I}}(\mathfrak{A}^*) \cong \mathbf{FiFam}(\mathfrak{A})$. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Consider $\mathbf{F}$-matrix families $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ and $\mathfrak{B} = \langle \mathcal{B}, T' \rangle$ and a surjective matrix morphism $\langle H, \gamma \rangle : \mathfrak{A} \to \mathfrak{B}$. By definition, we have $T \leq \gamma^{-1}(T')$. We call $\gamma^{-1}(T')$ the **filter kernel** of $\langle H, \gamma \rangle$. By the inclusion relation above, we can see that, if $\mathfrak{B} \in \mathrm{MatFam}(\mathcal{I})$, then $\gamma^{-1}(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$.

Given $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ and $X \in \mathrm{SenFam}(\mathcal{A})$, such that $T \leq X$, we define

$$\mathfrak{A}/X := \langle \mathcal{A}, X \rangle^* = \langle \mathcal{A}/\Omega^{\mathcal{A}}(X), X/\Omega^{\mathcal{A}}(X) \rangle.$$

We call $\mathfrak{A}/X$ the **quotient of $\mathfrak{A}$ by** X. We note that, if $\mathfrak{A} \in \mathrm{MatFam}(\mathcal{I})$, then

$$X \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}) \quad \text{iff} \quad \mathfrak{A}/X \in \mathrm{MatFam}^*(\mathcal{I}).$$

The following is an analog in the context of $\mathcal{I}$-matrix families of the Homomorphism Theorem of Universal Algebra.

Theorem 1341 (Homomorphism Theorem) *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a protoalgebraic π-institution based on $\mathbf{F}$. Let also $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, $\mathfrak{A}' = \langle \mathcal{A}', T' \rangle \in \mathrm{MatFam}(\mathcal{I})$ and $\langle H, \gamma \rangle : \mathfrak{A} \to \mathfrak{A}'$ a surjective morphism.*

(i) There exists a strict surjective morphism $\langle H,\gamma'\rangle : \mathfrak{A}/\gamma^{-1}(T') \to \mathfrak{A}'^*$ *with isomorphic components;*

(ii) If $X \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$ *and* $X \le \gamma^{-1}(T')$, *then, there exists a surjective morphism* $\langle H,\gamma^X\rangle : \mathfrak{A}/X \to \mathfrak{A}'^*$, *such that*

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\langle H,\gamma\rangle} & \mathfrak{A}' \\ {\scriptstyle\langle I,\pi^X\rangle}\downarrow & & \downarrow{\scriptstyle\langle I,\pi\rangle} \\ \mathfrak{A}/X & \xrightarrow[\langle H,\gamma^X\rangle]{} & \mathfrak{A}'^* \end{array}$$

$\langle H,\gamma^X\rangle \circ \langle I,\pi^X\rangle = \langle I,\pi\rangle \circ \langle H,\gamma\rangle$.

Proof:

(i) First, note that $\gamma^{-1}(T') \le \gamma^{-1}(T')$, whence, $\langle H,\gamma\rangle : \langle \mathcal{A},\gamma^{-1}(T')\rangle \to \mathfrak{A}'$ is also a surjective matrix morphism. Thus, taking into account that $T \le \gamma^{-1}(T')$, we get, by Theorem 1337, a surjective matrix morphism $\langle H,\gamma^*\rangle : \mathfrak{A}/\gamma^{-1}(T') \to \mathfrak{A}'^*$, such that the following diagram commutes.

$$\begin{array}{ccc} \langle \mathcal{A},\gamma^{-1}(T')\rangle & \xrightarrow{\langle H,\gamma\rangle} & \mathfrak{A}' \\ {\scriptstyle\langle I,\pi\rangle}\downarrow & & \downarrow{\scriptstyle\langle I,\pi\rangle} \\ \mathfrak{A}/\gamma^{-1}(T') & \xrightarrow[\langle H,\gamma^*\rangle]{} & \mathfrak{A}'^* \end{array}$$

It remains to show that, for every $\Sigma \in |\mathbf{Sign}|$,

$$\gamma^*_\Sigma : \mathrm{SEN}^{\gamma^{-1}(T')}(\Sigma) \to \mathrm{SEN}'^*(H(\Sigma))$$

is a bijection and that $\langle H,\gamma^*\rangle$ is strict. To see that γ^*_Σ is a bijection, let $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that

$$\gamma^*_\Sigma(\phi/\Omega^{\mathcal{A}}_\Sigma(\gamma^{-1}(T'))) = \gamma^*_\Sigma(\psi/\Omega^{\mathcal{A}}_\Sigma(\gamma^{-1}(T'))).$$

Then, by the commutativity of the rectangle,

$$\gamma_\Sigma(\phi)/\Omega^{\mathcal{A}'}_{H(\Sigma)}(T') = \gamma_\Sigma(\psi)/\Omega^{\mathcal{A}'}_{H(\Sigma)}(T').$$

This gives that $\langle\phi,\psi\rangle \in \gamma^{-1}_\Sigma(\Omega^{\mathcal{A}'}_{H(\Sigma)}(T'))$. Thus, $\langle\phi,\psi\rangle \in \Omega^{\mathcal{A}}_\Sigma(\gamma^{-1}(T'))$ and, hence,

$$\phi/\Omega^{\mathcal{A}}_\Sigma(\gamma^{-1}(T')) = \phi/\Omega^{\mathcal{A}}_\Sigma(\gamma^{-1}(T')).$$

Therefore, $\gamma^*_\Sigma : \mathrm{SEN}^{\gamma^{-1}(T')}(\Sigma) \to \mathrm{SEN}'^*(H(\Sigma))$ is a bijection, for all $\Sigma \in |\mathbf{Sign}|$.

