Chapter 23

The Frege Hierarchy

23.1 The Frege Hierarchy

23.2 Self Extensionality and Implication

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ a binary natural transformation in $N^\flat$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

We say that $\to^\flat$ has the **Deduction Detachment Property in** $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi, \psi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\psi \in C_\Sigma(\Phi, \phi) \quad \text{iff} \quad \phi \to^\flat_\Sigma \psi \in C_\Sigma(\Phi).$$

$\mathcal{I}$ has the **Uniterm Deduction Detachment Property with respect to** $\to^\flat$ if $\to^\flat$ has the Deduction Detachment Property in $\mathcal{I}$. $\mathcal{I}$ has the **Uniterm Deduction Detachment Property** if it has the Uniterm Deduction Detachment Property with respect to some $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$.

If a π-institution has the Uniterm Deduction Detachment Theorem with respect to two different binary natural transformations in $N^\flat$, then the two must be interderivable in the following precise sense.

Lemma 1711 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\to^\flat, \to'^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$, *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *has the Uniterm Deduction Detachment Property with respect to both* $\to^\flat$ *and* $\to'^\flat$, *then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C_\Sigma(\phi \to^\flat_\Sigma \psi) = C_\Sigma(\phi \to'^\flat_\Sigma \psi).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. We have $\phi \to^\flat_\Sigma \psi \in C_\Sigma(\phi \to^\flat_\Sigma \psi)$. By the Uniterm Deduction Detachment Property with respect to $\to^\flat$, we get $\psi \in C_\Sigma(\phi, \phi \to^\flat_\Sigma \psi)$. By the Uniterm Deduction Detachment Property with respect to $\to'^\flat$, we get $\phi \to'^\flat_\Sigma \psi \in C_\Sigma(\phi \to^\flat_\Sigma \psi)$. Using symmetry, we obtain that $C_\Sigma(\phi \to^\flat_\Sigma \psi) = C_\Sigma(\phi \to'^\flat_\Sigma \psi)$. ■

Thus, for self extensional π-institutions, we get immediately

Corollary 1712 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\to^\flat, \to'^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$, *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a self extensional* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *has the Uniterm Deduction Detachment Property with respect to both* $\to^\flat$ *and* $\to'^\flat$, *then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi \to^\flat_\Sigma \psi, \phi \to'^\flat_\Sigma \psi \rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I}).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. By Lemma 1711, $\langle \phi \to^\flat_\Sigma \psi, \phi \to'^\flat_\Sigma \psi \rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I})$. But, by self extensionality, $\widetilde{\lambda}(\mathcal{I}) = \widetilde{\Omega}(\mathcal{I})$. This yields the conclusion. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a class of $\mathbf{F}$-algebraic systems. The class K is said to be **Hilbert based with respect to** $\to^\flat$ if, for all $\mathcal{A} \in \mathsf{K}$, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi, \chi \in \mathrm{SEN}(\Sigma)$,

H1. $\phi \to_\Sigma^\mathcal{A} \phi = \psi \to_\Sigma^\mathcal{A} \psi$;

H2. $(\phi \to_\Sigma^\mathcal{A} \phi) \to_\Sigma^\mathcal{A} \phi = \phi$;

H3. $\phi \to_\Sigma^\mathcal{A} (\psi \to_\Sigma^\mathcal{A} \chi) = (\phi \to_\Sigma^\mathcal{A} \psi) \to_\Sigma^\mathcal{A} (\phi \to_\Sigma^\mathcal{A} \chi)$;

H4. $(\phi \to_\Sigma^\mathcal{A} \psi) \to_\Sigma^\mathcal{A} ((\psi \to_\Sigma^\mathcal{A} \phi) \to_\Sigma^\mathcal{A} \psi) = (\psi \to_\Sigma^\mathcal{A} \phi) \to_\Sigma^\mathcal{A} ((\phi \to_\Sigma^\mathcal{A} \psi) \to_\Sigma^\mathcal{A} \phi)$.

These equations are commonly referred to as the **Hilbert equations**. The class K is **Hilbert based** if it is Hilbert based with respect to $\to^\flat$, for some $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$.

A class K of $\mathbf{F}$-algebraic systems is called **pointed** if there exists $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, such that, for all $\mathcal{A} \in \mathsf{K}$, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)$,

$$\top_\Sigma^\mathcal{A}(\vec{\phi}) = \top_\Sigma^\mathcal{A}(\vec{\psi}).$$

$\top^\flat$ is then called a **constant in** K and we sometimes write $\top_\Sigma^\mathcal{A}$ for $\top_\Sigma^\mathcal{A}(\vec{\phi})$, since this value is independent of the argument $\vec{\phi} \in \mathrm{SEN}(\Sigma)$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a Hilbert based class with respect to $\to^\flat$. Then, by the Hilbert equation H1, the natural transformation $\top^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ in $N^\flat$ defined by

$$\top^\flat := \to^\flat \circ \langle p^{1,0}, p^{1,0} \rangle$$

(in abbreviated more readable form $\top^\flat(x) := x \to^\flat x$) is a constant in K. So in this case, it makes sense to write $\top_\Sigma^\mathcal{A}$ for the constant defined by this natural transformation in $\mathcal{A} \in \mathsf{K}$, for $\Sigma \in |\mathbf{Sign}|$.

Moreover, for $\mathcal{A} \in \mathsf{K}$, we define the relation family $\leq^\mathcal{A} = \{\leq_\Sigma^\mathcal{A}\}_{\Sigma \in |\mathbf{Sign}|}$ on $\mathcal{A}$ by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\phi \leq_\Sigma^\mathcal{A} \psi \quad \text{iff} \quad \phi \to_\Sigma^\mathcal{A} \psi = \top_\Sigma^\mathcal{A}.$$

It is not difficult to see that this is actually a partial order system on $\mathcal{A}$.

Lemma 1713 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with a binary* $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* K *a Hilbert based class with respect to* $\to^\flat$*. For all* $\mathcal{A} \in \mathsf{K}$*,* $\leq^\mathcal{A}$ *is a posystem on* $\mathcal{A}$*.*

Proof: We show, first, that, for all $\Sigma \in |\mathbf{Sign}|$, $\leq_\Sigma^\mathcal{A}$ is a partial order on $\mathrm{SEN}(\Sigma)$. Let $\phi, \psi, \chi \in \mathrm{SEN}(\Sigma)$.

- By definition $\phi \to_\Sigma^\mathcal{A} \phi = \top_\Sigma^\mathcal{A}$, whence $\phi \leq_\Sigma^\mathcal{A} \phi$ and $\leq_\Sigma^\mathcal{A}$ is reflexive;

- Suppose $\phi \leq^{\mathcal{A}}_{\Sigma} \psi$ and $\psi \leq^{\mathcal{A}}_{\Sigma} \phi$. Then, we get $\phi \to^{\mathcal{A}}_{\Sigma} \psi = \psi \to^{\mathcal{A}}_{\Sigma} \phi = \top^{\mathcal{A}}_{\Sigma}$. Thus, we get

$$\begin{aligned} \phi &= \top^{\mathcal{A}}_{\Sigma} \to^{\mathcal{A}}_{\Sigma} \phi \quad \text{(by H2)} \\ &= \top^{\mathcal{A}}_{\Sigma} \to^{\mathcal{A}}_{\Sigma} (\top^{\mathcal{A}}_{\Sigma} \to^{\mathcal{A}}_{\Sigma} \phi) \quad \text{(by H2)} \\ &= \top^{\mathcal{A}}_{\Sigma} \to^{\mathcal{A}}_{\Sigma} (\top^{\mathcal{A}}_{\Sigma} \to^{\mathcal{A}}_{\Sigma} \psi) \quad \text{(by H4)} \\ &= \top^{\mathcal{A}}_{\Sigma} \to^{\mathcal{A}}_{\Sigma} \psi \quad \text{(by H2)} \\ &= \psi. \quad \text{(by H2)} \end{aligned}$$

  Hence, $\leq^{\mathcal{A}}_{\Sigma}$ is antisymmetric;

- Suppose $\phi \leq^{\mathcal{A}}_{\Sigma} \psi$ and $\psi \leq^{\mathcal{A}}_{\Sigma} \chi$. Then $\phi \to^{\mathcal{A}}_{\Sigma} \psi = \psi \to^{\mathcal{A}}_{\Sigma} \chi = \top^{\mathcal{A}}_{\Sigma}$. Thus, we get

$$\begin{aligned} \phi \to^{\mathcal{A}}_{\Sigma} \chi &= \top^{\mathcal{A}}_{\Sigma} \to^{\mathcal{A}}_{\Sigma} (\phi \to^{\mathcal{A}}_{\Sigma} \chi) \quad \text{(by H2)} \\ &= (\phi \to^{\mathcal{A}}_{\Sigma} \psi) \to^{\mathcal{A}}_{\Sigma} (\phi \to^{\mathcal{A}}_{\Sigma} \chi) \quad \text{(hypothesis)} \\ &= \phi \to^{\mathcal{A}}_{\Sigma} (\psi \to^{\mathcal{A}}_{\Sigma} \chi) \quad \text{(by H3)} \\ &= \phi \to^{\mathcal{A}}_{\Sigma} \top^{\mathcal{A}}_{\Sigma} \quad \text{(hypothesis)} \\ &= \phi \to^{\mathcal{A}}_{\Sigma} (\phi \to^{\mathcal{A}}_{\Sigma} \phi) \quad \text{(definition)} \\ &= (\phi \to^{\mathcal{A}}_{\Sigma} \phi) \to^{\mathcal{A}}_{\Sigma} (\phi \to^{\mathcal{A}}_{\Sigma} \phi) \quad \text{(by H3)} \\ &= \top^{\mathcal{A}}_{\Sigma}. \quad \text{(definition)} \end{aligned}$$

  So $\leq^{\mathcal{A}}_{\Sigma}$ is also transitive.

Thus, $\leq^{\mathcal{A}}$ is a partial order family on $\mathcal{A}$. We show that, in addition, it is a system, i.e., it is invariant under signature morphisms. To this end, suppose $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\phi \leq^{\mathcal{A}}_{\Sigma} \psi$. Then $\phi \to^{\mathcal{A}}_{\Sigma} \psi = \top^{\mathcal{A}}_{\Sigma}$. Hence, $\mathrm{SEN}(f)(\phi \to^{\mathcal{A}}_{\Sigma} \psi) = \mathrm{SEN}(f)(\top^{\mathcal{A}}_{\Sigma})$. This gives

$$\mathrm{SEN}(f)(\phi) \to^{\mathcal{A}}_{\Sigma'} \mathrm{SEN}(f)(\psi) = \top^{\mathcal{A}}_{\Sigma'}.$$

We conclude that $\mathrm{SEN}(f)(\phi) \leq^{\mathcal{A}}_{\Sigma'} \mathrm{SEN}(f)(\psi)$. Therefore, $\leq^{\mathcal{A}}$ is indeed a posystem on $\mathcal{A}$. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, K a Hilbert based class with respect to $\to^\flat$, $\mathcal{A} \in \mathsf{K}$ and $T \in \mathrm{SenFam}(\mathcal{A})$. We say that T is an **$\to^\flat$-implicative filter family of $\mathcal{A}$** if

- $\top^{\mathcal{A}}_{\Sigma} \in T_\Sigma$, for all $\Sigma \in |\mathbf{Sign}|$;
- $\phi \to^{\mathcal{A}}_{\Sigma} \psi \in T_\Sigma$ and $\phi \in T_\Sigma$ imply $\psi \in T_\Sigma$, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$.

We write $\mathrm{FiFam}^{\to}(\mathcal{A})$ for the collection of all $\to^\flat$-implicative filter families on $\mathcal{A}$.

Next, we show that in any $\mathbf{F}$-algebraic system $\mathcal{A}$ in a Hilbert based class K, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi_0, \dots, \phi_{n-1}, \phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{aligned} &\phi_0 \to^{\mathcal{A}}_{\Sigma} (\phi_1 \to^{\mathcal{A}}_{\Sigma} \cdots \to^{\mathcal{A}}_{\Sigma} (\phi_{n-1} \to^{\mathcal{A}}_{\Sigma} \phi) \cdots) = \top^{\mathcal{A}}_{\Sigma} \\ &\quad \text{iff} \quad \phi_{\pi(0)} \to^{\mathcal{A}}_{\Sigma} (\phi_{\pi(1)} \to^{\mathcal{A}}_{\Sigma} \cdots \to^{\mathcal{A}}_{\Sigma} (\phi_{\pi(n-1)} \to^{\mathcal{A}}_{\Sigma} \phi) \cdots) = \top^{\mathcal{A}}_{\Sigma}, \end{aligned}$$

where π is any permutation of $\{0, \dots, n-1\}$.

Lemma 1714 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with a binary* $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* K *a Hilbert based class of* $\mathbf{F}$*-algebraic systems with respect to* $\to^\flat$*. For all* $\mathcal{A} \in \mathsf{K}$*, all* $\Sigma \in |\mathbf{Sign}|$*, all* $\phi_0, \phi_1, \dots, \phi_{n-1}, \phi \in \mathrm{SEN}(\Sigma)$ *and every permutation* π *of* $\{0, 1, \dots, n-1\}$*,*

$$\begin{array}{l}\phi_0 \to^{\mathcal{A}}_\Sigma (\phi_1 \to^{\mathcal{A}}_\Sigma \cdots \to^{\mathcal{A}}_\Sigma (\phi_{n-1} \to^{\mathcal{A}}_\Sigma \phi)\cdots) = \top^{\mathcal{A}}_\Sigma \\ \quad \textit{iff} \quad \phi_{\pi(0)} \to^{\mathcal{A}}_\Sigma (\phi_{\pi(1)} \to^{\mathcal{A}}_\Sigma \cdots \to^{\mathcal{A}}_\Sigma (\phi_{\pi(n-1)} \to^{\mathcal{A}}_\Sigma \phi)\cdots) = \top^{\mathcal{A}}_\Sigma.\end{array}$$

Proof:

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Lemma 1714 allows us to write

$$\vec{\vec{\Phi}} \to^{\mathcal{A}}_\Sigma \phi = \top^{\mathcal{A}}_\Sigma$$

for $\phi_0 \to^{\mathcal{A}}_\Sigma (\phi_1 \to^{\mathcal{A}}_\Sigma \cdots \to^{\mathcal{A}}_\Sigma (\phi_{n-1} \to^{\mathcal{A}}_\Sigma \phi)\cdots) = \top^{\mathcal{A}}_\Sigma$, where $\Phi = \{\phi_0, \dots, \phi_{n-1}\}$, when appropriate, since the equation does not depend on the order in which the elements of Φ are arranged in the implication expression. Moreover, for convenience, if $\Phi = \varnothing$, we take

$$\vec{\vec{\Phi}} \to^{\mathcal{A}}_\Sigma \phi := \phi.$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, K a Hilbert based class with respect to $\to^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a *finitary* π-institution based on $\mathbf{F}$. $\mathcal{I}$ is called **Hilbert based with respect to** K **and** $\to^\flat$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in C_\Sigma(\Phi) \quad \text{iff} \quad \text{for all } \mathcal{A} \in \mathsf{K},\ \alpha_\Sigma(\vec{\vec{\Phi}} \to^\flat_\Sigma \phi) = \top^{\mathcal{A}}_{F(\Sigma)}.$$

We say that $\mathcal{I}$ is **Hilbert based** if there exists $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$ and a Hilbert based class K of $\mathbf{F}$-algebraic systems with respect to $\to^\flat$, such that $\mathcal{I}$ is Hilbert based with respect to K and $\to^\flat$.

Corollary 1715 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with a binary* $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*,* K *a Hilbert based class with respect to* $\to^\flat$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Hilbert based* π*-institution with respect to* K *and* $\to^\flat$*. For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$C_\Sigma(\phi) = C_\Sigma(\psi) \quad \textit{iff} \quad \langle \phi, \psi \rangle \in \mathrm{Ker}_\Sigma(\mathsf{K}).$$

Proof: Suppose $\mathcal{I}$ is Hilbert based with respect to K and $\to^\flat$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $C_\Sigma(\phi) = C_\Sigma(\psi)$ if and only if, by definition, for all $\mathcal{A} \in \mathsf{K}$, $\alpha_\Sigma(\phi \to^\flat_\Sigma \psi) = \alpha_\Sigma(\psi \to^\flat_\Sigma \phi) = \top^{\mathcal{A}}_{F(\Sigma)}$, if and only if, by Lemma 1713, for all $\mathcal{A} \in \mathsf{K}$, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$, if and only if, $\langle \phi, \psi \rangle \in \mathrm{Ker}_\Sigma(\mathsf{K})$.
■

It is not difficult to see that if a π-institution is Hilbert based with respect to a Hilbert based class K, then it is also Hilbert based with respect to the semantic variety generated by K.

Lemma 1716 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with a binary* $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*,* K *a Hilbert based class with respect to* $\to^\flat$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Hilbert based* π*-institution with respect to* K *and* $\to^\flat$*. Then* $\mathcal{I}$ *is also Hilbert based with respect to* $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$ *and* $\to^\flat$*.*

Proof: Assume that $\mathcal{I}$ is Hilbert based with respect K and $\to^\flat$. First, note that, since, for all $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$, $\mathrm{Ker}(\mathsf{K}) \le \mathrm{Ker}(\mathcal{A})$, all $\mathbf{F}$-algebraic systems in $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$ satisfy the Hilbert equations and, hence, $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$ is a Hilbert based class with respect to $\to^\flat$.

Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$.

Suppose $\phi \in C_\Sigma(\Phi)$ and let $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. By hypothesis $\langle \vec{\vec{\Phi}} \to^\flat_\Sigma \phi, \top^\flat_\Sigma \rangle \in \mathrm{Ker}_\Sigma(\mathsf{K})$. Since $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$, $\mathrm{Ker}(\mathsf{K}) \le \mathrm{Ker}(\mathcal{A})$. Therefore, $\langle \vec{\vec{\Phi}} \to^\flat_\Sigma \phi, \top^\flat_\Sigma \rangle \in \mathrm{Ker}_\Sigma(\mathcal{A})$. This shows that $\alpha_\Sigma(\vec{\vec{\Phi}} \to^\flat_\Sigma \phi) = \top^{\mathcal{A}}_{F(\Sigma)}$. Conversely, if, for all $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$, $\alpha_\Sigma(\vec{\vec{\Phi}} \to^\flat_\Sigma \phi) = \top^{\mathcal{A}}_{F(\Sigma)}$, then this holds, a fortiori, for all $\mathcal{A} \in \mathsf{K}$ and, hence, by the hypothesis $\phi \in C_\Sigma(\Phi)$.

Thus, $\mathcal{I}$ is Hilbert based both with respect to $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$ and $\to^\flat$. ∎

Corollary 1717 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with a binary* $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*,* K *a Hilbert based class with respect to* $\to^\flat$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Hilbert based* π*-institution with respect to* K *and* $\to^\flat$*. For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$C_\Sigma(\phi) = C_\Sigma(\psi) \quad \textit{iff} \quad \langle \phi, \psi \rangle \in \mathrm{Ker}_\Sigma(\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})).$$

Proof: By Corollary 1715 and Lemma 1716. ∎

We can also show that, if K and K' are two Hilbert based classes of $\mathbf{F}$-algebraic systems with respect to binary transformations $\to^\flat$ and $\to'^\flat$ in $N^\flat$, respectively, and a π-institution $\mathcal{I}$ happens to be Hilbert based with respect to both K and $\to^\flat$ and K' and $\to'^\flat$, then, the two classes K and K' generate the same semantic variety of $\mathbf{F}$-algebraic systems.

Proposition 1718 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\to^\flat, \to'^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*,* K *a Hilbert class with respect to* $\to^\flat$ *and* K' *a Hilbert class with respect to* $\to'^\flat$*. If* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *is a* π*-institution that is Hilbert based with respect to* K *and* $\to^\flat$ *and Hilbert based with respect to* K' *and* $\to'^\flat$*, then* $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathbb{V}^{\mathsf{Sem}}(\mathsf{K}')$*.*

Proof: We show that $\mathsf{K}' \subseteq \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. Then the conclusion will follow by symmetry. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in \mathrm{Ker}_\Sigma(\mathsf{K})$, and $\mathcal{A}' \in \mathsf{K}'$. By hypothesis, for all $\mathcal{A}$, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. Hence, for all $\mathcal{A} \in \mathsf{K}$, $\alpha_\Sigma(\phi \to^\flat_\Sigma \psi) = \alpha_\Sigma(\psi \to^\flat_\Sigma \phi) = \top^{\mathcal{A}}_{F(\Sigma)}$. Thus, since $\mathcal{I}$ is Hilbert based with respect to K and $\to^\flat$, we get $C_\Sigma(\phi) = C_\Sigma(\psi)$. But, by hypothesis, $\mathcal{I}$ is also Hilbert based with respect to K' and $\to'^{\flat}$, whence $\alpha'_\Sigma(\phi \to'^{\flat}_\Sigma \psi) = \alpha'_\Sigma(\psi \to'^{\flat}_\Sigma \phi) = \top^{\mathcal{A}'}_{F'(\Sigma)}$. Hence, by Lemma 1713, $\alpha'_\Sigma(\phi) = \alpha'_\Sigma(\psi)$ or, equivalently, $\langle\phi,\psi\rangle \in \mathrm{Ker}_\Sigma(\mathcal{A}')$. This shows that $\mathcal{A}' \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. Thus, $\mathsf{K}' \subseteq \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. ■

We conclude that, if $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is a Hilbert based π-institution, there is a unique semantic variety of $\mathbf{F}$-algebraic systems, with respect to which it is Hilbert based. We denote this semantic variety by $\mathbb{V}^{\mathsf{Sem}}(\mathcal{I})$ and call it the **semantic variety of** $\mathcal{I}$.

A key result is that every Hilbert based π-institution is self extensional and has the Deduction Detachment Property. We also show that the semantic variety of $\mathcal{I}$ coincides with the class $\mathsf{K}^{\mathcal{I}}$, the semantic variety of $\mathcal{I}$.

