Chapter 27

Behavioral Algebraizability

27.1 Behavioral π-Institutions

Let **Sign** be an arbitrary category of **signatures**, S a nonempty set of **sorts** and, for each $s \in S$,

$$\mathrm{SEN}_s : \mathbf{Sign} \to \mathbf{Set}$$

a functor giving, for each signature Σ, a set of Σ-sentences **of sort** s. By a **multi-sorted sentence functor over set of sorts** S we understand the collection

$$\{\mathrm{SEN}_s : s \in S\},$$

where all sets $\mathrm{SEN}_s(\Sigma)$, $s \in S$, are assumed to be disjoint, i.e.,

$$\mathrm{SEN}_s(\Sigma) \cap \mathrm{SEN}_{s'}(\Sigma) = \varnothing, \text{ for all } s, s' \in S, s \neq s'.$$

Because of this condition, given $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\Phi \subseteq \bigcup_{s \in S} \mathrm{SEN}_s(\Sigma)$, we write

$$\mathrm{SEN}(f)(\Phi) = \bigcup_{s \in S} \{\mathrm{SEN}_s(f)(\phi) : \phi \in \Phi \text{ of sort } s\}.$$

A multi-sorted sentence functor over set of sorts S is called **behavioral** if a nonempty subset $V \subseteq S$ of **formula sorts** has been singled out and, moreover, there exists a companion subset $V^* = \{v^* : v \in V\}$ of **visible sorts**. In that case the (perhaps empty) set $H = S - (V \cup V^*)$ is called the set of **hidden sorts**. To denote a behavioral functor, making the set of visible and set of hidden sorts explicit, we write

$$\{\mathrm{SEN}_v, \mathrm{SEN}_{v^*}, \mathrm{SEN}_h : v \in V, h \in H\},$$

or sometimes, for the sake of succinctness,

$$\{\mathrm{SEN}_s\}_H^{V,V^*}.$$

Let **Sign** be a category and $\{\mathrm{SEN}_s\}_{s \in S}$ a multi-sorted sentence functor. The **clone of all natural transformations** on $\{\mathrm{SEN}_s\}_{s \in S}$ is the locally small category with:

- objects $\prod_{\kappa < \alpha} \mathrm{SEN}_{s_\kappa}$, with $s_\kappa \in S$, α an ordinal;
- morphisms $\tau : \prod_{\kappa < \alpha} \mathrm{SEN}_{s_\kappa} \to \prod_{\lambda < \beta} \mathrm{SEN}_{s'_\lambda}$ are β-sequences of natural transformations

$$\tau_\lambda : \prod_{\kappa < \alpha} \mathrm{SEN}_{s_\kappa} \to \mathrm{SEN}_{s'_\lambda}, \ \lambda < \beta.$$

Composition is defined as ordinary composition, i.e., by setting

$$\prod_{\kappa < \alpha} \mathrm{SEN}_{s_\kappa} \xrightarrow{\langle \tau_\lambda : \lambda < \beta \rangle} \prod_{\lambda < \beta} \mathrm{SEN}_{s'_\lambda} \xrightarrow{\langle \sigma_\mu : \mu < \gamma \rangle} \prod_{\mu < \gamma} \mathrm{SEN}_{s''_\mu}$$

$$\langle \sigma_\mu : \mu < \gamma \rangle \circ \langle \tau_\lambda : \lambda < \beta \rangle = \langle \sigma_\mu(\langle \tau_\lambda : \lambda < \beta \rangle) : \mu < \gamma \rangle.$$

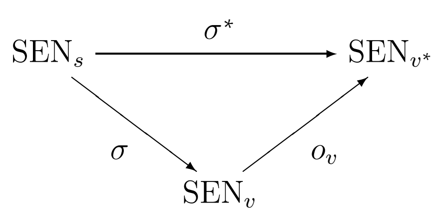
A subcategory N of the clone of all natural transformations on $\{\mathrm{SEN}_s\}_{s\in S}$, with objects all objects of the form $\prod_{i=1}^k \mathrm{SEN}_{s_i}$, $k < \omega$, is called a **category of natural transformations on** $\{\mathrm{SEN}_s\}_{s\in S}$ if the following conditions hold:

- It contains all natural projections
$$p^{s_1\cdots s_k \to s_i} : \prod_{i=1}^k \mathrm{SEN}_{s_i} \to \mathrm{SEN}_{s_i}, i < k, k < \omega.$$
- For every collection $\{\tau_i : \prod_{i=1}^k \mathrm{SEN}_{s_i} \to \mathrm{SEN}_{s_i'} : i < \ell\}$ of ℓ natural transformations in N, the tuple
$$\langle \tau_i : i < \ell \rangle : \prod_{i=1}^k \mathrm{SEN}_{s_i} \to \prod_{j=1}^{\ell} \mathrm{SEN}_{s_j'}$$
is also a natural transformation in N.

We refer to these conditions by saying that N "includes all projections" and is "closed under combinations" of natural transformations.

Let $\{\mathrm{SEN}_s\}_H^{V,V^*}$ be a behavioral sentence functor. A subcategory N of the clone of all natural transformations on $\{\mathrm{SEN}_s\}_{s\in S}$, with objects all objects of the form $\prod_{i=1}^k \mathrm{SEN}_{s_i}$, $k < \omega$, is called a **category of natural transformations on** $\{\mathrm{SEN}_s\}_H^{V,V^*}$ if, in addition to being a category of natural transformations on $\{\mathrm{SEN}_s\}_{s\in S}$, i.e., to including all projections and being closed under combinations, the following condition also holds:

- For all $v \in V$, there is no outgoing natural transformation from SEN_{v^*}, other than the identity, and there exists a unique surjective natural transformation
$$o_v : \mathrm{SEN}_v \to \mathrm{SEN}_{v^*},$$
called the v-**observation natural transformation**, or, simply, **observation**, when the formula sort v to which it corresponds is clear from context, such that, every incoming natural transformation $\sigma^* : \mathrm{SEN}_s \to \mathrm{SEN}_{v^*}$ factors through o_v:



We express this condition by saying that N "has observations".

Given a natural transformation $\sigma : \prod_{i=1}^{k} \mathrm{SEN}_{s_i} \to \mathrm{SEN}_s$ in N, we call $s_1 \cdots s_k \to s$ the **type of** σ and say that σ is **of sort** s (i.e., of the output sort).

A **multi-sorted algebraic system** $\mathbf{F} = \langle \mathbf{Sign}, \{\mathrm{SEN}_s\}_{s\in S}, N\rangle$ consists of a category of signatures, a multi-sorted sentence functor and a category N of natural transformations on $\{\mathrm{SEN}\}_{s\in S}$. It is called **behavioral** if $\{\mathrm{SEN}_s\}_H^{V,V^*}$ is a behavioral sentence functor and N is a category of natural transformations on $\{\mathrm{SEN}_s\}_H^{V,V^*}$ (i.e., has observations), and we then write

$$\mathbf{F} = \langle \mathbf{Sign}, \{\mathrm{SEN}_s\}_H^{V,V^*}, N\rangle.$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}_s^\flat\}_{s\in S}, N^\flat\rangle$ be a multi-sorted algebraic system. An $N^\flat$**-algebraic system** is a multi-sorted algebraic system

$$\mathbf{A} = \langle \mathbf{Sign}, \{\mathrm{SEN}_s\}_{s\in S}, N\rangle,$$

such that there exists a surjective functor $F : N^\flat \to N$ that preserves all natural projections (and, hence, the type of all natural transformations in $N^\flat$). We use $\sigma^{\mathbf{A}}$ to refer to the image of σ in $N^\flat$ under F.

Moreover, given two N-algebraic systems $\mathbf{A} = \langle \mathbf{Sign}, \{\mathrm{SEN}_s\}_{s\in S}, N\rangle$ and $\mathbf{B} = \langle \mathbf{Sign}', \{\mathrm{SEN}'_s\}_{s\in S}, N'\rangle$, a **morphism**

$$\langle F, \alpha\rangle : \mathbf{A} \to \mathbf{B}$$

consists of a functor $F : \mathbf{Sign} \to \mathbf{Sign}'$ and a collection $\alpha = \{\alpha^s\}_{s\in S}$ of natural transformations $\alpha^s : \mathrm{SEN}_s \to \mathrm{SEN}'_s \circ F$, $s \in S$, such that, for every $\sigma : \prod_{i=1}^{k} \mathrm{SEN}_{s_i}^\flat \to \mathrm{SEN}_s^\flat$ in N, all $\Sigma \in |\mathbf{Sign}|$, and all $\phi_i \in \mathrm{SEN}_{s_i}(\Sigma)$, $i \le k$,

$$\alpha_\Sigma^s(\sigma_\Sigma^{\mathbf{A}}(\phi_1, \dots \phi_k)) = \sigma_{F(\Sigma)}^{\mathbf{B}}(\alpha_\Sigma^{s_1}(\phi_1), \dots, \alpha_\Sigma^{s_k}(\phi_k)).$$

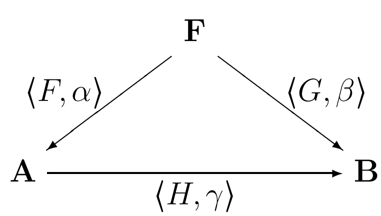
Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}_s^\flat\}_{s\in S}, N^\flat\rangle$ be a multi-sorted algebraic system. An **F-algebraic system** $\mathcal{A}$ is a pair $\langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, where

- $\mathbf{A} = \langle \mathbf{Sign}, \{\mathrm{SEN}_s\}_{s\in S}, N\rangle$ is an $N^\flat$-algebraic system;
- $\langle F, \alpha\rangle : \mathbf{F} \to \mathbf{A}$ is a surjective morphism, i.e., such that $F : \mathbf{Sign} \to \mathbf{Sign}'$ is surjective and full and $\alpha_\Sigma^s : \mathrm{SEN}_s^\flat(\Sigma) \to \mathrm{SEN}_s(F(\Sigma))$ is surjective, for all $\Sigma \in |\mathbf{Sign}|$ and all $s \in S$.

Given two **F**-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ and $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \{\mathrm{SEN}_s\}_{s\in S}, N\rangle$ and $\mathbf{B} = \langle \mathbf{Sign}', \{\mathrm{SEN}'_s\}_{s\in S}, N'\rangle$, a **morphism**

$$\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$$

is a morphism $\langle H, \gamma\rangle : \mathbf{A} \to \mathbf{B}$, such that



$$\langle H, \gamma \rangle \circ \langle F, \alpha \rangle = \langle G, \beta \rangle.$$

A **behavioral π-institution** is a pair $\mathcal{I} = \langle \mathbf{F}, C \rangle$, where

- $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ is a behavioral algebraic system;
- $C = \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ is a **closure system on** $\{\mathrm{SEN}^\flat_v\}_{v \in V}$, i.e., for all $\Sigma \in |\mathbf{Sign}^\flat|$,
$$C_\Sigma : \mathcal{P}(\bigcup_{v \in V} \mathrm{SEN}^\flat_v(\Sigma)) \to \mathcal{P}(\bigcup_{v \in V} \mathrm{SEN}^\flat_v(\Sigma))$$
is a closure operator and, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\Phi \subseteq \cup_{v \in V} \mathrm{SEN}^\flat_v(\Sigma)$,
$$\mathrm{SEN}^\flat(f)(C_\Sigma(\Phi)) \subseteq C_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)).$$

Given a behavioral algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$, a **behavioral sentence family** $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ **of F** consists of subsets

$$T_\Sigma \subseteq \bigcup_{v \in V} \mathrm{SEN}_v(\Sigma), \quad \Sigma \in |\mathbf{Sign}^\flat|.$$

Given a behavioral π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, based on $\mathbf{F}$, a **behavioral theory family** $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ **of** $\mathcal{I}$ is a behavioral sentence family of $\mathbf{F}$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$,
$$C_\Sigma(T_\Sigma) = T_\Sigma.$$
We write $\mathrm{ThFam}(\mathcal{I})$ for the collection of all behavioral theory families of $\mathcal{I}$.

27.2 Behavioral Algebra

Let $\mathbf{F} = \langle \mathbf{Sign}, \{\mathrm{SEN}_s\}_{s \in S}, N \rangle$ be a multi-sorted algebraic system. An **equivalence family** $\theta = \{\theta_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ **on F** is a family, such that, for all $\Sigma \in |\mathbf{Sign}|$, $\theta_\Sigma = \{\theta^s_\Sigma\}_{s \in S}$ consists of equivalence relations $\theta^s_\Sigma \subseteq \mathrm{SEN}_s(\Sigma)^2$. It is called an **equivalence system on F** if it is invariant under **Sign**-morphisms, i.e., such that, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $s \in S$,

$$\mathrm{SEN}_s(f)(\theta^s_\Sigma) \subseteq \theta^s_{\Sigma'}.$$

An equivalence family/system θ on $\mathbf{F}$ is called a **congruence family/system on F** if, for all $\sigma : \prod_{i=1}^k \mathrm{SEN}_{s_i} \to \mathrm{SEN}_s$ in N, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi}, \vec{\psi} \in \prod_{i=1}^k \mathrm{SEN}_{s_i}(\Sigma)$,

$$\vec{\phi} \ \prod_{i=1}^k \theta^{s_i}_\Sigma \ \vec{\psi} \quad \text{implies} \quad \sigma_\Sigma(\vec{\phi}) \ \theta^s_\Sigma \ \sigma_\Sigma(\vec{\psi}).$$

The collection of all congruence systems on $\mathbf{F}$ is denoted by $\text{ConSys}(\mathbf{F})$ and it forms a complete lattice under signature-wise and sort-wise inclusion $\leq$:

$$\mathbf{ConSys}(\mathbf{F}) = \langle \text{ConSys}(\mathbf{F}), \leq \rangle.$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\text{SEN}_s\}_H^{V,V^*}, N^\flat \rangle$ be a behavioral algebraic system and $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ a behavioral sentence family of $\mathbf{F}$. A congruence family $\theta = \{\theta_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ on $\mathbf{F}$ is **compatible with** T if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi, \psi \in \text{SEN}_v(\Sigma)$,

$$\langle \phi, \psi \rangle \in \theta_\Sigma^v \quad \text{and} \quad \phi \in T_\Sigma \quad \text{imply} \quad \psi \in T_\Sigma.$$

A fundamental result, akin to that allowing us to define Leibniz congruence systems in the context of ordinary π-institutions, is asserting that, given a behavioral sentence family, there exists a largest congruence system on $\mathbf{F}$ compatible with the theory family.

