Chapter 25

Order Algebraizability

25.1 Algebraic PoSystems

Let **Sign** be a category and SEN : **Sign** → **Set** be a sentence functor. A **qofamily** $\leq = \{\leq_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ on SEN is a relation family on SEN, such that, for all $\Sigma \in |\mathbf{Sign}|$, $\leq_\Sigma \subseteq \mathrm{SEN}(\Sigma)^2$ is a quasi-order on $\mathrm{SEN}(\Sigma)$. A **pofamily** $\leq = \{\leq_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ on SEN is a relation family on SEN, such that, for all $\Sigma \in |\mathbf{Sign}|$, $\leq_\Sigma \subseteq \mathrm{SEN}(\Sigma)^2$ is a partial order on $\mathrm{SEN}(\Sigma)$. A **qosystem** $\leq$ on SEN is a qofamily that is also a relation system. i.e., invariant under **Sign**-morphisms, that is, such that, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}(f)(\leq_\Sigma) \subseteq \leq_{\Sigma'}.$$

Similarly, a **posystem** $\leq$ on SEN is a pofamily that is also a relation system.

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system. A **qosystem** (**posystem**) on $\mathbf{A}$ is a qosystem (posystem, respectively) on SEN. The pair $\langle \mathbf{A}, \leq\rangle$ is then called an **algebraic qosystem** (**algebraic posystem**, respectively).

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ an $\mathbf{F}$-algebraic system. A **qosystem** (**posystem**) on $\mathcal{A}$ is a qosystem (posystem, respectively) on $\mathbf{A}$. We then term the pair $\langle \mathcal{A}, \leq\rangle$ an **F-algebraic qosystem** (**F-algebraic posystem**, respectively).

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system.

- The family of **F-inequations** $\mathrm{In}(\mathbf{F}) = \{\mathrm{In}_\Sigma(\mathbf{F})\}_{\Sigma\in|\mathbf{Sign}^\flat|}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{In}_\Sigma(\mathbf{F}) = \{\phi \preccurlyeq \psi : \phi, \psi \in \mathrm{SEN}^\flat(\Sigma)\};$$

- The family of **F-quasi inequations** $\mathrm{QIn}(\mathbf{F}) = \{\mathrm{QIn}_\Sigma(\mathbf{F})\}_{\Sigma\in|\mathbf{Sign}^\flat|}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{In}_\Sigma(\mathbf{F}) = \{\langle\{\phi_i \preccurlyeq \psi_i : i < k\}, \phi \preccurlyeq \psi\rangle : \vec{\phi}, \vec{\psi}, \phi, \psi \in \mathrm{SEN}^\flat(\Sigma)\};$$

- The family of **F-guasi inequations** $\mathrm{GIn}(\mathbf{F}) = \{\mathrm{GIn}_\Sigma(\mathbf{F})\}_{\Sigma\in|\mathbf{Sign}^\flat|}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{In}_\Sigma(\mathbf{F}) = \{\langle\{\phi_i \preccurlyeq \psi_i : i \in I\}, \phi \preccurlyeq \psi\rangle : \vec{\phi}, \vec{\psi}, \phi, \psi \in \mathrm{SEN}^\flat(\Sigma)\}.$$

As done previously, we sometimes abbreviate a guasi inequation $\langle\{\phi_i \preccurlyeq \psi_i : i \in I\}, \phi \preccurlyeq \psi\rangle$ by writing $\langle\vec{\phi} \preccurlyeq \vec{\psi}, \phi \preccurlyeq \psi\rangle$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and K a class of $\mathbf{F}$-algebraic posystems. We define the family $C^{\mathsf{K},\leq} : \mathcal{P}\mathrm{In}(\mathbf{F}) \to \mathcal{P}\mathrm{In}(\mathbf{F})$ by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $I \cup \{\phi \preccurlyeq \psi\} \subseteq \mathrm{In}_\Sigma(\mathbf{F})$,

$$\begin{aligned}&\phi \preccurlyeq \psi \in C^{\mathsf{K},\leq}_\Sigma(I) \text{ iff, for all } \langle \mathcal{A}, \leq^{\mathcal{A}}\rangle \in \mathsf{K}, \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma, \Sigma'),\\ &\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(I)) \subseteq \leq^{\mathcal{A}}_{F(\Sigma')}\\ &\qquad\qquad\text{implies } \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \leq^{\mathcal{A}}_{F(\Sigma')} \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi)).\end{aligned}$$

It is not difficult to see that $C^{\mathsf{K},\leq}$ is a closure system on $\mathrm{In}(\mathbf{F})$.

Lemma 1810 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic posystems. Then* $C^{\mathsf{K},\leq} : \mathcal{P}\mathrm{In}(\mathbf{F}) \to \mathcal{P}\mathrm{In}(\mathbf{F})$ *is a closure system on* $\mathrm{In}(\mathbf{F})$.

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$. It is straightforward from the definition of $C^{\mathsf{K},\leq}$ that $C^{\mathsf{K},\leq}_\Sigma$ is inflationary and monotone. We show that it is also idempotent. To this end, let $I \cup \{\phi \preccurlyeq \psi\} \subseteq \mathrm{In}_\Sigma(\mathbf{F})$, be such that $\phi \preccurlyeq \psi \in C^{\mathsf{K},\leq}_\Sigma(C^{\mathsf{K},\leq}_\Sigma(I))$. Then, for all$\langle \mathcal{A}, \leq^{\mathcal{A}} \rangle \in \mathsf{K}$, all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, we have $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(I)) \subseteq \leq^{\mathcal{A}}_{F(\Sigma')}$ implies, by definition,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(C^{\mathsf{K},\leq}_\Sigma(I))) \subseteq \leq^{\mathcal{A}}_{F(\Sigma')},$$

whence, by the hypothesis and the definition,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \leq^{\mathcal{A}}_{F(\Sigma')} \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi)).$$

We now get $\phi \preccurlyeq \psi \in C^{\mathsf{K},\leq}_\Sigma(I)$. Therefore, $C^{\mathsf{K},\leq}_\Sigma$ is also idempotent.

Finally, it only remains to show structurality. To this end, let $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $I \cup \{\phi \preccurlyeq \psi\} \subseteq \mathrm{In}_\Sigma(\mathbf{F})$, such that $\phi \preccurlyeq \psi \in C^{\mathsf{K},\leq}_\Sigma(I)$. Then, by definition, for every $\langle \mathcal{A}, \leq^{\mathcal{A}} \rangle \in \mathsf{K}$, all $\Sigma'' \in |\mathbf{Sign}^\flat|$ and all $h \in \mathbf{Sign}^\flat(\Sigma, \Sigma'')$, $\alpha_{\Sigma''}(\mathrm{SEN}^\flat(h)(I)) \subseteq \leq^{\mathcal{A}}_{F(\Sigma'')}$ implies $\alpha_{\Sigma''}(\mathrm{SEN}^\flat(h)(\phi)) \leq^{\mathcal{A}}_{F(\Sigma'')} \alpha_{\Sigma''}(\mathrm{SEN}^\flat(h)(\psi))$.

$$\begin{tikzcd} \Sigma \arrow[rr, "f"] \arrow[dr, "h"'] & & \Sigma' \arrow[dl, "g"] \\ & \Sigma'' & \end{tikzcd}$$

In particular, for all $\Sigma'' \in |\mathbf{Sign}^\flat|$ and all $g \in \mathbf{Sign}^\flat(\Sigma', \Sigma'')$,

$$\alpha_{\Sigma''}(\mathrm{SEN}^\flat(g)(\mathrm{SEN}^\flat(f)(I))) \subseteq \leq^{\mathcal{A}}_{F(\Sigma'')}$$

implies

$$\alpha_{\Sigma''}(\mathrm{SEN}^\flat(g)(\mathrm{SEN}^\flat(f)(\phi))) \leq^{\mathcal{A}}_{F(\Sigma'')} \alpha_{\Sigma''}(\mathrm{SEN}^\flat(g)(\mathrm{SEN}^\flat(f)(\psi))).$$

Therefore, $\mathrm{SEN}^\flat(f)(\phi) \preccurlyeq \mathrm{SEN}^\flat(f)(\psi) \in C^{\mathsf{K},\leq}_{\Sigma'}(\mathrm{SEN}^\flat(f)(I))$. We conclude that $C^{\mathsf{K},\leq}$ is also structural and, hence, a closure system on $\mathrm{In}(\mathbf{F})$. ■

As a result of Lemma 1810, it makes sense to define the **inequational** π**-institution** $\mathcal{I}^{\mathsf{K},\leq} = \langle \mathbf{F}, C^{\mathsf{K},\leq} \rangle$ **associated with** a class K of $\mathbf{F}$-algebraic posystems.

We can show that the π-institution $\mathcal{I}^{\mathsf{K},\leq}$ associated with the class K satisfies a reflexivity and transitivity property. On the other hand, antisymmetry is not expressible in the language under consideration, since it is a language without equality.

Lemma 1811 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic posystems. For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$*,*

(a) $\phi \preccurlyeq \phi \in C_\Sigma^{\mathsf{K},\leq}(\varnothing)$*;*

(b) $\phi \preccurlyeq \chi \in C_\Sigma^{\mathsf{K},\leq}(\phi \preccurlyeq \psi, \psi \preccurlyeq \chi)$*.*

Proof: Clearly, for all $\langle \mathcal{A}, \leq^{\mathcal{A}} \rangle \in \mathsf{K}$, all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, we get, by the reflexivity of $\leq^{\mathcal{A}}$, that

$$\mathrm{SEN}^\flat(f)(\phi) \leq^{\mathcal{A}}_{F(\Sigma')} \mathrm{SEN}^\flat(f)(\phi).$$

Thus, by definition, $\phi \preccurlyeq \phi \in C_\Sigma^{\mathsf{K},\leq}(\varnothing)$.

Similarly, for all $\langle \mathcal{A}, \leq^{\mathcal{A}} \rangle \in \mathsf{K}$, all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, by the transitivity of $\leq^{\mathcal{A}}$, we get that

$$\mathrm{SEN}^\flat(f)(\phi) \leq^{\mathcal{A}}_{F(\Sigma')} \mathrm{SEN}^\flat(f)(\psi) \text{ and } \mathrm{SEN}^\flat(f)(\psi) \leq^{\mathcal{A}}_{F(\Sigma')} \mathrm{SEN}^\flat(f)(\chi)$$

imply $\mathrm{SEN}^\flat(f)(\phi) \leq^{\mathcal{A}}_{F(\Sigma')} \mathrm{SEN}^\flat(f)(\chi)$. Therefore, by definition $\phi \preccurlyeq \chi \in C_\Sigma^{\mathsf{K},\leq}(\phi \preccurlyeq \psi, \psi \preccurlyeq \chi)$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\langle \mathcal{A}, \leq^{\mathcal{A}} \rangle$ an $\mathbf{F}$-algebraic posystem and $g = \langle \vec{\phi} \preccurlyeq \vec{\psi}, \phi \preccurlyeq \psi \rangle \in \mathrm{GIn}_\Sigma(\mathbf{F})$. We say $\langle \mathcal{A}, \leq^{\mathcal{A}} \rangle$ **satisfies** g or that g **holds** or **is valid in** $\langle \mathcal{A}, \leq^{\mathcal{A}} \rangle$, written

$$\langle \mathcal{A}, \leq^{\mathcal{A}} \rangle \vDash_\Sigma g,$$

if, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi_i)) \leq^{\mathcal{A}}_{F(\Sigma')} \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi_i)), \text{ for all } i \in I,$$

imply $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \leq^{\mathcal{A}}_{F(\Sigma')} \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi))$. Equivalently, $\langle \mathcal{A}, \leq^{\mathcal{A}} \rangle$ satisfies g if $\phi \preccurlyeq \psi \in C_\Sigma^{\{\langle \mathcal{A}, \leq^{\mathcal{A}} \rangle\},\leq}(\vec{\phi} \preccurlyeq \vec{\psi})$.

Given a class K of $\mathbf{F}$-algebraic posystems and a class G of $\mathbf{F}$-guasi inequations, we write $\mathrm{GIn}(\mathsf{K})$ for the class of all $\mathbf{F}$-guasi inequations satisfied by every $\mathbf{F}$-algebraic posystem in K and $\mathrm{PAlgSys}(G)$ for the class of all $\mathbf{F}$-algebraic posystems that satisfy all $\mathbf{F}$-guasi inequations in G.

We now turn to examining some operations on classes of $\mathbf{F}$-algebraic posystems. We first show that the inverse image of a posystem under an $\mathbf{F}$-algebraic system morphism is also a posystem.

Lemma 1812 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ *be two* $\mathbf{F}$*-algebraic systems and* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism.*

(a) If $\leq^{\mathcal{B}}$ *is a posystem on* $\mathcal{B}$*, then* $\gamma^{-1}(\leq^{\mathcal{B}})$ *is a qosystem on* $\mathcal{A}$*;*

(b) If $\leq^{\mathcal{B}}$ is a posystem on $\mathcal{B}$ and $\mathrm{Ker}(\langle H,\gamma\rangle) = \Delta^{\mathcal{A}}$, *then* $\gamma^{-1}(\leq^{\mathcal{B}})$ *is a posystem on* $\mathcal{A}$.

Proof: Let $\Sigma \in |\mathbf{Sign}|$, $\phi,\psi,\chi \in \mathrm{SEN}(\Sigma)$.

By the reflexivity of $\leq^{\mathcal{B}}$, we have $\gamma_\Sigma(\phi) \leq^{\mathcal{B}}_{H(\Sigma)} \gamma_\Sigma(\phi)$. Thus, $\phi\gamma_\Sigma^{-1}(\leq^{\mathcal{B}})\phi$ and, hence $\gamma_\Sigma^{-1}(\leq^{\mathcal{B}})$ is reflexive.

Suppose, next, that $\phi\gamma_\Sigma^{-1}(\leq^{\mathcal{B}})\psi$ and $\psi\gamma_\Sigma^{-1}(\leq^{\mathcal{B}})\chi$. Then, $\gamma_\Sigma(\phi) \leq^{\mathcal{B}}_{H(\Sigma)} \gamma_\Sigma(\psi)$ and $\gamma_\Sigma(\psi) \leq^{\mathcal{B}}_{H(\Sigma)} \gamma_\Sigma(\chi)$. Thus, by the transitivity of $\leq^{\mathcal{B}}$, we get $\gamma_\Sigma(\phi) \leq^{\mathcal{B}}_{H(\Sigma)} \gamma_\Sigma(\chi)$. Therefore, $\phi\gamma_\Sigma^{-1}(\leq^{\mathcal{B}})\chi$ and $\gamma_\Sigma^{-1}(\leq^{\mathcal{B}})$ is also transitive.

If $\phi\gamma_\Sigma^{-1}(\leq^{\mathcal{B}})\psi$, $\Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma,\Sigma')$, then we get $\gamma_\Sigma(\phi) \leq^{\mathcal{B}}_{H(\Sigma)} \gamma_\Sigma(\psi)$, whence $\mathrm{SEN}'(H(f))(\gamma_\Sigma(\phi)) \leq^{\mathcal{B}}_{H(\Sigma')} \mathrm{SEN}'(H(f))(\gamma_\Sigma(\psi))$. Thus,

$$\gamma_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \leq^{\mathcal{B}}_{H(\Sigma')} \gamma_{\Sigma'}(\mathrm{SEN}(f)(\psi)).$$

So we obtain $\mathrm{SEN}(f)(\phi)\gamma_{\Sigma'}^{-1}(\leq^{\mathcal{B}})\mathrm{SEN}(f)(\psi)$. This shows that $\gamma^{-1}(\leq^{\mathcal{B}})$ is a qosystem on $\mathcal{A}$.

Suppose, finally, for the sake of proving Part (b), that $\phi\gamma_\Sigma^{-1}(\leq^{\mathcal{B}})\psi$ and $\psi\gamma_\Sigma^{-1}(\leq^{\mathcal{B}})\phi$. Then, $\gamma_\Sigma(\phi) \leq^{\mathcal{B}}_{H(\Sigma)} \gamma_\Sigma(\psi)$ and $\gamma_\Sigma(\psi) \leq^{\mathcal{B}}_{H(\Sigma)} \gamma_\Sigma(\phi)$. Thus, by the antisymmetry of $\leq^{\mathcal{B}}$, we get $\gamma_\Sigma(\phi) = \gamma_\Sigma(\psi)$. Since, by hypothesis, $\mathrm{Ker}(\langle H,\gamma\rangle) = \Delta^{\mathcal{A}}$, we get $\phi = \psi$ and, hence, $\gamma^{-1}(\leq^{\mathcal{B}})$ is a posystem on $\mathcal{A}$ in this case. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, K a class of $\mathbf{F}$-algebraic posystems and $\langle \mathcal{A}, \leq^{\mathcal{A}}\rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, an $\mathbf{F}$-algebraic posystem.

- Given $\Sigma \in |\mathbf{Sign}^\flat|$, we say that $\langle \mathcal{A}, \leq\rangle$ is Σ-K-**order certified** if there exists $\langle \mathcal{A}^\Sigma, \leq^\Sigma\rangle \in \mathsf{K}$, such that $\mathrm{In}_\Sigma(\langle \mathcal{A}, \leq\rangle) = \mathrm{In}_\Sigma(\langle \mathcal{A}^\Sigma, \leq^\Sigma\rangle)$. In this case $\langle \mathcal{A}^\Sigma, \leq^\Sigma\rangle$ will be referred to as the Σ-K-**order certificate** of $\langle \mathcal{A}, \leq\rangle$.

- We say that $\langle \mathcal{A}, \leq\rangle$ is K-**order certified** if it is Σ-K-order certified, for all $\Sigma \in |\mathbf{Sign}^\flat|$. This, of course, means that

$$(\forall \Sigma \in |\mathbf{Sign}^\flat|)(\exists \langle \mathcal{A}^\Sigma, \leq^\Sigma\rangle \in \mathsf{K})(\mathrm{In}_\Sigma(\langle \mathcal{A}, \leq\rangle) = \mathrm{In}_\Sigma(\langle \mathcal{A}^\Sigma, \leq^\Sigma\rangle)).$$

We write $\mathbb{C}(\mathsf{K})$ for the class of all $\mathbf{F}$-algebraic posystems that are K-order certified. We say that K is an **abstract order class** whenever every K-order certified $\mathbf{F}$-algebraic posystem belongs to K, i.e., when $\mathbb{C}(\mathsf{K}) = \mathsf{K}$.

It is not difficult to show that $\mathbb{C}$ is a closure operator on classes of $\mathbf{F}$-algebraic systems.

Proposition 1813 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system. Then the operator* $\mathbb{C}$ *on classes of* $\mathbf{F}$*-algebraic posystems is a closure operator.*

Proof: Suppose K is a class of $\mathbf{F}$-algebraic posystems.

- Let $\langle\mathcal{A},\leq\rangle\in\mathsf{K}$. Then, since for all $\Sigma\in|\mathbf{Sign}^\flat|$, $\langle\mathcal{A}^\Sigma,\leq^\Sigma\rangle=\langle\mathcal{A},\leq\rangle\in\mathsf{K}$ is a Σ-K-order certificate for $\langle\mathcal{A},\leq\rangle$, we get that $\langle\mathcal{A},\leq\rangle\in\mathbb{C}(\mathsf{K})$. Thus, $\mathsf{K}\subseteq\mathbb{C}(\mathsf{K})$ and $\mathbb{C}$ is inflationary.
- If $\mathsf{K}\subseteq\mathsf{K}'$ and $\langle\mathcal{A},\leq\rangle\in\mathbb{C}(\mathsf{K})$, then, by definition, for every $\Sigma\in|\mathbf{Sign}^\flat|$, there exists a Σ-K-order certificate $\langle\mathcal{A}^\Sigma,\leq^\Sigma\rangle$. Since $\mathsf{K}\subseteq\mathsf{K}'$, $\langle\mathcal{A}^\Sigma,\leq^\Sigma\rangle\in\mathsf{K}'$ is also a Σ-K'-order certificate. Thus, $\langle\mathcal{A},\leq\rangle\in\mathbb{C}(\mathsf{K}')$ and $\mathbb{C}$ is also monotone.
- Finally, suppose that $\langle\mathcal{A},\leq\rangle\in\mathbb{C}(\mathbb{C}(\mathsf{K}))$. Then, there exists, for all $\Sigma\in|\mathbf{Sign}^\flat|$, a Σ-$\mathbb{C}(\mathsf{K})$-order certificate $\langle\mathcal{A}^\Sigma,\leq^\Sigma\rangle$ for $\mathcal{A}$. Therefore, for every $\Sigma'\in|\mathbf{Sign}^\flat|$, there exists a Σ'-K-order certificate $\langle\mathcal{A}^{\langle\Sigma,\Sigma'\rangle},\leq^{\langle\Sigma,\Sigma'\rangle}\rangle$ for $\langle\mathcal{A}^\Sigma,\leq^\Sigma\rangle$. Thus, for every $\Sigma\in|\mathbf{Sign}^\flat|$, there exists a Σ-K-order certificate $\langle\mathcal{A}^{\langle\Sigma,\Sigma\rangle},\leq^{\langle\Sigma,\Sigma\rangle}\rangle$ for $\langle\mathcal{A},\leq\rangle$, since, by hypothesis,
$$\mathrm{In}_\Sigma(\langle\mathcal{A},\leq\rangle)=\mathrm{In}_\Sigma(\langle\mathcal{A}^\Sigma,\leq^\Sigma\rangle)=\mathrm{In}_\Sigma(\langle\mathcal{A}^{\langle\Sigma,\Sigma\rangle},\leq^{\langle\Sigma,\Sigma\rangle}\rangle).$$

Thus $\mathbb{C}$ is a closure operator on classes of $\mathbf{F}$-algebraic posystems. ■

The importance of abstract classes of $\mathbf{F}$-algebraic posystems rests on the fact that the validity of an $\mathbf{F}$-guasi inequation transfers from K-order certificates to an $\mathbf{F}$-algebraic posystem itself.

Lemma 1814 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* K *a class of* $\mathbf{F}$*-algebraic posystems and* $\langle\mathcal{A},\leq\rangle$, $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$, *an* $\mathbf{F}$*-algebraic posystem. If* $\mathcal{A}\in\mathbb{C}(\mathsf{K})$, *then* $\mathrm{GIn}(\mathsf{K})\leq\mathrm{GIn}(\mathcal{A})$.

Proof: Suppose $\langle\mathcal{A},\leq\rangle\in\mathbb{C}(\mathsf{K})$, $\Sigma\in|\mathbf{Sign}^\flat|$ and $\langle\vec{\phi}\preccurlyeq\vec{\psi},\phi\preccurlyeq\psi\rangle\in\mathrm{GIn}_\Sigma(\mathsf{K})$, such that $\vec{\phi}\preccurlyeq\vec{\psi}\subseteq\mathrm{In}_\Sigma(\langle\mathcal{A},\leq\rangle)$. Let $\langle\mathcal{A}^\Sigma,\leq^\Sigma\rangle\in\mathsf{K}$ be a Σ-K-order certificate for $\mathcal{A}$. Then, by definition $\vec{\phi}\preccurlyeq\vec{\psi}\subseteq\mathrm{In}_\Sigma(\langle\mathcal{A}^\Sigma,\leq^\Sigma\rangle)$. Since $\langle\mathcal{A}^\Sigma,\leq^\Sigma\rangle\in\mathsf{K}$ and $\langle\vec{\phi}\preccurlyeq\vec{\psi},\phi\preccurlyeq\psi\rangle\in\mathrm{GIn}_\Sigma(\mathsf{K})$, we get $\phi\preccurlyeq\psi\in\mathrm{In}_\Sigma(\langle\mathcal{A}^\Sigma,\leq^\Sigma\rangle)=\mathrm{In}_\Sigma(\langle\mathcal{A},\leq\rangle)$. Therefore, $\langle\vec{\phi}\preccurlyeq\vec{\psi},\phi\preccurlyeq\psi\rangle\in\mathrm{GIn}_\Sigma(\langle\mathcal{A},\leq\rangle)$. We conclude that $\mathrm{GIn}(\mathsf{K})\leq\mathrm{GIn}(\langle\mathcal{A},\leq\rangle)$. ■

Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and K a class of $\mathbf{F}$-algebraic posystems.

- K is called an **inequational class** if there exists $I\leq\mathrm{In}(\mathbf{F})$, such that $\mathsf{K}=\mathrm{PAlgSys}(I)$;
- K is called a **quasi inequational class** if there exists $Q\leq\mathrm{QIn}(\mathbf{F})$, such that $\mathsf{K}=\mathrm{PAlgSys}(Q)$;
- K is called a **guasi inequational class** if there exists $G\leq\mathrm{GIn}(\mathbf{F})$, such that $\mathsf{K}=\mathrm{PAlgSys}(G)$.

Clearly, by definition, if K is an inequational class, then it is a quasi inequational class and, if it is a quasi inequational class, then it is a guasi inequational class.