Proposition 1719 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, K a Hilbert based class with respect to $\to^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a Hilbert based π-institution with respect to K and $\to^\flat$.*

(a) $\mathcal{I}$ is self extensional;

(b) $\mathcal{I}$ has the Deduction Detachment Property with respect to $\to^\flat$;

(c) $\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}) = \mathsf{K}^{\mathcal{I}}$; Thus, $\mathcal{I}$ is Hilbert based with respect to $\mathsf{K}^{\mathcal{I}}$ and $\to^\flat$.

Proof:

(a) We must show that $\widetilde{\Lambda}(\mathcal{I}) = \widetilde{\Omega}(\mathcal{I})$. Since $\widetilde{\Omega}(\mathcal{I})$ is the largest congruence system on $\mathbf{F}$ that is included in $\widetilde{\lambda}(\mathcal{I})$, it suffices to show that $\widetilde{\lambda}(\mathcal{I})$ is a congruence system. To this end, let $\sigma^\flat$ be a natural transformation in $N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi_i, \psi_i\rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I})$, for all $i < k$. Hence, by definition, $C_\Sigma(\phi_i) = C_\Sigma(\psi_i)$, for all $i \in I$. Since $\mathcal{I}$ is Hilbert based with respect to K and $\to^\flat$, we get, by Corollary 1715, $\langle \phi_i, \psi_i\rangle \in \mathrm{Ker}_\Sigma(\mathsf{K})$, for all $i < k$. But $\mathrm{Ker}(\mathsf{K})$ is a congruence system on $\mathbf{F}$, whence $\langle \sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi})\rangle \in \mathrm{Ker}_\Sigma(\mathsf{K})$. Again, by Corollary 1715, $C_\Sigma(\sigma^\flat_\Sigma(\vec{\phi})) = C_\Sigma(\sigma^\flat_\Sigma(\vec{\psi}))$ and, therefore, $\langle \sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi})\rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I})$. We conclude that $\widetilde{\lambda}(\mathcal{I}) = \widetilde{\Omega}(\mathcal{I})$ and, hence, $\mathcal{I}$ is self extensional.

(b) Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi, \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$.

- Suppose that $\psi \in C_\Sigma(\Phi, \phi)$. Since $\mathcal{I}$ is finitary, there exists $\Phi' \subseteq_f \Phi$, such that $\psi \in C_\Sigma(\Phi', \phi)$. Since $\mathcal{I}$ is Hilbert based with respect to K and $\to^\flat$, we get $\langle \overrightarrow{\Phi'} \to^\flat_\Sigma (\phi \to^\flat_\Sigma \psi), \top^\flat_\Sigma\rangle \in \mathrm{Ker}_\Sigma(\mathsf{K})$. Thus, again, since $\mathcal{I}$ is Hilbert based with respect to K and $\to^\flat$, $\phi \to^\flat_\Sigma \psi \in C_\Sigma(\Phi') \subseteq C_\Sigma(\Phi)$.
- Suppose, conversely, that $\phi \to^\flat_\Sigma \psi \in C_\Sigma(\Phi)$. Again, by finitarity, there exists $\Phi' \subseteq_f \Phi$, such that $\phi \to^\flat_\Sigma \psi \in C_\Sigma(\Phi')$. Hence, since $\mathcal{I}$ is Hilbert based with respect to K and $\to^\flat$, $\langle \overrightarrow{\Phi'} \to^\flat_\Sigma (\phi \to^\flat_\Sigma \psi), \top^\flat_\Sigma\rangle \in$

$\mathrm{Ker}_\Sigma(\mathsf{K})$. But, again by the fact that $\mathcal{I}$ is Hilbert based with respect to K and $\to^\flat$, we get that $\psi \in C_\Sigma(\Phi', \phi) \subseteq C_\Sigma(\Phi, \phi)$.

Hence $\mathcal{I}$ has the Deduction Detachment Property with respect to $\to^\flat$.

(c) Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. We have

$$\begin{array}{rll}\langle\phi,\psi\rangle \in \mathrm{Ker}_\Sigma(\mathbb{V}^{\mathsf{Sem}}(\mathcal{I})) & \text{iff} & C_\Sigma(\phi) = C_\Sigma(\psi) \quad \text{(by Corollary 1717)}\\ & \text{iff} & \langle\phi,\psi\rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}) \quad \text{(by definition)}\\ & \text{iff} & \langle\phi,\psi\rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I}) \quad \text{(by Part (a))}\\ & \text{iff} & \langle\phi,\psi\rangle \in \mathrm{Ker}_\Sigma(\mathsf{K}^{\mathcal{I}}). \quad \text{(by definition)}\end{array}$$

Therefore, since $\mathsf{K}^{\mathcal{I}}$ is a semantic variety by definition, we get that $\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}) = \mathsf{K}^{\mathcal{I}}$. The last statement follows now by Lemma 1716. ■

If $\mathcal{I}$ is Hilbert based, not only is the semantic variety with respect to which it is Hilbert based unique, but, in addition, any two binary natural transformations that serve as the Hilbert implications are in a sense interderivable and, hence, indistinguishable modeulo the Tarski congruence system of $\mathcal{I}$.

Corollary 1720 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\to^\flat$, $\to'^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *a Hilbert based* π*-institution with respect to* $\to^\flat$ *and with respect to* $\to'^\flat$*. Then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

(a) $C_\Sigma(\phi \to^\flat_\Sigma \psi) = C_\Sigma(\phi \to'^\flat_\Sigma \psi)$*;*

(b) $\langle\phi \to^\flat_\Sigma \psi, \phi \to'^\flat_\Sigma \psi\rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I})$*.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{rll}\phi \to^\flat_\Sigma \psi \in C_\Sigma(\phi \to^\flat_\Sigma \psi) & \text{iff} & \psi \in C_\Sigma(\phi, \phi \to^\flat_\Sigma \psi) \quad \text{(Proposition 1719)}\\ & \text{iff} & \phi \to'^\flat_\Sigma \psi \in C_\Sigma(\phi \to^\flat_\Sigma \psi). \quad \text{(Proposition 1719)}\end{array}$$

Therefore, $\phi \to'^\flat_\Sigma \psi \in C_\Sigma(\phi \to^\flat_\Sigma \psi)$. By symmetry, we get the conclusion of Part (a). Part (b) follows by Proposition 1719, which asserts that $\mathcal{I}$ is self extensional. ■

If $\mathcal{I}$ is self extensional and has the Deduction Detachment Property with respect to $\to^\flat$, it turns out that the singleton class $\mathsf{K} = \{\mathcal{F}/\widetilde{\Omega}(\mathcal{I})\}$, consisting of the Lindenbaum-Tarski $\mathbf{F}$-algebraic system of $\mathcal{I}$, is Hilbert based with respect to $\to^\flat$.

Lemma 1721 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with a binary* $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *a finitary self extensional* π*-institution, having the Deduction Detachment Property with respect to* $\to^\flat$*. The class* $\mathsf{K} = \{\mathcal{F}/\widetilde{\Omega}(\mathcal{I})\}$ *is Hilbert based with respect to* $\to^\flat$*.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$.

(H1) By the Deduction Detachment Property with respect to $\to^\flat$, we get $C_\Sigma(\phi \to^\flat_\Sigma \phi) = C_\Sigma(\psi \to^\flat_\Sigma \psi) = C_\Sigma(\varnothing)$. Therefore, by self extensionality, $\langle \phi \to^\flat_\Sigma \phi, \psi \to^\flat_\Sigma \psi\rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I})$;

(H2) We have $\phi \to^\flat_\Sigma \phi \in C_\Sigma(\phi \to^\flat_\Sigma \phi)$, whence $\phi \in C_\Sigma(\phi, \phi \to^\flat_\Sigma \phi)$ and, hence, $(\phi \to^\flat_\Sigma \phi) \to^\flat_\Sigma \phi \in C_\Sigma(\phi)$.

On the other hand, $(\phi \to^\flat_\Sigma \phi) \to^\flat_\Sigma \phi \in C_\Sigma((\phi \to^\flat_\Sigma \phi) \to^\flat_\Sigma \phi)$, whence $\phi \in C_\Sigma((\phi \to^\flat_\Sigma \phi) \to^\flat_\Sigma \phi, \phi \to^\flat_\Sigma \phi)$ and, since $\phi \to^\flat_\Sigma \phi \in C_\Sigma(\varnothing)$, $\phi \in C_\Sigma((\phi \to^\flat_\Sigma \phi) \to^\flat_\Sigma \phi)$.

This shows that $C_\Sigma((\phi \to^\flat_\Sigma \phi) \to^\flat_\Sigma \phi) = C_\Sigma(\phi)$ and, hence, by self extensionality, $\langle (\phi \to^\flat_\Sigma \phi) \to^\flat_\Sigma \phi, \phi\rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I})$.

(H3) By the Deduction Detachment Property with respect to $\to^\flat$, we get $\chi \in C_\Sigma(\phi, \phi \to^\flat_\Sigma \psi, (\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma (\phi \to^\flat_\Sigma \chi))$. Thus, since $\phi \to^\flat_\Sigma \psi \in C_\Sigma(\psi)$, we get $\chi \in C_\Sigma(\phi, \psi, (\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma (\phi \to^\flat_\Sigma \chi))$. This gives $\psi \to^\flat_\Sigma \chi \in C_\Sigma(\phi, (\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma (\phi \to^\flat_\Sigma \chi))$ and, hence, $\phi \to^\flat_\Sigma (\psi \to^\flat_\Sigma \chi) \in C_\Sigma((\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma (\phi \to^\flat_\Sigma \chi))$.

On the other hand, $\chi \in C_\Sigma(\phi, \phi \to^\flat_\Sigma \psi, \phi \to^\flat_\Sigma (\psi \to^\flat_\Sigma \chi))$. Therefore, $\phi \to^\flat_\Sigma \chi \in C_\Sigma(\phi \to^\flat_\Sigma \psi, \phi \to^\flat_\Sigma (\psi \to^\flat_\Sigma \chi))$, whence $(\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma (\phi \to^\flat_\Sigma \chi) \in C_\Sigma(\phi \to^\flat_\Sigma (\psi \to^\flat_\Sigma \chi))$.

We conclude that $C_\Sigma(\phi \to^\flat_\Sigma (\psi \to^\flat_\Sigma \chi)) = C_\Sigma((\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma (\phi \to^\flat_\Sigma \chi))$. Thus, by self extensionality, $\langle \phi \to^\flat_\Sigma (\psi \to^\flat_\Sigma \chi), (\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma (\phi \to^\flat_\Sigma \chi)\rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I})$.

(H4) By the Deduction Detachment Property with respect to $\to^\flat$,

$$\psi \in C_\Sigma(\psi \to^\flat_\Sigma \phi, \phi \to^\flat_\Sigma \psi, (\psi \to^\flat_\Sigma \phi) \to^\flat_\Sigma ((\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma \phi)).$$

Therefore, $(\psi \to^\flat_\Sigma \phi) \to^\flat_\Sigma \psi \in C_\Sigma(\phi \to^\flat_\Sigma \psi, (\psi \to^\flat_\Sigma \phi) \to^\flat_\Sigma ((\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma \phi))$, whence $(\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma ((\psi \to^\flat_\Sigma \phi) \to^\flat_\Sigma \psi) \in C_\Sigma((\psi \to^\flat_\Sigma \phi) \to^\flat_\Sigma ((\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma \phi))$. The other inclusion follows similarly and, then, we get the conclusion by self extensionality.

Therefore, $\{\mathcal{F}/\widetilde{\Omega}(\mathcal{I})\}$ is Hilbert based with respect to $\to^\flat$. ∎

Now we can fully characterize those finitary π-institutions which are Hilbert based.

Theorem 1722 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat$ *be an algebraic system and* $\mathcal{I}$ *a finitary* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is Hilbert based if and only if it is self extensional and has the Uniterm Deduction Detachment Property.*

Proof: The left-to-right implication is given by Proposition 1719. Assume, conversely, that $\mathcal{I}$ is self extensional and has the Uniterm Deduction Detachment Property with respect to some $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$. Consider $\mathsf{K} = \{\mathcal{F}/\widetilde{\Omega}(\mathcal{I})\}$. By Lemma 1721, K is Hilbert based with respect to $\to^\flat$. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{lll} \phi \in C_\Sigma(\Phi) & \text{iff} & \vec{\vec{\Phi}} \to^\flat_\Sigma \phi \in C_\Sigma(\varnothing) \quad \text{(Deduction Detachment)} \\ & \text{iff} & \langle \vec{\vec{\Phi}} \to^\flat_\Sigma \phi, \top^\flat_\Sigma \rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}) \quad \text{(definition of } \widetilde{\lambda}(\mathcal{I})) \\ & \text{iff} & \langle \vec{\vec{\Phi}} \to^\flat_\Sigma \phi, \top^\flat_\Sigma \rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I}). \quad \text{(self extensionality)} \end{array}$$

Therefore, $\mathcal{I}$ is Hilbert based with respect to K and, hence, by Lemma 1716, with respect to $\mathsf{K}^{\mathcal{I}} = \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$, and $\to^\flat$. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat$ be an algebraic system, with $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$ and K a Hilbert based semantic variety with respect to $\to^\flat$. We define the *finitary* π-**institution**

$$\mathcal{I}^{\mathsf{K},\to} = \langle \mathbf{F}, C^{\mathsf{K},\to} \rangle,$$

associated with K and $\to^\flat$, by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in C^{\mathsf{K},\to}_\Sigma(\Phi) \quad \text{iff} \quad \langle \vec{\vec{\Phi}} \to^\flat_\Sigma \phi, \top^\flat_\Sigma \rangle \in \mathrm{Ker}_\Sigma(\mathsf{K}).$$

We can see easily from the definition that $\mathcal{I}^{\mathsf{K},\to}$ is Hilbert based with respect to K and $\to^\flat$.

Corollary 1723 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$, *and* K *a Hilbert based semantic variety with respect to* $\to^\flat$. *Then* $\mathcal{I}^{\mathsf{K},\to}$ *is Hilbert based with respect to* K *and* $\to^\flat$ *and, moreover,* $\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}^{\mathsf{K},\to}) = \mathsf{K}$.

Proof: This follows directly from the definition of $\mathcal{I}^{\mathsf{K},\to}$ and by taking into account Lemma 1716 and the definition of $\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}^{\mathsf{K},\to})$. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat$ be an algebraic system, with $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$. Our next goal is to establish that the two mappings

$$\mathcal{I} \longmapsto \mathbb{V}^{\mathsf{Sem}}(\mathcal{I})$$
$$\mathcal{I}^{\mathsf{K},\to} \longleftarrow\!\!\!\!\dashv \mathsf{K}$$

form a dual order isomorphism from the collection of Hilbert based π-institutions with respect to $\to^\flat$, under $\le$, and Hilbert based semantic varieties with respect to $\to^\flat$, under $\subseteq$.

We first show that the Frege operator is both monotone and order reflecting on Hilbert based π-institutions with respect to $\to^\flat$.

Proposition 1724 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$*,* $\mathcal{I}' = \langle \mathbf{F}, C' \rangle$ *be Hilbert based* π*-institutions with respect to* $\to^\flat$*. Then*

$$\mathcal{I} \le \mathcal{I}' \quad \textit{iff} \quad \widetilde{\lambda}(\mathcal{I}) \le \widetilde{\lambda}(\mathcal{I}').$$

Proof: The left-to-right implication (monotonicity) is given by Lemma 1416. For the right-to-left implication, suppose $\widetilde{\lambda}(\mathcal{I}) \le \widetilde{\lambda}(\mathcal{I}')$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rll} \phi \in C_\Sigma(\Phi) & \text{iff} & C_\Sigma(\overset{\Rightarrow}{\Phi} \to^\flat_\Sigma \phi) = C_\Sigma(\varnothing) \quad \text{(by Theorem 1722)} \\ & \text{iff} & \langle \overset{\Rightarrow}{\Phi} \to^\flat_\Sigma \phi, \top^\flat_\Sigma \rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}) \quad \text{(definition)} \\ & \text{implies} & \langle \overset{\Rightarrow}{\Phi} \to^\flat_\Sigma \phi, \top^\flat_\Sigma \rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}') \quad \text{(hypothesis)} \\ & \text{iff} & C'_\Sigma(\overset{\Rightarrow}{\Phi} \to^\flat_\Sigma \phi) = C'_\Sigma(\varnothing) \quad \text{(definition)} \\ & \text{iff} & \phi \in C'_\Sigma(\Phi). \quad \text{(by Theorem 1722)} \end{array}$$

We conclude that $\mathcal{I} \le \mathcal{I}'$ and, hence $\widetilde{\lambda}$ is also order reflecting. ■

Now we present the preannounced order isomorphism theorem. For an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$, with $\to^\flat$ a binary natural transformation in $N^\flat$, we let

$$\mathsf{K}^{\mathbf{F},\to}$$

be the semantic variety consisting of all **F**-algebraic systems satisfying the Hilbert equations with respect to $\to^\flat$.

Theorem 1725 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, There exists a dual order isomorphism between the collection of all Hilbert based* π*-institutions with respect to* $\to^\flat$*, ordered under* $\le$*, and the collection of all semantic subvarieties of the semantic variety* $\mathsf{K}^{\mathbf{F},\to}$*, ordered under* $\subseteq$*, given by* $\mathcal{I} \mapsto \mathsf{K}^\mathcal{I}$*.*

Proof: The given mapping is onto, since, by Corollary 1723, for $\mathsf{K} \subseteq \mathsf{K}^{\mathbf{F},\to}$ a semantic subvariety of $\mathsf{K}^{\mathbf{F},\to}$, $\mathsf{K} = \mathsf{K}^{\mathcal{I}^{\mathsf{K},\to}}$. Moreover, it is 1-1, since $\mathsf{K}^\mathcal{I} = \mathsf{K}^{\mathcal{I}'}$ implies that $\widetilde{\lambda}(\mathcal{I}) = \widetilde{\Omega}(\mathcal{I}) = \widetilde{\Omega}(\mathcal{I}') = \widetilde{\lambda}(\mathcal{I}')$ and, hence, by Proposition 1724, $\mathcal{I} = \mathcal{I}'$. Finally, monotonicity and order reflectivity are both given by

$$\begin{array}{rll} \mathcal{I} \le \mathcal{I}' & \text{iff} & \widetilde{\lambda}(\mathcal{I}) \le \widetilde{\lambda}(\mathcal{I}') \quad \text{(Proposition 1724)} \\ & \text{iff} & \widetilde{\Omega}(\mathcal{I}) \le \widetilde{\Omega}(\mathcal{I}') \quad \text{(Theorem 1722)} \\ & \text{iff} & \mathcal{F}/\widetilde{\Omega}(\mathcal{I}') \in \mathbb{V}^{\mathsf{Sem}}(\mathcal{F}/\widetilde{\Omega}(\mathcal{I})) \\ & \text{iff} & \mathsf{K}^{\mathcal{I}'} \subseteq \mathsf{K}^{\mathcal{I}}. \end{array}$$

This establishes the order isomorphism. ■

Our next goal is to show that Hilbert based π-institutions, i.e., finitary self extensional π-institutions that have the Uniterm Deduction Detachment Property (by Theorem 1722) are fully self extensional.

We start by proving that on every **F**-algebraic system in the Hilbert class $\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}) = \mathsf{K}^\mathcal{I}$, $\mathcal{I}$-filter families and implicative filter families coincide.

Lemma 1726 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with a binary* $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$, *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Hilbert based* π*-institution with respect to* $\to^\flat$. *For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$,

$$\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \mathrm{FiFam}^{\to}(\mathcal{A}).$$

Proof: Let $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$.

Suppose that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$.

- We have $\phi \in C_\Sigma(\phi)$, whence, by Proposition 1719, $\phi \to^\flat_\Sigma \phi \in C_\Sigma(\varnothing)$. Hence, $\top^{\mathcal{A}}_{F(\Sigma)} \in T_{F(\Sigma)}$;
- Assume $\alpha_\Sigma(\phi \to^\flat_\Sigma \psi) \in T_{F(\Sigma)}$ and $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$. Since, again by Proposition 1719, $\psi \in C_\Sigma(\phi, \phi \to^\flat_\Sigma \psi)$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get $\alpha_\Sigma(\psi) \in T_{F(\Sigma)}$.

Therefore, taking into account the surjectivity of $\langle F, \alpha \rangle$, $T \in \mathrm{FiFam}^{\to}(\mathcal{A})$.

Suppose, conversely, that $T \in \mathrm{FiFam}^{\to}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$ and $\alpha_\Sigma(\Phi) \subseteq T_{F(\Sigma)}$. By Proposition 1719, $\vec{\vec{\Phi}} \to^\flat_\Sigma \phi \in C_\Sigma(\varnothing)$. Therefore, $C_\Sigma(\vec{\vec{\Phi}} \to^\flat_\Sigma \phi) = C_\Sigma(\top^\flat_\Sigma)$. Thus, by Proposition 1719 and the fact that $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$, we get that $\alpha_\Sigma(\vec{\vec{\Phi}} \to^\flat_\Sigma \phi) = \top^{\mathcal{A}}_{F(\Sigma)}$. Since, by hypothesis, $T \in \mathrm{FiFam}^{\to}(\mathcal{A})$, $\alpha_\Sigma(\vec{\vec{\Phi}} \to^\flat_\Sigma \phi) \in T_{F(\Sigma)}$. Thus, by the fact that $\alpha_\Sigma(\Phi) \subseteq T_{F(\Sigma)}$ and $T \in \mathrm{FiFam}^{\to}(\mathcal{A})$, we get that $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$. We conclude that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$, with $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, be an algebraic system, K be a Hilbert based class of $\mathbf{F}$-algebraic systems with respect to $\to^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a Hilbert based π-institution with respect to K and $\to^\flat$. For all $\mathcal{A} \in \mathsf{K}$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, define $T^{\langle \Sigma, \phi \rangle} = \{T^{\langle \Sigma, \phi \rangle}_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}|}$ by setting, for all $\Sigma' \in |\mathbf{Sign}|$,

$$T^{\langle \Sigma, \phi \rangle}_\Sigma = \{\chi \in \mathrm{SEN}(\Sigma) : \phi \to^{\mathcal{A}}_\Sigma \chi = \top^{\mathcal{A}}_\Sigma\}$$

and

$$T^{\langle \Sigma, \phi \rangle}_{\Sigma'} = \{\top^{\mathcal{A}}_{\Sigma'}\}, \quad \text{for all } \Sigma' \neq \Sigma.$$

It is not difficult to see that $T^{\langle \Sigma, \phi \rangle} \in \mathrm{FiFam}^{\to}(\mathcal{A})$.

