Chapter 22

The Strong Version of a π-Institution

22.1 The Strong Version of a π-Institution

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We define the following classes of $\mathcal{I}$-matrix families.

$$\begin{array}{rcl} \mathsf{M}^{\mathcal{I}*} & = & \{\langle \mathcal{A}, T\rangle : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}), T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})\}; \\ \mathsf{M}^{\mathcal{I},\mathsf{Su}} & = & \{\langle \mathcal{A}, T\rangle : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}), T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})\}; \\ \mathsf{M}^{\mathcal{I},m} & = & \{\langle \mathcal{A}, T\rangle : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}), T = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}. \end{array}$$

We show that all three classes of $\mathcal{I}$-matrix families generate the same closure system on $\mathbf{F}$.

Proposition 1662 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Then $\mathcal{I}^{\mathsf{M}^{\mathcal{I}*}} = \mathcal{I}^{\mathsf{M}^{\mathcal{I},m}}$.*

Proof: By Lemma 1568, we have that, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, $\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. Thus, $\mathsf{M}^{\mathcal{I},m} \subseteq \mathsf{M}^{\mathcal{I}*}$. This implies that $\mathcal{I}^{\mathsf{M}^{\mathcal{I}*}} \leq \mathcal{I}^{\mathsf{M}^{\mathcal{I},m}}$. To show the converse, assume that $\langle \mathcal{A}, T\rangle \in \mathsf{M}^{\mathcal{I}*}$ and consider the quotient morphism $\langle I, \pi\rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T)$. By Corollary 1554, $\pi(T^*)$ is the least $\mathcal{I}$-filter family of $\mathcal{A}/\Omega^{\mathcal{A}}(T)$. By hypothesis $T = T^*$, whence $\pi(T) = \pi(T^*)$ and, hence, since $\langle I, \pi\rangle : \langle \mathcal{A}, T\rangle \to \langle \mathcal{A}/\Omega^{\mathcal{A}}(T), \pi(T)\rangle$ is a strict surjective morphism,, we get that

$$\mathcal{I}^{\langle \mathcal{A}, T\rangle} = \mathcal{I}^{\langle \mathcal{A}/\Omega^{\mathcal{A}}(T), \pi(T)\rangle} = \mathcal{I}^{\langle \mathcal{A}/\Omega^{\mathcal{A}}(T), \pi(T^*)\rangle}$$

and $\langle \mathcal{A}/\Omega^{\mathcal{A}}(T), \pi(T^*)\rangle \in \mathsf{M}^{\mathcal{I},m}$. Putting things together, we finally obtain

$$\begin{array}{rcl} \mathcal{I}^{\mathsf{M}^{\mathcal{I},m}} & \leq & \bigcap\{\mathcal{I}^{\langle \mathcal{A}/\Omega^{\mathcal{A}}(T), \pi(T^*)\rangle} : T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})\} \\ & = & \bigcap\{\mathcal{I}^{\langle \mathcal{A}, T\rangle} : T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})\} \\ & = & \mathcal{I}^{\mathsf{M}^{\mathcal{I}*}}. \end{array}$$

Therefore, $\mathcal{I}^{\mathsf{M}^{\mathcal{I}*}} = \mathcal{I}^{\mathsf{M}^{\mathcal{I},m}}$. ■

Proposition 1662 enables us to show that $\mathsf{M}^{\mathcal{I}*}$ and $\mathsf{M}^{\mathcal{I},\mathsf{Su}}$ also generate the same closure system on $\mathbf{F}$.

Corollary 1663 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Then $\mathcal{I}^{\mathsf{M}^{\mathcal{I}*}} = \mathcal{I}^{\mathsf{M}^{\mathcal{I},\mathsf{Su}}}$.*

Proof: By Lemma 1583, $\mathsf{M}^{\mathcal{I},\mathsf{Su}} \subseteq \mathsf{M}^{\mathcal{I}*}$. Also by Lemma 1583, $\mathsf{M}^{\mathcal{I},m} \subseteq \mathsf{M}^{\mathcal{I},\mathsf{Su}}$. So we get $\mathcal{I}^{\mathsf{M}^{\mathcal{I}*}} \leq \mathcal{I}^{\mathsf{M}^{\mathcal{I},\mathsf{Su}}} \leq \mathcal{I}^{\mathsf{M}^{\mathcal{I},m}}$. Therefore, by Proposition 1662, $\mathcal{I}^{\mathsf{M}^{\mathcal{I}*}} = \mathcal{I}^{\mathsf{M}^{\mathcal{I},\mathsf{Su}}}$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Taking into account Proposition 1662 and Corollary 1663, we define the **strong version of** $\mathcal{I}$, denoted by $\mathcal{I}^+ = \langle \mathbf{F}, C^+\rangle$, by

$$\mathcal{I}^+ := \mathcal{I}^{\mathsf{M}^{\mathcal{I}*}} = \mathcal{I}^{\mathsf{M}^{\mathcal{I},\mathsf{Su}}} = \mathcal{I}^{\mathsf{M}^{\mathcal{I},m}}.$$

There are even more ways to characterize the π-institution $\mathcal{I}^+$. Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Given a class K of $\mathbf{F}$-algebraic systems, we define

$$\begin{array}{lcl} \mathsf{M}_{\mathsf{K}}^{\mathcal{I},m} & = & \{\langle \mathcal{A}, T \rangle : \mathcal{A} \in \mathsf{K}, T = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}; \\ \mathsf{M}_{\mathsf{K}}^{\mathcal{I}*} & = & \{\langle \mathcal{A}, T \rangle : \mathcal{A} \in \mathsf{K}, T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})\}; \\ \mathsf{M}_{\mathsf{K}}^{\mathcal{I},\mathsf{Su}} & = & \{\langle \mathcal{A}, T \rangle : \mathcal{A} \in \mathsf{K}, T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})\}. \end{array}$$

Proposition 1664 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $\mathsf{K} = \mathrm{AlgSys}^*(\mathcal{I})$ or $\mathsf{K} = \mathrm{AlgSys}(\mathcal{I})$. Then*

$$\mathcal{I}^+ = \mathcal{I}^{\mathsf{M}_{\mathsf{K}}^{\mathcal{I},m}} = \mathcal{I}^{\mathsf{M}_{\mathsf{K}}^{\mathcal{I}*}} = \mathcal{I}^{\mathsf{M}_{\mathsf{K}}^{\mathcal{I},\mathsf{Su}}}.$$

Proof: By definition and Lemma 1583, we have

$$\mathsf{M}_{\mathsf{K}}^{\mathcal{I},m} \subseteq \mathsf{M}_{\mathsf{K}}^{\mathcal{I},\mathsf{Su}} \subseteq \mathsf{M}_{\mathsf{K}}^{\mathcal{I}*} \subseteq \mathsf{M}^{\mathcal{I}*}.$$

Therefore, we get

$$\mathcal{I}^+ \leq \mathcal{I}^{\mathsf{M}_{\mathsf{K}}^{\mathcal{I}*}} \leq \mathcal{I}^{\mathsf{M}_{\mathsf{K}}^{\mathcal{I},\mathsf{Su}}} \leq \mathcal{I}^{\mathsf{M}_{\mathsf{K}}^{\mathcal{I},m}}.$$

For the converse, suppose $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$ and $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. By Proposition 1572, $T/\Omega^{\mathcal{A}}(T)$ is the least $\mathcal{I}$-filter family of $\mathcal{A}/\Omega^{\mathcal{A}}(T) \in \mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$. Therefore, we get

$$\begin{array}{lcl} \mathcal{I}^{\mathsf{M}_{\mathsf{K}}^{\mathcal{I},m}} & \leq & \bigcap\{\mathcal{I}^{\langle \mathcal{A}/\Omega^{\mathcal{A}}(T), T/\Omega^{\mathcal{A}}(T)\rangle} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}), T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})\} \\ & = & \bigcap\{\mathcal{I}^{\langle \mathcal{A}, T\rangle} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}), T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})\} \\ & = & \mathcal{I}^+. \end{array}$$

We conclude that $\mathcal{I}^+ = \mathcal{I}^{\mathsf{M}_{\mathsf{K}}^{\mathcal{I},m}} = \mathcal{I}^{\mathsf{M}_{\mathsf{K}}^{\mathcal{I}*}} = \mathcal{I}^{\mathsf{M}_{\mathsf{K}}^{\mathcal{I},\mathsf{Su}}}$. ∎

The following proposition lists some of the properties of the strong version $\mathcal{I}^+$ of a π-institution $\mathcal{I}$.

Proposition 1665 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I} \leq \mathcal{I}^+$;

(b) $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, *for every $\mathbf{F}$-algebraic system $\mathcal{A}$;*

(c) $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$, *for every $\mathbf{F}$-algebraic system $\mathcal{A}$;*

(d) If $\mathcal{I}$ is family reflective, then $\mathcal{I}^+ = \mathcal{I}$.

Proof:

(a) Since $\mathsf{M}^{\mathcal{I},m} \subseteq \mathrm{MatFam}(\mathcal{I})$, we get $\mathcal{I} = \mathcal{I}^{\mathrm{MatFam}(\mathcal{I})} \leq \mathcal{I}^{\mathsf{M}^{\mathcal{I},m}} = \mathcal{I}^+$.

(b) Since, by Part (a), $\mathcal{I} \leq \mathcal{I}^+$, we get that $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$.

