Chapter 26

Gentzen π-Institutions

26.1 Gentzen π-Institutions Revisited

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an $N^\flat$-algebraic system, $\Sigma \in |\mathbf{Sign}|$ and $m, n \in \omega$. An **$\langle m, n \rangle$-Σ-sequent of A** is an expression

$$\phi_0, \ldots, \phi_{m-1} \rhd_\Sigma \psi_0, \ldots, \psi_{n-1},$$

abbreviated $\vec{\phi} \rhd_\Sigma \vec{\psi}$, consisting of two finite (possibly empty) sequences $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)$. A $\langle 0, n \rangle$-Σ-sequent $\varnothing \rhd_\Sigma \vec{\psi}$ is abbreviated $\rhd_\Sigma \vec{\psi}$.

Given $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and an $\langle m, n \rangle$-Σ-sequent $\vec{\phi} \rhd_\Sigma \vec{\psi}$, we write

$$\mathrm{SEN}(f)(\vec{\phi} \rhd_\Sigma \vec{\psi}) \coloneqq \mathrm{SEN}(f)(\vec{\phi}) \rhd_{\Sigma'} \mathrm{SEN}(f)(\vec{\psi}),$$

where, as usual,

$$\begin{array}{rcl}\mathrm{SEN}(f)(\vec{\phi}) & \coloneqq & \langle \mathrm{SEN}(f)(\phi_0), \ldots, \mathrm{SEN}(f)(\phi_{m-1}) \rangle, \\ \mathrm{SEN}(f)(\vec{\psi}) & \coloneqq & \langle \mathrm{SEN}(f)(\psi_0), \ldots, \mathrm{SEN}(f)(\psi_{n-1}) \rangle.\end{array}$$

Sometimes, we denote a Σ-sequent by $\boldsymbol{\phi} \coloneqq \vec{\phi}^0 \rhd_\Sigma \vec{\phi}^1$ and a set of Σ-sequents by $\mathbf{\Phi}$. The notation for images under morphisms is then extended to sets of Σ-sequents by writing

$$\mathrm{SEN}(f)(\mathbf{\Phi}) = \{\mathrm{SEN}(f)(\boldsymbol{\phi}) : \boldsymbol{\phi} \in \mathbf{\Phi}\}.$$

A **trace** tr is a nonempty subset of $\omega \times \omega$. An $\langle m, n \rangle$-Σ-sequent is a **tr-Σ-sequent** if $\langle m, n \rangle \in \mathrm{tr}$. The collection of all tr-Σ-sequents of $\mathbf{A}$ is denoted by $\mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{A})$ and we set

$$\mathrm{Seq}^{\mathrm{tr}}(\mathbf{A}) = \{\mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{A})\}_{\Sigma \in |\mathbf{Sign}|}.$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and tr be a given trace. A **Gentzen π-institution $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ of trace** tr based on $\mathbf{F}$ consists of a closure system

$$G : \mathcal{P}\mathrm{Seq}^{\mathrm{tr}}(\mathbf{F}) \to \mathcal{P}\mathrm{Seq}^{\mathrm{tr}}(\mathbf{F}),$$

i.e., a collection of closure operators

$$G_\Sigma : \mathcal{P}\mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F}) \to \mathcal{P}\mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F}), \quad \Sigma \in |\mathbf{Sign}^\flat|,$$

that also satisfy structurality, that is, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, and all $\mathbf{\Phi} \subseteq \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$,

$$\mathrm{SEN}(f)(G_\Sigma(\mathbf{\Phi})) \subseteq G_{\Sigma'}(\mathrm{SEN}(f)(\mathbf{\Phi})).$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$. If, for some $\Sigma \in |\mathbf{Sign}^\flat|$,

$\mathbf{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$, such that $\boldsymbol{\phi} \in G_\Sigma(\mathbf{\Phi})$, we say that $\langle \mathbf{\Phi}, \boldsymbol{\phi}\rangle$ is a **Σ-rule of $\mathfrak{G}$** or a **Σ-derivable rule of $\mathfrak{G}$**, sometimes denoted

$$\frac{\mathbf{\Phi}}{\boldsymbol{\phi}}.$$

A Σ-rule of form $\langle \varnothing, \boldsymbol{\phi}\rangle$ is called a **Σ-derivable sequent** or a **Σ-theorem of $\mathfrak{G}$**.

$\mathfrak{G}$ is **inconsistent** if all elements in $\mathrm{Seq}^{\mathrm{tr}}(\mathbf{F})$ are derivable sequents in $\mathfrak{G}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, tr a trace and $\mathfrak{G}^i = \langle \mathbf{F}, G^i\rangle$, $i \in I$, a collection of Gentzen π-institutions, all of trace tr, based on $\mathbf{F}$. Then

$$\bigcap_{i\in I} \mathfrak{G}^i = \langle \mathbf{F}, \bigcap_{i\in I} G^i\rangle,$$

defined, by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathbf{\Phi} \subseteq \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$,

$$(\bigcap_{i\in I} G^i)_\Sigma(\mathbf{\Phi}) = \bigcap_{i\in I} G^i_\Sigma(\mathbf{\Phi}),$$

is also a Gentzen π-institution.

Therefore, given a family $\mathfrak{X} = \{\mathfrak{X}_\Sigma\}_{\Sigma\in|\mathbf{Sign}^\flat|}$ of rules, there is a smallest Gentzen π-institution $\mathfrak{G}^{\mathfrak{X}} = \langle \mathbf{F}, G^{\mathfrak{X}}\rangle$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\langle \mathbf{\Phi}, \boldsymbol{\phi}\rangle \in \mathfrak{X}_\Sigma$,

$$\boldsymbol{\phi} \in G^{\mathfrak{X}}_\Sigma(\mathbf{\Phi}).$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ a Gentzen π-institution based on $\mathbf{F}$, with $\langle 0, 1\rangle \in \mathrm{tr}$. Consider

$$G^0 : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}$$

defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, by

$$\phi \in G^0_\Sigma(\Phi) \quad \text{iff} \quad \rhd_\Sigma \phi \in G_\Sigma(\{\rhd_\Sigma \psi : \psi \in \Phi\}).$$

Lemma 1878 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ a Gentzen π-institution based on $\mathbf{F}$, with $\langle 0, 1\rangle \in \mathrm{tr}$. $G^0 : \mathcal{P}\mathrm{SEN}^\flat \to \mathcal{P}\mathrm{SEN}^\flat$ is a closure system on $\mathbf{F}$.*

Proof: Suppose, first, that $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \Phi$. Then, by the inflationarity of G, $\rhd_\Sigma \phi \in G_\Sigma(\{\rhd_\Sigma \psi : \psi \in \Phi\})$ and, hence, by definition of G^0, $\phi \in G^0_\Sigma(\Phi)$. Suppose, next, that $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \Psi \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\Phi \subseteq \Psi$. Then, by monotonicity of G, $G_\Sigma(\{\rhd_\Sigma \phi : \phi \in \Phi\}) \subseteq G_\Sigma(\{\rhd_\Sigma \psi : \psi \in \Psi\})$, whence, by the definition of G^0, $G^0_\Sigma(\Phi) \subseteq G^0_\Sigma(\Psi)$. Now assume that $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in G^0_\Sigma(G^0_\Sigma(\Phi))$. Then, taking into account the idempotency of G, we get

$$\begin{array}{rcl}\rhd_\Sigma \phi & \in & G_\Sigma(\{\rhd_\Sigma \psi : \psi \in G^0_\Sigma(\Phi)\}) \\ & \subseteq & G_\Sigma(G_\Sigma(\{\rhd_\Sigma \phi : \phi \in \Phi\})) \\ & \subseteq & G_\Sigma(\{\rhd_\Sigma \phi : \phi \in \Phi\}),\end{array}$$

whence $\phi \in G^0_\Sigma(\Phi)$. Finally, the structurality property of G^0 follows directly by the structurality property of G. ∎

According to Lemma 1878, the structure $\mathcal{G}^0 = \langle \mathbf{F}, G^0\rangle$ is a π-institution, called the **π-institution reduct of** the Gentzen π-institution $\mathfrak{G}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and K a class of $\mathbf{F}$-algebraic systems. Recall the closure system $C^\mathsf{K} : \mathcal{P}(\mathrm{SEN}^\flat)^2 \to \mathcal{P}(\mathrm{SEN}^\flat)^2$ defined, by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $E \cup \{\phi \approx \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)^2$,

$$\begin{array}{lll}\phi \approx \psi \in C^\mathsf{K}_\Sigma(E) & \text{iff} & \text{for all } \mathcal{A} \in \mathsf{K}, \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}(\Sigma,\Sigma'),\\ & & \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(E)) \subseteq \Delta^\mathcal{A}_{F(\Sigma')}\\ & & \text{implies } \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) = \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi)).\end{array}$$

The π-institution $\mathcal{I}^\mathsf{K} = \langle \mathbf{F}, C^\mathsf{K}\rangle$ was called the **equational π-institution associated with** the class K. This π-institution may be recast as a Gentzen π-institution of trace $\{\langle 1,1\rangle\}$. More precisely, we define the Gentzen π-institution $\mathfrak{G}^\mathsf{K} = \langle \mathbf{F}, G^\mathsf{K}\rangle$ by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\{\phi_i,\psi_i : i \in I\} \cup \{\phi,\psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \rhd_\Sigma \psi \in G^\mathsf{K}_\Sigma(\{\phi_i \rhd_\Sigma \psi_i : i \in I\}) \quad \text{iff} \quad \phi \approx \psi \in C^\mathsf{K}_\Sigma(\{\phi_i \approx \psi_i : i \in I\}).$$

We call $\mathfrak{G}^\mathsf{K}$ the **Gentzen π-institution associated with** the class K.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ may also be recast as a Gentzen π-institution of trace $\{\langle 0,1\rangle\}$. More precisely, given $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$, denote by

$$\rhd_\Sigma \Phi = \{\rhd_\Sigma \phi : \phi \in \Phi\}$$

and, similarly, given $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|} \in \mathrm{SenFam}(\mathbf{F})$, let

$$\rhd T = \{\rhd_\Sigma T_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}.$$

We define $\mathfrak{G}^\mathcal{I} = \langle \mathbf{F}, G^\mathcal{I}\rangle$ by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\rhd_\Sigma \phi \in G^\mathcal{I}_\Sigma(\rhd_\Sigma \Phi) \quad \text{iff} \quad \phi \in C_\Sigma(\Phi).$$

We call $\mathfrak{G}^\mathcal{I}$ the **Hilbert π-institution associated with** $\mathcal{I}$. In this terminology, a Hilbert π-institution is a Gentzen π-institution of trace $\{\langle 0,1\rangle\}$.

Given a Gentzen π-institution $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ of trace tr, such that $\langle 0,1\rangle \in \mathrm{tr}$, we call the Hilbert π-institution $\mathfrak{G}^{\mathcal{G}^0}$ associated with the π-institution reduct $\mathcal{G}^0$ of $\mathfrak{G}$ the **Hilbert π-institution reduct of** $\mathfrak{G}$ and we denote it by $\mathfrak{G}^0 = \langle \mathbf{F}, G^0\rangle$ (note the overloading of notation for G^0, used both for the closure system of the π-institution $\mathcal{G}^0$ and for the closure system of $\mathfrak{G}^{\mathcal{G}^0}$; hopefully, this will not result into any confusion, since it should be resolvable based on context).

26.2 Equivalence of Gentzen π-Institutions

Let $\mathbf{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{F}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ be algebraic systems and tr, tr′ be traces. A tr-tr′-**translation** is a collection of functions

$$\alpha = \{\alpha^{m,n} : \langle m,n\rangle \in \mathrm{tr}\},$$

where, for all $\langle m,n\rangle \in \mathrm{tr}$,

$$\alpha^{m,n} = \{\alpha^{m,n}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$$

is such that, for all $\Sigma \in |\mathbf{Sign}|$,

$$\alpha^{m,n}_\Sigma : \mathrm{SEN}(\Sigma)^{m,n} \to \mathcal{P}(\mathrm{Seq}^{\mathrm{tr}'}_\Sigma(\mathbf{F}'))$$

assigns to each $\langle m,n\rangle$-Σ-sequent $\vec{\phi} \rhd_\Sigma \vec{\psi}$ of $\mathbf{F}$ a set of tr′-Σ-sequents of $\mathbf{F}'$

$$\alpha^{m,n}_\Sigma[\vec{\phi};\vec{\psi}].$$

We extend the notation in a natural way in order to write expressions more concisely. Thus, given $\Sigma \in |\mathbf{Sign}|$ and $\mathbf{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$, we set

$$\alpha_\Sigma[\boldsymbol{\phi}] = \alpha_\Sigma[\vec{\phi};\vec{\psi}],$$

if $\boldsymbol{\phi} = \vec{\phi} \rhd_\Sigma \vec{\psi}$, and

$$\alpha_\Sigma[\mathbf{\Phi}] = \bigcup\{\alpha^{m,n}_\Sigma[\boldsymbol{\phi}] : \boldsymbol{\phi} \in \mathbf{\Phi}^{m,n},\ \langle m,n\rangle \in \mathrm{tr}\}.$$

Finally, if $\mathbf{\Phi} = \{\mathbf{\Phi}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} \le \mathrm{Seq}^{\mathrm{tr}}(\mathbf{F})$, we set

$$\alpha[\mathbf{\Phi}] = \bigcup\{\alpha_\Sigma[\mathbf{\Phi}_\Sigma] : \Sigma \in |\mathbf{Sign}|\}.$$

Even though we defined translations in a very general way, we will deal almost exclusively with a special kind of translation, called a transformation. To introduce those, we fix $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ and two traces tr and tr′. A tr-tr′-translation $\alpha = \{\alpha^{m,n} : \langle m,n\rangle \in \mathrm{tr}\}$ is called a tr-tr′-**transformation** if there exists a family

$$\tau = \{\tau^{m,n} : \langle m,n\rangle \in \mathrm{tr}\},$$

such that, for all $\langle m,n\rangle \in \mathrm{tr}$,

$$\tau^{m,n} : \mathrm{SEN}^\omega \to \bigcup\{\mathrm{SEN}^{k+\ell} : \langle k,\ell\rangle \in \mathrm{tr}'\}$$

is a collection of natural transformations in $N^\flat$, with $m+n$ distinguished arguments, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \rhd_\Sigma \vec{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m,n\rangle$,

$$\alpha^{m,n}_\Sigma[\vec{\phi};\vec{\psi}] = \tau^{m,n}_\Sigma[\vec{\phi};\vec{\psi}],$$

where, we let $\tau_\Sigma^{m,n}[\vec{\phi};\vec{\psi}]$ be defined, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, by

$$\tau_\Sigma^{m,n}[\vec{\phi};\vec{\psi}] \quad = \quad \bigcup\{\tau_\Sigma^{m,n}(\vec{\phi},\vec{\psi},\vec{\chi}) : \vec{\chi}' \in \mathrm{SEN}^\flat(\Sigma)\}.$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, tr and tr′ two traces and $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ and $\mathfrak{G}' = \langle \mathbf{F}, G'\rangle$ two Gentzen π-institutions of traces tr and tr′, respectively, both based on $\mathbf{F}$. A tr-tr′-transformation τ is an **interpretation from $\mathfrak{G}$ to $\mathfrak{G}'$**, written $\tau : \mathfrak{G} \to \mathfrak{G}'$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\mathbf{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$,

$$\boldsymbol{\phi} \in G_\Sigma(\mathbf{\Phi}) \quad \text{iff} \quad \tau_\Sigma[\boldsymbol{\phi}] \subseteq G'_\Sigma(\tau_\Sigma[\mathbf{\Phi}]).$$

The two π-institutions $\mathfrak{G}$ and $\mathfrak{G}'$ are **equivalent** if there exist a tr-tr′-transformation τ and a tr′-tr-transformation ρ, such that:

- $\tau : \mathfrak{G} \to \mathfrak{G}'$ is an interpretation;
- $\rho : \mathfrak{G}' \to \mathfrak{G}$ is an interpretation;
- for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$,
$$G_\Sigma(\boldsymbol{\phi}) = G_\Sigma(\rho_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]]);$$
- for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi}' \in \mathrm{Seq}_\Sigma^{\mathrm{tr}'}(\mathbf{F})$,
$$G'_\Sigma(\boldsymbol{\phi}') = G'_\Sigma(\tau_\Sigma[\rho_\Sigma[\boldsymbol{\phi}']]).$$

In this case the pair (τ, ρ) is called a **conjugate pair** of transformations and denoted by $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$.

As in Lemma 889, it suffices to check only the first and last conditions, or, equivalently, the middle two conditions to ensure that two Gentzen π-institutions are equivalent.

Lemma 1879 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr, tr′ *be traces,* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$, $\mathfrak{G}' = \langle \mathbf{F}, G'\rangle$ *two Gentzen π-institutions of traces* tr, tr′, *respectively, based on* $\mathbf{F}$, τ *a* tr-tr′*-transformation and* ρ *a* tr′-tr-*transformation. The following are equivalent:*

(i) $\tau : \mathfrak{G} \to \mathfrak{G}'$ *is an interpretation and, for all* $\Sigma \in |\mathbf{Sign}^\flat|$, $\boldsymbol{\phi}' \in \mathrm{Seq}_\Sigma^{\mathrm{tr}'}(\mathbf{F})$, $G'_\Sigma(\boldsymbol{\phi}') = G'_\Sigma(\tau_\Sigma[\rho_\Sigma[\boldsymbol{\phi}']])$;

(ii) $\rho : \mathfrak{G}' \to \mathfrak{G}$ *is an interpretation and, for all* $\Sigma \in |\mathbf{Sign}^\flat|$, $\boldsymbol{\phi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$, $G_\Sigma(\boldsymbol{\phi}) = G_\Sigma(\rho_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]])$.

Proof: Similar to the proof of Lemma 889. Suppose that the conditions in (i) hold. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\mathbf{\Phi}' \cup \{\boldsymbol{\phi}'\} \subseteq \mathrm{Seq}^{\mathrm{tr}'}_\Sigma(\mathbf{F})$, we have

$$\begin{array}{lll}\boldsymbol{\phi}' \in G'_\Sigma(\mathbf{\Phi}') & \text{iff} & \tau_\Sigma[\rho_\Sigma[\boldsymbol{\phi}']] \subseteq G'_\Sigma(\tau_\Sigma[\rho_\Sigma[\mathbf{\Phi}']]) \\ & \text{iff} & \rho_\Sigma[\boldsymbol{\phi}'] \subseteq G_\Sigma(\rho_\Sigma[\mathbf{\Phi}']).\end{array}$$

Hence, $\rho : \mathfrak{G}' \to \mathfrak{G}$ is also an interpretation. Finally, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$, we get, for all $\boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$,

$$\begin{array}{lll}\boldsymbol{\psi} \in G_\Sigma(\boldsymbol{\phi}) & \text{iff} & \tau_\Sigma[\boldsymbol{\psi}] \subseteq G'_\Sigma(\tau_\Sigma[\boldsymbol{\phi}]) \\ & \text{iff} & \tau_\Sigma[\boldsymbol{\psi}] \subseteq G_\Sigma(\tau_\Sigma[\rho_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]]]) \\ & \text{iff} & \boldsymbol{\psi} \in G_\Sigma(\rho_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]]).\end{array}$$

Thus, the second condition of (ii) is also satisfied. Thus (i) implies (ii) holds and, by symmetry, we conclude that (i) and (ii) are equivalent. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr, tr′ traces and τ a tr-tr′-transformation. Define

$$\tau^* : \mathrm{SenFam}(\mathrm{Seq}^{\mathrm{tr}'}(\mathbf{F})) \to \mathrm{SenFam}(\mathrm{Seq}^{\mathrm{tr}}(\mathbf{F}))$$

by setting, for all $\mathbf{\Phi}' \in \mathrm{SenFam}(\mathrm{Seq}^{\mathrm{tr}'}(\mathbf{F}))$,

$$\tau^*(\mathbf{\Phi}') = \{\tau^*_\Sigma(\mathbf{\Phi}')\}_{\Sigma \in |\mathbf{Sign}^\flat|}$$

be given, for all $\Sigma \in |\mathbf{Sign}^\flat|$, by

$$\tau^*_\Sigma(\mathbf{\Phi}') = \{\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F}) : \tau_\Sigma[\boldsymbol{\phi}] \subseteq \mathbf{\Phi}'_\Sigma\}.$$

Analogously to Theorem 893, we can show that, if $\mathfrak{G}$ and $\mathfrak{G}'$ are equivalent Gentzen π-institutions via a conjugate pair $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$, then $\rho^* : \mathbf{ThFam}(\mathfrak{G}) \to \mathbf{ThFam}(\mathfrak{G}')$ and $\tau^* : \mathbf{ThFam}(\mathfrak{G}') \to \mathbf{ThFam}(\mathfrak{G})$ form a pair of mutually inverse order isomorphisms between the complete lattices of the corresponding theory families.

Theorem 1880 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr*,* tr′ *traces,* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$*,* $\mathfrak{G}' = \langle \mathbf{F}, G' \rangle$ *Gentzen π-institutions of traces* tr*,* tr′*, respectively, and* $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ *a conjugate pair of transformations. Then*

$$\rho^* : \mathbf{ThFam}(\mathfrak{G}) \to \mathbf{ThFam}(\mathfrak{G}') \quad \textit{and} \quad \tau^* : \mathbf{ThFam}(\mathfrak{G}') \to \mathbf{ThFam}(\mathfrak{G})$$

are mutually inverse order isomorphisms.

Proof: Similar to the proof of Theorem 893. Let $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$, we get

$$\begin{array}{lll}\boldsymbol{\phi} \in \tau^*_\Sigma(\rho^*(\boldsymbol{T})) & \text{iff} & \tau_\Sigma[\boldsymbol{\phi}] \subseteq \rho^*_\Sigma(\boldsymbol{T}) \\ & \text{iff} & \rho_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]] \subseteq \boldsymbol{T}_\Sigma \\ & \text{iff} & \boldsymbol{\phi} \in \boldsymbol{T}_\Sigma.\end{array}$$

Thus, $\tau^*(\rho^*(\boldsymbol{T})) = \boldsymbol{T}$. By symmetry, for all $\boldsymbol{T}' \in \mathrm{ThFam}(\mathfrak{G}')$, $\rho^*(\tau^*(\boldsymbol{T}')) = \boldsymbol{T}'$. Thus, ρ^* and τ^* are mutually inverse bijections and, since they are both order preserving, they form a pair of mutually inverse order isomorphisms between $\mathbf{ThFam}(\mathfrak{G})$ and $\mathbf{ThFam}(\mathfrak{G}')$. ∎

Conversely, it is true that, under ceratin hypotheses, given mutually inverse order isomorphisms between the complete lattices of two Gentzen π-institutions, one may define a conjugate pair between the two that establishes the order-isomorphism via the process that was described above.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr, tr′ be traces, and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$, $\mathfrak{G}' = \langle \mathbf{F}, G' \rangle$ be Gentzen π-institutions of traces tr, tr′, respectively, based on $\mathbf{F}$. Consider an order isomorphism

$$h : \mathbf{ThFam}(\mathfrak{G}') \to \mathbf{ThFam}(\mathfrak{G})$$

between the corresponding complete lattices of theory families.

Define $\overrightarrow{h} = \{\overrightarrow{h}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by letting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\overrightarrow{h}_\Sigma : \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F}) \to \mathcal{P}(\mathrm{Seq}_\Sigma^{\mathrm{tr}'}(\mathbf{F}))$$

be given, for all $\boldsymbol{\phi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$, by

$$\overrightarrow{h}_\Sigma[\boldsymbol{\phi}] = h_\Sigma^{-1}(G(\boldsymbol{\phi})).$$

Further, define $\overleftarrow{h} = \{\overleftarrow{h}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\overleftarrow{h}_\Sigma : \mathrm{Seq}_\Sigma^{\mathrm{tr}'}(\mathbf{F}) \to \mathcal{P}(\mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F}))$$

be given, for all $\boldsymbol{\phi}' \in \mathrm{Seq}_\Sigma^{\mathrm{tr}'}(\mathbf{F})$, by

$$\overleftarrow{h}_\Sigma[\boldsymbol{\phi}'] = h_\Sigma(G'(\boldsymbol{\phi}')).$$

The order isomorphism $h : \mathbf{ThFam}(\mathfrak{G}') \to \mathbf{ThFam}(\mathfrak{G})$ is called **transformational** if there exist

- a tr-tr′-translation τ,
- a tr′-tr-translation ρ,

such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\boldsymbol{\phi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ and all $\boldsymbol{\phi}' \in \mathrm{Seq}_\Sigma^{\mathrm{tr}'}(\mathbf{F})$,

$$\overrightarrow{h}_\Sigma[\boldsymbol{\phi}] = G'_\Sigma(\tau_\Sigma[\boldsymbol{\phi}]) \quad \text{and} \quad \overleftarrow{h}_\Sigma[\boldsymbol{\phi}'] = G_\Sigma(\rho_\Sigma[\boldsymbol{\phi}']),$$

i.e., by definition of $\overrightarrow{h}$ and $\overleftarrow{h}$, if and only if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\boldsymbol{\phi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ and all $\boldsymbol{\phi}' \in \mathrm{Seq}_\Sigma^{\mathrm{tr}'}(\mathbf{F})$,

$$h_\Sigma^{-1}(G(\boldsymbol{\phi})) = G'_\Sigma(\tau_\Sigma[\boldsymbol{\phi}]) \quad \text{and} \quad h_\Sigma(G'(\boldsymbol{\phi}')) = G_\Sigma(\rho_\Sigma[\boldsymbol{\phi}']).$$

Here $G(\boldsymbol{\phi})$ and $G'(\boldsymbol{\phi}')$ denote the theory families of $\mathfrak{G}$ and $\mathfrak{G}$' generated by the Σ-sequents $\boldsymbol{\phi}$ and $\boldsymbol{\phi}'$, respectively. Since all components of these theory families other than the Σ-components consist of sets of theorems, we sometimes write by a slight abuse of notation

$$h_\Sigma^{-1}(G_\Sigma(\boldsymbol{\phi})) = G'_\Sigma(\tau_\Sigma[\boldsymbol{\phi}]) \quad \text{and} \quad h_\Sigma(G'_\Sigma(\boldsymbol{\phi}')) = G_\Sigma(\rho_\Sigma[\boldsymbol{\phi}']).$$

In this case, we say that h is **induced by** the pair of translations (τ,ρ).

We can show that the properties defining transformationality of an order isomorphism extend to sets of Σ-sequents.

Lemma 1881 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr, tr$'$ *be traces and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$, $\mathfrak{G}' = \langle \mathbf{F}, G'\rangle$ *Gentzen π-institutions of traces* tr, tr$'$, *respectively, based on* $\mathbf{F}$. *If* $h : \mathbf{ThFam}(\mathfrak{G}') \to \mathbf{ThFam}(\mathfrak{G})$ *a transformational order isomorphism induced by the pair* (τ,ρ) *of translations, then, for all for all* $\Sigma \in |\mathbf{Sign}^\flat|$, *all* $\mathbf{\Phi} \subseteq \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ *and all* $\mathbf{\Phi}' \subseteq \mathrm{Seq}_\Sigma^{\mathrm{tr}'}(\mathbf{F})$,

$$h_\Sigma^{-1}(G(\mathbf{\Phi})) = G'_\Sigma(\tau_\Sigma[\mathbf{\Phi}]) \quad \textit{and} \quad h_\Sigma(G'(\mathbf{\Phi}')) = G_\Sigma(\rho_\Sigma[\mathbf{\Phi}']).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$, and $\mathbf{\Phi} \subseteq \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$. Then, taking into account that both $\mathbf{ThFam}(\mathfrak{G})$ and $\mathbf{ThFam}(\mathfrak{G}')$ are ordered signature-wise, we have

$$\begin{array}{rcl} h_\Sigma^{-1}(G(\mathbf{\Phi})) &=& h_\Sigma^{-1}(\bigvee_{\boldsymbol{\phi}\in\mathbf{\Phi}} G(\boldsymbol{\phi}))\\ &=& \bigvee_{\boldsymbol{\phi}\in\mathbf{\Phi}} h_\Sigma^{-1}(G(\boldsymbol{\phi}))\\ &=& \bigvee_{\boldsymbol{\phi}\in\mathbf{\Phi}} G'_\Sigma(\tau_\Sigma[\boldsymbol{\phi}])\\ &=& G'_\Sigma(\bigcup_{\boldsymbol{\phi}\in\mathbf{\Phi}} \tau_\Sigma[\boldsymbol{\phi}])\\ &=& G'_\Sigma(\tau_\Sigma[\mathbf{\Phi}]).\end{array}$$

The second equality holds by symmetry. ■

Then the following result forms an analog of Theorem 900.

Theorem 1882 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr, tr$'$ *be traces and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$, $\mathfrak{G}' = \langle \mathbf{F}, G'\rangle$ *Gentzen π-institutions of traces* tr, tr$'$, *respectively, based on* $\mathbf{F}$. *If* $h : \mathbf{ThFam}(\mathfrak{G}') \to \mathbf{ThFam}(\mathfrak{G})$ *a transformational order isomorphism induced by the pair* (τ,ρ) *of translations, then* $(\tau,\rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ *is a conjugate pair of transformations.*

Proof: Similar to the proof of Theorem 900. Suppose $h : \mathbf{ThFam}(\mathfrak{G}') \to \mathbf{ThFam}(\mathfrak{G})$ is an order isomorphism and let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\mathbf{\Phi}' \cup \{\boldsymbol{\phi}'\} \subseteq \mathrm{Seq}_\Sigma^{\mathrm{tr}'}(\mathfrak{G}')$. Then we have

$$\begin{array}{rll} \boldsymbol{\phi}' \in G'_\Sigma(\mathbf{\Phi}') & \text{iff} & G'_\Sigma(\boldsymbol{\phi}') \subseteq G'_\Sigma(\mathbf{\Phi}')\\ & \text{iff} & h_\Sigma(G'(\boldsymbol{\phi}')) \subseteq h_\Sigma(G'(\mathbf{\Phi}'))\\ & \text{iff} & G_\Sigma(\rho_\Sigma[\boldsymbol{\phi}]) \subseteq G_\Sigma(\rho_\Sigma[\mathbf{\Phi}])\\ & \text{iff} & \rho_\Sigma[\boldsymbol{\phi}] \subseteq G_\Sigma(\rho_\Sigma[\mathbf{\Phi}]).\end{array}$$

Thus, $\rho : \mathfrak{G}' \to \mathfrak{G}$ is an interpretation. Furthermore, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$, we have

$$\begin{array}{rcl} G_\Sigma(\rho_\Sigma[\tau_\Sigma[\phi]]) & = & h_\Sigma(G'_\Sigma(\tau_\Sigma[\phi])) \\ & = & h_\Sigma(h^{-1}_\Sigma(G_\Sigma(\phi))) \\ & = & G_\Sigma(\phi). \end{array}$$

We conclude that $(\tau,\rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ is a conjugate pair. ■

Finally, we show that interpretations compose and the same holds for equivalences of Gentzen π-institutions.