Directly by these definitions and Lemma 1814, we get

Corollary 1815 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic posystems. If* K *is a guasi inequational class (and, hence, a fortiori, if it is a quasi inequational class or an inequational class), then it is abstract.*

Proof: Suppose K is a guasi inequational class defined by the $\mathbf{F}$-guasi inequations $G \leq \mathrm{GIn}(\mathbf{F})$ and let $\langle \mathcal{A}, \leq \rangle \in \mathbb{C}(\mathsf{K})$. Then, by Lemma 1814, $\mathrm{GIn}(\mathsf{K}) \leq \mathrm{GIn}(\langle \mathcal{A}, \leq \rangle)$, whence

$$\begin{array}{rcl}\langle \mathcal{A}, \leq \rangle & \in & \mathrm{PAlgSys}(\mathrm{GIn}(\mathcal{A}))\\ & \subseteq & \mathrm{PAlgSys}(\mathrm{GIn}(\mathsf{K}))\\ & = & \mathrm{PAlgSys}(\mathrm{GIn}(\mathrm{PAlgSys}(G)))\\ & = & \mathrm{PAlgSys}(G)\\ & = & \mathsf{K}.\end{array}$$

Thus, K is an abstract class. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and K a class of $\mathbf{F}$-algebraic posystems. We define:

- The **semantic order variety generated by** K

$$\mathbb{VO}^{\mathsf{Sem}}(\mathsf{K}) = \{\langle \mathcal{A}, \leq \rangle \in \mathrm{PAlgSys}(\mathbf{F}) : \mathrm{In}(\mathsf{K}) \leq \mathrm{In}(\langle \mathcal{A}), \leq \rangle\};$$

- The **semantic order quasivariety generated by** K

$$\mathbb{QO}^{\mathsf{Sem}}(\mathsf{K}) = \{\langle \mathcal{A}, \leq \rangle \in \mathrm{PAlgSys}(\mathbf{F}) : \mathrm{QIn}(\mathsf{K}) \leq \mathrm{QIn}(\langle \mathcal{A}, \leq \rangle)\};$$

- The **semantic order guasivariety generated by** K

$$\mathbb{GO}^{\mathsf{Sem}}(\mathsf{K}) = \{\langle \mathcal{A}, \leq \rangle \in \mathrm{PAlgSys}(\mathbf{F}) : \mathrm{GIn}(\mathsf{K}) \leq \mathrm{GIn}(\langle \mathcal{A}, \leq \rangle)\}.$$

We have the following obvious relations between these classes.

Lemma 1816 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic posystems. Then*

$$\mathsf{K} \subseteq \mathbb{GO}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{QO}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{VO}^{\mathsf{Sem}}(\mathsf{K}).$$

Proof: The essential observation is that

$$\mathrm{In}(\mathsf{K}) \leq \mathrm{QIn}(\mathsf{K}) \leq \mathrm{GIn}(\mathsf{K}).$$

Thus, we get

$$\begin{array}{l}\{\langle \mathcal{A}, \leq \rangle \in \mathrm{PAlgSys}(\mathbf{F}) : (\forall g \in \mathrm{GIn}(\mathsf{K}))(\langle \mathcal{A}, \leq \rangle \vDash g)\}\\ \qquad \subseteq \{\langle \mathcal{A}, \leq \rangle \in \mathrm{PAlgSys}(\mathbf{F}) : (\forall q \in \mathrm{QIn}(\mathsf{K}))(\langle \mathcal{A}, \leq \rangle \vDash q)\}\\ \qquad \subseteq \{\langle \mathcal{A}, \leq \rangle \in \mathrm{PAlgSys}(\mathbf{F}) : (\forall e \in \mathrm{In}(\mathsf{K}))(\langle \mathcal{A}, \leq \rangle \vDash e)\}.\end{array}$$

In other words, $\mathsf{K} \subseteq \mathbb{GO}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{QO}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{VO}^{\mathsf{Sem}}(\mathsf{K})$. ■

Given a class K of $\mathbf{F}$-algebraic posystems

- K is a **semantic order variety** if $\mathbb{VO}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$;
- K is a **semantic order quasivariety** if $\mathbb{QO}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$;
- K is a **semantic order guasivariety** if $\mathbb{GO}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$.

We have the following result identifying inequational classes with semantic order varieties, quasi inequational classes with semantic order quasivarieties and guasi inequational classes with semantic order guasivarieties.

Proposition 1817 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic posystems.*

(a) K *is an inequational class iff it is a semantic order variety;*

(b) K *is a quasi inequational class iff it is a semantic order quasivariety;*

(c) K *is a guasi inequational class iff it is a semantic order guasivariety.*

Proof:

(a) Suppose, first, that K is an inequational class. Then, there exists $I \leq \mathrm{In}(\mathbf{F})$, such that $\mathsf{K} = \mathrm{PAlgSys}(I)$. Let $\langle \mathcal{A}, \leq \rangle \in \mathrm{PAlgSys}(\mathbf{F})$, such that $\mathrm{In}(\mathsf{K}) \leq \mathrm{In}(\langle \mathcal{A}, \leq \rangle)$. Then we have $\langle \mathcal{A}, \leq \rangle \in \mathrm{PAlgSys}(\mathrm{In}(\langle \mathcal{A}, \leq \rangle)) \subseteq \mathrm{PAlgSys}(\mathrm{In}(\mathsf{K})) = \mathrm{PAlgSys}(\mathrm{In}(\mathrm{PAlgSys}(I))) = \mathrm{PAlgSys}(I) = \mathsf{K}$. Therefore, K is a semantic order variety.

Suppose, conversely, that K is a semantic order variety. Set $I = \mathrm{In}(\mathsf{K})$. Then $\mathsf{K} \subseteq \mathrm{PAlgSys}(\mathrm{In}(\mathsf{K})) = \mathrm{PAlgSys}(I)$. On the other hand, if $\langle \mathcal{A}, \leq \rangle \in \mathrm{PAlgSys}(I)$, then

$$\mathrm{In}(\mathsf{K}) = \mathrm{In}(\mathrm{PAlgSys}(\mathrm{In}(\mathsf{K}))) = \mathrm{In}(\mathrm{PAlgSys}(I)) \leq \mathrm{In}(\langle \mathcal{A}, \leq \rangle),$$

whence, by hypothesis, $\langle \mathcal{A}, \leq \rangle \in \mathsf{K}$. Therefore, $\mathsf{K} = \mathrm{PAlgSys}(I)$ and K is an inequational class.

(b) Suppose, first, that K is a quasi inequational class. Then, there exists $Q \leq \mathrm{QIn}(\mathbf{F})$, such that $\mathsf{K} = \mathrm{PAlgSys}(Q)$. Let $\langle \mathcal{A}, \leq \rangle \in \mathrm{PAlgSys}(\mathbf{F})$, such that $\mathrm{QIn}(\mathsf{K}) \leq \mathrm{QIn}(\langle \mathcal{A}, \leq \rangle)$. Then we have

$$\begin{array}{rcl}\langle \mathcal{A}, \leq \rangle & \in & \mathrm{PAlgSys}(\mathrm{QIn}(\langle \mathcal{A}, \leq \rangle)) \\ & \subseteq & \mathrm{PAlgSys}(\mathrm{QIn}(\mathsf{K})) \\ & = & \mathrm{PAlgSys}(\mathrm{QIn}(\mathrm{PAlgSys}(Q))) \\ & = & \mathrm{PAlgSys}(Q) \\ & = & \mathsf{K}.\end{array}$$

Therefore, K is a semantic order quasivariety.

Suppose, conversely, that K is a semantic order quasivariety. Set $Q = \mathrm{QIn}(\mathsf{K})$. Then $\mathsf{K} \subseteq \mathrm{PAlgSys}(\mathrm{QIn}(\mathsf{K})) = \mathrm{PAlgSys}(Q)$. On the other hand, if $\langle \mathcal{A}, \leq \rangle \in \mathrm{PAlgSys}(Q)$, then

$$\mathrm{QIn}(\mathsf{K}) = \mathrm{QIn}(\mathrm{PAlgSys}(\mathrm{QIn}(\mathsf{K}))) = \mathrm{QIn}(\mathrm{PAlgSys}(Q)) \leq \mathrm{QIn}(\langle \mathcal{A}, \leq \rangle),$$

whence, by hypothesis, $\langle \mathcal{A}, \leq \rangle \in \mathsf{K}$. Therefore, $\mathsf{K} = \mathrm{PAlgSys}(Q)$ and K is a quasi inequational class.

(c) Very similar to Part (b).

■

We introduce, next, some operators on classes of **F**-algebraic posystems, paralleling those introduced previously for classes of **F**-algebraic systems, that will serve to provide different characterizations to the inequational, quasi inequational and guasi inequational classes of **F**-algebraic posystems.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\langle \mathcal{A}, \leq \rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\langle \mathcal{A}^i, \leq^i \rangle$, with $\mathcal{A}^i = \langle \mathbf{A}^i, \langle F^i, \alpha^i \rangle \rangle$, $i \in I$, **F**-algebraic posystems and $\langle H^i, \gamma^i \rangle : \langle \mathcal{A}, \leq \rangle \to \langle \mathcal{A}^i, \leq^i \rangle$, $i \in I$, surjective morphisms. We say the collection

$$\langle H^i, \gamma^i \rangle : \langle \mathcal{A}, \leq \rangle \to \langle \mathcal{A}^i, \leq^i \rangle, \quad i \in I,$$

is a **subdirect intersection** if

$$\bigcap_{i \in I} (\gamma^i)^{-1}(\leq^i) = \leq.$$

Note that this implies that

$$\bigcap_{i \in I} \mathrm{Ker}(\langle H^i, \gamma^i \rangle) = \Delta^{\mathcal{A}}.$$

Indeed, we have

$$\begin{array}{rcl}
\bigcap_{i \in I} \mathrm{Ker}(\langle H^i, \gamma^i \rangle) & = & \bigcap_{i \in I}((\gamma^i)^{-1}(\leq^i) \cap (\gamma^i)^{-1}(\leq^i)^{-1}) \\
& = & \bigcap_{i \in I}(\gamma^i)^{-1}(\leq^i) \cap \bigcap_{i \in I}(\gamma^i)^{-1}(\leq^i)^{-1} \\
& = & \bigcap_{i \in I}(\gamma^i)^{-1}(\leq^i) \cap (\bigcap_{i \in I}(\gamma^i)^{-1}(\leq^i))^{-1} \\
& = & \leq \cap (\leq)^{-1} \\
& = & \Delta^{\mathcal{A}}.
\end{array}$$

Given a class K of **F**-algebraic posystems, we write $\langle \mathcal{A}, \leq \rangle \in \overset{\triangleleft}{\mathbb{I}}(\mathsf{K})$ in case there exists a subdirect intersection $\{\langle H^i, \gamma^i \rangle : \langle \mathcal{A}, \leq \rangle \to \langle \mathcal{A}^i, \leq^i \rangle, i \in I\}$, with $\langle \mathcal{A}^i, \leq^i \rangle \in \mathsf{K}$, for all $i \in I$. If $\overset{\triangleleft}{\mathbb{I}}(\mathsf{K}) = \mathsf{K}$, we say that K is **closed under subdirect intersections**.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an **F**-algebraic system and $\leq$ a posystem on $\mathcal{A}$. A congruence system $\theta \in$

$\mathrm{ConSys}(\mathcal{A})$ is said to be **compatible with** $\leq$ if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \phi', \psi, \psi' \in \mathrm{SEN}(\Sigma)$,

$$\begin{tikzcd}
\phi \arrow[r, dash, "\leq_\Sigma"] \arrow[d, dash, "\theta_\Sigma"'] & \psi \arrow[d, dash, "\theta_\Sigma"] \\
\phi' \arrow[r, dash, "\leq_\Sigma"'] & \psi'
\end{tikzcd}$$

$$\phi \leq_\Sigma \psi, \quad \phi \; \theta_\Sigma \; \phi' \quad \text{and} \quad \psi \; \theta_\Sigma \; \psi' \quad \text{imply} \quad \phi' \leq_\Sigma \psi'.$$

A congruence system $\theta \in \mathrm{ConSys}(\mathcal{A})$ is said to be a **congruence system on the F**-algebraic posystem $\langle \mathcal{A}, \leq^{\mathcal{A}} \rangle$ if it is compatible with $\leq^{\mathcal{A}}$. We write $\mathrm{ConSys}(\langle \mathcal{A}, \leq^{\mathcal{A}} \rangle)$ for the collection of all congruence systems on $\langle \mathcal{A}, \leq^{\mathcal{A}} \rangle$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\langle \mathcal{A}, \leq \rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, an $\mathbf{F}$-algebraic posystem and $\{\theta^i : i \in I\} \subseteq \mathrm{ConSys}(\langle \mathcal{A}, \leq \rangle)$ a (upward) directed collection of congruence systems on $\langle \mathcal{A}, \leq \rangle$. It is not difficult to show that $\bigcup_{i \in I} \theta^i \in \mathrm{ConSys}(\langle \mathcal{A}, \leq \rangle)$.

Lemma 1818 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\langle \mathcal{A}, \leq \rangle$ *an* $\mathbf{F}$*-algebraic posystem and* $\{\theta^i : i \in I\} \subseteq \mathrm{ConSys}(\langle \mathcal{A}, \leq \rangle)$ *a directed collection of congruence systems on* $\langle \mathcal{A}, \leq \rangle$*. Then* $\bigcup_{i \in I} \theta^i$ *is a congruence system on* $\langle \mathcal{A}, \leq \rangle$*.*

Proof: We know, by Lemma **??** that $\bigcup_{i \in I} \theta^i_\Sigma$ is a congruence system on $\mathcal{A}$. Thus, it suffices to show that it is compatible with $\leq$. To this end, suppose $\Sigma \in |\mathbf{Sign}|$, $\phi, \phi', \psi, \psi' \in \mathrm{SEN}(\Sigma)$, such that $\phi \leq_\Sigma \psi$, $\phi \; \bigcup_{i \in I} \theta^i_\Sigma \; \phi'$ and $\psi \; \bigcup_{i \in I} \theta^i_\Sigma \; \psi'$. Thus, there exist $j \in I$ and $j' \in I$, such that $\phi \; \theta^j_\Sigma \; \phi'$ and $\psi \; \theta^{j'}_\Sigma \; \psi'$. But $\{\theta^i\}_{i \in I}$ is directed, whence, there exists $k \in I$, such that $\phi \; \theta^k_\Sigma \; \phi'$ and $\psi \; \theta^k_\Sigma \; \psi'$. Therefore, since $\theta^k \in \mathrm{ConSys}(\langle \mathcal{A}, \leq \rangle)$, we get $\phi' \leq_\Sigma \psi'$. We conclude that $\bigcup_{i \in I} \theta^i \in \mathrm{ConSys}(\langle \mathcal{A}, \leq \rangle)$. ∎

Due to Lemma 1818, it makes sense to consider the quotient $\langle \mathcal{A}, \leq \rangle / \bigcup_{i \in I} \theta^i$. This $\mathbf{F}$-algebraic posystem is called the **directed union** of the collection $\langle \mathcal{A}, \leq \rangle / \theta^i$. Given a class K of $\mathbf{F}$-algebraic posystems, we write $\langle \mathcal{A}, \leq \rangle / \bigcup_{i \in I} \theta^i \in \mathbb{U}(\mathsf{K})$ in case $\langle \mathcal{A}, \leq \rangle / \theta^i \in \mathsf{K}$, for all $i \in I$. If $\mathbb{U}(\mathsf{K}) = \mathsf{K}$, we say that K is **closed under directed unions**.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\langle \mathcal{A}, \leq^{\mathcal{A}} \rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\langle \mathcal{B}, \leq^{\mathcal{B}} \rangle$, with $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$, $\mathbf{F}$-algebraic posystems and

$$\langle H, \gamma \rangle : \langle \mathcal{A}, \leq^{\mathcal{A}} \rangle \to \langle \mathcal{B}, \leq^{\mathcal{B}} \rangle$$

a surjective morphism. In this case we say $\langle \mathcal{B}, \leq^{\mathcal{B}} \rangle$ is an **morphic image** of $\langle \mathcal{A}, \leq^{\mathcal{A}} \rangle$. Given a class K of $\mathbf{F}$-algebraic posystems, we write $\langle \mathcal{B}, \leq^{\mathcal{B}} \rangle \in \mathbb{H}(\mathsf{K})$ in case there exists a surjective morphism

$$\langle H, \gamma \rangle : \langle \mathcal{A}, \leq^{\mathcal{A}} \rangle \to \langle \mathcal{B}, \leq^{\mathcal{B}} \rangle,$$

with $\langle\mathcal{A},\leq^{\mathcal{A}}\rangle\in\mathsf{K}$. If $\mathbb{H}(\mathsf{K})=\mathsf{K}$, we say that K is **closed under morphic images**.

It is not difficult to verify that all three operators are closure operators on classes of $\mathbf{F}$-algebraic posystems.

Proposition 1819 *Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system. Then the operators $\overset{\triangleleft}{\mathbb{P}}$, $\mathbb{U}$ and $\mathbb{H}$ on classes of $\mathbf{F}$-algebraic posystems are closure operators.*

Proof: We first look at $\overset{\triangleleft}{\mathbb{P}}$. Suppose K is a class of $\mathbf{F}$-algebraic posystems.

- If $\langle\mathcal{A},\leq\rangle\in\mathsf{K}$, then $\{\langle I,\iota\rangle:\langle\mathcal{A},\leq\rangle\to\langle\mathcal{A},\leq\rangle\}$, where $\langle I,\iota\rangle:\mathcal{A}\to\mathcal{A}$ is the identity morphism, is a subdirect intersection family. Thus, we get that $\langle\mathcal{A},\leq\rangle\in\overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$. Hence $\mathsf{K}\subseteq\overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$ and $\overset{\triangleleft}{\mathbb{P}}$ is inflationary;

- It is obvious that $\overset{\triangleleft}{\mathbb{P}}$ is monotonic;

- Suppose that $\langle\mathcal{A},\leq\rangle\in\overset{\triangleleft}{\mathbb{P}}(\overset{\triangleleft}{\mathbb{P}}(\mathsf{K}))$. Then, there exists a subdirect intersection family

$$\{\langle H^i,\gamma^i\rangle:\langle\mathcal{A},\leq\rangle\to\langle\mathcal{A}^i,\leq^i\rangle,\ i\in I\},$$

with $\langle\mathcal{A}^i,\leq^i\rangle\in\overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$, for all $i\in I$. Therefore, for each $i\in I$, there exists a sibdirect intersection family

$$\{\langle H^{ij},\gamma^{ij}\rangle:\langle\mathcal{A}^i,\leq^i\rangle\to\langle\mathcal{A}^{ij},\leq^{ij}\rangle,\ j\in J_i\},$$

with $\langle\mathcal{A}^{ij},\leq^{ij}\rangle\in\mathsf{K}$, for all $i\in I$ and all $j\in J_i$. Consider

$$\{\langle H^{ij},\gamma^{ij}\rangle\circ\langle H^i,\gamma^i\rangle:\langle\mathcal{A},\leq\rangle\to\langle\mathcal{A}^{ij},\leq^{ij}\rangle,\ i\in I,j\in J_i\}.$$

It is a subdirect intersection family, since

$$\begin{array}{lll}\bigcap_{i\in I,j\in J_i}(\gamma^{ij}\circ\gamma^i)^{-1}(\leq^{ij}) & = & \bigcap_{i\in I,j\in J_i}(\gamma^i)^{-1}((\gamma^{ij})^{-1}(\leq^{ij}))\\ & = & \bigcap_{i\in I}(\gamma^i)^{-1}(\bigcap_{j\in J_i}(\gamma^{ij})^{-1}(\leq^{ij}))\\ & = & \bigcap_{i\in I}(\gamma^i)^{-1}(\leq^i)\\ & = & \leq.\end{array}$$

Since $\langle\mathcal{A}^{ij},\leq^{ij}\rangle\in\mathsf{K}$, for all $i\in I$, $j\in J_i$, we get that $\overset{\triangleleft}{\mathbb{P}}(\overset{\triangleleft}{\mathbb{P}}(\mathsf{K}))\subseteq\overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$ and $\overset{\triangleleft}{\mathbb{P}}$ is idempotent.

Thus, $\overset{\triangleleft}{\mathbb{P}}$ is a closure operator.

Now we turn to $\mathbb{U}$. Suppose, again, that K is a class of $\mathbf{F}$-algebraic posystems.

- If $\langle\mathcal{A},\le\rangle\in\mathsf{K}$, we look at the singleton family $\{\Delta^{\mathcal{A}}\}$, consisting of the identity congruence system on $\langle\mathcal{A},\le\rangle$. Clearly, it is directed and its union is $\Delta^{\mathcal{A}}$. Therefore, $\langle\mathcal{A},\le\rangle\cong\langle\mathcal{A},\le\rangle/\Delta^{\mathcal{A}}\in\mathbb{U}(\mathsf{K})$;

- Monotonicity is obvious in this case as well;

- Suppose that $\langle\mathcal{A},\le\rangle/\theta\in\mathbb{U}(\mathbb{U}(\mathsf{K}))$. Then $\theta=\bigcup_{i\in I}\theta^i$ for a directed family $\{\theta^i:i\in I\}\subseteq\mathrm{ConSys}(\langle\mathcal{A},\le\rangle)$, such that $\langle\mathcal{A},\le\rangle/\theta^i\in\mathbb{U}(\mathsf{K})$, for all $i\in I$. Thus, for all $i\in I$, $\theta^i=\bigcup_{j\in J_i}\theta^{ij}$ for a directed family $\{\theta^{ij}: j\in J_i\}\subseteq\mathrm{ConSys}(\langle\mathcal{A},\le\rangle)$, such that $\langle\mathcal{A},\le\rangle/\theta^{ij}\in\mathsf{K}$, for all $j\in J_i$. Now, let $\Sigma\in|\mathbf{Sign}|$ and $\phi,\psi\in\mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle\in\theta^{ij_i}_\Sigma\cup\theta^{i'j_{i'}}_\Sigma$. By hypothesis, $\langle\phi,\psi\rangle\in\theta^i_\Sigma\cup\theta^{i'}_\Sigma$. Hence, since $\{\theta^i:i\in I\}$ is directed, there exists $k\in I$, such that $\langle\phi,\psi\rangle\in\theta^k_\Sigma$. Thus, again, by the hypothesis, there exists, $j_k\in J_k$, such that $\langle\phi,\psi\rangle\in\theta^{kj_k}$. We conclude that the collection $\{\theta^{ij}:j\in J_i,i\in I\}$ is directed, such that $\langle\mathcal{A},\le\rangle/\theta^{ij}\in\mathsf{K}$, for all $i\in I,j\in J_i$, and, moreover, $\theta=\bigcup_{i\in I}\theta^i=\bigcup_{i\in I}\bigcup_{j\in J_i}\theta^{ij}=\bigcup_{\substack{j\in J_i\\ i\in I}}\theta^{ij}$. Thus, $\langle\mathcal{A},\le\rangle/\theta\in\mathbb{U}(\mathsf{K})$ and $\mathbb{U}$ is also idempotent.

Therefore, $\mathbb{U}$ is also a closure operator.

Finally, we deal with $\mathbb{H}$, which is the easiest case. Let K be a class of $\mathbf{F}$-algebraic posystems. If $\langle\mathcal{A},\le\rangle\in\mathsf{K}$, then, using again the identity $\langle I,\iota\rangle:\langle\mathcal{A},\le\rangle\to\langle\mathcal{A},\le\rangle$, we see that $\langle\mathcal{A},\le\rangle\in\mathbb{H}(\mathsf{K})$, and, hence, $\mathbb{H}$ is inflationary. It is again obvious that it is monotonic. Finally, if $\mathcal{A}\in\mathbb{H}(\mathbb{H}(\mathsf{K}))$, then, there exists a surjective morphism $\langle G,\beta\rangle:\langle\mathcal{A}',\le'\rangle\to\langle\mathcal{A},\le\rangle$, with $\langle\mathcal{A}',\le'\rangle\in\mathbb{H}(\mathsf{K})$, whence, there also exists a surjective morphism $\langle H,\gamma\rangle:\langle\mathcal{A}'',\le''\rangle\to\langle\mathcal{A}',\le'\rangle$, with $\langle\mathcal{A}'',\le''\rangle\in\mathsf{K}$. Now the surjective morphism

$$\langle G,\beta\rangle\circ\langle H,\gamma\rangle:\langle\mathcal{A}'',\le''\rangle\to\langle\mathcal{A},\le\rangle$$

witnesses the fact that $\langle\mathcal{A},\le\rangle\in\mathbb{H}(\mathsf{K})$. Therefore, $\mathbb{H}(\mathbb{H}(\mathsf{K}))\subseteq\mathbb{H}(\mathsf{K})$, and $\mathbb{H}$ is idempotent. Thus, $\mathbb{H}$ is a closure operator. ∎

We show next that, if a class K of $\mathbf{F}$-algebraic posystems is closed under morphic images, then it is also closed under directed unions.

