Chapter 19

Full Models of π-Institutions

19.1 π-Structures Revisited

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system. An $N^\flat$-**structure** is a pair $\mathbb{L} = \langle \mathbf{A}, D\rangle$, where $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ is an $N^\flat$-algebraic system and $D : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}$ is a closure (operator) family (not necessarily a system, i.e., not necessarily structural) on $\mathbf{A}$. An $\mathbf{F}$-**structure** is a pair $\mathbb{L} = \langle \mathcal{A}, D\rangle$, where $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ is an $\mathbf{F}$-algebraic system and $D : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}$ is a closure family on $\mathcal{A}$.

We give a condition pinpointing exactly when a closure family is a closure system and, as a consequence, when a π-structure becomes a π-institution.

Proposition 1357 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system and* $D : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}$ *a closure family on* SEN. *Then* D *is a closure system, if and only if, for all* $\Sigma, \Sigma' \in |\mathbf{Sign}|$ *and all* $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\mathrm{SEN}(f)^{-1}(\mathcal{D}_{\Sigma'}) \subseteq \mathcal{D}_\Sigma$.

Proof: Suppose, first, that D is structural and let $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $X' \subseteq \mathrm{SEN}(\Sigma')$, such that $D_{\Sigma'}(X') = X'$. Then we have

$$\begin{array}{rcl}\mathrm{SEN}(f)(D_\Sigma(\mathrm{SEN}(f)^{-1}(X'))) & \subseteq & D_{\Sigma'}(\mathrm{SEN}(f)(\mathrm{SEN}(f)^{-1}(X'))) \\ & \subseteq & D_{\Sigma'}(X') \\ & = & X'.\end{array}$$

So $D_\Sigma(\mathrm{SEN}(f)^{-1}(X')) \subseteq \mathrm{SEN}(f)^{-1}(X')$ and $\mathrm{SEN}(f)^{-1}(X') \in \mathcal{D}_\Sigma$.

Suppose, conversely, that, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\mathrm{SEN}(f)^{-1}(\mathcal{D}_{\Sigma'}) \subseteq \mathcal{D}_\Sigma$. Let $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathrm{Sign}(\Sigma, \Sigma')$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\phi \in D_\Sigma(\Phi)$. Let $T' \in \mathcal{D}_{\Sigma'}$, such that $\mathrm{SEN}(f)(\Phi) \subseteq T'$. Thus, $\Phi \subseteq \mathrm{SEN}(f)^{-1}(T')$. By hypothesis, $\mathrm{SEN}(f)^{-1}(T') \in \mathcal{D}_\Sigma$, whence, since $\phi \in D_\Sigma(\Phi)$ and $\Phi \subseteq \mathrm{SEN}(f)^{-1}(T')$, we get that $\phi \in \mathrm{SEN}(f)^{-1}(T')$ and, therefore, $\mathrm{SEN}(f)(\phi) \in T'$. We conclude that $\mathrm{SEN}(f)(\phi) \in D_{\Sigma'}(\mathrm{SEN}(f)(\Phi))$. Thus, D is a structural closure family on SEN. ■

Let $\mathbb{L} = \langle \mathbf{A}, D\rangle$ and $\mathbb{L}' = \langle \mathbf{A}', D'\rangle$ be two $N^\flat$-structures, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ and $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$, and consider an $N^\flat$-algebraic system morphism $\langle F, \alpha\rangle : \mathbf{A} \to \mathbf{A}'$. We say that

> $\langle F, \alpha\rangle$ is a **logical morphism from** $\mathbb{L}$ **to** $\mathbb{L}'$, denoted $\langle F, \alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$, if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,
>
> $$\phi \in D_\Sigma(\Phi) \quad \text{implies} \quad \alpha_\Sigma(\phi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)),$$
>
> or, equivalently, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)$,
>
> $$\alpha_\Sigma(D_\Sigma(\Phi)) \subseteq D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

Proposition 1358 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *and* $\mathbb{L}' = \langle \mathbf{A}', D'\rangle$ *be two* $N^\flat$*-structures, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *and* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$*, and consider an* $N^\flat$*-algebraic system morphism* $\langle F, \alpha\rangle : \mathbf{A} \to \mathbf{A}'$. $\langle F, \alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$ *is a logical morphism if and only if, for every* $T' \in \mathrm{ThFam}(\mathbb{L}')$, $\alpha^{-1}(T') \in \mathrm{ThFam}(\mathbb{L})$.

Proof: Suppose, first, that $\langle F,\alpha\rangle : \mathbb{L}{\succ}\mathbb{L}'$ is a logical morphism and let $T' \in \mathrm{ThFam}(\mathbb{L}')$, $\Sigma \in |\mathbf{Sign}|$, $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in D_\Sigma(\alpha_\Sigma^{-1}(T'_{F(\Sigma)}))$. Then, we have

$$\begin{array}{rcl}\alpha_\Sigma(\phi) & \in & \alpha_\Sigma(D_\Sigma(\alpha_\Sigma^{-1}(T'_{F(\Sigma)})))\\ & \subseteq & D'_{F(\Sigma)}(\alpha_\Sigma(\alpha_\Sigma^{-1}(T'_{F(\Sigma)})))\\ & \subseteq & D'_{F(\Sigma)}(T'_{F(\Sigma)})\\ & = & T'_{F(\Sigma)}.\end{array}$$

Therefore, $\phi \in \alpha_\Sigma^{-1}(T'_{F(\Sigma)})$ and we conclude that $\alpha^{-1}(T') \in \mathrm{ThFam}(\mathbb{L})$.

Suppose, conversely, that, for every $T' \in \mathrm{ThFam}(\mathbb{L}')$, we have $\alpha^{-1}(T') \in \mathrm{ThFam}(\mathbb{L})$ and let $\Sigma \in |\mathbf{Sign}|$, $\Phi\cup\{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\phi \in D_\Sigma(\Phi)$. Then, for all $T' \in \mathrm{ThFam}(\mathbb{L}')$, such that $\alpha_\Sigma(\Phi) \subseteq T'_{F(\Sigma)}$, we get $\Phi \subseteq \alpha_\Sigma^{-1}(T'_{F(\Sigma)})$. Since $\phi \in D_\Sigma(\Phi)$ and, by hypothesis, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathbb{L})$, we get $\phi \in \alpha_\Sigma^{-1}(T'_{F(\Sigma)})$. Hence, $\alpha_\Sigma(\phi) \in T'_{F(\Sigma)}$. Since $T' \in \mathrm{ThFam}(\mathbb{L}')$ was arbitrary, we conclude that $\alpha_\Sigma(\phi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$. Thus, $\langle F,\alpha\rangle : \mathbb{L}{\succ}\mathbb{L}'$ is a logical morphism. ∎

Let $\mathbb{L} = \langle\mathbf{A}, D\rangle$ and $\mathbb{L}' = \langle\mathbf{A}', D'\rangle$ be two $N^\flat$-structures, with $\mathbf{A} = \langle\mathbf{Sign}, \mathrm{SEN}, N\rangle$ and $\mathbf{A}' = \langle\mathbf{Sign}', \mathrm{SEN}', N'\rangle$, and consider an $N^\flat$-algebraic system morphism $\langle F,\alpha\rangle : \mathbf{A} \to \mathbf{A}'$. We say that

> $\langle F,\alpha\rangle$ is a **bilogical morphism from $\mathbb{L}$ to $\mathbb{L}'$** , denoted $\langle F,\alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$, if $\langle F,\alpha\rangle : \mathbf{A} \to \mathbf{A}'$ is surjective and, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,
>
> $$\phi \in D_\Sigma(\Phi) \quad \text{iff} \quad \alpha_\Sigma(\phi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)),$$
>
> or, equivalently, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)$,
>
> $$\alpha_\Sigma(D_\Sigma(\Phi)) = D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

Proposition 1359 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathbb{L} = \langle\mathbf{A}, D\rangle$ *and* $\mathbb{L}' = \langle\mathbf{A}', D'\rangle$ *be two* $N^\flat$*-structures, with* $\mathbf{A} = \langle\mathbf{Sign}, \mathrm{SEN}, N\rangle$ *and* $\mathbf{A}' = \langle\mathbf{Sign}', \mathrm{SEN}', N'\rangle$*, and consider a surjective* $N^\flat$*-algebraic system morphism* $\langle F,\alpha\rangle : \mathbf{A} \to \mathbf{A}'$*.* $\langle F,\alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$ *is a bilogical morphism if and only if* $\mathrm{ThFam}(\mathbb{L}) = \alpha^{-1}(\mathrm{ThFam}(\mathbb{L}'))$*.*

Proof: Suppose, first, that $\langle F,\alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$ is a bilogical morphism. Then, by Proposition 1358, if $T' \in \mathrm{ThFam}(\mathbb{L}')$, then $\alpha^{-1}(T') \in \mathrm{ThFam}(\mathbb{L})$, whence $\alpha^{-1}(\mathrm{ThFam}(\mathbb{L}')) \subseteq \mathrm{ThFam}(\mathbb{L})$. To show the converse, suppose that $T \in \mathrm{ThFam}(\mathbb{L})$. For every $\Sigma' \in |\mathbf{Sign}'|$, choose $\Sigma \in |\mathbf{Sign}|$, such that $F(\Sigma) = \Sigma'$ and let $T'_{\Sigma'} = D'_{\Sigma'}(\alpha_\Sigma(T_\Sigma))$. Then set

$$T' = \{T'_{\Sigma'}\}_{\Sigma'\in|\mathbf{Sign}'|}.$$

Clearly, $T' \in \mathrm{ThFam}(\mathbb{L}')$ and, since $\langle F,\alpha\rangle$ is a bilogical morphism, $\alpha^{-1}(T') \in \mathrm{ThFam}(\mathbb{L})$. But we also have. for all $\Sigma \in |\mathbf{Sign}|$,

$$\alpha_\Sigma^{-1}(T'_{F(\Sigma)}) = \alpha_\Sigma^{-1}(D'_{F(\Sigma)}(\alpha_\Sigma(T_\Sigma))) = D_\Sigma(T_\Sigma) = T_\Sigma.$$

Therefore, we conclude that $\mathrm{ThFam}(\mathbb{L}) \subseteq \alpha^{-1}(\mathrm{ThFam}(\mathbb{L}'))$.

Suppose, conversely, that $\mathrm{ThFam}(\mathbb{L}) = \alpha^{-1}(\mathrm{ThFam}(\mathbb{L}'))$ and let $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$. We have $\phi \in D_\Sigma(\Phi)$ iff, for all $T \in \mathrm{ThFam}(\mathbb{L})$,

$$\Phi \subseteq T_\Sigma \quad \text{implies} \quad \phi \in T_\Sigma,$$

iff, for all $T' \in \mathrm{ThFam}(\mathbb{L}')$,

$$\Phi \subseteq \alpha_\Sigma^{-1}(T'_{F(\Sigma)}) \quad \text{implies} \quad \phi \in \alpha_\Sigma^{-1}(T'_{F(\Sigma)}),$$

iff, for all $T' \in \mathrm{ThFam}(\mathbb{L}')$,

$$\alpha_\Sigma(\Phi) \subseteq T'_{F(\Sigma)} \quad \text{implies} \quad \alpha_\Sigma(\phi) \in T'_{F(\Sigma)},$$

iff $\alpha_\Sigma(\phi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$. Therefore, $\langle F,\alpha\rangle$ is a bilogical morphism. ■

Let $\mathbb{L} = \langle \mathbf{A}, D\rangle$ and $\mathbb{L}' = \langle \mathbf{A}', D'\rangle$ be two $N^\flat$-structures, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ and $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$, and consider an $N^\flat$-algebraic system morphism $\langle F,\alpha\rangle : \mathbf{A} \to \mathbf{A}'$. We say that

> $\langle F,\alpha\rangle$ is an α-**isomorphism from** $\mathbb{L}$ **to** $\mathbb{L}'$, denoted $\langle F,\alpha\rangle : \mathbb{L} \vdash^\alpha \mathbb{L}'$, if it is a bilogical morphism $\langle F,\alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$, such that, for all $\Sigma \in |\mathbf{Sign}|$, $\alpha_\Sigma : \mathrm{SEN}(\Sigma) \to \mathrm{SEN}'(F(\Sigma))$ is a bijection.

Finally, $\langle F,\alpha\rangle : \mathbb{L} \to \mathbb{L}'$ is an **isomorphism**, denoted $\langle F,\alpha\rangle : \mathbb{L} \cong \mathbb{L}'$, if it is an α-isomorphism and $F : \mathbf{Sign} \to \mathbf{Sign}'$ is also an isomorphism.

In most instances, when a result holds for $F : \mathbf{Sign} \to \mathbf{Sign}'$ an isomorphism, we will formulate it, for simplicity, for the identity functor $I_{\mathbf{Sign}} : \mathbf{Sign} \to \mathbf{Sign}$, which will be sufficient for most of our purposes.

The following is an important characterization result for bilogical morphisms containing many equivalent formulations.

Given an $N^\flat$-structure $\mathbb{L} = \langle \mathbf{A}, D\rangle$, a congruence system $\theta \in \mathrm{ConSys}(\mathbf{A})$ is called a **logical congruence system of** $\mathbb{L}$ if it is compatible with every theory family of $\mathbb{L}$, i.e., if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi,\psi \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi,\psi\rangle \in \theta_\Sigma \quad \text{implies} \quad D_\Sigma(\phi) = D_\Sigma(\psi).$$

If this is the case, we write $\theta \in \mathrm{ConSys}(\mathbb{L})$.

Proposition 1360 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *and* $\mathbb{L}' = \langle \mathbf{A}', D'\rangle$ *be two* $N^\flat$*-structures, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *and* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$*, and consider a surjective* $N^\flat$*-algebraic system morphism* $\langle F,\alpha\rangle : \mathbf{A} \to \mathbf{A}'$*. Then the following are equivalent:*

(i) $\langle F,\alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$ *is a bilogical morphism;*

(ii) *For all* $\Sigma \in |\mathbf{Sign}|$, $\Phi \subseteq \mathrm{SEN}(\Sigma)$, $D_\Sigma(\Phi) = \alpha_\Sigma^{-1}(D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)))$;

(iii) *For all* $\Sigma \in |\mathbf{Sign}|$, $\Phi \subseteq \mathrm{SEN}(\Sigma)$, $\alpha_\Sigma(D_\Sigma(\Phi)) = D'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$ *and* $\mathrm{Ker}(\langle F,\alpha\rangle) \in \mathrm{ConSys}(\mathbb{L})$;

(iv) *For all* $\Sigma \in |\mathbf{Sign}|$, $\Psi \subseteq \mathrm{SEN}'(F(\Sigma))$, $D'_{F(\Sigma)}(\Psi) = \alpha_\Sigma(D_\Sigma(\alpha_\Sigma^{-1}(\Psi)))$ *and* $\mathrm{Ker}(\langle F,\alpha\rangle) \in \mathrm{ConSys}(\mathbb{L})$;

(v) *For all* $\Sigma \in |\mathbf{Sign}|$, $\mathrm{Th}_{F(\Sigma)}(\mathbb{L}') = \alpha_\Sigma(\mathrm{Th}_\Sigma(\mathbb{L}))$ *and, also,* $\mathrm{Ker}(\langle F,\alpha\rangle) \in \mathrm{ConSys}(\mathbb{L})$;

(vi) $\mathrm{ThFam}(\mathbb{L}) = \alpha^{-1}(\mathrm{ThFam}(\mathbb{L}'))$.

Proof:

(i)⇒(ii) Suppose $\langle F,\alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$ and let $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$. Then, we have

$$\begin{array}{rll} \phi \in D_\Sigma(\Phi) & \text{iff} & \alpha_\Sigma(\phi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \\ & \text{iff} & \phi \in \alpha_\Sigma^{-1}(D'_{F(\Sigma)}(\alpha_\Sigma(\Phi))). \end{array}$$

We conclude that $D_\Sigma(\Phi) = \alpha_\Sigma^{-1}(D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)))$.

(ii)⇒(iii) Let $\Sigma \in |\mathbf{Sign}|$ and $\Phi \subseteq \mathrm{SEN}(\Sigma)$. Then, by the hypothesis (ii), $D_\Sigma(\Phi) = \alpha_\Sigma^{-1}(D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)))$, whence, by surjectivity of $\langle F,\alpha\rangle$, $\alpha_\Sigma(D_\Sigma(\Phi)) = D'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$. For the second claim, suppose $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. Then

$$\alpha_\Sigma^{-1}(D'_{F(\Sigma)}(\alpha_\Sigma(\phi))) = \alpha_\Sigma^{-1}(D'_{F(\Sigma)}(\alpha_\Sigma(\psi))).$$

Thus, by hypothesis, $D_\Sigma(\phi) = D_\Sigma(\psi)$. It follows that $\mathrm{Ker}(\langle F,\alpha\rangle)$ is a logical congruence system of $\mathbb{L}$.

(iii)⇒(iv) Let $\Sigma \in |\mathbf{Sign}|$ and $\Psi \in \mathrm{SEN}'(F(\Sigma))$. Then we have

$$\begin{array}{rcl} D'_{F(\Sigma)}(\Psi) & = & D'_{F(\Sigma)}(\alpha_\Sigma(\alpha_\Sigma^{-1}(\Psi))) \\ & = & \alpha_\Sigma(D_\Sigma(\alpha_\Sigma^{-1}(\Psi))). \end{array}$$

(iv)⇒(v) Let $\Sigma \in |\mathbf{Sign}|$ and assume, first, that $T' \in \mathrm{Th}_{F(\Sigma)}(\mathbb{L}')$. Then, we have

$$T' = D'_{F(\Sigma)}(T') = \alpha_\Sigma(D_\Sigma(\alpha_\Sigma^{-1}(T'))) \in \alpha_\Sigma(\mathrm{Th}_\Sigma(\mathbb{L})).$$

Suppose, conversely, that $T \in \mathrm{Th}_\Sigma(\mathbb{L})$. Then, we have

$$\begin{array}{rcl} D'_{F(\Sigma)}(\alpha_\Sigma(T)) & = & \alpha_\Sigma(D_\Sigma(\alpha_\Sigma^{-1}(\alpha_\Sigma(T)))) \\ & = & \alpha_\Sigma(D_\Sigma(T)) \\ & = & \alpha_\Sigma(T). \end{array}$$

Therefore, $\alpha_\Sigma(T) \in \mathrm{Th}_{F(\Sigma)}(\mathbb{L}')$.

(v)⇒(vi) It suffices to show that, for all $\Sigma \in |\mathbf{Sign}|$, $\mathrm{Th}_\Sigma(\mathbb{L}) = \alpha_\Sigma^{-1}(\mathrm{Th}_{F(\Sigma)}(\mathbb{L}'))$.

Suppose, first, $T \in \mathrm{Th}_\Sigma(\mathbb{L})$. Then, by hypothesis, $\alpha_\Sigma(T) \in \mathrm{Th}_{F(\Sigma)}(\mathbb{L}')$. But, since $\mathrm{Ker}(\langle F,\alpha\rangle) \in \mathrm{ConSys}(\mathbb{L})$, we now get

$$T = \alpha_\Sigma^{-1}(\alpha_\Sigma(T)) \in \alpha_\Sigma^{-1}(\mathrm{Th}_{F(\Sigma)}(\mathbb{L}')).$$

Therefore, $\mathrm{Th}_\Sigma(\mathbb{L}) \subseteq \alpha_\Sigma^{-1}(\mathrm{Th}_{F(\Sigma)}(\mathbb{L}'))$.

Suppose, conversely, $T' \in \mathrm{Th}_{F(\Sigma)}(\mathbb{L}')$. Then, by hypothesis, there exists $T \in \mathrm{Th}_\Sigma(\mathbb{L})$, such that $T' = \alpha_\Sigma(T)$. Thus, since $\mathrm{Ker}(\langle F,\alpha\rangle) \in \mathrm{ConSys}(\mathbb{L})$, we now get

$$\alpha_\Sigma^{-1}(T') = \alpha_\Sigma^{-1}(\alpha_\Sigma(T)) = T \in \mathrm{Th}_\Sigma(\mathbb{L}).$$

We conclude that $\alpha_\Sigma^{-1}(\mathrm{Th}_{F(\Sigma)}(\mathbb{L}')) \subseteq \mathrm{Th}_\Sigma(\mathbb{L})$ and, hence, $\mathrm{ThFam}(\mathbb{L}) = \alpha^{-1}(\mathrm{ThFam}(\mathbb{L}'))$.

(vi)⇒(i) This is one part of Proposition 1359. ∎

A consequence of the preceding characterization is that, when the categories of signatures of the $N^\flat$-structures that are connected via a bilogical morphism are isomorphic, then the complete lattices of their theory families are order isomorphic.

Proposition 1361 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *and* $\mathbb{L}' = \langle \mathbf{A}', D'\rangle$ *be two* $N^\flat$*-structures, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *and* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *and consider a surjective* $N^\flat$*-algebraic system morphism* $\langle F,\alpha\rangle : \mathbf{A} \to \mathbf{A}'$*. Then* $\langle F,\alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$ *is a bilogical morphism if and only if, for all* $\Sigma \in |\mathbf{Sign}|$*,* $\alpha_\Sigma : \mathbf{Th}_\Sigma(\mathbb{L}) \to \mathbf{Th}_{F(\Sigma)}(\mathbb{L}')$ *is an order isomorphism.*

Proof: First, by Part (v) of Proposition 1360, α is a well defined surjection from $\mathrm{Th}_\Sigma(\mathbb{L})$ onto $\mathrm{Th}_{F(\Sigma)}(\mathbb{L}')$. Second, by Part (ii) of Proposition 1360, it is an injection. Therefore, it is a bijection, whose inverse, also by Part (ii) of Proposition 1360, is α_Σ^{-1}. That both α_Σ and α_Σ^{-1} are order preserving is straightforward.

Conversely, note that Part (vi) of Proposition 1360 is automatically satisfied in case $\alpha_\Sigma : \mathbf{Th}_\Sigma(\mathbb{L}) \to \mathbf{Th}_{F(\Sigma)}(\mathbb{L}')$ is an order isomorphism, for all $\Sigma \in |\mathbf{Sign}|$. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system and consider an $N^\flat$-algebraic system $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$. Extending the concept and notation from the framework of closure systems and π-institutions, given two closure families D and D' on $\mathbf{A}$ and corresponding $N^\flat$-structures $\mathbb{L} = \langle \mathbf{A}, D\rangle$ and $\mathbb{L}' = \langle \mathbf{A}, D'\rangle$, we write $D \leq D'$ and $\mathbb{L} \leq \mathbb{L}'$ to signify that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)$,

$$D_\Sigma(\Phi) \subseteq D'_\Sigma(\Phi).$$

Under this ordering, the collection of all closure families on the algebraic system $\mathbf{A}$ forms a complete lattice, which will be denoted by

$$\mathbf{ClFam}(\mathbf{A}) = \langle \mathrm{ClFam}(\mathbf{A}), \leq \rangle.$$

Given $D \in \mathrm{ClFam}(\mathbf{A})$ and corresponding $N^\flat$-structure $\mathbb{L} = \langle \mathbf{A}, D \rangle$, we write $\mathrm{ClFam}(\mathbb{L}) = \mathrm{ClFam}^D(\mathbf{A})$ to denote the principal filter of $\mathbf{ClFam}(\mathbf{A})$ generated by D, i.e., we set

$$\mathrm{ClFam}(\mathbb{L}) = \{D' \in \mathrm{ClFam}(\mathbf{A}) : D \leq D'\}.$$

Then, we have the following corollary, expressed partially in terms of the closed set families corresponding in the standard way with closure operator families.

Corollary 1362 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system,* $\mathbb{L} = \langle \mathbf{A}, D \rangle$ *and* $\mathbb{L}' = \langle \mathbf{A}', D' \rangle$ *be two* $N^\flat$*-structures, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *and* $\mathbf{A}' = \langle \mathbf{Sign}, \mathrm{SEN}', N' \rangle$*. If* $\langle I, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$*, where* $I : \mathbf{Sign} \to \mathbf{Sign}$ *is the identity functor, is a bilogical morphism, then*

$$\mathcal{T} \mapsto \alpha(\mathcal{T}) \coloneqq \{\alpha(T) : T \in \mathcal{T}\}$$

is an isomorphism between $\mathbf{ClFam}(\mathbb{L})$ *and* $\mathbf{ClFam}(\mathbb{L}')$*.*

Proof: Directly from Proposition 1361. ■

Recall that given an algebraic system $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ and a closure family $\mathcal{D}$ on $\mathbf{A}$, with corresponding $N^\flat$-structure $\mathbb{L} = \langle \mathbf{A}, \mathcal{D} \rangle$, and $T \in \mathrm{ThFam}(\mathbb{L})$, we denote by $\mathbb{L}^T = \langle \mathbf{A}, \mathcal{D}^T \rangle$ the $N^\flat$-structure whose theory families are those closure families of $\mathbb{L}$ that contain T. Moreover, we denote by $\widetilde{\Omega}(\mathbb{L}^T)$ or $\widetilde{\Omega}^{\mathbf{A}}(\mathcal{D}^T)$ the Tarski congruence system of $\mathbb{L}^T$, i.e., the large congruence system on $\mathbf{A}$ compatible with all theory families in $\mathcal{D}^T$.

Connecting Tarski congruence systems and bilogical morphisms, we obtain the following:

Proposition 1363 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system,* $\mathbb{L} = \langle \mathbf{A}, D \rangle$ *and* $\mathbb{L}' = \langle \mathbf{A}', D' \rangle$ *be two* $N^\flat$*-structures, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *and* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$*. If* $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ *is a bilogical morphism, then, for all* $T' \in \mathrm{ThFam}(\mathbb{L}')$*,*

$$\alpha^{-1}(\widetilde{\Omega}(\mathbb{L}'^{T'})) = \widetilde{\Omega}(\mathbb{L}^{\alpha^{-1}(T')}).$$

Proof: We have

$$\begin{array}{rcl}
\alpha^{-1}(\widetilde{\Omega}(\mathbb{L}'^{T'})) & = & \alpha^{-1}(\bigcap\{\Omega^{\mathbf{A}'}(T'') : T' \leq T'' \in \mathrm{ThFam}(\mathbb{L}')\}) \\
& = & \bigcap\{\alpha^{-1}(\Omega^{\mathbf{A}'}(T'')) : T' \leq T'' \in \mathrm{ThFam}(\mathbb{L}')\}) \\
& = & \bigcap\{\Omega^{\mathbf{A}}(\alpha^{-1}(T'')) : T' \leq T'' \in \mathrm{ThFam}(\mathbb{L}')\}) \\
& = & \bigcap\{\Omega^{\mathbf{A}}(T) : \alpha^{-1}(T') \leq T \in \mathrm{ThFam}(\mathbb{L})\}) \\
& = & \widetilde{\Omega}(\mathbb{L}^{\alpha^{-1}(T')}).
\end{array}$$

■

In particular, we obtain

Corollary 1364 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system,* $\mathbb{L} = \langle \mathbf{A}, D \rangle$ *and* $\mathbb{L}' = \langle \mathbf{A}', D' \rangle$ *be two* $N^\flat$*-structures, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *and* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$*. If* $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ *is a bilogical morphism, then*

$$\alpha^{-1}(\widetilde{\Omega}(\mathbb{L}')) = \widetilde{\Omega}(\mathbb{L}).$$

Proof: By Proposition 1361, we have that $\alpha^{-1}(\mathrm{Thm}(\mathbb{L}')) = \mathrm{Thm}(\mathbb{L})$. So the result follows by applying Proposition 1363 with $T' = \mathrm{Thm}(\mathbb{L}')$. ∎

We close the section by proving that two important properties of $N^\flat$-structures are preserved under bilogical morphisms.

First, we show that finitarity is preserved across bilogical morphisms. Given a base algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$, an $N^\flat$-algebraic system $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ and an $N^\flat$-structure $\mathbb{L} = \langle \mathbf{A}, D \rangle$, we say that $\mathbb{L}$ is **finitary** if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)$,

$$D_\Sigma(\Phi) = \bigcup \{ D_\Sigma(\Psi) : \Psi \subseteq_f \Phi \},$$

where $\subseteq_f$ denoted the finite subset relation.

Proposition 1365 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ $N^\flat$*-algebraic systems,* $\mathbb{L} = \langle \mathbf{A}, D \rangle$*,* $\mathbb{L}' = \langle \mathbf{A}', D' \rangle$ $N^\flat$*-structures, based on* $\mathbf{A}$*,* $\mathbf{A}'$*, respectively, and* $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ *a bilogical morphism. Then* $\mathbb{L}$ *is finitary if and only if* $\mathbb{L}'$ *is finitary.*

Proof: Since $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ is a bilogical morphism, it is surjective and, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\phi \in D_\Sigma(\Phi) \quad \text{iff} \quad \alpha_\Sigma(\phi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

Now we have $\mathbb{L}$ finitary iff, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\phi \in D_\Sigma(\Phi) \quad \text{implies} \quad \phi \in D_\Sigma(\Psi), \text{ some } \Psi \subseteq_f \Phi,$$

iff, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\alpha_\Sigma(\phi) \subseteq D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \quad \text{implies} \quad \alpha_\Sigma(\phi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Psi)), \text{ some } \Psi \subseteq_f \Phi,$$

which, taking into account the surjectivity of $\langle F, \alpha \rangle$, is equivalent to $\mathbb{L}'$ being finitary. ∎

Finally, we show that structurality is also preserved by bilogical morphisms.

Proposition 1366 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ $N^\flat$*-algebraic systems,* $\mathbb{L} = \langle \mathbf{A}, D \rangle$*,* $\mathbb{L}' = \langle \mathbf{A}', D' \rangle$ $N^\flat$*-structures, based on* $\mathbf{A}$*,* $\mathbf{A}'$*, respectively, and* $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ *a bilogical morphism. Then* D *is structural if and only if* D' *is structural.*

Proof: Suppose, first, that D is structural. We will use Proposition 1357. Consider $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $T' \in \mathcal{D}'_{F(\Sigma')}$. Then we have

$$\begin{array}{rcl} D'_{F(\Sigma)}(\mathrm{SEN}'(F(f))^{-1}(T')) & = & \alpha_\Sigma(D_\Sigma(\alpha_\Sigma^{-1}(\mathrm{SEN}'(F(f))^{-1}(T')))) \\ & & \text{(Proposition 1360)} \end{array}$$

$$\begin{array}{ccc} \mathrm{SEN}(\Sigma) & \xrightarrow{\alpha_\Sigma} & \mathrm{SEN}'(F(\Sigma)) \\ {\scriptstyle \mathrm{SEN}(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{SEN}'(F(f))} \\ \mathrm{SEN}(\Sigma') & \xrightarrow[\alpha_{\Sigma'}]{} & \mathrm{SEN}'(F(\Sigma')) \end{array}$$

$$\begin{array}{rl} = & \alpha_\Sigma(D_\Sigma(\mathrm{SEN}(f)^{-1}(\alpha_{\Sigma'}^{-1}(T')))) \\ & \text{(Commutativity of Rectangle)} \\ = & \alpha_\Sigma(\mathrm{SEN}(f)^{-1}(\alpha_{\Sigma'}^{-1}(T'))) \\ & \text{(Propositions 1360 and 1357)} \\ = & \alpha_\Sigma(\alpha_{\Sigma'}^{-1}(\mathrm{SEN}'(F(f))^{-1}(T'))) \\ & \text{(Commutativity of Rectangle)} \\ = & \mathrm{SEN}'(F(f))^{-1}(T'). \\ & \text{(Surjectivity of } \langle F, \alpha\rangle) \end{array}$$

By Proposition 1357 and taking into account the surjectivity of $\langle F, \alpha\rangle$, we conclude that D' is structural.

Assume, conversely, that D' is structural. Let $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $T \in \mathcal{D}_{\Sigma'}$. Then, there exists, by Proposition 1360, $T' \in \mathcal{D}'_{F(\Sigma')}$, such that $T = \alpha_{\Sigma'}^{-1}(T')$. So we have

$$\begin{array}{rcl} D_\Sigma(\mathrm{SEN}(f)^{-1}(T)) & = & D_\Sigma(\mathrm{SEN}(f)^{-1}(\alpha_{\Sigma'}^{-1}(T'))) \\ & = & D_\Sigma(\alpha_\Sigma^{-1}(\mathrm{SEN}'(F(f))^{-1}(T'))) \\ & & \text{(Commutativity of Rectangle)} \\ & = & \alpha_\Sigma^{-1}(\mathrm{SEN}'(F(f))^{-1}(T')) \\ & & \text{(Propositions 1357 and 1360)} \\ & = & \mathrm{SEN}(f)^{-1}(\alpha_{\Sigma'}^{-1}(T')) \\ & & \text{(Commutativity of Rectangle)} \\ & = & \mathrm{SEN}(f)^{-1}(T). \end{array}$$

We conclude, using Proposition 1357, that D is structural. ∎

19.2 Quotients and Morphisms

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system and $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an $N^\flat$-algebraic system. Given a congruence system $\theta \in \mathrm{ConSys}(\mathbf{A})$, we may define the quotient $\mathbf{A}^\theta := \mathbf{A}/\theta$ and the quotient morphism $\langle I, \pi^\theta\rangle : \mathbf{A} \to \mathbf{A}^\theta$. Moreover, given an $N^\flat$-structure $\mathbb{L} = \langle \mathbf{A}, D\rangle$, we define on the quotient $\mathbf{A}^\theta$ the closure family $D^\theta : \mathcal{P}\mathrm{SEN}^\theta \to \mathcal{P}\mathrm{SEN}^\theta$ by stipulating that its corresponding closure family $\mathcal{D}^\theta \subseteq \mathcal{P}\mathrm{SEN}^\theta$ is given by

$$\mathcal{D}^\theta := \{T \in \mathrm{SenFam}(\mathbf{A}^\theta) : (\pi^\theta)^{-1}(T) \in \mathcal{D}\}.$$

It is not difficult to see that $\mathcal{D}^\theta$ is indeed a closure family on $\mathbf{A}^\theta$. Indeed, for all $T^i \in \mathcal{D}^\theta$, $i \in I$, we have

$$(\pi^\theta)^{-1}(\bigcap_{i\in I} T^i) = \bigcap_{i\in I} (\pi^\theta)^{-1}(T^i) \in \mathcal{D},$$

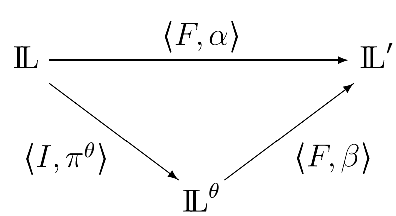
since $\mathcal{D}$ is, by hypothesis, a closure family on $\mathbf{A}$. The $N^\flat$-structure $\mathbb{L}^\theta = \langle \mathbf{A}^\theta, D^\theta\rangle$ is called the **quotient of** $\mathbb{L}$ **by** θ.

Consider, again, the quotient morphism $\langle I, \pi^\theta\rangle : \mathbf{A} \to \mathbf{A}^\theta$. It is not difficult to see either that $\langle I, \pi^\theta\rangle : \mathbb{L} \rangle\!\!-\mathbb{L}^\theta$ is a logical morphism. This simply follows from the definition of $\mathbb{L}^\theta$ and the characterization in Proposition 1358. This logical morphism is also termed the **quotient morphism** from $\mathbb{L}$ onto $\mathbb{L}^\theta$.

Suppose, now, that, in addition to being a congruence system on $\mathbf{A}$, θ is a logical congruence system of $\mathbb{L}$, $\theta \in \mathrm{ConSys}(\mathbb{L})$. An equivalent formalization is to say that $\theta \leq \widetilde{\Omega}(\mathbb{L})$. This hypothesis ensures that $\mathcal{D}^\theta = \pi^\theta(\mathcal{D})$ and that, moreover, $\mathrm{Ker}(\langle I, \pi^\theta\rangle) = \theta \in \mathrm{ConSys}(\mathbb{L})$. Therefore, by Part (v) of Proposition 1360, the quotient morphism $\langle I, \pi^\theta\rangle : \mathbb{L} \to \mathbb{L}^\theta$ becomes a bilogical morphism.

Having behind us this short introduction, we proceed to formulate and prove the Morphism Theorems, which correspond for $N^\flat$-structures to the Homomorphism, Second Isomorphism and Correspondence Theorems of Universal Algebra in forms reminiscent of the versions applicable in the context of abstract logics of abstract algebraic logic.

Theorem 1367 (Morphism Theorem) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *two* $N^\flat$*-algebraic systems, and* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *and* $\mathbb{L}' = \langle \mathbf{A}', D'\rangle$ *two* $N^\flat$*-structures. If* $\langle F, \alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$ *is a bilogical morphism, with* $\theta = \mathrm{Ker}(\langle F, \alpha\rangle)$*, then there exists an* α*-isomorphism* $\langle F, \beta\rangle : \mathbb{L}^\theta \vdash^\alpha \mathbb{L}'$*, that makes the following diagram commute*



Proof: Since $\langle F, \alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$ is a bilogical morphism, we get that $\theta = \mathrm{Ker}(\langle F, \alpha\rangle)$ is a congruence system of $\mathbb{L}$. Thus, $\langle I, \pi^\theta\rangle : \mathbb{L} \vdash \mathbb{L}^\theta$ is also a bilogical morphism. Define $\langle F,\beta\rangle : \mathbf{A}^\theta \to \mathbf{A}'$ by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\beta_\Sigma(\phi/\theta_\Sigma) = \alpha_\Sigma(\phi).$$

First, $\langle F,\beta\rangle$ is well-defined: This is straightforward, since, if $\langle \phi,\psi\rangle \in \theta_\Sigma = \mathrm{Ker}_\Sigma(\langle F,\alpha\rangle)$, then, by definition, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$.

Second, $\beta : \mathrm{SEN}^\theta \to \mathrm{SEN}' \circ F$ is natural: We have, for all $\Sigma,\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma,\Sigma')$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{ccc}
\mathrm{SEN}^\theta(\Sigma) & \xrightarrow{\beta_\Sigma} & \mathrm{SEN}'(F(\Sigma)) \\
{\scriptstyle \mathrm{SEN}^\theta(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{SEN}'(F(f))} \\
\mathrm{SEN}^\theta(\Sigma') & \xrightarrow[\beta_{\Sigma'}]{} & \mathrm{SEN}'(F(\Sigma'))
\end{array}$$

$$\begin{array}{rcl}
\mathrm{SEN}'(F(f))(\beta_\Sigma(\phi/\theta_\Sigma)) & = & \mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi)) \\
& = & \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \\
& = & \beta_{\Sigma'}(\mathrm{SEN}(f)(\phi)/\theta_{\Sigma'}) \\
& = & \beta_{\Sigma'}(\mathrm{SEN}^\theta(f)(\phi/\theta_\Sigma)).
\end{array}$$

Third, $\langle F,\beta\rangle : \mathbf{A}^\theta \to \mathbf{A}'$ is surjective: This is also clear, based on the fact that $\langle F,\alpha\rangle : \mathbf{A} \to \mathbf{A}'$ is surjective.