Lemma 1883 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr*,* tr$'$*,* tr$''$ *be traces and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$*,* $\mathfrak{G}' = \langle \mathbf{F}, G' \rangle$*,* $\mathfrak{G}'' = \langle \mathbf{F}, G'' \rangle$ *be Gentzen π-institutions of traces* tr*,* tr$'$*,* tr$''$*, respectively, based on* $\mathbf{F}$*.*

(a) If $\tau : \mathfrak{G} \to \mathfrak{G}'$ *and* $\tau' : \mathfrak{G}' \to \mathfrak{G}''$ *are interpretations, then* $\tau' \circ \tau : \mathfrak{G} \to \mathfrak{G}''$ *is also an interpretation;*

(b) If $(\tau,\rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ *and* $(\tau',\rho') : \mathfrak{G}' \rightleftarrows \mathfrak{G}''$ *are conjugate pairs, then* $(\tau' \circ \tau, \rho \circ \rho') : \mathfrak{G} \rightleftarrows \mathfrak{G}''$ *is also a conjugate pair.*

Proof:

(a) Suppose $\tau : \mathfrak{G} \to \mathfrak{G}'$ and $\tau' : \mathfrak{G}' \to \mathfrak{G}''$ are interpretations. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\mathbf{\Phi} \cup \{\phi\} \subseteq \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$, we get

$$\begin{array}{rll} \phi \in G_\Sigma(\mathbf{\Phi}) & \text{iff} & \tau_\Sigma[\phi] \subseteq G'_\Sigma(\tau_\Sigma[\mathbf{\Phi}]) \\ & \text{iff} & \tau'_\Sigma[\tau_\Sigma[\phi]] \subseteq G''_\Sigma(\tau'_\Sigma[\tau_\Sigma[\Phi]]). \end{array}$$

hence, $\tau' \circ \tau : \mathfrak{G} \to \mathfrak{G}''$ is also an interpretation.

(b) Now suppose that $(\tau,\rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ and $(\tau',\rho') : \mathfrak{G}' \rightleftarrows \mathfrak{G}''$ are conjugate pairs. Then, by Part (a), $\tau' \circ \tau : \mathfrak{G} \to \mathfrak{G}''$ is an interpretation. Moreover, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi'', \psi'' \in \mathrm{Seq}^{\mathrm{tr}''}_\Sigma(\mathbf{F})$, we have $\psi'' \in G''_\Sigma(\phi'')$ if and only if

$$\rho'_\Sigma[\psi''] \subseteq G'_\Sigma(\rho'_\Sigma[\phi'']) = G'_\Sigma(\tau_\Sigma[\rho_\Sigma[\rho'_\Sigma[\phi'']]]).$$

This holds if and only if

$$\tau'_\Sigma[\rho'_\Sigma[\psi'']] \subseteq G''_\Sigma(\tau'_\Sigma[\tau_\Sigma[\rho_\Sigma[\rho'_\Sigma[\phi'']]]]).$$

Equivalently,

$$\psi'' \in G''_\Sigma(\tau'_\Sigma[\tau_\Sigma[\rho_\Sigma[\rho'_\Sigma[\phi'']]]]).$$

We conclude that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi'' \in \mathrm{Seq}^{\mathrm{tr}''}_\Sigma(\mathbf{F})$,

$$G''_\Sigma(\phi'') = G''_\Sigma(\tau'_\Sigma[\tau_\Sigma[\rho_\Sigma[\rho'_\Sigma[\phi'']]]]).$$

Therefore, by Lemma 1879, $(\tau' \circ \tau, \rho \circ \rho') : \mathfrak{G} \rightleftarrows \mathfrak{G}''$ is also a conjugate pair. ■

26.3 Hilbertizability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$. $\mathfrak{G}$ is **Hilbertizable** if it is equivalent to a Hilbert π-institution based on $\mathbf{F}$. In other words, $\mathfrak{G}$ is Hilbertizable if there exists a Hilbert π-institution $\mathfrak{H} = \langle \mathbf{F}, H \rangle$, based on $\mathbf{F}$, and a conjugate pair of transformations $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{H}$.

We have the following proposition that follows directly from the relevant definitions.

Proposition 1884 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system,* tr *a trace and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace* tr *based on $\mathbf{F}$. $\mathfrak{G}$ is Hilbertizable if and only if there exist:*

(1) A Hilbert π-institution $\mathfrak{H} = \langle \mathbf{F}, H \rangle$;

(2) A collection $\rho : (\mathrm{SEN}^\flat)^\omega \to \bigcup_{\langle m,n \rangle \in \mathrm{tr}} \mathrm{SEN}^{m+m}$ in $N^\flat$ with a single distinguished argument;

(3) A family $\tau = \{\tau^{m,n} : \langle m,n \rangle \in \mathrm{tr}\}$, where, for all $\langle m,n \rangle \in \mathrm{tr}$, the collection $\tau^{m,n} : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}$ in $N^\flat$ has $m+n$ distinguished arguments;

such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\mathbf{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

(a) $\boldsymbol{\phi} \in G_\Sigma(\mathbf{\Phi})$ iff $\tau_\Sigma[\boldsymbol{\phi}] \subseteq H_\Sigma(\tau_\Sigma[\mathbf{\Phi}])$;

(b) $H_\Sigma(\phi) = H_\Sigma(\tau_\Sigma[\rho_\Sigma[\phi]])$;

or, equivalently, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$,

(c) $\rhd_\Sigma \phi \in H_\Sigma(\rhd_\Sigma \Phi)$ iff $\rho_\Sigma[\phi] \subseteq G_\Sigma(\rho_\Sigma[\Phi])$;

(d) $G_\Sigma(\boldsymbol{\phi}) = G_\Sigma(\rho_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]])$.

Proof: This is a rephrasing of the definition of Hilbertizability using the conditions establishing an equivalence between two Gentzen π-institutions and taking into account Lemma 1879. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution and $\mathfrak{H} = \langle \mathbf{F}, H \rangle$ a Hilbert π-institution both based on $\mathbf{F}$. Define the $\{\langle 0,1 \rangle\}$-tr-transformation ρ^0 by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\rho^0_\Sigma[\phi] = \{\rhd_\Sigma \phi\}.$$

We say that $\mathfrak{G}$ and $\mathfrak{H}$ are **simply equivalent** if they are equivalent via a conjugate pair of the form $(\tau, \rho^0) : \mathfrak{G} \rightleftarrows \mathfrak{H}$. The Gentzen π-institution $\mathfrak{G}$ is **simply Hilbertizable** if it is simply equivalent to some Hilbert π-institution $\mathfrak{H} = \langle \mathbf{F}, H \rangle$.

If $\mathfrak{G}$ is simply Hilbertizable, it turns out that there is a unique Hilbert π-institution simply equivalent to $\mathfrak{G}$, namely, the Hilbert π-institution reduct $\mathfrak{G}^0$ of $\mathfrak{G}$.

Proposition 1885 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen π-institution based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *is simply Hilbertizable, then it is simply equivalent to a unique Hilbert π-institution, namely, the Hilbert π-institution reduct* $\mathfrak{G}^0 = \langle \mathbf{F}, G^0 \rangle$ *of* $\mathfrak{G}$.

Proof: Suppose that $\mathfrak{G}$ is simply Hilbertizable via the conjugate pair $(\tau, \rho^0) : \mathfrak{G} \to \mathfrak{H}$, with $\mathfrak{H} = \langle \mathbf{F}, H \rangle$. It suffices to show that $H = G^0$. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{lll} \rhd_\Sigma \phi \in H_\Sigma(\rhd_\Sigma \Phi) & \text{iff} & \rho^0_\Sigma[\phi] \subseteq G_\Sigma(\rho^0_\Sigma[\Phi]) \quad \text{(by hypothesis)} \\ & \text{iff} & \rhd_\Sigma \phi \in G_\Sigma(\rhd_\Sigma \Phi) \quad \text{(definition of } \rho^0) \\ & \text{iff} & \rhd_\Sigma \phi \in G^0_\Sigma(\rhd_\Sigma \Phi). \quad \text{(definition of } G^0) \end{array}$$

Therefore $\mathfrak{H} = \mathfrak{G}^0$, whence it follows that $\mathfrak{G}$ is simply Hilbertizable via a simple equivalence involving the Hilbert π-institution reduct $\mathfrak{G}^0$ of $\mathfrak{G}$ ■

We have, further, the following simpler characterization of simple Hilbertizability, due to the fact that the interpretation in one of the two directions is required to be a fixed one.

Proposition 1886 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$*, be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$*.* $\mathfrak{G}$ *is simply Hilbertizable if and only if there exists a* tr-$\{\langle 0,1 \rangle\}$*-transformation* $\tau = \{\tau^{m,n} : \langle m,n \rangle \in \mathrm{tr}\}$*, such that, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$*,*

$$G_\Sigma(\boldsymbol{\phi}) = G_\Sigma(\rhd \tau_\Sigma[\boldsymbol{\phi}]).$$

Proof: If $\mathfrak{G}$ is simply Hilbertizable, then, by Proposition 1885, it is equivalent to the Hilbert π-institution reduct $\mathfrak{G}^0$ of $\mathfrak{G}$ via some conjugate pair $(\tau, \rho^0) : \mathfrak{G} \rightleftarrows \mathfrak{G}^0$. Thus, by the definition of equivalence, we get, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$,

$$\begin{array}{lll} G_\Sigma(\boldsymbol{\phi}) & = & G_\Sigma(\rho^0_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]]) \\ & = & G_\Sigma(\rhd_\Sigma \tau_\Sigma[\boldsymbol{\phi}]). \end{array}$$

Assume, conversely, that there exists a tr-$\{\langle 0,1 \rangle\}$-transformation τ, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$, $G_\Sigma(\boldsymbol{\phi}) = G_\Sigma(\rhd \tau_\Sigma[\boldsymbol{\phi}])$. To show that $\mathfrak{G}$ is simply Hilbertizable, it suffices, by Proposition 1885 and Proposition 1884, to show that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\rhd_\Sigma \phi \in G^0_\Sigma(\rhd_\Sigma \Phi) \quad \text{iff} \quad \rhd_\Sigma \phi \in G_\Sigma(\rhd_\Sigma \Phi).$$

This equivalence, however, holds by the definition of G^0. ■

26.4 Syntactic WF Algebraizability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$. $\mathfrak{G}$ is (**syntactically WF**) **algebraizable** if it is equivalent to the Gentzen π-institution $\mathfrak{G}^{\mathsf{K}} = \langle \mathbf{F}, G^{\mathsf{K}} \rangle$ associated with some class K of $\mathbf{F}$-algebraic systems.

Explicitly, using the definition of equivalence, this means that there exists a class K of $\mathbf{F}$-algebraic systems, a tr-$\{\langle 1,1\rangle\}$-transformation τ and a $\{\langle 1,1\rangle\}$-tr-transformation ρ, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\mathbf{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

(a) $\boldsymbol{\phi} \in G_\Sigma(\mathbf{\Phi})$ iff $\tau_\Sigma[\boldsymbol{\phi}] \subseteq G^{\mathsf{K}}_\Sigma(\tau_\Sigma[\mathbf{\Phi}])$;

(b) $G^{\mathsf{K}}_\Sigma(\phi \rhd_\Sigma \psi) = G^{\mathsf{K}}_\Sigma(\tau_\Sigma[\rho_\Sigma[\phi;\psi]])$;

or, equivalently, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$,

(c) $\phi \rhd_\Sigma \psi \in G^{\mathsf{K}}_\Sigma(E)$ iff $\rho_\Sigma[\phi;\psi] \subseteq G_\Sigma(\rho_\Sigma[E])$;

(d) $G_\Sigma(\boldsymbol{\phi}) = G_\Sigma(\rho_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]])$.

Recall that, given a class K of $\mathbf{F}$-algebraic systems, we denote by $\mathbb{G}(\mathsf{K})$, the guasivariety of $\mathbf{F}$-algebraic systems generated by K, i.e., the collection of all $\mathbf{F}$-algebraic systems that satisfy the $\mathbf{F}$-guasiequations that are satisfied by all $\mathcal{A} \in \mathsf{K}$.

It turns out that, when a Gentzen π-institution $\mathfrak{G}$ is algebraizable via two different classes K and K' of $\mathbf{F}$-algebraic systems, then both classes K and K' generate the same guasivariety and, hence, that there exists a unique guasivariety of $\mathbf{F}$-algebraic systems that serves as the algebraizing class of $\mathfrak{G}$. This is proven in Proposition 1888, following a needed lemma.

Lemma 1887 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system,* tr *a trace and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace* tr *based on $\mathbf{F}$. If $\mathfrak{G}$ is algebraizable via $(\tau,\rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}}$, then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m,n\rangle \in \mathrm{tr}$,*

$$\boldsymbol{\psi} \in G_\Sigma(\{\boldsymbol{\phi}\} \cup \bigcup\{\rho_\Sigma[\boldsymbol{\phi}_i;\boldsymbol{\psi}_i] : i < m+n\}).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m,n\rangle \in \mathrm{tr}$. By the definition of an equational Gentzen π-institution, we get

$$\tau_\Sigma[\boldsymbol{\psi}] \subseteq G^{\mathsf{K}}_\Sigma(\tau_\Sigma[\boldsymbol{\phi}] \cup \{\boldsymbol{\phi}_i \rhd_\Sigma \boldsymbol{\psi}_i : i < m+n\}).$$

Thus, since, by the definition of equivalence

$$G^{\mathsf{K}}_\Sigma(\boldsymbol{\phi}_i \rhd_\Sigma \boldsymbol{\psi}_i) = G^{\mathsf{K}}_\Sigma(\tau_\Sigma[\rho_\Sigma[\boldsymbol{\phi}_i;\boldsymbol{\psi}_i]]),$$

we get that

$$\tau_\Sigma[\boldsymbol{\psi}] \subseteq G^{\mathsf{K}}_\Sigma(\tau_\Sigma[\boldsymbol{\phi}] \cup \bigcup\{\tau_\Sigma[\rho_\Sigma[\boldsymbol{\phi}_i;\boldsymbol{\psi}_i]] : i < m+n\}).$$

Therefore, since τ is an interpretation,

$$\boldsymbol{\psi} \in G_\Sigma(\{\boldsymbol{\phi}\} \cup \bigcup\{\rho_\Sigma[\boldsymbol{\phi}_i;\boldsymbol{\psi}_i] : i < m+n\}).$$

This establishes the conclusion. ∎

Proposition 1888 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *is algebraizable via both the conjugate pair* $(\tau,\rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}}$ *of transformations and the conjugate pair* $(\tau',\rho') : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}'}$ *of transformations, then* $\mathbb{G}(\mathsf{K}) = \mathbb{G}(\mathsf{K}')$.

Proof: Suppose that $\mathfrak{G}$ is algebraizable via both the conjugate pair $(\tau,\rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}}$ of transformations and the conjugate pair $(\tau',\rho') : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}'}$ of transformations.

We show, first, that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$G_\Sigma(\rho_\Sigma[\phi;\psi]) = G_\Sigma(\rho'_\Sigma[\phi;\psi]).$$

Note that $\rho'_\Sigma[\phi;\phi] \subseteq G_\Sigma(\varnothing)$, since $\phi \rhd_\Sigma \phi \in G^{\mathsf{K}'}_\Sigma(\varnothing)$ and $\rho' : \mathfrak{G}^{\mathsf{K}'} \to \mathfrak{G}$ is an interpretation. Moreover, for all $\boldsymbol{\sigma} \in \rho'$ of trace $\langle m,n\rangle \in$ tr, all $i < m+n$,all $\Sigma \in |\mathbf{Sign}^\flat|$, and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\rho_\Sigma[\boldsymbol{\sigma}^i_\Sigma(\phi,\phi,\vec{\chi});\boldsymbol{\sigma}^i_\Sigma(\phi,\psi,\vec{\chi})] \subseteq G_\Sigma(\rho_\Sigma[\phi;\psi]).$$

Since, by Lemma 1887, ρ has the modus ponens in $\mathfrak{G}$, we get that $\rho'_\Sigma[\phi;\psi] \subseteq G_\Sigma(\rho_\Sigma[\phi;\psi])$. By symmetry, we conclude that $G_\Sigma(\rho_\Sigma[\phi;\psi]) = G_\Sigma(\rho'_\Sigma[\phi;\psi])$.

Finally, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rll} \phi \rhd_\Sigma \psi \in G^{\mathsf{K}}_\Sigma(E) & \text{iff} & \rho_\Sigma[\phi;\psi] \subseteq G_\Sigma(\rho_\Sigma[E]) \\ & \text{iff} & \rho'_\Sigma[\phi;\psi] \subseteq G_\Sigma(\rho'_\Sigma[E]) \\ & \text{iff} & \phi \rhd_\Sigma \psi \in G^{\mathsf{K}'}_\Sigma(E). \end{array}$$

Thus, we get that $\mathbb{G}(\mathsf{K}) = \mathbb{G}(\mathsf{K}')$. ∎

Given an algebraizable Gentzen π-institution $\mathfrak{G}$, there exists, by Proposition 1888, a unique guasivariety K that serves as the algebraic counterpart of $\mathfrak{G}$. It is called the **equivalent algebraic semantics of** $\mathfrak{G}$.

The next result asserts that equivalent Gentzen systems have the same status vis-à-vis algebraizability and, in case they are algebraizable, they share a common algebraic semantics. Moreover, they share the same Hilbertizability status and, in case they are Hilbertizable, they share the same Hilbertizations (which, however, are not unique).

Proposition 1889 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr*,* tr′ *traces and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$*,* $\mathfrak{G} = \langle \mathbf{F}, G' \rangle$ *two equivalent Gentzen* π*-institutions of traces* tr*,* tr′*, respectively, based on* $\mathbf{F}$*.*

(a) $\mathfrak{G}$ *is algebraizable if and only if* $\mathfrak{G}'$ *is algebraizable. If this is the case,* $\mathfrak{G}$ *and* $\mathfrak{G}'$ *have the same algebraic semantics.*

(b) $\mathfrak{G}$ *is Hilbertizable if and only if* $\mathfrak{G}'$ *is Hilbertizable. If this is the case, every Hilbertization of* $\mathfrak{G}$ *is one of* $\mathfrak{G}'$ *also.*

Proof:

(a) Suppose $\mathfrak{G}$ and $\mathfrak{G}'$ are equivalent via $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ and that $\mathfrak{G}'$ is algebraizable via $(\tau', \rho') : \mathfrak{G}' \rightleftarrows \mathfrak{G}^{\mathsf{K}'}$, for some class K' of $\mathbf{F}$-algebraic systems. Then, by Lemma 1883,

$$\mathfrak{G} \underset{\rho}{\overset{\tau}{\rightleftarrows}} \mathfrak{G}' \underset{\rho'}{\overset{\tau'}{\rightleftarrows}} \mathfrak{G}^{\mathsf{K}'}$$

$(\tau' \circ \tau, \rho \circ \rho') : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}'}$ is witnessing the algebraizability of $\mathfrak{G}$. By symmetry $\mathfrak{G}$ is algebraizable if and only if $\mathfrak{G}'$ is algebraizable. Since any algebraizing class K' for $\mathfrak{G}'$ is also an algebraizing class for $\mathfrak{G}$, and vice versa, we get that $\mathfrak{G}$ and $\mathfrak{G}'$ have the same equivalent algebraic semantics.

(b) Suppose $\mathfrak{G}$ and $\mathfrak{G}'$ are equivalent via $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ and that $\mathfrak{G}'$ is Hilbertizable via $(\tau', \rho') : \mathfrak{G}' \rightleftarrows \mathfrak{H}'$, for some Hilbert π-institution $\mathfrak{H}'$. Then, again by Lemma 1883,

$$\mathfrak{G} \underset{\rho}{\overset{\tau}{\rightleftarrows}} \mathfrak{G}' \underset{\rho'}{\overset{\tau'}{\rightleftarrows}} \mathfrak{H}'$$

$(\tau' \circ \tau, \rho \circ \rho') : \mathfrak{G} \rightleftarrows \mathfrak{H}'$ is witnessing the Hilbertizability of $\mathfrak{G}$. By symmetry, $\mathfrak{G}$ is Hilbertizable if and only if $\mathfrak{G}'$ is. Moreover, any Hilbertization $\mathfrak{H}'$ for $\mathfrak{G}'$ serves also as one for $\mathfrak{G}$, and vice versa, i.e., $\mathfrak{G}$ and $\mathfrak{G}'$ share the same Hilbertizations.

■

Suppose that a Gentzen π-institution $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ of trace tr, together with a trace tr′, are given. We give, next, a characterization of the existence of an equivalence $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ of $\mathfrak{G}$ with some Gentzen π-institution $\mathfrak{G}'$, having the given trace tr′.

Theorem 1890 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr*,* tr′ *be traces and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*.* $\mathfrak{G}$ *is equivalent to a Gentzen* π*-institution* $\mathfrak{G}' = \langle \mathbf{F}, G' \rangle$ *of trace* tr′ *based on* $\mathbf{F}$ *if and only if there exist a* tr*-*tr′*-transformation* τ *and a* tr′*-*tr*-transformation* ρ*, such that:*

(1) $\rho^* : \mathrm{ThFam}(\mathfrak{G}) \to \mathrm{SenFam}(\mathrm{Seq}^{\mathrm{tr}'}(\mathbf{F}))$ *is injective on* $\mathrm{ThFam}(\mathfrak{G})$*;*

(2) For all $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$, $\rho^*_\Sigma(G(\boldsymbol{\phi})) = G'_\Sigma(\tau_\Sigma[\boldsymbol{\phi}])$*, where* G' *is the closure system induced by* $\rho^*(\mathrm{ThFam}(\mathfrak{G}))$*.*

Proof: Suppose, first, that there exists an equivalence $(\tau,\rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$, where $\mathfrak{G}' = \langle \mathbf{F}, G' \rangle$ is a Gentzen π-institution of trace tr′ based on $\mathbf{F}$. By Theorem 1880, we know that $\rho^* : \mathbf{ThFam}(\mathfrak{G}) \to \mathbf{ThFam}(\mathfrak{G}')$ is an order isomorphism, whence, in particular, it is injective on $\mathrm{ThFam}(\mathfrak{G})$. Moreover, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$, we have

$$\begin{array}{lll} \boldsymbol{\psi} \in \rho^*_\Sigma(G(\boldsymbol{\phi})) & \text{iff} & \rho_\Sigma[\boldsymbol{\psi}] \subseteq G_\Sigma(\boldsymbol{\phi}) \\ & \text{iff} & \tau_\Sigma[\rho_\Sigma[\boldsymbol{\psi}]] \subseteq G'_\Sigma(\tau_\Sigma[\boldsymbol{\phi}]) \\ & \text{iff} & \boldsymbol{\psi} \in G'_\Sigma(\tau_\Sigma[\boldsymbol{\phi}]). \end{array}$$

Therefore, $\rho^*_\Sigma(G(\boldsymbol{\phi})) = G'_\Sigma(\tau_\Sigma[\boldsymbol{\phi}])$.

Suppose, conversely, that there exist a tr-tr′-transformation τ and a tr′-tr-transformation ρ, such that Conditions (1) and (2) of the statement hold. Since $\rho^*(\mathrm{ThFam}(\mathfrak{G}))$ is closed under intersection, it defines a closure system on $\mathrm{Seq}^{\mathrm{tr}'}(\mathbf{F})$, which we denote by G', writing $\mathfrak{G}' = \langle \mathbf{F}, G' \rangle$ for the corresponding Gentzen π-institution of trace tr′. It suffices now, by Theorem 1882, to show that $\rho^* : \mathbf{ThFam}(\mathfrak{G}) \to \mathbf{ThFam}(\mathfrak{G}')$ is a transformational order isomorphism induced by (τ,ρ). We know, by hypothesis, that ρ^* is injective. By definition of $\mathfrak{G}'$, it is surjective. By definition of ρ^*, it is order preserving. Finally, it is order reflecting, since, for all $\boldsymbol{T}, \boldsymbol{T}' \in \mathrm{ThFam}(\mathfrak{G})$,

$$\begin{array}{lll} \rho^*(\boldsymbol{T}) \le \rho^*(\boldsymbol{T}') & \text{iff} & \rho^*(\boldsymbol{T}) \cap \rho^*(\boldsymbol{T}') = \rho^*(\boldsymbol{T}) \\ & \text{iff} & \rho^*(\boldsymbol{T} \cap \boldsymbol{T}') = \rho^*(\boldsymbol{T}) \\ & \text{iff} & \boldsymbol{T} \cap \boldsymbol{T}' = \boldsymbol{T} \\ & \text{iff} & \boldsymbol{T} \le \boldsymbol{T}'. \end{array}$$

Therefore, $\rho^* : \mathbf{ThFam}(\mathfrak{G}) \to \mathbf{ThFam}(\mathfrak{G}')$ is, indeed, an order isomorphism. To show that $\rho^* : \mathbf{ThFam}(\mathfrak{G}) \to \mathbf{ThFam}(\mathfrak{G}')$ is transformational, it suffices to show that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ and all $\boldsymbol{\phi}' \in \mathrm{Seq}^{\mathrm{tr}'}_\Sigma(\mathbf{F})$,

$$\rho^*_\Sigma(G(\boldsymbol{\phi})) = G'_\Sigma(\tau_\Sigma[\boldsymbol{\phi}]) \quad \text{and} \quad (\rho^*)^{-1}_\Sigma(G'(\boldsymbol{\phi}')) = G_\Sigma(\rho_\Sigma[\boldsymbol{\phi}']).$$

The first holds by hypothesis and the second holds by the definition of G', since $\rho^*_\Sigma(G(\rho_\Sigma[\boldsymbol{\phi}']))$ is the least theory family of $\mathfrak{G}'$ containing $\boldsymbol{\phi}'$. ∎

26.5 Matrix Families and Algebraic Semantics

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr a trace and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, an $\mathbf{F}$-algebraic system. A tr-**filter family**

of $\mathcal{A}$ is a family $\boldsymbol{T} \le \mathrm{Seq}^{\mathrm{tr}}(\mathcal{A})$. The pair $\mathfrak{A} = \langle \mathcal{A}, \boldsymbol{T}\rangle$ is called a tr-**matrix family**. It defines a closure family $G^{\mathfrak{A}}$ of trace tr on $\mathbf{F}$ as follows: For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$,

$$\boldsymbol{\phi} \in G^{\mathfrak{A}}_\Sigma(\boldsymbol{\Phi}) \quad \text{iff} \quad \text{for all } \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma, \Sigma'),$$
$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\boldsymbol{\Phi})) \subseteq \boldsymbol{T}_{F(\Sigma')} \text{ implies } \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\boldsymbol{\phi})) \in \boldsymbol{T}_{F(\Sigma')}.$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$. Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, be an $\mathbf{F}$-algebraic system and $\boldsymbol{T}$ a tr-filter family of $\mathcal{A}$. $\boldsymbol{T}$ is called a **$\mathfrak{G}$-filter family of $\mathcal{A}$** if $G \le G^{\mathfrak{A}}$, i.e., if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$,

$$\boldsymbol{\phi} \in G_\Sigma(\boldsymbol{\Phi}) \quad \text{implies} \quad \boldsymbol{\phi} \in G^{\mathfrak{A}}_\Sigma(\boldsymbol{\Phi}).$$

Note that, as was pointed out previously, because of the structurality of G, it suffices to check that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$, such that $\boldsymbol{\phi} \in G_\Sigma(\boldsymbol{\Phi})$, we have

$$\alpha_\Sigma(\boldsymbol{\Phi}) \subseteq \boldsymbol{T}_{F(\Sigma)} \quad \text{implies} \quad \alpha_\Sigma(\boldsymbol{\phi}) \in \boldsymbol{T}_{F(\Sigma)}.$$

If $\boldsymbol{T}$ is a $\mathfrak{G}$-filter family of $\mathcal{A}$, then the pair $\mathfrak{A} = \langle \mathcal{A}, \boldsymbol{T}\rangle$ is called a **$\mathfrak{G}$-matrix family of $\mathcal{A}$**. We denote by $\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$ the collection of all $\mathfrak{G}$-filter families of $\mathcal{A}$ and by $\mathrm{MatFam}(\mathfrak{G})$ the collection of all $\mathfrak{G}$-matrix families.

Many facts, introduced previously in this work, that hold for $\mathcal{I}$-filter families and $\mathcal{I}$-matrix families, for a π-institution $\mathcal{I}$, have analogs for $\mathfrak{G}$-filter and $\mathfrak{G}$-matrix families, respectively. We list some of those that will be needed in the sequel.

Lemma 1891 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace,* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$ *and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta\rangle\rangle$ $\mathbf{F}$*-algebraic systems.*

(a) The collection $\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$ *forms a complete lattice*

$$\mathbf{FiFam}^{\mathfrak{G}}(\mathcal{A}) = \langle \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}), \le\rangle$$

under signature-wise inclusion $\le$*;*

(b) $\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{F}) = \mathrm{ThFam}(\mathfrak{G})$*;*

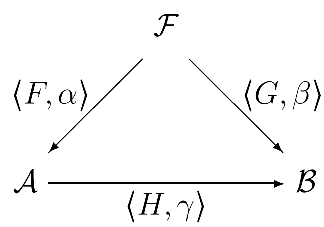
(c) If $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta\rangle\rangle$ *are* $\mathbf{F}$*-algebraic systems and* $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism, then* $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{B})$ *if and only if* $\gamma^{-1}(\boldsymbol{T}) \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$*.*

Proof:

(a) Let $\{\boldsymbol{T}^i : i \in I\} \subseteq \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$, such that $\boldsymbol{\phi} \subseteq G_\Sigma(\boldsymbol{\Phi})$. Then, if $\alpha_\Sigma(\boldsymbol{\Phi}) \subseteq \bigcap_{i\in I} \boldsymbol{T}^i_{F(\Sigma)}$, we get $\alpha_\Sigma(\boldsymbol{\Phi}) \subseteq \boldsymbol{T}^i_{F(\Sigma)}$, for all $i \in I$, whence, since $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, $\alpha_\Sigma(\boldsymbol{\phi}) \in \boldsymbol{T}^i_{F(\Sigma)}$, for all $i \in I$, i.e., $\alpha_\Sigma(\boldsymbol{\phi}) \in \bigcap_{i\in I} \boldsymbol{T}^i_{F(\Sigma)}$. We conclude that $\bigcap_{i\in I} \boldsymbol{T}^i \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$.

