Categorical Abstract Algebraic Logic: Hierarchies of π-Institutions

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Chapter 1

Introduction

1.1 Introduction

The field of *algebraic logic* assumed its modern systematic form, known as *abstract algebraic logic*, with the appearance of the pioneering "Memoirs" monograph of Blok and Pigozzi [35]. In this celebrated monograph one can find clearly discernible the seeds and the foundations of almost all subsequent developments in the field and, consequently, also, the foundations on which most parts of the work and of the developments detailed in the present monograph are based.

Related to the term "abstract algebraic logic", another of the pioneers of the field, Josep Maria Font, in a more recent textbook, titled "Abstract Algebraic Logic An Introductory Textbook" [86], advocates that the name should continue to be simply *algebraic logic* and that, as is the case with most other fields of Mathematics, Logic and Science, the abstraction, to which the term "abstract" refers, is part of the natural evolution of the same field, and should not be construed as constituting a special subfield justifying a special naming or rebranding.

In a similar sense, one may share the same belief for *categorical abstract algebraic logic*, which is also another natural evolution of algebraic logic and, therefore, according to this point of view, should also be referred to, simply, as *algebraic logic*. It may, in fact, be preferable to refer to the underlying formalizations of the logical systems treated in each particular context than to rebrand the entire field. So instead of referring to "abstract algebraic logic", we may say "algebraic logic as applied to sentential logics" (or "to deductive systems") and, similarly, "algebraic logic as applied to logics formalized as institutions or π-institutions", instead of using "categorical abstract algebraic logic" for the latter. For now, however, the traditional names have stuck and have been used widely, with well-discernible meanings, and we use them freely, as is also done in [86].

In "traditional" algebraic logic, which may be viewed to have started with the work of Tarski [5], the underlying formalism consists of *sentential logics* or *deductive systems*. These are pairs $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$, where $\mathcal{L}$ is an algebraic language (a set of operation symbols with specified finite arities) and $\vdash_{\mathcal{S}}$ is a *consequence relation* on the absolutely free algebra $\mathbf{Fm}_{\mathcal{L}}(V)$ generated by a countable set V of variables. That is, $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)) \times \mathrm{Fm}_{\mathcal{L}}(V)$, satisfies the following, for all $\Gamma \cup \Delta \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

Inflation: $\Gamma \vdash_{\mathcal{S}} \varphi$, for all $\varphi \in \Gamma$;

Monotonicity: $\Gamma \vdash_{\mathcal{S}} \varphi$ and $\Gamma \subseteq \Delta$ imply $\Delta \vdash_{\mathcal{S}} \varphi$;

Idempotency: $\Gamma \vdash_{\mathcal{S}} \varphi$ and $\Delta \vdash_{\mathcal{S}} \gamma$, for all $\gamma \in \Gamma$, imply, $\Delta \vdash_{\mathcal{S}} \varphi$;

Structurality: $\Gamma \vdash_{\mathcal{S}} \varphi$ implies $\sigma(\Gamma) \vdash_{\mathcal{S}} \sigma(\varphi)$, for all endomorphisms $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$.

Equivalently, $\mathcal{S}$ may be expressed in terms of a *structural closure operator* $C_{\mathcal{S}}$, i.e., a function $C_{\mathcal{S}} : \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)) \to \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V))$, satisfying, for all $\Gamma \cup \Delta \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$:

Inflation: $\Gamma \subseteq C_{\mathcal{S}}(\Gamma)$;

Monotonicity: $C_{\mathcal{S}}(\Gamma) \subseteq C_{\mathcal{S}}(\Delta)$, for all $\Gamma \subseteq \Delta$;

Idempotency: $C_{\mathcal{S}}(C_{\mathcal{S}}(\Gamma)) \subseteq C_{\mathcal{S}}(\Gamma)$;

Structurality: $\sigma(C_{\mathcal{S}}(\Gamma)) \subseteq C_{\mathcal{S}}(\sigma(\Gamma))$, for all endomorphisms $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$.

The equivalence is established by setting, on the one hand, for all $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

$$C_{\mathcal{S}}(\Gamma) = \{\varphi \in \mathrm{Fm}_{\mathcal{L}}(V) : \Gamma \vdash_{\mathcal{S}} \varphi\},$$

and, on the other, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Gamma \vdash_{\mathcal{S}} \varphi \quad \text{iff} \quad \phi \in C_{\mathcal{S}}(\Gamma).$$

The reliance on sentential logics as the underlying formalism of the theory persists when passing to abstract algebraic logic. The reader is referred to the aforementioned [35, 86], as well as to the standard reference [64] by Janusz Czelakowski, another pioneer in the field, all clearly showcasing the primary role of this framework in all related developments and investigations.

By contrast, in this monograph the underlying logical formalism consists of π-institutions [33]. This formalism encompasses systems with varying signatures and quantifiers in a more direct way than allowed by the formalism of sentential logics (see Appendix C of [35], as well as the work on cylindric [15, 27] and polyadic algebras [9] and related work at the institutional level [100, 101, 102, 103] based and/or closely related to these). The structure of a π-institution forms a modification of the structure of an institution [25, 41], which was introduced in computer science to formalize logical systems for specification and programming, based on semantics. Diaconescu's monograph [79] offers a comprehensive advanced study of institutions and presents a multitude of model theoretic results that can be abstracted from first-order, and other specific logical systems, to the institutional level. On the other hand, in π-institutions, the framework is stripped of the semantic, or model theoretic, aspects and the focus is on the syntax, thus recovering the essential features of the sentential logic framework, without, however, shedding the versatility afforded, and the advantage gained, by incorporating in the object language multiple signatures and signature-changing morphisms. In fact, this inclusion is what gives the area its distinctive and unique character inside (abstract) algebraic logic. This is apparent in all aspects of our studies.

To make clearer the exact relationship between sentential logics and π-institutions, and showcase the fact that the former constitute very narrow special cases of the latter, let us recall the definition of a π-institution. A *π-institution*, as originally defined in [33], is a triple $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, where

- $\mathbf{Sign}$ is an arbitrary category, whose objects are called *signatures* and its morphisms *signature morphisms*;
- $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is a functor giving, for each signature $\Sigma \in |\mathbf{Sign}|$, the set $\mathrm{SEN}(\Sigma)$ of Σ-*sentences*;
- For every $\Sigma \in |\mathbf{Sign}|$, $C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \to \mathcal{P}(\mathrm{SEN}(\Sigma))$ is a closure operator, such that the collection $C = \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ satisfies the property of *structurality*, i.e., for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)$,
$$\mathrm{SEN}(f)(C_\Sigma(\Phi)) \subseteq C_{\Sigma'}(\mathrm{SEN}(f)(\Phi)).$$

In the modified (enriched) form that is used in the present monograph, and which was (essentially) introduced in [106], there is an additional component N, which represents a *category of natural transformations* on the sentence functor SEN. Roughly speaking, this category corresponds to clones of algebraic operations on $\{\mathrm{SEN}(\Sigma) : \Sigma \in |\mathbf{Sign}|\}$, under the assumption that all operations are defined uniformly and naturally over all $\mathrm{SEN}(\Sigma)$, for $\Sigma \in |\mathbf{Sign}|$. This accords in style with the algebraic theories of Lawvere [10], which are closely related to the Eilenberg-Moore [11] and the Kleisli [12] constructions. For more details on these, one may consult the classic texts by Mac Lane [16], Pareigis [14], Borceux [45] and Barr and Wells [57]. Thus, we are studying logical systems formalized as quadruples $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, N, C\rangle$, which are further recast as pairs
$$\mathcal{I} = \langle \mathbf{F}, C\rangle,$$
where

- $\mathbf{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ expresses the algebraic structure, corresponding to the absolutely free algebra in the case of deductive systems, and
- $C = \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ is a family of closure operators, satisfying structurality, which is referred to as a *closure system*, and corresponds to the closure $C_{\mathcal{S}}$ in the case of sentential logics.

Suppose now that $\mathcal{S} = \langle \mathcal{L}, C_{\mathcal{S}}\rangle$ is a sentential logic. The standard rendering of it as a π-institution
$$\mathcal{I}^{\mathcal{S}} = \langle \mathbf{F}^{\mathcal{L}}, C^{\mathcal{S}}\rangle,$$
with $\mathbf{F}^{\mathcal{L}} = \langle \mathbf{Sign}^{\mathcal{L}}, \mathrm{SEN}^{\mathcal{L}}, N^{\mathcal{L}}\rangle$, is given by defining the four components as follows:

- $\mathbf{Sign}^{\mathcal{L}}$ is a trivial category, with object, say, V;

- $\mathrm{SEN}^{\mathcal{L}} : \mathbf{Sign}^{\mathcal{L}} \to \mathbf{Set}$ is given by $\mathrm{SEN}^{\mathcal{L}}(V) = \mathrm{Fm}_{\mathcal{L}}(V)$;
- $N^{\mathcal{L}}$ is the clone of all $\mathcal{L}$-operations on $\mathrm{Fm}_{\mathcal{L}}(V)$;
- $C_V^{\mathcal{S}} = C_{\mathcal{S}} : \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)) \to \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V))$.

It is worth noting that $\mathbf{F}^{\mathcal{L}}$ only depends on $\mathcal{L}$ and V, as was to be expected (since it was deemed to correspond to the algebraic structure), and the deductive apparatus is reflected entirely in the definition of $C^{\mathcal{S}}$. Moreover, the formalism on the logical side does not incorporate substitutions in the object language, even though, since $C_V^{\mathcal{S}} = C_{\mathcal{S}}$ and the latter is structural, we have, for every endomorphism $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$,

$$\sigma(C_V^{\mathcal{S}}(\Phi)) \subseteq C_V^{\mathcal{S}}(\sigma(\Phi)),$$

for all $\Phi \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$. On the algebraic side, on the other hand, e.g., when congruences are to be determined, the inclusion of the clone $N^{\mathcal{L}}$, reflecting the algebraic $\mathcal{L}$-structure, forces congruences at the institutional level to exactly correspond to the familiar $\mathcal{L}$-congruences on the formula algebra in the universal algebraic sense.

The reasons why one might want to develop a theory of algebraization for logical systems formalized as institutions or π-institutions parallel the motivations provided by Blok and Pigozzi [35] for developing a theory of algebraizability for sentential logics.

One of the main motivations is providing a classification of logical systems based on the strength of the ties of their deductive apparatuses with those corresponding to algebraic deductive systems, i.e., deductive systems whose closure systems are induced by algebraic structures. Preferably, when the definitions applicable in the context of logical systems formalized as π-institutions specialize in the way outlined above to π-institutions associated with deductive systems, one would be able to recover the well-known algebraic (or Leibniz) hierarchy of abstract algebraic logic [64, 86]. The finitary and finitely algebraizable sentential logics of [35] form a special class in this hierarchy. In [86], this property is termed *Blok-Pigozzi algebraizability* (see Definition 3.39 of [86]).

Another desideratum is that the definitions should be as general as possible so that, given virtually any π-institution, one would be able, at least in principle, to classify it in one or more of the classes of the hierarchy, based on the strength of its algebraic properties.

Further, an additional reassurance would be provided if the definitions supplied turned out to be robust in the sense that one would be able to obtain, at least for several, if not for most, of them, different characterizations depending on the various viewpoints taken. This was clearly and successfully undertaken in [35] for the class of algebraizable deductive systems. In fact, Blok and Pigozzi obtained several different characterizations whose variety and strength played a major role in convincing other researchers that their definitions were chosen wisely and, as a result, in establishing firmly the new trends in the field and, thus, contributing, in large part, to virtually all subsequent developments. It is hoped that pursuits along the same lines here will prove, at least moderately, successful with respect to similar criteria. In particular, it is hoped that the characterizations of many of the classes presented in this monograph in a variety of ways will prove to many of the readers and to, present and future, researchers in the field satisfactory and motivating, as was the case with the work of Blok and Pigozzi [35].

One last motivation, equally important, however, in significance, comes by taking an adversarial point of view. As Blok and Pigozzi realized when studying sentential logics, and is certainly true also for logics formalized as π-institutions, since they encompass sentential logics, is the fact that many logical systems of historical and/or practical significance failed to be amenable to classical methods of algebraization, such as, e.g., the Lindenbaum-Tarski process. Naturally one is inclined to ask whether those systems can be algebraized in some alternative way, using different techniques, or whether the failure in their algebraization is due to intrinsic reasons. That is, one would like to investigate whether those systems have some innate characteristics, e.g., pertaining to their structural properties, that many, if not all, of them share and that decide their algebraizability status. This is reminiscent of the extensive and intensive research in computational complexity theory in separating various complexity classes [94, 92, 95, 93, 96, 91], where common features and rigorous criteria are sought for classification of problems in hierarchies of complexity classes. As is the case there, such an analysis and rigorous classification presupposes the existence of a formal definition of algebraizability (and of other related properties) so as to delineate formal boundaries and establish criteria that could potentially be used to falsify claims of algebraizability for some logical systems. Such criteria would point to shortcomings and defects of some logical systems as related to qualitative requirements that a logic should satisfy in order to qualify for membership in a corresponding class. It is believed that the definitions adopted here are helpful in establishing such criteria and in setting up boundaries. The examples that are scattered throughout seem to support this assertion, but, of course, the jury is out as far as gathering further evidence in support of, or in criticism and opposition to, this claim.

The notion of algebraizability adopted in this monograph is inspired by the one established for deductive systems by Blok and Pigozzi in [35]. Apart from the technical complications inherent in passing from the sentential to the institutional framework, one substantial difference is that we distinguish between a treatment based on the Leibniz operator, referred to as **semantic**, as contrasted to the one based on interpretations from logic to algebra and vice-versa, which is termed **syntactic**, since it is based on natural transformations corresponding to term operations on the free algebra of terms. In the sentential logic framework, such a distinction is only apparent, since, as it turns out, the two approaches are equivalent and, hence, interchangeable. On the other hand, in π-institutions, the added flexibility afforded in the relation between morphisms (which are treated in the object language in the category of signatures) and clone operations (also part of the framework, but added a posteriori to enhance the algebraic character of the intended studies) means that the syntactic concepts dominate (i.e., are, in general, stronger) than their corresponding semantic counterparts.

The role that theories play in sentential logics is subsumed here by theory families, which consist of deductively closed sets of sentences, one for each signature. They form a complete lattice $\mathbf{ThFam}(\mathcal{I}) = \langle \mathrm{ThFam}(\mathcal{I}), \leq \rangle$, when ordered by signature-wise inclusion $\leq$. To each theory family is associated a congruence system, a collection of equivalence relations on formulas, one for each signature, that satisfy both the congruence property (or substitution property) and invariance under signature morphisms. These also form a complete lattice under signature-wise inclusions, which is denoted by $\mathbf{ConSys}(\mathcal{I}) = \langle \mathrm{ConSys}(\mathcal{I}), \leq \rangle$. The congruence system selected is the largest one compatible with the given theory family and is termed, by analogy with the sentential logic framework, the *Leibniz congruence system* associated with the theory family.

Starting from **semantics**, we say that a π-institution is *algebraizable* if it satisfies two conditions that impose very intimate ties between the lattice of theory families of the π-institution and that of the congruence systems determined by a class of algebraic systems. The first condition is that the Leibniz operator is monotone on theory families. The second is that it is order-reflecting.

On the **syntactic** side, a π-institution is *algebraizable* if, on the one hand, the Leibniz congruence systems are definable via a collection of natural transformations in two arguments and, on the other, if the theory families are definable via a collection of natural transformations in a single argument. In general, parametric arguments are allowed and, by restricting those, we obtain potentially narrower classes.

One of the main theorems established by Blok and Pigozzi in [35] is the characterization of algebraizable sentential logics via the existence of an isomorphism between the theory lattice of the deductive system and the equational theory lattice associated to a class of algebras, which also commutes with substitutions. A characterization along similar lines is established here for logical systems formalized as π-institutions (see, e.g., Section 4.3 or Section 12.3, even though other related forms appear in other places in the monograph, as will be discussed in the overview). In the literature several forms of this theorem and a host of generalizations of increasing power (or generality) have been discussed. A sample list includes [73, 35, 40, 99, 75, 81, 88]. The majority of these deal with deductive equivalence of logical systems, and related lattice-theoretic algebraic structures. They encompass the characterization of algebraizability mentioned above and deal with the case in which mutual interpretations between logical structures induce isomorphisms between lattices of theories and vice versa, under some constraints and special hypotheses, depending on the context under consideration.

Another major characterization theorem provided in [35] for the notion of algebraizability asserts that, roughly speaking, in the context of sentential logics, the aforementioned analogs of the semantic and of the syntactic notions are equivalent. That is, the algebraization attained via the definability of theories and congruences via sets of equations and formulas, respectively, coincides with that ensured by the Leibniz operator being monotone and order reflecting on the lattice of the theories of the logic. This characterization, when abstracted to logics formalized as π-institutions, continues to hold under special provisos, namely, under the hypotheses that the π-institution under consideration has a rich enough supply of natural transformations or, more formally, as will be studied in detail in the monograph, that it has a Leibniz binary reflexive core and an adequate Suszko core.

In [35] as well as in many other works in the field, a considerable amount of emphasis has been placed on, and a substantial amount of effort expended in, studying specific logical systems of historical and/or practical interest from the point of view of algebraizability. This was only natural, given, on the one hand, the desire to showcase the applicability of the theory on logics of particular interest in traditional studies, and, on the other, the urge to investigate the power of falsifiability that the theory provides for those concrete logical systems that had resisted previous attempts at algebraization.

Our point of view, however, is slightly different and, as a consequence, we do not deal with or present such examples. Firstly, the majority of logical systems of historical and/or practical interest have already been dealt with in existing literature. Secondly, since our treatment abstracts and subsumes that of sentential logics and, considerably generalizes it, as was shown above, our goal is not to look at the more concrete, already encompassed by the study of the algebraization of deductive systems, but, rather, to look into the more abstract and discern what can be carried over to that level and how its validity and its applicability compares when applied to new systems and new examples which do not fit exactly, or do not conform at all to, the sentential logic framework. However, these aims and the mode of treatment they motivate should in no way be construed as underestimating the significance, or underplaying the beauty and elegance, of the studies concerned with the concrete and the more specific. After all, it is on those studies that the abstract is based, to those studies' insights, ideas and methodology that an enormous scientific debt is due, and from those studies' successes, and widespread recognition and appreciation, that a relative confidence is drawn regarding the potential usefulness and applicability of the more general framework presented, and elaborated on, in this monograph.

1.2 Fin de siècle: The Golden Age

We give an account of some of the major developments in abstract algebraic logic that occurred mostly, but not exclusively, around the last two decades of the 20th century. This period may be thought of as constituting the golden age of algebraic logic, in the sense that, during this time, there is clearly discernible a passage from an ad-hoc, case-by-case algebraic treatment of logical systems to a well-organized field, with a powerful arsenal of universally applicable concepts, methods and techniques, culminating to the classification of logics in an algebraic hierarchy, known as the Leibniz hierarchy. Needless to say, the foundations for this success were laid much earlier. Likewise, the development continued, and many important results around, and complementing, the main theory were obtained later, into the new millennium, and the area continues to be active. In order to avoid, in our short exposition, reinventing the wheel, we base this account on preexisting sources. We draw the material primarily from the, perhaps best-known, survey of the field by Font, Jansana and Pigozzi [69] and, when needed and/or convenient, the two existing specialized books on the subject by Czelakowski [64] and by Font [86].

Algebraic logic has its origins in the work of George Boole [1, 2], who formalized the "laws of thought" in an algebraic way. The intuition governing this process was made mathematically precise by Tarski [5, 6, 8]. Tarski used the key idea of Lindenbaum of identifying formulas of a logical language with the terms of the absolutely free algebra formed using the logical connectives as operation symbols [3] to give a precise connection between classical propositional calculus and Boolean algebras. This formed the paradigmatic example from which significant inspiration was drawn and on which subsequent developments were based. Furthermore, it served as a kind of testbed for comparing, trying, modifying and calibrating new ideas, methods and techniques. The way Boolean algebras arose as the algebraic counterparts of classical propositional calculus has become known as the Lindenbaum-Tarski method. It has subsequently been used to "algebraize" a variety of propositional systems.

A conceptual shift occurred around 1950 when Rasiowa and Sikorski [7, 20] (see, also, the historical surveys [59, 74]), among others, realized that the Lindenbaum-Tarski method could be applied not only to isolated logics but, rather, to classes of logical systems with an implication connective satisfying certain properties. In passing from a "per logic" or "a la carte" treatment to one addressing classes specified by some abstract properties, one discerns clearly for the first time the seeds of what, later, became known as "abstract algebraic logic". Papers that may be thought of as protoabstract, in the sense that they advance further the main ideas of Rasiowa and Sikorski towards the modern truly abstract era, were the one by Prucnal and Wronski [19] introducing equivalential logics, the ones by Czelakowski introducing protoalgebraic logics [26, 29] and further studying equivalential logics [23, 24] and the one by Blok and Pigozzi [28] studying protoalgebraic logics.

The seismic shift, one might say, in firmly founding and establishing the modern era came in the 1980s with the work of Blok and Pigozzi, which led eventually to the publication of their famous, seminal "Memoirs" monograph [35]. In a way analogous to the preceding three passages, from classical logic and Boolean algebra to the Lindenbaum-Tarski method, from the Lindenbaum-Tarski method to dealing with implicative logics and from implicative logics to abstract properties of deduction, Blok and Pigozzi were able to distil the essential spirit of the association between logic and algebra and, thus, extract and formalize the concept of an algebraizable logic in modern abstract terms and provide landmark characterizations. On the way, they established a very general process of algebraization, applicable to arbitrary logical systems, which has been, since, further refined and used to create the Leibniz hierarchy, often considered the pinnacle - certainly a milestone and a gem - of algebraic logic in general.

Before returning to provide a more detailed account, we take a small break to recount those features of the theory that distinguish the abstract approach from the more traditional treatments and give it its special character. First, as alluded to previously, instead of applying the Lindenbaum-Tarski process in an ad-hoc way, on a case-by-case basis, or, as in Rasiowa's work, to a class of logics sharing a specific connective satisfying certain properties, it applies the abstract process to arbitrary sentential logics and, according to the outcome, classifies them into classes reflecting the closeness of the ties between them and the corresponding algebraic counterparts. In establishing this association and performing the resulting classification, it opens, in parallel, two distinct but closely interrelated directions. On the one hand, it motivates the study of classes of algebras arising as algebraic counterparts of either single or groups of logical systems. On the other, it allows investigating the exact correspondence between metalogical properties of the logical systems at hand and algebraic properties of the classes of their algebraic counterparts.

By now a plethora of works falling distinctly in each of these three directions exist and many will appear as references in the more detailed account that will follow. But to give some indication and pointers, we mention a few of the earliest ones that may be viewed as ground breaking. Concerning the process of algebraization itself and the classification, one should mention Blok and Pigozzi's [28, 35], Czelakowski's [23, 24], Herrmann's [43, 53, 54] and Font and Jansana's [52]. Concerning the study of classes of algebraic counterparts arising from the abstract algebraization process, one should mention [38, 39] dealing with the conjunction-disjunction fragment of classical propositional calculus, as well as Jansana's study of selfextensional logics in [71, 76], with clear precedents in Font and Jansana's [52]. Finally, paradigmatic examples of the study of metalogical and corresponding algebraic properties constitute several works addressing forms of the deduction-detachment theorem, e.g., Czelakowski's [26, 29] and Blok and Pigozzi's [32, 37, 63], the work of Blok and Hoogland on the Beth poperty [72], as well as the work of Czelakowski and Pigozzi concerning interpolation and amalgamation properties [58].

1.3 Outline of Contents by Chapter

We give an outline of the contents of the monograph focusing on the main points of each chapter and describing them by section, using some formal notation, but without providing formal definitions, which will be presented in the main body of the text. This section is very closely related to other sections. First, in Section 1.4, we give a very concise summary, only mentioning the main overarching topics discussed in each chapter. Second, at the beginning of each chapter, a similar overview is provided focusing only on the specific chapter, with the exception that, in those introductions, being closer to the formal treatment, an even more informal narrative is adopted and a concerted effort is made to keep notation at a minimum.

1.3.1 Chapter 2

Chapter 2 presents the basic elements of the theory of algebraic systems, of π-institutions and of the interaction between logical and algebraic structures. These constitute the foundations and form the backbone of our theory throughout the monograph.

Section 2.1 gives an informal introduction to the chapter, akin to the introduction presented here, only containing a little less of formal notation and being more on the narrative, informal, side.

Section 2.2 is the first main section of the chapter. Here, we start by introducing the notion of a *sentence functor* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, which is simply a set-valued functor on an arbitrary category of signatures. It formalizes the carriers on which both algebras and logical systems are based, akin to the underlying universe of a universal algebra. Then we consider *sentence families* of sentence functors, which are families $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ of subsets of sentences, one for each signature. These formalize distinguished sets of sentences when one considers logical structures, much like the distinguished sets in logical matrices. A *sentence system* is a sentence family T which is invariant under the action of signature morphisms. Two canonical ways of obtaining from a given sentence family T a sentence system consist of taking the largest sentence system $\overleftarrow{T}$ included in the family T and taking the smallest sentence system $\overrightarrow{T}$ that includes the sentence family T. Sentence functors are related via *sentence morphisms*, which are pairs $\langle F, \alpha \rangle$, F being a functor between the categories of signatures and α a natural transformation mapping sentences to sentences, taking into account the effect of

F. *Special morphisms* are those with surjective and full signature components and *surjective* ones are special ones whose sentence components are also surjective.

We then turn to *relation families* $R = \{R_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ over sentence functors. Those assume the place of binary relations. Of the highest interest and importance are *equivalence families* and *equivalence systems*, i.e., equivalence families invariant under the action of signature morphisms. They induce partitions on the components of sentence functors. Equivalence families and systems have important interactions and connections with both sentence families and with morphisms. The notion that relates an equivalence family with a sentence family is that of *compatibility*. An equivalence family R is *compatible* with a sentence family T if each component of the sentence family is a union of blocks of the equivalence family on the same component. The connection between equivalence systems and morphisms goes through the notion of kernels. Namely, the *kernel* $\mathrm{Ker}(\langle F,\alpha\rangle)$ of a morphism $\langle F,\alpha\rangle$ between two sentence functors forms an equivalence system on the domain.

If a set is equipped with operations, we get an algebraic structure. On this algebraic structure, one may reason in an algebraic way about any of the operations that are in its clone, i.e., that can be generated by applying the fundamental operations and the projections and composing them in arbitrary ways. In an analogous fashion, if a sentence functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is equipped with a category of natural transformations N, which corresponds to the clone of algebraic operations on an algebra, one obtains an *algebraic system* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$. As algebras play a fundamental role in both logical and algebraic aspects of the traditional theory, so do algebraic systems in the theory developed in the monograph. The role of free algebra is played in this context by that of a *base algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$. Moreover, the notion of morphism extends from the context of sentence functors to the context of algebraic systems. The additional stipulation is that they also preserve the algebraic structure that turns the sentence functor into an algebraic system, i.e., that they satisfy the well-known *replacement* or *congruence condition*.

In traditional treatments, in specific contexts, all algebras are considered to be over the same algebraic signature, which is fully captured by the absolutely free algebra over that signature. In the present context, this similarity is captured by fixing a base algebraic system $\mathbf{F}$, as above, and considering only $\mathbf{F}$-*algebraic systems*, which are algebraic systems that, roughly speaking, have similar clones of operations with $\mathbf{F}$ and whose sentences are all images of sentences of $\mathbf{F}$ under a fixed algebraic system morphism $\langle F,\alpha\rangle$. Formally, these are expressed as pairs $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, where $\langle F,\alpha\rangle : \mathbf{F} \to \mathbf{A}$ is a surjective algebraic system morphism. The notion of morphism extends further to morphisms between $\mathbf{F}$-algebraic systems.

In Section 2.3, the limelight falls on *congruence systems*, which play in this context the same role that congruences play in the context of universal algebras. The least congruence system on an algebraic system $\mathbf{A}$ is the identity congruence system $\Delta^{\mathbf{A}}$ and the largest one is the full relation system, written $\nabla^{\mathbf{A}}$. These form the min and max elements, respectively, of the complete lattice of congruence systems $\mathbf{ConSys(A)}$ on $\mathbf{A}$. The kernel $\mathrm{Ker}(\langle F, \alpha\rangle)$ of a morphism $\langle F, \alpha\rangle : \mathbf{A} \to \mathbf{B}$ between two algebraic systems forms a congruence system on $\mathbf{A}$. Moreover, congruence systems allow the definition of *quotient algebraic systems*. And, for every algebraic system $\mathbf{A}$ and every one of its quotient systems $\mathbf{A}^\theta := \mathbf{A}/\theta$, there is a canonical morphism $\langle I, \pi^\theta\rangle : \mathbf{A} \to \mathbf{A}^\theta$ onto the quotient algebraic system, whose kernel is exactly the congruence system θ that gave rise to the quotient. All these properties reflect well known properties from the context of congruences and quotients of universal algebras.

Congruence systems inherit from equivalence families the relation of compatibility with given sentence families. The critical property to be established is that for a given sentence family T on an algebraic system $\mathbf{A}$, there exists a largest congruence system on $\mathbf{A}$ that is compatible with T. This is called the *Leibniz congruence system of* T *on* $\mathbf{A}$, is denoted by $\Omega^{\mathbf{A}}(T)$ and plays the role that Leibniz congruences play in the context of traditional abstract algebraic logic. As such, its role in characterizing many of the classes in the algebraic hierarchy studied in the monograph is ubiquitous and, as a consequence, the whole hierarchy is known as the *Leibniz hierarchy*. After introducing the *Leibniz operator* on an algebraic system, we establish two important results concerning it. The first, inspired by a result from the traditional treatment, provides a characterization of the Leibniz operator in terms of the category of natural transformations (i.e., clone operations) of the algebraic system and the sentence family. Roughly speaking it asserts that a pair of sentences are Leibniz related if and only if they are indistinguishable modulo T with respect to the available algebraic apparatus. The second addresses specifically the categorical framework and asserts that the Leibniz congruence system of a sentence family T is dominated by the Leibniz congruence system of the largest sentence system $\overleftarrow{T}$ contained in the sentence family, i.e., that $\Omega^{\mathbf{A}}(T) \leq \Omega^{\mathbf{A}}(\overleftarrow{T})$. The value of this observation in establishing refinements of the traditional hierarchy, as reflected in the present context, is critical and hard to overestimate. Also of importance is the fact that the surjective morphisms between algebraic systems, which form the focus of our work, respect Leibniz congruence systems, in the sense that, if $\langle F, \alpha\rangle : \mathbf{A} \to \mathbf{B}$ is a surjective morphism and T is a sentence family on $\mathbf{B}$, then $\Omega^{\mathbf{A}}(\alpha^{-1}(T)) = \alpha^{-1}(\Omega^{\mathbf{B}}(T))$. Finally, it is worth noting that, in general, the intersection of the Leibniz congruence systems of a collection of sentence families is contained in the Leibniz congruence of the intersection of those sentence families. Significantly, though, the reverse inclusion holds universally on sentence families if and only if the Leibniz operator is monotone on sentence families, a property that does not always hold. In fact, the latter property is used in a critical way, when restricted to special kinds of sentence families, to determine some of the most important classes of logical systems located close to the bottom of the algebraic hierarchy. In addition, it is of great historical significance in many of the most important classical developments in the field.

In Section 2.4, we focus on *congruence systems relative to given classes of algebraic systems*. Given a class K of algebraic systems, all over the same base algebraic system, that is, possessing, in some sense, the same algebraic signature, a congruence system θ on an algebraic system $\mathbf{A}$, not necessarily belonging to K, is called a *congruence system relative to* K or a K-*congruence system* if the quotient $\mathbf{A}^\theta$ belongs to the class K. Naturally, if $\mathbf{A} \in \mathsf{K}$ and the class K happens to be closed under morphic images, then congruence systems relative to K coincide with arbitrary congruence systems. The section introduces another important notion in this context. That of an algebraic system $\mathbf{A}$ being a *subdirect intersection* of a collection of algebraic systems. This means that there exists surjective morphisms $\langle H^i, \gamma^i\rangle : \mathbf{A} \to \mathbf{A}^i$ from the algebraic system to each of the algebraic systems in the given collection and, moreover, the intersection of the kernels of those morphisms is the identity congruence on $\mathbf{A}$. Closure of a class of algebraic systems under subdirect intersections ensures that the collection of congruence systems relative to the class is closed under intersections. Additionally, if the class K contains a trivial algebraic system, then the nabla congruence system happens to be a relative congruence system. Therefore, possession of a trivial algebraic system together with closure under subdirect intersections ensures that the collection of all congruence systems relative to the class forms a complete lattice under signature-wise inclusion.

Suppose that the class K contains a trivial algebraic system and is closed under subdirect intersections so that it makes sense to associate with a given relation family X on its base algebraic system the least congruence system $\Theta^{\mathsf{K}}(X)$ relative to K containing X. An alternative, equally natural, way to associate a congruence system with X is to consider the closure $D^{\mathsf{K}}(X)$ under equational consequence relative to the algebraic systems in the class K. It is proven in this section that the two closures give rise to the same congruence system on the base algebraic system $\mathbf{F}$.

In Section 2.5, we study *varieties of algebraic systems*. There are two possibilities in adopting a choice for the entities that would play the role of equations in this context. The first is to view pairs of sentences as equations. The second is to adopt pairs of natural transformations in the clone as equations. The ones of the latter type are called *natural equations* to differentiate them from those of the former kind which are simply referred to as *equations*. We define formally the notion of *satisfiability* of a given equation and of a given natural equation in an algebraic system and that of validity of a natural equation in an algebraic system. Depending on whether we use equations or natural equations to determine a class of algebraic systems through satisfiability, we obtain two different kinds of varieties. Varieties determined by families of equations are called *semantic varieties*. Those determined by collections of natural equations are called *syntactic varieties*. It turns out that, in general, every syntactic variety is also a semantic variety. The opposite implication does not hold in general. The section concludes by presenting a sufficient condition on the structure of a base algebraic system that ensures that the classes of semantic and syntactic varieties over it coincide.

Much of the work in the first sections of Chapter 1 focuses on the algebraic framework that underlies both the logical and the algebraic aspects of the theory in the monograph. In Section 2.6, we turn to the study of π-*institutions*, the underlying structure of the logical aspects of our theory. The entire monograph assumes that all logical systems are formalized as π-institutions and its main goal is to study the process of their algebraization and to detail the various classes in the hierarchy that is formed by examining their algebraic character. It is needless, thus, to point out the importance of Section 2.6, as it presents the foundational aspects of the logical side of the theory.

We start, here, by defining the notion of π-*institution*. It is a pair $\mathcal{I} = \langle \mathbf{F}, C \rangle$ consisting of a base algebraic system $\mathbf{F}$ and a closure system C on the sentence functor of $\mathbf{F}$. It generalizes the Tarskian concept of a deductive system in that it allows multiple signatures and accommodates morphisms between signatures. To take into account the logical structure imposed on top of the underlying algebraic structure in this context, sentence families and systems are subsumed by *theory families* and *theory systems*. These are sentence families (systems, respectively) each of whose components is closed under logical deduction. The least among these is called the *theorem system* of $\mathcal{I}$. It turns out that, due to the property of structurality, which is key in the study of π-institutions, given a theory family T, $\overleftarrow{T}$ is also closed under deduction, whence it forms that largest theory system included in T. On the other hand, $\overrightarrow{T}$ fails to be closed under deduction in general. That is the reason why the smallest theory system including T is not simply $\overrightarrow{T}$ but, rather, $C(\overrightarrow{T})$.

An important derived concept is that of the π-institution that has as its theory families those theory families of $\mathcal{I} = \langle \mathbf{F}, C \rangle$ which include a given theory system T of $\mathcal{I}$. This is denoted by $\mathcal{I}^T = \langle \mathbf{F}, C^T \rangle$. The construction results in a π-institution whose theorem system is identical with the theory system T of $\mathcal{I}$.

As is the case in most mathematical contexts, objects are accompanied by morphisms between them that preserve the structure of interest in each particular context. *Morphisms between* π-*institutions* are algebraic morphisms between the underlying algebraic systems that, in addition, preserve the logical structure in the sense that the forward image of the logical closure of a set of sentences is included in the closure of the image of the same set of sentences. Among the most useful characterizations is that a given algebraic morphism is logical if and only if preimages of theory families of the target institution under the morphism constitute theory families of the domain π-institution.

In Section 2.7, we turn to those structures that are intermediate between logic and algebra and facilitate the interplay and the establishment of meaningful ties between the two domains. These are *matrix families*, which correspond to the ordinary logical matrices in the traditional treatment. Roughly speaking a *matrix family* $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ consists of an algebraic system $\mathcal{A}$ together with a sentence family T of the algebraic system. If the sentence family is a system, i.e., invariant under signature morphisms, then the matrix family is called a *matrix system*. Their role is twofold. On the one hand, a given collection of matrix families M, over a base algebraic system $\mathbf{F}$, may be used to define a closure system C^{M}, and hence a π-institution structure $\mathcal{I}^{\mathsf{M}} = \langle \mathbf{F}, C^{\mathsf{M}} \rangle$, on $\mathbf{F}$. On the other, given a π-institution structure $\mathcal{I}$ on $\mathbf{F}$, we may define the class $\mathrm{MatFam}(\mathcal{I})$ of all matrix families whose sentence families are closed under the deductive apparatus of the π-institution. Such sentence families are termed *$\mathcal{I}$-filter families* and, if they happen to be systems, then they are called *$\mathcal{I}$-filter systems*. The collection $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ of all filter families over the same underlying algebraic system $\mathcal{A}$, ordered by component-wise inclusion, forms a complete lattice and the collection of all filter systems on that same algebraic system forms a complete sublattice of the complete lattice of all filter families.

Among the main results presented in this section are the ones relating morphisms between algebraic systems with preservation of filter families. More precisely, the inverse image of a filter family is a filter family. The situation is more complicated when it comes to direct images. First of all, it only makes sense to define the direct image of a filter family in case the signature functor is an isomorphism. Second, it turns out that, in that case, for the image to also be a filter family on the target algebraic system, we must require additional restrictions. A sufficient condition is that the kernel system of the algebraic morphism be compatible with the filter family in the domain.

This result has particular consequences for the most important type of morphisms considered in the monograph, the canonical quotient morphisms associated with congruence systems on an algebraic system. It asserts that, given a filter family T on the quotient $\mathcal{A}^\theta$, the inverse image $\pi^{\theta^{-1}}(T)$ under the quotient morphism $\langle I, \pi^\theta \rangle : \mathcal{A} \to \mathcal{A}^\theta$ is a filter family on $\mathcal{A}$ and that, moreover, if the congruence system θ is compatible with a filter family T on $\mathcal{A}$, then the quotient T/θ is a filter family on $\mathcal{A}^\theta$.

We consider, by particularizing even further, the Leibniz quotient morphisms, which are those morphisms defined using the Leibniz congruence system that is compatible with a given filter family on the domain. Since, by definition, the Leibniz congruence system $\Omega^{\mathcal{A}}(T)$ associated with a given sentence family T is compatible with that sentence family, it follows that a filter family T on $\mathcal{A}$ gives rise, by passing to the Leibniz quotient $\mathcal{A}/\Omega^{\mathcal{A}}(T)$, to a filter family in the quotient. The corresponding matrix family $\mathfrak{A}/\theta = \langle\mathcal{A}/\Omega^{\mathcal{A}}(T), T/\Omega^{\mathcal{A}}(T)\rangle$ is called a (*Leibniz*) *reduced matrix family*.

The section closes by defining two classes of matrix families and two classes of algebraic systems that play a key role when investigating the algebraic nature of a given π-institution $\mathcal{I}$. The first is the class $\mathrm{MatFam}^*(\mathcal{I})$ of all *Leibniz reduced matrix families* associated with the given π-institution. The second is the class $\mathrm{MatSys}^*(\mathcal{I})$ of all *Leibniz reduced matrix systems*. Finally, on the algebraic side, by considering all algebraic system reducts of the reduced matrix families, we get the class $\mathrm{AlgSys}^*(\mathcal{I})$ of all *family reduced algebraic systems* and, by considering all algebraic system reducts of the reduced matrix systems, we get the class $\mathrm{AlgSys}^\bullet(\mathcal{I})$ of all *system reduced algebraic systems*.

Section 2.8 studies the two related concepts of *axiomatic extensions* and of *filter extensions*. An *axiomatic extension* $\mathcal{I}'$ of a given π-institution $\mathcal{I}$ is a π-institution over the same base algebraic system whose closure system is obtained by that of $\mathcal{I}$ by adding more axioms. More precisely, the consequences $C'(X)$ of a family of sentences X under $\mathcal{I}'$ are the consequences under $\mathcal{I}$ of the same family of sentences, augmented by some fixed family of sentences T, i.e., $C'(X) = C(X \cup T)$. The sentences in T are viewed as axioms in $\mathcal{I}'$. A *filter extension* arises in a similar way. The difference is that one considers filter families over arbitrary algebraic systems and not just theory families over the base algebraic system.

One of the first results in this section provides a characterization of axiomatic extensions. It asserts that axiomatic extensions are characterized by preservation of all those theories that include the theorem system of the extension. An alternative, lifting the condition to arbitrary algebraic systems, asserts that being an axiomatic extension is tantamount to the preservation of filterhood over all algebraic systems, for all those filters that include the least filter over the extension.

The last part of the section deals with *filter generation* over a given matrix family modulo a given π-institution $\mathcal{I}$. It defines the concept and formalizes, in a rather technical proposition, how generation of filters and surjective matrix family morphisms interact.

Section 2.9 turns the focus back to those structures that, like matrix families, play a critical role as intermediate structures in connecting the logical with the algebraic aspects of the theory. *Generalized matrix families* correspond to the generalized matrices of classical algebraic logic and, like generalized matrices, play a critical role in identifying classes of algebraic systems that may be naturally associated with given π-institutions (or classes of π-institutions). The way this association is established sheds light on the strength of ties between the two and on the nature of their interaction, e.g., by revealing which properties may be expected to be shared by the two or transferred from one to the other.

A *generalized matrix family* $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$ consists of an underlying algebraic system $\mathcal{A}$ and a collection of sentence families $\mathcal{T}$ of the algebraic system. Such structures may also be used in two ways. They may serve in defining a closure system on a base algebraic system and, therefore, a π-institution structure. On the other hand, given a π-institution $\mathcal{I}$, we may associate with it the collection $\mathrm{GMatFam}(\mathcal{I})$ of those generalized matrices all of whose sentence families are filter families of the π-institution. With any generalized matrix family $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$, one may associate its *Tarski congruence system* $\widetilde{\Omega}(\mathbb{A})$ or $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})$, an abstraction of the Tarski congruence systems associated with generalized matrices in classical abstract algebraic logic. *Tarski congruence systems* constitute the largest congruence systems on the base algebraic system compatible with all sentence families of the generalized matrix family. Taking the quotient $\mathbb{A}/\widetilde{\Omega}(\mathbb{A})$ of the generalized matrix family by its Tarski congruence system gives a new generalized matrix family $\mathbb{A}^*$, which is called the *Tarski reduction* of $\mathbb{A}$. A *Tarski reduced matrix family* is one that is isomorphic to its reduction, i.e., one whose Tarski congruence system is the identity congruence system on the underlying algebraic system.

There is a close connection between Tarski congruence systems and Leibniz congruence systems. Each generalized matrix system $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$ may be viewed as a bundle of matrix families $\{\langle \mathcal{A}, T \rangle : T \in \mathcal{T}\}$, i.e., of those matrix families whose sentence families belong to the collection of sentence families of the generalized matrix family. In that case, the Tarksi congruence system of the generalized matrix family is the intersection (in the component-wise sense) of the Leibniz congruence systems of all matrix families in the corresponding bundle, i.e., $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)$.

In a similar way to Tarski congruence systems, one may also consider *Suszko congruence systems* $\widetilde{\Omega}^{\mathcal{A},\mathcal{T}}(T)$ associated with ordinary matrix families $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, and these are also introduced in Section 2.9. Suszko congruence systems of matrix families are defined only in a relative way, by viewing the matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ as being part of a bundle expressed as a generalized matrix family $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$. Then the *Suszko congruence system* of the matrix family is identical to the Tarski congruence system $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}^T)$ of the bundle $\langle \mathcal{A}, \mathcal{T}^T \rangle$ consisting of only those sentence families that include (in the component-wise ordering) the sentence family T of the matrix family. Of course, expressed in terms of Lebniz congruence systems, the Suszko congruence system is the intersection of the Leibniz congruence systems of all matrix families determined by the sentence families in the given bundle that include that of the matrix family under consideration, i.e., $\widetilde{\Omega}^{\mathcal{A},\mathcal{T}}(T) = \bigcap_{T \leq T' \in \mathcal{T}} \Omega^{\mathcal{A}}(T')$. As was the case with Tarski congruence systems, we may consider the *Suszko reduction* $\mathfrak{A}^{\mathsf{Su}}$ of a given matrix family $\mathfrak{A}$, obtained by dividing out by the Suszko congruence system $\widetilde{\Omega}^{\mathcal{A},\mathcal{T}}(T)$. And, likewise, we call a matrix family *Suszko reduced*, when its Suszko congruence system is the identity congruence system on the underlying algebraic system.

Part of the significance of the Tarski and of the Suszko operators in algebraic logic is that they form one of the main mechanisms of selecting the "natural" class of algebraic systems to be associated with a given π-institution. Briefly and sketchily, starting from a π-institution $\mathcal{I}$, we obtain the collection $\mathrm{GMatFam}(\mathcal{I})$ of all generalized matrix families $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$ whose sentence families $T \in \mathcal{T}$ are filter families of the π-institution. We then compute the Tarski reductions $\mathbb{A}^*$ by dividing out by the corresponding Tarski congruences $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})$. This process gives rise to the class $\mathrm{GMatFam}^*(\mathcal{I})$ of all Tarski reduced generalized matrix families and to the class $\mathrm{AlgSys}(\mathcal{I})$ of all their underlying algebraic systems. This class subsumes, in the π-institution framework, the class of algebras which has long been viewed, in the traditional framework, as the most appropriate one to be associated with a given logic and, hence, as constituting the "natural" choice for the algebraic counterpart of the sentential logic. As it turns out, using a similar path, but relying on the Suszko operator, rather than on the Tarski operator, gives rise to exactly the same class of algebraic systems. Tracing the analogous process, one starts from a given π-institution $\mathcal{I}$ and considers all matrix families $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, viewed as part of the bundle $\mathbb{A} = \langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ of all matrix families associated with the π-institution. Then, one considers the Suszko reductions $\mathfrak{A}^{\mathsf{Su}}$ by dividing out by the corresponding Suszko congruence systems $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. The class of Suszko reduced matrix families obtained in this way is denoted by $\mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})$. It can then be shown that the class of all algebraic reducts of the matrix families in $\mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})$ coincides with the class $\mathrm{AlgSys}(\mathcal{I})$.

In Sections 2.7 and 2.9, using the classes of Leibniz reduced matrix families and of Tarski reduced generalized matrix families associated with a given π-institution $\mathcal{I}$, we are able to define the two classes $\mathrm{AlgSys}^*(\mathcal{I})$ and $\mathrm{AlgSys}(\mathcal{I})$ of algebraic systems associated with the π-institution. In Section 2.10, we take up the study of two additional classes of algebraic systems that may be perceived as counterparts of a given π-institution and compare them with those already defined.

Both new classes are based on a single algebraic system, namely the algebraic system $\mathcal{F}/\widetilde{\Omega}(\mathcal{I})$ resulting by considering the quotient of the base algebraic system $\mathcal{F}$ by the Tarski congruence of the collection of all theory families of $\mathcal{I}$. Using this quotient algebraic system, the two classes are formed as the two kinds of varieties that may be generated by it. The first type, called the *semantic variety*, denoted by $\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}) = \mathbb{V}^{\mathsf{Sem}}(\mathcal{F}/\widetilde{\Omega}(\mathcal{I}))$, is the class of all algebraic systems that satisfy all equations that are satisfied by $\mathcal{F}/\widetilde{\Omega}(\mathcal{I})$, i.e., all equations included in $\widetilde{\Omega}(\mathcal{I})$. The second type, called the *syntactic variety*, denoted by $\mathbb{V}^{\mathsf{Syn}}(\mathcal{I}) = \mathbb{V}^{\mathsf{Syn}}(\mathcal{F}/\widetilde{\Omega}(\mathcal{I}))$, is the class of all algebraic systems that satisfy all natural equations that are satisfied by $\mathcal{F}/\widetilde{\Omega}(\mathcal{I})$.

Some results relating the four classes are presented. There is a linear hierarchy of inclusions that is not very difficult to establish. The class $\mathrm{AlgSys}^*(\mathcal{I})$ is the smallest class, followed by $\mathrm{AlgSys}(\mathcal{I})$, which is, in turn, included in

$\mathbb{V}^{\mathsf{Sem}}(\mathcal{I})$, whereas $\mathbb{V}^{\mathsf{Syn}}(\mathcal{I})$ is the largest of the four classes considered. It turns out that all four classes generate the same syntactic variety of algebraic systems, which is identical to $\mathbb{V}^{\mathsf{Syn}}(\mathcal{I})$, since it constitutes already a syntactic variety by definition. The section concludes with an important result showing that the class $\mathrm{AlgSys}(\mathcal{I})$ - perhaps the most important class associated with $\mathcal{I}$ - is closed under subdirect intersections and contains a trivial algebraic system. The usefulness of this fact is that it enables consideration, on any given algebraic system, of the least congruence system relative to $\mathrm{AlgSys}(\mathcal{I})$ generated by a prespecified relation family.

In Section 2.11, we study *equivalence families* and *systems* that are induced by sentence families or collections of sentence families of an algebraic system. The most fundamental among these is the *Frege equivalence family* $\lambda^{\mathbf{A}}(T)$ associated with a sentence family T of an algebraic system $\mathbf{A}$. It identifies two sentences if they are both inside or both outside the sentence family. Its companion *Frege equivalence system* $\Lambda^{\mathbf{A}}(T)$ is the largest equivalence system included in $\lambda^{\mathbf{A}}(T)$. The two Frege equivalences are intimately connected with the Leibniz congruence system $\Omega^{\mathbf{A}}(T)$, the latter being the largest congruence system contained in either of $\lambda^{\mathbf{A}}(T)$ or $\Lambda^{\mathbf{A}}(T)$.

In a way analogous to the extensions of the Leibniz congruence system that give rise to the Tarski and Suszko congruence systems, the Frege relations give rise to two more equivalences with similar roles. Given a collection $\mathcal{T}$ of sentence families of $\mathbf{A}$, the *Carnap equivalence family* $\widetilde{\lambda}^{\mathbf{A}}(\mathcal{T})$ identifies two sentences if they are equivalent modulo T (in the Frege sense) for all $T \in \mathcal{T}$, i.e., $\widetilde{\lambda}^{\mathbf{A}}(\mathcal{T}) = \bigcap_{T \in \mathcal{T}} \lambda^{\mathbf{A}}(T)$. The *Carnap equivalence system* $\widetilde{\Lambda}^{\mathbf{A}}(\mathcal{T})$ is the largest equivalence system included in $\widetilde{\lambda}^{\mathbf{A}}(\mathcal{T})$. The relation connecting Leibniz congruence systems with the Frege equivalences persists here as well, but with the Suszko congruence system in place of the Leibniz one. That is, the Suszko congruence system $\widetilde{\Omega}^{\mathbf{A}}(\mathcal{T})$ is the largest congruence system contained in either of $\widetilde{\lambda}^{\mathbf{A}}(\mathcal{T})$ or $\widetilde{\Lambda}^{\mathbf{A}}(\mathcal{T})$.

Finally, reminiscent of the passage from the Tarski to the Suszko congruence system, given a collection of sentence families $\mathcal{T}$ and $T \in \mathcal{T}$, the *Lindenbaum equivalence family* $\widetilde{\lambda}^{\mathbf{A},\mathcal{T}}(T)$ is the relation family identifying two sentences if they are equivalent modulo every $T' \in \mathcal{T}$, such that $T \leq T'$. The *Lindenbaum equivalence system* $\widetilde{\Lambda}^{\mathbf{A},\mathcal{T}}(T)$ is the largest equivalence system contained in $\widetilde{\lambda}^{\mathbf{A},\mathcal{T}}(T)$, and the Suszko congruence system $\widetilde{\Omega}^{\mathbf{A},\mathcal{T}}(T)$ turns out to be the largest congruence system included in either of $\widetilde{\lambda}^{\mathbf{A},\mathcal{T}}(T)$ or $\widetilde{\Lambda}^{\mathbf{A},\mathcal{T}}(T)$.

The Carnap operators, viewed as operators on collections of sentence families on the same algebraic system, are monotone. The same applies to the Lindenbaum operators, viewed as operators on sentence families relative to the same collection of sentence families. However, the Frege operators do not satisfy a monotonicity property.

In Section 2.12, we are discussing *algebraic subsystems* and *π-subinstitutions*. The starting point is the observation that an algebraic system $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ may contain a *universe*, i.e., a functor $\mathrm{SEN}' : \mathbf{Sign} \to \mathbf{Set}$, such that $\mathrm{SEN}' \leq \mathrm{SEN}$ and closed under the action of natural transformations in N. Then, it is clear that this universe may be used to define an *algebraic subsystem* $\mathbf{A}'$ of $\mathbf{A}$ and, as it turns out, there exists a *canonical injection morphism* $\langle I, j\rangle : \mathbf{A}' \to \mathbf{A}$. Apart from detecting the existence of universes, there is a natural way to generate a universe starting from a given sentence family T of $\mathbf{A}$. This consists of passing, first, to the least sentence system $\overrightarrow{T}$ containing T and, then, closing $\overrightarrow{T}$ under the clone operations in N. This two-step process gives rise to a universe $\nu^{\mathbf{A}}(\overrightarrow{T})$. In case the algebraic system $\mathbf{A}$ supports a π-institution $\mathcal{I} = \langle \mathbf{A}, C\rangle$, then one obtains, for each algebraic subsystem $\mathbf{A}'$ of $\mathbf{A}$, a *π-subsinstitution* $\mathcal{I}' = \langle \mathbf{A}', C'\rangle$ by restricting the action of C on elements of $\mathbf{A}'$. It can be shown that the theory families of $\mathcal{I}'$ are exactly the restrictions of those of $\mathcal{I}$ on the universe giving rise to $\mathbf{A}'$. The section ends with some results relating Leibniz congruence systems of theory families of $\mathcal{I}$ with those of the corresponding theory families of $\mathcal{I}'$. A similar result also holds for Leibniz congruence systems of corresponding filter families of the two π-institutions.

Sections 2.13-2.15 deal with aspects of the "syntactic" apparatus of an algebraic system, i.e., with properties of the natural transformations viewed as term functions. Section 2.13 introduces the framework and studies some connections with the definability of the Leibniz congruence systems. Section 2.14 explores various modes of definability and details their relative power. Section 2.15 studies the effect of parameters and shows that two different possible ways of obtain a parameterless collection of natural transformations out of a given parametric one are essentially equivalent. We provide, next, some more details by section.

Section 2.13 introduces the concepts of *distinguished arguments* and of *parametric arguments* of a collection E of natural transformations. This is a conceptual distinction which becomes important in practice when one differentiates the role they each play when the collection of natural transformations is used to transform sentences, i.e., to produce new sets of sentences from tuples of given ones. The new family of sentences produced from a tuple of sentences $\vec{\phi}$ (possibly with the aid of parameters) is denoted by $E_\Sigma[\vec{\phi}]$, where Σ is the signature of $\vec{\phi}$. Another mode of transformation uses a dual or inverse construction. Namely, given a sentence family T, we consider the set $\overleftarrow{E}(T)$ consisting of all tuples $\vec{\phi}$, such that $E_\Sigma[\vec{\phi}] \leq T$. These tuples all share the same length, which equals the number of distinguished arguments of the transformations in E. The construction has some important properties, e.g., $\overleftarrow{E}$, viewed as an operator on sentence families is monotone and, moreover, commutes with inverse surjective morphisms. But, perhaps, its most important property is that, if E has two distinguished arguments and T is such that $\overleftarrow{E}(T)$ is reflexive, then $\overleftarrow{E}(T)$ includes the Leibniz congruence system $\Omega^{\mathbf{A}}(T)$ of T. Consequently, if $\overleftarrow{E}(T)$ is itself a congruence system compatible with T, then it coincides with $\Omega^{\mathbf{A}}(T)$. Thus, in this case, we may say that, in a specific sense, the Leibniz operator of T is *definable* using the natural transformations in E. We view this as a syntactic definability condition, which plays an important role in establishing the algebraic classification of π-institutions "by syntactic means" in subsequent chapters.

In Section 2.14 we continue the study of natural transformations as means of transforming tuples of sentences to sentences. We look as four possible ways of relating, via a fixed collection E of natural transformations with k distinguished arguments, a k-tuple of sentences $\vec{\phi}$ to a sentence family T. The simplest, *E-local membership*, asserts that $E_\Sigma(\vec{\phi}, \vec{\chi}) \subseteq T_\Sigma$, for all values $\vec{\chi}$ of the parametric arguments. The second, *E-global membership*, asserts that $E_{\Sigma'}(\mathrm{SEN}(f)(\vec{\phi}), \vec{\chi}) \subseteq T_{\Sigma'}$ holds for all signatures Σ', all morphisms $f : \Sigma \to \Sigma'$ and all appropriate values of the parameters $\vec{\chi}$. The remaining two, *left E-local membership* and *left E-global membership* mimic the preceding ones except that they use membership in $\overleftarrow{T}$ instead of membership in T. Closer scrutiny of the four modes reveals that the two global memberships are equivalent, followed in strength by left local membership, which, in turn, implies local membership. When a membership property holds for all $\vec{\phi}$, then we attribute it to the collection E itself. In this sense, it turns out that global, local, left global and left local memberships of E in T all coincide.