To prove strictness, assume $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi/\Omega^{\mathcal{A}}_{\Sigma}(\gamma^{-1}(T')) \in \gamma^{*-1}_{\Sigma}(T'_{H(\Sigma)}/\Omega^{\mathcal{A}'}_{H(\Sigma)}(T'))$. Then $\gamma^{*}_{\Sigma}(\phi/\Omega^{\mathcal{A}}_{\Sigma}(\gamma^{-1}(T'))) \in T'_{H(\Sigma)}/\Omega^{\mathcal{A}'}_{H(\Sigma)}(T')$. Hence, by the definition of γ^*, we get $\gamma_{\Sigma}(\phi)^* \in T'_{H(\Sigma)}/\Omega^{\mathcal{A}'}_{H(\Sigma)}(T')$. By compatibility, we obtain $\gamma_{\Sigma}(\phi) \in T'_{H(\Sigma)}$, whence $\phi \in \gamma^{-1}_{\Sigma}(T'_{H(\Sigma)})$. This, finally, yields

$$\phi/\Omega^{\mathcal{A}}_{\Sigma}(\gamma^{-1}(T')) \in \gamma^{-1}_{\Sigma}(T'_{H(\Sigma)})/\Omega^{\mathcal{A}}_{\Sigma}(\gamma^{-1}(T')),$$

proving strictness.

(ii) This part is proven by the following diagram chase:

$$\begin{array}{ccccc}
\mathfrak{A} & & \xrightarrow{\langle H,\gamma\rangle} & & \mathfrak{A}' \\
{\scriptstyle \langle I,\pi^X\rangle}\downarrow & & & & \downarrow{\scriptstyle \langle I,\pi\rangle} \\
\mathfrak{A}/X & \xrightarrow[\langle I,\pi\rangle]{} & \mathfrak{A}/\gamma^{-1}(T') & \xrightarrow[\langle H,\gamma'\rangle]{} & \mathfrak{A}'^{*}
\end{array}$$

where $\langle I,\pi\rangle : \mathfrak{A}/X \to \mathfrak{A}/\gamma^{-1}(T')$ is the canonical projection morphism, defined because of the hypothesis $X \le \gamma^{-1}(T')$ and protoalgebraicity, and

$$\langle H,\gamma'\rangle : \mathfrak{A}/\gamma^{-1}(T') \to \mathfrak{A}'^{*}$$

is the morphism obtained in Part (i).

■

18.2 Pointed Classes of Algebraic Systems

Proposition 1342 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a protoalgebraic π-institution based on $\mathbf{F}$, having theorems. Then the following conditions are equivalent:*

(i) $\mathcal{I}$ is family regular;

(ii) For all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in T_\Sigma$, $T_\Sigma = \phi/\Omega_\Sigma(T)$;

(iii) For all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$, $T_\Sigma = t/\Omega_\Sigma(T)$;

(iv) For all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and some $\phi \in T_\Sigma$, $T_\Sigma = \phi/\Omega_\Sigma(T)$;

(v) For all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and some $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$, $T_\Sigma = t/\Omega_\Sigma(T)$.

Proof:

(i)⇒(ii) Suppose $\mathcal{I}$ is family regular and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in T_\Sigma$. Then, we have, for all $\psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rlll} \psi \in T_\Sigma & \text{iff} & \phi, \psi \in T_\Sigma & (\phi \in T_\Sigma) \\ & \text{iff} & C(\phi,\psi) \le T & (\text{definition of } C(\phi,\psi)) \\ & \text{implies} & \Omega_\Sigma(C(\phi,\psi)) \le \Omega_\Sigma(T) & (\mathcal{I} \text{ protoalgebraic}) \\ & \text{implies} & \langle \phi,\psi\rangle \in \Omega_\Sigma(T) & (\mathcal{I} \text{ family regular}) \\ & \text{iff} & \psi \in \phi/\Omega_\Sigma(T). & (\text{definition of } \phi/\Omega_\Sigma(T)) \end{array}$$

On the other hand, if $\psi \in \phi/\Omega_\Sigma(T)$, then, since $\phi \in T_\Sigma$, by the compatibility of $\Omega(T)$ with T, $\psi \in T_\Sigma$. Thus, we conclude that $T_\Sigma = \phi/\Omega_\Sigma(T)$.

(ii)⇒(iii) Suppose (ii) holds and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$. Then, since $\mathrm{Thm}(\mathcal{I}) \le T$, we get that $t \in T_\Sigma$ and, hence, by hypothesis, $T_\Sigma = t/\Omega_\Sigma(T)$.

(iii)⇒(iv) Assume (iii) holds and let $T \in \mathrm{ThFam}(\mathcal{I})$ and $\Sigma \in |\mathbf{Sign}^\flat|$. Since $\mathcal{I}$ has theorems, there exists $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$. Then, $t \in T_\Sigma$ and, by hypothesis, $T_\Sigma = t/\Omega_\Sigma(T)$.

(iv)⇒(v) Assume (iv) holds and let $T \in \mathrm{ThFam}(\mathcal{I})$ and $\Sigma \in |\mathbf{Sign}^\flat|$. Then, by hypothesis, there exists $\phi \in T_\Sigma$, such that $T_\Sigma = \phi/\Omega_\Sigma(T)$. Moreover, $\mathcal{I}$ has theorems, whence, there exists $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$. Then, we have $t \in T_\Sigma = \phi/\Omega_\Sigma(T)$, whence $\langle \phi, t\rangle \in \Omega_\Sigma(T)$ and, therefore, $T_\Sigma = \phi/\Omega_\Sigma(T) = t/\Omega_\Sigma(T)$.