(b) For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$, such that $\boldsymbol{\phi} \in G_\Sigma(\boldsymbol{\Phi})$, we get that, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, $\boldsymbol{\Phi} \subseteq \boldsymbol{T}_\Sigma$ implies $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$. Therefore, $\mathrm{ThFam}(\mathfrak{G}) \subseteq \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{F})$. On the other hand, if $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{F})$, then, if $\boldsymbol{\phi} \in G_\Sigma(\boldsymbol{T}_\Sigma)$, then $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$, i.e., $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$. Therefore, $\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{F}) = \mathrm{ThFam}(\mathfrak{G})$.

(c) Assume, first, that $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{B})$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\boldsymbol{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$, such that $\boldsymbol{\phi} \in G_\Sigma(\boldsymbol{\Phi})$ and $\alpha_\Sigma(\boldsymbol{\Phi}) \subseteq \gamma^{-1}_{F(\Sigma)}(\boldsymbol{T}_{H(F(\Sigma))})$. Then



$\gamma_{F(\Sigma)}(\alpha_\Sigma(\boldsymbol{\Phi})) \subseteq \boldsymbol{T}_{H(F(\Sigma))}$, i.e., $\beta_\Sigma(\boldsymbol{\Phi}) \subseteq \boldsymbol{T}_{G(\Sigma)}$. Since $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{B})$, we now get $\beta_\Sigma(\boldsymbol{\phi}) \in \boldsymbol{T}_{G(\Sigma)}$. Reversing the steps above, we conclude that $\alpha_\Sigma(\boldsymbol{\phi}) \in \gamma^{-1}_{F(\Sigma)}(\boldsymbol{T}_{H(F(\Sigma))})$. Therefore, $\gamma^{-1}(\boldsymbol{T}) \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$.

Assume, conversely, that $\gamma^{-1}(\boldsymbol{T}) \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\boldsymbol{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$, such that $\boldsymbol{\phi} \in G_\Sigma(\boldsymbol{\Phi})$ and $\beta_\Sigma(\boldsymbol{\Phi}) \subseteq \boldsymbol{T}_{G(\Sigma)}$. Then $\gamma_{F(\Sigma)}(\alpha_\Sigma(\boldsymbol{\Phi})) \subseteq \boldsymbol{T}_{H(F(\Sigma))}$, whence $\alpha_\Sigma(\boldsymbol{\Phi}) \subseteq \gamma^{-1}_{F(\Sigma)}(\boldsymbol{T}_{H(F(\Sigma))})$. Since $\gamma^{-1}(\boldsymbol{T}) \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, we get $\alpha_\Sigma(\boldsymbol{\phi}) \in \gamma^{-1}_{F(\Sigma)}(\boldsymbol{T}_{H(F(\Sigma))})$. Reversing, once more, the preceding steps, we get that $\beta_\Sigma(\boldsymbol{\phi}) \in \boldsymbol{T}_{G(\Sigma)}$. Therefore, $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{B})$.

■

The isomorphism between the complete lattices of theory families induced by an equivalence extends to corresponding order isomorphisms between the complete lattices of filter families of the equivalent Gentzen π-institutions on the same algebraic system.

Proposition 1892 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr, tr′ *be traces, and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$, $\mathfrak{G}' = \langle \mathbf{F}, G'\rangle$ *two Gentzen π-institutions of traces* tr, tr′, *respectively, based on* $\mathbf{F}$. *If* $\mathfrak{G}$ *and* $\mathfrak{G}'$ *are equivalent via the conjugate pair* $(\tau,\rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$, *then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, *the mappings*

$$T \longmapsto \rho^{\mathcal{A}*}(T), \qquad T \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}),$$

$$\tau^{\mathcal{A}*}(T') \longleftarrow\!\!\dashv T', \qquad T' \in \mathrm{FiFam}^{\mathfrak{G}'}(\mathcal{A}),$$

are mutually inverse isomorphisms from $\mathbf{FiFam}^{\mathfrak{G}}(\mathcal{A})$ *onto* $\mathbf{FiFam}^{\mathfrak{G}'}(\mathcal{A})$.

Proof: We show, first, that, for all $\boldsymbol{T} \in \mathrm{FiFam}^{\mathcal{G}}(\mathcal{A})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$,

$$\rho^{\mathcal{A}}_{F(\Sigma)}[\tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\boldsymbol{\phi})]] \subseteq \boldsymbol{T}_{F(\Sigma)} \quad \text{iff} \quad \alpha_\Sigma(\boldsymbol{\phi}) \in \boldsymbol{T}_{F(\Sigma)}.$$

Indeed, taking into account the surjectivity of $\langle F, \alpha\rangle$, we obtain

$$\begin{array}{rll}\rho^{\mathcal{A}}_{F(\Sigma)}[\tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\boldsymbol{\phi})]] \subseteq \boldsymbol{T}_{F(\Sigma)} & \text{iff} & \rho^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\tau_\Sigma[\boldsymbol{\phi}])] \subseteq \boldsymbol{T}_{F(\Sigma)} \\ & \text{iff} & \alpha_\Sigma(\rho_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]]) \subseteq \boldsymbol{T}_{F(\Sigma)} \\ & \text{iff} & \rho_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]] \subseteq \alpha_\Sigma^{-1}(\boldsymbol{T}_{F(\Sigma)}) \\ & \text{iff} & \boldsymbol{\phi} \in \alpha_\Sigma^{-1}(\boldsymbol{T}_{F(\Sigma)}) \\ & \text{iff} & \alpha_\Sigma(\boldsymbol{\phi}) \in \boldsymbol{T}_{F(\Sigma)}.\end{array}$$

By symmetry, we also have, for all $\boldsymbol{T}' \in \mathrm{FiFam}^{\mathcal{G}'}(\mathcal{A})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi}' \in \mathrm{Seq}_\Sigma^{\mathrm{tr}'}(\mathbf{F})$,

$$\tau^{\mathcal{A}}_{F(\Sigma)}[\rho^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\boldsymbol{\phi}')]] \subseteq \boldsymbol{T}'_{F(\Sigma)} \quad \text{iff} \quad \alpha_\Sigma(\boldsymbol{\phi}') \in \boldsymbol{T}'_{F(\Sigma)}.$$

Using the first of these equivalences and, once again, taking into account the surjectivity of $\langle F, \alpha\rangle$, we get, for all $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\boldsymbol{\phi} \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{rll}\boldsymbol{\phi} \in \tau^{\mathcal{A}*}_\Sigma(\rho^{\mathcal{A}*}(\boldsymbol{T})) & \text{iff} & \tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}] \subseteq \rho^{\mathcal{A}*}_\Sigma(\boldsymbol{T}) \\ & \text{iff} & \rho^{\mathcal{A}}_\Sigma[\tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}]] \subseteq \boldsymbol{T}_\Sigma \\ & \text{iff} & \boldsymbol{\phi} \in \boldsymbol{T}_\Sigma.\end{array}$$

Thus, $\tau^{\mathcal{A}*}(\rho^{\mathcal{A}*}(\boldsymbol{T})) = \boldsymbol{T}$, for all $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$ and, by symmetry, we also have $\rho^{\mathcal{A}*}(\tau^{\mathcal{A}*}(\boldsymbol{T}')) = \boldsymbol{T}'$, for all $\boldsymbol{T}' \in \mathrm{FiFam}^{\mathfrak{G}'}(\mathcal{A})$. Therefore, $\rho^{\mathcal{A}*}$ and $\tau^{\mathcal{A}*}$ are mutually inverse bijections and reflect component-wise inclusion, since they are obviously order preserving undel $\leq$. We conclude that

$$\rho^{\mathcal{A}*} : \mathbf{FiFam}^{\mathfrak{G}}(\mathcal{A}) \rightleftarrows \mathbf{FiFam}^{\mathfrak{G}'}(\mathcal{A}) : \tau^{\mathcal{A}*}$$

are mutually inverse order isomorphisms. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, tr a trace, $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ an $N^\flat$-algebraic system and $\theta \in \mathrm{ConSys}(\mathbf{A})$.

Given $\boldsymbol{\phi} = \vec{\phi} \rhd_\Sigma \vec{\psi}$, $\boldsymbol{\phi}' = \vec{\phi}' \rhd_\Sigma \vec{\psi}' \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{A})$ of the same trace $\langle m, n\rangle$, we say that $\boldsymbol{\phi}$ is **θ-equivalent to $\boldsymbol{\phi}'$**, denoted $\boldsymbol{\phi}\ \theta_\Sigma\ \boldsymbol{\phi}'$, if, for all $i < m$ and all $j < n$,

$$\phi_i\ \theta_\Sigma\ \phi'_i \quad \text{and} \quad \psi_j\ \theta_\Sigma\ \psi'_j.$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, tr a trace, $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ an $N^\flat$-algebraic system, $\boldsymbol{T} \leq \mathrm{Seq}^{\mathrm{tr}}(\mathbf{A})$ and $\theta \in \mathrm{ConSys}(\mathbf{A})$.

We say that θ is **compatible with $\boldsymbol{T}$** if, for all $\Sigma \in |\mathbf{Sign}|$, and all $\boldsymbol{\phi}, \boldsymbol{\phi}' \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{A})$ (of the same trace),

$$\boldsymbol{\phi}\ \theta_\Sigma\ \boldsymbol{\phi}' \quad \text{and} \quad \boldsymbol{\phi} \in \boldsymbol{T}_\Sigma \quad \text{imply} \quad \boldsymbol{\phi}' \in \boldsymbol{T}_\Sigma.$$

An alternative characterization of compatibility is given in the following lemma.

Lemma 1893 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system,* $\boldsymbol{T} \leq \mathrm{Seq}^{\mathrm{tr}}(\mathcal{A})$ *and* $\theta \in \mathrm{ConSys}(\mathcal{A})$*.* θ *is compatible with* $\boldsymbol{T}$ *if and only if the quotient morphism* $\langle I, \pi^\theta \rangle : \mathcal{A} \to \mathcal{A}^\theta$ *induces a strict matrix family morphism*

$$\langle I, \pi^\theta \rangle : \langle \mathcal{A}, \boldsymbol{T} \rangle \to \langle \mathcal{A}^\theta, \pi^\theta(\boldsymbol{T}) \rangle,$$

i.e., if and only if $(\pi^\theta)^{-1}(\pi^\theta(\boldsymbol{T})) = \boldsymbol{T}$*.*

Proof: Suppose, first, that θ is compatible with $\boldsymbol{T}$ and let $\Sigma \in |\mathbf{Sign}|$, $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathcal{A})$, such that $\boldsymbol{\phi} \in (\pi^\theta_\Sigma)^{-1}(\pi^\theta_\Sigma(\boldsymbol{T}_\Sigma))$. Then, we get $\pi^\theta_\Sigma(\boldsymbol{\phi}) \in \pi^\theta_\Sigma(\boldsymbol{T}_\Sigma)$. Hence, there exists $\boldsymbol{\phi}' \in \boldsymbol{T}_\Sigma$, such that $\boldsymbol{\phi} \ \theta_\Sigma \ \boldsymbol{\phi}'$. Therefore, by the compatibility of θ with $\boldsymbol{T}$, we get that $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$. Thus, $(\pi^\theta)^{-1}(\pi^\theta(\boldsymbol{T})) \leq \boldsymbol{T}$ and, since the reverse inclusion always holds, we conclude that $(\pi^\theta)^{-1}(\pi^\theta(\boldsymbol{T})) = \boldsymbol{T}$.

Conversely, assume that $(\pi^\theta)^{-1}(\pi^\theta(\boldsymbol{T})) = \boldsymbol{T}$. Let $\Sigma \in |\mathbf{Sign}|$, $\boldsymbol{\phi}, \boldsymbol{\phi}' \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathcal{A})$, such that $\boldsymbol{\phi} \ \theta_\Sigma \ \boldsymbol{\phi}'$ and $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$. Then $\boldsymbol{\phi}' \in (\pi^\theta_\Sigma)^{-1}(\pi^\theta_\Sigma(\boldsymbol{T}_\Sigma)) = \boldsymbol{T}_\Sigma$ and, hence, θ is compatible with $\boldsymbol{T}$. ■

Given a Gentzen π-institution, taking the quotient of any filter family by a compatible congruence system results in a filter family on the quotient algebraic system.

Lemma 1894 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace,* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$, $\langle \mathcal{A}, \boldsymbol{T} \rangle$ *an* $\mathbf{F}$*-matrix family and* $\theta \in \mathrm{ConSys}(\mathcal{A})$*. If* θ *is compatible with* $\boldsymbol{T}$*, then*

$$\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}) \quad \textit{iff} \quad \boldsymbol{T}/\theta \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}^\theta).$$

Proof: Suppose that θ is compatible with $\boldsymbol{T}$. Then, using Lemmas 1891 and 1893, we have the following equivalences:

$$\begin{array}{rll} \boldsymbol{T}/\theta \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}/\theta) & \text{iff} & (\pi^\theta \circ \alpha)^{-1}(\boldsymbol{T}/\theta) \in \mathrm{ThFam}(\mathfrak{G}) \\ & \text{iff} & \alpha^{-1}((\pi^\theta)^{-1}(\boldsymbol{T}/\theta)) \in \mathrm{ThFam}(\mathfrak{G}) \\ & \text{iff} & \alpha^{-1}(\boldsymbol{T}) \in \mathrm{ThFam}(\mathfrak{G}) \\ & \text{iff} & \boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}). \end{array}$$

Hence $\boldsymbol{T}/\theta$ is a $\mathfrak{G}$-filter family of $\mathcal{A}^\theta$ iff $\boldsymbol{T}$ is a $\mathfrak{G}$-filter family of $\mathcal{A}$. ■

The following lemma forms an analog of the characterization of the Leibniz congruence system of a filter family on a given $\mathbf{F}$-algebraic system. It will also give rise to a corresponding operator, also termed the Leibniz operator, for theory families of Gentzen π-institutions.

Lemma 1895 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *an* $N^\flat$*-algebraic system,* $\boldsymbol{T} \leq \mathrm{Seq}^{\mathrm{tr}}(\mathbf{A})$ *and* $\theta \in \mathrm{ConSys}(\mathbf{A})$*.* θ *is compatible with* $\boldsymbol{T}$ *if and only if, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,*

$\langle\phi,\psi\rangle\in\theta_\Sigma$ *implies, for all* $\langle m,n\rangle\in\mathrm{tr}$*, all* $\vec{\sigma}=\langle\sigma^0,\dots,\sigma^{m-1}\rangle$*, and all* $\vec{\tau}=\langle\tau^0,\dots,\tau^{n-1}\rangle$ *in* $N^\flat$*, all* $\Sigma'\in|\mathbf{Sign}|$*, all* $f\in\mathbf{Sign}(\Sigma,\Sigma')$ *and all* $\vec{\chi}\in\mathrm{SEN}(\Sigma')$*,*

$$\begin{aligned}&\vec{\sigma}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi})\rhd_{\Sigma'}\vec{\tau}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi})\in\boldsymbol{T}_{\Sigma'}\\ &\qquad\textit{iff}\quad \vec{\sigma}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi})\rhd_{\Sigma'}\vec{\tau}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi})\in\boldsymbol{T}_{\Sigma'}.\end{aligned}$$

Proof: Suppose that $\Sigma\in|\mathbf{Sign}|$ and $\langle\phi,\psi\rangle\in\mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle\in\theta_\Sigma$. Since θ is a congruence *system*, for all $\Sigma'\in|\mathbf{Sign}|$ and all $f\in\mathbf{Sign}(\Sigma,\Sigma')$ $\langle\mathrm{SEN}(f)(\phi),\mathrm{SEN}(f)(\psi)\rangle\in\theta_{\Sigma'}$. Since θ is a *congruence* system, we get, for all $i<m$, all $j<n$ and all $\vec{\chi}\in\mathrm{SEN}(\Sigma')$,

$$\begin{aligned}&\langle\sigma^i_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}),\sigma^i_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi})\rangle\in\theta_{\Sigma'}\\ &\qquad\text{and}\quad\langle\tau^j_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}),\tau^j_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi})\rangle\in\theta_{\Sigma'}.\end{aligned}$$

The conclusion follows immediately by the assumption of compatibility of θ with $\boldsymbol{T}$. ■

Lemma 1895 serves to show that, given an algebraic system $\mathbf{A}$ and $\boldsymbol{T}\le\mathrm{Seq}^{\mathrm{tr}}(\mathbf{A})$, there exists a largest congruence system on $\mathbf{A}$ that is compatible with $\boldsymbol{T}$.

Corollary 1896 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* tr *a trace,* $\mathbf{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$ *an* $N^\flat$*-algebraic system and* $\boldsymbol{T}\le\mathrm{Seq}^{\mathrm{tr}}(\mathbf{A})$*. There exists a largest congruence system on* $\mathbf{A}$ *compatible with* $\boldsymbol{T}$*.*

Proof: Define $\theta=\{\theta_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ as follows: For all $\Sigma\in|\mathbf{Sign}|$ and all $\phi,\psi\in\mathrm{SEN}(\Sigma)$, $\langle\phi,\psi\rangle\in\theta_\Sigma$ iff, for all $\langle m,n\rangle\in\mathrm{tr}$, all $\vec{\sigma}=\langle\sigma^0,\dots,\sigma^{m-1}\rangle$, and all $\vec{\tau}=\langle\tau^0,\dots,\tau^{n-1}\rangle$ in $N^\flat$, all $\Sigma'\in|\mathbf{Sign}|$, all $f\in\mathbf{Sign}(\Sigma,\Sigma')$ and all $\vec{\chi}\in\mathrm{SEN}(\Sigma')$,

$$\begin{aligned}&\vec{\sigma}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi})\rhd_{\Sigma'}\vec{\tau}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi})\in\boldsymbol{T}_{\Sigma'}\\ &\qquad\text{iff}\quad \vec{\sigma}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi})\rhd_{\Sigma'}\vec{\tau}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi})\in\boldsymbol{T}_{\Sigma'}.\end{aligned}$$

It is easy to see that θ, thus defined, is a congruence system on $\mathbf{A}$ compatible with $\boldsymbol{T}$. By Lemma 1895, it is the largest one compatible with $\boldsymbol{T}$. ■

The largest congruence system on $\mathbf{A}$ compatible with $\boldsymbol{T}$ is denoted by $\Omega^{\mathbf{A}}(\boldsymbol{T})$ and called the **Leibniz congruence system of $\boldsymbol{T}$ on A**.

As a consequence of the definition of the Leibniz congruence system, given $\boldsymbol{T}\in\mathrm{Seq}^{\mathrm{tr}}(\mathbf{A})$ and $\theta\in\mathrm{ConSys}(\mathbf{A})$,

$$\theta\text{ is compatible with }\boldsymbol{T}\text{ if and only if }\theta\le\Omega^{\mathbf{A}}(\boldsymbol{T}).$$

Given an algebraic system $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$, a trace tr, a Gentzen π-institution $\mathfrak{G}=\langle\mathbf{F},G\rangle$ of trace tr based on $\mathbf{F}$ and an $\mathbf{F}$-algebraic system $\mathcal{A}$, the operator

$$\Omega^{\mathbf{A}}:\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\to\mathrm{ConSys}(\mathcal{A})$$

is called the **Leibniz operator of $\mathfrak{G}$ on $\mathcal{A}$**.

Recall from Proposition 1892 that, given two equivalent Gentzen π-institutions, the conjugate transformations establishing the equivalence induce an order isomorphism between the corresponding filter families of the gentzen π-institutions involved on arbitrary algebraic systems. It turns out that, under this isomorphism, corresponding filter families have identical Leibniz congruence systems.

Proposition 1897 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr, tr′ *traces and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$, $\mathfrak{G}' = \langle \mathbf{F}, G' \rangle$ *Gentzen π-institutions of traces* tr, tr′*, respectively, based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *and* $\mathfrak{G}'$ *are equivalent via a conjugate pair* $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ *of transformations, then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$*,*

$$\Omega^{\mathcal{A}}(\boldsymbol{T}) = \Omega^{\mathcal{A}}(\rho^{\mathcal{A}*}(\boldsymbol{T})).$$

Proof: Let $\Sigma \in |\mathbf{Sign}|$, $\boldsymbol{\phi}, \boldsymbol{\phi}' \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathcal{A})$, such that $\boldsymbol{\phi}\ \Omega_\Sigma^{\mathcal{A}}(\boldsymbol{T})\ \boldsymbol{\phi}'$ and suppose that $\boldsymbol{\phi} \in \rho_\Sigma^{\mathcal{A}*}(\boldsymbol{T})$. Then, we obtain $\rho_\Sigma^{\mathcal{A}}[\boldsymbol{\phi}] \subseteq \boldsymbol{T}_\Sigma$. Thus, since, by definition, $\Omega^{\mathcal{A}}(\boldsymbol{T})$ is a congruence system compatible with $\boldsymbol{T}$, we get that $\rho_\Sigma^{\mathcal{A}}[\boldsymbol{\phi}'] \subseteq \boldsymbol{T}_\Sigma$ and, therefore, $\boldsymbol{\phi}' \in \rho_\Sigma^{\mathcal{A}*}(\boldsymbol{T})$. Hence $\Omega^{\mathcal{A}}(\boldsymbol{T})$ is compatible with $\rho^{\mathcal{A}*}(\boldsymbol{T})$, showing that $\Omega^{\mathcal{A}}(\boldsymbol{T}) \leq \Omega^{\mathcal{A}}(\rho^{\mathcal{A}*}(\boldsymbol{T}))$.

Assume, conversely, that $\Sigma \in |\mathbf{Sign}|$, $\boldsymbol{\phi}, \boldsymbol{\phi}' \in \mathrm{Seq}_\Sigma^{\mathrm{tr}'}(\mathcal{A})$, such that

$$\boldsymbol{\phi}\ \Omega_\Sigma^{\mathcal{A}}(\rho^{\mathcal{A}*}(\boldsymbol{T}))\ \boldsymbol{\phi}'$$

and suppose that $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$. Then, we obtain $\rho_\Sigma^{\mathcal{A}}[\tau_\Sigma^{\mathcal{A}}[\boldsymbol{\phi}]] \subseteq \boldsymbol{T}_\Sigma$, i.e., $\tau_\Sigma^{\mathcal{A}}[\boldsymbol{\phi}] \subseteq \rho_\Sigma^{\mathcal{A}*}(\boldsymbol{T})$. Thus, since, by definition, $\Omega^{\mathcal{A}}(\rho^{\mathcal{A}*}(\boldsymbol{T}))$ is a congruence system compatible with $\rho^{\mathcal{A}*}(\boldsymbol{T})$, we get that $\tau_\Sigma^{\mathcal{A}}[\boldsymbol{\phi}'] \subseteq \rho^{\mathcal{A}*}(\boldsymbol{T})$. Therefore, $\rho_\Sigma^{\mathcal{A}}[\tau_\Sigma^{\mathcal{A}}[\boldsymbol{\phi}']] \subseteq \boldsymbol{T}_\Sigma$. So $\boldsymbol{\phi}' \in \boldsymbol{T}_\Sigma$ and, hence, $\Omega^{\mathcal{A}}(\rho^{\mathcal{A}*}(\boldsymbol{T}))$ is compatible with $\boldsymbol{T}$, showing that $\Omega^{\mathcal{A}}(\rho^{\mathcal{A}*}(\boldsymbol{T})) \leq \Omega^{\mathcal{A}}(\boldsymbol{T})$. ■

As was the case with ordinary π-institutions, the Suszko operator is a very useful tool in the study of the algebraization of Gentzen π-institutions.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr a trace, $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system. The **Suszko operator** $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}$ **of** $\mathfrak{G}$ **on** $\mathcal{A}$ is the operator

$$\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}} : \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}) \to \mathrm{ConSys}(\mathcal{A})$$

defined, for all $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, by

$$\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) = \bigcap \{\Omega^{\mathcal{A}}(\boldsymbol{T}') : \boldsymbol{T} \leq \boldsymbol{T}' \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\}.$$

Since, obviously, for all $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$,

$$\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) \leq \Omega^{\mathcal{A}}(\boldsymbol{T}),$$

$\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$ is also a congruence system on $\mathcal{A}$ compatible with $\boldsymbol{T}$. Moreover, the operator $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}$ is monotone on $\mathbf{FiFam}^{\mathfrak{G}}(\mathcal{A})$, for every $\mathbf{F}$-algebraic system $\mathcal{A}$.

Using the definition of the Suszko congruence system and Corollary 1896, it is not difficult to see that the following characterization of the Suszko congruence system of a filter family holds:

Proposition 1898 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace,* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$*. For all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,* $\langle \phi, \psi\rangle \in \widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}_\Sigma(\boldsymbol{T})$ *if and only if, for all* $\langle m, n\rangle \in \mathrm{tr}$*, all* $\sigma^0, \ldots, \sigma^{m-1}$*,* $\tau^0, \ldots, \tau^{n-1}$ *in* $N^\flat$*, all* $\Sigma' \in |\mathbf{Sign}|$*, all* $f \in \mathbf{Sign}(\Sigma, \Sigma')$ *and all* $\vec{\chi} \in \mathrm{SEN}(\Sigma')$*,*

$$\begin{aligned} &G^{\mathfrak{G},\mathcal{A}}_{\Sigma'}(\boldsymbol{T}_{\Sigma'}, \vec{\sigma}^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \rhd_{\Sigma'} \vec{\tau}^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}))\\ &\quad = G^{\mathfrak{G},\mathcal{A}}_{\Sigma'}(\boldsymbol{T}_{\Sigma'}, \vec{\sigma}^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}) \rhd_{\Sigma'} \vec{\tau}^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi})).\end{aligned}$$

Proof: The statement follows directly by combining the definition of the Suszko congruence system of $\boldsymbol{T}$ on $\mathcal{A}$ with the characterization of the Leibniz operator of each $\boldsymbol{T}'$, with $\boldsymbol{T} \le \boldsymbol{T}'$, given in the proof of Corollary 1896. ∎

Moreover, as is clear from the definition, we have

Lemma 1899 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *be a trace,* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$*, and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system. The Suszko and the Leibniz operators on* $\mathcal{A}$ *coincide, i.e.,* $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}} = \Omega^{\mathcal{A}}$*, if and only if* $\Omega^{\mathcal{A}}$ *is monotone on* $\mathbf{FiFam}^{\mathfrak{G}}(\mathcal{A})$*.*

Proof: Since $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}$ is monotone on $\mathbf{FiFam}^{\mathfrak{G}}(\mathcal{A})$, if the two operators coincide, $\Omega^{\mathcal{A}}$ is also monotone on $\mathbf{FiFam}^{\mathfrak{G}}(\mathcal{A})$.

On the other hand, if $\Omega^{\mathcal{A}}$ is monotone on $\mathbf{FiFam}^{\mathfrak{G}}(\mathcal{A})$, then, for all $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, we get

$$\begin{aligned} \widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) &= \bigcap\{\Omega^{\mathcal{A}}(\boldsymbol{T}') : \boldsymbol{T} \le \boldsymbol{T}' \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\}\\ &= \Omega^{\mathcal{A}}(\boldsymbol{T}).\end{aligned}$$

Therefore, $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}} = \Omega^{\mathcal{A}}$. ∎

An analog of Proposition 1897 holds also for the Suszko operator. That is, under the isomorphism between the corresponding filter families of two gentzen π-institutions that are equivalent, corresponding filter families have identical Suszko congruence systems.

Proposition 1900 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr, tr′ *traces and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$*,* $\mathfrak{G}' = \langle \mathbf{F}, G'\rangle$ *Gentzen π-institutions of traces* tr, tr′*, respectively, based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *and* $\mathfrak{G}'$ *are equivalent via the conjugate pair* $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$*, then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$*,*

$$\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) = \widetilde{\Omega}^{\mathfrak{G}',\mathcal{A}}(\rho^{\mathcal{A}*}(\boldsymbol{T})).$$

Proof: Since $\rho^{\mathcal{A}*} : \mathbf{FiFam}^{\mathfrak{G}}(\mathcal{A}) \to \mathbf{FiFam}^{\mathfrak{G}'}(\mathcal{A})$ is an order isomorphism, and taking into account Proposition 1897, we obtain, for all $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$,

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) & = & \bigcap\{\Omega^{\mathcal{A}}(\boldsymbol{T}') : \boldsymbol{T} \leq \boldsymbol{T}' \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\}\\ & = & \bigcap\{\Omega^{\mathcal{A}}(\rho^{\mathcal{A}*}(\boldsymbol{T}')) : \rho^{\mathcal{A}*}(\boldsymbol{T}) \leq \rho^{\mathcal{A}*}(\boldsymbol{T}') \in \mathrm{FiFam}^{\mathfrak{G}'}(\mathcal{A})\}\\ & = & \bigcap\{\Omega^{\mathcal{A}}(\boldsymbol{T}'') : \rho^{\mathcal{A}*}(\boldsymbol{T}) \leq \boldsymbol{T}'' \in \mathrm{FiFam}^{\mathfrak{G}'}(\mathcal{A})\}\\ & = & \widetilde{\Omega}^{\mathfrak{G}',\mathcal{A}}(\rho^{\mathcal{A}*}(\boldsymbol{T})).\end{array}$$

Thus, the conclusion holds. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$. A $\mathfrak{G}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, \boldsymbol{T} \rangle$ is called **Suszko reduced** if

$$\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) = \Delta^{\mathcal{A}}.$$

We denote by $\mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$ the class of all Suszko reduced $\mathfrak{G}$-matrix families.

Foe every $\mathfrak{G}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, \boldsymbol{T} \rangle$, the quotient structure

$$\langle \mathcal{A}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}), \boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) \rangle$$

is also a $\mathfrak{G}$-matrix family and it is Suszko reduced. Moreover, if a $\mathfrak{G}$-matrix family $\langle \mathcal{A}, \boldsymbol{T} \rangle$ is Suszko reduced, it is obviously isomorphic to a $\mathfrak{G}$-matrix family of this form.

Among other things, Suszko reduced $\mathfrak{G}$-matrix families are important because they form a class of structures with respect to which $\mathfrak{G}$ enjoys a completeness property.