In Section 2.15, starting from a given collection S of natural transformations, possibly including parametric arguments, we study ways of obtaining a collection that is parameter-free. Here, two of the most natural, for our purposes, ways of doing this turn out to be equivalent, and, hence, release us from the obligation to distinguish between which one is applied in any specific context. Let us assume that S is taken to have k distinguished arguments. Then one way of obtaining from S a parameter-free collection is to replace all parametric arguments with k-ary natural transformations. This results in a collection $\dot{S}$ of k-ary, i.e., parameter-free, natural transformations. The second method builds on the notion of an *anti-monotone property* of natural transformations. These are properties P that a natural transformation either does or does not satisfy and for which an anti-monotonicity property holds, namely, if for all tuples of sentences $\vec{\phi}$, the family of transforms of $\vec{\phi}$ under σ is included in the family of transforms of $\vec{\phi}$ under τ, then τ satisfying P implies that σ also satisfies P. If P also denotes the class of all natural transformations satisfying property P, then we let $\widehat{P}$ be the subclass of P consisting of the parameter-free members of P. The section concludes with the assertion that, for anti-monotone properties P, both constructions $\dot{P}$ and $\widehat{P}$ give the same class of parameter-free natural transformations associated with P.

In Section 2.16, we study *finitarity*. This property holds for a π-institution $\mathcal{I}$ if every sentence ϕ that is derivable from a set Φ of sentences can be derived from some finite subset Φ' of Φ. Finitarity holds for the overwhelming majority of the logics considered in the literature. So it has played a central role in algebraic logic, even though much of the more abstract body of the theory is formalized and developed in a way that encompasses arbitrary, that is, not necessarily finitary, logical systems. A characterization of finitarity using the property of *continuity* is provided. We say that a collection of theory families is *directed* if every finite subcollection is included in some theory family in the collection. A π-institution is *continuous* if the union of a directed collection of theory families is also a theory family. Finitarity and continuity, as it turns out, are equivalent properties.

In the second part of the section, given a finitary π-institution $\mathcal{I}$, we provide a construction of the filter family $C^{\mathcal{I},\mathcal{A}}(X)$ generated by a sentence family X of $\mathcal{A}$. Taking advantage of the finitarity of $\mathcal{I}$, the filter family may be obtained by an incremental process, each step of which adds in the filter family sentences of $\mathcal{A}$ which are derivable, in a certain sense, by finite subsets of sentences that have already been included in the filter family at previous stages of the construction. In this way, the family $\Xi^{\mathcal{I},\mathcal{A}}(X)$ is obtained as the union of the families obtained at all stages and it can be shown that $C^{\mathcal{I},\mathcal{A}}(X) = \Xi^{\mathcal{I},\mathcal{A}}(X)$.

In the last two sections, Sections 2.17 and 2.18, we study *equational consequences* and provide analogs of some well-known fundamental results of universal algebra for classes of algebraic systems.

In Section 2.17, we look at closure families on pairs of sentences, i.e., equations, over a base algebraic system $\mathbf{F}$ that are induced by classes of $\mathbf{F}$-algebraic systems. Given a class K of $\mathbf{F}$-algebraic systems, we say that an equation $\phi \approx \psi$ is a *consequence* of a set E of equations *relative to* K if every algebraic system in K satisfying E also satisfies $\phi \approx \psi$. The resulting consequence family is denoted by D^{K}. It is not necessarily a closure system since it may fail to be structural. It is shown, however, that its theory families are exactly the congruence systems on $\mathbf{F}$ relative to the class K.

The second part of the section deals with a process of generating the closure of a family of equations E relative to an equational axiomatic system Q in an incremental way. Roughly speaking, it formalizes the process of closing under reflexivity, symmetry and transitivity, as well as under replacement and the action of signature morphisms. The family of equations obtained under this step-wise process from axioms Q and hypotheses E is denoted by $\Xi^Q(E)$. In the final result of the section, it is shown that the operator Ξ^Q coincides with D^{K} when Q is taken to be the collection of all equations satisfied by all algebraic systems in K.

Section 2.18, the closing section of the chapter, is inspired by universal algebra. It provides characterizations, in the spirit of Birkhoff's variety and Mal'cev's quasivariety theorems, of classes of algebraic systems defined by equations, quasiequations and generalized quasiequations, also referred to as guasiequations. The section begins by formally defining *equations*, *quasiequations* and *guasiequations* in the context of π-institutions. The relation of *satisfaction* of a syntactic entity of either of the above types in an algebraic system is also formally defined. In the usual way, these satisfaction relations establish Galois connections. The closed sets on the syntactic side form *equational*, *quasiequational* and *guasiequational theories*, whereas, on the semantic side, one obtains *equational*, *quasiequational* and *guasiequational classes of algebraic systems*, respectively. These are, respectively, the classes closed under the semantic variety $\mathbb{V}^{\mathsf{Sem}}$, semantic quasivariety $\mathbb{Q}^{\mathsf{Sem}}$ and semantic guasivariety $\mathbb{G}^{\mathsf{Sem}}$ operators.

To formulate characterizations of these classes, we introduce and study four operators on classes of algebraic systems. Let K be a class of algebraic systems. First, we say that an algebraic system $\mathcal{A}$ is K-*certified* if, for each signature Σ, there exists an algebraic system $\mathcal{A}^\Sigma$ in the class K that satisfies exactly the same equations of signature Σ as $\mathcal{A}$. The class K is said to be *abstract* or *closed under* K-*certifications* if every K-certified algebraic system is in K. The operator $\mathbb{C}$ is a closure operator and, if $\mathcal{A} \in \mathbb{C}(\mathsf{K})$, then $\mathcal{A}$ satisfies all guasiequations satisfied by K. Moreover, if K is guasiequational, then it is an abstract class. Next, we say that an algebraic system $\mathcal{A}$ is *directedly* K-*certified* if, for each signature Σ, there exists a collection of algebraic systems $\{\mathcal{A}^{\Sigma,i} : i \in I\}$ in the class K that satisfy two conditions: On the one hand, the collection of all finite sets of equations satisfied by some $\mathcal{A}^{\Sigma,i}$, $i \in I$, is directed and, on the other, the union of all those sets is exactly the set of equations of signature Σ satisfied by $\mathcal{A}$. The class K is said to be *directedly abstract* or *closed under directed* K-*certifications* if every directedly K-certified algebraic system is in K. The operator $\mathbb{C}^*$ is a closure operator. It is shown that, if $\mathcal{A}$ is directedly K-certified, then it satisfies all quasiequations satisfied by K and, furthermore, that directed abstraction is a necessary condition for a class of algebraic systems to be a quasiequational class.

The third operator on classes of algebraic systems is that of taking *subdirect intersections* $\overset{\triangleleft}{\mathbb{I}\mathbb{I}}$. *Subdirect intersections* are collections of morphisms $\langle H^i, \gamma^i \rangle : \mathcal{A} \to \mathcal{A}^i$, $i \in I$, with the same domain, the intersection of whose kernels is the identity system on $\mathcal{A}$. In that case, we also say that $\mathcal{A}$ is a *subdirect intersection* of the $\mathcal{A}^i$'s. This also turns out to be a closure operator on classes of algebraic systems and, in fact, closure under $\overset{\triangleleft}{\mathbb{I}\mathbb{I}}$ is necessary for a class to be guasiequational. The last operator considered is that of taking *morphic images*, denoted by $\mathbb{H}$. It also forms a closure operator on classes of algebraic systems and closure under $\mathbb{H}$ is necessary for a class to be an equational class.

The four operators serve in formulating the Birkhoff-style characterizations referred to previously for equational, quasiequational and guasiequational classes. Guasiequational classes are characterized as those that are abstract and closed under subdirect intersections. Quasiequational classes are those that are directedly abstract and closed under subdirect intersections. Finally, equational classes are characterized as those that are closed under subdirect intersections and morphic images. The section concludes with some additional characterizations of these three classes involving the structure of the subcollection $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ of the complete lattice $\mathrm{ConSys}(\mathcal{F})$. All of those additional results are based on the main characterizations described above.

1.3.2 Chapter 3

In Chapter 3 we start in earnest the study of the Leibniz hierarchy of π-institutions. Chapters 3-9 deal with the *semantic Leibniz hierarchy*. Here the classes are defined using properties of the Leibniz operator on theory families/systems of a π-institution. Chapters 11-**??**, on the other hand, deal with the *syntactic Leibniz hierarchy* in which classes are defined using collections of natural transformations satisfying specific definability properties. We shall see that "corresponding" classes in the two hierarchies may not coincide, but, nevertheless, the two hierarchies are closely connected - in fact may be seen as forming parts of a single hierarchy - and they are both modeled on the Leibniz hierarchy of sentential logics.

In Section 3.2, we study three properties. The first two are fundamental because they introduce concepts and terminology that play a critical role throughout the monograph. The third is used to establish classes of π-institutions at the very bottom of the hierarchy which abstract all other classes considered later in this and in subsequent chapters.

The first property if *systemicity*. A π-institution $\mathcal{I}$ is called *systemic* if every theory family of $\mathcal{I}$ is actually a theory system, i.e., if $\mathrm{ThFam}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$. Recalling from Chapter 2 that, given a theory family T of $\mathcal{I}$, $\overleftarrow{T}$ is the largest theory system included in T, $\mathcal{I}$ is systemic if and only if, for every theory family T, $\overleftarrow{T} = T$. Yet another characterization asserts that, for every Σ-sentence ϕ of $\mathcal{I}$, the least theory family $C(\phi)$ of $\mathcal{I}$ generated by ϕ contains all translates of ϕ under arbitrary signature morphisms. One of the reasons why systemicity plays such a critical role is that, for a systemic π-institution, it suffices to restrict attention to theory systems, i.e., one may take invariance under signature morphisms for granted.

The second property is *stability*. It may be thought of as the counterpart of systemicity when focus shifts from theory families to corresponding Leibniz congruence systems. A π-institution $\mathcal{I}$ is *stable* if, for all theory families T, $\Omega(\overleftarrow{T}) = \Omega(T)$. Of course, every systemic π-institution is stable, and this implication is proper. Both systemicity and stability *transfer*. This means that a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is systemic if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, every $\mathcal{I}$-filter family of $\mathcal{A}$ is a filter system. Similarly, $\mathcal{I}$ is stable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and every $\mathcal{I}$-filter family T of $\mathcal{A}$, $\Omega^{\mathcal{A}}(\overleftarrow{T}) = \Omega^{\mathcal{A}}(T)$. These two transfer results are only the first of a host of, so-called, *transfer theorems* that are proved in the sequel for the majority of properties used to define classes in the Leibniz hierarchy. Having established the pattern and exhibited the main idea, we only mention such results briefly from now on, postponing the details for the main account in the relevant sections of the text.

The third property we study in Section 3.2 is *loyalty*. Unlike systemicity and stability, loyalty comes, as is typical for many subsequently introduced properties, in multiple flavors. To establish the pattern that will be followed in the presentation throughout, we introduce, first, the four versions, termed *family*, *left*, *right* and *system*. They may or may not be all different. So we study their properties, show which ones, if any, coincide, establish general implications between those that are not equivalent, and show, via examples, that these implications are proper, i.e., that no further collapsing of the subhierarchy based on these properties is possible.

A π-institution $\mathcal{I}$ is *family loyal* if, for all theory families T, T' of $\mathcal{I}$, $T \nless T'$ or $\Omega(T) \ngtr \Omega(T')$, or, equivalently, if it is not the case that $T < T'$ and $\Omega(T) > \Omega(T')$. If Ω, viewed as an operator mapping theory families to congruence systems, is either order preserving or order reflecting, then it is necessarily family loyal. So this property abstracts both monotonicity and reflectivity of Ω. Since both monotonicity and reflectivity play important roles in specifying classes in the Leibniz hierarchy, this observation provides partial justification for considering loyalty as a common abstraction. Here, as in all subsequently defined properties, once the family version is introduced, the other three versions follow by applying similar modifications. To obtain the *left version* one replaces, on the theory family side, all theory families by their arrow versions. So $\mathcal{I}$ is *left loyal* if, for all theory families T, T', $\overleftarrow{T} \nless \overleftarrow{T'}$ or $\Omega(T) \ngtr \Omega(T')$. To obtain the *right version*, a similar replacement is applied on the congruence system side. Thus, $\mathcal{I}$ is *right loyal* if, for all theory families T, T', $T \nless T'$ or $\Omega(\overleftarrow{T}) \ngtr \Omega(\overleftarrow{T'})$. Finally, the *system version* is obtained by imposing the same condition as in the family version, but restricting its application to theory systems, instead of insisting that it hold for all theory families. Accordingly, $\mathcal{I}$ is *system loyal* if, for all theory *systems* T, T' of $\mathcal{I}$, $T \nless T'$ or $\Omega(T) \ngtr \Omega(T')$.

Family loyalty properly implies stability. Moreover, family loyalty implies left loyalty, which, in turn, implies system and right loyalty, the latter two being equivalent properties. System loyalty, together with systemicity, imply family loyalty. That is, as is the case with virtually all properties introduced in the monograph, imposing systemicity has the effect of collapsing the entire four-class subhierarchy into a single class. This observation can be applied to obtain a backbone - or a bird's eye view - of the Leibniz hierarchy without worrying about the refinements and subdivisions due to the different flavors of each property. Section 3.2 concludes by showing that all three distinct versions of loyalty transfer, i.e., that a given π-institution has a certain loyalty property if the corresponding defining condition holds for all pairs of filter families (or systems) on arbitrary $\mathbf{F}$-algebraic systems.

In Section 3.3, we study *monotonicity properties*. A π-institution $\mathcal{I}$ is *family monotone* if, for all theory families T, T', $T \leq T'$ implies $\Omega(T) \leq \Omega(T')$, i.e., if the Leibniz operator on theory families is order preserving. In accordance with the general framework outlined above for loyalty, $\mathcal{I}$ is *left monotone* if, for all T, T', $\overleftarrow{T} \leq \overleftarrow{T'}$ implies $\Omega(T) \leq \Omega(T')$, *right monotone* if, for all T, T', $T \leq T'$ implies $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$ and *system monotone* if the same condition defining family monotonicity is restricted to theory systems, i.e., if the Leibniz operator on theory systems is order preserving. It is shown that family monotonicity implies stability. Most importantly, family and left monotonicity coincide as do system and right monotonicity. Following terminology inherited from sentential logics, we term π-institutions that satisfy family monotonicity *protoalgebraic* and those that satisfy the system version *prealgebraic*. Protoalgebraicity is equivalent to prealgebraicity plus stability. In particular, every protoalgebraic π-institution is prealgebraic, and this inclusion is proper. Both monotonicity properties transfer. Finally, pursuing connections with classes introduced in Section 3.2, we show that protoalgebraicity implies family loyalty, whereas prealgebraicity is sufficient for system loyalty.

In Sections 3.4 and 3.5, we study versions of a property called *complete monotonicity*. This is a property dual to complete order reflectivity, a property that characterizes truth equationality in the sentential framework. Given a sentential logic $\mathcal{S}$, complete order reflectivity stipulates that, for every collection $\mathcal{T} \cup \{T'\}$ of theories of $\mathcal{S}$, if $\bigcap_{T\in\mathcal{T}} \Omega(T) \subseteq \Omega(T')$, then $\bigcap \mathcal{T} \subseteq T'$. Since, in both the lattice of theories and that of congruences, meet and intersection coincide, but, on both theories and congruences, join is not the same as union, one may obtain two "dual" versions of complete order reflectivity. The first, following a set-theoretic approach, says that, for all $\mathcal{T} \cup \{T'\}$, $T' \subseteq \bigcup \mathcal{T}$ implies $\Omega(T') \subseteq \bigcup_{T\in\mathcal{T}} \Omega(T)$. The second, taking a lattice-theoretic point of view, asserts that, for all $\mathcal{T} \cup \{T'\}$, $T' \leq \bigvee \mathcal{T}$ implies $\Omega(T') \leq \bigvee_{T\in\mathcal{T}} \Omega(T)$, where the join in the hypothesis is taken in the complete lattice of theories of $\mathcal{S}$ and the one in the conclusion in the complete lattice of congruences on the formula algebra. In Section 3.4 we study an analog of the former property and in Section 3.5 an analog of the latter in the context of logics formalized as π-institutions. A few more details follow in the next two paragraphs.

In Section 3.4, we look at *complete* $\bigcup$*-monotonicity*, which is abbreviated as *c*$^\cup$*-monotonicity* or, simply, *c-monotonicity*. A π-institution is *family c*$^\cup$*-monotone* if, for every collection $\mathcal{T} \cup \{T'\}$ of theory families, $T' \leq \bigcup \mathcal{T}$ implies $\Omega(T') \leq \bigcup_{T\in\mathcal{T}} \Omega(T)$. *Left* and *right c*$^\cup$*-monotonicities* are obtained by replacing in the hypothesis and in the conclusion, respectively, every theory family occurring by its arrow version. Finally, *system c*$^\cup$*-monotonicity* is defined by the same condition as the family version, but applied exclusively to collections of theory systems. Family c$^\cup$-monotonicity implies stability, as does left c$^\cup$-monotonicity. Moreover, the family version is equivalent to the conjunction of the left and right versions and either of the latter implies system $c^\cup$-monotonicity. All four $c^\cup$-monotonicity properties transfer. And, whereas the left version is sufficient for protoalgebraicity, the system version implies only prealgebraicity.

In Section 3.5, we continue the study of complete monotonicity but switch from complete $\bigcup$-monotonicity to *complete* $\bigvee$*-monotonicity*, which is abbreviated as $c^\vee$*-monotonicity*. A π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is *family* $c^\vee$*-monotone* if, for every collection $\mathcal{T} \cup \{T'\}$ of theory families, $T' \leq \bigvee^{\mathcal{I}} \mathcal{T}$ implies $\Omega(T') \leq \bigvee^{\mathbf{F}}_{T \in \mathcal{T}} \Omega(T)$, where $\bigvee^{\mathcal{I}}$ denotes the join in the complete lattice of theory families of $\mathcal{I}$ and $\bigvee^{\mathbf{F}}$ the join in the complete lattice of congruence systems on $\mathbf{F}$. Again, following the general pattern, $\mathcal{I}$ is *left* $c^\vee$*-monotone* if, for all $\mathcal{T} \cup \{T'\}$, $\overleftarrow{T'} \leq \bigvee^{\mathcal{I}}_{T \in \mathcal{T}} \overleftarrow{T}$ implies $\Omega(T') \leq \bigvee^{\mathbf{F}}_{T \in \mathcal{T}} \Omega(T)$ and is *right* $c^\vee$*-monotone* if, for all $\mathcal{T} \cup \{T'\}$, $T' \leq \bigvee^{\mathcal{I}} \mathcal{T}$ implies $\Omega(\overleftarrow{T'}) \leq \bigvee^{\mathbf{F}}_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$. Finally, $\mathcal{I}$ is *system* $c^\vee$*-monotone* if, for every collection $\mathcal{T} \cup \{T'\}$ of theory *systems* of $\mathcal{I}$, $T' \leq \bigvee^{\mathcal{I}} \mathcal{T}$ implies $\Omega(T') \leq \bigvee^{\mathbf{F}}_{T \in \mathcal{T}} \Omega(T)$. Again, either family or left $c^\vee$-monotonicity implies stability. The family version is equivalent to the conjunction of the left and right versions and either of those two implies system $c^\vee$-monotonicity. Left $c^\vee$-monotonicity implies protoalgebraicity and system $c^\vee$-monotonicity implies prealgebraicity.

Contrary to what the similarities of results pertaining to $c^\vee$-monotonicity with those of Section 3.4 on $c^\cup$-monotonicity may suggest, there are also significant differences between the two complete monotonicity properties. One instance concerns transfer theorems. Unlike $c^\cup$-monotonicity, $c^\vee$-monotonicity properties do not transfer in general. This is due to the fact that, unlike unions, joins do not commute with inverse surjective morphisms between algebraic systems. A second difference, which affords, perhaps, partial justification for introducing and discussing both types of properties in some detail, is that corresponding classes of π-institutions are incomparable. E.g., there exists a family $c^\vee$-monotone π-institution which is not family $c^\cup$-monotone and vice-versa.

In Section 3.6, we study *injectivity*. A π-institution $\mathcal{I}$ is *family injective* if, for all theory families T, T', $\Omega(T) = \Omega(T')$ implies $T = T'$, i.e., if the Leibniz operator is injective on theory families. It is *left injective* if, for all T, T', $\Omega(T) = \Omega(T')$ implies $\overleftarrow{T} = \overleftarrow{T'}$ and *right injective* if, for all T, T', $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$ implies $T = T'$. Finally, it is *system injective* if the Leibniz operator is injective on theory systems. Right injectivity is the strongest of the four injectivity properties and it implies systemicity. It is followed by family injectivity, then left injectivity, which implies system injectivity. System injectivity together with systemicity is equivalent to right injectivity, whereas, together with stability, which is weaker than systemicity, it implies left injectivity. All four injectivity properties transfer.

In Section 3.7, we turn to *reflectivity properties*. A π-institution $\mathcal{I}$ is *family reflective* if, for all theory families T, T' of $\mathcal{I}$, $\Omega(T) \leq \Omega(T')$ implies

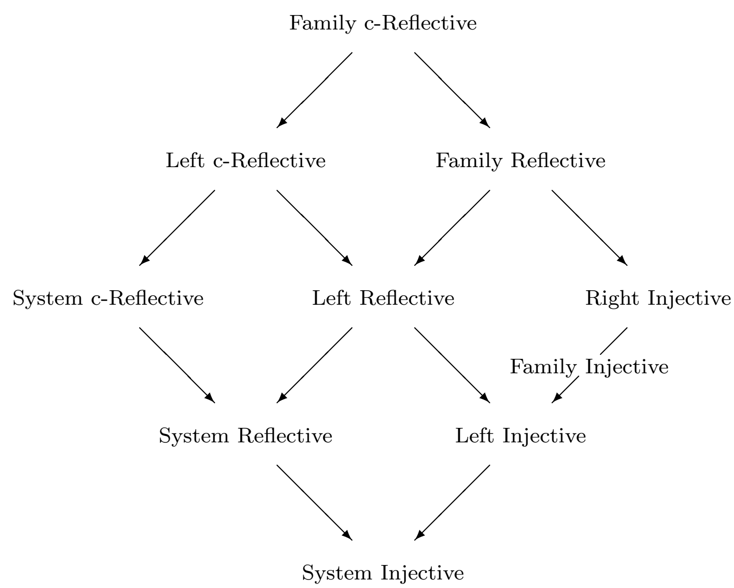
$T \leq T'$, i.e., if the Leibniz operator on theory families is order reflecting. If, for all T, T', $\Omega(T) \leq \Omega(T')$ implies $\overleftarrow{T} \leq \overleftarrow{T'}$, then $\mathcal{I}$ is *left reflective*, whereas, if, for all T, T', $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$ implies $T \leq T'$, $\mathcal{I}$ is *right reflective*. *System reflectivity* stipulates the order reflectivity of the Leibniz operator on theory systems. It turns out that family or right reflectivity imply systemicity. This allows showing that the two are actually equivalent properties. They imply left reflectivity, which, in turn, implies system reflectivity. System reflectivity, coupled with stability, implies left reflectivity, whereas, together with systemicity, it becomes equivalent to family reflectivity. All four versions transfer. Section 3.7 ends by relating reflectivity with the injectivity properties, introduced in Section 3.6, and with the loyalty properties, introduced in Section 3.2. More precisely, it is shown that family/right, left and system reflectivity imply, respectively, right, left and system injectivity and that family/right, left and system reflectivity imply, respectively, family, left and system loyalty.

Section 3.8, the last section of Chapter 3, introduces *complete reflectivity properties*, abbreviated to *c-reflectivity*. These form a generalization of the reflectivity properties of Section 3.7. Complete reflectivity originates in the work of Raftery, where it is used to characterize truth equationality in the context of sentential logics. A π-institution is *family c-reflective* if, for every collection $\mathcal{T} \cup \{T'\}$ of theory families of $\mathcal{I}$, $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$ implies $\bigcap \mathcal{T} \leq T'$. It is *left c-reflective* if, for all $\mathcal{T} \cup \{T'\}$, $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$ implies $\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$ and *right c-reflective* if, for all $\mathcal{T} \cup \{T'\}$, $\bigcap_{T\in\mathcal{T}} \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$ implies $\bigcap_{T\in\mathcal{T}} \mathcal{T} \leq T'$. *System c-reflectivity* is defined using the same condition as family c-reflectivity restricted to collections of theory systems. As was the case with reflectivity, either family or right c-reflectivity implies systemicity and this enables showing that the family and right versions are equivalent. They imply left c-reflectivity, which, in turn, implies the system version. System c-reflectivity and systemicity are jointly equivalent to family c-reflectivity, whereas system c-reflectivity, augmented with stability, implies left c-reflectivity. All complete reflectivity properties transfer and, as is apparent from the relevant definitions, each version of c-reflectivity implies the corresponding reflectivity version.

1.3.3 Chapter 4

In Chapter 4, we visit weak prealgebraizability and weak algebraizability properties of π-institutions. These create a subhierarchy of π-institutions whose members roughly correspond to the weakly algebraizable logics in the sentential logic framework. Weak prealgebraizability classes arise when coupling family monotonicity with either of injectivity, reflectivity or complete reflectivity properties. Analogously, weak algebraizability results by combining system monotonicity with injectivity, reflectivity or complete reflectivity.

Before describing the versions of weak prealgebraizability and algebraizability in more detail, we mention, firstly, that the term "weak" refers to the use of monotonicity, as opposed to the stronger notion of equivalentiality, in the definitions, and remind, secondly, the reader of the hierarchy, established in Chapter 3, of the various flavors of injectivity, reflectivity and c-reflectivity properties, which assumed the form depicted in the diagram.



In Section 4.2, we define the classes of *weakly prealgebraizable π-institutions*. Each class results by imposing prealgebraicity (system monotonicity) and one of the ten flavors of injectivity, reflectivity and complete reflectivity shown in the preceding hierarchy. Since prealgebraicity is shared by all classes, the deciding factor in the subhierarchy is the type of injectivity, reflectivity or c-reflectivity imposed. Thus, a priori, one obtains ten potentially distinct classes whose hierarchy reflects that shown in the preceding diagram. We name the corresponding property "weak X prealgebraizability", or "WX prealgebraizability" for short, where the string X stands for one of SI, LI, FI, RI for system, left, family, right injectivity, respectively, SR, LR, FR for system, left, family reflectivity, respectively, or SC, LC, FC for system, left, family c-reflectivity, respectively.

In our first result, we show that prealgebraicity is sufficient to identify all system versions, which forces the collapsing of the classes of WSI, WSR and WSC prealgebraizable π-institutions. We call the corresponding property *WS prealgebraizability*. In what sets a pattern for subsequent work in this chapter, it is shown that WS prealgebraizability transfers and, further, a characterization is obtained via properties of the Leibniz operator $\Omega^{\mathcal{A}}$, viewed as a mapping between ordered sets, for arbitrary $\mathbf{F}$-algebraic systems $\mathcal{A}$. More precisely, it is shown that a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is WS prealgebraizable iff, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$

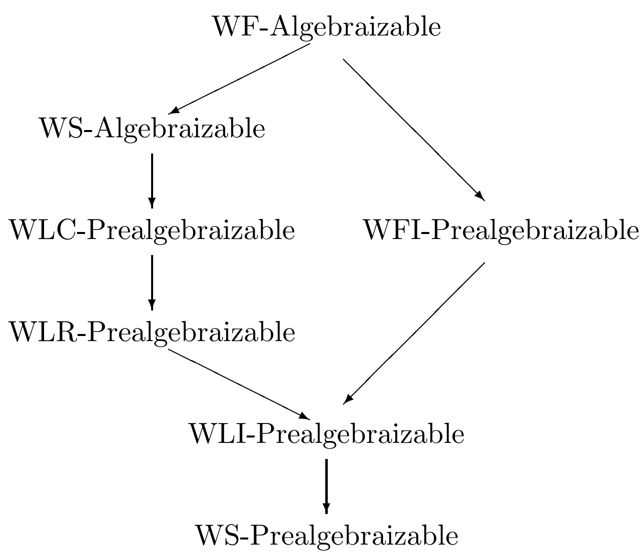
is an order embedding. Next, it is shown that, in view of prealgebraicity, family reflectivity implies family c-reflectivity and this leads to the identification of WFR prealgebraizability and WFC prealgebraizability. Moreover, under protoalgebraicity, family injectivity implies family reflectivity. This enables showing that both WFR and WRI prealgebraizability are characterized as the conjunction of WFI prealgebraizability and systemicity and, hence, are identical properties. Both WFR and WFI prealgebraizability transfer. Moreover, the WFI version is characterized by the property that, for all $\mathcal{A}$, $\Omega^{\mathcal{A}}$ is a bijection on filter families, restricting to an order embedding on filter systems, whereas the WFR version is characterized by the condition that, for all $\mathcal{A}$, $\Omega^{\mathcal{A}}$ is an order isomorphism.

At this point, the hierarchy has been reduced to six classes, since, as it turned out, all three system classes are identical and the three family plus the WRI prealgebraizability collapse down to two classes. The only classes not put under the microscope yet are those defined using the left versions of injectivity, reflectivity and c-reflectivity. We return to them after a short break that gives a glimpse of further possible reductions under special circumstances. Namely, it is proven that, under systemicity, the entire hierarchy collapses to a single class and that, under stability, it collapses down to two classes, as the only properties that can be distinguished are the family (but including also WRI prealgebraizability) from the remaining versions.

Returning to the left properties, Section 4.2 concludes by showing that all three transfer and by providing characterizations along the lines outlined previously, using $\Omega^{\mathcal{A}}$. More precisely, it is shown that $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is WLC (WLR, WLI, respectively) prealgebraizable iff, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is a left completely order reflecting (left order reflecting, left injective, respectively) surjection, restricting to an order embedding on theory systems.

In Section 4.3, we study versions of weak algebraizability. These combine protoalgebraicity (family monotonicity) with the various versions of injectivity, reflectivity and complete reflectivity. Since protoalgebraicity dominates prealgebraicity, it is clear that one obtains at least as many identifications between the ten apparent weak algebraizability properties as those established between corresponding weak prealgebraizability properties in Section 4.2. However, the situation under closer scrutiny turns out to be much more radical. Since protoalgebraicity is strong enough to yield stability, the emerging landscape was anticipated by the previously mentioned collapse of the weak prealgebraizability hierarchy down to two classes in the presence of stability. Similarly, under protoalgebraicity and, hence, stability, all three weak family algebraizability properties together with WRI algebraizability collapse to a single property, termed *WF algebraizability*. Further, all remaining six left and system versions also collapse to a single property we call *WS algebraizability*. Both of these properties transfer. Also, for both one may obtain Leibniz operator type characterizations. More specifically, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is WS algebraizable iff it is stable and, for all $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism, whereas it is WF algebraizable iff, for all $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism.

Observing that the characterization of WF algebraizability is identical with that obtained for WF prealgebraizability, we conclude that the top classes in the weak prealgebraizability and weak algebraizability subhierarchies actually coincide. Thus, by fusing these two subhierarchies, one obtains a total of seven potentially distinct classes, which form the combined hierarchy depicted in the diagram.



1.3.4 Chapter 5

In Chapter 5, we deal with classes of π-institutions that result from weakly prealgebraizable and weakly algebraizable π-institutions when the properties of prealgebraicity (system monotonicity) and protoalgebraicity (family monotonicity) are strengthened to preequivalentiality and equivalentiality, respectively. The strengthening, i.e., the passage from proto- (or pre-) algebraicity to (pre)equivalentiality, involves adding the condition of either family or system extensionality. Depending on which of these two properties is imposed, one obtains two parallel hierarchies, one on top of the other, both of which reflect the structure of the weak (pre)algebraizability hierarchy, described in Chapter 4.

In Section 5.2, we introduce and study *extensionality*. The definition requires the notion of subsystem of an algebraic system $\mathbf{F}$ generated by a given sentence family X, which is denoted by $\langle X\rangle$ and was introduced in Section 2.12. A π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is called *family extensional* if, for all sentence families X of $\mathbf{F}$ and all theory families T of $\mathcal{I}$, $\Omega(T) \cap \langle X\rangle^2 = \Omega^{\langle X\rangle}(T \cap \langle X\rangle)$.

It is called *system extensional* if the same condition holds, but T is quantified over all theory systems of $\mathcal{I}$, instead of ranging over arbitrary theory families. Since system extensionality specializes family extensionality, every family extensional π-institution is also system extensional. It is, moreover, the case that system extensionality, coupled with stability, implies family extensionality. Extensionality is very useful because, when satisfied, it causes certain properties that hold in a π-institution to be inherited by all its subinstitutions. For instance, under system extensionality, stability propagates from a π-institution $\mathcal{I}$ to all its subinstitutions $\mathcal{I}' \leq \mathcal{I}$. Additionally, system or family extensionality causes prealgebraicity or protoalgebraicity, respectively, to be inherited by all subinstitutions of a given π-institution. Both versions of extensionality transfer. The section closes by looking at *2-extensionality*, an apparently weaker condition than extensionality, which, however, turns out to be equivalent to it. A π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is *family 2-extensional* if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$ and every theory family T of $\mathcal{I}$, $\langle \phi, \psi\rangle \in \Omega_\Sigma(T)$ if and only if $\langle \phi, \psi\rangle \in \Omega_\Sigma^{\langle \phi,\psi\rangle}(T \cap \langle \phi, \psi\rangle)$. *System 2-extensionality* is defined by the same condition in which T is quantified over theory systems. A π-institution is family/system extensional if and only if it is family/system 2-extensional, respectively.

In Section 5.3, we study *Leibniz commutativity*. The notion relies on the concepts of *extension* and *logical extension*. Given an algebraic system $\mathbf{F}$ and a sentence family X of $\mathbf{F}$, an *extension* is an algebraic system morphism of the form $\langle I, \alpha\rangle : \langle X\rangle \to \mathbf{F}$, where $\langle X\rangle$ is the algebraic subsystem of $\mathbf{F}$ generated by X and I is the identity functor on signatures. Given a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$, an extension $\langle I, \alpha\rangle : \langle X\rangle \to \mathbf{F}$ is called *logical*, denoted $\langle I, \alpha\rangle : \mathcal{I}^{\langle X\rangle} \to \mathcal{I}$, if, for every signature Σ and all $\Phi \subseteq \langle X\rangle_\Sigma$, $\alpha_\Sigma(C_\Sigma^{\langle X\rangle}(\Phi)) \subseteq C_\Sigma(\alpha_\Sigma(\Phi))$, where $C^{\langle X\rangle}$ is the restriction of C on $\langle X\rangle$, discussed in detail in Section 2.12. A characterization of this notion asserts that $\langle I, \alpha\rangle$ is logical if and only if α^{-1} preserves theory families, i.e., if $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I}^{\langle X\rangle})$, for every $T \in \mathrm{ThFam}(\mathcal{I})$.

Logical extensions form the background for introducing the property of *Leibniz commutativity*, or, simply, *commutativity*. A π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is called *family commuting* if the Leibniz operator on theory families commutes with logical extensions, i.e., if, for every sentence family X of $\mathbf{F}$, every logical extension $\langle I, \alpha\rangle : \mathcal{I}^{\langle X\rangle} \to \mathcal{I}$ and all $T' \in \mathrm{ThFam}(\mathcal{I}^{\langle X\rangle})$, $\alpha(\Omega^{\langle X\rangle}(T')) \leq \Omega(C(\alpha(T')))$. Applying the same condition, where T' ranges over all theory systems of $\mathcal{I}^{\langle X\rangle}$, defines *system commutativity*. A closely related concept is that of *inverse Leibniz commutativity*, or, simply, *inverse commutativity*. A π-institution $\mathcal{I}$ is *family inverse commuting* if, for every sentence family X, every logical extension $\langle I, \alpha\rangle : \mathcal{I}^{\langle X\rangle} \to \mathcal{I}$, and all $T \in \mathrm{ThFam}(\mathcal{I})$, $\alpha^{-1}(\Omega(T)) = \Omega^{\langle X\rangle}(\alpha^{-1}(T))$. The same condition, imposed on theory systems only, defines *system inverse commutativity*. The fact that injection morphisms $\langle I, j\rangle : \mathcal{I}^{\langle X\rangle} \to \mathcal{I}$ of subinstitutions into their parent institutions are logical extensions allows us to show that family/system inverse commutativity implies family/system extensionality, respectively. It is clear that the family version implies the system version and, as it turns out, the system version augmented by stability implies the family version. What is important for our purposes, and the reason why both direct and inverse commutativity properties are studied, is that under pre/proto-algebraicity, respectively, system/family commutativity is equivalent to system/family inverse commutativity. Moreover, in a result strengthening the relationship mentioned above, it is proven that family/system inverse commutativity and family/system extensionality, respectively, are actually equivalent properties. This section concludes by showing that both versions of inverse commutativity transfer.

In Section 5.4, we introduce *equivalentiality*. This is the section we have been preparing for by studying extensionality and commutativity in Sections 5.2 and 5.3, respectively. *Equivalentiality* is the result of coupling monotonicity with extensionality. Since each of those two properties comes in two flavors, there are, a priori, four possible versions of equivalentiality. *Family equivalentiality* combines protoalgebraicity with family extensionality. *System equivalentiality* keeps protoalgebraicity but uses system extensionality. *Family* and *system preequivalentiality* are defined analogously, but here one uses prealgebraicity instead of protoalgebraicity. Since protoalgebraicity is strong enough to imply stability, it turns out that family and system equivalentiality coincide. This property is referred to simply as *equivalentiality*. Thus, we get three properties in this hierarchy, namely, in decreasing order of potency, equivalentiality, family preequivalentiality and system preequivalentiality. Moreover, equivalentiality is equivalent to system preequivalentiality plus stability. All three properties transfer. There also exist characterizations of equivalentiality and preequivalentiality by conditions imposed on the Leibniz operator on filter families/systems, respectively, on arbitrary **F**-algebraic systems. Finally, as is clear by the corresponding definitions, equivalentiality dominates protoalgebraicity and preequivalentiality dominates prealgebraicity.

In Section 5.5, by replacing prealgebraicity by preequivalentiality, we obtain from the weak prealgebraizability hierarchy of Section 4.2 two parallel *prealgebraizability hierarchies*. The term "prealgebraizability" in both refers to the fact that preequivalentiality, as opposed to equivalentiality, is applied. In one of the two hierarchies, "family prealgebraizability" refers to the application of family preequivalentiality, whereas in "prealgebraizability", it is understood that (system) preequivalentiality is applied. The five classes in the first hierarchy are termed *XF prealgebraizable* and in the second *X prealgebraizable*, where X is one of the following strings, suggesting the imposition of an additional property on the Leibniz operator.

- LC for left completely reflective;
- LR for left reflective;

- FI for family injective;
- LI for left injective; and
- S for system (system completely reflective, system reflective or system injective, which are all equivalent in view of prealgebraicity).

It is shown that systemicity causes the total collapse of the hierarchy into a single class, whereas stability collapses the two family injectivity classes, FI and FIF prealgebraizability, and, also, all eight remaining classes and, therefore, leads to a 2-class hierarchy. Moreover, it is proven that all ten properties transfer. The remainder of this section is devoted to providing characterizations of each of the ten classes using order theoretic properties of the Leibniz operator viewed as a mapping from lattices of filters systems/families to lattices of congruence systems over arbitrary **F**-algebraic systems. We focus only on a couple of pairs to give a flavor of the type of results obtained, and refer the reader to the main text for a full account. A π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is FIF prealgebraizable if and only if, for all **F**-algebraic systems $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is a bijection commuting with inverse logical extensions, which restricts to an order embedding on filter systems. A similar characterization is obtained for FI prealgebraizability, but with a subtle important change: $\mathcal{I}$ is FI prealgebraizable if and only if, for all $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is a bijection, which restricts to an order embedding commuting with inverse logical morphisms on filter systems. Analogously, for the left reflectivity classes, we get, on the one hand, that $\mathcal{I}$ is LRF prealgebraizable if and only if, for all $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is a left order reflecting surjection commuting with inverse logical extensions, which restricts to an order embedding on filter systems, and, on the other, noting again the same subtle change, $\mathcal{I}$ is LR prealgebraizable if and only if, for all $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is a left order reflecting surjection, which restricts to an order embedding commuting with inverse logical extensions on filter systems. Characterizations of the remaining six classes follow a similar pattern.

In Section 5.6, we switch from prealgebraizability to *algebraizability*. Dropping "pre" signifies using equivalentiality instead of the weaker preequivalentiality property. Equivalentiality encompasses protoalgebraicity and, under protoalgebraicity, only two classes of the ten potentially different ones are actually distinct. Accordingly, we get *family algebraizability*, or, simply, *F algebraizability*, when family injectivity is added, and *system algebraizability*, or, simply, *algebraizability*, when system injectivity is added. Family algebraizability is equivalent to algebraizability plus systemicity. Both properties transfer. Finally, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is algebraizable if and only if it is stable and, for every **F**-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism commuting with inverse logical extensions, whereas $\mathcal{I}$ is family algebraizable if and only if, for all $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism commuting with inverse logical extensions.

1.3.5 Chapter 6

The motivating force behind the considerations in this chapter is the observation that, since for a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$, with $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$, $\Omega(\overline{\varnothing}) = \nabla^{\mathbf{F}} = \Omega(\mathrm{SEN}^\flat)$, no π-institution without theorems can satisfy any of the injectivity, reflectivity of complete reflectivity properties introduced in Chapter 3. The question naturally arises whether, in that case, the existence of theory families with empty components is the only reason causing the lack of these properties or whether the π-institution in question would still not satisfy them even if theory families with empty components were in some way "discarded" or "bypassed". We choose two ways in which this circumvention may be accomplished, and study the various flavors of injectivity, reflectivity and complete reflectivity properties that result.

In Section 6.2, we introduce and study the relation of rough equivalence between theory families of a π-institution. Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Given a theory family T of $\mathcal{I}$, we define the *rough companion* (*rough associate* or *rough representative*) $\widetilde{T}$ of T as the theory family resulting from T by replacing all empty Σ-components of T by the corresponding set $\mathrm{SEN}^\flat(\Sigma)$ of Σ-sentences. We say that two theory families T and T' are *roughly equivalent*, written $T \sim T'$, if $\widetilde{T} = \widetilde{T'}$. The rough equivalence class of T is denoted by $\widetilde{[T]}$ and $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ denotes the collection of all rough equivalence classes. When one considers the restriction of rough equivalence on theory systems, the corresponding rough equivalence class is denoted by $\widetilde{\lfloor T\rfloor}$ and the collection of all these classes by $\widetilde{\mathrm{ThSys}}(\mathcal{I})$. Reasoning with rough equivalence classes is one way of bypassing theory families with empty components. An alternative way is to ignore those theory families that have at least one empty component. This is accomplished by considering the collections $\mathrm{ThFam}^{\not\flat}(\mathcal{I})$ and $\mathrm{ThSys}^{\not\flat}(\mathcal{I})$ of all theory families and theory systems, respectively, none of whose components is empty.

The usefulness of rough equivalence in considering properties of the Leibniz operator stems from the fact that, for every theory family T, $\Omega(T) = \Omega(\widetilde{T})$. As a consequence, the Leibniz operator is constant on each rough equivalence class. It is fairly obvious that the rough companion $\widetilde{T}$ of a theory family T is the maximum element in the class $\widetilde{[T]}$. However, even if T happens to be a theory system, $\widetilde{T}$ may not be one. On the other hand, it can be shown that, even in that case, $\widetilde{\lfloor T\rfloor}$ has a maximum element, which, of course, does not coincide with $\widetilde{T}$. An unfortunate fact, when considering the operators $\overleftarrow{\ }$ and $\widetilde{\ }$ in the same context is that, even if two theory families T and T' are roughly equivalent, the same may not hold for $\overleftarrow{T}$ and $\overleftarrow{T'}$. On the positive side, if $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ is an $\mathbf{F}$-algebraic system and T is an $\mathcal{I}$-filter family of $\mathcal{A}$, we do have $\alpha^{-1}(\widetilde{T}) = \widetilde{\alpha^{-1}(T)}$. This implies that the action of α^{-1} preserves rough equivalence, i.e., if T and T' are $\mathcal{I}$-filter families of $\mathcal{A}$, with $T \sim T'$, then $\alpha^{-1}(T) \sim \alpha^{-1}(T')$, the latter being roughly equivalent theory families of $\mathcal{I}$.

In Section 6.3, we look at some notions combining systemicity with rough equivalence. They form a hierarchy weakening systemicity in the absence of theorems. In the presence of theorems, however, all concepts considered coincide. We say that a π-institution $\mathcal{I}$ is *roughly systemic* if, for every theory family T, $\overleftarrow{T}$ is roughly equivalent to T, i.e., $\overleftarrow{T} \sim T$. We say $\mathcal{I}$ is *narrowly systemic* if, for every theory family T in $\mathrm{ThFam}^{\nmid}(\mathcal{I})$ (i.e., with all components nonempty), $\overleftarrow{T} = T$. Finally, we say that $\mathcal{I}$ is *exclusively systemic* if, for all $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, such that $\overleftarrow{T} \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, $\overleftarrow{T} = T$. Systemicity is the strongest of these four conditions, followed by rough and narrow systemicity, which are incomparable in strength, and each of these two implies exclusive systemicity. Moreover, as mentioned previously, exclusive systemicity in the presence of theorems implies systemicity and, therefore, in that case, the entire hierarchy collapses to a single class.

In Section 6.4, we formalize and study various versions of rough injectivity, resulting by combining injectivity of the Leibniz operator with rough equivalence. The easiest to grasp is rough family injectivity. A π-institution $\mathcal{I}$ is *roughly family injective* if, for all theory families T, T', $\Omega(T) = \Omega(T')$ implies $T \sim T'$. *Rough left injectivity* results by replacing in the conclusion of the implication defining rough family injectivity T and T' by $\overleftarrow{T}$ and $\overleftarrow{T'}$, respectively. *Rough right injectivity* arises by a similar replacement in the hypothesis. Finally, *rough system injectivity* imposes the same condition as the family version, but restricts its application to theory systems. Rough right injectivity implies rough systemicity, but the converse fails in general. The rough injectivity hierarchy turns out to be more complex than the injectivity hierarchy studied in Section 3.6. There, it was shown that right injectivity implies family injectivity, which implies left injectivity, which, in turn, implies system injectivity, giving rise to a linear injectivity hierarchy. On the other hand, in the rough case, it is shown that rough right injectivity implies rough family injectivity, which implies the system version, and, in addition, rough left injectivity also implies the system version. Moreover, rough right injectivity is equivalent to rough system injectivity plus rough systemicity. Rough system injectivity, supplemented with stability, implies rough left injectivity. Each of the four rough injectivity properties, together with the availability of theorems, is equivalent to the corresponding injectivity property. The section concludes by establishing that all four rough injectivity properties transfer and by providing characterizations of rough family and rough system injectivity via the Leibniz operator Ω, viewed as a mapping from $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ and $\widetilde{\mathrm{ThSys}}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

In Section 6.5, we switch to a different version of injectivity properties, the overarching motivation still remaining that of bypassing theory families with empty components. *Narrow family injectivity* is defined by imposing the injectivity of the Leibniz operator on $\mathrm{ThFam}^{\nleq}(\mathcal{I})$, i.e., by stipulating that, for all $T, T' \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, $\Omega(T) = \Omega(T')$ implies $T = T'$. *Narrow left injectivity* replaces T, T' in the conclusion by $\overleftarrow{T}, \overleftarrow{T'}$, respectively, whereas *narrow right injectivity* applies the same replacement in the hypothesis. Finally, *narrow system injectivity* enforces the same condition as that of narrow family injectivity, but restricts its scope on theory systems in $\mathrm{ThSys}^{\nleq}(\mathcal{I})$. Narrow right injectivity implies exclusive systemicity, but does not imply any of the stronger versions of rough or narrow systemicity. With narrow injectivity, we recover the linearity of the injectivity hierarchy that was lost in passing to rough injectivity. That is, narrow right injectivity implies narrow family injectivity, which implies narrow left injectivity, which, in turn, implies the system version. Moreover, narrow system injectivity, supplemented by narrow systemicity, implies narrow right injectivity. It turns out that narrow family injectivity is equivalent to rough family injectivity. On the other hand, the two left injectivity properties, narrow left and rough left injectivity, ar incomparable, i.e., none implies the other. Some order is regained when looking at the right versions, where rough right injectivity implies narrow right injectivity. This order is maintained at the system level in which rough system injectivity also implies narrow system injectivity. As was the case with rough injectivity, each narrow injectivity property, supplemented with the existence of theorems, is equivalent to the corresponding injectivity property. Moreover, all four narrow injectivity properties transfer. Finally, the family and system versions have characterizations in terms of the injectivity of Ω, viewed as a mapping from $\mathrm{ThFam}^{\nleq}(\mathcal{I})$ and $\mathrm{ThSys}^{\nleq}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

In Sections 6.4 and 6.5, we looked at the rough and narrow injectivity hierarchies. Following this paradigm, in Sections 6.6 and 6.7, we introduce and study the rough and narrow reflectivity properties and, then, in Sections 6.8 and 6.9, the rough and narrow complete reflectivity properties.

In Section 6.6, we turn to rough reflectivity. Once more, the family version is the easiest to describe. A π-institution is called *roughly family reflective* if, for all theory families T, T', $\Omega(T) \leq \Omega(T')$ implies $\widetilde{T} \leq \widetilde{T'}$. *Rough left reflectivity* results by replacing T, T' in the conclusion by $\overleftarrow{T}$, $\overleftarrow{T'}$, respectively. *Rough right reflectivity* applies the same change in the hypothesis. Finally, *rough system reflectivity* imposes the same implication as the family version, but only on theory systems. Rough right reflectivity implies rough systemicity. It also implies rough family reflectivity, which implies rough system reflectivity. Rough left reflectivity also implies the system version. Rough right reflectivity is actually equivalent to the system version plus rough systemicity. On the other hand, rough system reflectivity and stability imply rough left reflectivity. It is straightforward to see, based on the relevant defining conditions, that each of the four rough reflectivity versions implies the corresponding rough injectivity version. Furthermore, each rough reflectivity version, supplemented by the existence of theorems, is equivalent to the corresponding reflectivity property. The section concludes with a proof that all four rough reflectivity properties transfer and with characterizations of rough family and rough system reflectivity in terms of the Leibniz operator, viewed as a mapping from $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ and $\widetilde{\mathrm{ThSys}}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

In Section 6.7, we look at narrow reflectivity properties. These constitute alternatives to rough reflectivity when dealing with reflectivity properties while attempting to bypass theory families with empty components. A π-institution is *narrowly family reflective* if, for all theory families T, T' in $\mathrm{ThFam}^{\nleftrightarrow}(\mathcal{I})$, $\Omega(T) \leq \Omega(T')$ implies $T \leq T'$. As before, *narrow left reflectivity* results by replacing T, T' in the conclusion by $\overleftarrow{T}$, $\overleftarrow{T'}$, respectively, and *narrow right reflectivity* by performing the same replacement in the hypothesis instead. Finally, *narrow system reflectivity* stipulates that, for all $T, T' \in \mathrm{ThSys}^{\nleftrightarrow}(\mathcal{I})$, $\Omega(T) \leq \Omega(T')$ implies $T \leq T'$. Narrow family reflectivity implies exclusive systemicity. As was the case with narrow injectivity properties, narrow reflectivity properties also align into a linear hierarchy. The strongest is narrow right reflectivity, followed by narrow family reflectivity, then by the left version and, at the tail, by narrow system reflectivity. The weakest one, narrow system reflectivity, supplemented by narrow systemicity, implies narrow right reflectivity. The relationships between corresponding rough and narrow versions of reflectivity follow those established in Section 6.5 between corresponding rough and narrow injectivity properties. First, rough family and narrow family reflectivity are equivalent. On the opposite end, the left versions turn out to be incomparable. Somewhere in between, for both the right and system versions, it turns out that the rough property implies the narrow one. Not surprisingly, each narrow reflectivity property implies the corresponding narrow injectivity property. Moreover, a given narrow reflectivity property is equivalent to the corresponding reflectivity property in the presence of theorems. All four narrow reflectivity properties transfer. Finally, characterizations are provided of narrow family and narrow system reflectivity in terms of the Leibniz operator seen as a mapping from $\mathrm{ThFam}^{\nleftrightarrow}(\mathcal{I})$ and $\mathrm{ThSys}^{\nleftrightarrow}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

In Section 6.8, we turn to complete reflectivity (c-reflectivity) properties starting with rough complete reflectivity. A π-institution $\mathcal{I}$ is *roughly family c-reflective* if, for every collection $\mathcal{T} \cup \{T'\}$ of theory families, $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$ implies $\bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T}'$. The *left version* results by replacing each theory family by its arrow counterpart in the conclusion, whereas the *right one* by applying the same change in the hypothesis instead. Finally, the *system version* stipulates that the same condition as that defining the family version applies, but $\mathcal{T} \cup \{T'\}$ is allowed to range over collections of theory systems instead of arbitrary theory families. Paralleling the rough reflectivity hierarchy, rough right c-reflectivity implies rough family c-reflectivity, which implies rough system c-reflectivity, while the left version also implies the system version. In fact, rough right c-reflectivity is equivalent to rough system c-reflectivity plus rough systemicity, whereas rough system c-reflectivity, together with stability, imply rough left c-reflectivity. It is clear that each rough c-reflectivity property generalizes the corresponding rough reflectivity property. It is also not difficult to show that each rough c-reflectivity property, in the presence of theorems, coincides with the corresponding c-reflectivity property. All four rough c-reflectivity properties transfer and, as before, characterizations may be formulated of the family and system versions in terms of the Leibniz operator, perceived as a mapping from $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ and $\widetilde{\mathrm{ThSys}}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

Section 6.9 deals with narrow complete reflectivity. A π-institution $\mathcal{I}$ is *narrowly family c-reflective* if, for every collection $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\lightning}(\mathcal{I})$, $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$ implies $\bigcap \mathcal{T} \leq T'$. Once more, the *left version* arises by replacing all theory families in the conclusion by their arrow counterparts and, similarly, the *right version* by performing the same change in the hypothesis. *Narrow system c-reflectivity* imposes the same condition as the family version, but restricted to collections $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^{\lightning}(\mathcal{I})$. As with narrow reflectivity, the narrow c-reflectivity hierarchy is linear. The right version is the strongest, followed by the family, then the left and, finally, the system version. In addition, narrow system c-reflectivity, together with narrow systemicity, implies the right version. Comparisons between the rough c-reflectivity and the narrow c-reflectivity classes also follow the pattern revealed for corresponding reflectivity properties. In accordance, rough family and narrow family c-reflectivity are equivalent, rough left and narrow left c-reflectivity are incomparable, whereas the rough right and rough system versions imply, respectively, the narrow right and narrow system versions. As with their rough counterparts in Section 6.8, all four narrow c-reflectivity properties coincide with the corresponding c-reflectivity properties in the presence of theorems. Furthermore, all four narrow c-reflectivity properties transfer. The family and system versions have characterizations via the Leibniz operator seen as a mapping from $\mathrm{ThFam}^{\lightning}(\mathcal{I})$ and $\mathrm{ThSys}^{\lightning}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$, analogous to the ones obtained for both narrow injectivity and narrow reflectivity.

The last section of the chapter, Section 6.10, contains some characterizations of the property of a π-institution possessing theorems. This is closely connected to the overarching ideas governing the properties investigated in Sections 6.2-6.9, which aimed at rectifying the "pathologies" introduced by the absence of theorems. The availability of theorems is characterized by the injectivity of the Frege equivalence family operator, as well as by both the injectivity and the complete reflectivity of the Lindenbaum equivalence family operator, both applied to the collection of theory families of the π-

institution. These operators were introduced in Section 2.11. Possession of theorems transfers to the collections of all $\mathcal{I}$-filter families over arbitrary **F**-algebraic systems.

1.3.6 Chapter 7

In Chapter 7, we further pursue our endeavor of making properties in the lower bottom of the algebraic hierarchy suitable for the study of π-institutions that do not have theorems. Similarly to Chapter 6, we employ rough equivalence and narrowness to achieve this goal, but, unlike in Chapter 6, the focus here is on monotonicity and complete monotonicity properties, rather than on injectivity, reflectivity and complete reflectivity properties.

In Section 7.2, we define a *stability hierarchy*, which serves, in the sequel, to formalize properties of some of the classes in the monotonicity and complete monotonicity hierarchies. Recall that a π-institution $\mathcal{I}$ is *stable* if, for all theory families $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(\overleftarrow{T}) = \Omega(T)$. Weakening this notion, we call $\mathcal{I}$ *narrowly stable* if the same equation holds, provided $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, i.e., the scope is restricted to theory families all of whose components are nonempty. A further weakening insists that the same equation hold for all $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, such that $\overleftarrow{T} \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, i.e., it further restricts the scope of the quantification to theory families all of whose components are nonempty and whose arrow counterparts also have all components nonempty. Clearly, stability implies narrow stability, which, in turn, implies the last property, which is termed *exclusive stability*. It is shown that both implications are strict.