Lemma 1727 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$, *with* $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$, *be an algebraic system,* K *be a Hilbert based class of* $\mathbf{F}$*-algebraic systems with respect to* $\to^\flat$, *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Hilbert based* π*-institution with respect to* K *and* $\to^\flat$. *For all* $\mathcal{A} \in \mathsf{K}$, *all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi \in \mathrm{SEN}(\Sigma)$, $T^{\langle \Sigma, \phi \rangle} \in \mathrm{FiFam}^{\to}(\mathcal{A})$.

Proof: Consider, first, $\Sigma' \neq \Sigma$. By definition, $\top^{\mathcal{A}}_{\Sigma'} \in T^{\langle\Sigma,\phi}_{\Sigma'}$. Moreover, if $\top^{\mathcal{A}}_{\Sigma'} \to^{\mathcal{A}}_{\Sigma'} \in T^{\langle\Sigma,\phi\rangle}_{\Sigma'}$, then, by H2, $\phi \in T^{\langle\Sigma,\phi\rangle}_{\Sigma'}$.

Consider, next, $\Sigma' = \Sigma$. Note that we have

$$\begin{array}{rcl}\phi \to^{\mathcal{A}}_{\Sigma} \top^{\mathcal{A}}_{\Sigma} & = & \phi \to^{\mathcal{A}}_{\Sigma} (\phi \to^{\mathcal{A}}_{\Sigma} \phi) \quad \text{(definition)}\\ & = & (\phi \to^{\mathcal{A}}_{\Sigma} \phi) \to^{\mathcal{A}}_{\Sigma} (\phi \to^{\mathcal{A}}_{\Sigma} \phi) \quad \text{(by H3)}\\ & = & \top^{\mathcal{A}}_{\Sigma} \to^{\mathcal{A}}_{\Sigma} \top^{\mathcal{A}}_{\Sigma} \quad \text{(definition)}\\ & = & \top^{\mathcal{A}}_{\Sigma}. \quad \text{(by H2)}\end{array}$$

Hence, $\top^{\mathcal{A}}_{\Sigma} \in T^{\langle\Sigma,\phi\rangle}_{\Sigma}$. Moreover, if $\psi, \psi \to^{\mathcal{A}}_{\Sigma} \chi \in T^{\langle\Sigma,\phi\rangle}_{\Sigma}$, then, we get, by definition, $\phi \to^{\mathcal{A}}_{\Sigma} \psi = \top^{\mathcal{A}}_{\Sigma}$ and $\phi \to^{\mathcal{A}}_{\Sigma} (\psi \to^{\mathcal{A}}_{\Sigma} \chi) \in \top^{\mathcal{A}}_{\Sigma}$. Therefore, we get

$$\begin{array}{rcl}\phi \to^{\mathcal{A}}_{\Sigma} \chi & = & \top^{\mathcal{A}}_{\Sigma} \to^{\mathcal{A}}_{\Sigma} (\phi \to^{\mathcal{A}}_{\Sigma} \chi) \quad \text{(by H2)}\\ & = & (\phi \to^{\mathcal{A}}_{\Sigma} \psi) \to^{\mathcal{A}}_{\Sigma} (\phi \to^{\mathcal{A}}_{\Sigma} \chi) \quad \text{(hypothesis)}\\ & = & \phi \to^{\mathcal{A}}_{\Sigma} (\psi \to^{\mathcal{A}}_{\Sigma} \chi) \quad \text{(by H3)}\\ & = & \top^{\mathcal{A}}_{\Sigma}, \quad \text{(hypothesis)}\end{array}$$

We conclude that $\chi \in T^{\langle\Sigma,\phi\rangle}_{\Sigma}$. Thus, $T^{\langle\Sigma,\phi\rangle} \in \mathrm{FiFam}^{\to}(\mathcal{A})$. ■

We can now prove that the Frege equivalence system of every full $\mathcal{I}$-structure of the form $\langle\mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$, with $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$, is the identity congruence system. This shows, in particular that every $\mathbf{F}$-algebraic system in $\mathsf{K}^{\mathcal{I}}$ is an $\mathcal{I}$-algebraic system and, moreover, satisfies the Congruence Property.

Lemma 1728 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *a Hilbert based* π*-institution based on* $\mathbf{F}$*. For every* $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$*,* $\widetilde{\lambda}^{\mathcal{I}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}$*. Thus,* $\langle\mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ *is reduced and satisfies the Congruence Property.*

Proof: Let $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$, $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle \notin \Delta^{\mathcal{A}}_{\Sigma}$. Thus, $\phi \neq \psi$. Taking into account Lemma 1727, consider $T^{\langle\Sigma,\phi\rangle}, T^{\langle\Sigma,\psi\rangle} \in \mathrm{FiFam}^{\to}(\mathcal{A})$. By Lemma 1726, $T^{\langle\Sigma,\phi\rangle}, T^{\langle\Sigma,\psi\rangle} \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Moreover, by definition, $\phi \in T^{\langle\Sigma,\phi\rangle}_{\Sigma}$ and $\psi \in T^{\langle\Sigma,\psi\rangle}_{\Sigma}$. On the other hand, if it was the case that $\psi \in T^{\langle\Sigma,\phi\rangle}_{\Sigma}$ and $\phi \in T^{\langle\Sigma,\psi\rangle}_{\Sigma}$, then, by definition, $\phi \to^{\mathcal{A}}_{\Sigma} \psi = \psi \to^{\mathcal{A}}_{\Sigma} \phi = \top^{\mathcal{A}}_{\Sigma}$, whence, by Lemma 1713, $\phi = \psi$, contrary to hypothesis. Thus, it must be the case that $\psi \notin T^{\langle\Sigma,\phi\rangle}_{\Sigma}$ or $\phi \notin T^{\langle\Sigma,\psi\rangle}_{\Sigma}$. We can now conclude that $\langle\phi,\psi\rangle \notin \widetilde{\lambda}^{\mathcal{A}}_{\Sigma}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$. This shows that $\widetilde{\lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}$. Since $\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \leq \widetilde{\lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$, we get that $\langle\mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ is a reduced $\mathcal{I}$-structure and that it satisfies the Congruence Property. ■

Lemma 1728 allows us to conclude that, for Hilbert based π-institutions $\mathcal{I}$, the semantic variety of $\mathcal{I}$ coincides with the class of all $\mathcal{I}$-algebraic systems.

Theorem 1729 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *a Hilbert based* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathrm{AlgSys}(\mathcal{I}) = \mathsf{K}^{\mathcal{I}} = \mathbb{V}^{\mathsf{Sem}}(\mathcal{I})$*;*

(b) $\mathrm{AlgSys}(\mathcal{I})$ *is a semantic variety;*

(c) $\mathcal{I}$ *is Hilbert based with respect to* $\mathrm{AlgSys}(\mathcal{I})$.

Proof: By Proposition 65, we have $\mathrm{AlgSys}(\mathcal{I}) \subseteq \mathsf{K}^{\mathcal{I}}$. On the other hand, Lemma 1728 gives $\mathsf{K}^{\mathcal{I}} \subseteq \mathrm{AlgSys}(\mathcal{I})$. Therefore, $\mathrm{AlgSys}(\mathcal{I}) = \mathsf{K}^{\mathcal{I}}$. Since $\mathsf{K}^{\mathcal{I}}$ is a semantic variety, we conclude that $\mathrm{AlgSys}(\mathcal{I})$ is also a semantic variety. Finally, since, by Proposition 1719, $\mathcal{I}$ is Hilbert based with respect to $\mathsf{K}^{\mathcal{I}}$, we conclude that $\mathcal{I}$ is Hilbert based with respect to $\mathrm{AlgSys}(\mathcal{I})$. ■

In one of the main theorems of the section, we show that a finitary self extensional π-institution with the Uniterm Deduction Detachment Property is necessarily fully self extensional.

Theorem 1730 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a finitary self extensional π-institution with the Uniterm Deduction Detachment Property. Then $\mathcal{I}$ is fully self extensional.*

Proof: Suppose that $\mathcal{I}$ is finitary self extensional and that it has the Uniterm Deduction Detachment Property with respect to $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$. By Theorem 1722 and Proposition 1719, $\mathcal{I}$ is Hilbert based with respect to $\mathsf{K}^{\mathcal{I}}$ and $\to^\flat$. By Theorem 1729, $\mathsf{K}^{\mathcal{I}} = \mathrm{AlgSys}(\mathcal{I})$, whence, by Lemma 1728, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$,

$$\widetilde{\lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}.$$

Now to prove full self extensionality, we use Proposition 1428. To this end, assume $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ is a full $\mathcal{I}$-structure. Then $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \in \mathrm{AlgSys}(\mathcal{I})$ and, by definition,

$$\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})).$$

Thus, by what was shown above,

$$\widetilde{\lambda}^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))) = \Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))}.$$

By Proposition 1426, we infer that $\langle \mathcal{A}.\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ also has the Congruence Property. We now conclude, by Proposition 1428, that $\mathcal{I}$ is fully self extensional. ■

We finish the section by looking at some connections with the theory of Gentzen π-institutions, that is presented in another chapter.

Recall that, given an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ and a finitary π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ based on $\mathbf{F}$, a finitary Gentzen π-institution $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ is said to be **adequate for** $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in C_\Sigma(\Phi) \quad \text{iff} \quad \Phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing).$$

We say that the Gentzen π-institution $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ has the **Congruence Property** if, for all $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\sigma^\flat_\Sigma(\vec{\phi}) \vdash \sigma^\flat_\Sigma(\vec{\psi}) \in G_\Sigma(\{\phi_i \vdash_\Sigma \psi_i,\ \psi_i \vdash_\Sigma \phi_i : i \in I\}).$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ a finitary Gentzen π-institution based on $\mathbf{F}$. We say that:

- $\mathfrak{G}$ has the **Deduction Rule with respect to** $\to^\flat$, if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi, \psi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$,

$$\Phi \vdash_\Sigma \phi \to^\flat_\Sigma \psi \in G_\Sigma(\Phi, \phi \vdash_\Sigma \psi);$$

- $\mathfrak{G}$ has the **Detachment Rule with respect to** $\to^\flat$, if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi, \psi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$,

$$\Phi, \phi \vdash_\Sigma \psi \in G_\Sigma(\Phi \vdash_\Sigma \phi \to^\flat_\Sigma \psi);$$

- $\mathfrak{G}$ has the **Deduction Detachment Rule with respect to** $\to^\flat$, if it has both the Deduction and the Detachment Property with respect to $\to^\flat$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a finitary self extensional π-institution, having the Deduction Detachment Property with respect to $\to^\flat$.

- Define $\mathrm{Ax}^{\mathcal{I}} = \{\mathrm{Ax}^{\mathcal{I}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{Ax}^{\mathcal{I}}_\Sigma = \{\Phi \vdash_\Sigma \phi : \phi \in C_\Sigma(\Phi)\};$$

- Define $\mathrm{Ir}^{\mathcal{I}} = \{\mathrm{Ir}^{\mathcal{I}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\begin{aligned}\mathrm{Ir}^{\mathcal{I}}_\Sigma = {} & \{\langle\{\phi_i \vdash_\Sigma \psi_i, \psi_i \vdash_\Sigma \phi_i : i \in I\}, \sigma^\flat_\Sigma(\vec{\phi}) \vdash_\Sigma \sigma^\flat_\Sigma(\vec{\psi})\rangle : \\ & \qquad \sigma^\flat \in N^\flat, \vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)\} \\ & \cup \{\langle\{\Phi, \phi \vdash_\Sigma \psi\}, \Phi \vdash_\Sigma \phi \to^\flat_\Sigma \psi\rangle : \Phi \cup \{\phi, \psi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)\} \\ & \cup \{\langle\{\Phi \vdash_\Sigma \phi \to^\flat_\Sigma \psi\}, \Phi, \phi \vdash_\Sigma \psi\rangle : \Phi \cup \{\phi, \psi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)\}.\end{aligned}$$

- $R^{\mathcal{I}} := \mathrm{Ax}^{\mathcal{I}} \cup \mathrm{Ir}^{\mathcal{I}}$.

Finally, define $\mathfrak{G}^{\mathcal{I}} = \langle \mathbf{F}, C^{\mathcal{I}}\rangle := \mathfrak{G}^{R^{\mathcal{I}}}$ be the Gentzen π-institution generated by the system $R^{\mathcal{I}}$ of Gentzen rules. Recall, by Proposition 1482, that $G^{\mathcal{I}} = \Xi^{R^{\mathcal{I}}}$.

We are almost ready to establish the existence of a fully adequate Gentzen π-institution for any given Hilbert based π-institution. Recall, again, from work in a different chapter, that a Gentzen π-institution $\mathfrak{G}$ is **fully adequate**

for a π-institution $\mathcal{I}$ (with theorems) if, for every **F**-structure $\mathbb{L} = \langle \mathcal{A}, D \rangle$, $\mathbb{L}$ is a full $\mathcal{I}$-structure if and only if it is a $\mathfrak{G}$-structure.

For Hilbert based π-institutions, it turns out that any $\mathcal{I}$-structure whose Frege equivalence system is the identity is of the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$, for some $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$.

Lemma 1731 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Hilbert based* π*-institution based on* $\mathbf{F}$*. If* $\langle \mathcal{A}, D \rangle \in \mathrm{Str}(\mathcal{I})$*, such that* $\widetilde{\lambda}^{\mathcal{A}}(D) = \Delta^{\mathcal{A}}$*, then* $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and* $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$*.*

Proof: Let $\langle \mathcal{A}, D \rangle \in \mathrm{Str}(\mathcal{I})$, such that $\widetilde{\lambda}^{\mathcal{A}}(D) = \Delta^{\mathcal{A}}$. Then, we have

$$\widetilde{\Omega}^{\mathcal{A}}(D) \le \widetilde{\lambda}^{\mathcal{A}}(D) = \Delta^{\mathcal{A}},$$

i.e., $\widetilde{\Omega}^{\mathcal{A}}(D) = \Delta^{\mathcal{A}}$ and, therefore, $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I}) \subseteq \mathsf{K}^{\mathcal{I}}$.

Suppose, next, that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. By Lemma 1726, $T \in \mathrm{FiFam}^{\to}(\mathcal{A})$, where $\to^\flat$ in $N^\flat$ is the binary transformation with respect to which $\mathcal{I}$ is Hilbert based. Let $\Sigma \in |\mathbf{Sign}|$, $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in D_\Sigma(T_\Sigma)$. By finitarity and Proposition 114, there exists $\Phi \subseteq_f T_\Sigma$, such that $\phi \in D_\Sigma(\Phi)$. Thus, by the hypothesis, Proposition 1719 and Corollary 1440, $\vec{\vec{\Phi}} \to^{\mathcal{A}}_\Sigma \phi \in D_\Sigma(\top^{\mathcal{A}}_\Sigma)$, i.e., $D_\Sigma(\vec{\vec{\Phi}} \to^{\mathcal{A}}_\Sigma \phi) = D_\Sigma(\top^{\mathcal{A}}_\Sigma)$. By hypothesis, $\vec{\vec{\Phi}} \to^{\mathcal{A}}_\Sigma \phi = \top^{\mathcal{A}}_\Sigma$. Since $T \in \mathrm{FiFam}^{\to}(\mathcal{A})$, we have $\vec{\vec{\Phi}} \to^{\mathcal{A}}_\Sigma \phi \in T_\Sigma$ and, since, also, $\Phi \subseteq T_\Sigma$, we infer that $\phi \in T_\Sigma$. Therefore, $T = D(T)$, showing that $T \in \mathcal{D}$ and, hence, $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. ∎

We finally show that any Hilbert based π-institution $\mathcal{I}$ has a fully adequate Gentzen π-institution, namely, the π-institution $\mathfrak{G}^{\mathcal{I}}$, generated by the Gentzen rules $R^{\mathcal{I}}$, which encode the rules of $\mathcal{I}$, the Congruence Property and the Deduction Detachment Property.

Theorem 1732 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a finitary* π*-institution based on* $\mathbf{F}$*, having the Uniterm Deduction Detachment Property.* $\mathcal{I}$ *is self extensional if and only if the Gentzen* π*-institution* $\mathfrak{G} = \langle \mathbf{F}, G^{\mathcal{I}} \rangle$ *if fully adequate for* $\mathcal{I}$*.*

Proof: Suppose that $\mathfrak{G}^{\mathcal{I}}$ is fully adequate for $\mathcal{I}$. We know that $\langle \mathcal{F}, C \rangle$ is a full $\mathcal{I}$-structure. Thus, by full adequacy, $\langle \mathcal{F}, C \rangle \in \mathrm{Str}(\mathfrak{G}^{\mathcal{I}})$. Therefore, $\langle \mathcal{F}, C \rangle$ satisfies all the Gentzen rules that hold in $\mathfrak{G}^{\mathcal{I}}$. In particular, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \subseteq \mathrm{SEN}^\flat(\Sigma)$, $C_\Sigma(\phi_i) = C_\Sigma(\psi_i)$, for all $i \in I$, imply $C_\Sigma(\sigma^\flat_\Sigma(\vec{\phi})) = C_\Sigma(\sigma^\flat_\Sigma(\vec{\psi}))$. Thus, $\widetilde{\lambda}(\mathcal{I})$ is a congruence system and, hence $\mathcal{I}$ is self extensional.

Suppose, conversely, that $\mathcal{I}$ is self extensional. By Theorem 1730, it is fully self extensional. Let $\langle \mathcal{A}, D \rangle \in \mathrm{FStr}(\mathcal{I})$. Then, by Theorem 1444 and the definition of $\mathfrak{G}^{\mathcal{I}}$, we get that $\langle \mathcal{A}, D \rangle \in \mathrm{Str}(\mathfrak{G}^{\mathcal{I}})$. Assume, conversely, that $\langle \mathcal{A}, D \rangle \in \mathrm{Str}(\mathfrak{G}^{\mathcal{I}})$. By considering, if necessary, $\langle \mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(D), \mathcal{D}/\widetilde{\Omega}^{\mathcal{A}}(D) \rangle$, we may assume that $\widetilde{\Omega}^{\mathcal{A}}(D) = \Delta^{\mathcal{A}}$ and must show that $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. By the definition of $\mathfrak{G}^{\mathcal{I}}$, we get $\langle \mathcal{A}, D\rangle \in \mathrm{Str}(\mathcal{I})$. For the same reason, $\langle \mathcal{A}, D\rangle$ satisfies the Congruence Property. Therefore, $\widetilde{\lambda}^{\mathcal{A}}(D) = \widetilde{\Omega}^{\mathcal{A}}(D) = \Delta^{\mathcal{A}}$. Since, by hypothesis, it has the Deduction Detachment Property, we get, by Theorem 1722, that it is Hilbert based. Now, applying Lemma 1731, we conclude that $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Therefore, $\langle \mathcal{A}, \mathcal{D}\rangle \in \mathrm{FStr}(\mathcal{I})$.

This shows that $\mathrm{FStr}(\mathcal{I}) = \mathrm{Str}(\mathfrak{G}^{\mathcal{I}})$ and, hence, $\mathfrak{G}^{\mathcal{I}}$ is, indeed, fully adequate for $\mathcal{I}$. ■

23.3 Self Extensionality and Conjunction

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. We say that $\mathcal{I}$ has the **Conjunction Property with respect to** $\wedge^\flat$ and that $\wedge^\flat$ is a **conjunction for** $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

- $\phi \wedge^\flat_\Sigma \psi \in C_\Sigma(\phi, \psi)$;
- $\phi, \psi \in C_\Sigma(\phi \wedge^\flat_\Sigma \psi)$.

Equivalently, $\wedge^\flat$ is a conjunction for $\mathcal{I}$ if and only if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C_\Sigma(\phi \wedge^\flat_\Sigma \psi) = C_\Sigma(\phi, \psi).$$

We say $\mathcal{I}$ is **conjunctive** if it has the Conjunction Property with respect to some $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$.

If a π-institution has the Conjunction Property with respect to two different binary natural transformations in $N^\flat$, then the two concjunctions must be interderivable in an obvious sense.

Lemma 1733 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\wedge^\flat, \wedge'^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *has the Conjunction Property with respect to both* $\wedge^\flat$ *and* $\wedge'^\flat$*, then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$C_\Sigma(\phi \wedge^\flat_\Sigma \psi) = C_\Sigma(\phi \wedge'^\flat_\Sigma \psi).$$

Proof: Suppose that $\wedge^\flat$ and $\wedge'^\flat$ are both conjunctions for $\mathcal{I}$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\psi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{rcll} C_\Sigma(\phi \wedge^\flat_\Sigma \psi) & = & C_\Sigma(\phi, \psi) & (\wedge^\flat \text{ a conjunction}) \\ & = & C_\Sigma(\phi \wedge'^\flat_\Sigma \psi). & (\wedge'^\flat \text{ a conjunction}) \end{array}$$

This proves the statement. ■

Corollary 1734 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\wedge^\flat, \wedge'^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is self extensional and has the Conjunction Property with respect to both* $\wedge^\flat$ *and* $\wedge'^\flat$*, then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\langle \phi \wedge^\flat_\Sigma \psi, \phi \wedge'^\flat_\Sigma \psi \rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I}).$$

Proof: Suppose that $\mathcal{I}$ is self extensional and $\wedge^\flat$ and $\wedge'^\flat$ are both conjunctions for $\mathcal{I}$. Then, if $\Sigma \in |\mathbf{Sign}^\flat|$, $\psi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, by Lemma 1733, $\langle \phi \wedge^\flat_\Sigma \psi, \phi \wedge'^\flat_\Sigma \psi \rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I})$, whence, by self extensionality, $\langle \phi \wedge^\flat_\Sigma \psi, \phi \wedge'^\flat_\Sigma \psi \rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I})$. ∎

We also know, by Proposition 1434 that the Conjunction Property transfers from a π-institution $\mathcal{I}$ to all $\mathcal{I}$-structures.