(c) By definition of $\mathcal{I}^+$, we have, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, all $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$ and all $T' \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$, $C^+ \leq C^{\langle\mathcal{A},T\rangle}$ and $C^+ \leq C^{\langle\mathcal{A},T'\rangle}$. Moreover, by Lemma 1583, every Suszko filter family is a Leibniz filter family. We conclude that $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$.

(d) By the hypothesis and Proposition 1573, $\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, for every $\mathbf{F}$-algebraic system $\mathcal{A}$. Therefore, $\mathcal{I}^+ = \mathcal{I}^{\mathsf{M}^{\mathcal{I}*}} = \mathcal{I}^{\mathrm{MatFam}(\mathcal{I})} = \mathcal{I}$. ■

It turns out that the strong version $\mathcal{I}^+$ is mostly interesting when $\mathcal{I}$ itself has theorems. In the absence of theorems $\mathcal{I}^+$ has only trivial theory families.

Proposition 1666 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *does not have theorems, then* $\mathcal{I}$ *is almost inconsistent.*

Proof: Assume that $\mathcal{I}$ does not have theorems. Then, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\varnothing \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Therefore, by definition $\mathcal{I}^+ = \bigcap\{\mathcal{I}^{\langle\mathcal{A},\varnothing\rangle} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$. This implies that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, we have, vacuously, for all $\psi \in \mathrm{SEN}^\flat(\Sigma)$, $\psi \in C^+_\Sigma(\phi)$. Therefore, the only Σ-theory families of $\mathcal{I}^+$ are $\varnothing$ and $\mathrm{SEN}^\flat(\Sigma)$. So $\mathcal{I}^+$ is almost inconsistent. ■

The least $\mathcal{I}$-filter family on every algebraic system $\mathcal{A}$ coincides with the least $\mathcal{I}^+$-filter family. As a consequence $\mathcal{I}$ and $\mathcal{I}^+$ share the same theorems.

Lemma 1667 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,*

$$\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \bigcap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}).$$

In particular, $\mathrm{ThFam}(\mathcal{I}^+) = \mathrm{ThFam}(\mathcal{I})$*.*

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system. By Proposition 1665, $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Thus, we have $\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \leq \bigcap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. On the other hand, by Lemma 1568, $\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$, whence, by Proposition 1665, $\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. Therefore, $\bigcap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \leq \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Equality now follows. ■

Lemma 1667 implies the idempotency of the strong version operator on π-institutions.

Corollary 1668 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then* $(\mathcal{I}^+)^+ = \mathcal{I}^+$*.*

Proof: We have

$$\begin{aligned}(\mathcal{I}^+)^+ &= \bigcap\{\mathcal{I}^{\langle\mathcal{A},T\rangle} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}), T = \bigcap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{I})\}\\ &= \bigcap\{\mathcal{I}^{\langle\mathcal{A},T\rangle} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}), T = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{I})\}\\ &= \mathcal{I}^+.\end{aligned}$$

The first and last equalities follow by the definition of $^+$, and the main equality is due to Lemma 1667. ∎

The next proposition provides sufficient conditions for recognizing that a given π-institution is the strong version of another π-institution based on the same algebraic system.

Proposition 1669 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$, $\mathcal{I}' = \langle \mathbf{F}, C'\rangle$ π-institutions based on $\mathbf{F}$, such that*

1. *$\mathcal{I}'$ is family reflective;*
2. $\mathrm{AlgSys}(\mathcal{I}') = \mathrm{AlgSys}(\mathcal{I})$*;*
3. *For all $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I}')$, $\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \bigcap \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A})$.*

Then $\mathcal{I}' = \mathcal{I}^+$.

Proof: We have

$$\begin{aligned}\mathcal{I}' &= \mathcal{I}'^+ \quad \text{(by 1 and Proposition 1665)}\\ &= \bigcap\{\mathcal{I}^{\langle\mathcal{A},T\rangle} : \mathcal{A} \in \mathrm{AlgSys}(\mathcal{I}'), T = \bigcap \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A})\}\\ &\quad \text{(by Proposition 1664)}\\ &= \bigcap\{\mathcal{I}^{\langle\mathcal{A},T\rangle} : \mathcal{A} \in \mathrm{AlgSys}(\mathcal{I}), T = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}\\ &\quad \text{(by 2 and 3)}\\ &= \mathcal{I}^+. \quad \text{(by Proposition 1664)}\end{aligned}$$

This proves the claim. ∎

We now show that Suszko and Leibniz $\mathcal{I}$-filter families form subclasses, respectively, of the classes of Suszko and Leibniz $\mathcal{I}^+$-filter families on every $\mathbf{F}$-algebraic system.

Proposition 1670 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. For every $\mathbf{F}$-algebraic system $\mathcal{A}$,*

$$\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}^+,\mathsf{Su}}(\mathcal{A}) \quad \textit{and} \quad \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A}).$$

Proof: By Proposition 1665, $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \mathrm{FFam}^{\mathcal{I}}(\mathcal{A})$. Thus, for all $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$, $[\![T]\!]^{\mathcal{I}^+*} \subseteq [\![T]\!]^{\mathcal{I}*}$ and $[\![T]\!]^{\mathcal{I}^+,\mathsf{Su}} \subseteq [\![T]\!]^{\mathcal{I},\mathsf{Su}}$.

Suppose that $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. Then, by Proposition 1665, $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$ and, moreover, $T = \bigcap [\![T]\!]^{\mathcal{I},\mathsf{Su}} \leq \bigcap [\![T]\!]^{\mathcal{I}^+,\mathsf{Su}}$. Thus, since $T \in [\![T]\!]^{\mathcal{I}^+,\mathsf{Su}}$, we get that $T = \bigcap [\![T]\!]^{\mathcal{I}^+,\mathsf{Su}} \in \mathrm{FiFam}^{\mathcal{I}^+,\mathsf{Su}}(\mathcal{A})$.

The second inclusion may be shown similarly. ∎

But the Leibniz counterpart of an $\mathcal{I}^+$-filter family is identical whether it be considered with respect to $\mathcal{I}$ or with respect to $\mathcal{I}^+$.

Lemma 1671 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, and all* $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$*,* $T^{\mathcal{I}*} = T^{\mathcal{I}^+*}$*.*

Proof: By Proposition 1665, $[\![T]\!]^{\mathcal{I}^+*} \subseteq [\![T]\!]^{\mathcal{I}*}$. Therefore, $T^{\mathcal{I}*} \leq T^{\mathcal{I}^+*}$. On the other hand,

$$\begin{array}{rcll} T^{\mathcal{I}*} & \in & \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) & \text{(by Proposition 1570)} \\ & \subseteq & \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) & \text{(by Proposition 1670)} \end{array}$$

and, since $T^{\mathcal{I}*} \in [\![T]\!]^{\mathcal{I}*}$, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T^{\mathcal{I}*})$. Thus, $T^{\mathcal{I}*} \subseteq [\![T]\!]^{\mathcal{I}^+*}$, which gives $T^{\mathcal{I}^+*} \leq T^{\mathcal{I}*}$. We conclude that $T^{\mathcal{I}*} = T^{\mathcal{I}^+*}$. ■

And this implies that the Leibniz $\mathcal{I}$-filter families and the Leibniz $\mathcal{I}^+$-filter families coincide on every $\mathbf{F}$-algebraic system.

Corollary 1672 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,*

$$\mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: The right-to-left inclusion was shown in Proposition 1670. For the reverse, assume that $T \in \mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A})$. Then, by Proposition 1665, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and, by Lemma 1671, $T = T^{\mathcal{I}^+*} = T^{\mathcal{I}*}$. Therefore, $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. ■

22.2 Leibniz and Suszko $\mathcal{I}^+$-Filter Families

There is a relation between the $\mathcal{I}^+$-filter families on algebraic systems and the Leibniz and Suszko $\mathcal{I}$-filter families on the same algebraic systems. The following proposition shows how these relations interplay with family c-reflectivity.

Proposition 1673 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If, for all $\mathbf{F}$*-algebraic systems* $\mathcal{A}$*,* $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$*, then* $\mathcal{I}^+$ *is family c-reflective.*

(b) If $\mathcal{I}^+$ *is family c-reflective, then* $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$*, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A}$*.*

Proof:

(a) Suppose, for all $\mathbf{F}$-algebraic systems $\mathcal{A}$, $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system. By Proposition 1670, $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}^+,\mathsf{Su}}(\mathcal{A})$. Hence, by hypothesis, $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}^+,\mathsf{Su}}(\mathcal{A})$. Thus, $\mathrm{FiFam}^{\mathcal{I}^+,\mathsf{Su}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. By Theorem 1590, $\mathcal{I}^+$ is family c-reflective.

(b) Suppose $\mathcal{I}^+$ is family c-reflective and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system. By Theorem 1590, $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}^+,\mathsf{Su}}(\mathcal{A})$. Since, by Lemma 1583 and Corollary 1672, $\mathrm{FiFam}^{\mathcal{I}^+,\mathsf{Su}}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$, we get that $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. The reverse inclusion holds by Proposition 1665.

■

A necessary and sufficient condition for the $\mathcal{I}^+$-filter families to coincide with the Leibniz $\mathcal{I}$-filter families is the universal reflectivity of the Leibniz operator on $\mathcal{I}^+$-filter families.

Proposition 1674 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. For every $\mathbf{F}$-algebraic system $\mathcal{A}$,*

$$\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$$

if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}}$ is order reflecting on $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$.