Theorem 1958 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\text{SEN}_s^\flat\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system,* $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ *a behavioral sentence family of* $\mathbf{F}$. *There exists a largest congruence system on* $\mathbf{F}$ *compatible with* T.

Proof: We define $\theta = \{\theta_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, where $\theta_\Sigma = \{\theta_\Sigma^s\}_{s \in S}$ by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $s \in S$ and all $\phi, \psi \in \text{SEN}_s^\flat(\Sigma)$, $\langle \phi, \psi \rangle \in \theta_\Sigma^s$ if and only if, for all $\sigma : \text{SEN}_s^\flat \times \prod_{i=1}^k \text{SEN}_{s_i}^\flat \to \text{SEN}_v^\flat$ in $N^\flat$, with $v \in V$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \prod_{i=1}^k \text{SEN}_{s_i}^\flat(\Sigma')$,

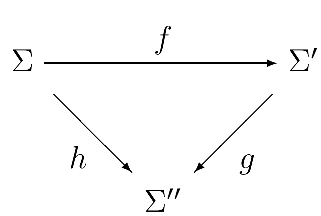
$$\sigma_{\Sigma'}(\text{SEN}_s^\flat(f)(\phi), \vec{\chi}) \in T_{\Sigma'} \quad \text{iff} \quad \sigma_{\Sigma'}(\text{SEN}_s^\flat(f)(\psi), \vec{\chi}) \in T_{\Sigma'}.$$

We show that θ, thus defined, is a congruence system on $\mathbf{F}$ that is compatible with T.

First, it is straightforward by the definition that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $s \in S$, θ_Σ^s is reflexive, symmetric and transitive. So θ is an equivalence family on $\mathbf{F}$. To see that it is a system, let $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, $s \in S$ and $\phi, \psi \in \text{SEN}_s^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in \theta_\Sigma^s$. Then, for all $\sigma : \text{SEN}_s^\flat \times \prod_{i=1}^k \text{SEN}_{s_i}^\flat \to \text{SEN}_v^\flat$ in $N^\flat$, with $v \in V$, all $\Sigma'' \in |\mathbf{Sign}^\flat|$, all $h \in \mathbf{Sign}^\flat(\Sigma, \Sigma'')$ and all $\vec{\chi} \in \prod_{i=1}^k \text{SEN}_{s_i}^\flat(\Sigma'')$,

$$\sigma_{\Sigma''}(\text{SEN}_s^\flat(h)(\phi), \vec{\chi}) \in T_{\Sigma''} \quad \text{iff} \quad \sigma_{\Sigma''}(\text{SEN}_s^\flat(f)(\psi), \vec{\chi}) \in T_{\Sigma''}.$$

Thus, as fortiori, for all $\sigma : \text{SEN}_s^\flat \times \prod_{i=1}^k \text{SEN}_{s_i}^\flat \to \text{SEN}_v^\flat$ in $N^\flat$, with $v \in V$, all $\Sigma'' \in |\mathbf{Sign}^\flat|$, all $g \in \mathbf{Sign}^\flat(\Sigma', \Sigma'')$ and all $\vec{\chi} \in \prod_{i=1}^k \text{SEN}_{s_i}^\flat(\Sigma'')$,



$$\sigma_{\Sigma''}(\mathrm{SEN}^\flat_s(g)(\mathrm{SEN}^\flat_s(f)(\phi)),\vec{\chi}) \in T_{\Sigma''}$$
$$\text{iff} \quad \sigma_{\Sigma''}(\mathrm{SEN}^\flat_s(g)(\mathrm{SEN}^\flat_s(f)(\psi)),\vec{\chi}) \in T_{\Sigma''}.$$

This shows that $\langle \mathrm{SEN}^\flat_s(f)(\phi), \mathrm{SEN}^\flat_s(f)(\psi)\rangle \in \theta^s_{\Sigma'}$ and, hence, θ is an equivalence system.

To see that θ is a congruence system, let $\tau : \prod_{j=1}^\ell \mathrm{SEN}^\flat_{s'_j} \to \mathrm{SEN}^\flat_s$ be in $N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi},\vec{\psi} \in \prod_{j=1}^\ell \mathrm{SEN}^\flat_{s'_j}(\Sigma)$, such that $\vec{\phi} \prod_{j=1}^\ell \theta^{s'_j}_\Sigma \vec{\psi}$. Then, we have, for all $\sigma : \mathrm{SEN}^\flat_s \times \prod_{i=1}^k \to \mathrm{SEN}^\flat_v$ in $N^\flat$, with $v \in V$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i}(\Sigma')$,

$$\begin{array}{ll}
& \sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\tau_\Sigma(\vec{\phi})),\vec{\chi}) \in T_{\Sigma'} \\
\text{iff} & \sigma_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}^\flat_{s'_1}(f)(\phi_1), \mathrm{SEN}^\flat_{s'_2}(f)(\phi_2), \mathrm{SEN}^\flat_{s'_3}(f)(\phi_3),\ldots, \\
& \qquad \mathrm{SEN}^\flat_{s'_{\ell-1}}(f)(\phi_{\ell-1}), \mathrm{SEN}^\flat_{s'_\ell}(f)(\phi_\ell)),\vec{\chi}) \in T_{\Sigma'} \\
\text{iff} & \sigma_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}^\flat_{s'_1}(f)(\psi_1), \mathrm{SEN}^\flat_{s'_2}(f)(\phi_2), \mathrm{SEN}^\flat_{s'_3}(f)(\phi_3),\ldots, \\
& \qquad \mathrm{SEN}^\flat_{s'_{\ell-1}}(f)(\phi_{\ell-1}), \mathrm{SEN}^\flat_{s'_\ell}(f)(\phi_\ell)),\vec{\chi}) \in T_{\Sigma'} \\
\text{iff} & \sigma_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}^\flat_{s'_1}(f)(\psi_1), \mathrm{SEN}^\flat_{s'_2}(f)(\psi_2), \mathrm{SEN}^\flat_{s'_3}(f)(\phi_3),\ldots, \\
& \qquad \mathrm{SEN}^\flat_{s'_{\ell-1}}(f)(\phi_{\ell-1}), \mathrm{SEN}^\flat_{s'_\ell}(f)(\phi_\ell)),\vec{\chi}) \in T_{\Sigma'} \\
\text{iff}\cdots & \\
\text{iff} & \sigma_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}^\flat_{s'_1}(f)(\psi_1), \mathrm{SEN}^\flat_{s'_2}(f)(\psi_2), \mathrm{SEN}^\flat_{s'_3}(f)(\psi_3),\ldots, \\
& \qquad \mathrm{SEN}^\flat_{s'_{\ell-1}}(f)(\psi_{\ell-1}), \mathrm{SEN}^\flat_{s'_\ell}(f)(\phi_\ell)),\vec{\chi}) \in T_{\Sigma'} \\
\text{iff} & \sigma_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}_{s'_1}(f)(\psi_1), \mathrm{SEN}^\flat_{s'_2}(f)(\psi_2), \mathrm{SEN}^\flat_{s'_3}(f)(\psi_3),\ldots, \\
& \qquad \mathrm{SEN}^\flat_{s'_{\ell-1}}(f)(\psi_{\ell-1}), \mathrm{SEN}^\flat_{s'_\ell}(f)(\psi_\ell)),\vec{\chi}) \in T_{\Sigma'} \\
\text{iff} & \sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\tau_\Sigma(\vec{\psi})),\vec{\chi}) \in T_{\Sigma'}.
\end{array}$$

Hence, $\langle \tau_\Sigma(\vec{\phi}), \tau_\Sigma(\vec{\psi})\rangle \in \theta^s_\Sigma$, showing that θ is a congruence system on $\mathbf{F}$.

θ is also compatible with T, since, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\psi,\psi \in \mathrm{SEN}^\flat_v(\Sigma)$, such that $\phi \in T_\Sigma$ and $\langle\phi,\psi\rangle \in \theta^v_\Sigma$, we get, as a special instance in the definition by taking $\sigma = \iota : \mathrm{SEN}^\flat_v \to \mathrm{SEN}^\flat_v$ in $N^\flat$, $\Sigma' = \Sigma$ and $f = i_\Sigma$, $\phi \in T_\Sigma$ iff $\psi \in T_\Sigma$. Therefore, $\psi \in T_\Sigma$ and θ is, in fact, a congruence system on $\mathbf{F}$ compatible with T.

Finally, we show that, if θ' is a congruence system on $\mathbf{F}$ compatible with T, then $\theta' \leq \theta$. Suppose, to this end, that θ' is a congruence system on $\mathbf{F}$ compatible with T and let $\Sigma \in |\mathbf{Sign}^\flat|$, $s \in S$ and $\phi,\psi \in \mathrm{SEN}^\flat_s(\Sigma)$, such that $\langle\phi,\psi\rangle \in \theta'^s_\Sigma$. Then, since θ' is a congruence system, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$, $\langle \mathrm{SEN}^\flat_s(f)(\phi), \mathrm{SEN}^\flat_s(f)(\psi)\rangle \in \theta'^s_{\Sigma'}$. Thus, since θ' is a congruence system, for all $\sigma : \mathrm{SEN}^\flat_s \times \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i} \to \mathrm{SEN}^\flat_v$ in $N^\flat$, with $v \in V$, and all $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i}(\Sigma')$,

$$\langle \sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\phi),\vec{\chi}), \sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\psi),\vec{\chi})\rangle \in \theta'^v_{\Sigma'}.$$

Therefore, by the compatibility of θ' with T, we get

$$\sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\phi),\vec{\chi}) \in T_{\Sigma'} \quad \text{iff} \quad \sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\psi),\vec{\chi}) \in T_{\Sigma'}.$$

This shows that $\langle\phi,\psi\rangle \in \theta^s_\Sigma$ and, hence, $\theta' \leq \theta$. Thus, θ is indeed the largest congruence system on $\mathbf{F}$ compatible with T. ∎

The largest congruence system on $\mathbf{F}$ compatible with T is called the **behavioral Leibniz congruence system of** T **on** $\mathbf{F}$ and is denoted by $\Upsilon(T)$. Moreover, given a behavioral π-institution $\mathcal{I} = \langle\mathbf{F}, C\rangle$, and a behavioral theory family $T \in \mathrm{ThFam}(\mathcal{I})$, we define the **behavioral Suszko congruence system of** T **on** $\mathbf{F}$ by

$$\widetilde{\Upsilon}^{\mathcal{I}}(T) = \bigcap\{\Upsilon(T') : T \leq T' \in \mathrm{ThFam}(\mathcal{I})\}.$$

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}^{V,V^*}_H, N^\flat\rangle$ be a behavioral algebraic system and $\mathcal{A} = \langle\mathbf{A}, \langle F, \alpha\rangle\rangle$, with $\mathbf{A} = \langle\mathbf{Sign}, \{\mathrm{SEN}_s\}_{s\in S}, N\rangle$, an $\mathbf{F}$-algebraic system. We define $\equiv^{\mathcal{A}} = \{\equiv^{\mathcal{A}}_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$, where, for all $\Sigma \in |\mathbf{Sign}|$, $\equiv^{\mathcal{A}}_\Sigma = \{\equiv^{\mathcal{A},s}_\Sigma\}_{s\in S}$ is given, for all $s \in S$, all $\phi, \psi \in \mathrm{SEN}_s(\Sigma)$, by $\phi \equiv^{\mathcal{A},s}_\Sigma \psi$ if and only if, for all $\sigma : \mathrm{SEN}_s \times \prod_{i=1}^k \mathrm{SEN}_{s_i} \to \mathrm{SEN}_{v^*}$, with $v \in V$, all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}_{s_i}(\Sigma')$,

$$\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}_s(f)(\phi), \vec{\chi}) = \sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}_s(f)(\psi), \vec{\chi}).$$

We show that $\equiv^{\mathcal{A}}$ is a congruence system on $\mathcal{A}$.

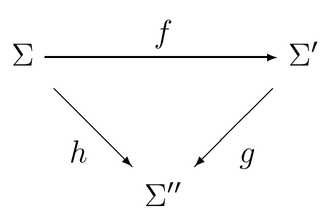
Proposition 1959 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}^{V,V^*}_H, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{A} = \langle\mathbf{A}, \langle F, \alpha\rangle\rangle$*, with* $\mathbf{A} = \langle\mathbf{Sign}, \{\mathrm{SEN}_s\}^{V,V^*}_H, N\rangle$*, an* $\mathbf{F}$*-algebraic system. The relation family* $\equiv^{\mathcal{A}}$ *is a congruence system on* $\mathcal{A}$*.*

Proof: By the definition, it is obvious that, for all $\Sigma \in |\mathbf{Sign}|$ and all $s \in S$, $\equiv^{\mathcal{A},s}_\Sigma$ is an equivalence family on $\mathrm{SEN}_s(\Sigma)$. We show that $\equiv^{\mathcal{A}}$ is a system and that it satisfies the congruence property.

Let $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $s \in S$ and $\phi, \psi \in \mathrm{SEN}_s(\Sigma)$, such that $\phi \equiv^{\mathcal{A},s}_\Sigma \psi$. Then, for all $\sigma : \mathrm{SEN}_s \times \prod_{i=1}^k \mathrm{SEN}_{s_i} \to \mathrm{SEN}_{v^*}$, with $v \in V$, all $\Sigma'' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma'')$ and $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}_{s_i}(\Sigma'')$,

$$\sigma^{\mathcal{A}}_{\Sigma''}(\mathrm{SEN}_s(h)(\phi), \vec{\chi}) = \sigma^{\mathcal{A}}_{\Sigma''}(\mathrm{SEN}_s(h)(\psi), \vec{\chi}).$$

In particular, for all $g \in \mathbf{Sign}(\Sigma', \Sigma'')$ and all $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}_{s_i}(\Sigma'')$,



$$\sigma^{\mathcal{A}}_{\Sigma''}(\mathrm{SEN}_s(g)(\mathrm{SEN}_s(f)(\phi)), \vec{\chi}) = \sigma^{\mathcal{A}}_{\Sigma''}(\mathrm{SEN}_s(g)(\mathrm{SEN}_s(f)(\psi)), \vec{\chi}).$$

Thus, by definition, $\mathrm{SEN}_s(f)(\phi) \equiv^{\mathcal{A},s}_{\Sigma'} \mathrm{SEN}_s(f)(\psi)$ and, therefore, $\equiv^{\mathcal{A}}$ is an equivalence system.