Corollary 1735 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, having the Conjunction Property with respect to* $\wedge^\flat$*. For every* $\mathcal{I}$*-structure* $\langle \mathcal{A}, D \rangle$*, all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,*

$$D_\Sigma(\phi \wedge^\mathcal{A}_\Sigma \psi) = D_\Sigma(\phi, \psi).$$

Proof: This is simply a restatement of Proposition 1434. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a class of $\mathbf{F}$-algebraic systems. K is **semilattice based with respect to** $\wedge^\flat$ if, for all $\mathcal{A} \in \mathsf{K}$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi, \chi \in \mathrm{SEN}(\Sigma)$,

- L1. $\phi \wedge^\mathcal{A}_\Sigma \phi = \phi$;
- L2. $\phi \wedge^\mathcal{A}_\Sigma \psi = \psi \wedge^\mathcal{A}_\Sigma \phi$;
- L3. $(\phi \wedge^\mathcal{A}_\Sigma \psi) \wedge^\mathcal{A}_\Sigma \chi = \phi \wedge^\mathcal{A}_\Sigma (\psi \wedge^\mathcal{A}_\Sigma \chi)$.

L1-L3 are referred to as the **semilattice equations**. We say that K is **semilattice based** if it is semilattice based with respect to some $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$.

If a class of $\mathbf{F}$-algebraic systems is semilattices based, then the semantic variety generated by the class is also semilattice based with respect to the same binary natural transformation.

Lemma 1736 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* K *a class of* $\mathbf{F}$*-algebraic systems. If* K *is semilattice based with respect to* $\wedge^\flat$*, then* $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$ *is also semilattice based with respect to* $\wedge^\flat$*.*

Proof: Let $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. We show $\mathcal{A}$ satisfies L2. The work for L1 and L3 follows along the same lines. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Since K is semilattice based with respect to $\wedge^\flat$, $\langle \phi \wedge^\flat_\Sigma \psi, \psi \wedge^\flat_\Sigma \phi\rangle \in \mathrm{Ker}_\Sigma(\mathsf{K})$. Since, by hypothesis, $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$, we get $\mathrm{Ker}(\mathsf{K}) \leq \mathrm{Ker}(\mathcal{A})$, whence $\langle \phi \wedge^\flat_\Sigma \psi, \psi \wedge^\flat_\Sigma \phi\rangle \in \mathrm{Ker}_\Sigma(\mathcal{A})$. This shows that $\alpha_\Sigma(\phi \wedge^\flat_\Sigma \psi) = \alpha_\Sigma(\psi \wedge^\flat_\Sigma \phi)$, i.e., $\alpha_\Sigma(\phi) \wedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\psi) = \alpha_\Sigma(\psi) \wedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi)$. Thus, by the surjectivity of $\langle F, \alpha\rangle$, we conclude that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $\phi \wedge^{\mathcal{A}}_\Sigma \psi = \psi \wedge^{\mathcal{A}}_\Sigma \phi$. Therefore, $\mathcal{A}$ satisfies L2. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a semilattice based class of $\mathbf{F}$-algebraic systems with respect to $\wedge^\flat$. For every $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$, define the relation family $\leq^{\mathcal{A}} = \{\leq^{\mathcal{A}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ on $\mathcal{A}$ by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\phi \leq^{\mathcal{A}}_\Sigma \psi \quad \text{iff} \quad \phi \wedge^{\mathcal{A}}_\Sigma \psi = \phi.$$

It is easily shown that $\leq^{\mathcal{A}}$ is a partial order system on $\mathcal{A}$.

Lemma 1737 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a semilattice based class with respect to $\wedge^\flat$. For all $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$, $\leq^{\mathcal{A}}$ is a posystem on $\mathcal{A}$.*

Proof: First, fix $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$, $\Sigma \in |\mathbf{Sign}|$. We show that $\leq^{\mathcal{A}}_\Sigma$ is a partial order on $\mathrm{SEN}(\Sigma)$. To this end, let $\phi, \psi, \chi \in \mathrm{SEN}(\Sigma)$.

- By L1, $\phi = \phi \wedge^{\mathcal{A}}_\Sigma \phi$, whence, by definition, $\phi \leq^{\mathcal{A}}_\Sigma \phi$ and $\leq^{\mathcal{A}}_\Sigma$ is reflexive;
- If $\phi \leq^{\mathcal{A}}_\Sigma \psi$ and $\psi \leq^{\mathcal{A}}_\Sigma \phi$, then, we get

$$\begin{array}{rcll} \phi & = & \phi \wedge^{\mathcal{A}}_\Sigma \psi & (\phi \leq^{\mathcal{A}}_\Sigma \psi) \\ & = & \psi \wedge^{\mathcal{A}}_\Sigma \phi & (\text{by L2}) \\ & = & \psi. & (\psi \leq^{\mathcal{A}}_\Sigma \phi) \end{array}$$

  Thus, $\leq^{\mathcal{A}}_\Sigma$ is antisymmetric.

- If $\phi \leq^{\mathcal{A}}_\Sigma \psi$ and $\psi \leq^{\mathcal{A}}_\Sigma \chi$, then

$$\begin{array}{rcll} \phi & = & \phi \wedge^{\mathcal{A}}_\Sigma \psi & (\phi \leq^{\mathcal{A}}_\Sigma \psi) \\ & = & \phi \wedge^{\mathcal{A}}_\Sigma (\psi \wedge^{\mathcal{A}}_\Sigma \chi) & (\psi \leq^{\mathcal{A}}_\Sigma \chi) \\ & = & (\phi \wedge^{\mathcal{A}}_\Sigma \psi) \wedge^{\mathcal{A}}_\Sigma \chi & (\text{by L3}) \\ & = & \phi \wedge^{\mathcal{A}}_\Sigma \chi. & (\phi \leq^{\mathcal{A}}_\Sigma \psi) \end{array}$$

  Hence $\phi \leq^{\mathcal{A}}_\Sigma \chi$ and $\leq^{\mathcal{A}}_\Sigma$ is also transitive.

Thus, $\leq^{\mathcal{A}}$ is a partial order on $\mathrm{SEN}(\Sigma)$. Suppose, now, that $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\phi \leq^{\mathcal{A}}_\Sigma \psi$. Then, by definition, $\phi = \phi \wedge^{\mathcal{A}}_\Sigma \psi$. Thus, $\mathrm{SEN}^\flat(f)(\phi) = \mathrm{SEN}^\flat(f)(\phi \wedge^{\mathcal{A}}_\Sigma \psi) = \mathrm{SEN}^\flat(f)(\phi) \wedge^{\mathcal{A}}_{\Sigma'} \mathrm{SEN}^\flat(f)(\psi)$.

This shows that $\mathrm{SEN}^\flat(f)(\phi) \leq^{\mathcal{A}}_{\Sigma'} \mathrm{SEN}^\flat(f)(\psi)$. Thus, $\leq^{\mathcal{A}}$ is a partial order system on $\mathcal{A}$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, K a semilattice based class of $\mathbf{F}$-algebraic systems with respect to $\wedge^\flat$, $\mathcal{A} \in \mathsf{K}$ and $T \in \mathrm{SenFam}(\mathcal{A})$. We say that T is a **semilattice filter family of $\mathcal{A}$** if the following conditions hold, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$:

- $T_\Sigma \neq \varnothing$;
- $\phi, \psi \in T_\Sigma$ implies $\phi \wedge^{\mathcal{A}}_\Sigma \psi \in T_\Sigma$;
- $\phi \in T_\Sigma$ and $\phi \leq^{\mathcal{A}}_\Sigma \psi$ imply $\psi \in T_\Sigma$.

We denote by $\mathrm{FiFam}^\wedge(\mathcal{A})$ the collection of all semilattice filter families on $\mathcal{A}$. Moreover, we write $\mathrm{FiFam}^{\wedge,\varnothing}(\mathcal{A})$ for the same collection augmented by those sentence families resulting from semilattice filter families after one or more components are replaced by the empty set.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, K a semilattice based class of $\mathbf{F}$-algebraic systems with respect to $\wedge^\flat$, $\mathcal{A} \in \mathsf{K}$, $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. Define

$$T^{\langle \Sigma, \phi \rangle} = \{T^{\langle \Sigma, \phi \rangle}_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}|}$$

by setting,

- $T^{\langle \Sigma, \phi \rangle}_\Sigma = \{\chi \in \mathrm{SEN}(\Sigma) : \phi \leq^{\mathcal{A}}_\Sigma \chi\}$;
- $T^{\langle \Sigma, \phi \rangle}_{\Sigma'} = \begin{cases} \{\chi \in \mathrm{SEN}(\Sigma) : 1_\Sigma \leq^{\mathcal{A}}_\Sigma \chi\}, & \text{if } 1_\Sigma \text{ is a maximum in } \leq^{\mathcal{A}}_\Sigma \\ \varnothing, & \text{if } \leq^{\mathcal{A}}_\Sigma \text{ has no maximum} \end{cases}$, for all $\Sigma \neq \Sigma' \in |\mathbf{Sign}|$,

We show that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $T^{\langle \Sigma, \phi \rangle} \in \mathrm{FiFam}^\wedge(\mathcal{A})$ or $T^{\langle \Sigma, \phi \rangle} \in \mathrm{FiFam}^{\wedge,\varnothing}(\mathcal{A})$.

Lemma 1738 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, K a semilattice based class of $\mathbf{F}$-algebraic systems with respect to $\wedge^\flat$, $\mathcal{A} \in \mathsf{K}$, $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. Then, $T^{\langle \Sigma, \phi \rangle} \in \mathrm{FiFam}^\wedge(\mathcal{A})$ or $T^{\langle \Sigma, \phi \rangle} \in \mathrm{FiFam}^{\wedge,\varnothing}(\mathcal{A})$.*

Proof: It suffices to show that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, the collection $\{\chi \in \mathrm{SEN}(\Sigma) : \phi \leq^{\mathcal{A}}_\Sigma \chi\}$ is an upset under $\leq^{\mathcal{A}}_\Sigma$, closed under $\wedge^{\mathcal{A}}_\Sigma$. Let $\psi, \chi \in \mathrm{SEN}(\Sigma)$.

- If $\psi, \chi \in T^{\langle \Sigma, \phi \rangle}_\Sigma$, then, by definition, $\phi \leq^{\mathcal{A}}_\Sigma \psi$ and $\phi \leq^{\mathcal{A}}_\Sigma \chi$. Thus, we get

$$\begin{array}{rcll} \phi & = & \phi \wedge^{\mathcal{A}}_\Sigma \chi & (\phi \leq^{\mathcal{A}}_\Sigma \chi) \\ & = & (\phi \wedge^{\mathcal{A}}_\Sigma \psi) \wedge^{\mathcal{A}}_\Sigma \chi & (\phi \leq^{\mathcal{A}}_\Sigma \psi) \\ & = & \phi \wedge^{\mathcal{A}}_\Sigma (\psi \wedge^{\mathcal{A}}_\Sigma \chi). & (\text{by L3}) \end{array}$$

Therefore, $\phi \leq^{\mathcal{A}}_\Sigma \psi \wedge^{\mathcal{A}}_\Sigma \chi$ and, hence $\psi \wedge^{\mathcal{A}}_\Sigma \chi \in T^{\langle \Sigma, \phi \rangle}_\Sigma$.

- If $\psi \in T^{\langle\Sigma,\phi\rangle}_\Sigma$ and $\psi \leq^{\mathcal{A}}_\Sigma \chi$, then $\phi \leq^{\mathcal{A}}_\Sigma \psi$ and $\psi \leq^{\mathcal{A}}_\Sigma \chi$, whence, by Lemma 1737, $\phi \leq^{\mathcal{A}}_\Sigma \chi$, i.e., $\chi \in T^{\langle\Sigma,\phi\rangle}_\Sigma$.

Thus, $T^{\langle\Sigma,\phi\rangle} \in \mathrm{FiFam}^\wedge(\mathcal{A})$ or $T^{\langle\Sigma,\phi\rangle} \in \mathrm{FiFam}^{\wedge,\varnothing}(\mathcal{A})$. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, K a semilattice based class of $\mathbf{F}$-algebraic systems with respect to $\wedge^\flat$ and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a *finitary* π-institution based on $\mathbf{F}$. We say $\mathcal{I}$ is **semilattice based with respect to** K and $\wedge^\flat$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$, with $\Phi \neq \varnothing$,

$$\phi \in C_\Sigma(\Phi) \quad \text{iff} \quad \text{for all } \mathcal{A} \in \mathsf{K}, \ \textstyle\bigwedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\Phi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi).$$

We say that $\mathcal{I}$ is **semilattice based** if it is semilattice based with respect to K and $\wedge^\flat$, for some semilattice based class K with respect to some $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$.

We get immediately from the definition that interderivability in $\mathcal{I}$ is reflected into equality in all algebraic systems in the defining class K.

Lemma 1739 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, K a semilattice based class of $\mathbf{F}$-algebraic systems with respect to $\wedge^\flat$ and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a semilattice based π-institution with respect to K and $\wedge^\flat$. For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,*

$$C_\Sigma(\phi) = C_\Sigma(\psi) \quad \textit{iff} \quad \textit{for all } \mathcal{A} \in \mathsf{K}, \ \alpha_\Sigma(\phi) = \alpha_\Sigma(\psi).$$

Proof: By the definition, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $C_\Sigma(\phi) = C_\Sigma(\psi)$ iff, for all $\mathcal{A} \in \mathsf{K}$, $\alpha_\Sigma(\phi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\psi)$ and $\alpha_\Sigma(\psi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi)$ iff, by Lemma 1737, for all $\mathcal{A} \in \mathsf{K}$, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. ∎

Moreover, in case $\mathcal{I}$ is semilattice based with respect to a class K, then it is also semilattice based with respect to the semantic variety generated by K, with respect to the same binary transformation.

Lemma 1740 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, K a semilattice based class of $\mathbf{F}$-algebraic systems with respect to $\wedge^\flat$ and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a semilattice based π-institution with respect to K and $\wedge^\flat$. Then $\mathcal{I}$ is semilattice based with respect to $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$ and $\wedge^\flat$.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$ and $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. Since $\mathcal{I}$ is semilattice based with respect to K and $\wedge^\flat$, $\langle \bigwedge^\flat_\Sigma \Phi \wedge^\flat_\Sigma \phi, \bigwedge^\flat_\Sigma \Phi\rangle \in \mathrm{Ker}_\Sigma(\mathsf{K})$. Since $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$, we get $\mathrm{Ker}(\mathsf{K}) \leq \mathrm{Ker}(\mathcal{A})$. This gives $\bigwedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\Phi) \wedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi) = \bigwedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\Phi)$, and, therefore, $\bigwedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\Phi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi)$.

Conversely, if, for all $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$, $\bigvee^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\Phi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi)$, then, a fortiori, for all $\mathcal{A} \in \mathsf{K}$, we have $\bigvee^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\Phi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi)$. Therefore, by hypothesis, $\phi \in C_\Sigma(\Phi)$. We conclude that $\mathcal{I}$ is semilattice based with respect to $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$ and $\wedge^\flat$. ■

We can now show that, if a π-institution is semilattice based with respect to two different classes of semilattice based $\mathbf{F}$-algebraic systems, then, they both have to generate the same semantic variety.

Lemma 1741 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\wedge^\flat, \wedge'^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, K a semilattice based class with respect to $\wedge^\flat$ and K' a semilattice based class with respect to $\wedge'^\flat$. If $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is a semilattice based π-institution both with respect to K and $\wedge^\flat$ and with respect to K' and $\wedge'^\flat$, then $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathbb{V}^{\mathsf{Sem}}(\mathsf{K}')$.*

Proof: Suppose $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is a semilattice based π-institution both with respect to K and $\wedge^\flat$ and with respect to K' and $\wedge'^\flat$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in \mathrm{Ker}_\Sigma(\mathsf{K})$ and $\mathcal{A} \in \mathsf{K}'$. Then, since $\mathcal{I}$ is semilattice based with respect to K, by Lemma 1739, $C_\Sigma(\phi) = C_\Sigma(\psi)$. Thus, since $\mathcal{I}$ is semilattice based with respect to K', again, by Lemma 1739, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. Hence, $\mathrm{Ker}(\mathsf{K}) \le \mathrm{Ker}(\mathcal{A})$, which gives that $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. We conclude that $\mathsf{K}' \subseteq \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$ and, by symmetry, $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathbb{V}^{\mathsf{Sem}}(\mathsf{K}')$. ■

Thus, if $\mathcal{I}$ is semilattice based with respect to some K and $\wedge^\flat$, it makes sense, based on Lemma 1741 and Lemma 1740, to denote by $\mathbb{V}^{\mathsf{Sem}}(\mathcal{I})$ the unique semantic variety of $\mathbf{F}$-algebraic systems with respect to which it is semilattice based.

In one of the cornerstone results of the section, we show that, if a π-institution is semilattice based, then it is self extensional and has the Conjunction Property, and, in addition, its semantic variety coincides with the semantic variety $\mathsf{K}^{\mathcal{I}}$ generated by the Lindenbaum-Tarski algebraic system of $\mathcal{I}$.

Proposition 1742 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\wedge^\flat$ in $N^\flat$, K a semilattice based class with respect to $\wedge^\flat$, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a semilattice based π-institution with respect to K and $\wedge^\flat$.*

(a) $\mathcal{I}$ is self extensional;

(b) $\mathcal{I}$ has the Conjunction Property with respect to $\wedge^\flat$;

(c) $\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}) = \mathsf{K}^{\mathcal{I}}$; Hence $\mathcal{I}$ is semilattice based with respect to $\mathsf{K}^{\mathcal{I}}$.

Proof:

(a) For self extensionality, it suffices to show that the Frege equivalence system $\widetilde{\lambda}(\mathcal{I})$ is a congruence system. To this end, let $\sigma^\flat$ be in $N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi_i, \psi_i \rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I})$, for all $i < k$. By definition, we get $C_\Sigma(\phi_i) = C_\Sigma(\psi_i)$, for all $i < k$. Hence, since

$\mathcal{I}$ is semilattice based with respect to K and $\wedge^\flat$, $\langle\phi_i,\psi_i\rangle \in \mathrm{Ker}_\Sigma(\mathsf{K})$, for all $i < k$. But $\mathrm{Ker}(\mathsf{K})$ is a congruence system on $\mathbf{F}$, whence, $\langle\sigma^\flat_\Sigma(\vec{\phi}),\sigma^\flat_\Sigma(\vec{\psi})\rangle \in \mathrm{Ker}_\Sigma(\mathsf{K})$. By Lemma 1739, $C_\Sigma(\sigma^\flat_\Sigma(\vec{\phi})) = C_\Sigma(\sigma^\flat_\Sigma(\vec{\psi}))$, whence $\langle\sigma^\flat_\Sigma(\vec{\phi}),\sigma^\flat_\Sigma(\vec{\psi})\rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I})$. Therefore, $\widetilde{\lambda}(\mathcal{I})$ is a congruence system on $\mathbf{F}$ and $\mathcal{I}$ is self extensional.

(b) Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $\mathcal{A} \in \mathsf{K}$,

$$\alpha_\Sigma(\phi \wedge^\flat_\Sigma \psi) = \alpha_\Sigma(\phi) \wedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\psi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi),\alpha_\Sigma(\psi).$$

Thus, since $\mathcal{I}$ is semilattice based with respect to K and $\wedge^\flat$, we get that $\phi\wedge^\flat_\Sigma\psi \in C_\Sigma(\phi,\psi)$ and $\phi,\psi \in C_\Sigma(\phi\wedge^\flat_\Sigma\psi)$. Hence, $\wedge^\flat$ is a conjunction for $\mathcal{I}$.

(c) We have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}\langle\phi,\psi\rangle \in \mathrm{Ker}_\Sigma(\mathbb{V}^{\mathsf{Sem}}(\mathcal{I})) & \text{iff} & \langle\phi,\psi\rangle \in \mathrm{Ker}_\Sigma(\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})) \\ & & \text{(definition of } \mathbb{V}^{\mathsf{Sem}}(\mathcal{I})) \\ & \text{iff} & \langle\phi,\psi\rangle \in \mathrm{Ker}_\Sigma(\mathsf{K}) \\ & & \text{(definition of } \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})) \\ & \text{iff} & C_\Sigma(\phi) = C_\Sigma(\psi) \\ & & \text{(Lemma 1739)} \\ & \text{iff} & \langle\phi,\psi\rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}) \\ & & \text{(definition of } \widetilde{\lambda}(\mathcal{I})) \\ & \text{iff} & \langle\phi,\psi\rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I}) \\ & & \text{(Part (a))} \\ & \text{iff} & \langle\phi,\psi\rangle \in \mathrm{Ker}_\Sigma(\mathsf{K}^{\mathcal{I}}). \\ & & \text{(definition of } \mathsf{K}^{\mathcal{I}})\end{array}$$

Therefore, since both classes are semantic varieties, we conclude that $\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}) = \mathsf{K}^{\mathcal{I}}$.

■

If $\mathcal{I}$ is a finitary self extensional π-institution $\mathcal{I}$, having the Conjunction Property with respect to $\wedge^\flat$, then the singleton class $\mathsf{K} = \{\mathcal{F}/\widetilde{\Omega}(\mathcal{I})\}$, consisting of its Lindenbaum-Tarski $\mathbf{F}$-algebraic system $\mathcal{F}/\widetilde{\Omega}(\mathcal{I})$, is semilattice based with respect to $\wedge^\flat$.