Proof: By Corollary 1672, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. By Proposition 1573, $\Omega^{\mathcal{A}}$ is reflective on $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$, for all $\mathcal{A}$, if and only if $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A})$, for all $\mathcal{A}$. Thus, we get that $\Omega^{\mathcal{A}}$ is reflective on $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$, for all $\mathcal{A}$, if and only if $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$, for all $\mathcal{A}$. ■

Under the stipulation that the strong version of $\mathcal{I}$ be protoalgebraic, the identification of $\mathcal{I}^+$-filter families with the Leibniz $\mathcal{I}$-families have several characterizations.

Proposition 1675 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, such that $\mathcal{I}^+$ is protoalgebraic. The following conditions are equivalent:*

(i) $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$, *for every $\mathbf{F}$-algebraic system $\mathcal{A}$;*

(ii) $\mathrm{ThFam}(\mathcal{I}^+) = \mathrm{ThFam}^*(\mathcal{I})$*;*

(iii) $\mathcal{I}^+$ is weakly family algebraizable;

(iv) $\mathcal{I}^+$ is family c-reflective;

Proof:

(i)⇒(ii) Trivial.

(ii)⇒(iii) Suppose that $\mathrm{ThFam}(\mathcal{I}^+) = \mathrm{ThFam}^*(\mathcal{I})$. By Proposition 1528, Ω is injective on $\mathrm{ThFam}^*(\mathcal{I})$. By definition it is onto $\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{F})$. Thus, by hypothesis and Corollary 1672, $\Omega : \mathrm{FiFam}(\mathcal{I}^+) \to \mathrm{ConSys}^{\mathcal{I}^+*}(\mathcal{F})$ is a bijection. By hypothesis it is monotone and, by Proposition 1528, it is order reflecting. Therefore, it is an order isomorphism. By Theorem 296, $\mathcal{I}^+$ is weakly family algebraizable.

(iii)⇒(iv) Every weakly family algebraizable π-institution is a fortiori family c-reflective.

(iv)⇒(i) By hypothesis, $\mathcal{I}^+$ is protoalgebraic, whence, by Proposition 1601 and Corollary 1672,

$$\mathrm{FiFam}^{\mathcal{I}^+,\mathsf{Su}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}).$$

By hypothesis and Theorem 1590, $\mathrm{FiFam}^{\mathcal{I}^+,\mathsf{Su}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. Therefore, we get that $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. ■

We close the section by looking at various consequences of the condition imposed on a π-institution $\mathcal{I}$ that $\Omega^{\mathcal{A}}$ be an order isomorphism from the Leibniz $\mathcal{I}$-filter families of $\mathcal{A}$ onto the $\mathcal{I}^*$-congruence systems on $\mathcal{A}$, for every $\mathcal{I}$-algebraic system $\mathcal{A}$. First, we show that this condition ensures that $\mathcal{I}$-algebraic systems, $\mathcal{I}^*$-algebraic systems, $\mathcal{I}^+$-algebraic systems and $(\mathcal{I}^+)^*$-algebraic systems all coincide.

Lemma 1676 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathcal{I}$, such that, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{AlgSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism. Then

$$\mathrm{AlgSys}(\mathcal{I}^+) = \mathrm{AlgSys}^*(\mathcal{I}^+) = \mathrm{AlgSys}^*(\mathcal{I}) = \mathrm{AlgSys}(\mathcal{I}).$$

Proof: We show, first, that $\mathrm{AlgSys}^*(\mathcal{I}^+) = \mathrm{AlgSys}^*(\mathcal{I})$. The left-to-right inclusion holds because, by Proposition 1665, $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, for every $\mathbf{F}$-algebraic system $\mathcal{I}$. Assume, conversely, that $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I})$. Then $\Delta^{\mathcal{A}} \in \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. By hypothesis, then, there exists $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. By Proposition 1665 again, $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. Hence, $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I}^+)$.

Now we have

$$\begin{array}{rcll}
\mathrm{AlgSys}(\mathcal{I}) & = & \mathrm{AlgSys}^*(\mathcal{I}) & \text{(by Lemma 1623)}\\
& = & \mathrm{AlgSys}^*(\mathcal{I}^+) & \text{(shown above)}\\
& \subseteq & \mathrm{AlgSys}(\mathcal{I}^+) & \text{(by Proposition 65)}\\
& \subseteq & \mathrm{AlgSys}(\mathcal{I}). & \text{(by Proposition 1665).}
\end{array}$$

We conclude that all four classes of algebraic system coincide. ■

Next we show that, under the same hypothesis the Leibniz congruence systems of a filter family and its Leibniz counterpart coincide and that the Suszko congruence system of a filter family coincides with the Leibniz congruence system of its Suszko counterpart.

Proposition 1677 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathcal{I}$*, such that, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{AlgSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism. Then, for every $\mathbf{F}$*-algebraic system and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T^*) \quad \textit{and} \quad \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T^{\mathcal{I},\mathsf{Su}}).$$

Proof: By Proposition 1622, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism.

Let $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Since $\Omega^{\mathcal{A}}(T) \in \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$, there exists $T' \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T') = \Omega^{\mathcal{A}}(T)$. Hence, $[\![T]\!]^* = [\![T']\!]^*$, which gives $T^* = T'^* = T'$. Thus, we get $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T') = \Omega^{\mathcal{A}}(T^*)$.

By hypothesis and Lemma 1623, $\mathrm{AlgSys}^*(\mathcal{I}) = \mathrm{AlgSys}(\mathcal{I})$. Since we have $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$, there exists $T'' \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T'') = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. Thus, we get $[\![T]\!]^{\mathsf{Su}} = [\![T'']\!]^*$ and, therefore, $T^{\mathcal{I},\mathsf{Su}} = T''^* = T''$. This gives $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T'') = \Omega^{\mathcal{A}}(T^{\mathcal{I},\mathsf{Su}})$. ■

Under the same hypothesis, it turns out that the coincidence of the class of Leibniz filter families with Suszko filter families on every algebraic system characterizes protoalgebraicity.

Corollary 1678 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathcal{I}$*, such that, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{AlgSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism. $\mathcal{I}$ *is protoalgebraic if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$*.*

Proof: If $\mathcal{I}$ is protoalgebraic, then, by Proposition 1601, Leibniz and Suszko classes coincide and, therefore, $\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$.

Suppose, conversely, that, for all $\mathbf{F}$-algebraic systems $\mathcal{A}$, $\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. Let $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. By Lemma 1583, $T^{\mathcal{I},\mathsf{Su}} \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. By the hypothesis and Lemma 1586, $T^{\mathcal{I},\mathsf{Su}}$ is the largest Leibniz $\mathcal{I}$-filter family included in T. Since, by Lemma

1583, $T^{\mathcal{I},\mathsf{Su}} \leq T^* \leq T$ and, by Proposition 1570, T^* is a Leibniz $\mathcal{I}$-filter family, we get $T^{\mathcal{I},\mathsf{Su}} = T^*$. Therefore, using Proposition 1570, we get

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T^{\mathcal{I},\mathsf{Su}}) = \Omega^{\mathcal{A}}(T^*) = \Omega^{\mathcal{A}}(T).$$

Thus, on every **F**-algebraic system $\mathcal{A}$, the Suszko and the Leibniz operators coincide and, therefore, by Lemma 1518, $\mathcal{I}$ is protoalgebraic. ■

We already have the tools to show that the property that $\Omega^{\mathcal{A}}$ be an isomorphism between Leibniz filter families and reduced algebraic systems is bequeathed by a π-institution $\mathcal{I}$ to its strong version $\mathcal{I}^+$.

Lemma 1679 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathcal{I}$*, such that, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{AlgSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism. Then, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I}^+)$*,* $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^+*}(\mathcal{A})$ *is also an order isomorphism.*

Proof: By Corollary 1672, we have $\mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. By Lemma 1676, $\mathrm{AlgSys}^*(\mathcal{I})) = \mathrm{AlgSys}^*(\mathcal{I}^+)$. Now, taking into account the hypothesis, we get the conclusion. ■

In a proposition analogous to Proposition 1675, we provide under our working hypothesis, of the Leibniz operator being an order isomorphism, a characterization of the property of $\mathcal{I}^+$ being weakly family algebraizable.

Proposition 1680 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathcal{I}$*, such that, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{AlgSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism. The following conditions are equivalent:

(i) $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$*, for every* **F***-algebraic system* $\mathcal{A}$*;*

(ii) $\mathrm{ThFam}(\mathcal{I}^+) = \mathrm{ThFam}^*(\mathcal{I})$*;*

(iii) $\mathcal{I}^+$ *is weakly family algebraizable;*

(iv) $\mathcal{I}^+$ *is family c-reflective;*

(v) Ω *is injective on the collection of reduced* $\mathcal{I}^+$*-filter families.*

Proof:

(i)⇒(ii) Trivial.

(ii)⇒(iii) By hypothesis and Lemma 1676, $\Omega : \mathrm{ThFam}(\mathcal{I}^+) \to \mathrm{ConSys}^{\mathcal{I}^{+*}}(\mathcal{F})$ is an order isomorphism. Thus Ω is both monotone and family c-reflective, whence $\mathcal{I}^+$ is weakly family algebraizable.

(iii)⇒(iv) Weak family algebraizability implies family c-reflectivity.