Finally, let $\sigma : \prod_{i=1}^k \mathrm{SEN}_{s_i} \to \mathrm{SEN}_s$ be in N, $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi}, \vec{\psi} \in \prod_{i=1}^k \mathrm{SEN}_{s_i}(\Sigma)$, such that $\vec{\phi} \prod_{i=1}^k \equiv_\Sigma^{\mathcal{A},s_i} \vec{\psi}$. Then, we have, for all $\tau : \mathrm{SEN}_s \times \prod_{j=1}^\ell \mathrm{SEN}_{s'_j} \to \mathrm{SEN}_{v^*}$, with $v \in V$, all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\vec{\chi} \in \prod_{j=1}^\ell \mathrm{SEN}_{s'_j}(\Sigma')$,

$$\begin{array}{l}
\tau_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}_s(f)(\sigma_\Sigma^{\mathcal{A}}(\vec{\phi})), \vec{\chi}) \\
= \tau_{\Sigma'}^{\mathcal{A}}(\sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}_{s_1}(f)(\phi_1), \mathrm{SEN}_{s_2}(f)(\phi_2), \mathrm{SEN}_{s_3}(f)(\phi_3), \ldots, \\
\qquad\qquad \mathrm{SEN}_{s_{k-1}}(f)(\phi_{k-1}), \mathrm{SEN}_{s_k}(f)(\phi_k)), \vec{\chi}) \\
= \tau_{\Sigma'}^{\mathcal{A}}(\sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}_{s_1}(f)(\psi_1), \mathrm{SEN}_{s_2}(f)(\phi_2), \mathrm{SEN}_{s_3}(f)(\phi_3), \ldots, \\
\qquad\qquad \mathrm{SEN}_{s_{k-1}}(f)(\phi_{k-1}), \mathrm{SEN}_{s_k}(f)(\phi_k)), \vec{\chi}) \\
= \tau_{\Sigma'}^{\mathcal{A}}(\sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}_{s_1}(f)(\psi_1), \mathrm{SEN}_{s_2}(f)(\psi_2), \mathrm{SEN}_{s_3}(f)(\phi_3), \ldots, \\
\qquad\qquad \mathrm{SEN}_{s_{k-1}}(f)(\phi_{k-1}), \mathrm{SEN}_{s_k}(f)(\phi_k)), \vec{\chi}) \\
= \cdots \\
= \tau_{\Sigma'}^{\mathcal{A}}(\sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}_{s_1}(f)(\psi_1), \mathrm{SEN}_{s_2}(f)(\psi_2), \mathrm{SEN}_{s_3}(f)(\psi_3), \ldots, \\
\qquad\qquad \mathrm{SEN}_{s_{k-1}}(f)(\psi_{k-1}), \mathrm{SEN}_{s_k}(f)(\phi_k)), \vec{\chi}) \\
= \tau_{\Sigma'}^{\mathcal{A}}(\sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}_{s_1}(f)(\psi_1), \mathrm{SEN}_{s_2}(f)(\psi_2), \mathrm{SEN}_{s_3}(f)(\psi_3), \ldots, \\
\qquad\qquad \mathrm{SEN}_{s_{k-1}}(f)(\psi_{k-1}), \mathrm{SEN}_{s_k}(f)(\psi_k)), \vec{\chi}) \\
= \tau_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}_s(f)(\sigma_\Sigma^{\mathcal{A}}(\vec{\psi})), \vec{\chi})
\end{array}$$

Hence, $\sigma_\Sigma^{\mathcal{A}}(\vec{\phi}) \equiv_\Sigma^{\mathcal{A},s} \sigma_\Sigma^{\mathcal{A}}(\vec{\phi})$ and $\equiv^{\mathcal{A}}$ is a congruence system on $\{\mathrm{SEN}_s\}_H^{V,V^*}$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}_s^\flat\}_H^{V,V^*}, N^\flat \rangle$ be a behavioral algebraic system and K a class of $\mathbf{F}$-algebraic systems. We define the closure system $C^{\mathsf{K}} = \{C_\Sigma^{\mathsf{K}}\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$C_\Sigma^{\mathsf{K}} : \mathcal{P}(\bigcup_{v \in V} \mathrm{SEN}_v^\flat(\Sigma)^2) \to \mathcal{P}(\bigcup_{v \in V} \mathrm{SEN}_v^\flat(\Sigma)^2)$$

be given, for all $E \cup \{\phi \approx \psi\} \subseteq \bigcup_{v \in V} \mathrm{SEN}_v(\Sigma)^2$,

$$\begin{array}{lll}
\phi \approx \psi \in C_\Sigma^{\mathsf{K}}(E) & \text{iff} & \text{for all } \mathcal{A} \in \mathsf{K}, \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma, \Sigma'), \\
& & \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(E)) \subseteq \equiv_{F(\Sigma')}^{\mathcal{A}} \text{ implies} \\
& & \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi)) \equiv_{F(\Sigma')}^{\mathcal{A}} \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi)).
\end{array}$$

Proposition 1960 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}_s^\flat\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems. Then* C^{K} *is a closure system on* $\bigcup_{v \in V} (\mathrm{SEN}_v^\flat)^2$.

Proof: It is straightforward to check that C_Σ^{K} is inflationary, monotone and idempotent, for all $\Sigma \in |\mathbf{Sign}^\flat|$. The fact that it is invariant under $\mathbf{Sign}^\flat$-morphisms can be shown in a way similar to that in the proof of Proposition 1959. ■

We call $\mathcal{I}^{\mathsf{K}} = \langle \mathbf{F}, C^{\mathsf{K}} \rangle$ the **behavioral equational π-institution associated with** the class K of $\mathbf{F}$-algebraic systems.

27.3 Behavioral Algebraizability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ be a behavioral algebraic system, K a class of $\mathbf{F}$-algebraic systems and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a behavioral π-institution based on $\mathbf{F}$.

- A **transformation** τ from $\mathcal{I}$ to $\mathcal{I}^\mathsf{K}$ is a collection $\tau = \{\tau^v : v \in V\}$, where, for every $v \in V$, $\tau^v = \{\tau^{v,u} : u \in V\}$ is such that
$$\tau^{v,u} : \mathrm{SEN}^\flat_v \times \prod_{i<\omega} \mathrm{SEN}^\flat_{s_i} \to (\mathrm{SEN}^\flat_u)^2$$
is a collection of natural transformations in $N^\flat$;
- A **transformation** ρ from $\mathcal{I}^\mathsf{K}$ to $\mathcal{I}$ is a collection $\rho = \{\rho^v : v \in V\}$, where, for every $v \in V$, $\rho^v = \{\rho^{v,u} : u \in V\}$ is such that
$$\rho^{v,u} : (\mathrm{SEN}^\flat_v)^2 \times \prod_{i<\omega} \mathrm{SEN}^\flat_{s_i} \to \mathrm{SEN}^\flat_u$$
is a collection of natural transformations in $N^\flat$.

A transformation τ from $\mathcal{I}$ to $\mathcal{I}^\mathsf{K}$ is called an **interpretation**, written $\tau : \mathcal{I} \to \mathcal{I}^\mathsf{K}$, if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\Phi \cup \{\phi\} \subseteq \bigcup_{v\in V} \mathrm{SEN}_v(\Sigma)$,
$$\phi \in C_\Sigma(\Phi) \quad \text{iff} \quad \tau_\Sigma[\phi] \subseteq C^\mathsf{K}_\Sigma(\tau_\Sigma[\Phi]).$$
Similarly, a transformation ρ from $\mathcal{I}^\mathsf{K}$ to $\mathcal{I}$ is called an **interpretation**, written $\rho : \mathcal{I}^\mathsf{K} \to \mathcal{I}$, if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $E \cup \{\phi \approx \psi\} \subseteq \bigcup_{v\in V} \mathrm{SEN}^\flat_v(\Sigma)^2$,
$$\phi \approx \psi \in C^\mathsf{K}_\Sigma(E) \quad \text{iff} \quad \rho_\Sigma[\phi, \psi] \subseteq C_\Sigma(\rho_\Sigma[E]).$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a behavioral π-institution based on $\mathbf{F}$. $\mathcal{I}$ is said to be **behaviorally** (**syntactically WF**) **algebraizable** if there exists a class K of $\mathbf{F}$-algebraic systems and interpretations $\tau : \mathcal{I} \to \mathcal{I}^\mathsf{K}$, $\rho : \mathcal{I}^\mathsf{K} \to \mathcal{I}$, that form a **conjugate pair**, i.e., such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$,

- $C_\Sigma(\phi) = C_\Sigma(\rho_\Sigma[\tau_\Sigma[\phi]])$;
- $C^\mathsf{K}_\Sigma(\phi \approx \psi) = C^\mathsf{K}_\Sigma(\tau_\Sigma[\rho_\Sigma[\phi, \psi]])$.

In this case we also say that $\mathcal{I}$ and $\mathcal{I}^\mathsf{K}$ are **equivalent via** (τ, ρ) and we write $(\tau, \rho) : \mathcal{I} \rightleftarrows \mathcal{I}^\mathsf{K}$.

Explicitly, $\mathcal{I}$ is behaviorally algebraizable if and only if, there exists a class K of $\mathbf{F}$-algebraic systems and translations τ from $\mathcal{I}$ to $\mathcal{I}^\mathsf{K}$ and ρ from $\mathcal{I}^\mathsf{K}$ to $\mathcal{I}$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\Phi \cup \{\phi\} \subseteq \bigcup_{v\in V} \mathrm{SEN}^\flat_v(\Sigma)$ and all $E \cup \{\phi \approx \psi\} \subseteq \bigcup_{v\in V} \mathrm{SEN}^\flat_v(\Sigma)^2$,

(1) $\phi \in C_\Sigma(\Phi)$ if and only if $\tau_\Sigma[\phi] \subseteq C^{\mathsf{K}}_\Sigma(\tau_\Sigma[\Phi])$;

(2) $\phi \approx \psi \in C^{\mathsf{K}}_\Sigma(E)$ if and only if $\rho_\Sigma[\phi, \psi] \subseteq C_\Sigma(\rho_\Sigma[E])$;

(3) $C_\Sigma(\phi) = C_\Sigma(\rho_\Sigma[\tau_\Sigma[\phi]])$;

(4) $C^{\mathsf{K}}_\Sigma(\phi \approx \psi) = C^{\mathsf{K}}_\Sigma(\tau_\Sigma[\rho_\Sigma[\phi, \psi]])$.

As in normal syntactic WF algebraizability, it turns out that, in this case as well, Conditions (1) and (4), or dually, Conditions (2) and (3) suffice to establish behavioral algebraizability.

Proposition 1961 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}^{V,V^*}_H, N^\flat \rangle$ *be a behavioral algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*,* K *a class of* $\mathbf{F}$*-algebraic systems and* τ*,* ρ *translations from* $\mathcal{I}$ *to* $\mathcal{I}^{\mathsf{K}}$ *and from* $\mathcal{I}^{\mathsf{K}}$ *to* $\mathcal{I}$*, respectively. The following statements are equivalent:*

(i) $\tau : \mathcal{I} \to \mathcal{I}^{\mathsf{K}}$ *is an interpretation and, for all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $v \in V$ *and all* $\phi \approx \psi \in \mathrm{SEN}^\flat_v(\Sigma)$*,* $C^{\mathsf{K}}_\Sigma(\phi \approx \psi) = C^{\mathsf{K}}_\Sigma(\tau_\Sigma[\rho_\Sigma[\phi, \psi]])$*;*

(ii) $\rho : \mathcal{I}^{\mathsf{K}} \to \mathcal{I}$ *is an interpretation and, for all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $v \in V$ *and all* $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$*,* $C_\Sigma(\phi) = C_\Sigma(\rho_\Sigma[\tau_\Sigma[\phi]])$*.*

Proof: We only prove that (i) implies (ii), since the converse then follows by the symmetry of the notion of equivalence. Suppose that (i) holds and let $\Sigma \in |\mathbf{Sign}^\flat|$ and $E \cup \{\phi \approx \psi\} \subseteq \bigcup_{v\in V} \mathrm{SEN}^\flat_v(\Sigma)^2$. Then we have

$$\begin{array}{rll} \phi \approx \psi \in C^{\mathsf{K}}_\Sigma(E) & \text{iff} & \tau_\Sigma[\rho_\Sigma[\phi, \psi]] \subseteq C^{\mathsf{K}}_\Sigma(\tau_\Sigma[\rho_\Sigma[E]]) \\ & \text{iff} & \rho_\Sigma[\phi, \psi] \subseteq C_\Sigma(\rho_\Sigma[E]). \end{array}$$

Moreover, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$, we have, for all $\psi \in \bigcup_{v\in V} \mathrm{SEN}^\flat_v(\Sigma)$,

$$\begin{array}{rll} \psi \in C_\Sigma(\rho_\Sigma[\tau_\Sigma[\phi]]) & \text{iff} & \tau_\Sigma[\psi] \subseteq C^{\mathsf{K}}_\Sigma(\tau_\Sigma[\rho_\Sigma[\tau_\Sigma[\phi]]]) \\ & \text{iff} & \tau_\Sigma[\psi] \subseteq C^{\mathsf{K}}_\Sigma(\tau_\Sigma[\phi]) \\ & \text{iff} & \psi \in C_\Sigma(\phi). \end{array}$$

Hence, $C_\Sigma(\phi) = C_\Sigma(\rho_\Sigma[\tau_\Sigma[\phi]])$. This shows that Condition (ii) holds. ■

We look next at some properties that are entailed by behavioral algebraizability.

Proposition 1962 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}^{V,V^*}_H, N^\flat \rangle$ *be a behavioral algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$ *and* K *a class of* $\mathbf{F}$*-algebraic systems. If* $(\tau, \rho) : \mathcal{I} \rightleftarrows \mathcal{I}^{\mathsf{K}}$ *is a conjugate pair, then, for all* $v, u \in V$*, all* $\sigma : \mathrm{SEN}_v \times \prod_{i=1}^k \mathrm{SEN}_{s_i} \to \mathrm{SEN}_u$ *in* $N^\flat$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $\phi, \psi, \chi \in \mathrm{SEN}^\flat_v(\Sigma)$ *and all* $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i}(\Sigma)$*,*

(a) $\rho_\Sigma[\phi,\phi] \subseteq \mathrm{Thm}_\Sigma(\mathcal{I})$;

(b) $\rho_\Sigma[\psi,\phi] \subseteq C_\Sigma(\rho_\Sigma[\phi,\psi])$;

(c) $\rho_\Sigma[\phi,\chi] \subseteq C_\Sigma[\rho_\Sigma[\phi,\psi],\rho_\Sigma[\psi,\chi])$;

(d) $\rho_\Sigma[\sigma_\Sigma(\phi,\vec{\chi}),\sigma_\Sigma(\psi,\vec{\chi})] \subseteq C_\Sigma(\rho_\Sigma[\phi,\psi])$;

(e) $\psi \in C_\Sigma(\phi,\rho_\Sigma[\phi,\psi])$.