Fourth, $\langle F,\beta\rangle : \mathbb{L}^\theta \vdash \mathbb{L}'$ is bilogical: Since surjectivity was pointed out above, we only have to show Part (iii) of Proposition 1360. First, note that $\mathrm{Ker}(\langle F,\beta\rangle) \in \mathrm{ConSys}(\mathbb{L}^\theta)$, since, for all $\Sigma \in |\mathbf{Sign}|$, $\phi,\psi \in \mathrm{SEN}(\Sigma)$, if $\langle \phi/\theta_\Sigma,\psi/\theta_\Sigma\rangle \in \mathrm{Ker}_\Sigma(\langle F,\beta\rangle)$, then $\beta_\Sigma(\phi/\theta_\Sigma) = \beta_\Sigma(\psi/\theta_\Sigma)$, whence $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. Thus, since $\mathrm{Ker}(\langle F,\alpha\rangle) \in \mathrm{ConSys}(\mathbb{L})$, $D_\Sigma(\phi) = D_\Sigma(\psi)$ and, hence, $D^\theta_\Sigma(\phi/\theta_\Sigma) = D^\theta_\Sigma(\psi/\theta_\Sigma)$. This proves that $\mathrm{Ker}(\langle F,\beta\rangle) \in \mathrm{ConSys}(\mathbb{L}^\theta)$. Finally, we have, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)$,

$$\begin{array}{rcl}
\beta_\Sigma(D^\theta_\Sigma(\Phi/\theta_\Sigma)) & = & \beta_\Sigma(D_\Sigma(\Phi)/\theta_\Sigma) \\
& = & \alpha_\Sigma(D_\Sigma(\Phi)) \\
& = & D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \\
& = & D'_{F(\Sigma)}(\beta_\Sigma(\Phi/\theta_\Sigma)).
\end{array}$$

This shows that both conditions in Part (iii) of Proposition 1360 are satisfied and, hence, $\langle F,\beta\rangle : \mathbb{L}^\theta \vdash \mathbb{L}'$ is a bilogical morphism.

Fifth, for all $\Sigma \in |\mathbf{Sign}|$ $\beta_\Sigma : \mathrm{SEN}^\theta(\Sigma) \to \mathrm{SEN}'(F(\Sigma))$ is injective. If $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\beta_\Sigma(\phi/\theta_\Sigma) = \beta_\Sigma(\psi/\theta_\Sigma)$, then $\alpha_\Sigma(\phi) = \beta_\Sigma(\phi)$, whence $\phi/\theta_\Sigma = \psi/\theta_\Sigma$. We now conclude that $\langle F,\beta\rangle : \mathbb{L}^\theta \vdash^\alpha \mathbb{L}'$ is an α-isomorphism.

Finally, the triangle commutes: This is clear, since, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $\beta_\Sigma(\pi^\theta_\Sigma(\phi)) = \beta_\Sigma(\phi/\theta_\Sigma) = \alpha_\Sigma(\phi)$. ∎

Theorem 1368 (Isomorphism Theorem) *Suppose* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *is a base algebraic system and* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *an* $N^\flat$*-algebraic system. If* $\mathbb{L} = \langle \mathbf{A}, D \rangle$ *is an* $N^\flat$*-structure and* $\theta, \theta' \in \mathrm{ConSys}(\mathbb{L})$*, such that* $\theta \leq \theta'$*, then* $\theta'/\theta \in \mathrm{ConSys}(\mathbb{L}^\theta)$ *and* $(\mathbb{L}^\theta)^{\theta'/\theta} \cong \mathbb{L}^{\theta'}$*.*

Proof: First, we show that $\theta'/\theta \in \mathrm{ConSys}(\mathbb{L}^\theta)$. To this end, let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi/\theta_\Sigma, \psi/\theta_\Sigma \rangle \in \theta'_\Sigma/\theta_\Sigma$. Then $\langle \phi, \psi \rangle \in \theta'_\Sigma$, whence, since $\theta' \in \mathrm{ConSys}(\mathbb{L})$, $D_\Sigma(\phi) = D_\Sigma(\psi)$. Hence, $D^\theta_\Sigma(\phi/\theta_\Sigma) = D^\theta_\Sigma(\psi/\theta_\Sigma)$. So $\theta'/\theta \in \mathrm{ConSys}(\mathbb{L}^\theta)$.

To finish the proof, we define $\langle I, \alpha \rangle : \mathbb{L}^\theta \vdash \mathbb{L}^{\theta'}$, by setting, for all $\Sigma \in |\mathbf{Sign}|$, $\phi \in \mathrm{SEN}(\Sigma)$,

$$\alpha_\Sigma(\phi/\theta_\Sigma) = \phi/\theta'_\Sigma.$$

If we show that $\langle I, \alpha \rangle : \mathbb{L}^\theta \vdash \mathbb{L}^{\theta'}$ is a bilogical morphism, then, by noting that $\mathrm{Ker}(\langle I, \alpha \rangle) = \theta'/\theta$ and applying Theorem 1367,

$$\begin{array}{ccc} \mathbb{L}^\theta & \xrightarrow{\langle I, \alpha \rangle} & \mathbb{L}^{\theta'} \\ {\scriptstyle \langle I, \pi^{\theta'/\theta} \rangle} \searrow & & \nearrow {\scriptstyle \langle I, \beta \rangle} \\ & (\mathbb{L}^\theta)^{\theta'/\theta} & \end{array}$$

we will have the sought after isomorphism $\langle I, \beta \rangle : (\mathbb{L}^\theta)^{\theta'/\theta} \cong \mathbb{L}^{\theta'}$.

First, $\langle I, \alpha \rangle : \mathrm{SEN}^\theta \to \mathrm{SEN}^{\theta'}$ is well-defined, since, for all $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, if $\langle \phi, \psi \rangle \in \theta_\Sigma$, then, by hypothesis, $\langle \phi, \psi \rangle \in \theta'_\Sigma$, showing that $\alpha_\Sigma(\phi/\theta_\Sigma) = \alpha_\Sigma(\psi/\theta_\Sigma)$.

Second, $\alpha : \mathrm{SEN}^\theta \to \mathrm{SEN}^{\theta'}$ is natural, since, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{ccc} \mathrm{SEN}^\theta(\Sigma) & \xrightarrow{\alpha_\Sigma} & \mathrm{SEN}^{\theta'}(\Sigma) \\ {\scriptstyle \mathrm{SEN}^\theta(f)} \downarrow & & \downarrow {\scriptstyle \mathrm{SEN}^{\theta'}(f)} \\ \mathrm{SEN}^\theta(\Sigma') & \xrightarrow[\alpha_{\Sigma'}]{} & \mathrm{SEN}^{\theta'}(\Sigma') \end{array}$$

$$\begin{array}{rcl} \mathrm{SEN}^{\theta'}(f)(\alpha_\Sigma(\phi/\theta_\Sigma)) & = & \mathrm{SEN}^{\theta'}(f)(\phi/\theta'_\Sigma) \\ & = & \mathrm{SEN}(f)(\phi)/\theta'_{\Sigma'} \\ & = & \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)/\theta_{\Sigma'}) \\ & = & \alpha_{\Sigma'}(\mathrm{SEN}^\theta(\phi/\theta_\Sigma)). \end{array}$$

Third, it is clear that $\langle I, \alpha \rangle : \mathbf{A}^\theta \to \mathbf{A}^{\theta'}$ is surjective. So it suffices now to show that the conditions in Part (iii) of Proposition 1360 are satisfied.

First, $\text{Ker}(\langle I,\alpha\rangle) = \theta'/\theta \in \text{ConSys}(\mathbb{L}^\theta)$, as was shown above. Finally, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \subseteq \text{SEN}(\Sigma)$, we have

$$\begin{array}{rcl}\alpha_\Sigma(D^\theta_\Sigma(\Phi/\theta_\Sigma)) & = & \alpha_\Sigma(D_\Sigma(\Phi)/\theta_\Sigma)\\ & = & D_\Sigma(\Phi)/\theta'_\Sigma\\ & = & D^{\theta'}_\Sigma(\Phi/\theta'_\Sigma)\\ & = & D^{\theta'}_\Sigma(\alpha_\Sigma(\Phi/\theta_\Sigma)).\end{array}$$

Therefore, $\langle I,\alpha\rangle : \mathbb{L}^\theta \vdash \mathbb{L}^{\theta'}$ is indeed a bilogical morphism. ∎

Theorem 1369 (Correspondence Theorem) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \text{SEN}, N\rangle$ *an* $N^\flat$*-algebraic system,* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *an* $N^\flat$*-structure and* $\theta \in \text{ConSys}(\mathbb{L})$*. Then* $\theta' \mapsto \theta'/\theta$ *defines an order isomorphism between the principal filter* $[\theta, \widetilde{\Omega}(\mathbb{L})]$ *in* $\mathbf{ConSys}(\mathbb{L})$ *and the complete lattice* $\mathbf{ConSys}(\mathbb{L}^\theta)$*.*

Proof: By Theorem 1368, the mapping $\theta' \mapsto \theta'/\theta$ is a well defined mapping from $[\theta, \widetilde{\Omega}(\mathbb{L})]$ into $\text{ConSys}(\mathbb{L}^\theta)$. The mapping is also one-to-one. To see this, assume $\theta'/\theta = \theta''/\theta$ and let $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi \in \text{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle \in \theta'_\Sigma$. Then $\langle\phi/\theta_\Sigma, \psi/\theta_\Sigma\rangle \in \theta'_\Sigma/\theta_\Sigma = \theta''_\Sigma/\theta_\Sigma$ and, therefore, $\langle\phi,\psi\rangle \in \theta''_\Sigma$. Thus, $\theta' \le \theta''$ and, hence, by symmetry, $\theta' = \theta''$. The mapping is also surjective. To prove surjectivity, Let $\eta \in \text{ConSys}(\mathbb{L}^\theta)$. Define $\theta' = \{\theta'_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\theta'_\Sigma = \{\langle\phi,\psi\rangle \in \text{SEN}(\Sigma)^2 : \langle\phi/\theta_\Sigma, \psi/\theta_\Sigma\rangle \in \eta_\Sigma\}.$$

It is easy to see that θ' is a congruence system on $\mathbf{A}$. It is also easy to see that $\theta \le \theta'$. Furthermore, θ' is a congruence system of $\mathbb{L}$, since, if $\Sigma \in |\mathbf{Sign}|$, $\phi,\psi \in \text{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle \in \theta'_\Sigma$, we get $\langle\phi/\theta_\Sigma, \psi/\theta_\Sigma\rangle \in \eta_\Sigma \in \text{ConSys}(\mathbb{L}^\theta)$, whence $D^\theta_\Sigma(\phi/\theta_\Sigma) = D^\theta_\Sigma(\psi/\theta_\Sigma)$, i.e., $D_\Sigma(\phi)/\theta_\Sigma = D_\Sigma(\psi)/\theta_\Sigma$ and, since $\theta \in \text{ConSys}(\mathbb{L})$, $D_\Sigma(\phi) = D_\Sigma(\psi)$. Since $\theta' \mapsto \theta'/\theta = \eta$, it follows that the mapping is also surjective. Finally, it is obvious that both it and its inverse are monotone, which establishes that it is an order isomorphism. ∎

The Correspondence Theorem implies immediately a relation between the quotient of a Tarski congruence system and the Tarski system of the corresponding quotient structure.

Corollary 1370 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \text{SEN}, N\rangle$ *an* $N^\flat$*-algebraic system,* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *an* $N^\flat$*-structure and* $\theta \in \text{ConSys}(\mathbb{L})$*. Then*

$$\widetilde{\Omega}(\mathbb{L}^\theta) = \widetilde{\Omega}(\mathbb{L})/\theta.$$

Proof: We take $\theta' = \widetilde{\Omega}(\mathbb{L})$ and apply the Correspondence Theorem 1369. ∎

Corollary 1370 allows us also to conclude that the quotient of any $N^\flat$-structure by its Tarski congruence system has an identity Tarski congruence system. More precisely,

$$\widetilde{\Omega}(\mathbb{L}^{\widetilde{\Omega}(\mathbb{L})}) = \widetilde{\Omega}(\mathbb{L})/\widetilde{\Omega}(\mathbb{L}) = \Delta^{\mathbf{A}/\widetilde{\Omega}(\mathbb{L})}.$$

This leads to the definition of a reduced $N^\flat$-structure and of the reduction of an $N^\flat$-structure.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system, $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ an $N^\flat$-algebraic system and $\mathbb{L} = \langle \mathbf{A}, D \rangle$ an $N^\flat$-structure. We call $\mathbb{L}$ **reduced** if $\widetilde{\Omega}(\mathbb{L}) = \Delta^{\mathbf{A}}$. More generally, we set $\mathbb{L}^* = \mathbb{L}^{\widetilde{\Omega}(\mathbb{L})}$ and call $\mathbb{L}^*$ the **reduction** of $\mathbb{L}$. Moreover, for a class L of $N^\flat$-structures, we set

$$\mathsf{L}^* = \{\mathbb{L}^* : \mathbb{L} \in \mathsf{L}\}.$$

By the comments following Corollary 1370, $\mathbb{L}^*$ is reduced for any $N^\flat$-structure $\mathbb{L}$. In case $\mathbb{L}$ is reduced to start with, then $\mathbb{L}^* \cong \mathbb{L}$ and, in this case, $\mathbb{L}^*$ will be identified with $\mathbb{L}$.

Another important consequence of the Correspondence Theorem is that reducing a quotient of a structure results in a reduced structure that is isomorphic (and, thus, can be identified) with the reduction of the originally given structure.

Proposition 1371 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *an* $N^\flat$*-algebraic system,* $\mathbb{L} = \langle \mathbf{A}, D \rangle$ *an* $N^\flat$*-structure and* $\theta \in \mathrm{ConSys}(\mathbb{L})$*. Then*

$$(\mathbb{L}^\theta)^* \cong \mathbb{L}^*.$$

Proof: We have

$$\begin{array}{rcl}(\mathbb{L}^\theta)^* & = & (\mathbb{L}^\theta)^{\widetilde{\Omega}(\mathbb{L}^\theta)} \quad \text{(Definition of Reduction)}\\ & = & (\mathbb{L}^\theta)^{\widetilde{\Omega}(\mathbb{L})/\theta} \quad \text{(Corollary 1370)}\\ & \cong & \mathbb{L}^{\widetilde{\Omega}(\mathbb{L})} \quad \text{(Theorem 1368)}\\ & = & \mathbb{L}^*. \quad \text{(Definition of Reduction)}\end{array}$$

■

Generalizing Proposition 1371, we can show that a similar relation holds between the reductions of two $N^\flat$-structures that are related via a bilogical morphism.

Proposition 1372 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *and* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ $N^\flat$*-algebraic systems and* $\mathbb{L} = \langle \mathbf{A}, D \rangle$ *and* $\mathbb{L}' = \langle \mathbf{A}, D' \rangle$ $N^\flat$*-structures, based on* $\mathbf{A}$ *and* $\mathbf{A}'$*, respectively. If there exists a bilogical morphism* $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$*, then there exists an* α*-isomorphism*

$$\mathbb{L}^* \vdash^\alpha \mathbb{L}'^*.$$

Proof: We define $\langle F,\beta\rangle : \mathbf{A}^* \to \mathbf{A}'^*$ by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\beta_\Sigma(\phi/\widetilde{\Omega}_\Sigma(\mathbb{L})) = \alpha_\Sigma(\phi)/\widetilde{\Omega}_{F(\Sigma)}(\mathbb{L}'),$$

i.e., $\langle F,\beta\rangle$ is the morphism that makes the following rectangle commute

$$\begin{array}{ccc}
\mathbf{A} & \xrightarrow{\langle F,\alpha\rangle} & \mathbf{A}' \\
{\scriptstyle \langle I,\pi^{\widetilde{\Omega}(\mathbb{L})}\rangle}\downarrow & & \downarrow{\scriptstyle \langle I,\pi^{\widetilde{\Omega}(\mathbb{L}')}\rangle} \\
\mathbf{A}^* & \xrightarrow[\langle F,\beta\rangle]{} & \mathbf{A}'^*
\end{array}$$

First, $\langle I,\beta\rangle$ is well-defined: In fact, if $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle \in \widetilde{\Omega}_\Sigma(\mathbb{L})$, then, since, by Corollary 1364, $\Omega(\mathbb{L}) = \alpha^{-1}(\widetilde{\Omega}(\mathbb{L}'))$, we get that $\langle\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\rangle \in \widetilde{\Omega}_{F(\Sigma)}(\mathbb{L}')$.

Second $\beta : \mathrm{SEN}^{\widetilde{\Omega}(\mathbb{L})} \to \mathrm{SEN}'^{\widetilde{\Omega}(\mathbb{L}')} \circ F$ is a natural transformation: Let $\Sigma,\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma,\Sigma')$ and $\phi \in \mathrm{SEN}(\Sigma)$. We have

$$\begin{array}{ccc}
\mathrm{SEN}^{\widetilde{\Omega}(\mathbb{L})}(\Sigma) & \xrightarrow{\beta_\Sigma} & \mathrm{SEN}'^{\widetilde{\Omega}(\mathbb{L}')}(F(\Sigma)) \\
{\scriptstyle \mathrm{SEN}^{\widetilde{\Omega}(\mathbb{L})}(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{SEN}'^{\widetilde{\Omega}(\mathbb{L}')}(F(f))} \\
\mathrm{SEN}^{\widetilde{\Omega}(\mathbb{L})}(\Sigma') & \xrightarrow[\beta_{\Sigma'}]{} & \mathrm{SEN}'^{\widetilde{\Omega}(\mathbb{L}')}(F(\Sigma'))
\end{array}$$

$$\begin{aligned}
&\mathrm{SEN}'^{\widetilde{\Omega}(\mathbb{L}')}(F(f))(\beta_\Sigma(\phi/\widetilde{\Omega}_\Sigma(\mathbb{L}))) \\
&\quad = \mathrm{SEN}'^{\widetilde{\Omega}(\mathbb{L}')}(F(f))(\alpha_\Sigma(\phi)/\widetilde{\Omega}_{F(\Sigma)}(\mathbb{L}')) \\
&\quad = \mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi))/\widetilde{\Omega}_{F(\Sigma')}(\mathbb{L}') \\
&\quad = \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi))/\widetilde{\Omega}_{F(\Sigma')}(\mathbb{L}') \\
&\quad = \beta_{\Sigma'}(\mathrm{SEN}(f)(\phi)/\widetilde{\Omega}_{\Sigma'}(\mathbb{L})) \\
&\quad = \beta_{\Sigma'}(\mathrm{SEN}^{\widetilde{\Omega}(\mathbb{L})}(f)(\phi/\widetilde{\Omega}_\Sigma(\mathbb{L})).
\end{aligned}$$

Third, for every $\Sigma \in |\mathbf{Sign}|$, $\beta_\Sigma : \mathrm{SEN}^{\widetilde{\Omega}(\mathbb{L})}(\Sigma) \to \mathrm{SEN}'^{\widetilde{\Omega}(\mathbb{L}')}(F(\Sigma))$ is a bijection. Surjectivity is immediate and follows from the fat that both $\langle F,\alpha\rangle$ and $\langle I,\pi^{\widetilde{\Omega}(\mathbb{L}')}\rangle$ are surjective. For injectivity, if $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\beta_\Sigma(\phi/\widetilde{\Omega}_\Sigma(\mathbb{L})) = \beta_\Sigma(\psi/\widetilde{\Omega}_\Sigma(\mathbb{L}))$, then $\alpha_\Sigma(\phi)/\widetilde{\Omega}_{F(\Sigma)}(\mathbb{L}') = \alpha_\Sigma(\psi)/\widetilde{\Omega}_{F(\Sigma)}(\mathbb{L}')$, whence

$$\langle\phi,\psi\rangle \in \alpha_\Sigma^{-1}(\widetilde{\Omega}_{F(\Sigma)}(\mathbb{L}')) = \widetilde{\Omega}_\Sigma(\mathbb{L}).$$

This proves that β_Σ is indeed a bijection.

Finally, we use Part (iii) of Proposition 1360 to show that it is a bilogical morphism. Of course, since $\langle F,\beta\rangle$ has injective components, we get

$\mathrm{Ker}(\langle F,\beta\rangle) = \Delta^{\mathbf{A}^*}$ and, hence it is a congruence system of $\mathbb{L}^*$. Finally, if $\Sigma \in |\mathbf{Sign}|$ and $\Phi \subseteq \mathrm{SEN}(\Sigma)$, we have

$$\begin{array}{rcl}\beta_\Sigma(D^*_\Sigma(\Phi/\widetilde{\Omega}_\Sigma(\mathbb{L}))) & = & \beta_\Sigma(D_\Sigma(\Phi)/\widetilde{\Omega}_\Sigma(\mathbb{L}))\\ & = & \alpha_\Sigma(D_\Sigma(\Phi))/\widetilde{\Omega}_{F(\Sigma)}(\mathbb{L}')\\ & = & D'_{F(\Sigma)}(\alpha_\Sigma(\Phi))/\widetilde{\Omega}_{F(\Sigma)}(\mathbb{L}')\\ & = & D'^*_{F(\Sigma)}(\alpha_\Sigma(\Phi)/\widetilde{\Omega}_{F(\Sigma)}(\mathbb{L}'))\\ & = & D'^*_{F(\Sigma)}(\beta_\Sigma(\Phi/\widetilde{\Omega}_\Sigma(\mathbb{L})).\end{array}$$

Thus, $\langle F,\beta\rangle : \mathbb{L}^* \vdash^\alpha \mathbb{L}'^*$ is an α-isomorphism, as claimed. ■

In case $\mathbf{Sign}' = \mathbf{Sign}$ and $\langle I,\alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$ is a bilogical morphism, with $I : \mathbf{Sign} \to \mathbf{Sign}$ the identity functor, then we obtain

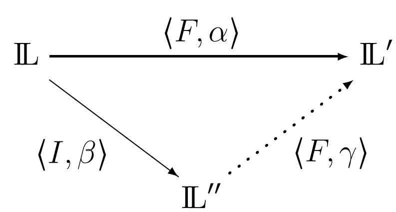
Corollary 1373 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *and* $\mathbf{A}' = \langle \mathbf{Sign}, \mathrm{SEN}', N'\rangle$ $N^\flat$*-algebraic systems and* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *and* $\mathbb{L}' = \langle \mathbf{A}, D'\rangle$ $N^\flat$*-structures, based on* $\mathbf{A}$ *and* $\mathbf{A}'$*, respectively. If there exists a bilogical morphism* $\langle I,\alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$*, then*

$$\mathbb{L}^* \cong \mathbb{L}'^*.$$

Proof: Immediate by Proposition 1372. ■

The next result is a "fill-in" lemma that provides sufficient conditions under which one can find a morphism that "fills-in" the third side of a commutative triangle, given arrows emanating from one of its vertices.

Proposition 1374 (Fill-In Lemma) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *and* $\mathbf{A}'' = \langle \mathbf{Sign}, \mathrm{SEN}'', N''\rangle$ $N^\flat$*-algebraic systems and* $\mathbb{L} = \langle \mathbf{A}, D\rangle$*,* $\mathbb{L}' = \langle \mathbf{A}', D'\rangle$ *and* $\mathbb{L}'' = \langle \mathbf{A}'', D''\rangle$ $N^\flat$*-structures, based on* $\mathbf{A}$*,* $\mathbf{A}'$ *and* $\mathbf{A}''$*, respectively. Given a logical morphism* $\langle F,\alpha\rangle : \mathbb{L} \rangle\!\!- \mathbb{L}'$ *and a bilogical morphism* $\langle I,\beta\rangle : \mathbb{L} \vdash \mathbb{L}''$*, such that* $\ker(\langle I,\beta\rangle) \leq \mathrm{Ker}(\langle F,\alpha\rangle)$*,*



there exists a unique logical morphism $\langle F,\gamma\rangle : \mathbb{L}'' \rangle\!\!- \mathbb{L}'$*, such that the triangle commutes. Moreover,* $\langle F,\gamma\rangle$ *is bilogical if and only if* $\langle F,\alpha\rangle$ *is bilogical.*

Proof: Define, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}''(\Sigma)$,

$$\gamma_\Sigma(\phi) = \alpha_\Sigma(\psi),$$

where $\psi \in \mathrm{SEN}(\Sigma)$, such that $\beta_\Sigma(\psi) = \phi$.

First, since, for all $\Sigma \in |\mathbf{Sign}|$, $\psi, \psi' \in \mathrm{SEN}(\Sigma)$, such that $\beta_\Sigma(\psi) = \beta_\Sigma(\psi')$, we have, by hypothesis, $\alpha_\Sigma(\psi) = \alpha_\Sigma(\psi')$, this definition is sound.

Second, $\gamma : \mathrm{SEN}'' \to \mathrm{SEN}' \circ F$ is a natural transformation: For all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\phi \in \mathrm{SEN}''(\Sigma)$, such that $\phi = \beta_\Sigma(\psi)$, for some $\psi \in \mathrm{SEN}(\Sigma)$, we have

$$\begin{array}{ccc}
\mathrm{SEN}''(\Sigma) & \xrightarrow{\gamma_\Sigma} & \mathrm{SEN}'(F(\Sigma)) \\
{\scriptstyle \mathrm{SEN}''(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{SEN}'(F(f))} \\
\mathrm{SEN}''(\Sigma') & \xrightarrow[\gamma_{\Sigma'}]{} & \mathrm{SEN}'(F(\Sigma'))
\end{array}$$

$$\begin{array}{rcl}
\mathrm{SEN}'(F(f))(\gamma_\Sigma(\phi)) & = & \mathrm{SEN}'(F(f))(\alpha_\Sigma(\psi)) \\
& = & \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi)) \\
& = & \gamma_{\Sigma'}(\mathrm{SEN}''(f)(\phi)),
\end{array}$$

where the last equality follows from

$$\beta_{\Sigma'}(\mathrm{SEN}(f)(\psi)) = \mathrm{SEN}''(f)(\beta_\Sigma(\psi)) = \mathrm{SEN}''(f)(\phi)$$

and the definition of $\gamma_{\Sigma'}$.

Now it is clear that the triangle of the diagram commutes. Moreover, for all $T' \in \mathrm{ThFam}(\mathcal{I}')$, since $\langle F, \alpha \rangle$ is a logical morphism, $\alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$ and, hence, by commutativity, $\beta^{-1}(\gamma^{-1}(T')) \in \mathrm{ThFam}(\mathcal{I})$. Hence, since $\langle I, \beta \rangle$ is a bilogical morphism, $\gamma^{-1}(T') \in \mathrm{ThFam}(\mathcal{I}'')$. This proves that $\langle F, \gamma \rangle$ is also a logical morphism.

Finally, for the last statement, note that $\langle F, \gamma \rangle$ is surjective if and only if $\langle F, \alpha \rangle$ is surjective, and, furthermore, $\mathrm{ThFam}(\mathcal{I}'') = \gamma^{-1}(\mathrm{ThFam}(\mathcal{I}'))$ if and only if $\beta^{-1}(\mathrm{ThFam}(\mathcal{I}'')) = \alpha^{-1}(\mathrm{ThFam}(\mathcal{I}'))$ if and only if $\mathrm{ThFam}(\mathcal{I}) = \alpha^{-1}(\mathrm{ThFam}(\mathcal{I}'))$. We conclude, taking into account Part (vi) of Proposition 1360, that $\langle F, \gamma \rangle$ is bilogical if and only if $\langle F, \alpha \rangle$ is. ■

19.3 Filter Families and π-Structures

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ an $\mathbf{F}$-algebraic system. We have seen that the collection $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ of $\mathcal{I}$-filter families on $\mathcal{A}$ forms a complete lattice $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A}) = \langle \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}), \leq \rangle$ under signature-wise inclusion. Therefore, the pair $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ constitutes an $\mathbf{F}$-structure. This $\mathbf{F}$-structure will also be denoted interchangeably by $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ or $\langle \mathcal{A}, \mathcal{C}^{\mathcal{I},\mathcal{A}} \rangle$, with reference to the closure (operator) family or the closed set family corresponding to $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Such pairs will play an important role in this chapter, since they will be used as models of $\mathcal{I}$ that provide a semantics for the logical system formalized by $\mathcal{I}$.

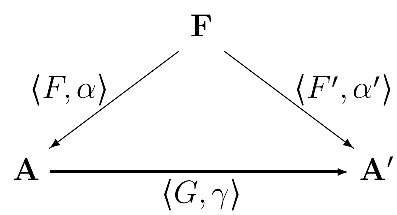
It is clear that the closure families of $\mathbf{F}$-structures of this form are structural and, hence, $\mathbf{F}$-structures of this form are actually π-institutions.

Proposition 1375 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, an $\mathbf{F}$-algebraic system. Then $C^{\mathcal{I},\mathcal{A}} : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}$ is a structural closure operator on SEN.*

Proof: We use Proposition 1357. Let $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, $T \in \mathcal{C}^{\mathcal{I},\mathcal{A}}_{F(\Sigma')}$. Then, by Lemma 51, $\alpha^{-1}_{\Sigma'}(T) \in \mathcal{C}_{\Sigma'}$. Since C is structural, by Proposition 1357, $\mathrm{SEN}(f)^{-1}(\alpha^{-1}_{\Sigma'}(T)) \in \mathcal{C}_\Sigma$. By the naturality of $\alpha : \mathrm{SEN}^\flat \to \mathrm{SEN} \circ F$, we get $\alpha^{-1}_\Sigma(\mathrm{SEN}(F(f))^{-1}(T)) \in \mathcal{C}_\Sigma$, whence, again by Lemma 51, $\mathrm{SEN}(F(f))^{-1}(T) \in \mathcal{C}^{\mathcal{I},\mathcal{A}}_\Sigma$. Using the surjectivity of $\langle F, \alpha\rangle$ and Proposition 1357, we conclude that $C^{\mathcal{I},\mathcal{A}}$ is structural. ■

Our next result characterizes bilogical morphisms between $\mathbf{F}$-structures of the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$.

Proposition 1376 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha'\rangle\rangle$ with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ $\mathbf{F}$-algebraic systems, such that, there exists a surjective $\langle G, \gamma\rangle : \mathbf{A} \to \mathbf{A}'$, such that*



$\langle G, \gamma\rangle \circ \langle F, \alpha\rangle = \langle F', \alpha'\rangle$. Then the following statements are equivalent:

(i) *$\langle G, \gamma\rangle : \langle \mathcal{A}, \mathcal{C}^{\mathcal{I},\mathcal{A}}\rangle \vdash \langle \mathcal{A}', \mathcal{C}^{\mathcal{I},\mathcal{A}'}\rangle$ is a bilogical morphism;*

(ii) *For every $\Sigma \in |\mathbf{Sign}|$, $\gamma_\Sigma : \mathcal{C}^{\mathcal{I},\mathcal{A}}_\Sigma \to \mathcal{C}^{\mathcal{I},\mathcal{A}'}_{G(\Sigma)}$ is an order isomorphism;*

(iii) *For every $\Sigma \in |\mathbf{Sign}|$, and all $T \in \mathcal{C}^{\mathcal{I},\mathcal{A}}_\Sigma$, $\gamma_\Sigma(T) \in \mathcal{C}^{\mathcal{I},\mathcal{A}'}_{G(\Sigma)}$ and, in addition, we have $\mathrm{Ker}(\langle G, \gamma\rangle) \in \mathrm{ConSys}(\langle \mathcal{A}, \mathcal{C}^{\mathcal{I},\mathcal{A}}\rangle)$.*

Proof:

(i)⇒(ii) This is a special case of Proposition 1361.

(ii)⇒(iii) The first assertion is obvious. For the second, if $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\gamma_\Sigma(\phi) = \gamma_\Sigma(\psi)$, then, for every $T' \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A}')$,

$$\gamma_\Sigma(\phi) \in T'_{G(\Sigma)} \quad \text{iff} \quad \gamma_\Sigma(\psi) \in T'_{G(\Sigma)}.$$

Hence, for every $T' \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A}')$,

$$\phi \in \gamma_\Sigma^{-1}(T'_{G(\Sigma)}) \quad \text{iff} \quad \phi \in \gamma_\Sigma^{-1}(T'_{G(\Sigma)}).$$

Therefore, by hypothesis, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

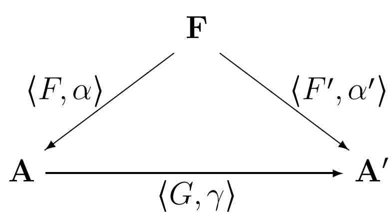
$$\phi \in T_\Sigma \quad \text{iff} \quad \psi \in T_\Sigma.$$

It now follows that $\mathrm{Ker}(\langle I, \gamma\rangle) \in \mathrm{ConSys}(\langle \mathcal{A}, \mathcal{C}^{\mathcal{I},\mathcal{A}}\rangle)$.

(iii)⇒(i) By hypothesis, we get $\gamma(\mathcal{C}^{\mathcal{I},\mathcal{A}}_\Sigma) \subseteq \mathcal{C}^{\mathcal{I},\mathcal{A}'}_{G(\Sigma)}$ and also that $\mathrm{Ker}(\langle I, \gamma\rangle) \in \mathrm{ConSys}(\langle \mathcal{A}, \mathcal{C}^{\mathcal{I},\mathcal{A}}\rangle)$. Therefore, by Part (v) of Proposition 1360, it suffices to show that $\mathcal{C}^{\mathcal{I},\mathcal{A}'}_{G(\Sigma)} \subseteq \gamma(\mathcal{C}^{\mathcal{I},\mathcal{A}}_\Sigma)$. But this follows from the fact that, if $T' \in \mathcal{C}^{\mathcal{I},\mathcal{A}'}_{G(\Sigma)}$, then by Corollary 55, $\gamma_\Sigma^{-1}(T') \in \mathcal{C}^{\mathcal{I},\mathcal{A}}_\Sigma$ and, then, by surjectivity, $T' = \gamma_\Sigma(\gamma_\Sigma^{-1}(T'))$. ■

We also have the following related result that, roughly speaking, forces the closure family of a structure that is the bilogical morphism image of a structure whose closure family consists of all filter families to also consist of the entirely of filter families.

Proposition 1377 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*,* $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha'\rangle\rangle$ *with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ $\mathbf{F}$*-algebraic systems, such that, there exists a surjective* $\langle G, \gamma\rangle : \mathbf{A} \to \mathbf{A}'$*, such that*



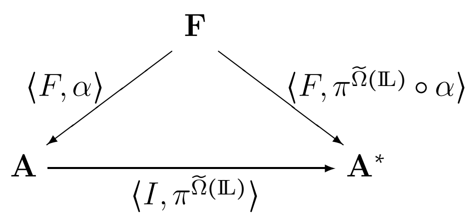
$\langle G, \gamma\rangle \circ \langle F, \alpha\rangle = \langle F', \alpha'\rangle$*. If* $\langle G, \gamma\rangle : \langle \mathcal{A}, \mathcal{C}^{\mathcal{I},\mathcal{A}}\rangle \vdash \langle \mathcal{A}', \mathcal{C}'\rangle$ *is a bilogical morphism, then* $\mathcal{C}' = \mathcal{C}^{\mathcal{I},\mathcal{A}'}$*.*

Proof: First, since $\langle G, \gamma\rangle$ is a bilogical morphism, for all $T' \in \mathcal{C}'$, we have $\gamma^{-1}(T') \in \mathcal{C}^{\mathcal{I},\mathcal{A}}$. Thus, by Corollary 55, $T' \in \mathcal{C}^{\mathcal{I},\mathcal{A}'}$. This proves that $\mathcal{C}' \subseteq \mathcal{C}^{\mathcal{I},\mathcal{A}'}$. Suppose, conversely, that $T' \in \mathcal{C}^{\mathcal{I},\mathcal{A}'}$. Then, again by Corollary 55, $\gamma^{-1}(T') \in \mathcal{C}^{\mathcal{I},\mathcal{A}}$. Therefore, since $\langle G, \gamma\rangle$ is a bilogical morphism, $T' = \gamma(\gamma^{-1}(T')) \in \mathcal{C}'$. We conclude that $\mathcal{C}' = \mathcal{C}^{\mathcal{I},\mathcal{A}'}$. ■

This result has the following immediate corollaries, one addressing reductions and the other isomorphisms.

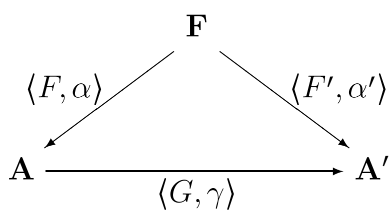
Corollary 1378 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, an* $\mathbf{F}$*-algebraic system. Then* $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$*.*

Proof: Let $\mathbb{L} = \langle \mathcal{A}, \mathcal{C}^{\mathcal{I},\mathcal{A}} \rangle$ and apply Proposition 1377 to the special configuration of morphisms depicted in the diagram:



■

Corollary 1379 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$ *with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ $\mathbf{F}$*-algebraic systems, such that, there exists an isomorphism* $\langle G, \gamma \rangle : \mathbf{A} \cong \mathbf{A}'$*, such that*

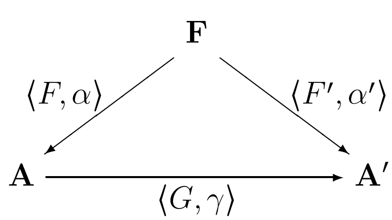


$\langle G, \gamma \rangle \circ \langle F, \alpha \rangle = \langle F', \alpha' \rangle$*. If* $\langle G, \gamma \rangle : \langle \mathcal{A}, \mathcal{D} \rangle \vdash \langle \mathcal{A}', \mathcal{D}' \rangle$ *is a bilogical morphism, then* $\mathcal{D} = \mathcal{C}^{\mathcal{I},\mathcal{A}}$ *if and only if* $\mathcal{D}' = \mathcal{C}^{\mathcal{I},\mathcal{A}'}$*.*

Proof: We apply Proposition 1377 twice; once using $\langle G, \gamma \rangle : \mathcal{A} \to \mathcal{A}'$ and once using $\langle G, \gamma \rangle^{-1} : \mathcal{A}' \to \mathcal{A}$. ■

Corollary 1379 can be strengthened slightly but, to accomplish this, we need the following proposition, which is a sort of symmetric to Proposition 1377.

Proposition 1380 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$ *two* $\mathbf{F}$*-algebraic systems.*



If $\langle G, \gamma \rangle : \langle \mathcal{A}, \mathcal{D} \rangle \vdash^\alpha \langle \mathcal{A}', C^{\mathcal{I},\mathcal{A}'} \rangle$ *is an* α*-isomorphism, then* $\mathcal{D} = \mathcal{C}^{\mathcal{I},\mathcal{A}}$*.*

Proof: We show, first, that $\mathcal{D} \subseteq \mathcal{C}^{\mathcal{I},\mathcal{A}}$. Suppose $T \in \mathcal{D}$. Then, since $\langle G,\gamma\rangle$ is an α-isomorphism, there exists, by Proposition 1360, $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$, such that $T = \gamma^{-1}(T')$. Now, by Corollary 55, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Hence, $\mathcal{D} \subseteq \mathcal{C}^{\mathcal{I},\mathcal{A}}$.