(iv)⇒(v) If $\mathcal{I}^+$ is family c-reflective, then it is a fortiori injective. Therefore, by Theorem 214, $\Omega^{\mathcal{A}}$ is injective on the $\mathcal{I}$-filter families of every $\mathbf{F}$-algebraic system $\mathcal{A}$.

(v)⇒(i) Suppose (v) holds and let $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$. By Proposition 1665, we have $\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. So it suffices to prove the reverse inclusion. To this end, suppose $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. Consider the quotient morphism

$$\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T).$$

$\mathrm{Ker}(\langle I, \pi \rangle) = \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T^*)$, the last inclusion, since, by Proposition 1525, $T^* \in [\![T]\!]^{\mathcal{I}*}$. Hence, by Corollary 56,

$$\pi(T), \pi(T^*) \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}/\Omega^{\mathcal{A}}(T))$$

and, by compatibility, $\pi^{-1}(\pi(T)) = T$ and $\pi^{-1}(\pi(T^*)) = T^*$. By Corollary 1554, $\pi(T^*) = \pi(T)^*$. Now we get

$$\begin{array}{rcll} \Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)} & = & \Omega^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\pi(T)) & \text{(by Lemma 1557)} \\ & = & \Omega^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\pi(T)^*) & \text{(by Proposition 1677)} \\ & = & \Omega^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\pi(T^*)). & \end{array}$$

This, both $\pi(T)$ and $\pi(T^*)$ are reduced $\mathcal{I}^+$-filter families and, therefore, by the injectivity hypothesis, $\pi(T) = \pi(T^*)$. Now we conclude that $T = \pi^{-1}(\pi(T)) = \pi^{-1}(\pi(T^*)) = T^*$. This proves that, for all $\mathcal{A}$, $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. Equality now follows.

■

22.3 Full $\mathcal{I}^+$-Structures

We now explore the relation between full $\mathcal{I}$-structures and full $\mathcal{I}^+$-structures.

Proposition 1681 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\mathcal{A}$ *an* $\mathbf{F}$*-algebraic system.* $\langle \mathcal{A}, \mathcal{D} \rangle \in \mathrm{FStr}^{\mathcal{I}^+}(\mathcal{A})$ *if and only if, there exists* $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, such that* $\langle \mathcal{A}, \mathcal{T} \rangle \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$ *and* $\mathcal{D} = \mathcal{T} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$*, i.e.,*

$$\mathrm{FStr}(\mathcal{I}^+) = \{\langle \mathcal{A}, \mathcal{T} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \rangle : \langle \mathcal{A}, \mathcal{T} \rangle \in \mathrm{FStr}(\mathcal{I})\}.$$

Proof:

($\Rightarrow$) Suppose that $\langle\mathcal{A},\mathcal{D}\rangle \in \mathrm{FStr}(\mathcal{I}^+)$. Set

$$\mathcal{T} = \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) \leq \Omega^{\mathcal{A}}(T)\}.$$

If $T \in \mathcal{D}$, then $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) \leq \Omega^{\mathcal{A}}(T)$ and $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(T)$. Thus, $T \in \mathcal{T} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. On the other hand, let $T \in \mathcal{T} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. Then $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) \leq \Omega^{\mathcal{A}}(T)$ and, since $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$ and $\langle\mathcal{A},\mathcal{D}\rangle \in \mathrm{FStr}(\mathcal{I}^+)$, we must have, by Theorem 1395, $T \in \mathcal{D}$. We conclude that $\mathcal{D} = \mathcal{T} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. Thus, it only remains to show that $\langle\mathcal{A},\mathcal{T}\rangle \in \mathrm{FStr}(\mathcal{I})$.

To this end, let $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \leq \Omega^{\mathcal{A}}(T)$. Then, we get $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \leq \bigcap_{T' \in \mathcal{D}} \Omega^{\mathcal{A}}(T') = \widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) \leq \Omega^{\mathcal{A}}(T)$. Thus, by definition, $T \in \mathcal{T}$. We conclude, using Theorem 1395, that $\langle\mathcal{A},\mathcal{T}\rangle \in \mathrm{FStr}(\mathcal{I})$.

($\Leftarrow$) Suppose, now, that $\langle\mathcal{A},\mathcal{T}\rangle \in \mathrm{FStr}(\mathcal{I})$ and $\mathcal{D} = \mathcal{T} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. Since, by Proposition 1563, the least element of a full $\mathcal{I}$-structure is a Leibniz $\mathcal{I}$-filter family, we get that $\bigcap\mathcal{T} \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. To see that $\langle\mathcal{A},\mathcal{D}\rangle$ is a dull $\mathcal{I}^+$-structure, let $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) \leq \Omega^{\mathcal{A}}(T)$. Then, we infer

$$\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \leq \widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) \leq \Omega^{\mathcal{A}}(T).$$

Since $\langle\mathcal{A},\mathcal{T}\rangle \in \mathrm{FStr}(\mathcal{I})$, then, by Theorem 1395, $T \in \mathcal{T}$. Since, in addition, by hypothesis, $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$, we get $T \in \mathcal{D}$. Thus, again by Theorem1395, $\langle\mathcal{A},\mathcal{D}\rangle \in \mathrm{FStr}(\mathcal{I}^+)$.

■

Next, we show that the association

$$\langle\mathcal{A},\mathcal{T}\rangle \mapsto \langle\mathcal{A},\mathcal{T} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})\rangle$$

of full $\mathcal{I}^+$-structures to full $\mathcal{I}$-structures, given in Proposition 1681, is one-to-one, provided that $\mathcal{I}$- and $\mathcal{I}^+$-algebraic systems coincide.

Proposition 1682 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*, such that* $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}(\mathcal{I}^+)$*, and* $\mathcal{A}$ *an* $\mathbf{F}$*-algebraic system. For all* $\langle\mathcal{A},\mathcal{T}\rangle$*,* $\langle\mathcal{A},\mathcal{T}'\rangle \in \mathrm{FStr}(\mathcal{I})$*,*

$$\mathcal{T} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathcal{T}' \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \quad \textit{implies} \quad \mathcal{T} = \mathcal{T}'.$$

Proof: We start with some preparatory remarks. Suppose $\mathcal{A}$ is an $\mathbf{F}$-algebraic system. Since, by hypothesis, $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}(\mathcal{I}^+)$, we get that $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}) = \mathrm{ConSys}^{\mathcal{I}^+}(\mathcal{A})$. Now, using Theorem 1408 (or, alternatively, Corollary 1565), we have that $\mathbf{FStr}^{\mathcal{I}}(\mathcal{A}) \cong \mathrm{FStr}^{\mathcal{I}^+}(\mathcal{A})$, through

$$\mathcal{T} \mapsto \overline{\mathcal{T}} = \{T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \leq \Omega^{\mathcal{A}}(T)\}.$$

This is obtained, by applying Theorem 1408 to get an isomorphism

$$\begin{array}{rcl}\gamma : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) & \to & \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}); \\ \mathcal{T} & \stackrel{\gamma}{\mapsto} & \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}),\end{array}$$

then, applying Theorem 1408 to get an isomorphism

$$\begin{array}{rcl}\delta : \mathrm{ConSys}^{\mathcal{I}^+}(\mathcal{A}) & \to & \mathrm{FStr}^{\mathcal{I}^+}(\mathcal{A}); \\ \theta & \stackrel{\delta}{\mapsto} & \{T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) : \theta \leq \Omega^{\mathcal{A}}(T)\}\end{array}$$

and, finally, composing these two, taking into account the hypothesis.

Now let $\mathcal{T}, \mathcal{T}' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\langle \mathcal{A}, \mathcal{T} \rangle$, $\langle \mathcal{A}, \mathcal{T}' \rangle \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$, and suppose that $\mathcal{T} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathcal{T}' \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$.

Claim 1: $\overline{\mathcal{T}} = \mathcal{T} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$ and $\overline{\mathcal{T}'} = \mathcal{T}' \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$.

We show the first equality. The second one is shown in exactly the same way. First, if $T \in \overline{\mathcal{T}}$, then $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$ and $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \leq \Omega^{\mathcal{A}}(T)$. Since $\langle \mathcal{A}, \mathcal{T} \rangle$ is a full $\mathcal{I}$-structure, by Theorem 1395, $T \in \mathcal{T}$. Thus, $T \in \mathcal{Y} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. If, on the other hand, $T \in \mathcal{T} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$, then $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$ and $T \in \mathcal{T}$. Thus, $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$ and $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \leq \Omega^{\mathcal{A}}(T)$. Therefore, $T \in \overline{\mathcal{T}}$.

Claim 2: $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \widetilde{\Omega}^{\mathcal{A}}(\overline{\mathcal{T}})$ and $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}') = \widetilde{\Omega}^{\mathcal{A}}(\overline{\mathcal{T}'})$.

Again, it suffices to show the first equality, since the second is proven in exactly the same way. By Claim 1 and Proposition 1681, $\langle \mathcal{A}, \overline{\mathcal{T}} \rangle \in \mathrm{FStr}^{\mathcal{I}^+}(\mathcal{A})$. Therefore, by Theorem 1395, $\overline{\mathcal{T}} = \{T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{A}}(\overline{\mathcal{T}}) \leq \Omega^{\mathcal{A}}(T)\}$. Thus, we get $\delta(\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})) = \delta(\gamma(\mathcal{T})) = \overline{\mathcal{T}} = \delta(\widetilde{\Omega}^{\mathcal{A}}(\overline{\mathcal{T}}))$. Since δ is an isomorphism, we get that $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \widetilde{\Omega}^{\mathcal{A}}(\overline{\mathcal{T}})$.