In Section 7.3, we study the *rough monotonicity hierarchy*. Recall that, given a π-institution $\mathcal{I}$ and a theory family T of $\mathcal{I}$, $\widetilde{T}$ denotes the *rough companion* of the theory family T, which is the theory family resulting from T by replacing all empty Σ-components of T by $\mathrm{SEN}^\flat(\Sigma)$. Two theory families T and T' are *roughly equivalent* if they have the same rough companion. This is equivalent to saying that if T and T' differ at some signature Σ, they one has an empty Σ-component, whereas the other has $\mathrm{SEN}^\flat(\Sigma)$ as its Σ-component. A π-institution $\mathcal{I}$ is *roughly family monotone* if, for all theory families $T, T' \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{T} \leq \widetilde{T'}$ implies $\Omega(T) \leq \Omega(T')$. *Rough left monotonicity* results by replacing T, T' in the hypothesis by $\overleftarrow{T}$, $\overleftarrow{T'}$, respectively, and *rough right monotonicity* by applying the same replacement in the conclusion. *Rough system monotonicity* stipulates that the original implication hold, for all $T, T' \in \mathrm{ThSys}(\mathcal{I})$. It turns out that rough left monotonicity implies both rough family and rough right monotonicity and that each of the latter two implies the system version. Additionally, the strongest version, rough left monotonicity, is equivalent to the weakest, system, version, together with stability. Recall from Section 3.3 that family and left monotonicity are equivalent and this property was termed *protoalgebraicity*.

Recall also, from the same section, that system and right monotonicity are equivalent and this property was called *prealgebraicity*. Protoalgebraicity implies rough left monotonicity, whereas prealgebraicity implies rough right monotonicity. Tighter connections can be established under some fairly general hypotheses. For non almost inconsistent π-institutions, protoalgebraicity is equivalent to rough family or rough left monotonicity, coupled with the availability of theorems. Similarly, for π-institutions having a theory family $T \neq \mathrm{SEN}^\flat$, with $\overleftarrow{T} \neq \overline{\varnothing}$, prealgebraicity is equivalent to rough right or rough system monotonicity, supplemented with the availability of theorems. All four rough monotonicity properties transfer. E.g., $\mathcal{I}$ is roughly family monotone if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $\mathcal{I}$-filter families $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{T} \leq \widetilde{T}'$ implies $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. Both rough family and rough system monotonicity can be characterized using properties of the Leibniz operator viewed as a mapping from $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ and $\widetilde{\mathrm{ThSys}}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

In Section 7.4, we switch from rough monotonicity to *narrow monotonicity properties*. These constitute an alternative approach to bypassing theory families and theory systems with one or more empty components. We say that a π-institution $\mathcal{I}$ is *narrowly family monotone* if, for all theory families T, T', with all components nonempty, $T \leq T'$ implies $\Omega(T) \leq \Omega(T')$. The *left version* results by replacing T, T' by $\overleftarrow{T}$, $\overleftarrow{T'}$, respectively, in the hypothesis and the *right version* by performing the same replacement in the conclusion instead. *Narrow system monotonicity* stipulates that, for all $T, T' \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$, $T \leq T'$ implies $\Omega(T) \leq \Omega(T')$. Narrow left monotonicity implies narrow family monotonicity, which implies narrow system monotonicity, while the latter is also a consequence of narrow right monotonicity. Narrow left monotonicity is strong enough to yield exclusive stability, which, however, is the weakest of the three stability versions studied in Section 7.2. Under narrow systemicity, introduced in Section 6.3, the narrow monotonicity hierarchy collapses to a single class. Protoalgebraicity implies narrow left monotonicity and prealgebraicity implies the right version. In this case as well, tighter connections are possible under additional, fairly general, hypotheses, as was the case with rough monotonicity properties. Namely, under the hypothesis that $\mathcal{I}$ is not almost inconsistent, protoalgebraicity is equivalent to narrow left or narrow family monotonicity, coupled with the existence of theorems. And, provided that $\mathcal{I}$ possess a theory system $T \neq \overline{\varnothing}, \mathrm{SEN}^\flat$, prealgebraicity is equivalent to narrow right or narrow system monotonicity, together with the availability of theorems. Of central interest here is whether and how the rough monotonicity properties are related to the narrow monotonicity properties. In comparing the two hierarchies, we discover that the two family versions are equivalent, whereas each of the three remaining rough monotonicity properties implies the corresponding narrow monotonicity property. All four narrow monotonicity properties transfer. Finally, characterizations of the family and the system versions may be formulated in terms of the Leibniz operator seen as a mapping from $\mathrm{ThFam}^{\lightning}(\mathcal{I})$ and $\mathrm{ThSys}^{\lightning}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

In Section 7.5, we return to roughness, but study complete monotonicity (c-monotonicity) instead of monotonicity properties. *Rough family c-monotonicity* stipulates that, for all collections $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T}$ implies $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. *Rough left c-monotonicity* and *rough right c-monotonicity* result by replacing in the hypothesis and in the conclusion, respectively, all theory families by their arrow versions. *Rough system c-monotonicity* imposes the same condition as does the family version, but restricts its applicability on collections $\mathcal{T} \cup \{T'\}$ consisting of theory systems. Here, it turns out that each of the left, family and right versions implies the system version. Moreover, rough left c-monotonicity is equivalent to the conjunction of rough system c-monotonicity and stability. It is also the case that, under stability, the rough family and rough right c-monotonicity properties coincide and that, under rough systemicity, the entire rough c-monotonicity hierarchy collapses to a single class. From the definitions, it is obvious that each of the four rough c-monotonicity properties implies the corresponding rough monotonicity version. It is also the case that each c-monotonicity property implies its rough c-monotonicity counterpart. Once more, for non almost inconsistent π-institutions, family (left c-monotonicity, respectively) is equivalent to the conjunction of rough family (rough left, respectively) c-monotonicity and the existence of theorems. Furthermore, if $\mathcal{I}$ possesses a theory family $T \neq \mathrm{SEN}^\flat$, such that $\overleftarrow{T} \neq \overline{\varnothing}$, then system (right, respectively) c-monotonicity is equivalent to rough system (right, respectively) c-monotonicity plus the existence of theorems. All four rough c-monotonicity properties transfer and one may, in this case also, recast the family and system versions in terms of properties of the Leibniz operator seen as a mapping from $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ and $\widetilde{\mathrm{ThSys}}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

In Section 7.6, we switch from rough versions of c-monotonicity to narrow versions of the same property. A π-institution $\mathcal{I}$ is called *narrowly family c-monotone* if, for all collections $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\lightning}(\mathcal{I})$, $T' \leq \bigcup_{T \in \mathcal{T}} T$ implies $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. In the *left version*, all theory families are replaced in the hypothesis by their arrow counterparts and, in the *right version*, the same change is applied in the conclusion. The *system version* stipulates that the implication above hold for all collections $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^{\lightning}(\mathcal{I})$. Each of the left, family and right versions implies the system version. Moreover, each of the four c-monotonicity versions implies the corresponding narrow c-monotonicity version. As was the case in relating rough and narrow monotonicity classes in Section 7.4, rough family c-monotonicity is equivalent to narrow family c-monotonicity, whereas each of the other three rough c-monotonicity properties implies the corresponding narrow c-monotonicity version. From the definitions, it is clear that a narrow c-monotonicity property implies its narrow monotonicity counterpart, the latter being a specialization of the former. All four narrow c-monotonicity properties transfer. In closing, both the family and the system versions have characterizations in terms of properties of the Leibniz operator perceived as a mapping from $\mathrm{ThFam}^{\nmid}(\mathcal{I})$ and $\mathrm{ThSys}^{\nmid}(\mathcal{I})$, respectively, into $\mathrm{ConSys}^*(\mathcal{I})$.

1.3.7 Chapter 8

In Chapter 8, we undertake the study of *regularity*. Roughly speaking, it is the property stipulating that, whenever two sentences belong to a theory family of a given π-institution, they must be identified modulo the Leibniz congruence system relative to that theory family. When, in addition to regularity, availability of theorems is also postulated, the property of *assertionality* is obtained. Assertionality strengthens complete reflectivity and, as a result, it can be used to strengthen (weak) (pre)algebraizability properties. These strengthenings and their associated hierarchies are under the microscope in Sections 8.4-8.7. The classes of π-institutions obtained here are among the most powerful classes in the semantic hierarchy of π-institutions, i.e., satisfy the strongest properties and are included in most of the other classes in the hierarchy.

In Section 8.2, we introduce *regularity*. As was the case with other properties in preceding chapters, regularity comes in four different versions. Once more, we begin from the easiest to describe, the family version. A π-institution $\mathcal{I}$ is *family regular* if, for all theory families T, all signatures Σ and all Σ-sentences ϕ an ψ, if $\phi,\psi \in T_\Sigma$, then $\langle\phi,\psi\rangle \in \Omega_\Sigma(T)$. *Left regularity* results by replacing T in the hypothesis by $\overleftarrow{T}$, *right regularity* by performing the same replacement in the conclusion instead, whereas *system regularity* stipulates that the implication hold for all theory systems T. Family regularity is the strongest of the four properties, followed by right regularity, which implies left regularity, which, in turn, implies the system version. Thus, regularity properties are stratified into a linear hierarchy. System regularity plus stability imply left regularity, and right regularity plus stability yield family regularity. It follows that, under stability, the four-class hierarchy is reduced to two classes. On the other hand, system regularity plus systemicity clearly yield family regularity, whence, systemicity leads to a collapse of the regularity hierarchy into a single class. The family, left and system versions have elegant characterizations in terms of the Suszko operator and one of its variants. E.g., a π-institution $\mathcal{I}$ is family regular if and only if, for every signature Σ and all Σ-sentences ϕ and ψ, $\langle\phi,\psi\rangle \in \widetilde{\Omega}^{\mathcal{I}}_\Sigma(C(\phi,\psi))$, where $C(\phi,\psi)$ is the least theory family of $\mathcal{I}$ containing ϕ and ψ. All four regularity properties transfer. For instance, with regards to the right version, $\mathcal{I}$ is right regular if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, all $\mathcal{I}$-filter families T of $\mathcal{A}$, all signatures Σ in $\mathcal{A}$ and all Σ-sentences ϕ and ψ, $\phi,\psi \in T_\Sigma$ implies $\langle\phi,\psi\rangle \in \Omega^{\mathcal{A}}_\Sigma(\overleftarrow{T})$. The other three transfer results are formalized similarly.

Finally, the family and system versions may be characterized by the property that the filter family (system, respectively) in any reduced matrix family (system, respectively) is at most a singleton, in the sense that it consists of components with at most one element each.

In Section 8.3, we study *assertionality*. This is the property resulting from regularity by adding the requirement that theorems exist. Accordingly, four versions of assertionality are a priori obtained, depending on which of the four versions of regularity is postulated. They are termed *family*, *right*, *left* and *system assertionality* and, based on the hierarchy of regularity properties of Section 8.2, these also form a linear hierarchy, with the family version at the top, followed by the right, then the left and, finally, the system version at the bottom of the hierarchy. Assertionality is characterized by asserting, roughly speaking, that each theory family is fully determined by its Leibniz congruence system as the equivalence class of any theorem. Even though, a priori, there are four assertionality versions, there is a reduction holding without proviso. More precisely, it can be shown that right assertionality implies systemicity and this entails that right and family assertionality are equivalent. This property implies left assertionality, which, in turn, implies the system version. Moreover, the latter supplied with systemicity, implies family assertionality. By the definitions, it is clear that each assertionality version implies the corresponding regularity version. What is, however, more interesting, albeit not much more challenging to demonstrate, is that each assertionality property implies the corresponding complete reflectivity (c-reflectivity) property (see Section 3.8). So the assertionality properties may be viewed as further strengthening the hierarchy of reflectivity and c-reflectivity properties, studied in Sections 3.7 and 3.8. All three different assertionality properties transfer. Again, indicative of the flavor, a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is, e.g., left assertional if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, the π-institution $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ is left assertional, meaning that, on the one hand, the least $\mathcal{I}$-filter family of $\mathcal{A}$ has all components nonempty and, on the other, that, for all $\mathcal{I}$-filter families T of $\mathcal{A}$, all signatures Σ and all Σ-sentences ϕ and ψ, such that $\phi, \psi \in T_\Sigma$, one has $\langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_\Sigma(T)$. The section concludes with characterizations of the family and system versions, analogous to the ones provided in the conclusion of Section 8.2 for regularity. Namely, it is shown that $\mathcal{I}$ is family (system) assertional if and only if the filter family (system, respectively) of every reduced matrix family (system, respectively) is a singleton (i.e., consists of singleton components).

In Sections 8.4-8.7, we take advantage of the role of assertionality in strengthening of c-reflectivity to obtain strengthened versions of weak (pre)-algebraizability and (pre)algebraizability properties. The first two are obtained by combining assertionality properties with pre- or protoalgebraicity, whereas the latter are obtained by using (pre)equivalentiality instead.

In Section 8.4, we look at *regular weak prealgebraizability* properties. These result from adding to prealgebraicity (i.e., system monotonicity) a version of assertionality. Since there are three distinct versions of assertionality, one obtains three distinct corresponding versions of regular weak prealgebraizability. A π-institution $\mathcal{I}$ is *regularly weakly family* (RWF) *prealgebraizable* if it is prealgebraic and family assertional. It is *regularly weakly left* (RWL) *prealgebraizable* if it is prealgebraic and left assertional and it is *regularly weakly system* (RWS) *prealgebraizable* if it is prealgebraic and system assertional. Since the distinguishing feature between these three properties is the type of assertionality imposed, the assertionality hierarchy immediately yields that RWF prealgebraizability implies RWL prealgebraizability, which, in turn, implies RWS prealgebraizability. Equally clear from the definitions is the fact that RWF/L/S prealgebraizability implies, respectively, family/left/system assertionality. Additionally, the fact that each assertionality property implies its c-reflectivity counterpart entails that RWF/L/S prealgebraizability implies, respectively, WF/L/SC prealgebraizability (see Section 4.2). All three versions of regular weak prealgebraizability transfer. The section concludes with characterizations of the three versions based on the Leibniz operator viewed as a mapping between ordered sets of filter families/systems and congruence systems. To provide a flavor, we look at RWF prealgebraizability. The characterization states that $\mathcal{I}$ is RWF prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism, such that, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T/\Omega^{\mathcal{A}}(T)$ is a singleton.

In Section 8.5, we study *regular weak algebraizability*. The properties here are obtained from the regular weak prealgebraizability properties of Section 8.4 by upgrading prealgebraicity to protoalgebraicity. Accordingly, a π-institution $\mathcal{I}$ is *regularly weakly family* (RWF) *algebraizable* if it is protoalgebraic and family assertional, it is *regularly weakly left* (RWL) *algebraizable* if it is protoalgebraic and left assertional and it is *regularly weakly system* (RWS) *algebraizable* if it is protoalgebraic and system assertional. Notice that, since these properties constitute enhancements of the properties of Section 8.4, the right version has been absorbed within the family version. Here, however, protoalgebraicity, which, unlike prealgebraicity, implies stability, forces, in addition, the identification of the left and the system versions. Thus, there are only two distinct regular weak algebraizability properties, regular weak family (equivalently, right) algebraizability being the strongest and regular weak system (equivalently, left) algebraizability the weakest of the two. In comparing this two-step hierarchy with that of regular weak prealgebraizability properties, we discover that the two family versions coincide, whereas regular weak system algebraizability implies regular weak left prealgebraizability. As a consequence, the combined regular weak (pre)algebraizability hierarchy consists of four classes that are linearly ordered. Moreover, essentially due to the fact that assertionality properties imply c-reflectivity properties, each of the two regular weak algebraizability classes are included in the corresponding weak algebraizability classes. Both regular weak algebraizability properties transfer. Finally, both have characterizations in terms of the Leibniz operator seen as a mapping between ordered sets. E.g., $\mathcal{I}$ is RWS algebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphim, such that, for all $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, $T/\Omega^{\mathcal{A}}(T)$ is a singleton.

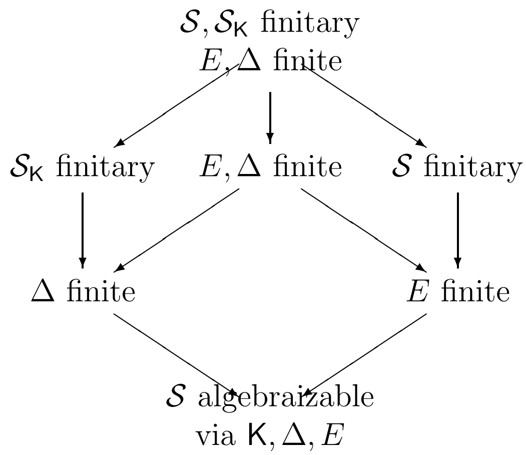
In Section 8.6, we introduce *regular prealgebraizability* properties. These are obtained by combining assertionality properties with preequivalentiality. Recalling that preequivalentiality is obtained by adding system extensionality to prealgebraicity, an alternative point of view is that regular prealgebraizability is obtained from regular weak prealgebraizability, studied in Section 8.4, by adding system extensionality. A π-institution $\mathcal{I}$ is *regularly family* (RF) *prealgebraizable* if it is preequivalential and family assertional, it is *regularly left* (RL) *prealgebraizable* if it is preequivalential and left assertional and it is *regularly system* (RS) *prealgebraizable* if it is preequivalential and system assertional. Based on the linear hierarchy of assertionality properties, we obtain a linear hierarchy of regular prealgebraizability properties, with RF prealgebraizability at the apex, followed by RL prealgebraizability, while RS prealgebraizability is at the bottom. Since preequivalentiality strengthens prealgebraicity, RF/L/S prealgebraizability implies, respectively, RWF/L/S prealgebraizability. Moreover, since each version of assertionality implies the corresponding c-reflectivity version, RF/L/S prealgebraizability implies, respectively, family/ left c-reflective/ system prealgebraizability (see Section 5.5). All three versions transfer. Finally, characterization theorems may be formulated for each of the three properties in terms of the Leibniz operator viewed as a mapping between ordered sets. To provide, once more, a preview, we mention the form this characterization takes in the case of regular left prealgebraizability. A π-institution $\mathcal{I}$ is regularly left prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system, $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order embedding, commuting with inverse logical extensions, such that, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\overleftarrow{T}/\Omega^{\mathcal{A}}(T)$ is a singleton.

In Section 8.7, the last section of Chapter 8, we turn to the study of *regular algebraizability* properties, which combine equivalentiality with assertionality. Equivalentiality forms a common strengthening of both protoalgebraicity and preequivalentiality. Even though one obtains, a priori, three versions of regular algebraizability, only two are distinct. We say that $\mathcal{I}$ is *regularly family* (RF) *algebraizable* if it is equivalential and family assertional, *regularly left* (RL) *algebraizable* if it is equivalential and left assertional, and *regularly system* (RS) *algebraizable* if it is equivalential and system assertional. Regular left and regular system algebraizability coincide and, as a result, the regular algebraizability hierarchy consists of the class of RF algebraizable π-institutions and its proper subclass of RS algebraizable π-institutions. In comparing regular algebraizability with regular prealgebraizability properties, we discover that the two family versions are equivalent and that regular system algebraizability implies regular left prealgebraizability. Further, in comparing regular algebraizability with regular weak algebraizability properties, we obtain, based on equivalentiality's dominant position over protoalgebraicity, that RF/S algebraizability implies, respectively, RWF/S algebraizability. In the ultimate comparison between subhierarchies, based on the fact that assertionality implies c-reflectivity, we obtain that RF/S algebraizability implies, respectively, F/S algebraizability. The section closes with the same type of theorems as previous sections. Namely, it is shown that both versions of regular algebraizability transfer from a π-institution to the filter families/systems over arbitrary $\mathbf{F}$-algebraic systems and characterizations of both versions are obtained in terms of the Leibniz operator perceived as a mapping between ordered sets. The family version, e.g., asserts that $\mathcal{I}$ is regularly family algebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism commuting with inverse logical extensions, such that, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T/\Omega^{\mathcal{A}}(T)$ is a singleton.

1.3.8 Chapter 9

In Chapter 9, we undertake the study of finitarity properties of weakly family algebraizable π-institutions. Here we draw inspiration by the analysis of corresponding properties of algebraizable sentential logics.

According to the theory of algebraization of sentential logics, a, not necessarily finitary, algebraizable sentential logic $\mathcal{S}$ is algebraized via an equivalence that relates its consequence relation with the equational consequence of a generalized quasivariety K. The relation of equivalence is established via a possibly infinite set of defining equations $E(x)$ in a single variable x, which serve to translate formulas into equations, and a possibly infinite set $\Delta(x,y)$ of equivalence formulas in two variables x and y, which serve to translate equations into formulas. Besides constituting interpretations between the two consequences, they should be mutually inverse in a specific sense. In examining the relationships between the various finitarity conditions that may hold, namely, $\mathcal{S}$ finitary, $\mathcal{S}_{\mathsf{K}}$ (the equational deductive system induced by K) finitary, $E(x)$ finite and $\Delta(x,y)$ finite, one may show that they are related by the implications depicted in the following diagram (see p. 137 in Section 3.4 of [86]).

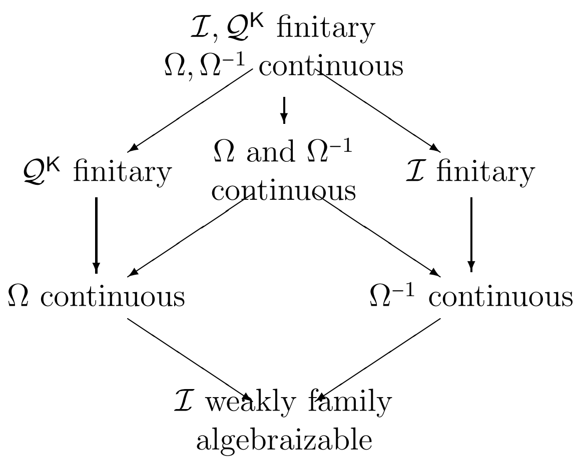


In the framework of sentential logics, roughly speaking, syntactic and semantic properties, i.e., those imposing the existence of transformations, such as $E(x)$ and $\Delta(x, y)$, satisfying certain properties, and those defined by order-theoretic properties of the Leibniz operator go hand-in-hand, in a tight correspondence. This is not the case in the framework of logics formalized as π-institutions. So in this chapter, the goal is to translate the sentential finitarity conditions to corresponding semantic properties and to establish an analogous hierarchy for weakly family algebraizable π-institutions. We also use examples from the sentential framework, recasting them as π-institutions, to obtain logical systems that serve to separate the classes of π-institutions specified by these finitarity properties.

In Section 9.2, the concept of *π-structure* is introduced, which abstracts that of a π-institution by removing the requirement of structurality. For π-structures, and, hence, also for π-institutions, the *finitary companion* is constructed, which is the π-structure over the same base algebraic system that has the largest finitary closure family included in the closure family of the given π-structure. *Locally finitely generated theory families* are defined and they are used to characterize those sentence families of a π-structure that are theory families of its finitary companion. These turn out to be exactly those sentence families that are unions of directed collections of locally finitely generated theory families of the given π-structure.

In Section 9.3, we investigate under which provisos, if any, the properties that define weak family algebraizability, i.e., protoalgebraicity and family reflectivity, are inherited by the finitay companion from the original π-structure and vice-versa. It is shown, first, that protoalgebraicity and family reflectivity are propagated from the finitary companion up to the parent π-structure unconditionally. On the other hand, the reverse inheritance requires additional conditions. To this end, the concept of *continuity* of the Leibniz and of the inverse Leibniz operator are introduced. The latter, of course, makes sense only if the π-institution under consideration is such that its Leibniz operator is an isomorphism, e.g., when it is weakly family algebraizable, which is precisely the case we focus on. If the Leibniz operator is continuous, it is easy to see that the π-institution is protoalgebraic. So continuity of the Leibniz operator actually strengthens protoalgebraicity. If, in addition to continuity, finiteness of the signature category is postulated, then the finitary companion is also protoalgebraic. Finally, it is shown that, if a π-institution, with a finite category of signatures, is weakly family algebraizable and both its Leibniz and inverse Leibniz operators are continuous, then its finitary companion is also weakly family algebraizable.

In Section 9.4, we undertake a detailed study of the interrelationships of the four finitarity properties pertaining to weakly family algebraizable π-institutions. These are the finitarity of the π-institution itself, the finitarity of its equational counterpart, the continuity of the Leibniz operator and the continuity of the inverse Leibniz operator, which is well-defined precisely because the π-institution is assumed to be weakly family algebraizable. These four properties are appropriate abstractions in the semantical institutional context of the properties of an algebraizable sentential logic being finitary, of its equivalent algebraic semantics being a quasivariety, of the set of equivalence formulas being finite and of the set of defining equations being finite, respectively. The close analogy is reflected in the fact that the results and hierarchy obtained here parallel the ones that hold for the corresponding properties in the sentential context. Our results come, as do their sentential counterparts, in dual pairs. In the first, it is shown that, for a weakly family algebraizable π-institution $\mathcal{I}$, the finitarity of $\mathcal{I}$ implies the continuity of its inverse Leibniz operator and, dually, the finitarity of the equational π-structure $\mathcal{Q}^{\mathsf{K}}$ induced by $\mathsf{K} := \mathrm{AlgSys}(\mathcal{I})$ implies the continuity of the Leibniz operator itself. Next, it is shown that, under weak family algebraizability, the finitarity of $\mathcal{I}$ and the continuity of the Leibniz operator imply that the equational counterpart is also finitary. Dually, the finitarity of the equational counterpart and the continuity of the inverse Leibniz operator imply that $\mathcal{I}$ itself is finitary. These implications lead to the following conditional equivalences, all applying to weakly family algebraizable π-institutions. For continuous Leibniz and inverse Leibniz operators, a π-institution is finitary if and only if its algebraic counterpart is finitary. For a finitary π-institution, its counterpart is finitary if and only if its Leibniz operator is continuous. Finally, if the algebraic counterpart of a π-institution is finitary, then the π-institution itself is finitary if and only if its inverse Leibniz operator is continuous. These outcomes lead to a finitarity hierarchy for weakly family algebraizable π-institutions paralleling the hierarchy depicted above for sentential logics.



What remains to be done is separate the classes of π-institutions constituting the finitarity hierarchy. For this task, given the analogies established with the sentential framework, we seek inspiration from the realm of sentential logics.

In Section 9.5, we revisit three sentential logics that serve in separating the classes that form the finitarity hierarchy in the sentential framework. The classes related by the vertical arrows are separated by Łukasiewicz's infinite valued logic. This is a non-finitary, semantically defined sentential logic. It is algebraizable with a non-finitary equivalent algebraic semantics. On the other hand, both sets of defining equations and equivalence formulas are finite. The classes connected by the southeast arrows are separated using a finitary logic introduced by Dellunde and defined via a Hilbert calculus. It is regularly algebraizable via a singleton set of defining equations but a necessarily infinite set of equivalence formulas. Finally, the classes related by the southwest arrows of the diagram are separated using a non-finitary logic semantically defined by Raftery. This logic has a finitary equivalent algebraic semantics (actually a variety) and is algebraized via a finite set of equivalence formulas but a necessarily infinite set of defining equations. Even though we could certainly rely on well-written accounts from the literature to simply refer to these logics, we chose to recount all details, based on those original references. The Introduction to Chapter 9 and the main body contain more information, as well as appropriate references.

In Section 9.6, the three sentential logics of Section 9.5 are recast as π-institutions, according to the general procedure outlined in Section 1.1. The resulting π-institutions serve, in turn, in separating the corresponding classes appearing in the finitarity hierarchy of weakly family algebraizable π-institutions. Further evidencing the analogies described between the two finitarity hierarchies, the π-institution encapsulating Łukasiewicz's logic separates the classes of π-institutions connected via vertical arrows, the one incorporating Dellunde's logic separates classes along the southeast arrows, while the one arising from Raftery's logic separates classes related by the southwest arrows in the institutional finitarity hierarchy.

1.4 A Very Concise Summary of Contents

In Chapter 2, we introduce the basic definitions and fundamental results of algebra and logic and some indispensable notions and results pertaining to their interaction. These form the necessary background and the prerequisites for the general theory of algebraization of logics formalized as π-institutions that is presented in the monograph.

In Chapter 3, we introduce fundamental classes of the semantic Leibniz hierarchy. The term semantic alludes to the fact that they are defined purely by properties of the Leibniz operator on the complete lattices of the theory families or theory systems of π-institutions. Very central to our studies throughout, partly because they equip us with indispensable terminology regarding crucial properties, are the classes of systemic and stable π-institutions. At the bottom center of the hierarchy lie the loyalty properties. These simultaneously abstract monotonicity properties, on the one side, and reflectivity properties, on the other side. On the monotonicity side, we study monotonicity and two kinds of complete monotonicity, complete $\bigcup$-monotonicity, using the union operation, and complete $\bigvee$-monotonicity, using the join operation. In crossing over to the reflectivity side, we pass through, and study, injectivity properties. On the other side, we look, first, at reflectivity and, finally, at complete reflectivity properties. In Chapter 3, we not only define various flavors of each of these properties and compare their various strengths, but we also investigate the relations across those different kinds of properties. On the way, we also present many concrete examples, some of which are reused throughout the monograph to illustrate concepts, but, also - and mainly - to separate classes in the various hierarchies.

In Chapter 4, we study weak prealgebraizability and weak algebraizability properties. Weak prealgebraizability arises by combining prealgebraicity (system monotonicity) with one of the ten possible versions of injectivity, reflectivity or complete reflectivity. On the other hand, weal algebraizability results when combining protoalgebraiciy (family monotonicity) with one of those ten versions. Taking into account the combined hierarchy of injectivity, reflectivity and complete reflectivity properties, established in Chapter 3, we obtain a hierarchy of ten potentially different classes of weak prealgebraizability and a similar one consisting of ten potentially different classes of weak algebraizability. However, it is shown that the weak prealgebraizability hierarchy collapses down to six classes, whereas the one of weak algebraizability down to only two. Moreover, the top classes in the two hierarchies are identical. Therefore, when the two hierarchies are merged, a combined hierarchy consisting of seven distinct classes is obtained. The chapter includes, inter alia, characterizations of these seven classes using the Leibniz operator perceived as a mapping from the lattice of filter families to the poset of congruence systems over arbitrary algebraic systems.

In Chapter 5, we study the hierarchies of prealgebraizable and of algebraizable π-institutions. We look, first, at the property of extensionality and the seemingly weaker property of 2-extensionality and show that they are equivalent. Roughly speaking, extensionality relates Leibniz congruence systems of theories of an institution with those of corresponding theories of subinstitutions. Then, we look at the closely related properties of (Leibniz) commutativity and inverse (Leibniz) commutativity. These two properties are equivalent under monotonicity and, moreover, inverse commutativity is equivalent to extensionality. By combining monotonicity with extensionality properties, we build the hierarchy of equivalential π-institutions. Depending on which of the available versions of monotonicity or extensionality are imposed, three versions of equivalentiality arise, namely, equivalentiality, family preequivalentiality and (system) preequivalentiality in decreasing strength. By combining versions of preequivalentiality with injectivity, reflectivity or complete reflectivity properties, the ten classes of the prealgebraizability hierarchy are obtained. Similarly, by combining equivalentiality with injectivity properties (which are, in the presence of equivalentiality, equivalent to corresponding reflectivity or complete reflectivity properties), we get the two classes of algebraizable π-institutions.

In Chapter 6, we look at classes of the Leibniz hierarchy lying below the classes of injective, reflective and completely reflective π-institutions, which were introduced in Chapter 3. The motivating observation is that, if a π-institution satisfies injectivity or, a fortiori, reflectivity or complete reflectivity, then it must possess theorems. Thus, π-institutions without theorems are automatically excluded from consideration in contexts where these properties are postulated or studied. To bypass this hurdle, we define and study weakened versions of injectivity, reflectivity and complete reflectivity that can accommodate absence of theorems, but are equivalent to injectivity, reflectivity and complete reflectivity, respectively, in the presence of theorems. For each of those three properties, we study the rough versions and the narrow versions and carefully compare them to the original versions, as well as to each other, to obtain the hierarchies of injectivity, rough injectivity, narrow injectivity, reflectivity, rough reflectivity and narrow reflectivity, and c-reflectivity, rough c-reflectivity and narrow c-reflectivity classes of π-institutions. Roughly speaking, roughness identifies two theory families if their Σ-components are either equal or one is $\varnothing$ and the other is $\mathrm{SEN}^\flat(\Sigma)$. Those turn out to have identical Leibniz congruence systems. On the other hand, narrowness excludes from consideration altogether theory families with at least one empty component.

In Chapter 7, we continue the study of properties of π-institutions obtained by combining properties lying at the bottom of the Leibniz hierarchy with rough equivalence, on the one hand, and with narrowness, on the other. As opposed to Chapter 6, which considered properties lying below injectivity, reflectivity and complete reflectivity, this chapter undertakes the study of properties lying below monotonicity and complete monotonicity (c-monotonicity) properties. In a nutshell, roughly monotone and roughly c-monotone π-institutions form super classes, respectively, of the classes of monotone and c-monotone π-institutions. Additionally, narrowly monotone and narrow c-monotone π-institutions encompass respectively, roughly monotone and roughly c-monotone ones. By studying all four versions of each of these properties, we obtain a mixed hierarchy of rough and narrow monotonicity and rough and narrow c-monotonicity properties.

In Chapter 8, we study properties obtained by combining pre- or protoalgebraicity or (pre)equivalentiality, on the one hand, with assertionality, on the other. The latter, a property that strengthen complete reflectivity asserts, roughly speaking, that a π-institution has theorems and, in addition, each of its theory families is determined by its associated Leibniz congruence system as the equivalence class of a theorem. The chapter starts with the study of regularity, a property similar to assertionality, except that it does not require existence of theorems. It holds when any two sentences belonging to a theory family are identified modulo the Leibniz congruence system relative to that theory family. Assertionality properties are formalized next. The hierarchy they form and its interrelationships with the classes of the regularity hierarchy are explored in detail. Prealgebraicity, coupled with asserrtionality, gives rise to regular weak prealgebraizability, strengthening the classes of weak prealgebraizability properties. Protoalgebraicity, together with assertionality, leads to regular weak algebraizability properties. This hierarchy strengthens both regular weak prealgebraizability and weak algebraizability properties. Preequivalentiality and assertionality give rise to regular prealgebraizability, which strengthens both regular weak prealgebraizability and prealgebraizability. The chapter concludes with the study of regular algebraizability, which combines equivalentiality with assertionality. The classes of this hierarchy form subclasses of both those consisting of regularly prealgebraizable and those consisting of algebraizable π-institutions.

Chapter 9 starts with the introduction of the finitary companion of a π-institution. It is the largest finitary π-institution below the given one in the $\leq$ ordering of π-institutions based on the same algebraic system. The focus is on those properties defining weak family algebraizability, namely protoalgebraicity and family reflectivity. We investigate under which conditions, if any, those properties are passed from a π-institution to its finitary companion and vice-versa. In the second part, the focus shifts to the study of finitarity properties of weakly family algebraizable π-institutions. This class of π-institutions is chosen because, on its members, the Leibniz operator is an isomorphism and, hence, it makes sense to consider the inverse Leibniz operator. The four finitarity properties under investigation are the finitarity of the π-institution itself, the finitarity of its algebraic counterpart and the continuity of the Leibniz operator and of the inverse Leibniz operator. The implications holding between these properties give rise to the finitarity hierarchy of weakly family algebraizable π-institutions. The chapter also revisits some examples of sentential logics and formalizes them as π-institutions. The latter are then used to separate the various classes in the finitarity hierarchy. The three examples are Łukasiewicz's infinite valued logic, Dellunde's logic and a logic due to Raftery.

1.5 Further Reading

This is the first attempt to systematize the body of knowledge gathered over the years concerning the algebraization of logics formalized as π-institutions. However, for the readers interested in learning much more about the origins, history, concepts, results and developments in algebraic logic as applied to deductive systems, i.e., "abstract algebraic logic", there are a few excellent sources available that have served well over the years in educating the second and third generations of "abstract algebraic logicians".

Starting tangentially to the subject, but of interest, since they provide a comprehensive study of logical calculi and of institutions, respectively, the latter being the precursors of π-institutions used here, are the monographs by Wójcicki [34] and Diaconescu [79].

Two of the first sources that played a critical role in establishing and solidifying the discipline in its present form were the seminal "Memoirs" monograph of Blok and Pigozzi [35], in which algebraizable logics were introduced, and the pioneering monograph of Font and Jansana [52], in which generalized matrices were studied in a systematic way and the notion of Tarski congruence and accompanying reduced class of generalized matrices and underlying class of algebras were defined and studied in detail.

More at the textbook, rather than at the research, level, are the books of Czelakowski [64] and the more recent textbook by Font [86]. These are the only two books, to my knowledge, that are focused on systematically treating and presenting the most important results in the abstract setting. It goes, of course, without saying, that they both contain a plethora of concrete examples that have been studied in the literature, showcasing various aspects of the general theory and exemplifying the wide reach of its applicability.

Apart from research monographs and books, a few surveys have also appeared that provide overviews of, and/or details on, significant parts of the theory. Among them are [40], [68], [69, 80] and [90].

Finally, there have been a few, as far as I am aware, Ph.D. Dissertations which have dealt, either in their introductions or in their main corpus, with expositions and/or overviews of significant parts of the theory. Among them, some that have helped my own understanding and enhanced and/or diversified my point of view of various aspects of the theory are, in chronological order, those of Herrmann [43], Elgueta [47], Rebagliato [49], Dellunde [51], Gyuris [60], Martins [70], Russo [78], Albuquerque [85] and Moraschini [87].

The algebraization of logics formalized as π-institutions may be said to have started with the Ph.D. Dissertation by the author [97] (see, also, [98]), under the influence of preceding unpublished work by Zinovy Diskin [46] (see, also, [51]), which had been communicated to Professor Don Pigozzi, the author's Ph.D. Dissertation advisor, and used with Zinovy's kind permission and encouragement.

Chapter 2

Algebra and Logic

2.1 Introduction

In Section 2.2, we introduce the basic algebraic machinery that underlies all structures considered in the monograph. We start with *sentence functors*, which are arbitrary **Set**-valued functors on a category of signatures. *Sentence families* are families of sets over sentence functors. They are called *systems* in case they are invariant under signature morphisms. Associated with a sentence family T is the largest sentence system $\overleftarrow{T}$ included in T and the smallest sentence system $\overrightarrow{T}$ which includes T. We also introduce and discuss *morphisms* between sentence functors and, in particular, distinguish the key class of *surjective morphisms*. By analogy to sentence families, one may also consider *relation families* over sentence functors, i.e., families of relations on sentences. Relation families satisfying the requisite properties constitute *equivalence families*. A fundamental notion, pervasive throughout our treatise, is that of *compatibility* of an equivalence family with a given sentence family. The importance of compatibility was exemplified in [35] (see, e.g., Section 1.4 of [35], where the notion is defined). Whereas sentence functors capture the underlying carriers of all algebraic and logical structures we consider, the earnest algebraic treatment begins when they get endowed with *categories of natural transformations* which correspond to clones of algebraic operations [31, 44]. These enriched structures are termed *algebraic systems*. Appropriate mappings, preserving the relevant features, are also called *morphisms* (of algebraic systems). In most contexts, it is required that all algebraic systems under consideration are over the same algebraic signature. This is ensured by adopting a base algebraic system $\mathbf{F}$, which fixes the signature, and, then, considering only algebraic systems whose sentences and clones of operations are, in a certain sense, interpretations of the basic one. These play an important role and are termed *interpreted algebraic systems* or $\mathbf{F}$-*algebraic systems*.

In Section 2.3, we introduce and study *congruence systems*. These are equivalence systems on an underlying algebraic system that satisfy a suitably adapted version of the congruence (sometimes also called compatibility or replacement) property. They play in this context the role that congruences play in universal algebra [22, 13, 21, 30, 84]. The collection of congruence systems on a given algebraic system forms a complete lattice. Of utmost importance is the process of constructing the *quotient* of an algebraic system by a congruence system and of the accompanying *canonical quotient morphism*. Equally important, in fact indispensable for the development of the theory, is the fact that the collection of congruence systems on a given algebraic system $\mathbf{A}$ that are compatible with a given sentence family T of $\mathbf{A}$ form a complete lattice. This fact allows considering the largest congruence system on $\mathbf{A}$ compatible with T, which is denoted by $\Omega^{\mathbf{A}}(T)$ and termed the *Leibniz congruence system of T on $\mathbf{A}$* [35]. A property that is worth mentioning, since it plays a critical role in establishing pieces of the various hierarchies considered in subsequent chapters, is that the Leibniz congruence system of a sentence family T is always included in that of the largest sentence system contained in T, i.e., $\Omega^{\mathbf{A}}(T) \leq \Omega^{\mathbf{A}}(\overleftarrow{T})$.

In Section 2.4, we look at a special class of congruence systems whose definition presupposes fixed in the background a class K of algebraic systems. Given an arbitrary algebraic system, a congruence system on it is said to be a K-*congruence system* or a *congruence system relative to* K if the quotient algebraic system it induces belongs to the class K (see, e.g., Chapter Q of [64]). Two important concepts in this context are *closure of a class under morphic images* and *closure under subdirect intersections*. If the class K is closed under morphic images, then, for every algebraic system in K, the absolute and relative concepts of congruence system coincide. On the other hand, if K happens to be closed under subdirect intersections and contains a trivial algebraic system, then the collection of all K-congruence systems on any algebraic system forms a complete lattice. In this case, it makes sense to consider, given a relation system X on an algebraic system $\mathcal{A}$, the *least* K-*congruence system on* $\mathcal{A}$ *including* X, also known as the K- *congruence system generated by* X, and denoted by $\Theta^{\mathsf{K},\mathcal{A}}(X)$. In the main result of the section, it is shown that this congruence system coincides with the equational closure of X relative to the class K.

In Section 2.5, we introduce *semantic* and *syntactic varieties* of algebraic systems. These play the role that varieties play in universal algebra (see, e.g., [21, 30, 84]). All algebraic systems are understood to be over a fixed signature specified by a base algebraic system $\mathbf{F}$. To define the two types of varieties, we look at *equations*, consisting of pairs of sentences, and at *natural equations*, which are pairs of natural transformations. Given a class K of algebraic systems, the *semantic variety* generated by K is the class of all algebraic systems satisfying all equations valid in all members of K. The *syntactic variety* generated by K is defined analogously with reference to natural equations. It turns out that the semantic variety generated by K is subsumed by the corresponding syntactic one. A technical definition, that of a *transformational algebraic system*, is introduced as a way to establish a sufficient condition for semantic and syntactic varieties to coincide.

In Section 2.6, we switch from purely algebraic to logical considerations. We define *systems of closure operators* on algebraic systems, which give rise to π-*institutions* [33] (see, also, [25, 41]). Those constitute the basic underlying logical structures on which all subsequent studies will be founded. Many well-known fundamental logical concepts are adapted to this framework, among them, *theorem systems*, *theory families* and *inconsistent*, *almost inconsistent* and *trivial* π-*institutions* (see, e.g., [64, 86] for the counterparts in abstract algebraic logic). Concerning theory families, it is worth mentioning that in case T is a theory family of a given π-institution, the construction of $\overleftarrow{T}$ gives rise to a theory system, and not merely a sentence system, but this is not the case for $\overrightarrow{T}$. Therefore, $\overleftarrow{T}$ does constitute the largest theory system included in T, but to construct the smallest theory system including T, one has to apply the closure operator and obtain $C(\overrightarrow{T})$. Comparing closure systems over the same underlying algebraic system, the notions of *extension* and *weakening* are introduced, as well as that of the closure system obtained as the intersection of a family of closure systems. Given a closure system C and one of its theory systems T, we also consider the extension C^T of C that is induced by adopting the given theory system as a system of axioms. Finally, we look at *logical morphisms* between π-institutions. These are morphisms that preserve the logical structure, i.e., map closures into closures in a formal sense, or, what turns out to be equivalent, morphisms whose inverses preserve theory families.

In Section 2.7, after having discussed the algebraic and logical prerequisites, we turn into developing the first rudiments of their interaction. We look at *matrix families* which serve both to define closure systems, and, hence, also, π-institutions, but also as algebraically based models of given π-institutions. They are pairs consisting of an underlying algebraic system together with a sentence family over it and correspond to the ordinary logical matrices of abstract algebraic logic [64, 86]. For a given π-institution $\mathcal{I}$, its matrix family models are termed *$\mathcal{I}$-matrix families* and the corresponding sentence families are called *$\mathcal{I}$-filter families*. Some characterizations of these families are provided along with the observation that the collection of all $\mathcal{I}$-filter families on a given algebraic system forms a complete lattice. A discussion follows on when and under which conditions morphisms between the underlying algebraic systems preserve, under taking direct or inverse images, $\mathcal{I}$-filter families. In closing the Section, we look at *quotients of matrix families* under the Leibniz congruence systems of their filter families. These are referred to as *Leibniz reductions* (see, e.g., Section 4.3 of [86]). We say that a matrix family is *Leibniz reduced* when the Leibniz congruence system of its filter family is the identity. Leibniz reductions give rise to the fundamental collection of *Leibniz reduced $\mathcal{I}$-matrix families* and the accompanying collection of their algebraic system reducts. Two more related subcollections are obtained if one restricts attention to *$\mathcal{I}$-filter systems* and *$\mathcal{I}$-matrix systems*, i.e., those that consist of filter families that are invariant under the action of signature morphisms.

In Section 2.8, continuing the study of filter families and matrix families, we introduce *axiomatic extensions*, or *axiomatic strengthenings*, and the closely related concept of *filter extension* (see Section 0.8 of [64] and Sections 1.3 and 1.4 of [86]). We provide characterizations and study interactions with morphisms, looking, in particular, into some preservation properties.

In Section 2.9, a generalization of matrix families and filter families is introduced. Namely, we consider structures consisting of an underlying algebraic system together with a collection of sentence families over it. These are called *generalized matrix families* or *gmatrix families*, for short. They play the role that generalized matrices play in the traditional treatment [52] (see, also, Chapter 5 of [86]). As was the case with matrix families, gmatrix families serve a dual purpose. They may be used to define closure systems, but they also serve as models of π-institutions. In the latter case, if a gmatrix family is a model of a given π-institution $\mathcal{I}$, we say that it is an *$\mathcal{I}$-gmatrix family*. By analogy with $\mathcal{I}$-matrix families, one may consider reductions of gmatrix families. The *Tarski congruence system* of a gmatrix family is the largest congruence system on its underlying algebraic system which is compatible with all filter families of the gmatrix family [52]. Equivalently, it may be characterized as the intersection of all Leibniz congruence systems of its constituent filter families. The process of taking the quotient of a gmatrix family by its Tarski congruence system is called *Tarski reduction*. We say that a gmatrix family is *Tarski reduced* if its Tarski congruence system in the identity. The construction gives rise to the class of all Tarski reduced $\mathcal{I}$-gmatrix families and the class of the corresponding algebraic system reducts. Both are of critical importance in the study of algebraization of π-institutional logics. Very intimately related to Tarski congruence systems is the notion of *Suszko congruence systems* [67] (see, also, Section 1.5 of [64] and Section 5.3 of [86]). Here, one considers the filter family subcollection $\mathcal{T}^T$ of a filter family collection $\mathcal{T}$ by keeping only those filter families containing a fixed filter family $T \in \mathcal{T}$. The *Suszko congruence system of T relative to $\mathcal{T}$* is the Tarski congruence system of $\mathcal{T}^T$. Conversely, assuming that $\mathcal{T}$ has a smallest filter family T, the Tarski congruence system of $\mathcal{T}$ coincides with the Susko congruence system of T in $\mathcal{T}$. As before, one may consider *Suszko reductions* and *Suszko reduced $\mathcal{I}$-matrix families*, where the reductions are taken relative to the collection of all $\mathcal{I}$-filter families. Even though, given a π-institution $\mathcal{I}$, this process results in the new class of Suszko reduced $\mathcal{I}$-matrix families, the class of corresponding algebraic system reducts turns out to be identical with that obtained from the process of Tarski reduction.

In Section 2.10, we continue the study of classes of algebraic systems associated with a given π-institution $\mathcal{I}$. In Section 2.7, we introduced the class of all algebraic system reducts of all Leibniz reduced $\mathcal{I}$-matrix families. This class is known as the class of *$\mathcal{I}^*$-algebraic systems*. In Section 2.9, we looked at the class of all algebraic system reducts of all Tarski reduced $\mathcal{I}$-gmatrix families. These are known as *$\mathcal{I}$-algebraic systems*. The two classes correspond, respectively, to the classes $\mathrm{Alg}^*\mathcal{S}$ and $\mathrm{Alg}\mathcal{S}$ in the case of a sentential logic $\mathcal{S}$ [52]. On top of these two classes of algebraic systems, two more classes considered in relation to a π-institution $\mathcal{I}$ are the semantic and syntactic varieties generated by the underlying algebraic system of the Tarski reduction of the $\mathcal{I}$-gmatrix system consisting of the collection of all theory families of $\mathcal{I}$. The first is termed the *semantic* and the second the *syntactic variety of $\mathcal{I}$*. It turns out that, in general, the class of $\mathcal{I}^*$-algebraic systems forms the smallest class, followed by the class of $\mathcal{I}$-algebraic systems, followed by the semantic variety of $\mathcal{I}$, while the syntactic variety of $\mathcal{I}$ constitutes the largest of these four classes. An interesting result is that any of these four classes generates the same syntactic variety, namely, the syntactic variety of $\mathcal{I}$. The section concludes with the observation that the class of all $\mathcal{I}$-algebraic systems is closed under subdirect intersections and contains a trivial algebraic system. Consequently, one is justified in considering congruence systems generated by any relation system on any given algebraic system relative to this class.

In Section 2.11, we switch from the study of congruence systems associated with a given π-institution and of their quotients to the study of equivalence families and systems resulting by considering mutual membership or non-membership in theory families. The reader is warned that the terminology here deviates from the standard one for sentential logics (Section 2.4 of [52] and Section 1.3 of [86]). This is done in an attempt to streamline the theory of these equivalence families with the theory based on the Leibniz, Tarski and Suszko congruence systems. The most basic equivalence family is the *Frege equivalence family* of a given theory family, which identifies sentences if they are both inside or both outside the given theory family. Sometimes, this is expressed by saying that the sentences are *equivalent modulo the theory family*. The *Frege relation system* is the largest equivalence system included in the Frege equivalence family. There is a close connection between Leibniz congruence systems and Frege relation families/systems. The Leibniz congruence system of a given theory family is the largest congruence system contained in the Frege equivalence family or system associated with the theory family. In a way analogous to the passage from Leibniz congruence systems of single theory families to the Tarski congruence systems of collections of theory families, one transitions from Frege equivalence families to *Carnap equivalence families*. These express equivalence of sentences modulo collections of theory families. The Carnap equivalence family turns out to be the intersection of the Frege equivalence families of all theory families in the collection. Here, again, the *Carnap equivalence system* is the largest equivalence system included in the Carnap equivalence family. Further, extending the relation between Leibniz congruence systems and Frege equivalence families, the Tarski congruence system of a collection is the largest congruence system included in either the Carnap equivalence family or the Carnap equivalence system of the same collection. The same paradigm gives rise to *Lindenbaum equivalence families/systems*, which formalize the equivalence of sentences modulo a theory family, relative to a given collection of theory families. This is identical to the intersection of all Frege equivalence families/systems of those theory families in the collection including the given one. Similar relations as before hold in this case as well, with the role of Leibniz and Tarski congruence systems played by Suszko congruence systems. A small table at the end of the section summarizes the three congruence systems and the three corresponding pairs of equivalence families/systems that are considered in this context. Hopefully, the analogies outlined between congruence systems and equivalence families/systems provide some justification for introducing distinct names for the Carnap equivalences and the Lindenbaum equivalences, which are all referred to as Frege equivalences in the literature.

In Section 2.12, we look at *subsystems* of algebraic systems and induced π-*subsinstitutions*. Given an algebraic system, a *universe* is a sentence subfunctor over the same category of signatures that is also closed under the algebraic operations. In a natural way, a universe gives rise to an *algebraic subsystem*. With each subsystem, there is associated a *natural injection morphism*. Given a sentence family of an algebraic system, by closing successively under the action of signature morphisms and under the action of natural transformations, one obtains the *universe* of the algebraic system *generated by* the given sentence family. If the given algebraic system happens to be the underlying system of a π-institution, which is a case of central interest, then, by restricting the action of the closure system of the π-institution on sentences of the universe, we obtain a π-*subinstitution*. Its theory families turn out to be exactly the restrictions of the theory families of the original π-institution on the universe. The section concludes by establishing some connections between the Leibniz congruences of theory families of the original π-institution and those of the induced theory families of the subinstitution. These relations extend in a natural way to filter families of the two institutions.

Up to Section 2.12, only cursory attention is paid to natural transformations. They are used in establishing syntactic varieties of algebraic systems via natural equations, but they are not thoroughly studied as "syntactic" objects of interest in their own right. This deficiency is rectified by devoting Sections 2.13-2.15 to their study and to particular aspects of their properties and behavior that are of interest for subsequent considerations.

In Section 2.13, we consider the role played by collections of natural transformations. In general, in the context of collections of natural transformations, a number of arguments is fixed and they are considered as *primary* or *distinguished arguments*. The remaining positions play an auxiliary role and are perceived as *parametric* (see, e.g., Section 1.2 of [64] and Section 6.2 of [86]). In accordance with this paradigm, if E is a collection of natural transformations, of which k positions are considered distinguished, then, for any k-tuple of sentences $\vec{\phi}$ over a signature Σ, $E_\Sigma[\vec{\phi}]$ denotes the sentence family consisting of all sentences of the form $\varepsilon_{\Sigma'}(\mathrm{SEN}(f)(\vec{\phi}),\vec{\chi})$, for $\varepsilon \in E$, $f : \Sigma \to \Sigma'$ a signature morphism and $\vec{\chi}$ an arbitrary tuple of sentences over Σ'. In this way a tuple, or a collection of tuples, of sentences gives rise to a sentence family. Dually, given a sentence family T, one may consider the family of all k-tuples $\vec{\phi}$, such that $\varepsilon_{\Sigma'}(\mathrm{SEN}(f)(\vec{\phi}),\vec{\chi}) \in T_{\Sigma'}$, for all ε, f and $\vec{\chi}$. This gives rise to a k-ary relation system, depending on both E and T, denoted $\overleftarrow{E}(T)$. $\overleftarrow{E}$, viewed as an operator from sentence families to relation systems, is monotone and commutes with inverse surjective morphisms. For the purposes of relating logical with algebraic systems, critical is the role played by $\overleftarrow{E}$ as a potential means of defining Leibniz congruence systems of theory families. Along those lines, it is shown that, if $k = 2$ and $\overleftarrow{E}(T)$ defines a reflexive relation system, then this includes the Leibniz congruence system of T. Consequently, if $\overleftarrow{E}(T)$ is itself a congruence system compatible with T, then it necessarily coincides with the Leibniz congruence system of T (see, e.g., Theorem 1.6 of [35]).

In Section 2.14, taking a cue from the definition of the operator $\overleftarrow{E}$ in Section 2.13, we investigate membership relations of k-tuples of sentences in theory families of a π-institution induced by a fixed set E of natural transformations, taken to possess k distinguished arguments. Four modes are considered, namely, *E-local*, *E-global*, *left E-local* and *left E-global* membership. It is shown that E-global and left E-global memberships are equivalent, that they imply left E-local membership, which, in turn, implies E-local membership. Both implications are shown to be strict in general. If a membership property holds for all k-tuples of sentences (for the same E), then that property is attributed to the set E itself. It turns out that, in that case, all three resulting modes of membership of E in a theory family T are actually equivalent properties.

Section 2.15 is the last of the three sections that are devoted exclusively to the analysis of syntactic definability properties via sets of natural transformations. In this section, we consider two possible ways which may be used to obtain, starting from a parametric collection S of natural transformations, a related one that is parameter-free. The first is effectuated by replacing all parametric arguments by k-ary terms, where k is the number of distinguished arguments of S. This process gives rise to a new collection $\dot{S}$ of natural transformations with k arguments altogether and, therefore, without parameters. The second process is more abstract. It is defined via the use of, so called, *anti-monotone global properties* of natural transformations. These are properties that satisfy a technical anti-monotonicity condition. Given such a property P, by slightly abusing notation, we also denote by P the collection of all natural transformations (possibly with parameters) satisfying P. Then $\widehat{P}$ denotes the subcollection of P of parameter-free natural transformations satisfying P. In the main result of Section 2.15, it is shown that, given such a property P, both constructors $\dot{P}$ and $\widehat{P}$ result in identical de-parameterizations of the collection P.

The last three sections of Chapter 2 deal with more specialized topics. Section 2.16 addresses the special case of π-institutions whose closure systems are *finitary*. Most applied logical systems encountered in the literature fall under this case. Section 2.17 deals with *equational π-institutions*. These are π-institutions whose sets of sentences are pairs of sentences drawn from a base algebraic system and whose closure operators reflect the equational consequence determined by a class of algebraic systems. Finally, Section 2.18 adapts some of the rudiments pertaining to *varieties*, *quasivarieties* and *generalized quasivarieties* of universal algebra and their generation to the context of algebraic systems.