Proposition 1820 *Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and K be a class of $\mathbf{F}$-algebraic posystems. If K is closed under morphic images, then it is closed under directed unions.*

Proof: Suppose K is closed under $\overset{\triangleleft}{\mathbb{P}}$ and $\mathbb{H}$ and let $\langle\mathcal{A}.\le\rangle$, with $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$, be an $\mathbf{F}$-algebraic posystem and $\{\theta^i:i\in I\}\subseteq\mathrm{ConSys}(\langle\mathcal{A},\le\rangle)$, a directed family of congruence systems, such that $\langle\mathcal{A},\le\rangle/\theta^i\in\mathsf{K}$. Consider a morphism

$$\langle I,\pi^i\rangle:\langle\mathcal{A},\le\rangle/\theta^i\to\langle\mathcal{A},\le\rangle/\bigcup_{i\in I}\theta^i,$$

given, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, by

$$\pi^i_\Sigma(\phi/\theta^i_\Sigma) = \phi/\bigcup_{i\in I}\theta^i_\Sigma.$$

It is well defined, since, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi,\psi \in \mathrm{SEN}(\Sigma)$, if $\langle\phi,\psi\rangle \in \theta^i_\Sigma$, then, automatically, $\langle\phi,\psi\rangle \in \bigcup_{i\in I}\theta^i_\Sigma$. Therefore, since $\langle\mathcal{A},\leq\rangle/\theta^i \in \mathsf{K}$, we get, by hypothesis, $\langle\mathcal{A},\leq\rangle/\bigcup_{i\in I}\theta^i \in \mathbb{H}(\mathsf{K}) = \mathsf{K}$. We conclude that $\mathbb{U}(\mathsf{K}) \subseteq \mathsf{K}$ and, hence, K is closed under directed unions. ■

We are now ready to provide alternative characterizations of inequational, quasi inequational and guasi inequational classes of **F**-algebraic posystems. Namely, we show that an abstract class of **F**-algebraic posystems is a guasi inequational class if and only if it is closed under subdirect intersections, that it is a quasi inequational class if and only if it is closed under subdirect intersections and directed unions and that it is an inequational class if and only if it is closed under subdirect intersections and morphic images.

Theorem 1821 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* K *an abstract class of* **F***-algebraic posystems.* K *is a guasi inequational class if and only if it is closed under subdirect intersections.*

Proof: Suppose, first, that K is a guasi inequational class and consider a subdirect intersection

$$\{\langle H^i,\gamma^i\rangle : \langle\mathcal{A},\leq\rangle \to \langle\mathcal{A}^i,\leq^i\rangle,\ i\in I\},$$

with $\langle\mathcal{A}^i,\leq^i\rangle \in \mathsf{K}$. Let G be the set of guasi inequations defining K and $\Sigma \in |\mathbf{Sign}^\flat|$, $\langle\vec{\phi} \preccurlyeq \vec{\psi}, \phi \preccurlyeq \psi\rangle \in G_\Sigma$, such that, for some $\Sigma' \in |\mathbf{Sign}^\flat|$ and $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi_j)) \leq_{F(\Sigma')} \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi_j)),\ \text{for all } j\in J.$$

Then we get $\gamma^i_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi_j))) \leq^i_{H^i(F(\Sigma'))} \gamma^i_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi_j)))$, for all $i\in I$, $j\in J$. This gives $\alpha^i_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi_j)) \leq^i_{F^i(\Sigma')} \alpha^i_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi_j))$, for all $i\in I$, $j\in J$. Since $\langle\mathcal{A}^i,\leq^i\rangle \in \mathsf{K}$, for all $i\in I$, and $\langle\vec{\phi} \preccurlyeq \vec{\psi}, \phi \preccurlyeq \psi\rangle \in G_\Sigma$, we get that $\alpha^i_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \leq^i_{F^i(\Sigma')} \alpha^i_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi))$, for all $i\in I$. Equivalently, $\gamma^i_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi))) \leq^i_{H^i(F(\Sigma'))} \gamma^i_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi)))$, for all $i\in I$, i.e.,

$$\langle\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)), \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi))\rangle \in \bigcap_{i\in I}(\gamma^i_{F(\Sigma')})^{-1}(\leq^i).$$

Since $\{\langle H^i,\gamma^i\rangle : i\in I\}$ is a subdirect intersection, we get

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \leq_{F(\Sigma')} \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi)).$$

We conclude that $\langle\mathcal{A},\leq\rangle \in \mathrm{PAlgSys}(G) = \mathsf{K}$. Hence, K is closed under subdirect intersections.

Assume, conversely, that K is closed under subdirect intersections and set $G = \mathrm{GIn}(\mathsf{K})$. Let $\langle\mathcal{A},\leq\rangle \in \mathrm{PAlgSys}(\mathbf{F})$, such that $G \leq \mathrm{GIn}(\langle\mathcal{A},\leq\rangle)$. Let $\Sigma \in |\mathbf{Sign}^\flat|$, such that $\phi \preccurlyeq \psi \notin \mathrm{In}_\Sigma(\langle\mathcal{A},\leq\rangle)$, i.e., for some $\Sigma' \in |\mathbf{Sign}^\flat|$ and some $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$, $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \nleq_{F(\Sigma')} \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi))$. Thus, by definition, the guasi inequation $\langle\mathrm{In}_\Sigma(\langle\mathcal{A},\leq\rangle),\phi \preccurlyeq \psi\rangle \notin \mathrm{GIn}_\Sigma(\langle\mathcal{A},\leq\rangle)$. Therefore, since $G \leq \mathrm{GIn}(\langle\mathcal{A},\leq\rangle)$, $\langle\mathrm{In}_\Sigma(\mathcal{A}),\phi \preccurlyeq \psi\rangle \notin \mathrm{GIn}_\Sigma(\mathsf{K})$. Hence, for every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \preccurlyeq \psi \notin \mathrm{In}_\Sigma(\langle\mathcal{A},\leq\rangle)$, there exists $\langle\mathcal{K}^{\langle\Sigma,\phi,\psi\rangle},\leq^{\langle\Sigma,\phi,\psi\rangle}\rangle \in \mathsf{K}$, such that

$$\begin{array}{l}\mathrm{In}_\Sigma(\langle\mathcal{A},\leq\rangle) \subseteq \mathrm{In}_\Sigma(\langle\mathcal{K}^{\langle\Sigma,\phi,\psi\rangle},\leq^{\langle\Sigma,\phi,\psi\rangle}\rangle),\\ \phi \preccurlyeq \psi \notin \mathrm{In}_\Sigma(\langle\mathcal{K}^{\langle\Sigma,\phi,\psi\rangle},\leq^{\langle\Sigma,\phi,\psi\rangle}\rangle).\end{array}$$

We conclude that

$$\mathrm{In}_\Sigma(\langle\mathcal{A},\leq\rangle) = \bigcap\{\mathrm{In}_\Sigma(\langle\mathcal{K}^{\langle\Sigma,\phi,\psi\rangle},\leq^{\langle\Sigma,\phi,\psi\rangle}\rangle) : \phi \preccurlyeq \psi \notin \mathrm{In}_\Sigma(\langle\mathcal{A},\leq\rangle)\}.$$

Denote, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathsf{K}^\Sigma = \{\langle\mathcal{K}^{\langle\Sigma,\phi,\psi\rangle},\leq^{\langle\Sigma,\phi,\psi\rangle}\rangle : \phi \preccurlyeq \psi \notin \mathrm{In}_\Sigma(\langle\mathcal{A},\leq\rangle)\}$, for brevity.

- Since K is closed under subdirect intersections, and

  $$\{\langle F^{\mathcal{K}},\alpha^{\mathcal{K}}\rangle : \langle\mathcal{F},\bigcap_{\langle\mathcal{K},\leq^{\mathcal{K}}\rangle\in\mathsf{K}^\Sigma}(\alpha^{\mathcal{K}})^{-1}(\leq^{\mathcal{K}})\rangle/\mathrm{Eq}(\mathsf{K}^\Sigma) \to \langle\mathcal{K},\leq^{\mathcal{K}}\rangle, \langle\mathcal{K},\leq^{\mathcal{K}}\rangle \in \mathsf{K}^\Sigma\}$$

  is a subdirect intersection, we get that

  $$\langle\mathcal{F},\bigcap_{\langle\mathcal{K},\leq^{\mathcal{K}}\rangle\in\mathsf{K}^\Sigma}(\alpha^{\mathcal{K}})^{-1}(\leq^{\mathcal{K}})\rangle/\mathrm{Eq}(\mathsf{K}^\Sigma) \in \mathsf{K}.$$

- Since, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

  $$\begin{array}{rcl}\mathrm{In}_\Sigma(\langle\mathcal{A},\leq\rangle) &=& \mathrm{In}_\Sigma(\mathsf{K}^\Sigma)\\ &=& \mathrm{In}_\Sigma(\langle\mathcal{F},\bigcap_{\langle\mathcal{K},\leq^{\mathcal{K}}\rangle\in\mathsf{K}^\Sigma}(\alpha^{\mathcal{K}})^{-1}(\leq^{\mathcal{K}})\rangle/\mathrm{Eq}(\mathsf{K}^\Sigma))\end{array}$$

  and $\langle\mathcal{F},\bigcap_{\langle\mathcal{K},\leq^{\mathcal{K}}\rangle\in\mathsf{K}^\Sigma}(\alpha^{\mathcal{K}})^{-1}(\leq^{\mathcal{K}})\rangle/\mathrm{Eq}(\mathsf{K}^\Sigma) \in \mathsf{K}$, $\langle\mathcal{A},\leq\rangle \in \mathbb{C}(\mathsf{K})$. Since K is abstract, we conclude that $\langle\mathcal{A},\leq\rangle \in \mathsf{K}$.

Hence, K is indeed a guasi inequational class of $\mathbf{F}$-algebraic posystems. ■

Theorem 1822 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* K *an abstract class of* $\mathbf{F}$*-algebraic posystems.* K *is a quasi inequational class if and only if it is closed under subdirect intersections and directed unions.*

Proof: Suppose, first, that K is a quasi inequational class, defined by a collection Q of $\mathbf{F}$-quasi inequations. Then it is a guasi inequational class and, therefore, by Theorem 1821, closed under subdirect intersections. Let

$\langle\mathcal{A},\le\rangle$ be an $\mathbf{F}$-algebraic posystem and $\{\theta^i : i\in I\}\subseteq \mathrm{ConSys}(\langle\mathcal{A},\le\rangle)$ a directed union of congruence systems on $\langle\mathcal{A},\le\rangle$, such that $\langle\mathcal{A},\le\rangle/\theta^i\in\mathsf{K}$, for all $i\in I$. Let $\Sigma\in|\mathbf{Sign}^\flat|$, $\langle\vec{\psi}\preccurlyeq\vec{\psi},\phi\preccurlyeq\psi\rangle\in Q_\Sigma$, such that, for some $\Sigma'\in|\mathbf{Sign}^\flat|$ and $f\in\mathbf{Sign}^\flat(\Sigma,\Sigma')$,

$$\alpha_{\Sigma'}^{\bigcup_{i\in I}\theta^i}(\mathrm{SEN}^\flat(f)(\phi_j))\le_{F(\Sigma')}^{\bigcup_{i\in I}\theta^i}\alpha_{\Sigma'}^{\bigcup_{i\in I}\theta^i}(\mathrm{SEN}^\flat(f)(\psi_j)),\quad j<n.$$

Thus, for every $j<n$, there exists $k_j\in I$, such that

$$\alpha_{\Sigma'}^{\theta^{k_j}}(\mathrm{SEN}^\flat(f)(\phi_j))\le_{F(\Sigma')}^{\theta^{k_j}}\alpha_{\Sigma'}^{\theta^{k_j}}(\mathrm{SEN}^\flat(f)(\psi_j)).$$

Since $\{\theta^i : i\in I\}$ is directed, there exists $k\in I$, such that

$$\alpha_{\Sigma'}^{\theta^k}(\mathrm{SEN}^\flat(f)(\phi_j))\le_{F(\Sigma')}^{\theta^k}\alpha_{\Sigma'}^{\theta^k}(\mathrm{SEN}^\flat(f)(\psi_j)),\quad j<n.$$

Since $\langle\mathcal{A},\le\rangle/\theta^k\in\mathsf{K}$ and $\langle\vec{\psi}\preccurlyeq\vec{\psi},\phi\preccurlyeq\psi\rangle\in Q_\Sigma$, we get that

$$\alpha_{\Sigma'}^{\bigcup_{i\in I}\theta^i}(\mathrm{SEN}^\flat(f)(\phi))\le_{F(\Sigma')}^{\bigcup_{i\in I}\theta^i}\alpha_{\Sigma'}^{\bigcup_{i\in I}\theta^i}(\mathrm{SEN}^\flat(f)(\psi)).$$

Therefore, $\langle\mathcal{A},\le\rangle/\bigcup_{i\in I}\theta^i\in\mathrm{PAlgSys}(Q)=\mathsf{K}$ and K is closed under directed unions.

Suppose, conversely, that K is an abstract class of $\mathbf{F}$-algebraic posystems closed under subdirect intersections and directed unions. Set $Q=\mathrm{QIn}(\mathsf{K})$ and let $\langle\mathcal{A},\le\rangle\in\mathrm{PAlgSys}(\mathbf{F})$, such that $Q\le\mathrm{QIn}(\mathcal{A})$. Let $\Sigma\in|\mathbf{Sign}^\flat|$, such that $\phi\preccurlyeq\psi\notin\mathrm{In}_\Sigma(\langle\mathcal{A},\le\rangle)$, i.e., for some $\Sigma'\in|\mathbf{Sign}^\flat|$ and some $f\in\mathbf{Sign}^\flat(\Sigma,\Sigma')$, $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi))\nleq_{F(\Sigma')}\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi))$. Thus, by definition, for every finite $I\le\mathrm{In}_\Sigma(\mathcal{A})$ the quasi inequation $\langle I,\phi\preccurlyeq\psi\rangle\notin\mathrm{QIn}_\Sigma(\langle\mathcal{A},\le\rangle)$. Therefore, since $Q\le\mathrm{QIn}(\langle\mathcal{A},\le\rangle)$, $\langle I,\phi\preccurlyeq\psi\rangle\notin\mathrm{QIn}_\Sigma(\mathsf{K})$. Hence, for every $\Sigma\in|\mathbf{Sign}^\flat|$, all $I\subseteq_f\mathrm{In}_\Sigma(\langle\mathcal{A},\le\rangle)$ and all $\phi\preccurlyeq\psi\notin\mathrm{In}_\Sigma(\langle\mathcal{A},\le\rangle)$, there exists

$$\langle\mathcal{K}^{\langle\Sigma,I,\phi,\psi\rangle},\le^{\langle\Sigma,I,\phi,\psi\rangle}\rangle\in\mathsf{K},$$

such that

- $I\subseteq\mathrm{In}_\Sigma(\langle\mathcal{K}^{\langle\Sigma,I,\phi,\psi\rangle},\le^{\langle\Sigma,I,\phi,\psi\rangle}\rangle)$;
- $\phi\preccurlyeq\psi\notin\mathrm{In}_\Sigma(\langle\mathcal{K}^{\langle\Sigma,I,\phi,\psi\rangle},\le^{\langle\Sigma,I,\phi,\psi\rangle}\rangle)$.

We conclude that, for all $\Sigma\in|\mathbf{Sign}^\flat|$,

$$\begin{array}{r}\mathrm{In}_\Sigma(\langle\mathcal{A},\le\rangle)=\bigcap\{\bigcup\{\mathrm{In}_\Sigma(\mathcal{K}^{\langle\Sigma,I,\phi,\psi\rangle},\le^{\langle\Sigma,I,\phi,\psi\rangle}\rangle):\\ I\subseteq_f\mathrm{In}_\Sigma(\langle\mathcal{A},\le\rangle)\}:\phi\preccurlyeq\psi\notin\mathrm{In}_\Sigma(\langle\mathcal{A},\le\rangle)\}.\end{array}$$

Denote, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\phi\preccurlyeq\psi\notin\mathrm{In}_\Sigma(\langle\mathcal{A},\le\rangle)$,

$$\mathsf{K}^{\langle\Sigma,\phi,\psi\rangle}=\{\langle\mathcal{F},(\alpha^{\langle\Sigma,I,\phi,\psi\rangle})^{-1}(\le^{\langle\Sigma,I,\phi,\psi\rangle})\rangle/\mathrm{Eq}(\mathcal{K}^{\langle\Sigma,I,\phi,\psi\rangle}):I\subseteq_f\mathrm{In}_\Sigma(\langle\mathcal{A},\le\rangle)\},$$

and, for all $\Sigma\in|\mathbf{Sign}^\flat|$,

$$\begin{array}{c}\mathsf{K}^\Sigma=\{\langle\mathcal{F},\bigcup_{\mathcal{K}\in\mathsf{K}^{\langle\Sigma,\phi,\psi\rangle}}(\alpha^{\mathcal{K}})^{-1}(\le^{\mathcal{K}})\rangle/\bigcup_{\mathcal{K}\in\mathsf{K}^{\langle\Sigma,\phi,\psi\rangle}}\mathrm{Eq}(\mathcal{K}):\\ \phi\preccurlyeq\psi\notin\mathrm{In}_\Sigma(\langle\mathcal{A},\le\rangle)\},\end{array}$$

for brevity.

- Since K is closed under directed unions, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \lessdot \psi \notin \mathrm{In}_\Sigma(\mathcal{A})$, we have $\langle \mathcal{F}, \bigcup_{\mathcal{K} \in \mathsf{K}^{\langle \Sigma, \phi, \psi \rangle}} (\alpha^{\mathcal{K}})^{-1}(\leq^{\mathcal{K}}) \rangle / \bigcup_{\mathcal{K} \in \mathsf{K}^{\langle \Sigma, \phi, \psi \rangle}} \mathrm{Eq}(\mathcal{K}) \in \mathsf{K}$.

- Since K is closed under subdirect intersections,
$$\langle \mathcal{F}, \bigcap_{\mathcal{K} \in \mathsf{K}^\Sigma} (\alpha^{\mathcal{K}})^{-1}(\leq^{\mathcal{K}}) \rangle / \mathrm{Eq}(\mathsf{K}^\Sigma) \in \mathsf{K},$$
for all $\Sigma \in |\mathbf{Sign}^\flat|$.

- Finally, noting that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\langle \mathcal{F}, \bigcap_{\mathcal{K} \in \mathsf{K}^\Sigma} (\alpha^{\mathcal{K}})^{-1}(\leq^{\mathcal{K}}) \rangle / \mathrm{Eq}(\mathsf{K}^\Sigma)$ is a Σ-K-certificate for $\mathcal{A}$, and, taking into account that K is abstract, we conclude that $\mathcal{A} \in \mathsf{K}$.

Therefore K is indeed a quasiequational class of $\mathbf{F}$-algebraic systems. ∎

25.2 Syntactic Order Algebraizability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and K a class of $\mathbf{F}$-algebraic posystems. If $\mathcal{I}$ is equivalent to $\mathcal{I}^{\mathsf{K},\leq}$ via a conjugate pair $(\alpha, \beta) : \mathcal{I} \rightleftarrows \mathcal{I}^{\mathsf{K},\leq}$, we say that the class K **β-order algebraizes** the π-institution $\mathcal{I}$. Recall, in more detail, that this means that there exist a collection $\alpha : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with a single distinguished argument and a collection $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with two distinguished arguments, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$ and $I \subseteq \mathrm{In}_\Sigma(\mathbf{F})$,

1. $\phi \in C_\Sigma(\Phi)$ if and only if $\alpha_\Sigma[\phi] \leq C_\Sigma^{\mathsf{K},\leq}(\alpha_\Sigma[\Phi])$;
2. $\phi \lessdot \psi \in C_\Sigma^{\mathsf{K},\leq}(I)$ if and only if $\beta_\Sigma[\phi, \psi] \leq C(\beta_\Sigma[I])$;
3. $C^{\mathsf{K},\leq}(\phi \lessdot \psi) = C^{\mathsf{K},\leq}(\alpha[\beta_\Sigma[\phi, \psi]])$;
4. $C(\phi) = C(\beta[\alpha_\Sigma[\phi]])$.

Moreover, we say that $\mathcal{I}$ is **β-order algebraizable** if there exists a class K of $\mathbf{F}$-algebraic posystems, such that K β-order algebraizes $\mathcal{I}$. In this case, we call the least order guasivariety including K the **β-ordered class of $\mathcal{I}$** and denote it by $\mathrm{PAlgSys}(\mathcal{I})$.

Lemma 1823 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$ having two distinguished arguments, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and K, K' two classes of $\mathbf{F}$-algebraic posystems. If both K and K' β-order algebraize $\mathcal{I}$, then $\mathcal{I}^{\mathsf{K},\leq} = \mathcal{I}^{\mathsf{K}',\leq}$. Therefore, $\mathbb{GO}^{\mathsf{Sem}}(\mathsf{K}) = \mathbb{GO}^{\mathsf{Sem}}(\mathsf{K}')$.*

Proof: Suppose both K and K' β-order algebraize $\mathcal{I}$. Then, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $I \cup \{\phi \preccurlyeq \psi\} \in \mathrm{In}_\Sigma(\mathbf{F})$,

$$\begin{array}{rll}\phi \preccurlyeq \psi \in C_\Sigma^{\mathsf{K},\leq}(I) & \text{iff} & \beta_\Sigma[\phi,\psi] \leq C(\beta_\Sigma[I]) \\ & \text{iff} & \phi \preccurlyeq \psi \in C_\Sigma^{\mathsf{K}',\leq}(I).\end{array}$$

We conclude that $\mathcal{I}^{\mathsf{K},\leq} = \mathcal{I}^{\mathsf{K}',\leq}$, whence the semantic order guasivarieties generated by K and K' coincide. ■

We call the unique semantic order guasivariety that β-order algebraizes $\mathcal{I}$ the β-**order class of** $\mathcal{I}$ and denote it by $\mathrm{PAlgSys}^\beta(\mathcal{I})$.

Next we show that if two families $\beta,\beta' : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$ are deductively equivalent, in the sense that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, $\beta_\Sigma[\phi,\psi]$ and $\beta'_\Sigma[\phi,\psi]$ are interderivable, then $\mathcal{I}$ is β-order algebraizable if and only if it is β'-order algebraizable and, in fact, in that case, the corresponding order classes of $\mathcal{I}$ coincide.

Lemma 1824 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with β,β' : $(\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with two distinguished arguments, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$. If, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, $C(\beta_\Sigma[\phi,\psi]) = C(\beta'_\Sigma[\phi,\psi])$, then $\mathcal{I}$ is β-order algebraizable if and only if $\mathcal{I}$ is β'-order algebraizable. In that case, the β- and β'-order classes of $\mathcal{I}$ coincide, i.e.,* $\mathrm{PAlgSys}^\beta(\mathcal{I}) = \mathrm{PAlgSys}^{\beta'}(\mathcal{I})$.

Proof: Suppose that $\mathcal{I}$ is β-order algebraizable. Then, there exists a conjugate pair $(\alpha,\beta) : \mathcal{I} \rightleftarrows \mathcal{I}^{\mathsf{K},\leq}$. We show that $\mathcal{I}$ is also β'-order algebraizable via the conjugate pair $(\alpha,\beta') : \mathcal{I} \rightleftarrows \mathcal{I}^{\mathsf{K},\leq}$.

- We have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $I \cup \{\phi \preccurlyeq \psi\} \subseteq \mathrm{In}_\Sigma(\mathbf{F})$,

$$\begin{array}{rll}\phi \preccurlyeq \psi \in C_\Sigma^{\mathsf{K},\leq}(I) & \text{iff} & \beta_\Sigma[\phi,\psi] \leq C(\beta_\Sigma[I]) \\ & \text{iff} & \beta'_\Sigma[\phi,\psi] \leq C(\beta'_\Sigma[I]).\end{array}$$

- $C(\phi) = C(\beta[\alpha_\Sigma[\phi]]) = C(\beta'[\alpha_\Sigma[\phi]])$.

Thus, by Proposition 898, $\mathcal{I}$ and $\mathcal{I}^{\mathsf{K},\leq}$ are equivalent via (α,β'). By symmetry, we infer the first statement of the lemma. The second conclusion now follows directly from Lemma 1823, since the same class K both β- and β'-order algebraizes $\mathcal{I}$. ■

Moreover, we can show that the conjugate transformation α in a β-order algebraization is essentially unique, in the sense that any two of them are deductively equivalent modulo inequational derivability.

Lemma 1825 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with β : $(\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$ having two distinguished arguments, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$ and K a class of $\mathbf{F}$-algebraic posystems. If K β-order* *algebraizes $\mathcal{I}$ via a conjugate pair $(\alpha,\beta):\mathcal{I}\rightleftarrows\mathcal{I}^{\mathsf{K},\le}$ and via a conjugate pair $(\alpha',\beta):\mathcal{I}\rightleftarrows\mathcal{I}^{\mathsf{K},\le}$, then, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\phi\in\mathrm{SEN}^\flat(\Sigma)$,*

$$C^{\mathsf{K},\le}(\alpha_\Sigma[\phi]) = C^{\mathsf{K},\le}(\alpha'_\Sigma[\phi]).$$

Proof: We have, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\phi\in\mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}\alpha_\Sigma[\phi]\le C^{\mathsf{K},\le}(\alpha'_\Sigma[\phi]) & \text{iff} & \beta[\alpha_\Sigma[\phi]]\le C(\beta[\alpha'_\Sigma[\phi]])\\ & \text{iff} & \phi\in C_\Sigma(\phi).\end{array}$$

Therefore, $\alpha_\Sigma[\phi]\le C^{\mathsf{K},\le}(\alpha'_\Sigma[\phi])$ and, hence, by symmetry, $C^{\mathsf{K},\le}(\alpha_\Sigma[\phi]) = C^{\mathsf{K},\le}(\alpha'_\Sigma[\phi])$. ■

We give next some conditions that are equivalent to $\overleftrightarrow{\beta}$ defining Leibniz congruence systems of theory families of $\mathcal{I}$. Recall that this is tantamount to $\mathcal{I}$ being syntactically protoalgebraic.