Theorem 1901 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system,* tr *be a trace, and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace* tr *based on $\mathbf{F}$. For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$, $\boldsymbol{\phi} \in G_\Sigma(\boldsymbol{\Phi})$ if and only if, for all $\langle \mathcal{A}, \boldsymbol{T} \rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$, all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,*

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\boldsymbol{\Phi})) \subseteq \boldsymbol{T}_{F(\Sigma')} \quad \textit{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\boldsymbol{\phi})) \in \boldsymbol{T}_{F(\Sigma')}.$$

Proof: Suppose $\boldsymbol{\phi} \in G_\Sigma(\boldsymbol{\Phi})$ and let $\langle \mathcal{A}, \boldsymbol{T} \rangle$ be a Suszko reduced $\mathfrak{G}$-matrix family. Then $\langle \mathcal{A}, \boldsymbol{T} \rangle$ is, in particular, a $\mathfrak{G}$-matrix family, whence the conclusion holds by applying the definition of a $\mathfrak{G}$-filter family to the $\mathfrak{G}$-filter family $\boldsymbol{T}$. Suppose, conversely, that, for all $\langle \mathcal{A}, \boldsymbol{T} \rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$, all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\boldsymbol{\Phi})) \subseteq \boldsymbol{T}_{F(\Sigma')} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\boldsymbol{\phi})) \in \boldsymbol{T}_{F(\Sigma')}.$$

Let $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$ and consider the Suszko reduced $\mathfrak{G}$-matrix family

$$\langle \mathcal{F}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}), \boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}) \rangle.$$

Then, we have, by hypothesis, taking $\Sigma' = \Sigma$ and $f = i_\Sigma$,

$$\boldsymbol{\Phi}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}_\Sigma(\boldsymbol{T}) \subseteq \boldsymbol{T}_\Sigma/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}_\Sigma(\boldsymbol{T}) \quad \text{implies} \quad \boldsymbol{\phi}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}_\Sigma(\boldsymbol{T}) \in \boldsymbol{T}_\Sigma/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}_\Sigma(\boldsymbol{T}),$$

i.e., using the compatibility of $\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})$ with $\boldsymbol{T}$, $\boldsymbol{\Phi} \subseteq \boldsymbol{T}_\Sigma$ implies $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$. Equivalently, since $\boldsymbol{T} \in \text{ThFam}(\mathfrak{G})$ was arbitrary, $\boldsymbol{\phi} \in G_\Sigma(\boldsymbol{\Phi})$. ■

If two Gentzen π-institutions are equivalent, then the classes of algebraic system reducts of their Suszko reduced matrix families coincide.

Theorem 1902 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr*,* tr′ *traces and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$*,* $\mathfrak{G}' = \langle \mathbf{F}, G' \rangle$ *Gentzen π-institutions of traces* tr*,* tr′*, respectively, based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *and* $\mathfrak{G}'$ *are equivalent via the conjugate pair* $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$*, then* $\text{MatFam}^{\mathsf{Su}}(\mathfrak{G})$ *and* $\text{MatFam}^{\mathsf{Su}}(\mathfrak{G}')$ *have the same class of* $\mathbf{F}$*-algebraic system reducts.*

Proof: Suppose that $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ is the $\mathbf{F}$-algebraic system reduct of $\langle \mathcal{A}, \boldsymbol{T} \rangle \in \text{MatFam}^{\mathsf{Su}}(\mathfrak{G})$. Then, by definition, we have $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) = \Delta^{\mathcal{A}}$. Therefore, by Proposition 1900, we obtain $\widetilde{\Omega}^{\mathfrak{G}',\mathcal{A}}(\rho^{\mathcal{A}*}(\boldsymbol{T})) = \Delta^{\mathcal{A}}$. Since, $\rho^{\mathcal{A}*}(\boldsymbol{T}) \in \text{FiFam}^{\mathfrak{G}}(\mathcal{A})$, we conclude that $\langle \mathcal{A}, \rho^{\mathcal{A}*}(\boldsymbol{T}) \rangle \in \text{MatFam}^{\mathsf{Su}}(\mathfrak{G}')$ and, hence, $\mathcal{A}$ is also the $\mathbf{F}$-algebraic system reduct of a Suszko reduced $\mathfrak{G}'$-matrix family. By symmetry of equivalence, every $\mathbf{F}$-algebraic system reduct of a Suszko reduced $\mathfrak{G}'$-matrix family is also one of a Suszko reduced $\mathfrak{G}$-matrix family. Therefore, the two classes of $\mathbf{F}$-algebraic system reducts coincide. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr be a trace, and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$. The class of all $\mathbf{F}$-algebraic system reducts of Suszko reduced $\mathfrak{G}$-matrix families is denoted by $\text{AlgSys}(\mathfrak{G})$, i.e., we have, by definition,

$$\begin{array}{rcl} \text{AlgSys}(\mathfrak{G}) & = & \{\mathcal{A} : (\exists \boldsymbol{T} \leq \text{Seq}^{\text{tr}}(\mathcal{A}))(\langle \mathcal{A}, \boldsymbol{T} \rangle \in \text{MatFam}^{\mathsf{Su}}(\mathfrak{G})\} \\ & = & \{\mathcal{A} : (\exists \boldsymbol{T} \in \text{FiFam}^{\mathfrak{G}}(\mathcal{A}))(\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) = \Delta^{\mathcal{A}})\}. \end{array}$$

It is not difficult to show that the class $\text{AlgSys}(\mathfrak{G})$ is closed under $\overset{\triangleleft}{\text{I}\Pi}$ and, thence, conclude that the class of all $\text{AlgSys}(\mathfrak{G})$-congruence systems on every $\mathbf{F}$-algebraic system $\mathcal{A}$ forms a complete lattice under signature-wise inclusion.

Proposition 1903 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$*. Then* $\text{AlgSys}(\mathfrak{G})$ *is closed under subdirect intersections, i.e.,*

$$\overset{\triangleleft}{\text{I}\Pi}(\text{AlgSys}(\mathfrak{G})) \subseteq \text{AlgSys}(\mathfrak{G}).$$

Proof: Suppose that $\mathcal{A}^i = \langle \mathbf{A}^i, \langle F^i, \alpha^i\rangle\rangle \in \mathrm{AlgSys}(\mathfrak{G})$, for all $i \in I$, and let

$$\langle H^i, \gamma^i\rangle : \mathfrak{A} \to \mathcal{A}^i, \quad i \in I,$$

be a subdirect intersection, i.e., such that $\bigcap_{i\in I} \mathrm{Ker}(\langle H^i, \gamma^i\rangle) = \Delta^{\mathcal{A}}$. Then, for all $i \in I$, there exists $\boldsymbol{T}^i \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}^i)$, such that $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}^i}(\boldsymbol{T}^i) = \Delta^{\mathcal{A}^i}$. We consider the least $\mathfrak{G}$-filter family on $\mathcal{A}$, namely $\bigcap \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$. We have

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\bigcap \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})) & = & \bigcap\{\Omega^{\mathcal{A}}(\boldsymbol{X}) : \boldsymbol{X} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\}\\ & \le & \bigcap_{i\in I}\bigcap_{\boldsymbol{X}^i \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}^i)}(\gamma^i)^{-1}(\Omega^{\mathcal{A}^i}(\boldsymbol{X}^i))\\ & = & \bigcap_{i\in I}(\gamma^i)^{-1}(\bigcap_{\boldsymbol{X}^i \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}^i)}\Omega^{\mathcal{A}^i}(\boldsymbol{X}^i))\\ & \le & \bigcap_{i\in I}(\gamma^i)^{-1}(\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}^i}(\boldsymbol{T}^i))\\ & = & \bigcap_{i\in I}(\gamma^i)^{-1}(\Delta^{\mathcal{A}^i})\\ & = & \Delta^{\mathcal{A}}.\end{array}$$

Hence, we get that $\mathcal{A} \in \mathrm{AlgSys}(\mathfrak{G})$. Therefore, $\mathrm{AlgSys}(\mathfrak{G})$ is indeed closed under subdirect intersections. ∎

26.6 Equivalence and Algebraic Counterpart

In Theorem 1890, given a π-institution $\mathfrak{G}$ and a trace tr', we gave a characterization of the existence of an equivalence $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ of $\mathfrak{G}$ with some Gentzen π-institution $\mathfrak{G}'$, having the given trace tr'. We strengthen this result here, by considering only $\mathfrak{G}$-filter families on algebraic systems belonging to $\mathrm{AlgSys}(\mathfrak{G})$.

Theorem 1904 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr*,* tr' *be traces and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*.* $\mathfrak{G}$ *is equivalent to a Gentzen* π*-institution* $\mathfrak{G}' = \langle \mathbf{F}, G'\rangle$ *of trace* tr' *based on* $\mathbf{F}$ *if and only if there exist a* tr-tr'*-transformation* τ *and a* tr'-tr*-transformation* ρ*, such that, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathfrak{G})$*:*

(1) $\rho^{\mathcal{A}*} : \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}) \to \mathrm{SenFam}(\mathrm{Seq}^{\mathrm{tr}'}(\mathcal{A}))$ *is injective on* $\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$*;*

(2) For all $\Sigma \in |\mathbf{Sign}|$ *and all* $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathcal{A})$*,* $\rho^{\mathcal{A}*}_\Sigma(G^{\mathfrak{G},\mathcal{A}}(\boldsymbol{\phi})) = G'_\Sigma(\tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}])$*, where* G' *is the closure system on* $\mathcal{A}$ *induced by* $\rho^{\mathcal{A}*}(\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}))$*.*

Proof: Suppose, first, that there exists an equivalence $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$, where $\mathfrak{G}' = \langle \mathbf{F}, G'\rangle$ is a Gentzen π-institution of trace tr' based on $\mathbf{F}$. By Proposition 1892, we know that $\rho^{\mathcal{A}*} : \mathbf{FiFam}^{\mathfrak{G}}(\mathcal{A}) \to \mathbf{FiFam}^{\mathfrak{G}'}(\mathcal{A})$ is an order isomorphism, whence, in particular, it is injective on $\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$. Moreover, for all $\Sigma \in |\mathbf{Sign}|$ and all $\boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathcal{A})$, we have

$$\begin{array}{rcl}\boldsymbol{\psi} \in \rho^{\mathcal{A}*}_\Sigma(G^{\mathfrak{G},\mathcal{A}}(\boldsymbol{\phi})) & \text{iff} & \rho^{\mathcal{A}}_\Sigma[\boldsymbol{\psi}] \subseteq G^{\mathfrak{G},\mathcal{A}}_\Sigma(\boldsymbol{\phi})\\ & \text{iff} & \tau^{\mathcal{A}}_\Sigma[\rho^{\mathcal{A}}_\Sigma[\boldsymbol{\psi}]] \subseteq G^{\mathfrak{G}',\mathcal{A}}_\Sigma(\tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}])\\ & \text{iff} & \boldsymbol{\psi} \in G^{\mathfrak{G}',\mathcal{A}}_\Sigma(\tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}]).\end{array}$$

Therefore, $\rho^{\mathcal{A}*}_\Sigma(G^{\mathfrak{G},\mathcal{A}}(\boldsymbol{\phi})) = G^{\mathfrak{G}',\mathcal{A}}_\Sigma(\tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}])$ and, again by Proposition 1892, $G^{\mathfrak{G}',\mathcal{A}}$ is the closure system on $\mathcal{A}$ induced by $\rho^{\mathcal{A}*}(\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}))$.

Suppose, conversely, that there exist a tr-tr′-transformation τ and a tr′-tr-transformation ρ, such that Conditions (1) and (2) of the statement hold. The function $\rho^{\mathcal{A}*}$ commutes with intersections of $\mathfrak{G}$-filter families on $\mathcal{A}$. As a consequence, we obtain, on the one hand, that $\rho^{\mathcal{A}*}$ is order reflecting on $\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$ and, on the other, that $\rho^{\mathcal{A}*}(\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}))$ is closed under intersection, and, hence, defines a closure system on $\mathrm{Seq}^{\mathrm{tr}'}(\mathcal{A})$, which we denote by G'. It suffices now, to prove the two conditions of Theorem 1890.

Assume, first, that $\boldsymbol{T},\boldsymbol{T}' \in \mathrm{ThFam}(\mathfrak{G})$, such that $\rho^*(\boldsymbol{T}) \le \rho^*(\boldsymbol{T}')$. Let $\boldsymbol{X} = G(\rho[\rho^*(\boldsymbol{T})])$ and $\mathcal{A} = \mathcal{F}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X})$. Since $\boldsymbol{X}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X}) \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, we get that $\langle\mathcal{A},\boldsymbol{X}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X})\rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$. Therefore, $\mathcal{A} \in \mathrm{AlgSys}(\mathfrak{G})$. Further, $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$ and $\rho[\rho^*(\boldsymbol{T})] \le \boldsymbol{T}$, which give $\boldsymbol{X} \le \boldsymbol{T}$. Moreover, $\rho[\rho^*(\boldsymbol{T})] \le \rho[\rho^*(\boldsymbol{T}')] \le \boldsymbol{T}'$. Hence, $\boldsymbol{X} \le \boldsymbol{T}'$. Thus, by the monotonicity of the Suszko operator, $\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X}) \le \widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})$ and $\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X}) \le \widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}')$. These imply that $\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X})$ is compatible with both $\boldsymbol{T}$ and $\boldsymbol{T}'$. This, in turn, gives that both $\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X})$ and $\boldsymbol{T}'/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X})$ are $\mathfrak{G}$-filter families on $\mathcal{A}$ and, furthermore, that

$$\rho^{\mathcal{A}*}(\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X})) = \rho^*(\boldsymbol{T})/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X})$$

and, similarly, $\rho^{\mathcal{A}*}(\boldsymbol{T}'/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X})) = \rho^*(\boldsymbol{T}')/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X})$. Since, by hypothesis, $\rho^*(\boldsymbol{T}) \le \rho^*(\boldsymbol{T}')$, we get

$$\rho^{\mathcal{A}*}(\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X})) \le \rho^{\mathcal{A}*}(\boldsymbol{T}'/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X})).$$

Thus, by Condition (1) in the hypothesis, we get $\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X}) \le \boldsymbol{T}'/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X})$, whence, using again the compatibility of $\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X})$ with both $\boldsymbol{T}$ and $\boldsymbol{T}'$, we obtain $\boldsymbol{T} \le \boldsymbol{T}'$. We conclude that ρ^* is order reflecting and, therefore, a fortiori, injective on $\mathrm{ThFam}(\mathfrak{G})$.

Finally, let $\Sigma \in |\mathbf{Sign}^\flat|$, $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ and consider $\theta = \widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\mathrm{Thm}(\mathfrak{G}))$. Then $\mathcal{F}/\theta \in \mathrm{AlgSys}(\mathfrak{G})$, whence, by hypothesis,

$$\rho^{(\mathcal{F}/\theta)*}(G^{\mathfrak{G},\mathcal{F}/\theta}(\boldsymbol{\phi}/\theta_\Sigma)) = G'_\Sigma(\tau^{\mathcal{F}/\theta}_\Sigma[\boldsymbol{\phi}/\theta_\Sigma]),$$

where G' is the closure system on $\mathcal{F}/\theta$ generated by

$$\begin{array}{rcl}\rho^{(\mathcal{F}/\theta)*}(\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{F}/\theta)) & = & \rho^{(\mathcal{F}/\theta)*}(\mathrm{ThFam}(\mathfrak{G})/\theta)\\ & = & \rho^*(\mathrm{ThFam}(\mathfrak{G}))/\theta.\end{array}$$

Thus, we get $\rho^*_\Sigma(G(\boldsymbol{\phi}))/\theta = G'_\Sigma(\tau_\Sigma[\boldsymbol{\phi}]/\theta_\Sigma)$, whence

$$\begin{array}{rcl}\rho^{(\mathcal{F}/\theta)*}_\Sigma(G(\boldsymbol{\phi})/\theta) & = & \bigcap\{\rho^*(\boldsymbol{X})/\theta : \tau_\Sigma[\boldsymbol{\phi}]/\theta \subseteq \rho^*(\boldsymbol{X})/\theta\}\\ & = & \bigcap\{\rho^*(\boldsymbol{X}) : \tau_\Sigma[\boldsymbol{\phi}] \subseteq \rho^*(\boldsymbol{X})\}/\theta.\end{array}$$

Therefore, $\rho^*_\Sigma(G(\boldsymbol{\phi})) = \bigcap\{\rho^*(\boldsymbol{X}) : \tau_\Sigma[\boldsymbol{\phi}] \subseteq \rho^*_\Sigma(\boldsymbol{X})\} = G''(\tau_\Sigma[\boldsymbol{\phi}])$, where G'' is the closure system on $\mathcal{F}$ generated by $\rho^*(\mathrm{ThFam}(\mathfrak{G}))$. ■

Theorem 1904 may be used to provide a characterization of equivalence based on the coincidence of the algebraic counterparts of two Gentzen π-institutions.

Theorem 1905 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr, tr′ *traces and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$, $\mathfrak{G}' = \langle \mathbf{F}, G'\rangle$ *Gentzen* π*-institutions of traces* tr, tr′*, respectively, based on* $\mathbf{F}$*.* $\mathfrak{G}$ *and* $\mathfrak{G}'$ *are equivalent if and only if*

- $\mathrm{AlgSys}(\mathfrak{G}) = \mathrm{AlgSys}(\mathfrak{G}')$ *and*
- *there exist a* tr-tr′*-transformation* τ *and a* tr′-tr*-transformation* ρ*, such that, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathfrak{G})$*,*
  - $\rho^{\mathcal{A}*} : \mathbf{FiFam}^{\mathfrak{G}}(\mathcal{A}) \to \mathbf{FiFam}^{\mathfrak{G}'}(\mathcal{A})$ *is an order isomorphism and*
  - *for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}(\mathcal{A})$*,*
  $$\rho^{\mathcal{A}*}_\Sigma(G^{\mathfrak{G},\mathcal{A}}(\boldsymbol{\phi})) = G'^{\mathcal{A}}_\Sigma(\tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}]),$$
  where $G'^{\mathcal{A}}$ *is the closure system on* $\mathcal{A}$ *induced by* $\rho^{\mathcal{A}*}(\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}))$*.*

Proof: Suppose that $\mathfrak{G}$ and $\mathfrak{G}'$ are equivalent via the conjugate pair $(\tau,\rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$. Then, by Theorem 1902, $\mathrm{AlgSys}(\mathfrak{G}) = \mathrm{AlgSys}(\mathfrak{G}')$. By Proposition 1892, $\rho^{\mathcal{A}*}$ is an order isomorphism and, finally, for all $\Sigma \in |\mathbf{Sign}|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathcal{A})$ and all $\boldsymbol{\phi}' \in \mathrm{Seq}^{\mathrm{tr}'}_\Sigma(\mathcal{A})$,

$$\begin{array}{rll}\boldsymbol{\phi}' \in \rho^{\mathcal{A}*}_\Sigma(G^{\mathfrak{G},\mathcal{A}}(\boldsymbol{\phi})) & \text{iff} & \rho^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}'] \subseteq G^{\mathfrak{G},\mathcal{A}}_\Sigma(\boldsymbol{\phi})\\ & \text{iff} & \tau^{\mathcal{A}}_\Sigma[\rho^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}']] \subseteq G^{\mathfrak{G}',\mathcal{A}}_\Sigma(\tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}])\\ & \text{iff} & \boldsymbol{\phi}' \in G^{\mathfrak{G}',\mathcal{A}}_\Sigma(\tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}]),\end{array}$$

i.e., for all $\Sigma \in |\mathbf{Sign}|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}(\mathcal{A})$, $\rho^{\mathcal{A}*}_\Sigma(G^{\mathfrak{G},\mathcal{A}}(\boldsymbol{\phi})) = G^{\mathfrak{G}',\mathcal{A}}_\Sigma(\tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}])$.

Conversely, assume that the conditions in the claimed characterization of equivalence hold. Then, by Theorem 1904, there exists a Gentzen π-institution $\mathfrak{X}'$ of trace tr′ to which $\mathfrak{G}$ is equivalent, such that $\rho^* : \mathbf{ThFam}(\mathfrak{G}) \to \mathbf{ThFam}(\mathfrak{X})$ is an order isomorphism. Thus, ρ^* is both order preserving and order reflecting and, hence, injective, on $\mathrm{ThFam}(\mathfrak{G})$. Thus, it suffices to show that it is onto $\mathrm{ThFam}(\mathfrak{G}')$.

Suppose $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$. Set $\mathcal{A} = \mathcal{F}/\Omega(\boldsymbol{T})$ and let $\langle I,\pi\rangle : \mathcal{F} \to \mathcal{A}$ be the quotient morphism. Then, since $\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}) \leq \Omega(\boldsymbol{T})$, we get, by the definition of $\mathrm{AlgSys}(\mathfrak{G})$ and the hypothesis, $\mathcal{A} \in \mathrm{AlgSys}(\mathfrak{G}) = \mathrm{AlgSys}(\mathfrak{G}')$. By the compatibility of $\Omega(\boldsymbol{T})$ with $\boldsymbol{T}$, we get that $\boldsymbol{T}/\Omega(\boldsymbol{T}) \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$ and $\pi^{-1}(\boldsymbol{T}/\Omega(\boldsymbol{T})) = \boldsymbol{T}$. By hypothesis, $\rho^{\mathcal{A}*}(\boldsymbol{T}/\Omega(\boldsymbol{T})) \in \mathrm{FiFam}^{\mathfrak{G}'}(\mathcal{A})$, whence $\pi^{-1}(\rho^{\mathcal{A}*}(\boldsymbol{T}/\Omega(\boldsymbol{T}))) \in \mathrm{ThFam}(\mathfrak{G}')$. On the other hand, we have

$$\rho^*(\boldsymbol{T}) = \rho^*(\pi^{-1}(\boldsymbol{T}/\Omega(\boldsymbol{T}))) = \pi^{-1}(\rho^{\mathcal{A}*}(\boldsymbol{T}/\Omega(\boldsymbol{T}))).$$

Hence, we obtain $\rho^*(\boldsymbol{T}) \in \mathrm{ThFam}(\mathfrak{G}')$.

Finally, consider $\boldsymbol{T}' \in \text{ThFam}(\mathfrak{G}')$. Set $\mathfrak{B} = \mathcal{F}/\Omega(\boldsymbol{T}') \in \text{AlgSys}(\mathfrak{G}') = \text{AlgSys}(\mathfrak{G})$ and let $\langle I, \pi' \rangle : \mathcal{F} \to \mathcal{B}$ be the quotient morphism. Then we have $\boldsymbol{T}'/\Omega(\boldsymbol{T}') \in \text{FiFam}^{\mathfrak{G}'}(\mathcal{B})$ and, by compatibility, $\pi'^{-1}(\boldsymbol{T}'/\Omega(\boldsymbol{T}')) = \boldsymbol{T}'$. By hypothesis, there exists $\boldsymbol{T} \in \text{FiFam}^{\mathfrak{G}}(\mathcal{B})$, such that $\boldsymbol{T}'/\Omega(\boldsymbol{T}') = \rho^{\mathcal{B}*}(\boldsymbol{T})$. On the other hand, $\pi'^{-1}(\boldsymbol{T}) \in \text{ThFam}(\mathfrak{G})$ and

$$\rho^*(\pi'^{-1}(\boldsymbol{T})) = \pi'^{-1}(\rho^{\mathcal{B}*}(\boldsymbol{T})) = \pi'^{-1}(\boldsymbol{T}'/\Omega(\boldsymbol{T}')) = \boldsymbol{T}'.$$

thus, ρ^* maps $\text{ThFam}(\mathfrak{G})$ onto $\text{ThFam}(\mathfrak{G}')$ and, hence, it is an order isomorphism from $\mathbf{ThFam}(\mathfrak{G})$ onto $\mathbf{ThFam}(\mathfrak{G}')$. Therefore, $\mathfrak{G}' = \mathfrak{X}$ and $\mathfrak{G}'$ is equivalent to $\mathfrak{G}$. ■

Directly from Theorem 1904, we get the following

Corollary 1906 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$. $\mathfrak{G}$ *is Hilbertizable if and only if there exist a* tr-$\{\langle 0,1 \rangle\}$*-transformation* τ *and a* $\{\langle 0,1 \rangle\}$-tr*-transformation* ρ*, such that, for all* $\mathcal{A} \in \text{AlgSys}(\mathfrak{G})$*:*

(1) $\rho^{\mathcal{A}*} : \text{FiFam}^{\mathfrak{G}}(\mathcal{A}) \to \text{SenFam}(\mathcal{A})$ *is injective on* $\text{FiFam}^{\mathfrak{G}}(\mathcal{A})$*;*

(2) For all $\Sigma \in |\mathbf{Sign}|$ *and all* $\boldsymbol{\phi} \in \text{Seq}^{\text{tr}}_\Sigma(\mathcal{A})$*,*

$$\rho^{\mathcal{A}*}_\Sigma(G^{\mathfrak{G},\mathcal{A}}(\boldsymbol{\phi})) = G'^{\mathcal{A}}_\Sigma(\tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}]),$$

where $G'^{\mathcal{A}}$ *is the closure system on* $\mathcal{A}$ *induced by* $\rho^{\mathcal{A}*}(\text{FiFam}^{\mathfrak{G}}(\mathcal{A}))$.

Proof: This is a special case of Theorem 1904. ■

Specializing further, we get the following result characterizing simple Hilbertizability.

Corollary 1907 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$. $\mathfrak{G}$ *is simply Hilbertizable if and only if there exists a* tr-$\{\langle 0,1 \rangle\}$*-transformation* τ*, such that:*

(1) For all $\mathcal{A} \in \text{AlgSys}(\mathfrak{G})$ *and all* $\boldsymbol{T}, \boldsymbol{T}' \in \text{FiFam}^{\mathfrak{G}}(\mathcal{A})$*,*

$$\boldsymbol{T} \cap \rhd \mathcal{A} = \boldsymbol{T}' \cap \rhd \mathcal{A} \quad \textit{implies} \quad \boldsymbol{T} = \boldsymbol{T}';$$

(2) For all $\Sigma \in |\mathbf{Sign}|$ *and all* $\boldsymbol{\phi} \in \text{Seq}^{\text{tr}}_\Sigma(\mathcal{A})$*,*

$$G^{\mathfrak{G},\mathcal{A}}_\Sigma(\boldsymbol{\phi}) \cap \rhd \mathcal{A} = \bigcap\{\rhd \boldsymbol{T}_\Sigma : \tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}] \subseteq \boldsymbol{T}_\Sigma, \boldsymbol{T} \in \text{FiFam}^{\mathfrak{G}}(\mathcal{A})\}.$$

Proof: It suffices to see that Conditions (1) and (2) in the statement reflect exactly Conditions (1) and (2) in the statement of Corollary 1906, where the role of ρ is assumed by the special $\{\langle 0,1 \rangle\}$-tr-transformation ρ^0. ■

Finally, we obtain a characterization of those algebraic Gentzen π-institutions, i.e., Gentzen π-institutions associated with guasivarieties of algebraic systems, which are equivalent to some Hilbert π-institution.

Corollary 1908 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a guasivariety of* $\mathbf{F}$*-algebraic systems.* $\mathfrak{G}^\mathsf{K} = \langle \mathbf{F}, G^\mathsf{K} \rangle$ *is Hilbertizable if and only if there exists a* $\{\langle 1,1\rangle\}$-$\{\langle 0,1\rangle\}$*-transformation* τ *and a* $\{\langle 0,1\rangle\}$-$\{\langle 1,1\rangle\}$*-transformation* ρ*, such that, for all* $\mathcal{A} \in \mathsf{K}$*:*

(1) $\rho^{\mathcal{A}*} : \mathrm{ConSys}^\mathsf{K}(\mathcal{A}) \to \mathrm{SenFam}(\underset{\sim}{A})$ *is injective on* $\mathrm{ConSys}^\mathsf{K}(\mathcal{A})$*;*

(2) For all $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,*

$$\rho_\Sigma^{\mathcal{A}*}(\Theta^{\mathsf{K},\mathcal{A}}(\phi \approx \psi)) = G'^{\mathcal{A}}(\tau_\Sigma^{\mathcal{A}}[\phi;\psi]),$$

where $G'^{\mathcal{A}}$ *is the closure system on* $\mathcal{A}$ *induced by* $\rho^{\mathcal{A}*}(\mathrm{ConSys}^\mathsf{K}(\mathcal{A}))$*.*

Proof: This is again a specialization of Theorem 1904 for $\mathfrak{G} = \mathfrak{G}^\mathsf{K}$, where we take into account the facts $\mathrm{FiFam}^{\mathfrak{G}^\mathsf{K}}(\mathcal{A}) = \mathrm{ConSys}^\mathsf{K}(\mathcal{A})$, $\mathrm{AlgSys}(\mathfrak{G}^\mathsf{K}) = \mathsf{K}$ and, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, we have, under appropriate identifications, $G^{\mathfrak{G}^\mathsf{K},\mathcal{A}}(\phi \rhd_\Sigma \psi) = \Theta^{\mathsf{K},\mathcal{A}}(\phi \approx \psi)$. ∎

26.7 Protoalgebraicity

We now start a relatively brief tour of analogs of some of the classes in the algebraic hierarchy of π-institutions that were introduced in the earlier chapters of this work, as adapted and generalized for Gentzen π-institutions. Even though we revisit and recast only very few of the classes considered previously for π-institutions, the observant reader would realize that all other classes have similarly adapted analogs that have analogous properties.

In this section, we define protoalgebraic and syntactically protoalgebraic Gentzen π-institutions and study some of their properties. In the following section, we shall take a look at order algebraizable Gentzen π-institutions, which parallel the order algebraizable π-institutions of Chapter 25. In the last section, we look at completely reflective and truth equational Gentzen π-institutions.

We look, first, at some properties of the Leibniz operator, whose analogs for π-institutions have been established in Chapter 2.

Lemma 1909 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* $\mathbf{B} = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ *be* $N^\flat$*-algebraic systems and* $\langle H, \gamma \rangle : \mathbf{A} \to \mathbf{B}$ *a morphism. For every trace* tr *and all* $\boldsymbol{T} \leq \mathrm{Seq}^{\mathrm{tr}}(\mathbf{B})$*,*

(a) $\gamma^{-1}(\Omega^\mathbf{B}(\boldsymbol{T})) \leq \Omega^\mathbf{A}(\gamma^{-1}(\boldsymbol{T}))$*;*

(b) $\gamma^{-1}(\Omega^\mathbf{B}(\boldsymbol{T})) = \Omega^\mathbf{A}(\gamma^{-1}(\boldsymbol{T}))$*, if* $\langle H, \gamma \rangle$ *is surjective.*

Proof:

(a) It is straightforward to check that $\gamma^{-1}(\Omega^{\mathbf{B}}(\boldsymbol{T}))$ is a congruence system on $\mathbf{A}$ compatible with $\gamma^{-1}(\boldsymbol{T})$. Hence, by the maximality property of $\Omega^{\mathbf{A}}(\gamma^{-1}(\boldsymbol{T}))$, we get that $\gamma^{-1}(\Omega^{\mathbf{B}}(\boldsymbol{T})) \leq \Omega^{\mathbf{A}}(\gamma^{-1}(\boldsymbol{T}))$.