(v)⇒(i) Assume that (v) holds and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in T_\Sigma$. By hypothesis, for some $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$, $T_\Sigma = t/\Omega_\Sigma(T)$. Hence, $\phi, \psi \in t/\Omega_\Sigma(T)$, i.e., $\langle \phi, t\rangle \in \Omega_\Sigma(T)$ and $\langle t, \psi\rangle \in \Omega_\Sigma(T)$. By transitivity, $\langle \phi, \psi\rangle \in \Omega_\Sigma(T)$. Therefore, $\mathcal{I}$ is family regular. ■

We show now that a protoalgebraic family assertional π-institution $\mathcal{I}$ is weakly family algebraizable.

Theorem 1343 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a protoalgebraic π-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is family assertional, then it is weakly family algebraizable.*

Proof: By Definition 613, protoalgebraicity and family assertionality are equivalent to regular weak family algebraizability. By Proposition 620, this entails weak family algebraizability.

More directly, assume $\mathcal{I}$ is family assertional. Since it is protoalgebraic, by hypothesis, it suffices to show that $\mathcal{I}$ is family injective. To this end, let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$,

$$T_\Sigma = t/\Omega_\Sigma(T) = t/\Omega_\Sigma(T') = T'_\Sigma.$$

Therefore $T = T'$. Hence $\mathcal{I}$ is family injective and, therefore, it is weakly family algebraizable. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a class of $\mathbf{F}$-algebraic systems. We say that K is $\top^\flat$-**pointed** if, for all $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle \in \mathsf{K}$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)$,

$$\top^{\mathcal{A}}_\Sigma(\vec{\phi}) = \top^{\mathcal{A}}_\Sigma(\vec{\psi}).$$

K is called **pointed** if it is $\top^\flat$-pointed with respect to some $\top^\flat$ in $N^\flat$.

If a class K is pointed, then, for every $\mathcal{A} \in \mathsf{K}$, we write $\top^{\mathcal{A}} = \{\top^{\mathcal{A}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, where $\top^{\mathcal{A}}_\Sigma \coloneqq \top^{\mathcal{A}}_\Sigma(\vec{\phi})$, for some $\vec{\phi} \in \mathrm{SEN}(\Sigma)$, this value being independent of the choice of $\vec{\phi} \in \mathrm{SEN}(\Sigma)$.

We focus now on protoalgebraic, family regular π-institutions that have natural theorems. Recall that this means that there exists a natural transformation $\top^\flat$ in $N^\flat$, such that $\top^\flat$ is evaluated to a theorem in every signature and at all tuples of sentences. Of course, by definition, all π-institutions that fit this description are regularly weakly family algebraizable. We show that for such π-institutions, the class $\mathrm{AlgSys}^*(\mathcal{I})$ of their reduced algebraic systems is a pointed class of $\mathbf{F}$-algebraic systems, where any natural theorem may serve as the "point".

Proposition 1344 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a protoalgebraic family regular π-institution based on $\mathbf{F}$, having natural theorems. Then, the class $\mathrm{AlgSys}^*(\mathcal{I})$ is a pointed class of $\mathbf{F}$-algebraic systems.*

Proof: Suppose $\mathcal{I}$ is protoalgebraic and family regular, with a natural theorem $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$, i.e., such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$, $\top^\flat_\Sigma(\vec{\phi}) \in \mathrm{Thm}_\Sigma(\mathcal{I})$. By family regularity, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$, $\langle \top^\flat_\Sigma(\vec{\phi}), \top^\flat_\Sigma(\vec{\psi}) \rangle \in \Omega_\Sigma(\mathrm{Thm}(\mathcal{I}))$. Now, let $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I})$. Thus, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. Therefore, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$, we get, by what was shown above and protoalgebraicity, and taking into account Lemma 51, $\langle \top^\flat_\Sigma(\vec{\phi}), \top^\flat_\Sigma(\vec{\psi}) \rangle \in \Omega_\Sigma(\alpha^{-1}(T))$, whence, by Proposition 24, $\langle \top^\flat_\Sigma(\vec{\phi}), \top^\flat_\Sigma(\vec{\psi}) \rangle \in \alpha^{-1}_\Sigma(\Omega^{\mathcal{A}}_{F(\Sigma)}(T))$. Thus,

$$\langle \alpha_\Sigma(\top^\flat_\Sigma(\vec{\phi})), \alpha_\Sigma(\top^\flat_\Sigma(\vec{\psi})) \rangle \in \Omega^{\mathcal{A}}_{F(\Sigma)}(T) = \Delta^{\mathcal{A}}_{F(\Sigma)},$$

i.e., since $\top^\flat$ is a natural transformation, $\top^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})) = \top^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\psi}))$. Taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that $\mathrm{AlgSys}^*(\mathcal{I})$ is a pointed class of $\mathbf{F}$-algebraic systems, with any natural theorem serving as the "point" natural transformation. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$ and K a $\top^\flat$-pointed class of $\mathbf{F}$-algebraic systems. We say that K

is **relatively point regular** if, for every $\theta,\theta' \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$,

$$\top^\flat/\theta = \top^\flat/\theta' \quad \text{implies} \quad \theta = \theta'.$$

It is not difficult to show that the defining property transfers from K-congruence systems on $\mathcal{F}$ to K-congruence systems on every $\mathbf{F}$-algebraic system, under the proviso that K be an abstract class.