Theorem 1826 *Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, with $\beta:(\mathrm{SEN}^\flat)^\omega\to\mathrm{SEN}^\flat$ in $N^\flat$ having two distinguished arguments and $\mathcal{I}=\langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$. The following conditions are equivalent:*

(i) For all $\langle\mathcal{A},T\rangle\in\mathrm{MatFam}^*(\mathcal{I})$, $\beta^{\mathcal{A}}(T)$ is reflexive and antisymmetric;

(ii) For all $\sigma^\flat$ in $N^\flat$, all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\phi,\psi,\vec{\chi}\in\mathrm{SEN}^\flat(\Sigma)$,

- *$\beta_\Sigma[\phi,\phi]\le\mathrm{Thm}(\mathcal{I})$;*
- *$\sigma^\flat_\Sigma(\psi,\vec{\chi})\in C_\Sigma(\beta_\Sigma[\phi,\psi],\beta_\Sigma[\psi,\phi],\sigma^\flat_\Sigma(\phi,\vec{\chi}))$;*

(iii) For every $\mathbf{F}$-algebraic system $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$ and every $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T)=\overleftrightarrow{\beta}^{\mathcal{A}}(T)$.

Proof:

(i)⇒(ii) Suppose that, for all $\langle\mathcal{A},T\rangle\in\mathrm{MatFam}^*(\mathcal{I})$, $\beta^{\mathcal{A}}(T)$ is reflexive and antisymmetric and let $\sigma^\flat\in N^\flat$, $\Sigma\in|\mathbf{Sign}^\flat|$, and $\phi,\psi,\vec{\chi}\in\mathrm{SEN}^\flat(\Sigma)$. Consider $\langle\mathcal{F}/\Omega(\mathrm{Thm}(\mathcal{I})),\mathrm{Thm}(\mathcal{I})/\Omega(\mathrm{Thm}(\mathcal{I}))\rangle\in\mathrm{MatFam}^*(\mathcal{I})$. Then, by hypothesis, $\langle\phi,\psi\rangle\in\beta_\Sigma^{\mathcal{F}/\Omega(\mathrm{Thm}(\mathcal{I}))}(\mathrm{Thm}(\mathcal{I}))$, i.e.,

$$\beta_\Sigma^{\mathcal{F}/\Omega(\mathrm{Thm}(\mathcal{I}))}[\phi/\Omega_\Sigma(\mathrm{Thm}(\mathcal{I})),\phi/\Omega_\Sigma(\mathrm{Thm}(\mathcal{I}))]\le\mathrm{Thm}(\mathcal{I})/\Omega(\mathrm{Thm}(\mathcal{I})).$$

This is equivalent to $\beta_\Sigma[\phi,\phi]\le\mathrm{Thm}(\mathcal{I})$.

Assume, next, that for some $T\in\mathrm{ThFam}(\mathcal{I})$, $\beta_\Sigma[\phi,\psi],\beta_\Sigma[\psi,\phi]\le T$ and $\sigma^\flat_\Sigma(\phi,\vec{\chi})\in T_\Sigma$. Then, we get

$$\begin{array}{l}\beta_\Sigma^{\mathcal{F}/\Omega(T)}[\phi/\Omega_\Sigma(T),\psi/\Omega_\Sigma(T)]\le T/\Omega(T),\\ \beta_\Sigma^{\mathcal{F}/\Omega(T)}[\psi/\Omega_\Sigma(T),\phi/\Omega_\Sigma(T)]\le T/\Omega(T),\end{array}$$

i.e.,

$$\langle \phi/\Omega_\Sigma(T), \psi/\Omega_\Sigma(T)\rangle \in \beta_\Sigma^{\mathcal{F}/\Omega(T)}(T/\Omega(T)),$$
$$\langle \psi/\Omega_\Sigma(T), \phi/\Omega_\Sigma(T)\rangle \in \beta_\Sigma^{\mathcal{F}/\Omega(T)}(T/\Omega(T)).$$

But on $\langle \mathcal{F}/\Omega(T), T/\Omega(T)\rangle \in \mathrm{MatFam}^*(\mathcal{I})$, $\beta^{\mathcal{F}/\Omega(T)}(T/\Omega(T))$ is, by hypothesis, antisymmetric, whence we get $\langle \phi,\psi\rangle \in \Omega_\Sigma(T)$. Therefore, since $\sigma_\Sigma^\flat(\phi,\vec{\chi}) \in T_\Sigma$, we have, by compatibility, that $\sigma_\Sigma^\flat(\psi,\vec{\chi}) \in T_\Sigma$.

(ii)⇒(iii) Suppose that Condition (ii) holds and let $\mathcal{A}$ be an **F**-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Let $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi \in \mathrm{SEN}(\Sigma)$.

- If $\langle \phi,\psi\rangle \in \Omega_\Sigma^{\mathcal{A}}(T)$, then, for all $\sigma^\flat \in \beta$ and all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma,\Sigma')$, $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

$$\begin{aligned}&\langle \sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi), \vec{\chi}),\\ &\qquad\qquad \sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\phi), \vec{\chi})\rangle \in \Omega_{\Sigma'}^{\mathcal{A}}(T),\\ &\langle \sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}(f)(\psi), \mathrm{SEN}(f)(\phi), \vec{\chi}),\\ &\qquad\qquad \sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\phi), \vec{\chi})\rangle \in \Omega_{\Sigma'}^{\mathcal{A}}(T).\end{aligned}$$

  Thus, by compatibility, $\beta_\Sigma^{\mathcal{A}}[\phi,\psi] \le T$ and $\beta_\Sigma^{\mathcal{A}}[\psi,\phi] \le T$, i.e., $\overset{\leftrightarrow}{\beta}{}_\Sigma^{\mathcal{A}}[\phi,\psi] \le T$. Thus, $\langle \phi,\psi\rangle \in \overset{\leftrightarrow}{\beta}_\Sigma(T)$.

- Suppose, conversely, that $\langle \phi,\psi\rangle \in \overset{\leftrightarrow}{\beta}{}_\Sigma^{\mathcal{A}}(T)$, i.e., $\beta_\Sigma^{\mathcal{A}}[\phi,\psi] \le T$ and $\beta_\Sigma^{\mathcal{A}}[\psi,\phi] \le T$. Then, by hypothesis, we have, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma,\Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

$$\sigma\mathcal{A}_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}) \in T_{\Sigma'} \quad \text{iff} \quad \sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}(f)(\psi),\vec{\chi}) \in T_{\Sigma'}.$$

  Therefore, $\langle \phi,\psi\rangle \in \Omega^{\mathcal{A}}(T)$.

we conclude that $\overset{\leftrightarrow}{\beta}{}^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T)$.

(iii)⇒(i) Finally, suppose that, for every **F**-algebraic system $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\overset{\leftrightarrow}{\beta}{}^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T)$ and let $\langle \mathcal{A}, T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$. Then we get that $\overset{\leftrightarrow}{\beta}{}^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. Thus, clearly, $\beta^{\mathcal{A}}(T)$ is reflexive and antisymmetric.

■

We obtain as a corollary characterizing those collection of natural transformations with two distinguished arguments that define posystems on the class $\mathrm{AlgSys}^*(\mathcal{I})$, i.e., on the algebraic system reducts of the reduced matrix families of a π-institution $\mathcal{I}$.

Corollary 1827 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$ *having two distinguished arguments, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $\langle \mathcal{A}, T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$*,* $\beta^{\mathcal{A}}(T)$ *is a posystem on* $\mathcal{A}$ *if and only if the following conditions hold, for all* $\sigma^\flat$ *in* $N^\flat$*,* $\Sigma \in |\mathbf{Sign}^\flat|$*,* $\phi,\psi,\chi,\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$*:*

1. $\beta_\Sigma[\phi,\phi] \le \mathrm{Thm}(\mathcal{I})$;

2. $\beta_\Sigma[\phi,\chi] \le C(\beta_\Sigma[\phi,\psi],\beta_\Sigma[\psi,\chi])$;

3. $\sigma^\flat_\Sigma(\psi,\vec{\chi}) \in C_\Sigma(\overleftrightarrow{\beta}_\Sigma[\phi,\psi],\sigma^\flat(\phi,\vec{\chi}))$.

Proof: Suppose that, for all $\langle\mathcal{A},T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$, $\beta^{\mathcal{A}}(T)$ is a posystem on $\mathcal{A}$. Then, by Theorem 1826, Conditions 1 and 3 hold. By considering all reduced matrix families $\langle/\Omega(T),T/\Omega(T)\rangle$, with $t \in \mathrm{ThFam}(\mathcal{I})$, we get, by the transitivity of $\beta^{\mathcal{F}/\Omega(T)}(T/\Omega(T))$, $\beta^{\mathcal{F}/\Omega(T)}_\Sigma[\phi/\Omega_\Sigma(T),\psi/\Omega_\Sigma(T)] \le T/\Omega(T)$ and $\beta^{\mathcal{F}/\Omega(T)}_\Sigma[\psi/\Omega_\Sigma(T),\chi/\Omega_\Sigma(T)] \le T/\Omega(T)$ imply $\beta^{\mathcal{F}/\Omega(T)}_\Sigma[\phi/\Omega_\Sigma(T),\chi/\Omega_\Sigma(T)] \le T/\Omega(T)$, i.e., $\beta_\Sigma[\phi,\psi] \le T$ and $\beta_\Sigma[\psi,\chi] \le T$ imply $\beta_\Sigma[\phi,\chi] \le T$. This proves that $\beta_\Sigma[\phi,\chi] \le C(\beta_\Sigma[\phi,\psi],\beta_\Sigma[\psi,\chi])$, i.e., that Condition 2 also holds.

Conversely, suppose Conditions 1-3 hold and let $\langle\mathcal{A},T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$. Then, by Theorem 1826, the relation system $\beta^{\mathcal{A}}(T)$ is reflexive and antisymmetric. But, by Condition 2 of the hypothesis, it is also transitive and, therefore, it is a posystem on $\mathcal{A}$. ∎

Given a π-institution $\mathcal{I} - \langle\mathbf{F},C\rangle$, we term any collection $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$, with two distinguished arguments, that satisfies 1-3 of Corollary 1827 a **semi-equivalence system for $\mathcal{I}$**.

Now we are in a position to provide a characterization of syntactic order algebraizability.

Theorem 1828 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$. The following conditions are equivalent:*

(i) $\mathcal{I}$ is syntactically order algebraizable;

(ii) There exists $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$ having two distinguishes arguments and $\alpha : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ with a single distinguished argument, such that, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi,\chi,\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

- $\beta_\Sigma[\phi,\phi] \le \mathrm{Thm}(\mathcal{I})$;
- $\beta_\Sigma[\phi,\chi] \le C(\beta_\Sigma[\phi,\psi],\beta_\Sigma[\psi,\chi])$;
- $\beta_\Sigma[\sigma^\flat_\Sigma(\psi,\vec{\chi}),\tau^\flat_\Sigma(\psi,\vec{\chi})] \le C(\overleftrightarrow{\beta}_\Sigma[\phi,\psi],\beta_\Sigma[\sigma^\flat_\Sigma(\phi,\vec{\chi}),\tau^\flat_\Sigma(\phi,\vec{\chi})])$;
- $C(\phi) = C(\beta[\alpha_\Sigma[\phi]])$;

(iii) $\mathcal{I}$ has a semi-equivalence system β and there exists $\alpha : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ with a single distinguished argument, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $C(\phi) = C(\beta[\alpha_\Sigma[\phi]])$.

If any of Conditions (i)-(iii) holds, then $\mathcal{I}$ is β-order algebraizable, with β in $N^\flat$ any collection satisfying Condition (ii) or (iii).

Proof:

(i)⇒(ii) Suppose $\mathcal{I}$ is syntactically order algebraizable. Then, by definition, it is equivalent to the inequational π-institution $\mathcal{I}^{\mathsf{K},\le} = \langle \mathbf{F}, C^{\mathsf{K},\le}\rangle$ associated with some class K of $\mathbf{F}$-algebraic posystems, via a conjugate pair $(\alpha,\beta) : \mathcal{I} \rightleftarrows \mathcal{I}^{\mathsf{K},\le}$. Let $\sigma^\flat$ in $N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi,\chi,\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$.

- We have, by Lemma 1811, $\phi \preccurlyeq \phi \in C^{\mathsf{K},\le}_\Sigma(\varnothing)$. Thus, we get $\beta_\Sigma[\phi,\phi] \le C(\varnothing)$.
- Similarly, by Lemma 1811, $\phi \preccurlyeq \chi \in C^{\mathsf{K},\le}_\Sigma(\phi \preccurlyeq \psi, \psi \preccurlyeq \chi)$. Therefore, $\beta_\Sigma[\phi,\chi] \le C(\beta_\Sigma[\phi,\psi],\beta_\Sigma[\psi,\chi])$.
- Since K is a class of $\mathbf{F}$-algebraic posystems, we have

$$\sigma^\flat_\Sigma(\psi,\vec{\chi}) \preccurlyeq \tau^\flat_\Sigma(\psi,\vec{\chi}) \in C^{\mathsf{K},\le}_\Sigma(\phi \preccurlyeq \psi, \psi \preccurlyeq \phi, \sigma^\flat_\Sigma(\phi,\vec{\chi}) \preccurlyeq \tau^\flat_\Sigma(\phi,\vec{\chi})).$$

  From this, we get

$$\beta_\Sigma[\sigma^\flat_\Sigma(\psi,\vec{\chi}),\tau^\flat_\Sigma(\psi,\vec{\chi})] \le C(\overleftrightarrow{\beta}_\Sigma[\phi,\psi],\beta_\Sigma[\sigma^\flat_\Sigma(\phi,\vec{\chi}),\tau^\flat_\Sigma(\phi,\vec{\chi})]).$$

- $C(\phi) = C(\beta[\alpha_\Sigma[\phi]])$ holds by the definition of equivalence.

(ii)⇒(iii) Assume that $\alpha : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ with one distinguished argument and $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ with two distinguished arguments satisfy the Conditions in (ii). According to the definition of a semi-equivalence system, it suffices to show that, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi,\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\sigma^\flat_\Sigma(\psi,\vec{\chi}) \in C_\Sigma(\overleftrightarrow{\beta}_\Sigma[\phi,\psi],\sigma^\flat_\Sigma(\phi,\vec{\chi})).$$

By the last condition in the hypothesis, we have

$$\beta[\alpha_\Sigma[\sigma^\flat_\Sigma(\phi,\vec{\chi})]] \le C(\sigma^\flat_\Sigma(\phi,\vec{\chi})).$$

By the third condition in the hypothesis, we get

$$\beta[\alpha_\Sigma[\sigma^\flat_\Sigma(\psi,\vec{\chi})]] \le C(\overleftrightarrow{\beta}_\Sigma[\phi,\psi],\beta[\alpha_\Sigma[\sigma^\flat_\Sigma(\phi,\vec{\chi})]]).$$

Again, using the last condition in the hypothesis, we get

$$\sigma^\flat_\Sigma(\psi,\vec{\chi}) \in C_\Sigma(\beta[\alpha_\Sigma[\sigma^\flat_\Sigma(\psi,\vec{\chi})]]).$$

Combining these, we get $\sigma^\flat_\Sigma(\psi,\vec{\chi}) \in C_\Sigma(\overleftrightarrow{\beta}_\Sigma[\phi,\psi],\sigma^\flat_\Sigma(\phi,\vec{\chi}))$.

(iii)⇒(i) Suppose $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with two distinguished arguments, is a semi-equivalence system for $\mathcal{I}$ and $\alpha : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ satisfies the condition in (iii). We have to construct a class of

$\mathbf{F}$-algebraic posystems that will serve as the basis for the syntactic order algebraization of $\mathcal{I}$. Consider $\langle\mathcal{A},T\rangle\in\mathrm{MatFam}^*(\mathcal{I})$. Define on $\mathcal{A}$, $\leq^{\mathcal{A},T}=\{\leq^{\mathcal{A},T}_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ by setting, for all $\Sigma\in|\mathbf{Sign}|$, $\phi,\psi\in\mathrm{SEN}(\Sigma)$,

$$\phi\leq^{\mathcal{A},T}_\Sigma\psi\quad\text{iff}\quad\beta^{\mathcal{A}}_\Sigma[\phi,\psi]\leq T.$$

By Corollary 1827, $\leq^{\mathcal{A},T}$ is a posystem on $\mathcal{A}$. Set

$$\mathsf{K}=\{\langle\mathcal{A},\leq^{\mathcal{A},T}\rangle:\langle\mathcal{A},T\rangle\in\mathrm{MatFam}^*(\mathcal{I})\}.$$

It now suffices to show that $\mathcal{I}$ is equivalent to $\mathcal{I}^{\mathsf{K},\leq}$ via the conjugate pair $(\alpha,\beta):\mathcal{I}\rightleftarrows\mathcal{I}^{\mathsf{K},\leq}$. One of the two requirements demanded by Proposition 898 is fulfilled by the hypothesis. It suffices, therefore, to show that, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $I\cup\{\phi\preccurlyeq\psi\}\subseteq\mathrm{In}_\Sigma(\mathbf{F})$,

$$\phi\preccurlyeq\psi\in C^{\mathsf{K},\leq}_\Sigma(I)\quad\text{iff}\quad\beta_\Sigma[\phi,\psi]\leq C(\beta_\Sigma[I]).$$

We have $\phi\preccurlyeq\psi\in C^{\mathsf{K},\leq}_\Sigma(I)$ if and only if, for all $\langle\mathcal{A},\leq^{\mathcal{A},T}\rangle\in\mathsf{K}$, $\Sigma'\in|\mathbf{Sign}^\flat|$ $f\in\mathbf{Sign}^\flat(\Sigma,\Sigma')$,

$$\begin{aligned}&\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(I))\subseteq\leq^{\mathcal{A},T}_{F(\Sigma')}\\&\qquad\text{implies}\quad\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi))\leq^{\mathcal{A},T}_{F(\Sigma')}\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi))\end{aligned}$$

if and only if, for all $\langle\mathcal{A},T\rangle\in\mathrm{MatFam}^*(\mathcal{I})$, all $\Sigma'\in|\mathbf{Sign}^\flat|$ and all $f\in\mathbf{Sign}^\flat(\Sigma,\Sigma')$,

$$\begin{aligned}&\beta[\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(I))]\leq T\\&\qquad\text{implies}\quad\beta[\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)),\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi))]\leq T\end{aligned}$$

if and only if, for all $\langle\mathcal{A},T\rangle\in\mathrm{MatFam}^*(\mathcal{I})$, all $\Sigma'\in|\mathbf{Sign}^\flat|$ and all $f\in\mathbf{Sign}^\flat(\Sigma,\Sigma')$,

$$\begin{aligned}&\alpha(\beta_{\Sigma'}[\mathrm{SEN}^\flat(f)(I)])\leq T\\&\qquad\text{implies}\quad\alpha(\beta_{\Sigma'}[\mathrm{SEN}^\flat(f)(\phi),\mathrm{SEN}^\flat(f)(\psi)])\leq T\end{aligned}$$

iff, by the completeness of $\mathcal{I}$ with respect to $\mathrm{MatFam}^*(\mathcal{I})$, $\beta_\Sigma[\phi,\psi]\leq C(\beta_\Sigma[I])$. ■

This characterization allows us to obtain several properties pertaining to syntactic order algebraizability.

Theorem 1829 *Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, with $\beta:(\mathrm{SEN}^\flat)^\omega\to\mathrm{SEN}^\flat$ in $N^\flat$, having two distinguished arguments, and $\mathcal{I}=\langle\mathbf{F},C\rangle$ a β-order algebraizable π-institution based on $\mathbf{F}$.*

(a) For every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$\Omega^{\mathcal{A}}(T)=\overset{\leftrightarrow}{\beta}{}^{\mathcal{A}}(T);$$

(b) *The β-order class of $\mathcal{I}$ is the semantic order guasivariety generated by* $\mathsf{K} = \{\langle\mathcal{A}, \leq^{\mathcal{A},T}\rangle : \langle\mathcal{A},T\rangle \in \mathrm{MatFam}^*(\mathcal{I})\}$, *where, for all* $\langle\mathcal{A},T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}|$, $\phi,\psi \in \mathrm{SEN}(\Sigma)$,

$$\phi \leq_\Sigma^{\mathcal{A},T} \psi \quad \textit{iff} \quad \beta_\Sigma^{\mathcal{A}}[\phi,\psi] \leq T;$$

(c) *For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$, *the mapping* $T \mapsto \beta^{\mathcal{A}}(T)$ *in injective on* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

Proof:

(a) The conclusion follows from Theorems 1828 and 1826.

(b) This also follows from Theorem 1828.

(c) Suppose $\alpha : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, having one distinguished argument, be as in Theorem 1828 and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system, $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. Then we have

$$\begin{array}{lll} \phi \in T_\Sigma & \text{iff} & \beta^{\mathcal{A}}[\alpha_\Sigma^{\mathcal{A}}[\phi]] \leq T \\ & \text{iff} & \alpha_\Sigma^{\mathcal{A}}[\phi] \leq \beta^{\mathcal{A}}(T), \end{array}$$

and, similarly, $\phi \in T'_\Sigma$ if and only if $\alpha_\Sigma^{\mathcal{A}}[\phi] \leq \beta^{\mathcal{A}}(T')$. We conclude that, if $\beta^{\mathcal{A}}(T) = \beta^{\mathcal{A}}(T')$, then $T = T'$ and, hence, $T \mapsto \beta^{\mathcal{A}}(T)$ is injective.

■

We can now establish some connections between syntactic order algebraizability and syntactic protoalgebraicity.

Proposition 1830 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *has a semi-equivalence system if and only if it is syntactically protoalgebraic;*

(b) *If* $\mathcal{I}$ *is syntactically order algebraizable, then it is syntactically protoalgebraic;*

(c) *If* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$, *with two distinguished arguments, witnesses the syntactic protoalgebraicity of* $\mathcal{I}$ *and* $\mathcal{I}$ *is* β*-order algebraizable, then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C(\overset{\leftrightarrow}{I}{}^\flat_\Sigma[\phi,\psi]) = C(\overset{\leftrightarrow}{\beta}_\Sigma[\phi,\psi]).$$

Proof:

(a) $\mathcal{I}$ has a semi-equivalence system if and only if, by Corollary 1827, for all $\langle\mathcal{A},T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$, $\beta^{\mathcal{A}}(T)$ is a posystem on $\mathcal{A}$, implies that, for all $\langle\mathcal{A},T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$, $\beta^{\mathcal{A}}(T)$ is reflexive and antisymmetric, if and only if, by Theorem 1826, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) = \overrightarrow{\beta}^{\mathcal{A}}(T)$, if and only if $\mathcal{I}$ is syntactically protoalgebraic.

On the other hand, if $\mathcal{I}$ is syntactically protoalgebraic, with witnessing transformations $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$, having two distinguished arguments, then, for all $\langle\mathcal{A},T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$, $\overleftrightarrow{I}^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}$, whence for all $\langle\mathcal{A},T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$, $\overleftrightarrow{I}^{\mathcal{A}}(T)$ is a posystem on $\mathcal{A}$ and, hence, by Corollary 1827, $\overleftrightarrow{I}^\flat$ is a semi-equivalence system for $\mathcal{I}$.

(b) By Part (a) of Theorem 1829.

(c) This follows from the fact that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, $\overleftrightarrow{I}^\flat_\Sigma[\phi,\psi] \leq T$ if and only if $\langle\phi,\psi\rangle \in \Omega(T)$ if and only if $\overleftrightarrow{\beta}_\Sigma[\phi,\psi] \leq T$. ∎

Theorem 1830 reveals two important properties. First, that syntactic order algebraizability implies syntactic protoalgebraicity and, second, that the latter is equivalent to the existence of a semi-equivalence system for $\mathcal{I}$. Recall that syntactic protoalgebraicity is one component in syntactic WF algebraizability. We turn now to investigating how far syntactic WF algebraizability is from syntactic order algebraizability.

Theorem 1831 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, having two distinguished arguments, $\mathcal{I} = \langle\mathbf{F},C\rangle$ a β-order algebraizable π-institution, based on $\mathbf{F}$, and K the β-order class of $\mathcal{I}$. Then, the following conditions are equivalent:*

(i) $\mathcal{I}$ is syntactically WF algebraizable;

(ii) There exists $\gamma : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with two distinguished arguments, such that, for all $\langle\mathcal{A},\leq^{\mathcal{A}}\rangle \in \mathsf{K}$, $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi \in \mathrm{SEN}(\Sigma)$,

$$\phi \preccurlyeq^{\mathcal{A}}_\Sigma \psi \quad \text{iff} \quad \gamma^{\mathcal{A}}_\Sigma[\phi,\psi] \leq \Delta^{\mathcal{A}};$$

(iii) There exists $\gamma : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with two distinguished arguments, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C(\beta_\Sigma[\phi,\psi]) = C(\overleftrightarrow{\beta}[\gamma_\Sigma[\phi,\psi]]);$$

(iv) $\mathcal{I}$ is $S^{\mathcal{I}}$-fortified and for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}}$ is injective on $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$;

(v) $\mathcal{I}$ is $S^{\mathcal{I}}$-fortified and Ω is injective on $\mathrm{ThFam}(\mathcal{I})$.