In Section 2.16, we study *finitarity* (see, e.g., Section 0.1 of [64] and Section 1.4 of [86]). This is the property of a closure system (or π-institution) that holds when every sentence which is a consequence of a set of sentences is also a consequence of some finite subset of that set. Some characterizations of finitarity are provided based on the properties of *local continuity* and *continuity* of a π-institution which, in turn, are defined using *local directedness* and *directedness* of collections of theory families. The last part of the section provides a step-wise, inductive construction of the filter family of a finitary π-institution on an arbitrary algebraic system generated by a give sentence family of the algebraic system.

In Section 2.17, we introduce *equational consequences* based on fixed classes of algebraic systems and show that all their theory families happen to be theory systems and that, moreover, they coincide with the congruence systems relative to the class of algebraic systems inducing the equational consequence. Then, as in Section 2.16, we present a step-wise construction of the equational consequence generated by a given family of equations, considered as axioms. We show that, if this defining family of equations is taken to be the family of equations that holds in a class K of algebraic systems, then the equational consequence they generate, according to this step-wise process, coincides with the equational consequence induced by the class K.

The final section, Section 2.18, translates some of the classical results of universal algebra pertaining to *varieties*, *quasivarieties* and *generalized quasivarieties* [21, 30, 84] (see, also, Chapter Q of [64]) to the context of classes of algebraic systems. We revisit *equations* and, in addition, consider *quasiequations* and *generalized quasiequations*, referred to as *guasiequations*. *Satisfaction* of an equation, quasiequation or guasiequation by a given algebraic system is defined. These relations give rise to Galois connections (see, e.g., Chapter 11 of [36]). The closed sets on the algebraic side form, respectively, *equational*, *quasiequational* and *guasiequational classes* of algebraic systems. Equivalently, these are the classes of algebraic systems defined by equations, quasiequations and guasiequations. When they are thought of as classes generated by given collections of algebraic systems, they are termed *varieties*, *quasivarieties* and *guasivarieties*, respectively. The second part of Section 2.18 is dedicated to proving Birkhoff [4] and Mal'cev [18] style characterization theorems of these classes using closures under class operators (see, also, [21, 30, 84]). The four operators considered are taking *certifications*, *directed certifications*, *subdirect intersections* and *morphic images*. It is shown that a given class of algebraic systems is a variety if it is closed under subdirect intersections and morphic images, it is a quasivariety if it is closed under directed certifications and subdirect intersections and it is a guasivariety if it is closed under certifications and subdirect intersections. In the last part of the section, we translate the conditions of closure under subdirect intersections and morphic images into properties of the subcollection of the collection of all congruence systems on the base algebraic system relative to the class under consideration. On the other hand, certifications and directed certifications are abstraction conditions (akin to closure under isomorphisms) and do not seem to have such intrinsic equivalent formalizations.

Chapter 2, in a nutshell, includes the majority of the very basic concepts and results that constitute the prerequisites for following the developments recounted in subsequent chapters of the monograph.

2.2 Algebraic Systems

A **sentence functor** $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is a **Set**-valued functor, with the property that, for every $\Sigma \in |\mathbf{Sign}|$, $\mathrm{SEN}(\Sigma) \neq \varnothing$. We say that a sentence functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is **trivial** if $|\mathrm{SEN}(\Sigma)| = 1$, for all $\Sigma \in |\mathbf{Sign}|$.

A **sentence family** of SEN is a collection $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, such that $T_\Sigma \subseteq \mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$. The collection of all sentence families of SEN is denoted by SenFam(SEN). Sentence families can be ordered by signature-wise inclusion. More precisely, given $T, T' \in \mathrm{SenFam}(\mathrm{SEN})$, we define

$$T \leq T' \text{ iff } \quad T_\Sigma \subseteq T'_\Sigma, \text{ for all } \Sigma \in |\mathbf{Sign}|.$$

Under this ordering sentence families form a complete lattice which is denoted by $\mathbf{SenFam}(\mathrm{SEN}) = \langle \mathrm{SenFam}(\mathrm{SEN}), \leq \rangle$.

A sentence family T of SEN is called a **sentence system** if it is invariant under signature morphisms, i.e., if, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, we have

$$\mathrm{SEN}(f)(T_\Sigma) \subseteq T_{\Sigma'}.$$

The collection of all sentence systems of SEN is denoted by SenSys(SEN). It forms a complete sublattice of the lattice of sentence families under $\leq$, denoted by $\mathbf{SenSys}(\mathrm{SEN}) = \langle \mathrm{SenSys}(\mathrm{SEN}), \leq \rangle$.

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a sentence functor and $T \in \mathrm{SenFam}(\mathrm{SEN})$. We define, based on T, two important sentence families of SEN:

- $\overleftarrow{T} = \{\overleftarrow{T}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\overleftarrow{T}_\Sigma = \{\phi \in \mathrm{SEN}(\Sigma) : \text{for all } \Sigma' \in |\mathbf{Sign}| \text{ and all } f \in \mathbf{Sign}(\Sigma, \Sigma'), \ \mathrm{SEN}(f)(\phi) \in T_{\Sigma'}\}.$$

  Sometimes, we abbreviate this using the notation

$$\overleftarrow{T}_\Sigma = \{\phi \in \mathrm{SEN}(\Sigma) : (\forall f)(\mathrm{SEN}(f)(\phi) \in T_{\Sigma'})\}$$

- $\overrightarrow{T} = \{\overrightarrow{T}_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\overrightarrow{T}_\Sigma = \{\mathrm{SEN}(f)(\phi) : \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma',\Sigma), \phi \in T_{\Sigma'}\}$$

First, it is clear that both operators on sentence families are monotone.

Lemma 1 *Let* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a sentence functor and consider* $T, T' \in \mathrm{SenFam}(\mathrm{SEN})$. *If* $T \leq T'$, *then* $\overleftarrow{T} \leq \overleftarrow{T'}$ *and* $\overrightarrow{T} \leq \overrightarrow{T'}$.

Proof: Both implications are quite obvious. For the second, e.g., consider $\Sigma \in |\mathbf{Sign}|$, $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in \overrightarrow{T}_\Sigma$. Thus, there exists $\Sigma_0 \in |\mathbf{Sign}|$, $\phi_0 \in T_{\Sigma_0}$ and $f_0 \in \mathbf{Sign}(\Sigma_0,\Sigma)$ such that $\phi = \mathrm{SEN}(f_0)(\phi_0)$.

$$\begin{array}{ccc} \Sigma_0 & \xrightarrow{f_0} & \Sigma \\ T'_{\Sigma_0} \supseteq T_{\Sigma_0} \ni \phi_0 & \longmapsto & \phi \end{array}$$

Since $T_{\Sigma_0} \subseteq T'_{\Sigma_0}$, $\phi_0 \in T'_{\Sigma_0}$ and we conclude that $\phi \in \overrightarrow{T'}_\Sigma$. ∎

The importance of $\overleftarrow{T}$ and $\overrightarrow{T}$ stems, in part, from their relationship with T, which is described in the following proposition, but also from the critical role they play in the theory presented here.

Proposition 2 *Let* $\mathbf{Sign}$ *be a category,* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *a sentence functor and suppose that* $T \in \mathrm{SenFam}(\mathrm{SEN})$.

(a) $\overleftarrow{T}$ *is the largest sentence system of* SEN *included in* T;

(b) $\overrightarrow{T}$ *is the smallest sentence system of* SEN *that contains* T.

Proof:

(a) It is obvious that $\overleftarrow{T} \leq T$. We must show that $\overleftarrow{T}$ is a sentence system and that it is the largest one included in T.

To show that it is a sentence system, consider $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma,\Sigma')$ and $\phi \in \overleftarrow{T}_\Sigma$. We must show that $\mathrm{SEN}(f)(\phi) \in \overleftarrow{T}_{\Sigma'}$. To this end, let $\Sigma'' \in |\mathbf{Sign}|$ and $g \in \mathbf{Sign}(\Sigma',\Sigma'')$.

$$\Sigma \xrightarrow{f} \Sigma' \xrightarrow{g} \Sigma''$$

Then we have

$$\mathrm{SEN}(g)(\mathrm{SEN}(f)(\phi)) = \mathrm{SEN}(gf)(\phi) \overset{\phi\in\overleftarrow{T}_\Sigma}{\in} T_{\Sigma''}.$$

Since this holds for all $\Sigma'' \in |\mathbf{Sign}|$ and all $g \in \mathbf{Sign}(\Sigma', \Sigma'')$, we conclude that $\mathrm{SEN}(f)(\phi) \in \overleftarrow{T}_{\Sigma'}$.

To show that $\overleftarrow{T}$ is the largest sentence system in T, consider $T' \in \mathrm{SenSys}(\mathrm{SEN})$, such that $T' \le T$ and let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in T'_\Sigma$. Since T' is a sentence system, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, we get $\mathrm{SEN}(f)(\phi) \in T'_{\Sigma'}$. Now since $T' \le T$, we get that, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\mathrm{SEN}(f)(\phi) \in T_{\Sigma'}$. But this shows that $\phi \in \overleftarrow{T}_\Sigma$. Thus, $T' \le \overleftarrow{T}$ and $\overleftarrow{T}$ is the largest sentence system included in T.

(b) It is obvious that $T \le \overrightarrow{T}$. We must show that $\overrightarrow{T}$ is a sentence system and that it is the smallest one containing T.

To show that $\overrightarrow{T}$ is a sentence system, consider $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in \overrightarrow{T}_\Sigma$. Let $\Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma, \Sigma')$. We must show that $\mathrm{SEN}(f)(\phi) \in \overrightarrow{T}_{\Sigma'}$. Since $\phi \in \overrightarrow{T}_\Sigma$, there exists $\Sigma_0 \in |\mathbf{Sign}|$, $f_0 \in \mathbf{Sign}(\Sigma_0, \Sigma)$ and $\phi_0 \in T_{\Sigma_0}$, such that $\mathrm{SEN}(f_0)(\phi_0) = \phi$.

$$\Sigma_0 \xrightarrow{f_0} \Sigma \xrightarrow{f} \Sigma'$$

Thus, we get

$$\mathrm{SEN}(f)(\phi) = \mathrm{SEN}(f)(\mathrm{SEN}(f_0)(\phi_0)) = \mathrm{SEN}(f f_0)(\phi_0) \in \overrightarrow{T}_{\Sigma'}.$$

Finally, we must show that $\overrightarrow{T}$ is the smallest sentence system that contains T. To this end, suppose that $T' \in \mathrm{SenSys}(\mathrm{SEN})$, such that $T \le T'$. Let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in \overrightarrow{T}_\Sigma$. Then, there exist $\Sigma_0 \in |\mathbf{Sign}|$, $f_0 \in \mathbf{Sign}(\Sigma_0, \Sigma)$ and $\phi_0 \in T_{\Sigma_0}$, such that $\phi = \mathrm{SEN}(f_0)(\phi_0)$. Now, since $T \le T'$, we get $\phi_0 \in T'_{\Sigma_0}$. Moreover, since T' is a sentence system, we get $\mathrm{SEN}(f_0)(\phi_0) \in T'_\Sigma$. But this means $\phi = \mathrm{SEN}(f_0)(\phi_0) \in T'_\Sigma$. This proves that $\overrightarrow{T} \le T'$ and, hence, $\overrightarrow{T}$ is the least sentence system that contains T.

■

It is also of interest to observe that the back arrow operator commutes with intersections:

Lemma 3 *Let* **Sign** *be a category,* SEN : **Sign** → **Set** *a sentence functor and consider* $\mathcal{T} \subseteq \mathrm{SenFam}(\mathrm{SEN})$*. Then*

$$\overleftarrow{\bigcap_{T \in \mathcal{T}} T} = \bigcap_{T \in \mathcal{T}} \overleftarrow{T}.$$

Proof: First, by Lemma 1, we have, for all $T \in \mathcal{T}$, $\overleftarrow{\bigcap_{T \in \mathcal{T}} T} \leq \overleftarrow{T}$. Therefore, we conclude that $\overleftarrow{\bigcap_{T \in \mathcal{T}} T} \leq \bigcap_{T \in \mathcal{T}} \overleftarrow{T}$.

On the other hand, we have, by Proposition 2, $\overleftarrow{T} \leq T$, for all $T \in \mathcal{T}$. Therefore $\bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \bigcap_{T \in \mathcal{T}} T$. Now, since $\bigcap_{T \in \mathcal{T}} \overleftarrow{T}$ is a sentence system (Proposition 2) included in $\bigcap_{T \in \mathcal{T}} T$, it must lie below the largest such, which, by Proposition 2, is $\overleftarrow{\bigcap_{T \in \mathcal{T}} T}$. Thus, we have $\bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{\bigcap_{T \in \mathcal{T}} T}$. ■

On the other hand, the back arrow does not commute, in general, with unions. We first prove a lemma showing the there is an inclusion relation governing the interaction between the back arrow and unions and, then, provide an example to show that this inclusion may be proper.

Lemma 4 *Let* **Sign** *be a category,* SEN : **Sign** → **Set** *a sentence functor and consider* $\mathcal{T} \subseteq \mathrm{SenFam}(\mathrm{SEN})$. *Then*

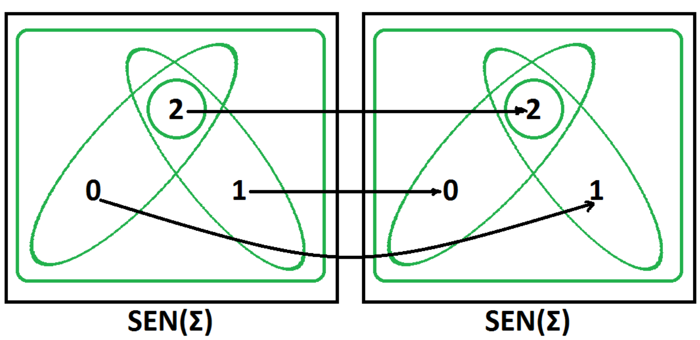
$$\bigcup_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{\bigcup_{T \in \mathcal{T}} T}.$$

Proof: Since, for all $T \in \mathcal{T}$, $T \leq \bigcup_{T \in \mathcal{T}} T$, we get, by Lemma 1, $\overleftarrow{T} \leq \overleftarrow{\bigcup_{T \in \mathcal{T}} T}$. Since this holds for all $T \in \mathcal{T}$, we conclude that $\bigcup_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{\bigcup_{T \in \mathcal{T}} T}$. ■

That the inclusion of Lemma 4 is, in general, a proper inclusion is showcased by the following example.

Example 5 *Let* **Sign** *be the category with a single object* Σ *and a single (non-identity) arrow* $f : \Sigma \to \Sigma$, *such that* $f \circ f = i_\Sigma$.

Let SEN : **Sign** → **Set** *be the functor defined by setting* $\mathrm{SEN}(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}(f)(0) = 1$, $\mathrm{SEN}(f)(1) = 0$ *and* $\mathrm{SEN}(f)(2) = 2$. *Consider the col-*



lection $\{T, T'\} \subseteq \mathrm{SenFam}(\mathrm{SEN})$, *with* $T_\Sigma = \{0, 2\}$ *and* $T'_\Sigma = \{1, 2\}$. *Then we have* $\overleftarrow{T}_\Sigma = \{2\} = \overleftarrow{T}_{\Sigma'}$ *and, therefore*

$$\overleftarrow{T}_\Sigma \cup \overleftarrow{T'}_\Sigma = \{2\} \cup \{2\} = \{2\}.$$

On the other hand,

$$\overleftarrow{T \cup T'}_\Sigma = \overleftarrow{\{\{0,1,2\}\}}_\Sigma = \{0,1,2\}.$$

Thus, we get $\bigcup_{T \in \mathcal{T}} \overleftarrow{T} \lneq \overleftarrow{\bigcup_{T \in \mathcal{T}} T}$.

Let **Sign**, **Sign′** be categories and $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ and $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ be two sentence functors. A **morphism** (of sentence functors) $\langle F, \alpha \rangle : \mathrm{SEN} \to \mathrm{SEN}'$ consists of:

- A functor $F : \mathbf{Sign} \to \mathbf{Sign}'$;
- A natural transformation $\alpha : \mathrm{SEN} \to \mathrm{SEN}' \circ F$.

We will make heavy use of the following particular types of morphisms:

- A morphism $\langle F, \alpha \rangle : \mathrm{SEN} \to \mathrm{SEN}'$ is **special** if $F : \mathbf{Sign} \to \mathbf{Sign}'$ is surjective on objects and full.
- A morphism $\langle F, \alpha \rangle : \mathrm{SEN} \to \mathrm{SEN}'$ is **surjective** if it is special and $\alpha_\Sigma : \mathrm{SEN}(\Sigma) \to \mathrm{SEN}'(F(\Sigma))$ is surjective, for all $\Sigma \in |\mathbf{Sign}|$.

Let **Sign**, **Sign′** be categories, $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ and $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ be two sentence functors and $\langle F, \alpha \rangle : \mathrm{SEN} \to \mathrm{SEN}'$ be a morphism. Given a sentence family $T \in \mathrm{SenFam}(\mathrm{SEN}')$, define the sentence family $\alpha^{-1}(T) = \{\alpha^{-1}(T)_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} \in \mathrm{SenFam}(\mathrm{SEN})$ by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\alpha^{-1}(T)_\Sigma = \alpha_\Sigma^{-1}(T_{F(\Sigma)}).$$

In the next lemma, we prove some useful properties concerning this operator.

Lemma 6 *Let* **Sign**, **Sign′** *be categories,* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *and* $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ *be two sentence functors,* $\langle F, \alpha \rangle : \mathrm{SEN} \to \mathrm{SEN}'$ *be a morphism and* $T \in \mathrm{SenFam}(\mathrm{SEN}')$.

(a) If $T \in \mathrm{SenSys}(\mathrm{SEN}')$, *then* $\alpha^{-1}(T) \in \mathrm{SenSys}(\mathrm{SEN})$, *with equivalence holding if* $\langle F, \alpha \rangle : \mathrm{SEN} \to \mathrm{SEN}'$ *is surjective;*

(b) $\alpha^{-1}(\overleftarrow{T}) \leq \overleftarrow{\alpha^{-1}(T)}$, *with equality holding if* $\langle F, \alpha \rangle$ *is special;*

(c) $\overrightarrow{\alpha^{-1}(T)} \leq \alpha^{-1}(\overrightarrow{T})$, *with equality holding if* $\langle F, \alpha \rangle$ *is surjective.*

Proof:

(a) Let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})$. Then, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, we have

$$\begin{array}{rcl}\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) & = & \mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi)) \\ & & (\alpha \text{ natural transformation}) \\ & \in & \mathrm{SEN}'(F(f))(T_{F(\Sigma)}) \quad (\phi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})) \\ & \subseteq & T_{F(\Sigma')} \quad (T \in \mathrm{SenSys}(\mathrm{SEN}')).\end{array}$$

This shows that $\mathrm{SEN}(f)(\phi) \in \alpha_{\Sigma'}^{-1}(T_{F(\Sigma')})$. We now conclude that $\alpha^{-1}(T) \in \mathrm{SenSys}(\mathrm{SEN})$.

Suppose, next, that $\langle F, \alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$ is surjective and $\alpha^{-1}(T) \in \mathrm{SenSys}(\mathrm{SEN})$. Let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$. Note that this implies that $\phi \in \alpha^{-1}(T_{F(\Sigma)})$. So, by hypothesis, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}(f)(\phi) \in \alpha_{\Sigma'}^{-1}(T_{F(\Sigma')}).$$

Therefore,

$$\begin{array}{rcl}\mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi)) & = & \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \\ & \in & \alpha_{\Sigma'}(\alpha_{\Sigma'}^{-1}(T_{F(\Sigma')})) \\ & \subseteq & T_{F(\Sigma')}.\end{array}$$

Since $\langle F, \alpha\rangle$ is surjective, we conclude that, for all $\Sigma, \Sigma' \in |\mathbf{Sign}'|$ and all $f \in \mathbf{Sign}'(\Sigma, \Sigma')$,

$$\mathrm{SEN}'(f)(T_\Sigma) \subseteq T_{\Sigma'}.$$

Therefore, $T \in \mathrm{SenSys}(\mathrm{SEN}')$.

(b) Let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in \alpha_\Sigma^{-1}(\overleftarrow{T}_{F(\Sigma)})$. Then we get that $\alpha_\Sigma(\phi) \in \overleftarrow{T}_{F(\Sigma)}$. Thus, by definition of $\overleftarrow{T}$, for all $\Sigma' \in |\mathbf{Sign}'|$ and all $f \in \mathbf{Sign}'(F(\Sigma), \Sigma')$,

$$\mathrm{SEN}'(f)(\alpha_\Sigma(\phi)) \in T_{\Sigma'}.$$

This implies, in particular, that, for all $\Sigma'' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma'')$, $\mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi)) \in T_{F(\Sigma'')}$. So we get $\alpha_{\Sigma''}(\mathrm{SEN}(f)(\phi)) \in T_{F(\Sigma'')}$, i.e., $\mathrm{SEN}(f)(\phi) \in \alpha_{\Sigma''}^{-1}(T_{F(\Sigma'')})$. Since Σ'' and f were arbitrary, we finally obtain $\phi \in \overleftarrow{\alpha_\Sigma^{-1}(T_{F(\Sigma)})}$.

It is straightforward to see that, if $\langle F, \alpha\rangle$ is special, then the above chain of implications is reversible and, by following it, we get the reverse inclusion.

(c) Let $\Sigma \in |\mathbf{Sign}|$, Then we have

$$\begin{array}{l}\alpha_\Sigma(\overrightarrow{\alpha_\Sigma^{-1}(T_{F(\Sigma)})})\\ = \alpha_\Sigma(\{\mathrm{SEN}(f_0)(\phi_0) : f_0 \in \mathbf{Sign}(\Sigma_0,\Sigma), \phi_0 \in \alpha_{\Sigma_0}^{-1}(T_{F(\Sigma_0)})\})\\ = \{\alpha_\Sigma(\mathrm{SEN}(f_0)(\phi_0)) : f_0 \in \mathbf{Sign}(\Sigma_0,\Sigma), \phi_0 \in \alpha_{\Sigma_0}^{-1}(T_{F(\Sigma_0)})\}\\ = \{\mathrm{SEN}'(F(f_0))(\alpha_{\Sigma_0}(\phi_0)) : f_0 \in \mathbf{Sign}(\Sigma_0,\Sigma), \phi_0 \in \alpha_{\Sigma_0}^{-1}(T_{F(\Sigma_0)})\}\\ \subseteq \{\mathrm{SEN}'(f_0')(\phi_0') : f_0' \in \mathbf{Sign}'(\Sigma_0', F(\Sigma)), \phi_0' \in T_{\Sigma_0'}\}\\ = \overrightarrow{T}_{F(\Sigma)}.\end{array}$$

Again, it is easy to see that the only inclusion becomes an equality in case $\langle F,\alpha\rangle$ is a surjective morphism.

■

Let **Sign** be a category and $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a sentence functor. A **relation family on** SEN is a collection $R = \{R_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$, such that $R_\Sigma \subseteq \mathrm{SEN}(\Sigma)^2$, for all $\Sigma \in |\mathbf{Sign}|$. A relation family is a **relation system** if it is invariant under **Sign**-morphisms, i.e., if for all $\Sigma,\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma,\Sigma')$,

$$\mathrm{SEN}(f)(R_\Sigma) \subseteq R_{\Sigma'}.$$

The collection of all relation families on SEN is denoted by RelFam(SEN) and, similarly, the collection of all relation systems by RelSys(SEN). A relation family/system on SEN is an **equivalence family/system on** SEN if, for all $\Sigma \in |\mathbf{Sign}|$, R_Σ is an equivalence relation on $\mathrm{SEN}(\Sigma)$. As with relation families/systems, we denote the collection of all equivalence families on SEN by EqvFam(SEN) and the collection of all equivalence systems on SEN by EqvSys(SEN).

Given a sentence family $T \in \mathrm{SenFam(SEN)}$, we say that the equivalence family R on SEN is **compatible with** T, if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi,\psi \in \mathrm{SEN}(\Sigma)$,

$$\langle\phi,\psi\rangle \in R_\Sigma \quad\text{and}\quad \phi \in T_\Sigma \quad\text{imply}\quad \psi \in T_\Sigma.$$

Lemma 7 *Let* **Sign** *be a category,* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *a sentence functor,* $T \in \mathrm{SenFam(SEN)}$ *and* θ *a relation system on* SEN*. If* θ *is compatible with* T*, then it is also compatible with* $\overleftarrow{T}$*.*

Proof: Suppose that θ is compatible with T. Let $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle \in \theta_\Sigma$ and $\phi \in \overleftarrow{T}_\Sigma$. Let $\Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma,\Sigma')$. Since θ is a relation system, we get $\langle\mathrm{SEN}(f)(\phi),\mathrm{SEN}(f)(\psi)\rangle \in \theta_{\Sigma'}$. Since $\phi \in \overleftarrow{T}_\Sigma$, $\mathrm{SEN}(f)(\phi) \in T_{\Sigma'}$. Thus, by compatibility, we get $\mathrm{SEN}(f)(\psi) \in T_{\Sigma'}$. Since $\Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma,\Sigma')$ were arbitrary, we conclude that $\psi \in \overleftarrow{T}_\Sigma$, showing that θ is also compatible with $\overleftarrow{T}$. ■

Let **Sign**, **Sign′** be categories, $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ and $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ be sentence functors and $\langle F,\alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$ be a morphism. Define the

kernel system of $\langle F,\alpha\rangle$, denoted $\mathrm{Ker}(\langle F,\alpha\rangle)=\{\mathrm{Ker}_\Sigma(\langle F,\alpha\rangle)\}_{\Sigma\in|\mathbf{Sign}|}$, by letting, for all $\Sigma\in|\mathbf{Sign}|$ and all $\phi,\psi\in\mathrm{SEN}(\Sigma)$,

$$\langle\phi,\psi\rangle\in\mathrm{Ker}_\Sigma(\langle F,\alpha\rangle)\quad\text{iff}\quad\alpha_\Sigma(\phi)=\alpha_\Sigma(\psi).$$

The kernel system $\mathrm{Ker}(\langle F,\alpha\rangle)$ is sometimes denoted more compactly by $\theta^{\langle F,\alpha\rangle}=\{\theta^{\langle F,\alpha\rangle}_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$.

Lemma 8 *Let* **Sign**, **Sign′** *be categories,* $\mathrm{SEN}:\mathbf{Sign}\to\mathbf{Set}$, $\mathrm{SEN}':\mathbf{Sign}'\to\mathbf{Set}$ *be sentence functors and* $\langle F,\alpha\rangle:\mathrm{SEN}\to\mathrm{SEN}'$ *a morphism. Then* $\mathrm{Ker}(\langle F,\alpha\rangle)$ *is an equivalence system on* SEN*.*

Proof: It is obvious from the definition that $\mathrm{Ker}(\langle F,\alpha\rangle)$ is an equivalence family of SEN. The system property follows from the fact that α is a natural transformation. Let $\Sigma\in|\mathbf{Sign}|$ and $\phi,\psi\in\mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle\in\mathrm{Ker}_\Sigma(\langle F,\alpha\rangle)$. Then, for all $\Sigma'\in|\mathbf{Sign}|$ and all $f\in\mathbf{Sign}(\Sigma,\Sigma')$,

$$\begin{array}{rcll}\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) & = & \mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi)) & \text{(naturality of }\alpha)\\ & = & \mathrm{SEN}'(F(f))(\alpha_\Sigma(\psi)) & (\langle\phi,\psi\rangle\in\mathrm{Ker}_\Sigma(\langle F,\alpha\rangle))\\ & = & \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi)) & \text{(naturality of }\alpha).\end{array}$$

Therefore, we get that $\langle\mathrm{SEN}(f)(\phi),\mathrm{SEN}(f)(\psi)\rangle\in\mathrm{Ker}_{\Sigma'}(\langle F,\alpha\rangle)$, showing that $\mathrm{Ker}(\langle F,\alpha\rangle)$ is an equivalence system. ∎

Let **Sign**, **Sign′** be categories, $\mathrm{SEN}:\mathbf{Sign}\to\mathbf{Set}$ and $\mathrm{SEN}':\mathbf{Sign}'\to\mathbf{Set}$ be sentence functors and $\langle F,\alpha\rangle:\mathrm{SEN}\to\mathrm{SEN}'$ be a morphism, with F an isomorphism. Given a sentence family $T\in\mathrm{SenFam}(\mathrm{SEN})$, define the sentence family $\alpha(T)=\{\alpha(T)_{F(\Sigma)}\}_{\Sigma\in|\mathbf{Sign}|}\in\mathrm{SenFam}(\mathrm{SEN}')$ by setting, for all $\Sigma\in|\mathbf{Sign}|$,

$$\alpha(T)_{F(\Sigma)}=\alpha_\Sigma(T_\Sigma).$$

In the next lemma, we prove some useful properties concerning this operator.

Lemma 9 *Let* **Sign**, **Sign′** *be categories,* $\mathrm{SEN}:\mathbf{Sign}\to\mathbf{Set}$ *and* $\mathrm{SEN}':\mathbf{Sign}'\to\mathbf{Set}$ *be sentence functors,* $\langle F,\alpha\rangle:\mathrm{SEN}\to\mathrm{SEN}'$ *a surjective morphism, with* F *an isomorphism, and* $T\in\mathrm{SenFam}(\mathrm{SEN})$*, such that the kernel* $\mathrm{Ker}(\langle F,\alpha\rangle)$ *of* $\langle F,\alpha\rangle$ *is compatible with* T*.*

(a) $\alpha(T)\in\mathrm{SenSys}(\mathrm{SEN}')$ *iff* $T\in\mathrm{SenSys}(\mathrm{SEN})$*;*

(b) $\alpha(\overleftarrow{T})=\overleftarrow{\alpha(T)}$*;*

(c) $\overrightarrow{\alpha(T)}=\alpha(\overrightarrow{T})$*.*

Proof:

(a) Exploiting the surjectivity of $\langle F,\alpha\rangle$, $\alpha(T)\in\mathrm{SenSys}(\mathrm{SEN}')$ holds if and only if, for all $\Sigma\in|\mathbf{Sign}|$, all $\phi\in\mathrm{SEN}(\Sigma)$, all $\Sigma'\in|\mathbf{Sign}|$ and all $f\in\mathbf{Sign}(\Sigma,\Sigma')$,

$$\mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi))\in\alpha_{\Sigma'}(T_{\Sigma'}).$$

By the naturality of α, the latter is equivalent to

$$\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi))\in\alpha_{\Sigma'}(T_{\Sigma'}).$$

Finally, by compatibility of $\mathrm{Ker}(\langle F,\alpha\rangle)$ with T, this is equivalent to $\mathrm{SEN}(f)(\phi)\in T_{\Sigma'}$. But this holds for all $\Sigma\in|\mathbf{Sign}|$, all $\phi\in\mathrm{SEN}(\Sigma)$, all $\Sigma'\in|\mathbf{Sign}|$ and all $f\in\mathbf{Sign}(\Sigma,\Sigma')$ if and only if $T\in\mathrm{SenSys}(\mathrm{SEN})$.

(b) Again we exploit the surjectivity of $\langle F,\alpha\rangle$. We have, for all $\Sigma\in|\mathbf{Sign}|$ and all $\phi\in\mathrm{SEN}(\Sigma)$, $\alpha_\Sigma(\phi)\in\overleftarrow{\alpha_\Sigma(T_\Sigma)}$ iff, for all $\Sigma'\in|\mathbf{Sign}|$ and all $f\in\mathbf{Sign}(\Sigma,\Sigma')$, $\mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi))\in\alpha_{\Sigma'}(T_{\Sigma'})$ iff, by the naturality of α, $\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi))\in\alpha_{\Sigma'}(T_{\Sigma'})$ iff, by the compatibility of $\mathrm{Ker}(\langle F,\alpha\rangle)$ with T, $\mathrm{SEN}(f)(\phi)\in T_{\Sigma'}$ iff, by the definition of $\overleftarrow{T}$, $\phi\in\overleftarrow{T}_\Sigma$ iff, by the compatibility of $\mathrm{Ker}(\langle F,\alpha\rangle)$ with $\overleftarrow{T}$, which follows from Lemmas 7 and 8, $\alpha_\Sigma(\phi)\in\alpha_\Sigma(\overleftarrow{T}_\Sigma)$. Thus, we conclude that $\alpha(\overleftarrow{T})=\overleftarrow{\alpha(T)}$.

(c) Suppose, first, that $\alpha_\Sigma(\phi)\in\overrightarrow{\alpha_\Sigma(T_\Sigma)}$. Then, there exist, by surjectivity, $\Sigma_0\in|\mathbf{Sign}|$, $f_0\in\mathbf{Sign}(\Sigma_0,\Sigma)$ and $\phi_0\in T_{\Sigma_0}$, such that

$$\begin{array}{rcl}\alpha_\Sigma(\phi)&=&\mathrm{SEN}'(F(f_0))(\alpha_{\Sigma_0}(\phi_0))\\&=&\alpha_\Sigma(\mathrm{SEN}(f_0)(\phi_0))\\&\in&\alpha_\Sigma(\overrightarrow{T}_\Sigma).\end{array}$$

Suppose, conversely, that $\alpha_\Sigma(\phi)\in\alpha_\Sigma(\overrightarrow{T}_\Sigma)$. Then, there exist $\Sigma_0\in|\mathbf{Sign}|$, $f_0\in\mathbf{Sign}(\Sigma_0,\Sigma)$ and $\phi_0\in T_{\Sigma_0}$, such that

$$\begin{array}{rcl}\alpha_\Sigma(\phi)&=&\alpha_\Sigma(\mathrm{SEN}(f_0)(\phi_0))\\&=&\mathrm{SEN}'(F(f_0))(\alpha_{\Sigma_0}(\phi_0))\\&\in&\overrightarrow{\alpha_\Sigma(T_\Sigma)}.\end{array}$$

■

By analogy to the case of sentence families, we may also define the inverse of a relation family under a morphism of sentence functors. Let $\mathbf{Sign}$, $\mathbf{Sign}'$ be categories, $\mathrm{SEN}:\mathbf{Sign}\to\mathbf{Set}$ and $\mathrm{SEN}':\mathbf{Sign}'\to\mathbf{Set}$ be sentence functors and $\langle F,\alpha\rangle:\mathrm{SEN}\to\mathrm{SEN}'$ a morphism. Let, also, $R=\{R_\Sigma\}_{\Sigma\in|\mathbf{Sign}'|}$ be a relation family on SEN'. Define the relation family $\alpha^{-1}(R)=\{\alpha^{-1}(R)_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ on SEN by setting, for all $\Sigma\in|\mathbf{Sign}|$,

$$\alpha^{-1}(R)_\Sigma=\alpha_\Sigma^{-1}(R_{F(\Sigma)}).$$

Proposition 10 *Let* **Sign**, **Sign**$'$ *be categories,* SEN : **Sign** → **Set** *and* SEN$'$: **Sign**$'$ → **Set** *be sentence functors,* $\langle F,\alpha\rangle$: SEN → SEN$'$ *a morphism and* R *a relation family on* SEN$'$.

(a) If R is a relation system, then $\alpha^{-1}(R)$ is also a relation system;

(b) If R is an equivalence family, then α^{-1} is also an equivalence family.

Proof:

(a) Let $\Sigma\in|\mathbf{Sign}|$, $\phi,\psi\in\mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle\in\alpha_\Sigma^{-1}(R_{F(\Sigma)})$. Then, we have $\langle\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\rangle\in R_{F(\Sigma)}$. Since R is a relation system, for all $\Sigma'\in|\mathbf{Sign}|$ and all $f\in\mathbf{Sign}(\Sigma,\Sigma')$, we get

$$\langle\mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi)),\mathrm{SEN}'(F(f))(\alpha_\Sigma(\psi))\rangle\in R_{F(\Sigma')}.$$

Thus, by the naturality of α,

$$\langle\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)),\alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi))\rangle\in R_{F(\Sigma')}.$$

Now we get $\langle\mathrm{SEN}(f)(\phi),\mathrm{SEN}(f)(\psi)\rangle\in\alpha_{\Sigma'}^{-1}(R_{F(\Sigma')})$. This proves that $\alpha^{-1}(R)$ is a relation system on SEN.

(b) Let $\Sigma\in|\mathbf{Sign}|$ and $\phi,\chi,\psi\in\mathrm{SEN}(\Sigma)$ be arbitrary. Then we have:

Reflexivity By the reflexivity of R, $\langle\alpha_\Sigma(\phi),\alpha_\Sigma(\phi)\rangle\in R_{F(\Sigma)}$. Therefore, $\langle\phi,\phi\rangle\in\alpha_\Sigma^{-1}(R_{F(\Sigma)})$.

Symmetry If $\langle\phi,\psi\rangle\in\alpha_\Sigma^{-1}(R_{F(\Sigma)})$, then $\langle\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\rangle\in R_{F(\Sigma)}$, whence, by the symmetry of R, $\langle\alpha_\Sigma(\psi),\alpha_\Sigma(\phi)\rangle\in R_{F(\Sigma)}$, showing that $\langle\psi,\phi\rangle\in\alpha_\Sigma^{-1}(R_{F(\Sigma)})$.

Transitivity If $\langle\phi,\chi\rangle,\langle\chi,\psi\rangle\in\alpha_\Sigma^{-1}(R_{F(\Sigma)})$, then, we get

$$\langle\alpha_\Sigma(\phi),\alpha_\Sigma(\chi)\rangle,\langle\alpha_\Sigma(\chi),\alpha_\Sigma(\psi)\rangle\in R_{F(\Sigma)},$$

whence, by the transitivity of R, we get $\langle\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\rangle\in R_{F(\Sigma)}$, showing that $\langle\phi,\psi\rangle\in\alpha_\Sigma^{-1}(R_{F(\Sigma)})$.

■

Let **Sign** be a category and SEN : **Sign** → **Set** a sentence functor. The **clone of all natural transformations** on SEN is the category **Cln**(SEN) with collection of objects $\{\mathrm{SEN}^\alpha:\alpha\text{ an ordinal}\}$ and collection of morphisms $\tau:\mathrm{SEN}^\alpha\to\mathrm{SEN}^\beta$ β-sequences of natural transformations $\tau^i:\mathrm{SEN}^\alpha\to\mathrm{SEN}$, $i<\beta$. Composition of $\langle\tau^i:i<\beta\rangle:\mathrm{SEN}^\alpha\to\mathrm{SEN}^\beta$ with $\langle\sigma^j:j<\gamma\rangle:\mathrm{SEN}^\beta\to\mathrm{SEN}^\gamma$

$$\mathrm{SEN}^\alpha\xrightarrow{\langle\tau^i:i<\beta\rangle}\mathrm{SEN}^\beta\xrightarrow{\langle\sigma^j:j<\gamma\rangle}\mathrm{SEN}^\gamma$$

is defined by

$$\langle\sigma^j:j<\gamma\rangle\circ\langle\tau^i:i<\beta\rangle=\langle\sigma^j(\langle\tau^i:i<\beta\rangle:j<\gamma\rangle.$$

A **clone** (or a **category**) **of natural transformations** on SEN is a subcategory N of the category **Cln**(SEN), such that:

- Its objects are those in $\{\mathrm{SEN}^k : k < \omega\}$;
- Its morphisms include all projection natural transformations

  $$p^{k,i} : \mathrm{SEN}^k \to \mathrm{SEN}, i < k, k < \omega,$$

  with $p^{k,i}_\Sigma : \mathrm{SEN}(\Sigma)^k \to \mathrm{SEN}(\Sigma)$ given by

  $$p^{k,i}_\Sigma(\vec{\phi}) = \phi_i, \text{ for all } \vec{\phi} \in \mathrm{SEN}(\Sigma)^k,$$

  and are such that, for every family $\{\tau^i : \mathrm{SEN}^k \to \mathrm{SEN} : i < \ell\}$ of natural transformations in N, $\langle\tau^i : i < \ell\rangle : \mathrm{SEN}^k \to \mathrm{SEN}^\ell$ is also in N.

This definition has two important consequences that we now make explicit. Let **Sign** be a category, SEN : **Sign** → **Set** a sentence functor and $k \in \omega$. Consider a function

$$\pi : \{0, 1, \dots, k-1\} \to \{0, 1, \dots, k-1\}.$$

Given $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi} = \langle\phi_0, \phi_1, \dots, \phi_{k-1}\rangle \in \mathrm{SEN}(\Sigma)^k$, we define

$$\vec{\phi}^\pi = \langle\phi_{\pi(0)}, \phi_{\pi(1)}, \dots, \phi_{\pi(k-1)}\rangle.$$

Now, consider, in addition, a clone N of natural transformations on SEN and $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N. Define the natural transformation

$$\sigma^\pi : \mathrm{SEN}^k \to \mathrm{SEN}$$

by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)$,

$$\sigma^\pi_\Sigma(\vec{\phi}) = \sigma_\Sigma(\vec{\phi}^\pi).$$

That this is a natural transformation is easy to see: For all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)$, we have

$$\begin{array}{ccc}
\mathrm{SEN}(\Sigma)^k & \xrightarrow{\sigma^\pi_\Sigma} & \mathrm{SEN}(\Sigma) \\
{\scriptstyle \mathrm{SEN}(f)^k}\downarrow & & \downarrow{\scriptstyle \mathrm{SEN}(f)} \\
\mathrm{SEN}(\Sigma')^k & \xrightarrow[\sigma^\pi_{\Sigma'}]{} & \mathrm{SEN}(\Sigma')
\end{array}$$

$$\begin{array}{rcl}
\mathrm{SEN}(f)(\sigma^\pi_\Sigma(\vec{\phi})) & = & \mathrm{SEN}(f)(\sigma_\Sigma(\vec{\phi}^\pi)) \\
& = & \sigma_{\Sigma'}(\mathrm{SEN}(f)^k(\vec{\phi}^\pi)) \\
& = & \sigma_{\Sigma'}(\mathrm{SEN}(f)^k(\vec{\phi})^\pi) \\
& = & \sigma^\pi_{\Sigma'}(\mathrm{SEN}(f)^k(\vec{\phi})).
\end{array}$$

Proposition 11 *Let* **Sign** *be a category,* SEN : **Sign** → **Set** *a sentence functor and* N *a clone of natural transformations on* SEN*. If* $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ *is in* N*, then, for all functions* $\pi : \{0, \dots, k-1\} \to \{0, \dots, k-1\}$*,* $\sigma^\pi : \mathrm{SEN}^k \to \mathrm{SEN}$ *is also in* N*.*

Proof: The key is to observe that

$$\sigma^\pi = \sigma \circ \langle p^{k,\pi(0)}, \dots, p^{k,\pi(k-1)} \rangle.$$

Since all projections are in N and N is closed under formation of tuples, we get that $\langle p^{k,\pi(0)}, \dots, p^{k,\pi(k-1)} \rangle : \mathrm{SEN}^k \to \mathrm{SEN}^k$ is in N. Therefore, since N is a category and, by hypothesis, σ is in N, we get that σ^π is also in N. ■

The following is a very useful consequence that allows simplifying quantifications.

Corollary 12 *Let* **Sign** *be a category,* SEN : **Sign** → **Set** *a sentence functor and* N *a clone of natural transformations on* SEN*. The statement*

$$\text{For all } \sigma : \mathrm{SEN}^k \to \mathrm{SEN} \text{ in } N, \text{ all } i < k \text{ and all } \Sigma \in |\mathbf{Sign}|, \ \phi, \vec{\chi} \in \mathrm{SEN}(\Sigma),$$
$$\mathrm{Property}(\sigma_\Sigma(\chi_0, \dots, \chi_{i-1}, \phi, \chi_{i+1}, \dots, \chi_{k-1}))$$

is equivalent to the simpler statement

$$\text{For all } \sigma : \mathrm{SEN}^k \to \mathrm{SEN} \text{ in } N \text{ and all } \Sigma \in |\mathbf{Sign}|, \ \phi, \vec{\chi} \in \mathrm{SEN}(\Sigma),$$
$$\mathrm{Property}(\sigma_\Sigma(\phi, \vec{\chi})).$$

Proof: The left-to-right implication is trivial. The right-to-left implication follows from Proposition 11, since $\sigma^{\pi_i} : \mathrm{SEN}^k \to \mathrm{SEN}$, with π_i being the permutation

$$\begin{pmatrix} 0 & 1 & \cdots & i-1 & i & i+1 & \cdots & k-1 \\ 1 & 2 & \cdots & i & 0 & i+1 & \cdots & k-1 \end{pmatrix},$$

is also in N, for every $i < k$. ■

An **algebraic system** is a triple $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, where:

- **Sign** is an arbitrary category;
- SEN : **Sign** → **Set** is a sentence functor;
- N is a clone on SEN.

An algebraic system $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ is said to be **trivial** if its underlying sentence functor SEN : **Sign** → **Set** is trivial, i.e., if all its sets of sentences are singletons.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. An $N^\flat$**-algebraic system** $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ is an algebraic system, such that there exists a surjective functor $\Xi : N^\flat \to N$ that preserves all projection natural transformations, i.e., such that, for all $k < \omega$ and all $i < k$, if $p^{k,i^\flat} : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ denotes the i-th projection natural transformation on $(\mathrm{SEN}^\flat)^k$, then $\Xi(p^{k,i^\flat}) : \mathrm{SEN}^k \to \mathrm{SEN}$ is the i-th projection $p^{k,i}$ on SEN^k.

This condition implies that Ξ also preserves the arities of all natural transformations involved. Given $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, the image $\Xi(\sigma^\flat) : \mathrm{SEN}^k \to \mathrm{SEN}$ in N will be denoted by σ, keeping the same lowercase Greek letter, but adjusting superscripts, subscripts, primes, etc., as demanded by context. Occasionally, to simplify notation, we might drop superscripts, subscripts, etc., overloading the notation of the lowercase Greek letter, allowing the context to make the interpretation of each occurrence clear (and hoping that, because of this, confusion can be avoided).

In the context where $N^\flat$-algebraic systems are under consideration, the algebraic system $\mathbf{F}$ will be referred to as the **base algebraic system**, since the clones on all other systems under consideration are images of the clone of $\mathbf{F}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system and $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ be $N^\flat$-algebraic systems. A **morphism** (of $N^\flat$-algebraic systems) $\langle F, \alpha \rangle : \mathbf{A} \to \mathbf{A}'$ is a morphism of sentence functors $\langle F, \alpha \rangle : \mathrm{SEN} \to \mathrm{SEN}'$, such that, for all $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)$ (meaning $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$),

$$\begin{array}{ccc}
\mathrm{SEN}(\Sigma)^k & \xrightarrow{\sigma_\Sigma} & \mathrm{SEN}(\Sigma) \\
\downarrow{\scriptstyle \alpha_\Sigma^k} & & \downarrow{\scriptstyle \alpha_\Sigma} \\
\mathrm{SEN}'(F(\Sigma))^k & \xrightarrow[\sigma'_{F(\Sigma)}]{} & \mathrm{SEN}'(F(\Sigma))
\end{array}$$

$$\alpha_\Sigma(\sigma_\Sigma(\vec{\phi})) = \sigma'_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})).$$

We call this the **morphism property**.

Concerning algebraic systems, we will have occasion to make use of the following useful construction and properties.

Let again $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system and $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ and $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ be $N^\flat$-algebraic systems and $\langle F, \alpha \rangle : \mathbf{A} \to \mathbf{A}'$ an algebraic system morphism, with $F : \mathbf{Sign} \to \mathbf{Sign}'$ an isomorphism. We define the algebraic system $\alpha(\mathbf{A}) = \langle \mathbf{Sign}', \mathrm{SEN}'^\alpha, N'^\alpha \rangle$ as follows:

- For all $\Sigma \in |\mathbf{Sign}|$,

$$\mathrm{SEN}'^\alpha(F(\Sigma)) = \alpha_\Sigma(\mathrm{SEN}(\Sigma));$$

For all $\Sigma, \Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}'^{\alpha}(F(f)) : \mathrm{SEN}'^{\alpha}(F(\Sigma)) \to \mathrm{SEN}'^{\alpha}(F(\Sigma'))$$

is given by setting, for all $\phi \in \mathrm{SEN}'^{\alpha}(F(\Sigma))$,

$$\mathrm{SEN}'^{\alpha}(F(f))(\phi) = \mathrm{SEN}'(F(f))(\phi).$$

- For every $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, we let $\sigma'^{\alpha} : (\mathrm{SEN}'^{\alpha})^k \to \mathrm{SEN}'^{\alpha}$ be the restriction of $\sigma' : \mathrm{SEN}'^k \to \mathrm{SEN}'$ to SEN'^{α}.

  Composition works as expected, i.e., for all $\tau^\flat : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^\ell$ and all $\sigma^\flat : (\mathrm{SEN}^\flat)^\ell \to (\mathrm{SEN}^\flat)^m$ in $N^\flat$,

$$\sigma'^{\alpha} \circ \tau'^{\alpha} = (\sigma' \circ \tau')^{\alpha}.$$

It is not difficult to see that $\alpha(\mathbf{A})$, thus defined, is an $N^\flat$-algebraic system.

Lemma 13 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *and* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ *be* $N^\flat$*-algebraic systems and* $\langle F, \alpha \rangle : \mathbf{A} \to \mathbf{A}'$ *an algebraic system morphism, with* $F : \mathbf{Sign} \to \mathbf{Sign}'$ *an isomorphism. Then* $\alpha(\mathbf{A}) = \langle \mathbf{Sign}', \mathrm{SEN}'^{\alpha}, N'^{\alpha} \rangle$ *is an* $N^\flat$*-algebraic system.*

Proof: The critical step is to show that $\mathrm{SEN}'^{\alpha} : \mathbf{Sign}' \to \mathbf{Set}$ is a well-defined functor and that N'^{α} consists in fact of natural transformations on SEN'^{α}.

For the first, let $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\phi \in \mathrm{SEN}(\Sigma)$. Then we have

$$\begin{array}{rcl}
\mathrm{SEN}'^{\alpha}(F(f))(\alpha_\Sigma(\phi)) & = & \mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi)) \\
& = & \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \\
& \in & \mathrm{SEN}'^{\alpha}(F(\Sigma')).
\end{array}$$

So SEN'^{α} is a well-defined functor.

Similarly, for $\sigma^\flat \in N^\flat$, $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\vec{\phi} \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{ccc}
\mathrm{SEN}'^{\alpha}(F(\Sigma))^k & \xrightarrow{\sigma'^{\alpha}_{F(\Sigma)}} & \mathrm{SEN}'^{\alpha}(F(\Sigma)) \\
\mathrm{SEN}'^{\alpha}(F(f))^k \downarrow & & \downarrow \mathrm{SEN}'^{\alpha}(F(f)) \\
\mathrm{SEN}'^{\alpha}(F(\Sigma'))^k & \xrightarrow[\sigma'^{\alpha}_{F(\Sigma')}]{} & \mathrm{SEN}'^{\alpha}(F(\Sigma'))
\end{array}$$

$$\begin{array}{rcl}
\mathrm{SEN}'^{\alpha}(F(f))(\sigma'^{\alpha}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi}))) & = & \mathrm{SEN}'^{\alpha}(F(f))(\sigma'_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi}))) \\
& = & \mathrm{SEN}'^{\alpha}(F(f))(\alpha_\Sigma(\sigma_\Sigma(\vec{\phi}))) \\
& = & \mathrm{SEN}'(F(f))(\alpha_\Sigma(\sigma_\Sigma(\vec{\phi}))) \\
& = & \alpha_{\Sigma'}(\mathrm{SEN}(f)(\sigma_\Sigma(\vec{\phi}))) \\
& = & \alpha_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\vec{\phi}))) \\
& = & \sigma'_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}(f)(\vec{\phi}))) \\
& = & \sigma'^{\alpha}_{F(\Sigma')}(\mathrm{SEN}'(F(f))(\alpha_\Sigma(\vec{\phi}))) \\
& = & \sigma'^{\alpha}_{F(\Sigma')}(\mathrm{SEN}'^{\alpha}(F(f))(\alpha_\Sigma(\vec{\phi}))).
\end{array}$$

Thus, $\sigma'^{\alpha} : (\mathrm{SEN}'^{\alpha})^k \to (\mathrm{SEN}'^{\alpha})$ is a well-defined natural transformation on SEN'^{α}. ■

We call $\alpha(\mathbf{A})$ the **image algebraic system** of $\mathbf{A}$ under $\langle F, \alpha\rangle$.

It is not difficult to see that, additionally, one may construct a surjective morphism from $\mathbf{A}$ to $\alpha(\mathbf{A})$. In fact, we define $\langle F, \alpha'\rangle : \mathbf{A} \to \alpha(\mathbf{A})$ by letting $\alpha' : \mathrm{SEN} \to \mathrm{SEN}'^{\alpha} \circ F$ be given, for all $\Sigma \in |\mathbf{Sign}|$, by

$$\alpha'_\Sigma(\phi) = \alpha_\Sigma(\phi), \text{ for all } \phi \in \mathrm{SEN}(\Sigma).$$

Lemma 14 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *and* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *be* $N^\flat$*-algebraic systems and* $\langle F, \alpha\rangle : \mathbf{A} \to \mathbf{A}'$ *an algebraic system morphism, with* $F : \mathbf{Sign} \to \mathbf{Sign}'$ *an isomorphism. Then* $\langle F, \alpha'\rangle : \mathbf{A} \to \alpha(\mathbf{A})$ *is a surjective algebraic system morphism.*

Proof: The fact that $\alpha' : \mathrm{SEN} \to \mathrm{SEN}'^{\alpha} \circ F$ is a natural transformation follows from the corresponding property of α. Moreover, the fact that $\langle F, \alpha'\rangle$ has the morphism property also follows from the corresponding property of $\langle F, \alpha\rangle$. Finally, surjectivity of $\alpha'_\Sigma : \mathrm{SEN}(\Sigma) \to \mathrm{SEN}'^{\alpha}(F(\Sigma))$, for all $\Sigma \in |\mathbf{Sign}|$, follows by the definition of SEN'^{α}. ■

Let again $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system. An **F-algebraic system** (or an **interpreted algebraic system**) $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ consists of:

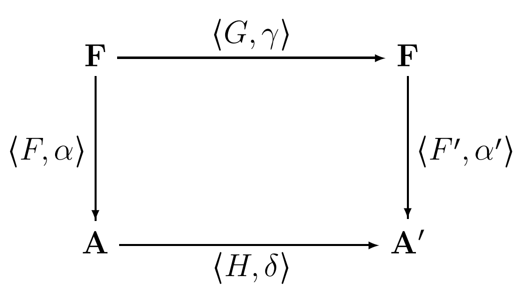
- An $N^\flat$-algebraic system $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$;
- A surjective algebraic system morphism $\langle F, \alpha\rangle : \mathbf{F} \to \mathbf{A}$.

We denote the class of all $\mathbf{F}$-algebraic systems by $\mathrm{AlgSys}(\mathbf{F})$.

Given two $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ and $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha'\rangle\rangle$, a **morphism** (of $\mathbf{F}$-algebraic systems) $\langle\langle G, \gamma\rangle, \langle H, \delta\rangle\rangle : \mathcal{A} \to \mathcal{A}'$ consists of:

- A morphism of $N^\flat$-algebraic systems $\langle G, \gamma\rangle : \mathbf{F} \to \mathbf{F}$;
- A morphism of $N^\flat$-algebraic systems $\langle H, \delta\rangle : \mathbf{A} \to \mathbf{A}'$

such that the following diagram commutes



We call a morphism $\langle\langle G, \gamma\rangle, \langle H, \delta\rangle\rangle : \mathcal{A} \to \mathcal{A}'$ **special** if $\langle G, \gamma\rangle : \mathbf{F} \to \mathbf{F}$ is special and we call it **surjective** if $\langle G, \gamma\rangle : \mathbf{F} \to \mathbf{F}$ is surjective.

We show that these properties propagate to $\langle H, \delta\rangle : \mathbf{A} \to \mathbf{A}'$.

Lemma 15 *Consider a base algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$. *Let* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and* $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$ *be* $\mathbf{F}$*-algebraic systems and* $\langle \langle G, \gamma \rangle, \langle H, \delta \rangle \rangle : \mathcal{A} \to \mathcal{A}'$ *a morphism.*

(a) If $\langle \langle G, \gamma \rangle, \langle H, \delta \rangle \rangle : \mathcal{A} \to \mathcal{A}'$ *is special, then* $\langle H, \delta \rangle : \mathbf{A} \to \mathbf{A}'$ *is special;*

(b) If $\langle \langle G, \gamma \rangle, \langle H, \delta \rangle \rangle : \mathcal{A} \to \mathcal{A}'$ *is surjective, then* $\langle H, \delta \rangle : \mathbf{A} \to \mathbf{A}'$ *is surjective.*

Proof:

(a) Suppose $\langle G, \gamma \rangle$ is special. We show, first, that H is surjective on objects and, then, that it is full. Surjectivity on objects is easy. Since F' and G are surjective on objects, $H \circ F = F' \circ G$ is also surjective on objects. This implies that H is surjective on objects.

For fullness, recall that it suffices to show that, for all $Y, Y' \in |\mathbf{Sign}|$,

$$H : \mathbf{Sign}(Y, Y') \to \mathbf{Sign}'(H(Y), H(Y'))$$

is surjective. So let $k \in \mathbf{Sign}'(H(Y), H(Y'))$. Then, by the surjectivity of F, there exist $X, X' \in |\mathbf{Sign}^\flat|$, such that $F(X) = Y$ and $F(X') = Y'$. Thus, we get

$$k \in \mathbf{Sign}'(H(F(X)), H(F(X'))) = \mathbf{Sign}'(F'(G(X)), F'(G(X'))).$$

Since G and F' are full, there exists $f \in \mathbf{Sign}^\flat(X, X')$, such that $F'(G(f)) = k$. So we have that $H(F(f)) = k$ and $F(f) \in \mathbf{Sign}(F(X), F(X')) = \mathbf{Sign}(Y, Y')$. Therefore H is full.