Lemma 1345 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ a natural transformation in $N^\flat$, and K a $\top^\flat$-pointed abstract class of $\mathbf{F}$-algebraic systems. If K is relatively point regular, then, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $\theta,\theta' \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$,*

$$\top^{\mathcal{A}}/\theta = \top^{\mathcal{A}}/\theta' \quad \text{implies} \quad \theta = \theta'.$$

Proof: Suppose $\mathcal{A}$ is an $\mathbf{F}$-algebraic system, $\theta,\theta' \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$, such that $\top^{\mathcal{A}}/\theta = \top^{\mathcal{A}}/\theta'$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}\langle \phi, \top^\flat_\Sigma\rangle \in \alpha_\Sigma^{-1}(\theta_{F(\Sigma)}) & \text{iff} & \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\top^\flat_\Sigma)\rangle \in \theta_{F(\Sigma)}\\ & \text{iff} & \langle \alpha_\Sigma(\phi), \top^{\mathcal{A}}_{F(\Sigma)}\rangle \in \theta_{F(\Sigma)}\\ & \text{iff} & \langle \alpha_\Sigma(\phi), \top^{\mathcal{A}}_{F(\Sigma)}\rangle \in \theta'_{F(\Sigma)}\\ & \text{iff} & \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\top^\flat_\Sigma)\rangle \in \theta'_{F(\Sigma)}\\ & \text{iff} & \langle \phi, \top^\flat_\Sigma\rangle \in \alpha_\Sigma^{-1}(\theta'_{F(\Sigma)}).\end{array}$$

Thus, $\top^\flat/\alpha^{-1}(\theta) = \top^\flat/\alpha^{-1}(\theta')$. Since K is abstract and $\mathcal{A}/\theta, \mathcal{A}/\theta' \in \mathsf{K}$, we get that $\mathcal{F}/\alpha^{-1}(\theta), \mathcal{F}/\alpha^{-1}(\theta') \in \mathsf{K}$. It follows that $\alpha^{-1}(\theta), \alpha^{-1}(\theta') \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$. Since K is relatively point regular, by definition, $\alpha^{-1}(\theta) = \alpha^{-1}(\theta')$. Therefore, by surjectivity of $\langle F, \alpha\rangle$, $\theta = \theta'$. ∎

Moreover, we can show that, for a protoalgebraic family regular π-institution $\mathcal{I}$, having natural theorems, the associated class $\mathrm{AlgSys}^*(\mathcal{I})$ of its reduced algebraic systems is a relatively point regular class.

Proposition 1346 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a protoalgebraic family regular π-institution based on $\mathbf{F}$, having natural theorems. Then, the class $\mathrm{AlgSys}^*(\mathcal{I})$ is a relatively point regular class of $\mathbf{F}$-algebraic systems.*

Proof: We know, by Proposition 1344, that $\mathrm{AlgSys}^*(\mathcal{I})$ is pointed, with any natural theorem $\top^\flat$ serving as a "point". Consider $\theta,\theta' \in \mathrm{ConSys}^*(\mathcal{I})$, such that $\top^\flat/\theta = \top^\flat/\theta'$. Since $\theta,\theta' \in \mathrm{ConSys}^*(\mathcal{I})$, there exist $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\theta = \Omega(T)$ and $\theta' = \Omega(T')$. But then, since $\mathcal{I}$ is protoalgebraic and family regular, with theorems, we get, by Proposition 1342,

$$\begin{array}{lll}\theta & = & \Omega(T)\\ & = & \Omega(\top^\flat/\Omega(T)) \quad \text{(Proposition 1342)}\\ & = & \Omega(\top^\flat/\theta)\\ & = & \Omega(\top^\flat/\theta') \quad \text{(hypothesis)}\\ & = & \Omega(\top^\flat/\Omega(T'))\\ & = & \Omega(T') \quad \text{(Proposition 1342)}\\ & = & \theta'.\end{array}$$

Hence $\mathrm{AlgSys}^*(\mathcal{I})$ is indeed relatively point regular. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a $\top^\flat$-pointed class of $\mathbf{F}$-algebraic systems. Define on $\mathbf{F}$ the family $C^{\mathsf{K},\top} = \{C^{\mathsf{K},\top}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$C^{\mathsf{K},\top}_\Sigma : \mathcal{P}(\mathrm{SEN}^\flat(\Sigma)) \to \mathcal{P}(\mathrm{SEN}^\flat(\Sigma)),$$

be given, for all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, by

$$\phi \in C^{\mathsf{K},\top}_\Sigma(\Phi) \quad \text{iff} \quad \phi \approx \top^\flat_\Sigma \in C^{\mathsf{K}}_\Sigma(\Phi \approx \top^\flat_\Sigma),$$

i.e., $\phi \in C^{\mathsf{K},\top}_\Sigma(\Phi)$ if and only if, for all $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle \in \mathsf{K}$, all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq \{\top^{\mathcal{A}}_{F(\Sigma')}\} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) = \top^{\mathcal{A}}_{F(\Sigma')}.$$

In the next proposition, it is shown that $C^{\mathsf{K},\top}$ is a closure system on $\mathbf{F}$. In this way the pointed class K of $\mathbf{F}$-algebraic systems defines a bona fide π-institution based on $\mathbf{F}$.

Proposition 1347 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a $\top^\flat$-pointed class of $\mathbf{F}$-algebraic systems. $C^{\mathsf{K},\top}$ is a closure system on $\mathbf{F}$.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$. It is obvious from the definition that

$$C^{\mathsf{K},\top}_\Sigma : \mathcal{P}(\mathrm{SEN}^\flat(\Sigma)) \to \mathcal{P}(\mathrm{SEN}^\flat(\Sigma))$$

is inflationary and monotone. To show that it is also idempotent, let $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C^{\mathsf{K},\top}_\Sigma(C^{\mathsf{K},\top}_\Sigma(\Phi))$. Thus, we have, by definition, for all $\mathcal{A} \in \mathsf{K}$, all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(C^{\mathsf{K},\top}_\Sigma(\Phi))) \subseteq \{\top^{\mathcal{A}}_{F(\Sigma')}\} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) = \top^{\mathcal{A}}_{F(\Sigma')}.$$

But, also by definition, we have, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq \{\top^{\mathcal{A}}_{F(\Sigma')}\} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(C^{\mathsf{K},\top}_\Sigma(\Phi))) \subseteq \{\top^{\mathcal{A}}_{F(\Sigma')}\}.$$

Therefore, we get that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq \{\top^{\mathcal{A}}_{F(\Sigma')}\} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) = \top^{\mathcal{A}}_{F(\Sigma')},$$

showing that $\phi \in C^{\mathsf{K},\top}_\Sigma(\Phi)$.