Proof:

(a) We have $\phi \approx \phi \in C^{\mathsf{K}}_\Sigma(\varnothing)$, whence, since $\rho : \mathcal{I}^{\mathsf{K}} \to \mathcal{I}$ is an interpretation, $\rho_\Sigma[\phi,\phi] \subseteq C_\Sigma(\varnothing)$.

(b) Since $\psi \approx \phi \in C^{\mathsf{K}}_\Sigma(\phi \approx \psi)$, we get, again by the fact $\rho : \mathcal{I}^{\mathsf{K}} \to \mathcal{I}$ is an interpretation, $\rho_\Sigma[\psi,\phi] \subseteq C_\Sigma(\rho_\Sigma[\phi,\psi])$.

(c) Since $\phi \approx \chi \in C^{\mathsf{K}}_\Sigma(\phi \approx \psi, \psi \approx \chi)$ and $\rho : \mathcal{I}^{\mathsf{K}} \to \mathcal{I}$ is an interpretation, we get that $\rho_\Sigma[\phi,\chi] \subseteq C_\Sigma(\rho_\Sigma[\phi,\psi],\rho_\Sigma[\psi,\chi])$.

(d) By Proposition 1959, we have, for all $\sigma : \mathrm{SEN}_v \times \prod_{i=1}^k \mathrm{SEN}_{s_i} \to \mathrm{SEN}_u$ and all $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i}(\Sigma)$, $\sigma_\Sigma(\phi,\vec{\chi}) \approx \sigma_\Sigma(\psi,\vec{\chi}) \in C^{\mathsf{K}}_\Sigma(\phi \approx \psi)$. Hence, again by the fact that $\rho : \mathcal{I}^{\mathsf{K}} \to \mathcal{I}$ is an interpretation, we get that $\rho_\Sigma[\sigma_\Sigma(\phi,\vec{\chi}),\sigma_\Sigma(\psi,\vec{\chi})] \subseteq C_\Sigma(\rho_\Sigma[\phi,\psi])$.

(e) In $\mathcal{I}^{\mathsf{K}}$, we have $\tau_\Sigma[\psi] \subseteq C^{\mathsf{K}}_\Sigma(\tau_\Sigma[\phi],\phi \approx \psi)$. Hence, by Property (4) of equivalence, $\tau_\Sigma[\psi] \subseteq C^{\mathsf{K}}_\Sigma(\tau_\Sigma[\phi],\tau_\Sigma[\rho_\Sigma[\phi,\psi]])$. Thus, by Property (1) of equivalence, we get that $\psi \in C_\Sigma(\phi,\rho_\Sigma[\phi,\psi])$.

■

We can also prove that, if a behavioral π-institution $\mathcal{I}$ is behaviorally algebraizable in two different ways, then the interpretations are, roughly speaking, interderivable and the classes of **F**-algebraic systems serving as behavioral algebraic semantics generate the same behavioral consequence operators.

Theorem 1963 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally algebraizable via conjugate pairs* $(\tau,\rho) : \mathcal{I} \rightleftarrows \mathcal{I}^{\mathsf{K}}$ *and* $(\tau',\rho') : \mathcal{I} \rightleftarrows \mathcal{I}^{\mathsf{K}'}$*, then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $v \in V$ *and all* $\phi,\psi \in \mathrm{SEN}^\flat_v(\Sigma)$*,*

(a) $C_\Sigma(\rho_\Sigma[\phi,\psi]) = C_\Sigma(\rho'_\Sigma[\phi,\psi])$;

(b) $C^{\mathsf{K}} = C^{\mathsf{K}'}$;

(c) $C^{\mathsf{K}}_\Sigma(\tau_\Sigma[\phi]) = C^{\mathsf{K}}_\Sigma(\tau'_\Sigma[\phi])$.

Proof:

(a) For all $\sigma' \in \rho'$ and all $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i}(\Sigma)$, we get

$$\rho_\Sigma[\sigma'_\Sigma(\phi,\phi,\vec{\chi}),\sigma'_\Sigma(\phi,\psi,\vec{\chi})] \subseteq C_\Sigma(\rho_\Sigma[\phi,\psi]).$$

But we also have $\rho'_\Sigma[\phi,\phi] \subseteq C_\Sigma(\varnothing) \subseteq C_\Sigma(\rho_\Sigma[\phi,\psi])$. Thus, by Proposition 1962, Part (e), $\rho'_\Sigma[\phi,\psi] \subseteq C_\Sigma(\rho_\Sigma[\phi,\psi])$. By symmetry, we now get $C_\Sigma(\rho_\Sigma[\phi,\psi]) = C_\Sigma(\rho'_\Sigma[\phi,\psi])$.

(b) Using Part (a), we get, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $E \cup \{\phi \approx \psi\} \subseteq \bigcup_{v \in V} \mathrm{SEN}^\flat_v(\Sigma)^2$,

$$\begin{array}{rll}\phi \approx \psi \in C^{\mathsf{K}}_\Sigma(E) & \text{iff} & \rho_\Sigma[\phi,\psi] \subseteq C_\Sigma(\rho_\Sigma[E]) \\ & \text{iff} & \rho'_\Sigma[\phi,\psi] \subseteq C_\Sigma(\rho'_\Sigma[E]) \\ & \text{iff} & \phi \approx \psi \in C^{\mathsf{K}'}_\Sigma(E).\end{array}$$

(c) Using Parts (a) and (b), we get

$$\begin{array}{rll}C_\Sigma(\phi) = C_\Sigma(\phi) & \text{iff} & C_\Sigma(\rho_\Sigma[\tau_\Sigma[\phi]]) = C_\Sigma(\rho'_\Sigma[\tau'_\Sigma[\phi]]) \\ & \text{iff} & C_\Sigma(\rho_\Sigma[\tau_\Sigma[\phi]]) = C_\Sigma(\rho_\Sigma[\tau'_\Sigma[\phi]]) \\ & \text{iff} & C^{\mathsf{K}}_\Sigma(\tau_\Sigma[\phi]) = C^{\mathsf{K}}_\Sigma(\tau'_\Sigma[\phi]).\end{array}$$

■

27.4 Behavioral Protoalgebraicity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a behavioral π-institution based on $\mathbf{F}$.

- $\mathcal{I}$ is **behaviorally protoalgebraic** if the behavioral Leibniz operator $\Upsilon : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}(\mathbf{F})$ is monotone on the behavioral theory families of $\mathcal{I}$, i.e., for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$T \leq T' \quad \text{implies} \quad \Upsilon(T) \leq \Upsilon(T').$$

- $\mathcal{I}$ is **behaviorally syntactically protoalgebraic** if there exists a collection $\rho = \{\rho^v : v \in V\}$, where, for all $v \in V$, $\rho^v = \{\rho^{v,u} : u \in V\}$ is such that $\rho^{v,u} : (\mathrm{SEN}^\flat_v)^2 \times \prod_{i<\omega} \mathrm{SEN}^\flat_{s_i} \to \mathrm{SEN}^\flat_u$ in $N^\flat$ is a collection of natural transformations satisfying, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$,

$$\langle \phi,\psi\rangle \in \Upsilon_\Sigma(T) \quad \text{iff} \quad \rho^v_\Sigma[\phi,\psi] \leq T.$$

  The set ρ is referred to as the set of **witnessing transformations** for the behavioral syntactic protoalgebraicity of $\mathcal{I}$.

We have the following characterization of behavioral protoalgebraicity.

Proposition 1964 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is behaviorally protoalgebraic if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $v \in V$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$*,*

$$\langle \phi, \psi\rangle \in \Upsilon_\Sigma(T) \quad \textit{implies} \quad C_\Sigma(T_\Sigma, \phi) = C_\Sigma(T_\Sigma, \psi).$$

Proof: Suppose, first, that $\mathcal{I}$ is behaviorally protoalgebraic and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$, such that $\langle \phi, \psi\rangle \in \Upsilon_\Sigma(T)$. Let $T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T_\Sigma \subseteq T'_\Sigma$ and $\psi \in T'_\Sigma$. Then, by behavioral protoalgebraicity, we have $\langle \phi, \psi\rangle \in \Upsilon_\Sigma(T) \subseteq \Upsilon_\Sigma(T')$. Since, by hypothesis, $\psi \in T'_\Sigma$, we get, by compatibility of $\Upsilon(T')$ with T', that $\phi \in T'_\Sigma$. Thus, $\phi \in C_\Sigma(T_\Sigma, \psi)$ and, by symmetry, $C_\Sigma(T_\Sigma, \phi) = C_\Sigma(T_\Sigma, \psi)$.

Suppose, conversely, that the condition in the statement holds and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \le T'$ and $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$, such that $\langle \phi, \psi\rangle \in \Upsilon_\Sigma(T)$. Then, since $\Upsilon(T)$ is a congruence system on $\mathbf{F}$, we get, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, all $\sigma : \mathrm{SEN}^\flat_v \times \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i} \to \mathrm{SEN}^\flat_u$ in $N^\flat$, with $u \in V$, and all $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i}(\Sigma')$,

$$\langle \sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\phi), \vec{\chi}), \sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\psi), \vec{\chi})\rangle \in \Upsilon_{\Sigma'}(T).$$

By hypothesis,

$$C_{\Sigma'}(T_{\Sigma'}, \sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\phi), \vec{\chi})) = C_{\Sigma'}(T_{\Sigma'}, \sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\psi), \vec{\chi})).$$

Hence, since $T \le T'$,

$$C_{\Sigma'}(T'_{\Sigma'}, \sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\phi), \vec{\chi})) = C_{\Sigma'}(T'_{\Sigma'}, \sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\psi), \vec{\chi})).$$

We now get

$$\sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\phi), \vec{\chi}) \in T'_{\Sigma'} \quad \text{iff} \quad \sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\psi), \vec{\chi}) \in T'_{\Sigma'}.$$

Therefore, by the characterization in the proof of Theorem 1958, we get that $\langle \phi, \psi\rangle \in \Upsilon_\Sigma(T')$. Hence $\Upsilon(T) \le \Upsilon(T')$ and it follows that $\mathcal{I}$ is behaviorally protoalgebraic. ∎

Behavioral syntactic protoalgebraicity implies behavioral protoalgebraicity.

Theorem 1965 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally syntactically protoalgebraic, then it is behaviorally protoalgebraic.*

Proof: Suppose $\mathcal{I}$ is behaviorally syntactically protoalgebraic, with witnessing transformations ρ, and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \le T'$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$, we have

$$\begin{array}{rcl}\langle \phi, \psi\rangle \in \Upsilon_\Sigma(T) & \text{iff} & \rho^v_\Sigma[\phi, \psi] \le T\\ & \text{implies} & \rho^v_\Sigma[\phi, \psi] \le T'\\ & \text{iff} & \langle \phi, \psi\rangle \in \Upsilon_\Sigma(T').\end{array}$$

Hence, $\Upsilon(T) \le \Upsilon(T')$ and, therefore, $\mathcal{I}$ is behaviorally protoalgebraic. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a behavioral π-institution based on $\mathbf{F}$. We define the **behavioral reflexive core of** $\mathcal{I}$

$$A^{\mathcal{I}} = \{A^{\mathcal{I},s} : s \in S\},$$

by letting, for all $s \in S$, $A^{\mathcal{I},s}$ be the collection of all natural transformations $\sigma : (\mathrm{SEN}^\flat_s)^2 \times \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i} \to \mathrm{SEN}^\flat_v$ in $N^\flat$, with $v \in V$, such that:

For all $\Sigma \in |\mathbf{Sign}^\flat|$, all $s \in S$, all $\phi \in \mathrm{SEN}^\flat_s(\Sigma)$,

$$\sigma_\Sigma[\phi, \phi] \le \mathrm{Thm}(\mathcal{I}).$$

The importance of the behavioral reflexive core lies, as in previous cases, in the fact that it forms a pool of candidates for drawing witnessing transformations for the behavioral syntactic protoalgebraicity of $\mathcal{I}$.

Lemma 1966 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a behavioral π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is behaviorally syntactically protoalgebraic, with witnessing transformations ρ, then $\rho \subseteq A^{\mathcal{I}}$.*

Proof: Suppose $\mathcal{I}$ is behaviorally syntactically protoalgebraic, with witnessing transformations ρ. Let $\sigma \in \rho$, $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$. Since $\langle \phi, \phi \rangle \in \Upsilon_\Sigma(\mathrm{Thm}(\mathcal{I}))$, we get that $\sigma_\Sigma[\phi, \phi] \le \rho_\Sigma[\phi, \phi] \le \mathrm{Thm}(\mathcal{I})$. Therefore, we get that $\rho \subseteq A^{\mathcal{I}}$. ∎

Moreover, if $\mathcal{I}$ is behaviorally syntactically protoalgebraic, then $A^{\mathcal{I}}$ satisfies a modus ponens property in $\mathcal{I}$.