Lemma 1743 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle\mathbf{F},C\rangle$ a finitary self extensional π-institution, having the Conjunction Property with respect to $\wedge^\flat$. Then the class $\mathsf{K} = \{\mathcal{F}/\widetilde{\Omega}(\mathcal{I})\}$ is semilattice based with respect to $\wedge^\flat$.*

Proof: We have to verify that the class K satisfies the semilattice identities. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi,\psi,\chi \in \mathrm{SEN}^\flat(\Sigma)$.

- By the Conjunction Property, $C_\Sigma(\phi\wedge^\flat_\Sigma\phi) = C_\Sigma(\phi)$. Thus, using self extensionality, $\langle\phi\wedge^\flat_\Sigma\phi,\phi\rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}) = \widetilde{\Omega}_\Sigma(\mathcal{I})$. Hence K satisfies L1.

- By the Conjunction Property, $C_\Sigma(\phi \wedge^\flat_\Sigma \psi) = C_\Sigma(\phi, \psi) = C_\Sigma(\psi \wedge^\flat_\Sigma \phi)$. Thus, again using self extensionality, $\langle \phi \wedge^\flat_\Sigma \psi, \psi \wedge^\flat_\Sigma \phi\rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}) = \widetilde{\Omega}_\Sigma(\mathcal{I})$. Hence K satisfies L2.

- Finally, we have

$$\begin{array}{rcl} C_\Sigma((\phi \wedge^\flat_\Sigma \psi) \wedge^\flat_\Sigma \chi) & = & C_\Sigma(\phi \wedge^\flat_\Sigma \psi, \chi) \\ & = & C_\Sigma(\phi, \psi, \chi) \\ & = & C_\Sigma(\phi, \psi \wedge^\flat_\Sigma \chi) \\ & = & C_\Sigma(\phi \wedge^\flat_\Sigma (\psi \wedge^\flat_\Sigma \chi)). \end{array}$$

  Thus, again using self extensionality,

$$\langle (\phi \wedge^\flat_\Sigma \psi) \wedge^\flat_\Sigma \chi, \phi \wedge^\flat_\Sigma (\psi \wedge^\flat_\Sigma \chi)\rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}) = \widetilde{\Omega}_\Sigma(\mathcal{I})$$

  and K also satisfies L3.

Thus, K is semilattice based with respect to $\wedge^\flat$. ■

In one of our main theorems, we characterize semilattice based π-institutions as those that are self extensional and conjunctive.

Theorem 1744 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a finitary* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is semilattice based if and only if it is self extensional and conjunctive.*

Proof: The left-to-right implication is by Proposition 1742. Suppose, conversely, that $\mathcal{I}$ is self extensional and conjunctive, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$ a conjunction for $\mathcal{I}$. By Lemma 1743, $\mathsf{K} = \{\mathcal{F}/\widetilde{\Omega}(\mathcal{I})\}$ is semilattice based with respect to $\wedge^\flat$. Moreover, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rll} \phi \in C_\Sigma(\Phi) & \text{iff} & \phi \in C_\Sigma(\bigwedge^\flat_\Sigma \Phi) \quad \text{(Conjunction Property)} \\ & \text{iff} & C_\Sigma(\bigwedge^\flat_\Sigma \Phi \wedge^\flat_\Sigma \phi) = C_\Sigma(\bigwedge^\flat_\Sigma \Phi) \quad \text{(Conjunction Property)} \\ & \text{iff} & \langle \bigwedge^\flat_\Sigma \Phi \wedge^\flat_\Sigma \phi, \bigwedge^\flat_\Sigma \Phi\rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}) \quad \text{(definition of } \widetilde{\lambda}(\mathcal{I})) \\ & \text{iff} & \langle \bigwedge^\flat_\Sigma \Phi \wedge^\flat_\Sigma \phi, \bigwedge^\flat_\Sigma \Phi\rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I}) \quad \text{(self extensionality)} \\ & \text{iff} & \bigwedge_\Sigma^{\mathcal{F}/\widetilde{\Omega}(\mathcal{I})} \Phi/\widetilde{\Omega}_\Sigma(\mathcal{I}) \leq_\Sigma^{\mathcal{F}/\widetilde{\Omega}(\mathcal{I})} \phi/\widetilde{\Omega}_\Sigma(\mathcal{I}). \quad \text{(def. of } \leq^{\mathcal{F}/\widetilde{\Omega}(\mathcal{I})}) \end{array}$$

Therefore, $\mathcal{I}$ is indeed semilattice based with respect to K and $\wedge^\flat$. ■

In some contexts it is desirable to have a specification of the theorems of a π-institution under discussion. However, the hypothesis that $\mathcal{I}$ is semilattice based by itself does not provide information about the theorems of $\mathcal{I}$, since it only specifies, based on properties of the defining class K, entailments with non empty sets of hypotheses. We discuss the *property of being non psudo-axiomatic*, which serves to streamline this ambiguity concerning theorems.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a finitary π-institution based on $\mathbf{F}$. We say that $\mathcal{I}$ is **non pseudo-axiomatic** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{Thm}_\Sigma(\mathcal{I}) = \bigcap\{C_\Sigma(\phi) : \phi \in \mathrm{SEN}^\flat(\Sigma)\}.$$

The property may be equivalently expressed by the condition

$$\mathrm{Thm}(\mathcal{I}) = \bigcap\{T \in \mathrm{ThFam}(\mathcal{I}) : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(T_\Sigma \neq \varnothing)\}.$$

Non pseudo-axiomatic semilattice based π-institutions form a generalization of π-institutions with theorems.

Lemma 1745 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a semilattice based π-institution with theorems, based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is non pseudo-axiomatic;

(b) For all $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$, all $\Sigma \in |\mathbf{Sign}^\flat|$, all $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\alpha_\Sigma(\phi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(t).$$

Proof:

(a) Let $\Sigma \in |\mathbf{Sign}^\flat|$. On the one hand, $\mathrm{Thm}_\Sigma(\mathcal{I}) \subseteq \bigcap\{C_\Sigma(\phi) : \phi \in \mathrm{SEN}^\flat(\Sigma)\}$. On the other, $\bigcap\{C_\Sigma(\phi) : \phi \in \mathrm{SEN}^\flat(\Sigma)\} \subseteq \bigcap\{C_\Sigma(\phi) : \phi \in \mathrm{Thm}_\Sigma(\mathcal{I})\} = \mathrm{Thm}_\Sigma(\mathcal{I})$. Hence, $\mathcal{I}$ is non pseudo-axiomatic.

(b) Suppose $\Sigma \in |\mathbf{Sign}^\flat|$ and $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$. Then, for all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $t \in C_\Sigma(\phi)$. Since, by Proposition 1742, $\mathcal{I}$ is semilattice based with respect to $\mathsf{K}^{\mathcal{I}}$, we get that, for all $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$, $\alpha_\Sigma(\phi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(t)$. ■

By Lemma 1739, for all $\Sigma \in |\mathbf{Sign}^\flat|$, and all $t, t' \in \mathrm{Thm}_\Sigma(\mathcal{I})$, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$, for all $\mathcal{A} \in \mathsf{K}$. Furthermore, the common value of all theorems in $\mathrm{SEN}(\Sigma)$ is, by Lemma 1745, a maximum element under $\leq^{\mathcal{A}}_\Sigma$. This element will be denoted by $1^{\mathcal{A}}_\Sigma$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a semilattice based semantic variety with respect to $\wedge^\flat$. We assume that K is **conditionally max natural**, i.e., for all $\mathcal{A} \in \mathsf{K}$, either for no $\Sigma \in |\mathbf{Sign}|$ is there a maximum under $\leq^{\mathcal{A}}_\Sigma$ or, for every $\Sigma \in |\mathbf{Sign}|$, there exists a maximum $1^{\mathcal{A}}_\Sigma$ under $\leq^{\mathcal{A}}_\Sigma$, and moreover, $1^{\mathcal{A}} = \{1^{\mathcal{A}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ is natural, i.e., it satisfies, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}(f)(1^{\mathcal{A}}_\Sigma) = 1^{\mathcal{A}}_{\Sigma'}.$$

Define a finitary closure system

$$C^{\mathsf{K},\wedge} : \mathcal{P}\mathrm{SEN}^\flat \to \mathcal{P}\mathrm{SEN}^\flat$$

on $\mathbf{F}$ by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$,

- $\phi \in C^{\mathsf{K},\wedge}_\Sigma(\varnothing)$ if and only if, for all $\mathcal{A} \in \mathsf{K}$ and all $\chi \in \mathrm{SEN}(F(\Sigma))$, $\chi \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi)$;
- $\phi \in C^{\mathsf{K},\wedge}_\Sigma(\Phi)$ if and only if, for all $\mathcal{A} \in \mathsf{K}$, $\bigwedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\Phi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi)$.

We note that conditional max naturality is essential in guaranteeing the structurality of $C^{\mathsf{K},\wedge}$.

Lemma 1746 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* K *a conditionally max natural semilattice based semantic variety with respect to* $\wedge^\flat$*. Then* $\mathcal{I}^{\mathsf{K},\wedge} = \langle \mathbf{F}, C^{\mathsf{K},\wedge}\rangle$ *is a non pseudo-axiomatic* π*-institution.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \bigcap\{C_\Sigma(\psi) : \psi \in \mathrm{SEN}^\flat(\Sigma)\}$. Then, by definition, for all $\mathcal{A} \in \mathsf{K}$, $\alpha_\Sigma(\psi) \leq \alpha_\Sigma(\phi)$, for all $\psi \in \mathrm{SEN}^\flat(\Sigma)$. By the surjectivity of $\langle F, \alpha\rangle$, we get that $\phi \in \mathrm{Thm}_\Sigma(\mathcal{I}^{\mathsf{K},\wedge})$. Therefore, $\mathcal{I}^{\mathsf{K},\wedge}$ is non pseudo-axiomatic. ■

The semantic variety of $\mathcal{I}^{\mathsf{K},\wedge}$ turns out to be the class K.

Proposition 1747 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* K *a conditionally max natural semilattice based semantic variety with respect to* $\wedge^\flat$*. Then* $\mathcal{I}^{\mathsf{K},\wedge} = \langle \mathbf{F}, C^{\mathsf{K},\wedge}\rangle$ *is semilattice based with respect to* K *and* $\wedge^\flat$ *and* $\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}^{\mathsf{K},\wedge}) = \mathsf{K}$*.*

Proof: By the second condition in the definition of $C^{\mathsf{K},\wedge}$, we conclude that $\mathcal{I}^{\mathsf{K},\wedge}$ is semilattice based with respect to K and $\wedge^\flat$. Then, by definition $\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}^{\mathsf{K},\wedge}) = \mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$, since, by hypothesis, K is a semantic variety. ■

Proposition 1748 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a semilattice based non pseudo-axiomatic* π*-institution. Then* $\mathcal{I}^{\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}),\wedge} = \mathcal{I}$*.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, with $\Phi \neq \varnothing$. Then, we have:

$$\begin{array}{lll}
\phi \in C^{\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}),\wedge}_\Sigma(\varnothing) & \text{iff} & \text{for all } \mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathcal{I}), \chi \in \mathrm{SEN}(F(\Sigma)), \\
& & \chi \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi), \\
& \text{iff} & \text{for all } \mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathcal{I}), \psi \in \mathrm{SEN}^\flat(\Sigma), \\
& & \alpha_\Sigma(\psi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi), \\
& \text{iff} & \text{for all } \mathcal{A} \in \mathsf{K}, \psi \in \mathrm{SEN}^\flat(\Sigma), \\
& & \alpha_\Sigma(\psi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi), \\
& \text{iff} & \phi \in \bigcap\{C_\Sigma(\psi) : \psi \in \mathrm{SEN}^\flat(\Sigma)\} \\
& \text{iff} & \phi \in C_\Sigma(\varnothing).
\end{array}$$

Moreover,

$$\begin{array}{lll}\phi \in C_\Sigma^{\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}),\wedge}(\Phi) & \text{iff} & \text{for all } \mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathcal{I}),\\ & & \bigwedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\Phi) \le^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi)\\ & \text{iff} & \text{for all } \mathcal{A} \in \mathsf{K},\\ & & \bigwedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\Phi) \le^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi)\\ & \text{iff} & \phi \in C_\Sigma(\Phi).\end{array}$$

Thus, we get $\mathcal{I}^{\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}),\wedge} = \mathcal{I}$. ■

For non pseudo-axiomatic semilattice based π-institutions on the same algebraic system, the Frege relations reflect the $\le$ ordering on their closure systems.

Proposition 1749 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$, *and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$, $\mathcal{I}' = \langle \mathbf{F}, C'\rangle$ *non pseudo-axiomatic semilattice based* π*-institutions with respect to* $\wedge^\flat$. *Then*

$$\mathcal{I} \le \mathcal{I}' \quad \textit{iff} \quad \widetilde{\lambda}(\mathcal{I}) \le \widetilde{\lambda}(\mathcal{I}').$$

Proof: The left-to-right implication is by Lemma 1416. Assume, conversely, that $\mathcal{I}$, $\mathcal{I}'$ are non pseudo-axiomatic semilattice based with respect to $\wedge^\flat$, such that $\widetilde{\lambda}(\mathcal{I}) \le \widetilde{\lambda}(\mathcal{I}')$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$, with $\Phi \ne \varnothing$,

$$\begin{array}{lll}\phi \in C_\Sigma(\Phi) & \text{iff} & \phi \in C_\Sigma(\bigwedge^\flat_\Sigma \Phi)\\ & \text{iff} & C_\Sigma(\bigwedge^\flat_\Sigma \Phi \wedge^\flat_\Sigma \phi) = C_\Sigma(\bigwedge^\flat_\Sigma \Phi)\\ & \text{iff} & \langle \bigwedge^\flat_\Sigma \Phi \wedge^\flat_\Sigma \phi, \bigwedge^\flat_\Sigma \Phi\rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I})\\ & \text{implies} & \langle \bigwedge^\flat_\Sigma \Phi \wedge^\flat_\Sigma \phi, \bigwedge^\flat_\Sigma \Phi\rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}')\\ & \text{iff} & C'_\Sigma(\bigwedge^\flat_\Sigma \Phi \wedge^\flat_\Sigma \phi) = C'_\Sigma(\bigwedge^\flat_\Sigma \Phi)\\ & \text{iff} & \phi \in C'_\Sigma(\bigwedge^\flat_\Sigma \Phi)\\ & \text{iff} & \phi \in C'_\Sigma(\Phi).\end{array}$$

Moreover, taking into account what was just demonstrated,

$$\begin{array}{lll}\phi \in C_\Sigma(\varnothing) & \text{iff} & \phi \in \bigcap\{C_\Sigma(\psi) : \psi \in \mathrm{SEN}^\flat(\Sigma)\}\\ & \text{implies} & \phi \in \bigcap\{C'_\Sigma(\psi) : \psi \in \mathrm{SEN}^\flat(\Sigma)\}\\ & \text{iff} & \phi \in C'_\Sigma(\varnothing).\end{array}$$

We conclude that $\mathcal{I} \le \mathcal{I}'$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$. Denote by $\mathsf{K}^{\mathbf{F},\wedge}$ the semantic variety of $\mathbf{F}$-algebraic systems generated by the semilattice equations L1-L4.

Theorem 1750 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$. *There exists a dual isomorphism between the collection of semilattice based non pseudo-axiomatic* π*-institutions with respect to* $\wedge^f$*lat, ordered under* $\le$, *and the collection of all conditionally max natural semantic subvarieties of* $\mathsf{K}^{\mathbf{F},\wedge}$, *ordered under* $\subseteq$, *given by* $\mathcal{I} \mapsto \mathsf{K}^{\mathcal{I}}$.

Proof: Consider the mapping $\mathcal{I} \mapsto \mathsf{K}^{\mathcal{I}}$.

Suppose, first, that $\mathcal{I}, \mathcal{I}'$ are non pseudo-axiomatic and semilattice based with respect to $\wedge^\flat$, such that $\mathsf{K}^{\mathcal{I}} = \mathsf{K}^{\mathcal{I}'}$. Then $\mathcal{I}^{\mathsf{K}^{\mathcal{I}},\wedge} = \mathcal{I}^{\mathsf{K}^{\mathcal{I}'},\wedge}$. By Proposition 1742 and Proposition 1748, we get $\mathcal{I} = \mathcal{I}'$. Therefore, the mapping is one-to-one.

Assume, now, that K is conditionally max natural and semilattice based with respect to $\wedge^\flat$. Then, by Lemma 1746 and Proposition 1747, $\mathcal{I}^{\mathsf{K},\wedge}$ is a non pseudo-axiomatic and semilattice based π-institution with respect to K and $\wedge^\flat$, such that $\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}^{\mathsf{K},\wedge}) = \mathsf{K}$. Therefore, by Proposition 1742, $\mathsf{K}^{\mathcal{I}^{\mathsf{K},\wedge}} = \mathsf{K}$ and the mapping is also onto. Thus, it is a bijection from the collection of semilattice based non pseudo-axiomatic π-institutions with respect to $\wedge^f lat$ onto the collection of all conditionally max natural semantic subvarieties of $\mathsf{K}^{\mathbf{F},\wedge}$.

Finally, for all non pseudo-axiomatic and semilattice based π-institutions $\mathcal{I}, \mathcal{I}'$, with respect to $\wedge^\flat$, we have

$$\begin{array}{lll}\mathcal{I} \leq \mathcal{I}' & \text{iff} & \widetilde{\lambda}(\mathcal{I}) \leq \widetilde{\lambda}(\mathcal{I}') \quad \text{(by Proposition 1749)}\\ & \text{iff} & \widetilde{\Omega}(\mathcal{I}) \leq \widetilde{\Omega}(\mathcal{I}') \quad \text{(by Proposition 1742)}\\ & \text{iff} & \mathsf{K}^{\mathcal{I}'} \leq \mathsf{K}^{\mathcal{I}}. \quad \text{(definition of } \mathsf{K}^{\mathcal{I}}, \mathsf{K}^{\mathcal{I}'})\end{array}$$

Thus, the bijection is also order reversing and dual order reflecting and, therefore, it is a dual order isomorphism, as claimed. ■

Our next goal is to show that for semilattice based π-institutions, their semantic variety coincides with the class of all $\mathcal{I}$-algebraic systems. We start by showing that, for such a π-institution, the $\mathcal{I}$-filter families on any algebraic system in their semantic variety coincides (roughly) with the collection of all semilattice filter families.

Lemma 1751 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a semilattice based* π*-institution based on* $\mathbf{F}$*. For all* $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$*,*

$$\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \begin{cases} \mathrm{FiFam}^{\wedge}(\mathcal{A}), & \textit{if } \mathcal{I} \textit{ has theorems}\\ \mathrm{FiFam}^{\wedge,\varnothing}(\mathcal{A}), & \textit{otherwise}\end{cases}$$

Proof: It suffices to show that, for all $T \in \mathrm{SenFam}(\mathcal{A})$, such that $T_\Sigma \neq \varnothing$, for all $\Sigma \in |\mathbf{Sign}|$, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ if and only if $T \in \mathrm{FiFam}^{\wedge}(\mathcal{A})$.

Suppose, first, that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$.

- Suppose that $\alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \in T_{F(\Sigma)}$. Then, since $\phi \wedge^\flat_\Sigma \psi \in C_\Sigma(\phi, \psi)$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get $\alpha_\Sigma(\phi) \wedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\psi) = \alpha_\Sigma(\phi \wedge^\flat_\Sigma \psi) \in T_{F(\Sigma)}$;
- If $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$ and $\alpha_\Sigma(\phi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\psi)$, then $\alpha_\Sigma(\phi \wedge^\flat_\Sigma \psi) = \alpha_\Sigma(\phi) \wedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\psi) = \alpha_\Sigma(\phi) \in T_{F(\Sigma)}$. Since $\psi \in C_\Sigma(\phi \wedge^\flat_\Sigma \psi)$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get $\alpha_\Sigma(\psi) \in T_{F(\Sigma)}$.

Taking into account the surjectivity of $\langle F,\alpha\rangle$, we get $T \in \mathrm{FiFam}^{\wedge}(\mathcal{A})$.

Assume, conversely, that $T \in \mathrm{FiFam}^{\wedge}(\mathcal{A})$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$ and $\alpha_\Sigma(\Phi) \subseteq T_{F(\Sigma)}$. Since, by Proposition 1742, $\mathcal{I}$ is semilattice based with respect to $\mathsf{K}^{\mathcal{I}}$, we get that $\bigwedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\Phi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi)$ and $\alpha_\Sigma(\Phi) \subseteq T_{F(\Sigma)}$. Since, by hypothesis, $T \in \mathrm{FiFam}^{\wedge}(\mathcal{A})$, $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$.

Finally, if $\phi \in C_\Sigma(\varnothing)$, then $\phi \in C_\Sigma(\psi)$, for all $\psi \in \mathrm{SEN}^\flat(\Sigma)$. Since $\mathcal{I}$ has theorems, $T_{F(\Sigma)} \neq \varnothing$, whence, by the surjectivity of $\langle F,\alpha\rangle$, for some $\psi \in \mathrm{SEN}^\flat(\Sigma)$, $\alpha_\Sigma(\psi) \in T_{F(\Sigma)}$. For this chosen ψ, we also have, since $\mathcal{I}$ is semilattice based with respect to $\mathsf{K}^{\mathcal{I}}$, that $\alpha_\Sigma(\psi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi)$. Thus, since $T \in \mathrm{FiFam}^{\wedge}(\mathcal{A})$, we conclude that $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$.

Since in all cases $\phi \in C_\Sigma(\Phi)$ and $\alpha_\Sigma(\Phi) \subseteq T_{F(\Sigma)}$ imply $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. ■

In the next step, we show that, for a semilattice based π-institution, the Frege congruence system of any $\mathcal{I}$-structure of the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$, with $\mathcal{A}$ in the semantic variety of $\mathcal{I}$, is reduced.