It remains, finally, to show that $C^{\mathsf{K},\top}$ is structural. Let $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C^{\mathsf{K},\top}_\Sigma(\Phi)$. Consider $\mathcal{A} \in \mathsf{K}$, such that, for all $\Sigma'' \in |\mathbf{Sign}^\flat|$ and all $g \in \mathbf{Sign}^\flat(\Sigma', \Sigma'')$,

$$\Sigma \xrightarrow{\ f\ } \Sigma' \xrightarrow{\ g\ } \Sigma''$$

$\alpha_{\Sigma''}(\mathrm{SEN}^\flat(g)(\mathrm{SEN}^\flat(f)(\Phi))) \subseteq \{\top^{\mathcal{A}}_{F(\Sigma'')}\}$. This gives $\alpha_{\Sigma''}(\mathrm{SEN}^\flat(gf)(\Phi)) \subseteq \{\top^{\mathcal{A}}_{F(\Sigma'')}\}$, whence, by hypothesis, $\alpha_{\Sigma''}(\mathrm{SEN}^\flat(gf)(\phi)) = \top^{\mathcal{A}}_{F(\Sigma'')}$. Thus, for all $\Sigma'' \in |\mathbf{Sign}^\flat|$ and all $g \in \mathbf{Sign}^\flat(\Sigma',\Sigma'')$, $\alpha_{\Sigma''}(\mathrm{SEN}^\flat(g)(\mathrm{SEN}^\flat(f)(\phi))) = \top^{\mathcal{A}}_{F(\Sigma'')}$. We conclude that $\mathrm{SEN}^\flat(f)(\phi) \in C^{\mathsf{K},\top}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))$ and, therefore, $C^{\mathsf{K},\top}$ is also structural. ■

Based on Proposition 1347, it makes sense, given an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$, with $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a $\top^\flat$-pointed class of $\mathbf{F}$-algebraic systems, to define the **assertional π-institution of K** as the pair

$$\mathcal{I}^{\mathsf{K},\top} = \langle \mathbf{F}, C^{\mathsf{K},\top} \rangle.$$

We have seen in Proposition 1346 that, if $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is a protoalgebraic and family regular π-institution, having natural theorems, then its class $\mathrm{AlgSys}^*(\mathcal{I})$ of reduced $\mathbf{F}$-algebraic systems is a relatively point regular class. We show next, in a form of converse, that if K is a relatively point regular guasivariety of $\mathbf{F}$-algebraic systems, then the assertional π-institution $\mathcal{I}^{\mathsf{K},\top}$, associated with K, is a protoalgebraic family regular π-institution that has natural theorems.

First, we establish possession of natural theorems, under the assumption that K is pointed.

Proposition 1348 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a pointed class of $\mathbf{F}$-algebraic systems. Then $\mathcal{I}^{\mathsf{K},\top}$ has natural theorems.*

Proof: Let K be a pointed class of $\mathbf{F}$-algebraic systems. Since K is pointed, there exists $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$, all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$,

$$\top^{\mathcal{A}}_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}))) = \top^{\mathcal{A}}_{F(\Sigma')}.$$

This implies that $\alpha_{\Sigma'}(\top^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}))) = \top^{\mathcal{A}}_{F(\Sigma')}$ and, hence, we obtain $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\top^\flat_{\Sigma}(\vec{\phi}))) = \top^{\mathcal{A}}_{F(\Sigma')}$. Thus, by definition, $\top^\flat_\Sigma(\vec{\phi}) \in C^{\mathsf{K},\top}_\Sigma(\varnothing)$ and, therefore, $\top^\flat$ is a natural theorem. ■

Next, we turn to proving family regularity, again under the assumption of pointedness.

Proposition 1349 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a pointed class of $\mathbf{F}$-algebraic systems. Then $\mathcal{I}^{\mathsf{K},\top}$ is a family regular π-institution.*

Proof: Let K be a pointed class of $\mathbf{F}$-algebraic systems. We know, by Proposition 1348, that $\mathcal{I}^{\mathsf{K},\top}$ has a natural theorem $\top^\flat$, where $\top^\flat$ is a point in K. We show that $\mathcal{I}^{\mathsf{K},\top}$ is family regular. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, we have

$$\mathrm{SEN}^\flat(f)(\phi) \approx \top^\flat_{\Sigma'}, \mathrm{SEN}^\flat(f)(\psi) \approx \top^\flat_{\Sigma'} \in C^{\mathsf{K}}_{\Sigma'}(\phi \approx \top^\flat_\Sigma, \psi \approx \top^\flat_\Sigma).$$

This implies that, for all $\sigma^\flat$ in $N^\flat$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \approx \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}) \in C^{\mathsf{K}}_{\Sigma'}(\phi \approx \top^\flat_\Sigma, \psi \approx \top^\flat_\Sigma).$$

Now we get

$$\begin{array}{l}\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \approx \top^\flat_{\Sigma'} \in C^{\mathsf{K}}_{\Sigma'}(\phi \approx \top^\flat_\Sigma, \psi \approx \top^\flat_\Sigma)\\ \qquad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}) \approx \top^\flat_{\Sigma'} \in C^{\mathsf{K}}_{\Sigma'}(\phi \approx \top^\flat_\Sigma, \psi \approx \top^\flat_\Sigma).\end{array}$$

Hence, by definition,

$$\begin{array}{l}\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \in C^{\mathsf{K},\top}_{\Sigma'}(\phi, \psi)\\ \qquad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}) \in C^{\mathsf{K},\top}_{\Sigma'}(\phi, \psi).\end{array}$$

Therefore, by Theorem 19, $\langle \phi, \psi \rangle \in \Omega_\Sigma(C(\phi, \psi))$. ■

Before establishing protoalgebraicity, we need a couple of lemmas. We show, first, that, if K is a pointed class, then all theory families of $\mathcal{I}^{\mathsf{K},\top}$ are fully determined by the corresponding Leibniz class of the point.