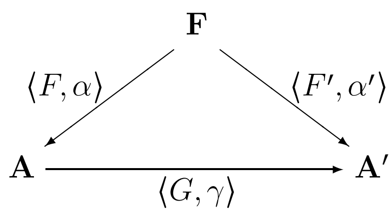
Suppose, conversely, that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Since $\langle G,\gamma\rangle$ is an α-isomorphism, there exists a unique $T' \in \mathrm{SenFam}(\mathcal{A}')$, such that $T = \gamma^{-1}(T')$. Thus, we have

$$\alpha'^{-1}(T') = \alpha^{-1}(\gamma^{-1}(T')) = \alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I}).$$

Hence, $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$ and, since $\langle G,\gamma\rangle$ is a bilogical morphism, $T = \gamma^{-1}(T') \in \mathcal{D}$. We conclude that $\mathcal{C}^{\mathcal{I},\mathcal{A}} \subseteq \mathcal{D}$ and equality follows. ∎

A generalization of Corollary 1379 relaxes the requirement that there exists an isomorphism between **F**-algebraic systems to the requirement that there exists an α-isomorphism.

Corollary 1381 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* **F***,* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$*,* $\mathcal{A}' = \langle \mathbf{A}', \langle F',\alpha'\rangle\rangle$ *with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ **F***-algebraic systems, such that, there exists a surjective morphism* $\langle G,\gamma\rangle : \mathbf{A} \cong \mathbf{A}'$*, such that*



$\langle G,\gamma\rangle \circ \langle F,\alpha\rangle = \langle F',\alpha'\rangle$. *If* $\langle G,\gamma\rangle : \langle \mathcal{A},\mathcal{D}\rangle \vdash^\alpha \langle \mathcal{A}',\mathcal{D}'\rangle$ *is an* α*-isomorphism, then* $\mathcal{D} = \mathcal{C}^{\mathcal{I},\mathcal{A}}$ *if and only if* $\mathcal{D}' = \mathcal{C}^{\mathcal{I},\mathcal{A}'}$.

Proof: We put together Proposition 1377 and Proposition 1380. ∎

19.4 $\mathcal{I}$-Structures

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, an **F**-algebraic system and $\mathbb{L} = \langle \mathcal{A}, D\rangle$ an **F**-structure. Define $C^{\mathbb{L}} = \{C^{\mathbb{L}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$C^{\mathbb{L}}_\Sigma : \mathcal{P}\mathrm{SEN}^\flat \to \mathcal{P}\mathrm{SEN}^\flat$$

be defined, for all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in C^{\mathbb{L}}_\Sigma(\Phi) \quad \text{iff} \quad \begin{array}{l} \text{for all } \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma,\Sigma'), \\ \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \subseteq D_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))). \end{array}$$

More generally, given a class L of **F**-structures, we set

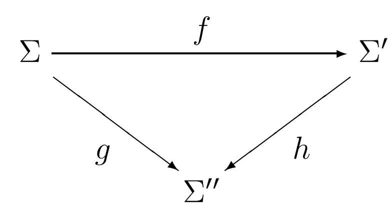
$$C^{\mathsf{L}} = \bigcap\{C^{\mathbb{L}} : \mathbb{L} \in \mathsf{L}\}.$$

We show that C^{L} is a closure system on **F** and, as a result, $\mathcal{I}^{\mathsf{L}} = \langle \mathbf{F}, C^{\mathsf{L}}\rangle$ qualifies as a π-institution.

Proposition 1382 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* L *a class of* $\mathbf{F}$*-structures. The collection* $C^{\mathsf{L}} : \mathcal{P}\mathrm{SEN}^\flat \to \mathcal{P}\mathrm{SEN}^\flat$ *is a closure system on* $\mathbf{F}$*.*

Proof: Inflationarity, monotonicity and idempotency of C^{L} follow immediately from the corresponding properties of each of the operators of the $\mathbf{F}$-structures in L. We show structurality in more detail. Suppose $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C^{\mathsf{L}}_\Sigma(\Phi)$. Then, for all $\langle \mathcal{A}, D \rangle \in \mathsf{L}$, all $\Sigma'' \in |\mathbf{Sign}^\flat|$ and all $g \in \mathbf{Sign}^\flat(\Sigma, \Sigma'')$,

$$\alpha_{\Sigma''}(\mathrm{SEN}^\flat(g)(\phi)) \in D_{F(\Sigma'')}(\alpha_{\Sigma''}(\mathrm{SEN}^\flat(g)(\Phi))).$$



Thus, for all $\langle \mathcal{A}, D \rangle \in \mathsf{L}$, all $\Sigma'' \in |\mathbf{Sign}^\flat|$ and all $h \in \mathbf{Sign}^\flat(\Sigma', \Sigma'')$,

$$\alpha_{\Sigma''}(\mathrm{SEN}^\flat(h)(\mathrm{SEN}^\flat(f)(\phi))) \in D_{F(\Sigma'')}(\alpha_{\Sigma''}(\mathrm{SEN}^\flat(h)(\mathrm{SEN}^\flat(f)(\Phi)))).$$

This proves that $\mathrm{SEN}^\flat(f)(\phi) \in C^{\mathsf{L}}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))$, and, hence, that C^{L} is structural. ∎

C^{L} is termed the **closure system on F generated by** L and we denote by $\mathcal{I}^{\mathsf{L}} = \langle \mathbf{F}, C^{\mathsf{L}} \rangle$ the π-institution corresponding to C^{L}.

Next, it is shown that $\mathbf{F}$-structures related by bilogical morphism generate identical closure systems.

Proposition 1383 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$ *two* $\mathbf{F}$*-algebraic systems,* $\mathbb{L} = \langle \mathcal{A}, D \rangle$*,* $\mathbb{L}' = \langle \mathcal{A}', D' \rangle$ *two* $\mathbf{F}$*-structures and* $\langle G, \gamma \rangle : \mathbb{L} \vdash \mathbb{L}'$ *a bilogical morphism, such that* $\langle F', \alpha' \rangle = \langle G, \gamma \rangle \circ \langle F, \alpha \rangle$*. Then* $C^{\mathbb{L}} = C^{\mathbb{L}'}$*.*

Proof: We have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

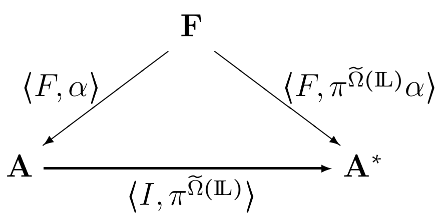
$$\begin{array}{lll}
\phi \in C^{\mathbb{L}}_\Sigma(\Phi) & \text{iff} & \text{for all } \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma, \Sigma') \\
& & \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \subseteq D_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))) \\
& \text{iff} & \text{for all } \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma, \Sigma') \\
& & \gamma_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi))) \\
& & \qquad \subseteq D'_{G(F(\Sigma'))}(\gamma_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)))) \\
& \text{iff} & \text{for all } \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma, \Sigma') \\
& & \alpha'_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \subseteq D'_{F'(\Sigma')}(\alpha'_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))) \\
& \text{iff} & \phi \in C^{\mathbb{L}'}_\Sigma(\Phi).
\end{array}$$

We conclude that $C^{\mathbb{L}} = C^{\mathbb{L}'}$. ∎

As a special case of Proposition 1383, we get that both an $\mathbf{F}$-structure and its reduction generate the same closure system on the base algebraic system $\mathbf{F}$.

Corollary 1384 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $\mathbb{L} = \langle \mathcal{A}, D \rangle$ *an* $\mathbf{F}$*-structure. Then* $C^{\mathbb{L}} = C^{\mathbb{L}^*}$.

Proof: This is obtained directly by Proposition 1383 once we recall that, since $\widetilde{\Omega}(\mathbb{L})$ is a congruence system of $\mathbb{L}$, $\langle I, \pi^{\widetilde{\Omega}(\mathbb{L})} \rangle : \mathbb{L} \to \mathbb{L}^*$ is a bilogical morphism.



And this gives the configuration of the diagram that matches the setup in the hypothesis of Proposition 1383. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, an $\mathbf{F}$-algebraic system and $\mathbb{L} = \langle \mathcal{A}, D \rangle$ an $\mathbf{F}$-structure. We say that $\mathbb{L}$ is an $\mathcal{I}$-**structure** or a **model of** $\mathcal{I}$ if $C \leq C^{\mathbb{L}}$, i.e., if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in C_\Sigma(\Phi) \quad \text{implies} \quad \phi \in C^{\mathbb{L}}_\Sigma(\Phi).$$

Of course, $C \leq C^{\mathbb{L}}$ requires that, for all $T \in \mathrm{ThFam}(\mathbb{L})$, all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq T_{F(\Sigma')} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \in T_{F(\Sigma')},$$

i.e., that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Therefore, we obtain the following characterization:

Proposition 1385 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, an* $\mathbf{F}$*-algebraic system and* $\mathbb{L} = \langle \mathcal{A}, D \rangle$ *an* $\mathbf{F}$*-structure.* $\mathbb{L}$ *is an* $\mathcal{I}$*-structure if and only if, for all* $T \in \mathrm{ThFam}(\mathbb{L})$*,* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, i.e., if and only if* $\mathrm{ThFam}(\mathbb{L}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

Again, the defining condition of an $\mathcal{I}$-structure may be simplified due to the structurality of $\mathcal{I}$. More precisely, based on Lemma 50, we have:

Lemma 1386 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, an* $\mathbf{F}$*-algebraic system and* $\mathbb{L} = \langle \mathcal{A}, D \rangle$ *an* $\mathbf{F}$*-structure. Then, the following conditions are equivalent:*

(a) $\mathbb{L}$ *is an* $\mathcal{I}$*-structure;*

(b) For all $T \in \mathrm{ThFam}(\mathbb{L})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$*, such that* $\phi \in C_\Sigma(\Phi)$*,*

$$\alpha_\Sigma(\Phi) \subseteq T_{F(\Sigma)} \quad \textit{implies} \quad \alpha_\Sigma(\phi) \in T_{F(\Sigma)};$$

(c) For all $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$*,* $\alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi))$*.*

Proof: Condition (a) clearly implies (b) and (b) and (c) are equivalent. So it remains to show that (b) implies (a). But, if Condition (*b*) holds, then, by Lemma 50, $\mathrm{ThFam}(\mathbb{L}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, whence, by Proposition 1385, $\mathbb{L}$ is an $\mathcal{I}$-structure. ■

We denote by $\mathrm{Str}(\mathcal{I})$ the class of all $\mathcal{I}$-structures and let

$$\mathrm{Str}^*(\mathcal{I}) = (\mathrm{Str}(\mathcal{I}))^*$$

be the class of all reduced $\mathcal{I}$-structures.

Since we know that $\mathbb{L} \in \mathrm{Str}(\mathcal{I})$ is and only if $\mathrm{ThFam}(\mathbb{L}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, it follows that, given an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ is the weakest $\mathcal{I}$-structure of $\mathcal{A}$, i.e., the one with the finest closure family.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that $\mathcal{I}$ is **complete with respect to a given class** L of $\mathbf{F}$-structures if $C = C^{\mathsf{L}}$, i.e., if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in C_\Sigma(\Phi) \quad \text{iff} \quad \phi \in C^{\mathsf{L}}_\Sigma(\Phi).$$

As consequences of Proposition 1383 and of its Corollary 1384, we have the following results about models of π-institutions and about classes of structures with respect to which a π-institution is complete.

Proposition 1387 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\langle \mathcal{A}, D \rangle$*,* $\mathbb{L}' = \langle \mathcal{A}', D' \rangle$ *are* $\mathbf{F}$*-structures and* $\langle G, \gamma \rangle : \mathbb{L} \vdash \mathbb{L}'$ *a bilogical morphism, the* $\mathbb{L}$ *is an* $\mathcal{I}$*-structure if and only if* $\mathbb{L}'$ *is an* $\mathcal{I}$*-structure.*

(b) If $\mathbb{L} = \langle \mathcal{A}, D \rangle$ *is an* $\mathbf{F}$*-structure, then* $\mathbb{L}$ *is an* $\mathcal{I}$*-structure if and only if* $\mathbb{L}^*$ *is an* $\mathcal{I}$*-structure.*

(c) If $\mathcal{I}$ *is complete with respect to a class* L *of* $\mathbf{F}$*-structures, then it is also complete with respect to* L^**.*

Proof: the first part is a consequence of Proposition 1383, whereas Parts (b) and (c) follow directly from Corollary 1384. ∎

Proposition 1388 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* L *a class of* $\mathcal{I}$*-structures. If* L *includes* $\langle \mathcal{F}, C \rangle$ *or* $\langle \mathcal{F}, C \rangle^*$*, then* $\mathcal{I}$ *is complete with respect to both* L *and* L^**. In particular* $\mathcal{I}$ *is complete with respect to both* $\mathrm{Str}(\mathcal{I})$ *and* $\mathrm{Str}^*(\mathcal{I})$*.*

Proof: The key here is to notice that $C = C^{\langle \mathcal{F}, C \rangle} = C^{\langle \mathcal{F}, C \rangle^*}$. Then, the rest is easy because we have

$$C \le C^{\mathsf{L}} = C^{\mathsf{L}^*} \le C^{\langle \mathcal{F}, C \rangle} = C^{\langle \mathcal{F}, C \rangle^*} = C.$$

Therefore, we conclude $C = C^{\mathsf{L}} = C^{\mathsf{L}^*}$ and, hence, $\mathcal{I}$ is complete with respect to both L and L^*. ∎

19.5 Full $\mathcal{I}$-Structures

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and $\mathbb{L} = \langle \mathcal{A}, D \rangle$ an $\mathbf{F}$-structure. $\mathbb{L}$ is a **full** $\mathcal{I}$**-structure** or a **full model of** $\mathcal{I}$ if

$$\mathbb{L}^* = \langle \mathcal{A}^*, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*) \rangle,$$

i.e., if the closure family of the reduction of $\mathbb{L}$ consists of all $\mathcal{I}$-filter families on the $\mathbf{F}$-algebraic system $\mathcal{A}/\widetilde{\Omega}(\mathbb{L})$.

We denote the class of all full $\mathcal{I}$-structures by $\mathrm{FStr}(\mathcal{I})$ and the class of all reduced full $\mathcal{I}$-structures by $\mathrm{FStr}^*(\mathcal{I})$. We also write $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$ for the collection of all full $\mathcal{I}$-structures on the $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$.

We show that full $\mathcal{I}$-structures are fully deserving of the name $\mathcal{I}$-structures.

Proposition 1389 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $\mathbb{L} = \langle \mathcal{A}, D \rangle$ *a full* $\mathcal{I}$*-structure.*

(a) D *is structural;*

(b) $\mathbb{L}$ *is an* $\mathcal{I}$*-structure;*

(c) $\mathbb{L}$ *has theorems if and only if* $\mathcal{I}$ *has theorems.*

Proof:

(a) By Proposition 1375, D^* is structural. Therefore, by Proposition 1366, D is also structural.

(b) Suppose $\mathbb{L} \in \mathrm{FStr}(\mathcal{I})$. Then, by definition, $\mathrm{ThFam}(\mathbb{L}^*) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$. Thus, by Proposition 1385, $\mathbb{L}^* \in \mathrm{Str}(\mathcal{I})$. Therefore, by Proposition 1387, $\mathbb{L} \in \mathrm{Str}(\mathcal{I})$.

(c) If $\mathcal{I}$ does not have theorems, then $\varnothing \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$. Therefore, by the definition of a full $\mathcal{I}$-structure, $\varnothing \in \mathrm{ThFam}(\mathbb{L}^*)$ and, hence $\varnothing \in \mathrm{ThFam}(\mathbb{L})$. Conversely, if $\varnothing \notin \mathrm{ThFam}(\mathcal{I})$, then $\varnothing \notin \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and, hence, $\varnothing \notin \mathrm{ThFam}(\mathbb{L})$.

■

We now show that, for every **F**-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, the pair $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ is always a full $\mathcal{I}$-structure and, thus, the weakest such structure on $\mathcal{A}$.

Proposition 1390 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ *is the weakest full* $\mathcal{I}$*-structure on* $\mathcal{A}$*.*

Proof: Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an **F**-algebraic system. Then, by Corollary 1378,

$$(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*).$$

So $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ is a full $\mathcal{I}$-structure. Moreover, since, by Proposition 1389, every full $\mathcal{I}$-structure is an $\mathcal{I}$-structure, by Proposition 1385, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is the largest possible set of theory families of a full $\mathcal{I}$-structure. So $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ is the weakest full $\mathcal{I}$-structure. ■

Specializing to the algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$, where $\langle I, \iota \rangle : \mathbf{F} \to \mathbf{F}$ is the identity morphism, we get

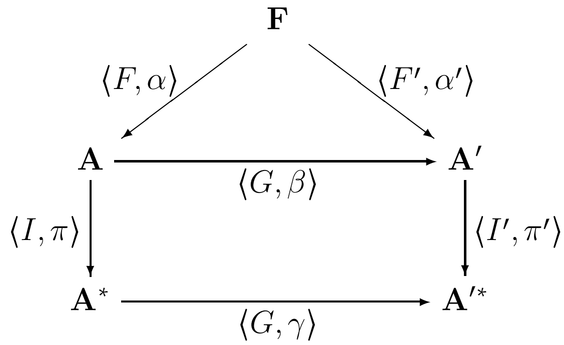
Corollary 1391 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. Then* $\langle \mathcal{F}, C \rangle$ *is the weakest full* $\mathcal{I}$*-structure on* $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$*.*

Proof: By taking $\mathcal{A} = \mathcal{F}$ in Proposition 1390. ■

Next, we see that bilogical morphisms between **F**-structures preserve the property of being a full model in both directions.

Proposition 1392 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$ *two* $\mathbf{F}$*-algebraic systems and* $\mathbb{L} = \langle \mathcal{A}, D \rangle$*,* $\mathbb{L}' = \langle \mathcal{A}', D' \rangle$ *two* $\mathbf{F}$*-structures. If there exists a bilogical morphism* $\langle G, \beta \rangle : \mathbb{L} \vdash \mathbb{L}'$*, then* $\mathbb{L}$ *is a full* $\mathcal{I}$*-structure if and only if* $\mathbb{L}'$ *is a full* $\mathcal{I}$*-structure.*

Proof: Suppose $\mathbb{L} = \langle \mathcal{A}, D \rangle$ and $\mathbb{L}' = \langle \mathcal{A}', D' \rangle$ are two $\mathbf{F}$-structures and let $\langle G, \beta \rangle : \mathbb{L} \vdash \mathbb{L}'$ be a bilogical morphism. Then, by Proposition 1372, there exists an α-isomorphism $\langle G, \gamma \rangle : \mathbb{L}^* \vdash^\alpha \mathbb{L}'^*$, such that the following diagram commutes.



where $\langle I, \pi \rangle : \mathbf{A} \to \mathbf{A}/\widetilde{\Omega}(\mathbb{L})$ and $\langle I', \pi' \rangle : \mathbf{A}' \to \mathbf{A}'/\widetilde{\Omega}(\mathbb{L}')$ denote the quotient morphisms. If $\mathbb{L}$ is a full $\mathcal{I}$-structure, then, by definition, $\mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$. Thus, by Proposition 1377, $\mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}'^*)$. This shows that $\mathbb{L}'$ is a full $\mathcal{I}$-structure. If, conversely, $\mathbb{L}'$ is a full $\mathcal{I}$-structure, then, by definition $\mathcal{D}'^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}'^*)$. Thus, by Proposition 1380, $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$ and, therefore, $\mathbb{L}$ is a full $\mathcal{I}$-structure, by definition. ∎

Proposition 1392 allows the formulation of a characterizing property of full $\mathcal{I}$-structures in terms of bilogical morphisms and weakest full $\mathcal{I}$-structures.

Corollary 1393 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and $\mathbb{L} = \langle \mathcal{A}, D \rangle$ an $\mathbf{F}$-structure. $\mathbb{L}$ is a full $\mathcal{I}$-structure if and only if there exists a bilogical morphism from $\mathbb{L}$ onto an $\mathbf{F}$-structure of the form $\langle \mathcal{A}', \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}') \rangle$, for some $\mathbf{F}$-algebraic system $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$.*

Proof: The "only if" is clear, since, if $\mathbb{L} = \langle \mathcal{A}, D \rangle$ is a full $\mathcal{I}$-structure, then $\langle I, \pi \rangle : \mathbb{L} \vdash \mathbb{L}^*$ is a bilogical morphism and, moreover, by the definition of fullness, $\mathbb{L}^* = \langle \mathcal{A}^*, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*) \rangle$.

Assume, conversely, that $\langle H, \gamma \rangle : \mathbb{L} \vdash \langle \mathcal{A}', \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}') \rangle$ is a bilogical morphism. By Proposition 1390, $\langle \mathcal{A}', \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}') \rangle \in \mathrm{FStr}(\mathcal{I})$. Therefore, by Proposition 1392, $\mathbb{L} \in \mathrm{FStr}(\mathcal{I})$, as well. ∎

We now formulate a result that can be used to show that a property of $\mathbf{F}$-structures for every full $\mathcal{I}$-structure of a π-institution $\mathcal{I}$ based on $\mathbf{F}$. It characterizes $\mathrm{FStr}(\mathcal{I})$ as the smallest class of $\mathbf{F}$-structures containing all $\mathbf{F}$-structures of the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$, with $\mathcal{A}$ ranging over all $\mathbf{F}$-algebraic systems, and closed under bilogical morphisms. It follows that to prove that a property holds for all members of $\mathrm{FStr}(\mathcal{I})$ it suffices to show that it holds for all $\mathbf{F}$-structures of the specific form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ and that it is preserved under bilogical morphisms.

Corollary 1394 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathrm{FStr}(\mathcal{I})$ *is the smallest class containing* $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$*, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and closed under both images and preimages under bilogical morphisms.*

Proof: By Proposition 1390, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, the pair $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle \in \mathrm{FStr}(\mathcal{I})$. Moreover, by Proposition 1392, $\mathrm{FStr}(\mathcal{I})$ is closed under both images and preimages under bilogical morphisms. On the other hand, let L be a class satisfying these properties and let $\langle \mathcal{A}, D \rangle \in \mathrm{FStr}(\mathcal{I})$. By Corollary 1393, there exists an $\mathbf{F}$-algebraic system $\mathcal{A}'$ and a bilogical morphism

$$\langle H, \gamma \rangle : \langle \mathcal{A}, D \rangle \vdash \langle \mathcal{A}', \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}') \rangle.$$

By hypothesis, $\langle \mathcal{A}', \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}') \rangle \in \mathsf{L}$ and, again by hypothesis, $\langle \mathcal{A}, D \rangle \in \mathsf{L}$. Thus, we conclude that $\mathrm{FStr}(\mathcal{I}) \subseteq \mathsf{L}$. This proves that $\mathrm{FStr}(\mathcal{I})$ is indeed the smallest class satisfying the given properties. ■

An alternative characterization of full $\mathcal{I}$-structures uses both the Leibniz and the Tarski operators.

Theorem 1395 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $\mathbb{L} = \langle \mathcal{A}, D \rangle$ *an* $\mathbf{F}$*-structure. Then* $\mathbb{L}$ *is a full* $\mathcal{I}$*-structure if and only if*

$$\mathcal{D} = \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}(\mathbb{L}) \leq \Omega^{\mathcal{A}}(T)\}.$$

Proof: Let $\mathbb{L}$ be an $\mathbf{F}$-structure and set

$$\mathcal{T} = \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}(\mathbb{L}) \leq \Omega^{\mathcal{A}}(T)\}.$$

Suppose, first, that $\mathbb{L} = \langle \mathcal{A}, D \rangle \in \mathrm{FStr}(\mathcal{I})$. We must show $\mathcal{D} = \mathcal{T}$. To this end, let $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, by Proposition 1385, $T \in \mathrm{FIFam}^{\mathcal{I}}(\mathcal{A})$ and, by the definition of the Tarski congruence system, $\widetilde{\Omega}(\mathbb{L}) \leq \Omega^{\mathcal{A}}(T)$. Thus, $\mathcal{D} \subseteq \mathcal{T}$. Conversely, if $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}(\mathbb{L}) \leq \Omega^{\mathcal{A}}(T)$, then $\widetilde{\Omega}(\mathbb{L})$ is compatible with T. Setting $T' = T/\widetilde{\Omega}(\mathbb{L})$, we have, by Corollary 56, that $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}(\mathbb{L}))$ and $T = \pi^{-1}(T')$, where $\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\widetilde{\Omega}(\mathbb{L})$ is the quotient morphism, which is also a bilogical morphism $\langle I, \pi \rangle : \mathbb{L} \vdash \mathbb{L}^*$. Since, by hypothesis $\mathbb{L}$ is full, we get that $\mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}(\mathbb{L}))$, whence, $T = \pi^{-1}(T') \in \pi^{-1}(\mathcal{D}^*) = \mathcal{D}$. We conclude that $\mathcal{T} \subseteq \mathcal{D}$.

Assume, conversely, that $\mathcal{D} = \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}(\mathbb{L}) \leq \Omega^{\mathcal{A}}(T)\}$. Then, by Proposition 1360,

$$\langle I, \pi \rangle : \mathbb{L} \vdash \langle \mathcal{A}/\widetilde{\Omega}(\mathbb{L}), \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}(\mathbb{L})) \rangle$$

is a bilogical morphism. Therefore, $\mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}(\mathbb{L}))$, showing that $\mathbb{L} \in \mathrm{FStr}(\mathcal{I})$. ■

19.6 $\mathcal{I}$-Algebraic Systems

Since $\langle\mathcal{A},\mathcal{D}\rangle$ is a full $\mathcal{I}$-structure if and only if $\mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$, we conclude that the reduced full $\mathcal{I}$-structures are exactly those structures of the form $\langle\mathcal{A},\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$, which are reduced.

Definition 1396 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F},C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*. An* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle$ *if an* $\mathcal{I}$-**algebraic system** *if and only if the* $\mathbf{F}$*-structure* $\langle\mathcal{A},\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ *is reduced, i.e., if* $\mathcal{A}$ *is the underlying* $\mathbf{F}$*-algebraic system of a reduced full* $\mathcal{I}$*-structure.*

We denote by $\mathrm{AlgSys}(\mathcal{I})$ *the class of all* $\mathcal{I}$*-algebraic systems.*

Since $\mathcal{I}$-algebraic systems are determined based on reduced full $\mathcal{I}$-structures, the following characterization is useful in this context.

Proposition 1397 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $\mathbb{L} = \langle\mathcal{A},D\rangle$ *an* $\mathbf{F}$*-structure. Then the following are equivalent:*

(i) $\mathbb{L}$ *is a reduced full* $\mathcal{I}$*-structure;*

(ii) $\mathbb{L}$ *is reduced and* $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*;*

(iii) $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$ *and* $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

Proof:

(i)⇒(ii) Suppose that $\mathbb{L} = \langle\mathcal{A},D\rangle$ is a reduced full $\mathcal{I}$-structure. Since $\mathbb{L}$ is full, $\mathbb{L}^* = \langle\mathcal{A}^*,\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)\rangle$. Since $\mathbb{L}$ is reduced, $\mathbb{L}^* = \mathbb{L}$. Thus, $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

(ii)⇒(ii) Assume $\mathbb{L} = \langle\mathcal{A},D\rangle$ is reduced and $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Since $\mathbb{L}$ is reduced, $\mathbb{L}^* = \mathbb{L} = \langle\mathcal{A},\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$. Therefore, $\mathbb{L}$ is also full and, consequently, $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$.

(iii)⇒(i) Let $\mathbb{L} = \langle\mathcal{A},D\rangle$, with $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$ and $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Since $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, there exists a closure family D' on $\mathcal{A}$, such that $\langle\mathcal{A},D'\rangle$ is a reduced full $\mathcal{I}$-structure. Since $\langle\mathcal{A},D'\rangle$ is full and reduced, we have $\mathcal{D}' = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Since, by hypothesis, $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get that $\mathcal{D}' = \mathcal{D}$. Hence, $\mathbb{L} = \langle\mathcal{A},\mathcal{D}\rangle = \langle\mathcal{A},\mathcal{D}'\rangle$ is a reduced full $\mathcal{I}$-structure.

■

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$. Let, also, $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle$ an $\mathbf{F}$-algebraic system. We denote by $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ the collection of all congruence systems θ on $\mathcal{A}$, such that the quotient algebraic system $\mathcal{A}^\theta$ is in $\mathrm{AlgSys}(\mathcal{I})$:

$$\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}) = \{\theta \in \mathrm{ConSys}(\mathcal{A}) : \mathcal{A}^\theta \in \mathrm{AlgSys}(\mathcal{I})\}.$$

A congruence system $\theta \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is called an $\mathcal{I}$-**congruence system on** $\mathcal{A}$.

It turns out that the Tarski congruence systems of full $\mathcal{I}$-structures are all $\mathcal{I}$-congruence systems.

Proposition 1398 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $\mathbb{L} = \langle \mathcal{A}, D \rangle$ *an* $\mathbf{F}$*-structure. If* $\mathbb{L}$ *is full, then* $\mathcal{A}^* \in \mathrm{AlgSys}(\mathcal{I})$ *and, therefore,* $\widetilde{\Omega}(\mathbb{L}) \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$.

Proof: Suppose that $\mathbb{L} = \langle \mathcal{A}, D \rangle$ is a full $\mathcal{I}$-structure. Then, by definition, $\mathbb{L}^* = \langle \mathcal{A}^*, \mathcal{D}^* \rangle = \langle \mathcal{A}^*, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*) \rangle$ is a reduced full $\mathcal{I}$-structure. Hence $\mathcal{A}^* \in \mathrm{AlgSys}(\mathcal{I})$ and, therefore, by definition, $\widetilde{\Omega}(\mathbb{L}) \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$. ■

Even though, according to the definition, $\mathcal{I}$-algebraic systems are determined as the $\mathbf{F}$-algebraic system reducts of reduced full $\mathcal{I}$-structures, they can also be characterized without reference to fullness.

Proposition 1399 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathrm{AlgSys}(\mathcal{I})$ *is the class of all underlying* $\mathbf{F}$*-algebraic systems of all reduced* $\mathcal{I}$*-structures:*

$$\mathrm{AlgSys}(\mathcal{I}) = \{\mathcal{A} : (\exists \langle \mathcal{A}, D \rangle \in \mathrm{Str}^{\mathcal{I}}(\mathcal{A}))(\widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) = \Delta^{\mathcal{A}})\}.$$

Proof: If $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, then, by Proposition 1397, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ is a reduced full $\mathcal{I}$-structure. Conversely, if $\mathbb{L} = \langle \mathcal{A}, D \rangle$ is a reduced $\mathcal{I}$-structure, then $\mathcal{D} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and, therefore,

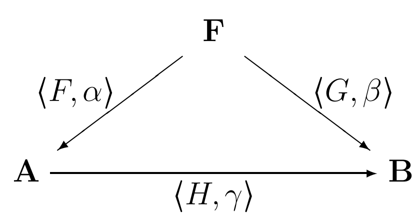
$$\widetilde{\Omega}(\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle) \leq \widetilde{\Omega}(\mathbb{L}) = \Delta^{\mathcal{A}}.$$

Thus, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ is a reduced full $\mathcal{I}$-structure and $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$. ■

The class of all $\mathcal{I}$-algebraic systems is closed under isomorphisms.

Proposition 1400 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathrm{AlgSys}(\mathcal{I})$ *is closed under isomorphisms.*

Proof: Assume $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle \in \mathrm{AlgSys}(\mathcal{I})$ and let $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ be an isomorphism.



Then, we have that $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \gamma^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}))$, which shows that

$$\langle H,\gamma\rangle : \langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle \cong \langle \mathcal{B}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})\rangle.$$

Now we can use Proposition 1363 to see that $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ is reduced if and only if $\langle \mathcal{B}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})\rangle$ is reduced and, therefore, by Proposition 1399, $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$ if and only if $\mathcal{B} \in \mathrm{AlgSys}(\mathcal{I})$. ■

Proposition 1397 gave characterizing conditions for reduced full $\mathcal{I}$-structures. An analog for full $\mathcal{I}$-structures is the following:

Proposition 1401 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $\mathbb{L} = \langle \mathcal{A}, D\rangle$ *an* $\mathbf{F}$*-structure. Then the following are equivalent:*

(i) $\mathbb{L}$ *is a full* $\mathcal{I}$*-structure;*

(ii) $\mathcal{A}^* \in \mathrm{AlgSys}(\mathcal{I})$ *and* $\mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$*;*

(iii) There exists a bilogical morphism $\langle H,\gamma\rangle : \mathbb{L} \vdash \langle \mathcal{A}', D'\rangle$*, such that* $\mathcal{A}' \in \mathrm{AlgSys}(\mathcal{I})$ *and* $\mathcal{D}' = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$*.*

Proof:

(i)⇒(ii) Suppose $\mathbb{L} = \langle \mathcal{A}, D\rangle$ is a full $\mathcal{I}$-structure. Then, by definition, $\mathbb{L}^* = \langle \mathcal{A}^*, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)\rangle$. Thus, $\mathcal{A}^\in \mathrm{AlgSys}(\mathcal{I})$ and $\mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$.

(ii)⇒(iii) Obvious, since $\langle H,\gamma\rangle : \mathbb{L} \vdash \mathbb{L}^*$ is a bilogical morphism.

(iii)⇒(i) Assume that $\langle H,\gamma\rangle : \mathbb{L} \to \langle \mathcal{A}', \mathcal{D}'\rangle$ is a bilogical morphism, such that $\mathcal{A}' \in \mathrm{AlgSys}(\mathcal{I})$ and $\mathcal{D}' = \mathrm{Fifam}^{\mathcal{I}}(\mathcal{A}')$. Then, by Proposition 1372, there exists an α-isomorphism $\mathbb{L}^* \vdash^\alpha \langle \mathcal{A}'^*, \mathcal{D}'^*\rangle$. Since $\mathcal{A}' \in \mathrm{AlgSys}(\mathcal{I})$ and $\mathcal{D}' = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$, it follows, by Proposition 1397, that $\langle \mathcal{A}', \mathcal{D}'\rangle$ is a reduced full $\mathcal{I}$-structure. So we have

$$\langle \mathcal{A}'^*, \mathcal{D}'^*\rangle = \langle \mathcal{A}', \mathcal{D}'\rangle = \langle \mathcal{A}', \mathrm{FiFam}(\mathcal{A}')\rangle.$$

Hence, by Proposition 1380, $\mathbb{L}^* = \langle \mathcal{A}^*, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)\rangle$ and, therefore, $\mathbb{L}$ is a full $\mathcal{I}$-structure.

■

It turns out that, given a π-institution $\mathcal{I}$, the class of all full $\mathcal{I}$-structures, the class of all $\mathbf{F}$-structures of the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$, where $\mathcal{A}$ ranges over all $\mathbf{F}$-algebraic systems, as well as the class of all reduced full $\mathcal{I}$-structures are complete $\mathbf{F}$-structure semantics for $\mathcal{I}$.

Theorem 1402 (Completeness Theorem) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is complete with respect to the following classes of* $\mathbf{F}$*-structures:*

(i) The class of all full $\mathcal{I}$-structures;

(ii) The class of all $\mathbf{F}$-structures of the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$;

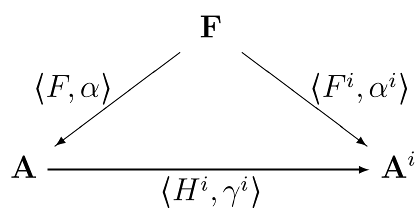
(iii) The class of all reduced full $\mathcal{I}$-structures, i.e., structures of the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$, with $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$.

Proof: Note that all three classes of $\mathbf{F}$-structures consist of $\mathcal{I}$-structures and include $\langle \mathcal{F}, C\rangle^*$. Therefore, by Proposition 1388, $\mathcal{I}$ is complete with respect to each one of them. ∎

Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution. To $\mathcal{I}$ we have associated (among others) two classes of $\mathbf{F}$-algebraic systems. One is the class $\mathrm{AlgSys}^*(\mathcal{I})$ of underlying $\mathbf{F}$-algebraic systems of reduced $\mathcal{I}$-matrix families. The other is the class $\mathrm{AlgSys}(\mathcal{I})$ of underlying $\mathbf{F}$-algebraic systems of reduced $\mathcal{I}$-structures (according to Proposition 1399). To explore an important relationship between these two classes, we introduce an operator on $\mathbf{F}$-algebraic systems, which is related to an operator on $\mathbf{F}$-matrix families, given the same name.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and consider $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ and $\mathcal{A}^i = \langle \mathbf{A}^i, \langle F^i, \alpha^i\rangle\rangle$, $i \in I$, and a system of surjective morphisms

$$\langle H^i, \gamma^i\rangle : \mathcal{A} \to \mathcal{A}^i, \quad i \in I.$$



We say $\{\langle H^i, \gamma^i\rangle : i \in I\}$ is a **subdirect intersection (system)** and call the $\langle H^i, \gamma^i\rangle$ **subdirect intersection morphisms** if

$$\bigcap_{i \in I} \mathrm{Ker}(\langle H^i, \gamma^i\rangle) = \Delta^{\mathcal{A}}.$$

If such a system exists, we say that $\mathcal{A}$ is a **subdirect intersection** of the $\mathbf{F}$-algebraic systems $\{\mathcal{A}^i : i \in I\}$. Given a class K of $\mathbf{F}$-algebraic systems and an $\mathbf{F}$-algebraic system $\mathcal{A}$, we write

$$\mathcal{A} \in \overset{\triangleleft}{\mathbb{\Pi}}(\mathsf{K})$$

to signify that $\mathcal{A}$ is a subdirect intersection of a collection $\{\mathcal{A}^i : i \in I\}$, with $\mathcal{A}^i \in \mathsf{K}$, for all $i \in I$.

We can show that the operator $\overset{\triangleleft}{\mathbb{\Pi}}$ is a closure operator on classes of $\mathbf{F}$-algebraic systems.

Proposition 1403 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system. Then*

$$\overset{\triangleleft}{\mathbb{II}} : \mathcal{P}(\mathrm{AlgSys}(\mathbf{F})) \to \mathcal{P}(\mathrm{AlgSys}(\mathbf{F}))$$

is a closure operator.

Proof: Suppose, first, that $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$ and $\mathcal{A} \in \mathsf{K}$. Since the identity morphism $\langle I, \iota \rangle : \mathcal{A} \to \mathcal{A}$ is a subdirect intersection, we get that $\mathcal{A} \in \overset{\triangleleft}{\mathbb{II}}(\mathsf{K})$. Thus, $\overset{\triangleleft}{\mathbb{II}}$ is inflationary.

Suppose, next, that $\mathsf{K} \subseteq \mathsf{K}' \subseteq \mathrm{AlgSys}(\mathbf{F})$ and $\mathcal{A} \in \overset{\triangleleft}{\mathbb{II}}(\mathsf{K})$. Thus, $\mathcal{A}$ is a subdirect intersection of a collection $\{\mathcal{A}^i : i \in I\} \subseteq \mathsf{K}$. Then $\mathcal{A}$ is a subdirect intersection of $\{\mathcal{A}^i : i \in I\} \subseteq \mathsf{K}'$. Hence, $\overset{\triangleleft}{\mathbb{II}}(\mathsf{K}) \subseteq \overset{\triangleleft}{\mathbb{II}}(\mathsf{K}')$ and, therefore, $\overset{\triangleleft}{\mathbb{II}}$ is also monotone.