To finish the proof, we get $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \widetilde{\Omega}^{\mathcal{A}}(\overline{\mathcal{T}}) = \widetilde{\Omega}^{\mathcal{A}}(\overline{\mathcal{T}'}) = \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}')$. Therefore, by Theorem 1408, $\mathcal{T} = \mathcal{T}'$. ■

Now we can formulate an order isomorphism between full $\mathcal{I}$- and full $\mathcal{I}^+$-structures, subject to the condition that $\mathcal{I}$- and $\mathcal{I}^+$-algebraic systems coincide.

Corollary 1683 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, such that* $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}(\mathcal{I}^+)$*, and* $\mathcal{A}$ *an* $\mathbf{F}$*-algebraic system.*

$$\begin{array}{rcl}h : \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) & \to & \mathrm{FStr}^{\mathcal{I}^+}(\mathcal{A}); \\ \langle \mathcal{A}, \mathcal{T} \rangle & \stackrel{h}{\mapsto} & \langle \mathcal{A}, \mathcal{T} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \rangle\end{array}$$

is an order isomorphism.

Proof: By Propositions 1681 and 1682. ■

We turn next to relationships between full classes of filter families with respect to a π-institution $\mathcal{I}$ and its strong version $\mathcal{I}^+$. Recall that, given any

$\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, we have $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. So we get immediately the following inclusions, for all $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$.

$$\begin{array}{rcl}[\![T]\!]^{\mathcal{I}^+*} & = & \{T' \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) : \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\} \\ & \subseteq & \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\} \\ & = & [\![T]\!]^{\mathcal{I}*}.\end{array}$$

Moreover, taking into account

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T) \leq \widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})^T) = \widetilde{\Omega}^{\mathcal{I}^+,\mathcal{A}}(T),$$

we infer

$$\begin{array}{rcl}[\![T]\!]^{\mathcal{I}^+,\mathsf{Su}} & = & \{T' \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{I}^+,\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\} \\ & \subseteq & \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\} \\ & = & [\![T]\!]^{\mathcal{I},\mathsf{Su}}.\end{array}$$

These relationships may be strengthened to apply to all extensions to a π-institution rather that only its strong version. More precisely, we obtain

Lemma 1684 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ and $\mathcal{I}' = \langle \mathbf{F}, C' \rangle$ be π-institutions based on $\mathbf{F}$, such that $\mathcal{I} \leq \mathcal{I}'$, $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A})$. Then*

$$[\![T]\!]^{\mathcal{I}'*} = [\![T]\!]^{\mathcal{I}*} \cap \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A}) \quad \textit{and} \quad [\![T]\!]^{\mathcal{I}',\mathsf{Su}} \subseteq [\![T]\!]^{\mathcal{I},\mathsf{Su}} \cap \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A}).$$

Proof: We have, mimicking the process preceding the statement, applied to the extension $\mathcal{I}'$ rather than specifically $\mathcal{I}^+$:

$$\begin{array}{rcl}[\![T]\!]^{\mathcal{I}'*} & = & \{T' \in \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A}) : \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\} \\ & = & \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\} \cap \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A}) \\ & = & [\![T]\!]^{\mathcal{I}*} \cap \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A}).\end{array}$$

Moreover, taking into account

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T) \leq \widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A})^T) = \widetilde{\Omega}^{\mathcal{I}',\mathcal{A}}(T),$$

we infer

$$\begin{array}{rcl}[\![T]\!]^{\mathcal{I}',\mathsf{Su}} & = & \{T' \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{I}',\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\} \\ & \subseteq & \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\} \cap \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A}) \\ & = & [\![T]\!]^{\mathcal{I},\mathsf{Su}} \cap \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A}).\end{array}$$

Thus, we have the equality and the inclusion claimed. ■

Since $\mathcal{I}^+$ is an extension of $\mathcal{I}$, then we immediately deduce

Corollary 1685 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A}$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$*. Then*

$$[\![T]\!]^{\mathcal{I}^+*} = [\![T]\!]^{\mathcal{I}*} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \quad \text{and} \quad [\![T]\!]^{\mathcal{I}^+,\mathsf{Su}} \subseteq [\![T]\!]^{\mathcal{I},\mathsf{Su}} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}).$$

Proof: By Lemma 1684, since $\mathcal{I} \leq \mathcal{I}^+$. ■

Finally, we strengthen the preceding relation between Suszko classes to an equality, in the special case, where T happens to be a Suszko $\mathcal{I}$-filter family of $\mathcal{I}$ (recalling that $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{I}) \subseteq \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$).

Lemma 1686 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A}$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$*. Then* $[\![T]\!]^{\mathcal{I}^+,\mathsf{Su}} = [\![T]\!]^{\mathcal{I},\mathsf{Su}} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$*.*

Proof: Let $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. Then, by Lemma 1583, $[\![T]\!]^{\mathcal{I},\mathsf{Su}} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$. Since $T = \bigcap [\![T]\!]^{\mathcal{I},\mathsf{Su}}$, $[\![T]\!]^{\mathcal{I}^+,\mathsf{Su}} \subseteq [\![T]\!]^{\mathcal{I},\mathsf{Su}}$ and $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$, we get $T = \bigcap [\![T]\!]^{\mathcal{I}^+,\mathsf{Su}}$. Hence $T \in \mathrm{FiFam}^{\mathcal{I}^+,\mathsf{Su}}(\mathcal{A})$. Again, using Lemma 1583, we get $[\![T]\!]^{\mathcal{I}^+,\mathsf{Su}} = \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})^T$. Therefore, we conclude that

$$\begin{array}{rcl} [\![T]\!]^{\mathcal{I}^+,\mathsf{Su}} & = & \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})^T \\ & = & \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \\ & = & [\![T]\!]^{\mathcal{I},\mathsf{Su}} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}). \end{array}$$

■

22.4 Leibniz Truth Equationality

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is **Leibniz truth equational** if there exists $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, such that, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$T^* = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T)),$$

i.e., for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T^*_\Sigma \quad \text{iff} \quad \tau^{\mathcal{A}}_\Sigma[\phi] \leq \Omega^{\mathcal{A}}(T).$$

It follows directly by the definition that, if $\mathcal{I}$ is Leibniz truth equational, then, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \quad \text{iff} \quad T = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T)).$$

Moreover, we can easily see that family truth equationality implies Leibniz truth equationality.

Lemma 1687 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is family truth equational, then* $\mathcal{I}$ *is Leibniz truth equational.*

Proof: Suppose that $\mathcal{I}$ is family truth equational, with witnessing transformations $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$. Thus, by Theorem 848, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$. Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. We have

$$\begin{array}{rll} \phi \in T_\Sigma & \text{iff} & \tau_\Sigma^{\mathcal{A}}[\phi] \leq \Omega^{\mathcal{A}}(T) \quad (\mathcal{I} \text{ truth equational}) \\ & \text{implies} & \tau^{\mathcal{A}}[\phi] \leq \Omega^{\mathcal{A}}(T^*) \quad (T^* \in [\![T]\!]^*) \\ & \text{iff} & \phi \in T_\Sigma^*. \quad (\mathcal{I} \text{ truth equational}) \end{array}$$

Thus, we get $T \leq T^*$. On the other hand, by Lemma 1568, $T^* \leq T$, whence $T - T^*$. This gives $T^* = T$ and, hence $T^* = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$, showing that $\mathcal{I}$ is Leibniz truth equational. ■

If $\mathcal{I}$ is Leibniz truth equational, then the collection of all its Leibniz filters on every algebraic system forms a closure family.

Proposition 1688 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Leibniz truth equational* π*-institution based on* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$ *is closed under signature-wise intersections and, hence, forms a closure family on* $\mathcal{A}$*.*

Proof: Suppose $\mathcal{I}$ is Leibniz truth-equational, with witnssing transformations $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$. Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $\{T^i : i \in I\} \subseteq \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$ be a collection of Leibniz $\mathcal{I}$-filter families. Then

$$\begin{array}{rll} \bigcap_{i\in I} T^i & = & \bigcap_{i\in I} (T^i)^* \quad (T^i \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})) \\ & = & \bigcap_{i\in I} \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T^i)) \quad (\mathcal{I} \text{ Leibniz truth equational}) \\ & \leq & \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(\bigcap_{i\in I} T^i)) \quad (\bigcap_{i\in I} \Omega^{\mathcal{A}}(T^i) \leq \Omega^{\mathcal{A}}(\bigcap_{i\in I} T^i)) \\ & = & (\bigcap_{i\in I} T^i)^*. \quad (\mathcal{I} \text{ Leibniz truth equational}) \end{array}$$

Since, by Lemma 1568, $(\bigcap_{i\in I} T^i)^* \leq \bigcap_{i\in I} T^i$, we get that $(\bigcap_{i\in I} T^i)^* = \bigcap_{i\in I} T^i$ and, therefore, $\bigcap_{i\in I} T^i \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. ■

The next proposition shows that to check that a given π-institution $\mathcal{I}$ is Leibniz truth equational, it is sufficient to work with $\mathcal{I}^*$-algebraic systems only. That is, if the defining property holds for all Leibniz filters of $\mathcal{I}^*$-algebraic systems, then it extends to Leibniz filters over arbitrary $\mathbf{F}$-algebraic systems.