(b) By Part (a), it suffices to show that, for all $Y \in |\mathbf{Sign}|$, $\delta_Y : \mathrm{SEN}(Y) \to \mathrm{SEN}'(H(Y))$ is surjective. Let $\chi \in \mathrm{SEN}'(H(Y))$. Since F is surjective, there exists $X \in |\mathbf{Sign}^\flat|$, such that $F(X) = Y$. So we get $\chi \in \mathrm{SEN}'(H(F(X))) = \mathrm{SEN}'(F'(G(X)))$. Since both $\gamma_X : \mathrm{SEN}^\flat(X) \to \mathrm{SEN}^\flat(G(X))$ and $\alpha'_{G(X)} : \mathrm{SEN}^\flat(G(X)) \to \mathrm{SEN}'(F'(G(X)))$ are surjective, we get that $\alpha'_{G(X)} \circ \gamma_X : \mathrm{SEN}^\flat(X) \to \mathrm{SEN}'(F'(G(X)))$ is surjective. Thus, there exists $\phi \in \mathrm{SEN}^\flat(X)$, such that

$$\chi = \alpha'_{G(X)}(\gamma_X(\phi)) = \delta_{F(X)}(\alpha_X(\phi)) = \delta_Y(\alpha_X(\phi)).$$

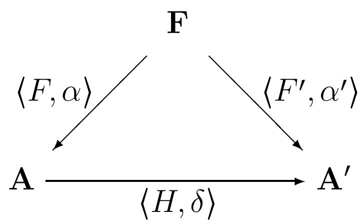
So $\delta_Y : \mathrm{SEN}(Y) \to \mathrm{SEN}'(H(Y))$ is also surjective.

■

In the future, we will restrict attention mostly to $\mathbf{F}$-algebraic system morphisms $\langle \langle G, \gamma \rangle, \langle H, \delta \rangle \rangle : \mathcal{A} \to \mathcal{A}'$, with

$$\langle G, \gamma \rangle = \langle I, \iota \rangle : \mathbf{F} \to \mathbf{F},$$

where $\langle I, \iota\rangle : \mathbf{F} \to \mathbf{F}$ denotes the identity morphism on $\mathbf{F}$. Since this morphism is surjective, by Lemma 15, this property will automatically hold for $\langle H, \delta\rangle : \mathbf{A} \to \mathbf{A}'$ as well. In this case, we also use the simplified notation $\langle H, \delta\rangle : \mathcal{A} \to \mathcal{A}'$



and even though we might say a "surjective" morphism $\langle H, \delta\rangle : \mathcal{A} \to \mathcal{A}'$ for emphasis, it is understood that this will always be the case, even without this specification.

2.3 Congruence Systems

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system. A **relation family on A** is a relation family on SEN, i.e., a collection $R = \{R_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, such that $R_\Sigma \subseteq \mathrm{SEN}(\Sigma)^2$, for all $\Sigma \in |\mathbf{Sign}|$. A relation family on $\mathbf{A}$ is a **relation system** if it is a relation system on SEN, i.e., if it is invariant under **Sign**-morphisms; that is, if for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}(f)(R_\Sigma) \subseteq R_{\Sigma'}.$$

A relation family/system on $\mathbf{A}$ is an **equivalence family/system on A** if it is an equivalence family/system on SEN, i.e., for all $\Sigma \in |\mathbf{Sign}|$, R_Σ is an equivalence relation on $\mathrm{SEN}(\Sigma)$. Finally, an equivalence system is called a **congruence system on A** if, for all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi_i, \psi_i\rangle \in R_\Sigma, i < k, \text{ implies } \langle \sigma_\Sigma(\vec{\phi}), \sigma_\Sigma(\vec{\psi})\rangle \in R_\Sigma.$$

We call this the **congruence property**.

The collection of all congruence systems on the algebraic system $\mathbf{A}$ will be denoted by $\mathrm{ConSys}(\mathbf{A})$. Ordered under signature-wise inclusion $\leq$, it forms a complete lattice, which is denoted by $\mathbf{ConSys(A)} = \langle \mathrm{ConSys}(\mathbf{A}), \leq\rangle$.

The least congruence system on $\mathbf{A}$ is the **identity congruence system**, which denoted by $\Delta^{\mathbf{A}} = \{\Delta^{\mathbf{A}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, where, for all $\Sigma \in |\mathbf{Sign}|$,

$$\Delta^{\mathbf{A}}_\Sigma = \{\langle \phi, \phi\rangle : \phi \in \mathrm{SEN}(\Sigma)\}.$$

The largest congruence system is the **nabla congruence system**, denoted $\nabla^{\mathbf{A}}$ or SEN^2, and defined by $\nabla^{\mathbf{A}} = \{\nabla^{\mathbf{A}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, such that, for all $\Sigma \in |\mathbf{Sign}|$,

$$\nabla^{\mathbf{A}}_\Sigma = \{\langle \phi, \psi\rangle : \phi, \psi \in \mathrm{SEN}(\Sigma)\} = \mathrm{SEN}(\Sigma)^2.$$

The infimum of a family $\{\theta^i : i \in I\} \subseteq \mathrm{ConSys}(\mathbf{A})$ is given by signature-wise intersection $\bigcap_{i\in I}\theta^i$, while the supremum is the congruence system generated by the signature-wise union of the θ^i, $\bigvee_{i\in I}\theta^i = \{\Theta(\bigcup_{i\in I}\theta^i_\Sigma)\}_{\Sigma\in|\mathbf{Sign}|}$, where $\Theta(\bigcup_{i\in I}\theta^i_\Sigma)$ denotes the congruence on $\mathrm{SEN}(\Sigma)$ (viewed as an ordinary algebra with operations $\sigma_\Sigma : \mathrm{SEN}(\Sigma)^k \to \mathrm{SEN}(\Sigma)$, for $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N) generated by $\bigcup_{i\in I}\theta^i$.

Proposition 16 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *two* $N^\flat$*-algebraic systems and* $\langle F, \alpha\rangle : \mathbf{A} \to \mathbf{A}'$ *a morphism of* $N^\flat$*-algebraic systems. If* $\theta \in \mathrm{ConSys}(\mathbf{A}')$*, then* $\alpha^{-1}(\theta) \in \mathrm{ConSys}(\mathbf{A})$*.*

Proof: By Proposition 10 it suffices to show that, if θ has the congruence property, then $\alpha^{-1}(\theta)$ also has the congruence property. To see this, consider $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi_i, \psi_i\rangle \in \alpha_\Sigma^{-1}(\theta_{F(\Sigma)})$, for all $i < k$. Then we get $\langle \alpha_\Sigma(\phi_i), \alpha_\Sigma(\psi_i)\rangle \in \theta_{F(\Sigma)}$, for all $i < k$. Thus, by the congruence property of θ,

$$\langle \sigma'_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})), \sigma'_{F(\Sigma)}(\alpha_\Sigma(\vec{\psi}))\rangle \in \theta_{F(\Sigma)}.$$

By the morphism property, we get

$$\langle \alpha_\Sigma(\sigma_\Sigma(\vec{\phi})), \alpha_\Sigma(\sigma_\Sigma(\vec{\psi}))\rangle \in \theta_{F(\Sigma)}.$$

Hence $\langle \sigma_\Sigma(\vec{\phi}), \sigma_\Sigma(\vec{\psi})\rangle \in \alpha_\Sigma^{-1}(\theta_{F(\Sigma)})$, showing that $\alpha^{-1}(\theta)$ also satisfies the congruence property. ∎

As a special case of Proposition 16, we obtain that kernels of morphisms are congruence systems.

Corollary 17 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *two* $N^\flat$*-algebraic systems and* $\langle F, \alpha\rangle : \mathbf{A} \to \mathbf{A}'$ *a morphism of* $N^\flat$*-algebraic systems. Then* $\mathrm{Ker}(\langle F, \alpha\rangle) \in \mathrm{ConSys}(\mathbf{A})$*.*

Proof: This follows by Proposition 16 by taking $\theta = \Delta^{\mathcal{A}'}$. Then, obviously, $\alpha^{-1}(\theta) = \mathrm{Ker}(\langle F, \alpha\rangle)$. ∎

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system and $\theta \in \mathrm{ConSys}(\mathbf{A})$. The **quotient** $\mathbf{A}^\theta$ (or $\mathbf{A}/\theta$) **of A by** θ is the algebraic system $\mathbf{A}^\theta = \langle \mathbf{Sign}, \mathrm{SEN}^\theta, N^\theta\rangle$, defined as follows:

- For all $\Sigma \in |\mathbf{Sign}|$,

$$\mathrm{SEN}^\theta(\Sigma) = \mathrm{SEN}(\Sigma)/\theta_\Sigma = \{\phi/\theta_\Sigma : \phi \in \mathrm{SEN}(\Sigma)\}.$$

  For all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\mathrm{SEN}^\theta(f)(\phi/\theta_\Sigma) = \mathrm{SEN}(f)(\phi)/\theta_{\Sigma'}.$$

- N^θ is the category of natural transformations on SEN^θ of the form $\sigma^\theta : (\mathrm{SEN}^\theta)^k \to \mathrm{SEN}^\theta$, where $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ is in N, defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)$, by
$$\sigma^\theta_\Sigma(\vec{\phi}/\theta_\Sigma) = \sigma_\Sigma(\vec{\phi})/\theta_\Sigma.$$

The fact that θ is an equivalence system makes the functor SEN^θ well-defined at both the object and the morphism level. Moreover, the fact that θ is a congruence system makes the definition of each natural transformation in N^θ sound. Finally, the identities, projections and the composition in N^θ are the images of the corresponding operations and of the composition in N under $\cdot \mapsto \cdot^\theta$: For all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $\vec{\phi} \in \mathrm{SEN}(\Sigma)$,

- $i^\theta_\Sigma(\phi/\theta_\Sigma) = \phi/\theta_\Sigma = i_\Sigma(\phi)/\theta_\Sigma$;
- $p^{k,i^\theta}_\Sigma(\vec{\phi}/\theta_\Sigma) = \phi_i/\theta_\Sigma = p^{k,i}_\Sigma(\vec{\phi})/\theta_\Sigma$;
- $\tau^\theta_\Sigma(\sigma^{0^\theta}_\Sigma(\vec{\phi}/\theta_\Sigma), \dots, \sigma^{k-1^\theta}_\Sigma(\vec{\phi}/\theta_\Sigma)) = \tau^\theta_\Sigma(\sigma^0_\Sigma(\vec{\phi})/\theta_\Sigma, \dots, \sigma^{k-1}_\Sigma(\vec{\phi})/\theta_\Sigma)$
$= \tau_\Sigma(\sigma^0_\Sigma(\vec{\phi}), \dots, \sigma^{k-1}_\Sigma(\vec{\phi}))/\theta_\Sigma$.

We denote by $\langle I, \pi^\theta\rangle : \mathbf{A} \to \mathbf{A}^\theta$ the **quotient morphism**, defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, by
$$\pi^\theta_\Sigma(\phi) = \phi/\theta_\Sigma.$$
To see that it is well-defined, we must show that $\pi^\theta : \mathrm{SEN} \to \mathrm{SEN}^\theta$ is a natural transformation and that it satisfies the morphism property. In fact, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{ccc}
\mathrm{SEN}(\Sigma) & \xrightarrow{\pi^\theta_\Sigma} & \mathrm{SEN}^\theta(\Sigma) \\
{\scriptstyle \mathrm{SEN}(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{SEN}^\theta(f)} \\
\mathrm{SEN}(\Sigma') & \xrightarrow[\pi^\theta_{\Sigma'}]{} & \mathrm{SEN}^\theta(\Sigma')
\end{array}$$

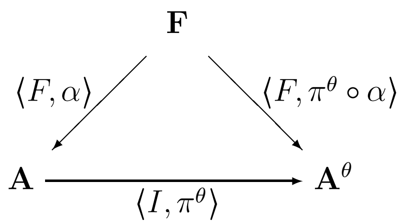
$$\begin{array}{rcl}
\pi^\theta_{\Sigma'}(\mathrm{SEN}(f)(\phi)) & = & \mathrm{SEN}(f)(\phi)/\theta_{\Sigma'} \\
& = & \mathrm{SEN}^\theta(f)(\phi/\theta_\Sigma) \\
& = & \mathrm{SEN}^\theta(f)(\pi^\theta_\Sigma(\phi)).
\end{array}$$

And for all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{ccc}
\mathrm{SEN}(\Sigma)^k & \xrightarrow{\pi^{\theta^k}_\Sigma} & \mathrm{SEN}^\theta(\Sigma)^k \\
{\scriptstyle \sigma_\Sigma}\downarrow & & \downarrow{\scriptstyle \sigma^\theta_\Sigma} \\
\mathrm{SEN}(\Sigma) & \xrightarrow[\pi^\theta_\Sigma]{} & \mathrm{SEN}^\theta(\Sigma)
\end{array}$$

$$\pi^\theta_\Sigma(\sigma_\Sigma(\vec{\phi})) = \sigma_\Sigma(\vec{\phi})/\theta_\Sigma = \sigma^\theta_\Sigma(\vec{\phi}/\theta_\Sigma) = \sigma^\theta_\Sigma(\pi^\theta_\Sigma(\vec{\phi})).$$

Note that this construction allows us to discuss also quotients of **F**-algebraic systems. More precisely, consider a base algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$. Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an **F**-algebraic system and let $\theta \in \mathrm{ConSys}(\mathcal{A}) \coloneqq \mathrm{ConSys}(\mathbf{A})$. The **quotient F-algebraic system of $\mathcal{A}$ by θ** is defined as $\mathcal{A}^\theta = \langle \mathbf{A}^\theta, \langle F, \pi^\theta \circ \alpha \rangle \rangle$.



Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an algebraic system and let $T \in \mathrm{SenFam}(\mathbf{A})$. We say that a congruence system θ on $\mathbf{A}$ is **compatible with** T if it is compatible with T as an equivalence system on SEN, i.e., if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi, \psi \rangle \in \theta_\Sigma \quad \text{and} \quad \phi \in T_\Sigma \quad \text{imply} \quad \psi \in T_\Sigma.$$

Note that, for every $T \in \mathrm{SenFam}(\mathbf{A})$, $\Delta^{\mathbf{A}}$ is compatible with T. We denote the collection of all congruence systems on $\mathbf{A}$ that are compatible with T by $\mathrm{ConSys}^{\mathbf{A}}(T)$.

Proposition 18 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *be an algebraic system and* $T \in \mathrm{SenFam}(\mathbf{A})$. *The collection* $\mathrm{ConSys}^{\mathbf{A}}(T)$, *of all congruence systems on* $\mathbf{A}$ *that are compatible with* T, *forms a complete lattice*

$$\mathbf{ConSys}^{\mathbf{A}}(T) = \langle \mathrm{ConSys}^{\mathbf{A}}(T), \leq \rangle$$

under signature-wise inclusion.

Proof: First, the collection $\mathrm{ConSys}^{\mathbf{A}}(T)$ is closed under arbitrary intersections: Let θ^i, $i \in I$, be in $\mathrm{ConSys}^{\mathbf{A}}(T)$. Suppose that $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \bigcap_{i \in I} \theta^i_\Sigma$ and $\phi \in T_\Sigma$. Then $\langle \phi, \psi \rangle \in \theta^i_\Sigma$, for all $i \in I$. Since θ^i is compatible with T, we get $\psi \in T_\Sigma$. This shows that $\bigcap_{i \in I} \theta^i$ is compatible with T.

It suffices, therefore, to show that $\mathrm{ConSys}^{\mathbf{A}}(T)$ has a greatest element. The signature-wise union of every directed subset of $\mathrm{ConSys}^{\mathbf{A}}(T)$ is an upper bound for the subset in $\mathrm{ConSys}(\mathbf{A})$. Moreover, it is in $\mathrm{ConSys}^{\mathbf{A}}(T)$ since every member of the subset is. So, by Zorn's Lemma, $\mathrm{ConSys}^{\mathbf{A}}(T)$ has a maximal element.

Suppose, for the sake of obtaining a contradiction, that $\theta \neq \theta'$ are two such maximal elements. Recall that their join $\theta \vee \theta'$ is given by $\theta \vee \theta' =$

$\{\theta_\Sigma \vee \theta'_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$, where

$$\theta_\Sigma \vee \theta'_\Sigma = \bigcup_{k=0}^{\infty} \underbrace{\theta_\Sigma \circ \theta'_\Sigma \circ \cdots \circ \theta_\Sigma}_{k \text{ factors}}.$$

Thus, their join $\theta \vee \theta'$ as congruence systems on $\mathbf{A}$ is also compatible with T. This, however, contradicts the maximality of θ and θ', since, clearly, $\theta < \theta \vee \theta'$ and $\theta' < \theta \vee \theta'$. Therefore, the unique maximal element of $\mathrm{ConSys}^{\mathbf{A}}(T)$ is a largest element. ■

The largest congruence system on an algebraic system $\mathbf{A}$ compatible with $T \in \mathrm{SenFam}(\mathbf{A})$ is called the **Leibniz congruence system** of T on $\mathbf{A}$ and is denoted by $\Omega^{\mathbf{A}}(T)$.

The following theorem provides an explicit characterization of the Leibniz congruence system of a sentence family T on an algebraic system $\mathbf{A}$.

Theorem 19 *Suppose that $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ is an algebraic system and $T \in \mathrm{SenFam}(\mathbf{A})$. Then, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,*

$$\begin{array}{lll}\langle\phi,\psi\rangle \in \Omega^{\mathbf{A}}_\Sigma(T) & \textit{iff} & \textit{for all } \sigma : \mathrm{SEN}^k \to \mathrm{SEN} \textit{ in } N, \textit{ all } \Sigma' \in |\mathbf{Sign}|,\\ & & \textit{all } f \in \mathbf{Sign}(\Sigma,\Sigma') \textit{ and all } \vec{\chi} \in \mathrm{SEN}(\Sigma'), \textit{ we have}\\ & & \sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}) \in T_{\Sigma'} \textit{ iff } \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi}) \in T_{\Sigma'}.\end{array}$$

Proof: Let $R = \{R_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ be the relation system on $\mathbf{A}$ defined by the given condition, i.e., for all $\Sigma \in |\mathbf{Sign}|$,

$$\begin{array}{ll}R_\Sigma = \{\langle\phi,\psi\rangle \in \mathrm{SEN}(\Sigma)^2 : & \\ & \text{for all } \sigma : \mathrm{SEN}^k \to \mathrm{SEN} \text{ in } N, \text{ all } \Sigma' \in |\mathbf{Sign}|,\\ & \text{all } f \in \mathbf{Sign}(\Sigma,\Sigma') \text{ and all } \vec{\chi} \in \mathrm{SEN}(\Sigma'),\\ & \sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}) \in T_{\Sigma'} \text{ iff } \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi}) \in T_{\Sigma'}\}.\end{array}$$

First, we show that $\Omega^{\mathbf{A}}(T) \le R$. Let $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle \in \Omega^{\mathbf{A}}_\Sigma(T)$. Since $\Omega^{\mathbf{A}}(T)$ is a congruence system, we get that, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma,\Sigma')$, $\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)\rangle \in \Omega^{\mathbf{A}}_{\Sigma'}(T)$. Now, since $\Omega^{\mathbf{A}}(T)$ is a congruence system, we get that, for all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

$$\langle \sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}), \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi})\rangle \in \Omega^{\mathbf{A}}_{\Sigma'}(T).$$

Finally, since $\Omega^{\mathbf{A}}(T)$ is compatible with T, we get that

$$\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}) \in T_{\Sigma'} \quad \text{iff} \quad \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi}) \in T_{\Sigma'}.$$

But the last condition, being universally quantified on $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma,\Sigma')$, σ in N and $\vec{\chi} \in \mathrm{SEN}(\Sigma')$, yields $\langle\phi,\psi\rangle \in R_\Sigma$. Therefore, we get that $\Omega^{\mathbf{A}}(T) \le R$.

Finally, we show that $R \leq \Omega^{\mathbf{A}}(T)$. For this inclusion, it suffices to show that R is a congruence system on $\mathbf{A}$ that is compatible with T. Then the conclusion would follow from the fact that $\Omega^{\mathbf{A}}(T)$ is, by definition, the largest congruence system on $\mathbf{A}$ that is compatible with T.

It is clear from its definition that R is an equivalence family on $\mathbf{A}$.

To see that it is an equivalence system, let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in R_\Sigma$. Consider $\Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma, \Sigma')$. Then, for all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, all $\Sigma'' \in |\mathbf{Sign}|$, all $g \in \mathbf{Sign}(\Sigma', \Sigma'')$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma'')$,

$$\Sigma \xrightarrow{f} \Sigma' \xrightarrow{g} \Sigma''$$

we have

$$\begin{array}{ll} \multicolumn{2}{l}{\sigma_{\Sigma''}(\mathrm{SEN}(g)(\mathrm{SEN}(f)(\phi)), \vec{\chi}) \in T_{\Sigma''}} \\ \text{iff} & \sigma_{\Sigma''}(\mathrm{SEN}(gf)(\phi), \vec{\chi}) \in T_{\Sigma''} \\ \text{iff} & \sigma_{\Sigma''}(\mathrm{SEN}(gf)(\psi), \vec{\chi}) \in T_{\Sigma''} \quad (\text{since } \langle \phi, \psi \rangle \in R_\Sigma) \\ \text{iff} & \sigma_{\Sigma''}(\mathrm{SEN}(g)(\mathrm{SEN}(f)(\psi)), \vec{\chi}) \in T_{\Sigma''}. \end{array}$$

Thus, we conclude that $\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi) \rangle \in R_{\Sigma'}$, showing that R is an equivalence system.

Next, to see that R is a congruence system, consider $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, $\Sigma \in |\mathbf{Sign}|$, and $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi_i, \psi_i \rangle \in R_\Sigma$, $i < k$. Let $\tau : \mathrm{SEN}^\ell \to \mathrm{SEN}$ be in N, $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\vec{\chi} \in \mathrm{SEN}(\Sigma')$. Then, we have

$$\begin{array}{ll} \multicolumn{2}{l}{\tau_{\Sigma'}(\mathrm{SEN}(f)(\sigma_\Sigma(\vec{\phi})), \vec{\chi}) \in T_{\Sigma'}} \\ \text{iff} & \tau_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\vec{\phi})), \vec{\chi}) \in T_{\Sigma'} \\ \text{iff} & \tau_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\vec{\psi})), \vec{\chi}) \in T_{\Sigma'} \\ & (\tau \circ \langle \sigma \circ \langle p^{k+\ell-1,0}, \ldots, p^{k+\ell-1,k} \rangle, p^{k+\ell-1,k+1}, \ldots, p^{k+\ell-1,k+\ell-2} \rangle \text{ in } N \\ & \text{together with Corollary 12, applied } k \text{ times}) \\ \text{iff} & \tau_{\Sigma'}(\mathrm{SEN}(f)(\sigma_\Sigma(\vec{\psi})), \vec{\chi}) \in T_{\Sigma'}. \end{array}$$

This shows that $\langle \sigma_\Sigma(\vec{\phi}), \sigma_\Sigma(\vec{\psi}) \rangle \in R_\Sigma$, whence R is a congruence system.

Finally, upon setting in the defining condition $\sigma = p^{1,0} : \mathrm{SEN} \to \mathrm{SEN}$ in N, $\Sigma' = \Sigma$, $f = i_\Sigma$, the identity **Sign**-morphism, we get that for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, with $\langle \phi, \psi \rangle \in R_\Sigma$

$$\phi \in T_\Sigma \quad \text{iff} \quad \psi \in T_\Sigma.$$

Thus, R is compatible with T. ∎

The characterization of the Leibniz congruence system, presented in Theorem 19, provides a justification for an alternative name that is sometimes attributed to the Leibniz congruence system of a sentence family T on an algebraic system $\mathbf{A}$. Given $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, we say that ϕ and ψ are **indiscernible modulo** T if

$$\langle \phi, \psi \rangle \in \Omega_\Sigma^{\mathbf{A}}(T).$$

Therefore $\Omega^{\mathbf{A}}(T)$ is also referred to as the **indiscernibility relation on A modulo** T.

We can now prove a proposition asserting that the Leibniz congruence system of a sentence family T is included in that of the sentence system $\overleftarrow{T}$.

Proposition 20 *Suppose that* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *is an algebraic system and* $T \in \mathrm{SenFam}(\mathbf{A})$. *Then*

$$\Omega^{\mathbf{A}}(T) \leq \Omega^{\mathbf{A}}(\overleftarrow{T}).$$

Proof: To prove this inclusion, it suffices to show that $\Omega^{\mathbf{A}}(T)$ is compatible with $\overleftarrow{T}$. We can invoke Lemma 7, but we also give a direct proof due to the heavy significance of this result. Let $\Sigma \in |\mathbf{Sign}|$, and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \Omega_\Sigma^{\mathbf{A}}(T)$ and $\phi \in \overleftarrow{T}_\Sigma$. Since $\Omega^{\mathbf{A}}(T)$ is a congruence system and by the definition of $\overleftarrow{T}$, we get that, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi) \rangle \in \Omega_{\Sigma'}^{\mathbf{A}}(T) \quad \text{and} \quad \mathrm{SEN}(f)(\phi) \in T_{\Sigma'}.$$

Thus, by the compatibility of $\Omega^{\mathbf{A}}(T)$ with T, we obtain $\mathrm{SEN}(f)(\psi) \in T_{\Sigma'}$. Since this holds for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, we get $\psi \in \overleftarrow{T}_\Sigma$. Thus, $\Omega^{\mathbf{A}}(T)$ is compatible with $\overleftarrow{T}$, showing that $\Omega^{\mathbf{A}}(T) \leq \Omega^{\mathbf{A}}(\overleftarrow{T})$. ■

We exhibit, next, an algebraic system $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ together with a sentence family $T \in \mathrm{SenFam}(\mathbf{A})$, such that $\Omega^{\mathbf{A}}(T) \lneq \Omega^{\mathbf{A}}(\overleftarrow{T})$.

Example 21 *Define* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *as follows:*

- $\mathbf{Sign}$ *is a category with two objects* Σ, Σ' *and a single (non-identity) morphism* $f : \Sigma \to \Sigma'$.
- $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *is defined by setting* $\mathrm{SEN}(\Sigma) = \{0, 1\}$, $\mathrm{SEN}(\Sigma') = \{a, b\}$, $\mathrm{SEN}(f)(0) = a$ *and* $\mathrm{SEN}(f)(1) = b$.
- *The clone* N *of natural transformations is trivial, i.e., consists of the projection natural transformations only.*

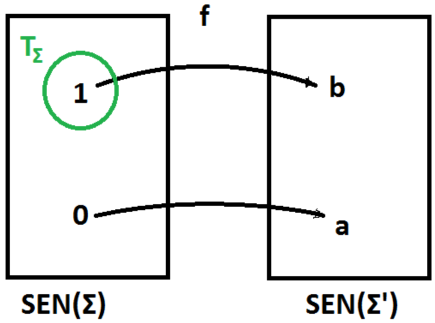
Finally, let $T = \{T_\Sigma, T_{\Sigma'}\}$ *be specified by setting* $T_\Sigma = \{1\}$ *and* $T_{\Sigma'} = \varnothing$. *Then it is not difficult to see that* $\overleftarrow{T}_\Sigma = \varnothing = \overleftarrow{T}_{\Sigma'}$ *and, therefore, that*

$$\Omega_\Sigma^{\mathbf{A}}(\overleftarrow{T}) = \nabla_\Sigma^{\mathbf{A}} \quad \textit{and} \quad \Omega_{\Sigma'}^{\mathbf{A}}(\overleftarrow{T}) = \nabla_{\Sigma'}^{\mathbf{A}},$$

whereas

$$\Omega_\Sigma^{\mathbf{A}}(T) = \Delta_\Sigma^{\mathbf{A}} \quad \textit{and} \quad \Omega_{\Sigma'}^{\mathbf{A}}(T) = \nabla_{\Sigma'}^{\mathbf{A}}.$$

Hence, we have $\Omega^{\mathbf{A}}(T) \lneq \Omega^{\mathbf{A}}(\overleftarrow{T})$.



Proposition 20 and Example 21 have important consequences. We give a brief account here, as is proper after proving these facts, but postpone further treatment for subsequent chapters.

1. Note that, given an algebraic system $\mathbf{A}$, for any sentence family T of $\mathbf{A}$, both $T, \overleftarrow{T}$ are sentence families of $\mathbf{A}$, such that, in general,
$$\overleftarrow{T} \leq T \quad \text{and} \quad \Omega^{\mathbf{A}}(T) \leq \Omega^{\mathbf{A}}(\overleftarrow{T}).$$
But it is an accepted wisdom in abstract algebraic logic that a logic is amenable to a meaningful algebraic treatment and, thus, deserves a place in the algebraic (Leibniz) hierarchy, if it is at least *protoalgebraic* or *truth-equational*, meaning that the Leibniz operator on its collection of theories is at least monotone of completely order reflecting. The displayed relations between T and $\overleftarrow{T}$, therefore, force us to define a new class of π-institutional logics, fulfilling a minimum, in some sense, condition for amenability to algebraic treatment and techniques, which we shall call **stable**, if their Leibniz operator satisfies, for all theory families T of the π-institution,
$$\Omega(T) = \Omega(\overleftarrow{T}).$$
The term "stable" is adopted to insinuate contrast to *inverting* or *changing* the order, since, given that $\overleftarrow{T} \leq T$ and that $\Omega(T) \leq \Omega(\overleftarrow{T})$, for all theory families T, an inversion in the order would occur in case $\Omega(T) \neq \Omega(\overleftarrow{T})$ for some theory family T.

2. Now note the remarkable fact that, for a stable π-institution, the range of the Leibniz operator is entirely covered by its values on theory systems of the π-institution, since, given a theory family T, one can work with its Leibniz congruence system by working with the congruence system $\Omega(\overleftarrow{T})$ of the theory system $\overleftarrow{T}$.

These two remarkable facts form an enticement, a preview and a justification for some of the upcoming definitions and concepts regarding classes of π-institutions, forming the semantic Leibniz hierarchy, in subsequent chapters.

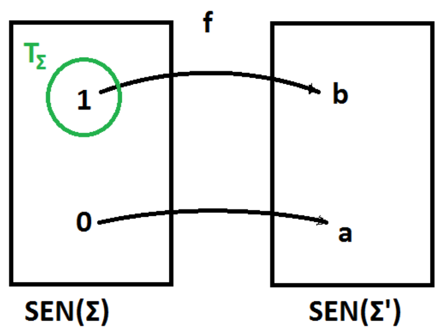
In the next example it is shown that the Leibniz congruence system of a sentence family T of an algebraic system $\mathbf{A}$ does not stand in a definite relationship with that of the sentence system $\overrightarrow{T}$.

Example 22 *We exhibit, first, an algebraic system* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *together with a sentence family* $T \in \mathrm{SenFam}(\mathbf{A})$, *such that* $\Omega^{\mathbf{A}}(\overrightarrow{T}) \subsetneq \Omega^{\mathbf{A}}(T)$.

We use the same algebraic system and the same sentence family as in Example 21. Define $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *as follows:*

- $\mathbf{Sign}$ *is a category with two objects* Σ, Σ' *and a single (non-identity) morphism* $f : \Sigma \to \Sigma'$.
- $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *is defined by setting* $\mathrm{SEN}(\Sigma) = \{0,1\}$, $\mathrm{SEN}(\Sigma') = \{a,b\}$, $\mathrm{SEN}(f)(0) = a$ *and* $\mathrm{SEN}(f)(1) = b$.
- *The clone* N *of natural transformations is trivial, i.e., consists of the projection natural transformations only.*

Finally, let $T = \{T_\Sigma, T_{\Sigma'}\}$ *be specified by setting* $T_\Sigma = \{1\}$ *and* $T_{\Sigma'} = \varnothing$.



Note that $\overrightarrow{T}_\Sigma = \{1\}$ *and* $\overrightarrow{T}_{\Sigma'} = \{b\}$. *So in this case we have*

$$\Omega^{\mathbf{A}}_\Sigma(\overrightarrow{T}) = \Delta^{\mathbf{A}}_\Sigma \quad \textit{and} \quad \Omega^{\mathbf{A}}_{\Sigma'}(\overrightarrow{T}) = \Delta^{\mathbf{A}}_{\Sigma'},$$

whereas, as pointed out in Example 21,

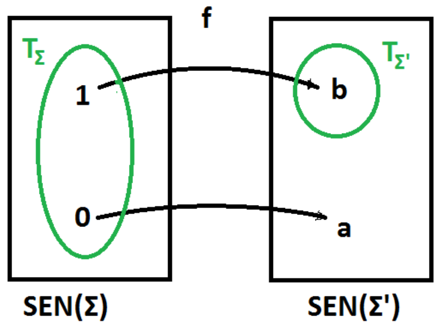
$$\Omega^{\mathbf{A}}_\Sigma(T) = \Delta^{\mathbf{A}}_\Sigma \quad \textit{and} \quad \Omega^{\mathbf{A}}_{\Sigma'}(T) = \nabla^{\mathbf{A}}_{\Sigma'}.$$

So we see that $\Omega^{\mathbf{A}}(\overrightarrow{T}) \subsetneq \Omega^{\mathbf{A}}(T)$.

Finally, we construct an algebraic system $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *together with a sentence family* $T \in \mathrm{SenFam}(T)$, *such that* $\Omega^{\mathbf{A}}(T) \subsetneq \Omega^{\mathbf{A}}(\overrightarrow{T})$.

The algebraic system is the same algebraic system $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, *defined above. But now the sentence family* $T = \{T_\Sigma, T_{\Sigma'}\}$ *is defined by*

$$T_\Sigma = \{0,1\} \quad \textit{and} \quad T_{\Sigma'} = \{b\}.$$



It is clear that $\overrightarrow{T}_\Sigma = \{0,1\}$ *and* $\overrightarrow{T}_{\Sigma'} = \{a,b\}$. *Thus, we have*

$$\Omega^{\mathbf{A}}(T) = \Delta^{\mathbf{A}}_\Sigma \quad \textit{and} \quad \Omega^{\mathbf{A}}_{\Sigma'}(T) = \Delta^{\mathbf{A}}_{\Sigma'},$$

whereas

$$\Omega^{\mathbf{A}}_\Sigma(\overrightarrow{T}) = \nabla^{\mathbf{A}}_\Sigma \quad \textit{and} \quad \Omega^{\mathbf{A}}_{\Sigma'}(\overrightarrow{T}) = \nabla^{\mathbf{A}}_{\Sigma'}.$$

Thus we see that, in this case, $\Omega^{\mathbf{A}}(T) \lneq \Omega^{\mathbf{A}}(\overrightarrow{T})$.

It turns out that the Leibniz congruence system of the intersection of two sentence families of an algebraic system is at least as large as the intersection of the corresponding Leibniz congruence systems.

Lemma 23 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system and let* $\mathcal{T} \subseteq \mathrm{SenFam}(\mathbf{A})$. *Then*

$$\bigcap_{T\in\mathcal{T}} \Omega^{\mathbf{A}}(T) \leq \Omega^{\mathbf{A}}(\bigcap_{T\in\mathcal{T}} T).$$

Proof: The Leibniz congruence system of $\bigcap_{T\in\mathcal{T}} T$ is, by definition, the largest congruence system on $\mathbf{A}$ that is compatible with $\bigcap_{T\in\mathcal{T}} T \in \mathrm{SenFam}(\mathbf{A})$. So to prove the conclusion it suffices to show that $\bigcap_{T\in\mathcal{T}} \Omega^{\mathbf{A}}(T)$ is a congruence system on $\mathbf{A}$ that is compatible with $\bigcap_{T\in\mathcal{T}} T$. That it is a congruence system follows from the fact that $\mathbf{ConSys}(\mathbf{A})$ has the structure of a complete lattice with signature-wise intersection as its infimum. For compatibility, Let $\Sigma \in |\mathbf{Sign}|$, $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle \in \bigcap_{T\in\mathcal{T}} \Omega^{\mathbf{A}}_\Sigma(T)$ and $\phi \in \bigcap_{T\in\mathcal{T}} T_\Sigma$. These two imply the following relations:

$$\langle\phi,\psi\rangle \in \Omega^{\mathbf{A}}_\Sigma(T), \quad \phi \in T_\Sigma, \quad \text{for all } T \in \mathcal{T}.$$

Now, using the compatibility property of $\Omega^{\mathbf{A}}(T)$, $T \in \mathcal{T}$, we get $\psi \in T_\Sigma$, for all $T \in \mathcal{T}$. So $\psi \in \bigcap_{T\in\mathcal{T}} T_\Sigma$ and, therefore, $\bigcap_{T\in\mathcal{T}} \Omega^{\mathbf{A}}(T)$ is compatible with $\bigcap_{T\in\mathcal{T}} T$. ∎

An important property of the Leibniz operator is that it commutes with inverse surjective morphisms of $N^\flat$-algebraic systems.

Proposition 24 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ *two* $N^\flat$*-algebraic systems,* $\langle F, \alpha \rangle : \mathbf{A} \to \mathbf{A}'$ *an algebraic system morphism and* $T \in \mathrm{SenFam}(\mathbf{A}')$*. We have:*

(a) $\alpha^{-1}(\Omega^{\mathbf{A}'}(T)) \leq \Omega^{\mathbf{A}}(\alpha^{-1}(T))$;

(b) If $\langle F, \alpha \rangle$ *is surjective,* $\alpha^{-1}(\Omega^{\mathbf{A}'}(T)) = \Omega^{\mathbf{A}}(\alpha^{-1}(T))$.

Proof:

(a) Since $\Omega^{\mathbf{A}}(\alpha^{-1}(T))$ is the largest congruence system that is compatible with $\alpha^{-1}(T)$, it suffices to show that $\alpha^{-1}(\Omega^{\mathbf{A}'}(T))$ is a congruence system on $\mathbf{A}$ that is compatible with $\alpha^{-1}(T)$. The fact that it is a congruence system on $\mathbf{A}$ is guaranteed by Proposition 16. So it suffices to show its compatibility with $\alpha^{-1}(T)$. Let $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \alpha_\Sigma^{-1}(\Omega_{F(\Sigma)}^{\mathbf{A}'}(T))$ and $\phi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})$. Now we get $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \Omega_{F(\Sigma)}^{\mathbf{A}'}(T)$ and $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$. By compatibility of $\Omega^{\mathbf{A}'}(T)$ with T, we get $\alpha_\Sigma(\psi) \in T_{F(\Sigma)}$. Therefore $\psi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})$, which proves compatibility of $\alpha^{-1}(\Omega^{\mathbf{A}'}(T))$ with $\alpha^{-1}(T)$.

(b) By Part (a), it suffices to prove, under the hypothesis that $\langle F, \alpha \rangle : \mathbf{A} \to \mathbf{A}'$ is surjective, the inclusion $\Omega^{\mathbf{A}}(\alpha^{-1}(T)) \leq \alpha^{-1}(\Omega^{\mathbf{A}'}(T))$. Let $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \Omega_\Sigma^{\mathbf{A}}(\alpha^{-1}(T))$. Then, by Theorem 19, we get that, for all $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

$$\begin{array}{l}\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi}) \in \alpha_{\Sigma'}^{-1}(T_{F(\Sigma')}) \\ \qquad\text{iff}\quad \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}) \in \alpha_{\Sigma'}^{-1}(T_{F(\Sigma')}).\end{array}$$

Equivalently,

$$\begin{array}{l}\alpha_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi})) \in T_{F(\Sigma')} \\ \qquad\text{iff}\quad \alpha_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})) \in T_{F(\Sigma')}.\end{array}$$

Equivalently, by the morphism property,

$$\begin{array}{l}\sigma'_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)), \alpha_{\Sigma'}(\vec{\chi})) \in T_{F(\Sigma')} \\ \qquad\text{iff}\quad \sigma'_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi)), \alpha_{\Sigma'}(\vec{\chi})) \in T_{F(\Sigma')}.\end{array}$$

Equivalently, by the naturality of α,

$$\begin{array}{l}\sigma'_{F(\Sigma')}(\mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi)), \alpha_{\Sigma'}(\vec{\chi})) \in T_{F(\Sigma')} \\ \qquad\text{iff}\quad \sigma'_{F(\Sigma')}(\mathrm{SEN}'(F(f))(\alpha_\Sigma(\psi)), \alpha_{\Sigma'}(\vec{\chi})) \in T_{F(\Sigma')}.\end{array}$$

Equivalently, by Theorem 19 and the surjectivity of $\langle F, \alpha \rangle$, we get that

$$\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \Omega_{F(\Sigma)}^{\mathbf{A}'}(T),$$

which finishes the proof.

■

2.4 Relative Congruence Systems

We look at a variety of results related to congruence systems in this section. First, we give a condition that ensures that, given a morphism $\langle H,\gamma\rangle : \mathbf{A} \to \mathbf{A}'$ of $N^\flat$-algebraic systems, with an isomorphic functor component, and an equivalence family θ on $\mathbf{A}$, we have $\gamma^{-1}(\gamma(\theta)) = \theta$.

Lemma 25 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *be* $N^\flat$*-algebraic systems,* $\langle F,\alpha\rangle : \mathbf{A} \to \mathbf{A}'$ *a morphism, with* F *an isomorphism, and* $\theta \in \mathrm{EqvFam}(\mathbf{A})$*. Then*

$$\mathrm{Ker}(\langle F,\alpha\rangle) \leq \theta \quad \textit{iff} \quad \alpha^{-1}(\alpha(\theta)) = \theta.$$

Proof: Suppose, first, that $\mathrm{Ker}(\langle F,\alpha\rangle) \leq \theta$ and let $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle \in \alpha_\Sigma^{-1}(\alpha_\Sigma(\theta_\Sigma))$. Then, by definition, we get

$$\langle\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\rangle \in \alpha_\Sigma(\theta_\Sigma).$$

Thus, there exist $\phi',\psi' \in \mathrm{SEN}(\Sigma)$, such that

$$\langle\phi',\psi'\rangle \in \theta_\Sigma \quad \text{and} \quad \langle\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\rangle = \langle\alpha_\Sigma(\phi'),\alpha_\Sigma(\psi')\rangle.$$

Thus, we get

$$\langle\phi',\psi'\rangle \in \theta_\Sigma \quad \text{and} \quad \langle\phi,\phi'\rangle,\langle\psi,\psi'\rangle \in \mathrm{Ker}_\Sigma(\langle F,\alpha\rangle).$$

Since $\mathrm{Ker}(\langle F,\alpha\rangle) \leq \theta$ and θ is an equivalence family, we get that $\langle\phi,\psi\rangle \in \theta_\Sigma$. Thus, we conclude that $\alpha^{-1}(\alpha(\theta)) \leq \theta$. Since the reverse inclusion always holds, $\alpha^{-1}(\alpha(\theta)) = \theta$.

Assume, conversely, that $\alpha^{-1}(\alpha(\theta)) = \theta$ and let $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle \in \mathrm{Ker}_\Sigma(\langle F,\alpha\rangle)$. Then, by definition, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. Therefore, since θ is an equivalence family, we get

$$\langle\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\rangle = \langle\alpha_\Sigma(\phi),\alpha_\Sigma(\phi)\rangle \in \alpha_\Sigma(\theta).$$

Now we get $\langle\phi,\psi\rangle \in \alpha_\Sigma^{-1}(\alpha_\Sigma(\theta_\Sigma))$ and, by hypothesis, $\langle\phi,\psi\rangle \in \theta_\Sigma$. We conclude that $\mathrm{Ker}(\langle F,\alpha\rangle) \leq \theta$. ∎

Next we show that, given algebraic systems $\mathbf{A}$ and $\mathbf{A}'$, a surjective morphism $\langle F,\alpha\rangle : \mathbf{A} \to \mathbf{A}'$, with an isomorphic functor component, and a congruence system θ on $\mathbf{A}$, its image under $\langle F,\alpha\rangle$ is a congruence system on $\mathbf{A}'$, provided that θ contains the kernel system of $\langle F,\alpha\rangle$.

Lemma 26 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *be* $N^\flat$*-algebraic systems,* $\langle F,\alpha\rangle : \mathbf{A} \to \mathbf{A}'$ *a surjective morphism, with* F *an isomorphism, and* $\theta \in \mathrm{ConSys}(\mathbf{A})$*, such that* $\mathrm{Ker}(\langle F,\alpha\rangle) \leq \theta$*. Then* $\alpha(\theta) \in \mathrm{ConSys}(\mathbf{A}')$*.*

Proof: We first show that $\alpha(\theta)$ is an equivalence family on $\mathbf{A}'$. To this end, let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi, \psi', \chi \in \mathrm{SEN}(\Sigma)$.

- By hypothesis, $\theta \in \mathrm{ConSys}(\mathbf{A})$. Hence, $\langle \phi, \phi \rangle \in \theta_\Sigma$. Thus, $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\phi) \rangle \in \alpha_\Sigma(\theta_\Sigma)$. Since $\langle F, \alpha \rangle$ is surjective, $\alpha(\theta)$ is reflexive.

- Suppose $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \alpha_\Sigma(\theta_\Sigma)$. Then $\langle \phi, \psi \rangle \in \alpha_\Sigma^{-1}(\alpha_\Sigma(\theta_\Sigma))$. By Lemma 25, $\langle \phi, \psi \rangle \in \theta_\Sigma$. Since $\theta \in \mathrm{ConSys}(\mathbf{A})$, $\langle \psi, \phi \rangle \in \theta_\Sigma$. Hence, $\langle \alpha_\Sigma(\psi), \alpha_\Sigma(\phi) \rangle \in \alpha_\Sigma(\theta_\Sigma)$. Thus, by the surjectivity of $\langle F, \alpha \rangle$, $\alpha(\theta)$ is also symmetric.

- Finally, suppose that $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \alpha_\Sigma(\theta_\Sigma)$ and $\langle \alpha_\Sigma(\psi'), \alpha_\Sigma(\chi) \rangle \in \alpha_\Sigma(\theta_\Sigma)$, with $\alpha_\Sigma(\psi) = \alpha_\Sigma(\psi')$. Then, by Lemma 25, $\langle \phi, \psi \rangle \in \theta_\Sigma$ and $\langle \psi', \chi \rangle \in \theta_\Sigma$. Moreover, by hypothesis, $\langle \psi, \psi' \rangle \in \mathrm{Ker}_\Sigma(\langle F, \alpha \rangle) \subseteq \theta_\Sigma$. Since $\theta \in \mathrm{ConSys}(\mathbf{A})$, we get $\langle \phi, \chi \rangle \in \theta_\Sigma$ and, therefore, $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\chi) \rangle \in \alpha_\Sigma(\theta_\Sigma)$. Taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that $\alpha(\theta)$ is also transitive.

We showed that $\alpha(\theta) \in \mathrm{EqvFam}(\mathbf{A}')$.

Next, we show that $\alpha(\theta)$ is also a system. To this end, suppose $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \alpha_\Sigma(\theta_\Sigma)$. Then, by Lemma 25, $\langle \phi, \psi \rangle \in \theta_\Sigma$. Since $\theta \in \mathrm{ConSys}(\mathbf{A})$, we get $\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi) \rangle \in \theta_{\Sigma'}$. Thus,

$$\begin{aligned}&\langle \mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi)), \mathrm{SEN}'(F(f))(\alpha_\Sigma(\psi)) \rangle \\ &\quad = \langle \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)), \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi)) \rangle \in \alpha_{\Sigma'}(\theta_{\Sigma'}).\end{aligned}$$

Since $\langle F, \alpha \rangle$ is surjective, we get that $\alpha(\theta)$ is invariant under $\mathbf{Sign}'$-morphisms. Now we have that $\alpha(\theta) \in \mathrm{EqvSys}(\mathbf{A}')$.

Finally, it remains to see that it is also a congruence system. To this end, let $\sigma^\flat$ be a natural transformation in $N^\flat$, $\Sigma \in |\mathbf{Sign}|$, $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)$, such that $\langle \alpha_\Sigma(\phi_i), \alpha_\Sigma(\psi_i) \rangle \in \alpha_\Sigma(\theta_\Sigma)$, for all $i < k$. We get, by Lemma 25, $\langle \phi_i, \psi_i \rangle \in \theta_\Sigma$, whence, since $\theta \in \mathrm{ConSys}(\mathbf{A})$, $\langle \sigma^{\mathbf{A}}_\Sigma(\vec{\phi}), \sigma^{\mathbf{A}}_\Sigma(\vec{\psi}) \rangle \in \theta_\Sigma$. Now, applying the morphism property, we get

$$\langle \sigma^{\mathbf{A}'}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})), \sigma^{\mathbf{A}'}_{F(\Sigma)}(\alpha_\Sigma(\vec{\psi})) \rangle = \langle \alpha_\Sigma(\sigma^{\mathbf{A}}_\Sigma(\vec{\phi})), \alpha_\Sigma(\sigma^{\mathbf{A}}_\Sigma(\vec{\psi})) \rangle \in \alpha_\Sigma(\theta_\Sigma).$$

Again, taking into account the surjectivity of $\langle F, \alpha \rangle$, we get that $\alpha(\theta)$ has the congruence property. We conclude that $\alpha(\theta) \in \mathrm{ConSys}(\mathbf{A}')$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system, K be a class of $\mathbf{F}$-algebraic systems and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system. A congruence system θ on $\mathbf{A}$ is called a K-**congruence system**, or a **congruence system relative to** K, if the quotient algebraic system $\mathcal{A}/\theta$ is a member of the class K, i.e., $\mathcal{A}/\theta = \mathcal{A}^\theta \in \mathsf{K}$. Given an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, we denote by $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ the collection of all K-congruence systems on $\mathcal{A}$:

$$\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A}) = \{\theta \in \mathrm{ConSys}(\mathcal{A}) : \mathcal{A}/\theta \in \mathsf{K}\}.$$

Let K be a class of $\mathbf{F}$-algebraic systems. We write $\mathbb{H}(\mathsf{K})$ for the class of all $\mathbf{F}$-algebraic systems $\mathcal{B}$, such that, there exists $\mathcal{A} \in \mathsf{K}$ and a (surjective) $\mathbf{F}$-algebraic system morphism $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$:

$$\mathbb{H}(\mathsf{K}) = \{\mathcal{B} \in \mathrm{AlgSys}(\mathbf{F}) : (\exists \mathcal{A} \in \mathsf{K})(\exists \langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B})\}.$$

We show that, if K is a class that is closed under the operator $\mathbb{H}$, then the K-congruence systems on any $\mathbf{F}$-algebraic system in K coincide with the ordinary congruence systems on $\mathcal{A}$.

Proposition 27 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems, such that* $\mathbb{H}(\mathsf{K}) \subseteq \mathsf{K}$*. Then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} \in \mathsf{K}$, $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A}) = \mathrm{ConSys}(\mathcal{A})$.

Proof: Clearly, $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A}) \subseteq \mathrm{ConSys}(\mathcal{A})$. Suppose $\theta \in \mathrm{ConSys}(\mathcal{A})$. Consider the quotient morphism

$$\langle I, \pi^\theta\rangle : \mathcal{A} \to \mathcal{A}/\theta.$$

Since $\mathcal{A} \in \mathsf{K}$, $\mathcal{A}/\theta \in \mathbb{H}(\mathsf{K}) \subseteq \mathsf{K}$. Thus, by definition, $\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ and $\mathcal{A}^i = \langle \mathbf{A}^i, \langle F^i, \alpha^i\rangle\rangle$, $i \in I$, be $\mathbf{F}$-algebraic systems and, for all $i \in I$,

$$\langle H^i, \gamma^i\rangle : \mathcal{A} \to \mathcal{A}^i$$

a surjective morphism. We say that $\{\langle H^i, \gamma^i\rangle : \mathcal{A} \to \mathcal{A}^i : i \in I\}$ is a **subdirect intersection** if

$$\bigcap_{i\in I} \mathrm{Ker}(\langle H^i, \gamma^i\rangle) = \Delta^{\mathcal{A}}.$$

Given a class K of $\mathbf{F}$-algebraic systems, we write $\overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$ to denote the class of all $\mathbf{F}$-algebraic systems $\mathcal{A}$, for which there exists a subdirect intersection $\{\langle H^i, \gamma^i\rangle : \mathcal{A} \to \mathcal{A}^i : i \in I\}$, with $\mathcal{A}^i \in \mathsf{K}$, for all $i \in I$.

We show that if a class K is closed under subdirect intersections, then the collection of all K-congruence systems on any $\mathbf{F}$-algebraic system $\mathcal{A}$ is closed under intersections. If, in addition, K contains a trivial $\mathbf{F}$-algebraic system, then $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ becomes a closure family on $\mathcal{A}^2$, for every $\mathbf{F}$-algebraic system $\mathcal{A}$.

Proposition 28 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* K *be a class of* $\mathbf{F}$*-algebraic systems, such that* $\overset{\triangleleft}{\mathbb{P}}(\mathsf{K}) \subseteq \mathsf{K}$.

(a) For every $\mathbf{F}$*-algebraic system* $\mathcal{A}$, $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ *is closed under signature-wise intersections;*

(b) If, in addition, K *contains a trivial* $\mathbf{F}$*-algebraic system, then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$, $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ *is a closure family on* $\mathcal{A}^2$.

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $\{\theta^i : i \in I\} \subseteq \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$. Then $\mathcal{A}/\theta^i \in \mathsf{K}$, for all $i \in I$. Consider the canonical morphisms

$$\langle I, \pi^i\rangle : \mathcal{A}/\bigcap_{i\in I}\theta^i \to \mathcal{A}/\theta^i, \quad i \in I.$$

Clearly, we have

$$\bigcap_{i\in I}\mathrm{Ker}(\langle I,\pi^i\rangle) = \bigcap_{i\in I}(\theta^i/\bigcap_{i\in I}\theta^i) = \bigcap_{i\in I}\theta^i/\bigcap_{i\in I}\theta^i = \Delta^{\mathcal{A}/\bigcap_{i\in I}\theta^i}.$$

Thus, $\{\langle I,\pi^i\rangle : \mathcal{A}/\bigcap_{i\in I}\theta^i \to \mathcal{A}/\theta^i : i \in I\}$ is a subdirect intersection. Since $\mathcal{A}/\theta^i \in \mathsf{K}$, for all $i \in I$, we get $\mathcal{A}/\bigcap_{i\in I}\theta^i \in \overset{\triangleleft}{\mathbb{I}\mathrm{I}}(\mathsf{K}) \subseteq \mathsf{K}$. Therefore, $\bigcap_{i\in I}\theta^i \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$.

Suppose, in addition, that K contains a trivial $\mathbf{F}$-algebraic system. Then $\nabla^{\mathcal{A}} \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$, whence $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ is a closure family on $\mathcal{A}^2$. ■

By Proposition 28, for a class K of $\mathbf{F}$-algebraic systems closed under $\overset{\triangleleft}{\mathbb{I}\mathrm{I}}$ and containing a trivial $\mathbf{F}$-algebraic system, it makes sense to define, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $X \in \mathrm{SenFam}(\mathcal{A}^2)$,

$$\Theta^{\mathsf{K},\mathcal{A}}(X) = \bigcap\{\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A}) : X \leq \theta\}.$$

When $\mathcal{A}$ coincides with the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$, where $\langle I,\iota\rangle : \mathbf{F} \to \mathbf{F}$ is the identity morphism, we write simply Θ^{K}.

We now provide a different characterization of the operator $\Theta^{\mathsf{K},\mathcal{A}}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system and K a class of $\mathbf{F}$-algebraic systems. Define the operator $D^{\mathsf{K}} : \mathcal{P}(\mathrm{SEN}^\flat)^2 \to \mathcal{P}(\mathrm{SEN}^\flat)^2$, by letting, for all $X \leq (\mathrm{SEN}^\flat)^2$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\langle \phi,\psi\rangle \in \mathrm{SEN}^\flat(\Sigma)^2$,

$$\langle \phi,\psi\rangle \in D^{\mathsf{K}}_\Sigma(X) \quad \text{iff} \quad \begin{array}{l}\text{for all } \mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle \in \mathsf{K},\\ \alpha(X) \leq \Delta^{\mathcal{A}} \text{ implies } \alpha_\Sigma(\phi) = \alpha_\Sigma(\psi).\end{array}$$

We show that D^{K} is a closure family on $(\mathrm{SEN}^\flat)^2$.

Proposition 29 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems.* D^{K} *is a closure family on* $(\mathrm{SEN}^\flat)^2$.

Proof: We must show that D^{K} is inflationary, monotone and idempotent.

Let $X \leq (\mathrm{SEN}^\flat)^2$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\langle \phi,\psi\rangle \in X_\Sigma$. Then, for all $\mathcal{A} \in \mathsf{K}$, if $\alpha(X) \leq \Delta^{\mathcal{A}}$, we clearly have $\alpha(\phi) = \alpha(\psi)$. Hence, $\langle \phi,\psi\rangle \in D^{\mathsf{K}}_\Sigma(X)$ and D^{K} is inflationary.

Suppose $X \leq Y \leq (\mathrm{SEN}^\flat)^2$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\langle \phi,\psi\rangle \in \mathrm{SEN}^\flat(\Sigma)^2$, such that $\langle \phi,\psi\rangle \in D^{\mathsf{K}}_\Sigma(X)$. Let $\mathcal{A} \in \mathsf{K}$, such that $\alpha(Y) \leq \Delta^{\mathcal{A}}$. Then, we get $\alpha(X) \leq \alpha(Y) \leq \Delta^{\mathcal{A}}$, whence, by hypothesis, $\alpha(\phi) = \alpha(\psi)$. Therefore, $\langle \phi,\psi\rangle \in D^{\mathsf{K}}_\Sigma(Y)$ and D^{K} is also monotone.