Assume, finally, that $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$ and let $\mathcal{A} \in \overset{\triangleleft}{\mathbb{II}}(\overset{\triangleleft}{\mathbb{II}}(\mathsf{K}))$. Thus, there exists a subdirect intersection system

$$\langle H^i, \gamma^i \rangle : \mathcal{A} \to \mathcal{A}^i, \quad i \in I,$$

where $\mathcal{A}^i \in \overset{\triangleleft}{\mathbb{II}}(\mathsf{K})$, for all $i \in I$. Consequently, for all $i \in I$, there exists a subdirect intersection system

$$\langle H^{ij}, \gamma^{ij} \rangle : \mathcal{A}^i \to \mathcal{A}^{ij}, \quad j \in J_i,$$

where $\mathcal{A}^{ij} \in \mathsf{K}$, for all $i \in I, j \in J_i$. We consider the collection

$$\langle H^{ij}, \gamma^{ij} \rangle \circ \langle H^i, \gamma^i \rangle : \mathcal{A} \to \mathcal{A}^{ij}, \quad i \in I, j \in J_i.$$

We have

$$\begin{array}{rcl}
\bigcap_{i\in I} \bigcap_{j \in J_i} \mathrm{Ker}(\langle H^{ij}, \gamma^{ij} \rangle \circ \langle H^i, \gamma^i \rangle) & = & \bigcap_{i \in I} \bigcap_{j \in J_i} (\gamma^i)^{-1}(\mathrm{Ker}(\langle H^{ij}, \gamma^{ij} \rangle)) \\
& = & \bigcap_{i\in I} (\gamma^i)^{-1}(\bigcap_{j \in J_i}(\mathrm{Ker}(\langle H^{ij}, \gamma^{ij} \rangle))) \\
& = & \bigcap_{i \in I} (\gamma^i)^{-1}(\Delta^{\mathcal{A}^i}) \\
& = & \bigcap_{i \in I} \mathrm{Ker}(\langle H^i, \gamma^i \rangle) \\
& = & \Delta^{\mathcal{A}}.
\end{array}$$

Thus, the system $\{\langle H^{ij}, \gamma^{ij} \rangle \circ \langle H^i, \gamma^o \rangle : i \in I, j \in J_i\}$ is a subdirect intersection system, showing that $\mathcal{A} \in \overset{\triangleleft}{\mathbb{II}}(\mathsf{K})$. We conclude that $\overset{\triangleleft}{\mathbb{II}}$ is also idempotent. ■

Using subdirect intersections, we can give the exact relationship between the classes $\mathrm{AlgSys}(\mathcal{I})$ and $\mathrm{AlgSys}^*(\mathcal{I})$. Namely, we show that the former is the class of all subdirect intersections of collections of algebraic systems in the latter class. In particular $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$.

Theorem 1404 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *Then*

$$\mathrm{AlgSys}(\mathcal{I}) = \overset{\triangleleft}{\mathbb{II}}(\mathrm{AlgSys}^*(\mathcal{I})).$$

Proof: Assume, first, that $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$. Then, we have

$$\bigcap\{\Omega^{\mathcal{A}}(T) : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} = \widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}.$$

Now, consider the collection

$$\langle I, \pi^{\Omega^{\mathcal{A}}(T)}\rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T), \quad T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}).$$

By the preceding equation, this collection constitutes a subdirect intersection. Moreover, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we have $\langle \mathcal{A}/\Omega^{\mathcal{A}}(T), T/\Omega^{\mathcal{A}}(T)\rangle \in \mathrm{MatFam}^*(\mathcal{I})$ and, hence, $\mathcal{A}/\Omega^{\mathcal{A}}(T) \in \mathrm{AlgSys}^*(\mathcal{I})$. Therefore, we get that $\mathcal{A} \in \overset{\triangleleft}{\mathbb{I}\Pi}(\mathrm{AlgSys}^*(\mathcal{I}))$.

Suppose, conversely, that $\mathcal{A} \in \overset{\triangleleft}{\mathbb{I}\Pi}(\mathrm{AlgSys}^*(\mathcal{I}))$. Thus, there exists a subdirect intersection

$$\langle H^i \gamma^i\rangle : \mathcal{A} \to \mathcal{A}^i, \quad i \in I,$$

with $\mathcal{A}^i \in \mathrm{AlgSys}^*(\mathcal{I})$, for all $i \in I$. Thus, for all $i \in I$, there exists $T^i \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^i)$, such that $\Omega^{\mathcal{A}^i}(T^i) = \Delta^{\mathcal{A}^i}$. Now, we calculate:

$$\begin{array}{rcl}
\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) & = & \bigcap\{\Omega^{\mathcal{A}}(T) : T \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A})\} \\
& \subseteq & \bigcap_{i\in I} \Omega^{\mathcal{A}}((\gamma^i)^{-1}(T^i)) \\
& = & \bigcap_{i\in I} (\gamma^i)^{-1}(\Omega^{\mathcal{A}^i}(T^i)) \\
& = & \bigcap_{i\in I} (\gamma^i)^{-1}(\Delta^{\mathcal{A}^i}) \\
& = & \bigcap_{i\in I} \mathrm{Ker}(\langle H^i, \gamma^i\rangle) \\
& = & \Delta^{\mathcal{A}}.
\end{array}$$

We conclude that $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$. Thus, $\mathrm{AlgSys}(\mathcal{I}) \subseteq \overset{\triangleleft}{\mathbb{I}\Pi}(\mathrm{AlgSys}^*(\mathcal{I}))$. ∎

Corollary 1405 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then* $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$ *and, moreover,* $\mathrm{ALgSys}^*(\mathcal{I}) = \mathrm{AlgSys}(\mathcal{I})$ *if and only if* $\mathrm{AlgSys}^*(\mathcal{I})$ *is closed under subdirect intersections.*

Proof: We have

$$\begin{array}{rcll}
\mathrm{AlgSys}^*(\mathcal{I}) & \subseteq & \overset{\triangleleft}{\mathbb{I}\Pi}(\mathrm{AlgSys}^*(\mathcal{I})) & \text{(by Proposition 1403)} \\
& = & \mathrm{AlgSys}(\mathcal{I}). & \text{(by Theorem 1404)}
\end{array}$$

If $\mathrm{AlgSys}^*(\mathcal{I})$ is closed under subdirect intersections,

$$\mathrm{AlgSys}(\mathcal{I}) = \overset{\triangleleft}{\mathbb{I}\Pi}(\mathrm{AlgSys}^*(\mathcal{I})) \subseteq \mathrm{AlgSys}^*(\mathcal{I}).$$

Conversely, if $\mathrm{AlgSys}^*(\mathcal{I}) = \mathrm{AlgSys}(\mathcal{I})$, then $\overset{\triangleleft}{\mathbb{I}\Pi}(\mathrm{AlgSys}^*(\mathcal{I})) = \mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$. ∎

Finally, we give a relation between the classes of algebraic systems associated in this way with π-institutions based on the same algebraic system that are related by $\leq$. Recall that, given $\mathcal{I} = \langle \mathbf{F}, C\rangle$ and $\mathcal{I}' = \langle \mathbf{F}, C'\rangle$, we write $C \leq C'$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$, $C_\Sigma(\Phi) \subseteq C'_\Sigma(\Phi)$. If this is the case, we also write $\mathcal{I} \leq \mathcal{I}'$ and say that $\mathcal{I}'$ is stronger than $\mathcal{I}$ and that $\mathcal{I}$ is weaker than $\mathcal{I}'$. Recall, also, that, $\mathcal{I} \leq \mathcal{I}'$ if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

Proposition 1406 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$, $\mathcal{I}' = \langle \mathbf{F}, C'\rangle$ be two π-institutions based on $\mathbf{F}$. If $\mathcal{I} \leq \mathcal{I}'$, then*

$$\mathrm{AlgSys}(\mathcal{I}') \subseteq \mathrm{AlgSys}(\mathcal{I}) \quad \textit{and} \quad \mathrm{AlgSys}^*(\mathcal{I}') \subseteq \mathrm{AlgSys}^*(\mathcal{I}).$$

Proof: If $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I}')$, then, there exists $T' \in \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T') = \Delta^{\mathcal{A}}$. But, since $\mathcal{I} \leq \mathcal{I}'$, we have $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Therefore $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I})$. We now conclude that $\mathrm{AlgSys}^*(\mathcal{I}') \subseteq \mathrm{AlgSys}^*(\mathcal{I})$.

For the second inclusion, we get

$$\begin{array}{rcll} \mathrm{AlgSys}(\mathcal{I}') & = & \overset{\triangleleft}{\mathrm{I}\mathrm{P}}(\mathrm{AlgSys}^*(\mathcal{I}')) & \text{(Theorem 1404)}\\ & \subseteq & \overset{\triangleleft}{\mathrm{I}\mathrm{P}}(\mathrm{AlgSys}^*(\mathcal{I})) & \text{(Proposition 1403)}\\ & = & \mathrm{AlgSys}(\mathcal{I}). & \text{(Theorem 1404)} \end{array}$$

■

19.7 Lattice of Full $\mathcal{I}$-Structures

In this section we show that, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ and an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{F}, \langle F, \alpha\rangle\rangle$, the poset $\langle \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}), \leq\rangle$ of full $\mathcal{I}$-structures on $\mathcal{A}$ and the poset $\langle \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}), \leq\rangle$ of $\mathcal{I}$-congruence systems on $\mathcal{A}$ are isomorphic through the Tarski operator

$$\langle \mathcal{A}, D\rangle \mapsto \widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}).$$

We start by defining an operator which will turn out to be the inverse of the Tarski operator.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ an $\mathbf{F}$-algebraic system. Given $\theta \in \mathrm{ConSys}(\mathcal{A})$, define

$$\widetilde{H}^{\mathcal{A}}(\theta) = \langle \mathcal{A}, \mathcal{D}^\theta\rangle,$$

by setting

$$\mathcal{D}^\theta = (\pi^\theta)^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta)),$$

where $\langle I, \pi^\theta\rangle : \mathcal{A} \to \mathcal{A}^\theta$ is the quotient morphism.

Note that, by definition of $\widetilde{H}^{\mathcal{A}}(\theta)$, the morphism

$$\langle I, \pi^\theta\rangle : \widetilde{H}^{\mathcal{A}}(\theta) \to \langle \mathcal{A}^\theta, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta)\rangle$$

is a bilogical morphism.

We have the following properties concerning the operator $\widetilde{H}$.

Lemma 1407 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system.*

(a) For every $\theta \in \mathrm{ConSys}(\mathcal{A})$*,*

(i) $\theta \in \mathrm{ConSys}(\widetilde{H}^{\mathcal{A}}(\theta))$*;*

(ii) $\widetilde{H}^{\mathcal{A}}(\theta)/\theta = \langle \mathcal{A}^\theta, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta) \rangle$*;*

(iii) $\widetilde{H}^{\mathcal{A}}(\theta) \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$*;*

(b) $\theta \mapsto \widetilde{H}^{\mathcal{A}}(\theta)$ *is order preserving, i.e., for all* $\theta, \theta' \in \mathrm{ConSys}(\mathcal{A})$*,* $\theta \le \theta'$ *implies* $\widetilde{H}^{\mathcal{A}}(\theta) \le \widetilde{H}^{\mathcal{A}}(\theta')$*.*

Proof:

(a) For Part (i) we must show that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $\langle \phi, \psi \rangle \in \theta_\Sigma$ implies that $D^\theta_\Sigma(\phi) = D^\theta_\Sigma(\psi)$. Suppose, to this end, that $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \theta_\Sigma$. Then, we have

$$C^{\mathcal{I},\mathcal{A}^\theta}_\Sigma(\phi/\theta_\Sigma) = C^{\mathcal{I},\mathcal{A}^\theta}_\Sigma(\psi/\theta_\Sigma),$$

i.e., $C^{\mathcal{I},\mathcal{A}^\theta}_\Sigma(\pi^\theta_\Sigma(\phi)) = C^{\mathcal{I},\mathcal{A}^\theta}_\Sigma(\pi^\theta_\Sigma(\psi))$. This gives that

$$(\pi^\theta_\Sigma)^{-1}(C^{\mathcal{I},\mathcal{A}^\theta}_\Sigma(\pi^\theta_\Sigma(\phi))) = (\pi^\theta_\Sigma)^{-1}(C^{\mathcal{I},\mathcal{A}^\theta}_\Sigma(\pi^\theta_\Sigma(\psi))).$$

Since $\langle I, \pi^\theta \rangle : \widetilde{H}^{\mathcal{A}}(\theta) \to \langle \mathcal{A}^\theta, C^{\mathcal{I},\mathcal{A}^\theta} \rangle$ is a bilogical morphism, we get by Proposition 1360, $D^\theta_\Sigma(\phi) = D^\theta_\Sigma(\psi)$. We conclude that $\theta \in \mathrm{ConSys}(\widetilde{H}^{\mathcal{A}}(\theta))$.

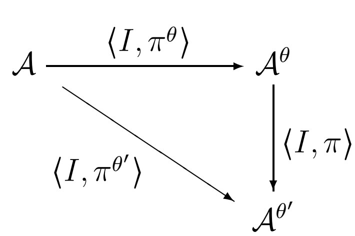
For Part (ii), we have

$$\pi^\theta(\mathcal{D}^\theta) = \pi^\theta((\pi^\theta)^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta))) = \mathrm{FiFam}^{\mathcal{A}}(\mathcal{A}^\theta),$$

where the last equality follows from the fact that $\langle I, \pi^\theta \rangle$ is a bilogical morphism, by applying Proposition 1360.

Part (iii) follows from the fact that the morphism $\langle I, \pi^\theta \rangle : \widetilde{H}^{\mathcal{A}}(\theta) \to \langle \mathcal{A}^\theta, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta) \rangle$ is a bilogical morphism and Corollary 1393.

(b) Suppose $\theta, \theta' \in \mathrm{ConSys}(\mathcal{A})$, such that $\theta \le \theta'$. Then we have the following commutative diagram of $\mathbf{F}$-algebraic systems.



where, $\langle I, \pi \rangle : \mathcal{A}^\theta \to \mathcal{A}^{\theta'}$ is defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, by

$$\pi_\Sigma(\phi/\theta_\Sigma) = \phi/\theta'_\Sigma.$$

Taking this diagram into account, we have

$$\begin{array}{rcl}\mathcal{D}^{\theta'} & = & (\pi^{\theta'})^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^{\theta'})) \quad (\text{Definition of } \mathcal{D}^{\theta'})\\ & = & (\pi^{\theta})^{-1}(\pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^{\theta'}))) \quad (\pi\circ\pi^{\theta}=\pi^{\theta'})\\ & \subseteq & (\pi^{\theta})^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^{\theta})) \quad (\text{Corollary 55})\\ & = & \mathcal{D}^{\theta}. \quad (\text{Definition of } \mathcal{D}^{\theta})\end{array}$$

Thus, we get $\widetilde{H}^{\mathcal{A}}(\theta)\leq\widetilde{H}^{\mathcal{A}}(\theta')$. ∎

We are ready now for the main isomorphism theorem that was promised at the beginning of the section.

Theorem 1408 (Isomorphism Theorem) *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$*, with* $\mathbf{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$*, an* $\mathbf{F}$*-algebraic system. Then*

$$\widetilde{\Omega}^{\mathcal{A}}:\langle\mathrm{FStr}^{\mathcal{I}}(\mathcal{A}),\leq\rangle\to\langle\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}),\leq\rangle$$

is an order isomorphism, with inverse

$$\widetilde{H}^{\mathcal{A}}:\langle\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}),\leq\rangle\to\langle\mathrm{FStr}^{\mathcal{I}}(\mathcal{A}),\leq\rangle.$$

Proof: By Proposition 1398, if $\mathbb{L}\in\mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$, then $\widetilde{\Omega}^{\mathcal{A}}(\mathbb{L})\in\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$. Moreover, by Lemma 1407, if $\theta\in\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$, then $\widetilde{H}^{\mathcal{A}}(\theta)\in\mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$. So, both $\widetilde{\Omega}^{\mathcal{A}}$ and $\widetilde{H}^{\mathcal{A}}$ are well-defined, with domains and codomains as indicated.

We show, next, that they are mutually inverse mappings.

Suppose, first, that $\mathbb{L}=\langle\mathcal{A},D\rangle\in\mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$. Then, by Proposition 1398, $\mathcal{A}^*\in\mathrm{AlgSys}^{\mathcal{I}}(\mathcal{A})$ and $\widetilde{\Omega}^{\mathcal{A}}(\mathbb{L})\in\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$. By fullness, $\mathcal{D}=\pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*))$, where $\langle I,\pi\rangle:\mathbb{L}\vdash\mathbb{L}^*$ is the quotient bilogical morphism. Then, by definition of $\widetilde{H}^{\mathcal{A}}$, we get that $\widetilde{H}^{\mathcal{A}}(\widetilde{\Omega}^{\mathcal{A}}(\mathbb{L}))=\mathbb{L}$.

Suppose, on the other hand, that $\theta\in\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$. By definition, $\mathcal{A}^\theta\in\mathrm{AlgSys}(\mathcal{I})$. Thus, by definition,

$$\langle\mathcal{A}^\theta,\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta)\rangle\in\mathrm{FStr}(\mathcal{I})\quad\text{and}\quad\widetilde{\Omega}^{\mathcal{A}^\theta}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta))=\Delta^{\mathcal{A}^\theta}.$$

Now, we get

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathcal{A}}(\widetilde{H}^{\mathcal{A}}(\theta)) & = & \widetilde{\Omega}^{\mathcal{A}}((\pi^\theta)^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta))) \quad (\text{Definition of } \widetilde{H}^{\mathcal{A}}(\theta))\\ & = & (\pi^\theta)^{-1}(\widetilde{\Omega}^{\mathcal{A}^\theta}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta))) \quad (\text{Corollary 1364})\\ & = & (\pi^\theta)^{-1}(\Delta^{\mathcal{A}^\theta}) \quad (\text{Hypothesis})\\ & = & \theta. \quad (\text{Set Theory})\end{array}$$

Since, by definition $\widetilde{\Omega}^{\mathcal{A}}$ is order preserving and, by Lemma 1407, $\widetilde{H}^{\mathcal{A}}$ is also order preserving, we conclude that $\widetilde{\Omega}^{\mathcal{A}}$ is an order isomorphism with inverse $\widetilde{H}^{\mathcal{A}}$. ∎

We show next that the poset of $\mathcal{I}$-congruence systems on an $\mathbf{F}$-algebraic system $\mathcal{A}$ is a complete lattice with infimum given by signature-wise intersection. In conjunction with the Isomorphism Theorem, this will yield that the poset of full $\mathcal{I}$-structures on $\mathcal{A}$ is also a complete lattice.

Theorem 1409 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, the poset* $\langle \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}), \leq \rangle$ *is a complete lattice with infimum given by signature-wise intersection.*

Proof: First, note that $\nabla^{\mathcal{A}} \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$, since $\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\nabla^{\mathcal{A}})) = \Delta^{\mathcal{A}/\nabla^{\mathcal{A}}}$ and, hence, $\mathcal{A}/\nabla^{\mathcal{A}} \in \mathrm{AlgSys}(\mathcal{I})$. So $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ has a largest element. Assume, next, that, for all $i \in I$, $\theta^i \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$. We must show that $\bigcap_{i\in I} \theta^i \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$. To this end, set $\theta := \bigcap_{i\in I} \theta^i$ and consider the projection morphisms

$$\langle I, \pi^i \rangle : \mathcal{A}^\theta \to \mathcal{A}^{\theta^i}, \quad i \in I,$$

which are bilogical morphisms

$$\langle I, \pi^i \rangle : \langle \mathcal{A}^\theta, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta) \rangle \vdash \langle \mathcal{A}^{\theta^i}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^{\theta^i}) \rangle, \quad i \in I.$$

By hypothesis, $\langle \mathcal{A}^{\theta^i}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^{\theta^i}) \rangle$ is reduced, i.e.,

$$\widetilde{\Omega}^{\mathcal{A}^{\theta^i}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^{\theta^i})) = \Delta^{\mathcal{A}^{\theta^i}}, \quad i \in I.$$

Now we have, for all $i \in I$,

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathcal{A}^\theta}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta)) & \leq & \widetilde{\Omega}^{\mathcal{A}^\theta}((\pi^i)^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^{\theta^i}))) \\ & = & (\pi^i)^{-1}(\widetilde{\Omega}^{\mathcal{A}^{\theta^i}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^{\theta^i}))) \\ & = & (\pi^i)^{-1}(\Delta^{\mathcal{A}^{\theta^i}}) \\ & = & \theta^i/\theta.\end{array}$$

Thus, we get

$$\widetilde{\Omega}^{\mathcal{A}^\theta}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta)) \leq \bigcap_{i\in I}(\theta^i/\theta) = (\bigcap_{i\in I}\theta^i)/\theta = \theta/\theta = \Delta^{\mathcal{A}^\theta}.$$

We conclude that $\langle \mathcal{A}^\theta, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta) \rangle$ is reduced and, hence, $\mathcal{A}^\theta \in \mathrm{AlgSys}(\mathcal{I})$, giving that $\theta \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$.

The conclusion of the theorem now follows. ■

As a consequence of the Isomorphism Theorem 1408 and Theorem 1409, we get

Corollary 1410 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* $\langle \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}), \leq \rangle$ *is a complete lattice and*

$$\widetilde{\Omega}^{\mathcal{A}} : \langle \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}), \leq \rangle \to \langle \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}), \leq \rangle$$

is a lattice isomorphism.

Proof: By Theorems 1408 and 1409. ■

It follows from the preceding results that, given a collection $\{\mathbb{L}^i : i \in I\} \subseteq \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$, with $\mathbb{L}^i = \langle \mathcal{A}, D^i \rangle$, $i \in I$, its infimum in $\langle \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}), \leq \rangle$ is the $\mathcal{I}$-structure

$$\langle \mathcal{A}, (\pi^\theta)^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta)) \rangle,$$

where $\theta = \bigcap_{i \in I} \widetilde{\Omega}^{\mathcal{A}}(\mathbb{L}^i)$, and $\langle I, \pi^\theta \rangle : \mathcal{A} \to \mathcal{A}^\theta$ is the quotient morphism. It is not necessarily the case, however, that this system be the signature-wise intersection of the $\mathbb{L}^i$'s. In other words, $\langle \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}), \leq \rangle$ is not, in general, a sublattice of the complete lattice of all $\mathcal{I}$-structures on $\mathcal{A}$.

It turns out that bilogical morphisms with isomorphic functor components induce isomorphisms between principal ideals of the corresponding full structure lattices and, similarly isomorphisms between principal ideals of the corresponding lattices of congruence systems.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. Let also $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, be an $\mathbf{F}$-algebraic system. Consider the complete lattice $\langle \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}), \leq \rangle$ and let $\mathbb{L} = \langle \mathcal{A}, D \rangle \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$. Recall that the ordering $\leq$ reflects the ordering of the closure operators on $\mathcal{A}$, which is dual to the inclusion ordering of the corresponding closure set systems. So, when we refer to an ideal in $\langle \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}), \leq \rangle$ we mean in the form of closure (operator) families and this translates to a filter, when one views structures in the form of their theory families. Keeping this in mind, we introduce the notation $\mathrm{FStr}^{\mathcal{I}}(\mathbb{L})$ to refer to the principal ideal of all full $\mathcal{I}$-structures on $\mathcal{A}$ generated by $\mathbb{L}$. These are full $\mathcal{I}$-structures whose collection of theory families include $\mathcal{D}$.

$$\begin{array}{rcl}\mathrm{FStr}^{\mathcal{I}}(\langle \mathcal{A}, D \rangle) & = & \{\langle \mathcal{A}, D' \rangle \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) : D' \leq D\} \\ & = & \{\langle \mathcal{A}, \mathcal{D}' \rangle \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) : \mathcal{D} \leq \mathcal{D}'\}.\end{array}$$

Then we have the following.

Proposition 1411 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$. *Let also* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{A}' = \langle \mathbf{Sign}, \mathrm{SEN}', N' \rangle$ *be* $N^\flat$*-algebraic systems, over the same category of signatures,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$ $\mathbf{F}$*-algebraic systems,* $\mathbb{L} = \langle \mathcal{A}, D \rangle$, $\mathbb{L}' = \langle \mathcal{A}', D' \rangle$ *full* $\mathcal{I}$*-structures and* $\langle I, \gamma \rangle : \mathbb{L} \vdash \mathbb{L}'$ *a bilogical morphism. Then*

$$\langle \mathcal{A}, \mathcal{X} \rangle \mapsto \langle \mathcal{A}', \gamma(\mathcal{X}) \rangle$$

is an isomorphism from $\mathrm{FStr}^{\mathcal{I}}(\mathbb{L})$ *to* $\mathrm{FStr}^{\mathcal{I}}(\mathbb{L}')$.

Moreover, the principal ideals of $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ *and of* $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}')$*, generated by* $\widetilde{\Omega}^{\mathcal{A}}(\mathbb{L})$ *and* $\widetilde{\Omega}^{\mathcal{A}'}(\mathbb{L}')$*, respectively, are isomorphic.*

Proof: By Corollary 1362, the displayed mapping is an isomorphism between $\mathbf{ClFam}(\mathbb{L})$ and $\mathbf{ClFam}(\mathbb{L}')$. Proposition 1392 gives the statement, since

$\langle I,\gamma\rangle$ induces bilogical morphisms between the corresponding elements in $\mathrm{ClFam}(\mathbb{L})$ and $\mathrm{ClFam}(\mathbb{L}')$. The second statement now follows by applying the Isomorphism Theorem 1408. ∎

Corollary 1412 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*. Let also* $\mathbf{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$*,* $\mathbf{A}'=\langle\mathbf{Sign},\mathrm{SEN}',N'\rangle$ *be* $N^\flat$*-algebraic systems, over the same category of signatures,* $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$*,* $\mathcal{A}'=\langle\mathbf{A}',\langle F',\alpha'\rangle\rangle$ $\mathbf{F}$*-algebraic systems and* $\langle I,\gamma\rangle:\mathcal{A}\to\mathcal{A}'$ *a surjective morphism, such that*

$$\langle I,\gamma\rangle:\langle\mathcal{A},\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle\vdash\langle\mathcal{A}',\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')\rangle$$

is a bilogical morphism. Then $\langle\mathcal{A},\mathcal{X}\rangle\mapsto\langle\mathcal{A}',\gamma(\mathcal{X})\rangle$ *is an isomorphism from* $\mathbf{FStr}^{\mathcal{I}}(\mathcal{A})$ *to* $\mathbf{FStr}^{\mathcal{I}}(\mathcal{A}')$*. Moreover,* $\mathbf{ConSys}^{\mathcal{I}}(\mathcal{A})\cong\mathbf{ConSys}^{\mathcal{I}}(\mathcal{A}')$*.*

Proof: This follows by Proposition 1411, since, by Proposition 1390, the $\mathcal{I}$-structures $\langle\mathcal{A},\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ and $\langle\mathcal{A}',\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')\rangle$ are the weakest full $\mathcal{I}$-structures on $\mathcal{A}$ and $\mathcal{A}'$, respectively. The last isomorphism follows by the second statement of Proposition 1411. ∎

We close the section by looking at some functors that relate the categories having as objects the structures that we have focused upon and with surjective homomorphism running between them.

Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and $\mathcal{I}=\langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$. We describe three categories related to $\mathcal{I}$.

- The category $\mathbf{FStr}(\mathcal{I})$:
  - The objects are full $\mathcal{I}$-structures $\mathbb{L}=\langle\mathcal{A},D\rangle$;
  - Given objects $\mathbb{L}=\langle\mathcal{A},D\rangle$ and $\mathbb{L}'=\langle\mathcal{A}',D'\rangle$, a morphism in $\mathbf{FStr}(\mathcal{I})$
  $$\langle H,\gamma\rangle:\mathbb{L}\to\mathbb{L}'$$
  is a surjective morphism $\langle H,\gamma\rangle:\mathcal{A}\to\mathcal{A}'$, which is also a logical morphism $\langle H,\gamma\rangle:\mathbb{L}\rangle\!\!-\mathbb{L}'$.

  It is not difficult to verify that these two clauses specify indeed a category, with composition being ordinary composition of morphisms.

- The category $\mathbf{FStr}^*(\mathcal{I})$:

  This is the full subcategory of $\mathbf{FStr}(\mathcal{I})$, with objects all full $\mathcal{I}$-structures.

- The category $\mathbf{AlgSys}(\mathcal{I})$:
  - The objects are $\mathcal{I}$-algebraic systems $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$;

– Given objects $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle\rangle$ and $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle\rangle$, a morphism in $\mathbf{AlgSys}(\mathcal{I})$

$$\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{A}'$$

is a surjective $\mathbf{F}$-algebraic system morphism.

It is not difficult in this case either to verify that these two clauses specify indeed a category, with composition being ordinary composition of morphisms.

The following picture gives an overview of the relationships that hold between these categories and will be established shortly. The categories $\mathbf{AlgSys}(\mathcal{I})$ and $\mathbf{FStr}^*(\mathcal{I})$ are isomorphic through an isomorphism

$$\Phi : \mathbf{AlgSys}(\mathcal{I}) \cong \mathbf{FStr}^*(\mathcal{I}),$$

which will be defined in the upcoming Theorem 1413. Moreover, the category $\mathbf{FStr}^*(\mathcal{I})$ is a reflective subcategory of the category $\mathbf{FStr}(\mathcal{I})$, with reflector the reduction functor ${}^* : \mathbf{FStr}(\mathcal{I}) \to \mathbf{FStr}^*(\mathcal{I})$ that will be visited in detail in the last Theorem 1414 of the section.

$$\mathbf{AlgSys}(\mathcal{I}) \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} \mathbf{FStr}^*(\mathcal{I}) \underset{*}{\overset{J}{\hookrightarrow\!\!\!\leftarrow}} \mathbf{FStr}(\mathcal{I})$$

Theorem 1413 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *Then the categories* $\mathbf{AlgSys}(\mathcal{I})$ *and* $\mathbf{FStr}^*(\mathcal{I})$ *are isomorphic.*

Proof: We define the functor

$$\Phi : \mathbf{AlgSys}(\mathcal{I}) \to \mathbf{FStr}^*(\mathcal{I})$$

by setting:

- For all $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle\rangle \in \mathrm{AlgSys}(\mathcal{I})$,

$$\Phi(\mathcal{A}) = \langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle;$$

- For all $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{A}'$ in $\mathbf{AlgSys}(\mathcal{I})$,

$$\Phi(\langle H, \gamma \rangle) = \langle H, \gamma \rangle : \langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle \vdash \langle \mathcal{A}', \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')\rangle.$$

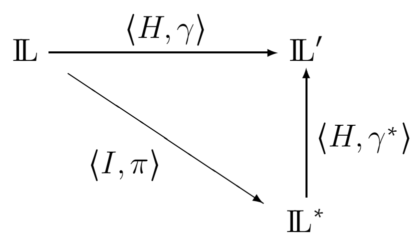
First, observe that, by Proposition 1397, if $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, then $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ is a reduced full $\mathcal{I}$-structure. So Φ is correctly defined. Moreover, if $\langle \mathcal{A}, D \rangle \in \mathrm{FStr}^*(\mathcal{I})$, then, again by Proposition 1397, $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$. Thus, Φ is a bijection on objects.

Finally, by Corollary 55, if $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{A}'$ is a surjective morphism, then, for all $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$, $\gamma^{-1}(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Therefore, $\Phi(\langle H,\gamma\rangle)$ is a well-defined logical morphism, by Proposition 1358. Since it is clear that Φ is bijective on morphisms as well, we get that $\Phi : \mathbf{AlgSys}(\mathcal{I}) \to \mathbf{FStr}^*(\mathcal{I})$ is indeed an isomorphism of categories. ∎

Finally, for the reflection:

Theorem 1414 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a π-institution based on* $\mathbf{F}$. *Then* $\mathbf{FStr}^*(\mathcal{I})$ *is a full reflective subcategory of* $\mathbf{FStr}(\mathcal{I})$ *with reflector the reduction functor* $^* : \mathbf{FStr}(\mathcal{I}) \to \mathbf{FStr}^*(\mathcal{I})$.

Proof: It is obvious that $\mathbf{FStr}^*(\mathcal{I})$ is a full subcategory of $\mathbf{FStr}(\mathcal{I})$. We must show that, given $\mathbb{L} = \langle \mathcal{A}, D\rangle \in \mathrm{FStr}(\mathcal{I})$, the pair $\langle \mathbb{L}^*, \langle I,\pi\rangle : \mathbb{L} \to \mathbb{L}^*\rangle$ is a reflector, i.e., that, given $\mathbb{L}' = \langle \mathcal{A}', D'\rangle \in \mathrm{FStr}^*(\mathcal{I})$ and $\langle H,\gamma\rangle : \mathbb{L} \to \mathbb{L}'$ in $\mathbf{FStr}(\mathcal{I})$, there exists a unique $\langle H,\gamma^*\rangle : \mathbb{L}^* \to \mathbb{L}'$ in $\mathbf{FStr}^*(\mathcal{I})$, such that the following diagram commutes.



Consider $\mathbb{L}^\gamma := \langle \mathcal{A}, \gamma^{-1}(\mathcal{D}')\rangle$. Clearly, since, by hypothesis and Proposition 1358, $\gamma^{-1}(\mathcal{D}') \subseteq \mathcal{D}$, we have that $\mathbb{L} \leq \mathbb{L}^\gamma$. Now we have

$$\begin{array}{rcll} \mathrm{Ker}(\langle I,\pi\rangle) & = & \widetilde{\Omega}^{\mathcal{A}}(\mathbb{L}) & \text{(Set Theory)} \\ & \leq & \widetilde{\Omega}^{\mathcal{A}}(\mathbb{L}^\gamma) & (\mathbb{L} \leq \mathbb{L}^\gamma) \\ & = & \gamma^{-1}(\widetilde{\Omega}^{\mathcal{A}'}(\mathbb{L}')) & \text{(Corollary 1364)} \\ & = & \gamma^{-1}(\Delta^{\mathcal{A}'}) & (\mathbb{L}' \in \mathrm{FStr}^*(\mathcal{I})) \\ & = & \mathrm{Ker}(\langle H,\gamma\rangle). & \text{(Set Theory)} \end{array}$$

By the Fill-in Lemma (Proposition 1374), there exists a unique logical morphism $\langle H,\gamma^*\rangle : \mathbb{L}^* \to \mathbb{L}'$, such that the displayed diagram commutes, which, in addition, is surjective by the commutativity of the triangle. ∎

19.8 Frege Relations Revisited

We revisit here in more detail the types of Frege relations and Frege operators one may consider in conjunction with π-institutions or π-structures, more generally.

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system and $T \in \mathrm{SenFam}(\mathbf{A})$.

- The **local Frege relation family** $\lambda^{\mathbf{A}}(T) = \{\lambda^{\mathbf{A}}_\Sigma(T)\}_{\Sigma\in|\mathbf{Sign}|}$ **of** T **on** $\mathbf{A}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\lambda^{\mathbf{A}}_\Sigma(T) = \{\langle\phi,\psi\rangle \in \mathrm{SEN}(\Sigma)^2 : \phi \in T_\Sigma \text{ iff } \psi \in T_\Sigma\}.$$

- The **global Frege relation family** $\Lambda^{\mathbf{A}}(T) = \{\Lambda^{\mathbf{A}}_\Sigma(T)\}_{\Sigma\in|\mathbf{Sign}|}$ **of** T **on** $\mathbf{A}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\begin{array}{rcl}\Lambda^{\mathbf{A}}_\Sigma(T) & = & \{\langle\phi,\psi\rangle \in \mathrm{SEN}(\Sigma)^2 : \text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma'),\\ & & \mathrm{SEN}(f)(\phi) \in T_{\Sigma'} \text{ iff } \mathrm{SEN}(f)(\psi) \in T_{\Sigma'}\}.\end{array}$$

The operators $\lambda^{\mathbf{A}}, \Lambda^{\mathbf{A}} : \mathrm{SenFam}(\mathbf{A}) \to \mathrm{RelFam}(\mathbf{A})$ are called the **local** and **global Frege operators on A**, respectively.

Let now $\mathbf{A} = \langle\mathbf{Sign},\mathrm{SEN},N\rangle$ be an algebraic system and $\mathbb{L} = \langle\mathbf{A},D\rangle$ be a π-structure.

- The **local Frege relation family** $\widetilde{\lambda}^{\mathbf{A}}(\mathbb{L}) = \widetilde{\lambda}^{\mathbf{A}}(D) = \{\widetilde{\lambda}^{\mathbf{A}}_\Sigma(D)\}_{\Sigma\in|\mathbf{Sign}|}$ **of** $\mathbb{L}$, or **of** D **on A**, is defined by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\widetilde{\lambda}^{\mathbf{A}}_\Sigma(D) = \{\langle\phi,\psi\rangle \in \mathrm{SEN}(\Sigma)^2 : D_\Sigma(\phi) = D_\Sigma(\psi)\}.$$

- The **global Frege relation family** $\widetilde{\Lambda}^{\mathbf{A}}(\mathbb{L}) = \widetilde{\Lambda}^{\mathbf{A}}(D) = \{\widetilde{\Lambda}^{\mathbf{A}}_\Sigma(D)\}_{\Sigma\in|\mathbf{Sign}|}$ **of** $\mathbb{L}$, or **of** D **on A**, is defined by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\begin{array}{rcl}\widetilde{\Lambda}^{\mathbf{A}}_\Sigma(D) & = & \{\langle\phi,\psi\rangle \in \mathrm{SEN}(\Sigma)^2 : \text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma'),\\ & & D_{\Sigma'}(\mathrm{SEN}(f)(\phi)) = D_{\Sigma'}(\mathrm{SEN}(f)(\psi))\}.\end{array}$$

Consider again an algebraic system $\mathbf{A} = \langle\mathbf{Sign},\mathrm{SEN},N\rangle$, a π-structure $\mathbb{L} = \langle\mathbf{A},D\rangle$ and $X \in \mathrm{SenFam}(\mathbf{A})$. Recall the notation $D^X : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}$ denoting the closure family on $\mathbf{A}$ that is defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)$, by

$$D^X_\Sigma(\Phi) = D_\Sigma(X_\Sigma \cup \Phi).$$

- The **local Frege relation family**

$$\widetilde{\lambda}^{\mathbb{L}}(X) = \widetilde{\lambda}^{\mathbf{A},D}(X) = \{\widetilde{\lambda}^{\mathbf{A},D}_\Sigma(X)\}_{\Sigma\in|\mathbf{Sign}|}$$

of X **in** $\mathbb{L}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\widetilde{\lambda}^{\mathbf{A},D}_\Sigma(X) = \{\langle\phi,\psi\rangle \in \mathrm{SEN}(\Sigma)^2 : D^X_\Sigma(\phi) = D^X_\Sigma(\psi)\}.$$

- The **global Frege relation family**

$$\widetilde{\Lambda}^{\mathbb{L}}(X) = \widetilde{\Lambda}^{\mathbf{A},D}(X) = \{\widetilde{\Lambda}^{\mathbf{A},D}_\Sigma(X)\}_{\Sigma\in|\mathbf{Sign}|}$$

of X **in** $\mathbb{L}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\begin{array}{rcl}\widetilde{\Lambda}^{\mathbf{A},D}_\Sigma(X) & = & \{\langle\phi,\psi\rangle \in \mathrm{SEN}(\Sigma)^2 : \text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma'),\\ & & D^X_{\Sigma'}(\mathrm{SEN}(f)(\phi)) = D^X_{\Sigma'}(\mathrm{SEN}(f)(\psi))\}.\end{array}$$

The operators $\widetilde{\lambda}^{\mathbf{A},D}, \widetilde{\Lambda}^{\mathbf{A},D} : \mathrm{SenFam}(\mathbf{A}) \to \mathrm{RelFam}(\mathbf{A})$ are called the **local** and **global Frege operators on** $\mathbb{L}$, respectively.