Lemma 1752 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a semilattice based π-institution based on* $\mathbf{F}$. *For every* $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$,

$$\widetilde{\lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}.$$

Hence, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ *is reduced.*

Proof: Let $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$, $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi,\psi\rangle \notin \Delta^{\mathcal{A}}_\Sigma$. Then, by definition, $\phi \neq \psi$ and, by Lemma 1737 and Proposition 1742, we get $\psi \notin T^{\langle \Sigma,\phi\rangle}_\Sigma$ or $\phi \in T^{\langle \Sigma,\psi\rangle}_\Sigma$. Since, by Lemmas 1738 and 1751, $T^{\langle \Sigma,\phi\rangle}, T^{\langle \Sigma,\psi\rangle} \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get that $\langle \phi,\psi\rangle \notin \widetilde{\lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$. Thus, $\widetilde{\lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}$.

Finally, $\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \leq \widetilde{\lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$, whence, $\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}$ and, therefore, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ is a reduced $\mathcal{I}$-structure. ■

In the last step before the main theorem, we show that for a semilattice based π-institution $\mathcal{I}$, if $\langle \mathcal{A}, D\rangle$ is any $\mathcal{I}$-structure, such that $\widetilde{\lambda}^{\mathcal{A}}(D) = \Delta^{\mathcal{A}}$, then $\mathcal{D}$ is either $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, if $\mathcal{I}$ has theorems, or $\mathcal{D}^\varnothing$ is $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, if $\mathcal{I}$ does not have theorems, where $\mathcal{D}^\varnothing$ consists of the filter families in $\mathcal{D}$, (potentially) augmented by filter families in $\mathcal{D}$, in which one or more components have been replaced by $\varnothing$.

Lemma 1753 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a semilattice based π-institution based on* $\mathbf{F}$, *and* $\langle \mathcal{A}, D\rangle \in \mathrm{Str}(\mathcal{I})$, *such that* $\widetilde{\lambda}^{\mathcal{A}}(D) = \Delta^{\mathcal{A}}$. *If $\mathcal{I}$ has theorems, then* $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. *If $\mathcal{I}$ does not have theorems, then* $\mathcal{D}^\varnothing = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

Proof: Let $\mathcal{A}$ be an **F**-algebraic system and suppose $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, with $T_\Sigma \neq \varnothing$, for all $\Sigma \in |\mathbf{Sign}|$. Let $\Sigma \in |\mathbf{Sign}|$, $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in D_\Sigma(T_\Sigma)$. By Proposition 114, there exists $\Phi \subseteq_f T_\Sigma$, such that $\phi \in D_\Sigma(\Phi)$. Since $\langle \mathcal{A}, D\rangle \in \mathrm{Str}(\mathcal{I})$, by Corollary 1735, $D_\Sigma(\bigwedge_\Sigma^{\mathcal{A}} \Phi \wedge_\Sigma^{\mathcal{A}} \phi) = D_\Sigma(\bigwedge_\Sigma^{\mathcal{A}} \Phi)$. Hence, by hypothesis, $\bigwedge_\Sigma^{\mathcal{A}} \Phi \wedge_\Sigma^{\mathcal{A}} \phi = \bigwedge_\Sigma^{\mathcal{A}} \Phi$. Since $\Phi \subseteq_f T_\Sigma$ and, by Lemma 1751, $T \in \mathrm{FiFam}^{\wedge}(\mathcal{A})$, $\bigwedge_\Sigma^{\mathcal{A}} \Phi \in T_\Sigma$. By the preceding equation, $\bigwedge_\Sigma^{\mathcal{A}} \Phi \wedge_\Sigma^{\mathcal{A}} \phi \in T_\Sigma$ and, therefore, $\phi \in T_\Sigma$. We conclude that $T = D(T)$ and, hence, $T \in \mathcal{D}$.

If $\mathcal{I}$ has theorems, then, for every $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T_\Sigma \neq \varnothing$, for all $\Sigma \in |\mathbf{Sign}|$. By what was proven above, $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. On the other hand, if $\mathcal{I}$ does not have theorems, then any of the components of $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is allowed to be empty and, therefore, $\mathcal{D}^\varnothing = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. ■

In one of the main theorems, we show that, for a semilattice based π-institution $\mathcal{I}$, the semantic variety $\mathsf{K}^{\mathcal{I}}$ of $\mathcal{I}$ coincides with the class $\mathrm{AlgSys}(\mathcal{I})$ of all $\mathcal{I}$-algebraic systems.

Theorem 1754 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a semilattice based* π*-institution based on* $\mathbf{F}$.

(a) $\mathrm{AlgSys}(\mathcal{I}) = \mathsf{K}^{\mathcal{I}}$*;*

(b) $\mathrm{AlgSys}(\mathcal{I})$ *is a semantic variety;*

(c) $\mathcal{I}$ *is semilattice based with respect to* $\mathrm{AlgSys}(\mathcal{I})$.

Proof: We have, by Proposition 65, that, in general, $\mathrm{AlgSys}(\mathcal{I}) \subseteq \mathsf{K}^{\mathcal{I}}$. Suppose, conversely, that $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$. Then, by Lema 1752, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ is reduced. Therefore, $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$. We conclude that $\mathrm{AlgSys}(\mathcal{I}) = \mathsf{K}^{\mathcal{I}}$. Since $\mathsf{K}^{\mathcal{I}}$ is, by definition, a semantic variety, then so is $\mathrm{AlgSys}(\mathcal{I})$. Finally, since, by Proposition 1742, $\mathcal{I}$ is semilattice based with respect to $\mathsf{K}^{\mathcal{I}}$, it is semilattice based with respect to $\mathrm{AlgSys}(\mathcal{I})$. ■

In another main theorem, it is shown that a finitary self extensional and conjunctive π-institution $\mathcal{I}$ is necessarily fully self extensional, i.e., that every $\langle \mathcal{A}, D\rangle \in \mathrm{FStr}(\mathcal{I})$ satisfies the Congruence Property.

Theorem 1755 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a finitary* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is self extensional and conjunctive, then it is fully self extensional.*

Proof: Suppose that $\mathcal{I}$ is finitary, self extensional and has the Conjunction Property with respect to $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$. By Theorem 1744 and Proposition 1742, $\mathcal{I}$ is semilattice based with respect to $\mathsf{K}^{\mathcal{I}}$ and $\wedge^\flat$. By Proposition 65, $\mathrm{AlgSys}(\mathcal{I}) \subseteq \mathsf{K}^{\mathcal{I}}$, whence, by Lemma 1752, if $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, then $\widetilde{\lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}$.

Suppose, now, that $\langle \mathcal{A}, D\rangle \in \mathrm{FStr}(\mathcal{I})$. Then, by definition,

$$\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(D)) = \mathcal{D}/\widetilde{\Omega}^{\mathcal{A}}(D).$$

Since $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) \in \mathrm{AlgSys}(\mathcal{I})$, by what was shown in the preceding paragraph,

$$\widetilde{\lambda}^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(D)}(\mathcal{D}/\widetilde{\Omega}^{\mathcal{A}}(D)) = \Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(D)}.$$

Therefore, we get that $\langle \mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(D), \mathcal{D}/\widetilde{\Omega}^{\mathcal{A}}(D)\rangle$ has the Congruence Property. Therefore, by Proposition 1426, $\langle \mathcal{A}, D\rangle$ also has the Congruence Property. We conclude that $\mathcal{I}$ is fully self extensional. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ a finitary Gentzen π-institution based on $\mathbf{F}$.

$\mathfrak{G}$ **has congruence** if, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\sigma^\flat_\Sigma(\vec{\phi}) \vdash_\Sigma \sigma^\flat_\Sigma(\vec{\psi}) \in G_\Sigma(\{\phi_i \vdash_\Sigma \psi_i, \psi_i \vdash_\Sigma \phi_i : i < k\}).$$

Moreover, $\mathfrak{G}$ **has conjunction with respect to** $\wedge^\flat$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi, \psi \vdash_\Sigma \phi \wedge^\flat_\Sigma \psi,\ \phi \wedge^\flat_\Sigma \psi \vdash_\Sigma \phi,\ \phi \wedge^\flat_\Sigma \psi \vdash_\Sigma \psi \in G_\Sigma(\varnothing).$$

Let, also, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a finitary π-institutuon based on $\mathbf{F}$. Recall that:

- If $\mathcal{I}$ has theorems, $\mathfrak{G}$ is **fully adequate for** $\mathcal{I}$ if $\mathrm{Str}(\mathfrak{G}) = \mathrm{FStr}(\mathcal{I})$;
- If $\mathcal{I}$ does not have theorems, then $\mathfrak{G}$ is **fully adequate for** $\mathcal{I}$ if $\mathrm{Str}(\mathfrak{G})^\varnothing = \mathrm{FStr}(\mathcal{I})$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a finitary self extensional π-institution, having the Conjunction Property with respect to $\wedge^\flat$.

- Define $\mathrm{Ax}^{\mathcal{I}} = \{\mathrm{Ax}^{\mathcal{I}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{Ax}^{\mathcal{I}}_\Sigma = \{\Phi \vdash_\Sigma \phi : \phi \in C_\Sigma(\Phi)\};$$

- Define $\mathrm{Ir}^{\mathcal{I}} = \{\mathrm{Ir}^{\mathcal{I}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\begin{aligned}\mathrm{Ir}^{\mathcal{I}}_\Sigma &= \{\langle\{\phi_i \vdash_\Sigma \psi_i, \psi_i \vdash_\Sigma \phi_i : i \in I\}, \sigma^\flat_\Sigma(\vec{\phi}) \vdash_\Sigma \sigma^\flat_\Sigma(\vec{\psi})\rangle : \\ &\qquad \sigma^\flat \in N^\flat, \vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)\}.\end{aligned}$$

- $R^{\mathcal{I}} \coloneqq \mathrm{Ax}^{\mathcal{I}} \cup \mathrm{Ir}^{\mathcal{I}}$.

Finally, define $\mathfrak{G}^{\mathcal{I}} = \langle \mathbf{F}, C^{\mathcal{I}}\rangle \coloneqq \mathfrak{G}^{R^{\mathcal{I}}}$ as the Gentzen π-institution generated by the system $R^{\mathcal{I}}$ of Gentzen rules. Recall, by Proposition 1482, that $G^{\mathcal{I}} = \Xi^{R^{\mathcal{I}}}$.

This Gentzen π-institution turns out to be fully adequate for the π-institution $\mathcal{I}$:

Theorem 1756 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a finitary conjunctive* π*-institution.* $\mathcal{I}$ *is self extensional if and only if* $\mathfrak{G}^{\mathcal{I}} = \langle \mathbf{F}, G^{\mathcal{I}} \rangle$ *is fully adequate for* $\mathcal{I}$*.*

Proof: Suppose, first, that $\mathfrak{G}$ is fully adequate for $\mathcal{I}$. Since $\langle \mathcal{F}, C \rangle \in \mathrm{FStr}(\mathcal{I})$, we get that $\langle \mathcal{F}, C \rangle \in \mathrm{Str}(\mathfrak{G}^{\mathcal{I}})$, if $\mathcal{I}$ has theorems, and that $\langle \mathcal{F}, C \rangle \in \mathrm{Str}(\mathfrak{G}^{\mathcal{I}})^\varnothing$, otherwise. Since, by definition, $\mathfrak{G}^{\mathcal{I}}$ has congruence, we get that $\langle \mathcal{F}, C \rangle$ has the Congruence Property, which amounts to $\mathcal{I}$ having the Congruence Property. Thus, $\mathcal{I}$ is self extensional.

Assume, conversely, that $\mathcal{I}$ is finitary, self extensional and has the Conjunction Property with respect to $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$.

- Suppose, first, that $\langle \mathcal{A}, D \rangle \in \mathrm{FStr}(\mathcal{I})$. Then, by Theorem 1755, $\langle \mathcal{A}, D \rangle$ has the Congruence Property. Moreover, by definition, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$, if $\phi \in C_\Sigma(\Phi)$, then, $\alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi))$. We conclude that, if $\mathcal{I}$ has theorems, then $\langle \mathcal{A}, D \rangle \in \mathrm{Str}(\mathfrak{G}^{\mathcal{I}})$ and that, otherwise, $\langle \mathcal{A}, D \rangle \in \mathrm{Str}(\mathfrak{G}^{\mathcal{I}})^\varnothing$.

- Suppose, conversely, that $\langle \mathcal{A}, D \rangle \in \mathrm{Str}(\mathfrak{G}^{\mathcal{I}})$, if $\mathcal{I}$ has theorems, or $\langle \mathcal{A}, D \rangle \in \mathrm{Str}(\mathfrak{G}^{\mathcal{I}})^\varnothing$, otherwise. Consider the reduction

  $$\langle \mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})), \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))\rangle.$$

  This reduction is an $\mathcal{I}$-structure and, by hypothesis and Proposition 1426, it has the Congruence Property. Thus, we get

  $$\widetilde{\lambda}^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))) = \Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))}.$$

  Now Lemma 1753 allows us to conclude that

  $$\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))).$$

  Therefore, $\langle \mathcal{A}, D \rangle \in \mathrm{FStr}(\mathcal{I})$.

We conclude that $\mathfrak{G}^{\mathcal{I}}$ is fully adequate for $\mathcal{I}$. ■

23.4 Fregeanity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with a binary natural transformation $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

Recall that $\mathcal{I}$ is called:

- **strongly Fregean** if, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\lambda}(T) = \widetilde{\Omega}(T)$, i.e., if and only if the strong Frege equivalence family $\widetilde{\lambda}(T)$ is a congruence system;

- **congruential** if, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\lambda}(T)$ satisfies the congruence property, i.e., $\widetilde{\lambda}(T)$ is a congruence family (but not necessarily a system);
- **Fregean** if, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\Lambda}(T) = \widetilde{\Omega}(T)$, i.e., if its Frege equivalence system $\widetilde{\Lambda}$ is a congruence system.

Strong Fregeanity implies congruentiality, which, in turn, implies Fregeanity.

Recall, also, that $\mathcal{I}$ is said to have the **Deduction Detachment Theorem with respect to** $\to^\flat$ if, for every $\Sigma \in |\mathbf{Sign}^\flat|$, all $\Phi \cup \{\phi, \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\psi \in C_\Sigma(\Phi, \phi) \quad \text{iff} \quad \phi \to^\flat_\Sigma \psi \in C_\Sigma(\Phi).$$

In the following proposition, it is shown that every strongly Fregean π-institution with the Deduction Detachment Theorem satisfies certain axioms and the rule of Modus Ponens.

Proposition 1757 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with a binary natural transformation* $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a congruential* π*-institution having the Deduction Detachment Theorem with respect to* $\to^\flat$*. Then, for all* $\sigma^\flat$ *in* $N^\flat$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi, \chi, \vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$*,*

(a) $\phi \to^\flat_\Sigma (\psi \to^\flat_\Sigma \phi) \in C_\Sigma(\varnothing)$;

(b) $(\phi \to^\flat_\Sigma (\psi \to^\flat_\Sigma \chi)) \to^\flat_\Sigma ((\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma (\phi \to^\flat_\Sigma \chi)) \in C_\Sigma(\varnothing)$;

(c) $(\phi_0 \to^\flat_\Sigma \psi_0) \to^\flat_\Sigma ((\psi_0 \to^\flat_\Sigma \phi_0) \to^\flat_\Sigma (\cdots((\phi_{k-1} \to^\flat_\Sigma \psi_{k-1}) \to^\flat_\Sigma ((\psi_{k-1} \to^\flat_\Sigma \phi_{k-1}) \to^\flat_\Sigma (\sigma^\flat_\Sigma(\vec{\phi}) \to^\flat_\Sigma \sigma^\flat_\Sigma(\vec{\psi}))))\cdots)) \in C_\Sigma(\varnothing)$;

(d) $\psi \in C_\Sigma(\phi, \phi \to^\flat_\Sigma \psi)$.

Proof:

(a) We have, by inflationarity, $\phi \in C_\Sigma(\phi, \psi)$, whence, by two applications of the Deduction Theorem, $\phi \to^\flat_\Sigma (\psi \to^\flat_\Sigma \phi) \in C_\Sigma(\varnothing)$.

(b) We have, using the Detachment Theorem,

$$\chi \in C_\Sigma(\psi, \psi \to^\flat_\Sigma \chi) \subseteq C_\Sigma(\phi, \phi \to^\flat_\Sigma \psi, \phi \to^\flat_\Sigma (\psi \to^\flat_\Sigma \chi)).$$

Thus, using the Deduction Theorem, we get

$$(\phi \to^\flat_\Sigma (\psi \to^\flat_\Sigma \chi)) \to^\flat_\Sigma ((\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma (\phi \to^\flat_\Sigma \chi)) \in C_\Sigma(\varnothing).$$

(c) Since $\mathcal{I}$ is congruential, we get, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$C_\Sigma(T_\Sigma,\phi_i) = C_\Sigma(T_\Sigma,\psi_i),\ i < k, \quad \text{imply} \quad C_\Sigma(T_\Sigma,\sigma^\flat_\Sigma(\vec{\phi})) = C_\Sigma(T_\Sigma,\sigma^\flat_\Sigma(\vec{\psi})).$$

But $C_\Sigma(T_\Sigma,\phi_i) = C_\Sigma(T_\Sigma,\psi_i)$ is equivalent, by the Deduction Detachment Theorem, to $\phi \to^\flat_\Sigma \psi$, $\psi \to^\flat_\Sigma \phi \in C_\Sigma(T_\Sigma) = T_\Sigma$. Similarly,

$$C_\Sigma(T_\Sigma,\sigma^\flat_\Sigma(\vec{\phi})) = C_\Sigma(T_\Sigma,\sigma^\flat_\Sigma(\vec{\psi}))$$

is equivalent to $\sigma^\flat_\Sigma(\vec{\phi}) \to^\flat_\Sigma \sigma^\flat_\Sigma(\vec{\psi})$, $\sigma^\flat_\Sigma(\vec{\psi}) \to^\flat_\Sigma \sigma^\flat_\Sigma(\vec{\phi}) \in C_\Sigma(T_\Sigma) = T_\Sigma$. Therefore, we get

$$\sigma^\flat_\Sigma(\vec{\phi}) \to^\flat_\Sigma \sigma^\flat_\Sigma(\vec{\psi}) \in C_\Sigma(\{\phi_i \to^\flat_\Sigma \psi_i, \psi_i \to^\flat_\Sigma \phi_i : i < k\}).$$

Now (c) follows by several applications of the Deduction Theorem.

(d) By inflationarity, $\phi \to^\flat_\Sigma \psi \in C_\Sigma(\phi \to^\flat_\Sigma \psi)$, whence by the Detachment Theorem, $\psi \in C_\Sigma(\phi,\phi \to^\flat_\Sigma \psi)$.

■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$. Define $\mathrm{Ax}^0 = \{\mathrm{Ax}^0_\Sigma\}_{\Sigma\in|\mathbf{Sign}^\flat|}$, by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$, Ax^0_Σ is the set consisting, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi,\chi,\vec{\phi},\vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,

- $\phi \to^\flat_\Sigma (\psi \to^\flat_\Sigma \phi)$;
- $(\phi \to^\flat_\Sigma (\psi \to^\flat_\Sigma \chi)) \to^\flat_\Sigma ((\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma (\phi \to^\flat_\Sigma \chi))$;
- $(\phi_0 \to^\flat_\Sigma \psi_0) \to^\flat_\Sigma ((\psi_0 \to^\flat_\Sigma \phi_0) \to^\flat_\Sigma (\cdots((\phi_{k-1} \to^\flat_\Sigma \psi_{k-1}) \to^\flat_\Sigma ((\psi_{k-1} \to^\flat_\Sigma \phi_{k-1}) \to^\flat_\Sigma (\sigma^\flat_\Sigma(\vec{\phi}) \to^\flat_\Sigma \sigma^\flat_\Sigma(\vec{\psi}))))\cdots))$.

Furthermore, define $\mathrm{Ir}^0 = \{\mathrm{Ir}^0_\Sigma\}_{\Sigma\in|\mathbf{Sign}^\flat|}$ by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{Ir}^0_\Sigma = \{\langle\{\phi,\phi \to^\flat_\Sigma \psi\},\psi\rangle : \phi,\psi \in \mathrm{SEN}^\flat(\Sigma)\}.$$

Finally, let $R^0 = \mathrm{Ax}^0 \cup \mathrm{Ir}^0$. Set $\mathcal{I}^0 = \langle \mathbf{F}, C^0\rangle$ be the finitary π-institution, based on $\mathbf{F}$, with $C^0 = C^{R^0}$ the closure system on $\mathbf{F}$ generated by the collection R^0 of $\mathbf{F}$-axioms and $\mathbf{F}$-rules of inference.

Our work in Proposition 1757 allows us to formalize the fact that a congruential finitary π-institution having the Deduction Detachment Theorem with respect to $\to^\flat$ is an axiomatic extension of $\mathcal{I}^0$. Recall that $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is an axiomatic extension of $\mathcal{I}^0$ if there exists an axiom family Ax', such that $C = C^{R^0 \cup \mathrm{Ax}'}$.

Theorem 1758 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with a binary natural transformation $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a congruential finitary π-institution having the Deduction Detachment Theorem with respect to $\to^\flat$. Then $\mathcal{I}$ is an axiomatic extension of $\mathcal{I}^0$.*

Proof: Since $\mathcal{I}$ is finitary, its closure system is specified by a collection $R = \mathrm{Ax} \cup \mathrm{Ir}$ of $\mathbf{F}$-axioms and $\mathbf{F}$-rules. We define $R' = \mathrm{Ax}' \cup \mathrm{Ir}'$, where, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

- $\mathrm{Ax}'_\Sigma = \mathrm{Ax}_\Sigma \cup \{\phi_0 \to^\flat_\Sigma (\phi_i \to^\flat_\Sigma \cdots \to^\flat_\Sigma (\phi_{n-1} \to^\flat_\Sigma \phi)\cdots) : \langle\{\phi_0,\ldots,\phi_{n-1}\},\phi\rangle \in \mathrm{Ir}_\Sigma\}$;
- $\mathrm{Ir}'_\Sigma = \{\langle\{\langle\phi, \phi \to^\flat_\Sigma \psi\},\psi\rangle : \phi,\psi \in \mathrm{SEN}^\flat(\Sigma)\}$.