(b) Suppose, now, that $\langle H,\gamma\rangle$ is surjective and let $\Sigma \in |\mathbf{Sign}|$, $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle \in \Omega^{\mathbf{A}}_{\Sigma}(\gamma^{-1}(\boldsymbol{T}))$. Then, by Lemma 1895, we get that, for all $\langle m,n\rangle \in \mathrm{tr}$, all $\vec{\sigma} = \langle\sigma^0,\dots,\sigma^{m-1}\rangle$, and all $\vec{\tau} = \langle\tau^0,\dots,\tau^{n-1}\rangle$ in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma,\Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

$$\begin{array}{l}\vec{\sigma}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}) \rhd_{\Sigma'} \vec{\tau}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}) \in \gamma^{-1}_{\Sigma'}(\boldsymbol{T}_{H(\Sigma')})\\ \quad\text{iff}\quad \vec{\sigma}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi}) \rhd_{\Sigma'} \vec{\tau}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi}) \in \gamma^{-1}_{\Sigma'}(\boldsymbol{T}_{H(\Sigma')}).\end{array}$$

Equivalently,

$$\begin{array}{l}\gamma_{\Sigma'}(\vec{\sigma}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}) \rhd_{\Sigma'} \vec{\tau}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi})) \in \boldsymbol{T}_{H(\Sigma')}\\ \quad\text{iff}\quad \gamma_{\Sigma'}(\vec{\sigma}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi}) \rhd_{\Sigma'} \vec{\tau}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi})) \in \boldsymbol{T}_{H(\Sigma')}.\end{array}$$

This holds if and only if, by the morphism property,

$$\begin{array}{l}\vec{\sigma}^{\mathbf{B}}_{H(\Sigma')}(\gamma_{\Sigma'}(\mathrm{SEN}(f)(\phi)),\gamma_{\Sigma'}(\vec{\chi}))\\ \qquad\qquad \rhd_{H(\Sigma')} \vec{\tau}^{\mathbf{B}}_{H(\Sigma')}(\gamma_{\Sigma'}(\mathrm{SEN}(f)(\phi)),\gamma_{\Sigma'}(\vec{\chi})) \in \boldsymbol{T}_{H(\Sigma')}\\ \quad\text{iff}\quad \vec{\sigma}^{\mathbf{B}}_{H(\Sigma')}(\gamma_{\Sigma'}(\mathrm{SEN}(f)(\psi)),\gamma_{\Sigma'}(\vec{\chi}))\\ \qquad\qquad \rhd_{H(\Sigma')} \vec{\tau}^{\mathbf{B}}_{H(\Sigma')}(\gamma_{\Sigma'}(\mathrm{SEN}(f)(\psi)),\gamma_{\Sigma'}(\vec{\chi})) \in \boldsymbol{T}_{H(\Sigma')}.\end{array}$$

Equivalently, by the naturality of γ,

$$\begin{array}{l}\vec{\sigma}^{\mathbf{B}}_{H(\Sigma')}(\mathrm{SEN}'(H(f))(\gamma_{\Sigma}(\phi)),\gamma_{\Sigma'}(\vec{\chi}))\\ \qquad\qquad \rhd_{H(\Sigma')} \vec{\tau}^{\mathbf{B}}_{H(\Sigma')}(\mathrm{SEN}'(H(f))(\gamma_{\Sigma}(\phi)),\gamma_{\Sigma'}(\vec{\chi})) \in \boldsymbol{T}_{H(\Sigma')}\\ \quad\text{iff}\quad \vec{\sigma}^{\mathbf{B}}_{H(\Sigma')}(\mathrm{SEN}'(H(f))(\gamma_{\Sigma}(\psi)),\gamma_{\Sigma'}(\vec{\chi}))\\ \qquad\qquad \rhd_{H(\Sigma')} \vec{\tau}^{\mathbf{B}}_{H(\Sigma')}(\mathrm{SEN}'(H(f))(\gamma_{\Sigma}(\psi)),\gamma_{\Sigma'}(\vec{\chi})) \in \boldsymbol{T}_{H(\Sigma')}.\end{array}$$

Hence, taking into account the surjectivity of $\langle H,\gamma\rangle$, by Lemma 1895, we get $\langle\gamma_{\Sigma}(\phi),\gamma_{\Sigma}(\psi)\rangle \in \Omega^{\mathbf{B}}_{H(\Sigma)}(\boldsymbol{T})$, i.e., $\gamma_{\Sigma}(\Omega^{\mathbf{A}}_{\Sigma}(\gamma^{-1}(\boldsymbol{T}))) \subseteq \Omega^{\mathbf{B}}_{H(\Sigma)}(\boldsymbol{T})$. We conclude that $\Omega^{\mathbf{A}}(\gamma^{-1}(\boldsymbol{T})) \leq \gamma^{-1}(\Omega^{\mathbf{B}}(\boldsymbol{T}))$.

■

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle\mathbf{F},G\rangle$ a Gentzen π-institution based on $\mathbf{F}$.

- We say $\mathfrak{G}$ is **protoalgebraic** if the Leibniz operator $\Omega : \mathrm{ThFam}(\mathfrak{G}) \to \mathrm{ConSys}(\mathcal{F})$ is monotone on $\mathrm{ThFam}(\mathfrak{G})$;

- We say $\mathfrak{G}$ is **syntactically protoalgebraic** if, for all $\langle m,n\rangle \in \mathrm{tr}$, there exists $I^{\langle m,n\rangle} : (\mathrm{SEN}^\flat)^\omega \to \bigcup_{\langle k,\ell\rangle\in\mathrm{tr}}(\mathrm{SEN}^\flat)^{k+\ell}$ in $N^\flat$ with $(m+n)+(m+n)$ distinguished arguments, such that, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi},\boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}(\mathbf{F})$ of trace $\langle m,n\rangle$,

$$\langle\boldsymbol{\phi},\boldsymbol{\psi}\rangle \in \Omega_{\Sigma}(\boldsymbol{T}) \quad\text{iff}\quad I^{\langle m,n\rangle}_{\Sigma}[\boldsymbol{\phi},\boldsymbol{\psi}] \subseteq \boldsymbol{T}_{\Sigma}.$$

In this case the collection $I = \{I^{\langle m,n\rangle} : \langle m,n\rangle \in \mathrm{tr}\}$ is called a collection of **witnessing transformations** of the syntactic protoalgebraicity of $\mathfrak{G}$.

We give an alternative characterization of syntactic protoalgebraicity that comes handy in what follows.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and tr a trace. Given $\langle m,n\rangle \in \mathrm{tr}$, we say that a collection $I : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^k$ of natural transformations in $N^\flat$, with $(m+n)+(m+n)$ distinguished variables is **(pairwise) permutable** if and only if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$ and all $\{i_1,\dots,i_{m+n}\} = \{0,\dots,m+n-1\}$,

$$I_\Sigma[\phi_{i_1},\dots,\phi_{i_{m+n}},\psi_{i_1},\dots,\psi_{i_{m+n}}] = I_\Sigma[\phi_0,\dots,\phi_{(m+n)-1},\psi_0,\dots,\psi_{(n+m)-1}].$$

When we want to refer to an arbitrary pairwise permutation of two sequences $\vec{\phi},\vec{\psi}$ of the same length as above, we write $\vec{\phi}^\pi,\vec{\psi}^\pi$, the meaning being that $\vec{\phi},\vec{\psi}$ have the same length and that in $\vec{\phi}^\pi,\vec{\psi}^\pi$, their elements have been permuted both by applying the same arbitrary permutation π.

Theorem 1910 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen π-institution based on* $\mathbf{F}$*.* $\mathfrak{G}$ *is syntactically protoalgebraic if and only if, for all* $\langle m,n\rangle \in \mathrm{tr}$*, there exists* $\hat{I}^{\langle m,n\rangle} : (\mathrm{SEN}^\flat)^\omega \to \bigcup_{\langle k,\ell\rangle\in\mathrm{tr}}(\mathrm{SEN}^\flat)^{k+\ell}$ *in* $N^\flat$*, with* $(m+n)+(m+n)$ *distinguished arguments, which is permutable, such that, for all* $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi,\psi,\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\langle\phi,\psi\rangle \in \Omega_\Sigma(\boldsymbol{T}) \quad \textit{iff} \quad \hat{I}^{\langle m,n\rangle}_\Sigma[\langle\phi,\vec{\chi}\rangle,\langle\psi,\vec{\chi}\rangle] \subseteq \boldsymbol{T}_\Sigma.$$

Proof: Suppose, first, that $\mathfrak{G}$ is syntactically protoalgebraic, with witnessing transformations $I = \{I^{\langle m,n\rangle} : \langle m,n\rangle \in \mathrm{tr}\}$. For all $\langle m,n\rangle \in \mathrm{tr}$, we symmetrize $I^{\langle m,n\rangle}$ by defining $\hat{I}^{\langle m,n\rangle}$ in $N^\flat$, with $(m+n)+(m+n)$ distinguished arguments, by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi},\boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m,n\rangle$,

$$\hat{I}^{\langle m,n\rangle}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}] = \bigcup\{I^{\langle m,n\rangle}_\Sigma[\boldsymbol{\phi}^\pi,\boldsymbol{\psi}^\pi] : \pi \text{ a permutation}\}.$$

Then, we have, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi},\boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of the same trace $\langle m,n\rangle$,

$$\begin{array}{lll} I^{\langle m,n\rangle}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma & \text{iff} & \langle\boldsymbol{\phi},\boldsymbol{\psi}\rangle \in \Omega_\Sigma(\boldsymbol{T}) \\ & \text{iff} & \langle\boldsymbol{\phi}^\pi,\boldsymbol{\psi}^\pi\rangle \in \Omega_\Sigma(\boldsymbol{T}), \text{ for all } \pi, \\ & \text{iff} & \hat{I}^{\langle m,n\rangle}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}] \subseteq_\Sigma \boldsymbol{T}_\Sigma. \end{array}$$

Therefore, we obtain, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi,\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll} \langle\phi,\psi\rangle \in \Omega_\Sigma(\boldsymbol{T}) & \text{iff} & I^{\langle m,n\rangle}_\Sigma[\langle\phi,\vec{\chi}\rangle,\langle\psi,\vec{\chi}\rangle] \subseteq \boldsymbol{T}_\Sigma \\ & \text{iff} & \hat{I}^{\langle m,n\rangle}_\Sigma[\langle\phi,\vec{\chi}\rangle,\langle\psi,\vec{\chi}\rangle] \subseteq \boldsymbol{T}_\Sigma. \end{array}$$

Suppose, conversely, that there exists a permutable $I = \{I^{\langle m,n\rangle} : \langle m,n\rangle \in \mathrm{tr}\}$ that satisfies the condition in the statement of the theorem. Define a collection $\check{I} = \{\check{I}^{\langle m,n\rangle} : \langle m,n\rangle \in \mathrm{tr}\}$ in $N^\flat$ having $(m+n)+(m+n)$ distinguished arguments by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi},\boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m,n\rangle$,

$$\check{I}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}] = \bigcup\{I^{\langle m,n\rangle}_\Sigma[(\boldsymbol{\phi}\boldsymbol{\psi})^{i+1},(\boldsymbol{\phi}\boldsymbol{\psi})^i] : i < m+n-1\},$$

where

$$(\boldsymbol{\phi}\boldsymbol{\psi})^i \coloneqq \langle \phi_0,\dots,\phi_{i-1},\psi_i,\dots,\psi_{m+n-1}\rangle.$$

Then we have, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi},\boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m,n\rangle$,

$$\begin{array}{lll}\langle\boldsymbol{\phi},\boldsymbol{\psi}\rangle \in \Omega_\Sigma(\boldsymbol{T}) & \text{iff} & \langle\phi_i,\psi_i\rangle \in \Omega_\Sigma(\boldsymbol{T}),\ i < m+n-1,\\ & \text{iff} & I^{\langle m,n\rangle}[(\boldsymbol{\phi}\boldsymbol{\psi})^{i+1},(\boldsymbol{\phi}\boldsymbol{\psi})^i] \subseteq \boldsymbol{T}_\Sigma,\ i < m+n-1,\\ & \text{iff} & \check{I}^{\langle m,n\rangle}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma.\end{array}$$

Therefore, $\mathfrak{G}$ is syntactically protoalgebraic with witnessing transformations $\check{I}$. ∎

Before embarking on a characterization of the exact relationship between syntactic protoalgebraicity and protoalgebraicity, we look at some properties related to notions that have been studied in this chapter, namely, the algebraic counterpart of a Gentzen π-institution and equivalence between Gentzen π-institutions.

The first property states that it suffices to check monotonicity of the Leibniz operator only on the filter families of algebraic systems belonging to the algebraic counterpart.

Lemma 1911 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system,* tr *a trace and $\mathfrak{G} = \langle\mathbf{F},G\rangle$ a Gentzen π-institution of trace* tr *based on $\mathbf{F}$. If, for all $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle \in \mathrm{AlgSys}(\mathfrak{G})$, $\Omega^{\mathcal{A}}$ is monotone, then $\mathfrak{G}$ is protoalgebraic.*

Proof: Suppose that $\Omega^{\mathcal{A}}$ is monotone, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathfrak{G})$ and let $\boldsymbol{T},\boldsymbol{T}' \in \mathrm{ThFam}(\mathfrak{G})$, such that $\boldsymbol{T} \le \boldsymbol{T}'$. Then, by the monotonicity of the Suszko operator, $\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}) \le \widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}')$. Thus, the congruence system $\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})$ is compatible with both $\boldsymbol{T}$ and $\boldsymbol{T}'$. Hence, both $\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})$ and $\boldsymbol{T}'/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})$ are $\mathfrak{G}$-filter families of $\mathcal{F}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})$, such that $\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}) \le \boldsymbol{T}'/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})$. By hypothesis, since $\mathcal{F}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}) \in \mathrm{AlgSys}(\mathfrak{G})$,

$$\Omega^{\mathcal{F}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})}(\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})) \le \Omega^{\mathcal{F}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})}(\boldsymbol{T}'/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})).$$

Thus, applying the inverse of the quotient morphism $\langle I,\pi\rangle : \mathcal{F} \to \mathcal{F}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})$, we get that

$$\pi^{-1}(\Omega^{\mathcal{F}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})}(\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}))) \le \pi^{-1}(\Omega^{\mathcal{F}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})}(\boldsymbol{T}'/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}))),$$

whence, by Lemma 1909,

$$\Omega(\pi^{-1}(\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}))) \leq \Omega(\pi^{-1}(\boldsymbol{T}'/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}))).$$

Thus, since $\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})$ is compatible with both $\boldsymbol{T}$ and $\boldsymbol{T}'$, we get that $\Omega(\boldsymbol{T}) \leq \Omega(\boldsymbol{T}')$. Therefore, $\mathfrak{G}$ is protoalgebraic. ■

Now we prove that protoalgebraicity is preserved under equivalence.

Theorem 1912 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr*,* tr′ *traces and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$*,* $\mathfrak{G}' = \langle \mathbf{F}, G' \rangle$ *Gentzen π-institutions of traces* tr*,* tr′*, respectively, based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *and* $\mathfrak{G}'$ *are equivalent, then* $\mathfrak{G}$ *is protoalgebraic if and only if* $\mathfrak{G}'$ *is also.*

Proof: Suppose $\mathfrak{G}$ and $\mathfrak{G}'$ are equivalent via the conjugate pair $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ and that $\mathfrak{G}'$ is protoalgebraic. Let $\boldsymbol{T}, \boldsymbol{T}' \in \mathrm{ThFam}(\mathfrak{G})$, such that $\boldsymbol{T} \leq \boldsymbol{T}'$. Then, by Theorem 1880, $\rho^*(\boldsymbol{T}) \leq \rho^*(\boldsymbol{T}')$. Thus, by hypothesis, $\Omega(\rho^*(\boldsymbol{T})) \leq \Omega(\rho^*(\boldsymbol{T}'))$. Hence, by Proposition 1897, $\Omega(\boldsymbol{T}) \leq \Omega(\boldsymbol{T}')$. Therefore, $\mathfrak{G}$ is also protoalgebraic. The converse follows by the symmetry of equivalence. ■

Finally, it is shown that the same applies to syntactic protoalgebraicity, i.e., if two Gentzen π-institutions are equivalent, then one is syntactically protoalgebraic if and only if the other is also.

Theorem 1913 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr*,* tr′ *traces and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$*,* $\mathfrak{G}' = \langle \mathbf{F}, G' \rangle$ *Gentzen π-institutions of traces* tr*,* tr′*, respectively, based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *and* $\mathfrak{G}'$ *are equivalent, then* $\mathfrak{G}$ *is syntactically protoalgebraic if and only if* $\mathfrak{G}'$ *is also.*

Proof: Suppose that $\mathfrak{G}$ and $\mathfrak{G}'$ are equivalent via a conjugate pair $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ and that $\mathfrak{G}'$ is syntactically protoalgebraic, with witnessing transformations $I := \{I^{\langle m,m \rangle} : \langle m, n \rangle \in \mathrm{tr}'\}$. Then, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$, we get, setting, according to Theorem 1880, $\boldsymbol{T}' \in \mathrm{ThFam}(\mathfrak{G}')$ be such that $\boldsymbol{T} \underset{\tau^*}{\overset{\rho^*}{\rightleftarrows}} \boldsymbol{T}'$, and taking into account Theorem 1919,

$$\begin{array}{lll} \langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle \in \Omega_\Sigma(\boldsymbol{T}) & \text{iff} & \langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle \in \Omega_\Sigma(\rho^*(\boldsymbol{T})) \\ & \text{iff} & \langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle \in \Omega_\Sigma(\boldsymbol{T}') \\ & \text{iff} & \hat{I}_\Sigma[\langle \phi_i, \vec{\chi} \rangle, \langle \psi_i, \vec{\chi} \rangle] \subseteq \boldsymbol{T}'_\Sigma, i < m + n, \\ & \text{iff} & \hat{I}_\Sigma[\langle \phi_i, \vec{\chi} \rangle, \langle \psi_i, \vec{\chi} \rangle] \subseteq \tau^*_\Sigma(\boldsymbol{T}), i < m + n, \\ & \text{iff} & \tau_\Sigma[\hat{I}_\Sigma[\langle \phi_i, \vec{\chi} \rangle, \langle \psi_i, \vec{\chi} \rangle]] \subseteq \boldsymbol{T}_\Sigma, i < m + n. \end{array}$$

Therefore, $(\tau \circ \hat{I})^{\check{}}$ witnesses the syntactic protoalgebraicity of $\mathfrak{G}$. The converse follows by the symmetry of equivalence. ■

It is relatively easy to see that, if a Gentzen π-institution $\mathfrak{G}$ is syntactically protoalgebraic, then it is protoalgebraic.

Theorem 1914 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$. *If* $\mathfrak{G}$ *is syntactically protoalgebraic, then it is protoalgebraic.*

Proof: Suppose $\mathfrak{G}$ is syntactically protoalgebraic, with witnessing transformations $I = \{I^{\langle m,n \rangle} : \langle m,n \rangle \in \mathrm{tr}\}$ in $N^\flat$, and let $\boldsymbol{T}, \boldsymbol{T}' \in \mathrm{ThFam}(\mathfrak{G})$, such that $\boldsymbol{T} \le \boldsymbol{T}'$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ of the same trace $\langle m,n \rangle$, we have

$$\begin{array}{rcl} \langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle \in \Omega_\Sigma(\boldsymbol{T}) & \text{iff} & I_\Sigma^{\langle m,n \rangle}[\boldsymbol{\phi}, \boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma \\ & \text{implies} & I_\Sigma^{\langle m,n \rangle}[\boldsymbol{\phi}, \boldsymbol{\psi}] \subseteq \boldsymbol{T}'_\Sigma \\ & \text{iff} & \langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle \in \Omega_\Sigma(\boldsymbol{T}'). \end{array}$$

Hence $\Omega(\boldsymbol{T}) \le \Omega(\boldsymbol{T}')$ and $\mathfrak{G}$ is protoalgebraic. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$. The **reflexive core** $R^{\mathfrak{G}}$ **of** $\mathfrak{G}$ is the collection

$$R^{\mathfrak{G}} = \{R^{\mathfrak{G},\langle m,n \rangle} : \langle m,n \rangle \in \mathrm{tr}\},$$

where, for all $\langle m,n \rangle \in \mathrm{tr}$, $R^{\mathfrak{G},\langle m,n \rangle}$ consists of all natural transformations $\rho : (\mathrm{SEN}^\flat)^\omega \to \bigcup_{\langle k,\ell \rangle \in \mathrm{tr}} (\mathrm{SEN}^\flat)^{k+\ell}$ in $N^\flat$ with $(m+n)+(m+n)$ distinguished arguments that satisfy:

1. For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\rho_\Sigma[\langle \phi, \vec{\chi} \rangle, \langle \phi, \vec{\chi} \rangle] \subseteq \mathrm{Thm}_\Sigma(\mathfrak{G});$$

2. For all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ of trace $\langle m,n \rangle$,

$$\rho_{\Sigma'}[\mathrm{SEN}^\flat(f)(\boldsymbol{\phi}), \mathrm{SEN}^\flat(f)(\boldsymbol{\psi})] \subseteq G_{\Sigma'}(\rho_\Sigma[\boldsymbol{\phi}, \boldsymbol{\psi}]).$$

Using the notation in the proof of Theorem 1919, we observe that, $\hat{R}^{\mathfrak{G}} \subseteq R^{\mathfrak{G}}$ and that $\check{R}^{\mathfrak{G}} \subseteq R^{\mathfrak{G}}$:

- If $\rho \in R^{\mathfrak{G}}$, then, for

$$\sigma_\Sigma(\boldsymbol{\phi}, \boldsymbol{\psi}, \vec{\chi}) \coloneqq \rho_\Sigma(\boldsymbol{\phi}^\pi, \boldsymbol{\psi}^\pi, \vec{\chi}),$$

  we get $\sigma_\Sigma[\boldsymbol{\phi}, \boldsymbol{\phi}] = \rho_\Sigma[\boldsymbol{\phi}^\pi, \boldsymbol{\phi}^\pi] \subseteq \mathrm{Thm}_\Sigma(\mathfrak{G})$;

- If $\rho \in R^{\mathfrak{G}}$, then, for

$$\sigma_\Sigma(\boldsymbol{\phi}, \boldsymbol{\psi}, \vec{\chi}) \coloneqq \rho_\Sigma((\boldsymbol{\phi}\boldsymbol{\psi})^{i+1}, (\boldsymbol{\phi}\boldsymbol{\psi})^i, \vec{\chi}),$$

  we get $\sigma_\Sigma[\boldsymbol{\phi}, \boldsymbol{\phi}] = \rho_\Sigma[\boldsymbol{\phi}, \boldsymbol{\phi}] \subseteq \mathrm{Thm}_\Sigma(\mathfrak{G})$.

If a Gentzen π-institution $\mathfrak{G}$ of trace tr is syntactically protoalgebraic with witnessing transformations I, then $I^{\langle m,n\rangle} \subseteq R^{\mathfrak{G},\langle m,n\rangle}$, for all $\langle m,n\rangle \in \mathrm{tr}$.

Lemma 1915 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *is syntactically protoalgebraic with witnessing transformations* $I = \{I^{\langle m,n\rangle} : \langle m,n\rangle \in \mathrm{tr}\}$*, then* $I \subseteq R^{\mathfrak{G}}$*.*

Proof: Suppose that $\mathfrak{G}$ is syntactically protoalgebraic, with witnessing transformations I.

- Since, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$, $\langle \boldsymbol{\phi}, \boldsymbol{\phi}\rangle \in \Omega_\Sigma(\mathrm{Thm}(\mathfrak{G}))$, we get that $I_\Sigma[\boldsymbol{\phi}, \boldsymbol{\phi}] \subseteq \mathrm{Thm}_\Sigma(\mathfrak{G})$.
- If, for some $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$, we have $I_\Sigma[\boldsymbol{\phi}, \boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma$, then we get $\langle \boldsymbol{\phi}, \boldsymbol{\psi}\rangle \in \Omega_\Sigma(\boldsymbol{T})$, whence, since $\Omega(\boldsymbol{T})$ is a congruence system on $\mathbf{F}$, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, we get $\langle \mathrm{SEN}^\flat(f)(\boldsymbol{\phi}), \mathrm{SEN}^\flat(f)(\boldsymbol{\psi})\rangle \in \Omega_{\Sigma'}(\boldsymbol{T})$, showing that
$$I_{\Sigma'}[\mathrm{SEN}^\flat(f)(\boldsymbol{\phi}), \mathrm{SEN}^\flat(f)(\boldsymbol{\psi})] \subseteq \boldsymbol{T}_{\Sigma'}.$$

Thus, by definition of $R^{\mathfrak{G}}$, we get that $I \subseteq R^{\mathfrak{G}}$. ■

Another important property of syntactic protoalgebraicity is that it guarantees that the reflexive core of $\mathfrak{G}$ possesses a modus ponens property in $\mathfrak{G}$.

Theorem 1916 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *is syntactically protoalgebraic, then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ *of the same trace,*
$$\boldsymbol{\psi} \in G_\Sigma(\boldsymbol{\phi}, R_\Sigma^{\mathfrak{G}}[\boldsymbol{\phi}, \boldsymbol{\psi}]).$$

Proof: Suppose $\mathfrak{G}$ is syntactically protoalgebraic, with witnessing transformations I and let $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$, such that $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$ and $R_\Sigma^{\mathfrak{G}}[\boldsymbol{\phi}, \boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma$. Then, by Lemma 1915, we get $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$ and $I_\Sigma[\boldsymbol{\phi}, \boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma$, that is, by syntactic protoalgebraicity, $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$ and $\langle \boldsymbol{\phi}, \boldsymbol{\psi}\rangle \in \Omega_\Sigma(\boldsymbol{T})$. Therefore, by compatibility, we get $\boldsymbol{\psi} \in \boldsymbol{T}_\Sigma$, showing that $\boldsymbol{\psi} \in G_\Sigma(\boldsymbol{\phi}, R_\Sigma^{\mathfrak{G}}[\boldsymbol{\phi}, \boldsymbol{\psi}])$. ■

Conversely, if the reflexive core $R^{\mathfrak{G}}$ of a Gentzen π-institution $\mathfrak{G}$ has the modus ponens property in $\mathfrak{G}$, then $\mathfrak{G}$ is syntactically protoalgebraic, with witnessing transformations $R^{\mathfrak{G}}$. First, a lemma of a technical nature. For a Gentzen π-institution $\mathfrak{G}$ and $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, we set
$$R^{\mathfrak{G}}(\boldsymbol{T}) = \{R_\Sigma^{\mathfrak{G}}(\boldsymbol{T})\}_{\Sigma \in |\mathbf{Sign}^\flat|},$$

where, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$R^{\mathfrak{G}}_\Sigma(\boldsymbol{T}) = \{\langle\phi,\psi\rangle \in \mathrm{SEN}^\flat(\Sigma) : (\forall\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma))(R^{\mathfrak{G}}_\Sigma[\langle\phi,\vec{\chi}\rangle,\langle\psi,\vec{\chi}\rangle] \subseteq \boldsymbol{T}_\Sigma)\}.$$

Of course, by the symmetry of the transformations in $N^\flat$, in this definition, ϕ and ψ may appear, equivalently, in any position of the sequents on the right, as long as they appear in the same position in both of the first sequent arguments of $R^{\mathfrak{G}}$.

Lemma 1917 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle\mathbf{F},G\rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$*. If, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\boldsymbol{\phi},\boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ *of the same trace,*

$$\boldsymbol{\psi} \in G_\Sigma(\boldsymbol{\phi},R^{\mathfrak{G}}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}]),$$

the $R^{\mathfrak{G}}(\boldsymbol{T})$ *is a congruence family on* $\mathbf{F}$ *compatible with* $\boldsymbol{T}$*.*

Proof: We start by showing that $R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$ is an equivalence family on $\mathbf{F}$.

- By the definition of $R^{\mathfrak{G}}$, we get, for all $\phi,\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

  $$R^{\mathfrak{G}}_\Sigma[\langle\phi,\vec{\chi}\rangle,\langle\phi,\vec{\chi}\rangle] \subseteq \mathrm{Thm}_\Sigma(\mathfrak{G}) \subseteq \boldsymbol{T}_\Sigma.$$

  Thus, $\langle\phi,\phi\rangle \in R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$ and $R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$ is reflexive.

- Suppose $\langle\phi,\psi\rangle \in R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$. Then, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, we have

  $$R^{\mathfrak{G}}_\Sigma[\langle\phi,\vec{\chi}\rangle,\langle\psi,\vec{\chi}\rangle] \subseteq \boldsymbol{T}_\Sigma.$$

  But then, by the definition of $R^{\mathfrak{G}}$ and the symmetry of $N^\flat$, we get

  $$R^{\mathfrak{G}}_\Sigma[\langle\psi,\vec{\chi}\rangle,\langle\phi,\vec{\chi}\rangle] \subseteq R^{\mathfrak{G}}_\Sigma[\langle\phi,\vec{\chi}\rangle,\langle\psi,\vec{\chi}\rangle] \subseteq \boldsymbol{T}_\Sigma.$$

  Therefore, $\langle\psi,\phi\rangle \in R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$ and $R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$ is also symmetric.

- Suppose, now, that $\langle\phi,\psi\rangle,\langle\psi,\chi\rangle \in R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$. Thus, we get, for all $\vec{\chi} \in \mathrm{SEN}(\Sigma)$,

  $$R^{\mathfrak{G}}_\Sigma[\langle\phi,\vec{\chi}\rangle,\langle\psi,\vec{\chi}\rangle] \subseteq \boldsymbol{T}_\Sigma \text{ and } R^{\mathfrak{G}}_\Sigma[\langle\psi,\vec{\chi}\rangle,\langle\chi,\vec{\chi}\rangle] \subseteq \boldsymbol{T}_\Sigma.$$

  By hypothesis, we have, for all $\rho \in R^{\mathfrak{G}}$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$,

  $$\begin{aligned} \rho_\Sigma(\langle\phi,\vec{\chi}\rangle,\langle\chi,\vec{\chi}\rangle,\vec{\xi}) &\subseteq G_\Sigma(\rho_\Sigma(\langle\phi,\vec{\chi}\rangle,\langle\psi,\vec{\chi}\rangle,\vec{\xi}),\\ &\qquad R^{\mathfrak{G}}_\Sigma[\rho_\Sigma(\langle\phi,\vec{\chi}\rangle,\langle\psi,\vec{\chi}\rangle,\vec{\xi}),\rho_\Sigma(\langle\phi,\vec{\chi}\rangle,\langle\chi,\vec{\chi}\rangle,\vec{\xi})])\\ &\subseteq G_\Sigma(R^{\mathfrak{G}}_\Sigma[\langle\phi,\vec{\chi}\rangle,\langle\psi,\vec{\chi}\rangle],R^{\mathfrak{G}}_\Sigma[\langle\psi,\vec{\chi}\rangle,\langle\chi,\vec{\chi}\rangle])\\ &\subseteq G_\Sigma(\boldsymbol{T}_\Sigma) = \boldsymbol{T}_\Sigma. \end{aligned}$$

  Therefore, $R^{\mathfrak{G}}_\Sigma[\langle\phi,\vec{\chi}\rangle,\langle\chi,\vec{\chi}\rangle] \subseteq \boldsymbol{T}_\Sigma$, showing that $\langle\phi,\chi\rangle \in R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$ and, hence, $R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$ is also transitive.