Theorem 1967 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a behavioral π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is behaviorally syntactically protoalgebraic, then, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$,*

$$\psi \in C_\Sigma(\phi, A^{\mathcal{I}}_\Sigma[\phi, \psi]).$$

Proof: Assume $\mathcal{I}$ is behaviorally syntactically protoalgebraic, with witnessing transformations ρ and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$, such that $\phi \in T_\Sigma$ and $A^{\mathcal{I}}_\Sigma[\phi, \psi] \le T$. Then we have $\phi \in T_\Sigma$ and, by Lemma 1966, $\rho_\Sigma[\phi, \psi] \le A^{\mathcal{I}}_\Sigma[\phi, \psi] \le T$, whence $\phi \in T_\Sigma$ and $\langle \phi, \psi \rangle \in \Upsilon_\Sigma(T)$. Thus, by compatibility of $\Upsilon(T)$ with T, we conclude that $\psi \in T_\Sigma$. Therefore, $\psi \in C_\Sigma(\phi, \mathcal{A}^{\mathcal{I}}_\Sigma[\phi, \psi])$. ∎

Define, for all $T \in \mathrm{ThFam}(\mathcal{I})$, a relation family $A^{\mathcal{I}}(T) = \{A^{\mathcal{I}}_\Sigma(T)\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ on $\mathbf{F}$, by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $s \in S$ and all $\phi, \psi \in \mathrm{SEN}^\flat_s(\Sigma)$,

$$\langle \phi, \psi \rangle \in A^{\mathcal{I}}_\Sigma(T) \quad \text{iff} \quad A^{\mathcal{I},s}_\Sigma[\phi, \psi] \le T.$$

Then we have

Lemma 1968 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If, for all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $v \in V$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$*,*

$$\psi \in C_\Sigma(\phi, A^{\mathcal{I}}_\Sigma[\phi, \psi]),$$

then, for all $T \in \mathrm{ThFam}(\mathcal{I})$*,* $A^{\mathcal{I}}(T)$ *is a congruence system on* $\mathbf{F}$ *compatible with* T*.*

Proof: Fix $T \in \mathrm{ThFam}(\mathcal{I})$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $s \in S$ and $\phi \in \mathrm{SEN}^\flat_s(\Sigma)$. By definition of $\mathcal{A}^{\mathcal{I}}$, we have $A^{\mathcal{I}}_\Sigma[\phi,\phi] \le \mathrm{Thm}(\mathcal{I}) \le T$. Therefore, $\langle \phi,\phi\rangle \in A^{\mathcal{I}}_\Sigma(T)$ and, hence, $A^{\mathcal{I}}(T)$ is reflexive.

Let $\Sigma \in |\mathbf{Sign}^\flat|$, $s \in S$ and $\phi, \psi \in \mathrm{SEN}^\flat_s(\Sigma)$, such that $\langle \phi, \psi\rangle \in A^{\mathcal{I}}_\Sigma(T)$. Then $A^{\mathcal{I}}_\Sigma[\phi,\psi] \le T$. Again by the definition of $A^{\mathcal{I}}$, we get that $A^{\mathcal{I}}_\Sigma[\psi,\phi] = A^{\mathcal{I}}_\Sigma[\phi,\psi] \le T$, whence $\langle \psi,\phi\rangle \in A^{\mathcal{I}}_\Sigma(T)$ and $A^{\mathcal{I}}(T)$ is symmetric.

Let $\Sigma \in |\mathbf{Sign}^\flat|$, $s \in S$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat_s(\Sigma)$, such that $\langle \phi,\psi\rangle \in A^{\mathcal{I}}_\Sigma(T)$ and $\langle \psi, \chi\rangle \in A^{\mathcal{I}}_\Sigma(T)$. Then, we have $A^{\mathcal{I}}_\Sigma[\phi,\psi] \le T$ and $A^{\mathcal{I}}_\Sigma[\psi,\chi] \le T$. Thus, by hypothesis, we have, for all $\alpha \in A^{\mathcal{I}}$, all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$ of appropriate sorts,

$$\begin{array}{l}\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\chi})\\ \in C_{\Sigma'}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), \vec{\chi}),\\ \qquad A^{\mathcal{I}}_{\Sigma'}[\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), \vec{\chi}),\\ \qquad\qquad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\chi})])\\ \subseteq C_{\Sigma'}(A^{\mathcal{I}}_\Sigma[\phi,\psi], A^{\mathcal{I}}_\Sigma[\psi,\chi])\\ \subseteq T_{\Sigma'}.\end{array}$$

Hence $A^{\mathcal{I}}_\Sigma[\phi,\chi] \le T$ and, therefore, $\langle \phi,\chi\rangle \in A^{\mathcal{I}}_\Sigma(T)$ and $A^{\mathcal{I}}(T)$ is also transitive. It is, by its definition, a system. To see that it is a congruence system, suppose $\sigma : \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i} \to \mathrm{SEN}_s$ is in N $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi}, \vec{\psi} \in \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i}(\Sigma)$, such that $\vec{\phi} \prod_{i=1}^k A^{\mathcal{I},s_i}_\Sigma(T) \vec{\psi}$. Then we have

$$\begin{array}{rll}A^{\mathcal{I}}_\Sigma[\sigma_\Sigma(\vec{\phi}), \sigma_\Sigma(\vec{\phi})] \le T & \text{iff} & A^{\mathcal{I}}_\Sigma[\sigma_\Sigma(\vec{\phi}), \sigma_\Sigma(\psi_1,\phi_2,\phi_3,\dots,\phi_{k-1},\phi_k)] \le T\\ & \text{iff} & A^{\mathcal{I}}_\Sigma[\sigma_\Sigma(\vec{\phi}), \sigma_\Sigma(\psi_1,\psi_2,\phi_3,\dots,\phi_{k-1},\phi_k)] \le T\\ & \text{iff} & \cdots\\ & \text{iff} & A^{\mathcal{I}}_\Sigma[\sigma_\Sigma(\vec{\phi}), \sigma_\Sigma(\psi_1,\psi_2,\psi_3,\dots,\psi_{k-1},\phi_k)] \le T\\ & \text{iff} & A^{\mathcal{I}}_\Sigma[\sigma_\Sigma(\vec{\phi}), \sigma_\Sigma(\vec{\psi})] \le T.\end{array}$$

Therefore, $A^{\mathcal{I}}(T)$ is a congruence system. Finally, by hypothesis, it is compatible with T. ■

Lemma 1968 enables us to show that, if the behavioral reflexive core of a behavioral π-institution satisfies the modus ponens property postulated in its hypothesis, then it is behaviorally syntactically protoalgebraic, with witnessing transformations $A^{\mathcal{I},V} = \{A^{\mathcal{I},v} : v \in V\}$.

Theorem 1969 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If, for all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $v \in V$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$*,*

$$\psi \in C_\Sigma(\phi, A^{\mathcal{I}}_\Sigma[\phi, \psi]),$$

then $\mathcal{I}$ *is behaviorally syntactically protoalgebraic, with witnessing transformations* $A^{\mathcal{I},V} = \{A^{\mathcal{I},v} : v \in V\}$*.*

Proof: Suppose that $\mathcal{I}$ satisfies the condition in the hypothesis. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$.

- Assume, first, that $\langle \phi, \psi \rangle \in \Upsilon_\Sigma(T)$. Then, since $\Upsilon(T)$ is a congruence system on $\mathbf{F}$, we have, for all $\sigma : (\mathrm{SEN}^\flat_v)^2 \times \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i} \to \mathrm{SEN}^\flat_u$ in $A^{\mathcal{I},v}$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i}(\Sigma')$,

  $$\begin{array}{l}\langle \sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\phi), \mathrm{SEN}^\flat_v(f)(\phi), \vec{\chi}),\\ \qquad \sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\phi), \mathrm{SEN}^\flat_v(f)(\psi), \vec{\chi})\rangle \in \Upsilon_{\Sigma'}(T).\end{array}$$

  But, by definition of $A^{\mathcal{I}}$, we also have that

  $$\sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\phi), \mathrm{SEN}^\flat_v(f)(\phi), \vec{\chi}) \in \mathrm{Thm}_{\Sigma'}(\mathcal{I}) \subseteq T_{\Sigma'}.$$

  Hence, by the compatibility property of $\Upsilon(T)$ with T, we get that $\sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\phi), \mathrm{SEN}^\flat_v(f)(\psi), \vec{\chi}) \in T_{\Sigma'}$. Thus, $A^{\mathcal{I},v}_\Sigma[\phi, \psi] \leq T$.

- Assume, conversely, that $A^{\mathcal{I},v}_\Sigma[\phi, \psi] \leq T$. Then, we get $\langle \phi, \psi \rangle \in A^{\mathcal{I}}_\Sigma(T)$. But, by Lemma 1968 and the hypothesis, $A^{\mathcal{I}}(T)$ is a congruence system on $\mathbf{F}$ compatible with T, whence, by the maximality of $\Upsilon(T)$, $A^{\mathcal{I}}(T) \leq \Upsilon(T)$. Thus, $\langle \phi, \psi \rangle \in \Upsilon_\Sigma(T)$.

We now conclude that $\mathcal{I}$ is behaviorally syntactically protoalgebraic. ■

We have now the essential ingredients for formulating a characterization of behavioral syntactic protoalgebraicity.

$\mathcal{I}$ is behaviorally syntactically protoalgebraic $\longleftrightarrow$ $A^{\mathcal{I},V}$ has the MP.

Theorem 1970 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is behaviorally syntactically protoalgebraic if and only if, for all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $v \in V$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$*,* $\psi \in C_\Sigma(\phi, A^{\mathcal{I},v}_\Sigma[\phi, \psi])$*.*

Proof: The "only if" is by Theorem 1967. The "if" is by Theorem 1969. ■

It is not difficult to show now that, if a behavioral π-institution is behaviorally syntactically protoalgebraic, then any set of witnessing transformations is deductively equivalent to the visible part of the behavioral reflexive core.

Corollary 1971 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally syntactically protoalgebraic, with witnessing transformations* ρ*, then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $v \in V$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$*,*

$$C(A^{\mathcal{I},v}_\Sigma[\phi,\psi]) = C(\rho^v_\Sigma[\phi,\psi]).$$

Proof: Suppose $\mathcal{I}$ is behaviorally syntactically protoalgebraic, with witnessing transformations ρ. Then, by Theorems 1967 and 1969, $A^{\mathcal{I},V}$ is also a collection of witnessing transformations for the behavioral syntactic protoalgebraicity of $\mathcal{I}$. Therefore, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$, we get

$$\begin{array}{lll} A^{\mathcal{I},v}_\Sigma[\phi,\psi] \leq T & \text{iff} & \langle \phi,\psi \rangle \in \Upsilon_\Sigma(T) \\ & \text{iff} & \rho^v_\Sigma[\phi,\psi] \leq T. \end{array}$$

Hence, we get $C(A^{\mathcal{I},v}_\Sigma[\phi,\psi]) = C(\rho^v_\Sigma[\phi,\psi])$. ■

Another characterizing property, therefore, of behavioral syntactic protoalgebraicity is that the behavioral reflexive core define behavioral Leibniz congruence systems in $\mathcal{I}$.

$\mathcal{I}$ is behaviorally syntactically protoalgebraic
$\longleftrightarrow$ $A^{\mathcal{I},V}$ defines behavioral Leibniz congruence systems.

Theorem 1972 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is behaviorally syntactically protoalgebraic if and only if* $A^{\mathcal{I},V}$ *defines behavioral Leibniz congruence systems of theory families in* $\mathcal{I}$*, i.e., for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,* $A^{\mathcal{I},V}(T) = \Upsilon(T)$*.*

Proof: Suppose, first, that $\mathcal{I}$ is behaviorally syntactically protoalgebraic. Then, by Theorems 1967 and 1969, $A^{\mathcal{I},V}$ is a collection of witnessing transformations for the behavioral syntactic protoalgebraicity of $\mathcal{I}$. Therefore, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $A^{\mathcal{I},V}(T) = \Upsilon(T)$. Conversely, if $A^{\mathcal{I},V}(T) = \Upsilon(T)$, for all $T \in \mathrm{ThFam}(\mathcal{I})$, then $\mathcal{I}$ is behaviorally syntactically protoalgebraic, with witnessing transformations $A^{\mathcal{I},V}$. ■

The connection between behavioral syntactic protoalgebraicity and behavioral protoalgebraicity passes through another property of the behavioral Suszko core that we term *Leibniz*.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a behavioral π-institution based on $\mathbf{F}$. We say that the behavioral reflexive core $A^{\mathcal{I}}$ of $\mathcal{I}$ is **Leibniz** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$,

$$\langle \phi,\psi \rangle \in \Upsilon_\Sigma(C(A^{\mathcal{I},v}_\Sigma[\phi,\psi])).$$

It is straightforward to show that, if $A^{\mathcal{I},V}$ has the modus ponens property in $\mathcal{I}$, then it is also Leibniz.

Proposition 1973 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $A^{\mathcal{I}}$ *has the modus ponens in* $\mathcal{I}$*, then it is Leibniz.*

Proof: Suppose that $A^{\mathcal{I}}$ has the modus ponens in $\mathcal{I}$. Then, by Theorem 1969, $\mathcal{I}$ is behaviorally syntactically protoalgebraic, with witnessing transformations $A^{\mathcal{I},V}$. Thus, we obtain, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Upsilon_\Sigma(C(A^{\mathcal{I},v}_\Sigma[\phi, \psi])) \quad \text{iff} \quad A^{\mathcal{I},v}_\Sigma[\phi, \psi] \leq C(A^{\mathcal{I},v}_\Sigma[\phi, \psi]).$$

However, the condition of the right always holds, whence, we get that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$, $\langle \phi, \psi \rangle \in \Upsilon_\Sigma(C(A^{\mathcal{I},v}_\Sigma[\phi, \psi]))$, i.e., $A^{\mathcal{I}}$ is Leibniz. ∎

The opposite implication is not true in general. It holds, however, in behaviorally protoalgebraic behavioral π-institutions.

Proposition 1974 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behaviorally protoalgebraic behavioral* π*-institution based on* $\mathbf{F}$*. If* $A^{\mathcal{I}}$ *is Leibniz, then it has the modus ponens in* $\mathcal{I}$*.*

Proof: Suppose that $\mathcal{I}$ is behaviorally protoalgebraic and $A^{\mathcal{I}}$ is Leibniz. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$, such that $\phi \in T_\Sigma$ and $A^{\mathcal{I},v}_\Sigma[\phi, \psi] \leq T$. Since $A^{\mathcal{I}}$ is Leibniz, we get that $\langle \phi, \psi \rangle \in \Upsilon_\Sigma(C(A^{\mathcal{I},v}_\Sigma[\phi, \psi]))$. Since $A^{\mathcal{I},v}_\Sigma[\phi, \psi] \leq T$, we get, by the hypothesis of behavioral protoalgebraicity, $\Upsilon(C(A^{\mathcal{I},v}_\Sigma[\phi, \psi])) \leq \Upsilon(T)$, whence, $\langle \phi, \psi \rangle \in \Upsilon_\Sigma(T)$. Hence, by the compatibility of $\Upsilon(T)$, with T, we get $\psi \in T_\Sigma$. We conclude that $\psi \in C_\Sigma(\phi, A^{\mathcal{I},v}_\Sigma[\phi, \psi])$ and, thus, $A^{\mathcal{I}}$ has the modus ponens in $\mathcal{I}$. ∎

We close by formulating the exact relation between behavioral syntactic protoalgebraicity and behavioral protoalgebraicity.