Lemma 1350 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with a natural transformation* $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* K *a pointed class of* $\mathbf{F}$*-algebraic systems. Then, for all* $T \in \mathrm{ThFam}(\mathcal{I}^{\mathsf{K},\top})$*,*

$$T = \top^\flat / \Omega(T).$$

Proof: Let K be a pointed class of $\mathbf{F}$-algebraic systems, $T \in \mathrm{ThFam}(\mathcal{I}^{\mathsf{K},\top})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$.

- Suppose $\phi \in \top^\flat_\Sigma / \Omega_\Sigma(T)$. This means that $\langle \phi, \top^\flat_\Sigma \rangle \in \Omega_\Sigma(T)$. But, by definition, $\top^\flat_\Sigma \in \mathrm{Thm}_\Sigma(\mathcal{I}^{\mathsf{K},\top}) \subseteq T_\Sigma$, whence, by the compatibility property of $\Omega(T)$ with T, we get that $\phi \in T_\Sigma$.

- Suppose $\phi \in T_\Sigma$. Then $\phi \approx \top^\flat_\Sigma \in C^{\mathsf{K}}_\Sigma(T \approx \top^\flat)$. This implies that, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

  $$\begin{array}{l}\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \approx \top^\flat_{\Sigma'} \in C^{\mathsf{K}}_{\Sigma'}(T \approx \top^\flat)\\ \qquad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\top^\flat_\Sigma), \vec{\chi}) \approx \top^\flat_{\Sigma'} \in C^{\mathsf{K}}_{\Sigma'}(T \approx \top^\flat).\end{array}$$

  This is, by definition, equivalent to the statement that, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

  $$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \in C^{\mathsf{K},\top}_{\Sigma'}(T) \quad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\top^\flat_\Sigma), \vec{\chi}) \in C^{\mathsf{K},\top}_{\Sigma'}(T).$$

  We conclude that $\langle \phi, \top^\flat_\Sigma \rangle \in \Omega_\Sigma(T)$, i.e., that $\phi \in \top^\flat_\Sigma / \Omega_\Sigma(T)$.

Thus, we get that $T = \top^\flat/\Omega(T)$. ■

Next, we show that, if K is a relatively point regular guasivariety, then, for every theory family of $\mathcal{I}^{\mathsf{K},\top}$, the quotient of $\mathcal{F}$ by the Leibniz congruence system of T, belongs to K and, therefore, for every theory family T of $\mathcal{I}^{\mathsf{K},\top}$, the Leibniz congruence system $\Omega(T)$ is a K-congruence system on $\mathcal{F}$.

Lemma 1351 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with a natural transformation* $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* K *a relatively point regular guasivariety of* $\mathbf{F}$*-algebraic systems. Then, for all* $T \in \mathrm{ThFam}(\mathcal{I}^{\mathsf{K},\top})$, $\mathcal{F}/\Omega(T) \in \mathsf{K}$.

Proof: Suppose that K is a relatively point regular guasivariety of $\mathbf{F}$-algebraic systems and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi_i, \psi_i \in \mathrm{SEN}^\flat(\Sigma)$, $i \in I$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$\langle \{\phi_i \approx \psi_i : i \in I\}, \phi \approx \psi \rangle \in \mathrm{GEq}_\Sigma(\mathsf{K}).$$

This is equivalent to the statement $\langle \phi, \psi \rangle \in \Theta_\Sigma^{\mathsf{K},\mathcal{F}}(\{\langle \phi_i, \psi_i \rangle : i \in I\})$. Since $\Theta^{\mathsf{K},\mathcal{F}}(\{\langle \phi_i, \psi_i \rangle : i \in I\}) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ and K is relatively point regular, $\Theta^{\mathsf{K},\mathcal{F}}(\{\langle \phi_i, \psi_i \rangle : i \in I\})$ is completely determined by its $\top^\flat$-equivalence class. So it suffices to consider guasiequations of the form

$$\langle \{\phi_i \approx \top_\Sigma^\flat : i \in I\}, \phi \approx \top_\Sigma^\flat \rangle \in \mathrm{GEq}_\Sigma(\mathsf{K}).$$

Now, let $T \in \mathrm{ThFam}(\mathcal{I}^{\mathsf{K},\top})$, such that $\langle \phi_i, \top_\Sigma^\flat \rangle \in \Omega_\Sigma(T)$, for all $i \in I$. Then, taking into account Lemma 1350, $\phi_i \in \top_\Sigma^\flat/\Omega_\Sigma(T) = T_\Sigma$, for all $i \in I$. Therefore, by definition, $\phi_i \approx \top_\Sigma^\flat \in C_\Sigma^{\mathsf{K}}(T \approx \top^\flat)$, for all $i \in I$. Since, by hypothesis, $\langle \{\phi_i \approx \top_\Sigma^\flat : i \in I\}, \phi \approx \top_\Sigma^\flat \rangle \in \mathrm{GEq}_\Sigma(\mathsf{K})$, we get $\phi \approx \top_\Sigma^\flat \in C_\Sigma^{\mathsf{K}}(T \approx \top^\flat)$, i.e., $\phi \in C_\Sigma^{\mathsf{K},\top}(T)$. Since $T \in \mathrm{ThFam}(\mathcal{I}^{\mathsf{K},\top})$, $\phi \in T_\Sigma = \top_\Sigma^\flat/\Omega_\Sigma(T)$. Therefore, $\langle \phi, \top_\Sigma^\flat \rangle \in \Omega_\Sigma(T)$. We conclude that $\mathcal{F}/\Omega(T)$ satisfies all guasiequations of K and, hence, since K is a guasivariety, $\mathcal{F}/\Omega(T) \in \mathsf{K}$. ■

Finally, we establish protoalgebraicity of $\mathcal{I}^{\mathsf{K},\top}$, under the hypotheses that K is a relatively point regular guasivariety of $\mathbf{F}$-algebraic systems.