We show, next, that $R^{\mathfrak{G}}(\boldsymbol{T})$ is a congruence family. Let σ be in $N^\flat$, $\vec{\phi},\vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$, such that, for all $i < k$, $\langle\phi_i,\psi_i\rangle \in R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$. Then, for all $i < k$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $R^{\mathfrak{G}}_\Sigma[\langle\phi_i,\vec{\chi}\rangle,\langle\psi_i,\vec{\chi}\rangle] \subseteq \boldsymbol{T}_\Sigma$. But, then, for all $i < k$,

$$R^{\mathfrak{G}}_\Sigma[\langle\sigma_\Sigma((\vec{\phi}\vec{\psi})^{i+1}),\vec{\chi}\rangle,\langle\sigma_\Sigma((\vec{\phi}\vec{\psi})^{i}),\vec{\chi}\rangle] \subseteq R^{\mathfrak{G}}_\Sigma[\langle\phi_i,\vec{\chi}\rangle,\langle\psi_i,\vec{\chi}\rangle] \subseteq \boldsymbol{T}_\Sigma,$$

i.e., $\langle\sigma_\Sigma((\vec{\phi}\vec{\psi})^{i+1}),\sigma_\Sigma((\vec{\phi}\vec{\psi})^{i})\rangle \in R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$. Since this holds for all $i < k$, we get by the transitivity of $R^{\mathfrak{G}}(\boldsymbol{T})$ proven above, that $\langle\sigma_\Sigma(\vec{\phi}),\sigma_\Sigma(\vec{\psi})\rangle \in R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$ and, therefore, $R^{\mathfrak{G}}(\boldsymbol{T})$ is also a congruence family.

Finally, $R^{\mathfrak{G}}(\boldsymbol{T})$ is a congruence system by the definition of $R^{\mathfrak{G}}$. Compatibility of $R^{\mathfrak{G}}(\boldsymbol{T})$ with $\boldsymbol{T}$ is also readily obtainable by the hypothesis, since $\langle\boldsymbol{\phi},\boldsymbol{\psi}\rangle \in R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$ implies $R^{\mathfrak{G}}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma$. Therefore, if $\boldsymbol{\phi} \in T_\Sigma$ and $\langle\boldsymbol{\phi},\boldsymbol{\psi}\rangle \in R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$, we get

$$\boldsymbol{\psi} \in G_\Sigma(\boldsymbol{\phi},R^{\mathfrak{G}}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}]) \subseteq G_\Sigma(\boldsymbol{T}_\Sigma) = \boldsymbol{T}_\Sigma.$$

Hence, $R^{\mathfrak{G}}(\boldsymbol{T})$ is a congruence system on $\mathbf{F}$ compatible with $\boldsymbol{T}$. ■

Now we are ready for the promised theorem.

Theorem 1918 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle\mathbf{F},G\rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$. *If, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\boldsymbol{\phi},\boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ *of the same trace,*

$$\boldsymbol{\psi} \in G_\Sigma(\boldsymbol{\phi},R^{\mathfrak{G}}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}]),$$

then $\mathfrak{G}$ *is syntactically protoalgebraic, with witnessing transformations* $R^{\mathfrak{G}}$.

Proof: Suppose that $R^{\mathfrak{G}}$ satisfies the displayed condition. We must show that, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi},\boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$,

$$\langle\boldsymbol{\phi},\boldsymbol{\psi}\rangle \in \Omega_\Sigma(\boldsymbol{T}) \quad \text{iff} \quad R^{\mathfrak{G}}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma.$$

Suppose, first, that $\langle\boldsymbol{\phi},\boldsymbol{\psi}\rangle \in \Omega_\Sigma(\boldsymbol{T})$. Then,, since $\Omega(\boldsymbol{T})$ is a congruence system on $\mathbf{F}$, we get, for all $\rho \in R^{\mathfrak{G}}$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle\rho_\Sigma(\boldsymbol{\phi},\boldsymbol{\phi},\vec{\chi}),\rho_\Sigma(\boldsymbol{\phi},\boldsymbol{\psi},\vec{\chi})\rangle \in \Omega_\Sigma(\boldsymbol{T}).$$

Moreover, $R^{\mathfrak{G}}_\Sigma[\boldsymbol{\phi},\boldsymbol{\phi}] \subseteq \mathrm{Thm}_\Sigma(\mathfrak{G}) \subseteq \boldsymbol{T}_\Sigma$, by the definition of the reflexive core. Therefore, by the compatibility of $\Omega(\boldsymbol{T})$, with $\boldsymbol{T}$, we get that, for all $\rho \in R^{\mathfrak{G}}$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $\rho_\Sigma(\boldsymbol{\phi},\boldsymbol{\psi},\vec{\chi}) \in \boldsymbol{T}_\Sigma$. We conclude that $R^{\mathfrak{G}}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma$.

Assume, conversely, that $R^{\mathfrak{G}}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}] \subseteq T_\Sigma$. Since, by Lemma 1917, $R^{\mathfrak{G}}(\boldsymbol{T})$ is a congruence system on $\mathbf{F}$ compatible with $\boldsymbol{T}$, we get, by the maximality of $\Omega(\boldsymbol{T})$, that $R^{\mathfrak{G}}(\boldsymbol{T}) \leq \Omega(\boldsymbol{T})$. But the hypothesis implies that $\langle\boldsymbol{\phi},\boldsymbol{\psi}\rangle \in R^{\mathfrak{G}}_\Sigma(T)$. Therefore, we conclude that $\langle\boldsymbol{\phi},\boldsymbol{\psi}\rangle \in \Omega_\Sigma(\boldsymbol{T})$. ■

We now have a characterization of syntactic protoalgebraicity in terms of the property of modus ponens of the reflexive core $R^{\mathfrak{G}}$ of the Gentzen π-institution $\mathfrak{G}$.

$$\mathfrak{G} \text{ is syntactically protoalgebraic} \longleftrightarrow R^{\mathfrak{G}} \text{ has the MP.}$$

Theorem 1919 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*.* $\mathfrak{G}$ *is syntactically protoalgebraic if and only if* $R^{\mathfrak{G}}$ *has the modus ponens in* $\mathfrak{G}$*.*

Proof: Theorem 1916 gives the "only if" and the "if" is by Theorem 1918. ∎

As a corollary, we obtain

Corollary 1920 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *is syntactically protoalgebraic with witnessing transformations* $I = \{I^{\langle m,n\rangle} : \langle m,n\rangle \in \mathrm{tr}\}$*, then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ *of trace* $\langle m,n\rangle$*,*

$$G_\Sigma(R^{\mathfrak{G},\langle m,n\rangle}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}]) = G_\Sigma(I^{\langle m,n\rangle}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}]).$$

Proof: If $\mathfrak{G}$ is syntactically protoalgebraic, with witnessing transformations I, then, by Theorems 1919 and 1918, both I and $R^{\mathfrak{G}}$ are families of witnessing transformations for the syntactic protoalgebraicity of $\mathfrak{G}$. Therefore, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m,n\rangle$,

$$\begin{array}{rll} R^{\mathfrak{G},\langle m,n\rangle}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma & \text{iff} & \langle \boldsymbol{\phi},\boldsymbol{\psi}\rangle \in \Omega_\Sigma(\boldsymbol{T}) \\ & \text{iff} & I^{\langle m,n\rangle}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma. \end{array}$$

Therefore, $G_\Sigma(R^{\mathfrak{G},\langle m,n\rangle}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}]) = G_\Sigma(I^{\langle m,n\rangle}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}])$. ∎

We get relatively easily another related characterization of syntactic protoalgberaicity.

$$\begin{array}{l} \mathfrak{G} \text{ is syntactically protoalgebraic} \\ \qquad\qquad \longleftrightarrow R^{\mathfrak{G}} \text{ Defines Leibniz Congruence Systems.} \end{array}$$

Theorem 1921 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*.* $\mathfrak{G}$ *is syntactically protoalgebraic if and only if, for every* $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$*,*

$$\Omega(\boldsymbol{T}) = R^{\mathfrak{G}}(\boldsymbol{T}).$$

Proof: If $\mathfrak{G}$ is syntactically protoalgebraic, then, by Theorems 1919 and 1918, $R^{\mathfrak{G}}$ constitutes a collection of witnessing transformations, whence, for every $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$ $\Omega(\boldsymbol{T}) = \hat{R}^{\mathfrak{G}}(\boldsymbol{T}) = R^{\mathfrak{G}}(\boldsymbol{T})$.

The converse follows by the definition of syntactic protoalgberaicity, since, in that case, $\check{R}^{\mathfrak{G}} = R^{\mathfrak{G}}$ forms a collection of witnessing transformations. ∎

We finally show that the property that separates protoalgebraicity from syntactic protoalgebraicity is the compatibility property with respect to the theory family generated by the reflexive core.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$. We say that the reflexive core $R^\mathfrak{G}$ is **Leibniz** if, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$,

$$\boldsymbol{\phi} \ \Omega_\Sigma(G(R_\Sigma^\mathfrak{G}[\boldsymbol{\phi}, \boldsymbol{\psi}])) \ \boldsymbol{\psi}.$$

This property is weaker than $R^\mathfrak{G}$ having the modus ponens, i.e., if $R^\mathfrak{G}$ has the modus ponens, then it is Leibniz.

Proposition 1922 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$. *If* $R^\mathfrak{G}$ *has the modus ponens, then it is Leibniz.*

Proof: If $R^\mathfrak{G}$ has the modus ponens, then, by Theorem 1919, we get, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ of trace $\langle m, n \rangle \in \mathrm{tr}$,

$$\boldsymbol{\phi} \ \Omega_\Sigma(\boldsymbol{T}) \ \boldsymbol{\psi} \quad \text{iff} \quad R_\Sigma^\mathfrak{G}[\boldsymbol{\phi}, \boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma.$$

Therefore, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$, by considering, in particular, $\boldsymbol{T} = G(R_\Sigma^\mathfrak{G}[\boldsymbol{\phi}, \boldsymbol{\psi}])$, and taking into account that

$$R_\Sigma^\mathfrak{G}[\boldsymbol{\phi}, \boldsymbol{\psi}] \subseteq G_\Sigma(R_\Sigma^\mathfrak{G}[\boldsymbol{\phi}, \boldsymbol{\psi}]),$$

we get that $\boldsymbol{\phi} \ \Omega_\Sigma(G(R_\Sigma^\mathfrak{G}[\boldsymbol{\phi}, \boldsymbol{\psi}])) \ \boldsymbol{\psi}$. Thus, $R^\mathfrak{G}$ is Leibniz. ■

In the opposite direction, in a protoalgebraic Gentzen π-institution $\mathfrak{G}$, if the reflexive core $R^\mathfrak{G}$ is Leibniz, then it has the modus ponens in $\mathfrak{G}$.

Proposition 1923 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a protoalgebraic Gentzen π-institution of trace* tr *based on* $\mathbf{F}$. *If* $R^\mathfrak{G}$ *is Leibniz, then it has the modus ponens in* $\mathfrak{G}$.

Proof: Suppose that $\mathfrak{G}$ is protoalgebraic and that $R^\mathfrak{G}$ is Leibniz. Let $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ of trace $\langle m, n \rangle \in \mathrm{tr}$, such that $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$ and $R_\Sigma^\mathfrak{G}[\boldsymbol{\phi}, \boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma$. Since $R^\mathfrak{G}$ is Leibniz, we have

$$\boldsymbol{\phi} \ \Omega_\Sigma(G(R_\Sigma^\mathfrak{G}[\boldsymbol{\phi}, \boldsymbol{\psi}])) \ \boldsymbol{\psi},$$

whence, since $\mathfrak{G}$ is protoalgebraic and $R_\Sigma^\mathfrak{G}[\boldsymbol{\phi}, \boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma$, we get $\boldsymbol{\phi} \ \Omega_\Sigma(\boldsymbol{T}) \ \boldsymbol{\psi}$. Therefore, since $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$, we get, by the compatibility of $\Omega(\boldsymbol{T})$ with $\boldsymbol{T}$, that $\boldsymbol{\psi} \in \boldsymbol{T}_\Sigma$. We conclude that $R^\mathfrak{G}$ has the modus ponens in $\mathfrak{G}$. ■

We now show that a Gentzen π-institution is syntactically protoalgebraic if and only if it is protoalgebraic and it has a Leibniz reflexive core.

Syntactic Protoalgebraicity	=	$R^\mathfrak{G}$ has the Modus Ponens
	=	$R^\mathfrak{G}$ Defines Leibniz Congruence Systems
	=	Protoalgebraicity + $R^\mathfrak{G}$ Leibniz

Theorem 1924 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically protoalgebraic if and only if it is protoalgebraic and has a Leibniz reflexive core.*

Proof: Suppose, first, that $\mathfrak{G}$ is syntactically protoalgebraic. Then it is protoalgebraic by Theorem 1914. Moreover, its reflexive core has the modus ponens by Theorem 1916 and, hence, by Proposition 1922, its reflexive core is Leibniz.

Suppose, conversely, that $\mathfrak{G}$ is protoalgebraic with a Leibniz reflexive core. Then, by Proposition 1923, its reflexive core has the modus ponens and, therefore, by Theorem 1919, $\mathfrak{G}$ is syntactically protoalgebraic. ∎

26.8 Order Algebraizability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and K a class of $\mathbf{F}$-algebraic posystems. Recall the inequational π-institution $\mathcal{I}^{\mathsf{K}} = \langle \mathbf{F}, C^{\mathsf{K}} \rangle$ associated with the class K, i.e., in which, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $I \cup \{\phi \preccurlyeq \psi\} \subseteq \mathrm{In}_\Sigma(\mathbf{F})$,

$$\begin{array}{lll} \phi \preccurlyeq \psi \in C^{\mathsf{K}}_\Sigma(I) & \text{iff} & \text{for all } \langle \mathcal{A}, \leq \rangle \in \mathsf{K}, \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma, \Sigma'), \\ & & \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(I)) \subseteq \leq_{F(\Sigma')} \text{ implies} \\ & & \qquad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \leq_{F(\Sigma')} \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi)). \end{array}$$

To $\mathcal{I}^{\mathsf{K}}$ we associate the Gentzen π-institution $\mathfrak{G}^{\mathsf{K}} = \langle \mathbf{F}, G^{\mathsf{K}} \rangle$ of trace $\{\langle 1, 1 \rangle\}$ defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\{\phi_i, \psi_i : i \in I\} \cup \{\phi, \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \rhd_\Sigma \psi \in G^{\mathsf{K}}_\Sigma(\{\phi_i \rhd_\Sigma \psi_i : i \in I\}) \quad \text{iff} \quad \phi \preccurlyeq \psi \in C^{\mathsf{K}}_\Sigma(\{\phi_i \preccurlyeq \psi_i : i \in I\}).$$

We call $\mathfrak{G}^{\mathsf{K}}$ the **inequational Gentzen π-institution associated with** K.

It turns out that, for every class K of $\mathbf{F}$-algebraic posystems, the associated inequational Gentzen π-institution $\mathfrak{G}^{\mathsf{K}}$ is syntactically protoalgebraic.

Theorem 1925 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic posystems. Then* $\mathfrak{G}^{\mathsf{K}} = \langle \mathbf{F}, G^{\mathsf{K}} \rangle$ *is syntactically protoalgebraic.*

Proof: Consider $I = \{I^{\langle 1,1 \rangle}\}$, where $I^{\langle 1,1 \rangle} : (\mathrm{SEN}^\flat)^4 \to (\mathrm{SEN}^\flat)^2$ is given, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \phi', \psi' \in \mathrm{SEN}^\flat(\Sigma)$, by

$$I^{\langle 1,1 \rangle}_\Sigma[\langle \phi, \psi \rangle, \langle \phi', \psi' \rangle] = \{\phi \rhd_\Sigma \phi',\ \phi' \rhd_\Sigma \phi\ \psi \rhd_\Sigma \psi',\ \psi' \rhd_\Sigma \psi\}.$$

Then, we have, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G}^{\mathsf{K}}$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \rhd_\Sigma \psi \in \mathrm{Seq}^{\{\langle 1,1 \rangle\}}_\Sigma(\mathbf{F})$,

$$\begin{array}{lll} \langle \phi, \phi' \rangle, \langle \psi, \psi' \rangle \in \Omega_\Sigma(\boldsymbol{T}) & \text{iff} & \{\phi \rhd_\Sigma \phi',\ \phi' \rhd_\Sigma \phi,\ \psi \rhd_\Sigma \psi',\ \psi' \rhd_\Sigma \psi\} \subseteq \boldsymbol{T}_\Sigma \\ & \text{iff} & I^{\langle 1,1 \rangle}_\Sigma[\phi \rhd_\Sigma \psi, \phi' \rhd_\Sigma \psi'] \subseteq \boldsymbol{T}_\Sigma. \end{array}$$

Therefore, $\mathfrak{G}^{\mathsf{K}}$ is syntactically protoalgebraic, with witnessing transformations I. ∎

Note, also, how $I^{\langle 1,1\rangle}$ satisfies the modus ponens property in $\mathfrak{G}^{\mathsf{K}}$, i.e., for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi,\phi',\psi' \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi' \rhd_\Sigma \psi' \in G^{\mathsf{K}}_\Sigma(\phi \rhd_\Sigma \psi, I^{\langle 1,1\rangle}_\Sigma[\phi \rhd_\Sigma \psi, \phi' \rhd_\Sigma \psi']).$$

We now show that, if the class K happens to be an order guasivariety of $\mathbf{F}$-algebraic posystems, then the Leibniz reduced $\mathfrak{G}^{\mathsf{K}}$-matrix families coincide with the class K.

Proposition 1926 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* K *an ordered guasivariety of* $\mathbf{F}$*-algebraic posystems. Then* $\mathrm{MatFam}^*(\mathfrak{G}^{\mathsf{K}}) = \mathsf{K}$.

Proof: Suppose $\langle \mathcal{A}, \leq\rangle \in \mathsf{K}$ and let $\Sigma \in |\mathbf{Sign}|$ $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi,\psi\rangle \in \Omega^{\mathcal{A}}_\Sigma(\leq)$. Since $\phi \leq_\Sigma \phi$, we get, by compatibility of $\Omega^{\mathcal{A}}(\leq)$ with $\leq$, that $\phi \leq_\Sigma \psi$ and $\psi \leq_\Sigma \phi$. Thus, since $\leq$ is a posystem on $\mathcal{A}$ and, therefore, antisymmetric, we get that $\phi = \psi$. Hence, $\Omega^{\mathcal{A}}(\leq) = \Delta^{\mathcal{A}}$. We conclude that $\langle \mathcal{A}, \leq\rangle \in \mathrm{MatFam}^*(\mathfrak{G}^{\mathsf{K}})$. Thus, $\mathsf{K} \subseteq \mathrm{MatFam}^*(\mathfrak{G}^{\mathsf{K}})$.

Suppose, conversely, that $\langle \mathcal{A}, \leq\rangle \in \mathrm{MatFam}^*(\mathfrak{G}^{\mathsf{K}})$. Then $\Omega^{\mathcal{A}}(\leq) = \Delta^{\mathcal{A}}$. Since K is a class of $\mathbf{F}$-algebraic posystems, we get that, for all σ,τ in $N^\flat$, all $\Sigma,\Sigma' \in |\mathbf{Sign}|$, all $f \in |\mathbf{Sign}(\Sigma,\Sigma')$ and all $\phi,\psi,\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

$$\begin{aligned}&\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}) \leq_{\Sigma'} \tau^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi})\\ &\in G^{\mathsf{K},\mathcal{A}}_\Sigma(\phi \leq_\Sigma \psi, \psi \leq_\Sigma \phi, \sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}) \leq_{\Sigma'} \tau^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi})).\end{aligned}$$

Therefore, if $\phi \leq_\Sigma \psi$ and $\psi \leq_\Sigma \phi$, then we get that $\langle \phi,\psi\rangle \in \Omega^{\mathcal{A}}_\Sigma(\leq) = \Delta^{\mathcal{A}}_\Sigma$, i.e., that $\phi = \psi$. Therefore, $\leq$ is antisymmetric, i.e., $\langle \mathcal{A}, \leq\rangle \in \mathbb{GO}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$. We conclude that $\mathrm{MatFam}^*(\mathfrak{G}^{\mathsf{K}}) \subseteq \mathsf{K}$. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$. $\mathfrak{G}$ is **order algebraizable** if it is equivalent to the inequational Gentzen π-institution $\mathfrak{G}^{\mathsf{K}}$ associated with some class K of $\mathbf{F}$-algebraic posystems.

Order algebraizability implies protoalgebraicity.

Proposition 1927 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *is order algebraizable, then it is protoalgebraic.*

Proof: Suppose that $\mathfrak{G}$ is equivalent to $\mathfrak{G}^{\mathsf{K}}$, for some class K of $\mathbf{F}$-algebraic posystems. Then, since, by Theorem 1925, $\mathfrak{G}^{\mathsf{K}}$ is syntactically protoalgebraic, it is, by Theorem 1914, protoalgebraic. Therefore, by Theorem 1912, $\mathfrak{G}$ is protoalgebraic as well. ∎

The following result provides a characterization of order algebraizability.

Theorem 1928 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*.* $\mathfrak{G}$ *is order algebraizable if and only if there exist a* tr-$\{\langle 1,1\rangle\}$*-transformation* τ *and an* $\{\langle 1,1\rangle\}$-tr*-transformation* ρ*, such that, for all* σ, σ' *in* $N^\flat$*, all* $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$*, all* $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$*, all* $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$*, all* $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$*, all* $\{\phi_i, \psi_i : i \in I\} \subseteq \mathrm{SEN}^\flat(\Sigma)$*, and all* $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$*:*

(1) $\rho_\Sigma[\phi, \phi] \subseteq \mathrm{Thm}_\Sigma(\mathfrak{G})$*;*

(2) $\rho_\Sigma[\phi, \chi] \subseteq G_\Sigma(\rho_\Sigma[\phi, \psi], \rho_\Sigma[\psi, \chi])$*;*

(3) $\rho_\Sigma[\sigma_\Sigma(\psi, \vec{\chi}), \sigma'_\Sigma(\psi, \vec{\chi})]$
$\subseteq G_\Sigma(\rho_\Sigma[\phi, \psi], \rho_\Sigma[\psi, \phi], \rho_\Sigma[\sigma_\Sigma(\phi, \vec{\chi}), \sigma'_\Sigma(\phi, \vec{\chi})])$*;*

(4) $\rho_\Sigma[\phi, \psi] \subseteq G_\Sigma(\bigcup_{i\in I} \rho_\Sigma[\phi_i, \psi_i])$ *implies*

$$\rho_{\Sigma'}[\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(\psi)] \subseteq G_{\Sigma'}(\bigcup_{i\in I} \rho_{\Sigma'}[\mathrm{SEN}^\flat(f)(\phi_i), \mathrm{SEN}^\flat(f)(\psi_i)]);$$

(5) $G_\Sigma(\boldsymbol{\phi}) = G_\Sigma(\rho_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]])$*.*

Proof: Suppose, first, that $\mathfrak{G}$ is order algebraizable. Then there exist τ and ρ as postulated and a class K of $\mathbf{F}$-algebraic posystems, such that $\mathfrak{G}$ is equivalent to $\mathfrak{G}^{\mathsf{K}}$ via the conjugate pair $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}}$. Since, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \rhd_\Sigma \phi \in \mathrm{Thm}_\Sigma(\mathfrak{G}^{\mathsf{K}})$, we get that $\rho_\Sigma[\phi, \phi] \subseteq \mathrm{Thm}_\Sigma(\mathfrak{G})$. So Condition (1) holds. Since, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \rhd_\Sigma \chi \in G^{\mathsf{K}}_\Sigma(\phi \rhd \psi, \psi \rhd_\Sigma \chi)$, we get that

$$\rho_\Sigma[\phi, \chi] \subseteq G_\Sigma(\rho_\Sigma[\phi, \psi], \rho_\Sigma[\psi, \chi]).$$

Hence, Condition (2) is also satisfied. If, for some $\langle \mathcal{A}, \leq \rangle \in \mathsf{K}$, we have, for some $\Sigma \in |\mathbf{Sign}|$ and some $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $\phi \leq_\Sigma \psi$ and $\psi \leq_\Sigma \phi$, then, since K is a class of $\mathbf{F}$-algebraic posystems, we get that $\phi = \psi$. Hence, it follows that, if, for σ, σ' in $N^\flat$, and $\vec{\chi} \in \mathrm{SEN}(\Sigma)$, $\sigma^{\mathcal{A}}_\Sigma(\phi, \vec{\chi}) \leq_\Sigma \sigma'^{\mathcal{A}}_\Sigma(\phi, \vec{\chi})$, then, we will also have $\sigma^{\mathcal{A}}_\Sigma(\psi, \vec{\chi}) \leq_\Sigma \sigma'^{\mathcal{A}}_\Sigma(\psi, \vec{\chi})$. In other words, we get that, for all σ, σ' in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$, and all $\phi, \psi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\sigma_\Sigma(\psi, \vec{\chi}) \rhd_\Sigma \sigma'_\Sigma(\psi, \vec{\chi}) \in G^{\mathsf{K}}_\Sigma(\phi \rhd_\Sigma \psi, \psi \rhd_\Sigma \phi, \sigma_\Sigma(\phi, \vec{\chi}) \rhd_\Sigma \sigma'_\Sigma(\phi, \vec{\chi})).$$

Again, by applying ρ we get that Condition (3) holds. Suppose, now, that for some $\Sigma \in |\mathbf{Sign}^\flat|$ and $\{\phi_i, \psi_i : i \in I\} \cup \{\phi, \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, $\rho_\Sigma[\phi, \psi] \subseteq G_\Sigma(\bigcup_{i\in I} \rho_\Sigma[\phi_i, \psi_i])$. Then, we get $\phi \rhd_\Sigma \psi \in G^{\mathsf{K}}_\Sigma(\{\phi_i \rhd_\Sigma \psi_i : i \in I\})$. Therefore, since $\mathfrak{G}^{\mathsf{K}}$ is structural, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\mathrm{SEN}^\flat(f)(\phi \rhd_\Sigma \psi) \in G^{\mathsf{K}}_{\Sigma'}(\{\mathrm{SEN}^\flat(f)(\phi_i \rhd_\Sigma \psi_i) : i \in I\}).$$

By applying ρ again, we get that Condition (4) holds. Finally, Condition (5) holds directly by the definition of equivalence.

Assume, conversely, that ρ and τ, as postulated in the statement, exist and that they satisfy Conditions (1)-(5). Define $\mathfrak{G}' = \langle \mathbf{F}, G' \rangle$ of trace $\{\langle 1,1\rangle\}$ by setting, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $\{\phi_i, \psi_i : i \in I\} \cup \{\phi, \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \rhd_\Sigma \psi \in G'_\Sigma(\{\phi_i \rhd_\Sigma \psi_i : i \in I\}) \quad \text{iff} \quad \rho_\Sigma[\phi, \psi] \subseteq G_\Sigma(\bigcup\{\rho_\Sigma[\phi_i, \psi_i] : i \in I\}).$$

Then, by the fact that $\mathfrak{G}$ is a Gentzen π-institution and Property (4), we get that $\mathfrak{G}'$ is also a Gentzen π-institution. Moreover, by its definition and Condition (5), taking into account Lemma 1879, $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ is an equivalence. Thus, it suffices to show that $\mathfrak{G}' = \mathfrak{G}^{\mathsf{K}}$, for some class K of $\mathbf{F}$-algebraic posystems. For this, in turn, it suffices, by Theorem 1901, to show that $\mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G}')$ is a class of $\mathbf{F}$-algebraic posystems.