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system, $\mathbb{L} = \langle \mathbf{A}, D\rangle$ be a π-structure and $X \in \mathrm{SenFam}(\mathbf{A})$. Some obvious relationships hold between several of the notions defined above. We denote by $\mathrm{Thm}(\mathbb{L}) = \{\mathrm{Thm}_\Sigma(\mathbb{L})\}_{\Sigma \in |\mathbf{Sign}|}$, where $\mathrm{Thm}_\Sigma(\mathbb{L}) = D_\Sigma(\varnothing)$, an obvious generalization of the corresponding notion from π-institutions. Note, however, that, since D is not necessarily structural, in this case $\mathrm{Thm}(\mathbb{L})$ is a theory family, but not necessarily a theory system. Then, we have the following:

$$\begin{array}{rcl}\widetilde{\lambda}^{\mathbf{A}}(D) & = & \widetilde{\lambda}^{\mathbf{A},D}(\mathrm{Thm}(\mathbb{L}));\\ \widetilde{\Lambda}^{\mathbf{A}}(D) & = & \widetilde{\Lambda}^{\mathbf{A},D}(\mathrm{Thm}(\mathbb{L}));\\ & & \\ \widetilde{\lambda}^{\mathbf{A},D}(X) & = & \bigcap\{\lambda^{\mathbf{A}}(T) : X \le T \in \mathrm{ThFam}(\mathbb{L})\};\\ \widetilde{\Lambda}^{\mathbf{A},D}(X) & = & \bigcap\{\Lambda^{\mathbf{A}}(T) : X \le T \in \mathrm{ThFam}(\mathbb{L})\}.\end{array}$$

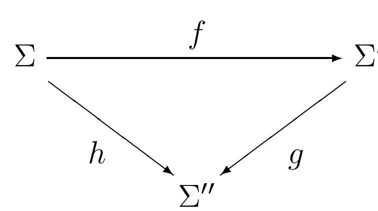
We show that all three local Frege operators give rise to equivalence families, whereas all three global operators give rise to equivalence systems on the underlying algebraic system $\mathbf{A}$.

Lemma 1415 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system,* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *a* π*-structure and* $X \in \mathrm{SenFam}(\mathbf{A})$*.*

(a) $\lambda^{\mathbf{A}}(X)$*,* $\widetilde{\lambda}^{\mathbf{A}}(D)$ *and* $\widetilde{\lambda}^{\mathbf{A},D}(X)$ *are equivalence families on* $\mathbf{A}$*;*

(b) $\Lambda^{\mathbf{A}}(X)$*,* $\widetilde{\Lambda}^{\mathbf{A}}(D)$ *and* $\widetilde{\Lambda}^{\mathbf{A},D}(X)$ *are equivalence systems on* $\mathbf{A}$*.*

Proof: Because of the interdependencies between these concepts, pointed out before the lemma, it suffices to prove the statements only for $\lambda^{\mathbf{A}}(X)$ and $\Lambda^{\mathbf{A}}(X)$. That both $\lambda^{\mathbf{A}}(X)$ and $\Lambda^{\mathbf{A}}(X)$ are equivalence families is obvious because of the properties of the equivalence connective used in their definitions. So it suffices to show only that $\Lambda^{\mathbf{A}}(X)$ is a system, i.e., that it is invariant under signature morphisms. So suppose $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi\rangle \in \Lambda^{\mathbf{A}}_\Sigma(X)$ and let $\Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma, \Sigma')$.



By the definition of $\Lambda^{\mathbf{A}}(X)$, we have that, for all $\Sigma'' \in |\mathbf{Sign}|$ and all $h \in \mathbf{Sign}(\Sigma, \Sigma'')$,

$$D_{\Sigma''}(\mathrm{SEN}(h)(\phi)) = D_{\Sigma''}(\mathrm{SEN}(h)(\psi)).$$

A fortiori, for all $\Sigma'' \in |\mathbf{Sign}|$ and all $g \in \mathbf{Sign}(\Sigma', \Sigma'')$, we have

$$D_{\Sigma''}(\mathrm{SEN}(g)(\mathrm{SEN}(f)(\phi))) = D_{\Sigma''}(\mathrm{SEN}(g)(\mathrm{SEN}(f)(\psi))).$$

By the definition of $\Lambda^{\mathbf{A}}(X)$, this proves that $\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)\rangle \in \Lambda^{\mathbf{A}}_{\Sigma'}(X)$. Thus $\Lambda^{\mathbf{A}}(X)$ is indeed an equivalence system on $\mathbf{A}$. ■

The next lemma shows that all four "tilde" Frege operators are monotone.

Lemma 1416 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system and* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *a* π*-structure on* $\mathbf{A}$*.*

(a) $\widetilde{\lambda}^{\mathbf{A}}, \widetilde{\Lambda}^{\mathbf{A}} : \mathbf{ClFam}(\mathbf{A}) \to \mathbf{EqvFam}(\mathbf{A})$ *are monotone;*

(b) $\widetilde{\lambda}^{\mathbf{A},D}, \widetilde{\Lambda}^{\mathbf{A},D} : \mathbf{SenFam}(\mathbf{A}) \to \mathbf{EqvFam}(\mathbf{A})$ *are monotone.*

Proof: Suppose $D, D' \in \mathrm{ClFam}(\mathbf{A})$, such that $D \le D'$, and let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi\rangle \in \widetilde{\lambda}^{\mathbf{A}}_{\Sigma}(D)$. Then $D_\Sigma(\phi) = D_\Sigma(\psi)$, whence $D'_\Sigma(\phi) = D'_\Sigma(\psi)$. Thus, $\langle \phi, \psi\rangle \in \widetilde{\lambda}^{\mathbf{A}}_{\Sigma}(D')$. We conclude that $\widetilde{\lambda}^{\mathbf{A}}(D) \le \widetilde{\lambda}^{\mathbf{A}}(D')$. The proof for $\widetilde{\Lambda}^{\mathbf{A}} : \mathbf{ClFam}(\mathbf{A}) \to \mathbf{EqvFam}(\mathbf{A})$ is similar.

Suppose, next, that $X, X' \in \mathrm{SenFam}(\mathbf{A})$, such that $X \le X'$. Note that, in this situation, we have

$$\{T \in \mathrm{ThFam}(\mathbb{L}) : X' \le T\} \subseteq \{T \in \mathrm{ThFam}(\mathbb{L}) : X \le T\}.$$

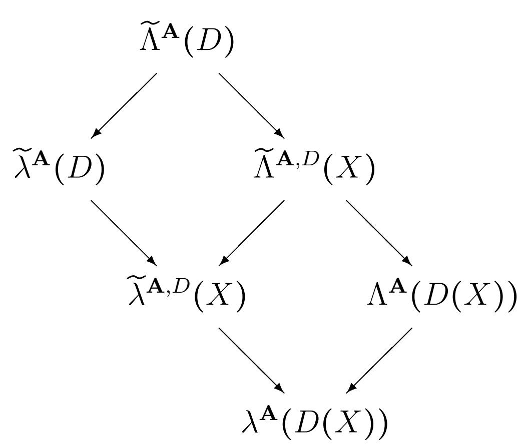
Therefore, we have

$$\begin{array}{rcl}\widetilde{\lambda}^{\mathbf{A},D}(X) & = & \bigcap\{\lambda^{\mathbf{A}}(T) : X \le T \in \mathrm{ThFam}(\mathbb{L})\}\\ & \le & \bigcap\{\lambda^{\mathbf{A}}(T) : X' \le T \in \mathrm{ThFam}(\mathbb{L})\}\\ & = & \widetilde{\lambda}^{\mathbf{A},D}(X').\end{array}$$

The proof for $\widetilde{\Lambda}^{\mathbf{A},D} : \mathbf{SenFam}(\mathbf{A}) \to \mathbf{EqvFam}(\mathbf{A})$ is similar. ■

The equivalence families produced by applying the six Frege operators form a hierarchy under inclusion that we now make explicit.

Proposition 1417 *Let* $\mathbf{A} = \langle \mathbf{A}, \mathrm{SEN}, N\rangle$ *be an algebraic system,* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *a* π*-structure and* $X \in \mathrm{SenFam}(\mathbf{A})$*. Then, we have the following inclusions between equivalence families on* $\mathbf{A}$*:*



Proof: First, note that the three southwest inclusions are obvious, since the conditions defining $\widetilde{\lambda}^{\mathbf{A}}$, $\widetilde{\lambda}^{\mathbf{A},D}$ and $\lambda^{\mathbf{A}}$ are special cases of the ones defining $\widetilde{\Lambda}^{\mathbf{A}}$, $\widetilde{\Lambda}^{\mathbf{A},D}$ and $\Lambda^{\mathbf{A}}$, respectively.

We show, next, the southeast inclusions between the λ's, since the ones between the Λ's may shown similarly. We have

$$\widetilde{\lambda}^{\mathbf{A}}(D) = \widetilde{\lambda}^{\mathbf{A},D}(\mathrm{Thm}(\mathbb{L})) \leq \widetilde{\lambda}^{\mathbf{A},D}(D(X)) = \widetilde{\lambda}^{\mathbf{A},D}(X).$$

Moreover,

$$\begin{array}{rcl}\widetilde{\lambda}^{\mathbf{A},D}(X) & = & \bigcap\{\lambda^{\mathbf{A}}(T) : X \leq T \in \mathrm{ThFam}(\mathbb{L})\}\\ & \leq & \lambda^{\mathbf{A}}(D(X)).\end{array}$$

Therefore, we have $\widetilde{\lambda}^{\mathbf{A}}(D) \leq \widetilde{\lambda}^{\mathbf{A},D}(X) \leq \lambda^{\mathbf{A}}(D(X))$. ∎

In the case of structural π-structures, i.e., π-institutions, the hierarchy collapses to a smaller one, the top pair collapses and in the case of a sentence system, the middle pair does also. More precisely, we have

Proposition 1418 *Let* $\mathbf{A} = \langle \mathbf{A}, \mathrm{SEN}, N\rangle$ *be an algebraic system,* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *a* π*-structure and* $X \in \mathrm{SenSys}(\mathbf{A})$. *If* D *is structural, then*

$$\widetilde{\Lambda}^{\mathbf{A}}(D) = \widetilde{\lambda}^{\mathbf{A}}(D) \quad \textit{and} \quad \widetilde{\Lambda}^{\mathbf{A},D}(X) = \widetilde{\lambda}^{\mathbf{A},D}(X).$$

Proof: By the remarks preceding Lemma 1415, it suffices to show that the second equation holds. Since it is always the case that $\widetilde{\Lambda}^{\mathbf{A},D}(X) \leq \widetilde{\lambda}^{\mathbf{A},D}(X)$, we must prove the opposite inclusion. Let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi\rangle \in \widetilde{\lambda}^{\mathbf{A},D}_{\Sigma}(X)$. Then, we have, by definition, $D_\Sigma(X_\Sigma, \phi) = D_\Sigma(X_\Sigma, \psi)$. By the structurality of D, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

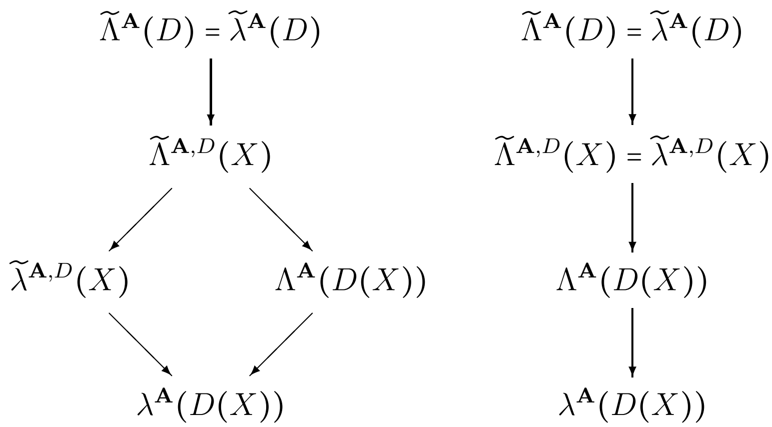
$$D_{\Sigma'}(\mathrm{SEN}(f)(X_\Sigma), \mathrm{SEN}(f)(\phi)) = D_{\Sigma'}(\mathrm{SEN}(f)(X_\Sigma), \mathrm{SEN}(f)(\psi)).$$

Therefore, is X is a sentence system, we get

$$D_{\Sigma'}(X_{\Sigma'}, \mathrm{SEN}(f)(\phi)) = D_{\Sigma'}(X_{\Sigma'}, \mathrm{SEN}(f)(\psi)).$$

We conclude that $\langle \phi, \psi\rangle \in \widetilde{\Lambda}^{\mathbf{A},D}_{\Sigma}(X)$. ∎

Thus, if $\mathbb{L} = \langle \mathbf{A}, D\rangle$ is a π-institution, and $X \in \mathrm{SenFam}(\mathbf{A})$, we obtain the simplified hierarchy of Frege relation families shown on the left and, if, in addition, $X \in \text{SenSys}(\mathbf{A})$, we get the linear hierarchy shown on the right.



We look next at how finitarity of a closure family relates to continuity of Frege operators.

Let $\mathbf{A} = \langle \mathbf{Sign}, \text{SEN}, N \rangle$ be an algebraic system. Recall that:

- An $X \in \text{SenFam}(\mathbf{A})$ is called **locally finite** if, for all $\Sigma \in |\mathbf{Sign}|$, X_Σ is finite. We write $Y \leq_{lf} X$ to suggest that Y is a locally finite sentence subfamily of X.

- A collection $\mathcal{X} \subseteq \text{SenFam}(\mathbf{A})$ is said to be **locally directed** if, for every $\Sigma \in |\mathbf{Sign}|$ and finite $\mathcal{Y} \subseteq \mathcal{X}$, there exists $X \in \mathcal{X}$, such that $Y_\Sigma \leq X_\Sigma$, for all $Y \in \mathcal{Y}$.

Let $\mathbb{L} = \langle \mathbf{A}, D \rangle$ be a π-structure based on $\mathbf{A}$.

- $\mathbb{L}$ is **finitary** if, for all $X \in \text{SenFam}(\mathbf{A})$,

$$D(X) = \bigcup \{D(Y) : Y \leq_{lf} X\}.$$

- The operator $\widetilde{\lambda}^{\mathbf{A},D} : \text{SenFam}(\mathbf{A}) \to \text{EqvFam}(\mathbf{A})$ is **locally continuous** if, for every locally directed $\{X^i : i \in I\} \subseteq \text{SenFam}(\mathbf{A})$,

$$\widetilde{\lambda}^{\mathbf{A},D}(\bigcup_{i \in I} X^i) = \bigcup_{i \in I} \widetilde{\lambda}^{\mathbf{A},D}(X^i).$$

Proposition 1419 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \text{SEN}, N \rangle$ *be an algebraic system and* $\mathbb{L} = \langle \mathbf{A}, D \rangle$ *a* π*-structure based on* $\mathbf{A}$*.* $\mathbb{L}$ *is finitary if and only if*

$$\widetilde{\lambda}^{\mathbf{A},D} : \text{SenFam}(\mathbf{A}) \to \text{EqvFam}(\mathbf{A})$$

is locally continuous.

Proof: Suppose, first, that $\mathbb{L}$ is finitary and let $\{X^i : i \in I\} \subseteq \mathrm{SenFam}(\mathbf{A})$ be locally directed. Since, by Lemma 1416, $\widetilde{\lambda}^{\mathbf{A},D}$ is monotone, we have

$$\bigcup_{i\in I} \widetilde{\lambda}^{\mathbf{A},D}(X^i) \leq \widetilde{\lambda}^{\mathbf{A},D}(\bigcup_{i\in I} X^i).$$

To show the reverse inclusion, let $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi\rangle \in \widetilde{\lambda}^{\mathbf{A},D}_\Sigma(\bigcup_{i\in I} X^i)$. Then, by definition, $D_\Sigma(\bigcup_{i\in I} X^i_\Sigma, \phi) = D_\Sigma(\bigcup_{i\in I} X^i_\Sigma, \psi)$. Since $\mathbb{L}$ is finitary, there exists finite $\Phi \leq_f \bigcup_{i\in I} X^i_\Sigma$, such that $D_\Sigma(\Phi, \phi) = D_\Sigma(\Phi, \psi)$. Hence, since $\{X^i : i \in I\}$ is locally directed, there exists $i \in I$, such that $\Phi \subseteq X^i_\Sigma$. Hence, $D_\Sigma(X^i_\Sigma, \phi) = D_\Sigma(X^i_\Sigma, \psi)$, i.e., $\langle \phi, \psi\rangle \in \widetilde{\lambda}^{\mathbf{A},D}_\Sigma(X^i)$. We conclude that $\widetilde{\lambda}^{\mathbf{A},D}(\bigcup_{i\in I} X^i) \leq \bigcup_{i\in I} \widetilde{\lambda}^{\mathbf{A},D}(X^i)$.

Assume, conversely, that $\widetilde{\lambda}^{\mathbf{A},D}$ is locally continuous and consider $X \in \mathrm{SenFam}(\mathbf{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in D_\Sigma(X_\Sigma)$. Let $\mathcal{Z} = \{Z \in \mathrm{SenFam}(\mathbf{A}) : Z \leq_{lf} X\}$. $\mathcal{Z}$ is a locally directed family, such that $\bigcup \mathcal{Z} = X$. For all $\psi \in X_\Sigma$, we have $D_\Sigma(X_\Sigma, \phi) = D_\Sigma(X_\Sigma, \psi) = D_\Sigma(X_\Sigma)$. So we get

$$\langle \phi, \psi\rangle \in \widetilde{\lambda}^{\mathbf{A},D}_\Sigma(X) = \widetilde{\lambda}^{\mathbf{A},D}_\Sigma(\bigcup \mathcal{Z}).$$

By the local directedness of $\mathcal{Z}$ and local continuity of $\widetilde{\lambda}^{\mathbf{A},D}$, we get $\langle \phi, \psi\rangle \in \bigcup_{Z\in\mathcal{Z}} \widetilde{\lambda}^{\mathbf{A},D}_\Sigma(Z)$. Therefore, we get, for some $Z \leq_{lf} X$, $\phi \in D_\Sigma(Z_\Sigma, \psi) = D_\Sigma(Z_\Sigma)$. This shows that $\mathbb{L}$ is finitary. ■

Among the key properties of Frege relations, which partly explains their usefulness in the algebraic study of logical systems, is that, loosely speaking, they are approximated from below by the Leibniz, the Tarski and the Suszko congruence systems.

Proposition 1420 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system and* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *a* π*-structure based on* $\mathbf{A}$ *and* $T \in \mathrm{ThFam}(\mathbb{L})$.

(a) The Leibniz congruence system $\Omega^{\mathbf{A}}(T)$ *is the largest congruence system on* $\mathbf{A}$ *included in* $\Lambda^{\mathbf{A}}(T)$ *and in* $\lambda^{\mathbf{A}}(T)$*;*

(b) The Tarski congruence system $\widetilde{\Omega}^{\mathbf{A}}(D)$ *is the largest congruence system on* $\mathbf{A}$ *included in* $\widetilde{\Lambda}^{\mathbf{A}}(D)$ *and in* $\widetilde{\lambda}^{\mathbf{A}}(D)$*;*

(c) The Suszko congruence system $\widetilde{\Omega}^{\mathbf{A},D}(T)$ *is the largest congruence system on* $\mathbf{A}$ *included in* $\widetilde{\Lambda}^{\mathbf{A},D}(T)$ *and in* $\widetilde{\lambda}^{\mathbf{A},D}(T)$*.*

Proof: Let $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi\rangle \in \Omega^{\mathbf{A}}_\Sigma(T)$. Since $\Omega^{\mathbf{A}}(T)$ is a congruence system, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)\rangle \in \Omega^{\mathbf{A}}_{\Sigma'}(T)$. Thus, by the compatibility property of $\Omega^{\mathbf{A}}(T)$ with T, we get

$$\mathrm{SEN}(f)(\phi) \in T_{\Sigma'} \quad \text{iff} \quad \mathrm{SEN}(f)(\psi) \in T_{\Sigma'},$$

i.e., $\langle \phi, \psi\rangle \in \Lambda^{\mathbf{A}}_\Sigma(T)$. We conclude that $\Omega^{\mathbf{A}}(T) \leq \Lambda^{\mathbf{A}}(T) \leq \lambda^{\mathbf{A}}(T)$.

Suppose, next, that $\theta \in \mathrm{ConSys}(\mathbf{A})$, such that $\theta \leq \lambda^{\mathbf{A}}(T)$. If $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \theta_\Sigma$ and $\phi \in T_\Sigma$, then $\langle \phi, \psi \rangle \in \lambda^{\mathbf{A}}(T)$ and $\phi \in T_\Sigma$, whence by the definition of $\lambda^{\mathbf{A}}(T)$, $\psi \in T_\Sigma$. Thus, θ is a congruence system on $\mathbf{A}$ compatible with T and, therefore, $\theta \leq \Omega^{\mathbf{A}}(T)$, by the maximality property of $\Omega^{\mathbf{A}}(T)$.

Parts (b) and (c) can be proved similarly. ■

We show next that Frege relations are preserved under inverse surjective morphisms.

Lemma 1421 *Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ be algebraic systems, $\mathbb{L} = \langle \mathbf{A}, D \rangle$, $\mathbb{L}' = \langle \mathbf{A}', D' \rangle$ be π-structures based on $\mathbf{A}$, $\mathbf{A}'$, respectively, and $\langle F, \alpha \rangle : \mathbf{A} \to \mathbf{A}'$ a surjective morphism.*

(a) For every $X \in \mathrm{SenFam}(\mathbf{A}')$, $\Lambda^{\mathbf{A}}(\alpha^{-1}(X)) = \alpha^{-1}(\Lambda^{\mathbf{A}'}(X))$ and, also, $\lambda^{\mathbf{A}}(\alpha^{-1}(X)) = \alpha^{-1}(\lambda^{\mathbf{A}'}(X))$;

(b) If $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ is a bilogical morphism, then $\widetilde{\Lambda}^{\mathbf{A}}(D) = \alpha^{-1}(\widetilde{\Lambda}^{\mathbf{A}'}(D'))$ and, also, $\widetilde{\lambda}^{\mathbf{A}}(D) = \alpha^{-1}(\widetilde{\lambda}^{\mathbf{A}'}(D'))$.

Proof:

(a) Let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$. We have $\langle \phi, \psi \rangle \in \alpha_\Sigma^{-1}(\Lambda^{\mathbf{A}'}(X))$ iff $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \Lambda^{\mathbf{A}'}_{F(\Sigma)}(X)$ iff, by surjectivity, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi)) \in X_{F(\Sigma')} \quad \text{iff} \quad \mathrm{SEN}'(F(f))(\alpha_\Sigma(\psi)) \in X_{F(\Sigma')},$$

iff, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \in X_{F(\Sigma')} \quad \text{iff} \quad \alpha_{\Sigma'}(\mathrm{SEN}'(f)(\psi)) \in X_{F(\Sigma')},$$

iff, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}(f)(\phi) \in \alpha_{\Sigma'}^{-1}(X_{F(\Sigma')}) \quad \text{iff} \quad \mathrm{SEN}(f)(\psi) \in \alpha_{\Sigma'}^{-1}(X_{F(\Sigma')}),$$

iff, $\langle \phi, \psi \rangle \in \Lambda^{\mathbf{A}}_\Sigma(\alpha^{-1}(X))$.

The proof of $\lambda^{\mathbf{A}}(\alpha^{-1}(X)) = \alpha^{-1}(\lambda^{\mathbf{A}'}(X))$ is similar.

(b) We have

$$\begin{array}{rcl}
\widetilde{\Lambda}(D) & = & \bigcap\{\Lambda^{\mathbf{A}}(T) : T \in \mathrm{ThFam}(\mathbb{L})\} \\
 & = & \bigcap\{\Lambda(\alpha^{-1}(T')) : T' \in \mathrm{ThFam}(\mathbb{L}')\} \\
 & = & \bigcap\{\alpha^{-1}(\Lambda^{\mathbf{A}'}(T')) : T' \in \mathrm{ThFam}(\mathbb{L}')\} \\
 & = & \alpha^{-1}(\bigcap\{\Lambda^{\mathbf{A}'}(T') : T' \in \mathrm{ThFam}(\mathbb{L}')\}) \\
 & = & \alpha^{-1}(\widetilde{\Lambda}^{\mathbf{A}'}(D')).
\end{array}$$

The proof of $\widetilde{\lambda}^{\mathbf{A}}(D) = \alpha^{-1}(\widetilde{\lambda}^{\mathbf{A}'}(D'))$ is similar.

■

19.9 Fullness and Metalogical Properties

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system.

An **F-rule** is a pair $\langle P, \rho \rangle$, with $P \cup \{\rho\} : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a finite set of natural transformations in $N^\flat$.

A **generalized** or **Gentzen F-rule**, or **F-grule** for short, is a pair

$$\langle \{\langle P^i, \rho^i \rangle : i \in I\}, \langle P, \rho \rangle \rangle,$$

where $\{\langle P^i, \rho^i \rangle : i \in I\} \cup \{\langle P, \rho \rangle\}$ is a finite set of $\mathbf{F}$-rules. We sometimes write an $\mathbf{F}$-grule in the "two-line" format

$$\frac{\langle P^i, \rho^i \rangle : i \in I}{\langle P, \rho \rangle} \quad \text{or} \quad \frac{P^i \vdash \rho^i : i \in I}{P \vdash \rho}.$$

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. $\mathcal{I}$ **satisfies** $\frac{\langle P^i, \rho^i \rangle : i \in I}{\langle P, \rho \rangle}$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\rho^i_\Sigma(\vec{\chi}) \in C_\Sigma(P^i_\Sigma(\vec{\chi})), \; i \in I, \quad \text{impies} \quad \rho_\Sigma(\vec{\chi}) \in C_\Sigma(P_\Sigma(\vec{\chi})).$$

Similarly, an $\mathbf{F}$-structure $\mathbb{L} = \langle \mathcal{A}, D \rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, **satisfies** $\frac{\langle P^i, \rho^i \rangle : i \in I}{\langle P, \rho \rangle}$ if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma)$,

$$\rho^i_\Sigma(\vec{\chi}) \in D_\Sigma(P^i_\Sigma(\vec{\chi})), \; i \in I, \quad \text{impies} \quad \rho_\Sigma(\vec{\chi}) \in D_\Sigma(P_\Sigma(\vec{\chi})).$$

Since an $\mathbf{F}$-rule can be perceived as a a special case of an $\mathbf{F}$-grule (with empty set of premises), this notion of satisfaction applies in particular to $\mathbf{F}$-rules.

It turns out that satisfaction of $\mathbf{F}$-rules is transferred from a π-institution to all its structure models.

Proposition 1422 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ satisfies an $\mathbf{F}$-rule $\langle P, \rho \rangle$, then every $\mathcal{I}$-structure satisfies the same $\mathbf{F}$-rule.*

Proof: Suppose $\mathcal{I}$ satisfies $\langle P, \rho \rangle$ and let $\mathbb{L} = \langle \mathcal{A}, D \rangle$ be an $\mathcal{I}$-structure, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$. Then, for all $T \in \mathrm{ThFam}(\mathbb{L})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, we have $P_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi})) \subseteq T_{F(\Sigma)}$ if and only if $\alpha_\Sigma(P_\Sigma(\vec{\chi})) \subseteq T_{F(\Sigma)}$ if and only if $P_\Sigma(\vec{\chi}) \subseteq \alpha_\Sigma^{-1}(T_{F(\Sigma)})$. Thus, since, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, we get, by hypothesis, $\rho_\Sigma(\vec{\chi}) \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})$. This is equivalent to $\alpha_\Sigma(\rho_\Sigma(\vec{\chi})) \in T_{F(\Sigma)}$ and, in turn, to $\rho_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi})) \in T_{F(\Sigma)}$. We conclude, by the surjectivity of $\langle F, \alpha \rangle$, that $\mathbb{L}$ satisfies $\langle P, \rho \rangle$ as well. ■

Moreover, it turns out that satisfaction of an $\mathbf{F}$-grule, in general, is preserved by bilogical morphisms.

Proposition 1423 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ $N^\flat$*-algebraic systems,* $\mathbb{L} = \langle \mathbf{A}, D \rangle$*,* $\mathbb{L}' = \langle \mathbf{A}', D' \rangle$ π*-structures based on* $\mathbf{A}$*,* $\mathbf{A}'$*, respectively, and* $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ *a bilogical morphism. Then* $\mathbb{L}$ *satisfies an* $\mathbf{F}$*-grule* $\langle \{\langle P^i, \rho^i \rangle : i \in I\}, \langle P, \rho \rangle \rangle$ *if and only if* $\mathbb{L}'$ *satisfies the same* $\mathbf{F}$*-grule.*

Proof: Suppose, first, that $\mathbb{L}$ satisfies the $\mathbf{F}$-grule and let $\Sigma \in |\mathbf{Sign}|$, $\vec{\chi} \in \mathrm{SEN}(\Sigma)$, such that, for all $i \in I$,

$$\rho'^i_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi})) \in D'_{F(\Sigma)}(P'^i_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi}))).$$

This is equivalent to

$$\alpha_\Sigma(\rho^i_\Sigma(\vec{\chi})) \in D'_{F(\Sigma)}(\alpha_\Sigma(P^i_{F(\Sigma)}(\vec{\chi}))).$$

Since $\langle F, \alpha \rangle$ is a bilogical morphism, we get $\rho^i_\Sigma(\vec{\chi}) \in D_\Sigma(P^i_\Sigma(\vec{\chi}))$. Since, by hypothesis, $\mathbb{L}$ satisfies the given $\mathbf{F}$-grule, we get that $\rho_\Sigma(\vec{\chi}) \in D_\Sigma(P_\Sigma(\vec{\chi}))$. Reversing the steps above, we conclude that

$$\rho'_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi})) \in D'_{F(\Sigma)}(P'_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi}))).$$

Since $\langle F, \alpha \rangle$ is surjective, this shows that $\mathbb{L}'$ satisfies the $\mathbf{F}$-grule as well.

Suppose, conversely, that $\mathbb{L}'$ satisfies the $\mathbf{F}$-grule $\langle \{\langle P^i, \rho^i \rangle : i \in I\}, \langle P, \rho \rangle \rangle$. Let $\Sigma \in |\mathbf{Sign}|$ and $\vec{\chi} \in \mathrm{SEN}(\Sigma)$, such that, for all $i \in I$,

$$\rho^i_\Sigma(\vec{\chi}) \in D_\Sigma(P^i_\Sigma(\vec{\chi})).$$

Since, $\langle F, \alpha \rangle$ is a bilogical morphism, we get

$$\alpha_\Sigma(\rho^i_\Sigma(\vec{\chi})) \in D'_{F(\Sigma)}(\alpha_\Sigma(P^i_\Sigma(\vec{\chi}))),$$

which gives $\rho'^i_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi})) \in D'_{F(\Sigma)}(P'^i_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi})))$. Since, by hypothesis, $\mathbb{L}'$ satisfies the given $\mathbf{F}$-grule, we now get

$$\rho'_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi})) \in D'_{F(\Sigma)}(P'_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi}))).$$

Reversing again the preceding steps, we finally obtain that

$$\rho_\Sigma(\vec{\chi}) \in D_\Sigma(P_\Sigma(\vec{\chi})).$$

Thus, $\mathbb{L}$ satisfies the same $\mathbf{F}$-grule as well. ∎

19.9.1 The Congruence Property

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system and $\mathbb{L} = \langle \mathbf{A}, D\rangle$ a π-structure.

- $\mathbb{L}$ has the **Congruence Property** if $\widetilde{\Lambda}^{\mathbf{A}}(D)$ is a congruence system on $\mathbf{A}$, i.e., by Proposition 1420, if and only if
$$\widetilde{\Lambda}^{\mathbf{A}}(D) = \widetilde{\Omega}^{\mathbf{A}}(D);$$
- $\mathbb{L}$ has the **strong Congruence Property** if $\widetilde{\lambda}^{\mathbf{A}}(D)$ is a congruence system on $\mathbf{A}$, i.e., by Proposition 1420, if and only if
$$\widetilde{\lambda}^{\mathbf{A}}(D) = \widetilde{\Omega}^{\mathbf{A}}(D).$$

Of course, the strong Congruence Property implies the Congruence Property.

Proposition 1424 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system and* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *a* π*-structure. If* $\mathcal{I}$ *has the strong Congruence Property, then it has the Congruence Property.*

Proof: We know that $\widetilde{\Omega}^{\mathbf{A}}(D) \leq \widetilde{\Lambda}^{\mathbf{A}}(T) \leq \widetilde{\lambda}^{\mathbf{A}}(T)$. If $\mathbb{L}$ has the strong Congruence Property, $\widetilde{\Omega}^{\mathbf{A}}(D) = \widetilde{\lambda}^{\mathbf{A}}(T)$, whence, also, $\widetilde{\Omega}^{\mathbf{A}}(D) = \widetilde{\Lambda}^{\mathbf{A}}(T)$. Thus, $\mathbb{L}$ has the Congruence Property. ■

Corollary 1425 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system and* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *a reduced* π*-structure based on* $\mathbf{A}$*.*

(a) $\mathbb{L}$ *has the Congruence Property if and only if* $\widetilde{\Lambda}^{\mathbf{A}}(D) = \Delta^{\mathbf{A}}$*.*

(b) $\mathbb{L}$ *has the strong Congruence Property if and only if* $\widetilde{\lambda}^{\mathbf{A}}(D) = \Delta^{\mathbf{A}}$*.*

Proof: $\mathbb{L}$ has the Congruence Property if and only if $\widetilde{\Lambda}^{\mathbf{A}}(D) = \widetilde{\Omega}^{\mathbf{A}}(D)$ if and only if, since $\mathbb{L}$ is reduced, $\widetilde{\Lambda}^{\mathbf{A}}(D) = \Delta^{\mathbf{A}}$. Part (b) is similar. ■

It turns out that the Congruence Property is preserved in both directions under bilogical morphisms.

Proposition 1426 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *be algebraic systems,* $\mathbb{L} = \langle \mathbf{A}, D\rangle$, $\mathbb{L}' = \langle \mathbf{A}', D'\rangle$ π*-structures based on* $\mathbf{A}$, $\mathbf{A}'$*, respectively, and* $\langle F, \alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$ *a bilogical morphism.*

(a) $\mathbb{L}$ *has the Congruence Property if and only if* $\mathbb{L}'$ *has the Congruence Property;*

(b) $\mathbb{L}$ *has the strong Congruence Property if and only if* $\mathbb{L}'$ *has the strong Congruence Property.*

Proof:

(a) We have

$$\begin{aligned}\widetilde{\Lambda}^{\mathbf{A}}(D) = \widetilde{\Omega}^{\mathbf{A}}(D) &\text{ iff } \alpha^{-1}(\widetilde{\Lambda}^{\mathbf{A}'}(D')) = \alpha^{-1}(\widetilde{\Omega}^{\mathbf{A}'}(D'))\\ &\text{ iff } \widetilde{\Lambda}^{\mathbf{A}'}(D') = \widetilde{\Omega}^{\mathbf{A}'}(D'),\end{aligned}$$

the first equivalence by Corollary 1364 and Lemma 1421, and the second equivalence by the surjectivity of $\langle F, \alpha\rangle$. Therefore, $\mathbb{L}$ has the Congruence Property if and only if $\mathbb{L}'$ has the Congruence Property.

(b) Similarly,

$$\begin{aligned}\widetilde{\lambda}^{\mathbf{A}}(D) = \widetilde{\Omega}^{\mathbf{A}}(D) &\text{ iff } \alpha^{-1}(\widetilde{\lambda}^{\mathbf{A}'}(D')) = \alpha^{-1}(\widetilde{\Omega}^{\mathbf{A}'}(D'))\\ &\text{ iff } \widetilde{\lambda}^{\mathbf{A}'}(D') = \widetilde{\Omega}^{\mathbf{A}'}(D'),\end{aligned}$$

the first equivalence by Corollary 1364 and Lemma 1421, and the second equivalence by the surjectivity of $\langle F, \alpha\rangle$. Therefore, $\mathbb{L}$ has the strong Congruence Property if and only if $\mathbb{L}'$ has the strong Congruence Property.

■

Using the Congruence Property, we are now able to introduce the first two classes of the Frege hierarchy of π-institutions, which will be looked more closely at in a subsequent chapter.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, \mathbb{N}^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.

- $\mathcal{I}$ is **selfextensional** if it has the Congruence Property, i.e., if

$$\widetilde{\Lambda}(\mathcal{I}) = \widetilde{\Omega}(\mathcal{I}).$$

  Recall from Proposition 1418, that, since $\mathcal{I}$, is structural, this is equivalent to having the strong Congruence Property.

- $\mathcal{I}$ is **fully selfextensional** if every full $\mathcal{I}$-structure $\mathbb{L} = \langle \mathcal{A}, D\rangle$ has the Congruence Property, i.e., if, for all $\mathbb{L} = \langle \mathcal{A}, D\rangle \in \mathrm{FStr}(\mathcal{I})$,

$$\widetilde{\Lambda}^{\mathcal{A}}(D) = \widetilde{\Omega}^{\mathcal{A}}(D).$$

  Recall, also, from Proposition 1389 and Proposition 1418, that, since every full $\mathcal{I}$-structure is structural, this amounts to every full $\mathcal{I}$-structure having the strong Congruence Property.

We give a characterization of selfextensional π-institutions next.

Proposition 1427 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is selfextensional if and only if, for all* $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi_i, \psi_i \in \mathrm{SEN}^\flat(\Sigma)$*,* $i < k$*,*

$$C_\Sigma(\phi_i) = C_\Sigma(\psi_i),\ i < k,\quad \textit{imply}\quad C_\Sigma(\sigma^\flat_\Sigma(\phi_0, \dots, \phi_{k-1})) = C_\Sigma(\sigma^\flat_\Sigma(\psi_0, \dots, \psi_{k-1})).$$

Proof: Suppose that $\mathcal{I}$ is selfextensional and let $\sigma^\flat \in N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi_i, \psi_i \in \mathrm{SEN}^\flat(\Sigma)$, $i < k$, such that $C_\Sigma(\phi_i) = C_\Sigma(\psi_i)$, $i < k$. Thus, for all $i < k$, $\langle \phi_i, \psi_i \rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}) = \widetilde{\Omega}_\Sigma(\mathcal{I})$, by selfextensionality. Since $\widetilde{\Omega}(\mathcal{I})$ is a congruence system,

$$\langle \sigma^\flat_\Sigma(\phi_0, \dots, \phi_{k-1}), \sigma^\flat_\Sigma(\psi_0, \dots, \psi_{k-1}) \rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I}) = \widetilde{\lambda}_\Sigma(\mathcal{I}).$$

We conclude that $C_\Sigma(\sigma^\flat_\Sigma(\phi_0, \dots, \phi_{k-1})) = C_\Sigma(\sigma^\flat_\Sigma(\psi_0, \dots, \psi_{k-1}))$.