Proof:

(iv)$\Leftrightarrow$(v) We know that injectivity of the Leibniz operator transfers from Theory families to all filter families over arbitrary algebraic systems.

(i)$\Leftrightarrow$(iv) If $\mathcal{I}$ is syntactically WF algebraizable, then it is $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified, protoalgebraic and family injective. Suppose, conversely, that $\mathcal{I}$ is $S^{\mathcal{I}}$-fortified and family injective. This implies that $\mathcal{I}$ is family truth equational. Together with the syntactic protoalgebraicity following from the hypothesis and Proposition 1830, we get that $\mathcal{I}$ is syntactically WF algebraizable.

(ii)$\Leftrightarrow$(iii) Let $\gamma : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with two distinguished arguments. Suppose, first, that, for all $\langle \mathcal{A}, \leq^{\mathcal{A}}\rangle \in \mathsf{K}$, $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $\phi \leq^{\mathcal{A}}_\Sigma \psi$ iff $\gamma^{\mathcal{A}}_\Sigma[\phi,\psi] \leq \Delta^{\mathcal{A}}$. Then, we have, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\beta_\Sigma[\phi,\psi] \leq T \quad \text{iff} \quad \overleftrightarrow{\beta}[\gamma_\Sigma[\phi,\psi]] \leq T,$$

which yields the conclusion. Conversely, if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $C(\beta_\Sigma[\phi,\psi]) = C(\overleftrightarrow{\beta}[\gamma_\Sigma[\phi,\psi]])$, then, we get

$$C^{\mathsf{K},\leq}(\alpha[\beta_\Sigma[\phi,\psi]]) = C^{\mathsf{K},\leq}(\alpha[\overleftrightarrow{\beta}[\gamma_\Sigma[\phi,\psi]]]).$$

Thus, $C^{\mathsf{K},\leq}(\phi \preccurlyeq \psi) = C^{\mathsf{K},\leq}(\gamma_\Sigma[\phi,\psi] \cup \gamma_\Sigma[\phi,\psi]^{-1})$. This yields the conclusion if we take into account that K consists of $\mathbf{F}$-algebraic posystems.

(i)$\Rightarrow$(ii) Suppose that $\mathcal{I}$ is syntactically WF algebraizable via the conjugate pair $(\tau, I) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathrm{AlgSys}^*(\mathcal{I})}$. Then, by Proposition 1830, we get that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $C(\overleftrightarrow{I}^\flat_\Sigma[\phi,\psi]) = C(\overleftrightarrow{\beta}_\Sigma[\phi,\psi])$. Thus, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $C(\phi) = C(\overleftrightarrow{I}^\flat[\tau^\flat_\Sigma[\phi]]) = C(\overleftrightarrow{\beta}[\tau^\flat_\Sigma[\phi]])$. Therefore, in particular, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C(\beta_\Sigma[\phi,\psi]) = C(\overleftrightarrow{\beta}[\tau^\flat[\beta_\Sigma[\phi,\psi]]]).$$

Now consider any $\langle \mathcal{A}, \leq^{\mathcal{A},T}\rangle$, where $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is such that $\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. Then, we get, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll}\phi \leq^{\mathcal{A},T}_\Sigma \psi & \text{iff} & \beta^{\mathcal{A}}_\Sigma[\phi,\psi] \leq T\\ & \text{iff} & \overleftrightarrow{\beta}^{\mathcal{A}}[\tau^{\mathcal{A}}[\beta^{\mathcal{A}}_\Sigma[\phi,\psi]]] \leq T\\ & \text{iff} & \tau^{\mathcal{A}}[\beta^{\mathcal{A}}_\Sigma[\phi,\psi]] \leq \Delta^{\mathcal{A}}.\end{array}$$

Thus, taking into account the fact that K is the semantic order guasivariety generated by the class $\{\langle \mathcal{A}, \leq^{\mathcal{A},T}\rangle : \langle \mathcal{A}, T\rangle \in \mathrm{MatFam}^*(\mathcal{I})\}$, we conclude that $\gamma := \tau \circ \beta$ is witnessing the property asserted in Part (ii).

(iii)⇒(v) Finally, assume $(\alpha,\beta):\mathcal{I}\rightleftarrows\mathcal{I}^{\mathsf{K},\le}$ witnesses the β-order algebraizability of $\mathcal{I}$ and that $\gamma:(\mathrm{SEN}^\flat)^\omega\to(\mathrm{SEN}^\flat)^2$, with two distinguished arguments satisfies the property in Condition (iii). Let $T\in\mathrm{ThFam}(\mathcal{I})$, $\Sigma\in|\mathbf{Sign}^\flat|$ and $\phi\in\mathrm{SEN}^\flat(\Sigma)$. Then, we have

$$\begin{array}{rll}\phi\in T_\Sigma & \text{iff} & \beta[\alpha_\Sigma[\phi]]\le T\\ & \text{iff} & \overleftrightarrow{\beta}[\gamma[\alpha_\Sigma[\phi]]]\le T\\ & \text{iff} & \gamma[\alpha_\Sigma[\phi]]\le\Omega^{\mathcal{A}}(T).\end{array}$$

This shows that $\mathcal{I}$ is truth equational, which implies that it is $S^{\mathcal{I}}$-fortified and family injective.

■

25.3 Polarities

Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system. A **polarity for F** is a pair $M=(M^+,M^-)$, where M^+ and M^- are subsets of $N^\flat$.

The intuition behind the definition is that

- if $\sigma^\flat\in M^+$, then it is monotone in the first argument and
- if $\sigma^\flat\in M^-$, then it is antimonotone in the first argument.

Why only referring to the first argument? The reason is that it suffices to refer to the first argument to cover all arguments. Suppose, e.g., that $\sigma^\flat:(\mathrm{SEN}^\flat)^2\to\mathrm{SEN}^\flat$ is in $N^\flat$. Then $\sigma^\flat\circ\langle p^{2,1},p^{2,0}\rangle$ is also in $N^\flat$. If we denote $\sigma^\flat$ informally by $\sigma^\flat(x,y)$, then we may denote $\sigma^\flat\circ\langle p^{2,1},p^{2,0}\rangle$ by $\sigma^\flat(y,x)$. Since both transformations are in $N^\flat$, if we wanted to declare that $\sigma^\flat$ is, say, antimonotone in the second argument, then we would assign $\sigma^\flat(y,x)$ in M^-, getting away with referring only to the first argument of some natural transformation in $N^\flat$. The same trick may be used for any argument position and, hence, the expression "$\sigma^\flat$ **has positive** (or **negative polarity**) **in the k-th argument**" should come as no surprise, even though the formal assignment is done only by classifying leading arguments.

Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, with $\beta:(\mathrm{SEN}^\flat)^\omega\to\mathrm{SEN}^\flat$ in $N^\flat$, with two distinguished arguments, and $\mathcal{I}=\langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$. Define the **polarity** $B=(B^+,B^-)$ **induced by** β (the letter B here is chosen to correspond to the transformation β) by setting, for all $\sigma^\flat$ in $N^\flat$:

(+) $\sigma^\flat\in B^+$ if and only if, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\phi,\psi,\vec{\chi}\in\mathrm{SEN}^\flat(\Sigma)$,

$$\sigma^\flat_\Sigma(\psi,\vec{\chi})\in C_\Sigma(\beta_\Sigma[\phi,\psi],\sigma^\flat_\Sigma(\phi,\vec{\chi}));$$

(−) $\sigma^\flat\in B^-$ if and only if, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\phi,\psi,\vec{\chi}\in\mathrm{SEN}^\flat(\Sigma)$,

$$\sigma^\flat_\Sigma(\phi,\vec{\chi})\in C_\Sigma(\beta_\Sigma[\phi,\psi],\sigma^\flat_\Sigma(\psi,\vec{\chi})).$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system, $\leq$ a relation system on $\mathcal{A}$ and $T \in \mathrm{SenFam}(\mathcal{A})$. We say that $\leq$ is M-**compatible with** T if, for all $\sigma^\flat$ in $N^\flat$, $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi, \vec{\chi} \in \mathrm{SEN}(\Sigma)$,

- if $\sigma^\flat \in M^+$, $\phi \leq_\Sigma \psi$, then $\sigma_\Sigma^{\mathcal{A}}(\phi, \vec{\chi}) \in T_\Sigma$ imply $\sigma_\Sigma^{\mathcal{A}}(\psi, \vec{\chi}) \in T_\Sigma$;
- if $\sigma^\flat \in M^-$, $\phi \leq_\Sigma \psi$, then $\sigma_\Sigma^{\mathcal{A}}(\psi, \vec{\chi}) \in T_\Sigma$ imply $\sigma_\Sigma^{\mathcal{A}}(\phi, \vec{\chi}) \in T_\Sigma$.

Proposition 1832 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$. *For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T \in \mathrm{SenFam}(\mathcal{A})$*, there exists a largest qosystem on* $\mathcal{A}$ *that is* M*-compatible with* T.

Proof: We consider the class $\mathrm{QoSys}^{\mathcal{A}}(T)$ of all qosystems on $\mathcal{A}$ that are M-compatible with T. We take the transitive closure of the union of all qosystems in $\mathrm{QoSys}^{\mathcal{A}}(T)$,

$$\mathrm{tc}(\bigcup \mathrm{QoSys}^{\mathcal{A}}(T)) = \{\mathrm{tc}_\Sigma(\bigcup \mathrm{QoSys}^{\mathcal{A}}(T)\}_{\Sigma \in |\mathbf{Sign}|}.$$

It suffices to show that this is also a qosystem on $\mathcal{A}$ M-compatible with T, i.e., it is itself a member of $\mathrm{QoSys}^{\mathcal{A}}(T)$. It will then follow that it is its largest member. It is clear by the definition that $\mathrm{tr}(\bigcup \mathrm{QoSys}^{\mathcal{A}}(T))$ is a qosystem on $\mathcal{A}$. So it suffices to show that it is M-compatible with T. Suppose $\sigma^\flat$ in M^+, $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi, \vec{\chi} \in \mathrm{SEN}(\Sigma)$, such that $\phi \; \mathrm{tr}_\Sigma(\bigcup \mathrm{QoSys}^{\mathcal{A}}(T)) \; \psi$ and $\sigma_\Sigma^\flat(\phi, \vec{\chi}) \in T_\Sigma$. Then, there exist $q^0, \ldots, q^k \in \mathrm{QoSys}^{\mathcal{A}}(T)$ and $\xi_1, \ldots, \xi_k \in \mathrm{SEN}(\Sigma)$, such that

$$\phi \; q_\Sigma^0 \; \xi_1 \; q_\Sigma^1 \; \xi_2 \; q_\Sigma^2 \cdots q_\Sigma^{k-1} \; \xi_k \; q_\Sigma^k \; \psi.$$

Since $\phi \; q_\Sigma^0 \; \xi_1$ and $\sigma_\Sigma^\flat(\phi, \vec{\chi}) \in T_\Sigma$, we get $\sigma_\Sigma^\flat(\xi_1, \vec{\chi}) \in T_\Sigma$. Similarly, since $\xi_1 \; q_\Sigma^1 \; \xi^2$ and $\sigma_\Sigma^\flat(\xi_1, \vec{\chi}) \in T_\Sigma$, we get $\sigma_\Sigma^\flat(\xi_2, \vec{\chi}) \in T_\Sigma$. We move one step to the right at a time in a similar fashion until we obtain $\sigma_\Sigma^\flat(\psi, \vec{\chi}) \in T_\Sigma$. A similar argument is used to handle the case of negative polarity for $\sigma^\flat$. This proves that $\mathrm{tr}(\bigcup \mathrm{QoSys}^{\mathcal{A}}(T)) \in \mathrm{QoSys}^{\mathcal{A}}(T)$ and, therefore, that it is its largest member. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and $T \in \mathrm{SenFam}(\mathcal{A})$. The M-**Leibniz order of** T **on** $\mathcal{A}$ is the largest qosystem $\preccurlyeq^{M,\mathcal{A}}(T)$ on $\mathcal{A}$ that is M-compatible with T.

The next theorem provides a characterization of the M-Leibniz order of a sentence family.

Theorem 1833 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *be a polarity for* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{SenFam}(\mathcal{A})$. *For all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,* $\phi \preccurlyeq_\Sigma^{M,\mathcal{A}}(T) \; \psi$ *if and only if, for all* $\sigma^\flat$ *in* $N^\flat$*,* $\Sigma' \in |\mathbf{Sign}|$*,* $f \in \mathbf{Sign}(\Sigma, \Sigma')$*,* $\vec{\chi} \in \mathrm{SEN}(\Sigma')$*,*

- $\sigma^\flat \in M^+$ *and* $\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}) \in T_{\Sigma'}$ *imply* $\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi}) \in T_{\Sigma'}$*;*
- $\sigma^\flat \in M^-$ *and* $\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi}) \in T_{\Sigma'}$ *imply* $\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}) \in T_{\Sigma'}$*.*

Proof: We let $\preccurlyeq^{\mathcal{A}} = \{\preccurlyeq^{\mathcal{A}}_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ be defined by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi,\psi \in \mathrm{SEN}(\Sigma)$, $\phi \preccurlyeq^{\mathcal{A}}_\Sigma \psi$ if and only if, for all $\sigma^\flat$ in $N^\flat$, $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma,\Sigma')$, $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

- $\sigma^\flat \in M^+$ and $\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}) \in T_{\Sigma'}$ imply $\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi}) \in T_{\Sigma'}$;
- $\sigma^\flat \in M^-$ and $\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi}) \in T_{\Sigma'}$ imply $\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}) \in T_{\Sigma'}$.

Then it it clear that $\preccurlyeq^{\mathcal{A}}$ is a qosystem on $\mathcal{A}$. Moreover, by its definition, it is compatible with T. Hence, by the maximality of the M-Leibniz order of T on $\mathcal{A}$, $\preccurlyeq^{\mathcal{A}} \leq \preccurlyeq^{M,\mathcal{A}}(T)$. On the other hand, if $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\phi \preccurlyeq^{M,\mathcal{A}}_\Sigma(T)\psi$, then, since $\preccurlyeq^{M,\mathcal{A}}(T)$ is a qosystem, we get for all $\Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma,\Sigma')$, $\mathrm{SEN}^\flat(f)(\phi) \preccurlyeq^{M,\mathcal{A}}_{\Sigma'}(T)\mathrm{SEN}(f)(\psi)$. Thus, since $\preccurlyeq^{M,\mathcal{A}}(T)$ is M-compatible with T, we get $\phi \preccurlyeq^{\mathcal{A}}_\Sigma \psi$. Therefore, $\preccurlyeq^{M,\mathcal{A}}(T) \leq \preccurlyeq^{\mathcal{A}}$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. The pair $\langle \mathcal{I}, M\rangle$ is called a **polar π-institution**.

Given an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, the qosystem $\preccurlyeq^{M,\mathcal{A}}(T)$ is called the M-**Leibniz order of** T **on** $\mathcal{A}$. The collection of maps

$$T \mapsto \preccurlyeq^{M,\mathcal{A}}(T), \quad T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}),$$

for all $\mathcal{A}$, constitute the M-**Leibniz order operator** $\preccurlyeq^M$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system. We denote by $O = (O^+, O^-)$ the **total polarity for F**, i.e., the polarity consisting of

$$O^+ = O^- = N^\flat.$$

Corollary 1834 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $O = (O^+, O^-)$ *the total polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. The* O*-Leibniz order operator* $\preccurlyeq^O$ *of* $\mathcal{I}$ *coincides with the Leibniz operator* Ω *of* $\mathcal{I}$*.*

Proof: This follows directly from the definition of O, Theorem 1833 and Theorem 19. ■

Next we give two properties of the operator $\preccurlyeq^M$. The first is commutativity with inverse surjective morphisms and the second is a characterization of monotonicity. Both properties take after similar properties of the Leibniz operator that were established in previous chapters.

Lemma 1835 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$*,* $\mathcal{B} = \langle \mathbf{B}, \langle G,\beta\rangle\rangle$*, all surjective morphisms* $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$*,*

$$\gamma^{-1}(\preccurlyeq^{M,\mathcal{B}}(T)) = \preccurlyeq^{M,\mathcal{A}}(\gamma^{-1}(T)).$$

Proof: Let $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi \in \mathrm{SEN}(\Sigma)$. We have $\phi \preccurlyeq^{M,\mathcal{A}}_\Sigma(\gamma^{-1}(T)\psi$ if and only if, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma,\Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

- if $\sigma^\flat \in M^+$, then $\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}) \in \gamma^{-1}_{\Sigma'}(T_{H(\Sigma')})$ implies $\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi}) \in \gamma^{-1}_{\Sigma'}(T_{H(\Sigma')})$;
- if $\sigma^\flat \in M^-$, then $\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi}) \in \gamma^{-1}_{\Sigma'}(T_{H(\Sigma')})$ implies $\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}) \in \gamma^{-1}_{\Sigma'}(T_{H(\Sigma')})$;

if and only if for all $\sigma^\flat$ in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma,\Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

- if $\sigma^\flat \in M^+$, then $\gamma_{\Sigma'}(\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi})) \in T_{H(\Sigma')}$ implies $\gamma_{\Sigma'}(\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi}) \in T_{H(\Sigma')}$;
- if $\sigma^\flat \in M^-$, then $\gamma_{\Sigma'}(\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi})) \in T_{H(\Sigma')}$ implies $\gamma_{\Sigma'}(\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}) \in T_{H(\Sigma')}$;

if and only if for all $\sigma^\flat$ in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma,\Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

- if $\sigma^\flat \in M^+$, then $\sigma^{\mathcal{B}}_{H(\Sigma')}(\mathrm{SEN}'(H(f))(\gamma_\Sigma(\phi)),\gamma_{\Sigma'}(\vec{\chi})) \in T_{H(\Sigma')}$ implies $\sigma^{\mathcal{B}}_{H(\Sigma')}(\mathrm{SEN}'(H(f))(\gamma_\Sigma(\psi)),\gamma_{\Sigma'}(\vec{\chi})) \in T_{H(\Sigma')}$;
- if $\sigma^\flat \in M^-$, then $\sigma^{\mathcal{B}}_{H(\Sigma')}(\mathrm{SEN}'(H(f))(\gamma_\Sigma(\psi)),\gamma_{\Sigma'}(\vec{\chi})) \in T_{H(\Sigma')}$ implies $\sigma^{\mathcal{B}}_{H(\Sigma')}(\mathrm{SEN}'(H(f))(\gamma_\Sigma(\phi)),\gamma_{\Sigma'}(\vec{\chi})) \in T_{H(\Sigma')}$;

if and only, by the surjectivity of $\langle H,\gamma\rangle$, $\gamma_\Sigma(\phi) \preccurlyeq^{M,\mathcal{B}}_{H(\Sigma)} \gamma_\Sigma(\psi)$ if and only if $\phi\gamma^{-1}_\Sigma(\preccurlyeq^{M,\mathcal{B}}_{H(\Sigma)}(T))\psi$. ∎

Lemma 1836 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. For every $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, $\preccurlyeq^{M,\mathcal{A}}$ is monotone if and only if it commutes with arbitrary intersections.*

Proof: Suppose, first, that $\preccurlyeq^{M,\mathcal{A}}$ is monotone and let $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, by monotonicity, $\preccurlyeq^{M,\mathcal{A}}(\bigcap_{T\in\mathcal{T}} T) \leq \bigcap_{T\in\mathcal{T}} \preccurlyeq^{M,\mathcal{A}}(T)$. On the other hand, $\bigcap_{T\in\mathcal{T}} \preccurlyeq^{M,\mathcal{A}}(T)$ is a qosystem on $\mathcal{A}$, which can be easily seen to be M-compatible with $\bigcap\mathcal{T}$. Thus, by the maximality property of $\preccurlyeq^{M,\mathcal{A}}(\bigcap\mathcal{T})$, we get $\bigcap_{T\in\mathcal{T}} \preccurlyeq^{M,\mathcal{A}}(T) \leq \preccurlyeq^{M,\mathcal{A}}(\bigcap_{T\in\mathcal{T}} T)$. Therefore, the two qosystems are equal and $\preccurlyeq^{M,\mathcal{A}}$ commutes with arbitrary intersections.

Suppose, conversely, $\preccurlyeq^{M,\mathcal{A}}$ commutes with arbitrary intersections and let $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \leq T'$. Then, we have

$$\preccurlyeq^{M,\mathcal{A}}(T) = \preccurlyeq^{M,\mathcal{A}}(T \cap T') = \preccurlyeq^{M,\mathcal{A}}(T) \cap \preccurlyeq^{M,\mathcal{A}}(T'),$$

whence, we get $\preccurlyeq^{M,\mathcal{A}}(T) \leq \preccurlyeq^{M,\mathcal{A}}(T')$ and, therefore, $\preccurlyeq^{M,\mathcal{A}}$ is monotone. ∎

25.4 Directional Systems

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The polar π-institution $\langle \mathcal{I}, M \rangle$ is called **directional** and the π-institution $\mathcal{I}$ is called M-**directional** if there exists $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with two distinguished arguments, such that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\phi \preccurlyeq_\Sigma^{M,\mathcal{A}}(T)\ \psi \quad \text{iff} \quad \beta_\Sigma^{\mathcal{A}}[\phi, \psi] \leq T,$$

The collection β in $N^\flat$ will be called a family of **witnessing transformations for the M-directionality of $\mathcal{I}$**.

Here are a couple of direct consequences of the definition. The first asserts that any two set of witnessing transformations for the M-directionality of a given π-institution are deductively equivalent. The second asserts that M-directionality is preserved under extensions.

Lemma 1837 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\beta, \beta' : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, having two distinguished arguments, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is M-directional with witnessing transformations β and β', then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,*

$$C(\beta_\Sigma[\phi, \psi]) = C(\beta'_\Sigma[\phi, \psi]).$$

Proof: We have, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$ and all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\begin{array}{rll} \beta_\Sigma[\phi, \psi] \leq T & \text{iff} & \phi \preccurlyeq_\Sigma^{M,\mathcal{F}}(T)\ \psi \\ & \text{iff} & \beta'_\Sigma[\phi, \psi] \leq T. \end{array}$$

Therefore, $C(\beta_\Sigma[\phi, \psi]) = C(\beta'_\Sigma[\phi, \psi])$. ∎

Lemma 1838 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, having two distinguished arguments, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$, $\mathcal{I}' = \langle \mathbf{F}, C' \rangle$ two π-institutions based on $\mathbf{F}$. If $\mathcal{I}$ is M-directional with witnessing transformations β and $\mathcal{I} \leq \mathcal{I}'$, then $\mathcal{I}'$ is also M-directional with witnessing transformations β.*

Proof: Suppose $\mathcal{I}$ is M-directional with witnessing transformations β and $\mathcal{I} \leq \mathcal{I}'$. Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$. Then, since every $\mathcal{I}'$-filter family of $\mathcal{A}$ is also an $\mathcal{I}$-filter family, we have, by hypothesis, $\phi \preccurlyeq_\Sigma^{M,\mathcal{A}}(T)\ \psi$ iff $\beta_\Sigma^{\mathcal{A}}[\phi, \psi] \leq T$. We conclude that $\mathcal{I}'$ is also M-directional, with witnessing transformations β. ∎

We give, next, sufficient conditions for the M-directionality of a given π-institution.

Theorem 1839 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$ *having two distinguished arguments, and* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$*, satisfying the following conditions:*

1. $\beta_\Sigma[\phi, \phi] \leq \mathrm{Thm}(\mathcal{I})$*, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*;*
2. $M \leq B$*, where* $B = (B^+, B^-)$ *is the polarity induced by* β*;*
3. *For all* $\sigma^\flat \in \beta$*,* $\sigma^\flat(x, y, \vec{z}) \in M^-$ *or* $\sigma^\flat(y, x, \vec{z}) \in M^+$*.*

Then $\mathcal{I}$ *is* M*-directional, with witnessing transformations* β*.*

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$.

Suppose, first, that $\phi \preccurlyeq_\Sigma^{M,\mathcal{A}}(T)\ \psi$ and $\sigma^\flat \in \beta$. Then, by Condition 3, either $\sigma^\flat$ is of negative M-polarity in the first argument or of positive M-polarity in the second argument.

- Assume $\sigma^\flat$ has negative polarity in the first argument. By Condition 1, we have $\sigma_\Sigma^{\mathcal{A}}[\psi, \psi] \leq T$. Therefore, by Condition 2 and the hypothesis, we get $\sigma_\Sigma^{\mathcal{A}}[\phi, \psi] \leq T$.
- Assume $\sigma^\flat$ has positive polarity in the second argument. By Condition 1, we have $\sigma_\Sigma^{\mathcal{A}}[\phi, \phi] \leq T$. Therefore, by Condition 2 and the hypothesis, we get $\sigma_\Sigma^{\mathcal{A}}[\phi, \psi] \leq T$.