Finally, suppose $X \leq (\mathrm{SEN}^\flat)^2$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\langle\phi,\psi\rangle \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle\phi,\psi\rangle \in D^{\mathsf{K}}_\Sigma(D^{\mathsf{K}}(X))$. Let $\mathcal{A} \in \mathsf{K}$, such that $\alpha(X) \leq \Delta^{\mathcal{A}}$. Then, by definition, $\alpha(D^{\mathsf{K}}(X)) \leq \Delta^{\mathcal{A}}$, whence, by hypothesis, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. Thus, $\langle\phi,\psi\rangle \in D^{\mathsf{K}}_\Sigma(X)$ and D^{K} is also idempotent.

We conclude that D^{K} is a closure family on $(\mathrm{SEN}^\flat)^2$. ■

We show, next, that, for all $X \leq (\mathrm{SEN}^\flat)^2$, the sentence family $D^{\mathsf{K}}(X)$ is a congruence system on the algebraic system $\mathbf{F}$ and that, moreover, it is a congruence system relative to the class K.

Proposition 30 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system and K a class of $\mathbf{F}$-algebraic systems. For all $X \leq (\mathrm{SEN}^\flat)^2$, $D^{\mathsf{K}}(X) \in \mathrm{ConSys}(\mathbf{F})$.*

Proof: We first show that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $D^{\mathsf{K}}_\Sigma(X)$ is an equivalence family.

- Let $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Since, for all $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle \in \mathsf{K}$, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\phi)$, we get that $\langle\phi,\phi\rangle \in D^{\mathsf{K}}_\Sigma(X)$, whence $D^{\mathsf{K}}_\Sigma(X)$ is reflexive.

- Suppose $\langle\phi,\psi\rangle \in D^{\mathsf{K}}_\Sigma(X)$ and let $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle \in \mathsf{K}$, such that $\alpha(X) \leq \Delta^{\mathcal{A}}$. Then, by hypothesis, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$, giving $\alpha_\Sigma(\psi) = \alpha_\Sigma(\phi)$. Hence, $\langle\psi,\phi\rangle \in D^{\mathsf{K}}_\Sigma(X)$, showing that $D^{\mathsf{K}}_\Sigma(X)$ is also symmetric.

- Finally, suppose $\langle\phi,\psi\rangle$, $\langle\psi,\chi\rangle \in D^{\mathsf{K}}_\Sigma(X)$. Let $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle \in \mathsf{K}$, such that $\alpha(X) \leq \Delta^{\mathcal{A}}$. By hypothesis, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$ and $\alpha_\Sigma(\psi) = \alpha_\Sigma(\chi)$. Therefore, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\chi)$, showing that $\langle\phi,\chi\rangle \in D^{\mathsf{K}}_\Sigma(X)$. Hence, $D^{\mathsf{K}}_\Sigma(X)$ is also transitive.

We show, next, that $D^{\mathsf{K}}(X)$ is an equivalence system, i.e., invariant under signature morphisms. Let $\Sigma,\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle\phi,\psi\rangle \in D^{\mathsf{K}}_\Sigma(X)$. Let $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle \in \mathsf{K}$, such that $\alpha(X) \leq \Delta^{\mathcal{A}}$. Then, by hypothesis, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. Thus, we get

$$\begin{array}{rcl}\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) &=& \mathrm{SEN}(F(f))(\alpha_\Sigma(\phi))\\ &=& \mathrm{SEN}(F(f))(\alpha_\Sigma(\psi))\\ &=& \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi)).\end{array}$$

Hence, $\langle\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi)\rangle \in D^{\mathsf{K}}_{\Sigma'}(X)$.

Finally, to see that it also satisfies the congruence property, let $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ be in $N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi},\vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle\phi_i,\psi_i\rangle \in D^{\mathsf{K}}_\Sigma(X)$, for all $i < k$. Let $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle \in \mathsf{K}$, such that $\alpha(X) \leq \Delta^{\mathcal{A}}$. then, by hypothesis, $\alpha_\Sigma(\phi_i) = \alpha_\Sigma(\psi_i)$, for all $i < k$. Therefore,

$$\begin{array}{rcl}\alpha_\Sigma(\sigma^\flat_\Sigma(\vec{\phi})) &=& \sigma^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi}))\\ &=& \sigma^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\psi}))\\ &=& \alpha_\Sigma(\sigma^\flat_\Sigma(\vec{\psi})).\end{array}$$

We conclude that $\langle\sigma^\flat_\Sigma(\vec{\phi}),\sigma^\flat_\Sigma(\vec{\psi})\rangle \in D^{\mathsf{K}}_\Sigma(X)$ and, therefore, $D^{\mathsf{K}}(X)$ is indeed a congruence system on $\mathbf{F}$. ■

Furthermore, if K happens to contain a trivial $\mathbf{F}$-algebraic system and be closed under subdirect intersections, we can show that $D^{\mathsf{K}}(X)$ is a K-congruence system on $\mathbf{F}$.

Proposition 31 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems, containing a trivial* $\mathbf{F}$*-algebraic system and closed under* $\overset{\triangleleft}{\amalg}$. *For all* $X \leq (\mathrm{SEN}^\flat)^2$, $D^{\mathsf{K}}(X) \in \mathrm{ConSys}^{\mathsf{K}}(\mathbf{F})$.

Proof: By Proposition 30, we know that $D^{\mathsf{K}}(X)$ is a congruence system on $\mathbf{F}$. Therefore, it suffices to show that it is a congruence system relative to K. For this, let $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle \in \mathsf{K}$, such that $X \leq \mathrm{Ker}(\langle F,\alpha\rangle)$. Define the morphism

$$\langle F,\alpha^{\mathsf{K}}\rangle : \mathcal{F}/D^{\mathsf{K}}(X) \to \mathcal{A}$$

by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)/D^{\mathsf{K}}_\Sigma(X)$,

$$\alpha^{\mathsf{K}}_\Sigma(\phi/D^{\mathsf{K}}_\Sigma(X)) = \alpha_\Sigma(\phi).$$

This morphism is well defined, since, if $\mathcal{A} \in \mathsf{K}$, with $X \leq \mathrm{Ker}(\langle F,\alpha\rangle)$, then, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle\phi,\psi\rangle \in D^{\mathsf{K}}_\Sigma(X) \quad \text{implies} \quad \alpha_\Sigma(\phi) = \alpha_\Sigma(\psi).$$

Now consider the collection

$$\langle F,\alpha^{\mathsf{K}}\rangle : \mathcal{F}/D^{\mathsf{K}}(X) \to \mathcal{A},\ \mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle \in \mathsf{K},\ X \leq \mathrm{Ker}(\langle F,\alpha\rangle).$$

We have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{aligned}
&\langle\phi/D^{\mathsf{K}}_\Sigma(X),\psi/D^{\mathsf{K}}_\Sigma(X)\rangle \in \textstyle\bigcap_{\langle F,\alpha^{\mathsf{K}}\rangle}\mathrm{Ker}_\Sigma(\langle F,\alpha^{\mathsf{K}}\rangle)\\
&\quad\text{iff}\quad \alpha^{\mathsf{K}}_\Sigma(\phi/D^{\mathsf{K}}_\Sigma(X)) = \alpha^{\mathsf{K}}_\Sigma(\psi/D^{\mathsf{K}}_\Sigma(X)),\ \text{for all } \langle F,\alpha^{\mathsf{K}}\rangle\\
&\quad\text{iff}\quad \alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)\ \text{for all } \langle F,\alpha^{\mathsf{K}}\rangle\\
&\quad\text{iff}\quad \langle\phi,\psi\rangle \in D^{\mathsf{K}}_\Sigma(X).
\end{aligned}$$

Therefore, the displayed collection above constitutes a subdirect intersection and, since $\mathcal{A} \in \mathsf{K}$, for all $\langle F,\alpha^{\mathsf{K}}\rangle$, and K is closed under subdirect intersections, we get that $\mathcal{F}/D^{\mathsf{K}}(X) \in \mathsf{K}$, and, therefore, $D^{\mathsf{K}}(X) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$. ■

We are now in a position to show the promised alternative characterization of the operator Θ^{K}. It turns out that it coincides with D^{K}.

Theorem 32 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems, containing a trivial* $\mathbf{F}$*-algebraic system and closed under subdirect intersections. For all* $X \leq (\mathrm{SEN}^\flat)^2$, $\Theta^{\mathsf{K}}(X) = D^{\mathsf{K}}(X)$.

Proof: Let $X \leq (\mathrm{SEN}^\flat)^2$. By Proposition 31, $D^{\mathsf{K}}(X) \in \mathrm{ConSys}^{\mathsf{K}}(X)$ and, by Proposition 29, $X \leq D^{\mathsf{K}}(X)$. Therefore, by the minimality of $\Theta^{\mathsf{K}}(X)$, we get that $\Theta^{\mathsf{K}}(X) \leq D^{\mathsf{K}}(X)$. To show the reverse inclusion, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in D^{\mathsf{K}}_\Sigma(X)$. Consider $\mathcal{F}/\Theta^{\mathsf{K}}(X) \in \mathsf{K}$. Since $\pi^{\Theta^{\mathsf{K}}(X)}(X) \leq \Delta^{\mathcal{F}/\Theta^{\mathsf{K}}(X)}$, we get, by hypothesis, $\pi^{\Theta^{\mathsf{K}}(X)}_\Sigma(\phi) = \pi^{\Theta^{\mathsf{K}}(X)}_\Sigma(\psi)$, i.e., $\langle \phi, \psi \rangle \in \Theta^{\mathsf{K}}_\Sigma(X)$. We conclude that $D^{\mathsf{K}}(X) \leq \Theta^{\mathsf{K}}(X)$. ∎

We look, next, at how the operator Θ^{K} interacts with morphisms.

Proposition 33 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and K be a class of $\mathbf{F}$-algebraic systems, containing a trivial $\mathbf{F}$-algebraic system and such that $\overset{\triangleleft}{\mathbb{IP}}(\mathsf{K}) \subseteq \mathsf{K}$. Let also $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ be $\mathbf{F}$-algebraic systems and $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism.*

(a) If $\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{B})$, then $\gamma^{-1}(\theta) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$;

(b) If H is an isomorphism, $\mathrm{Ker}(\langle H, \gamma \rangle) \leq \theta$ and $\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$, then $\gamma(\theta) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{B})$.

Proof:

(a) By Proposition 16, $\gamma^{-1}(\theta) \in \mathrm{ConSys}(\mathcal{A})$. Consider the morphism

$$\langle H, \gamma^* \rangle : \mathcal{A}/\gamma^{-1}(\theta) \to \mathcal{B}/\theta,$$

defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, by

$$\gamma^*_\Sigma(\phi/\gamma^{-1}_\Sigma(\theta_{H(\Sigma)})) = \gamma_\Sigma(\phi)/\theta_{H(\Sigma)}.$$

This is well-defined, since, if $\langle \phi, \psi \rangle \in \gamma^{-1}_\Sigma(\theta_{H(\Sigma)})$, then $\langle \gamma_\Sigma(\phi), \gamma_\Sigma(\psi) \rangle \in \theta_{H(\Sigma)}$. Moreover,

$$\mathrm{Ker}(\langle H, \gamma^* \rangle) = {\gamma^*}^{-1}(\Delta^{\mathcal{B}/\theta}) = \Delta^{\mathcal{A}/\gamma^{-1}(\theta)}.$$

Thus, $\{\langle H, \gamma^* \rangle : \mathcal{A}/\gamma^{-1}(\theta) \to \mathcal{B}/\theta\}$ is a subdirect intersection and, since, by hypothesis, $\mathcal{B}/\theta \in \mathsf{K}$, $\mathcal{A}/\gamma^{-1}(\theta) \in \overset{\triangleleft}{\mathbb{IP}}(\mathsf{K}) \subseteq \mathsf{K}$. Therefore, $\gamma^{-1}(\theta) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$.

(b) By Lemma 26, $\gamma(\theta) \in \mathrm{ConSys}(\mathcal{B})$. Moreover, it is not difficult to see that

$$\langle H, \gamma^* \rangle : \mathcal{A}/\theta \to \mathcal{B}/\gamma(\theta),$$

defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, by

$$\gamma^*_\Sigma(\phi/\theta_\Sigma) = \gamma_\Sigma(\phi)/\gamma_\Sigma(\theta_\Sigma)$$

is an isomorphism of $\mathbf{F}$-algebraic systems, since, by Lemma 25, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi, \psi \rangle \in \theta_\Sigma \quad \text{iff} \quad \langle \gamma_\Sigma(\phi), \gamma_\Sigma(\psi) \rangle \in \gamma_\Sigma(\theta_\Sigma).$$

Therefore, $\{\langle H,\gamma\rangle^{-1} : \mathcal{B}/\gamma(\theta) \to \mathcal{A}/\theta\}$ is a subdirect intersection. Since $\mathcal{A}/\theta \in \mathsf{K}$, it follows that $\mathcal{B}/\gamma(\theta) \in \overset{\triangleleft}{\amalg}(\mathsf{K}) \subseteq \mathsf{K}$. Therefore, $\gamma(\theta) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{B})$.

■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system, $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ be a collection of natural transformations in $N^\flat$ and $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ an $\mathbf{F}$-algebraic system. If $\tau^\flat$ is perceived as having a single distinguished argument, with the remaining arguments as parameters, we define, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, the sentence family

$$\tau^{\mathcal{A}}_\Sigma[\phi] = \{\tau^{\mathcal{A}}_{\Sigma,\Sigma'}[\phi]\}_{\Sigma'\in|\mathbf{Sign}|},$$

by setting, for all $\Sigma' \in |\mathbf{Sign}|$,

$$\tau^{\mathcal{A}}_{\Sigma,\Sigma'}[\phi] = \bigcup\{\tau^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi}) : f \in \mathbf{Sign}(\Sigma,\Sigma'), \vec{\chi} \in \mathrm{SEN}(\Sigma')\}.$$

Given $\Phi \subseteq \mathrm{SEN}(\Sigma)$, we set

$$\tau^{\mathcal{A}}_\Sigma[\Phi] = \bigcup\{\tau^{\mathcal{A}}_\Sigma[\phi] : \phi \in \Phi\}$$

and, given a sentence family $X \in \mathrm{SenFam}(\mathcal{A})$, we set

$$\tau^{\mathcal{A}}[X] = \bigcup\{\tau^{\mathcal{A}}_\Sigma[X_\Sigma] : \Sigma \in |\mathbf{Sign}|\}.$$

We will revisit these and similar definitions in more depth in Section 2.13. For now, we only use them to establish a result that involves the relative congruence system operator Θ^{K}, introduced in this section, and direct images under morphisms of $\mathbf{F}$-algebraic systems with isomorphic functor components.

Proposition 34 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ and K be a class of $\mathbf{F}$-algebraic systems, containing a trivial $\mathbf{F}$-algebraic system and such that $\overset{\triangleleft}{\amalg}(\mathsf{K}) \subseteq \mathsf{K}$. Let also $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G,\beta\rangle\rangle$ be $\mathbf{F}$-algebraic systems and $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism, with H an isomorphism. Then, for all $X \in \mathrm{SenFam}(\mathcal{A})$,*

$$\Theta^{\mathsf{K},\mathcal{B}}(\gamma(\Theta^{\mathsf{K},\mathcal{A}}(\tau^{\mathcal{A}}[X]))) = \Theta^{\mathsf{K},\mathcal{B}}(\tau^{\mathcal{B}}[\gamma(X)]).$$

Proof: Taking into account the surjectivity of $\langle H,\gamma\rangle$, we have $\tau^{\mathcal{B}}[\gamma(X)] = \gamma(\tau^{\mathcal{A}}[X]) \leq \gamma(\Theta^{\mathsf{K},\mathcal{A}}(\tau^{\mathcal{A}}[X]))$. Hence

$$\Theta^{\mathsf{K},\mathcal{B}}(\tau^{\mathcal{B}}[\gamma(X)]) \leq \Theta^{\mathsf{K},\mathcal{B}}(\gamma(\Theta^{\mathsf{K},\mathcal{A}}(\tau^{\mathcal{A}}[X]))).$$

On the other hand, $\gamma^{-1}(\Theta^{\mathsf{K},\mathcal{B}}(\tau^{\mathcal{B}}[\gamma(X)]))$ is, by Proposition 33, a K-congruence system on $\mathcal{A}$, and, moreover, it contains $\tau^{\mathcal{A}}[X]$, since

$$\gamma(\tau^{\mathcal{A}}[X]) = \tau^{\mathcal{B}}[\gamma(X)] \leq \Theta^{\mathsf{K},\mathcal{B}}(\tau^{\mathcal{B}}[\gamma(X)]).$$

Hence, $\Theta^{\mathsf{K},\mathcal{A}}(\tau^{\mathcal{A}}[X]) \leq \gamma^{-1}(\Theta^{\mathsf{K},\mathcal{B}}(\tau^{\mathcal{B}}[\gamma(X)]))$, i.e.,

$$\gamma(\Theta^{\mathsf{K},\mathcal{A}}(\tau^{\mathcal{A}}[X])) \leq \Theta^{\mathsf{K},\mathcal{B}}(\tau^{\mathcal{B}}[\gamma(X)]).$$

This yields $\Theta^{\mathsf{K},\mathcal{B}}(\gamma(\Theta^{\mathsf{K},\mathcal{A}}(\tau^{\mathcal{A}}[X]))) \leq \Theta^{\mathsf{K},\mathcal{B}}(\tau^{\mathcal{B}}[\gamma(X)])$. ■

We conclude the section by showing that the relative congruence system generated by a family of pairs may be expressed as the join in the complete lattice of relative congruence systems of those relative congruence systems generated by the single pairs of elements in the generating family.

Proposition 35 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *be a class of* $\mathbf{F}$*-algebraic systems, containing a trivial* $\mathbf{F}$*-algebraic system and such that* $\overset{\triangleleft}{\mathbb{II}}(\mathsf{K}) \subseteq \mathsf{K}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $X \in \mathrm{SenFam}(\mathcal{A}^2)$*,*

$$\Theta^{\mathsf{K},\mathcal{A}}(X) = \bigvee \{\Theta^{\mathsf{K},\mathcal{A}}(\phi, \psi) : \langle \phi, \psi \rangle \in X_\Sigma, \Sigma \in |\mathbf{Sign}|\}.$$

Proof: Set

$$\theta := \bigvee \{\Theta^{\mathsf{K},\mathcal{A}}(\phi, \psi) : \langle \phi, \psi \rangle \in X_\Sigma, \Sigma \in |\mathbf{Sign}|\}.$$

For all $\Sigma \in |\mathbf{Sign}|$ and all $\langle \phi, \psi \rangle \in X_\Sigma$, we have $\langle \phi, \psi \rangle \in \Theta^{\mathsf{K},\mathcal{A}}_\Sigma(X)$. So $\Theta^{\mathsf{K},\mathcal{A}}(\phi, \psi) \leq \Theta^{\mathsf{K},\mathcal{A}}(X)$ and, therefore, $\theta \leq \Theta^{\mathsf{K},\mathcal{A}}(X)$. Conversely, for all $\Sigma \in |\mathbf{Sign}|$ and all $\langle \phi, \psi \rangle \in X_\Sigma$, we have $\langle \phi, \psi \rangle \in \Theta^{\mathsf{K},\mathcal{A}}_\Sigma(\phi, \psi) \subseteq \theta_\Sigma$. Hence, $X \leq \theta$, which implies that $\Theta^{\mathsf{K},\mathcal{A}}(X) \leq \theta$. ■

2.5 Varieties of F-Algebraic Systems

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system.

An **natural F-equation** (sometimes, referred to, simply, as **natural equation**, **F-equation** or just **equation**, if the meaning is made clear from context) is a pair $\langle \sigma^\flat, \tau^\flat \rangle$, where $\sigma^\flat, \tau^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ are natural transformations in $N^\flat$. The $\mathbf{F}$-equation $\langle \sigma^\flat, \tau^\flat \rangle$ will be denoted also by $\sigma^\flat \approx \tau^\flat$. Sometimes notation such as $\tau^\flat := \tau^{0\flat} \approx \tau^{1\flat}$ may also become handy. We denote by $\mathrm{NEq}(\mathbf{F})$ the collection of all natural $\mathbf{F}$-equations.

Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, be an $\mathbf{F}$-algebraic system. Then, given $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$, we write $\mathcal{A} \vDash_\Sigma \sigma^\flat \approx \tau^\flat[\vec{\phi}]$ and say that $\vec{\phi}$ **Σ-satisfies** $\sigma^\flat \approx \tau^\flat$ **in** $\mathcal{A}$ if

$$\alpha_\Sigma(\sigma^\flat_\Sigma(\vec{\phi})) = \alpha_\Sigma(\tau^\flat_\Sigma(\vec{\phi})).$$

The following is a useful lemma concerning satisfiability of an equation.

Lemma 36 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\sigma^\flat \approx \tau^\flat$ *a natural* $\mathbf{F}$*-equation and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system. The following statements are equivalent:*

(a) $\mathcal{A} \vDash_\Sigma \sigma^\flat \approx \tau^\flat[\vec{\phi}]$;

(b) $\sigma^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})) = \tau^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi}))$;

(c) *For all* $\Sigma' \in |\mathbf{Sign}^\flat|$, *all* $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}))) = \alpha_{\Sigma'}(\tau^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}))).$$

Proof:

(a)⇔(b) By the homomorphism property,

$$\alpha_\Sigma(\sigma^\flat_\Sigma(\vec{\phi})) = \sigma^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})) \quad \text{and} \quad \alpha_\Sigma(\tau^\flat_\Sigma(\vec{\phi})) = \tau^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})).$$

So we get

$$\begin{array}{lll} \mathcal{A} \vDash_\Sigma \sigma^\flat \approx \tau^\flat[\vec{\phi}] & \text{iff} & \alpha_\Sigma(\sigma^\flat_\Sigma(\vec{\phi})) = \alpha_\Sigma(\tau^\flat_\Sigma(\vec{\phi})) \\ & \text{iff} & \sigma^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})) = \tau^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})). \end{array}$$

(c)⇒(a) This implication is trivial by taking $\Sigma' = \Sigma$ and $f = i_\Sigma$.

(b)⇒(c) We have

$$\begin{array}{l} \sigma^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})) = \tau^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})) \\ \text{implies} \quad \alpha_\Sigma(\sigma^\flat_\Sigma(\vec{\phi})) = \alpha_\Sigma(\tau^\flat_\Sigma(\vec{\phi})) \\ \text{implies, for all } \Sigma' \in |\mathbf{Sign}^\flat| \text{ and all } f \in \mathbf{Sign}^\flat(\Sigma, \Sigma'), \\ \qquad \mathrm{SEN}(F(f))(\alpha_\Sigma(\sigma^\flat_\Sigma(\vec{\phi}))) = \mathrm{SEN}(F(f))(\alpha_\Sigma(\tau^\flat_\Sigma(\vec{\phi}))) \\ \text{implies, for all } \Sigma' \in |\mathbf{Sign}^\flat| \text{ and all } f \in \mathbf{Sign}^\flat(\Sigma, \Sigma'), \\ \qquad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\sigma^\flat_\Sigma(\vec{\phi}))) = \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\tau^\flat_\Sigma(\vec{\phi}))) \\ \text{implies, for all } \Sigma' \in |\mathbf{Sign}^\flat| \text{ and all } f \in \mathbf{Sign}^\flat(\Sigma, \Sigma'), \\ \qquad \alpha_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}))) = \alpha_{\Sigma'}(\tau^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}))). \end{array}$$

■

Given a natural **F**-equation $\sigma^\flat \approx \tau^\flat$ and an **F**-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ we write

$$\mathcal{A} \vDash \sigma^\flat \approx \tau^\flat$$

and say that $\mathcal{A}$ **satisfies** $\sigma^\flat \approx \tau^\flat$ or that $\sigma^\flat \approx \tau^\flat$ is **satisfied in** $\mathcal{A}$ or is **valid in** $\mathcal{A}$, if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$, $\mathcal{A} \vDash_\Sigma \sigma^\flat \approx \tau^\flat[\vec{\phi}]$.

Let K be a class of **F**-algebraic systems and $E^\flat$ a set of natural **F**-equations. We write $\mathsf{K} \vDash E^\flat$ for

$$\mathcal{A} \vDash \sigma^\flat \approx \tau^\flat, \text{ for all } \mathcal{A} \in \mathsf{K} \text{ and all } \sigma^\flat \approx \tau^\flat \in E^\flat.$$

Given an **F**-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, we define the **kernel** $\mathrm{Ker}(\mathcal{A})$ **of** $\mathcal{A}$ to be the kernel of the morphism $\langle F, \alpha \rangle : \mathbf{F} \to \mathbf{A}$, i.e., we let

$$\mathrm{Ker}(\mathcal{A}) := \mathrm{Ker}(\langle F, \alpha \rangle).$$

Moreover, given a class K of $\mathbf{F}$-algebraic systems, we let

$$\mathrm{Ker}(\mathsf{K}) = \bigcap_{\mathcal{A}\in\mathsf{K}} \mathrm{Ker}(\mathcal{A}).$$

Now we are in a position to define two kinds of classes of $\mathbf{F}$-algebraic systems generated by a given class K of $\mathbf{F}$-algebraic systems. In other words, we introduce two class operators on classes of $\mathbf{F}$-algebraic systems.

Definition 37 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system and* K *be a class of* $\mathbf{F}$*-algebraic systems.*

- *The* **semantic variety** $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$ **generated by** K *is defined by*

$$\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : \mathrm{Ker}(\mathsf{K}) \le \mathrm{Ker}(\mathcal{A})\};$$

- *The* **syntactic variety** $\mathbb{V}^{\mathsf{Syn}}(\mathsf{K})$ **generated by** K *is defined by*

$$\mathbb{V}^{\mathsf{Syn}}(\mathsf{K}) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : (\forall \sigma^\flat \approx \tau^\flat)(\mathsf{K} \vDash \sigma^\flat \approx \tau^\flat \;\Rightarrow\; \mathcal{A} \vDash \sigma^\flat \approx \tau^\flat)\}.$$

It is relatively easy to see that both $\mathbb{V}^{\mathsf{Sem}}$ and $\mathbb{V}^{\mathsf{Syn}}$ are closure operators on the class of $\mathbf{F}$-algebraic systems.

Proposition 38 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system. Then* $\mathbb{V}^{\mathsf{Sem}}$ *and* $\mathbb{V}^{\mathsf{Syn}}$ *are closure operators on* $\mathrm{AlgSys}(\mathbf{F})$*.*

Proof: We work, first, with $\mathbb{V}^{\mathsf{Sem}}$.

- If $\mathcal{A} \in \mathsf{K}$, then, by definition, we have $\mathrm{Ker}(\mathsf{K}) \le \mathrm{Ker}(\mathcal{A})$. Thus, $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. So $\mathsf{K} \subseteq \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$.

- Suppose $\mathsf{K} \subseteq \mathsf{L}$ and $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. Then we have

$$\mathrm{Ker}(\mathsf{L}) \le \mathrm{Ker}(\mathsf{K}) \le \mathrm{Ker}(\mathcal{A}).$$

  So $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{L})$. Hence, if $\mathsf{K} \subseteq \mathsf{L}$ then $\mathbb{V}^{\mathsf{Sem}}(\mathrm{K}) \subseteq \mathbb{V}^{\mathsf{Sem}}(\mathrm{L})$.

- Finally, suppose $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}))$. Then $\mathrm{Ker}(\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})) \le \mathrm{Ker}(\mathcal{A})$. But, note that, for all $\mathcal{B} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$, we have $\mathrm{Ker}(\mathsf{K}) \le \mathrm{Ker}(\mathcal{B})$, whence $\mathrm{Ker}(\mathsf{K}) \le \mathrm{Ker}(\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}))$. Combining the two inclusions, we get

$$\mathrm{Ker}(\mathsf{K}) \le \mathrm{Ker}(\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})) \le \mathrm{Ker}(\mathcal{A}).$$

  Thus, $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. We conclude that $\mathbb{V}^{\mathsf{Sem}}(\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})) \subseteq \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$.

We work, next, with $\mathbb{V}^{\mathsf{Syn}}$. Consider the two mappings

$$\begin{array}{c}\mathrm{NEq}:\mathcal{P}(\mathrm{AlgSys}(\mathbf{F}))\to\mathcal{P}(\mathrm{NEq}(\mathbf{F})),\\ \mathrm{NMod}:\mathcal{P}(\mathrm{NEq}(\mathbf{F}))\to\mathcal{P}(\mathrm{AlgSys}(\mathbf{F})),\end{array}$$

defined by

$$\begin{array}{rcl}\mathrm{NEq}(\mathsf{K}) & = & \{\sigma^\flat\approx\tau^\flat\in\mathrm{NEq}(\mathbf{F}):\mathsf{K}\vDash\sigma^\flat\approx\tau^\flat\},\quad \mathsf{K}\subseteq\mathrm{AlgSys}(\mathbf{F});\\ \mathrm{NMod}(E) & = & \{\mathcal{A}\in\mathrm{AlgSys}(\mathbf{F}):\mathcal{A}\vDash E\},\quad E\subseteq\mathrm{NEq}(\mathbf{F}).\end{array}$$

It is not difficult to see that NEq and NMod form a Galois connection. Thus, $\mathbb{V}^{\mathsf{Syn}}=\mathrm{NMod}\circ\mathrm{NEq}$ is a closure operator on $\mathrm{AlgSys}(\mathbf{F})$. ■

We prove that the semantic variety is always included in the syntactic variety generated by the same class of $\mathbf{F}$-algebraic systems.

Theorem 39 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be a base algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems. Then*

$$\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})\subseteq\mathbb{V}^{\mathsf{Syn}}(\mathsf{K}).$$

Proof: Suppose that $\mathcal{A}\in\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. Let $\sigma^\flat\approx\tau^\flat$ be a natural $\mathbf{F}$-equation, such that $\mathsf{K}\vDash\sigma^\flat\approx\tau^\flat$. We must show that $\mathcal{A}\vDash\sigma^\flat\approx\tau^\flat$. To this end, let $\Sigma\in|\mathbf{Sign}^\flat|$ and $\vec{\phi}\in\mathrm{SEN}^\flat(\Sigma)$. Since $\mathsf{K}\vDash\sigma^\flat\approx\tau^\flat$, we have, for all $\mathcal{K}=\langle\mathbf{K},\langle K,\kappa\rangle\rangle\in\mathsf{K}$,

$$\kappa_\Sigma(\sigma^\flat_\Sigma(\vec{\phi}))=\kappa_\Sigma(\tau^\flat_\Sigma(\vec{\phi})).$$

This means that $\langle\sigma^\flat_\Sigma(\vec{\phi}),\tau^\flat_\Sigma(\vec{\phi})\rangle\in\mathrm{Ker}_\Sigma(\mathcal{K})$. Since this holds for all $\mathcal{K}\in\mathsf{K}$, we conclude that $\langle\sigma^\flat_\Sigma(\vec{\phi}),\tau^\flat_\Sigma(\vec{\phi})\rangle\in\mathrm{Ker}_\Sigma(\mathsf{K})$. But, by hypothesis, $\mathrm{Ker}(\mathsf{K})\leq\mathrm{Ker}(\mathcal{A})$. Therefore, we get $\langle\sigma^\flat_\Sigma(\vec{\phi}),\tau^\flat_\Sigma(\vec{\phi})\rangle\in\mathrm{Ker}_\Sigma(\mathcal{A})$. This means that

$$\alpha_\Sigma(\sigma^\flat_\Sigma(\vec{\phi}))=\alpha_\Sigma(\tau^\flat_\Sigma(\vec{\phi})).$$

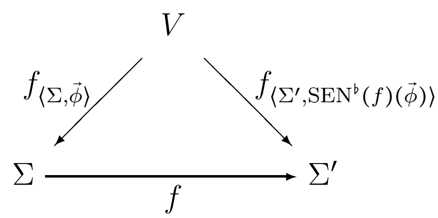
Since $\Sigma\in|\mathbf{Sign}^\flat|$ and $\vec{\phi}\in\mathrm{SEN}^\flat(\Sigma)$ were arbitrary, we get that $\mathcal{A}\vDash\sigma^\flat\approx\tau^\flat$. Now we conclude that $\mathcal{A}\in\mathbb{V}^{\mathsf{Syn}}(\mathsf{K})$. Thus, $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})\subseteq\mathbb{V}^{\mathsf{Syn}}(\mathsf{K})$. ■

Now we look at some sufficient conditions that ensure that these two variety operators generate the same class of $\mathbf{F}$-algebraic systems. However, the terminology, methodology and work presented in the rest of the section have proven very useful in many contexts and can be used to reconcile results that hold in more restricted contexts with partial analogs that hold in this very abstract setting.

Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be a base algebraic system and consider a cardinal κ (which will usually be taken to be either finite or ω). A **source signature κ-variable pair** (**ssv**$^\kappa$ for short) $\langle V,\vec{v}\rangle$ consists of a signature $V\in|\mathbf{Sign}^\flat|$ and a vector $\vec{v}\in\mathrm{SEN}^\flat(V)^\kappa$, satisfying the following conditions:

1. For all $\Sigma \in |\mathbf{Sign}^\flat|$, $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^\kappa$, there exists $f_{\langle\Sigma,\vec{\phi}\rangle} \in \mathbf{Sign}^\flat(V,\Sigma)$, such that
$$\mathrm{SEN}^\flat(f_{\langle\Sigma,\vec{\phi}\rangle})(\vec{v}) = \vec{\phi};$$

2. For all $\Sigma,\Sigma' \in |\mathbf{Sign}^\flat|$, $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^\kappa$ and $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$,



$$f \circ f_{\langle\Sigma,\vec{\phi}\rangle} = f_{\langle\Sigma',\mathrm{SEN}^\flat(f)(\vec{\phi})\rangle}.$$

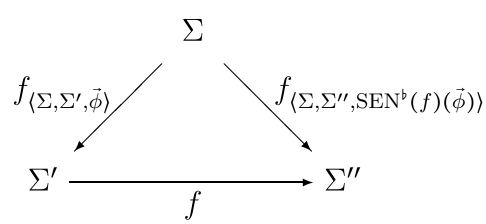
An algebraic system $\mathbf{F}$ is called κ-**term** if it has an ssv$^\kappa$. The morphisms $f_{\langle\Sigma,\vec{\phi}\rangle} : V \to \Sigma$ are referred to as the **ssv**$^\kappa$ **maps**.

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system. We say that $\mathbf{F}$ **has** κ**-variables** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, there exists $\vec{v}^\Sigma \in \mathrm{SEN}^\flat(\Sigma)^\kappa$, such that $\langle\Sigma,\vec{v}^\Sigma\rangle$ is an ssv$^\kappa$, with ssv$^\kappa$ maps $f_{\langle\Sigma,\Sigma',\vec{\phi}\rangle} : \Sigma \to \Sigma'$, for all $\Sigma,\Sigma' \in |\mathbf{Sign}^\flat|$ and $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma')^\kappa$. The algebraic system $\mathbf{F}$ is called κ-**formulaic** if it has κ-variables.

It follows, according to the preceding definitions, that $\mathbf{F}$ is κ-formulaic, with Σ-κ-variables $\vec{v}^\Sigma$ and ssv$^\kappa$ maps $f_{\langle\Sigma,\Sigma',\vec{\phi}\rangle}$ if:

- For all $\Sigma,\Sigma' \in |\mathbf{Sign}^\flat|$, $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma')^\kappa$,
$$f_{\langle\Sigma,\Sigma',\vec{\phi}\rangle}(\vec{v}^\Sigma) = \vec{\phi};$$

- For all $\Sigma,\Sigma',\Sigma'' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma',\Sigma'')$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma')^\kappa$,



$$f \circ f_{\langle\Sigma,\Sigma',\vec{\phi}\rangle} = f_{\langle\Sigma,\Sigma'',\mathrm{SEN}^\flat(f)(\vec{\phi})\rangle}.$$

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a κ-formulaic algebraic system, with κ-variables $\vec{v}^\Sigma$, $\Sigma \in |\mathbf{Sign}^\flat|$. $\mathbf{F}$ will be called κ-**transformational** (**modulo the given** κ**-variables**) if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, there exists $\sigma^{\langle\Sigma,\phi\rangle} : (\mathrm{SEN}^\flat)^\kappa \to \mathrm{SEN}^\flat$, such that:

- $\sigma^{\langle\Sigma,\phi\rangle}$ depends on only finitely may variables;

- $\phi = \sigma^{\langle\Sigma,\phi\rangle}_\Sigma(\vec{v}^\Sigma)$.

We have the following relation now that serves, so to speak, in bridging the gap between the semantical and syntactical definitions of varieties of algebraic systems.

Lemma 40 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a transformational algebraic system and* $\mathcal{A} = \langle\mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system. Then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle\phi,\psi\rangle \in \mathrm{Ker}_\Sigma(\mathcal{A}) \quad \textit{iff} \quad \mathcal{A} \vDash \sigma^{\langle\Sigma,\phi\rangle} \approx \sigma^{\langle\Sigma,\psi\rangle}.$$

Proof: Suppose, first, that $\mathcal{A} \vDash \sigma^{\langle\Sigma,\phi\rangle} \approx \sigma^{\langle\Sigma,\psi\rangle}$. This means that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\alpha_{\Sigma'}(\sigma^{\langle\Sigma,\phi\rangle}_{\Sigma'}(\vec{\phi})) = \alpha_{\Sigma'}(\sigma^{\langle\Sigma,\psi\rangle}_{\Sigma'}(\vec{\phi})).$$

Taking $\Sigma' = \Sigma$ and $\vec{\phi} = \vec{v}^\Sigma$, we get $\alpha_\Sigma(\sigma^{\langle\Sigma,\phi\rangle}_\Sigma(\vec{v}^\Sigma)) = \alpha_\Sigma(\sigma^{\langle\Sigma,\psi\rangle}_\Sigma(\vec{v}^\Sigma))$, or, what amounts to the same, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. Hence, $\langle\phi,\psi\rangle \in \mathrm{Ker}_\Sigma(\mathcal{A})$.

Suppose, conversely, that $\langle\phi,\psi\rangle \in \mathrm{Ker}_\Sigma(\mathcal{A})$. This means that $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. Since $\mathbf{F}$ is assumed to be transformational, there exist $\sigma^{\langle\Sigma,\phi\rangle}$ and $\sigma^{\langle\Sigma,\psi\rangle}$ in $N^\flat$, such that $\sigma^{\langle\Sigma,\phi\rangle}_\Sigma(\vec{v}^\Sigma) = \phi$ and $\sigma^{\langle\Sigma,\psi\rangle}_\Sigma(\vec{v}^\Sigma) = \psi$. Thus, we get

$$\alpha_\Sigma(\sigma^{\langle\Sigma,\phi\rangle}_\Sigma(\vec{v}^\Sigma)) = \alpha_\Sigma(\sigma^{\langle\Sigma,\psi\rangle}_\Sigma(\vec{v}^\Sigma)).$$

Now, by formulaicity, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma')$, we get an ssv$^\kappa$ map $f_{\langle\Sigma,\Sigma',\vec{\phi}\rangle} : \Sigma \to \Sigma'$, for which we have

$$\mathrm{SEN}(F(f_{\langle\Sigma,\Sigma',\vec{\phi}\rangle}))(\alpha_\Sigma(\sigma^{\langle\Sigma,\phi\rangle}_\Sigma(\vec{v}^\Sigma))) = \mathrm{SEN}(F(f_{\langle\Sigma,\Sigma',\vec{\phi}\rangle}))(\alpha_\Sigma(\sigma^{\langle\Sigma,\psi\rangle}_\Sigma(\vec{v}^\Sigma))).$$

Hence, since α is a natural transformation,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f_{\langle\Sigma,\Sigma',\vec{\phi}\rangle})(\sigma^{\langle\Sigma,\phi\rangle}_\Sigma(\vec{v}^\Sigma))) = \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f_{\langle\Sigma,\Sigma',\vec{\phi}\rangle})(\sigma^{\langle\Sigma,\psi\rangle}_\Sigma(\vec{v}^\Sigma))).$$

And since $\sigma^{\langle\Sigma,\phi\rangle}$, $\sigma^{\langle\Sigma,\psi\rangle}$ are also natural transformations, we get

$$\alpha_{\Sigma'}(\sigma^{\langle\Sigma,\phi\rangle}_{\Sigma'}(\mathrm{SEN}^\flat(f_{\langle\Sigma,\Sigma',\vec{\phi}\rangle})(\vec{v}^\Sigma))) = \alpha_{\Sigma'}(\sigma^{\langle\Sigma,\psi\rangle}_{\Sigma'}(\mathrm{SEN}^\flat(f_{\langle\Sigma,\Sigma',\vec{\phi}\rangle})(\vec{v}^\Sigma))).$$

Finally, by the κ-variable property, we get

$$\alpha_{\Sigma'}(\sigma^{\langle\Sigma,\phi\rangle}_{\Sigma'}(\vec{\phi})) = \alpha_{\Sigma'}(\sigma^{\langle\Sigma,\psi\rangle}_{\Sigma'}(\vec{\phi})).$$

Since $\Sigma' \in |\mathbf{Sign}^\flat|$ and $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma')$ were arbitrary, we conclude that $\mathcal{A} \vDash \sigma^{\langle\Sigma,\phi\rangle} \approx \sigma^{\langle\Sigma,\psi\rangle}$. ■

Now we are in a position to prove that, for algebraic systems over transformational base algebraic systems, the semantic and syntactic variety operators coincide.

Theorem 41 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a transformational algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems. Then*

$$\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathbb{V}^{\mathsf{Syn}}(\mathsf{K}).$$

Proof: By Theorem 39, $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{V}^{\mathsf{Syn}}(\mathsf{K})$ always holds. For the reverse inclusion, suppose that $\mathcal{A} \in \mathbb{V}^{\mathsf{Syn}}(\mathsf{K})$. We must show that $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$, i.e., that $\mathrm{Ker}(\mathsf{K}) \leq \mathrm{Ker}(\mathcal{A})$. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in \mathrm{Ker}_\Sigma(\mathsf{K})$. Then, by Lemma 40, $\mathsf{K} \vDash \sigma^{\langle \Sigma, \phi \rangle} \approx \sigma^{\langle \Sigma, \psi \rangle}$. Since $\mathcal{A} \in \mathbb{V}^{\mathsf{Syn}}(\mathsf{K})$, we get that $\mathcal{A} \vDash \sigma^{\langle \Sigma, \phi \rangle} \approx \sigma^{\langle \Sigma, \psi \rangle}$. Using Lemma 40 again, we infer that $\langle \phi, \psi \rangle \in \mathrm{Ker}_\Sigma(\mathcal{A})$. Thus, $\mathrm{Ker}(\mathsf{K}) \leq \mathrm{Ker}(\mathcal{A})$. Hence, $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. ■

2.6 π-Institutions

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. A **closure** (**operator**) **system** on $\mathbf{F}$ is a collection $C = \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, such that

$$C_\Sigma : \mathcal{P}(\mathrm{SEN}^\flat(\Sigma)) \to \mathcal{P}(\mathrm{SEN}^\flat(\Sigma))$$

is a closure operator on $\mathrm{SEN}^\flat(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}^\flat|$, and, moreover, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, and all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\mathrm{SEN}^\flat(f)(C_\Sigma(\Phi)) \subseteq C_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)).$$

This condition is often referred to as **structurality**.

A **π-institution** is a pair $\mathcal{I} = \langle \mathbf{F}, C \rangle$, where $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ is an algebraic system and C is a closure system on $\mathbf{F}$. We say that the π-institution $\mathcal{I}$ is **based on** the algebraic system $\mathbf{F}$. The following assumption is adopted throughout our treatise, unless explicitly stated otherwise:

> **Global Assumption**: If, for some $\Sigma \in |\mathbf{Sign}^\flat|$, $C_\Sigma(\varnothing) \neq \varnothing$, then, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $C_\Sigma(\varnothing) \neq \varnothing$. (2.1)

The set of **Σ-theorems**, denoted $\mathrm{Thm}_\Sigma(\mathcal{I})$, is defined by

$$\mathrm{Thm}_\Sigma(\mathcal{I}) = C_\Sigma(\varnothing).$$

We then set $\mathrm{Thm}(\mathcal{I}) = \{\mathrm{Thm}_\Sigma(\mathcal{I})\}_{\Sigma \in |\mathbf{Sign}|}$. We denote by $\overline{\varnothing}$ the $|\mathbf{Sign}^\flat|$-indexed collection $\overline{\varnothing} = \{\varnothing\}_{\Sigma \in |\mathbf{Sign}^\flat|}$. The Global Assumption (2.1), adopted above, says that, if a π-institution has Σ-theorems, for some signature Σ, then it has Σ-theorems, for every signature Σ.

A **natural theorem of** $\mathcal{I}$ is a natural transformation

$$\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$$

in $N^\flat$, for some $k \geq 0$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^k$,

$$\top^\flat_\Sigma(\vec{\phi}) \in \mathrm{Thm}_\Sigma(\mathcal{I}).$$

That is, a natural theorem of $\mathcal{I}$ is a natural transformation in $N^\flat$ all of whose values are theorems. We denote by $\mathrm{NThm}(\mathcal{I})$ the collection of all natural theorems of a π-institution $\mathcal{I}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution, based on $\mathbf{F}$, and $\Sigma \in |\mathbf{Sign}^\flat|$. A subset $T_\Sigma \subseteq \mathrm{SEN}^\flat(\Sigma)$ is called a **Σ-theory** if

$$C_\Sigma(T_\Sigma) = T_\Sigma.$$

We use $\mathrm{Th}_\Sigma(\mathcal{I})$ to denote the collection of all Σ-theories of the π-institution $\mathcal{I}$. A **theory family** of $\mathcal{I}$ is a sentence family $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ of $\mathbf{F}$, such that $T_\Sigma \in \mathrm{Th}_\Sigma(\mathcal{I})$, for all $\Sigma \in |\mathbf{Sign}^\flat|$. The collection of all theory families of $\mathcal{I}$ will be denoted by $\mathrm{ThFam}(\mathcal{I})$. Ordered by signature-wise inclusion $\leq$, it forms a complete lattice, denoted $\mathbf{ThFam}(\mathcal{I}) = \langle \mathrm{ThFam}(\mathcal{I}), \leq \rangle$.

A theory family of $\mathcal{I}$ is called a **theory system** of $\mathcal{I}$ if it is a sentence system, i.e., if it is invariant under signature morphisms. We denote by $\mathrm{ThSys}(\mathcal{I})$, the collection of all theory systems of $\mathcal{I}$. This collection forms a complete sublattice $\mathbf{ThSys}(\mathcal{I}) = \langle \mathrm{ThSys}(\mathcal{I}), \leq \rangle$ of the complete lattice $\mathbf{ThFam}(\mathcal{I})$.

Note that the minimum element of both $\mathbf{ThFam}(\mathcal{I})$ and $\mathbf{ThSys}(\mathcal{I})$ is $\mathrm{Thm}(\mathcal{I})$, the **theorem system** of $\mathcal{I}$, and the maximum element is

$$\mathrm{SEN}^\flat = \{\mathrm{SEN}^\flat(\Sigma)\}_{\Sigma \in |\mathbf{Sign}^\flat|}.$$

Thus, $\mathrm{SEN}^\flat$ is used to denote both the sentence functor of the base algebraic system $\mathbf{F}$ of the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ and the maximum theory family (system) $\mathrm{SEN}^\flat = \{\mathrm{SEN}^\flat(\Sigma)\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ of $\mathcal{I}$. This overloading will not, hopefully, cause any confusion, since the context can be used to clarify the meaning.

Proposition 42 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution and $T \in \mathrm{ThFam}(\mathcal{I})$. Then $\overleftarrow{T}$ is the largest theory system of $\mathcal{I}$ included in T.*

Proof: Since, by Proposition 2, $\overleftarrow{T}$ is the largest sentence system included in T, it suffices to show that $\overleftarrow{T}$ is a theory system. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\overleftarrow{T}_\Sigma)$. We must show that $\phi \in \overleftarrow{T}_\Sigma$. So let $\Sigma' \in |\mathbf{Sign}^\flat|$ and $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$. Then we have

$$\begin{array}{lll}\mathrm{SEN}^\flat(f)(\phi) & \in & \mathrm{SEN}(f)(C_\Sigma(\overleftarrow{T}_\Sigma)) \quad \text{(hypothesis)}\\ & \subseteq & C_{\Sigma'}(\mathrm{SEN}(f)(\overleftarrow{T}_\Sigma)) \quad \text{(structurality)}\\ & \subseteq & C_{\Sigma'}(T_{\Sigma'}) \quad (\text{definition of } \overleftarrow{T})\\ & = & T_{\Sigma'} \quad (T \in \mathrm{ThFam}(\mathcal{I})).\end{array}$$

We now conclude, by the definition of $\overleftarrow{T}$, that $\phi \in \overleftarrow{T}_\Sigma$. ■

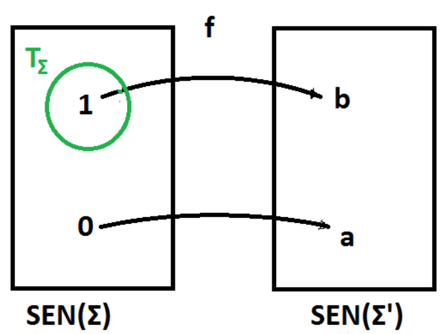
On the negative side, it is not true, in general, that, given a theory family T of a π-institution $\mathcal{I}$, the least sentence system $\overrightarrow{T}$, containing T, is a theory system. We show that this is the case via an example.

Example 43 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *consists of two signatures* Σ *and* Σ' *and the only (non-identity) morphism is* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign} \to \mathbf{Set}$ *is defined by setting*

$$\mathrm{SEN}^\flat(\Sigma) = \{0,1\}, \quad \mathrm{SEN}^\flat(\Sigma') = \{a,b\}$$

  and $\mathrm{SEN}^\flat(f)(0) = a$*,* $\mathrm{SEN}^\flat(f)(1) = b$*;*
- $N^\flat$ *consists of only the projection natural transformations.*



Consider the closure system C *on* $\mathbf{F}$ *defined by setting*

$$\mathcal{C}_\Sigma = \{\varnothing, \{0\}, \{1\}, \{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{a,b\}\}$$

and let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the associated* π*-institution.*

Finally, take $T = \{T_\Sigma, T_{\Sigma'}\} \in \mathrm{ThFam}(\mathcal{I})$ *to be the theory family specified by*

$$T_\Sigma = \{1\} \quad \textit{and} \quad T_{\Sigma'} = \varnothing.$$

Then we have

$$\overrightarrow{T}_\Sigma = \{1\} \quad \textit{and} \quad \overrightarrow{T}_{\Sigma'} = \{b\}.$$

Since clearly

$$C_{\Sigma'}(\overrightarrow{T}_{\Sigma'}) = C_{\Sigma'}(\{b\}) = \{a,b\} \neq \overrightarrow{T}_{\Sigma'},$$

it follows that $\overrightarrow{T}$ *is not a theory system of* $\mathcal{I}$*.*

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We define two operators

$$\begin{array}{rlcl} C: & \mathrm{SenFam}(\mathbf{F}) & \to & \mathrm{ThFam}(\mathcal{I}); \\ \overrightarrow{C}: & \mathrm{SenFam}(\mathbf{F}) & \to & \mathrm{ThSys}(\mathcal{I}); \end{array}$$

as follows. Consider a sentence family $T \in \mathrm{SenFam}(\mathbf{F})$.

- $C(T) = \{C(T)_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,
$$C(T)_\Sigma = C_\Sigma(T_\Sigma);$$
- $\overrightarrow{C}(T) = \{\overrightarrow{C}(T)_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,
$$\overrightarrow{C}(T)_\Sigma = C_\Sigma(\overrightarrow{T}_\Sigma).$$

It is clear that $C(T)$ is the smallest theory family of $\mathcal{I}$ containing T. We show in the next proposition that $\overrightarrow{C}(T)$ is the smallest theory system of $\mathcal{I}$ that contains the sentence family T. Note that this implies, in particular, that $\overrightarrow{C}(T)$ is the smallest theory system of $\mathcal{I}$ that contains a given theory family T of $\mathcal{I}$. Note, also, that $\overrightarrow{C}(T) = C(\overrightarrow{T})$ should not be confused with $\overrightarrow{C(T)}$, which, as shown in Example 43, may not be a theory family of $\mathcal{I}$.

Proposition 44 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution, based on $\mathbf{F}$, and $T \in \mathrm{SenFam}(\mathbf{F})$. Then $\overrightarrow{C}(T)$ is the smallest theory system of $\mathcal{I}$ that includes T.*

Proof: It is clear by the definition that $\overrightarrow{C}(T) = C(\overrightarrow{T}) \in \mathrm{ThFam}(\mathcal{I})$. We show that it is a theory system. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\overrightarrow{T}_\Sigma)$. Consider $\Sigma' \in |\mathbf{Sign}^\flat|$ and $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$. Then we have

$$\begin{array}{rcll} \mathrm{SEN}^\flat(f)(\phi) & \in & \mathrm{SEN}^\flat(f)(C_\Sigma(\overrightarrow{T}_\Sigma)) & (\text{definition of } \overrightarrow{C}(T)) \\ & \subseteq & C_{\Sigma'}(\mathrm{SEN}^\flat(f)(\overrightarrow{T}_\Sigma)) & (\text{structurality}) \\ & \subseteq & C_{\Sigma'}(\overrightarrow{T}_{\Sigma'}) & (\text{definition of } \overrightarrow{T}) \\ & = & \overrightarrow{C}(T)_{\Sigma'} & (\text{definition of } \overrightarrow{C}(T)). \end{array}$$

It remains to show that $C(\overrightarrow{T})$ is the smallest theory system containing T. To this end, let $T' \in \mathrm{ThSys}(\mathcal{I})$, such that $T \leq T'$. Since, by Proposition 2, $\overrightarrow{T}$ is the least sentence system containing T, we get $\overrightarrow{T} \leq T'$. Therefore, since $C(\overrightarrow{T})$ is the least theory family containing $\overrightarrow{T}$, $C(\overrightarrow{T}) \leq T'$. Thus, we conclude that $\overrightarrow{C}(T) = C(\overrightarrow{T}) \leq T'$ and $\overrightarrow{C}(T)$ is the least theory system of $\mathcal{I}$ that includes T. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. We say that $\mathcal{I}$ is:

- **inconsistent** if $\mathrm{ThFam}(\mathcal{I}) = \{\mathrm{SEN}^\flat\}$, i.e., if, for all $\Sigma \in |\mathbf{Sign}^\flat|$,
$$C_\Sigma(\varnothing) = \mathrm{SEN}^\flat(\Sigma);$$
- **almost inconsistent** if
$$\mathrm{ThFam}(\mathcal{I}) = \{T : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(T_\Sigma = \varnothing \text{ or } T_\Sigma = \mathrm{SEN}^\flat(\Sigma))\};$$
- **trivial** if it is either inconsistent or almost inconsistent.

Lemma 45 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is trivial if and only if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $\psi \in C_\Sigma(\phi)$.*

Proof: Suppose, first, that $\mathcal{I}$ is trivial and let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$. Since $\phi \in C_\Sigma(\phi)$, we have $C_\Sigma(\phi) \neq \varnothing$, which implies that $C_\Sigma(\phi) = \mathrm{SEN}^\flat(\Sigma)$. Therefore, $\psi \in C_\Sigma(\phi)$.

Suppose, conversely, that the given condition holds. Let $T \in \mathrm{ThFam}(\mathcal{I})$ and $\Sigma \in |\mathbf{Sign}^\flat|$, such that $T_\Sigma \neq \varnothing$. Then, there exists $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$. But then, by hypothesis, for all $\psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\psi \in C_\Sigma(\phi) \subseteq C_\Sigma(T_\Sigma) = T_\Sigma.$$
Therefore, we get that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $T_\Sigma = \varnothing$ or $T_\Sigma = \mathrm{SEN}^\flat(\Sigma)$, showing that T is almost inconsistent. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system. We can order π-institutions based on $\mathbf{F}$ by comparing their closure systems. Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ and $\mathcal{I}' = \langle \mathbf{F}, C'\rangle$ two π-institutions based on $\mathbf{F}$. We say that $\mathcal{I}'$ is an **extension** of $\mathcal{I}$ and that $\mathcal{I}$ is **weaker** than $\mathcal{I}'$, written $\mathcal{I} \leq \mathcal{I}'$ (or $C \leq C'$) if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$,
$$C_\Sigma(\Phi) \subseteq C'_\Sigma(\Phi).$$
Given a collection $\mathcal{I}^i = \langle \mathbf{F}, C^i\rangle$, $i \in I$, of π-institutions based on the same algebraic system $\mathbf{F}$, the **intersection** $\bigcap_{i\in I} \mathcal{I}^i = \langle \mathbf{F}, \bigcap_{i\in I} C^i\rangle$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$,
$$(\bigcap_{i\in I} C^i)_\Sigma(\Phi) = \bigcap_{i\in I} C^i_\Sigma(\Phi).$$
It can be shown that $\bigcap_{i\in I} C^i$ is a closure system on $\mathbf{F}$ and that it forms the meet with respect to the $\leq$ order of the closure systems C^i, $i \in I$, on $\mathbf{F}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution. Given a theory system $T \in \mathrm{ThSys}(\mathcal{I})$, we define the family $C^T = \{C^T_\Sigma\}_{\Sigma\in|\mathbf{Sign}^\flat|}$ of operators $C^T_\Sigma : \mathcal{P}(\mathrm{SEN}^\flat(\Sigma)) \to \mathcal{P}(\mathrm{SEN}^\flat(\Sigma))$ by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$,
$$C^T_\Sigma(\Phi) = C_\Sigma(T_\Sigma \cup \Phi).$$
We show that C^T is a closure system on $\mathbf{F}$.