Behavioral Syntactic Protoalgebraicity
= $A^{\mathcal{I}}$ has the Modus Ponens
= $A^{\mathcal{I}}$ Defines Behavioral Leibniz Congruence Systems
= Behavioral Protoalgebraicity + $A^{\mathcal{I}}$ Leibniz

Theorem 1975 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is behaviorally syntactically protoalgebraic if and only if it is behaviorally protoalgebraic and has a Leibniz behavioral reflexive core.*

Proof: If $\mathcal{I}$ is behaviorally syntactically protoalgebraic, then, by Theorem 1965, it is behaviorally protoalgebraic, by Theorem 1967, $A^{\mathcal{I}}$ has the modus ponens and, hence, by Proposition 1973, $A^{\mathcal{I}}$ is Leibniz.

Conversely, if $\mathcal{I}$ is behaviorally protoalgebraic and $A^{\mathcal{I}}$ is Leibniz, then, by Proposition 1974, $A^{\mathcal{I}}$ has the modus ponens in $\mathcal{I}$, whence, by Theorem 1969, $\mathcal{I}$ is behaviorally syntactically protoalgebraic. ∎

27.5 Behavioral Truth Equationality

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a behavioral π-institution based on $\mathbf{F}$.

- $\mathcal{I}$ is **behaviorally completely reflective** (or **behaviorally c-reflective**, for short), if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,
$$\bigcap_{T \in \mathcal{T}} \Upsilon(T) \leq \Upsilon(T') \quad \text{implies} \quad \bigcap_{T \in \mathcal{T}} T \leq T'.$$
- $\mathcal{I}$ is **behaviorally truth equational** if there exists $\tau = \{\tau^v : v \in V\}$, where, for all $v \in V$, $\tau^v = \{\tau^{v,u} : u \in V\}$ is a collection of natural transformations $\tau^{v,u} : \mathrm{SEN}^\flat_v \times \prod_{i<\omega} \mathrm{SEN}^\flat_{s_i} \to (\mathrm{SEN}^\flat_u)^2$ in $N^\flat$, such that, for all $T \in \mathbf{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$,
$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^v_\Sigma[\phi] \leq \Upsilon(T).$$
In this case, the collection τ forms a set of **witnessing transformations for the behavioral truth equationality of** $\mathcal{I}$.

We have the following alternative characterization of behavioral c-reflectivity, involving both the behavioral Suszko and the behavioral Leibniz operator.

Lemma 1976 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a behavioral π-institution based on $\mathbf{F}$. $\mathcal{I}$ is behaviorally c-reflective if and only if, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,*
$$\widetilde{\Upsilon}(T) \leq \Upsilon(T') \quad \textit{implies} \quad T \leq T'.$$

Proof: Suppose, first, that $\mathcal{I}$ is behaviorally c-reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{\Upsilon}(T) \leq \Upsilon(T')$. Then, we have $\bigcap\{\Upsilon(X) : T \leq X \in \mathrm{FiFam}(\mathcal{I})\} \leq \Upsilon(T')$, whence, by behavioral c-reflectivity, $\bigcap\{X : T \leq X \in \mathrm{ThFam}(\mathcal{I})\} \leq T'$, i.e., $T \leq T'$. Thus, the condition of the statement holds.

Assume, conversely, that the condition of the statement holds and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Upsilon(T) \leq \Upsilon(T')$. Then we get
$$\widetilde{\Upsilon}(\bigcap \mathcal{T}) \leq \bigcap\{\Upsilon(T) : T \in \mathcal{T}\} \leq \Upsilon(T').$$

Therefore, by the hypothesis, $\bigcap \mathcal{T} \leq T'$ and, hence, $\mathcal{I}$ is behaviorally c-reflective. ■

It is easy to see that behavioral truth equationality implies behavioral c-reflectivity.

Proposition 1977 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally truth equational, then it is behaviorally c-reflective.*

Proof: Suppose that $\mathcal{I}$ is behaviorally truth equational, with witnessing transformations $\tau = \{\tau^v : v \in V\}$, and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Upsilon(T) \leq \Upsilon(T')$. Then, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$,

$$\begin{array}{rcl} \phi \in \bigcap_{T \in \mathcal{T}} T_\Sigma & \text{iff} & \phi \in T_\Sigma,\ T \in \mathcal{T}, \\ & \text{iff} & \tau^v_\Sigma[\phi] \leq \Upsilon(T),\ T \in \mathcal{T}, \\ & \text{iff} & \tau^v_\Sigma[\phi] \leq \bigcap_{T \in \mathcal{T}} \Upsilon(T) \\ & \text{implies} & \tau^v_\Sigma[\phi] \leq \Upsilon(T') \\ & \text{iff} & \phi \in T'_\Sigma. \end{array}$$

Therefore, $\bigcap \mathcal{T} \leq T'$ and $\mathcal{I}$ is indeed behaviorally c-reflective. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a behavioral π-institution based on $\mathbf{F}$. We define the **behavioral Suszko core** $\Sigma^{\mathcal{I}} = \{\Sigma^{\mathcal{I},v} : v \in V\}$ **of** $\mathcal{I}$ by setting, for all $v \in V$,

$$\begin{array}{l} \Sigma^{\mathcal{I},v} = \{\sigma : \mathrm{SEN}^\flat_v \times \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i} \to (\mathrm{SEN}^\flat_u)^2, u \in V : \\ \qquad (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \phi \in \mathrm{SEN}^\flat_v(\Sigma))(\sigma_\Sigma[\phi] \leq \widetilde{\Upsilon}(C(\phi)))\} \end{array}$$

$\Sigma^{\mathcal{I}}$ is a pool for possible candidates witnessing the potential behavioral truth equationality of $\mathcal{I}$.

Lemma 1978 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally truth equational, with witnessing transformations* τ*, then* $\tau \subseteq \Sigma^{\mathcal{I}}$*.*

Proof: Suppose $\mathcal{I}$ is behaviorally truth equational, with witnessing transformations $\tau = \{\tau^v : v \in V\}$ and let $v \in V$, $\sigma \in \tau^v$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$. Then, we have, for all $T \in \mathrm{ThFam}(\mathcal{I})$, such that $\phi \in T_\Sigma$, $\sigma_\Sigma[\phi] \leq \Upsilon(T)$, whence

$$\sigma_\Sigma[\phi] \leq \bigcap\{\Upsilon(T) : \phi \in T_\Sigma\} = \widetilde{\Upsilon}(C(\phi)).$$

We conclude that $\sigma \in \Sigma^{\mathcal{I},v}$. Therefore, $\tau \subseteq \Sigma^{\mathcal{I}}$. ■

The behavioral Suszko core $\Sigma^{\mathcal{I}}$ was devised to carry a sentence of visible sort into the behavioral Suszko congruence system of the theory family generated by it. Because of the monotonicity of the behavioral Suszko operator and the fact that the behavioral Suszko operator is universally subsumed by the behavioral Leibniz operator, however, it turns out that the image of any behavioral theory family under the behavioral Suszko core always lies inside the behavioral Leibniz congruence system of the family.

Proposition 1979 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*,* $T \in \mathrm{ThFam}(\mathcal{I})$*,* $\Sigma \in |\mathbf{Sign}^\flat|$*,* $v \in V$ *and* $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$*. If* $\phi \in T_\Sigma$*, then*

$$\Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \Upsilon(T).$$

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$, such that $\phi \in T_\Sigma$. Then, we have

$$\begin{array}{rcll} \Sigma^{\mathcal{I},v}_\Sigma[\phi] & \leq & \widetilde{\Upsilon}(C(\phi)) & (\text{definition of } \Sigma^{\mathcal{I}}) \\ & \leq & \widetilde{\Upsilon}(T) & (\text{monotonicity of } \widetilde{\Upsilon}) \\ & \leq & \Upsilon(T). & (\widetilde{\Upsilon} \leq \Upsilon) \end{array}$$

This proves the conclusion. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a behavioral π-institution based on $\mathbf{F}$. We say that the behavioral Suszko core $\Sigma^{\mathcal{I}}$ of $\mathcal{I}$ is **soluble** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$,

$$\Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \Upsilon(T) \quad \text{implies} \quad \phi \in T_\Sigma.$$

The solubility of the behavioral Suszko core is a necessary condition for a behavioral π-institution to be behaviorally truth equational.

Theorem 1980 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally truth equational, then* $\Sigma^{\mathcal{I}}$ *is soluble.*

Proof: Suppose $\mathcal{I}$ is behaviorally truth equational, with witnessing transformations $\tau = \{\tau^v : v \in V\}$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$, such that $\Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \Upsilon(T)$, we have, by Lemma 1978,

$$\tau^v_\Sigma[\phi] \leq \Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \Upsilon(T),$$

whence, by the fact that τ witnesses the behavioral truth equationality of $\mathcal{I}$, $\phi \in T_\Sigma$. Therefore, $\Sigma^{\mathcal{I}}$ is indeed soluble. ∎

Conversely, the solubility of the behavioral Suszko core ensures that it can serve as a collection of witnessing transformations for the behavioral truth equationality of $\mathcal{I}$.

Theorem 1981 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}_s^\flat\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If the behavioral Suszko core* $\Sigma^{\mathcal{I}}$ *is soluble, then* $\mathcal{I}$ *is behaviorally truth equational, with witnessing transformations* $\Sigma^{\mathcal{I}}$*.*

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi \in \mathrm{SEN}_v^\flat(\Sigma)$. If $\phi \in T_\Sigma$, then, by Proposition 1979, $\Sigma_\Sigma^{\mathcal{I},v}[\phi] \le \Upsilon(T)$. On the other hand, if $\Sigma_\Sigma^{\mathcal{I},v}[\phi] \le \Upsilon(T)$, then, by the postulated solubility of $\Sigma^{\mathcal{I}}$, we get that $\phi \in T_\Sigma$. Hence, we have $\phi \in T_\Sigma$ if and only if $\Sigma_\Sigma^{\mathcal{I},v}[\phi] \le \Upsilon(T)$, showing that $\Sigma^{\mathcal{I}}$ witnesses the behavioral truth equationality of $\mathcal{I}$. ∎

We now have the following characterization of behavioral truth equationality depending on the behavior (in the ordinary sense) of the behavioral Suszko core.

$$\mathcal{I} \text{ is Behaviorally Truth Equational} \longleftrightarrow \Sigma^{\mathcal{I}} \text{ is Soluble.}$$

Theorem 1982 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}_s^\flat\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is behaviorally truth equational if and only if it has a soluble behavioral Suszko core.*

Proof: Necessity is by Theorem 1980, whereas sufficiency is proved in Theorem 1981. ∎

We say that the behavioral Suszko core $\Sigma^{\mathcal{I}}$ of a behavioral π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ **defines theory families** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi \in \mathrm{SEN}_v^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \Sigma_\Sigma^{\mathcal{I},v}[\phi] \le \Upsilon(T).$$

Then, another characterization of behavioral truth equationality is the following:

$$\mathcal{I} \text{ is Behaviorally Truth Equational} \longleftrightarrow \Sigma^{\mathcal{I}} \text{ Defines Theory Families.}$$

Theorem 1983 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}_s^\flat\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is behaviorally truth equational if and only if its behavioral Suszko core* $\Sigma^{\mathcal{I}}$ *defines theory families.*

Proof: $\mathcal{I}$ is behaviorally truth equational if and only if, by Theorem 1982 $\Sigma^{\mathcal{I}}$ is soluble if and only if, by Proposition 1979 and the definition of solubility, $\Sigma^{\mathcal{I}}$ defines theory families in $\mathcal{I}$. ∎

We have just seen that behavioral truth equationality of a behavioral π-institution is equivalent to the solubility property of its behavioral Suszko core. The solubility property implies another property, which, taking after similar work in preceding chapters, we call *adequacy*. It says, roughly speaking, that in a behavioral π-institution the category of natural transformations is rich enough to determine behavioral Suszko congruence systems in terms of the behavioral Leibniz congruence systems that it selects by inclusion. The property of adequacy is motivated by the following property that holds in every behavioral π-institution.

Proposition 1984 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a behavioral π-institution based on $\mathbf{F}$. For all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$,*

$$\bigcap\{\Upsilon(T) : \Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \Upsilon(T)\} \leq \widetilde{\Upsilon}(C(\phi)).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$. Then we have, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\begin{array}{lll}\phi \in T_\Sigma & \text{implies} & \Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \widetilde{\Upsilon}(T)\\ & \text{implies} & \Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \Upsilon(T).\end{array}$$

Thus, we get

$$\begin{array}{lll}\bigcap\{\Upsilon(T) : \Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \Upsilon(T)\} & \leq & \bigcap\{\Upsilon(T) : \Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \widetilde{\Upsilon}(T)\}\\ & \leq & \bigcap\{\Upsilon(T) : \phi \in T_\Sigma\}\\ & = & \widetilde{\Upsilon}(C(\phi)).\end{array}$$

Hence, the inclusion in the statement holds. ■

The reverse inclusion is not always guaranteed, but, when it holds, we say that the behavioral Suszko core of $\mathcal{I}$ is *adequate*. The terminology is intended to convey the idea that $\Sigma^{\mathcal{I},v}_\Sigma[\phi]$ suffices to determine the theory families whose behavioral Leibniz congruence systems form a "covering" of the behavioral Suszko congruence system corresponding to the theory family $C(\phi)$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a behavioral π-institution based on $\mathbf{F}$. The behavioral Suszko core $\Sigma^\mathcal{I}$ of $\mathcal{I}$ is **adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$,

$$\widetilde{\Upsilon}(C(\phi)) \leq \bigcap\{\Upsilon(T) : \Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \Upsilon(T)\}.$$

We can prove immediately that the solubility of the behavioral Suszko core implies its adequacy.

Proposition 1985 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a behavioral π-institution based on $\mathbf{F}$. If the behavioral Suszko core $\Sigma^\mathcal{I}$ is soluble, then it is adequate.*

Proof: Suppose that $\Sigma^{\mathcal{I}}$ is soluble. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$. By solubility, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \Upsilon(T) \quad \text{implies} \quad \phi \in T_\Sigma.$$

Hence, we get

$$\widetilde{\Upsilon}(C(\phi)) \leq \widetilde{\Upsilon}(T) \leq \Upsilon(T).$$

Since this holds, for all $T \in \mathrm{ThFam}(\mathcal{I})$, such that $\Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \Upsilon(T)$, we get that

$$\widetilde{\Upsilon}(C(\phi)) \leq \bigcap\{\Upsilon(T) : \Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \Upsilon(T)\}.$$

Therefore, $\Sigma^{\mathcal{I}}$ is adequate. ∎

Conversely, if a behavioral π-institution is behaviorally c-reflective, then the adequacy of its behavioral Suszko core is sufficient to give its solubility.