Proposition 1352 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* K *a relatively point regular guasivariety of* $\mathbf{F}$*-algebraic systems. Then* $\mathcal{I}^{\mathsf{K},\top}$ *is a protoalgebraic* π*-institution.*

Proof: Let K be a relatively point regular guasivariety of $\mathbf{F}$-algebraic systems. We know, by Proposition 1348, that $\mathcal{I}^{\mathsf{K},\top}$ has a natural theorem $\top^\flat$, where $\top^\flat$ is a point in K, and, by Proposition 1349, that $\mathcal{I}^{\mathsf{K},\top}$ is a family regular π-institution.

Now we show that $\mathcal{I}^{\mathsf{K},\top}$ is protoalgebraic. Suppose that $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$. Then, by Lemma 1350, we get $\top^\flat/\Omega(T) \leq \top^\flat/\Omega(T')$. Since, by Lemma 1351, $\Omega(T)$ and $\Omega(T')$ are K-congruence systems on $\mathcal{F}$ and K is relatively point regular, they are completely determined (generated) by their $\top^\flat$-classes and, hence, we get $\Omega(T) \leq \Omega(T')$. Thus, $\mathcal{I}^{\mathsf{K},\top}$ is protoalgebraic. ■

We show, next, that, for a protoalgebraic family regular π-institution $\mathcal{I}$, having natural theorems, the assertional π-institution of its class $\mathrm{AlgSys}^*(\mathcal{I})$ of reduced $\mathbf{F}$-algebraic systems coincides with $\mathcal{I}$.

Theorem 1353 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a family regular protoalgebraic* π*-institution based on* $\mathbf{F}$*, having a natural theorem* $\top$*. Then*

$$\mathcal{I}^{\mathrm{AlgSys}^*(\mathcal{I}),\top} = \mathcal{I}.$$

Proof: Set, for brevity in the course of this proof, $\mathsf{K} := \mathrm{AlgSys}^*(\mathcal{I})$. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{lll} \phi \in C^{\mathsf{K},\top}_\Sigma(\Phi) & \text{iff} & \phi \approx \top^\flat_\Sigma \in C^{\mathsf{K}}_\Sigma(\Phi \approx \top^\flat_\Sigma) \\ & \text{iff} & \text{for all } T \in \mathrm{ThFam}(\mathcal{I}), \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma,\Sigma'), \\ & & \mathrm{SEN}^\flat(f)(\Phi) \approx \top^\flat_{\Sigma'} \in \Omega_{\Sigma'}(T) \\ & & \qquad\qquad \text{implies } \mathrm{SEN}^\flat(f)(\phi) \approx \top^\flat_{\Sigma'} \in \Omega_{\Sigma'}(T) \\ & \text{iff} & \text{for all } T \in \mathrm{ThFam}(\mathcal{I}), \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma,\Sigma'), \\ & & \mathrm{SEN}^\flat(f)(\Phi) \in T_{\Sigma'} \text{ implies } \mathrm{SEN}^\flat(f)(\phi) \in T_{\Sigma'} \\ & \text{iff} & \phi \in C_\Sigma(\Phi). \end{array}$$

We conclude that $C^{\mathsf{K},\top} = C$ and, therefore, $\mathcal{I}^{\mathrm{AlgSys}^*(\mathcal{I}),\top} = \mathcal{I}$. ■

Moreover, starting with a relatively point regular guasivariety of $\mathbf{F}$-algebraic systems, the class of all reduced $\mathbf{F}$-algebraic systems of its assertional π-institution coincides with the original class.

Theorem 1354 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* K *a relatively point regular guasivariety of* $\mathbf{F}$*-algebraic systems. Then*

$$\mathrm{AlgSys}^*(\mathcal{I}^{\mathsf{K},\top}) = \mathsf{K}.$$

Proof: Let K be a relatively point regular guasivariety of $\mathbf{F}$-algebraic systems. Assume that $\mathcal{A} \in \mathsf{K}$ and consider $\{\top^\mathcal{A}\} := \{\top^\mathcal{A}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} \in \mathrm{SenFam}(\mathcal{A})$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C^{\mathsf{K},\top}_\Sigma(\Phi)$ and $\alpha_\Sigma(\Phi) \subseteq \{\top^\mathcal{A}_{F(\Sigma)}\}$, we get, by the definition of $C^{\mathsf{K},\top}$, $\alpha_\Sigma(\phi) = \top^\mathcal{A}_{F(\Sigma)}$. Therefore, $\{\top^\mathcal{A}\} \in \mathrm{ThFam}(\mathcal{I}^{\mathsf{K},\top})$. Moreover, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll} \langle \phi, \top^\mathcal{A}_\Sigma \rangle \in \Omega^\mathcal{A}_\Sigma(\{\top^\mathcal{A}\}) & \text{iff} & \phi = \top^\mathcal{A}_\Sigma \quad \text{(Lemma 1350)} \\ & \text{iff} & \langle \phi, \top^\mathcal{A}_\Sigma \rangle \in \Delta^\mathcal{A}_\Sigma. \end{array}$$

Thus, $\top^\mathcal{A}/\Omega^\mathcal{A}(\{\top^\mathcal{A}\}) = \top^\mathcal{A}/\Delta^\mathcal{A}$. Therefore, by relative point regularity, we obtain $\Omega^\mathcal{A}(\{\top^\mathcal{A}\}) = \Delta^\mathcal{A}$. This yields $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I}^{\mathsf{K},\top})$.