Suppose, conversely, that the displayed condition holds. Then $\widetilde{\lambda}(\mathcal{I})$ is a congruence system on $\mathbf{F}$. Therefore, by Proposition 1420, we have that $\widetilde{\Omega}(\mathcal{I}) = \widetilde{\lambda}(\mathcal{I})$. We conclude that $\mathcal{I}$ is selfextensional. ■

And also a characterization of fully selfextensional π-institutions.

Proposition 1428 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is fully selfextensional if and only if every $\mathbf{F}$-structure of the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ has the Congruence Property.*

Proof: Assume, first, that $\mathcal{I}$ is fully selfextensional. By definition, every full $\mathcal{I}$-structure has the Congruence Property. By Proposition 1390, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ is full, for every $\mathbf{F}$-algebraic system $\mathcal{A}$. Therefore, every $\mathbf{F}$-structure of the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ has the Congruence Property.

Assume, conversely, that every $\mathbf{F}$-structure of the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ has the Congruence Property. Let $\mathbb{L} = \langle \mathcal{A}, D \rangle$ be a full $\mathcal{I}$-structure. Then, by definition, the reduction morphism is a bilogical morphism

$$\langle I, \pi \rangle : \langle \mathcal{A}, D \rangle \vdash \langle \mathcal{A}^*, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*) \rangle.$$

By hypothesis, $\langle \mathcal{A}^*, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*) \rangle$ has the Congruence Property. Thus, by Proposition 1426, $\mathbb{L}$ also has the Congruence Property. Therefore, every full $\mathcal{I}$-structure has the Congruence Property and we conclude that $\mathcal{I}$ is fully selfextensional. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. Recall that the **Lindenbaum-Tarski algebraic system of** $\mathcal{I}$ is the algebraic system $\mathcal{F}/\widetilde{\Omega}(\mathcal{I})$, where $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. Recall, also, that, given a class of $\mathbf{F}$-algebraic systems, $Q(\mathsf{K})$ denotes the syntactic variety generated by K, i.e., those $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, such that $\bigcap \{\mathrm{Ker}(\mathcal{K}) : \mathcal{K} \in \mathsf{K}\} \leq \mathrm{Ker}(\mathcal{A})$. We denoted $\mathsf{Q}(\mathcal{I}) = Q(\mathcal{F}/\widetilde{\Omega}(\mathcal{I}))$, the syntactic variety generated by the Lindenbaum-Tarski $\mathbf{F}$-algebraic system of $\mathcal{I}$. Moreover, given $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, we write $\mathsf{K} \vDash_\Sigma \phi \approx \psi$ for $\langle \phi, \psi \rangle \in \bigcap_{\mathcal{K} \in \mathsf{K}} \mathrm{Ker}_\Sigma(\mathcal{K})$.

Using these conventions, we can formulate a proposition to the effect that, for a selfextensional π-institution $\mathcal{I}$, an equation is satisfied in $\mathsf{Q}(\mathcal{I})$ if and only if it is in the Frege equivalence family $\widetilde{\lambda}(\mathcal{I})$.

Proposition 1429 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is selfextensional, then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\mathsf{Q}(\mathcal{I}) \vDash_\Sigma \phi \approx \psi \quad \textit{if and only if} \quad \langle \phi, \psi \rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}).$$

Proof: By the definition of $\mathsf{Q}(\mathcal{I})$, it is easy to see that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\mathsf{Q}(\mathcal{I}) \vDash_\Sigma \phi \approx \psi \quad \text{iff} \quad \langle \phi, \psi \rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I}).$$

Since $\mathcal{I}$ is assumed selfextensional, this happens if and only if $\langle \phi, \psi \rangle \in \widetilde{\Lambda}_\Sigma(\mathcal{I})$, i.e., due to the structurality of C, if and only if $\langle \phi, \psi \rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I})$. ■

19.9.2 The Property of Conjunction

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an algebraic system, such that, in N, there exists a binary natural transformation

$$\wedge : \mathrm{SEN}^2 \to \mathrm{SEN},$$

and $\mathbb{L} = \langle \mathbf{A}, D \rangle$ a π-structure. We say that $\mathbb{L}$ has the **Conjunction Property with respect to** $\wedge$ if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$D_\Sigma(\phi \wedge_\Sigma \psi) = D_\Sigma(\phi, \psi),$$

where $\phi \wedge_\Sigma \psi := \wedge_\Sigma(\phi, \psi)$. In this case, we also say $\wedge$ is a **conjunction** for $\mathbb{L}$ and that $\mathbb{L}$ is **conjunctive**.

Lemma 1430 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *be an algebraic system, with* $\wedge : \mathrm{SEN}^2 \to \mathrm{SEN}$ *in* N*, and* $\mathbb{L} = \langle \mathbf{A}, D \rangle$ *a* π*-structure based on* $\mathbf{A}$*.* $\mathbb{L}$ *has the Conjunction Property with respect to* $\wedge$ *if and only if, for every* $T \in \mathrm{ThFam}(\mathbb{L})$*, all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,*

$$\phi \wedge_\Sigma \psi \in T_\Sigma \quad \textit{iff} \quad \phi \in T_\Sigma \textit{ and } \psi \in T_\Sigma.$$

Proof: Suppose that $\mathbb{L}$ has the Conjunction Property with respect to $\wedge$ and let $T \in \mathrm{ThFam}(\mathbb{L})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$.

If $\phi \wedge_\Sigma \psi \in T_\Sigma$, then

$$\phi \in D_\Sigma(\phi, \psi) = D_\Sigma(\phi \wedge_\Sigma \psi) \subseteq D_\Sigma(T_\Sigma) = T_\Sigma.$$

Similarly, $\psi \in T_\Sigma$.

Conversely, if $\phi, \psi \in T_\Sigma$, then

$$\phi \wedge_\Sigma \psi \in D_\Sigma(\phi \wedge_\Sigma \psi) = D_\Sigma(\phi, \psi) \subseteq D_\Sigma(T_\Sigma) = T_\Sigma.$$

Thus, $\phi \wedge_\Sigma \psi \in T_\Sigma$ if and only if $\phi, \psi \in T_\Sigma$.

Suppose, conversely, that the displayed condition in the statement is satisfied. Then, for all $\Sigma \in |\mathbf{Sign}|$, and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{rcl} D_\Sigma(\phi \wedge_\Sigma \psi) & = & \bigcap\{T_\Sigma : T \in \mathrm{ThFam}(\mathbb{L}) \text{ and } \phi \wedge_\Sigma \psi \in T_\Sigma\} \\ & = & \bigcap\{T_\Sigma : T \in \mathrm{Thfam}(\mathbb{L}) \text{ and } \phi, \psi \in T_\Sigma\} \\ & = & D_\Sigma(\phi, \psi). \end{array}$$

So $\wedge : \mathrm{SEN}^2 \to \mathrm{SEN}$ is a conjunction for $\mathbb{L}$. ■

In terms of **F**-rules one can characterize the Conjunction Property as follows.

Proposition 1431 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *based on* $\mathbf{F}$. $\mathcal{I}$ *has the Conjunction Property if and only if it satisfies*

$$p^{2,0}, p^{2,1} \vdash \wedge^\flat \circ \langle p^{2,0}, p^{2,1} \rangle, \quad \wedge^\flat \circ \langle p^{2,0}, p^{2,1} \rangle \vdash p^{2,0}, \quad \wedge^\flat \circ \langle p^{2,0}, p^{2,1} \rangle \vdash p^{2,1}.$$

Note that, in practice, we write these **F**-grules in the more familiar form

$$x, y \vdash x \wedge^\flat y, \quad x \wedge^\flat y \vdash x, \quad x \wedge^\flat y \vdash y,$$

where $x, y, z, \ldots$ stand for the corresponding projection natural transformations in $N^\flat$.

Proof: We have that $\mathcal{I}$ satisfies

$$x, y \vdash x \wedge^\flat y, \quad x \wedge^\flat y \vdash x, \quad x \wedge^\flat y \vdash y,$$

iff, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \wedge^\flat_\Sigma \psi \in C_\Sigma(\phi, \psi), \quad \phi \in C_\Sigma(\phi \wedge^\flat_\Sigma \psi), \quad \psi \in C_\Sigma(\phi \wedge^\flat_\Sigma \psi),$$

iff, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C_\Sigma(\phi \wedge^\flat_\Sigma \psi) = C_\Sigma(\phi, \psi)$$

iff $\mathcal{I}$ has the Conjunction Property with respect to $\wedge^\flat$. ■

Having the Conjunction Property is preserved under bilogical morphisms in both directions.

Proposition 1432 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system with a binary* $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$. *Suppose* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ *are* $N^\flat$*-algebraic systems* $\mathbb{L} = \langle \mathbf{A}, D \rangle$, $\mathbb{L}' = \langle \mathbf{A}', D' \rangle$ π*-structures based on* $\mathbf{A}$, $\mathbf{A}'$, *respectively, and* $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ *a bilogical morphism.* $\mathbb{L}$ *has the Conjunction Property with respect to* $\wedge$ *if and only if* $\mathbb{L}'$ *has the Conjunction Property with respect to* $\wedge'$.

Proof: This follows by Proposition 1431 and Proposition 1423. ∎

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an algebraic system and $\wedge : \mathrm{SEN}^2 \to \mathrm{SEN}$ a binary natural transformation in N. We denote by $N^\wedge$ the category of natural transformations on SEN generated by $\wedge$. Clearly, since $\wedge$ is in N, we have that $N^\wedge$ is a wide subcategory of N. Moreover, we denote

$$\mathbf{A}^\wedge = \langle \mathbf{Sign}, \mathrm{SEN}, N^\wedge \rangle$$

the algebraic system that results by taking $N^\wedge$ instead of N as its category of natural transformations. This corresponds to the well-known operation of reducing the type of an algebra.

Proposition 1433 *Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an algebraic system, with $\wedge : \mathrm{SEN}^2 \to \mathrm{SEN}$ in N, and $\mathbb{L} = \langle \mathbf{A}, D \rangle$ be a π-structure based on $\mathbf{A}$. If $\mathbb{L}$ has the Conjunction Property with respect to $\wedge$, then $\widetilde{\Lambda}^{\mathbf{A}}(D)$ is a congruence system on $\mathbf{A}^\wedge$. Moreover, for all $X \in \mathrm{SenFam}(\mathbf{A})$, $\widetilde{\Lambda}^{\mathbf{A},D}(X)$ is also a congruence system on $\mathbf{A}^\wedge$.*

Proof: It suffices to show that, for all $T \in \mathrm{ThFam}(\mathbb{L})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \phi', \psi, \psi' \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi, \phi' \rangle, \langle \psi, \psi' \rangle \in \widetilde{\Lambda}^{\mathbf{A}}_\Sigma(T) \quad \text{implies} \quad \langle \phi \wedge_\Sigma \psi, \phi' \wedge_\Sigma \psi' \rangle \in \widetilde{\Lambda}^{\mathbf{A}}_\Sigma(T).$$

To this end, suppose $\Sigma \in |\mathbf{Sign}|$, $\phi, \phi', \psi, \psi' \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \phi' \rangle$, $\langle \psi, \psi' \rangle \in \widetilde{\Lambda}^{\mathbf{A}}_\Sigma(T)$. Then, by definition, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\begin{array}{rcl} \mathrm{SEN}(f)(\phi) \in T_{\Sigma'} & \text{iff} & \mathrm{SEN}(f)(\phi') \in T_{\Sigma'}, \\ \mathrm{SEN}(f)(\psi) \in T_{\Sigma'} & \text{if} & \mathrm{SEN}(f)(\psi') \in T_{\Sigma'}. \end{array}$$

Thus, we have, by the Conjunction Property and Lemma 1430,

$$\begin{array}{rll} \mathrm{SEN}(f)(\phi \wedge_\Sigma \psi) \in T_{\Sigma'} & \text{iff} & \mathrm{SEN}(f)(\phi) \wedge_{\Sigma'} \mathrm{SEN}(f)(\psi) \in T_{\Sigma'} \\ & \text{iff} & \mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi) \in T_{\Sigma'} \\ & \text{iff} & \mathrm{SEN}(f)(\phi'), \mathrm{SEN}(f)(\psi') \in T_{\Sigma'} \\ & \text{iff} & \mathrm{SEN}(f)(\phi') \wedge_{\Sigma'} \mathrm{SEN}(f)(\psi') \in T_{\Sigma'} \\ & \text{iff} & \mathrm{SEN}(f)(\phi' \wedge_\Sigma \psi') \in T_{\Sigma'}. \end{array}$$

Thus, $\langle \phi \wedge_\Sigma \psi, \phi' \wedge_\Sigma \psi' \rangle \in \widetilde{\Lambda}^{\mathbf{A}}_\Sigma(T)$ and $\widetilde{\Lambda}^{\mathbf{A}}(T)$ is a congruence system on $\mathbf{A}^\wedge$.

The fact that $\widetilde{\Lambda}^{\mathbf{A}}(D)$ and $\widetilde{\Lambda}^{\mathbf{A},D}(X)$ are congruence systems on $\mathbf{A}^\wedge$ now follow from the relationships outlined before Lemma 1415. ∎

The Conjunction Property also satisfies a transfer property to the effect that a given π-institution has the Conjunction Property if and only if all its π-structure models have the Conjunction Property.

Proposition 1434 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *has the Conjunction Property with respect to* $\wedge^\flat$ *if and only if, for every* $\mathcal{I}$*-structure* $\mathbb{L} = \langle \mathcal{A}, D \rangle$*,* $\mathbb{L}$ *has the Conjunction Property with respect to* $\wedge$*.*

Proof: Suppose that $\mathcal{I}$ has the Conjunction Property with respect to $\wedge^\flat$. Then, by Propositions 1431 and 1422, $\mathbb{L}$ has the Conjunction Property with respect to $\wedge$.

The converse is trivial, since $\langle \mathcal{F}, C \rangle \in \mathrm{Str}(\mathcal{I})$, where $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. ∎

Finally, we show that if a π-institution $\mathcal{I}$ has the Conjunction Property, then any finitary $\mathcal{I}$-structure with the Congruence Property is full.

Proposition 1435 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, and* $\mathbb{L} = \langle \mathcal{A}, D \rangle$ *a finitary* $\mathcal{I}$*-structure, which has no theorems if* $\mathcal{I}$ *has no theorems. If* $\mathcal{I}$ *has the Conjunction Property with respect to* $\wedge^\flat$ *and* $\mathbb{L}$ *has the strong Congruence Property, then* $\mathbb{L}$ *is a full* $\mathcal{I}$*-structure.*

Proof: Suppose $\mathbb{L} = \langle \mathcal{A}, D \rangle$ is a finitary $\mathcal{I}$-structure, without theorems, if $\mathcal{I}$ has no theorems, satisfying the strong Congruence Property. Our goal is to show that $\mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$. We denote by $\langle I, \pi \rangle : \mathbb{L} \vdash \mathbb{L}^*$ the bilogical quotient morphism.

Assume, first, that $T \in \mathcal{D}^*$. Then $\pi^{-1}(T) \in \mathcal{D} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{D})$, by Proposition 1385. Hence, by Corollary 55, $T \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A}^*)$. We conclude that $\mathcal{D}^* \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$.

Conversely, assume that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$. If $T = \varnothing$, then $\mathcal{I}$ does not have theorems. Thus, by hypothesis, $\varnothing \in \mathcal{D}$ and, therefore, $\varnothing \in \mathcal{D}^*$. Suppose, next, that $T \neq \varnothing$. Let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in D^*_\Sigma(T_\Sigma)$. By hypothesis and Proposition 1365, there exist $\phi_0, \ldots, \phi_{n-1} \in T_\Sigma$, such that $\phi \in D^*_\Sigma(\phi_0, \ldots, \phi_{n-1})$. By the Conjunction Property and Proposition 1434, $\phi \in D^*_\Sigma(\phi_0 \wedge_\Sigma (\cdots \wedge_\Sigma \phi_{n-1}))$. Therefore,

$$D^*_\Sigma(\phi \wedge_\Sigma (\phi_0 \wedge_\Sigma (\cdots \wedge_\Sigma \phi_{n-1}))) = D^*_\Sigma(\phi_0 \wedge_\Sigma (\cdots \wedge_\Sigma \phi_{n-1})).$$

By hypothesis, Proposition 1426 and Corollary 1425, we get that

$$\phi \wedge_\Sigma (\phi_0 \wedge_\Sigma (\cdots \wedge_\Sigma \phi_{n-1})) = \phi_0 \wedge_\Sigma (\cdots \wedge_\Sigma \phi_{n-1}).$$

Since $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$ and $\mathcal{I}$ has the Conjunction Property with respect to $\wedge^\flat$, $\phi_0, \ldots, \phi_{n-1} \in T_\Sigma$ imply that $\phi_0 \wedge_\Sigma (\cdots \wedge \phi_{n-1}) \in T_\Sigma$. Thus, by the displayed equation above, $\phi \wedge_\Sigma (\phi_0 \wedge_\Sigma (\cdots \wedge_\Sigma \phi_{n-1})) \in T_\Sigma$. By Proposition 1434, $\phi \in T_\Sigma$. So we have $D^*_\Sigma(T_\Sigma) = T_\Sigma$ and, hence, $T \in \mathcal{D}^*$. Thus, $\mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$ and $\mathbb{L}$ is a full $\mathcal{I}$-structure. ∎

19.9.3 The Deduction-Detachment Theorem

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an algebraic system, with $\rightarrow: \mathrm{SEN}^2 \to \mathrm{SEN}$ a binary natural transformation in N, and $\mathbb{L} = \langle \mathbf{A}, D \rangle$ a π-structure.

- $\mathbb{L}$ has the **Modus Ponens** or **Detachment with respect to** $\rightarrow$ if, for all $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\phi, \psi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\phi \rightarrow_\Sigma \psi \in D_\Sigma(\Phi) \quad \text{implies} \quad \psi \in D_\Sigma(\Phi, \phi).$$

- $\mathbb{L}$ has the **Deduction Theorem with respect to** $\rightarrow$ if, for all $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\phi, \psi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\psi \in D_\Sigma(\Phi, \phi) \quad \text{implies} \quad \phi \rightarrow_\Sigma \psi \in D_\Sigma(\Phi).$$

- $\mathbb{L}$ has the **Deduction Detachment Theorem with respect to** $\rightarrow$ if it has both the Modus Ponens and the Deduction Theorem with respect to $\rightarrow$.

Structures that have the Deduction Detachment Theorem always have theorems. The following proposition gives a few of those theorems that are inspired by classical propositional calculus.

Proposition 1436 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *be an algebraic system, with* $\rightarrow: \mathrm{SEN}^2 \to \mathrm{SEN}$ *a binary natural transformation in* N*, and* $\mathbb{L} = \langle \mathbf{A}, D \rangle$ *a* π*-structure that has the Deduction Detachment Theorem with respect to* $\rightarrow$*. Then, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi, \chi \in \mathrm{SEN}(\Sigma)$*,*

(a) $\phi \rightarrow_\Sigma \phi \in \mathrm{Thm}_\Sigma(\mathbb{L})$*;*

(b) $\phi \rightarrow_\Sigma (\psi \rightarrow_\Sigma \phi) \in \mathrm{Thm}_\Sigma(\mathbb{L})$*;*

(c) $(\phi \rightarrow_\Sigma (\psi \rightarrow_\Sigma \chi)) \rightarrow_\Sigma ((\phi \rightarrow_\Sigma \psi) \rightarrow_\Sigma (\phi \rightarrow_\Sigma \chi)) \in \mathrm{Thm}_\Sigma(\mathbb{L})$*.*

Proof: Let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi, \chi \in \mathrm{SEN}(\Sigma)$.

(a) Since $\phi \in D_\Sigma(\phi)$, we get by the Deduction Theorem, $\phi \rightarrow_\Sigma \phi \in D_\Sigma(\varnothing)$. So $\phi \rightarrow_\Sigma \phi \in \mathrm{Thm}_\Sigma(\mathbb{L})$.

(b) Since $\phi \in D_\Sigma(\phi, \psi)$, we get, by the Deduction Theorem, $\psi \rightarrow_\Sigma \phi \in D_\Sigma(\phi)$. By yet another application of the Deduction Theorem, we conclude that $\phi \rightarrow_\Sigma (\psi \rightarrow_\Sigma \phi) \in D_\Sigma(\varnothing)$. Therefore, $\phi \rightarrow_\Sigma (\psi \rightarrow_\Sigma \phi) \in \mathrm{Thm}_\Sigma(\mathbb{L})$.

(c) Since $\phi \rightarrow_\Sigma \psi \in D_\Sigma(\phi \rightarrow_\Sigma \psi)$ and $\phi \rightarrow_\Sigma (\psi \rightarrow_\Sigma \chi) \in D_\Sigma(\phi \rightarrow_\Sigma (\psi \rightarrow_\Sigma \chi))$, we get, by Modus Ponens, $\psi \in D_\Sigma(\phi \rightarrow_\Sigma \psi, \phi)$ and $\psi \rightarrow_\Sigma \chi \in$

$D_\Sigma(\phi \to_\Sigma (\psi \to_\Sigma \chi), \phi)$. Moreover, since $\psi \to_\Sigma \chi \in D_\Sigma(\psi \to_\Sigma \chi)$, we get, by Modus Ponens, $\chi \in D_\Sigma(\psi \to_\Sigma \chi, \psi)$. Thus, we obtain

$$\chi \in D_\Sigma(\psi \to_\Sigma \chi, \psi) \subseteq D_\Sigma(\phi \to_\Sigma (\psi \to_\Sigma \chi), \phi \to_\Sigma \psi, \phi).$$

By the Deduction Theorem, $\phi \to_\Sigma \chi \in D_\Sigma(\phi \to_\Sigma (\psi \to_\Sigma \chi), \phi \to_\Sigma \psi)$. By another application of the Deduction Theorem, $(\phi \to_\Sigma \psi) \to_\Sigma (\phi \to_\Sigma \chi) \in D_\Sigma(\phi \to_\Sigma (\psi \to_\Sigma \chi))$. A final application of the Deduction Theorem yields

$$(\phi \to_\Sigma (\psi \to_\Sigma \chi)) \to_\Sigma ((\phi \to_\Sigma \psi) \to_\Sigma (\phi \to_\Sigma \chi)) \in D_\Sigma(\varnothing).$$

Therefore, $(\phi \to_\Sigma (\psi \to_\Sigma \chi)) \to_\Sigma ((\phi \to_\Sigma \psi) \to_\Sigma (\phi \to_\Sigma \chi)) \in \mathrm{Thm}_\Sigma(\mathbb{L})$. ∎

Corollary 1437 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *a binary natural transformation in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, which has the Deduction Detachment Theorem with respect to* $\to^\flat$*. Then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, and every* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $T \neq \varnothing$*. Consequently, for every* $\mathbb{L} = \langle \mathcal{A}, D \rangle \in \mathrm{Str}(\mathcal{I})$*,* $\mathrm{Thm}(\mathbb{L}) \neq \varnothing$*.*

Proof: Clear by Proposition 1436. ∎

We now give a characterization of the Podus Ponens property.

Proposition 1438 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *be an algebraic system, with* $\to: \mathrm{SEN}^2 \to \mathrm{SEN}$ *in* N*, and* $\mathbb{L} = \langle \mathcal{A}, D \rangle$ *a* π*-structure.* $\mathbb{L}$ *has the Modus Ponens with respect to* $\to$ *if and only if, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,*

$$\psi \in D_\Sigma(\phi, \phi \to_\Sigma \psi)$$

if and only if, for all $T \in \mathrm{ThFam}(\mathbb{L})$*, all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,*

$$\phi \in T_\Sigma \quad \textit{and} \quad \phi \to_\Sigma \psi \in T_\Sigma \quad \textit{imply} \quad \psi \in T_\Sigma.$$

Proof: Suppose, first, that $\mathbb{L}$ has the Modus Ponens with respect to $\to$ and let $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$. Then $\phi \to_\Sigma \psi \in D_\Sigma(\phi \to_\Sigma \psi)$, whence, by the Modus Ponens, $\psi \in D_\Sigma(\phi \to_\Sigma \psi, \phi)$. Conversely, suppose, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $\psi \in D_\Sigma(\phi, \phi \to_\Sigma \psi)$ and let $\Phi \subseteq \mathrm{SEN}(\Sigma)$, such that $\phi \to_\Sigma \psi \in D_\Sigma(\Phi)$. Then, we have

$$\psi \in D_\Sigma(\phi, \phi \to_\Sigma \psi) \subseteq D_\Sigma(\Phi, \phi).$$

So $\mathbb{L}$ has the Modus Ponens with respect to $\to$.

The second equivalence is straightforward. ∎

The Modus Ponens in a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ may also be characterized in terms of $\mathbf{F}$-rules.

Proposition 1439 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *has the Modus Ponens with respect to* $\to^\flat$ *if and only if it satisfies the* $\mathbf{F}$*-rule*

$$x, x \to^\flat y \vdash y.$$

Proof: $\mathcal{I}$ satisfies $x, x \to^\flat y \vdash y$ if and only if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\psi \in C_\Sigma(\phi, \phi \to^\flat_\Sigma \psi)$$

if and only if, by Proposition 1438, $\mathcal{I}$ has the Modus Ponens with respect to $\to^\flat$. ■

Since the Modus Ponens in a π-institution is expressible in terms of $\mathbf{F}$-rules, we may use Propositions 1422 and 1423 to draw the conclusions that the Modus Ponens transfers to all models and, moreover, that it is preserved by all bilogical morphisms.

Corollary 1440 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *has the Modus Ponens with respect to* $\to^\flat$*, then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and every* $\mathbb{L} = \langle \mathcal{A}, D \rangle \in \mathrm{Str}(\mathcal{I})$*,* $\mathbb{L}$ *has the Modus Ponens with respect to* $\to$*.*

Proof: This follows by combing Proposition 1439 with Proposition 1422. ■

Corollary 1441 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ *be* $N^\flat$*-algebraic systems,* $\mathbb{L} = \langle \mathbf{A}, D \rangle$*,* $\mathbb{L}' = \langle \mathbf{A}', D' \rangle$ *be* $N^\flat$*-structures based on* $\mathbf{A}$*,* $\mathbf{A}'$*, respectively, and* $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ *a bilogical morphism.* $\mathbb{L}$ *has the Modus Ponens with respect to* $\to$ *if and only if* $\mathbb{L}'$ *has the Modus Ponens with respect to* $\to'$*.*

Proof: The conclusion follows by combining Proposition 1439 with Proposition 1423. ■

It turns out that the Deduction Theorem also transfers under bilogical morphisms.

Proposition 1442 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat$ *be an algebraic system, with* $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ *be* $N^\flat$*-algebraic systems,* $\mathbb{L} = \langle \mathbf{A}, D \rangle$*,* $\mathbb{L}' = \langle \mathbf{A}', D' \rangle$ *be* $N^\flat$*-structures based on* $\mathbf{A}$*,* $\mathbf{A}'$*, respectively, and* $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ *a bilogical morphism.* $\mathbb{L}$ *has the Deduction Theorem with respect to* $\to$ *if and only if* $\mathbb{L}'$ *has the Deduction Theorem with respect to* $\to'$*.*

Proof: Suppose, first, that $\mathbb{L}$ has the Deduction Theorem with respect to $\rightarrow$ and let $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\phi, \psi\} \subseteq \mathrm{SEN}(\Sigma)$, such that

$$\alpha_\Sigma(\psi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi)).$$

Then, since $\langle F, \alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$ is a bilogical morphism, $\psi \in D_\Sigma(\Phi, \phi)$. Thus, since $\mathbb{L}$ has the Deduction Theorem, $\phi \rightarrow_\Sigma \psi \in D_\Sigma(\Phi)$. Again, by the fact that $\langle F, \alpha\rangle$ is a bilogical morphism, we obtain $\alpha_\Sigma(\phi \rightarrow_\Sigma \psi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$ or, equivalently, $\alpha_\Sigma(\phi) \rightarrow'_{F(\Sigma)} \alpha_\Sigma(\psi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$. Since $\langle F, \alpha\rangle$ is surjective, we conclude that $\mathbb{L}'$ also has the Deduction Theorem with respect to $\rightarrow'$.

Suppose, conversely, that $\mathbb{L}'$ has the Deduction Theorem with respect to $\rightarrow'$ and let $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\phi, \psi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\psi \in D_\Sigma(\Phi, \phi)$. Since $\langle F, \alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$ is a bilogical morphism, we get that

$$\alpha_\Sigma(\phi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi)).$$

Since $\mathbb{L}'$ has the Deduction Theorem with respect to $\rightarrow'$, $\alpha_\Sigma(\phi) \rightarrow'_{F(\Sigma)} \alpha_\Sigma(\psi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$ or, equivalently, $\alpha_\Sigma(\phi \rightarrow_\Sigma \psi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$. Again by the fact that $\langle F, \alpha\rangle$ is a bilogical morphism, we get $\phi \rightarrow_\Sigma \psi \in D_\Sigma(\Phi)$. Therefore, $\mathbb{L}$ also has the Deduction Theorem with respect to $\rightarrow$. ∎

In an analog of Theorem 1433, we prove that, in case a π-structure has the Deduction Detachment Theorem, with respect to a binary natural transformation, then, Frege relation systems defined on the π-structure are congruence systems if one restricts to the category of natural transformations generated by the binary natural transformation.

Proposition 1443 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system, with* $\rightarrow: \mathrm{SEN}^2 \rightarrow \mathrm{SEN}$ *in* N*, and* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *be a* π*-structure based on* $\mathbf{A}$*. If* $\mathbb{L}$ *has the Deduction Detachment Property with respect to* $\rightarrow$*, then* $\widetilde{\Lambda}^{\mathbf{A}}(D)$ *is a congruence system on* $\mathbf{A}^\rightarrow$*. Moreover, for all* $X \in \mathrm{SenFam}(\mathbf{A})$*,* $\widetilde{\Lambda}^{\mathbf{A},D}(X)$ *is also a congruence system on* $\mathbf{A}^\rightarrow$*.*

Proof: It suffices to show that, for all $X \in \mathrm{SenFam}(\mathbf{A})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \phi', \psi, \psi' \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi, \phi'\rangle, \langle \psi, \psi'\rangle \in \widetilde{\Lambda}^{\mathbf{A},D}_\Sigma(X) \quad \text{implies} \quad \langle \phi \rightarrow_\Sigma \psi, \phi' \rightarrow_\Sigma \psi'\rangle \in \widetilde{\Lambda}^{\mathbf{A},D}_\Sigma(X).$$

So, suppose $X \in \mathrm{SenFam}(\mathbf{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi, \phi', \psi, \psi' \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \phi'\rangle, \langle \psi, \psi'\rangle \in \widetilde{\Lambda}^{\mathbf{A},D}_\Sigma(X)$. Thus, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\begin{array}{rcl} D_{\Sigma'}(X_{\Sigma'}, \mathrm{SEN}(f)(\phi)) & = & D_{\Sigma'}(X_{\Sigma'}, \mathrm{SEN}(f)(\phi')), \\ D_{\Sigma'}(X_{\Sigma'}, \mathrm{SEN}(f)(\psi)) & = & D_{\Sigma'}(X_{\Sigma'}, \mathrm{SEN}(f)(\psi')). \end{array}$$

Now, using the Modus Ponens with respect to $\rightarrow$ and the displayed equations, we get

$$\mathrm{SEN}(f)(\psi) \in D_{\Sigma'}(X_{\Sigma'}, \mathrm{SEN}(f)(\phi) \rightarrow_{\Sigma'} \mathrm{SEN}(f)(\psi), \mathrm{SEN}(f)(\phi)),$$
$$\mathrm{SEN}(f)(\phi) \in D_{\Sigma'}(X_{\Sigma'}, \mathrm{SEN}(f)(\phi')).$$

Therefore,

$$\begin{array}{rcl}\mathrm{SEN}(f)(\psi') & \in & D_{\Sigma'}(X_{\Sigma'},\mathrm{SEN}(f)(\psi))\\ & \subseteq & D_{\Sigma'}(X_{\Sigma'},\mathrm{SEN}(f)(\phi)\to_{\Sigma'}\mathrm{SEN}(f)(\psi),\mathrm{SEN}(f)(\phi))\\ & \subseteq & D_{\Sigma'}(X_{\Sigma'},\mathrm{SEN}(f)(\phi)\to_{\Sigma'}\mathrm{SEN}(f)(\psi),\mathrm{SEN}(f)(\phi')).\end{array}$$

By the Deduction Theorem, we now get

$$\mathrm{SEN}(f)(\phi')\to_{\Sigma'}\mathrm{SEN}(f)(\psi')\in D_{\Sigma'}(X_{\Sigma'},\mathrm{SEN}(f)(\phi)\to_{\Sigma'}\mathrm{SEN}(f)(\psi)).$$

This is equivalent to $\mathrm{SEN}(f)(\phi'\to_\Sigma\psi')\in D_{\Sigma'}(X_{\Sigma'},\mathrm{SEN}(f)(\phi\to_\Sigma\psi))$. Thus, by symmetry, we get that, for all $\Sigma'\in|\mathbf{Sign}|$ and all $f\in\mathbf{Sign}(\Sigma,\Sigma')$, $D_{\Sigma'}(X_{\Sigma'},\mathrm{SEN}(f)(\phi\to_\Sigma\psi))=D_{\Sigma'}(X_{\Sigma'},\mathrm{SEN}(f)(\phi'\to_\Sigma\psi'))$ and, we conclude that $\langle\phi\to_\Sigma\psi,\phi'\to_\Sigma\psi'\rangle\in\widetilde{\Lambda}^{\mathbf{A},D}_\Sigma(X)$. ■

Our next goal is to show that, if a finitary π-institution $\mathcal{I}$ has the Deduction Detachment Theorem, then every full $\mathcal{I}$-structure also has the Deduction Detachment Theorem.

Theorem 1444 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system, with* $\to^\flat$: $(\mathrm{SEN}^\flat)^2\to\mathrm{SEN}^\flat$ *in* $N^\flat$, *and* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a finitary* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *has the Deduction Detachment Theorem with respect to* $\to^\flat$, *then every full* $\mathcal{I}$*-structure* $\mathbb{L}=\langle\mathcal{A},D\rangle$ *has the Deduction Detachment Theorem with respect to* $\to$.

Proof: Suppose $\mathcal{I}$ has the Deduction Detachment Theorem with respect to $\to^\flat$. By Corollary 1393, Corollary 1441 and Proposition 1442, it is enough to show that every $\mathcal{I}$-structure of the form $\langle\mathcal{A},\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ has the Deduction Detachment Theorem with respect to $\to$. By Corollary 1440, every $\mathcal{I}$-structure has the Modus Ponens with respect to $\to$. So it suffices to show that $\langle\mathcal{A},\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ has the Deduction Theorem with respect to $\to$, i.e., that, for all $\Sigma'\in|\mathbf{Sign}|$ and all $\Phi'\cup\{\phi',\psi'\}\subseteq\mathrm{SEN}(\Sigma')$,

$$\psi'\in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi',\phi')\quad\text{implies}\quad\phi'\to_{\Sigma'}\psi'\in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi').$$

We do this, using Proposition 114, by applying induction on $n<\omega$ to show that, for all $n<\omega$,

$$\psi'\in\Xi^n_{\Sigma'}(\Phi',\phi')\quad\text{implies}\quad\phi'\to_{\Sigma'}\psi'\in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi').$$

For $n=0$, we get $\psi'\in\Xi^0_{\Sigma'}(\Phi',\phi')=\Phi'\cup\{\phi'\}$.

- If $\psi'=\phi'$, then $\phi'\to_{\Sigma'}\psi'=\phi'\to_{\Sigma'}\phi'\in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\varnothing)\subseteq C^{\mathcal{A},\mathcal{I}}_{\Sigma'}(\Phi')$, because of Proposition 1436.
- If $\psi'\in\Phi'$, then $\psi'\to_{\Sigma'}(\phi'\to_{\Sigma'}\psi')\in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\varnothing)\subseteq C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi')$, again by Proposition 1436. Since $\psi'\in\Phi'\subseteq C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi')$, we get by the Modus Ponens, $\phi'\to_{\Sigma'}\psi'\in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi')$.

Assume, next that, if, for some $i < n$, $\psi' \in \Xi^i_\Sigma(\Phi', \phi')$, then $\phi' \to_{\Sigma'} \psi' \in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi')$. Consider $\psi' \in \Xi^n_{\Sigma'}(\Phi', \phi')$. By definition, there exists $\Sigma \in |\mathbf{Sign}^\flat|$, such that $F(\Sigma) = \Sigma'$, and $\Phi \cup \{\phi, \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that

$$\psi \in C_\Sigma(\Phi, \phi), \quad \alpha_\Sigma(\Phi) \subseteq \Xi^{n-1}_{\Sigma'}(\Phi', \phi'), \quad \alpha_\Sigma(\phi) = \phi', \quad \alpha_\Sigma(\psi) = \psi'.$$

Now, we have, on the one hand, by the Induction Hypothesis, $\phi' \to_{\Sigma'} \alpha_\Sigma(\chi) \in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi')$, for all $\chi \in \Phi$. On the other hand, since $\psi \in C_\Sigma(\Phi, \phi)$, we get, using Modus Ponens,

$$\psi \in C_\Sigma(\Phi, \phi) \subseteq C_\Sigma(\{\phi \to^\flat_\Sigma \chi : \chi \in \Phi\}, \phi)$$

and, therefore, by the Deduction Theorem, $\phi \to^\flat_\Sigma \psi \in C_\Sigma(\{\phi \to^\flat_\Sigma \chi : \chi \in \Phi\})$. Therefore, we obtain

$$\phi' \to_{\Sigma'} \psi' \in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\{\phi' \to_{\Sigma'} \alpha_\Sigma(\chi) : \chi \in \Phi\}).$$

Finally, we obtain

$$\begin{array}{rcl} \phi' \to_{\Sigma'} \psi' & \in & C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\{\phi' \to_{\Sigma'} \alpha_\Sigma(\chi) : \chi \in \Phi\}) \\ & \subseteq & C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi')) \\ & = & C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi'). \end{array}$$

We conclude that $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ has the Deduction Detachment Theorem and therefore, every full $\mathcal{I}$-structure does also. ∎

Finally, we show that if a π-institution has the Deduction Detachment Theorem, then every finitary $\mathcal{I}$-structure, with the Deduction Theorem and the Congruence Property is a full $\mathcal{I}$-structure.

Proposition 1445 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution that has the Deduction Detachment Theorem with respect to $\to^\flat$. If $\mathbb{L} = \langle \mathcal{A}, D\rangle$ is a finitary $\mathcal{I}$-structure with the Deduction Theorem and the strong Congruence Property, then $\mathbb{L}$ is a full $\mathcal{I}$-structure.*

Proof: Suppose $\mathbb{L} = \langle \mathcal{A}, D\rangle$ is a finitary $\mathcal{I}$-structure with the Deduction Theorem and the Congruence Property. Then, by Proposition 1385, $\mathcal{D} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and our goal is to show that $\mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$.