In either case $\sigma_\Sigma^{\mathcal{A}}[\phi, \psi] \leq T$, whence, $\beta_\Sigma^{\mathcal{A}}[\phi, \psi] \leq T$.

Assume, conversely, that $\beta_\Sigma^{\mathcal{A}}[\phi, \psi] \leq T$. Let $\sigma^\flat$ in $N^\flat$, viewed as having one distinguished argument.

- If $\sigma^\flat \in M^+$, then, by Condition 2, $\sigma^\flat \in B^+$. Hence, by definition of B and the hypothesis, $\sigma_\Sigma^{\mathcal{A}}[\phi] \leq T$ implies $\sigma_\Sigma^{\mathcal{A}}[\psi] \leq T$.
- If $\sigma^\flat \in M^-$, then, by Condition 2, $\sigma^\flat \in B^-$. Hence, by definition of B and the hypothesis, $\sigma_\Sigma^{\mathcal{A}}[\psi] \leq T$ implies $\sigma_\Sigma^{\mathcal{A}}[\phi] \leq T$.

Therefore, by Theorem 1833, we conclude that $\phi \preccurlyeq_\Sigma^{M,\mathcal{A}}(T)\ \psi$. Hence, $\mathcal{I}$ is M-directional with witnessing transformations β. ∎

Now we look at some properties of M-directional π-institutions.

Theorem 1840 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$ *having two distinguished arguments,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is* M*-directional, with witnessing transformations* β*, then the following properties hold:*

(a) For all $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\beta_\Sigma[\phi, \phi] \leq \mathrm{Thm}(\mathcal{I}) \quad \text{and} \quad \beta_\Sigma[\phi, \chi] \leq C(\beta_\Sigma[\phi, \psi], \beta_\Sigma[\psi, \chi]);$$

(b) $M \leq B$, *where* B *is the polarity for* $\mathbf{F}$ *induced by* β*;*

(c) *For all* $\sigma^\flat \in \beta$,

$$\sigma^\flat(x,y,\vec{z}) \in B^- \quad \text{and} \quad \sigma^\flat(y,x,\vec{z}) \in B^+;$$

(d) $\mathcal{I}$ *is* B*-directional, with witnessing transformations* β*;*

(e) *For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$, $\preccurlyeq^{M,\mathcal{A}} = \preccurlyeq^{B,\mathcal{A}}$*;*

(f) B *is the largest polarity* M' *for* $\mathbf{F}$*, such that* $\preccurlyeq^{M'} = \preccurlyeq^M$.

Proof:

(a) Since $\preccurlyeq^M$ is a qosystem, it is reflexive and transitive. Thus, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi,\chi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{gathered}\phi \preccurlyeq^{M,\mathcal{F}}_\Sigma (\mathrm{Thm}(\mathcal{I}))\ \phi,\\ \phi \preccurlyeq^{M,\mathcal{F}}_\Sigma (T)\ \psi \text{ and } \psi \preccurlyeq^{M,\mathcal{F}}_\Sigma (T)\ \chi \text{ imply } \phi \preccurlyeq^{M,\mathcal{F}}_\Sigma (T)\ \chi.\end{gathered}$$

Hence, by M-directionality, we get $\beta_\Sigma[\phi,\phi] \leq \mathrm{Thm}(\mathcal{I})$ and

$$\beta_\Sigma[\phi,\psi] \leq T \text{ and } \beta_\Sigma[\psi,\chi] \leq T \text{ imply } \beta_\Sigma[\phi,\chi] \leq T.$$

The latter gives $\beta_\Sigma[\phi,\chi] \leq C(\beta_\Sigma[\phi,\psi],\beta_\Sigma[\psi,\chi])$.

(b) Suppose $\sigma^\flat \in M^+$ and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi,\vec{\chi} \in \mathrm{SEN}(\Sigma)$, such that

$$\beta^{\mathcal{A}}_\Sigma[\phi,\psi] \leq T \quad \text{and} \quad \sigma^{\mathcal{A}}_\Sigma(\phi,\vec{\chi}) \in T_\Sigma.$$

By M-directionality, we get $\phi \preccurlyeq^{M,\mathcal{A}}_\Sigma (T)\ \psi$ and $\sigma^{\mathcal{A}}_\Sigma(\phi,\vec{\chi}) \in T_\Sigma$. Thus, since $\sigma^\flat \in M^+$, we get $\sigma^{\mathcal{A}}_\Sigma(\psi,\vec{\chi}) \in T_\Sigma$. We conclude that $\sigma^\flat \in B^+$ and, hence, $M^+ \subseteq B^+$. Similarly, we get that $M^- \subseteq B^-$ and, therefore, $M \leq B$.

(c) This follows directly by the second assertion of Part (a) and the definition of B.

(d) This follows from Parts (a), (c) and Theorem 1839.

(e) By the hypothesis and Part (d), we have, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and all $\phi,\psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{aligned}\phi \preccurlyeq^{M,\mathcal{A}}_\Sigma (T)\ \psi \quad &\text{iff} \quad \beta^{\mathcal{A}}_\Sigma[\phi,\psi] \leq T\\ &\text{iff} \quad \phi \preccurlyeq^{B,\mathcal{A}}_\Sigma (T)\ \psi.\end{aligned}$$

Therefore, $\preccurlyeq^{M,\mathcal{A}} = \preccurlyeq^{B,\mathcal{A}}$.

(f) We have that $\preccurlyeq^{M'} = \preccurlyeq^M$ if and only β witnesses the M'-directionality of $\mathcal{I}$. This implies, by Part (b), that $M' \leq B$.

■

We now obtain the following characterization of the existence of a polarity M for which $\mathcal{I}$ is M-directional with a predetermined set $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ of natural transformations in $N^\flat$ as its witnessing set.

Corollary 1841 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$ *having two distinguished arguments, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following conditions are equivalent:*

(i) There exists a polarity $M = (M^+, M^-)$ *for* $\mathbf{F}$*, such that* $\mathcal{I}$ *is* M*-directional with witnessing transformations* β*;*

(ii) For all $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\beta_\Sigma[\phi, \phi] \leq \mathrm{Thm}(\mathcal{I}) \quad \textit{and} \quad \beta_\Sigma[\phi, \chi] \leq C(\beta_\Sigma[\phi, \psi], \beta_\Sigma[\psi, \chi]);$$

(iii) $\mathcal{I}$ *is* B*-directional with witnessing transformations* β*.*

Proof: If Condition (i) holds, then Part (a) of Theorem 1840 ensures that Condition (ii) holds. If Condition (ii) holds, then, we get Part (c) of Theorem 1840 and, from Part (a) (our hypothesis) and Part (c) of Theorem 1840, we get, using Theorem 1839, Part (d) of Theorem 1840, which is Condition (iii). Finally, if (iii) holds, then B is a polarity on $\mathbf{F}$, such that $\mathcal{I}$ is B-directional, with witnessing transformations β and, thus, Condition (i) holds. ■

Our results allow us to show that families of collections of natural transformations in $N^\flat$ with two distinguished arguments, satisfying Condition (ii) of Corollary 1841 and polarities on $\mathbf{F}$ are in correspondence under appropriate identifications of deductively equivalent collections of transformations and of polarities giving rise to the same Leibniz order operators.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

- Let $\mathfrak{B}(\mathcal{I})$ be the collection of all families $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with two distinguished arguments, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$,

  $$\beta_\Sigma[\phi, \phi] \leq \mathrm{Thm}(\mathcal{I}) \quad \text{and} \quad \beta_\Sigma[\phi, \chi] \leq C(\beta_\Sigma[\phi, \psi], \beta_\Sigma[\psi, \chi]).$$

  Moreover, we declare two collections $\beta, \beta' \in \mathfrak{B}(\mathcal{I})$ to be equivalent, written $\beta \equiv^{\mathcal{I}} \beta'$ if and only if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

  $$C(\beta_\Sigma[\phi, \psi]) = C(\beta'_\Sigma[\phi, \psi]).$$

- Let $\mathfrak{M}(\mathcal{I})$ be the collection of all polarities for $\mathbf{F}$, such that $\mathcal{I}$ is M-directional.

  Moreover, we declare two polarities M, M' in $\mathfrak{M}(\mathcal{I})$ to be equivalent, written $M \sim^{\mathcal{I}} M'$, if and only if $\preccurlyeq^M = \preccurlyeq^{M'}$.

Then we have the following correspondence.

Theorem 1842 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *There exists a bijection from* $\mathcal{B}(\mathcal{I})/{\equiv^{\mathcal{I}}}$ *onto* $\mathfrak{M}(\mathcal{I})/{\sim^{\mathcal{I}}}$, *such that*

$$\beta/{\equiv^{\mathcal{I}}} \mapsto B/{\sim^{\mathcal{I}}}, \quad \beta \in \mathfrak{B}(\mathcal{I}),$$

and such that every $\beta' \in \mathfrak{B}(\mathcal{I})$, *such that* $\beta' \equiv^{\mathcal{I}} \beta$, *witnesses the* M*-directionality of* $\mathcal{I}$, *for all* $M \in \mathfrak{M}(\mathcal{I})$, *such that* $M \sim^{\mathcal{I}} B$.

Proof: Let $\beta, \beta' \in \mathfrak{B}(\mathcal{I})$, such that $\beta \equiv^{\mathcal{I}} \beta'$. Then, for every $\mathbf{F}$-algebraic system, all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll} \phi \preccurlyeq_\Sigma^{B,\mathcal{A}}(T)\ \psi & \text{iff} & \beta_\Sigma[\phi,\psi] \leq T \\ & \text{iff} & \beta'_\Sigma[\phi,\psi] \leq T \\ & \text{iff} & \phi \preccurlyeq_\Sigma^{B',\mathcal{A}}\ \psi. \end{array}$$

Thus, $B \sim^{\mathcal{I}} B'$ and the mapping in the statement of the theorem is well-defined.

By definition of $\mathfrak{M}(\mathcal{I})$ and Theorem 1840, it is onto.

Finally, if $\beta, \beta' \in \mathfrak{B}(\mathcal{I})$, such that $B \sim^{\mathcal{I}} B'$, then, by definition, $\preccurlyeq^B = \preccurlyeq^{B'}$. Thus, for all $\langle \mathcal{A}, T \rangle \in \mathrm{MatFam}^*(\mathcal{I})$, we get, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll} \beta_\Sigma[\phi,\psi] \leq T & \text{iff} & \phi \preccurlyeq_\Sigma^{B,\mathcal{A}}(T)\ \psi \\ & \text{iff} & \phi \preccurlyeq_\Sigma^{B',\mathcal{A}}(T)\ \psi \\ & \text{iff} & \beta'_\Sigma[\phi,\psi] \leq T. \end{array}$$

Therefore, by the completeness of $\mathcal{I}$ with respect to $\mathrm{MatFam}^*(\mathcal{I})$, we get that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $C(\beta_\Sigma[\phi,\psi]) = C(\beta'_\Sigma[\phi,\psi])$, i.e., $\beta \equiv^{\mathcal{I}} \beta'$ and, therefore, the map in the statement of the theorem is also injective. ∎

Corollary 1843 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$ *having two distinguished arguments, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* β *is a semi-equivalence system for* $\mathcal{I}$ *(in particular, if* $\mathcal{I}$ *is* β*-order algebraizable), then* $\mathcal{I}$ *is* B*-directional, with witnessing transformations* β.

Proof: By Theorem 1828 and Corollary 1841. ∎

25.5 Monotonicity and Directionaility

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I}$ be a π-institution based on $\mathbf{F}$.

- We say that $\mathcal{I}$ is M-**order monotone** if the M-Leibniz order operator $\preccurlyeq^M$ is monotone.
- We say that $\mathcal{I}$ is M-**directional** if there exists $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, having two distinguished arguments, such that, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,
$$\phi \preccurlyeq^{M,\mathcal{A}}_\Sigma(T)\ \psi \quad \text{iff} \quad \beta^{\mathcal{A}}_\Sigma[\phi, \psi] \leq T.$$

Our goal is to connect these two notions.

We have the following obvious relationship.

Theorem 1844 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I}$ be a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is M-directional, then $\mathcal{I}$ is M-order monotone.*

Proof: Suppose $\mathcal{I}$ is M-directional, with witnessing transformations β. Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \leq T'$. Then, we get, by the M-directionality of $\mathcal{I}$, we get

$$\preccurlyeq^{M,\mathcal{A}}(T) = \beta(T) \leq \beta(T') = \preccurlyeq^{M,\mathcal{A}}(T').$$

Therefore, $\mathcal{I}$ is M-order monotone. ■

We introduce a collection of natural transformations associated with $\mathcal{I}$ that play in the present context a role analog to the role that the reflexive core $R^{\mathcal{I}}$ played in the case of syntactic protoalgebraicity. In fact the collection we introduce is a subcollection of the reflexive core of a π-institution $\mathcal{I}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The M-**quasicore** $Q^{\mathcal{I},M}$ of $\mathcal{I}$ is the collection

$$\begin{array}{rcl} Q^{\mathcal{I},M} & = & \{\kappa^\flat \in N^\flat : \kappa^\flat(x, y, \vec{z}) \in M^- \text{ and } \kappa^\flat(y, x, \vec{z}) \in M^+ \text{ and} \\ & & (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\kappa^\flat_\Sigma[\phi, \phi] \leq \mathrm{Thm}(\mathcal{I}))\}. \end{array}$$

It turns out that, if $\mathcal{I}$ is M-directional with witnessing transformations β, then $\beta \subseteq Q^{\mathcal{I},M}$.

Lemma 1845 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$ having two distinguished arguments, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is M-directional, with witnessing transformations β, then $\beta \subseteq Q^{\mathcal{I},M}$.*

Proof: The conclusion follows directly from Parts (a) and (c) of Theorem 1840 and the definition of $Q^{\mathcal{I},M}$. ∎

The M-directionality of a π-institution $\mathcal{I}$ guarantees that the M-quasicore of $\mathcal{I}$ has the global family modus ponens.

Theorem 1846 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is M-directional, then $Q^{\mathcal{I},M}$ has the global family modus ponens.*

Proof: Suppose $\mathcal{I}$ is M-directional with witnessing transformations β and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $Q^{\mathcal{I},M}_\Sigma[\phi,\psi] \leq T$. Then, by Lemma 1845, $\phi \in T_\Sigma$ and $\beta_\Sigma[\phi,\psi] \leq T$. By M-directionality, $\phi \in T_\Sigma$ and $\phi \preccurlyeq^{M,\mathcal{F}}_\Sigma(T)\ \psi$. Therefore, by the definition of $\preccurlyeq^{M,\mathcal{F}}(T)$, $\psi \in T_\Sigma$. We conclude that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\psi \in C_\Sigma(Q^{\mathcal{I},M}_\Sigma[\phi,\psi], \phi),$$

i.e., $Q^{\mathcal{I},M}$ has the global family modus ponens in $\mathcal{I}$. ∎

Conversely, it turns out that, if the M-quasicore $Q^{\mathcal{I},M}$ of $\mathcal{I}$ has the global family modus ponens, then $\mathcal{I}$ is M-directional, with $Q^{\mathcal{I},M}$ as its set of witnessing transformations.

Theorem 1847 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $Q^{\mathcal{I},M}$ has the global family modus ponens in $\mathcal{I}$, then $\mathcal{I}$ is M-directional with witnessing transformations $Q^{\mathcal{I},M}$.*

Proof: We must show that, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\preccurlyeq^{M,\mathcal{A}}(T) = Q^{\mathcal{I},M,\mathcal{A}}(T)$.

Let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \preccurlyeq^{M,\mathcal{A}}_\Sigma(T)\ \psi$ and $\sigma \in Q^{\mathcal{I},M}$. Then, by the definition of the M-quasicore, $\sigma^{\mathcal{A}}_\Sigma[\psi,\psi] \leq T$. Since $\phi \preccurlyeq^{M,\mathcal{A}}_\Sigma(T)\ \psi$ and $\sigma^\flat \in M^-$, we get that $\sigma^{\mathcal{A}}_\Sigma[\phi,\psi] \leq T$. Therefore, $Q^{\mathcal{I},M,\mathcal{A}}_\Sigma[\phi,\psi] \leq T$, which gives $\langle \phi,\psi \rangle \in Q^{\mathcal{I},M,\mathcal{A}}_\Sigma(T)$. Thus, $\preccurlyeq^{M,\mathcal{A}}(T) \leq Q^{\mathcal{I},M,\mathcal{A}}(T)$.

Conversely, to see that $Q^{\mathcal{I},M,\mathcal{A}}(T) \leq \preccurlyeq^{M,\mathcal{A}}(T)$, it suffices to show that $Q^{\mathcal{I},M,\mathcal{A}}(T)$ is a qosystem on $\mathcal{A}$ that is M-compatible with T.

- By definition of the M-quasicore, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $Q^{\mathcal{I},M,\mathcal{A}}_\Sigma[\phi,\phi] \leq T$, whence $Q^{\mathcal{I},M,\mathcal{A}}(T)$ is reflexive.
- Next let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi, \chi \in \mathrm{SEN}(\Sigma)$, such that $Q^{\mathcal{I},M,\mathcal{A}}_\Sigma[\phi,\psi] \leq T$ and $Q^{\mathcal{I},M,\mathcal{A}}_\Sigma[\psi,\chi] \leq T$. For $\sigma^\flat, \tau^\flat \in Q^{\mathcal{I},M}$, note that, by the definition of the M-quasicore, the transformation $\tau^\flat(\sigma^\flat(z,x,\vec{p}), \sigma^\flat(z,y,\vec{p}), \vec{q}) \in$

$Q^{\mathcal{I},M}$. Hence, using modus ponens, we get, for all $\sigma^\flat \in Q^{\mathcal{I},M}$ and all $\vec{\xi} \in \mathrm{SEN}(\Sigma')$,

$$\begin{array}{l}\sigma_\Sigma^{\mathcal{A}}(\phi,\chi,\vec{\xi})\\ \in C_{\Sigma}^{\mathcal{I},\mathcal{A}}(Q_{\Sigma'}^{\mathcal{I},M,\mathcal{A}}[\sigma_\Sigma^{\mathcal{A}}(\phi,\psi,\vec{\xi}),\sigma_\Sigma^{\mathcal{A}}(\phi,\chi,\vec{\xi})],\sigma_\Sigma^{\mathcal{A}}(\phi,\psi,\vec{\xi}))\\ \leq C_{\Sigma'}^{\mathcal{I},\mathcal{A}}(Q_{\Sigma'}^{\mathcal{I},M,\mathcal{A}}[\psi,\chi],\sigma_\Sigma^{\mathcal{A}}(\phi,\psi,\vec{\xi})).\end{array}$$

We conclude that $Q_\Sigma^{\mathcal{I},M,\mathcal{A}}[\phi,\chi] \leq C^{\mathcal{I},\mathcal{A}}(Q_\Sigma^{\mathcal{I},M,\mathcal{A}}[\phi,\psi],Q_\Sigma^{\mathcal{I},M,\mathcal{A}}[\psi,\chi])$ and, therefore, $Q^{\mathcal{I},M,\mathcal{A}}(T)$ is also transitive.

- Suppose, next, that $\sigma^\flat \in M^+$, $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi,\vec{\chi} \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle \in Q_\Sigma^{\mathcal{I},M,\mathcal{A}}(T)$ and $\sigma_\Sigma^\flat(\phi,\vec{\chi}) \in T_\Sigma$. Then $Q_\Sigma^{\mathcal{I},M,\mathcal{A}}[\phi,\psi] \leq T$ and $\sigma^{\mathcal{A}}(\phi,\vec{\chi}) \in T_\Sigma$. So we get

$$\begin{array}{rcl}\sigma_\Sigma^{\mathcal{A}}(\psi,\vec{\chi}) & \in & C_\Sigma^{\mathcal{I},\mathcal{A}}(Q_\Sigma^{\mathcal{I},M,\mathcal{A}}[\sigma_\Sigma^{\mathcal{A}}(\phi,\vec{\chi}),\sigma_\Sigma^{\mathcal{A}}(\psi,\vec{\chi})],\sigma_\Sigma^{\mathcal{A}}(\phi,\vec{\chi}))\\ & \subseteq & C^{\mathcal{I},\mathcal{A}}(Q_\Sigma^{\mathcal{I},M,\mathcal{A}}[\phi,\psi],\sigma_\Sigma^{\mathcal{A}}(\phi,\vec{\chi}))\\ & \subseteq & T_\Sigma.\end{array}$$

  Similarly, consider $\sigma^\flat \in M^-$, $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi,\vec{\chi} \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle \in Q_\Sigma^{\mathcal{I},M,\mathcal{A}}(T)$ and $\sigma_\Sigma^\flat(\psi,\vec{\chi}) \in T_\Sigma$. Then $Q_\Sigma^{\mathcal{I},M,\mathcal{A}}[\phi,\psi] \leq T$ and $\sigma^{\mathcal{A}}(\psi,\vec{\chi}) \in T_\Sigma$. So we get

$$\begin{array}{rcl}\sigma_\Sigma^{\mathcal{A}}(\phi,\vec{\chi}) & \in & C_\Sigma^{\mathcal{I},\mathcal{A}}(Q_\Sigma^{\mathcal{I},M,\mathcal{A}}[\sigma_\Sigma^{\mathcal{A}}(\psi,\vec{\chi}),\sigma_\Sigma^{\mathcal{A}}(\phi,\vec{\chi})],\sigma_\Sigma^{\mathcal{A}}(\psi,\vec{\chi}))\\ & \subseteq & C^{\mathcal{I},\mathcal{A}}(Q_\Sigma^{\mathcal{I},M,\mathcal{A}}[\phi,\psi],\sigma_\Sigma^{\mathcal{A}}(\psi,\vec{\chi}))\\ & \subseteq & T_\Sigma.\end{array}$$

  Thus, $Q^{\mathcal{I},M,\mathcal{A}(T)}$ is M-compatible with T.

We conclude that $Q^{\mathcal{I},M,\mathcal{A}}(T)$ is a qosystem on $\mathcal{A}$ that is M-compatible with T, whence, by the maximality of $\preccurlyeq^{M,\mathcal{A}}(T)$, we get $Q^{\mathcal{I},M,\mathcal{A}}(T) \leq \preccurlyeq^{M,\mathcal{A}}(T)$. ∎

We now have a characterization of M-directionality in terms of the property of modus ponens of the M-quasicore $Q^{\mathcal{I},M}$ of the π-institution $\mathcal{I}$.

$$\mathcal{I} \text{ is } M\text{-directional} \longleftrightarrow Q^{\mathcal{I},M} \text{ has Global Family MP.}$$

Theorem 1848 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is* M*-directional if and only if* $Q^{\mathcal{I},M}$ *has the global family modus ponens in* $\mathcal{I}$.

Proof: Theorem 1846 gives the "only if" and the "if" is by Theorem 1847. ∎

As a corollary, we obtain

Corollary 1849 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$ *having two distinguished arguments,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is* M*-directional with witnessing transformations* β*, then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$C(Q_\Sigma^{\mathcal{I},M}[\phi,\psi]) = C(\beta_{\Sigma|}[\phi,\psi]).$$

Proof: If $\mathcal{I}$ is M-directional, with witnessing transformations β, then, by Theorems 1846 and 1847, both β and $Q^{\mathcal{I},M}$ are families of witnessing transformations for the M-directionality of $\mathcal{I}$. Therefore, by Lemma 1837, we get the conclusion. ∎

We get relatively easily another related characterization of M-directionality.

$\mathcal{I}$ is M-directional $\longleftrightarrow$ $Q^{\mathcal{I},M}$ Defines M-Leibniz QoSystems.

Theorem 1850 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is* M*-directional if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$\preccurlyeq^{M,\mathcal{A}}(T) = Q^{\mathcal{I},M,\mathcal{A}}(T).$$

Proof: If $\mathcal{I}$ is M-directional, then, by Theorem 1846 and Theorem 1847, $Q^{\mathcal{I},M}$ constitutes a collection of witnessing transformations, whence, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ $\preccurlyeq^{M,\mathcal{A}}(T) = Q^{\mathcal{I},M,\mathcal{A}}(T)$.

The converse follows by the definition of M-directionality, since, in that case, $Q^{\mathcal{I},M}$ forms a collection of witnessing transformations. ∎

We finally show that the property that separates M-order monotonicity from M-directionality is the M-order compatibility property with respect to the theory family generated by the M-quasicore.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. In analogy with the property of the reflexive core being Leibniz, we say that the M-quasicore $Q^{\mathcal{I},M}$ is **order Leibniz** if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\phi \preccurlyeq_\Sigma^{M,\mathcal{A}}(C^{\mathcal{I},\mathcal{A}}(Q_\Sigma^{\mathcal{I},M,\mathcal{A}}[\phi,\psi]))\ \psi.$$

This property is weaker than $Q^{\mathcal{I},M}$ having the global family modus ponens, i.e., if $Q^{\mathcal{I},M}$ has the global family modus ponens, then it is order Leibniz.