Assume, conversely, that $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I}^{\mathsf{K},\top})$. Then, by definition, there exists $T \in \mathrm{FiFam}^{\mathcal{I}^{\mathsf{K},\top}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. Suppose that $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that

$$\langle \Phi \approx \top^\flat_\Sigma, \phi \approx \top^\flat_\Sigma\rangle \in \mathrm{GEq}_\Sigma(\mathsf{K})$$

and $\alpha_\Sigma(\Phi) \subseteq \{\top^{\mathcal{A}}_{F(\Sigma)}\}$. Then, since $T \in \mathrm{FiFam}^{\mathcal{I}^{\mathsf{K},\top}}(\mathcal{A})$, $\alpha_\Sigma(\Phi) \subseteq T_{F(\Sigma)}$. Hence, $\Phi \subseteq \alpha^{-1}_\Sigma(T_{F(\Sigma)})$. Since $T \in \mathrm{ThFam}^{\mathcal{I}^{\mathsf{K},\top}}(\mathcal{A})$, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I}^{\mathsf{K},\top})$, whence, by Lemma 1350, $\alpha^{-1}(T) = \top^\flat/\Omega(\alpha^{-1}(T))$. Thus, we get $\Phi \subseteq \top^\flat_\Sigma/\Omega_\Sigma(\alpha^{-1}(T))$. Hence, $\Phi \approx \top^\flat_\Sigma \in \Omega_\Sigma(\alpha^{-1}(T))$. By Lemma 1351, $\Omega(\alpha^{-1}(T)) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$, whence, since $\langle \Phi \approx \top^\flat_\Sigma, \phi \approx \top^\flat_\Sigma\rangle \in \mathrm{GEq}_\Sigma(\mathsf{K})$, $\phi \approx \top^\flat_\Sigma \in \Omega_\Sigma(\alpha^{-1}(T))$. By Proposition 24, $\phi \approx \top^\flat_\Sigma \in \alpha^{-1}_\Sigma(\Omega^{\mathcal{A}}_{F(\Sigma)}(T))$, i.e., $\alpha_\Sigma(\phi) \approx \top^{\mathcal{A}}_{F(\Sigma)} \in \Omega^{\mathcal{A}}_{F(\Sigma)}(T) = \Delta^{\mathcal{A}}_{F(\Sigma)}$. Thus, $\alpha_\Sigma(\phi) = \top^{\mathcal{A}}_{F(\Sigma)}$. We conclude that $\langle \Phi \approx \top^\flat_\Sigma, \phi \approx \top^\flat_\Sigma\rangle \in \mathrm{GEq}_\Sigma(\mathcal{A})$. Since $\mathcal{A}$ satisfies all guasiequations in $\mathrm{GEq}(\mathsf{K})$ and K is, by hypothesis, a guasivariety, we get that $\mathcal{A} \in \mathsf{K}$. Therefore, $\mathrm{AlgSys}^*(\mathcal{I}^{\mathsf{K},\top}) = \mathsf{K}$. ■

Now we can formulate the main theorems of the section.

Theorem 1355 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a π-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is protoalgebraic family regular, with natural theorems, if and only if it is the assertional π-institution of a relatively point regular guasivariety of* $\mathbf{F}$*-algebraic systems.*

More precisely, $\mathcal{I}$ *is protoalgebraic family regular, with natural theorems, if and only if* $\mathrm{AlgSys}^*(\mathcal{I})$ *is a relatively point regular guasivariety and* $\mathcal{I} = \mathcal{I}^{\mathrm{AlgSys}^*(\mathcal{I}),\top}$*, where* $\top^\flat$ *is any natural theorem.*

Proof: Suppose $\mathcal{I}$ is protoalgebraic family regular, with natural theorems. Then, by Proposition 1346, $\mathrm{AlgSys}^*(\mathcal{I})$ is a relatively point regular class of $\mathbf{F}$-algebraic systems and, by protoalgebraicity, Proposition 68 and Theorem **??**, it is a guasivariety. Moreover, by Theorem 1353, $\mathcal{I} = \mathcal{I}^{\mathrm{AlgSys}^*(\mathcal{I}),\top}$.

Assume, conversely, that $\mathcal{I}^{\mathsf{K},\top}$ is the assertional π-institution of a relatively point regular guasivariety K of $\mathbf{F}$-algebraic systems. Then, by Proposition 1349, it is family regular, by Proposition 1352, it is protoalgebraic and, by Proposition 1348, it has natural theorems. ■

Theorem 1356 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ *in* $N^\flat$*. Then, there exists a one to one correspondence between relatively point regular guasivarieties, with point* $\top^\flat$*, and family regular protoalgebraic π-institutions, with a natural theorem* $\top^\flat$*.*

Every relatively point regular guasivariety with point $\top^\flat$ *determines a unique family regular protoalgebraic π-institution with natural theorems, its assertional π-institution.*

Every family regular protoalgebraic π-institution with natural theorems is the assertional π-institution of a unique relatively point regular guasivariety, the guasivariety $\mathrm{AlgSys}^*(\mathcal{I})$ *of all its reduced* $\mathbf{F}$*-algebraic systems.*

For each family regular protoalgebraic π-institution, with a natural theorem $\top^\flat$*, we have* $\mathcal{I} = \mathcal{I}^{\mathrm{AlgSys}^*(\mathcal{I}),\top}$ *and, conversely, for every relatively point regular guasivariety* K*, with point* $\top^\flat$*, we have* $\mathsf{K} = \mathrm{AlgSys}^*(\mathcal{I}^{\mathsf{K},\top})$*.*

Proof: This is a recap of Theorems 1353 and 1354. ∎