Note that, by the Deduction Theorem of $\mathcal{I}$, for every $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathrm{Ax}'_\Sigma \subseteq C_\Sigma(\varnothing)$. Moreover, by the Detachment Theorem for $\mathcal{I}$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, $\psi \in C_\Sigma(\phi, \phi \to^\flat_\Sigma \psi)$. Therefore, we conclude that $C^{\mathrm{R}'} \le C$.

Conversely, note that, by definition, $\mathrm{Ax} \le \mathrm{Ax}'$. Moreover, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\langle\{\phi_0,\ldots,\phi_{n-1}\},\phi\rangle \in \mathrm{Ir}_\Sigma$,

$$\phi \in C^{R'}_\Sigma(\Phi, \phi_0 \to^\flat_\Sigma (\phi_1 \to^\flat_\Sigma \cdots \to^\flat_\Sigma (\phi_{n-1} \to^\flat_\Sigma \phi)\cdots)) \subseteq C^{R'}_\Sigma(\Phi).$$

Hence, $C = C^R \le C^{R'}$. We now conclude that $C = C^{R'}$. ■

Next, we show that, if $\mathcal{I}$ is an axiomatic extension of $\mathcal{I}^0$, then it has the Deduction Detachment Theorem with respect to $\to^\flat$.

Proposition 1759 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with a binary natural transformation $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ an axiomatic extension of $\mathcal{I}^0$. Then $\mathcal{I}$ has the Deduction Detachment Theorem with respect to $\to^\flat$.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi,\psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \to^\flat_\Sigma \psi \in C_\Sigma(\Phi)$. Then, since, by hypothesis, $C^{R^0} \le C$, we get

$$\psi \in C_\Sigma(\phi, \phi \to^\flat_\Sigma \psi) \subseteq C_\Sigma(\Phi, \phi).$$

Suppose, conversely, that $\psi \in C_\Sigma(\Phi,\phi)$. Then, there exists in $\mathcal{I}$ a proof $\phi_0,\phi_1,\ldots,\phi_n = \psi$ of ψ from premises $\Phi \cup \{\phi\}$. We show by induction on $k \le n$ that there exists a proof in $\mathcal{I}$ of $\phi \to^\flat_\Sigma \psi$ from premises Φ in $\mathcal{I}$.

- If $k = 0$, then $\phi_0 \in \mathrm{Ax}_\Sigma$ or $\phi_0 \in \Phi \cup \{\phi\}$.
  - If $\phi_0 \in \mathrm{Ax}_\Sigma$, then $\phi_0 \to^\flat_\Sigma (\phi \to^\flat_\Sigma \phi_0), \phi_0, \phi \to^\flat_\Sigma \phi_0$ is a proof in $\mathcal{I}$ of $\phi \to^\flat_\Sigma \phi_0$ from Φ;
  - If $\phi_0 \in \Phi$, then $\phi_0 \to^\flat_\Sigma (\phi \to^\flat_\Sigma \phi_0), \phi_0, \phi \to^\flat_\Sigma \phi_0$ is a proof in $\mathcal{I}$ of $\phi \to^\flat_\Sigma \phi_0$ from premises Φ.
  - If $\phi_0 = \phi$, then

$$\begin{array}{l}(\phi_0 \to^\flat_\Sigma ((\phi_0 \to^\flat_\Sigma \phi_0) \to^\flat_\Sigma \phi_0)) \\ \qquad\qquad \to^\flat_\Sigma ((\phi_0 \to^\flat_\Sigma (\phi_0 \to^\flat_\Sigma \phi_0)) \to^\flat_\Sigma (\phi_0 \to^\flat_\Sigma \phi_0)) \\ \phi_0 \to^\flat_\Sigma ((\phi_0 \to^\flat_\Sigma \phi_0) \to^\flat_\Sigma \phi_0) \\ (\phi_0 \to^\flat_\Sigma (\phi_0 \to^\flat_\Sigma \phi_0)) \to^\flat_\Sigma (\phi_0 \to^\flat_\Sigma \phi_0) \\ \phi_0 \to^\flat_\Sigma (\phi_0 \to^\flat_\Sigma \phi_0) \\ \phi_0 \to^\flat_\Sigma \phi_0\end{array}$$

    is a proof in $\mathcal{I}$ of $\phi_0 \to^\flat_\Sigma \phi_0$ from Φ

- If $k > 0$, assume that, for all $\ell < k$, $\phi \to^\flat_\Sigma \phi_\ell \in C_\Sigma(\Phi)$. If ϕ_k is either an axiom or in $\Phi \cup \{\phi\}$, then the treatment is the same as in the Induction Basis. So assume, for the final case, that ϕ_k follows from preceding Σ-sentences in the sequel by an application of the only **F**-rule available, i.e., that, for some $i, j < k$, $\phi_i = \phi_j \to^\flat_\Sigma \phi_k$. Then, by the Induction Hypothesis, $\phi \to^\flat_\Sigma (\phi_j \to^\flat_\Sigma \phi_k) \in C_\Sigma(\Phi)$ and $\phi \to^\flat_\Sigma \phi_j \in C_\Sigma(\Phi)$. Then by adjoining the following Σ-sentences to the sequence consisting of the proofs in $\mathcal{I}$ from Φ of $\phi \to^\flat_\Sigma (\phi_j \to^\flat_\Sigma \phi_k)$ and $\phi \to^\flat_\Sigma \phi_j$, we obtain a proof in $\mathcal{I}$ from Φ of $\phi \to^\flat_\Sigma \phi_k$:

$$\begin{array}{l} \vdots \\ \phi \to^\flat_\Sigma (\phi_j \to^\flat_\Sigma \phi_k) \\ \vdots \\ \phi \to^\flat_\Sigma \phi_j \\ (\phi \to^\flat_\Sigma (\phi_j \to^\flat_\Sigma \phi_k)) \to^\flat_\Sigma ((\phi \to^\flat_\Sigma \phi_j) \to^\flat_\Sigma (\phi \to^\flat_\Sigma \phi_k)) \\ (\phi \to^\flat_\Sigma \phi_j) \to^\flat_\Sigma (\phi \to^\flat_\Sigma \phi_k) \\ \phi \to^\flat_\Sigma \phi_k \end{array}$$

  This completes the Induction Step.

Thus, we conclude that $\phi \to^\flat_\Sigma \psi \in C_\Sigma(\Phi)$ and, therefore, $\mathcal{I}$ has the Deduction Detachment Theorem with respect to $\to^\flat$. ■

Moreover, under the same hypotheses, $\mathcal{I}$ turns out to be strongly Fregean.

Proposition 1760 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with a binary natural transformation* $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *an axiomatic extension of* $\mathcal{I}^0$*. Then* $\mathcal{I}$ *is congruential.*

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ be in $N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$, such that $C_\Sigma(T_\Sigma, \phi_i) = C_\Sigma(T_\Sigma, \psi_i)$, for all $i < k$. Then, by Proposition 1759, we get that

$$\phi_i \to^\flat_\Sigma \psi_i, \psi_i \to^\flat_\Sigma \phi_i \in C_\Sigma(T_\Sigma) = T_\Sigma,\ i < k.$$

Since $C^{R^0} \le C$, we get, by multiple applications of the Detachment Theorem, $\sigma^\flat_\Sigma(\vec{\phi}) \to^\flat_\Sigma \sigma^\flat_\Sigma(\vec{\psi}), \sigma^\flat_\Sigma(\vec{\psi}) \to^\flat_\Sigma \sigma^\flat_\Sigma(\vec{\phi}) \in C_\Sigma(T_\Sigma)$. Hence, $C_\Sigma(T_\Sigma, \sigma^\flat_\Sigma(\vec{\phi})) = C_\Sigma(T_\Sigma, \sigma^\flat_\Sigma(\vec{\psi}))$. Thus, $\widetilde{\lambda}(T)$ is a congruence family on $\mathbf{F}$ and, therefore, $\mathcal{I}$ is congruential. ■

Thus, we have obtained an exact characterization of those π-institutions that are congruential and possess the Deduction Detachment Property with respect to a binary natural transformation $\to^\flat$.

Theorem 1761 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with a binary natural transformation* $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a finitary* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is congruential and has the Deduction Detachment Theorem with respect to* $\to^\flat$ *if and only if it is an axiomatic extension of* $\mathcal{I}^0$*.*

Proof: The implication left-to-right is by Theorem 1758. The converse is given by Propositions 1759 and 1760. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. Recall that a Σ-**sequent** is an expression of the form $\Phi \vdash_\Sigma \phi$, where $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. It is **finite** if Φ is a finite set. Moreover, a **Gentzen F-rule** is an expression of the form

$$\langle \{\Phi_i \vdash_\Sigma \phi_i : i \in I\}, \Phi \vdash_\Sigma \phi \rangle,$$

where $\Phi_i \vdash_\Sigma \phi_i$, $i \in I$, and $\Phi \vdash_\Sigma \phi$ are Σ-sequents. We say the rule is **finitary** if I is finite and all sequents in the rule are finite.

Let $\mathbb{L} = \langle \mathcal{A}, D \rangle$ be an **F**-structure, $\Sigma \in |\mathbf{Sign}^\flat|$, $s = \Phi \vdash_\Sigma \phi$ a Σ-sequent and $r = \langle \{\Phi_i \vdash_\Sigma \phi_i : i \in I\}, \Phi \vdash_\Sigma \phi \rangle$ a Gentzen **F**-rule.

- $\mathbb{L}$ **satisfies** s or s is **true** or **valid** or **holds in** $\mathbb{L}$, written $\mathbb{L} \vDash_\Sigma s$, if $\alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi))$;
- $\mathbb{L}$ **satisfies** r or r is **true** or **valid** or **holds in** $\mathbb{L}$, written $\mathbb{L} \vDash_\Sigma r$, if

$$\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi_i)),\ i \in I, \quad \text{imply} \quad \alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

These definitions are extended in the ordinary way to sets of rules and sets of structures.

Let M be a class of **F**-structures. We say that M is a (**finitary**) **Gentzen class** if it is specified by a collection $R = \{R_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ of (finitary) Gentzen **F**-rules (including sequents, viewed as rules with empty sets of premises).

The following examples illustrate the definition.

- The class of all finitary **F**-structures having the Deduction Detachment Theorem with respect to a binary natural transformation $\to^\flat$: $(\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$ is a finitary Gentzen class specified by $R = \{R_\Sigma\}_{\Sigma \in |\mathbf{Sign}|^\flat|}$, where

$$\begin{array}{rcl} R_\Sigma & = & \{\phi, \phi \to^\flat_\Sigma \psi \vdash_\Sigma \psi : \phi, \psi \in \mathrm{SEN}^\flat(\Sigma)\} \\ & & \cup \{\langle \{\Phi, \phi \vdash_\Sigma \psi\}, \Phi \vdash_\Sigma \phi \to^\flat_\Sigma \psi \rangle : \Phi \cup \{\phi, \psi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)\}. \end{array}$$

- The class of all finitary self extensional **F**-structures is also a finitary Gentzen class specified by $R = \{R_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, with

$$R_\Sigma = \{\langle \{\phi_i \vdash_\Sigma \psi_i,\ \psi_i \vdash_\Sigma \phi_i : i < k\}, \sigma^\flat_\Sigma(\vec{\phi}) \vdash_\Sigma \sigma^\flat_\Sigma(\vec{\psi}) \rangle : \sigma^\flat \in N^\flat, \vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)\}.$$

- The class of all finitary congruential **F**-structures is also a finitary Gentzen class specified by $R = \{R_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, with

$$R_\Sigma = \{\langle \{\Phi, \phi_i \vdash_\Sigma \psi_i,\ \Phi, \psi_i \vdash_\Sigma \phi_i : i < k\},\ \Phi, \sigma^\flat_\Sigma(\vec{\phi}) \vdash_\Sigma \sigma^\flat_\Sigma(\vec{\psi}) \rangle : \sigma^\flat \in N^\flat, \Phi \subseteq_f \mathrm{SEN}^\flat(\Sigma), \vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)\}.$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\Sigma \in |\mathbf{Sign}^\flat|$ and $r = \langle \{\Phi_i \vdash_\Sigma \phi_i : i \in I\}, \Phi \vdash_\Sigma \phi \rangle$ a finitary Gentzen $\mathbf{F}$-rule. The **accumulation of** r, denoted $\mathrm{acm}(r)$, is the collection of finitary Gentzen $\mathbf{F}$-rules

$$\mathrm{acm}(r) = \{\langle \{X, \Phi_i \vdash_\Sigma \phi_i : i \in I\},\ X, \Phi \vdash_\Sigma \phi \rangle : X \subseteq_f \mathrm{SEN}^\flat(\Sigma)\}.$$

We say that a collection R of Gentzen rules is **accumulative** if it is the union of accumulations. We say that a class M of $\mathbf{F}$-structures is an **accumulative class** if it is a Gentzen class specified by an accumulative collection of Gentzen $\mathbf{F}$-rules.

Note, e.g., that both the class of all finitary $\mathbf{F}$-structures having the Deduction Detachment Theorem with respect to $\to^\flat$ and the class of congruential finitary $\mathbf{F}$-structures are accumulative classes. On the other hand, the class of of all self extensional finitary $\mathbf{F}$-structures is not accumulative.

It is not difficult to see that satisfaction of Gentzen $\mathbf{F}$-rules is preserved under bilogical morphisms between $\mathbf{F}$-structures.

Proposition 1762 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathbb{L} = \langle \mathcal{A}, D \rangle$, $\mathbb{L}' = \langle \mathcal{A}', D' \rangle$ two $\mathbf{F}$-structures, $\langle H, \gamma \rangle : \mathbb{L} \vdash \mathbb{L}'$ a bilogical morphism, $\Sigma \in |\mathbf{Sign}^\flat|$ and $r = \langle \{\Phi_i \vdash \phi_i : i \in I\}, \Phi \vdash_\Sigma \phi \rangle$ a Gentzen $\mathbf{F}$-rule. Then*

$$\mathbb{L} \vDash_\Sigma r \quad \textit{iff} \quad \mathbb{L}' \vDash_\Sigma r.$$

Proof: We have, by the definition of satisfaction and that of bilogical morphism, $\mathbb{L} \vDash_\Sigma r$ if and only if

$$\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi_i)),\ i \in I, \quad \text{imply} \quad \alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi)),$$

if and only if

$$\begin{aligned} &\gamma_{F(\Sigma)}(\alpha_\Sigma(\phi_i)) \in D'_{H(F(\Sigma))}(\gamma_{F(\Sigma)}(\alpha_\Sigma(\Phi_i))),\ i \in I, \\ &\qquad\qquad \text{imply} \quad \gamma_{F(\Sigma)}(\alpha_\Sigma(\phi)) \in D'_{H(F(\Sigma))}(\gamma_{F(\Sigma)}(\alpha_\Sigma(\Phi))), \end{aligned}$$

if and only if

$$\alpha'_\Sigma(\phi_i) \in D'_{F'(\Sigma)}(\alpha'_\Sigma(\Phi_i)),\ i \in I, \quad \text{imply} \quad \alpha'_\Sigma(\phi) \in D'_{F'(\Sigma)}(\alpha'_\Sigma(\Phi)),$$

if and only if $\mathbb{L}' \vDash_\Sigma r$. ■

Additionally, we can show that the accumulation of a Gentzen rule holding in a finitary $\mathbf{F}$-structure $\mathbb{L} = \langle \mathcal{A}, D \rangle$ necessarily holds in all structures of the form $\mathbb{L}^T = \langle \mathcal{A}, D^T \rangle$, where, for all $T \in \mathrm{ThFam}(\mathbb{L})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in D^T_\Sigma(\Phi) \quad \text{iff} \quad \phi \in D_\Sigma(T_\Sigma, \Phi).$$

Lemma 1763 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\Sigma \in |\mathbf{Sign}^\flat|$, $r = \langle \{\Phi_i \vdash_\Sigma \phi_i : i \in I\}, \Phi \vdash_\Sigma \phi \rangle$ *be an* $\mathbf{F}$*-rule and* $\mathbb{L} = \langle \mathcal{A}, D \rangle$ *a finitary* $\mathbf{F}$*-structure. If* $\mathbb{L} \vDash_\Sigma \mathrm{acm}(r)$*, then, for all* $T \in \mathrm{ThFam}(\mathbb{L})$*,* $\mathbb{L}^T \vDash_\Sigma \mathrm{acm}(r)$*.*

Proof: Let $X \subseteq_f \mathrm{SEN}^\flat(\Sigma)$ and assume $\alpha_\Sigma(\phi_i) \in D^T_{F(\Sigma)}(\alpha_\Sigma(X), \alpha_\Sigma(\Phi_i))$, for all $i \in I$. By definition,

$$\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(T_\Sigma, \alpha_\Sigma(X), \alpha_\Sigma(\Phi_i)),\ i \in I.$$

But $\langle F, \alpha \rangle$ is surjective, whence there exists $\Psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\alpha_\Sigma(\Psi) = T_\Sigma$. Therefore, we get

$$\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Psi), \alpha_\Sigma(X), \alpha_\Sigma(\Phi_i)),\ i \in I.$$

By finitarity of $\mathbb{L}$, we get that there exists $\Psi' \subseteq_f \Psi$, such that

$$\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Psi'), \alpha_\Sigma(X), \alpha_\Sigma(\Phi_i)),\ i \in I.$$

By the hypothesis, $\alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Psi'), \alpha_\Sigma(X), \alpha_\Sigma(\Phi))$. Since $\Psi' \subseteq \Psi$, we get $\alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Psi), \alpha_\Sigma(X), \alpha_\Sigma(\Phi))$. Thus,

$$\alpha_\Sigma(\phi) \in D_{F(\Sigma)}(T_\Sigma, \alpha_\Sigma(X), \alpha_\Sigma(\Phi)),$$

i.e., $\alpha_\Sigma(\phi) \in D^T_{F(\Sigma)}(\alpha_\Sigma(X), \alpha_\Sigma(\Phi)$. We conclude that $\mathbb{L}^T \vDash_\Sigma \mathrm{acm}(r)$. ∎

Proposition 1764 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* M *an accumulative class of finitary* $\mathbf{F}$*-structures. If* $\mathbb{L} = \langle \mathcal{A}, D \rangle \in \mathsf{M}$*, then, for all* $T \in \mathrm{ThFam}(\mathbb{L})$*,* $\mathbb{L}^T = \langle \mathcal{A}, D^T \rangle \in \mathsf{M}$*.*

Proof: Directly by Lemma 1763. ∎

Next, we show that, if $\mathcal{I}$ is an accumulative protoalgebraic finitary π-institution, then all full $\mathcal{I}$-structures satisfy the defining Gentzen $\mathbf{F}$-rules of $\mathcal{I}$.

Theorem 1765 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* M *an accumulative class of finitary* $\mathbf{F}$*-structures. If* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *is a protoalgebraic* π*-institution in* M*, then the full* $\mathcal{I}$*-structures of the form* $\mathbb{L} = \langle \mathcal{A}, D \rangle$*, where* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* F *an isomorphism, are in* M*.*

Proof: Suppose $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is a protoalgebraic π-institution in M. By Proposition 1762, it suffices to show that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with F an isomorphism, $\mathbb{L} = \langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle \in \mathsf{M}$. By protoalgebraicity and Theorem 1577, we get

$$\alpha^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \mathrm{ThFam}(\mathcal{I})^T,$$

where $T = \alpha^{-1}(C^{\mathcal{I},\mathcal{A}}(\varnothing))$. But

$$\langle F,\alpha\rangle : \langle \mathcal{F}, \alpha^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))\rangle \to \langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$$

is a bilogical morphism. Since, by the hypothesis and Proposition 1764, we have $\langle \mathcal{F}, \mathrm{ThFam}(\mathcal{I})^T\rangle \in \mathsf{M}$, we get, by Proposition 1762. $\mathbb{L} \in \mathsf{M}$. ■

Now we get easily the following results concerning the Deduction Detachment Theorem and congruentiality, respectively.

Corollary 1766 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ a binary natural transformations in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a finitary π-institution based on $\mathbf{F}$ that has the Deduction Detachment Theorem with respect to $\to^\flat$. Then, every full $\mathcal{I}$-structure of the form $\mathbb{L} = \langle \mathcal{A}, D\rangle$, where $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with F an isomorphism, has the Deduction Detachment Theorem with respect to $\to^\flat$.*

Proof: This follows from Theorem 1765 once it is show that if $\mathcal{I}$ has the Deduction Detachment Property with respect to $\to^\flat$, then it is protoalgebraic. Suppose $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \le T'$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi,\psi\rangle \in \Omega_\Sigma(T)$. Then, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$, we have

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}) \in T_{\Sigma'} \quad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}) \in T_{\Sigma'}.$$

But, by the Deduction Detachment Theorem, this holds if and only if,

$$\begin{array}{l}\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}) \to^\flat_{\Sigma'} \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}),\\ \qquad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}) \to^\flat_{\Sigma'} \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}) \in T_{\Sigma'}.\end{array}$$

Since $T \le T'$, we get that

$$\begin{array}{l}\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}) \to^\flat_{\Sigma'} \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}),\\ \qquad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}) \to^\flat_{\Sigma'} \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}) \in T'_{\Sigma'}.\end{array}$$

Hence, again by the Deduction Detachment Theorem,

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}) \in T'_{\Sigma'} \quad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}) \in T'_{\Sigma'}.$$

This gives $\langle \phi,\psi\rangle \in \Omega_\Sigma(T')$. Therefore, $\Omega(T) \le \Omega(T')$ and, hence, $\mathcal{I}$ is protoalgebraic. ■

Corollary 1767 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a protoalgebraic finitary π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is congruential, then the full $\mathcal{I}$-structures of the form $\mathbb{L} = \langle \mathcal{A}, D\rangle$, where $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with F an isomorphism, are also congruential.*

Proof: This follows from Theorem 1765 and the fact that the class of all congruential finitary $\mathbf{F}$-structures is accumulative. ■

Now we look at the converse, in a certain sense, of the inheritance problem of properties specified by Gentzen $\mathbf{F}$-rules. Namely, we identify a type of properties that are bequeathed to the π-institution specified by classes of $\mathbf{F}$-structures, when all structures in the class satisfy the property.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and M be a class of $\mathbf{F}$-structures. Recall that the π-institution $\mathcal{I}^{\mathsf{M}} = \langle \mathbf{F}, C^{\mathsf{M}} \rangle$ determined by, or specified by or generated by, M is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}\phi \in C^{\mathsf{M}}_\Sigma(\Phi) & \text{iff} & \text{for all } \langle \mathcal{A}, D \rangle \in \mathsf{M}, \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma, \Sigma'),\\ & & \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \in D_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))).\end{array}$$

Let, now, $\Sigma \in |\mathbf{Sign}^\flat|$ and $r = \langle \{\Phi_i \vdash_\Sigma \phi_i : i \in I\}, \Phi \vdash_\Sigma \phi \rangle$ be a Gentzen $\mathbf{F}$-rule. The **structure of** r, denoted $\mathrm{str}(r)$ is the family of all Gentzen $\mathbf{F}$-rules of the form

$$\begin{array}{rl}\mathrm{SEN}^\flat(f)(r) := \langle \{\mathrm{SEN}^\flat(f)(\Phi_i) \vdash_{\Sigma'} \mathrm{SEN}^\flat(f)(\phi_i) : i \in I\}, & \\ \mathrm{SEN}^\flat(f)(\Phi) \vdash_{\Sigma'} \mathrm{SEN}^\flat(f)(\phi) \rangle, &\end{array}$$

where $\Sigma' \in |\mathbf{Sign}^\flat|$ and $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$. We say that a collection R of Gentzen $\mathbf{F}$-rules is **structural** if it is the union of structures. We say that a class M of $\mathbf{F}$-structures is a **structural class** if it is a Gentzen class specified by a structural collection of Gentzen $\mathbf{F}$-rules.