Note, first, that $I = \{I^{\langle 1,1\rangle}\}$, defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \phi', \psi' \in \mathrm{SEN}^\flat(\Sigma)$,

$$I_\Sigma^{\langle 1,1\rangle}[\langle \phi, \psi\rangle, \langle \phi', \psi'\rangle] \coloneqq \{\phi \rhd_\Sigma \phi',\ \phi' \rhd_\Sigma \phi,\ \psi \rhd_\Sigma \psi',\ \psi' \rhd_\Sigma \psi\}$$

is a subset of $R^{\mathfrak{G}'}$, which, by Condition (2) and the definition of $\mathfrak{G}'$ satisfies the Modus Ponens in $\mathfrak{G}'$. Therefore, by Theorem 1918, $\mathfrak{G}'$ is syntactically protoalgebraic and, hence, by Theorem 1914, it is protoalgebraic. Thus, by Lemma 1899, the Leibniz and the Suszko operator coincide. Moreover, by Conditions (1) and (2) and the definition of $\mathfrak{G}'$, for all $\langle \mathcal{A}, \le\rangle \in \mathrm{MatFam}(\mathfrak{G}')$, the relation family $\le$ is reflexive and transitive. Also, by Condition (3) and the definition of $\mathfrak{G}'$, we get that, for all $\langle \mathcal{A}, \le\rangle \in \mathrm{MatFam}(\mathfrak{G}')$, all σ, σ' in $N^\flat$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi, \vec{\chi} \in \mathrm{SEN}(\Sigma)$,

$$\phi \le_\Sigma \psi,\ \psi \le_\Sigma \phi,\ \sigma_\Sigma^{\mathcal{A}}(\phi, \vec{\chi}) \le_\Sigma \sigma_\Sigma'^{\mathcal{A}}(\phi, \vec{\chi}) \text{ imply } \sigma_\Sigma^{\mathcal{A}}(\psi, \vec{\chi}) \le_\Sigma \sigma_\Sigma'^{\mathcal{A}}(\psi, \vec{\chi}).$$

We finish the proof by showing that, for all $\langle \mathcal{A}, \le\rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G}')$, $\le$ is also antisymmetric. To this end, let $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\phi \le_\Sigma \psi$ and $\psi \le_\Sigma \phi$. Then, by Property (4) and the definition of $\mathfrak{G}'$, we get that, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}(f)(\phi) \le_{\Sigma'} \mathrm{SEN}(f)(\psi) \text{ and } \mathrm{SEN}(f)(\psi) \le_{\Sigma'} \mathrm{SEN}(f)(\phi).$$

Then, by what was shown above, we have, for all σ, σ' in $N^\flat$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

$$\begin{aligned} &\sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}(f)(\phi), \vec{\chi}) \le_{\Sigma'} \sigma_{\Sigma'}'^{\mathcal{A}}(\mathrm{SEN}(f)(\phi), \vec{\chi}) \\ &\qquad \text{iff} \quad \sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}(f)(\psi), \vec{\chi}) \le_{\Sigma'} \sigma_{\Sigma'}'^{\mathcal{A}}(\mathrm{SEN}(f)(\psi), \vec{\chi}). \end{aligned}$$

Therefore, by Corollary 1896, we get $\langle \phi, \psi\rangle \in \Omega_\Sigma^{\mathcal{A}}(\le) = \widetilde{\Omega}_\Sigma^{\mathfrak{G}', \mathcal{A}}(\le) = \Delta_\Sigma^{\mathcal{A}}$. We conclude that $\langle \mathcal{A}, \le\rangle$ is indeed an $\mathbf{F}$-algebraic posystem. Hence, $\mathfrak{G}'$ is an inequational Gentzen π-institution associated with the class $\mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G}')$ of $\mathbf{F}$-algebraic posystems and, as a consequence, the Gentzen π-institution $\mathfrak{G}$ is indeed order algebraizable. ∎

Specializing Theorem 1928 to the case of Hilbert π-institutions, we get

Corollary 1929 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathfrak{H} = \langle \mathbf{F}, H \rangle$ *a Hilbert* π*-institution based on* $\mathbf{F}$. $\mathfrak{H}$ *is order algebraizable if and only if there exist a* $\{\langle 0,1\rangle\}$-$\{\langle 1,1\rangle\}$*-transformation* τ *and an* $\{\langle 1,1\rangle\}$-$\{\langle 0,1\rangle\}$*-transformation* ρ*, such that, for all* σ, σ' *in* $N^\flat$*, all* $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$*, all* $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$*, all* $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$ *all* $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$ *and all* $\{\phi_i, \psi_i : i \in I\} \subseteq \mathrm{SEN}^\flat(\Sigma)$*:*

(1) $\rho_\Sigma[\phi, \phi] \subseteq \mathrm{Thm}_\Sigma(\mathfrak{H})$*;*

(2) $\rho_\Sigma[\phi, \chi] \subseteq H_\Sigma(\rho_\Sigma[\phi, \psi], \rho_\Sigma[\psi, \chi])$*;*

(3) $\rho_\Sigma[\sigma_\Sigma(\psi, \vec{\chi}), \sigma'_\Sigma(\psi, \vec{\chi})]$
$\subseteq H_\Sigma(\rho_\Sigma[\phi, \psi], \rho_\Sigma[\psi, \phi], \rho_\Sigma[\sigma_\Sigma(\phi, \vec{\chi}), \sigma'_\Sigma(\phi, \vec{\chi})])$*;*

(4) $\rho_\Sigma[\phi, \psi] \subseteq H_\Sigma(\bigcup_{i\in I} \rho_\Sigma[\phi_i, \psi_i])$ *implies*

$$\rho_{\Sigma'}[\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(\psi)] \subseteq H_{\Sigma'}(\bigcup_{i\in I} \rho_{\Sigma'}[\mathrm{SEN}^\flat(f)(\phi_i), \mathrm{SEN}^\flat(f)(\psi_i)]);$$

(5) $H_\Sigma(\triangleright_\Sigma \phi) = H_\Sigma(\rho_\Sigma[\tau_\Sigma[\phi]])$.

Proof: Directly from Theorem 1928. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$. $\mathfrak{G}$ is **simply order algebraizable** if it is equivalent to the inequational Gentzen π-institution $\mathfrak{G}^{\mathsf{K}}$, associated with some class K of $\mathbf{F}$-algebraic posystems, via a conjugate pair $(\tau, \rho^0) : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}}$, where, as before, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\rho^0_\Sigma(\phi; \psi) = \phi \triangleright_\Sigma \psi.$$

We have the following analog of Lemma 1823.

Lemma 1930 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *is simply order algebraizable via both* $(\tau, \rho^0) : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}}$ *and* $(\tau', \rho^0) : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}'}$*, then* $\mathbb{GO}^{\mathsf{Sem}}(\mathsf{K}) = \mathbb{GO}^{\mathsf{Sem}}(\mathsf{K}')$*.*

Proof: Suppose $\mathfrak{G}$ is simply order algebraizable via both $(\tau, \rho^0) : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}}$ and $(\tau', \rho^0) : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}'}$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $I \cup \{\phi \preccurlyeq \psi\} \subseteq \mathrm{In}_\Sigma(\mathbf{F})$, we have

$$\begin{array}{rll} \phi \preccurlyeq \psi \in G^{\mathsf{K}}_\Sigma(I) & \text{iff} & \rho^0_\Sigma[\phi; \psi] \subseteq G_\Sigma(\rho^0_\Sigma[I]) \\ & \text{iff} & \phi \preccurlyeq \psi \in G^{\mathsf{K}'}_\Sigma(I). \end{array}$$

Thus, K and K' satisfy exactly the same $\mathbf{F}$-guasiinequations. ■

The unique order guasivariety K that simply order algebraizes a simply order algebraizable Gentzen π-institution $\mathfrak{G}$ is called the **order class of** $\mathfrak{G}$.

Specializing Theorem 1928, we get

Corollary 1931 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace, with* $\langle 1,1 \rangle \in \mathrm{tr}$*, and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$. $\mathfrak{G}$ *is simply order algebraizable if and only if there exists a* tr-$\{\langle 1,1 \rangle\}$*-transformation* τ*, such that, for all* σ, σ' *in* $N^\flat$*, all* $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$*, all* $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$*, all* $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$*, all* $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$ *and all* $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$*:*

(1) $\phi \rhd_\Sigma \phi \in \mathrm{Thm}_\Sigma(\mathfrak{G})$;

(2) $\phi \rhd_\Sigma \chi \in G_\Sigma(\phi \rhd_\Sigma \psi, \psi \rhd_\Sigma \chi)$;

(3) $\sigma_\Sigma(\psi, \vec{\chi}) \rhd_\Sigma \sigma'_\Sigma(\psi, \vec{\chi}) \in G_\Sigma(\phi \rhd_\Sigma \psi, \psi \rhd_\Sigma \phi, \sigma_\Sigma(\phi, \vec{\chi}) \rhd_\Sigma \sigma'_\Sigma(\phi, \vec{\chi}))$;

(4) $G_\Sigma(\boldsymbol{\phi}) = G_\Sigma(\rho^0_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]])$.

Proof: Directly by Theorem 1928. ∎

26.9 Truth Equationality

By Theorem 1901, the closure system G of a Gentzen π-institution $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ can be recovered by the class $\mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$ of its Suszko reduced matrix families. A related issue is to investigate when G can be recovered just from the class of underlying $\mathbf{F}$-algebraic systems of the class $\mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$, i.e., from the class $\mathrm{AlgSys}(\mathfrak{G})$. The algebraizability property of $\mathfrak{G}$ gives that

$$\mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G}) = \{\langle \mathcal{A}, \tau^{\mathcal{A}*}(\Delta^{\mathcal{A}}) : \mathcal{A} \in \mathrm{AlgSys}(\mathfrak{G})\},$$

where $\tau : \mathfrak{G} \to \mathfrak{G}^{\mathsf{K}}$ is the $\{\langle 1,1 \rangle\}$-tr-transformation witnessing the algebraizability. In this case, the $\mathbf{F}$-algebraic system $\mathcal{A} \in \mathrm{AlgSys}(\mathfrak{G})$ is the $\mathbf{F}$-algebraic system reduct of a unique Suszko reduced $\mathfrak{G}$-matrix family, i.e., the $\mathfrak{G}$-filter family of every Suszko reduced $\mathfrak{G}$-matrix family is uniquely determined by the $\mathbf{F}$-algebraic system $\mathcal{A}$, since it is exactly $\tau^{\mathcal{A}*}(\Delta^{\mathcal{A}})$ and this expression does not depend on the choice of τ witnessing the algebraizability of $\mathfrak{G}$.

Even in the absence of algebraizability, however, if each $\mathbf{F}$-algebraic system in $\mathrm{AlgSys}(\mathfrak{G})$ is the $\mathbf{F}$-algebraic system reduct of e unique Suszko reduced $\mathfrak{G}$-matrix family, then, there exists, modulo a technical condition, analogous to the adequacy of the Suszko core introduced in a preceding chapter, a $\{\langle 1,1 \rangle\}$-tr-transformation τ that determines the unique $\mathfrak{G}$-matrix filter on the $\mathbf{F}$-algebraic system, as described previously.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution os trace tr based on $\mathbf{F}$.

- $\mathfrak{G}$ is **completely reflective**, or **c-reflective** for short, if, for all $\boldsymbol{\mathcal{T}} \cup \{\boldsymbol{T}'\} \subseteq \mathrm{ThFam}(\mathfrak{G})$,

$$\bigcap_{\boldsymbol{T} \in \boldsymbol{\mathcal{T}}} \Omega^{\mathcal{A}}(\boldsymbol{T}) \leq \Omega^{\mathcal{A}}(\boldsymbol{T}') \quad \text{implies} \quad \bigcap \boldsymbol{\mathcal{T}} \leq \boldsymbol{T}';$$

- $\mathfrak{G}$ is **truth equational** if there exists $\tau = \{\tau^{\langle m,n\rangle} : \langle m,n\rangle \in \mathrm{tr}\}$, where $\tau^{\langle m,n\rangle} : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$, with $m+n$ distinguished arguments, such that, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m,n\rangle$,
$$\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma \quad \text{iff} \quad \tau^{\langle m,n\rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T}).$$

First, we provide a characterization of c-reflectivity.

Theorem 1932 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system,* tr *a trace and $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ a Gentzen π-institution os trace* tr *based on* $\mathbf{F}$*. Then, the following statements are equivalent:*

(i) For every $\mathcal{A} \in \mathrm{AlgSys}(\mathfrak{G})$, there exists unique $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, such that $\langle \mathcal{A}, \boldsymbol{T}\rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$;

(ii) For every $\mathbf{F}$-algebraic system $\mathcal{A}$, and all $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, $\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$ is the least $\mathfrak{G}$-filter family on $\mathcal{A}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$;

(iii) For every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}$ is injective on $\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$;

(iv) For every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $\mathcal{T} \cup \{\boldsymbol{T}'\} \subseteq \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$,
$$\bigcap_{\boldsymbol{T}\in\mathcal{T}} \Omega^{\mathcal{A}}(\boldsymbol{T}) \leq \Omega^{\mathcal{A}}(\boldsymbol{T}') \quad \text{implies} \quad \bigcap \mathcal{T} \leq \boldsymbol{T}';$$

(v) For all $\mathcal{T} \cup \{\boldsymbol{T}'\} \subseteq \mathrm{ThFam}(\mathfrak{G})$, $\bigcap_{\boldsymbol{T}\in\mathcal{T}} \Omega(\boldsymbol{T}) \leq \Omega(\boldsymbol{T}')$ implies $\bigcap \mathcal{T} \leq \boldsymbol{T}'$.

Proof:

(i)⇒(ii) Suppose (i) holds and let $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$. Consider the algebraic system $\mathcal{A}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$ and let $\boldsymbol{T}'$ be the least filter on $\mathcal{A}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$. Then, since $\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}))$, we get that $\boldsymbol{T}' \leq \boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$. Thus, by the monotonicity of the Suszko operator,
$$\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})}(\boldsymbol{T}') \leq \widetilde{\Omega}^{\mathfrak{G},\mathcal{A}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})}(\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})) = \Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})}.$$
But, noting that $\mathcal{A}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) \in \mathrm{AlgSys}(\mathfrak{G})$, we get, by hypothesis, that $\boldsymbol{T}' = \boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$. Therefore, $\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$ is the least $\mathfrak{G}$-filter family on $\mathcal{A}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$.

(ii)⇒(iii) Suppose that (ii) holds and let $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $\boldsymbol{T}, \boldsymbol{T}' \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) = \widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}')$. Then $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$ is compatible with both $\boldsymbol{T}$ and $\boldsymbol{T}'$ and, hence, $\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$ and $\boldsymbol{T}'/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$ are both $\mathfrak{G}$-filter families on $\mathcal{A}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$. Thus, by hypothesis, $\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) \leq \boldsymbol{T}'/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$. Therefore, taking into account the compatibility of $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$ with both $\boldsymbol{T}$ and $\boldsymbol{T}'$, we get $\boldsymbol{T} \leq \boldsymbol{T}'$. By symmetry, we also have $\boldsymbol{T}' \leq \boldsymbol{T}$, whence $\boldsymbol{T} = \boldsymbol{T}'$. Thus, $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}$ is injective on $\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$.

(iii)⇒(iv) Suppose (iii) holds and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $\boldsymbol{\mathcal{T}} \cup \{\boldsymbol{T}'\} \subseteq \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, such that

$$\bigcap_{\boldsymbol{T} \in \boldsymbol{\mathcal{T}}} \Omega^{\mathcal{A}}(\boldsymbol{T}) \leq \Omega^{\mathcal{A}}(\boldsymbol{T}').$$

Then, we have

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\bigcap \boldsymbol{\mathcal{T}} \cap \boldsymbol{T}') & = & \bigcap\{\Omega^{\mathcal{A}}(\boldsymbol{X}) : \boldsymbol{\mathcal{T}} \cap \boldsymbol{T}' \leq \boldsymbol{X} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\} \\ & = & \bigcap\{\Omega^{\mathcal{A}}(\boldsymbol{X}) : \bigcap \boldsymbol{\mathcal{T}} \leq \boldsymbol{X} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\} \\ & = & \widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\bigcap \boldsymbol{\mathcal{T}}).\end{array}$$

By hypothesis, we get $\bigcap \boldsymbol{\mathcal{T}} \cap \boldsymbol{T}' = \bigcap \boldsymbol{\mathcal{T}}$, whence $\bigcap \boldsymbol{\mathcal{T}} \leq \boldsymbol{T}'$.

(iv)⇒(v) Condition (v) is a special case of Condition (iv).

(v)⇒(i) Assume that (v) holds and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle \in \mathrm{AlgSys}(\mathfrak{G})$ and $\boldsymbol{T}, \boldsymbol{T}' \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) = \widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}') = \Delta^{\mathcal{A}}$. By Lemma 1891, $\alpha^{-1}(\boldsymbol{T})$ and $\alpha^{-1}(\boldsymbol{T}')$ are both theory families of $\mathfrak{G}$. Now we have, by hypothesis, $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) = \widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}')$, whence, by the definition of the Suszko operator,

$$\bigcap\{\Omega^{\mathcal{A}}(\boldsymbol{X}) : \boldsymbol{T} \leq \boldsymbol{X} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\} \leq \Omega^{\mathcal{A}}(\boldsymbol{T}').$$

Hence, applying α^{-1} to both sides,

$$\alpha^{-1}(\bigcap\{\Omega^{\mathcal{A}}(\boldsymbol{X}) : \boldsymbol{T} \leq \boldsymbol{X} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\}) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(\boldsymbol{T}')).$$

Equivalently,

$$\bigcap\{\alpha^{-1}(\Omega^{\mathcal{A}}(\boldsymbol{X})) : \boldsymbol{T} \leq \boldsymbol{X} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\} \leq \alpha^{-1}(\Omega^{\mathcal{A}}(\boldsymbol{T}')).$$

By Lemma 1909,

$$\bigcap\{\Omega(\alpha^{-1}(\boldsymbol{X})) : \boldsymbol{T} \leq \boldsymbol{X} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\} \leq \Omega(\alpha^{-1}(\boldsymbol{T}')).$$

By Condition (v),

$$\bigcap\{\alpha^{-1}(\boldsymbol{X}) : \boldsymbol{T} \leq \boldsymbol{X} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\} \leq \alpha^{-1}(\boldsymbol{T}').$$

Hence, $\alpha^{-1}(\boldsymbol{T}) \leq \alpha^{-1}(\boldsymbol{T}')$, which gives, by the surjectivity of $\langle F, \alpha \rangle$, $\boldsymbol{T} \leq \boldsymbol{T}'$. By symmetry, we get that $\boldsymbol{T} = \boldsymbol{T}'$ and, therefore, there exists only one $\mathfrak{G}$-filter family $\boldsymbol{T}$ on $\mathcal{A}$, such that $\langle \mathcal{A}, \boldsymbol{T} \rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$. ∎

It also turns out that a sufficient condition for the c-reflectivity of a Gentzen π-institution $\mathfrak{G}$ is the injectivity of the Suszko operator on all $\mathbf{F}$-algebraic systems in $\mathrm{AlgSys}(\mathfrak{G})$.

Lemma 1933 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. If, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathfrak{G})$*,* $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}$ *is injective on* $\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$*, then* $\mathfrak{G}$ *is c-reflective.*

Proof: By the hypothesis, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathfrak{G})$, there exists a unique $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, such that $\langle \mathcal{A}, \boldsymbol{T} \rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$. Therefore, by Theorem 1932, $\mathfrak{G}$ is c-reflective. ■

Next we provide an alternative characterization of truth equationality, forming an analog of Theorem 818.

Theorem 1934 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*.* $\mathfrak{G}$ *is truth equational if and only if, there exists* $\tau = \{\tau^{\langle m,n \rangle} : \langle m,n \rangle \in \mathrm{tr}\}$*, where* $\tau^{\langle m,n \rangle} : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$*, with* $m+n$ *distinguished arguments, such that, for all* $\langle \mathcal{A}, \boldsymbol{T} \rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$*,* $\boldsymbol{T} = \tau^{\mathcal{A}*}(\Delta^{\mathcal{A}})$*.*

Proof: Suppose $\mathfrak{G}$ is truth equational, with witnessing transformations $\tau = \{\tau^{\langle m,n \rangle} : \langle m,n \rangle \in \mathrm{tr}\}$. Let $\langle \mathcal{A}, \boldsymbol{T} \rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m,n \rangle$. Then we have

$$\begin{array}{lll} \alpha_\Sigma(\boldsymbol{\phi}) \in \boldsymbol{T}_{F(\Sigma)} & \text{iff} & \alpha_\Sigma(\boldsymbol{\phi}) \in \boldsymbol{T}'_{F(\Sigma)}, \text{ all } \boldsymbol{T} \leq \boldsymbol{T}' \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}) \\ & \text{iff} & \boldsymbol{\phi} \in \alpha_\Sigma^{-1}(\boldsymbol{T}'_{F(\Sigma)}), \text{ all } \boldsymbol{T} \leq \boldsymbol{T}' \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}) \\ & \text{iff} & \tau^{\langle m,n \rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\alpha^{-1}(\boldsymbol{T}')), \text{ all } \boldsymbol{T} \leq \boldsymbol{T}' \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}) \\ & \text{iff} & \tau^{\langle m,n \rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \alpha_\Sigma^{-1}(\Omega^{\mathcal{A}}_{F(\Sigma)}(\boldsymbol{T}')), \text{ all } \boldsymbol{T} \leq \boldsymbol{T}' \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}) \\ & \text{iff} & \alpha_\Sigma(\tau^{\langle m,n \rangle}_\Sigma[\boldsymbol{\phi}]) \subseteq \Omega^{\mathcal{A}}_{F(\Sigma)}(\boldsymbol{T}'), \text{ all } \boldsymbol{T} \leq \boldsymbol{T}' \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}) \\ & \text{iff} & \tau^{\mathcal{A},\langle m,n \rangle}_{F(\Sigma)}[\alpha_\Sigma(\boldsymbol{\phi})] \subseteq \widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}_{F(\Sigma)}(\boldsymbol{T}) \\ & \text{iff} & \tau^{\mathcal{A},\langle m,n \rangle}_{F(\Sigma)}[\alpha_\Sigma(\boldsymbol{\phi})] \subseteq \Delta^{\mathcal{A}}_{F(\Sigma)}. \end{array}$$

The conclusion follows by taking into account the surjectivity of $\langle F, \alpha \rangle$.

Conversely, assume that the condition in the statement holds and let $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m,n \rangle$. Then, since

$$\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}/\Omega(\boldsymbol{T})}(\boldsymbol{T}/\Omega(\boldsymbol{T})) \leq \Omega^{\mathcal{F}/\Omega(\boldsymbol{T})}(\boldsymbol{T}/\Omega * \boldsymbol{T}) = \Delta^{\mathcal{F}/\Omega(\boldsymbol{T})},$$

we get that $\langle \mathcal{F}/\Omega(\boldsymbol{T}), \boldsymbol{T}/\Omega(\boldsymbol{T}) \rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$. Therefore, by hypothesis,

$$\boldsymbol{\phi}/\Omega_\Sigma(\boldsymbol{T}) \in \boldsymbol{T}_\Sigma/\Omega_\Sigma(\boldsymbol{T}) \quad \text{iff} \quad \tau_\Sigma^{\mathcal{F}/\Omega(\boldsymbol{T}),\langle m,n \rangle}[\boldsymbol{\phi}/\Omega_\Sigma(\boldsymbol{T})] \subseteq \Delta_\Sigma^{\mathcal{F}/\Omega(\boldsymbol{T})},$$

i.e.,

$$\boldsymbol{\phi}/\Omega_\Sigma(\boldsymbol{T}) \in \boldsymbol{T}_\Sigma/\Omega_\Sigma(\boldsymbol{T}) \quad \text{iff} \quad \tau_\Sigma^{\langle m,n \rangle}[\boldsymbol{\phi}]/\Omega_\Sigma(\boldsymbol{T}) \subseteq \Delta_\Sigma^{\mathcal{F}/\Omega(\boldsymbol{T})}.$$

By the compatibility of $\Omega(\boldsymbol{T})$ with $\boldsymbol{T}$, we now get

$$\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma \quad \text{iff} \quad \tau_\Sigma^{\langle m,n \rangle}[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T}).$$

Therefore, $\mathfrak{G}$ is truth equational. ∎

Before turning into a characterization of the exact relationship between c-reflectivity and truth equationality, we prove that both c-reflectivity and truth equationality are preserved under equivalence of Gentzen π-institutions.

Theorem 1935 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr, tr′ *traces and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$, $\mathfrak{G}' = \langle \mathbf{F}, G' \rangle$ *Gentzen* π*-institutions of traces* tr, tr′, *respectively, based on* $\mathbf{F}$. *If* $\mathfrak{G}$ *and* $\mathfrak{G}'$ *are equivalent, then* $\mathfrak{G}$ *is c-reflective if and only if* $\mathfrak{G}'$ *is also.*

Proof: Suppose that $\mathfrak{G}'$ is c-reflective and let $\boldsymbol{\mathcal{T}} \cup \{\boldsymbol{T}'\} \subseteq \mathrm{ThFam}(\mathfrak{G})$, such that $\bigcap_{\boldsymbol{T} \in \boldsymbol{\mathcal{T}}} \Omega(\boldsymbol{T}) \leq \Omega(\boldsymbol{T}')$. Then, by Proposition 1897, $\bigcap_{\boldsymbol{T} \in \boldsymbol{\mathcal{T}}} \Omega(\rho^*(\boldsymbol{T})) \leq \Omega(\rho^*(\boldsymbol{T}'))$. Thus, by Theorem 1880 and the hypothesis, we get $\bigcap_{\boldsymbol{T} \in \boldsymbol{\mathcal{T}}} \rho^*(\boldsymbol{T}) \leq \rho^*(\boldsymbol{T}')$ and, then, $\rho^*(\bigcap \boldsymbol{\mathcal{T}}) \leq \rho^*(\boldsymbol{T}')$. As ρ^* is order reflecting, we conclude that $\bigcap \boldsymbol{\mathcal{T}} \leq \boldsymbol{T}'$ and, therefore, $\mathfrak{G}$ is c-reflective. The converse follows by the symmetry of the notion of equivalence. ∎

Theorem 1936 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr, tr′ *traces and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$, $\mathfrak{G}' = \langle \mathbf{F}, G' \rangle$ *Gentzen* π*-institutions of traces* tr, tr′, *respectively, based on* $\mathbf{F}$. *If* $\mathfrak{G}$ *and* $\mathfrak{G}'$ *are equivalent, then* $\mathfrak{G}$ *is truth equational if and only if* $\mathfrak{G}'$ *is also.*

Proof: Suppose that $\mathfrak{G}$ and $\mathfrak{G}'$ are equivalent via a conjugate pair $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ and that $\mathfrak{G}'$ is truth equational, with witnessing transformations $\sigma := \{\sigma^{\langle m,m \rangle} : \langle m, n \rangle \in \mathrm{tr}'\}$. Then, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ of trace $\langle m, n \rangle \in \mathrm{tr}$, we get, setting, according to Theorem 1880, $\boldsymbol{T}' \in \mathrm{ThFam}(\mathfrak{G}')$ be such that $\boldsymbol{T} \underset{\tau^*}{\overset{\rho^*}{\rightleftarrows}} \boldsymbol{T}'$,

$$\begin{array}{lll} \boldsymbol{\phi} \in \boldsymbol{T}_\Sigma & \text{iff} & \boldsymbol{\phi} \in \tau_\Sigma^*(\boldsymbol{T}') \quad (\text{definition of } \boldsymbol{T}') \\ & \text{iff} & \tau_\Sigma[\boldsymbol{\phi}] \subseteq \boldsymbol{T}'_\Sigma \quad (\text{definition of } \tau^*) \\ & \text{iff} & \sigma_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]] \subseteq \Omega_\Sigma(\boldsymbol{T}') \quad (\text{hypothesis}) \\ & \text{iff} & \sigma_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]] \subseteq \Omega_\Sigma(\rho^*(\boldsymbol{T})) \quad (\text{definition of } \boldsymbol{T}^*) \\ & \text{iff} & \sigma_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]] \subseteq \Omega_\Sigma(\boldsymbol{T}). \quad (\text{Proposition 1897}) \end{array}$$

Therefore, $\sigma \circ \tau$ witnesses the truth equationality of $\mathfrak{G}$. The converse follows by the symmetry of equivalence. ∎

We now turn to the investigation of the exact relationship between complete reflectivity and truth equationality. We will show that for a Gentzen π-institution to be truth equational, it must be c-reflective and, in addition satisfy a technical condition analogous to the adequacy of the Suszko core in the context of π-institutions, that ensures that there are enough natural transformations in its category of natural transformations to specify the Suszko operator in a precise sense.

We start by showing that truth equationality implies c-reflectivity.

Proposition 1937 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$. *if* $\mathfrak{G}$ *is truth equational, then it is c-reflective.*

Proof: Suppose $\mathfrak{G}$ is truth equational, with witnessing transformations $\tau = \{\tau^{\langle m,n \rangle} : \langle m,n \rangle \in \mathrm{tr}\}$, and let $\boldsymbol{\mathcal{T}} \cup \{\boldsymbol{T}'\} \subseteq \mathrm{ThFam}(\mathfrak{G})$, such that $\bigcap_{\boldsymbol{T} \in \boldsymbol{\mathcal{T}}} \Omega(\boldsymbol{T}) \leq \Omega(\boldsymbol{T}')$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ of trace $\langle m,n \rangle$, we have

$$\begin{array}{rcl}\boldsymbol{\phi} \in \bigcap_{\boldsymbol{T} \in \boldsymbol{\mathcal{T}}} \boldsymbol{T}_\Sigma & \text{iff} & \boldsymbol{\phi} \in \boldsymbol{T}_\Sigma,\ \boldsymbol{T} \in \boldsymbol{\mathcal{T}}, \\ & \text{iff} & \tau_\Sigma^{\langle m,n \rangle}[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T}),\ \boldsymbol{T} \in \boldsymbol{\mathcal{T}}, \\ & \text{iff} & \tau_\Sigma^{\langle m,n \rangle}[\boldsymbol{\phi}] \subseteq \bigcap_{\boldsymbol{T} \in \boldsymbol{\mathcal{T}}} \Omega_\Sigma(\boldsymbol{T}) \\ & \text{implies} & \tau_\Sigma^{\langle m,n \rangle}[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T}') \\ & \text{iff} & \boldsymbol{\phi} \in \boldsymbol{T}'_\Sigma.\end{array}$$

Thus, $\bigcap \boldsymbol{\mathcal{T}} \leq \boldsymbol{T}'$ and, hence, $\mathfrak{G}$ is c-reflective. ■

The property of c-reflectivity also has a characterization involving both the Suszko and the Leibniz operator. Namely, we obtain

Lemma 1938 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$. $\mathfrak{G}$ *is c-reflective if and only if, for all* $\boldsymbol{T}, \boldsymbol{T}' \in \mathrm{ThFam}(\mathfrak{G})$,

$$\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}) \leq \Omega(\boldsymbol{T}') \quad \textit{implies} \quad \boldsymbol{T} \leq \boldsymbol{T}'.$$

Proof: Suppose, first, that $\mathfrak{G}$ is c-reflective and let $\boldsymbol{T}, \boldsymbol{T}' \in \mathrm{ThFam}(\mathfrak{G})$, such that $\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}) \leq \Omega(\boldsymbol{T}')$. Then, we get

$$\bigcap \{\Omega(\boldsymbol{X}) : \boldsymbol{T} \leq \boldsymbol{X} \in \mathrm{ThFam}(\mathfrak{G})\} \leq \Omega(\boldsymbol{T}').$$

Hence, by hypothesis, $\bigcap \{\boldsymbol{X} : \boldsymbol{T} \leq \boldsymbol{X} \in \mathrm{ThFam}(\mathfrak{G})\} \leq \boldsymbol{T}'$, i.e., $\boldsymbol{T} \leq \boldsymbol{T}'$.