Proposition 1986 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}^{V,V^*}_H, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally c-reflective and the behavioral Suszko core* $\Sigma^{\mathcal{I}}$ *is adequate, then* $\Sigma^{\mathcal{I}}$ *is soluble.*

Proof: Suppose that $\mathcal{I}$ is behaviorally c-reflective and that the behavioral Suszko core $\Sigma^{\mathcal{I}}$ is adequate. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$, such that $\Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \Upsilon(T)$. Then, by the adequacy of the Suszko core, we get that $\widetilde{\Upsilon}(C(\phi)) \leq \Upsilon(T)$, whence, by behavioral c-reflectivity and Lemma 1976, we get $C(\phi) \leq T$, i.e., $\phi \in T_\Sigma$. We conclude that $\Sigma^{\mathcal{I}}$ is soluble.
∎

We can now show that a behavioral π-institution is behaviorally truth equational if and only if it is behaviorally c-reflective and has an adequate behavioral Suszko core.

Behavioral Truth Equationality
= $\Sigma^{\mathcal{I}}$ Soluble
= $\Sigma^{\mathcal{I}}$ Defines Theory Families
= Behavioral c-Reflectivity + $\Sigma^{\mathcal{I}}$ Adequate

Theorem 1987 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}^{V,V^*}_H, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is behaviorally truth equational if and only if it is behaviorally c-reflective and has an adequate behavioral Suszko core.*

Proof: If $\mathcal{I}$ is behaviorally truth equational, then, by Proposition 1977, it is behaviorally c-reflective, by Theorem 1980, its behavioral Suszko core is soluble and, by Proposition 1985, its behavioral Suszko core is adequate. Conversely, if $\mathcal{I}$ is behaviorally c-reflective with an adequate behavioral Suszko core, then, by Proposition 1986, its behavioral Suszko core is soluble and, hence, by Theorem 1981, $\mathcal{I}$ is behaviorally truth equational. ∎

27.6 Behavioral Weak Algebraizability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a behavioral π-institution based on $\mathbf{F}$. $\mathcal{I}$ is **behaviorally WF algebraizable** if it is behaviorally protoalgebraic and behaviorally c-reflective.

Lemma 1988 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally protoalgebraic, then, for all* $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$*,*

$$\Upsilon(\bigcap_{i\in I} T^i) = \bigcap_{i\in I} \Upsilon(T^i).$$

Proof: Suppose $\mathcal{I}$ is behaviorally protoalgebraic and let $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$. Then, by hypothesis, $\Upsilon(\bigcap_{i\in I} T^i) \le \bigcap_{i\in I} \Upsilon(T^i)$. On the other hand, $\bigcap_{i\in I} \Upsilon(T^i)$ is a congruence system on $\mathbf{F}$. Moreover, it is easy to see that it is compatible with $\bigcap_{i\in I} T^i$. Hence, by the maximality property of the behavioral Leibniz congruence system, we get $\bigcap_{i\in I} \Upsilon(T^i) \le \Upsilon(\bigcap_{i\in I} T^i)$. ■

Lemma 1989 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally protoalgebraic and the behavioral Leibniz operator is injective, then, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$*,*

$$\bigcap_{T\in\mathcal{T}} \Upsilon(T) \le \Upsilon(T') \quad \textit{implies} \quad \bigcap \mathcal{T} \le T'.$$

Proof: Suppose that $\mathcal{I}$ is behaviorally protoalgebraic and that the behavioral Leibniz operator is injective. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Upsilon(T) \le \Upsilon(T')$. Then we have

$$\begin{array}{rcll} \Upsilon(\bigcap \mathcal{T} \cap T') & = & \bigcap_{T\in\mathcal{T}} \Upsilon(T) \cap \Upsilon(T') & \text{(Lemma 1988)} \\ & = & \bigcap_{T\in\mathcal{T}} \Upsilon(T) & \text{(hypothesis)} \\ & = & \Upsilon(\bigcap \mathcal{T}). & \text{(Lemma 1988)} \end{array}$$

Hence, by the injectivity of the behavioral Leibniz operator, $\bigcap \mathcal{T} \cap T' = \bigcap \mathcal{T}$, showing that $\bigcap \mathcal{T} \le T'$. ■

Proposition 1990 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is behaviorally WF algebraizable if and only if the behavioral Leibniz operator is monotone and injective on* $\mathrm{ThFam}(\mathcal{I})$*.*

Proof: Suppose $\mathcal{I}$ is behaviorally WF algebraizable. Then by definition, it is behaviorally protoalgebraic and behaviorally c-reflective. Thus, the behavioral Leibniz operator is monotone and c-reflective on $\mathrm{ThFam}(\mathcal{I})$, whence it is monotone and, a fortiori, injective on $\mathrm{ThFam}(\mathcal{I})$.

If, conversely, Υ is monotone and injective on $\mathrm{ThFam}(\mathcal{I})$, then it is monotone and, by Lemma 1989, c-reflective on $\mathrm{ThFam}(\mathcal{I})$. Hence, $\mathcal{I}$ is behaviorally protoalgebraic and behaviorally c-reflective, i.e., by definition, it is behaviorally WF algebraizable. ∎

Another characterization of behavioral WF algebraizability asserts that it is equivalent to the existence of an isomorphism from the complete lattice of theory families of a behavioral π-institution to the complete lattice of the $\mathcal{I}$-congruence systems on its underlying behavioral algebraic system.

We need the following preparatory definitions.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}_s^\flat\}_H^{V,V^*}, N^\flat \rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a behavioral π-institution based on $\mathbf{F}$.

- Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system. A family $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, with $T_\Sigma \subseteq \bigcup_{v \in V} \mathrm{SEN}_v(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$, is called an **$\mathcal{I}$-filter family** of $\mathcal{A}$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\Phi \cup \{\phi\} \subseteq \bigcup_{v \in V} \mathrm{SEN}_v^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$,
$$\alpha_\Sigma(\Phi) \subseteq T_{F(\Sigma)} \quad \text{implies} \quad \alpha_\Sigma(\phi) \in T_{F(\Sigma)}.$$
The collection of all $\mathcal{I}$-filter families of $\mathcal{A}$ is denoted by $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. It is a complete lattice, whose corresponding closure operator will be denoted by $C^{\mathcal{I},\mathcal{A}}$.

- An $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ is an **$\mathcal{I}$-algebraic system** if there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Upsilon}(T) = \Delta^{\mathcal{A}}$. The collection of all $\mathcal{I}$-algebraic systems is denoted by $\mathrm{AlgSys}(\mathcal{I})$.

- Given an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, a congruence system $\theta \in \mathrm{ConSys}(\mathcal{A})$ is a **$\mathcal{I}$-congruence system on $\mathcal{A}$** if $\mathcal{A}/\theta \in \mathrm{AlgSys}(\mathcal{I})$. The collection of all $\mathcal{I}$-congruence systems on $\mathcal{A}$ is denoted by $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$.

Lemma 1991 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}_s^\flat\}_H^{V,V^*}, N^\flat \rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a behavioral π-institution based on $\mathbf{F}$. For every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, such that, for all $\Sigma \in |\mathbf{Sign}|$, $T_\Sigma \subseteq \bigcup_{v \in V} \mathrm{SEN}_v(\Sigma)$,*
$$T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \quad \textit{iff} \quad \alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I}).$$

Proof: Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, such that, for all $\Sigma \in |\mathbf{Sign}|$, $T_\Sigma \subseteq \bigcup_{v \in V} \mathrm{SEN}_v(\Sigma)$.

Assume, first, that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi \in \mathrm{SEN}_v^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\alpha_\Sigma^{-1}(T_{F(\Sigma)}))$. Then, by the definition of $C^{\mathcal{I},\mathcal{A}}$, we get
$$\alpha_\Sigma(\phi) \in C_{F(\Sigma)}^{\mathcal{I},\mathcal{A}}(\alpha_\Sigma(\alpha_\Sigma^{-1}(T_{F(\Sigma)}))) \subseteq C_{F(\Sigma)}^{\mathcal{I},\mathcal{A}}(T_{F(\Sigma)}) = T_{F(\Sigma)}.$$
Hence, we get $\phi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})$ and we conclude that $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$.

Suppose, conversely, that $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \bigcup_{v\in V} \mathrm{SEN}^\flat_v(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$ and $\alpha_\Sigma(\Phi) \subseteq T_{F(\Sigma)}$. Then $\Phi \subseteq \alpha^{-1}_\Sigma(T_{F(\Sigma)})$, whence, since $\phi \in C_\Sigma(\Phi)$ and $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, we get that $\phi \in \alpha^{-1}_\Sigma(T_{F(\Sigma)})$, i.e., $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$. We conclude that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. ■

Lemma 1992 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}^{V,V^*}_H, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$\Upsilon(\alpha^{-1}(T)) = \alpha^{-1}(\Upsilon^{\mathcal{A}}(T)).$$

Proof: Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $s \in S$ and $\phi, \psi \in \mathrm{SEN}^\flat_s(\Sigma)$, we have $\langle \phi, \psi\rangle \in \Upsilon_\Sigma(\alpha^{-1}(T))$ if and only if, for all $\sigma : \mathrm{SEN}^\flat_s \times \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i} \to \mathrm{SEN}^\flat_v$, with $v \in V$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i}(\Sigma')$,

$$\begin{array}{l}\sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\phi), \vec{\chi}) \in \alpha^{-1}_{\Sigma'}(T_{F(\Sigma')})\\ \qquad\text{iff}\quad \sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\psi), \vec{\chi}) \in \alpha^{-1}_{\Sigma'}(T_{F(\Sigma')})\end{array}$$

iff

$$\begin{array}{l}\alpha_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\phi), \vec{\chi})) \in T_{F(\Sigma')}\\ \qquad\text{iff}\quad \alpha_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\psi), \vec{\chi})) \in T_{F(\Sigma')}\end{array}$$

iff

$$\begin{array}{l}\sigma^{\mathcal{A}}_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\phi)), \alpha_{\Sigma'}(\vec{\chi})) \in T_{F(\Sigma')}\\ \qquad\text{iff}\quad \sigma^{\mathcal{A}}_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\psi)), \alpha_{\Sigma'}(\vec{\chi})) \in T_{F(\Sigma')}\end{array}$$

iff

$$\begin{array}{l}\sigma^{\mathcal{A}}_{F(\Sigma')}(\mathrm{SEN}_s(F(f))(\alpha_\Sigma(\phi)), \alpha_{\Sigma'}(\vec{\chi})) \in T_{F(\Sigma')}\\ \qquad\text{iff}\quad \sigma^{\mathcal{A}}_{F(\Sigma')}(\mathrm{SEN}_s(F(f))(\alpha_\Sigma(\psi)), \alpha_{\Sigma'}(\vec{\chi})) \in T_{F(\Sigma')}\end{array}$$

if and only if, by the surjectivity of $\langle F, \alpha\rangle$, $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in \Upsilon^{\mathcal{A}}_{F(\Sigma)}(T)$ if and only if $\langle \phi, \psi\rangle \in \alpha^{-1}_{F(\Sigma)}(\Upsilon^{\mathcal{A}}_{F(\Sigma)}(T))$. We now conclude that $\Upsilon(\alpha^{-1}(T)) = \alpha^{-1}(\Upsilon^{\mathcal{A}}(T))$. ■

Now we have the following characterization result for behavioral WF algebraizability.

Theorem 1993 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}^{V,V^*}_H, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is behaviorally WF algebraizable if and only if* $\Upsilon : \mathbf{ThFam}(\mathcal{I}) \to \mathbf{ConSys}^{\mathcal{I}}(\mathcal{F})$ *is an order isomorphism.*

Proof: Suppose that $\mathcal{I}$ is behaviorally WF algebraizable. Then, by Proposition 1990, Υ is monotone and injective on $\mathrm{ThFam}(\mathcal{I})$. Moreover, by definition of behavioral WF algebraizability Υ is c-reflective on $\mathrm{ThFam}(\mathcal{I})$ and, therefore, a fortiori, it is order reflecting. Thus, it suffices to show that it is surjective, i.e., onto $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{F})$. To this end, let $\theta \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{F})$. By definition, $\mathcal{F}/\theta \in \mathrm{AlgSys}(\mathcal{I})$. Thus, there exists $T^\theta \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}/\theta)$, such that $\widetilde{\Upsilon}^{\mathcal{F}/\theta}(T^\theta) = \Delta^{\mathcal{F}/\theta}$. Let $\langle I, \pi^\theta\rangle : \mathcal{F} \to \mathcal{F}/\theta$ be the quotient morphism. Now, by Lemma 1991, $\pi^{\theta^{-1}}(T^\theta) \in \mathrm{ThFam}(\mathcal{I})$ and

$$\begin{array}{rcll}\Upsilon(\pi^{\theta^{-1}}(T^\theta)) & = & \pi^{\theta^{-1}}(\Upsilon^{\mathcal{F}/\theta}(T^\theta)) & \text{(Lemma 1992)}\\ & = & \pi^{\theta^{-1}}(\Delta^{\mathcal{F}/\theta}) & \text{(hypothesis and protoalgebraicity)}\\ & = & \theta. & \text{(set theory)}\end{array}$$

Therefore, Υ is surjective and, hence, an order isomorphism from $\mathbf{ThFam}(\mathcal{I})$ onto $\mathbf{ConSys}^{\mathcal{I}}(\mathcal{F})$.

Conversely, if $\Upsilon : \mathbf{ThFam}(\mathcal{I}) \to \mathbf{ConSys}^{\mathcal{I}}(\mathcal{F})$ is an order isomorphism, then it is monotone and injective on $\mathrm{ThFam}(\mathcal{I})$ and, hence, by Proposition 1990, $\mathcal{I}$ is behaviorally WF algebraizable. ■

Finally, we close by providing a relation between behavioral algebraizability and behavioral WF algebraizability. Our first step in this direction is to show that behavioral algebraizability implies both behavioral syntactic protoalgebraicity and behavioral truth equationality. To be able to show this, we start by proving two technical results asserting that the binary relation family induced on the underlying behavioral algebraic system of a given behaviorally algebraizable π-institution by one of the two interpretations witnessing the behavioral algebraizability is a congruence system having a compatibility property.