Lemma 1768 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\Sigma \in |\mathbf{Sign}^\flat|$, $r = \langle \{\Phi_i \vdash_\Sigma \phi_i : i \in I\}, \Phi \vdash_\Sigma \phi \rangle$ *a Gentzen* $\mathbf{F}$*-rule and* M *a class* $\mathbf{F}$*-structures. If* $\mathsf{M} \vDash \mathrm{str}(r)$*, then* $\mathrm{str}(r)$ *holds in* $\mathcal{I}^{\mathsf{M}}$*.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$, $r = \langle \{\Phi_i \vdash_\Sigma \phi_i : i \in I\}, \Phi \vdash_\Sigma \phi \rangle$ and suppose $\mathsf{M} \vDash \mathrm{str}(r)$ and $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, such that

$$\mathrm{SEN}^\flat(f)(\phi_i) \in C^{\mathsf{M}}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi_i)), \ i \in I.$$

Then, by definition of $\mathcal{I}^{\mathsf{M}}$, for all $\langle \mathcal{A}, D \rangle \in \mathsf{M}$, all $\Sigma'' \in |\mathbf{Sign}^\flat|$ and all $g \in \mathbf{Sign}^\flat(\Sigma', \Sigma'')$,

$$\Sigma \xrightarrow{\ f\ } \Sigma' \xrightarrow{\ g\ } \Sigma''$$

$$\alpha_{\Sigma''}(\mathrm{SEN}^\flat(g)(\mathrm{SEN}^\flat(f)(\phi_i))) \in D_{F(\Sigma'')}(\alpha_{\Sigma''}(\mathrm{SEN}^\flat(g)(\mathrm{SEN}^\flat(f)(\Phi_i)))), \ i \in I,$$

i.e.,

$$\alpha_{\Sigma''}(\mathrm{SEN}^\flat(gf)(\phi_i)) \in D_{F(\Sigma'')}(\alpha_{\Sigma''}(\mathrm{SEN}^\flat(gf)(\Phi_i))), \ i \in I.$$

Since, by hypothesis, $\mathsf{M} \vDash \mathrm{str}(r)$ and $\langle \mathcal{A}, D\rangle \in \mathsf{M}$, we get

$$\alpha_{\Sigma''}(\mathrm{SEN}^\flat(gf)(\phi)) \in D_{F(\Sigma'')}(\alpha_{\Sigma''}(\mathrm{SEN}^\flat(gf)(\Phi)))$$

and, thus,

$$\alpha_{\Sigma''}(\mathrm{SEN}^\flat(g)(\mathrm{SEN}^\flat(f)(\phi))) \in D_{F(\Sigma'')}(\alpha_{\Sigma''}(\mathrm{SEN}^\flat(g)(\mathrm{SEN}^\flat(f)(\Phi)))).$$

By the definition of $\mathcal{I}^{\mathsf{M}}$, we conclude that $\mathrm{SEN}^\flat(f)(\phi) \in C^{\mathsf{M}}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))$. Therefore, $\mathcal{I}^{\mathsf{M}} \vDash \mathrm{str}(r)$. ■

Theorem 1769 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and P a structural class $\mathbf{F}$-structures. If $\mathsf{M} \subseteq \mathsf{P}$, then $\mathcal{I}^{\mathsf{M}} \in \mathsf{P}$.*

Proof: Suppose that $\mathrm{str}(r)$ is a rule of P. Since $\mathsf{M} \subseteq \mathsf{P}$, $\mathrm{str}(r)$ is a rule of M. Therefore, by Lemma 1768, $\mathrm{str}(r)$ is a rule of $\mathcal{I}^{\mathsf{M}}$. Thus, $\mathcal{I}^{\mathsf{M}}$ satisfies all Gentzen $\mathbf{F}$-rules determining P (since all of them are, by hypothesis, structural) and, therefore, $\mathcal{I}^{\mathsf{M}} \in \mathsf{P}$. ■

An application of Theorem 1769 gives that, if all $\mathbf{F}$-structures in a class M are congruential, then the π-institution determined by the class is also congruential.

Corollary 1770 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and M a class of congruential $\mathbf{F}$-structures. Then $\mathcal{I}^{\mathsf{M}}$ is congruential.*

Proof: It suffices, by Theorem 1769 to show that the class of congruential $\mathbf{F}$-structures is a structural class. This is easily seen by observing that it is the class of $\mathbf{F}$-structures specified by $R = \{R_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, with

$$R_\Sigma = \{\langle \{\Phi, \phi_i \vdash_\Sigma \psi_i,\ \Phi, \psi_i \vdash_\Sigma \phi_i : i < k\},\ \Phi, \sigma^\flat_\Sigma(\vec{\phi}) \vdash_\Sigma \sigma^\flat_\Sigma(\vec{\psi})\rangle : \sigma^\flat \in N^\flat, \Phi \subseteq \mathrm{SEN}^\flat(\Sigma), \vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)\}.$$

It is easy to check that R is a structural class of Gentzen $\mathbf{F}$-rules, whence the class of all congruential $\mathbf{F}$-structures is a structural class. ■

23.5 Fregeanity and Congruence Orderability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a $\top^\flat$-pointed guasivariety of $\mathbf{F}$-algebraic systems.

We say that K is **congruence orderable** if, for all $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle \in \mathsf{K}$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}\phi = \psi & \text{if} & \text{for all } \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma, \Sigma'),\\ & & \Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}^\flat(f)(\phi), \top^{\mathcal{A}}_{\Sigma'}) = \Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}^\flat(f)(\psi), \top^{\mathcal{A}}_{\Sigma'}).\end{array}$$

Moreover, we say that K is **Fregean** if it is both relatively point regular and congruence orderable.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a $\top^\flat$-pointed guasivariety of $\mathbf{F}$-algebraic systems. For every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, define the relation family

$$\leq^{\mathsf{K},\mathcal{A}} = \{\leq_\Sigma^{\mathsf{K},\mathcal{A}}\}_{\Sigma \in |\mathbf{Sign}|}$$

on $\mathcal{A}$ by letting, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll}\phi \leq_\Sigma^{\mathsf{K},\mathcal{A}} \psi & \text{iff} & \text{for all } \Sigma' \in |\mathbf{Sign}, f \in \mathbf{Sign}(\Sigma, \Sigma'),\\ & & \Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}(f)(\phi), \top_{\Sigma'}^{\mathcal{A}}) \geq \Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}(f)(\psi), \top_{\Sigma'}^{\mathcal{A}}).\end{array}$$

We show that $\leq^{\mathsf{K},\mathcal{A}}$ is in fact a quasiordering system (**qosystem**, for short) on $\mathcal{A}$.

Proposition 1771 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a $\top^\flat$-pointed guasivariety of $\mathbf{F}$-algebraic systems. For every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\leq^{\mathsf{K},\mathcal{A}}$ is a quasiordering system on $\mathcal{A}$.*

Proof: Let $\Sigma \in |\mathbf{Sign}|$. Since, for all $\phi \in \mathrm{SEN}(\Sigma)$, all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}(f)(\phi), \top_{\Sigma'}^{\mathcal{A}}) = \Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}(f)(\phi), \top_{\Sigma'}^{\mathcal{A}})$, we get that $\phi \leq_\Sigma^{\mathsf{K},\mathcal{A}} \phi$ and $\leq^{\mathsf{K},\mathcal{A}}$ is reflexive. Since, for all $\phi, \psi, \chi \in \mathrm{SEN}(\Sigma)$, if $\phi \leq_\Sigma^{\mathsf{K},\mathcal{A}} \psi$ and $\psi \leq_\Sigma^{\mathsf{K},\mathcal{A}} \chi$ imply, by definition, that, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

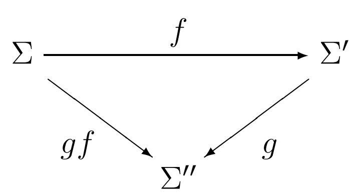
$$\Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}(f)(\phi), \top_{\Sigma'}^{\mathcal{A}}) \geq \Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}(f)(\psi), \top_{\Sigma'}^{\mathcal{A}}) \geq \Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}(f)(\chi), \top_{\Sigma'}^{\mathcal{A}}),$$

we, get, again by definition, $\phi \leq_\Sigma^{\mathsf{K},\mathcal{A}} \chi$. Thus, $\leq^{\mathsf{K},\mathcal{A}}$ is also transitive.

Finally, suppose $\phi \leq_\Sigma^{\mathsf{K},\mathcal{A}} \psi$ and let $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$. Then, by definition, for all $\Sigma'' \in |\mathbf{Sign}|$ and all $h \in \mathbf{Sign}(\Sigma, \Sigma'')$, we get

$$\Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}(h)(\phi), \top_{\Sigma''}^{\mathcal{A}}) \geq \Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}(h)(\psi), \top_{\Sigma''}^{\mathcal{A}}).$$

In particular, for all $\Sigma'' \in |\mathbf{Sign}|$ and all $g \in \mathbf{Sign}(\Sigma', \Sigma'')$,



$$\Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}(g)(\mathrm{SEN}(f)(\phi)), \top_{\Sigma''}^{\mathcal{A}}) \geq \Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}(g)(\mathrm{SEN}(f)(\psi)), \top_{\Sigma''}^{\mathcal{A}}),$$

i.e., $\mathrm{SEN}(f)(\phi) \leq_{\Sigma'}^{\mathsf{K},\mathcal{A}} \mathrm{SEN}(f)(\psi)$ and $\leq^{\mathsf{K},\mathcal{A}}$ is a system. ■

Corollary 1772 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* K *a* $\top^\flat$*-pointed guasivariety of* $\mathbf{F}$*-algebraic systems. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*, the qosystem* $\leq^{\mathsf{K},\mathcal{A}}$ *is a posystem if and only if* K *is congruence orderable.*

Proof: Clear, by Proposition 1772 and the definitions of $\leq^{\mathsf{K},\mathcal{A}}$ and of congruence orderability. ∎

Recall the assertional π-institution $\mathcal{I}^{\mathsf{K},\top}$ associated with a $\top^\flat$-pointed guasivariety K of $\mathbf{F}$-algebraic systems. Recall, also, that, if $\mathcal{I}^{\mathsf{K},\top}$ is family regular, protoalgebraic, with $\top^\flat$ a natural theorem, then the guasivariety K is relatively point regular.

We show, next, that, if $\mathcal{I}^{\mathsf{K},\top}$ is strongly Fregean, protoalgebraic, with $\top^\flat$ a natural theorem, then it is also family regular. Thus, the property of being strongly Freagean, protoalgebraic, with $\top^\flat$ a natural theorem is stronger than being family regular, protoalgebraic, with $\top^\flat$ a natural theorem. In terms of the $\top^\flat$-pointed guasivariety K, this is reflected, as we shall see in the following theorem, in the fact that, in additional to being relatively point regular, it is also congruence orderable, i.e., it is a Fregean guasivariety of $\mathbf{F}$-algebraic systems.

Lemma 1773 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* K *a* $\top^\flat$*-pointed guasivariety of* $\mathbf{F}$*-algebraic systems. If* $\mathcal{I}^{\mathsf{K},\top} = \langle \mathbf{F}, C^{\mathsf{K},\top}\rangle$ *is Fregean, then it is also family regular.*

Proof: Suppose $\mathcal{I}^{\mathsf{K},\top}$ is Fregean. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$ and consider the theory family $C^{\mathsf{K},\top}(\phi,\psi)$. We have, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$,

$$C^{\mathsf{K},\top}_{\Sigma'}(C^{\mathsf{K},\top}_{\Sigma'}(\phi,\psi), \mathrm{SEN}^\flat(f)(\phi)) = C^{\mathsf{K},\top}_{\Sigma'}(\phi,\psi) = C^{\mathsf{K},\top}_{\Sigma'}(C^{\mathsf{K},\top}_{\Sigma'}(\phi,\psi), \mathrm{SEN}^\flat(f)(\psi)).$$

Therefore, we get

$$\begin{array}{rcll}\langle\phi,\psi\rangle & \in & \widetilde{\Lambda}_\Sigma(C^{\mathsf{K},\top}(\phi,\psi)) & \\ & = & \widetilde{\Omega}_\Sigma(C^{\mathsf{K},\top}(\phi,\psi)) & \text{(by Fregeanity)} \\ & \subseteq & \Omega_\Sigma(C^{\mathsf{K},\top}(\phi,\psi)). & \end{array}$$

This shows that $\mathcal{I}^{\mathsf{K},\top}$ is family regular. ∎

Theorem 1774 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* K *a* $\top^\flat$*-pointed guasivariety of* $\mathbf{F}$*-algebraic systems. If* $\mathcal{I}^{\mathsf{K},\top} = \langle \mathbf{F}, C^{\mathsf{K},\top}\rangle$ *is Fregean, protoalgebraic, with* $\top^\flat$ *a natural theorem, then* K *is Fregean.*

Proof: Since $\mathcal{I}^{\mathsf{K},\top}$ is Fregean, by Lemma 1774, it is family regular. Since $\mathcal{I}^{\mathsf{K},\top}$ is family regular, protoalgebraic, with $\top^\flat$ a natural theorem, by Theorem 1356, K is a relatively point regular guasivariety of $\mathbf{F}$-algebraic systems. Thus, to show that K ir Fregean, it suffices, by definition, to show that it is also congruence orderable.

To this end, assume that $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle \in \mathsf{K}$, $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}(f)(\phi), \top^{\mathcal{A}}_{\Sigma'}) = \Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}(f)(\psi), \top^{\mathcal{A}}_{\Sigma'}).$$

This is equivalent to asserting that

$$C^{\mathcal{I}^{\mathsf{K},\top},\mathcal{A}}_\Sigma(\phi) = C^{\mathcal{I}^{\mathsf{K},\top},\mathcal{A}}_\Sigma(\psi).$$

Thus, we obtain

$$\begin{array}{lll}\langle\phi,\psi\rangle & \in & \widetilde{\Lambda}^{\mathcal{A}}_\Sigma(\mathrm{FiFam}^{\mathcal{I}^{\mathsf{K},\top}}(\mathcal{A})) \quad \text{(definition of Frege relation)}\\ & = & \widetilde{\Omega}^{\mathcal{A}}_\Sigma(\mathrm{FiFam}^{\mathcal{I}^{\mathsf{K},\top}}(\mathcal{A})) \quad \text{(Fregeanity)}\\ & = & \Omega^{\mathcal{A}}_\Sigma(\{\top^{\mathcal{A}}\}) \quad \text{(protoalgebraicity)}\\ & = & \Delta^{\mathcal{A}}_\Sigma.\end{array}$$

We conclude that $\phi = \psi$ and, therefore, K is also congruence orderable. ∎

To conclude the section, we would like to prove the converse of Theorem 1774, i.e., that, if K is a Fregean class of $\mathbf{F}$-algebraic systems, then the assertional π-institution $\mathcal{I}^{\mathsf{K},top}$ of K is a Fregean, protoalgebraic π-institution with $\top^\flat$ a natural theorem. Parts of the conclusion, we have already obtained in Theorem 1356. To obtain the full conclusion, we work towards the only remaining subgoal, expressed in the following proposition.

Proposition 1775 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and K a $\top^\flat$-pointed class of $\mathbf{F}$-algebraic systems. If K is Fregean, then the assertional π-institution $\mathcal{I}^{\mathsf{K},\top} = \langle \mathbf{F}, C^{\mathsf{K},\top}\rangle$ of K is Fregean.*

Proof: Suppose K is a Fregean guasivariety of $\mathbf{F}$-algebraic systems, i.e., relatively point regular and congruence orderable. We must show that, for all $T \in \mathrm{ThFam}(\mathcal{I}^{\mathsf{K},\top})$, $\widetilde{\Lambda}^{\mathcal{I}^{\mathsf{K},\top}}(T) = \widetilde{\Omega}^{\mathcal{I}^{\mathsf{K},\top}}(T)$. Let $T \in \mathrm{ThFam}(\mathcal{I}^{\mathsf{K},\top})$. Since $\widetilde{\Omega}^{\mathcal{I}^{\mathsf{K},\top}}(T) \leq \widetilde{\Lambda}^{\mathcal{I}^{\mathsf{K},\top}}(T)$ always holds, it suffices to show the reverse inclusion, i.e., that $\widetilde{\Lambda}^{\mathcal{I}^{\mathsf{K},\top}}(T) \leq \widetilde{\Omega}^{\mathcal{I}^{\mathsf{K},\top}}(T)$. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle\phi,\psi\rangle \notin \widetilde{\Omega}^{\mathcal{I}^{\mathsf{K},\top}}_\Sigma(T)$. Equivalently, $\phi/\widetilde{\Omega}^{\mathcal{I}^{\mathsf{K},\top}}_\Sigma(T) \neq \psi/\widetilde{\Omega}^{\mathcal{I}^{\mathsf{K},\top}}_\Sigma(T)$. Let us denote, for the sake of brevity $\theta \coloneqq \widetilde{\Omega}^{\mathcal{I}^{\mathsf{K},\top}}(T)$. Then, by Lemma 1351 and Proposition 1352, $\mathcal{F}/\theta \in \mathsf{K}$. Thus,y congruence orderability, there exists $\Sigma' \in |\mathbf{Sign}^\flat|$ and $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, such that

$$\Theta^{\mathsf{K},\mathcal{F}/\theta}(\mathrm{SEN}^\flat(f)(\phi)/\theta_{\Sigma'}, \top^{\mathcal{F}/\theta}_{\Sigma'}) \neq \Theta^{\mathsf{K},\mathcal{F}/\theta}(\mathrm{SEN}^\flat(f)(\psi)/\theta_{\Sigma'}, \top^{\mathcal{F}/\theta}_{\Sigma'}).$$

Thus, by relative point regularity, we must have

$$\top^{\mathcal{F}/\theta}/\Theta^{\mathsf{K},\mathcal{F}/\theta}(\mathrm{SEN}^\flat(f)(\phi)/\theta_{\Sigma'},\top_{\Sigma'}^{\mathcal{F}/\theta}) \neq \top^{\mathcal{F}/\theta}/\Theta^{\mathsf{K},\mathcal{F}/\theta}(\mathrm{SEN}^\flat(f)(\psi)/\theta_{\Sigma'},\top_{\Sigma'}^{\mathcal{F}/\theta}).$$

This gives that

$$C_\Sigma^{\mathsf{K},\top}(\mathrm{SEN}^\flat(f)(\phi),\top_{\Sigma'}^\flat/\Omega_{\Sigma'}(T)) \neq C_\Sigma^{\mathsf{K},\top}(\mathrm{SEN}^\flat(f)(\phi),\top_{\Sigma'}^\flat/\Omega_{\Sigma'}(T)),$$

which translates to $\langle\phi,\psi\rangle \notin \widetilde{\Lambda}_\Sigma^{\mathcal{I}^{\mathsf{K},\top}}(T)$. We conclude that $\mathcal{I}^{\mathsf{K},\top}$ is Fregean. ■

Finally, putting this together, we get

Theorem 1776 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* K *a* $\top^\flat$*-pointed class of* $\mathbf{F}$*-algebraic systems. If* K *is Fregean, then* $\mathcal{I}^{\mathsf{K},\top} = \langle\mathbf{F},C^{\mathsf{K},\top}\rangle$ *is a Fregean, protoalgebraic* π*-institution, with* $\top^\flat$ *a natural theorem.*

Proof: By Proposition 1348, $\top^\flat$ is a natural theorem of $\mathcal{I}^{\mathsf{K},\top}$. By Proposition 1352, $\mathcal{I}^{\mathsf{K},\top}$ is protoalgebraic. Finally, by Proposition 1775, $\mathcal{I}^{\mathsf{K},\top}$ is Fregean. ■

The main result proven in this section is summarized in

Theorem 1777 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* K *a* $\top^\flat$*-pointed class of* $\mathbf{F}$*-algebraic systems.* K *is Fregean if and only if* $\mathcal{I}^{\mathsf{K},\top} = \langle\mathbf{F},C^{\mathsf{K},\top}\rangle$ *is a Fregean, protoalgebraic* π*-institution, with* $\top^\flat$ *a natural theorem.*

Proof: The "if" by Theorem 1774. The "only if" by Theorem 1776. ■