Proposition 1851 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. If* $Q^{\mathcal{I},M}$ *has the global family modus ponens, then it is order Leibniz.*

Proof: If $Q^{\mathcal{I},M}$ has the global family modus ponens, then, by Theorem 1847, we get, for every **F**-algebraic system $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$\preccurlyeq^{M,\mathcal{A}}(T) = Q^{\mathcal{I},M,\mathcal{A}}(T).$$

Therefore, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, by considering, in particular, $T = C^{\mathcal{I},\mathcal{A}}(Q_\Sigma^{\mathcal{I},M,\mathcal{A}}[\phi,\psi])$, and taking into account that $Q_\Sigma^{\mathcal{I},M,\mathcal{A}}[\phi,\psi] \leq C^{\mathcal{I},\mathcal{A}}(Q_\Sigma^{\mathcal{I},M,\mathcal{A}}[\phi,\psi])$, we get that $\phi \preccurlyeq_\Sigma^{M,\mathcal{A}}(C^{\mathcal{I},\mathcal{A}}(Q_\Sigma^{\mathcal{I},M,\mathcal{A}}[\phi,\psi]))\ \psi$. Thus, $Q^{\mathcal{I},M}$ is order Leibniz. ∎

In the opposite direction, in an M-order monotone π-institution $\mathcal{I}$, if the M-quasicore is order Leibniz, then it has the global family modus ponens in $\mathcal{I}$.

Proposition 1852 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* **F** *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be an* M*-order monotone* π*-institution based on* **F**. *If* $Q^{\mathcal{I},M}$ *is order Leibniz, then it has the global family modus ponens in* $\mathcal{I}$.

Proof: Suppose that $\mathcal{I}$ is M-order monotone and that $Q^{\mathcal{I},M}$ is order Leibniz. Let $\mathcal{A}$ be an **F**-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in T_\Sigma$ and $Q_\Sigma^{\mathcal{I},M,\mathcal{A}}[\phi,\psi] \leq T$. Since $Q^{\mathcal{I},M}$ is order Leibniz, we have

$$\phi \preccurlyeq_\Sigma^{M,\mathcal{A}}(C(Q_\Sigma^{\mathcal{I},M,\mathcal{A}}[\phi,\psi]))\ \psi,$$

whence, since $\mathcal{I}$ is M-order monotone and $Q_\Sigma^{\mathcal{I},M,\mathcal{A}}[\phi,\psi] \leq T$,

$$\phi \preccurlyeq_\Sigma^{M,\mathcal{A}}(T)\ \psi.$$

Therefore, since $\phi \in T_\Sigma$, we get, by M-compatibility of $\preccurlyeq^{M,\mathcal{A}}(T)$ with T, that $\psi \in T_\Sigma$. We conclude that $Q^{\mathcal{I},M}$ has the global family modus ponens in $\mathcal{I}$. ∎

We now show that a π-institution is M-directional if and only if it is M-order monotone and it has an order Leibniz M-quasicore.

$$\begin{array}{rcl} M\text{-Directionality} & = & Q^{\mathcal{I},M} \text{ has Global Family MP} \\ & = & Q^{\mathcal{I},M} \text{ Defines Leibniz QoSystems} \\ & = & M\text{-Order Monotonicity} + Q^{\mathcal{I},M} \text{ Order Leibniz} \end{array}$$

Theorem 1853 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *be a polarity for* **F** *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* **F**. $\mathcal{I}$ *is* M*-directional if and only if it is* M*-order monotone and has an order Leibniz* M*-quasicore.*

Proof: Suppose, first, that $\mathcal{I}$ is M-directional. Then it is M-order monotone by Theorem 1844. Moreover, its M-quasicore has the global family modus ponens by Theorem 1846 and, hence, by Proposition 1851, its M-quasicore is order Leibniz.

Suppose, conversely, that $\mathcal{I}$ is M-order monotone with an order Leibniz M-quasicore. Then, by Proposition 1852, its M-quasicore has the global family modus ponens and, therefore, by Theorem 1848, $\mathcal{I}$ is M-directional. ∎

25.6 c-Reflectivity and Truth Inequationality

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The polar π-institution $\langle \mathcal{I}, M \rangle$ is called **truth inequational** and the π-institution $\mathcal{I}$ is called M-**truth inequational** if there exists a collection $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with a single distinguished argument, such that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \le \preccurlyeq^{M,\mathcal{F}}(T).$$

In this case $\tau^\flat$ is called a family of **witnessing transformations for the** M-**truth inequationality of** $\mathcal{I}$.

We can show, based on preceding work, that every β-order algebraizable π-institution $\mathcal{I}$ is B-truth inequational, where B is the polarity induced by β.

Proposition 1854 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, having two distinguished arguments, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is* β*-order algebraizable, then* $\mathcal{I}$ *is* B*-truth inequational.*

Proof: Suppose $\mathcal{I}$ is β-order algebraizable. Then, by Corollary 1843, it is B-directional, with witnessing transformations β. Thus, by Theorem 1828, there exists $\alpha : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$, with a single distinguished argument, such that, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll} \phi \in T_\Sigma & \text{iff} & \beta^{\mathcal{A}}[\alpha^{\mathcal{A}}_\Sigma[\phi]] \le T \\ & \text{iff} & \alpha^{\mathcal{A}}_\Sigma[\phi] \le \preccurlyeq^{B,\mathcal{A}}(T). \end{array}$$

Thus, $\mathcal{I}$ is B-truth inequational, with witnessing transformations α. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ be a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. We say that

$\preccurlyeq^M$ is **completely order reflecting** or **c-reflecting**, for short, if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,

$$\bigcap_{T\in\mathcal{T}} \preccurlyeq^{M,\mathcal{F}}(T) \leq \preccurlyeq^{M,\mathcal{F}}(T') \quad \text{implies} \quad \bigcap_{T\in\mathcal{T}} T \leq T'.$$

If this is the case, we call $\mathcal{I}$ M-**c-reflective**.

We formulate an equivalent condition to M-c-reflectivity.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Given an $\mathbf{F}$-algebraic system $\mathcal{A}$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we define the qosystem

$$\widetilde{\preccurlyeq}^{M,\mathcal{A}}(T) = \bigcap\{\preccurlyeq^{M,\mathcal{A}}(T') : T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}.$$

By analogy with the Suszko congruence system, we call $\widetilde{\preccurlyeq}^{M,\mathcal{A}}(T)$ the M-**Suszko qosystem of** T.

Lemma 1855 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is M-c-reflective if and only if, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,*

$$\widetilde{\preccurlyeq}^{M,\mathcal{F}}(T) \leq \preccurlyeq^{M,\mathcal{F}}(T') \quad \textit{implies} \quad T \leq T'.$$

Proof: Assume, first, that $\mathcal{I}$ is M-c-reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{\preccurlyeq}^{M,\mathcal{F}}(T) \leq \preccurlyeq^{M,\mathcal{F}}(T')$. Then, we have

$$\bigcap\{\preccurlyeq^{M,\mathcal{F}}(T'') : T \leq T'' \in \mathrm{ThFam}(\mathcal{I})\} \leq \preccurlyeq^{M,\mathcal{F}}(T').$$

Therefore, by M-c-reflectivity, $\bigcap\{T'' : T \leq T'' \in \mathrm{ThFam}(\mathcal{I})\} \leq T'$, i.e., $T \leq T'$.

Suppose, conversely, that the displayed condition holds and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \preccurlyeq^{M,\mathcal{F}}(T) \leq \preccurlyeq^{M,\mathcal{F}}(T')$. Then, we get

$$\begin{array}{rcl}\widetilde{\preccurlyeq}^{M,\mathcal{F}}(\bigcap\mathcal{T}) & = & \bigcap\{\preccurlyeq^{M,\mathcal{F}}(T) : \bigcap\mathcal{T} \leq T \in \mathrm{ThFam}(\mathcal{I})\}\\ & \leq & \bigcap\{\preccurlyeq^{M,\mathcal{F}}(T) : T \in \mathcal{T}\}\\ & \leq & \preccurlyeq^{M,\mathcal{F}}(T').\end{array}$$

Thus, by hypothesis, $\bigcap\mathcal{T} \leq T'$ and, therefore, $\mathcal{I}$ is M-c-reflective. ∎

Furthermore, under M-order monotonicity, it turns out that M-c-reflectivity is equivalent to the injectivity of the M-Leibniz order operator.

Lemma 1856 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is M-order monotone, then $\mathcal{I}$ is M-c-reflective if and only if $\preccurlyeq^{M,\mathcal{F}}$ is injective on theory families.*

Proof: Suppose that $\mathcal{I}$ is M-order monotone.

Assume, first, that $\mathcal{I}$ is M-c-reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\preccurlyeq^{M,\mathcal{F}}(T) = \preccurlyeq^{M,\mathcal{F}}(T')$. Then, we have

$$\preccurlyeq^{M,\mathcal{F}}(T) = \preccurlyeq^{M,\mathcal{F}}(T) \cap \preccurlyeq^{M,\mathcal{F}}(T') \leq \preccurlyeq^{M,\mathcal{F}}(T'),$$

whence, by M-c-reflectivity, $T \cap T' \leq T'$, i.e., $T \leq T'$. By symmetry, we get $T = T'$ and, therefore, $\preccurlyeq^{M,\mathcal{F}}$ is injective on theory families.

Assume, conversely, that $\preccurlyeq^{M,\mathcal{F}}$ is injective on theory families and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \preccurlyeq^{M,\mathcal{F}}(T) \leq \preccurlyeq^{M,\mathcal{F}}(T')$. Then we get

$$\begin{array}{rcll}
\preccurlyeq^{M,\mathcal{F}}(\bigcap_{T \in \mathcal{T}} T) & = & \bigcap_{T \in \mathcal{T}} \preccurlyeq^{M,\mathcal{F}}(T) & \text{(monotonicity)} \\
& = & \bigcap_{T \in \mathcal{T}} \preccurlyeq^{M,\mathcal{F}}(T) \cap \preccurlyeq^{M,\mathcal{F}}(T') & \text{(hypothesis)} \\
& = & \preccurlyeq^{M,\mathcal{F}}(\bigcap \mathcal{T} \cap T'). & \text{(monotonicity)}
\end{array}$$

Thus, by injectivity, $\bigcap \mathcal{T} = \bigcap \mathcal{T} \cap T'$, whence $\bigcap \mathcal{T} \leq T'$ and, therefore, $\mathcal{I}$ is M-c-reflective. ■

It is always the case that truth inequationality implies c-reflectivity.

Theorem 1857 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *be a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is* M*-truth inequational, then it is* M*-c-reflective.*

Proof: Suppose that $\mathcal{I}$ is M-truth inequational, with witnessing transformations $\tau^\flat$, and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \preccurlyeq^{M,\mathcal{F}}(T) \leq \preccurlyeq^{M,\mathcal{F}}(T')$. Then

$$\begin{array}{rcll}
\bigcap_{T \in \mathcal{T}} T & = & \bigcap_{T \in \mathcal{T}} \tau^\flat(\preccurlyeq^{M,\mathcal{F}}(T)) & \text{(Truth Inequationality)} \\
& = & \tau^\flat(\bigcap_{T \in \mathcal{T}} \preccurlyeq^{M,\mathcal{F}}(T)) & \text{(Set Theory)} \\
& \leq & \tau^\flat(\preccurlyeq^{M,\mathcal{F}}(T')) & \text{(Hypothesis and Lemma 94)} \\
& = & T'. & \text{(Truth Inequationality)}
\end{array}$$

Thus, $\mathcal{I}$ is M-c-reflective. ■

Recall the characterization of truth equationality in terms of the solubility property of the Suszko core of the π-institution. We now work to establish an analog for truth inequationaility. More precisely, we provide a characterization of truth inequationality in terms of the order solubility property of the order core of a π-institution. Then, we provide an exact description of those M-c-reflective π-institutions which are M-truth inequational.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We define the M-**order (Suszko) core** of $\mathcal{I}$ to be the collection

$$O^{\mathcal{I},M} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}(\mathcal{I}))(\sigma^\flat[T] \leq \widetilde{\preccurlyeq}^{M,\mathcal{F}}(T))\}.$$

Lemma 1858 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $\sigma^\flat$ *in* $N^\flat$*, the following conditions are equivalent:*

(i) For every $T \in \mathrm{ThFam}(\mathcal{I})$*,* $\sigma^\flat[T] \leq \,\widetilde{\lessapprox}^{M,\mathcal{F}}(T)$*;*

(ii) For every $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,* $\sigma^\flat_\Sigma[\phi] \leq \,\widetilde{\lessapprox}^{M,\mathcal{F}}(C(\phi))$*.*

Proof: Suppose Condition (i) holds and let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, setting $T = C(\phi)$ in (i), we obtain $\sigma^\flat[C(\phi)] \leq \,\widetilde{\lessapprox}^{M,\mathcal{F}}(C(\phi))$, whence, a fortiori, $\sigma^\flat_\Sigma[\phi] \leq \,\widetilde{\lessapprox}^{M,\mathcal{F}}(C(\phi))$. Assume, conversely, that Condition (ii) holds and let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, we get

$$\begin{array}{rcll}
\sigma^\flat[T] & = & \bigcup\{\sigma^\flat_\Sigma[\phi] : \phi \in T_\Sigma, \Sigma \in |\mathbf{Sign}^\flat|\} & \text{(definition)} \\
& \leq & \bigcup\{\widetilde{\lessapprox}^{M,\mathcal{F}}(C(\phi)) : \phi \in T_\Sigma, \Sigma \in |\mathbf{Sign}^\flat|\} & \text{(Condition (ii))} \\
& \leq & \bigcup\{\widetilde{\lessapprox}^{M,\mathcal{F}}(T) : \phi \in T_\Sigma, \Sigma \in |\mathbf{Sign}^\flat|\} & \text{(monotonicity of } \widetilde{\lessapprox}^{M,\mathcal{F}}) \\
& = & \widetilde{\lessapprox}^{M,\mathcal{F}}(T). &
\end{array}$$

Thus shows that Condition (i) holds and, therefore, that the two conditions are equivalent. ■

By Lemma 1858, this definition is equivalent to setting

$$\begin{array}{c}
O^{\mathcal{I},M} = \{\sigma^\flat \in N^\flat : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \phi \in \mathrm{SEN}^\flat(\Sigma)) \\
(\sigma^\flat_\Sigma[\phi] \leq \,\widetilde{\lessapprox}^{M,\mathcal{F}}(C(\phi)))\}.
\end{array}$$

It is clear, by definition that the M-order core of a π-institution satisfies the following property:

Proposition 1859 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. For every* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$T \leq O^{\mathcal{I},M}(\lessapprox^{M,\mathcal{F}}(T)).$$

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rll}
\phi \in T_\Sigma & \text{implies} & O^{\mathcal{I},M}_\Sigma[\phi] \leq \,\widetilde{\lessapprox}^{M,\mathcal{F}}(T) \quad \text{(definition of } O^{\mathcal{I},M}) \\
& \text{implies} & O^{\mathcal{I},M}_\Sigma[\phi] \leq \,\lessapprox^{M,\mathcal{F}}(T). \quad (\widetilde{\lessapprox}^{M,\mathcal{F}}(T) \leq \,\lessapprox^{M,\mathcal{F}}(T))
\end{array}$$

Thus, we get that $T \leq O^{\mathcal{I},M}(\lessapprox^{M,\mathcal{F}}(T))$. ■

It is possible, but not necessary, that the M-order core of a π-institution satisfies the reverse inclusion. We call this property order solubility.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. We say that the M-order core of $\mathcal{I}$ is **order soluble** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$O^{\mathcal{I},M}(\lessapprox^{M,\mathcal{F}}(T)) \leq T.$$

In other words $O^{\mathcal{I},M}$ is order soluble if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$O^{\mathcal{I},M}_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T) \quad \text{implies} \quad \phi \in T_\Sigma.$$

It turns out that possession of the order solubility property by the M-order core intrinsically characterizes M-truth inequationality. We show, first, that the M-order core being order soluble is necessary for M-truth inequationality. To see this, observe that, in case a π-institution is M-truth inequational, the witnessing transformations form a subset of the M-order core.

Lemma 1860 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is M-truth inequational, with witnessing transformations $\tau^\flat \subseteq N^\flat$, then $\tau^\flat \subseteq O^{\mathcal{I},M}$.*

Proof: By truth inequationality, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T).$$

Thus, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rll}
\phi \in T_\Sigma & \text{iff} & (\forall T \leq T' \in \mathrm{ThFam}(\mathcal{I}))(\phi \in T'_\Sigma) \\
& \text{iff} & (\forall T \leq T' \in \mathrm{ThFam}(\mathcal{I}))(\tau^\flat_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T')) \\
& \text{iff} & \tau^\flat_\Sigma[\phi] \leq \bigcap\{\preccurlyeq^{M,\mathcal{F}}(T') : T \leq T' \in \mathrm{ThFam}(\mathcal{I})\} \\
& \text{iff} & \tau^\flat_\Sigma[\phi] \leq \widetilde{\preccurlyeq}^{M,\mathcal{F}}(T).
\end{array}$$

We conclude, by the definition of $O^{\mathcal{I},M}$, that $\tau^\flat \subseteq O^{\mathcal{I},M}$. ■

Now we prove the necessity of order solubility for truth inequationality.

Theorem 1861 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is M-truth inequational, then $O^{\mathcal{I},M}$ is order soluble.*

Proof: Suppose that $\mathcal{I}$ is M-truth equational, with witnessing equations $\tau^\flat$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rlll}
O^{\mathcal{I},M}_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T) & \text{implies} & \tau^\flat_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T) & \text{(Lemma 1860)} \\
& \text{iff} & \phi \in T_\Sigma. & \text{(truth inequationality)}
\end{array}$$

Thus, $O^{\mathcal{I},M}$ is order soluble. ■

The reverse implication, which also holds and completes the promised characterization of M-truth inequationality in terms of the M-order core, is presented in the following result.

Theorem 1862 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. If $O^{\mathcal{I},M}$ is order soluble, then $\mathcal{I}$ is M-truth inequational, with witnessing equations $O^{\mathcal{I},M}$.*

Proof: It suffices to show that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad O^{\mathcal{I},M}_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T).$$

The left-to-right implication is given in Proposition 1859, whereas the converse is ensured by the postulated order solubility of $O^{\mathcal{I},M}$. ∎

Theorems 1861 and 1862 provide the promised characterization of M-truth inequationality in terms of the order solubility of the M-order core.

$$\mathcal{I} \text{ is } M\text{-Truth Inequational} \longleftrightarrow O^{\mathcal{I},M} \text{ is Soluble.}$$

Theorem 1863 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $M = (M^+,\ M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*. $\mathcal{I}$ is M-truth inequational if and only if $O^{\mathcal{I},M}$ is order soluble.*

Proof: Theorem 1861 gives the "only if" and the "if" is by Theorem 1862. ∎

If $\mathcal{I}$ is M-truth inequational, then the M-order core defines theory families in $\mathcal{I}$ in terms of their M-Leibniz qosystems.

Proposition 1864 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*. If $O^{\mathcal{I},M}$ is order soluble, then, for all $T \in \mathrm{ThFam}(\mathcal{I})$,*

$$T = O^{\mathcal{I},M}(\preccurlyeq^{M,\mathcal{F}}(T)).$$

Proof: If $O^{\mathcal{I},M}$ is order soluble, then, by Theorem 1862, $O^{\mathcal{I},M}$ forms a set of witnessing transformations for the M-truth inequationality of $\mathcal{I}$. Therefore, by definition, we get that, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $T = O^{\mathcal{I},M}(\preccurlyeq^{M,\mathcal{F}}(T))$. ∎

In fact, this property may also be restated as another characterization of truth inequationality. Let us say that $O^{\mathcal{I},M}$ **defines theory families** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T = O^{\mathcal{I},M}(\preccurlyeq^{M,\mathcal{F}}(T))$. Then we have:

$$\mathcal{I} \text{ is } M\text{-Truth Equational} \longleftrightarrow O^{\mathcal{I},M} \text{ Defines Theory Families.}$$

Theorem 1865 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $M = (M^+,\ M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*. $\mathcal{I}$ is M-truth inequational if and only if, for all $T \in \mathrm{ThFam}(\mathcal{I})$,*

$$T = O^{\mathcal{I},M}(\preccurlyeq^{M,\mathcal{F}}(T)).$$

Proof: If $\mathcal{I}$ is truth equational, then, by Theorem 1861, $O^{\mathcal{I},M}$ is order soluble. Thus, by Proposition 1864, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T = O^{\mathcal{I},M}(\preccurlyeq^{M,\mathcal{F}}(T))$.

Conversely, if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T = O^{\mathcal{I},M}(\preccurlyeq^{M,\mathcal{F}}(T))$, then, $O^{\mathcal{I},M}$ is order soluble. Thus, again by Theorem 1863, $O^{\mathcal{I},M}$ is a set of witnessing equations and $\mathcal{I}$ is M-truth inequational. ∎

We finally show that the property that separates M-complete reflectivity from M-truth inequationality is exactly the adequacy property of the M-order core. Roughly speaking, this property ensures that the M-order core is rich enough to define M-Suszko qosystems in terms of the M-Leibniz qosystems of theory families that it selects via inclusion.

We have the following relationship connecting the M-order core with both M-Leibniz quosystems and M-Suszko qosystems of enveloping theory families.

Proposition 1866 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,*

$$\bigcap\{\preccurlyeq^{M,\mathcal{F}}(T) : O^{\mathcal{I},M}_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T)\} \leq \widetilde{\preccurlyeq}^{M,\mathcal{F}}(C(\phi)).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\begin{array}{lll}\phi \in T_\Sigma & \text{implies} & O^{\mathcal{I},M}_\Sigma[\phi] \leq \widetilde{\preccurlyeq}^{M,\mathcal{F}}(T) \quad (M\text{-order core})\\ & \text{implies} & O^{\mathcal{I},M}_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T).\end{array}$$

Therefore, we have

$$\begin{array}{lll}\bigcap\{\preccurlyeq^{M,\mathcal{F}}(T) : O^{\mathcal{I},M}_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T)\} & \leq & \bigcap\{\preccurlyeq^{M,\mathcal{F}}(T) : O^{\mathcal{I},M}_\Sigma[\phi] \leq \widetilde{\preccurlyeq}^{M,\mathcal{F}}(T)\}\\ & \leq & \bigcap\{\preccurlyeq^{M,\mathcal{F}}(T) : \phi \in T_\Sigma\}\\ & = & \widetilde{\preccurlyeq}^{M,\mathcal{F}}(C(\phi)).\end{array}$$

■

It is possible, but not necessary, that the M-order core of a π-institution satisfies, for every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, the reverse inclusion of that given in Proposition 1866:

$$\widetilde{\preccurlyeq}^{M,\mathcal{F}}(C(\phi)) \leq \bigcap\{\preccurlyeq^{M,\mathcal{F}}(T) : O^{\mathcal{I},M}_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T)\}.$$

Intuitively speaking, this means that the M-order core $O^{\mathcal{I},M}$ is rich enough to allow, for every Σ-sentence ϕ, the determination of those theory families whose M-Leibniz qosystems form a covering of the M-Suszko qosystem of $C(\phi)$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. We say that the M-order core $O^{\mathcal{I},M}$ of $\mathcal{I}$ is **order adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widetilde{\preccurlyeq}^{M,\mathcal{F}}(C(\phi)) = \bigcap\{\preccurlyeq^{M,\mathcal{F}}(T) : O^{\mathcal{I},M}_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T)\}.$$

It is not difficult to see that, if $O^{\mathcal{I},M}$ is order soluble, then it is order adequate.

Corollary 1867 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $O^{\mathcal{I},M}$ *is order soluble, then it is order adequate.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{lll} \precapprox^{M,\mathcal{F}}(C(\phi)) & = & \bigcap\{\preccurlyeq^{M,\mathcal{F}}(T) : \phi \in T_\Sigma\} \quad (\text{definition of } \precapprox^{M,\mathcal{F}}(C(\phi))) \\ & = & \bigcap\{\preccurlyeq^{M,\mathcal{F}}(T) : O^{\mathcal{I},M}_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T)\}. \\ & & (\text{order solubility of } S^{\mathcal{I}} \text{ and Proposition 1864}) \end{array}$$

We conclude that $O^{\mathcal{I},M}$ is order adequate. ■

In the opposite direction, in an M-c-reflective π-institution $\mathcal{I}$, if the M-order core is order adequate, then it is also order soluble.

Proposition 1868 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *an* M*-c-reflective* π*-institution based on* $\mathbf{F}$*. If* $O^{\mathcal{I},M}$ *is order adequate, then it is order soluble.*

Proof: Suppose that $\mathcal{I}$ is M-c-reflective and that $O^{\mathcal{I},M}$ is order adequate. We must show that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$

$$\phi \in T_\Sigma \quad \text{iff} \quad O^{\mathcal{I},M}_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T).$$

The implication left-to-right is always satisfied by Proposition 1859. For the converse, assume that $O^{\mathcal{I},M}_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T)$. Then, by the adequacy of $O^{\mathcal{I},M}$, we get that $\precapprox^{M,\mathcal{F}}(C(\phi)) \leq \preccurlyeq^{M,\mathcal{F}}(T)$. Thus, by M-c-reflectivity, we conclude that $C(\phi) \leq T$, which gives $\phi \in T_\Sigma$. ■

We finally show that a π-institution is M-truth inequational if and only if it is M-c-reflective and it has an order adequate M-order core.

$$\begin{array}{lll} M\text{-Truth Inequationality} & = & O^{\mathcal{I},M} \text{ Order Soluble} \\ & = & O^{\mathcal{I},M} \text{ Defines Theory Families} \\ & = & M\text{-c-Reflectivity} + O^{\mathcal{I},M} \text{ Order Adequate} \end{array}$$

Theorem 1869 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is* M*-truth inequational if and only if it is* M*-c-reflective and has an order adequate* M*-order core.*

Proof: Suppose, first, that $\mathcal{I}$ is M-truth inequational. Then it is M-c-reflective by Theorem 1857. Moreover, its M-order core is order soluble by Theorem 1861 and, hence, by Corollary 1867, its M-order core is order adequate.