Proposition 1689 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is Leibniz truth equational if and only if, there exists* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*, such that, for all* $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I})$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $T^* = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$*.*

Proof: The implication left-to-right follows from the definition of Leibniz truth equationality. Suppose, conversely, that there exists $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, such that, for all $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I})$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T^* = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$. Let $\mathcal{A}$ be an arbitrary **F**-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, and consider the quotient morphism $\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T)$. Then, by Corollary 1554, $\pi(T^*) = \pi(T)^*$ and, by Proposition 1530, $\pi(T)^*$ is the least $\mathcal{I}$-filter family on $\mathcal{A}/\Omega^{\mathcal{A}}(T)$. Since $\mathcal{A}/\Omega^{\mathcal{A}}(T) \in \mathrm{AlgSys}^*(\mathcal{I})$, we get, by hypothesis,

$$\pi(T)^* = \tau^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(T/\Omega^{\mathcal{A}}(T)) = \tau^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}).$$

Hence, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{rll}\phi \in T^*_\Sigma & \text{iff} & \phi/\Omega^{\mathcal{A}}_\Sigma(T) \in \pi_\Sigma(T^*_\Sigma) \quad (\Omega^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T^*)) \\ & \text{iff} & \phi/\Omega^{\mathcal{A}}_\Sigma(T) \in \pi(T)^*_\Sigma \\ & \text{iff} & \phi/\Omega^{\mathcal{A}}_\Sigma(T) \in \tau_\Sigma^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}) \\ & \text{iff} & \phi \in \tau^{\mathcal{A}}_\Sigma(\Omega^{\mathcal{A}}(T)).\end{array}$$

Thus, $\mathcal{I}$ is Leibniz truth equational. ■

A fortiori, it suffices to show that the condition in the statement of Proposition 1689 holds for all $\mathcal{I}$-algebraic systems, since this class encompasses all $\mathcal{I}^*$-algebraic systems.

Corollary 1690 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* **F**. $\mathcal{I}$ *is Leibniz truth equational if and only if, there exists* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*, such that, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $T^* = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$.

Proof: The conclusion follows from Proposition 1689, taking into account the fact that $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$. ■

Next, we provide another characterization of Leibniz truth equationality by showing that it is equivalent to $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}})$ being the least $\mathcal{I}$-filter family on every $\mathcal{I}$- (or $\mathcal{I}*$-)algebraic system.

Proposition 1691 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* **F** *and* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*. The following conditions are equivalent.*

(i) $\mathcal{I}$ *is Leibniz truth equational, with witnessing transformations* $\tau^\flat$*;*

(ii) For all $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I})$*,* $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*;*

(iii) For all $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*,* $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

Proof:

(i)⇒(iii) Suppose $\mathcal{I}$ is Leibniz truth equational, with witnessing transformations $\tau^\flat$. Let $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$ and $T^m = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, by Lemma 1568, $T^m \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. Since $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \le \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T^m))$, we get, by hypothesis, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \le T^m$. On the other hand, since $T^m = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we have, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T^m \le T^*$, whence, by hypothesis, $T^m \le \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$. Since, this holds for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get, taking into account that $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$,

$$T^m \le \tau^{\mathcal{A}}(\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))) = \tau^{\mathcal{A}}(\Delta^{\mathcal{A}}).$$

Therefore, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) = T^m$.

(iii)⇒(ii) Trivial, since $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$.

(ii)⇒(i) Suppose, for all $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I})$, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and consider the quotient morphism

$$\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T).$$

Then, $\mathcal{A}/\Omega^{\mathcal{A}}(T) \in \mathrm{AlgSys}^*(\mathcal{I})$ and, by Corollary 1554, $\pi(T^*) = \pi(T)^*$ and, by Proposition 1530, $\pi(T)^* = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T))$. Thus, by hypothesis, $\pi(T^*) = \tau^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)})$. Therefore, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll}\phi \in T^*_\Sigma & \text{iff} & \phi/\Omega^{\mathcal{A}}_\Sigma(T) \in \pi_\Sigma(T^*_\Sigma)\\ & \text{iff} & \phi/\Omega^{\mathcal{A}}_\Sigma(T) \in \tau_\Sigma^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)})\\ & \text{iff} & \phi \in \tau^{\mathcal{A}}_\Sigma(\Omega^{\mathcal{A}}(T)).\end{array}$$

Hence, $\tau^\flat$ witnesses the Leibniz truth equationality of $\mathcal{I}$. ∎

If $\mathcal{I}$-algebraic systems and $\mathcal{I}^+$-algebraic systems coincide, then truth equationality of $\mathcal{I}^+$ guarantees the Leibniz truth equationality of $\mathcal{I}$.

Proposition 1692 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$ *and* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*. If* $\mathcal{I}^+$ *is family truth equational, with witnessing transformations* $\tau^\flat$ *and* $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}(\mathcal{I}^+)$*, then* $\mathcal{I}$ *is Leibniz truth equational, with witnessing transformations* $\tau^\flat$*.*

Proof: We use Proposition 1691. Suppose $\mathcal{I}^+$ is family truth equational via $\tau^\flat$ and $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}(\mathcal{I}^+)$. Let $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$. Since, by hypothesis $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I}^+)$, we get, by hypothesis, Lemma 1687 and Proposition 1691, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) = \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. By Lemma 1667, $\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \bigcap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. Hence, we get $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, whence, by Proposition 1691, $\mathcal{I}$ is Leibniz truth equational via $\tau^\flat$. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ and K a class of $\mathbf{F}$-algebraic systems. We define, as before, on $\mathbf{F}$ the closure system $C^{\mathsf{K},\tau} = \{C^{\mathsf{K},\tau}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, where, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $C^{\mathsf{K},\tau}_\Sigma : \mathcal{P}(\mathrm{SEN}^\flat(\Sigma)) \to \mathcal{P}(\mathrm{SEN}^\flat(\Sigma))$ is given, for all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, by

$$\phi \in C^{\mathsf{K},\tau}_\Sigma(\Phi) \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq C^{\mathsf{K}}(\tau^\flat_\Sigma[\Phi]).$$

Then we say that K is a **$\tau^\flat$-algebraic semantics for** $\mathcal{I}$ if $C = C^{\mathsf{K},\tau}$.

We show that, if a π-institution $\mathcal{I}$ is Leibniz truth equational, with witnessing transformations $\tau^\flat$, then any of the four classes $\mathrm{AlgSys}^*(\mathcal{I}^+)$, $\mathrm{AlgSys}(\mathcal{I}^+)$, $\mathrm{AlgSys}^*(\mathcal{I})$ or $\mathrm{AlgSys}(\mathcal{I})$ serves as a $\tau^\flat$-algebraic semantics for $\mathcal{I}^+$.

Theorem 1693 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a Leibniz truth equational π-institution based on $\mathbf{F}$, with witnessing transformations $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$. Set $\mathsf{K} = \mathrm{AlgSys}^*(\mathcal{I}^+)$ or $\mathrm{AlgSys}(\mathcal{I}^+)$ or $\mathrm{AlgSys}^*(\mathcal{I})$ or $\mathrm{AlgSys}(\mathcal{I})$. Then K is a $\tau^\flat$-algebraic semantics for $\mathcal{I}^+$.*

Proof: Let, first, $K = \mathrm{AlgSys}^*(\mathcal{I})$ or $\mathrm{AlgSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. Then, we have $\phi \in C^+_\Sigma(\Phi)$ if and only if, by Proposition 1664, $\phi \in C^{\mathsf{M}^{\mathcal{I},m}_{\mathsf{K}}}_\Sigma(\Phi)$ if and only if, for all $\mathcal{A} \in \mathsf{K}$,

$$\alpha_\Sigma(\Phi) \subseteq C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\varnothing) \quad \text{implies} \quad \alpha_\Sigma(\phi) \in C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\varnothing).$$

if and only if, by hypothesis and Proposition 1691,

$$\alpha_\Sigma(\Phi) \subseteq \tau^{\mathcal{A}}_{F(\Sigma)}(\Delta^{\mathcal{A}}) \quad \text{implies} \quad \alpha_\Sigma(\phi) \in \tau^{\mathcal{A}}_{F(\Sigma)}(\Delta^{\mathcal{A}}),$$

if and only if

$$\tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\Phi)] \leq \Delta^{\mathcal{A}} \quad \text{implies} \quad \tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi)] \leq \Delta^{\mathcal{A}},$$

if and only if

$$\alpha(\tau^\flat_\Sigma[\Phi]) \leq \Delta^{\mathcal{A}} \quad \text{implies} \quad \alpha(\tau^\flat_\Sigma[\phi]) \leq \Delta^{\mathcal{A}},$$

if and only if $\tau^\flat_\Sigma[\phi] \leq C^{\mathsf{K}}(\tau^\flat_\Sigma[\Phi])$ if and only if $\phi \in C^{\mathsf{K},\tau}_\Sigma(\Phi)$. Thus, K is a $\tau^\flat$-algebraic semantics of $\mathcal{I}^+$.