Lemma 1994 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally algebraizable via a conjugate pair* $(\tau, \rho) : \mathcal{I} \rightleftarrows \mathcal{I}^{\mathsf{K}}$*, for some class* K *of* $\mathbf{F}$*-algebraic systems, then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $v \in V$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$*,*

$$\psi \in C_\Sigma(\phi, \rho^v_\Sigma[\phi, \psi]).$$

Proof: Assume $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$, such that $\phi \in T_\Sigma$ and $\rho^v_\Sigma[\phi, \psi] \leq T$. Then we get that

$$\tau^v_\Sigma[\phi] \leq C^{\mathsf{K}}(\tau_\Sigma[T]) \quad \text{and} \quad \langle \phi, \psi\rangle \in C^{\mathsf{K}}(\tau_\Sigma[T]).$$

Hence, by the definition of C^{K}, we get that $\tau^v_\Sigma[\psi] \leq C^{\mathsf{K}}(\tau_\Sigma[T])$ and, therefore, $\psi \in C_\Sigma(T) = T_\Sigma$. We conclude that $\psi \in C_\Sigma(\phi, \rho^v_\Sigma[\phi, \psi])$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a behaviorally algebraizable π-institution, as witnessed by the conjugate pair $(\tau,\rho):\mathcal{I}\rightleftarrows\mathcal{I}^{\mathsf{K}}$, for some class K of $\mathbf{F}$-algebraic systems. We define a class $\rho^+=\{\rho^{+,s}:s\in S\}$ of natural transformations in $N^\flat$ by setting, for all $s\in S$, $\rho^{+,s}$ to be the collection of all natural transformations in $N^\flat$ of the form

$$\sigma^v(\sigma(x,\vec{z}),\sigma(y,\vec{z}),\vec{w}),$$

where

$$\sigma:\mathrm{SEN}^\flat_s\times\prod_{i=1}^k\mathrm{SEN}^\flat_{s_i}\to\mathrm{SEN}^\flat_v,\quad \sigma^v\in\rho^v,\quad v\in V.$$

Moreover, for all $T\in\mathrm{ThFam}(\mathcal{I})$, we define $\rho^{+*}(T)=\{\rho^{+*}_\Sigma(T)\}_{\Sigma\in|\mathbf{Sign}^\flat|}$, where, for all $\Sigma\in|\mathbf{Sign}^\flat|$, we set

$$\rho^{+*}_\Sigma(T)=\{\rho^{+*,s}_\Sigma(T):s\in S\}$$

by letting, for all $\Sigma\in|\mathbf{Sign}^\flat|$, all $s\in S$ and all $\phi,\psi\in\mathrm{SEN}^\flat_s(\Sigma)$,

$$\langle\phi,\psi\rangle\in\rho^{+*,s}_\Sigma(T)\quad\text{iff}\quad\rho^{+,s}_\Sigma[\phi,\psi]\le T.$$

Proposition 1995 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\{\mathrm{SEN}^\flat_s\}_H^{V,V^*},N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally algebraizable via a conjugate pair* $(\tau,\rho):\mathcal{I}\rightleftarrows\mathcal{I}^{\mathsf{K}}$*, for some class* K *of* $\mathbf{F}$*-algebraic systems, then, for all* $T\in\mathrm{ThFam}(\mathcal{I})$*,* $\rho^{+*}(T)$ *is a congruence system on* $\mathbf{F}$ *compatible with* T*.*

Proof: Let $T\in\mathrm{ThFam}(\mathcal{I})$ and $\Sigma\in|\mathbf{Sign}^\flat|$. Then $\rho^{+*}_\Sigma(T)$ is reflexive, symmetric and transitive, by the definition of $\mathcal{I}^{\mathsf{K}}$, the definition of ρ^+ and the fact that ρ is an interpretation.

E.g., to show symmetry, we let $\Sigma\in|\mathbf{Sign}^\flat|$, $s\in S$ and $\phi,\psi\in\mathrm{SEN}^\flat_s(\Sigma)$, such that $\langle\phi,\psi\rangle\in\rho^{+*,s}_\Sigma(T)$. Then, we have $\rho^{+,s}_\Sigma[\phi,\psi]\le T$ and, thus, for all $\sigma:\mathrm{SEN}^\flat_s\times\prod_{i=1}^k\mathrm{SEN}^\flat_{s_i}\to\mathrm{SEN}^\flat_v$ in $N^\flat$, $\Sigma'\in|\mathbf{Sign}^\flat|$, $f\in\mathbf{Sign}^\flat(\Sigma,\Sigma')$ and $\vec{\chi}\in\prod_{i=1}^k\mathrm{SEN}^\flat_{s_i}(\Sigma')$,

$$\rho^v_{\Sigma'}[\sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\phi),\vec{\chi}),\sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\psi),\vec{\chi})]\le T.$$

This, however, implies that

$$\rho^v_{\Sigma'}[\sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\psi),\vec{\chi}),\sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\phi),\vec{\chi})]\le T.$$

Reversing the steps above, we get that $\langle\psi,\phi\rangle\in\rho^{+*,s}_\Sigma(T)$. Hence, $\rho^{+*}_\Sigma(T)$ is symmetric.

Moreover, it has, by the same considerations, the congruence property. Finally, it is a system by the definition of $\rho^{+*}(T)$. It is compatible with T due to Lemma 1994. ■

Corollary 1996 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally algebraizable via a conjugate pair* $(\tau,\rho) : \mathcal{I} \rightleftarrows \mathcal{I}^{\mathsf{K}}$*, for some class* K *of* $\mathbf{F}$*-algebraic systems, then, for all* $v \in V$*,* $\rho^{+*,v}(T) = \rho^{*,v}(T)$*.*

Proof: Let $v \in V$ and suppose $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi,\psi \in \mathrm{SEN}^\flat_v(\Sigma)$, such that $\langle\phi,\psi\rangle \in \rho^{+*,v}_\Sigma(T)$. Then $\rho^{+,v}_\Sigma[\phi,\psi] \leq T$. But, by definition, $\rho \subseteq \rho^+$, whence, $\rho^v_\Sigma[\phi,\psi] \leq T$, Therefore, $\langle\phi,\psi\rangle \in \rho^{*,v}_\Sigma(T)$.

Suppose, conversely, that $\langle\phi,\psi\rangle \in \rho^{*,v}_\Sigma(T)$. Then $\rho^v_\Sigma[\phi,\psi] \leq T$. But this implies that, for all $\sigma : \mathrm{SEN}^\flat_v \times \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i} \to \mathrm{SEN}^\flat_u$, with $u \in V$, in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i}(\Sigma')$,

$$\rho^u_{\Sigma'}[\sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\phi),\vec{\chi}), \sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\psi),\vec{\chi})] \leq T.$$

Therefore, we conclude that $\rho^{+,v}_\Sigma[\phi,\psi] \leq T$, giving that $\langle\phi,\psi\rangle \in \rho^{+*,v}_\Sigma(T)$. ■

Proposition 1995 allows us to establish that the congruence system $\rho^{+*}(T)$ coincides with the behavioral Leibniz congruence system $\Upsilon(T)$ of T in $\mathcal{I}$.

Theorem 1997 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally algebraizable via a conjugate pair* $(\tau,\rho) : \mathcal{I} \rightleftarrows \mathcal{I}^{\mathsf{K}}$*, for some class* K *of* $\mathbf{F}$*-algebraic systems, then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\rho^{+*}(T) = \Upsilon(T).$$

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $s \in S$ and $\phi,\psi \in \mathrm{SEN}^\flat_s(\Sigma)$, such that $\langle\phi,\psi\rangle \in \Upsilon_\Sigma(T)$. Since $\Upsilon(T)$ is a congruence system, we get, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$, $\langle \mathrm{SEN}^\flat_s(f)(\phi), \mathrm{SEN}^\flat_s(f)(\psi)\rangle \in \Upsilon_{\Sigma'}(T)$. Since $\Upsilon(T)$ is a congruence system, we now get, for all $\sigma^v \in \rho^v$, all $\sigma : \mathrm{SEN}^\flat_s \times \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i} \to \mathrm{SEN}^\flat_v$ in $N^\flat$, and all $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i}(\Sigma')$, $\vec{\xi} \in \prod_{j<\omega} \mathrm{SEN}^\flat_{s_j}(\Sigma')$,

$$\begin{aligned}\langle &\sigma^v_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\phi),\vec{\chi}), \sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\phi),\vec{\chi}),\vec{\xi}),\\ &\sigma^v_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\phi),\vec{\chi}), \sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\psi),\vec{\chi}),\vec{\xi})\rangle \in \Upsilon_{\Sigma'}(T).\end{aligned}$$

On the other hand, we know that $\rho^{+,s}_\Sigma[\phi,\phi] \leq T$, whence, by the compatibility of $\Upsilon(T)$ with T, we get that $\rho^{+,s}_\Sigma[\phi,\psi] \leq T$. Therefore, $\langle\phi,\psi\rangle \in \rho^{+*}_\Sigma(T)$.

Conversely, since, by Proposition 1995, $\rho^{+*}(T)$ is a congruence system on $\mathbf{F}$ that is compatible with T, we get, by the maximality property of the behavioral Leibniz operator, $\rho^{+*}(T) \leq \Upsilon(T)$. ■

Now, we prove that behavioral algebraizability implies both behavioral syntactic protoalgebraicity and behavioral truth equationality.

Theorem 1998 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally algebraizable, then it is both behaviorally syntactically protoalgebraic and behaviorally truth equational.*

Proof: Suppose $\mathcal{I}$ is behaviorally algebraizable via the conjugate pair $(\tau,\rho) : \mathcal{I} \rightleftarrows \mathcal{I}^{\mathsf{K}}$, for some class K of $\mathbf{F}$-algebraic systems.

Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi,\psi \in \mathrm{SEN}^\flat_v(\Sigma)$. Then we have

$$\begin{array}{llll}\langle\phi,\psi\rangle \in \Upsilon_\Sigma(T) & \text{iff} & \langle\phi,\psi\rangle \in \rho^{+*}_\Sigma(T) & \text{(Theorem 1997)}\\ & \text{iff} & \langle\phi,\psi\rangle \in \rho^{*}_\Sigma(T) & \text{(by Corollary 1996)}\\ & \text{iff} & \rho_\Sigma[\phi,\psi] \le T. & \text{(definition of } \rho^*)\end{array}$$

Therefore, $\mathcal{I}$ is behaviorally syntactically protoalgebraic, with witnessing transformations ρ.

Finally, let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$. Then, we have

$$\begin{array}{llll}\phi \in T_\Sigma & \text{iff} & \rho_\Sigma[\tau_\Sigma[\phi]] \le T & ((\tau,\rho) \text{ conjugate pair})\\ & \text{iff} & \tau_\Sigma[\phi] \subseteq \rho^{*}_\Sigma(T) & \text{(definition of } \rho^*)\\ & \text{iff} & \tau_\Sigma[\phi] \subseteq \rho^{+*}_\Sigma(T) & \text{(by Corollary 1996)}\\ & \text{iff} & \tau_\Sigma[\phi] \le \Upsilon(T). & \text{(Theorem 1997)}\end{array}$$

Therefore, $\mathcal{I}$ is behaviorally truth equational, with witnessing transformations τ. ■

We show, next, that, conversely, behavioral syntactic protoalgebraicity and behavioral truth equationality guarantee behavioral algebraizability.

Theorem 1999 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally syntactically protoalgebraic and behaviorally truth equational, then it is behaviorally algebraizable.*

Proof: Suppose that $\mathcal{I}$ is behaviorally syntactically protoalgebraic, with witnessing transformations ρ, and behaviorally truth equational, with witnessing transformations τ. Then, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \bigcup_{v\in V} \mathrm{SEN}^\flat_v(\Sigma)$,

$$\begin{array}{lll}\phi \in C_\Sigma(\Phi) & \text{iff} & \phi \in \bigcap\{T_\Sigma : \Phi \subseteq T_\Sigma\}\\ & \text{iff} & \tau_\Sigma[\phi] \le \bigcap\{\Upsilon(T) : \tau_\Sigma[\Phi] \le \Upsilon(T)\}\\ & \text{iff} & \tau_\Sigma[\phi] \le C^{\mathsf{K}}(\tau_\Sigma[\Phi]).\end{array}$$

Moreover, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and all $\phi,\psi \in \mathrm{SEN}^\flat_v(\Sigma)$,

$$\begin{array}{lll}\langle\phi,\psi\rangle \in \Upsilon_\Sigma(T) & \text{iff} & \rho^v_\Sigma[\phi,\psi] \le T\\ & \text{iff} & \tau[\rho^v_\Sigma[\phi,\psi]] \le \Upsilon(T).\end{array}$$

Hence, we have that $C^{\mathsf{K}}(\phi \approx \psi) = C^{\mathsf{K}}(\tau[\rho^v_\Sigma[\phi,\psi]])$.

We conclude, by Proposition 1961, that $\mathcal{I}$ is equivalent to $\mathcal{I}^{\mathsf{K}}$ and, therefore, $\mathcal{I}$ is behaviorally algebraizable. ■

Now we can formulate the main characterization theorem:

Theorem 2000 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}_s^\flat\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(i) $\mathcal{I}$ *is behaviorally algebraizable;*

(ii) $\mathcal{I}$ *is behaviorally syntactically protoalgebraic and behaviorally truth equational;*

(iii) $\mathcal{I}$ *is behaviorally WF algebraizable (i.e., behaviorally protoalgebraic and behaviorally c-reflective) and has both a Leibniz behavioral reflexive core and an adequate behavioral Suszko core.*

Proof: If $\mathcal{I}$ is behaviorally algebraizable, then, by Theorem 1998, it is both behaviorally syntactically protoalgebraic and behaviorally truth equational. If $\mathcal{I}$ is behaviorally syntactically protoalgebraic and behaviorally truth equational, then, by Theorems 1975 and 1987, it is behaviorally protoalgebraic, behaviorally c-reflective and has both a Leibniz behavioral reflexive core and an adequate behavioral Suszko core. Finally, if $\mathcal{I}$ is behaviorally WF algebraizable, with a Leibniz behavioral reflexive core and an adequate behavioral Suszko core, then, by Theorems 1975 and 1987, it is behaviorally syntactically protoalgebraic and behaviorally truth equational, whence, by Theorem 1999, it is behaviorally algebraizable. ■