Suppose, first, that $T' \in \mathcal{D}^*$. Consider the bilgical quotient morphism $\langle I, \pi\rangle : \mathbb{L} \vdash \mathbb{L}^*$. Then we have $T = \pi^{-1}(T') \in \mathcal{D} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Thus, by Corollary 55, $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$. We conclude that $\mathcal{D}^* \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$.

Suppose, conversely, that $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$. Since $\mathcal{I}$ has the Deduction Detachment Theorem, by Corollary 1437, $T' \neq \varnothing$. Let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi^* \in D^*_\Sigma(T'_\Sigma)$. By the finitarity of $\mathbb{L}$ and Proposition 1365, there exist $\phi_0, \dots, \phi_{n-1} \in \mathrm{SEN}(\Sigma)$, such that $\phi^*_0, \dots, \phi^*_{n-1} \in T'_\Sigma$ and

$\phi^* \in D^*_\Sigma(\phi^*_0,\dots,\phi^*_{n-1})$. Since $\mathbb{L}$ has the Deduction Theorem, by Proposition 1442, so does $\mathbb{L}^*$, whence

$$\phi^*_0 \to^*_\Sigma (\cdots(\phi^*_{n-1} \to^*_\Sigma \phi^*)\cdots) \in D^*_\Sigma(\varnothing) = D^*_\Sigma(\phi^* \to^*_\Sigma \phi^*),$$

the last equality, by Proposition 1436. Now we get

$$D^*_\Sigma(\phi^*_0 \to^*_\Sigma (\cdots(\phi^*_{n-1} \to^*_\Sigma \phi^*)\cdots)) = D^*_\Sigma(\phi^* \to^*_\Sigma \phi^*).$$

Since $\mathbb{L}$ has the strong Congruence Property, by Proposition 1426, so does $\mathbb{L}^*$. Hence, by Corollary 1425,

$$\phi^*_0 \to^*_\Sigma (\cdots(\phi^*_{n-1} \to^*_\Sigma \phi^*)\cdots) = \phi^* \to^*_\Sigma \phi^* \in T'_\Sigma.$$

Since $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$ and $\phi^*_0,\dots,\phi^*_{n-1} \in T'_\Sigma$, we get by the Modus Ponens, $\phi^* \in T'_\Sigma$. We conclude that $D^*_\Sigma(T'_\Sigma) \subseteq T'_\Sigma$ and, therefore, $T' \in \mathcal{D}^*$. So $\mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$. Hence, $\mathbb{L} = \langle \mathcal{A}, D\rangle$ is a full $\mathcal{I}$-structure. ■

19.9.4 The Property of Disjunction

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system, with $\vee : \mathrm{SEN}^2 \to \mathrm{SEN}$ a binary natural transformation in N, and $\mathbb{L} = \langle \mathcal{A}, D\rangle$ a π-structure.

$\mathbb{L}$ has the **Disjunction Property with respect to** $\vee$ if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi,\psi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$D_\Sigma(\Phi, \phi \vee_\Sigma \psi) = D_\Sigma(\Phi,\phi) \cap D_\Sigma(\Phi,\psi).$$

In the next proposition, we discuss some of the $\mathbf{F}$(g)rules that a π-institution having the Disjunction Property satisfies.

Proposition 1446 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\vee^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ has the Disjunction Property with respect to $\vee^\flat$, then $\mathcal{I}$ satisfies the following $\mathbf{F}$-grules:*

(a) $x \vdash x \vee^\flat y$ *and* $y \vdash x \vee^\flat y$;

(b) $\dfrac{X, x \vdash z, \quad X, y \vdash z}{X, x \vee^\flat y \vdash z}$, *where X consists of any set of projections;*

(c) $x \vee^\flat y \vdash y \vee^\flat x$;

(d) $x \vdash x \vee^\flat x$ *and* $x \vee^\flat x \vdash x$.

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi,\psi,\chi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$.

(a) By the Disjunction Property, $\phi \vee^\flat_\Sigma \psi \in C_\Sigma(\phi) \cap C_\Sigma(\psi)$.

(b) Suppose $\chi \in C_\Sigma(\Phi,\phi)$ and $\chi \in C_\Sigma(\Phi,\psi)$. Then, by the Disjunction Property,

$$\chi \in C_\Sigma(\Phi,\phi) \cap C_\Sigma(\Phi,\psi) = C_\Sigma(\Phi,\phi \vee^\flat_\Sigma \psi).$$

(c) We have, using the Disjunction Property,

$$\begin{array}{rcl} \psi \vee^\flat_\Sigma \phi & \in & C_\Sigma(\psi \vee^\flat_\Sigma \phi) = C_\Sigma(\psi) \cap C_\Sigma(\phi) \\ & = & C_\Sigma(\phi) \cap C_\Sigma(\psi) = C_\Sigma(\phi \vee^\flat_\Sigma \psi). \end{array}$$

(d) We have

$$\phi \vee^\flat_\Sigma \phi \in C_\Sigma(\phi \vee^\flat_\Sigma \phi) = C_\Sigma(\phi) \cap C_\Sigma(\phi) = C_\Sigma(\phi)$$

and, also,

$$\phi \in C_\Sigma(\phi) = C_\Sigma(\phi) \cap C_\Sigma(\phi) = C_\Sigma(\phi \vee^\flat_\Sigma \phi).$$

■

It is not difficult to see that the Disjunction Property is also preserved under bilogical morphisms.

Proposition 1447 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\vee^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$, $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ *be* $N^\flat$*-algebraic systems,* $\mathbb{L} = \langle \mathbf{A}, D \rangle$, $\mathbb{L}' = \langle \mathbf{A}', D' \rangle$ *be* $N^\flat$*-structures based on* $\mathbf{A}$, $\mathbf{A}'$*, respectively, and* $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ *a bilogical morphism.* $\mathbb{L}$ *has the Disjunction Property with respect to* $\vee$ *if and only if* $\mathbb{L}'$ *has the Disjunction Property with respect to* $\vee'$*.*

Proof: Let $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\phi,\psi\} \subseteq \mathrm{SEN}(\Sigma)$. Then, since $\langle F,\alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ is a bilogical morphism, we have, using Proposition 1360,

$$\begin{array}{rcl} D_\Sigma(\Phi,\phi \vee_\Sigma \psi) & = & \alpha_\Sigma^{-1}(D'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi) \vee'_{F(\Sigma)} \alpha_\Sigma(\psi)); \\ D_\Sigma(\Phi,\phi) \cap D_\Sigma(\Phi,\psi) & = & \alpha_\Sigma^{-1}(D'_{F(\Sigma)}(\alpha_\Sigma(\Phi),\alpha_\Sigma(\phi)) \\ & & \quad \cap\, \alpha_\Sigma^{-1}(D'_{F(\Sigma)}(\alpha_\Sigma(\Phi),\alpha_\Sigma(\psi)) \\ & = & \alpha_\Sigma^{-1}(D'_{F(\Sigma)}(\alpha_\Sigma(\Phi),\alpha_\Sigma(\phi)) \\ & & \quad \cap\, D'_{F(\Sigma)}(\alpha_\Sigma(\Phi),\alpha_\Sigma(\psi))). \end{array}$$

Now, using the surjectivity of $\langle F,\alpha \rangle$, we get that

$$D_\Sigma(\Phi,\phi \vee_\Sigma \psi) = D_\Sigma(\Phi,\phi) \cap D_\Sigma(\Phi,\psi)$$

if and only if

$$\begin{array}{l} D'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi) \vee'_{F(\Sigma)} \alpha_\Sigma(\psi)) \\ \qquad = D'_{F(\Sigma)}(\alpha_\Sigma(\Phi),\alpha_\Sigma(\phi)) \cap D'_{F(\Sigma)}(\alpha_\Sigma(\Phi),\alpha_\Sigma(\psi)) \end{array}$$

Once more, taking into account the surjectivity of $\langle F,\alpha \rangle$, we conclude that $\mathbb{L}$ has the Disjunction Property with respect to $\vee$ if and only if $\mathbb{L}'$ has the Disjunction Property with respect to $\vee'$. ■

Using induction, we can extend the defining equation of the Disjunction Property so that we can accommodate a finite number of conjunctions instead of only a single conjunction.

Proposition 1448 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system, with* $\vee : \mathrm{SEN}^2 \to \mathrm{SEN}$ *in* N*, and* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *a* π*-structure. If* $\mathbb{L}$ *has the Disjunction Property with respect to* $\vee$*, then, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\Phi \cup \{\phi_0, \dots, \phi_{n-1}\} \subseteq \mathrm{SEN}(\Sigma)$*,*

$$D_\Sigma(\Phi, \phi_0 \vee_\Sigma \psi, \dots, \phi_{n-1} \vee_\Sigma \psi) = D_\Sigma(\Phi, \phi_0, \dots, \phi_{n-1}) \cap D_\Sigma(\Phi, \psi).$$

Proof: First, note that, for all $\Sigma \in |\mathbf{Sign}|$, all $\Phi \cup \{\phi_0, \dots, \phi_{n-1}, \psi\} \subseteq \mathrm{SEN}(\Sigma)$ and all $i < n$,

$$\begin{array}{rcl}\phi_i \vee_\Sigma \psi & \in & D_\Sigma(\Phi, \phi_i) \cap D_\Sigma(\Phi, \psi)\\ & \subseteq & D_\Sigma(\Phi, \phi_0, \dots, \phi_{n-1}) \cap D_\Sigma(\Phi, \psi).\end{array}$$

Thus, we get

$$D_\Sigma(\Phi, \phi_0 \vee_\Sigma \psi, \dots, \phi_{n-1} \vee_\Sigma \psi) \subseteq D_\Sigma(\Phi, \phi_0, \dots, \phi_{n-1}) \cap D_\Sigma(\Phi, \psi).$$

For the reverse inclusion, we use induction on n.

For $n = 1$, by the Disjunction Property, we get $D_\Sigma(\Phi, \phi_0) \cap D_\Sigma(\Phi, \psi) = D_\Sigma(\Phi, \phi_0 \vee_\Sigma \psi)$.

Assume that the inclusion holds for n.

Let $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\phi_0, \dots, \phi_n, \psi\} \subseteq \mathrm{SEN}(\Sigma)$. Then we have

$$\begin{array}{l}D_\Sigma(\Phi, \phi_0, \phi_1, \dots, \phi_n) \cap D_\Sigma(\Phi, \psi)\\ = D_\Sigma(\Phi, \phi_0, \phi_1, \dots, \phi_n) \cap D_\Sigma(\Phi, \psi) \cap D_\Sigma(\Phi, \psi)\\ \subseteq D_\Sigma(\Phi, \phi_0, \phi_1, \dots, \phi_n) \cap D_\Sigma(\Phi, \psi, \phi_1, \dots, \phi_n) \cap D_\Sigma(\Phi, \psi)\\ \subseteq D_\Sigma(\Phi, \phi_0 \vee_\Sigma \psi, \phi_1, \dots, \phi_n) \cap D_\Sigma(\Phi, \phi_0 \vee_\Sigma \psi, \psi)\\ \subseteq D_\Sigma(\Phi, \phi_0, \vee_\Sigma \psi, \phi_1 \vee_\Sigma \psi, \dots, \phi_n \vee_\Sigma \psi).\end{array}$$

Hence the inclusion - and, therefore, the equation - holds for all $n < \omega$. ■

Another property of structures having disjunction is that, if a specific entailment with a finite number of premises holds and the hypotheses are disjuncted with the same sentence, then the disjunction of the conclusion with the same sentence follows from the disjuncted hypotheses.

Lemma 1449 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system, with* $\vee : \mathrm{SEN}^2 \to \mathrm{SEN}$ *in* N*, and* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *a* π*-structure. If* $\mathbb{L}$ *has the Disjunction Property with respect to* $\vee$*, then, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi_0, \phi_{n-1}, \phi, \psi \in \mathrm{SEN}(\Sigma)$*,*

$$\phi \in D_\Sigma(\phi_0, \dots, \phi_{n-1}) \quad \textit{implies} \quad \phi \vee_\Sigma \psi \in D_\Sigma(\phi_0 \vee_\Sigma \psi, \dots, \phi_{n-1} \vee_\Sigma \psi).$$

Proof: Let $\Sigma \in |\mathbf{Sign}|$ and $\phi_0, \phi_{n-1}, \phi, \psi \in \mathrm{SEN}(\Sigma)$. By Proposition 1448,

$$D_\Sigma(\phi_0 \vee_\Sigma \psi, \dots, \phi_{n-1} \vee_\Sigma \psi) = D_\Sigma(\phi_0, \dots, \phi_{n-1}) \cap D_\Sigma(\psi).$$

But, by hypothesis, $\phi \in D_\Sigma(\phi_0,\dots,\phi_{n-1})$. So we get

$$\begin{array}{rcl}\phi \vee_\Sigma \psi & \in & D_\Sigma(\phi) \cap D_\Sigma(\psi)\\ & \subseteq & D_\Sigma(\phi_0,\dots,\phi_{n-1}) \cap D_\Sigma(\psi)\\ & = & D_\Sigma(\phi_0 \vee_\Sigma \psi,\dots,\phi_{n-1} \vee_\Sigma \psi).\end{array}$$

■

Our final goal is to show that, if a finitary π-institution has the Disjunction Property, then every full $\mathcal{I}$-structure also has the Disjunction Property. To accomplish this, we prove, first, a lemma to the effect that, if an entailment holds in a model then the disjunct of the conclusion with an arbitrary sentence is entailed by the same premises except for one, which is replaced by the disjunct of the original with the same sentence.

Lemma 1450 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\vee^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a finitary* π*-institution based on* $\mathbf{F}$*, which has the Disjunction Property with respect to* $\vee^\flat$*. Then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$*, all* $\Sigma' \in |\mathbf{Sign}|$ *and all* $\Phi' \cup \{\phi',\psi',\chi'\} \subseteq \mathrm{SEN}(\Sigma')$*,*

$$\chi' \in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi',\phi') \quad \textit{implies} \quad \chi' \vee_{\Sigma'} \psi' \in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi',\phi' \vee_{\Sigma'} \psi').$$

Proof: Let $\Sigma' \in |\mathbf{Sign}|$ and $\Phi' \cup \{\phi',\psi',\chi'\} \subseteq \mathrm{SEN}(\Sigma')$. By Proposition 114, $C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi',\phi') = \Xi_{\Sigma'}(\Phi',\phi') = \bigcup_{n<\omega} \Xi^n_{\Sigma'}(\Phi',\phi')$. We show by induction on $n < \omega$ that, for all $n < \omega$

$$\chi' \in \Xi^n_{\Sigma'}(\Phi',\phi') \quad \text{implies} \quad \chi' \vee_{\Sigma'} \psi' \in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi',\phi' \vee_{\Sigma'} \psi').$$

If $n = 0$, the hypothesis is $\chi' \in \Xi^0_{\Sigma'}(\Phi',\phi') = \Phi' \cup \{\phi'\}$.

- If $\chi' = \phi'$, then the conclusion follows trivially.
- Suppose that $\chi' \in \Phi'$. Then, we have $\chi' \vee_{\Sigma'} \psi' \in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\chi') \subseteq C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi') \subseteq C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi',\phi' \vee_{\Sigma'} \psi')$, where the first inclusion follows by Propositions 1446 and 1422.

Assume that the displayed implication holds for all $i < n$. Let $\chi' \in \Xi^n_{\Sigma'}(\Phi',\phi')$. By definition, there exists $\Sigma \in |\mathbf{Sign}^\flat|$, such that $F(\Sigma) = \Sigma'$, and $\phi_0,\dots,\phi_{k-1},\chi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$\chi \in C_\Sigma(\phi_0,\dots,\phi_{k-1}), \quad \alpha_\Sigma(\chi) = \chi', \quad \alpha_\Sigma(\phi_i) \in \Xi^{n-1}_{\Sigma'}(\Phi',\phi'),\ i < k.$$

By the induction hypothesis, $\alpha_\Sigma(\phi_i) \vee_{\Sigma'} \psi' \in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi',\phi' \vee_{\Sigma'} \psi')$, for all $i < k$. Note that, by the surjectivity of $\langle F,\alpha\rangle$, there exists $\psi \in \mathrm{SEN}(\Sigma)$, such that $\alpha_\Sigma(\psi) = \psi'$. Since $\chi \in C_\Sigma(\phi_0,\dots,\phi_{k-1})$, we get, by Lemma 1449, $\chi \vee^\flat_\Sigma \psi \in C_\Sigma(\phi_0 \vee^\flat_\Sigma \psi,\dots,\phi_{k-1} \vee^\flat_\Sigma \psi)$. Therefore, applying $\langle F,\alpha\rangle$,

$$\begin{array}{rcl}\chi' \vee_{\Sigma'} \psi' & \in & C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\alpha_\Sigma(\phi_0) \vee_{\Sigma'} \psi',\dots,\alpha_\Sigma(\phi_{k-1}) \vee_{\Sigma'} \psi')\\ & \subseteq & C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi',\phi' \vee_{\Sigma'} \psi').\end{array}$$

Thus, the displayed formula holds for all $n < \omega$, yielding the conclusion. ■

Finally, we show that all full models of a given finitary π-institution with the Disjunction Property also have the Disjunction Property.

Theorem 1451 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\vee^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a finitary* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *has the Disjunction Property with respect to* $\vee^\flat$*, then every full* $\mathcal{I}$*-structure* $\mathbb{L} = \langle \mathcal{A}, D \rangle$ *has the Disjunction Property with respect to* $\vee$*.*

Proof: By Corollary 1393 and Proposition 1447, it suffices to show that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi, \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{aligned} & C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi) \vee_{F(\Sigma)} \alpha_\Sigma(\psi)) \\ & \quad = C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi)) \cap C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\psi)). \end{aligned}$$

By Propositions Propositions 1446 and 1422, we have

$$\begin{aligned} & C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi) \vee_{F(\Sigma)} \alpha_\Sigma(\psi)) \\ & \quad C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi)) \cap C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\psi)). \end{aligned}$$

Conversely, suppose that, for some $\chi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\alpha_\Sigma(\chi) \in C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi)) \cap C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\psi)).$$

Then, by Lemma 1450,

$$\begin{aligned} \alpha_\Sigma(\chi) \vee_{F(\Sigma)} \alpha_\Sigma(\psi) &\in C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi) \vee_{F(\Sigma)} \alpha_\Sigma(\psi)), \\ \alpha_\Sigma(\chi) \vee_{F(\Sigma)} \alpha_\Sigma(\chi) &\in C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\psi) \vee_{F(\Sigma)} \alpha_\Sigma(\chi)). \end{aligned}$$

Now we get

$$\begin{array}{lll} \alpha_\Sigma(\chi) & \in & C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\chi) \vee_{F(\Sigma)} \alpha_\Sigma(\chi)) \\ & \subseteq & C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\psi) \vee_{F(\Sigma)} \alpha_\Sigma(\chi)) \\ & \subseteq & C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\chi) \vee_{F(\Sigma)} \alpha_\Sigma(\psi)) \\ & \subseteq & C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi) \vee_{F(\Sigma)} \alpha_\Sigma(\psi)). \end{array}$$

Taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that $\mathbb{L}$ has the Disjunction Property with respect to $\vee$. ■

19.9.5 Reductio ad Absurdum

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an algebraic system, with $\neg : \mathrm{SEN} \to \mathrm{SEN}$ a unary natural transformation in N, and $\mathbb{L} = \langle \mathbf{A}, D \rangle$ a π-structure based on $\mathbf{A}$.

$\mathbb{L}$ has the **Intuitionistic Reductio ad Absurdum with respect to** $\neg$ if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\neg_\Sigma \phi \in D_\Sigma(\Phi) \quad \text{if and only if} \quad D_\Sigma(\Phi, \phi) = \mathrm{SEN}(\Sigma).$$

$\mathbb{L}$ has the **Reductio ad Absurdum with respect to** $\neg$ if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\phi \in D_\Sigma(\Phi) \quad \text{if and only if} \quad D_\Sigma(\Phi, \neg_\Sigma \phi) = \mathrm{SEN}(\Sigma).$$

The two properties of the Reductio ad Absurdum are related.

Proposition 1452 *Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an algebraic system, with $\neg : \mathrm{SEN} \to \mathrm{SEN}$ in N, and $\mathbb{L} = \langle \mathbf{A}, D \rangle$ a π-structure based on $\mathbf{A}$. $\mathbb{L}$ has the Reductio as Absurdum with respect to $\neg$ if and only if it has the Intuitionistic Reductio ad Absurdum with respect to $\neg$ and, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $\phi \in D_\Sigma(\neg_\Sigma \neg_\Sigma \phi)$.*

Proof: Suppose, first, that $\mathbb{L}$ has the Reduction ad Absurdum with respect to $\neg$ and let $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$.

Since $\neg_\Sigma \phi \in D_\Sigma(\neg_\Sigma \phi)$, we get $D_\Sigma(\neg_\Sigma \phi, \neg_\Sigma \neg_\Sigma \phi) = \mathrm{SEN}(\Sigma)$. Therefore, $\phi \in D_\Sigma(\neg_\Sigma \neg_\Sigma \phi)$.

Suppose, next, that $\neg_\Sigma \phi \in D_\Sigma(\Phi)$. Note that, since $\phi \in D_\Sigma(\phi)$, we also have $D_\Sigma(\phi, \neg_\Sigma \phi) = \mathrm{SEN}(\Sigma)$. So, finally, $\mathrm{SEN}(\Sigma) = D_\Sigma(\phi, \neg_\Sigma \phi) \subseteq D_\Sigma(\Phi, \phi)$ and equality follows.

On the other hand, if $D_\Sigma(\Phi, \phi) = \mathrm{SEN}(\Sigma)$, then $D_\Sigma(\Phi, \neg_\Sigma \neg_\Sigma \phi) = \mathrm{SEN}(\Sigma)$, whence $\neg_\Sigma \phi \in D_\Sigma(\phi)$.

Suppose, conversely, that $\mathbb{L}$ has the Intuitionistic Reductio ad Absurdum and that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $\phi \in D_\Sigma(\neg_\Sigma \neg_\Sigma \phi)$, and let $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$.

Suppose, first, $\phi \in D_\Sigma(\Phi)$. Since $\neg_\Sigma \phi \in D_\Sigma(\neg_\Sigma \phi)$, we get $D_\Sigma(\phi, \neg_\Sigma \phi) = \mathrm{SEN}(\Sigma)$. Therefore, $\mathrm{SEN}(\Sigma) = D_\Sigma(\phi, \neg_\Sigma \phi) \subseteq D_\Sigma(\Phi, \neg_\Sigma \phi)$.

On the other hand, if $D_\Sigma(\Phi, \neg_\Sigma \phi) = \mathrm{SEN}(\Sigma)$, then $\neg_\Sigma \neg_\Sigma \phi \in D_\Sigma(\Phi)$, whence $\phi \in D_\Sigma(\neg_\Sigma \neg_\Sigma \phi) \subseteq D_\Sigma(\phi)$. ■

We also show for future reference that the Intuitionistic Reductio ad Absurdum is preserved under bilogical morphisms.

Proposition 1453 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\neg^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ in $N^\flat$, $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ be $N^\flat$-algebraic systems, $\mathbb{L} = \langle \mathbf{A}, D \rangle$, $\mathbb{L}' = \langle \mathbf{A}', D' \rangle$ be $N^\flat$-structures based on $\mathbf{A}$, $\mathbf{A}'$, respectively, and $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ a bilogical morphism. $\mathbb{L}$ has the Intuitionistic Reductio ad Absurdum with respect to $\neg$ if and only if $\mathbb{L}'$ has the Intuitionistic Reduction ad Absurdum with respect to $\neg'$.*

Proof: Suppose $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$. Then, we have

$$\begin{array}{lll}\neg_\Sigma \phi \in D_\Sigma(\Phi) & \text{iff} & \alpha_\Sigma(\neg_\Sigma \phi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \\ & \text{iff} & \neg'_{F(\Sigma)}(\alpha_\Sigma(\phi)) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)).\end{array}$$

Moreover, using the surjectivity of $\langle F, \alpha\rangle$,

$$D_\Sigma(\Phi, \phi) = \mathrm{SEN}(\Sigma) \quad \text{iff} \quad D'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi)) = \mathrm{SEN}'(F(\Sigma)).$$

Thus, the equivalence

$$\neg_\Sigma \phi \in D_\Sigma(\Phi) \quad \text{iff} \quad D_\Sigma(\Phi, \phi) = \mathrm{SEN}(\Sigma)$$

holds if and only if the equivalence

$$\neg'_{F(\Sigma)} \alpha_\Sigma(\phi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \quad \text{iff} \quad D'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi)) = \mathrm{SEN}'(F(\Sigma))$$

holds. In other words, $\mathbb{L}$ has the Intuitionistic Reduction ad Absurdum with respect to $\neg$ if and only if $\mathbb{L}'$ has the Intuitionistic Reductio ad Absurdum with respect to $\neg'$. ■

The Intuitionistic Reductio ad Absurdum is also closely related with are called inconsistent elements or inconsistencies.

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system, with $\bot : \mathrm{SEN} \to \mathrm{SEN}$ a unary natural transformation in N, and $\mathbb{L} = \langle \mathbf{A}, D\rangle$ a π-structure based on $\mathbf{A}$. $\bot$ is an **inconsistency** in $\mathbb{L}$ if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$D_\Sigma(\bot_\Sigma \phi) = \mathrm{SEN}(\Sigma).$$

Having an inconsistency is clearly expressible by an **F**-rule and, therefore, if a π-institution has an inconsistency then all its models do also.

Lemma 1454 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\bot^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\bot^\flat$ *is an inconsistency in* $\mathcal{I}$ *if and only if* $\mathcal{I}$ *satisfies the* **F***-rule* $\bot^\flat x \vdash y$*.*

Proof: We have that $\mathcal{I}$ satisfies $\bot^\flat x \vdash y$ if and only if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $\psi \in C_\Sigma(\bot^\flat_\Sigma \phi)$, if and only if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $C_\Sigma(\bot^\flat_\Sigma \phi) = \mathrm{SEN}(\Sigma)$, if and only if $\bot^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ is an inconsistency in $\mathcal{I}$. ■

Corollary 1455 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\bot^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\bot^\flat$ *is an inconsistency in* $\mathbb{L}$*, then,* $\bot$ *is an inconsistency in every* $\mathcal{I}$*-structure* $\mathbb{L} = \langle \mathcal{A}, D\rangle$*, for every* **F***-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*.*

Proof: By Lemma 1454 and Proposition 1422. ■

The following proposition exhibits the relation between the Intuitionistic Reductio ad Absurdum and inconsistencies.

Proposition 1456 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system, with* $\to : \mathrm{SEN}^2 \to \mathrm{SEN}$, *and* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *a* π*-structure that has the Deduction Detachment Theorem with respect to* $\to$. $\mathbb{L}$ *has the Intuitionistic Reductio ad Absurdum with respect to some* $\neg : \mathrm{SEN} \to \mathrm{SEN}$ *in* N *if and only if it has an inconsistency* $\bot : \mathrm{SEN} \to \mathrm{SEN}$ *in* N. *Moreover, in that case, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi \in \mathrm{SEN}(\Sigma)$,

$$D_\Sigma(\neg_\Sigma \phi) = D_\Sigma(\phi \to_\Sigma \bot_\Sigma \phi).$$

Proof: Suppose that $\mathbb{L}$ has the Deduction Detachment Theorem with respect to $\to$.

Assume, first, that $\mathbb{L}$ also has the Intuitionistic Reductio ad Absurdum with respect to $\neg$. Let $\bot : \mathrm{SEN} \to \mathrm{SEN}$ be defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, by

$$\bot_\Sigma \phi = \neg_\Sigma(\phi \to_\Sigma \phi).$$

First, note, that, since $\bot = \neg \circ \to \circ \langle p^{1,0}, p^{1,0}\rangle$ and both $\to$ and $\neg$ are in N, we get that $\bot$ is also in N. Moreover, we have, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{ll} & \neg_\Sigma(\phi \to_\Sigma \phi) \in D_\Sigma(\neg_\Sigma(\phi \to_\Sigma \phi)) \\ \text{iff} & D_\Sigma(\phi \to_\Sigma \phi, \neg_\Sigma(\phi \to_\Sigma \phi)) = \mathrm{SEN}(\Sigma) \\ & \text{(by the Intuitionistic Reductio ad Absurdum)} \\ \text{iff} & D_\Sigma(\neg_\Sigma(\phi \to_\Sigma \phi)) = \mathrm{SEN}(\Sigma). \\ & \text{(by Proposition 1436)} \end{array}$$

Thus, $\bot$ is an inconsistency in $\mathbb{L}$.

Assume, conversely, that $\bot : \mathrm{SEN} \to \mathrm{SEN}$ is an inconsistency in $\mathbb{L}$. Define $\neg : \mathrm{SEN} \to \mathrm{SEN}$, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, by

$$\neg_\Sigma \phi = \phi \to_\Sigma \bot_\Sigma \phi.$$

Since $\neg = \to \circ \langle p^{1,0}, \bot\rangle$ and, both $\to$ and $\bot$ are in N, it follows that $\neg$ is also in N. Moreover, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, we have

$$\begin{array}{lll} \neg_\Sigma \phi \in D_\Sigma(\Phi) & \text{iff} & \phi \to_\Sigma \bot_\Sigma \phi \in D_\Sigma(\Phi) \\ & \text{iff} & \bot_\Sigma \phi \in D_\Sigma(\Phi, \phi) \\ & \text{iff} & D_\Sigma(\Phi, \phi) = \mathrm{SEN}(\Sigma). \end{array}$$

Thus, $\mathbb{L}$ has the Intuitionistic Reductio ad Absurdum with respect to $\neg$.

Finally, it remains to prove the last equality. Let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. On the one hand, we have, by the Modus Ponens, $\bot_\Sigma \phi \in D_\Sigma(\phi, \phi \to_\Sigma$

$\perp_\Sigma\phi$), whence, since $\perp$ is an inconsistency, $D_\Sigma(\phi, \phi \to_\Sigma \perp_\Sigma\phi) = \mathrm{SEN}(\Sigma)$ and, hence, by the Intuitionistic Reductio ad Absurdum, $\neg_\Sigma\phi \in D_\Sigma(\phi \to_\Sigma \perp_\Sigma\phi)$. On the other hand, since $\neg_\Sigma\phi \in D_\Sigma(\neg_\Sigma\phi)$, by the Intuitionistic Reductio ad Absurdum, $D_\Sigma(\phi, \neg_\Sigma\phi) = \mathrm{SEN}(\Sigma)$ and, hence, $\neg_\Sigma\phi \in D_\Sigma(\phi, \neg_\Sigma\phi)$. Therefore, by the Deduction Theorem, $\phi \to_\Sigma \perp_\Sigma\phi \in D_\Sigma(\neg_\Sigma\phi)$. These two parts allow us to conclude that $D_\Sigma(\neg_\Sigma\phi) = D_\Sigma(\phi \to_\Sigma \perp_\Sigma\phi)$. ■

Since both the Deduction Detachment Theorem and inconsistencies are inherited by the full models of a finitary π-institution, we obtain the following concerning transference of the Intuitionistic Reductio ad Absurdum by full models.

Corollary 1457 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\to^\flat$: $(\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ and $\neg^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a finitary π-institution based on $\mathbf{F}$. If $\mathcal{I}$ has the Deduction Detachment Theorem with respect to $\to^\flat$ and the Intuitionistic Reductio ad Absurdum with respect to $\neg^\flat$, then, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, every full $\mathcal{I}$-structure $\mathbb{L} = \langle \mathcal{A}, D \rangle$ also has the Deduction Detachment Theorem with respect to $\to$ and the Intuitionistic Reductio ad Absurdum with respect to $\neg$.*

Proof: Assume the hypothesis and let $\mathbb{L} = \langle \mathcal{A}, D \rangle \in \mathrm{FStr}(\mathcal{I})$. By Theorem 1444, $\mathbb{L}$ has the Deduction Detachment Theorem with respect to $\to$. By Proposition 1456, $\perp^\flat = \neg^\flat \circ \to^\flat \circ \langle p^{1,0}, p^{1,0} \rangle$ is an inconsistency in $\mathcal{I}$. Therefore, by Corollary 1455, $\perp$ is an inconsistency in $\mathbb{L}$. Finally, we use again Proposition 1456 to conclude that $\mathbb{L}$ has both the Deduction Detachment Theorem with respect to $\to$ and the Intuitionistic Reductio ad Absurdum with respect to $\neg$. ■

19.9.6 Modality Introduction

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an algebraic system, with $\# : \mathrm{SEN} \to \mathrm{SEN}$ a unary natural transformation in N, and $\mathbb{L} = \langle \mathbf{A}, D \rangle$ a π-structure based on $\mathbf{A}$. $\mathbb{L}$ has **Modality Introduction with respect to** $\#$ if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\phi \in D_\Sigma(\Phi) \quad \text{implies} \quad \#_\Sigma\phi \in D_\Sigma(\#_\Sigma\Phi),$$

where $\#_\Sigma\Phi = \{\#_\Sigma\chi : \chi \in \Phi\}$.

It turns out that Modality Introduction is also preserved under bilogical morphisms.

Proposition 1458 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\#^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ in $N^\flat$, $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ be $N^\flat$-algebraic systems, $\mathbb{L} = \langle \mathbf{A}, D \rangle$, $\mathbb{L}' = \langle \mathbf{A}', D' \rangle$ be $N^\flat$-structures based on $\mathbf{A}$, $\mathbf{A}'$, respectively, and $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ a bilogical morphism. $\mathbb{L}$ has the Modality Introduction with respect to $\#$ if and only if $\mathbb{L}'$ has the Modality Introduction with respect to $\#'$.*

Proof: Let $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$. Then, since $\langle F, \alpha\rangle$ is a bilogical morphism, we have

$$\begin{array}{rcl}\phi \in D_\Sigma(\Phi) & \text{iff} & \alpha_\Sigma(\phi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi));\\ \#_\Sigma\phi \in D_\Sigma(\#_\Sigma\Phi) & \text{iff} & \#'_{F(\Sigma)}\alpha_\Sigma(\phi) \in D'_{F(\Sigma)}(\#'_{F(\Sigma)}\alpha_\Sigma(\Phi)).\end{array}$$

Therefore, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\phi \in D_\Sigma(\Phi) \quad \text{implies} \quad \#_\Sigma\phi \in D_\Sigma(\#_\Sigma\Phi)$$

is equivalent to

$$\alpha_\Sigma(\phi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \quad \text{implies} \quad \#'_{F(\Sigma)}\alpha_\Sigma(\phi) \in D'_{F(\Sigma)}(\#'_{F(\Sigma)}\alpha_\Sigma(\Phi)).$$

Taking into account the surjectivity of $\langle F, \alpha\rangle$, we conclude that $\mathbb{L}$ has Modality Introduction with respect to $\#$ if and only if $\mathbb{L}'$ has Modality Introduction with respect to $\#'$. ∎

We conclude this section by showing that modality introduction is inherited by all full $\mathcal{I}$-structures if $\mathcal{I}$ is a finitary π-institution possessing the property.

Proposition 1459 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\#^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a finitary* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *has Modality Introduction with respect to* $\#^\flat$*, then every full* $\mathcal{I}$*-structure* $\mathbb{L} = \langle \mathcal{A}, D\rangle$ *has the Modality Introduction with respect to* $\#$*.*

Proof: By Corollary 1393 and Proposition 1458, it suffices to show that every $\mathcal{I}$-structure of the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ has the Modality Introduction with respect to $\#$. Let $\Sigma' \in |\mathbf{Sign}|$ and $\Phi' \cup \{\phi'\} \subseteq \mathrm{SEN}(\Sigma')$. By Proposition 114, it suffices to show that

$$\phi' \in \Xi_{\Sigma'}(\Phi') \quad \text{implies} \quad \#_{\Sigma'}\phi' \in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\#_{\Sigma'}\Phi').$$

We do this by applying induction on $n < \omega$ to show that, for all $n < \omega$,

$$\phi' \in \Xi^n_{\Sigma'}(\Phi') \quad \text{implies} \quad \#_{\Sigma'}\phi' \in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\#_{\Sigma'}\Phi').$$

If $n = 0$, then the hypothesis is $\phi' \in \Xi^0_{\Sigma'}(\Phi') = \Phi'$ and the conclusion is trivial. Assume that the displayed formula holds, for all $i < n$ and assume $\Sigma' \in |\mathbf{Sign}|$ and $\Phi' \cup \{\phi'\} \subseteq \mathrm{SEN}(\Sigma')$, such that $\phi' \in \Xi^n_{\Sigma'}(\Phi')$. Then, by definition, there exists $\Sigma \in |\mathbf{Sign}^\flat|$, such that $F(\Sigma) = \Sigma'$, and $\phi_0, \ldots, \phi_{k-1}, \phi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$\phi \in C_\Sigma(\phi_0, \ldots, \phi_{k-1}), \quad \alpha_\Sigma(\phi) = \phi', \quad \alpha_\Sigma(\phi_i) \in \Xi^{n-1}_{\Sigma'}(\Phi'),\ i < k.$$

By the induction hypothesis, for all $i < k$, $\#_{\Sigma'}\alpha_\Sigma(\phi_i) \in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\#_{\Sigma'}\Phi')$. Moreover, since $\mathcal{I}$ has Modality Introduction with respect to $\#^\flat$, we get $\#^\flat_\Sigma\phi \in C_\Sigma(\#^\flat_\Sigma\phi_0,\dots,\#^\flat_\Sigma\phi_{k-1})$. Therefore, applying $\langle F,\alpha\rangle$,

$$\begin{array}{rcl}\#_{F(\Sigma)}\phi' & \in & C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\#_{F(\Sigma)}\alpha_\Sigma(\phi_0),\dots,\#_{F(\Sigma)}\alpha_\Sigma(\phi_{k-1}))\\ & \subseteq & C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\#_\Sigma\Phi').\end{array}$$

This proves the induction step and shows that $\mathbb{L}$ has the Modality Introduction with respect to $\#$. ∎

19.10 $\mathcal{I}$-Structures and Protoalgebraicity

We now work with an arbitrary π-institution $\mathcal{I}$ and look at its $\mathcal{I}$-structures and their properties. We start with a characterization of protoalgebraicity involving $\mathcal{I}$-structures.

Proposition 1460 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Then the following conditions are equivalent.*

(i) $\mathcal{I}$ is protoalgebraic;

(ii) For every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ and every $\mathcal{I}$-structure $\mathbb{L} = \langle \mathcal{A}, D\rangle$, $\widetilde{\Omega}^{\mathcal{A}}(D) = \Omega^{\mathcal{A}}(\mathrm{Thm}(\mathbb{L}))$;

(iii) For every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ and every $\mathcal{I}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, T\rangle$, $\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})) = \Omega^{\mathcal{A}}(T)$;

(iv) For every $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\Omega}(\mathcal{I}^T) = \Omega(T)$.