Proposition 46 *Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution and $T \in \mathrm{ThSys}(\mathcal{I})$. Then C^T is a closure system on $\mathbf{F}$.*

Proof: We must first show that $C^T_\Sigma : \mathcal{P}(\mathrm{SEN}^\flat(\Sigma)) \to \mathcal{P}(\mathrm{SEN}^\flat(\Sigma))$ is a closure operator. That it is inflationary and monotone follows directly from the corresponding properties of C_Σ. To see that it is idempotent, let $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{rcl} C^T_\Sigma(C^T_\Sigma(\Phi)) & = & C_\Sigma(T_\Sigma \cup C_\Sigma(T_\Sigma \cup \Phi)) \quad \text{(by definition)} \\ & = & C_\Sigma(C_\Sigma(T_\Sigma \cup \Phi)) \quad (\text{since } T_\Sigma \subseteq C_\Sigma(T_\Sigma \cup \Phi)) \\ & = & C_\Sigma(T_\Sigma \cup \Phi) \quad (\text{idempotency of } C) \\ & = & C^T_\Sigma(\Phi) \quad \text{(by definition).} \end{array}$$

Finally, we must show that C^T is structural. To this end, let $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$. We have

$$\begin{array}{rcl} \mathrm{SEN}^\flat(f)(C^T_\Sigma(\Phi)) & = & \mathrm{SEN}^\flat(f)(C_\Sigma(T_\Sigma \cup \Phi)) \quad \text{(by definition)} \\ & \subseteq & C_{\Sigma'}(\mathrm{SEN}^\flat(f)(T_\Sigma) \cup \mathrm{SEN}^\flat(f)(\Phi)) \\ & & \qquad\qquad (\text{by the structurality of } C) \\ & \subseteq & C_{\Sigma'}(T_{\Sigma'} \cup \mathrm{SEN}^\flat(f)(\Phi)) \quad (T \in \mathrm{ThSys}(\mathcal{I})) \\ & = & C^T_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \quad \text{(by definition).} \end{array}$$

We conclude that $C^T = \{C^T_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ is a closure system on $\mathbf{F}$. ■

Since C^T is a closure system on $\mathbf{F}$, we get, by definition, that the structure $\langle \mathbf{F}, C^T\rangle$ is a π-institution. We use the notation $\mathcal{I}^T = \langle \mathbf{F}, C^T\rangle$ to denote this π-institution.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$. An $\mathcal{I}$-**logical morphism** (or simply **logical morphism** if $\mathcal{I}$ is clear from context) is a morphism $\langle F, \alpha\rangle : \mathbf{F} \to \mathbf{F}$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\alpha_\Sigma(C_\Sigma(\Phi)) \subseteq C_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

More generally, let $\mathbf{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ and $\mathbf{F}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ be two algebraic systems and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ and $\mathcal{I}' = \langle \mathbf{F}', C'\rangle$ be π-institutions based on $\mathbf{F}$ and $\mathbf{F}'$, respectively. A **logical morphism** $\langle F, \alpha\rangle : \mathcal{I} \to \mathcal{I}'$ is an algebraic system morphism $\langle F, \alpha\rangle : \mathbf{F} \to \mathbf{F}'$, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)$,

$$\alpha_\Sigma(C_\Sigma(\Phi)) \subseteq C_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

The following lemma characterizes logical morphisms:

Lemma 47 *Let $\mathbf{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{F}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ be two algebraic systems and $\mathcal{I} = \langle \mathbf{F}, C\rangle$, $\mathcal{I}' = \langle \mathbf{F}', C'\rangle$ be π-institutions, based on $\mathbf{F}$, $\mathbf{F}'$, respectively. Suppose $\langle F, \alpha\rangle : \mathbf{F} \to \mathbf{F}$ is an algebraic system morphism. Then the following conditions are equivalent:*

(a) $\langle F,\alpha\rangle : \mathcal{I} \to \mathcal{I}'$ *is a logical morphism;*

(b) For all $\Sigma \in |\mathbf{Sign}|$ *and all* $\Psi \subseteq \mathrm{SEN}'(F(\Sigma))$,

$$C_\Sigma(\alpha_\Sigma^{-1}(\Psi)) \leq \alpha_\Sigma^{-1}(C'_{F(\Sigma)}(\Psi));$$

(c) For all $T' \in \mathrm{ThFam}(\mathcal{I}')$, $\alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$.

Proof:

(a)⇒(b) Let $\Sigma \in |\mathbf{Sign}|$ and $\Psi \subseteq \mathrm{SEN}'(F(\Sigma))$. Then, we have

$$\begin{array}{rcll}\alpha_\Sigma(C_\Sigma(\alpha_\Sigma^{-1}(\Psi))) & \subseteq & C_{F(\Sigma)}(\alpha_\Sigma(\alpha_\Sigma^{-1}(\Psi))) & \text{(hypothesis)}\\ & \subseteq & C_{F(\Sigma)}(\Psi). & \text{(set theory)}\end{array}$$

We conclude that $C_\Sigma(\alpha_\Sigma^{-1}(\Psi)) \subseteq \alpha_\Sigma^{-1}(C_{F(\Sigma)}(\Psi))$.

(b)⇒(c) Suppose that $T' \in \mathrm{ThFam}(\mathcal{I}')$. Then we have

$$\begin{array}{rcll}C(\alpha^{-1}(T')) & \leq & \alpha^{-1}(C'(T')) & \text{(hypothesis)}\\ & = & \alpha^{-1}(T'). & (T' \in \mathrm{ThFam}(\mathcal{I}'))\end{array}$$

Therefore, $\alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$.

(c)⇒(a) Let $\Sigma \in |\mathbf{Sign}|$ and $\Phi \subseteq \mathrm{SEN}(\Sigma)$. Then, we have, for all $T \in \mathrm{ThFam}(\mathcal{I}')$,

$$\begin{array}{rcll}\alpha_\Sigma(\Phi) \subseteq T_{F(\Sigma)} & \text{iff} & \Phi \subseteq \alpha_\Sigma^{-1}(T_{F(\Sigma)}) & \text{(set theory)}\\ & \text{implies} & C_\Sigma(\Phi) \subseteq \alpha_\Sigma^{-1}(T_{F(\Sigma)}) & \text{(hypothesis)}\\ & \text{iff} & \alpha_\Sigma(C_\Sigma(\Phi)) \subseteq T_{F(\Sigma)}. & \text{(set theory)}\end{array}$$

Since $T \in \mathrm{ThFam}(\mathcal{I}')$ was arbitrary, we get that

$$\alpha_\Sigma(C_\Sigma(\Phi)) \subseteq C'_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

So $\langle F,\alpha\rangle$ is a logical morphism.

■

In the special case of $\mathcal{I}$-logical morphisms, we obtain the following

Corollary 48 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution, based on* $\mathbf{F}$*, and* $\langle F,\alpha\rangle : \mathbf{F} \to \mathbf{F}$ *an algebraic system morphism. Then* $\langle F,\alpha\rangle$ *is an* $\mathcal{I}$*-logical morphism if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$.

Proof: Directly from Lemma 47. ■

2.7 Matrix Families and Systems

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. An **F-matrix family** is a pair $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, where $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ is an $\mathbf{F}$-algebraic system and $T \in \mathrm{SenFam}(\mathcal{A})$. The collection of all $\mathbf{F}$-matrix families is denoted by $\mathrm{MatFam}(\mathbf{F})$. An **F-matrix system** is an $\mathbf{F}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, such that $T \in \mathrm{SenSys}(\mathcal{A})$. The collection of all $\mathbf{F}$-matrix systems is denoted by $\mathrm{MatSys}(\mathbf{F})$.

An $\mathbf{F}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ defines a closure system $C^\mathfrak{A} = \{C^\mathfrak{A}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ on $\mathbf{F}$ by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{l} \phi \in C^\mathfrak{A}_\Sigma(\Phi) \text{ if and only if, for all } \Sigma' \in |\mathbf{Sign}^\flat| \text{ and all } f \in \mathbf{Sign}^\flat(\Sigma, \Sigma'), \\ \qquad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq T_{F(\Sigma')} \text{ imlpies } \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \in T_{F(\Sigma')}. \end{array}$$

Let, now, M be a class of $\mathbf{F}$-matrix families. We denote by

$$C^\mathsf{M} = \{C^\mathsf{M}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$$

the closure system on $\mathbf{F}$ that is the signature-wise intersection of the closure systems $C^\mathfrak{A}$, $\mathfrak{A} \in \mathsf{M}$, i.e.,

$$C^\mathsf{M} = \bigcap_{\mathfrak{A} \in \mathsf{M}} C^\mathfrak{A}.$$

We use the notation $\mathcal{I}^\mathsf{M} = \langle \mathbf{F}, C^\mathsf{M} \rangle$ to denote the associated π-institution based on $\mathbf{F}$.

We give a characterization of the closure system C^M on $\mathbf{F}$ generated by a class M of matrix families which shows how that closure system is constructed using the generating matrix families.

Proposition 49 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and M a class of $\mathbf{F}$-matrix families. Then $\mathcal{C}^\mathsf{M}$ is the least closure system on $\mathbf{F}$ containing the family*

$$\mathcal{T} = \{\alpha^{-1}(T) : \mathfrak{A} = \langle \langle \mathbf{A}, \langle F, \alpha \rangle \rangle, T \rangle \in \mathsf{M}\}.$$

Proof: First we show that $\mathcal{T} \subseteq \mathcal{C}^\mathsf{M}$. To this end, let $\mathfrak{A} = \langle \langle \mathbf{A}, \langle F, \alpha \rangle \rangle, T \rangle \in \mathsf{M}$. We must show that $\alpha^{-1}(T) \in \mathcal{C}^\mathsf{M}$. Suppose $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C^\mathsf{M}_\Sigma(\alpha^{-1}_\Sigma(T_{F(\Sigma)}))$. Then, by the definition of C^M and the fact that $\mathfrak{A} \in \mathsf{M}$, we get

$$\alpha_\Sigma(\alpha^{-1}_\Sigma(T_{F(\Sigma)})) \subseteq T_{F(\Sigma)} \quad \text{implies} \quad \alpha_\Sigma(\phi) \in T_{F(\Sigma)}.$$

Note, however, that the antecedent of the displayed implication always holds. So the consequent $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$ holds. Hence, $\phi \in \alpha^{-1}_\Sigma(T_{F(\Sigma)})$. Therefore, $C^\mathsf{M}(\alpha^{-1}(T)) \leq \alpha^{-1}(T)$, showing that $\alpha^{-1}(T) \in \mathcal{C}^\mathsf{M}$.

Next, we show that, if $\mathcal{C}$ is a closure system on $\mathbf{F}$, such that $\mathcal{T} \subseteq \mathcal{C}$, then $\mathcal{C}^\mathsf{M} \subseteq \mathcal{C}$. Equivalently, it suffices to show that $C \leq C^\mathsf{M}$. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$. Since $\mathcal{C}$ is a closure system on $\mathbf{F}$, we get, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$, $\mathrm{SEN}^\flat(f)(\phi) \in C_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))$. Thus, since $\mathcal{T} \subseteq \mathcal{C}$, we get, for all $\langle\mathcal{A},T\rangle \in \mathsf{M}$, with $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle$,

$$\mathrm{SEN}^\flat(f)(\Phi) \subseteq \alpha_{\Sigma'}^{-1}(T_{F(\Sigma')}) \quad \text{implies} \quad \mathrm{SEN}^\flat(f)(\phi) \in \alpha_{\Sigma'}^{-1}(T_{F(\Sigma')}),$$

i.e., for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq T_{F(\Sigma')} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \in T_{F(\Sigma')}.$$

Hence, for all $\mathfrak{A} = \langle\mathcal{A},T\rangle \in \mathsf{M}$, $\phi \in C_\Sigma^{\mathfrak{A}}(\Phi)$. We conclude that $\phi \in C_\Sigma^{\mathsf{M}}(\Phi)$. Therefore, $C \leq C^{\mathsf{M}}$, as was to be shown. ∎

Let again $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F},C\rangle$ be a π-institution based on $\mathbf{F}$. Let $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system, with $\mathbf{A} = \langle\mathbf{Sign},\mathrm{SEN},N\rangle$. A sentence family $T \in \mathrm{SenFam}(\mathcal{A})$ is called an **$\mathcal{I}$-filter family** and the $\mathbf{F}$-matrix family $\mathfrak{A} = \langle\mathcal{A},T\rangle$ an **$\mathcal{I}$-matrix family** if

$$C \leq C^{\mathfrak{A}}.$$

If T happens to be a sentence system, then we refer to T as an **$\mathcal{I}$-filter system** and to $\mathfrak{A} = \langle\mathcal{A},T\rangle$ as an **$\mathcal{I}$-matrix system**.

We have the following simpler characterization of $\mathcal{I}$-filter families, which follows from the structurality of the closure system of a π-institution.

Lemma 50 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle$ an $\mathbf{F}$-algebraic system, with $\mathbf{A} = \langle\mathbf{Sign},\mathrm{SEN},N\rangle$, and $T \in \mathrm{SenFam}(\mathcal{A})$. T is an $\mathcal{I}$-filter family if and only if, for every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$,*

$$\alpha_\Sigma(\Phi) \subseteq T_{F(\Sigma)} \quad \textit{implies} \quad \alpha_\Sigma(\phi) \in T_{F(\Sigma)}.$$

Proof: Suppose, first, that T is an $\mathcal{I}$-filter family and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$ and $\alpha_\Sigma(\Phi) \subseteq T_{F(\Sigma)}$. Since T is an $\mathcal{I}$-filter family, $C \leq C^{\langle\mathcal{A},T\rangle}$. Therefore, by taking in the definition of $C^{\langle\mathcal{A},C\rangle}$, $\Sigma' = \Sigma$ and $f : \Sigma \to \Sigma$ to be the identity morphism, we get that $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$.

Suppose, conversely, that the given condition holds and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$. Consider $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$, such that $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq T_{F(\Sigma')}$. Note, that, by structurality, $\mathrm{SEN}^\flat(f)(\phi) \in C_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))$. Therefore, by the assumption and the hypothesis, $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \in T_{F(\Sigma')}$. We conclude that T is an $\mathcal{I}$-filter family. ∎

The next lemma shows that the inverse image under an interpretation of an $\mathcal{I}$-filter family or system is a theory family or system, respectively, of $\mathcal{I}$. Moreover this property also characterizes $\mathcal{I}$-filter families/systems.

Lemma 51 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution, based on* $\mathbf{F}$*, and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *be an* $\mathbf{F}$*-algebraic system.*

(a) $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *if and only if* $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$*;*

(b) $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ *if and only if* $\alpha^{-1}(T) \in \mathrm{ThSys}(\mathcal{I})$*.*

Proof:

(a) Suppose, first, that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. We must show that $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$. To this end, suppose $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\alpha_\Sigma^{-1}(T_{F(\Sigma)}))$. Since $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we have, by definition,

$$\alpha_\Sigma(\alpha_\Sigma^{-1}(T_{F(\Sigma)})) \subseteq T_{F(\Sigma)} \quad \text{implies} \quad \alpha_\Sigma(\phi) \in T_{F(\Sigma)}.$$

But the hypothesis of this implication holds, whence the conclusion is also true and we get $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$ or, equivalently, $\phi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})$. Thus $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$.

Suppose, conversely, that $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$. To show that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$, and assume that $\alpha_\Sigma(\Phi) \subseteq T_{F(\Sigma)}$. Then, we have $\Phi \subseteq \alpha_\Sigma^{-1}(T_{F(\Sigma)})$. Since $\phi \in C_\Sigma(\Phi)$ and $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, we get that $\phi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})$ or, equivalently, $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$. This proves, by Lemma 50, that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

(b) This follows from Part (a) and from Part (a) of Lemma 6.

■

We denote by $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and by $\mathrm{MatFam}(\mathcal{I})$, respectively, the collection of all $\mathcal{I}$-filter families on $\mathcal{A}$ and the collection of all $\mathcal{I}$-matrix families. Note that $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is a complete lattice $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A}) = \langle \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}), \leq \rangle$, with the order $\leq$ inherited by the corresponding order on sentence families.

Similarly, we denote by $\mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ and by $\mathrm{MatSys}(\mathcal{I})$, respectively, the collection of all $\mathcal{I}$-filter systems on $\mathcal{A}$ and the collection of all $\mathcal{I}$-matrix systems. Note that $\mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ forms a complete lattice

$$\mathbf{FiSys}^{\mathcal{I}}(\mathcal{A}) = \langle \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}), \leq \rangle,$$

which is a complete sublattice of the complete lattice $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})$.

Moreover, given a $\mathbf{F}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, we say that T' is a **sentence family of** $\mathfrak{A}$, written $T' \in \mathrm{SenFam}(\mathfrak{A})$, if $T \leq T'$. Similarly, given an $\mathcal{I}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, we say that $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is an $\mathcal{I}$-**filter family of** $\mathfrak{A}$, written $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$, if $T \leq T'$.

Since $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $\mathbf{FiSys}^{\mathcal{I}}(\mathcal{A})$ are both complete lattices, it makes sense to define associated closure operators on $\mathrm{SenFam}(\mathcal{A})$.

- Denote by $C^{\mathcal{I},\mathcal{A}} : \mathrm{SenFam}(\mathcal{I}) \to \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ the operator that maps a given sentence family T of $\mathcal{A}$ to the least $\mathcal{I}$-filter family of $\mathcal{A}$ that includes T;

- Denote by $\overrightarrow{C}^{\mathcal{I},\mathcal{A}} : \mathrm{SenFam}(\mathcal{A}) \to \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ the operator that maps a given sentence family T of $\mathcal{A}$ to the least $\mathcal{I}$-theory system of $\mathcal{A}$ that includes T.

We look now at some relations between the pairs of operators $C^{\mathcal{I},\mathcal{A}}$, C on the one hand, and $\overrightarrow{C}^{\mathcal{I},\mathcal{A}}$, $\overrightarrow{C}$ on the other, established via the inverse interpretation α^{-1} of the $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$.

Proposition 52 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution, based on* $\mathbf{F}$*, and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *be an* $\mathbf{F}$*-algebraic system. Then, for all* $T \in \mathrm{SenFam}(\mathcal{A})$*, we have:*

(a) $C(\alpha^{-1}(T)) \leq \alpha^{-1}(C^{\mathcal{I},\mathcal{A}}(T))$*;*

(b) $\overrightarrow{C}(\alpha^{-1}(T)) \leq \alpha^{-1}(\overrightarrow{C}^{\mathcal{I},\mathcal{A}}(T))$.

Proof:

(a) Suppose $T \in \mathrm{SenFam}(\mathcal{A})$. We have $T \leq C^{\mathcal{I},\mathcal{A}}(T)$, whence $\alpha^{-1}(T) \leq \alpha^{-1}(C^{\mathcal{I},\mathcal{A}}(T))$. By Lemma 51, $\alpha^{-1}(C^{\mathcal{I},\mathcal{A}}(T))$ is a theory family of $\mathcal{I}$ and it includes $\alpha^{-1}(T)$. Therefore, by the definition of C, $C(\alpha^{-1}(T)) \leq \alpha^{-1}(C^{\mathcal{I},\mathcal{A}}(T))$.

(b) We have $T \leq \overrightarrow{C}^{\mathcal{I},\mathcal{A}}(T)$. Therefore, since, by Proposition 2, $\overrightarrow{T}$ is the least sentence system containing T, we get $\overrightarrow{T} \leq \overrightarrow{C}^{\mathcal{I},\mathcal{A}}(T)$. Now, taking into account Lemma 6, we get $\overrightarrow{\alpha^{-1}(T)} = \alpha^{-1}(\overrightarrow{T}) \leq \alpha^{-1}(\overrightarrow{C}^{\mathcal{I},\mathcal{A}}(T))$. By Lemma 51, $\alpha^{-1}(\overrightarrow{C}^{\mathcal{I},\mathcal{A}}(T))$ is a theory system of $\mathcal{I}$ including $\overrightarrow{\alpha^{-1}(T)}$ and, therefore, $C(\overrightarrow{\alpha^{-1}(T)}) \leq \alpha^{-1}(\overrightarrow{C}^{\mathcal{I},\mathcal{A}}(T))$, i.e., $\overrightarrow{C}(\alpha^{-1}(T)) \leq \alpha^{-1}(\overrightarrow{C}^{\mathcal{I},\mathcal{A}}(T))$. ■

We now exhibit a relation between the closure operators $C^{\mathcal{I},\mathcal{A}}$ and $\overrightarrow{C}^{\mathcal{I},\mathcal{A}}$ and the arrow operators, as applied to $\mathcal{I}$-filter families on an $\mathbf{F}$-algebraic system $\mathcal{A}$.

Proposition 53 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution, based on* $\mathbf{F}$*, and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*. Consider* $T \in \mathrm{SenFam}(\mathcal{A})$*. Then, we have:*

(a) If $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, then* $\overleftarrow{T} \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ *and it is the largest* $\mathcal{I}$*-filter system on* $\mathcal{A}$ *included in* T*;*

(b) $\overrightarrow{C}^{\mathcal{I},\mathcal{A}}(T) = \overrightarrow{C}^{\mathcal{I},\mathcal{A}}(\overrightarrow{T})$.

Proof:

(a) By Proposition 2, we know that $\overleftarrow{T}$ is a sentence system of $\mathcal{A}$ and that it is the largest one included in T. It suffices, thus, to show that $\overleftarrow{T}$ is an $\mathcal{I}$-filter system. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that

$$\phi \in C_\Sigma(\Phi) \quad \text{and} \quad \alpha_\Sigma(\Phi) \subseteq \overleftarrow{T}_{F(\Sigma)}.$$

Then, by definition of $\overleftarrow{T}$, we get that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$, $\mathrm{SEN}(F(f))(\alpha_\Sigma(\Phi)) \subseteq T_{F(\Sigma')}$. Since α is a natural transformation, $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq T_{F(\Sigma')}$. Since $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $\phi \in C_\Sigma(\Phi)$, we get that $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \in T_{F(\Sigma')}$. Therefore, $\mathrm{SEN}(F(f))(\alpha_\Sigma(\phi)) \in T_{F(\Sigma')}$. Now, noting that this holds for all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and that F is surjective, we conclude that $\alpha_\Sigma(\phi) \in \overleftarrow{T}_{F(\Sigma)}$. Therefore, we get that $\overleftarrow{T} \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$.

(b) The inclusion from left to right is clear, since $T \le \overrightarrow{T}$. On the other hand, since, by Proposition 2, $\overrightarrow{T}$ is the least sentence system including T, we have that every $\mathcal{I}$-filter system including T, also includes $\overrightarrow{T}$. Therefore,

$$\begin{array}{rcl}\overrightarrow{C}^{\mathcal{I},\mathcal{A}}(T) & = & \bigcap\{T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) : T \le T'\} \\ & = & \bigcap\{T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) : \overrightarrow{T} \le T'\} \\ & = & \overrightarrow{C}^{\mathcal{I},\mathcal{A}}(\overrightarrow{T}).\end{array}$$

■

We extend the definition of logical morphism to morphisms between $\mathbf{F}$-algebraic systems.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. An $\mathcal{I}$-**logical morphism** is a morphism

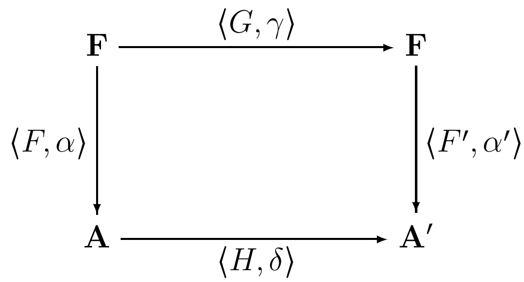
$$\langle\langle G,\gamma\rangle,\langle H,\delta\rangle\rangle : \mathcal{A} \to \mathcal{A}',$$

$$\begin{array}{ccc}\mathbf{F} & \xrightarrow{\langle G,\gamma\rangle} & \mathbf{F} \\ {\scriptstyle\langle F,\alpha\rangle}\downarrow & & \downarrow{\scriptstyle\langle F',\alpha'\rangle} \\ \mathbf{A} & \xrightarrow[\langle H,\delta\rangle]{} & \mathbf{A}'\end{array}$$

such that $\langle G,\gamma\rangle : \mathbf{F} \to \mathbf{F}$ is an $\mathcal{I}$-logical morphism $\langle G,\gamma\rangle : \mathcal{I} \to \mathcal{I}$.

Next, we prove a result relating $\mathcal{I}$-filter families/systems on algebraic systems related by morphisms. This result generalizes Lemma 51.

Proposition 54 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. Consider two* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and* $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$ *and a logical morphism* $\langle \langle G, \gamma \rangle, \langle H, \delta \rangle \rangle : \mathcal{A} \to \mathcal{A}'$*.*



(a) If $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathbf{A}')$*, then* $\delta^{-1}(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*;*

(b) If $\langle \langle G, \gamma \rangle, \langle H, \delta \rangle \rangle : \mathcal{A} \to \mathcal{A}'$ *is surjective and* $\delta^{-1}(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, then* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$*;*

(c) If $\langle \langle G, \gamma \rangle, \langle H, \delta \rangle \rangle : \mathcal{A} \to \mathcal{A}'$ *is surjective, with* G, H *isomorphisms and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *is such that* $\delta^{-1}(\delta(T)) = T$*, then* $\delta(T) \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A}')$*.*

Proof:

(a) Let $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$. Then, by Lemma 51, $\alpha'^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$. Thus, by Corollary 48, $\gamma^{-1}(\alpha'^{-1}(T)) \in \mathrm{ThFam}(\mathcal{I})$. Therefore, by the commutativity of the rectangle, $\alpha^{-1}(\delta^{-1}(T)) \in \mathrm{ThFam}(\mathcal{I})$. So, again by Lemma 51, we get that $\delta^{-1}(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

(b) Because of the surjectivity of $\langle G, \gamma \rangle$ and Lemma 50, it suffices to show that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$, we have

$$\alpha'_{G(\Sigma)}(\gamma_\Sigma(\Phi)) \subseteq T_{F'(G(\Sigma))} \quad \text{implies} \quad \alpha'_{G(\Sigma)}(\gamma_\Sigma(\phi)) \in T_{F'(G(\Sigma))}.$$

We have the following:

$$\begin{array}{ll} & \alpha'_{G(\Sigma)}(\gamma_\Sigma(\Phi)) \subseteq T_{F'(G(\Sigma))} \\ \Rightarrow & \delta_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \subseteq T_{H(F(\Sigma))} \\ \Rightarrow & \alpha_\Sigma(\Phi) \subseteq \delta^{-1}_{F(\Sigma)}(T_{H(F(\Sigma))}) \\ \Rightarrow & \alpha_\Sigma(\phi) \in \delta^{-1}_{F(\Sigma)}(T_{H(F(\Sigma))}) \\ \Rightarrow & \delta_{F(\Sigma)}(\alpha_\Sigma(\phi)) \in T_{H(F(\Sigma'))} \\ \Rightarrow & \alpha'_{G(\Sigma)}(\gamma_\Sigma(\phi)) \in T_{F'(G(\Sigma))}. \end{array}$$

(c) As in Part (b) because of the surjectivity of $\langle G, \gamma \rangle$ and Lemma 50, it suffices to show that for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$, we have

$$\alpha'_{G(\Sigma)}(\gamma_\Sigma(\Phi)) \subseteq \delta_{F(\Sigma)}(T_{F(\Sigma)}) \quad \text{implies} \quad \alpha'_{G(\Sigma)}(\gamma_\Sigma(\phi)) \in \delta_{F(\Sigma)}(T_{F(\Sigma)}).$$

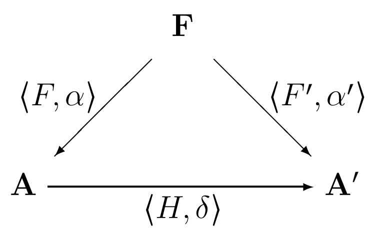
We have

$$
\begin{array}{ll}
& \alpha'_{G(\Sigma)}(\gamma_\Sigma(\Phi)) \subseteq \delta_{F(\Sigma)}(T_{F(\Sigma)}) \\
\Rightarrow & \delta_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \subseteq \delta_{F(\Sigma)}(T_{F(\Sigma)}) \\
\Rightarrow & \alpha_\Sigma(\Phi) \subseteq \delta^{-1}_{F(\Sigma)}(\delta_{F(\Sigma)}(T_{F(\Sigma)})) = T_{F(\Sigma)} \\
\Rightarrow & \alpha_\Sigma(\phi) \in \delta^{-1}_{F(\Sigma)}(\delta_{F(\Sigma)}(T_{F(\Sigma)})) \\
\Rightarrow & \delta_{F(\Sigma)}(\alpha_\Sigma(\phi)) \in \delta_{F(\Sigma)}(T_{F(\Sigma)}) \\
\Rightarrow & \alpha'_{G(\Sigma)}(\gamma_\Sigma(\phi)) \in \delta_{F(\Sigma)}(T_{F(\Sigma)}).
\end{array}
$$

■

This proposition has the following significant consequences.

Corollary 55 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. Let, also,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and* $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$ *be two* $\mathbf{F}$*-algebraic systems and* $\langle H, \delta \rangle : \mathcal{A} \to \mathcal{A}'$ *a (surjective) morphism (making the following diagram commute):*

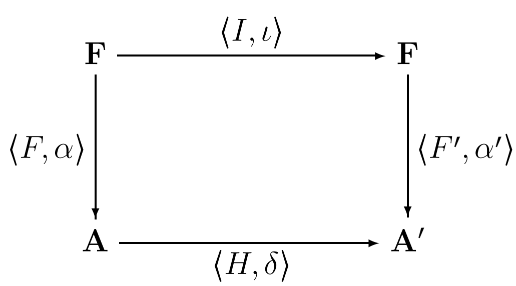


Consider $T \in \mathrm{SenFam}(\mathbf{A}')$.

(a) $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$ *iff* $\delta^{-1}(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*;*

(b) $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}')$ *iff* $\delta^{-1}(T) \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$*.*

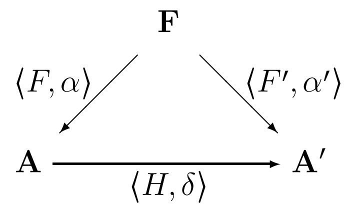
Proof: Follows immediately from Proposition 54 upon considering the commutative square,



where $\langle I, \iota \rangle : \mathbf{F} \to \mathbf{F}$ is the identity morphism. ■

Corollary 56 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. Let, also,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and* $\mathcal{A}' =$

$\langle \mathbf{A}', \langle F', \alpha' \rangle\rangle$ *be two* $\mathbf{F}$*-algebraic systems and* $\langle H, \delta\rangle : \mathcal{A} \to \mathcal{A}'$ *a morphism, with* H *an isomorphism:*



Suppose $T \in \mathrm{SenFam}(\mathbf{A})$ *and* $\mathrm{Ker}(\langle H, \delta\rangle)$ *is compatible with* T*.*

(a) $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *iff* $\delta(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$*;*

(b) $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ *iff* $\delta(T) \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}')$*.*

Proof: First we show that $\delta^{-1}(\delta(T)) = T$: The right to left inclusion is obvious. For the left to right inclusion, consider $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in \delta_\Sigma^{-1}(\delta_\Sigma(T_\Sigma))$. Then, we have $\delta_\Sigma(\phi) \in \delta_\Sigma(T_\Sigma)$. Thus, there exists $\psi \in T_\Sigma$, such that $\delta_\Sigma(\phi) = \delta_\Sigma(\psi)$. By hypothesis, $\mathrm{Ker}(\langle H, \delta\rangle)$ is compatible with T. Therefore, $\phi \in T_\Sigma$. Thus, we get $\delta^{-1}(\delta(T)) \leq T$.

Now the conclusion follows from Proposition 54, since $\delta^{-1}(\delta(T)) = T$. ■

Corollary 57 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*. Let, also,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *be an* $\mathbf{F}$*-algebraic system and* $\theta \in \mathrm{ConSys}(\mathbf{A})$*. Consider* $T \in \mathrm{SenFam}(\mathbf{A}^\theta)$*.*

(a) $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta)$ *iff* $\pi^{\theta^{-1}}(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*;*

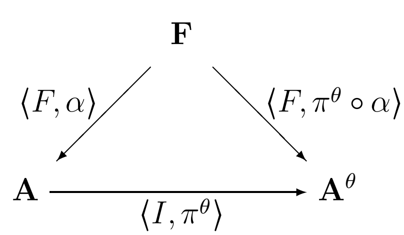
(b) $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}^\theta)$ *iff* $\pi^{\theta^{-1}}(T) \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$*.*

On the other hand, if $T \in \mathrm{SenFam}(\mathbf{A})$ *and* θ *is compatible with* T*, then we have:*

(c) $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *iff* $\pi^\theta(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta)$*;*

(d) $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ *iff* $\pi^\theta(T) \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}^\theta)$*.*

Proof: Parts (a) and (b) follow immediately from Corollary 55 upon considering the commutative diagram



Parts (c) and (d) follow from Corollary 56 upon noticing that $I : \mathbf{Sign} \to \mathbf{Sign}$ is an isomorphism and that, by hypothesis, $\mathrm{Ker}(\langle I, \pi^\theta\rangle) = \theta$ is compatible with T. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$. Consider two $\mathcal{I}$-matrix families $\mathfrak{A} = \langle \mathcal{A}, T\rangle$ and $\mathfrak{A}' = \langle \mathcal{A}', T'\rangle$, where $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ and $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha'\rangle\rangle$. A morphism $\langle\langle G, \gamma\rangle, \langle H, \delta\rangle\rangle : \mathcal{A} \to \mathcal{A}'$, is called a **matrix family morphism** $\langle\langle G, \gamma\rangle, \langle H, \delta\rangle\rangle : \mathfrak{A} \to \mathfrak{A}'$ if, for all $\Sigma \in |\mathbf{Sign}|$,

$$\delta_\Sigma(T_\Sigma) \subseteq T'_{H(\Sigma)}.$$

This matrix family morphism is said to be **strict** if, for all $\Sigma \in |\mathbf{Sign}|$,

$$\delta_\Sigma(T_\Sigma) \subseteq T'_{H(\Sigma)} \quad \text{and} \quad \delta_\Sigma(\mathrm{SEN}(\Sigma)\backslash T_\Sigma) \subseteq \mathrm{SEN}'(H(\Sigma))\backslash T'_{H(\Sigma)}.$$

These conditions can be equivalently expressed by saying that, for all $\Sigma \in |\mathbf{Sign}|$,

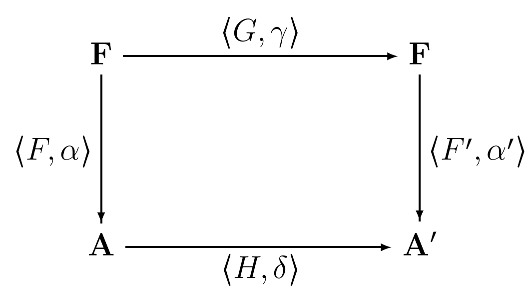
$$\delta_\Sigma^{-1}(T'_{H(\Sigma)}) = T_\Sigma.$$

They are also equivalent to the statement that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{if and only if} \quad \delta_\Sigma(\phi) \in T'_{H(\Sigma)}.$$

We have the following result relating strict morphisms between matrix families with strict morphisms between matrix families based on $\mathcal{F}$.

Lemma 58 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution, based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ and $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha'\rangle\rangle$ be two $\mathbf{F}$-algebraic systems and $\mathfrak{A} = \langle \mathcal{A}, T\rangle$ and $\mathfrak{A}' = \langle \mathcal{A}', T'\rangle$ two $\mathcal{I}$-matrix families. A matrix morphism $\langle\langle G, \gamma\rangle, \langle H, \delta\rangle\rangle : \mathfrak{A} \to \mathfrak{A}'$*



is strict if and only if $\langle G, \gamma\rangle : \langle \mathcal{F}, \alpha^{-1}(T)\rangle \to \langle \mathcal{F}, \alpha'^{-1}(T')\rangle$ is strict.

Proof: The statement follows by noticing that

$$\begin{array}{lll} \delta^{-1}(T') = T & \text{iff} & \alpha^{-1}(\delta^{-1}(T')) = \alpha^{-1}(T) \quad (\text{by the surjectivity of } \langle F, \alpha\rangle) \\ & \text{iff} & \gamma^{-1}(\alpha'^{-1}(T')) = \alpha^{-1}(T) \\ & & (\text{by the commutativity of the square}). \end{array}$$

Therefore $\langle\langle G,\gamma\rangle,\langle H,\delta\rangle\rangle : \mathfrak{A} \to \mathfrak{A}'$ is strict if and only if $\langle G,\gamma\rangle : \langle\mathcal{F},\alpha^{-1}(T)\rangle \to \langle\mathcal{F},\alpha'^{-1}(T')\rangle$ is strict. ■

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Given an $\mathbf{F}$-matrix family $\mathfrak{A} = \langle\mathcal{A}, T\rangle$, the **Leibniz reduction** of $\mathfrak{A}$, denoted $\mathfrak{A}^*$, is defined as

$$\mathfrak{A}^* = \langle\mathcal{A}^*, T^*\rangle = \langle\mathcal{A}^{\Omega^{\mathcal{A}}(T)}, T/\Omega^{\mathcal{A}}(T)\rangle,$$

where $\mathcal{A}^{\Omega^{\mathcal{A}}(T)}$ is the quotient $\mathbf{F}$-algebraic system of $\mathcal{A}$ by the congruence system $\Omega^{\mathcal{A}}(T)$ and $T/\Omega^{\mathcal{A}}(T) = \{T_\Sigma/\Omega^{\mathcal{A}}_\Sigma(T)\}_{\Sigma\in|\mathbf{Sign}|}$, with

$$T_\Sigma/\Omega^{\mathcal{A}}_\Sigma(T) = \{\phi/\Omega^{\mathcal{A}}_\Sigma(T) : \phi \in T_\Sigma\}.$$

An $\mathcal{I}$-matrix family $\mathfrak{A} = \langle\mathcal{A}, T\rangle$ is **Leibniz reduced** if

$$\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}.$$

An $\mathbf{F}$-algebraic system $\mathcal{A}$ is **Leibniz reduced** if it is the algebraic system reduct of a Leibniz reduced $\mathcal{I}$-matrix family.

We denote:

- the class of all Leibniz reduced $\mathcal{I}$-matrix families by $\mathrm{MatFam}^*(\mathcal{I})$;
- the class of all Leibniz reduced $\mathcal{I}$-matrix systems by $\mathrm{MatSys}^*(\mathcal{I})$;
- the class of all reduced $\mathbf{F}$-algebraic systems by $\mathrm{AlgSys}^*(\mathcal{I})$;
- the class of all system reduced $\mathbf{F}$-algebraic systems by $\mathrm{AlgSys}^\bullet(\mathcal{I})$;

i.e., we have:

$$\begin{array}{rcl}
\mathrm{MatFam}^*(\mathcal{I}) & = & \{\langle\mathcal{A},T\rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \text{ and } \Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}\};\\
\mathrm{MatSys}^*(\mathcal{I}) & = & \{\langle\mathcal{A},T\rangle : T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \text{ and } \Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}\};\\
\mathrm{AlgSys}^*(\mathcal{I}) & = & \{\mathcal{A} : (\exists T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}})\};\\
\mathrm{AlgSys}^\bullet(\mathcal{I}) & = & \{\mathcal{A} : (\exists T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}))(\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}})\}.
\end{array}$$

2.8 Axiomatic and Filter Extensions

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Given an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle\mathbf{A}, \langle F,\alpha\rangle\rangle$ and an $\mathbf{F}$-matrix family $\mathfrak{A} = \langle\mathcal{A}, T\rangle$, we set

$$\mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}) = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : T \leq T'\}.$$

$\mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$ is a complete sublattice of $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and we have $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$ if and only if $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. We call $\mathfrak{A}' = \langle\mathcal{A}, T'\rangle$ a **filter extension of**

$\mathfrak{A}$ if $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$. Sometimes, by slightly abusing notation, we write $\mathfrak{A}' \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$ in this case.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$, $\mathcal{I}' = \langle \mathbf{F}, C' \rangle$ be two π-institutions based on $\mathbf{F}$. $\mathcal{I}'$ is an **axiomatic extension** (or **axiomatic strengthening**) of $\mathcal{I}$ if there exists $X \in \mathrm{SenSys}(\mathbf{F})$, such that, for all $\Phi \in \mathrm{SenFam}(\mathbf{F})$,

$$C'(\Phi) = C(X \cup \Phi).$$

If this is the case, X is said to be a **system of axioms witnessing** the extension.

We provide now a characterization of axiomatic extensions.

Lemma 59 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$, $\mathcal{I}' = \langle \mathbf{F}, C' \rangle$ *be two* π*-institutions based on* $\mathbf{F}$. $\mathcal{I}'$ *is an axiomatic extension of* $\mathcal{I}$ *if and only if, for all* $\Phi \in \mathrm{SenFam}(\mathbf{F})$,

$$C'(\Phi) = C(\mathrm{Thm}(\mathcal{I}') \cup \Phi).$$

Proof: Assume, first, that $\mathcal{I}'$ is an axiomatic extension of $\mathcal{I}$, with witnessing system of axioms X. Then, we have $\mathrm{Thm}(\mathcal{I}') = C'(\varnothing) = C(X \cup \varnothing) = C(X)$. Therefore, for all $\Phi \in \mathrm{SenFam}(\mathbf{F})$,

$$C'(\Phi) = C(X \cup \Phi) = C(C(X) \cup \Phi) = C(\mathrm{Thm}(\mathcal{I}') \cup \Phi).$$

Assume conversely, that, for all $\Phi \in \mathrm{SenFam}(\mathbf{F})$, $C'(\Phi) = C(\mathrm{Thm}(\mathcal{I}') \cup \Phi)$. Then $X = \mathrm{Thm}(\mathcal{I}')$ is a system of axioms witnessing the fact that $\mathcal{I}'$ is an axiomatic extension of $\mathcal{I}$. ■

We also have the following characterization in terms of $\mathcal{I}$- and $\mathcal{I}'$-filter families and corresponding theory families.

Proposition 60 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$, $\mathcal{I}' = \langle \mathbf{F}, C' \rangle$ *be two* π*-institutions based on* $\mathbf{F}$. *The following statements are equivalent:*

(i) $\mathcal{I}'$ *is an axiomatic extension of* $\mathcal{I}$*;*

(ii) For all $\mathfrak{A} \in \mathrm{MatFam}(\mathcal{I}')$, $\mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}) = \mathrm{FiFam}^{\mathcal{I}'}(\mathfrak{A})$*;*

(iii) For all $T' \in \mathrm{ThFam}(\mathcal{I}')$ *and* $T' \le T \in \mathrm{SenFam}(\mathbf{F})$,

$$T \in \mathrm{ThFam}(\mathcal{I}) \quad \textit{if and only if} \quad T \in \mathrm{ThFam}(\mathcal{I}').$$

Proof:

(i)⇒(ii) Suppose that $\mathcal{I}'$ is an axiomatic extension of $\mathcal{I}$ and let $\mathfrak{A} = \langle \mathcal{A}, T' \rangle \in \mathrm{MatFam}(\mathcal{I}')$. Since $C \leq C'$, we have $\mathrm{FiFam}^{\mathcal{I}'}(\mathfrak{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$. So suppose that $T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$, i.e., $T' \leq T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C'_\Sigma(\Phi)$ and $\alpha_\Sigma(\Phi) \subseteq T''_{F(\Sigma)}$. Since $\phi \in C'_\Sigma(\Phi)$, by Lemma 59, $\phi \in C_\Sigma(\mathrm{Thm}_\Sigma(\mathcal{I}') \cup \Phi)$. Now observe that $\alpha_\Sigma(\mathrm{Thm}_\Sigma(\mathcal{I}')) \subseteq T'_{F(\Sigma)} \subseteq T''_{F(\Sigma)}$, since $T' \in \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A})$. Thus, we get

$$\alpha_\Sigma(\mathrm{Thm}_\Sigma(\mathcal{I}') \cup \Phi)) \subseteq T''_{F(\Sigma)}.$$

Hence, since $T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get that $\alpha_\Sigma(\phi) \in T''_{F(\Sigma)}$. So $T'' \in \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A})$. And, since $T' \leq T''$, $T'' \in \mathrm{FiFam}^{\mathcal{I}'}(\mathfrak{A})$.

(ii)⇒(iii) Let $\mathfrak{A} = \langle \mathcal{F}, T' \rangle \in \mathrm{MatFam}(\mathcal{I}')$. Then, by hypothesis, for all $T' \leq T$, we have $\langle \mathcal{F}, T \rangle \in \mathrm{MatFam}^{\mathcal{I}}(\mathfrak{A})$ iff $\langle \mathcal{F}, T \rangle \in \mathrm{MatFam}^{\mathcal{I}'}(\mathfrak{A})$, i.e., $T \in \mathrm{ThFam}(\mathcal{I})$ iff $T \in \mathrm{ThFam}(\mathcal{I}')$.

(iii)⇒(i) First, note that (iii) implies that $\mathrm{ThFam}(\mathcal{I}') \subseteq \mathrm{ThFam}(\mathcal{I})$ and, therefore, $C \leq C'$. We use this to show that, for all $X \in \mathrm{SenFam}(\mathbf{F})$,

$$C'(X) = C(\mathrm{Thm}(\mathcal{I}') \cup X).$$

From left to right, note that $\mathrm{Thm}(\mathcal{I}') \subseteq C(\mathrm{Thm}(\mathcal{I}')) \subseteq C(\mathrm{Thm}(\mathcal{I}') \cup X)$. So, by hypothesis, $C(\mathrm{Thm}(\mathcal{I}') \cup X) \in \mathrm{ThFam}(\mathcal{I}')$. Thus, we get

$$C'(X) \subseteq C'(C(\mathrm{Thm}(\mathcal{I}') \cup X)) = C(\mathrm{Thm}(\mathcal{I}') \cup X).$$

On the other hand, $C(\mathrm{Thm}(\mathcal{I}') \cup X) \subseteq C'(\mathrm{Thm}(\mathcal{I}') \cup X) = C'(X)$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and $\mathfrak{A} = \langle \mathcal{A}, T \rangle \in \mathrm{MatFam}(\mathcal{I})$. Define, for all $\Phi \in \mathrm{SenFam}(\mathcal{A})$,

$$C^{\mathcal{I},\mathfrak{A}}(\Phi) = C^{\mathcal{I},\mathcal{A}}(T \cup \Phi).$$

$C^{\mathcal{I},\mathfrak{A}}(\Phi)$ is the **$\mathcal{I}$-filter family of $\mathfrak{A}$ generated by** Φ.

We have, for all $\Phi \in \mathrm{SenFam}(\mathcal{A})$, $T \leq C^{\mathcal{I},\mathfrak{A}}(\Phi)$. In the special case where $\mathcal{A} = \mathcal{F}$ and $\mathfrak{A} = \mathfrak{F} = \langle \mathcal{F}, T \rangle \in \mathrm{MatFam}(\mathcal{I})$, we get, for all $\Phi \in \mathrm{SenFam}(\mathbf{F})$,

$$C^{\mathfrak{F}}(\Phi) = C(T \cup \Phi).$$

The following proposition gives many properties governing filter family generation and the interaction with surjective morphisms between $\mathcal{I}$-matrix families.

Proposition 61 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$, $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, $\mathfrak{A}' = \langle \mathcal{A}', T' \rangle$ *be* $\mathbf{F}$*-matrix families,* $\langle H, \gamma \rangle : \mathfrak{A} \to \mathfrak{A}'$ *a surjective morphism and* $X \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$, $Y, Y' \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}')$.

(a) $\gamma^{-1}(Y) \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$*;*

(b) If H *is an isomorphism,* $C^{\mathcal{I},\mathfrak{A}'}(\gamma(X)) \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}')$*;*

(c) If H *is an isomorphism,* $C^{\mathcal{I},\mathfrak{A}'}(\gamma(\gamma^{-1}(Y))) = \gamma(\gamma^{-1}(Y)) = Y$*;*

(d) If H *is an isomorphism,* $\gamma^{-1}(C^{\mathcal{I},\mathfrak{A}'}(\gamma(X))) = \gamma^{-1}(\gamma(X)) = X$ *if and only if* $\gamma^{-1}(T') \leq X$ *and* $\mathrm{Ker}(\langle H,\gamma\rangle)$ *is compatible with* X*;*

(e) $\gamma^{-1}(Y \cap Y') = \gamma^{-1}(Y) \cap \gamma^{-1}(Y')$*;*

(f) If H *is an isomorphism, for all* $\Phi \in \mathrm{SenFam}(\mathcal{A})$*,* $C^{\mathcal{I},\mathfrak{A}'}(\gamma(C^{\mathcal{I},\mathfrak{A}}(\Phi))) = C^{\mathcal{I},\mathfrak{A}'}(\gamma(\Phi))$*.*

Proof:

(a) We know, by Corollary 55, that $\gamma^{-1}(Y) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. In addition, $T \leq \gamma^{-1}(T') \leq \gamma^{-1}(Y)$. So we get $\gamma^{-1}(Y) \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$.

(b) It is obvious that $C^{\mathcal{I},\mathfrak{A}'}(\gamma(X)) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$. Moreover, by definition, $T' \leq C^{\mathcal{I},\mathfrak{A}'}(\gamma(X))$. So, we get $C^{\mathcal{I},\mathfrak{A}'}(\gamma(X)) \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}')$.

(c) We have

$$\begin{array}{rcll} C^{\mathcal{I},\mathfrak{A}'}(\gamma(\gamma^{-1}(Y))) & = & C^{\mathcal{I},\mathfrak{A}'}(Y) & (\langle H,\gamma\rangle \text{ surjective}) \\ & = & Y. & (Y \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}')) \end{array}$$

(d) Assume, first, that $\gamma^{-1}(C^{\mathcal{I},\mathfrak{A}'}(\gamma(X))) = \gamma^{-1}(\gamma(X)) = X$. Then, by surjectivity of $\langle H,\gamma\rangle$, $C^{\mathcal{I},\mathfrak{A}'}(\gamma(X)) = \gamma(X)$. This implies that $T' \leq \gamma(X)$, whence $\gamma^{-1}(T') \leq X$. To show compatibility, suppose $\Sigma \in |\mathbf{Sign}|$, $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\gamma_\Sigma(\phi) = \gamma_\Sigma(\psi)$ and $\phi \in X_\Sigma$. Then, we have

$$\psi \in \gamma_\Sigma^{-1}(\gamma_\Sigma(\phi)) \subseteq \gamma_\Sigma^{-1}(\gamma_\Sigma(X_\Sigma)) = X_\Sigma.$$

So $\mathrm{Ker}(\langle H,\gamma\rangle)$ is compatible with X.

Assume, conversely, that $\gamma^{-1}(T') \leq X$ and that $\mathrm{Ker}(\langle H,\gamma\rangle)$ is compatible with X. Then, by compatibility, $\gamma^{-1}(\gamma(X)) = X \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$. Thus, by Corollary 55, $\gamma(X) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$. But we also have $T' \leq \gamma(X)$, whence $\gamma(X) \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}')$. Now we get $\gamma^{-1}(C^{\mathcal{I},\mathfrak{A}'}(\gamma(X))) = \gamma^{-1}(\gamma(X)) = X$.

(e) This follows from set theory.

(f) Let $\Phi \in \mathrm{SenFam}(\mathcal{A})$. Clearly, $C^{\mathcal{I},\mathfrak{A}'}(\gamma(\Phi)) \leq C^{\mathcal{I},\mathfrak{A}'}(\gamma(C^{\mathcal{I},\mathfrak{A}}(\Phi)))$, since $C^{\mathcal{I},\mathfrak{A}'}(\gamma(C^{\mathcal{I},\mathfrak{A}}(\Phi)))$ is an $\mathcal{I}$-filter family of $\mathfrak{A}'$ including $\gamma(\Phi)$.

To show the reverse inclusion, assume $Y \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}')$, such that $\gamma(\Phi) \le Y$. Then $\Phi \le \gamma^{-1}(Y)$. Thus, $C^{\mathcal{I},\mathfrak{A}}(\Phi) \le C^{\mathcal{I},\mathfrak{A}}(\gamma^{-1}(Y)) = \gamma^{-1}(Y)$, the equality following by Part (a). Hence, we get

$$C^{\mathcal{I},\mathfrak{A}'}(\gamma(C^{\mathcal{I},\mathfrak{A}}(\Phi))) \le C^{\mathcal{I},\mathfrak{A}'}(\gamma(\gamma^{-1}(Y))) = C^{\mathcal{I},\mathfrak{A}'}(Y) = Y.$$

Since $C^{\mathcal{I},\mathfrak{A}'}(\gamma(C^{\mathcal{I},\mathfrak{A}}(\Phi))) \le Y$ holds, for all $Y \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}')$, such that $\gamma(\Phi) \le Y$, we get, in particular, $C^{\mathcal{I},\mathfrak{A}'}(\gamma(C^{\mathcal{I},\mathfrak{A}}(\Phi))) \le C^{\mathcal{I},\mathfrak{A}'}(\gamma(\Phi))$. ∎

2.9 Generalized Matrix Families and Systems

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. A **generalized F-matrix family**, or **F-gmatrix family** for short, is a pair $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$, where $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ is an $\mathbf{F}$-algebraic system and $\mathcal{T} \subseteq \mathrm{SenFam}(\mathcal{A})$ is a collection of sentence families of $\mathcal{A}$.

An $\mathbf{F}$-gmatrix family $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$ is said to be an **F-gmatrix system** if $\mathcal{T} \subseteq \mathrm{SenSys}(\mathcal{A})$.

Given an $\mathbf{F}$-gmatrix family $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$, the **Tarski congruence system of** $\mathbb{A}$ (or **of** $\mathcal{T}$ **on** $\mathcal{A}$), denoted $\widetilde{\Omega}(\mathbb{A})$ or $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})$, is the largest congruence system on $\mathcal{A}$ that is compatible with all sentence families in $\mathcal{T}$.

Lemma 62 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. Then, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $\mathcal{T} \subseteq \mathrm{SenFam}(\mathcal{A})$,*

$$\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T).$$

Proof: Note that, by definition, $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})$ is compatible with every $T \in \mathcal{T}$. Therefore, since $\Omega^{\mathcal{A}}(T)$ is the largest congruence system on $\mathcal{A}$ compatible with T, we get that

$$\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \le \Omega^{\mathcal{A}}(T), \text{ for all } T \in \mathcal{T}.$$

Thus, $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \le \bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)$.

For the reverse inclusion, note that $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)$ is a congruence system on $\mathcal{A}$ that is compatible with every $T \in \mathcal{T}$. Therefore, since $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})$ is the largest such congruence system, we get that $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \le \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})$. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. An $\mathbf{F}$-gmatrix family $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$ is called a **generalized $\mathcal{I}$-matrix family**, or **$\mathcal{I}$-gmatrix family** for short, if $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

We have a special notation for the Tarski congruence systems, when applied and/or relativized to the collection of all $\mathcal{I}$-filter families:

$$\widetilde{\Omega}^{\mathcal{A}}(\mathcal{I}) := \widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})).$$

Recall that $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$, where $\langle I, \iota \rangle : \mathbf{F} \to \mathbf{F}$ is the identity morphism. We set

$$\widetilde{\Omega}(\mathcal{I}) := \widetilde{\Omega}^{\mathcal{F}}(\mathcal{I}) = \widetilde{\Omega}^{\mathcal{F}}(\mathrm{ThFam}(\mathcal{I})).$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution, based on $\mathbf{F}$, and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system. Given an $\mathbf{F}$-gmatrix family $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$, the **Tarski reduction** of $\mathbb{A}$, denoted $\mathbb{A}^*$, is defined as

$$\mathbb{A}^* = \langle \mathcal{A}^*, \mathcal{T}^* \rangle = \langle \mathcal{A}^{\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})}, \mathcal{T}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \rangle,$$

where $\mathcal{A}^{\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})}$ is the quotient $\mathbf{F}$-algebraic system of $\mathcal{A}$ by the Tarski congruence system $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})$ and

$$\mathcal{T}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \{T/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) : T \in \mathcal{T}\},$$

with $T/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \{T_\Sigma/\widetilde{\Omega}^{\mathcal{A}}_\Sigma(\mathcal{T})\}_{\Sigma \in |\mathbf{Sign}|}$ such that, for all $\Sigma \in |\mathbf{Sign}|$,

$$T_\Sigma/\widetilde{\Omega}^{\mathcal{A}}_\Sigma(\mathcal{T}) = \{\phi/\widetilde{\Omega}^{\mathcal{A}}_\Sigma(\mathcal{T}) : \phi \in T_\Sigma\}.$$

An $\mathbf{F}$-gmatrix family $\mathbb{A} = \langle \mathcal{A}, \mathcal{T} \rangle$ is **Tarski reduced** if $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathcal{A}}$. An $\mathbf{F}$-algebraic system $\mathcal{A}$ is **Tarski reduced** if it is the algebraic system reduct of a Tarski reduced $\mathbf{F}$-gmatrix family.