Finally, note that, by hypothesis and Lemma 1671, $\mathcal{I}^+$ is Leibniz truth equational via $\tau^\flat$, as well. Moreover, by Corollary 1668, $(\mathcal{I}^+)^+ = \mathcal{I}^+$. Applying, therefore, what was shown above to $\mathcal{I}^+$, we get the result for $\mathsf{K} = \mathrm{AlgSys}^*(\mathcal{I}^+)$ or $\mathrm{AlgSys}(\mathcal{I}^+)$. ■

Theorem 1693 implies that for $\mathrm{AlgSys}(\mathcal{I})$ to be a $\tau^\flat$-algebraic semantics of a Leibniz truth equational π-institution $\mathcal{I}$, where $\tau^\flat$ is a set of witnessing transformations, $\mathcal{I}$ and $\mathcal{I}^+$ must be identical.

Corollary 1694 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Leibniz truth equational* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*.* $\mathrm{AlgSys}(\mathcal{I})$ *is a* $\tau^\flat$*-algebraic semantics for* $\mathcal{I}$ *if and only if* $\mathcal{I} = \mathcal{I}^+$*.*

Proof: By Theorem 1693, $C^+ = C^{\mathrm{AlgSys}(\mathcal{I}),\tau}$. Therefore, we get that $\mathrm{AlgSys}(\mathcal{I})$ is a $\tau^\flat$-algebraic semantics of $\mathcal{I}$ if and only if, by definition $C = C^{\mathrm{AlgSys}(\mathcal{I}),\tau}$ if and only if $C = C^+$. ∎

Moreover, we can show that Leibniz truth equationality of $\mathcal{I}$ implies the family truth equationality of $\mathcal{I}^+$.

Corollary 1695 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is Leibniz truth equational, with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$*, then* $\mathcal{I}^+$ *is family truth equational via* $\tau^\flat$*.*

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. By hypothesis and Theorem 1693, $\mathcal{I}^+$ has a $\tau^\flat$-algebraic semantics. Therefore, by Corollary 824, $T = \tau^{\mathcal{A}}(\widetilde{\Omega}^{\mathcal{I}^+,\mathcal{A}}(T)) \le \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$. Conversely, by hypothesis and the fact that, by Proposition 1665, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get, using Lemma 1568, $\tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T)) = T^* \le T$. We now conclude that $T = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$. Thus, $\mathcal{I}^+$ is family truth equational, with witnessing transformations $\tau^\flat$. ∎

As another consequence, we get that, under Leibniz truth equationality, $\mathcal{I}^+$ filter families coincide with Leibniz $\mathcal{I}$-filter families on any algebraic system.

Corollary 1696 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is Leibniz truth equational, then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$*.*

Proof: Suppose $\mathcal{I}$ is Leibniz truth equational. Then, by Corollary 1695, $\mathcal{I}^+$ is family truth equational. Thus, by Proposition 1673, $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$, for every $\mathbf{F}$-algebraic system $\mathcal{A}$. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a Leibniz truth equational π-institution, with witnessing transformations $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$. Let, also, $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, by definition $T^{\mathcal{I},\mathsf{Su}} = \bigcap [\![T]\!]^{\mathcal{I},\mathsf{Su}}$ and, by Proposition 1584, $\langle \mathcal{A}, [\![T]\!]^{\mathcal{I},\mathsf{Su}} \rangle \in \mathrm{FStr}(\mathcal{I})$. Thus, by Proposition 1584, $T^{\mathcal{I},\mathsf{Su}} \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. Now it follows, by hypothesis, that

$$T^{\mathcal{I},\mathsf{Su}} = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T^{\mathcal{I},\mathsf{Su}})).$$

There is also an additional characterization of the Suszko filter family, using the Suszko operator.

Proposition 1697 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Leibniz truth equational* π*-institution, with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$T^{\mathcal{I},\mathsf{Su}} = \tau^{\mathcal{A}}(\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)).$$

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and consider the quotient morphism

$$\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T).$$

Then $\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \mathrm{AlgSys}(\mathcal{I})$. Moreover, by Lemma 1557, $\pi(T^{\mathcal{I},\mathsf{Su}}) = \pi(T)^{\mathcal{I},\mathsf{Su}}$ and, by Proposition 1587, $\pi(T)^{\mathcal{I},\mathsf{Su}} = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))$. Thus, by Proposition 1691,

$$\pi(T^{\mathcal{I},\mathsf{Su}}) = \tau^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(\Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}).$$

Now we get

$$\begin{array}{rcl} T^{\mathcal{I},\mathsf{Su}} & = & \pi^{-1}(\pi(T^{\mathcal{I},\mathsf{Su}})) \\ & = & \pi^{-1}(\tau^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(\Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)})) \\ & = & \tau^{\mathcal{A}}(\pi^{-1}(\Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)})) \\ & = & \tau^{\mathcal{A}}(\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)). \end{array}$$

This proves the statement. ■

Proposition 1697 enables us to characterize the Suszko filter counterpart $T^{\mathcal{I},\mathsf{Su}}$ of a given filter family T as the intersection of all Leibniz filter family companions of filter families in the upset of T.

Corollary 1698 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Leibniz truth equational* π*-institution, with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$T^{\mathcal{I},\mathsf{Su}} = \bigcap \{T'^* : T \le T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}.$$

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then we have

$$\begin{array}{rcl} T^{\mathcal{I},\mathsf{Su}} & = & \tau^{\mathcal{A}}(\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)) \quad \text{(by Proposition 1697)} \\ & = & \tau^{\mathcal{A}}(\bigcap \{\Omega^{\mathcal{A}}(T') : T \le T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}) \\ & & \text{(definition of } \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)) \\ & = & \bigcap \{\tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T')) : T \le T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}) \\ & = & \bigcap \{T'^* : T \le T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}. \\ & & \text{(Leibniz truth equationality)} \end{array}$$

This proves the corollary. ■

We now get immediately

Corollary 1699 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Leibniz truth equational* π*-institution, with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{I}) \quad \textit{iff} \quad T \leq T'^*, \ \textit{for all } T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T.$$

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then we have $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$ if and only if, by definition, $T = T^{\mathcal{I},\mathsf{Su}}$ if and only if, by Corollary 1698, $T = \bigcap\{T'^* : T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T\}$, if and only if, taking into account that $T^* \leq T$, $T \leq T'^*$, for all $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$. ∎

We close the section with a characterization of weak family algebraizability of the strong version of $\mathcal{I}$ among those π-institutions that are Leibniz truth equational.

Proposition 1700 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Leibniz truth equational* π*-institution, with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*.* $\mathcal{I}^+$ *is weakly family algebraizable if and only if, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I}^+)$*,* $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^+*}(\mathcal{A})$ *is an order isomorphism.*

Proof: If $\mathcal{I}^+$ is weakly family algebraizable, then it is, a fortiori, protoalgebraic. Therefore, by Proposition 1621, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I}^+)$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^+*}(\mathcal{A})$ is an order isomorphism.

Assume, conversely, that the condition in the statement holds. Then, for every $\mathbf{F}$-algebraic system $\mathcal{A}$,

$$\begin{aligned} \mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A}) &= \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \quad \text{(by Corollary 1672)} \\ &= \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}). \quad \text{(by Corollary 1696)} \end{aligned}$$

Thus, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I}^+)$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^+*}(\mathcal{A})$ is an order isomorphism. Hence, by Theorem 296, $\mathcal{I}^+$ is weakly family algebraizable. ∎

Proposition 1700 gives a sufficient condition for the weak family algbebraizability of $\mathcal{I}^+$ that involves only $\mathcal{I}$-algebraic systems.

Corollary 1701 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Leibniz truth equational* π*-institution, with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*. If, for every* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*,* $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is an order isomorphism, then* $\mathcal{I}^+$ *is weakly family algebraizable.*

Proof: By hypothesis and Lemma 1679, for every $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I}^+)$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^+*}(\mathcal{A})$ is an order isomorphism. Hence, by Proposition 1700, $\mathcal{I}^+$ is weakly family algebraizable. ∎

22.5 Leibniz Definability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is **Leibniz definable** if, there exists $\mu^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, such that, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$T^* = \mu^{\mathcal{A}}(T),$$

i.e., for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\phi \in T^*_\Sigma \quad \text{iff} \quad \mu^{\mathcal{A}}_\Sigma[\phi] \leq T.$$

We show that it suffices to consider only $\mathcal{I}^*$-algebraic systems to establish Leibniz definability.

Proposition 1702 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is Leibniz definable if and only if, there exists $\mu^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, such that, for all $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I})$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T^* = \mu^{\mathcal{A}}(T)$.*

Proof: The "only if" is trivial. For the "if", suppose the stated condition holds and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Consider the quotient morphism

$$\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T).$$

Then $\mathcal{A}/\Omega^{\mathcal{A}}(T) \in \mathrm{AlgSys}^*(\mathcal{I})$ and, moreover, $\mathrm{Ker}(\langle I, \pi \rangle) = \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T^*)$, since $T^* \in [\![T]\!]^*$. Now we have

$$\begin{array}{rcll} T^* & = & \pi^{-1}(\pi(T^*)) & (\mathrm{Ker}(\langle I, \pi \rangle) \text{ compatible with } T^*) \\ & = & \pi^{-1}(\pi(T)^*) & (\text{by Lemma 1557}) \\ & = & \pi^{-1}(\mu^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\pi(T))) & (\text{by hypothesis}) \\ & = & \mu^{\mathcal{A}}(\pi^{-1}(\pi(T))) & (\text{algebra and surjectivity of } \langle I, \pi \rangle) \\ & = & \mu^{\mathcal{A}}(T). & (\mathrm{Ker}(\langle I, \pi \rangle) \text{ compatible with } T) \end{array}$$

Therefore, $\mathcal{I}$ is Leibniz definable via $\mu^\flat$. ■

Leibniz definability ensures that the mapping sending a filter family to it Leibniz counterpart is monotone and this, in turn, implies that T^* is the largest Leibniz filter family below T.