Proof:

(i)⇒(ii) Assume $\mathcal{I}$ is protoalgebraic and let $\mathbb{L} = \langle \mathcal{A}, D\rangle$ be an $\mathcal{I}$-structure. Then, by Proposition 1385, $\mathcal{D} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Moreover, by Theorem 179, $\Omega^{\mathcal{A}}$ is monotone on $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Hence, we get

$$\widetilde{\Omega}^{\mathcal{A}}(D) = \bigcap\{\Omega^{\mathcal{A}}(T) : T \in \mathcal{D}\} = \Omega^{\mathcal{A}}(\bigcap\mathcal{D}) = \Omega^{\mathcal{A}}(\mathrm{Thm}(\mathbb{L})).$$

(ii)⇒(iii) Follows by applying (ii) to $\mathbb{L} = \langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$.

(iii)⇒(iv) Follows by applying (iii) to $\mathcal{A} = \langle \mathcal{F}, T\rangle$, where, as usual, $\mathcal{F} = \langle \mathbf{F}, \langle I,\iota\rangle\rangle$, $\langle I,\iota\rangle : \mathbf{F} \to \mathbf{F}$ the identity morphism.

(iv)⇒(i) Suppose that, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\Omega}(\mathcal{I}^T) = \Omega(T)$ and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \le T'$. Then $T' \in \mathrm{ThFam}(\mathcal{I}^T)$, whence $\widetilde{\Omega}(\mathcal{I}^T) \le \Omega(T')$. But, by hypothesis, $\widetilde{\Omega}(\mathcal{I}^T) = \Omega(T)$. Thus, we get $\Omega(T) \le \Omega(T')$. We conclude that Ω is monotone on theory families and, therefore, $\mathcal{I}$ is protoalgebraic.

■

Recall that to a π-institution $\mathcal{I}$, we have associated two different classes of algebraic systems. On the one hand, the class $\mathrm{AlgSys}^*(\mathcal{I})$ consists of the $\mathbf{F}$-algebraic system reducts of the reduced $\mathcal{I}$-matrix families. On the other, the class $\mathrm{AlgSys}(\mathcal{I})$ consists of the $\mathbf{F}$-algebraic system reducts of the reduced full $\mathcal{I}$-structures, or, equivalently, as was shown in Proposition 1399, by the $\mathbf{F}$-algebraic system reducts of the reduced $\mathcal{I}$-structures. Under the hypothesis of protoalgebraicity, it turns out that the two classes $\mathrm{AlgSys}(\mathcal{I})$ and $\mathrm{AlgSys}^*(\mathcal{I})$ coincide.

Proposition 1461 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is protoalgebraic, then* $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$*.*

Proof: By Theorem 1404, we know that $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$. Assume, conversely, that $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle \in \mathrm{AlgSys}(\mathcal{I})$. Then, there exists, by Proposition 1399, $\mathcal{D} \in \mathrm{ClFam}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) = \Delta^{\mathcal{A}}$. Thus, by Proposition 1460, $\widetilde{\Omega}^{\mathcal{A}}(\bigcap \mathcal{D}) = \widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) = \Delta^{\mathcal{A}}$, whence, since $\bigcap \mathcal{D} \in \mathcal{D} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I})$. ■

Protoalgebraicity is strong enough to allow full $\mathcal{I}$-structures on an algebraic system to be determined by their theorem systems.

Lemma 1462 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system. If* $\mathcal{I}$ *is protoalgebraic and* $\mathbb{L} = \langle \mathcal{A}, D \rangle$*,* $\mathbb{L}' = \langle \mathcal{A}, D' \rangle$ *are full* $\mathcal{I}$ *structures based on* $\mathcal{A}$*, such that* $\mathrm{Thm}(\mathbb{L}) = \mathrm{Thm}(\mathbb{L}')$*, then* $\mathbb{L} = \mathbb{L}'$*.*

Proof: Since $\mathcal{I}$ is protoalgebraic, we have, by Proposition 1460,

$$\widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) = \Omega^{\mathcal{A}}(\mathrm{Thm}(\mathbb{L})) = \Omega^{\mathcal{A}}(\mathrm{Thm}(\mathbb{L}')) = \widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}').$$

By the Isomorphism Theorem 1408, $\widetilde{\Omega}^{\mathcal{A}} : \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order isomorphism, in particular one-to-one. So we get that $\mathbb{L} = \mathbb{L}'$. ■

For protoalgebraic π-institutions, it follows that all full $\mathcal{I}$-models have the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle = \langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{A}}(\langle \mathcal{A}, T \rangle) \rangle$, i.e., their closure systems are principal filters in the lattice of $\mathcal{I}$-filter families of the underlying algebraic system.

Theorem 1463 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is protoalgebraic if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, all* $\mathcal{I}$*-structures in* $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$ *have the form* $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}) \rangle$*, for some* $\mathfrak{A} = \langle \mathcal{A}, T \rangle \in \mathrm{MatFam}^{\mathcal{I}}(\mathcal{A})$*.*

Proof: Assume, first, that $\mathcal{I}$ is protoalgebraic and let $\mathbb{L} = \langle \mathcal{A}, D \rangle \in \mathrm{FStr}(\mathcal{I})$. Let $T = \mathrm{Thm}(\mathbb{L})$ and set $\mathfrak{A} = \langle \mathcal{A}, T \rangle$. Clearly, $\mathcal{D} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$. By protoalgebracity and Proposition 1460, $\widetilde{\Omega}^{\mathcal{A}}(\mathbb{L}) = \Omega^{\mathcal{A}}(T)$. Therefore, if $\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T)$ denotes the quotient morphism, $\langle I, \pi \rangle : \mathbb{L} \vdash \mathbb{L}^*$ is a bilogical morphism. Since $\mathbb{L} \in \mathrm{FStr}(\mathcal{I})$, $\mathcal{D}^* = \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A}^*)$. But then, if $T' \in \mathrm{ThFam}^{\mathcal{I}}(\mathfrak{A})$, $T \leq T'$, whence $\Omega^{\mathcal{A}}(T)$ is compatible with T' and, hence, by Corollary 56, $T'/\Omega^{\mathcal{A}}(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*) = \mathcal{D}^*$. Therefore, $T' = \pi^{-1}(T'/\Omega^{\mathcal{A}}(T)) \in \mathcal{D}$. So $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \mathcal{D}$.

Suppose, conversely, that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, all $\mathcal{I}$-structures in $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$ have the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}) \rangle$, for some $\mathfrak{A} = \langle \mathcal{A}, T \rangle \in \mathrm{MatFam}^{\mathcal{I}}(\mathcal{A})$. Let $T, T' \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \leq T'$. Since, by Theorem 1404, $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$, $\Omega^{\mathcal{A}}(T) \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$. Therefore, by the Isomorphism Theorem 1408, there exists $\mathbb{L} = \langle \mathcal{A}, \mathcal{D} \rangle \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) = \Omega^{\mathcal{A}}(T)$. Let $\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T)$ be the quotient morphism. Then $\langle I, \pi \rangle : \mathbb{L} \vdash \mathbb{L}^*$ is a bilogical morphism and, since $\mathbb{L}$ is full, $\mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$. Thus, $T = \pi^{-1}(T/\Omega^{\mathcal{A}}(T)) \in \mathcal{D}$. By hypothesis, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$, for some $\mathfrak{A} = \langle \mathcal{A}, T'' \rangle \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Thus, $T'' \leq T'$, whence $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}) = \mathcal{D}$. We now get $\Omega^{\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) \leq \Omega^{\mathcal{A}}(T')$. So $\Omega^{\mathcal{A}}$ is monotone on $\mathcal{A}$. Since $\mathcal{A}$ was arbitrary, we conclude that $\mathcal{I}$ is protoalgebraic. ■

We proved that, for a protoalgebraic π-institution $\mathcal{I}$, all full $\mathcal{I}$-structures have the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle$, for some $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. We seek now to characterize those $\mathcal{I}$-filter families T for which the pair $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle$ is a full $\mathcal{I}$-model, i.e., those $\mathcal{I}$-filter families T that give rise, through the principal filters they determine in $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})$ to full $\mathcal{I}$-structures.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system. We define

$$\mathrm{FiFam}^{\mathcal{I},f}(\mathcal{A}) = \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T) \rangle \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})\}.$$

Using the Isomorphism Theorem 1408, it is not difficult to see that, under protoalgebraicity, there exists an order isomorphism between the poset determined by $\mathrm{FiFam}^{\mathcal{I},f}(\mathcal{A})$ and the lattice of all $\mathcal{I}$-congruence systems on $\mathcal{A}$.

Proposition 1464 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is protoalgebraic, then, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,*

$$\Omega^{\mathcal{A}} : \mathbf{FiFam}^{\mathcal{I},f}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism.

Proof: Consider the mapping $T \mapsto \langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^T) \rangle$. This is a mapping from $\mathrm{FiFam}^{\mathcal{I},f}(\mathcal{A})$ into $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$, by the definition of $\mathrm{FiFam}^{\mathcal{I},f}(\mathcal{A})$. Clearly, it is one-to-one and both order preserving and order reflecting. If $\mathcal{I}$ is protoalgebraic, by Theorem 1463, it is also surjective. Hence, it is an order isomorphism. By the Isomorphism Theorem 1408, $\widetilde{\Omega}^{\mathcal{A}} : \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is also an order isomorphism. Thus, $T \mapsto \widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T)$ is an order isomorphism from $\mathrm{FiFam}^{\mathcal{I},f}(\mathcal{A})$ onto $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$. By Protoalgebraicity and Proposition 1460, we have $\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T) = \Omega^{\mathcal{A}}(T)$ and. moreover, by Proposition 1461, $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}) = \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. Hence, we conclude that $\Omega^{\mathcal{A}} : \mathbf{FiFam}^{\mathcal{I},f}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order-isomorphism. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system. Define

$$\sim^{\mathcal{I},\mathcal{A}} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^2$$

by setting, for all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$T \sim^{\mathcal{I},\mathcal{A}} T' \quad \text{iff} \quad \Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T'),$$

i.e., $\sim^{\mathcal{I},\mathcal{A}}$ is the kernel of the Leibniz operator on $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

It is clear from the definition that $\sim^{\mathcal{I},\mathcal{A}}$ is an equivalence relation on $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. In case $\mathcal{I}$ is protoalgebraic, we have another important property.

Lemma 1465 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, SEN^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system. If $\mathcal{I}$ is protoalgebraic, then each equivalence class of $\sim^{\mathcal{I},\mathcal{A}}$ has a minimum element.*

Proof: Let $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and consider the equivalence class $[T]$ of T under $\sim^{\mathcal{I},\mathcal{A}}$. Then we have $\bigcap [T] \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and, moreover,

$$\begin{array}{rcl}\Omega^{\mathcal{A}}(\bigcap [T]) & = & \bigcap \{\Omega^{\mathcal{A}}(T'); T' \in [T]\} \\ & = & \bigcap \{\Omega^{\mathcal{A}}(T) : T' \in [T]\} \\ & = & \Omega^{\mathcal{A}}(T).\end{array}$$

So $\bigcap [T] \in [T]$ and, therefore, $\bigcap [T]$ is the minimum element of $[T]$. ∎

The next proposition provides the promised characterization of those $\mathcal{I}$-filter families that determine full $\mathcal{I}$-structures through their principal filters in $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})$, in the case of a protoalgebraic $\mathcal{I}$.

Proposition 1466 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a protoalgebraic π-institution based on $\mathbf{F}$. For every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and every $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, the following conditions are equivalent:*

(i) $T \in \mathrm{FiFam}^{\mathcal{I},f}(\mathcal{A})$, *i.e.*, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle \in \mathrm{FStr}(\mathcal{I})$;

(ii) $T = \min [T]$, *where $[T]$ is the equivalence class of T under $\sim^{\mathcal{I},\mathcal{A}}$;*

(iii) $T/\Omega^{\mathcal{A}}(T) = \min \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T))$.

Proof:

(ii)⇒(iii) Assume that $T = \min [T]$ and let $Y \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T))$. Our goal is to show that $T/\Omega^{\mathcal{A}}(T) \le Y$. Consider the quotient morphism $\langle I, \pi\rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T)$ and let $X = \pi^{-1}(Y) \cap T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then

$$X = \pi^{-1}(Y) \cap \pi^{-1}(\pi(T)) = \pi^{-1}(Y \cap \pi(T)).$$

It follows that $\Omega^{\mathcal{A}}(T)$ is compatible with X and, hence, $\Omega^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(X)$. But, by definition, $X \le T$ and, thus, by protoalgebraicity, $\Omega^{\mathcal{A}}(X) \le \Omega^{\mathcal{A}}(T)$. We conclude that $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(X)$ and, hence, $T \sim^{\mathcal{I},\mathcal{A}} X$. By hypothesis, we now get $T \le \pi^{-1}(Y)$, i.e., $T/\Omega^{\mathcal{A}}(T) \le X$. Since $Y \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T))$ was arbitrary, we conclude that $T/\Omega^{\mathcal{A}}(T) = \min \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T))$.

(iii)⇒(i) Suppose that $T/\Omega^{\mathcal{A}}(T) = \min \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T))$. By protoalgebraicity (see the Correspondence Theorem 1336),

$$\pi : \mathbf{FiFam}^{\mathcal{I}}(\langle\mathcal{A}, T\rangle) \cong \mathbf{FiFam}^{\mathcal{I}}(\langle\mathcal{A}/\Omega^{\mathcal{A}}(T), T/\Omega^{\mathcal{A}}(T)\rangle).$$

By hypothesis,

$$\mathrm{FiFam}^{\mathcal{I}}(\langle\mathcal{A}/\Omega^{\mathcal{A}}(T), T/\Omega^{\mathcal{A}}(T)\rangle) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T)).$$

Since, by Proposition 1460, $\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\langle\mathcal{A}, T\rangle)) = \Omega^{\mathcal{A}}(T)$, we obtain

$$\mathrm{FiFam}^{\mathcal{I}}(\langle\mathcal{A}, T\rangle)^* = \mathrm{FiFam}^{\mathcal{I}}(\langle\mathcal{A}, T\rangle^*).$$

This proves that $\langle\mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\langle\mathcal{A}, T\rangle)\rangle$ is a full $\mathcal{I}$-structure. We conclude that $T \in \mathrm{FiFam}^{\mathcal{I},f}(\mathcal{A})$.

(i)⇒(ii) Suppose $T \in \mathrm{FiFam}^{\mathcal{I},f}(\mathcal{A})$. Since $\mathcal{I}$ is protoalgebraic, by Lemma 1465, there exists $T' = \min [T]$. By the proofs of the two preceding implications (ii)⇒(iii)⇒(i), $\langle\mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T'}\rangle$ is a full $\mathcal{I}$-structure. But, by hypothesis, $\langle\mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T}\rangle$ is also a full $\mathcal{I}$-structure. Now observe that, by Proposition 1460,

$$\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T'}) = \Omega^{\mathcal{A}}(T') = \Omega^{\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T}).$$

By the Isomorphism Theorem 1408, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T'} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T}$ and, therefore, $T' = T$. So we conclude that $T = \min [T]$.

■

Since, for a protoalgebraic π-institution $\mathcal{I}$, the filter families determining full $\mathcal{I}$-structures are the ones that are minimal in their equivalence classes under $\sim^{\mathcal{I},\mathcal{A}}$, we can easily conclude that the class of those filter families consists of all filter families just in case all equivalence classes are singletons.

Proposition 1467 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a protoalgebraic* π*-institution based on* $\mathbf{F}$*. Then* $\mathrm{FiFam}^{\mathcal{I},f}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *(i.e., for all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle$ *is a full* $\mathcal{I}$*-structure) if and only if* $\Omega^{\mathcal{A}}$ *is injective on* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

Proof: By Proposition 1466, $\mathrm{FiFam}^{\mathcal{I},f}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ if and only if, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T = \min[T]$, if and only if, for all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$ implies $T = T'$, if and only if $\Omega^{\mathcal{A}}$ is injective on $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. ■

Recall that a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is called **weakly family algebraizable**, or **WF algebraizable** for short, if the Leibniz operator Ω is monotone and injective on the theory families of $\mathcal{I}$. Equivalently, by Theorem 295, $\mathcal{I}$ is WF algebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, the Leibniz operator on $\mathcal{A}$ is monotone and injective on $\mathcal{I}$-filter families.

The following theorem provides additional characterizations in terms of $\mathcal{I}$-structures and $\mathcal{I}$-congruence systems.

Theorem 1468 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then the following conditions are equivalent.*

- *(i)* $\mathcal{I}$ *is protoalgebraic and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $T/\Omega^{\mathcal{A}}(T) = \min \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T))$*;*
- *(ii) For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\Omega^{\mathcal{A}}$ *is monotone and injective on* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*;*
- *(iii) For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $T \mapsto \langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle$ *is a bijection between* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and* $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$ *and, hence, an order isomorphism from* $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})$ *to* $\mathbf{FStr}^{\mathcal{I}}(\mathcal{A})$*;*
- *(iv) For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\Omega^{\mathcal{A}} : \mathbf{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}}(\mathcal{A})$ *is an order isomorphism;*
- *(v) For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\Omega^{\mathcal{A}} : \mathbf{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is an order isomorphism.*

Proof:

(i)⇔(ii) By Propositions 1466 and 1467.

(i)⇒(iii) It is clear that $T \to \langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle$ is injective. By Proposition 1466, it is into $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$ and, by Theorem 1463, it is onto $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$. Hence it is a bijection, as claimed.

(iii)⇒(iv) Since, by hypothesis, every full $\mathcal{I}$-structure is of the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle$, by Theorem 1463, $\mathcal{I}$ is protoalgebraic. The composition of the given isomorphism

$$\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A}) \cong \mathbf{FStr}^{\mathcal{I}}(\mathcal{A})$$

with the isomorphism established in the Isomorphism Theorem 1408,

$$\widetilde{\Omega}^{\mathcal{A}} : \mathbf{FStr}^{\mathcal{I}}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}}(\mathcal{A})$$

gives an isomorphism

$$\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A}) \cong \mathbf{ConSys}^{\mathcal{I}}(\mathcal{A}),$$

which by protoalgebraicity and Proposition 1460 is identical to the Leibniz operator.

(iv)⇒(v) By Corollary 1405, $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$. Thus, $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}) \subseteq \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$. By the hypothesis, every $\theta \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is of the form $\Omega^{\mathcal{A}}(T)$, for some $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Therefore, $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}) \subseteq \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$.

(v)⇒(ii) is trivial.

■

In the context of weakly family algebraizable π-institutions, we look, also at the local continuity of the Leibniz and the Tarski operators.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system.

- $\Omega^{\mathcal{A}}$ is **continuous** if, for every directed collection $\{T^i : i \in I\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\bigcup_{i \in I} T^i \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we have

$$\Omega^{\mathcal{A}}(\bigcup_{i \in I} T^i) = \bigcup_{i \in I} \Omega^{\mathcal{A}}(T^i).$$

- $\widetilde{\Omega}^{\mathcal{A}}$ is **continuous** if, for every directed family $\{\mathbb{L}^i : i \in I\} \subseteq \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$,

$$\widetilde{\Omega}^{\mathcal{A}}(\sup\{\mathbb{L}^i : i \in I\}) = \bigcup_{i \in I} \widetilde{\Omega}^{\mathcal{A}}(\mathbb{L}^i).$$

Proposition 1469 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a finitary π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is weakly family algebraizable, then, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ is continuous if and only if $\widetilde{\Omega}^{\mathcal{A}}$ is continuous.*

Proof: Let $\Phi : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$ be the bijection of Theorem 1468. Then, by Proposition 1460,

$$\Omega^{\mathcal{A}} = \widetilde{\Omega}^{\mathcal{A}} \circ \Phi \quad \text{and} \quad \widetilde{\Omega}^{\mathcal{A}} = \Omega^{\mathcal{A}} \circ \Phi^{-1}.$$

Suppose, first, that $\Omega^{\mathcal{A}}$ is continuous and let $\{\mathbb{L}^i : i \in I\} \subseteq \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$ be directed. If $T^i = \Phi^{-1}(\mathbb{L}^i)$, $i \in I$, then $\{T^i : i \in I\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is also directed. Directedness implies local directedness and, therefore, by Proposition 112, $\bigcup_{i \in I} T^i \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Now we get

$$\Phi(\bigcup_{i \in I} T^i) = \langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{\bigcup_{i \in I} T^i} \rangle = \langle \mathcal{A}, \bigcap_{i \in I} \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T^i} \rangle,$$

whence $\Phi(\bigcup_{i \in I} T^i) = \sup_{i \in I} \mathbb{L}^i$. Therefore, we get

$$\widetilde{\Omega}^{\mathcal{A}}(\sup_{i \in I} \mathbb{L}^i) = (\Omega^{\mathcal{A}} \circ \Phi^{-1})(\Phi(\bigcup_{i \in I} T^i)) = \Omega^{\mathcal{A}}(\bigcup_{i \in I} T^i) = \bigcup_{i \in I} \Omega^{\mathcal{A}}(T^i) = \bigcup_{i \in I} \widetilde{\Omega}^{\mathcal{A}}(\mathbb{L}^i).$$

So $\widetilde{\Omega}^{\mathcal{A}}$ is also continuous.

Assume, conversely, $\widetilde{\Omega}^{\mathcal{A}}$ is continuous. Let $\{T^i : i \in I\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ be directed. Then $\{\Phi(T^i) : i \in I\} \subseteq \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$ is also directed and we have

$$\Omega^{\mathcal{A}}(\bigcup_{i \in I} T^i) = \widetilde{\Omega}^{\mathcal{A}}(\Phi(\bigcup_{i \in I} T^i)) = \widetilde{\Omega}^{\mathcal{A}}(\sup_{i \in I} \mathbb{L}^i) = \bigcup_{i \in I} \widetilde{\Omega}^{\mathcal{A}}(\mathbb{L}^i) = \bigcup_{i \in I} \Omega^{\mathcal{A}}(T^i).$$

Therefore $\Omega^{\mathcal{A}}$ is also continuous. ■

19.11 $\mathcal{I}$-Structures and Fregeanity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is called **Fregean** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, the π-structure $\mathcal{I}^T$ has the Congruence Property, i.e., for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\widetilde{\Lambda}^{\mathcal{I}}(T) = \widetilde{\Omega}^{\mathcal{I}}(T).$$

Clearly, $\mathcal{I}$ is Fregean if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, if, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$C_{\Sigma'}(T_{\Sigma'}, \mathrm{SEN}^\flat(f)(\phi)) = C_{\Sigma'}(T_{\Sigma'}, \mathrm{SEN}^\flat(f)(\psi)),$$

then, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$C_{\Sigma'}(T_{\Sigma'}, \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi})) = C_{\Sigma'}(T_{\Sigma'}, \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi})).$$

$\mathcal{I}$ is called **strongly Fregean** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, the π-structure $\mathcal{I}^T$ has the strong Congruence Property, i.e., for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\widetilde{\lambda}^{\mathcal{I}}(T) = \widetilde{\Omega}^{\mathcal{I}}(T).$$

In this case, we get that $\mathcal{I}$ is strongly Fregean if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C_\Sigma(T_\Sigma, \phi) = C_\Sigma(T_\Sigma, \psi)$$

implies, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$C_{\Sigma'}(T_{\Sigma'},\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi})) = C_{\Sigma'}(T_{\Sigma'},\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi})).$$

A consequence of strong Fregeanity is that every reduced matrix family model has either an empty filter family or a filter family all of whose components are singletons.

Proposition 1470 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is strongly Fregean, then, for every reduced* $\mathcal{I}$*-matrix family* $\mathfrak{A} = \langle \mathcal{A}, T\rangle$*, with* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ *and* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*, we have, for all* $\Sigma \in |\mathbf{Sign}|$*,* $|T_\Sigma| = 0$*, or, for all* $\Sigma \in |\mathbf{Sign}|$*,* $|T_\Sigma| = 1$*.*

We abbreviate the first disjunct of the conclusion, as usual, by $T = \varnothing$ and the second by writing $|T| = 1$.

Proof: Assume that $T \neq \varnothing$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\alpha_\Sigma(\phi),\alpha_\Sigma(\psi) \in T_{F(\Sigma)}$. Then $\phi,\psi \in \alpha^{-1}_\Sigma(T_{F(\Sigma)})$. Hence,

$$C_\Sigma(\alpha^{-1}_\Sigma(T_{F(\Sigma)}),\phi) = C_\Sigma(\alpha^{-1}_\Sigma(T_{F(\Sigma)}),\psi),$$

i.e., $\langle\phi,\psi\rangle \in \widetilde{\lambda}^{\mathcal{I}}_\Sigma(\alpha^{-1}(T))$. By strong Fregeanity, $\langle\phi,\psi\rangle \in \widetilde{\Omega}^{\mathcal{I}}_\Sigma(\alpha^{-1}(T))$. Thus, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\begin{aligned}&C_{\Sigma'}(\alpha^{-1}_{\Sigma'}(T_{F(\Sigma')}),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}))\\ &\qquad = C_{\Sigma'}(\alpha^{-1}_{\Sigma'}(T_{F(\Sigma')}),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi})).\end{aligned}$$

Now we get

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}) \in \alpha^{-1}_{\Sigma'}(T_{F(\Sigma')}) \quad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}) \in \alpha^{-1}_{\Sigma'}(T_{F(\Sigma')}).$$

Equivalently,

$$\begin{aligned}&\sigma_{F(\Sigma')}(\mathrm{SEN}(F(f))(\alpha_\Sigma(\phi)),\alpha_{\Sigma'}(\vec{\chi})) \in T_{F(\Sigma')}\\ &\qquad \text{iff} \quad \sigma_{F(\Sigma')}(\mathrm{SEN}(F(f))(\alpha_\Sigma(\psi)),\alpha_{\Sigma'}(\vec{\chi})) \in T_{F(\Sigma')}.\end{aligned}$$

Taking into account the surjectivity of $\langle F,\alpha\rangle$, we get that $\langle\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\rangle \in \Omega^{\mathcal{A}}_{F(\Sigma)}(T) = \Delta^{\mathcal{A}}_{F(\Sigma)}$, the last equation holding since $\mathfrak{A}$ is reduced. Hence, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$ and, therefore, for all $\Sigma \in |\mathbf{Sign}|$, $|T_\Sigma| = 1$. ■

Of course, in the case of Fregeanity and protoalgebraicity, the role of the Tarski operator may be substituted by the Leibniz operator.

Proposition 1471 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) *$\mathcal{I}$ is protoalgebraic and Fregean if and only if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) = \widetilde{\Lambda}^{\mathcal{I}}(T)$;*

(b) *$\mathcal{I}$ is protoalgebraic and strongly Fregean if and only if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) = \widetilde{\lambda}^{\mathcal{I}}(T)$.*

Proof: We only prove Part (a), since Part (b) can be proven by following a similar reasoning.

If $\mathcal{I}$ is protoalgebraic, then, by Proposition 1460, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) = \widetilde{\Omega}^{\mathcal{I}}(T)$. If $\mathcal{I}$ is Fregean, then ,by definition, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\Omega}^{\mathcal{I}}(T) = \widetilde{\Lambda}^{\mathcal{I}}(T)$. Therefore, if $\mathcal{I}$ is protoalgebraic and Fregean, then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) = \widetilde{\Lambda}^{\mathcal{I}}(T)$.

Assume, conversely, that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, we have $\Omega(T) = \widetilde{\Lambda}^{\mathcal{I}}(T)$. Then, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$, we have

$$\Omega(T) = \widetilde{\Lambda}^{\mathcal{I}}(T) \stackrel{\text{Lemma 1416}}{\leq} \widetilde{\Lambda}^{\mathcal{I}}(T') = \Omega(T').$$

Thus, Ω is monotone on $\mathrm{ThFam}(\mathcal{I})$ and $\mathcal{I}$ is protoalgebraic. Moreover, since, by Proposition 1460, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\Omega}^{\mathcal{I}}(T) = \Omega(T)$, we get $\widetilde{\Omega}^{\mathcal{I}}(T) = \widetilde{\Lambda}^{\mathcal{I}}(T)$ and, therefore, $\mathcal{I}$ is also Fregean. ■

Recall that a π-institution $\mathcal{I}$ is self extensional if

$$\widetilde{\Omega}(\mathcal{I}) = \widetilde{\Lambda}(\mathcal{I}) \ (= \widetilde{\lambda}(\mathcal{I})).$$

It turns out that Fregeanity (and, therefore, strong Fregeanity) implies self extensionality.

Corollary 1472 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is Fregean, then it is self extensional.*

Proof: We have

$$\begin{array}{rcll}\widetilde{\Omega}(\mathcal{I}) & = & \widetilde{\Omega}^{\mathcal{I}}(\mathrm{Thm}(\mathcal{I})) & \text{(by definition)}\\ & = & \widetilde{\Lambda}^{\mathcal{I}}(\mathrm{Thm}(\mathcal{I})) & \text{(by Fregeanity)}\\ & = & \widetilde{\Lambda}(\mathcal{I}). & \text{(by definition)}\end{array}$$

So $\mathcal{I}$ is self extensional. ■

If a π-institution $\mathcal{I}$ is strongly Fregean and has theorems, then the mapping $T \mapsto \mathcal{I}^T$ establishes an order embedding from the lattice of the theory families of $\mathcal{I}$ into the lattice of full $\mathcal{I}$-structures on the algebraic system $\mathcal{F}$.

Proposition 1473 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is strongly Fregean with theorems, then $T \mapsto \mathcal{I}^T$ is an order embedding of $\mathbf{ThFam}(\mathcal{I})$ into $\mathbf{FStr}^{\mathcal{I}}(\mathcal{F})$.*

Proof: We start by showing that the proposed mapping is indeed well-defined into $\mathrm{FStr}^{\mathcal{I}}(\mathcal{F})$, i.e., that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\mathcal{I}^T = \langle \mathcal{F}, \mathrm{ThFam}(\mathcal{I})^T \rangle$ is a full $\mathcal{I}$-structure. To this end, let $T \in \mathrm{ThFam}(\mathcal{I})$ and set $\theta = \widetilde{\Omega}^{\mathcal{I}}(T) = \widetilde{\lambda}^{\mathcal{I}}(T)$. To verify that $\mathcal{I}^T$ is a full $\mathcal{I}$-structure, it suffices to show that $\mathrm{ThFam}(\mathcal{I}^T)/\theta = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}/\theta)$.

If $T' \in \mathrm{ThFam}(\mathcal{I}^T)$, then, by definition of θ, θ is compatible with T'. Therefore, by Corollary 56, $T'/\theta \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}/\theta)$. Thus, $\mathrm{ThFam}(\mathcal{I}^T)/\theta \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}/\theta)$.

If, on the other hand, $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}/\theta)$, then, setting $\langle I, \pi \rangle : \mathcal{F} \to \mathcal{F}/\theta$ the quotient morphism, we have, by Corollary 55, $\pi^{-1}(T') \in \mathrm{Fifam}^{\mathcal{I}}(\mathcal{F})$, i.e., $\pi^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$. We also have, taking into account that $\mathcal{I}$ has theorems, that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi \in T_\Sigma$ and $\psi \in T_\Sigma \cap \pi_\Sigma^{-1}(T'_\Sigma)$,

$$\langle \phi, \psi \rangle \in \widetilde{\lambda}^{\mathcal{I}}_\Sigma(T) = \theta_\Sigma.$$

So $\pi_\Sigma(\phi) = \pi_\Sigma(\psi)$, whence $\phi \in \pi_\Sigma^{-1}(\pi_\Sigma(\psi)) \in \pi_\Sigma^{-1}(T'_\Sigma)$. Since $\phi \in T_\Sigma$ was arbitrary, $T \le \pi^{-1}(T')$ and, hence, $\pi^{-1}(T') \in \mathrm{ThFam}(\mathcal{I}^T)$. This shows that $T' \in \mathrm{ThFam}(\mathcal{I}^T)/\theta$ and allows us to conclude that $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}/\theta) \subseteq \mathrm{ThFam}(\mathcal{I}^T)/\theta$.

As for the rest, everything follows, since $T \mapsto \mathcal{I}^T$ is clearly one-to-one and both order preserving and order reflecting. ∎

If one adds protoalgebraicity into the mix, then the order embedding of Proposition 1473 becomes an order isomorphism.

Proposition 1474 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a strongly Fregean protoalgebraic* π*-institution with theorems, based on* $\mathbf{F}$*. Then* $T \mapsto \mathcal{I}^T$ *is an isomorphism between* $\mathbf{ThFam}(\mathcal{I})$ *and* $\mathbf{FStr}^{\mathcal{I}}(\mathcal{F})$*.*

Proof: By Proposition 1473, it suffices to show that the mapping $T \mapsto \mathcal{I}^T$ is also onto $\mathrm{FStr}^{\mathcal{I}}(\mathcal{F})$. The latter follows from Theorem 1463. ∎

Strong Fregeanity, protoalgebraicity and the existence of theorems have very strong consequences for a π-institution. They ensure that the π-institution is weakly family algebraizable, that the Leibniz operator is continuous (in case of finitarity) and that all reduced matrix families have singleton filter families.

Proposition 1475 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a strongly Fregean protoalgebraic* π*-institution with theorems.*

(a) $\mathcal{I}$ *is family injective and hence weakly family algebraizable;*

(b) If $\mathcal{I}$ *is finitary, then* Ω *is locally continuous;*

(c) For every $\mathfrak{A} = \langle \mathcal{A}, T \rangle \in \mathrm{MatFam}^*(\mathcal{I})$, $|T| = 1$.

Proof: By Proposition 1460, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) = \widetilde{\Omega}^{\mathcal{I}}(T)$. Thus, composing the mapping $T \mapsto \mathcal{I}^T$ of Proposition 1474, with the isomorphism of Theorem 1408, we obtain an isomorphism $\Omega : \mathbf{ThFam}(\mathcal{I}) \to \mathbf{ConSys}^{\mathcal{I}}(\mathcal{F})$. By Proposition 1471, $\Omega = \widetilde{\lambda}^{\mathcal{I}}$ and, hence, by Proposition 1419, Ω is locally continuous. Finally, if $\mathfrak{A} = \langle \mathcal{A}, T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$, then, since $\mathcal{I}$ has theorems, $T \neq \varnothing$ and, therefore, by Proposition 1470, $|T| = 1$. ■

We saw in Corollary 1472 that Fregeanity implies self extensionality. On the other hand, even though we cannot prove that strong Fregeanity, coupled with protoalgebraicity, are strong enough to guarantee full self extensionality, we can show that they imply a weaker property, namely a version of full self extensionality applying only to full $\mathcal{I}$-structures with isomorphic functor components. We start with a technical lemma.

Lemma 1476 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a strongly Fregean protoalgebraic π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ an $\mathbf{F}$-algebraic system, with $F : \mathbf{Sign}^\flat \to \mathbf{Sign}$ an isomorphism. Then, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,*

$$\Omega^{\mathcal{A}}(T) = \widetilde{\lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T).$$

Proof: Note that, by protoalgebraicity, $\Omega^{\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T)$. Therefore, by compatibility, $\Omega^{\mathcal{A}}(T) \leq \widetilde{\lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T)$. It therefore suffices to show the reverse inclusion. To this end and taking into account the surjectivity of $\langle F, \alpha\rangle$, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in \widetilde{\lambda}^{\mathcal{A}}_{F(\Sigma)}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T)$. Then, by definition, for all $T \leq T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we have

$$\alpha_\Sigma(\phi) \in T''_{F(\Sigma)} \quad \text{iff} \quad \alpha_\Sigma(\psi) \in T''_{F(\Sigma)}.$$

However, since $\mathcal{I}$ is protoalgebraic, we get, by the Correspondence Theorem 1336, that, for all $\alpha^{-1}(T) \leq T' \in \mathrm{ThFam}(\mathcal{I})$,

$$\phi \in T'_\Sigma \quad \text{iff} \quad \psi \in T'_\Sigma.$$

Hence, by definition,

$$\begin{array}{rcll}\langle \phi, \psi\rangle & \in & \widetilde{\lambda}^{\mathcal{I}}_\Sigma(\alpha^{-1}(T)) & \\ & = & \Omega_\Sigma(\alpha^{-1}(T)) & \text{(strong Fregeanity)} \\ & = & \alpha^{-1}_\Sigma(\Omega^{\mathcal{A}}_{F(\Sigma)}(T)). & \text{(Proposition 24)}\end{array}$$

Hence $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in \Omega^{\mathcal{A}}_{F(\Sigma)}(T)$. Taking into account the surjectivity of $\langle F, \alpha\rangle$, we now conclude that $\widetilde{\lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T) \leq \Omega^{\mathcal{A}}(T)$. ■

Proposition 1477 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is strongly Fregean and protoalgebraic, then, every full $\mathcal{I}$-structure, with an isomorphic functor component, has the Congruence Property.*

Proof: We deal first with the case $\mathrm{Thm}(\mathcal{I}) = \varnothing$. Then, since $\mathcal{I}$ is protoalgebraic, the only option is

$$\mathrm{ThFam}(\mathcal{I}) = \{T : T_\Sigma = \varnothing \text{ or } \mathrm{SEN}^\flat(\Sigma), \text{ for all } \Sigma \in |\mathbf{Sign}^\flat|\}.$$

In this case, given an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, the only full $\mathcal{I}$-structures on $\mathcal{A}$ are of the form $\langle \mathcal{A}, \mathcal{D}\rangle$, with

$$\mathcal{D} = \{T : T_\Sigma = \varnothing \text{ or } \mathrm{SEN}(\Sigma), \text{ for all } \Sigma \in |\mathbf{Sign}|\}.$$

All those have the Congruence Property.

Assume, next, that $\mathcal{I}$ has theorems. By Lemma 1476, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, with $F : \mathbf{Sign}^\flat \to \mathbf{Sign}$ an isomorphism, and every $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$\Omega^{\mathcal{A}}(T) = \widetilde{\lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T).$$

Thus, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T\rangle$ has the strong Congruence Property. Since, by Theorem 1463, every full $\mathcal{I}$-structure, with an isomorphic functor component has this form, we conclude that every full $\mathcal{I}$-structure, with an isomorphic functor component, has the Congruence Property. ∎

For finitary fully self extensional π-institutions, we obtain the following characterizations of weak family algebraizability.

Proposition 1478 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a finitary, fully self extensional π-institution based on $\mathbf{F}$. Then the following conditions are equivalent.*

(i) $\mathcal{I}$ is strongly Fregean, protoalgebraic and has theorems;

(ii) $\mathcal{I}$ is weakly family algebraizable and Ω is locally continuous;

(iii) $\mathcal{I}$ is weakly family algebraizable.

Proof: (i)⇒(ii) follows from Proposition 1475. (ii)⇒(iii) is trivial. For (iii)⇒(i) note, first, that, by hypothesis $\mathcal{I}$ is family monotone and family injective. Thus, $\mathcal{I}$ is protoalgebraic. By Proposition 1468, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\mathcal{I}^T \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{F})$. Hence, by full self extensionality, $\mathcal{I}^T$ has the strong Congruence Property and, hence, $\mathcal{I}$ is strongly Fregean. Finally, since $\Omega(\varnothing) = \Omega(\mathrm{SEN}^\flat) = \nabla^{\mathcal{F}}$, we get, by injectivity, $\varnothing \notin \mathrm{ThFam}(\mathcal{I})$ and $\mathcal{I}$ has theorems. ∎