Assume, conversely, that the condition of the statement holds and let $\boldsymbol{\mathcal{T}} \cup \{\boldsymbol{T}'\} \subseteq \mathrm{ThFam}(\mathfrak{G})$, such that $\bigcap_{\boldsymbol{T} \in \boldsymbol{\mathcal{T}}} \Omega(\boldsymbol{T}) \leq \Omega(\boldsymbol{T}')$. Then we get

$$\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\bigcap \boldsymbol{\mathcal{T}}) \leq \bigcap \{\Omega(\boldsymbol{T}) : \boldsymbol{T} \in \boldsymbol{\mathcal{T}}\} \leq \Omega(\boldsymbol{T}').$$

Thus, by hypothesis, $\bigcap \boldsymbol{\mathcal{T}} \leq \boldsymbol{T}'$ and $\mathfrak{G}$ is c-reflective. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$. Define the **Suszko core**

$$S^{\mathfrak{G}} = \{S^{\mathfrak{G},\langle m,n \rangle} : \langle m,n \rangle \in \mathrm{tr}\}$$

of $\mathfrak{G}$, by setting, for all $\langle m,n \rangle \in \mathrm{tr}$,

$$\begin{array}{rl}S^{\mathfrak{G},\langle m,n \rangle} = \{\sigma : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2 \in N^\flat : & \\ (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \boldsymbol{\phi} \in \mathrm{Seq}_\Sigma^{\{\langle m,n \rangle\}}(\mathbf{F})) & \\ (\sigma_\Sigma[\boldsymbol{\phi}] \subseteq \widetilde{\Omega}_\Sigma^{\mathfrak{G},\mathcal{F}}(G(\boldsymbol{\phi})))\}. & \end{array}$$

$S^{\mathfrak{G}}$ is a set of natural candidates from which to seek witnesses for the truth equationality of $\mathfrak{G}$, if such exist, since it satisfies the following property.

Lemma 1939 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *is truth equational, with witnessing transformations* τ*, then* $\tau \subseteq S^{\mathfrak{G}}$*.*

Proof: Suppose $\mathfrak{G}$ is truth equational, with witnessing transformations $\tau = \{\tau^{\langle m,n\rangle} : \langle m,n\rangle \in \mathrm{tr}\}$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ of trace $\langle m,n\rangle$, we have $\boldsymbol{\phi} \in G_\Sigma(\boldsymbol{\phi})$, whence, $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$, for all $G(\boldsymbol{\phi}) \leq \boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$. Thus, by truth equationality, $\tau_\Sigma^{\langle m,n\rangle}[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T})$ and, therefore, $\tau_\Sigma^{\langle m,n\rangle}[\boldsymbol{\phi}] \subseteq \widetilde{\Omega}_\Sigma^{\mathfrak{G},\mathcal{F}}(G(\boldsymbol{\phi}))$. We conclude that $\tau^{\langle m,n\rangle} \subseteq S^{\mathfrak{G},\langle m,n\rangle}$. ∎

The Suszko core of $\mathfrak{G}$ always carries a theory family $\boldsymbol{T}$ of $\mathfrak{G}$ into the Leibniz congruence system $\Omega(\boldsymbol{T})$ of the theory family $\boldsymbol{T}$.

Proposition 1940 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. For all* $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$ *of trace* $\langle m,n\rangle \in \mathrm{tr}$*,*

$$S_\Sigma^{\mathfrak{G},\langle m,n\rangle}[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T}).$$

Proof: Let $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$ of trace $\langle m,n\rangle \in \mathrm{tr}$. Then, by the definition of $S^{\mathfrak{G}}$, we get

$$S_\Sigma^{\mathfrak{G},\langle m,n\rangle}[\boldsymbol{\phi}] \subseteq \widetilde{\Omega}_\Sigma^{\mathfrak{G},\mathcal{F}}(G(\boldsymbol{\phi})) \subseteq \widetilde{\Omega}_\Sigma^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}) \subseteq \Omega_\Sigma(\boldsymbol{T}).$$

This establishes the conclusion. ∎

The converse property, which does not always hold, is called *solubility of the Suszko core.*

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$. $S^{\mathfrak{G}}$ is **soluble** if, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ of trace $\langle m,n\rangle \in \mathrm{tr}$, we get

$$S_\Sigma^{\mathfrak{G},\langle m,n\rangle}[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T}) \quad \text{implies} \quad \boldsymbol{\phi} \in \boldsymbol{T}_\Sigma.$$

Truth equationality of a Gentzen π-institution guarantees the solubility of its Suszko core.

Theorem 1941 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *is truth equational, then the Suszko core* $S^{\mathfrak{G}}$ *is soluble.*

Proof: Suppose that $\mathfrak{G}$ is truth equational, with witnessing transformations τ, and let $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ of trace $\langle m,n\rangle$, such that $S_\Sigma^{\mathfrak{G},\langle m,n\rangle}[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T})$. Then, by Lemma 1939, $\tau_\Sigma^{\langle m,n\rangle}[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T})$. By truth equationality, $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$. Therefore, $S^{\mathfrak{G}}$ is indeed soluble. ∎

Conversely, if the Suszko core of a given Gentzen π-institution $\mathfrak{G}$ is soluble, then it acts as a set of witnessing transformations for the truth equationality of $\mathfrak{G}$.

Theorem 1942 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$*. If* $S^{\mathfrak{G}}$ *is soluble, then* $\mathfrak{G}$ *is truth equational, with witnessing transformations* $S^{\mathfrak{G}}$*.*

Proof: We must show that, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m, n \rangle$,

$$\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma \quad \text{iff} \quad S^{\mathfrak{G},\langle m,n\rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T}).$$

The left to right implication is by Proposition 1940, whereas the reverse is by the hypothesis of the solubility of the Suszko core. ∎

Theorems 1941 and 1942 allow two characterizations of truth equationality in terms of the solubility of the Suszko core and in terms of the definability of theory families by the Suszko core.

$$\mathfrak{G} \text{ is Truth Equational} \longleftrightarrow S^{\mathfrak{G}} \text{ is Soluble.}$$

Theorem 1943 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$*.* $\mathfrak{G}$ *is truth equational if and only if its Suszko core* $S^{\mathfrak{G}}$ *is soluble.*

Proof: The "only if" by Theorem 1941. The "if" by Theorem 1942. ∎

We say that $S^{\mathfrak{G}}$ **defines theory families** if, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$,

$$\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma \quad \text{iff} \quad S^{\mathfrak{G},\langle m,n\rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T}).$$

Then we can show

$$\mathfrak{G} \text{ is Truth Equational} \longleftrightarrow S^{\mathfrak{G}} \text{ Defines Theory Families.}$$

Theorem 1944 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$*.* $\mathfrak{G}$ *is truth equational if and only if* $S^{\mathfrak{G}}$ *defines theory families.*

Proof: If $\mathfrak{G}$ is truth equational, then, by Theorem 1943, $S^{\mathfrak{G}}$ is soluble, whence it defines theory families. On the other hand, if $S^{\mathfrak{G}}$ defines theory families, then it is soluble and, hence, by Theorem 1943, $\mathfrak{G}$ is truth equational. ∎

We now know that truth equationality of a Gentzen π-institution is equivalent to the solubility property of its Suszko core. The solubility property implies another property, which, in accordance with our previous work on π-institutions, we call adequacy. It says, roughly speaking, that in a Gentzen π-institution the category of natural transformations is rich enough to determine Suszko congruence systems in terms of the Leibniz congruence systems that it selects by inclusion. This property arises in a natural way by considering the following result relating the Suszko core with both Suszko and Leibniz congruence systems of theory families generated by single sequents.

Proposition 1945 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ *of trace* $\langle m, n \rangle$*,*

$$\bigcap \{\Omega(\boldsymbol{T}) : S^{\mathfrak{G},\langle m,n\rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T})\} \leq \widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(G(\boldsymbol{\phi})).$$

Proof: Note that, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m, n \rangle$, we have

$$\begin{array}{rcll}\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma & \Rightarrow & S^{\mathfrak{G},\langle m,n\rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}_\Sigma(\boldsymbol{T}) & \text{(Suszko core)}\\ & \Rightarrow & S^{\mathfrak{G},\langle m,n\rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T}). & (\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}) \leq \Omega(\boldsymbol{T}))\end{array}$$

Therefore, we get

$$\begin{array}{rcl}\bigcap \{\Omega(\boldsymbol{T}) : S^{\mathfrak{G},\langle m,n\rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T})\} & \leq & \bigcap \{\Omega(\boldsymbol{T}) : S^{\mathfrak{G},\langle m,n\rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}_\Sigma(\boldsymbol{T})\}\\ & \leq & \bigcap \{\Omega(\boldsymbol{T}) : \boldsymbol{\phi} \in \boldsymbol{T}_\Sigma\}\\ & = & \widetilde{\Omega}(G(\boldsymbol{\phi})).\end{array}$$

Thus, the displayed inclusion always holds. ■

The reverse inclusion is not always guaranteed, but, when it holds, we say that the Suszko core of $\mathfrak{G}$ is adequate. As the name suggests, the property somehow conveys the idea that $S^{\mathfrak{G}}[\boldsymbol{\phi}]$ suffices to determine the theory families whose Leibniz congruence systems form a covering of the Suszko congruence system corresponding to the theory family $G(\boldsymbol{\phi})$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$. The Suszko core $S^{\mathfrak{G}}$ is **adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m, n \rangle$,

$$\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}_\Sigma(G(\boldsymbol{\phi})) \leq \bigcap \{\Omega(\boldsymbol{T}) : S^{\mathfrak{G},\langle m,n\rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T})\}.$$

We can prove immediately that solubility implies adequacy.

Proposition 1946 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. If the Suszko core* $S^{\mathfrak{G}}$ *is soluble, then it is adequate.*

Proof: Suppose $S^{\mathfrak{G}}$ is soluble. We have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m, n \rangle$,

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(G(\boldsymbol{\phi})) & = & \bigcap \{\Omega(\boldsymbol{T}) : \boldsymbol{\phi} \in \boldsymbol{T}_\Sigma\}\\ & & \text{(Suszko congrunece system)}\\ & = & \bigcap \{\Omega(\boldsymbol{T}) : S^{\mathfrak{G},\langle m,n\rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T})\}.\\ & & (\text{solubility of } S^{\mathfrak{G}})\end{array}$$

Hence, the Suszko core of $\mathfrak{G}$ is adequate. ■

Conversely, if a Gentzen π-institution is c-reflective, then the adequacy of its Suszko core is sufficient to give its solubility.

Proposition 1947 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *is c-reflective and the Suszko core* $S^{\mathfrak{G}}$ *is adequate, then* $S^{\mathfrak{G}}$ *is soluble.*

Proof: Assume $\mathfrak{G}$ is c-reflective and $S^{\mathfrak{G}}$ is adequate. Let $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m, n \rangle$.

If $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$, then, by the definition of the Suszko core, we get

$$S^{\mathfrak{G},\langle m,n\rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}_\Sigma(G(\boldsymbol{\phi})) \subseteq \widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}_\Sigma(\boldsymbol{T}) \subseteq \Omega_\Sigma(\boldsymbol{T}).$$

Assume conversely, that $S^{\mathfrak{G},\langle m,n\rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T})$. Then, by adequacy of the Suszko core, $\widetilde{\Omega}^{\mathfrak{G},\mathfrak{F}}(G(\boldsymbol{\phi})) \leq \Omega(\boldsymbol{T})$. Hence, by c-reflectivity and Lemma 1938, $G(\boldsymbol{\phi}) \leq \boldsymbol{T}$, i.e., $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$. We conclude that $S^{\mathfrak{G}}$ is soluble. ∎

We finally show that a Gentzen π-institution is truth equational if and only if it is c-reflective and has an adequate Suszko core.

Truth Equationality	=	$S^{\mathfrak{G}}$ Soluble
	=	$S^{\mathfrak{G}}$ Defines Theory Families
	=	c-Reflectivity + $S^{\mathfrak{G}}$ Adequate

Theorem 1948 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*.* $\mathfrak{G}$ *is truth equational if and only if it is c-reflective and has an adequate Suszko core.*

Proof: If $\mathfrak{G}$ is truth equational, then, by Proposition 1937, it is c-reflective, by Theorem 1941, its Suszko core is soluble and, by Proposition 1946, its Suszko core is adequate. On the other hand, if $\mathfrak{G}$ is c-reflective with an adequate Suszko core, then, by Proposition 1947, its Suszko core is soluble and, hence, by Theorem 1942, $\mathfrak{G}$ is truth equational. ∎

We also obtain immediately

Corollary 1949 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a protoalgebraic Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*.* $\mathfrak{G}$ *is truth equational if and only if its Leibniz operator is injective on theory families and has an adequate Suszko core.*

Proof: If $\mathfrak{G}$ is truth equational, then, by Theorem 1948, it is c-reflective and has an adequate Suszko core, whence, it has, a fortiori, a Leibniz operator injective on theory families and an adequate Suszko core.

Conversely, by Theorem 1948, it suffices to show that monotonicity and injectivity of the Leibniz operator imply its c-reflectivity. In fact, given $\boldsymbol{T}, \boldsymbol{T}' \in \mathrm{ThFam}(\mathfrak{G})$, we have

$$\begin{array}{rcl}
\widetilde{\Omega}^{\mathfrak{G},\mathcal{G}}(\boldsymbol{T}) \leq \Omega(\boldsymbol{T}') & \Rightarrow & \Omega(\boldsymbol{T}) \leq \Omega(\boldsymbol{T}') \quad \text{(Protoalgebraicity)} \\
& \Rightarrow & \Omega(\boldsymbol{T} \cap \boldsymbol{T}') = \Omega(\boldsymbol{T}) \cap \Omega(\boldsymbol{T}') = \Omega(\boldsymbol{T}) \\
& & \text{(Protoalgebraicity)} \\
& \Rightarrow & \boldsymbol{T} \cap \boldsymbol{T}' = \boldsymbol{T} \quad \text{(Injectivity)} \\
& \Rightarrow & \boldsymbol{T} \leq \boldsymbol{T}'.
\end{array}$$

Thus, $\mathfrak{G}$ is c-reflective, by Lemma 1938. ∎

We close the section by a result asserting that truth equationality transfers from a Gentzen π-institution $\mathfrak{G}$ to all $\mathfrak{G}$-matrix families.

Theorem 1950 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$. $\mathfrak{G}$ *is truth equational, with witnessing transformations* $\tau = \{\tau^{\langle m,n\rangle} : \langle m,n\rangle \in \mathrm{tr}\}$ *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*, and all* $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$*,* $\Sigma \in |\mathbf{Sign}|$ *and* $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathcal{A})$ *of trace* $\langle m,n\rangle$*,*

$$\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma \quad \textit{iff} \quad \tau^{\mathcal{A},\langle m,n\rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \Omega^{\mathcal{A}}_\Sigma(\boldsymbol{T}).$$

Proof: Suppose $\mathfrak{G}$ is truth equational, with witnessing transformations $\tau = \{\tau^{\langle m,n\rangle} : \langle m,n\rangle \in \mathrm{tr}\}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system, $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m,n\rangle$. Then, we have

$$\begin{array}{rll}
\alpha_\Sigma(\boldsymbol{\phi}) \in \boldsymbol{T}_{F(\Sigma)} & \text{iff} & \boldsymbol{\phi} \in \alpha^{-1}_\Sigma(\boldsymbol{T}_{F(\Sigma)}) \\
& \text{iff} & \tau^{\langle m,n\rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\alpha^{-1}(\boldsymbol{T})) \\
& \text{iff} & \tau^{\langle m,n\rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \alpha^{-1}_\Sigma(\Omega^{\mathcal{A}}_{F(\Sigma)}(\boldsymbol{T})) \\
& \text{iff} & \alpha_\Sigma(\tau^{\langle m,n\rangle}_\Sigma[\boldsymbol{\phi}]) \subseteq \Omega^{\mathcal{A}}_{F(\Sigma)}(\boldsymbol{T}) \\
& \text{iff} & \tau^{\mathcal{A},\langle m,n\rangle}_{F(\Sigma)}[\alpha_\Sigma(\boldsymbol{\phi})] \subseteq \Omega^{\mathcal{A}}_{F(\Sigma)}(\boldsymbol{T}).
\end{array}$$

Taking into account the surjectivity of $\langle F, \alpha\rangle$, we have the conclusion. ∎

26.10 Weak Algebraizability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$. $\mathfrak{G}$ is called **WF algebraizable** if it is protoalgebraic and c-reflective.

Proposition 1951 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$. $\mathfrak{G}$ *is WF algebraizable if and only if the Leibniz operator is monotone and injective on* $\mathrm{ThFam}(\mathfrak{G})$.

Proof: It suffices to show that, under monotonicity, c-reflectivity and injectivity are equivalent properties. Indeed, c-reflectivity always implies injectivity because it implies order reflectivity. On the other hand, suppose that the Leibniz operator is monotone and injective. Then, we have, by monotonicity, for all $\mathcal{T} \cup \{\boldsymbol{T}\} \subseteq \mathrm{ThFam}(\mathfrak{G})$, such that $\bigcap_{\boldsymbol{T}\in\mathcal{T}} \Omega(\boldsymbol{T}) \leq \Omega(\boldsymbol{T}')$,

$$\Omega(\bigcap \mathcal{T} \cap \boldsymbol{T}') = \bigcap_{\boldsymbol{T}\in\mathcal{T}} \Omega(\boldsymbol{T}) \cap \Omega(\boldsymbol{T}') = \bigcap_{\boldsymbol{T}\in\mathcal{T}} \Omega(\boldsymbol{T}) = \Omega(\bigcap \mathcal{T}).$$

Thus, by injectivity, $\bigcap \mathcal{T} \cap \boldsymbol{T}' = \bigcap \mathcal{T}$ and, hence, $\bigcap \mathcal{T} \leq \boldsymbol{T}'$. Therefore $\mathfrak{G}$ is also c-reflective. ∎

The following theorem provides characterations of WF algebraizability.

Theorem 1952 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. Then the following statements are equivalent:*

(i) $\mathfrak{G}$ *is WF algebraizable;*

(ii) The Leibniz operator defines an order isomorphism from $\mathbf{ThFam}(\mathfrak{G})$ *onto the lattice of all* $\mathrm{AlgSys}(\mathfrak{G})$*-congruence families on* $\mathcal{F}$*;*

(iii) For every $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, the Leibniz operator defines an order isomorphism from* $\mathbf{FiFam}^{\mathfrak{G}}(\mathcal{A})$ *onto the lattice of all* $\mathrm{AlgSys}(\mathfrak{G})$*-congruence systems on* $\mathcal{A}$*;*

(iv) For every $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle \in \mathrm{AlgSys}(\mathfrak{G})$*, the Leibniz operator defines an order isomorphism from* $\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$ *onto the lattice of all* $\mathrm{AlgSys}(\mathfrak{G})$*-congruence systems on* $\mathcal{A}$*.*

Proof:

(i)⇒(ii) Suppose $\mathfrak{G}$ is WF algebraizable. Denote $\mathrm{ConSys}(\mathfrak{G})$ the collection of all $\mathrm{AlgSys}(\mathfrak{G})$-congruences on $\mathcal{F}$. Then, since, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$,

$$\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}/\Omega(\boldsymbol{T})}(\boldsymbol{T}/\Omega(\boldsymbol{T})) \le \Omega^{\mathcal{F}/\Omega(\boldsymbol{T})}(\boldsymbol{T}/\Omega(\boldsymbol{T})) = \Delta^{\mathcal{F}/\Omega(\boldsymbol{T})},$$

we get that $\langle \mathcal{F}/\Omega(\boldsymbol{T}), \boldsymbol{T}/\Omega(\boldsymbol{T}) \rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$. Thus, $\mathcal{F}/\Omega(\boldsymbol{T}) \in \mathrm{AlgSys}(\mathfrak{G})$ and, therefore, $\Omega(\boldsymbol{T}) \in \mathrm{ConSys}(\mathfrak{G})$. This shows that $\Omega : \mathrm{ThFam}(\mathfrak{G}) \to \mathrm{ConSys}(\mathfrak{G})$ is well defined. By Proposition 1951, it is injective. To see that it is surjective, consider $\theta \in \mathrm{ConSys}(\mathfrak{G})$. Then, by definition, $\mathcal{F}/\theta \in \mathrm{AlgSys}(\mathfrak{G})$, i.e., there exists $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{F}/\theta)$, such that $\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}/\theta}(\boldsymbol{T}) = \Delta^{\mathcal{F}/\theta}$. However, since the Leibniz operator is monotone, by hypothesis, we get that the Susko operator coincides with the Leibniz operator, whence $\Omega^{\mathcal{F}/\theta}(\boldsymbol{T}) = \Delta^{\mathcal{F}/\theta}$. Denoting by $\langle I, \pi \rangle : \mathcal{F} \to \mathcal{F}/\theta$ the quotient morphism, we now get

$$\Omega(\pi^{-1}(\boldsymbol{T})) = \pi^{-1}(\Omega^{\mathcal{F}/\theta}(\boldsymbol{T})) = \pi^{-1}(\Delta^{\mathcal{F}/\theta}) = \theta.$$

Thus, Ω is indeed surjective. It is monotone by hypothesis and it is order reflecting, since it is c-reflective. Thus, $\Omega : \mathbf{ThFam}(\mathfrak{G}) \to \mathbf{ConSys}(\mathfrak{G})$ is in fact an order isomorphism.

(ii)⇒(iii) It is not difficult to show that $\Omega^{\mathcal{A}}$ is also monotone and c-reflective. Therefore, one can work in the same way as in Part (ii) replacing the mapping Ω by $\Omega^{\mathcal{A}} : \mathbf{FiFam}^{\mathfrak{G}}(\mathcal{A}) \to \mathbf{ConSys}^{\mathfrak{G}}(\mathcal{A})$, where $\mathrm{ConSys}^{\mathfrak{G}}(\mathcal{A})$ denotes the collection of $\mathrm{AlgSys}(\mathfrak{G})$-congruence systems on $\mathcal{A}$.

(iii)⇒(iv) Condition (iv) is a special case of Condition (iii).

(iv)⇒(i) If Condition (iv) holds, the $\mathfrak{G}$ is protoalgebraic, by Lemma 1911. Hence the Leibniz and Suszko operators coincide on the $\mathfrak{G}$-filter families of all $\mathbf{F}$-algebraic systems. Thus, by Theorem 1932, $\mathfrak{G}$ is also truth equational. Therefore, it is WF algebraizable.

■

Finally, based on results of preceding sections, we can also give a relation between algebraizability and WF algebraizability.

We show, first, that, if $\mathfrak{G}$ is algebraizable, then it is both syntactically protoalgebraic and truth equational.

We start by giving a modus ponens property in the case of algebraizability.

Lemma 1953 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$. *If* $\mathfrak{G}$ *is algebraizable via the conjugate pair* $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}}$, *for some class* K *of* $\mathbf{F}$-*algebraic systems, then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\boldsymbol{\phi}, \boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ *of trace* $\langle m, n \rangle$,

$$\boldsymbol{\psi} \in G_\Sigma(\{\boldsymbol{\phi}\} \cup \bigcup_{i<m+n} \rho_\Sigma[\phi_i, \psi_i]).$$

Proof: We have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi}, \boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m, n \rangle$,

$$\tau_\Sigma[\boldsymbol{\psi}] \subseteq G^{\mathsf{K}}_\Sigma(\tau_\Sigma[\boldsymbol{\phi}] \cup \{\phi_i \rhd_\Sigma \psi_i : i < m + n\}).$$

Thus, we get

$$\rho_\Sigma[\tau_\Sigma[\boldsymbol{\psi}]] \subseteq G_\Sigma(\rho_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]] \cup \bigcup_{i<m+n} \rho_\Sigma[\phi_i, \psi_i]).$$

Therefore, $\boldsymbol{\psi} \in G_\Sigma(\{\boldsymbol{\phi}\} \cup \bigcup_{i<m+n} \rho_\Sigma[\phi_i, \psi_i])$. ■

Moreover, in case of algebraizability, the isomorphism ρ^* from the theory families of the Gentzen π-institution $\mathfrak{G}$ to the K-congruence systems on $\mathcal{F}$ coincides with the Leibniz operator Ω.

Proposition 1954 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$. *If* $\mathfrak{G}$ *is algebraizable via the conjugate pair* $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}}$, *for some class* K *of* $\mathbf{F}$-*algebraic systems, then, for all* $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$,

$$\rho^*(\boldsymbol{T}) = \Omega(\boldsymbol{T}).$$

Proof: Let $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$.

If $\langle \phi, \psi \rangle \in \Omega_\Sigma(\boldsymbol{T})$, then, since $\Omega(\boldsymbol{T})$ is a congruence system, we get, for all $\sigma \in \rho$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \sigma_\Sigma(\phi, \phi, \vec{\chi}), \sigma_\Sigma(\phi, \psi, \vec{\chi}) \rangle \in \Omega_\Sigma(\boldsymbol{T}).$$

But $\sigma_\Sigma(\phi,\phi,\vec{\chi}) \in \mathrm{Thm}_\Sigma(\mathfrak{G}) \subseteq \boldsymbol{T}_\Sigma$. Therefore, by the compatibility of $\Omega(\boldsymbol{T})$ with $\boldsymbol{T}$, we get that $\sigma_\Sigma(\phi,\psi,\vec{\chi}) \in \boldsymbol{T}_\Sigma$. Therefore, $\rho_\Sigma[\phi,\psi] \subseteq \boldsymbol{T}_\Sigma$, which gives that $\langle\phi,\psi\rangle \in \rho^*_\Sigma(\boldsymbol{T})$.

Conversely, to see that $\rho^*(\boldsymbol{T}) \leq \Omega(\boldsymbol{T})$ it suffices, by the maximality property of $\Omega(\boldsymbol{T})$, to show that $\rho^*(\boldsymbol{T})$ is compatible with $\boldsymbol{T}$. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi},\boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m,n\rangle$, such that $\langle\boldsymbol{\phi},\boldsymbol{\psi}\rangle \in \rho^*_\Sigma(\boldsymbol{T})$ and $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$. Then, we have $\rho_\Sigma[\phi_i,\psi_i] \subseteq \boldsymbol{T}_\Sigma$, for all $i < m+n$, and $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$, whence, by Lemma 1953, $\boldsymbol{\psi} \in \boldsymbol{T}_\Sigma$. We conclude that $\rho^*(\boldsymbol{T})$ is compatible with $\boldsymbol{T}$, giving $\rho^*(\boldsymbol{T}) \leq \Omega(\boldsymbol{T})$. ■

Now, we prove one of the main theorems of the section to the effect that algebraizability implies both syntactic protoalgebraicity and truth equationality.

Theorem 1955 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *is algebraizable via the conjugate pair* $(\tau,\rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}}$*, for some class* K *of* $\mathbf{F}$*-algebraic systems, then,* $\mathfrak{G}$ *is syntactically protoalgebraic and truth equational.*

Proof: Suppose $\mathfrak{G}$ is algebraizable via the conjugate pair $(\tau,\rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}}$, for some class K of $\mathbf{F}$-algebraic systems.

Let, first, $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{lll} \langle\phi,\psi\rangle \in \Omega_\Sigma(\boldsymbol{T}) & \text{iff} & \langle\phi,\psi\rangle \in \rho^*_\Sigma(\boldsymbol{T}) \quad \text{(Proposition 1954)} \\ & \text{iff} & \rho_\Sigma[\phi,\psi] \subseteq \boldsymbol{T}_\Sigma. \quad \text{(definition of } \rho^*) \end{array}$$

Therefore, $\mathfrak{G}$ is syntactically protoalgebraic, with witnessing transformations ρ.

Finally, let $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m,n\rangle$. Then, we have

$$\begin{array}{lll} \boldsymbol{\phi} \in \boldsymbol{T}_\Sigma & \text{iff} & \rho_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]] \subseteq \boldsymbol{T}_\Sigma \quad ((\tau,\rho) \text{ conjugate pair}) \\ & \text{iff} & \tau_\Sigma[\boldsymbol{\phi}] \subseteq \rho^*_\Sigma(\boldsymbol{T}) \quad \text{(definition of } \rho^*) \\ & \text{iff} & \tau_\Sigma[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T}). \quad \text{((Proposition 1954))} \end{array}$$

Therefore, $\mathfrak{G}$ is truth equational, with witnessing transformations τ. ■

We show, next, that, conversely, syntactic protoalgebraicity and truth equationality guarantee algebraizability.

Theorem 1956 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *is syntactically protoalgebraic and truth equational, then it is algebraizable.*

Proof: Suppose that $\mathfrak{G}$ is syntactically protoalgebraic, with witnessing transformations ρ, and truth equational, with witnessing transformations τ. Then, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\mathbf{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$,

$$\begin{array}{rll}\boldsymbol{\phi} \in G_\Sigma(\mathbf{\Phi}) & \text{iff} & \boldsymbol{\phi} \in \bigcap\{\boldsymbol{T}_\Sigma : \mathbf{\Phi} \subseteq \boldsymbol{T}_\Sigma\} \\ & \text{iff} & \tau_\Sigma[\boldsymbol{\phi}] \subseteq \bigcap\{\Omega_\Sigma(\boldsymbol{T}) : \tau_\Sigma[\mathbf{\Phi}] \subseteq \Omega_\Sigma(\boldsymbol{T})\} \\ & \text{iff} & \tau_\Sigma[\boldsymbol{\phi}] \subseteq G_\Sigma^{\mathsf{K}}(\tau_\Sigma[\mathbf{\Phi}]).\end{array}$$

Moreover, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rll}\langle \phi, \psi \rangle \in \Omega_\Sigma(\boldsymbol{T}) & \text{iff} & \rho_\Sigma[\phi, \psi] \subseteq \boldsymbol{T}_\Sigma \\ & \text{iff} & \tau_\Sigma[\rho_\Sigma[\phi, \psi]] \subseteq \Omega_\Sigma(\boldsymbol{T}).\end{array}$$

Hence, we have that $G_\Sigma^{\mathsf{K}}(\phi \rhd_\Sigma \psi) = G_\Sigma^{\mathsf{K}}(\tau_\Sigma[\rho_\Sigma[\phi, \psi]])$.

We conclude, by Lemma 1879, that $\mathfrak{G}$ is equivalent to $\mathfrak{G}^{\mathsf{K}}$ and, therefore, $\mathfrak{G}$ is algebraizable. ■

Now we can formulate the main characterization theorem:

Theorem 1957 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$. *The following statements are equivalent:*

(i) $\mathfrak{G}$ *is algebraizable;*

(ii) $\mathfrak{G}$ *is syntactically protoalgebraic and truth equational;*

(iii) $\mathfrak{G}$ *is WF algebraizable (i.e., protoalgebraic and c-reflective) and has both a Leibniz reflexive core and an adequate Suszko core.*

Proof: If $\mathfrak{G}$ is algebraizable, then, by Theorem 1955, it is syntactically protoalgebraic and truth equational. If $\mathfrak{G}$ is syntactically protoalgebraic and truth equational, then, by Theorems 1924 and 1948, it is protoalgebraic, c-reflective and has both a Leibniz reflexive core and an adequate Suszko core. Finally, if $\mathfrak{G}$ is WF algebraizable, with a Leibniz reflexive core and an adequate Suszko core, then, by Theorems 1924 and 1948, it is syntactically protoalgebraic and truth equational, whence, by Theorem 1956, it is algebraizable. ■