Suppose, conversely, that $\mathcal{I}$ is M-c-reflective with an order adequate M-order core. Then, by Proposition 1868, its M-order core is order soluble and, therefore, by Theorem 1863, $\mathcal{I}$ is M-truth inequational. ■

Taking into account Lemma 1856 we obtain the following

Corollary 1870 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *an* M*-order monotone* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is* M*-truth inequational if and only if it is* M*-order injective and has an order adequate* M*-order core.*

Proof: By Theorem 1869 and Lemma 1856. ■

Finally, it is not difficult to see that M-truth inequationality transfers from a π-institution to all $\mathcal{I}$-matrix families.

Theorem 1871 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathcal{I}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is* M*-truth inequational, with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*, if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $T = \tau^{\mathcal{A}}(\preccurlyeq^{M,\mathcal{A}}(T))$.

Proof: Suppose $\mathcal{I}$ is truth equational, with witnessing transformations $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, whence, by hypothesis, $\alpha^{-1}(T) = \tau^\flat(\preccurlyeq^{M,\mathcal{F}}(\alpha^{-1}(T)))$. Hence, by Lemma 1835, $\alpha^{-1}(T) = \tau^\flat(\alpha^{-1}(\preccurlyeq^{M,\mathcal{A}}(T)))$. Therefore, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, we get

$$\begin{array}{lll} \alpha_\Sigma(\phi) \in T_{F(\Sigma)} & \text{iff} & \phi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)}) \\ & \text{iff} & \tau_\Sigma^\flat[\phi] \leq \alpha^{-1}(\preccurlyeq^{M,\mathcal{A}}(T)) \\ & \text{iff} & \alpha(\tau_\Sigma^\flat[\phi]) \leq \preccurlyeq^{M,\mathcal{A}}(T) \\ & \text{iff} & \tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi)] \leq \preccurlyeq^{M,\mathcal{A}}(T). \quad (\langle F, \alpha \rangle \text{ surjective}) \end{array}$$

Taking again into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $\phi \in T_\Sigma$ if and only if $\tau_\Sigma^{\mathcal{A}}[\phi] \leq \preccurlyeq^{M,\mathcal{A}}(T)$, i.e., $T = \tau^{\mathcal{A}}(\preccurlyeq^{M,\mathcal{A}}(T))$. ■

25.7 Order Algebraizability

Theorem 1872 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$ *having two distinguished arguments,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *an* M*-directional* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* β*, such that, for all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*, all* $\sigma^\flat, \tau^\flat$ *in* $N^\flat$*, and all* $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\beta_\Sigma[\sigma_\Sigma^\flat(\psi, \vec{\chi}), \tau_\Sigma^\flat(\psi, \vec{\chi})] \leq C(\overleftrightarrow{\beta}_\Sigma[\phi, \psi], \beta_\Sigma[\sigma_\Sigma^\flat(\phi, \vec{\chi}), \tau_\Sigma^\flat(\phi, \vec{\chi})]).$$

Then the following conditions are equivalent:

(i) $\mathcal{I}$ *is* β*-order algebraized by* $\{\langle \mathcal{A}, \leq^{\mathcal{A},T}(T) \rangle : \langle \mathcal{A}, T \rangle \in \mathrm{MatFam}^*(\mathcal{I})\}$*;*

(ii) $\mathcal{I}$ is β-order algebraizable;

(iii) $\mathcal{I}$ is M-truth inequational;

(iv) $\mathcal{I}$ is M-order injective and has an order adequate M-order core.

Proof:

(i)⇒(ii) This implication is trivial.

(ii)⇒(iii) By hypothesis, β witnesses the M-directionality of $\mathcal{I}$. Therefore, by Theorem 1840, $\preccurlyeq^M = \preccurlyeq^B$. Thus, since, by hypothesis, $\mathcal{I}$ is β-order algebraizable, by Proposition 1854, $\mathcal{I}$ is M-truth equational, with witnessing transformations β.

(iii)⇒(i) Suppose $\mathcal{I}$ is M-truth inequational, with witnessing transformations $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$, having a single distinguished argument. Thus, we have, for every $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}_\Sigma(T).$$

Thus, by M-directionality, $\phi \in T_\Sigma$ if and only if $\beta[\tau^\flat_\Sigma[\phi]] \leq T$. Thus, we get that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C(\phi) = C(\beta[\tau^\flat_\Sigma[\phi]]). \tag{25.1}$$

Since, by hypothesis, $\mathcal{I}$ is M-directional, we have, by Theorem 1840, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{l} \beta_\Sigma[\phi, \phi] \leq \mathrm{Thm}(\mathcal{I}); \\ \beta_\Sigma[\phi, \chi] \leq C(\beta_\Sigma[\phi, \psi], \beta_\Sigma[\psi, \chi]). \end{array} \tag{25.2}$$

Given the hypothesis, Conditions (25.2) and Condition (25.1), we get, by Theorem 1828, that $\mathcal{I}$ is β-order algebraizable. Therefore, again by Theorem 1828, $\mathcal{I}$ is β-order algebraized by the class $\{\langle \mathcal{A}, \leq^{\mathcal{A},T}(T)\rangle : \langle \mathcal{A}, T\rangle \in \mathrm{MatFam}^*(\mathcal{I})\}$.

(iii)⇒(iv) Since $\mathcal{I}$ is M-truth inequational, by Theorem 1869, it is M-c-reflective and has an order adequate M-order core. Since $\mathcal{I}$ is M-directional, by Theorem 1853, it is M-order monotone. Hence, since it is M-c-reflective, by Lemma 1856, it is M-order injective, Thus, $\mathcal{I}$ is M-order injective and has an order adequate M-order core.

(iv)⇒(iii) Suppose $\mathcal{I}$ is M-order injective, with an order adequate M-order core. Since, by hypothesis, $\mathcal{I}$ is M-directional, it is, by Theorem 1853, M-order monotone. Thus, since it is, by hypothesis, M-order injective, it is by Lemma 1856, M-c-reflective. Being M-c-reflective with an M-order adequate M-order core, it is, by Theorem 1869, M-truth inequational.

■

Theorem 1873 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following conditions are equivalent:*

(i) There exists a polarity $M = (M^+, M^-)$ *for* $\mathbf{F}$*, such that* $\mathcal{I}$ *is* M*-order monotone,* M*-order injective,* $\preccurlyeq^M$ *is antisymmetric on* $\{T : \langle \mathcal{A}, T \rangle \in \mathrm{MatFam}^*(\mathcal{I})\}$*, with an order Leibniz* M*-quasicore and an order adequate* M*-order core;*

(ii) $\mathcal{I}$ *is order algebraizable, i.e., it is* β*-order algebraizable, for some* $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$ *having two distinguished arguments.*

If Condition (i) holds, then β *can be chosen so that* $\preccurlyeq^M = \preccurlyeq^B$ *and*

$$\{\langle \mathcal{A}, \leq^{\mathcal{A},T} \rangle : \langle \mathcal{A}, T \rangle \in \mathrm{MatFam}^*(\mathcal{I})\}$$

generates the β*-order class of* $\mathcal{I}$*.*

If Condition (ii) holds, then Condition (i) holds with $M = B$*.*

Proof:

(i)⇒(ii) Suppose Condition (i) holds. Since, by hypothesis, $\mathcal{I}$ is M-order monotone and has an order Leibniz M-quasicore, we get, by Theorem 1853, that $\mathcal{I}$ is M-directional, with some family β of witnessing transformations. Thus, by Theorem 1840, $\preccurlyeq^M = \preccurlyeq^B$. By hypothesis and Theorem 1869, we get that $\mathcal{I}$ is M-truth inequational. Therefore, by hypothesis, Theorem 1826 and Theorem 1872, we get that $\mathcal{I}$ is β-order algebraizable and that its β-order class is generated by $\{\langle \mathcal{A}, \leq^{\mathcal{A},T} \rangle : \langle \mathcal{A}, T \rangle \in \mathrm{MatFam}^*(\mathcal{I})\}$.

(ii)⇒(i) Suppose Condition (ii) holds. Then, by Corollary 1843, $\mathcal{I}$ is B-directional, with witnessing transformations β. Thus, by Theorem 1853, it is B-order monotone and has an order Leibniz B-quasicore. Moreover, by Proposition 1854, $\mathcal{I}$ is B-truth inequational and, therefore, by Theorem 1869, it is B-c-reflective and has on order adequate B-order core. Finally, taking into account Theorem 1828, we may apply Theorem 1872 to establish that $\preccurlyeq^B$ is antisymmetric on $\{T : \langle \mathcal{A}, T \rangle \in \mathrm{MatFam}^*(\mathcal{I})\}$. ■

Corollary 1874 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, having two distinguished arguments,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, such that, for all* $\sigma^\flat, \tau^\flat$ *in* $N^\flat$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $\phi, \psi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$*:*

1. $\beta_\Sigma[\phi, \phi] \leq \mathrm{Thm}(\mathcal{I})$*;*

2. $\sigma_\Sigma(\psi,\vec{\chi}) \in C_\Sigma(\beta_\Sigma[\phi,\psi],\sigma^\flat_\Sigma(\phi,\vec{\chi}))$, *if* $\sigma^\flat \in M^+$;

3. $\sigma_\Sigma(\phi,\vec{\chi}) \in C_\Sigma(\beta_\Sigma[\phi,\psi],\sigma^\flat_\Sigma(\psi,\vec{\chi}))$, *if* $\sigma^\flat \in M^-$;

4. $\beta_\Sigma[\sigma^\flat_\Sigma(\psi,\vec{\chi}),\tau^\flat_\Sigma(\psi,\vec{\chi})] \le C(\overleftrightarrow{\beta}_\Sigma[\phi,\psi],\beta_\Sigma[\sigma^\flat_\Sigma(\phi,\vec{\chi}),\tau^\flat_\Sigma(\phi,\vec{\chi})])$.

If, for all $\sigma^\flat \in \beta$, $\sigma^\flat(x,y,\vec{z}) \in M^-$ *or* $\sigma^\flat(y,x,\vec{z}) \in M^+$, *then* $\mathcal{I}$ *is* β*-order algbebraizable if and only if it is* M*-order injective and has an order adequate* M*-order core.*

Proof: By Theorem 1872, it suffices to show that $\mathcal{I}$ is M-directional. But this follows from Theorem 1839. ∎

25.8 Tonicity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $M = (M^+,M^-)$ a polarity for $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A},\langle F,\alpha\rangle\rangle$ an $\mathbf{F}$-algebraic system and $\le^\mathcal{A}$ a qosystem on $\mathcal{A}$. $\le^\mathcal{A}$ is called an M-**order** if, for all $\sigma^\flat$ in $N^\flat$, $\Sigma \in |\mathbf{Sign}|$, $\phi,\psi,\vec{\chi} \in \mathrm{SEN}(\Sigma)$,

- if $\sigma^\flat \in M^+$, then $\phi \le^\mathcal{A}_\Sigma \psi$ implies $\sigma^\mathcal{A}_\Sigma(\phi,\vec{\chi}) \le^\mathcal{A}_\Sigma \sigma^\mathcal{A}_\Sigma(\psi,\vec{\chi})$;
- if $\sigma^\flat \in M^-$, then $\phi \le^\mathcal{A}_\Sigma \psi$ implies $\sigma^\mathcal{A}_\Sigma(\psi,\vec{\chi}) \le^\mathcal{A}_\Sigma \sigma^\mathcal{A}_\Sigma(\phi,\vec{\chi})$.

In a way similar to the proof of the existence of $\preccurlyeq^{M,\mathcal{A}}(T)$ in Proposition 1832, we can also show that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A},\langle F,\alpha\rangle\rangle$ and all $T \in \mathrm{SenFam}(\mathcal{A})$, there always exists a largest M-order on $\mathcal{A}$, such that T is upward closed, i.e., for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi,\psi \in \mathrm{SEN}(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{and} \quad \phi \le^{M,\mathcal{A}}_\Sigma \psi \quad \text{imply} \quad \psi \in T_\Sigma.$$

Proposition 1875 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $M = (M^+,M^-)$ *a polarity for* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A},\langle F,\alpha\rangle\rangle$ *and all* $T \in \mathrm{SenFam}(\mathcal{A})$*, there exists a largest* M*-order on* $\mathcal{A}$*, such that* T *is upward closed.*

Proof: We consider the class $\mathrm{MOrd}^\mathcal{A}(T)$ of all M-orders on $\mathcal{A}$ with respect to which T is upward closed. We take the transitive closure of the union of all qosystems in $\mathrm{MOrd}^\mathcal{A}(T)$,

$$\mathrm{tc}(\bigcup \mathrm{MOrd}^\mathcal{A}(T)) = \{\mathrm{tc}_\Sigma(\bigcup \mathrm{MOrd}^\mathcal{A}(T)\}_{\Sigma\in|\mathbf{Sign}|}.$$

It suffices to show that this is also an M-order on $\mathcal{A}$ with respect to which T is upward closed. i.e., it is itself a member of $\mathrm{MOrd}^\mathcal{A}(T)$. It will then follow that it is its largest member.

It is clear by the definition that $\mathrm{tr}(\bigcup \mathrm{MOrd}^{\mathcal{A}}(T))$ is a qosystem on $\mathcal{A}$. So it suffices to show that it is an M-order with respect to which T is upward closed.

Suppose $\sigma^\flat$ in M^+, $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi, \vec{\chi} \in \mathrm{SEN}(\Sigma)$, such that

$$\phi \ \mathrm{tr}_\Sigma(\bigcup \mathrm{MOrd}^{\mathcal{A}}(T)) \ \psi.$$

Then, there exist $q^0, \dots, q^k \in \mathrm{MOrd}^{\mathcal{A}}(T)$ and $\xi_1, \dots, \xi_k \in \mathrm{SEN}(\Sigma)$, such that

$$\phi \ q^0_\Sigma \ \xi_1 \ q^1_\Sigma \ \xi_2 \ q^2_\Sigma \cdots q^{k-1}_\Sigma \ \xi_k \ q^k_\Sigma \ \psi.$$

Since $\phi \ q^0_\Sigma \ \xi_1$ and $q^0 \in \mathrm{MOrd}^{\mathcal{A}}(T)$, we get $\sigma^{\mathcal{A}}_\Sigma(\phi, \vec{\chi}) \ q^0_\Sigma \ \sigma^{\mathcal{A}}_\Sigma(\xi_1, \vec{\chi})$. Since $\xi_1 \ q^1_\Sigma \ \xi_2$ and $q^1 \in \mathrm{MOrd}^{\mathcal{A}}(T)$, we get $\sigma^{\mathcal{A}}_\Sigma(\xi_1, \vec{\chi}) \ q^1_\Sigma \ \sigma^{\mathcal{A}}_\Sigma(\xi_2, \vec{\chi})$. We move one step to the right at a time in a similar fashion until we obtain $\sigma^{\mathcal{A}}_\Sigma(\xi_k, \vec{\chi}) \ q^k_\Sigma \ \sigma^{\mathcal{A}}_\Sigma(\psi, \vec{\chi})$. Thus, we obtain

$$\sigma^{\mathcal{A}}_\Sigma(\phi, \vec{\chi}) \ \mathrm{tr}_\Sigma(\bigcup \mathrm{MOrd}^{\mathcal{A}}(T)) \ \sigma^{\mathcal{A}}_\Sigma(\psi, \vec{\chi}).$$

A similar argument is used to handle the case of negative polarity for $\sigma^\flat$. This proves that $\mathrm{tr}(\bigcup \mathrm{MOrd}^{\mathcal{A}}(T))$ is also an M-order on $\mathcal{A}$.

Finally, suppose $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in T_\Sigma$ and

$$\phi \ \mathrm{tr}_\Sigma(\bigcup \mathrm{MOrd}^{\mathcal{A}}(T)) \ \psi.$$

Then, there exist $q^0, \dots, q^k \in \mathrm{MOrd}^{\mathcal{A}}(T)$ and $\xi_1, \dots, \xi_k \in \mathrm{SEN}(\Sigma)$, such that

$$\phi \ q^0_\Sigma \ \xi_1 \ q^1_\Sigma \ \xi_2 \ q^2_\Sigma \cdots q^{k-1}_\Sigma \ \xi_k \ q^k_\Sigma \ \psi.$$

Since T is upward closed with respect to all elements in $\mathrm{MOrd}^{\mathcal{A}}(T)$ and $\phi \in T_\Sigma$, we get $\xi_1 \in T_\Sigma$, then $\xi_2 \in T_\Sigma$, then ..., until, in the last step, $\xi_k \in T_\Sigma$ implies $\psi \in T_\Sigma$. Therefore, T is also upward closed with respect to $\mathrm{tr}_\Sigma(\bigcup \mathrm{MOrd}^{\mathcal{A}}(T))$, showing that $\mathrm{tr}_\Sigma(\bigcup \mathrm{MOrd}^{\mathcal{A}}(T)) \in \mathrm{MOrd}^{\mathcal{A}}(T)$, whence it is its largest element. ■

Given an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and $T \in \mathrm{SenFam}(\mathcal{A})$, the **Leibniz M-order $\leq^{M,\mathcal{A}}(T)$ of $\langle \mathcal{A}, T \rangle$** is the largest M-order on $\mathcal{A}$, such that T is upward closed, whose existence is assured by Proposition 1875.

It turns out that the Leibniz M-order $\leq^{M,\mathcal{A}}(T)$ is included in the M-Leibniz qosystem $\preccurlyeq^{M,\mathcal{A}}(T)$ of T on $\mathcal{A}$.

Proposition 1876 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$. For every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T \in \mathrm{SenFam}(\mathcal{A})$,*

$$\leq^{M,\mathcal{A}}(T) \leq \preccurlyeq^{M,\mathcal{A}}(T).$$

Proof: It suffices to show that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T \in \mathrm{SenFam}(\mathcal{A})$, $\leq^{M,\mathcal{A}}(T)$ is M-compatible with T. To this end, let $\sigma^\flat$ in $N^\flat$, $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi, \vec{\chi} \in \mathrm{SEN}(\Sigma)$.

- Suppose $\sigma^\flat \in M^+$, $\phi \leq_\Sigma^{M,\mathcal{A}}(T)\psi$ and $\sigma_\Sigma^{\mathcal{A}}(\phi,\vec{\chi}) \in T_\Sigma$. Since $\phi \leq_\Sigma^{M,\mathcal{A}}(T)\psi$ and $\leq^{M,\mathcal{A}}(T)$ is an M-order, we get $\sigma_\Sigma^{\mathcal{A}}(\phi,\vec{\chi}) \leq \sigma_\Sigma^{\mathcal{A}}(\psi,\vec{\chi})$. Hence, since $\sigma_\Sigma^{\mathcal{A}}(\phi,\vec{\chi}) \in T_\Sigma$ and T is upward closed with respect to $\leq^{M,\mathcal{A}}(T)$, we get $\sigma_\Sigma^{\mathcal{A}}(\psi,\vec{\chi}) \in T_\Sigma$.

- Suppose $\sigma^\flat \in M^-$, $\phi \leq_\Sigma^{M,\mathcal{A}}(T)\psi$ and $\sigma_\Sigma^{\mathcal{A}}(\psi,\vec{\chi}) \in T_\Sigma$. Since $\phi \leq_\Sigma^{M,\mathcal{A}}(T)\psi$ and $\leq^{M,\mathcal{A}}(T)$ is an M-order, we get $\sigma_\Sigma^{\mathcal{A}}(\psi,\vec{\chi}) \leq \sigma_\Sigma^{\mathcal{A}}(\phi,\vec{\chi})$. Hence, since $\sigma_\Sigma^{\mathcal{A}}(\psi,\vec{\chi}) \in T_\Sigma$ and T is upward closed with respect to $\leq^{M,\mathcal{A}}(T)$, we get $\sigma_\Sigma^{\mathcal{A}}(\phi,\vec{\chi}) \in T_\Sigma$.

Thus, $\leq^{M,\mathcal{A}}(T)$ is M-compatible with T and, hence, by the maximality of $\preccurlyeq^{M,\mathcal{A}}(T)$, $\leq^{M,\mathcal{A}}(T) \leq \preccurlyeq^{M,\mathcal{A}}(T)$. ■

We finally provide sufficient conditions ensuring that the two orders on $\mathcal{A}$ associated with $\mathcal{I}$-filter families T of a π-institution $\mathcal{I}$, $\leq^{M,\mathcal{A}}(T)$ and $\preccurlyeq^{M,\mathcal{A}}(T)$, coincide.

Proposition 1877 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$ having two distinguished arguments, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$, such that $p^{1,0} \in M^+$, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Suppose $\mathcal{I}$ is M-directional, with witnessing transformations β, and that, for all σ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi,\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,*

- *if $\sigma^\flat \in M^+$, $\beta_\Sigma[\sigma_\Sigma^\flat(\phi,\vec{\chi}), \sigma_\Sigma^\flat(\psi,\vec{\chi})] \leq C(\beta_\Sigma[\phi,\psi])$;*

- *if $\sigma^\flat \in M^-$, $\beta_\Sigma[\sigma_\Sigma^\flat(\psi,\vec{\chi}), \sigma_\Sigma^\flat(\phi,\vec{\chi})] \leq C(\beta_\Sigma[\phi,\psi])$.*

Then, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\preccurlyeq^{M,\mathcal{A}}(T)$ is the largest M-order on $\mathcal{A}$ with respect to which T is upward closed, i.e., $\preccurlyeq^{M,\mathcal{A}}(T) = \leq^{M,\mathcal{A}}(T)$.

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. We show that $\preccurlyeq^{M,\mathcal{A}}(T)$ is an M-order on $\mathcal{A}$, with respect to which T is upward closed. Then, it will follow, by the maximality property of $\leq^{M,\mathcal{A}}(T)$, that $\preccurlyeq^{M,\mathcal{A}}(T) \leq \leq^{M,\mathcal{A}}(T)$.

Let $\sigma^\flat$ in $N^\flat$, $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi,\vec{\chi} \in \mathrm{SEN}(\Sigma)$.

- Suppose $\sigma^\flat \in M^+$ and $\phi \preccurlyeq_\Sigma^{M,\mathcal{A}}(T)\ \psi$. Thus, by M-directionality of $\mathcal{I}$, $\beta_\Sigma^{\mathcal{A}}[\phi,\psi] \leq T$, whence, by hypothesis, $\beta_\Sigma^{\mathcal{A}}[\sigma_\Sigma^{\mathcal{A}}(\phi,\vec{\chi}), \sigma_\Sigma^{\mathcal{A}}(\psi,\vec{\chi})] \leq T$. Thus, again by M-directionaility, $\sigma^{\mathcal{A}}(\phi,\vec{\chi}) \preccurlyeq_\Sigma^{M,\mathcal{A}}(T)\ \sigma_\Sigma^{\mathcal{A}}(\psi,\vec{\chi})$.

- Suppose $\sigma^\flat \in M^-$ and $\phi \preccurlyeq_\Sigma^{M,\mathcal{A}}(T)\ \psi$. Thus, by M-directionality of $\mathcal{I}$, $\beta_\Sigma^{\mathcal{A}}[\phi,\psi] \leq T$, whence, by hypothesis, $\beta_\Sigma^{\mathcal{A}}[\sigma_\Sigma^{\mathcal{A}}(\psi,\vec{\chi}), \sigma_\Sigma^{\mathcal{A}}(\phi,\vec{\chi})] \leq T$. Thus, again by M-directionaility, $\sigma^{\mathcal{A}}(\psi,\vec{\chi}) \preccurlyeq_\Sigma^{M,\mathcal{A}}(T)\ \sigma_\Sigma^{\mathcal{A}}(\phi,\vec{\chi})$.

Thus, $\preccurlyeq^{M,\mathcal{A}}(T)$ is an M-order on $\mathcal{A}$.

Finally, suppose $\phi \preccurlyeq^{M,\mathcal{A}}_\Sigma(T)\ \psi$ and $\phi \in T_\Sigma$. Then, since, by hypothesis, $p^{1,0} \in M^+$ and $\preccurlyeq^{M,\mathcal{A}}(T)$ is M-compatible with T, we get $\psi \in T_\Sigma$. Therefore, $\preccurlyeq^{M,\mathcal{A}}(T)$ is an M-order with respect to which T is upward closed. It now follows by the maximality of $\leq^{M,\mathcal{A}}(T)$, that $\preccurlyeq^{M,\mathcal{A}}(T) \leq \leq^{M,\mathcal{A}}(T)$ and, hence, by Proposition 1876, that $\preccurlyeq^{M,\mathcal{A}}(T) = \leq^{M,\mathcal{A}}(T)$. ■