Lemma 1703 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a Leibniz definable π-institution based on $\mathbf{F}$, with witnessing transformations $\mu^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$. For every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,*

$$T \leq T' \quad \textit{implies} \quad T^* \leq T'^*.$$

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \leq T'$. Then $T^* = \mu^{\mathcal{A}}(T) \leq \mu^{\mathcal{A}}(T') = T'^*$. ■

Corollary 1704 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Leibniz definable* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $\mu^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* T^* *is the largest Leibniz filter family below* T*.*

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Suppose $T' \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$, such that $T' \leq T$. Then we have $T' = T'^* \leq T^*$, where the last inclusion is due to Lemma 1703. ■

Under Leibniz definability, the condition that $\Omega^{\mathcal{A}}$ be an order isomorphism from Leibniz filter families of $\mathcal{A}$ onto $\mathcal{I}^*$-congruence systems on $\mathcal{A}$, for every $\mathcal{I}$-algebraic system yields protoalgebraicity.

Proposition 1705 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Leibniz definable* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $\mu^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*. If, for every* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*,* $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is an order isomorphism, then* $\mathcal{I}$ *is protoalgebraic.*

Proof: Suppose the stated condition holds and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then we have

$$\begin{aligned}
\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) &= \bigcap\{\Omega^{\mathcal{A}}(T') : T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \\
&\quad \text{(definition of } \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)) \\
&= \bigcap\{\Omega^{\mathcal{A}}(T'^*) : T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \\
&\quad \text{(by Proposition 1677)} \\
&= \Omega^{\mathcal{A}}(\bigcap\{T'^* : T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}) \\
&\quad \text{(by the hypothesis)} \\
&= \Omega^{\mathcal{A}}(T^*) \quad \text{(by Lemma 1703)} \\
&= \Omega^{\mathcal{A}}(T). \quad \text{(by Proposition 1677)}
\end{aligned}$$

Hence, the Leibniz and Suszko operators on every $\mathbf{F}$-algebraic system coincide, whence, by Lemma 1518, $\mathcal{I}$ is protoalgebraic. ■

We show, next, that, under Leibniz definability, the collection of Leibniz $\mathcal{I}$-filter families on every $\mathbf{F}$-algebraic system is closed under morphic images and preimages and under intersections.

Proposition 1706 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Leibniz definable* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $\mu^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*.*

(a) $\mathbb{M}(\mathsf{M}^{\mathcal{I}*}) \subseteq \mathsf{M}^{\mathcal{I}^*}$ *and* $\mathbb{M}^{-1}(\mathsf{M}^{\mathcal{I}*}) \subseteq \mathsf{M}^{\mathcal{I}^*}$*;*

(b) $\mathbb{I}\mathbb{I}(\mathsf{M}^{\mathcal{I}*}) \subseteq \mathsf{M}^{\mathcal{I}^*}$.

Proof:

(a) Let $\mathcal{A}$, $\mathcal{B}$ be $\mathbf{F}$-algebraic systems, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$ and $\langle H, \gamma\rangle : \langle \mathcal{A}, T\rangle \to \langle \mathcal{B}, T'\rangle$ a strict surjective morphism. We then have

$$\begin{array}{rll} T = T^* & \text{iff} & T = \mu^{\mathcal{A}}(T) \\ & \text{iff} & \gamma^{-1}(T') = \mu^{\mathcal{A}}(\gamma^{-1}(T')) \\ & \text{iff} & \gamma^{-1}(T') = \gamma^{-1}(\mu^{\mathcal{B}}(T')) \\ & \text{iff} & T' = \mu^{\mathcal{B}}(T') \\ & \text{iff} & T' = T'^*. \end{array}$$

Thus, $\langle \mathcal{A}, T\rangle \in \mathsf{M}^{\mathcal{I}*}$ if and only if $\langle \mathcal{B}, T'\rangle \in \mathsf{M}^{\mathcal{I}*}$.

(b) Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $\{T^i : i \in I\} \subseteq \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. Then we have

$$\begin{array}{rcl} \bigcap_{i\in I} T^i & = & \bigcap_{i\in I}(T^i)^* \\ & = & \bigcap_{i\in I} \mu^{\mathcal{A}}(T^i) \\ & = & \mu^{\mathcal{A}}(\bigcap_{i\in I} T^i) \\ & = & (\bigcap_{i\in I} T^i)^*. \end{array}$$

Therefore $\bigcap_{i\in I} T^i \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. Thus, if $\langle \mathcal{A}, T^i\rangle \in \mathsf{M}^{\mathcal{I}*}$, for all $i \in I$, then $\langle \mathcal{A}, \bigcap_{i\in I} T^i\rangle \in \mathsf{M}^{\mathcal{I}*}$.

■

Proposition 1706, in conjunction with the characterization Theorem 1787 of the $\mathcal{I}^{\mathsf{M}}$-matrix families for a class M of $\mathbf{F}$-matrix families, allow us to prove that, under Leibniz definability, $\mathcal{I}^+$-filter families and Leibniz $\mathcal{I}$-filter families on any $\mathbf{F}$-algebraic system coincide.

Theorem 1707 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a Leibniz definable π-institution based on $\mathbf{F}$, with witnessing transformations $\mu^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$. For every $\mathbf{F}$-algebraic system $\mathcal{A}$,*

$$\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: We have

$$\begin{array}{rcll} \mathrm{MatFam}(\mathcal{I}^+) & = & \mathrm{MatFam}(\mathcal{I}^{\mathsf{M}^{\mathcal{I}*}}) & (\mathcal{I}^+ = \mathcal{I}^{\mathsf{M}^{\mathcal{I}*}}, \text{ by definition}) \\ & = & \mathbb{M}\mathbb{I}\mathbb{P}\mathbb{I}\mathbb{M}^{-1}(\mathsf{M}^{\mathcal{I}*}) & (\text{by Theorem 1787}) \\ & \subseteq & \mathsf{M}^{\mathcal{I}*}. & (\text{by Proposition 1706}) \end{array}$$

This shows that $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. But, by Proposition 1665, the reverse inclusion always holds. Therefore, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. ■

We give several conditions involving the strong version of $\mathcal{I}$ that turn out to characterize both the protoalebraicity of $\mathcal{I}$ and the protoalgebraicity of $\mathcal{I}^+$, under the proviso that $\mathcal{I}$ be Leibniz definable.

Corollary 1708 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a Leibniz definable π-institution based on $\mathbf{F}$. The following conditions are equivalent:*

(i) $\mathcal{I}^+$ *is protoalgebraic;*

(ii) $\mathcal{I}$ *is protoalgebraic;*

(iii) For every $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is an order isomorphism;*

(iv) For every $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I}^+)$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^+*}(\mathcal{A})$ *is an order isomorphism;*

(v) $\mathcal{I}^+$ *is weakly family algebraizable.*

Proof:

(i)⇒(ii) Suppose $\mathcal{I}^+$ is protoalgebraic. Let $\mathcal{A}$ be an **F**-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \le T'$. By Lemma 1703, $T^* \le T'^*$. Hence, by Proposition 1665 and the hypothesis, $\Omega^{\mathcal{A}}(T^*) \le \Omega^{\mathcal{A}}(T'^*)$. By hypothesis, Proposition 1621 and Proposition 1677, $\Omega^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')$. Thus, the Leibniz operator is monotone on the $\mathcal{I}$-filter families of every **F**-algebraic system and, therefore, $\mathcal{I}$ is protoalgebraic.

(ii)⇒(iii) By Proposition 1621.

(iii)⇒(iv) By Lemma 1679.

(iv)⇒(v) We have, for every **F**-algebraic system $\mathcal{A}$,

$$\begin{array}{rcll} \mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A}) & = & \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) & \text{(by Corollary 1672)} \\ & = & \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}). & \text{(by Theorem 1707)} \end{array}$$

Therefore, by hypothesis, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^+*}(\mathcal{A})$ is an order isomorphism. By Theorem 296, $\mathcal{I}^+$ is weakly family algebraizable.

(v)⇒(i) If $\mathcal{I}^+$ is weakly family algebraizable, then it is, a fortiori, protoalgebraic. ■

Finally, we give some consequences of imposing both Leibniz definability and Leibniz truth equationality. The combination is strong enough to guarantee that Leibniz filter families and Suszko filter families coincide.

Proposition 1709 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Leibniz definable and Leibniz truth equational* π*-institution based on* **F**. *For every* **F***-algebraic system* $\mathcal{A}$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$T^* = T^{\mathcal{I},\mathsf{Su}}.$$

Proof: Let $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then

$$\begin{array}{lll} T^{\mathcal{I},\mathsf{Su}} & = & \bigcap\{T'^* : T \le T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \quad \text{(by Corollary 1698)} \\ & = & T^*. \quad \text{(by Lemma 1703)} \end{array}$$

This proves the statement. ∎

Corollary 1710 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Leibniz definable and Leibniz truth equational* π*-institution based on* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,*

$$\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}).$$

Proof: Let $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$. By Lemma 1583, $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. On the other hand, if $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$, then, by Proposition 1709, $T = T^* = T^{\mathcal{I},\mathsf{Su}}$. Thus, $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. ∎