We denote:

- the class of all Tarski reduced $\mathcal{I}$-gmatrix families by $\mathrm{GMatFam}^*(\mathcal{I})$;
- the corresponding class of all Tarski reduced $\mathbf{F}$-algebraic systems by $\mathrm{AlgSys}(\mathcal{I})$,

i.e., we have:

$$\begin{array}{rcl}\mathrm{GMatFam}^*(\mathcal{I}) & = & \{\langle \mathcal{A}, \mathcal{T} \rangle : \mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \text{ and } \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathcal{A}}\};\\ \mathrm{AlgSys}(\mathcal{I}) & = & \{\mathcal{A} : (\exists \mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathcal{A}})\}.\end{array}$$

Consider again an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$, an $\mathbf{F}$-algebraic system $\mathcal{A}$ and $\mathcal{T} \subseteq \mathrm{SenFam}(\mathcal{A})$. The **Suszko congruence system of** $T \in \mathcal{T}$ (**relative to** $\mathcal{T}$), denoted by $\widetilde{\Omega}^{\mathcal{A},\mathcal{T}}(T)$, is the largest congruence system on $\mathcal{A}$ that is compatible with all $T' \in \mathcal{T}$, such that $T \leq T'$.

Lemma 63 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system. Then, for all* $\mathbf{F}$-*algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, *all* $\mathcal{T} \subseteq \mathrm{SenFam}(\mathcal{A})$ *and all* $T \in \mathcal{T}$,

$$\widetilde{\Omega}^{\mathcal{A},\mathcal{T}}(T) = \bigcap_{T \leq T' \in \mathcal{T}} \Omega^{\mathcal{A}}(T').$$

Proof: The proof is similar to that of Lemma 62. ■

We note also the following relation between the Suszko congruence system of T relative to $\mathcal{T}$ and the Tarski congruence system of $\mathcal{T}^T = \{T' \in \mathcal{T} : T \leq T'\}$:

$$\widetilde{\Omega}^{\mathcal{A},\mathcal{T}}(T) = \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}^T).$$

We also have some special notations reserved for the Suszko congruence systems, when applied and/or relativized to $\mathrm{ThFam}(\mathcal{I})$ and to all $\mathcal{I}$-filter families.

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathcal{I}}(T) & := & \widetilde{\Omega}^{\mathcal{F},\mathrm{ThFam}(\mathcal{I})}(T), \text{ for all } T \in \mathrm{ThFam}(\mathcal{I});\\ \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) & := & \widetilde{\Omega}^{\mathcal{A},\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})}(T), \text{ for all } T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}).\end{array}$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Given an $\mathcal{I}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, the **Suszko reduction** of $\mathfrak{A}$, denoted $\mathfrak{A}^{\mathsf{Su}}$, is defined as

$$\mathfrak{A}^{\mathsf{Su}} = \langle \mathcal{A}^{\mathsf{Su}}, T^{\mathsf{Su}} \rangle = \langle \mathcal{A}^{\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}, T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \rangle,$$

where $\mathcal{A}^{\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}$ is the quotient $\mathbf{F}$-algebraic system of $\mathcal{A}$ by the Suszko congruence system $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$ and $T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \{T_\Sigma/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}_\Sigma(T)\}_{\Sigma \in |\mathbf{Sign}|}$, with

$$T_\Sigma/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}_\Sigma(T) = \{\phi/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}_\Sigma(T) : \phi \in T_\Sigma\}.$$

An $\mathcal{I}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ is **Suszko reduced** if $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. An $\mathbf{F}$-algebraic system $\mathcal{A}$ is **Suszko reduced** if it is the algebraic system reduct of a Suszko reduced $\mathcal{I}$-matrix family.

It turns out that, relative to a given π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, the classes of Tarski reduced $\mathbf{F}$-algebraic systems and of Suszko reduced $\mathbf{F}$-algebraic systems coincide.

Proposition 64 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution, based on* $\mathbf{F}$*, and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system.* $\mathcal{A}$ *is Suszko reduced if and only if* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*.*

Proof: Suppose, first, that $\mathcal{A}$ is a Suszko reduced $\mathbf{F}$-algebraic system. Then, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. But then we have

$$\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \subseteq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Delta^{\mathcal{A}}.$$

Hence $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle \in \mathrm{GMatFam}^*(\mathcal{I})$ and, consequently, $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$.

Suppose, conversely, that $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$. Thus, by definition, there exists $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathcal{A}}$. Now we get

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(\bigcap \mathcal{T}) & = & \widetilde{\Omega}^{\mathcal{A}}(\{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \bigcap \mathcal{T} \leq T\})\\ & \leq & \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathcal{A}}.\end{array}$$

Since $\bigcap\mathcal{T} \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get that $\langle\mathcal{A},\bigcap\mathcal{T}\rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})$ and, consequently, $\mathcal{A}$ is Suszko reduced. ■

We let $\mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})$ be the class of all Suszko reduced $\mathcal{I}$-matrix families, i.e., we have

$$\mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I}) = \{\langle\mathcal{A},T\rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \text{ and } \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Delta^{\mathcal{A}}\},$$

whereas, because of Proposition 64, there is no reason for introducing fresh notation for the class of all Suszko reduced **F**-algebraic systems, that class being $\mathrm{AlgSys}(\mathcal{I})$.

2.10 The Algebraic Systems of a π-Institution

We have introduced in Sections 2.7 and 2.9 two of the most important classes of **F**-algebraic systems associated to a π-institution $\mathcal{I} = \langle\mathbf{F},C\rangle$, namely, the classes $\mathrm{AlgSys}^*(\mathcal{I})$ and $\mathrm{AlgSys}(\mathcal{I})$. In this section, we introduce two more classes and consider some of the relationships that hold between them.

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F},C\rangle$ a π-institution based on **F**. The **semantic variety of** $\mathcal{I}$ is the semantic variety generated by the algebraic system $\mathcal{F}/\widetilde{\Omega}(\mathcal{I})$, i.e., the class

$$\begin{array}{rcl}\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}) & := & \mathbb{V}^{\mathsf{Sem}}(\mathcal{F}/\widetilde{\Omega}(\mathcal{I}))\\ & = & \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : \widetilde{\Omega}(\mathcal{I}) \le \mathrm{Ker}(\mathcal{A})\}.\end{array}$$

The **syntactic variety of** $\mathcal{I}$ is the syntactic variety generated by $\mathcal{F}/\widetilde{\Omega}(\mathcal{I})$, i.e., the class defined by

$$\begin{array}{rcl}\mathbb{V}^{\mathsf{Syn}}(\mathcal{I}) & := & \mathbb{V}^{\mathsf{Syn}}(\mathcal{F}/\widetilde{\Omega}(\mathcal{I}))\\ & = & \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : (\forall\sigma^\flat \approx \tau^\flat \in \mathrm{NEq}(\mathbf{F}))\\ & & \qquad (\mathcal{F}/\widetilde{\Omega}(\mathcal{I}) \vDash \sigma^\flat \approx \tau^\flat \;\Rightarrow\; \mathcal{A} \vDash \sigma^\flat \approx \tau^\flat)\}.\end{array}$$

We can say a few things about the relationships governing the four classes of **F**-algebraic systems associated with a given π-institution $\mathcal{I} = \langle\mathbf{F},C\rangle$.

Proposition 65 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F},C\rangle$ *a* π*-institution based on* **F**. *Then we have*

$$\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I}) \subseteq \mathbb{V}^{\mathsf{Sem}}(\mathcal{I}) \subseteq \mathbb{V}^{\mathsf{Syn}}(\mathcal{I}).$$

Proof: Suppose, first, that $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I})$. Then there exists an $\mathcal{I}$-filter family $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. Hence, we get

$$\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \le \Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}.$$

It follows that $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$.

Suppose, next, that $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$. Thus, there exists $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathcal{A}}$. This implies that $\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \leq \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathcal{A}}$, i.e., that $\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}$. Applying the inverse of the surjective morphism $\langle F, \alpha \rangle$, we get $\alpha^{-1}(\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \alpha^{-1}(\Delta^{\mathcal{A}}) = \mathrm{Ker}(\mathcal{A})$. Therefore, we obtain

$$\begin{array}{rcll}
\widetilde{\Omega}(\mathcal{I}) & = & \bigcap_{T \in \mathrm{ThFam}(\mathcal{I})} \Omega(T) & \text{(by Lemma 62)} \\
& \leq & \bigcap_{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})} \Omega(\alpha^{-1}(T)) & \text{(by Lemma 51)} \\
& = & \bigcap_{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})} \alpha^{-1}(\Omega^{\mathcal{A}}(T)) & \text{(by Proposition 24)} \\
& = & \alpha^{-1}(\bigcap_{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})} \Omega^{\mathcal{A}}(T)) & \text{(set theory))} \\
& = & \alpha^{-1}(\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))) & \text{(by Lemma 62)} \\
& = & \mathrm{Ker}(\mathcal{A}). & \text{(as shown above)}
\end{array}$$

Hence $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathcal{I})$.

The last inclusion follows from Theorem 39. ∎

Finally, it can be shown that all four classes generate the same syntactic variety. We first prove a technical lemma that simplifies some algebraic computations.

Lemma 66 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, an* $\mathbf{F}$*-algebraic system,* $\theta \in \mathrm{ConSys}(\mathcal{A})$ *and* $\sigma^\flat \approx \tau^\flat$ *an* $\mathbf{F}$*-equation. Then*

$$\begin{array}{rl}
\mathcal{A}/\theta \vDash \sigma^\flat \approx \tau^\flat \quad \textit{iff} & \textit{for all } \Sigma \in |\mathbf{Sign}^\flat|,\ \vec{\phi} \in \mathrm{SEN}^\flat(\Sigma), \\
& \langle \alpha_\Sigma(\sigma^\flat_\Sigma(\vec{\phi})), \alpha_\Sigma(\tau^\flat_\Sigma(\vec{\phi})) \rangle \in \theta_{F(\Sigma)}.
\end{array}$$

Proof: We have, by definition, $\mathcal{A}/\theta \vDash \sigma^\flat \approx \tau^\flat$ iff, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\alpha^\theta_\Sigma(\sigma^\flat_\Sigma(\vec{\phi})) = \alpha^\theta_\Sigma(\tau^\flat_\Sigma(\vec{\phi}))$$

iff, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\alpha_\Sigma(\sigma^\flat_\Sigma(\vec{\phi}))/\theta_{F(\Sigma)} = \alpha_\Sigma(\tau^\flat_\Sigma(\vec{\phi}))/\theta_{F(\Sigma)}$$

iff, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \alpha_\Sigma(\sigma^\flat_\Sigma(\vec{\phi})), \alpha_\Sigma(\tau^\flat_\Sigma(\vec{\phi})) \rangle \in \theta_{F(\Sigma)}.$$

∎

Theorem 67 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then*

$$\mathbb{V}^{\mathsf{Syn}}(\mathrm{AlgSys}^*(\mathcal{I})) = \mathbb{V}^{\mathsf{Syn}}(\mathrm{AlgSys}(\mathcal{I})) = \mathbb{V}^{\mathsf{Syn}}(\mathbb{V}^{\mathsf{Sem}}(\mathcal{I})) = \mathbb{V}^{\mathsf{Syn}}(\mathcal{I}).$$

Proof: By Propositions 65 and 38, we have

$$\mathbb{V}^{\mathsf{Syn}}(\mathrm{AlgSys}^*(\mathcal{I})) \subseteq \mathbb{V}^{\mathsf{Syn}}(\mathrm{AlgSys}(\mathcal{I})) \subseteq \mathbb{V}^{\mathsf{Syn}}(\mathbb{V}^{\mathsf{Sem}}(\mathcal{I})) \subseteq \mathbb{V}^{\mathsf{Syn}}(\mathcal{I}).$$

To conclude the proof we need to show that

$$\mathbb{V}^{\mathsf{Syn}}(\mathcal{I}) \subseteq \mathbb{V}^{\mathsf{Syn}}(\mathrm{AlgSys}^*(\mathcal{I})).$$

To this end, suppose $\mathcal{A} \in \mathbb{V}^{\mathsf{Syn}}(\mathcal{I})$, i.e., that, for every natural $\mathbf{F}$-equation $\sigma^\flat \approx \tau^\flat$,

$$\mathcal{F}/\widetilde{\Omega}(\mathcal{I}) \vDash \sigma^\flat \approx \tau^\flat \quad \text{implies} \quad \mathcal{A} \vDash \sigma^\flat \approx \tau^\flat.$$

To show that $\mathcal{A} \in \mathbb{V}^{\mathsf{Syn}}(\mathrm{AlgSys}^*(\mathcal{I}))$, suppose that $\sigma^\flat \approx \tau^\flat$ is an $\mathbf{F}$-equation, such that $\mathrm{AlgSys}^*(\mathcal{I}) \vDash \sigma^\flat \approx \tau^\flat$. In particular, for all $T \in \mathrm{ThFam}(\mathcal{I})$, we have that $\mathcal{F}/\Omega(T) \vDash \sigma^\flat \approx \tau^\flat$. This means, by Lemma 66, that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$, $\langle \sigma^\flat_\Sigma(\vec{\phi}), \tau^\flat_\Sigma(\vec{\phi})\rangle \in \Omega_\Sigma(T)$. Since this holds for all $T \in \mathrm{ThFam}(\mathcal{I})$, we get that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \sigma^\flat_\Sigma(\vec{\phi}), \tau^\flat_\Sigma(\vec{\phi})\rangle \in \bigcap_{T \in \mathrm{ThFam}(\mathcal{I})} \Omega_\Sigma(T) = \widetilde{\Omega}_\Sigma(\mathcal{I}).$$

Thus, again by Lemma 66, $\mathcal{F}/\widetilde{\Omega}(\mathcal{I}) \vDash \sigma^\flat \approx \tau^\flat$. Therefore, by hypothesis, $\mathcal{A} \vDash \sigma^\flat \approx \tau^\flat$. We conclude that $\mathcal{A} \in \mathbb{V}^{\mathsf{Syn}}(\mathrm{AlgSys}^*(\mathcal{I}))$ and, hence, $\mathbb{V}^{\mathsf{Syn}}(\mathcal{I}) \subseteq \mathbb{V}^{\mathsf{Syn}}(\mathrm{AlgSys}^*(\mathcal{I}))$. ■

We close this section by showing that, given a π-institution $\mathcal{I}$, the class of Tarski reduced algebraic systems $\mathrm{AlgSys}(\mathcal{I})$ is closed under the operator $\overset{\triangleleft}{\mathbb{P}}$ and contains a trivial $\mathbf{F}$-algebraic system and, therefore, by Proposition 28, it makes sense, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, to consider the relative congruence system $\Theta^{\mathrm{AlgSys}(\mathcal{I}),\mathcal{A}}(X)$ on $\mathcal{A}$ generated by a relation family $X \in \mathrm{RelFam}(\mathcal{A})$.

Proposition 68 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a π-institution based on* $\mathbf{F}$. *The class of* $\mathbf{F}$*-algebraic systems* $\mathrm{AlgSys}(\mathcal{I})$ *is closed under subdirect intersections and contains a trivial* $\mathbf{F}$*-algebraic system.*

Proof: It is clear that $\mathrm{AlgSys}(\mathcal{I})$ contains a trivial $\mathbf{F}$-algebraic system $\mathcal{A}$, since $\Delta^{\mathcal{A}} = \nabla^{\mathcal{A}}$ is the only congruence system on $\mathcal{A}$. So it suffices to show that $\mathrm{AlgSys}(\mathcal{I})$ is closed under subdirect intersections. To this end, let

$$\langle H^i, \gamma^i\rangle : \mathcal{A} \to \mathcal{A}^i, \quad i \in I,$$

be a subdirect intersection, with $\mathcal{A}^i \in \mathrm{AlgSys}(\mathcal{I})$, for all $i \in I$. Thus, by definition, we have, on the one hand, that

$$\bigcap_{i \in I} \mathrm{Ker}(\langle H^i, \gamma^i\rangle) = \Delta^{\mathcal{A}},$$

and on the other, that, for all $i \in I$, there exists $\mathcal{T}^i \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^i)$, such that

$$\widetilde{\Omega}^{\mathcal{A}^i}(\mathcal{T}^i) = \Delta^{\mathcal{A}^i}.$$

Now we obtain

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) & \leq & \widetilde{\Omega}^{\mathcal{A}}(\bigcup_{i\in I}(\gamma^i)^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^i))) \\ & = & \bigcap_{i\in I}\widetilde{\Omega}^{\mathcal{A}}((\gamma^i)^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^i))) \\ & = & \bigcap_{i\in I}(\gamma^i)^{-1}(\widetilde{\Omega}^{\mathcal{A}^i}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^i))) \\ & \leq & \bigcap_{i\in I}(\gamma^i)^{-1}(\widetilde{\Omega}^{\mathcal{A}^i}(\mathcal{T}^i)) \\ & = & \bigcap_{i\in I}(\gamma^i)^{-1}(\Delta^{\mathcal{A}^i}) \\ & = & \bigcap_{i\in I}\mathrm{Ker}(\langle H^i,\gamma^i\rangle) \\ & = & \Delta^{\mathcal{A}}.\end{array}$$

Therefore, $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, showing that $\overset{\triangleleft}{\mathbb{P}}(\mathrm{AlgSys}(\mathcal{I})) \subseteq \mathrm{AlgSys}(\mathcal{I})$. ■

Based on Proposition 68 and Proposition 28, we define, for every **F**-algebraic system $\mathcal{A}$, and all $X \in \mathrm{RelFam}(\mathcal{A})$,

$$\Theta^{\mathcal{I},\mathcal{A}}(X) := \Theta^{\mathrm{AlgSys}(\mathcal{I}),\mathcal{A}}(X).$$

2.11 Frege Relations

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a sentence functor and $T \in \mathrm{SenFam}(\mathrm{SEN})$. We define:

- The **Frege relation system** $\Lambda(T) = \{\Lambda_\Sigma(T)\}_{\Sigma\in|\mathbf{Sign}|}$ of T on SEN by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\begin{array}{rcl}\Lambda_\Sigma(T) & = & \{\langle\phi,\psi\rangle \in \mathrm{SEN}(\Sigma)^2 : \text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma'), \\ & & \mathrm{SEN}^\flat(f)(\phi) \in T_{\Sigma'} \iff \mathrm{SEN}^\flat(f)(\psi) \in T_{\Sigma'}\};\end{array}$$

- The **Frege relation family** $\lambda(T) = \{\lambda_\Sigma(T)\}_{\Sigma\in|\mathbf{Sign}|}$ of T on SEN by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\lambda_\Sigma(T) = \{\langle\phi,\psi\rangle \in \mathrm{SEN}(\Sigma)^2 : (\phi \in T_\Sigma \iff \psi \in T_\Sigma)\}.$$

It turns out that the Frege relation system of T on SEN is an equivalence system, the Frege relation family of T on SEN is an equivalence family and that the former is the largest equivalence system included in the latter.

Proposition 69 *Let* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a sentence functor and* $T \in \mathrm{SenFam}(\mathrm{SEN})$.

(a) $\Lambda(T)$ *is an equivalence system on* SEN*;*

(b) $\lambda(T)$ *is an equivalence family on* SEN*;*

(c) $\Lambda(T)$ *is the largest equivalence system included in* $\lambda(T)$*.*

Proof:

(a) That $\Lambda(T)$ is an equivalence family, i.e., that, for all $\Sigma \in |\mathbf{Sign}|$, $\Lambda_\Sigma(T)$ is an equivalence relation, is straightforward. To see that it is a system, i.e., invariant under signature morphisms, let $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \Lambda_\Sigma(T)$, and $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$. Then, we have, for all $\Sigma'' \in |\mathbf{Sign}|$ and all $g \in \mathbf{Sign}(\Sigma', \Sigma'')$,

$$\Sigma \xrightarrow{\ f\ } \Sigma' \xrightarrow{\ g\ } \Sigma''$$

$\mathrm{SEN}(gf)(\phi) \in T_{\Sigma''}$ iff $\mathrm{SEN}(gf)(\psi) \in T_{\Sigma''}$, whence, we derive that , for all $g \in \mathbf{Sign}(\Sigma', \Sigma'')$,

$$\mathrm{SEN}(g)(\mathrm{SEN}(f)(\phi)) \in T_{\Sigma''} \quad \text{iff} \quad \mathrm{SEN}(g)(\mathrm{SEN}(f)(\psi)) \in T_{\Sigma''}.$$

This shows that $\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi) \rangle \in T_{\Sigma'}$. Thus, $\Lambda(T)$ is an equivalence system.

(b) This part is straightforward.

(c) It is clear that $\Lambda(T) \leq \lambda(T)$, simply by considering, in the definition of $\Lambda(T)$, the particular case where $\Sigma' = \Sigma$ and $f = i_\Sigma : \Sigma \to \Sigma$ is the identity signature morphism. Suppose, next, that θ is an equivalence system, such that $\theta \leq \lambda(T)$. We must show that $\theta \leq \Lambda(T)$. To this end, let $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \theta_\Sigma$. Since θ is a system, we get, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi) \rangle \in \theta_{\Sigma'}$. Hence, since $\theta \leq \lambda(T)$, we conclude that $\mathrm{SEN}(f)(\phi) \in T_{\Sigma'}$ iff $\mathrm{SEN}(f)(\psi) \in T_{\Sigma'}$. Therefore, by definition, $\langle \phi, \psi \rangle \in \Lambda_\Sigma(T)$. Thus, $\theta \leq \Lambda(T)$ and $\Lambda(T)$ is indeed the largest equivalence system included in $\lambda(T)$.

■

There is also a close relationship between the two Frege equivalence families and the Leibniz congruence system of a sentence family. In case SEN is the underlying sentence functor of an algebraic system $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, we sometimes write $\Lambda^{\mathbf{A}}(T)$ and $\lambda^{\mathbf{A}}(T)$ for the relation families $\Lambda(T)$ and $\lambda(T)$, respectively.

Proposition 70 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *be an algebraic system and* $T \in \mathrm{SenFam}(\mathbf{A})$*.*

(a) $\Omega^{\mathbf{A}}(T)$ *is the largest congruence system contained in* $\lambda^{\mathbf{A}}(T)$*;*

(b) $\Omega^{\mathbf{A}}(T)$ *is the largest congruence system contained in* $\Lambda^{\mathbf{A}}(T)$.

Proof:

(a) By definition $\Omega^{\mathbf{A}}(T)$ is a congruence system on $\mathbf{A}$. So we must show that $\Omega^{\mathbf{A}}(T) \leq \lambda^{\mathbf{A}}(T)$ and that, moreover, it is the largest congruence system that satisfies this inclusion property.

To see that $\Omega^{\mathbf{A}}(T) \leq \lambda^{\mathbf{A}}(T)$, let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \Omega^{\mathbf{A}}_{\Sigma}(T)$. Then, by compatibility of $\Omega^{\mathbf{A}}(T)$ with T, we get that, $\phi \in T_\Sigma$ iff $\psi \in T_\Sigma$. So, by definition $\langle \phi, \psi \rangle \in \lambda^{\mathbf{A}}_{\Sigma}(T)$.

Finally, suppose, that $\theta \in \mathrm{ConSys}(\mathbf{A})$, such that $\theta \leq \lambda^{\mathbf{A}}(T)$. We must show that $\theta \leq \Omega^{\mathbf{A}}(T)$. Since, by definition $\Omega^{\mathbf{A}}(T)$ is the largest congruence system compatible with T, it suffices to show that θ is compatible with T. To this end, let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \theta_\Sigma$ and $\phi \in T_\Sigma$. Since $\theta \leq \lambda^{\mathbf{A}}(T)$, we get that $\langle \phi, \psi \rangle \in \lambda^{\mathbf{A}}_{\Sigma}(T)$ and $\phi \in T_\Sigma$. By the definition of $\lambda^{\mathbf{A}}(T)$, we conclude that $\psi \in T_\Sigma$. Therefore, θ is compatible with T and, hence, $\theta \leq \Omega^{\mathbf{A}}(T)$.

(b) Since $\Omega^{\mathbf{A}}(T)$ is, in particular, an equivalence system, we get, by Part (a) and Part (c) of Proposition 69, that $\Omega^{\mathbf{A}}(T) \leq \Lambda^{\mathbf{A}}(T)$. It is the largest congruence system satisfying this property, since $\Lambda^{\mathbf{A}}(T) \leq \lambda^{\mathbf{A}}(T)$ and, by Part (a), it is the largest congruence system in $\lambda^{\mathbf{A}}(T)$.

■

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a sentence functor and $\mathcal{T} \subseteq \mathrm{SenFam}(\mathrm{SEN})$ a collection of sentence families of SEN. The following relation systems are also known by the name of Frege in the literature, but we use the name "Carnap" instead to differentiate the two. In the present context, they have the same relation with Frege relation systems as Tarski congruence systems have with Leibniz congruence systems. We define:

- The **Carnap relation system** $\widetilde{\Lambda}(\mathcal{T}) = \{\widetilde{\Lambda}_\Sigma(\mathcal{T})\}_{\Sigma \in |\mathbf{Sign}|}$ of $\mathcal{T}$ on SEN, by
$$\widetilde{\Lambda}(\mathcal{T}) = \bigcap_{T \in \mathcal{T}} \Lambda(T),$$
where the intersection is taken signature-wise;

- The **Carnap relation family** $\widetilde{\lambda}(\mathcal{T}) = \{\widetilde{\lambda}_\Sigma(\mathcal{T})\}_{\Sigma \in |\mathbf{Sign}|}$ of $\mathcal{T}$ on SEN, by
$$\widetilde{\lambda}(\mathcal{T}) = \bigcap_{T \in \mathcal{T}} \lambda(T),$$
where the intersection is taken signature-wise.

That is, we have, for all $\Sigma \in |\mathbf{Sign}|$,

$$\begin{array}{rcl}\widetilde{\Lambda}_\Sigma(\mathcal{T}) & = & \{\langle \phi, \psi \rangle \in \mathrm{SEN}(\Sigma)^2 : \text{for all } T \in \mathcal{T}, \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma'), \\ & & \qquad \mathrm{SEN}(f)(\phi) \in T_{\Sigma'} \iff \mathrm{SEN}(f)(\psi) \in T_{\Sigma'}\}\end{array}$$

and, similarly,

$$\widetilde{\lambda}_\Sigma(\mathcal{T}) = \{\langle\phi,\psi\rangle \in \mathrm{SEN}(\Sigma)^2 : \text{for all } T \in \mathcal{T}, \quad \phi \in T_\Sigma \iff \psi \in T_\Sigma\}$$

We have analogs of Propositions 69 and 1420 for the case of $\widetilde{\Lambda}$ and $\widetilde{\lambda}$. The analog of Proposition 69 asserts that $\widetilde{\Lambda}(\mathcal{T})$ is an equivalence system on SEN, $\widetilde{\lambda}(\mathcal{T})$ is an equivalence family on SEN and that $\widetilde{\Lambda}(\mathcal{T})$ is the largest equivalence system included in $\widetilde{\lambda}(\mathcal{T})$.

Corollary 71 *Let* SEN : **Sign** → **Set** *be a sentence functor and consider* $\mathcal{T} \subseteq \mathrm{SenFam}(\mathrm{SEN})$.

- *(a)* $\widetilde{\Lambda}(\mathcal{T})$ *is an equivalence system on* SEN;
- *(b)* $\widetilde{\lambda}(\mathcal{T})$ *is an equivalence family on* SEN;
- *(c)* $\widetilde{\Lambda}(\mathcal{T})$ *is the largest equivalence system on* SEN *included in* $\widetilde{\lambda}(T)$.

Proof:

- (a) Since the intersection of equivalence relations is an equivalence relation, we get, by definition, that $\widetilde{\Lambda}(\mathcal{T})$ is an *equivalence* family. Moreover, since the intersection of relation systems is a relation system, we get, by Proposition 69, that $\widetilde{\Lambda}(\mathcal{T})$ is an equivalence *system*.
- (b) As in Part (a), Part (b) follows from the fact that $\lambda(T)$ is an equivalence family, for all $T \in \mathcal{T}$.
- (c) By Proposition 69, we get $\widetilde{\Lambda}(\mathcal{T}) = \bigcap_{T\in\mathcal{T}} \Lambda(T) \le \bigcap_{T\in\mathcal{T}} \lambda(T) = \widetilde{\lambda}(\mathcal{T})$. Let, now, θ be an equivalence system on SEN, such that $\theta \le \widetilde{\lambda}(\mathcal{T})$. We must show that $\theta \le \widetilde{\Lambda}(\mathcal{T})$. By hypothesis, $\theta \le \lambda(T)$, for all $T \in \mathcal{T}$. Therefore, by Proposition 69, $\theta \le \Lambda(T)$, for all $T \in \mathcal{T}$. Hence, $\theta \le \bigcap_{T\in\mathcal{T}} \Lambda(T) = \widetilde{\Lambda}(\mathcal{T})$. Thus, $\widetilde{\Lambda}(\mathcal{T})$ is indeed the largest equivalence system included in $\widetilde{\lambda}(\mathcal{T})$.

■

Once more, if SEN happens to be the underlying sentence functor of an algebraic system $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, we sometimes write $\widetilde{\Lambda}^{\mathbf{A}}(\mathcal{T})$ and $\widetilde{\lambda}^{\mathbf{A}}(\mathcal{T})$ for $\widetilde{\Lambda}(\mathcal{T})$ and $\widetilde{\lambda}(\mathcal{T})$, respectively.

The analog of Proposition 1420 asserts that both $\widetilde{\Lambda}^{\mathbf{A}}(\mathcal{T})$ and $\widetilde{\lambda}^{\mathbf{A}}(\mathcal{T})$ are in the same relation with $\widetilde{\Omega}^{\mathbf{A}}(\mathcal{T})$ as $\Lambda^{\mathbf{A}}(T)$ and $\lambda^{\mathbf{A}}(T)$ are with $\Omega^{\mathbf{A}}(T)$, i.e., that the Tarski congruence system of a collection of sentence families is the largest congruence system included in either the Carnap equivalence system or the Carnap equivalence family of the collection.

Proposition 72 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system and* $\mathcal{T} \subseteq \mathrm{SenFam}(\mathbf{A})$.

(a) $\widetilde{\Omega}^{\mathbf{A}}(\mathcal{T})$ *is the largest congruence system on* $\mathbf{A}$ *included in* $\widetilde{\lambda}^{\mathbf{A}}(\mathcal{T})$*;*

(b) $\widetilde{\Omega}^{\mathbf{A}}(\mathcal{T})$ *is the largest congruence system on* $\mathbf{A}$ *included in* $\widetilde{\Lambda}^{\mathbf{A}}(\mathcal{T})$*.*

Proof:

(a) To see that $\widetilde{\Omega}^{\mathbf{A}}(\mathcal{T}) \leq \widetilde{\lambda}^{\mathbf{A}}(\mathcal{T})$, let $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \widetilde{\Omega}^{\mathbf{A}}_{\Sigma}(\mathcal{T})$. Since $\widetilde{\Omega}^{\mathbf{A}}(\mathcal{T})$ is compatible with every $T \in \mathcal{T}$, we get that, for all $T \in \mathcal{T}$, $\phi \in T_\Sigma$ if and only if $\psi \in T_\Sigma$. Thus, $\langle \phi, \psi \rangle \in \widetilde{\lambda}^{\mathbf{A}}_{\Sigma}(\mathcal{T})$.

Suppose, next, that θ is a congruence system on $\mathbf{A}$, such that $\theta \leq \widetilde{\lambda}^{\mathbf{A}}(\mathcal{T})$. We must show that $\theta \leq \widetilde{\Omega}^{\mathbf{A}}(\mathcal{T})$. For this it suffices to show that θ is compatible with every $T \in \mathcal{T}$. Let $T \in \mathcal{T}$, $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \theta_\Sigma$ and $\phi \in T_\Sigma$. By hypothesis, $\langle \phi, \psi \rangle \in \widetilde{\lambda}^{\mathbf{A}}_{\Sigma}(\mathcal{T})$. By definition, $\langle \phi, \psi \rangle \in \lambda^{\mathbf{A}}_{\Sigma}(T)$. Therefore, since $\phi \in T_\Sigma$ we get $\psi \in T_\Sigma$ and, hence, θ is compatible with T.

(b) Since $\widetilde{\Omega}^{\mathbf{A}}(\mathcal{T})$ is an equivalence system and, by Part (a), $\widetilde{\Omega}^{\mathbf{A}}(\mathcal{T}) \leq \widetilde{\lambda}^{\mathbf{A}}(\mathcal{T})$, we get, by Corollary 71, $\widetilde{\Omega}^{\mathbf{A}}(\mathcal{T}) \leq \widetilde{\Lambda}^{\mathbf{A}}(\mathcal{T})$. Moreover, since, by Corollary 71, $\widetilde{\Lambda}^{\mathbf{A}}(\mathcal{T}) \leq \widetilde{\lambda}^{\mathbf{A}}(\mathcal{T})$ and, by Part (a), $\widetilde{\Omega}^{\mathbf{A}}(\mathcal{T})$ is the largest congruence system in $\widetilde{\lambda}^{\mathbf{A}}(\mathcal{T})$, it must also be the largest one in $\widetilde{\Lambda}^{\mathbf{A}}(\mathcal{T})$. ∎

Finally, consider a sentence functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, a collection $\mathcal{T}$ of sentence families of SEN and a sentence family $X \in \mathcal{T}$. The following is sometimes also termed Frege relation family, but, once more, to differentiate it from the preceding notions, we use the term “Lindenbaum” instead. we define:

- The **Lindenbaum relation system** $\widetilde{\Lambda}^{\mathcal{T}}(X) = \{\widetilde{\Lambda}^{\mathcal{T}}_{\Sigma}(X)\}_{\Sigma \in |\mathbf{Sign}|}$ **of** X **relative to** $\mathcal{T}$ by instantiating the definition of $\widetilde{\Lambda}$, given above, to the collection $\mathcal{T}^X$ of sentence families in $\mathcal{T}$ that include X, i.e.,

$$\widetilde{\Lambda}^{\mathcal{T}}(X) := \widetilde{\Lambda}(\mathcal{T}^X) = \bigcap \{\Lambda(T) : T \in \mathcal{T}, X \leq T\}.$$

- The **Lindenbaum relation family** $\widetilde{\lambda}^{\mathcal{T}}(X) = \{\widetilde{\lambda}^{\mathcal{T}}_{\Sigma}(X)\}_{\Sigma \in |\mathbf{Sign}|}$ **of** X **relative to** $\mathcal{T}$ by instantiating the definition of $\widetilde{\lambda}$ to the collection $\mathcal{T}^X$ of sentence families in $\mathcal{T}$ that include X, i.e.,

$$\widetilde{\lambda}^{\mathcal{T}}(X) := \widetilde{\lambda}(\mathcal{T}^X) = \bigcap \{\lambda(T) : T \in \mathcal{T}, X \leq T\}.$$

Using Corollary 71, we get immediately

Corollary 73 *Let* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a sentence functor,* $\mathcal{T} \subseteq \mathrm{SenFam}(\mathrm{SEN})$ *and* $X \in \mathcal{T}$*.*

(a) $\widetilde{\Lambda}^{\mathcal{T}}(X)$ *is an equivalence system on* SEN*;*

(b) $\widetilde{\lambda}^{\mathcal{T}}(X)$ *is an equivalence family on* SEN*;*

(c) $\widetilde{\Lambda}^{\mathcal{T}}(X)$ *is the largest equivalence system on* SEN *included in* $\widetilde{\lambda}^{\mathcal{T}}(X)$.

Proof: Directly by Corollary 71. ∎

When SEN happens to be the underlying sentence functor of an algebraic system $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, we sometimes write $\widetilde{\Lambda}^{\mathbf{A},\mathcal{T}}(X)$ and $\widetilde{\lambda}^{\mathbf{A},\mathcal{T}}(X)$ for the equivalence system $\widetilde{\Lambda}^{\mathcal{T}}(X)$ and the equivalence family $\widetilde{\lambda}^{\mathcal{T}}(X)$, respectively. Proposition 72 allows us to derive a relation between the Lindenbaum equivalence system $\widetilde{\Lambda}^{\mathbf{A},\mathcal{T}}(X)$ or the Lindenbaum equivalence family $\widetilde{\lambda}^{\mathbf{A},\mathcal{T}}(X)$ of a sentence family X relative to the collection $\mathcal{T}$ of sentence families and the Suszko congruence system $\widetilde{\Omega}^{\mathbf{A},\mathcal{T}}(X)$ of the family relative to the same collection.

Corollary 74 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system, consider* $\mathcal{T} \subseteq \mathrm{SenFam}(\mathbf{A})$ *and* $X \in \mathcal{T}$.

(a) $\widetilde{\Omega}^{\mathbf{A},\mathcal{T}}(X)$ *is the largest congruence system on* $\mathbf{A}$ *included in* $\widetilde{\lambda}^{\mathcal{T}}(X)$*;*

(b) $\widetilde{\Omega}^{\mathbf{A},\mathcal{T}}(X)$ *is the largest congruence system on* $\mathbf{A}$ *included in* $\widetilde{\Lambda}^{\mathcal{T}}(X)$.

Proof: We apply Proposition 72 to the collection $\mathcal{T}^X$. We get that $\widetilde{\Omega}^{\mathbf{A}}(\mathcal{T}^X)$ is the largest congruence system on $\mathbf{A}$ that is included in either $\widetilde{\lambda}(\mathcal{T}^X)$ or $\widetilde{\Lambda}(\mathcal{T}^X)$. The former is, by definition, equal to $\widetilde{\Omega}^{\mathbf{A},\mathcal{T}}(X)$ and the latter ones to $\widetilde{\lambda}^{\mathbf{A},\mathcal{T}}(X)$ and $\widetilde{\Lambda}^{\mathbf{A},\mathcal{T}}(X)$, respectively. So we get the conclusion. ∎

Consider now a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$, with $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$, and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system. The most common application of the Carnap operator will be to the collection $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ of all $\mathcal{I}$-filter families and that of the Lindenbaum operator to an $\mathcal{I}$-filter family T of $\mathcal{A}$ relative to $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. So we set the following notation:

$$\widetilde{\Lambda}^{\mathcal{A}}(\mathcal{I}) \coloneqq \widetilde{\Lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \quad \text{and} \quad \widetilde{\lambda}^{\mathcal{A}}(\mathcal{I}) \coloneqq \widetilde{\lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})).$$

Moreover, given $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we set

$$\widetilde{\Lambda}^{\mathcal{I},\mathcal{A}}(T) \coloneqq \widetilde{\Lambda}^{\mathcal{A},\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})}(T) \quad \text{and} \quad \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \coloneqq \widetilde{\lambda}^{\mathcal{A},\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})}(T).$$

When those notions specialize to the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$, the superscript referring to the algebraic system is often omitted. Thus, we have

$$\widetilde{\Lambda}(\mathcal{I}) = \widetilde{\Lambda}^{\mathcal{F}}(\mathcal{I}) \quad \text{and} \quad \widetilde{\lambda}(\mathcal{I}) = \widetilde{\lambda}^{\mathcal{F}}(\mathcal{I})$$

and, for $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\widetilde{\Lambda}^{\mathcal{I}}(T) = \widetilde{\Lambda}^{\mathcal{I},\mathcal{F}}(T) \quad \text{and} \quad \widetilde{\lambda}^{\mathcal{I}}(T) = \widetilde{\lambda}^{\mathcal{I},\mathcal{F}}(T).$$

We have the following characterizations of Lindenbaum equivalence systems and Lindenbaum equivalence families. We use those to derive characterizations of other relation families/systems as corollaries.

Theorem 75 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. Then, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,*

(a) $\langle \phi, \psi \rangle \in \widetilde{\Lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T)$ *if and only if, for all* $\Sigma' \in |\mathbf{Sign}|$*,* $f \in \mathbf{Sign}(\Sigma, \Sigma')$*,*

$$C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(T_{\Sigma'}, \mathrm{SEN}(f)(\phi)) = C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(T_{\Sigma'}, \mathrm{SEN}(f)(\psi));$$

(b) $\langle \phi, \psi \rangle \in \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T)$ *if and only if* $C^{\mathcal{I},\mathcal{A}}_\Sigma(T_\Sigma, \phi) = C^{\mathcal{I},\mathcal{A}}_\Sigma(T_\Sigma, \psi)$*.*

In particular, if $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$*, then* $\widetilde{\Lambda}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T)$*.*

Proof:

(a) We have, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $\langle \phi, \psi \rangle \in \widetilde{\Lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T)$ iff, for all $T \le T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}(f)(\phi) \in T'_{\Sigma'} \quad \text{iff} \quad \mathrm{SEN}(f)(\psi) \in T'_{\Sigma'}$$

iff, for all $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$T_\Sigma \cup \{\mathrm{SEN}(f)(\phi)\} \subseteq T'_{\Sigma'} \quad \text{iff} \quad T_\Sigma \cup \{\mathrm{SEN}(f)(\psi)\} \subseteq T'_{\Sigma'}$$

iff, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(T_\Sigma, \mathrm{SEN}(f)(\phi)) = C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(T_\Sigma, \mathrm{SEN}(f)(\psi)).$$

(b) We have, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $\langle \phi, \psi \rangle \in \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T)$ iff, for all $T \le T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\phi \in T'_\Sigma \Leftrightarrow \psi \in T'_\Sigma$ iff, for all $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T_\Sigma \cup \{\phi\} \subseteq T'_\Sigma \Leftrightarrow T_\Sigma \cup \{\psi\} \subseteq T'_\Sigma$ iff $C^{\mathcal{I},\mathcal{A}}_\Sigma(T_\Sigma, \phi) = C^{\mathcal{I},\mathcal{A}}_\Sigma(T_\Sigma, \psi)$.

The last statement follows from Parts (a) and (b) and the structurality property of $C^{\mathcal{I},\mathcal{A}}$. ∎

Specializing to the least $\mathcal{I}$-filter family on $\mathcal{A}$, which happens to be a theory system, we get

Corollary 76 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system. Then* $\widetilde{\Lambda}^{\mathcal{A}}(\mathcal{I}) = \widetilde{\lambda}^{\mathcal{A}}(\mathcal{I})$ *and, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,*

$$\langle \phi, \psi \rangle \in \widetilde{\lambda}^{\mathcal{A}}_\Sigma(\mathcal{I}) \quad \textit{iff} \quad C^{\mathcal{I},\mathcal{A}}_\Sigma(\phi) = C^{\mathcal{I},\mathcal{A}}_\Sigma(\psi).$$

Proof: Directly by Theorem 75, by taking $T = C^{\mathcal{I},\mathcal{A}}(\varnothing)$. ∎

Specializing to theory families, we get the following

Theorem 77 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $T \in \mathrm{ThFam}(\mathcal{I})$*. Then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

(a) $\langle \phi, \psi \rangle \in \widetilde{\Lambda}^{\mathcal{I}}_\Sigma(T)$ *if and only if, for all* $\Sigma' \in |\mathbf{Sign}^\flat|$*,* $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$*,*

$$C_{\Sigma'}(T_{\Sigma'}, \mathrm{SEN}^\flat(f)(\phi)) = C_{\Sigma'}(T_{\Sigma'}, \mathrm{SEN}^\flat(f)(\psi));$$

(b) $\langle \phi, \psi \rangle \in \widetilde{\lambda}^{\mathcal{I}}_\Sigma(T)$ *if and only if* $C_\Sigma(T_\Sigma, \phi) = C_\Sigma(T_\Sigma, \psi)$*.*

In particular, if $T \in \mathrm{ThSys}(\mathcal{I})$*, then* $\widetilde{\Lambda}^{\mathcal{I}}(T) = \widetilde{\lambda}^{\mathcal{I}}(T)$*.*

Proof: We apply Theorem 75 to the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. ∎

As a corollary, we also get

Corollary 78 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. Then* $\widetilde{\Lambda}(\mathcal{I}) = \widetilde{\lambda}(\mathcal{I})$ *and, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\langle \phi, \psi \rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}) \quad \text{iff} \quad C_\Sigma(\phi) = C_\Sigma(\psi).$$

Proof: Apply Theorem 77 to $T = \mathrm{Thm}(\mathcal{I})$, which happens to be a theory system. ∎

We record, finally, a couple of relatively straightforward monotonicity properties of the Carnap and Lindenbaum operators. The following theorem refers to collections of filter families and individual filter families and the subsequent corollary specializes this to theory families.

Theorem 79 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$*,* $\mathcal{I}' = \langle \mathbf{F}, C' \rangle$ π*-institutions based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

(a) *If* $\mathcal{I} \le \mathcal{I}'$*, then* $\widetilde{\Lambda}^{\mathcal{A}}(\mathcal{I}) \le \widetilde{\Lambda}^{\mathcal{A}}(\mathcal{I}')$ *and* $\widetilde{\lambda}^{\mathcal{A}}(\mathcal{I}) \le \widetilde{\lambda}^{\mathcal{A}}(\mathcal{I}')$*;*

(b) *If* $T \le T'$*, then* $\widetilde{\Lambda}^{\mathcal{A},\mathcal{I}}(T) \le \widetilde{\Lambda}^{\mathcal{A},\mathcal{I}}(T')$ *and* $\widetilde{\lambda}^{\mathcal{A},\mathcal{I}}(T) \le \widetilde{\lambda}^{\mathcal{A},\mathcal{I}}(T')$*.*

Proof:

(a) Since $\mathcal{I} \le \mathcal{I}'$, we have $\mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Hence,

$$\begin{array}{rcl}\widetilde{\Lambda}^{\mathcal{A}}(\mathcal{I}) & = & \bigcap\{\Lambda^{\mathcal{A}}(X) : X \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \\ & \le & \bigcap\{\Lambda^{\mathcal{A}}(X) : X \in \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A})\} \\ & = & \widetilde{\Lambda}^{\mathcal{A}}(\mathcal{I}').\end{array}$$

An almost identical reasoning yields the second inclusion.

(b) Since $T \leq T'$, we get

$$\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T'} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T},$$

whence we have

$$\begin{array}{rcl}\widetilde{\Lambda}^{\mathcal{I},\mathcal{A}}(T) & = & \bigcap\{\Lambda^{\mathcal{A}}(X) : T \leq X \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \\ & \leq & \bigcap\{\Lambda^{\mathcal{A}}(X) : T' \leq X \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \\ & = & \widetilde{\Lambda}^{\mathcal{I},\mathcal{A}}(T')\end{array}$$

and, similarly, $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \leq \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T')$.

■

Corollary 80 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$, $\mathcal{I}' = \langle \mathbf{F}, C' \rangle$ *π-institutions, based on* $\mathbf{F}$*, and* $T, T' \in \mathrm{ThFam}(\mathcal{I})$.

(a) If $\mathcal{I} \leq \mathcal{I}'$*, then* $\widetilde{\Lambda}(\mathcal{I}) \leq \widetilde{\Lambda}(\mathcal{I}')$ *and* $\widetilde{\lambda}(\mathcal{I}) \leq \widetilde{\lambda}(\mathcal{I}')$*;*

(b) If $T \leq T'$*, then* $\widetilde{\Lambda}^{\mathcal{I}}(T) \leq \widetilde{\Lambda}^{\mathcal{I}}(T')$ *and* $\widetilde{\lambda}^{\mathcal{I}}(T) \leq \widetilde{\lambda}^{\mathcal{I}}(T')$*.*

Proof: Apply Theorem 79 to $\mathcal{A} = \mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. ■

In closing, we provide the following table summarizing the correspondences between notions giving rise to congruence systems and notions giving rise to equivalence families and systems:

	$T \in \mathrm{SenFam}(\mathbf{A})$	$\mathcal{T} \subseteq \mathrm{SenFam}(\mathbf{A})$	$T \in \mathcal{T} \subseteq \mathrm{SenFam}(\mathbf{A})$
Congrunece Systems	Leibniz $\Omega^{\mathbf{A}}(T)$	Tarski $\widetilde{\Omega}^{\mathbf{A}}(\mathcal{T})$	Suszko $\widetilde{\Omega}^{\mathbf{A},\mathcal{T}}(T)$
Equivalence Families/Systems	Frege $\Lambda^{\mathbf{A}}(T), \lambda^{\mathbf{A}}(T)$	Carnap $\widetilde{\Lambda}^{\mathbf{A}}(\mathcal{T}), \widetilde{\lambda}^{\mathbf{A}}(\mathcal{T})$	Lindenbaum $\widetilde{\Lambda}^{\mathbf{A},\mathcal{T}}(T), \widetilde{\lambda}^{\mathbf{A},\mathcal{T}}(T)$

2.12 Subsystems and π-Subinstitutions

In this section, we look at $N^\flat$-algebraic subsystems. Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system and let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an $N^\flat$-algebraic system. A **universe** U **of** $\mathbf{A}$ is a sentence system of $\mathbf{A}$ that is closed under the operations in N, i.e., such that, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in U_\Sigma$,

$$\sigma_\Sigma(\vec{\phi}) \in U_\Sigma.$$

We denote by $\mathrm{Unv}(\mathbf{A})$ the collection of all universes of $\mathbf{A}$.

Given a universe $U \in \mathrm{Unv}(\mathbf{A})$, we may define a functor $\mathrm{SEN}' : \mathbf{Sign} \to \mathbf{Set}$, as follows:

- For all $\Sigma \in |\mathbf{Sign}|$, $\mathrm{SEN}'(\Sigma) = U_\Sigma$;

- For all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\phi \in \mathrm{SEN}'(\Sigma)$,

$$\mathrm{SEN}'(f)(\phi) = \mathrm{SEN}(f)(\phi).$$

Moreover, given a natural transformation $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, we may define the natural transformation $\sigma' : \mathrm{SEN}'^k \to \mathrm{SEN}'$ by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}'(\Sigma)$,

$$\sigma'_\Sigma(\vec{\phi}) = \sigma_\Sigma(\vec{\phi}).$$

In other words $\sigma' = \sigma \restriction_U = \sigma \restriction_{\mathrm{SEN}'}$.

We denote by N' the category of natural transformations on SEN' consisting of the restrictions $\sigma' = \sigma \restriction_{\mathrm{SEN}'}$, with the composition operation inherited by that of N, i.e., such that

$$\sigma' \circ \tau' = \sigma \restriction_{\mathrm{SEN}'} \circ \tau \restriction_{\mathrm{SEN}'} = (\sigma \circ \tau) \restriction_{\mathrm{SEN}'} = (\sigma \circ \tau)'.$$

Finally, we set $\mathbf{A}' = \langle \mathbf{Sign}, \mathrm{SEN}', N' \rangle$ and call $\mathbf{A}'$ the **algebraic subsystem of $\mathbf{A}$** on the universe U or on the functor SEN'. We write $\mathbf{A}' \le \mathbf{A}$ to signify that $\mathbf{A}'$ is an algebraic subsystem of $\mathbf{A}$.

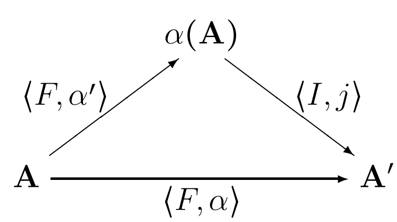
Note that the pair $\langle I, j \rangle : \mathbf{A}' \to \mathbf{A}$, where $I : \mathbf{Sign} \to \mathbf{Sign}$ and $j : \mathrm{SEN}' \to \mathrm{SEN}$, defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}'(\Sigma)$, by

$$j_\Sigma(\phi) = \phi,$$

becomes a morphism of $N^\flat$-algebraic systems, called the **injection morphism of $\mathbf{A}'$ into $\mathbf{A}$**.

Now we relate injection morphisms with the construction of the image algebraic system outlined in Lemma 13.

Proposition 81 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *and* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ $N^\flat$*-algebraic systems and* $\langle F, \alpha \rangle : \mathbf{A} \to \mathbf{A}'$ *an algebraic system morphism, with* $F : \mathbf{Sign} \to \mathbf{Sign}'$ *an isomorphism. Then, we have* $\langle F, \alpha \rangle = \langle I, j \rangle \circ \langle F, \alpha' \rangle$*,*



where $\langle F, \alpha' \rangle : \mathbf{A} \to \alpha(\mathbf{A})$ *is the surjective morphism defined in Lemma 14 and* $\langle I, j \rangle : \alpha(\mathbf{A}) \to \mathbf{A}'$ *is the injection morphism.*

Proof: We have, using the definitions, that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$j_{F(\Sigma)}(\alpha'_\Sigma(\phi)) = j_{F(\Sigma)}(\alpha_\Sigma(\phi)) = \alpha_\Sigma(\phi).$$

This proves the commutativity of the triangle. ∎

We call the decomposition of $\langle F, \alpha\rangle : \mathbf{A} \to \mathbf{A}'$ established in Proposition 81, the **(natural) epi-mono factorization of** $\langle F, \alpha\rangle$.

Of particular interest are the subuniverses of an algebraic system that are generated by a given sentence family X of the algebraic system. We detail this construction here and introduce some relevant notation.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system and let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an $N^\flat$-algebraic system. Consider a sentence family

$$X \in \mathrm{SenFam}(\mathbf{A}).$$

Of course, it is very likely that X is neither a system (i.e., invariant under signature morphisms) nor closed under the operations in N. But we have pertinent constructions that can be employed to obtain a closure of X with respect to those operations.

Recall, first, that, by Proposition 2, $\overrightarrow{X}$ is the least sentence system of $\mathbf{A}$ containing X.

Second, define $\nu^{\mathbf{A}}(X) = \{\nu^{\mathbf{A}}_\Sigma(X)\}_{\Sigma\in|\mathbf{Sign}|}$, by letting, for all $\Sigma \in |\mathbf{Sign}|$, $\nu^{\mathbf{A}}_\Sigma(X)$ be given by

$$\nu^{\mathbf{A}}_\Sigma(X) = \{\sigma^{\mathbf{A}}_\Sigma(\vec{\phi}) : \sigma \in N, \vec{\phi} \in X_\Sigma\}.$$

We can show that $\nu^{\mathbf{A}}(X)$ is the least sentence family of $\mathbf{A}$ containing X and closed under the operations in N and that, moreover, it happens to be a sentence system in case X is a sentence system. As a consequence, we obtain that $\nu^{\mathbf{A}}(\overrightarrow{X})$ is the least universe of $\mathbf{A}$ including X. These results are detailed in the following proposition and theorem.

Proposition 82 *Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system and $X \in \mathrm{SenFam}(\mathbf{A})$.*

(a) $\nu^{\mathbf{A}}(X)$ is the least sentence family of $\mathbf{A}$ including X and closed under the operations in N;

(b) If $X \in \mathrm{SenSys}(\mathbf{A})$, the $\nu^{\mathbf{A}}(X)$ is also a sentence system.

Proof: Note, first, that, since the identity $\iota : \mathrm{SEN} \to \mathrm{SEN}$ is a natural transformation in N, we have, by definition, that $X \le \nu^{\mathbf{A}}(X)$. Suppose, next, that $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ is in N, $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi} \in \nu^{\mathbf{A}}_\Sigma(X)$. Thus, for all $i < k$, there exists $\tau^i : \mathrm{SEN}^{n_i} \to \mathrm{SEN}$ and $\vec{\chi}^i \in X_\Sigma$, such that

$$\phi_i = \tau^i_\Sigma(\vec{\chi}^i).$$

Let $n = n_0 + n_1 + \cdots + n_{k-1}$ and $\vec{\chi} = \langle \vec{\chi}^0, \vec{\chi}^1, \dots, \vec{\chi}^{k-1} \rangle$ be the vector of length n resulting from the concatenation of the elements of the $\vec{\chi}^i$'s. Then we get that

$$\begin{array}{rcl}\sigma_\Sigma(\vec{\phi}) & = & \sigma_\Sigma(\tau^0_\Sigma(\vec{\chi}^0), \dots, \tau^{k-1}_\Sigma(\vec{\chi}^{k-1})) \\ & = & [\sigma \circ \langle \tau^0 \circ \langle p^{n,0}, \dots, p^{n,n_0-1} \rangle, \tau^1 \circ \langle p^{n,n_0}, \dots, p^{n,n_0+n_1-1} \rangle, \dots, \\ & & \qquad \tau^{k-1} \circ \langle p^{n,n_0+\cdots+n_{k-1}}, \dots, p^{n,n_0+\cdots+n_k-1} \rangle \rangle](\vec{\chi}).\end{array}$$

Since the natural transformation above is in N and $\vec{\chi} \in X_\Sigma$, we conclude that $\sigma_\Sigma(\vec{\phi}) \in \nu^{\mathbf{A}}_\Sigma(X)$, whence $\nu^{\mathbf{A}}(\Sigma)$ is closed under the operations in N.

To show minimality, suppose that $Y \in \mathrm{SenFam}(\mathbf{A})$, such that $X \leq Y$ and Y is closed under the operations in N. Consider $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \nu^{\mathbf{A}}_\Sigma(X)$. By definition, there exists σ in N and $\vec{\phi} \in X_\Sigma$, such that $\phi = \sigma_\Sigma(\vec{\phi})$. But, then, since $\vec{\phi} \in X_\Sigma \subseteq Y_\Sigma$ and Y is closed under the operations in N, we get that $\phi = \sigma_\Sigma(\vec{\phi}) \in Y_\Sigma$. Since this holds for all $\Sigma \in |\mathbf{Sign}|$, we get that $\nu^{\mathbf{A}}(X) \leq Y$ and, hence, $\nu^{\mathbf{A}}(X)$ is the least sentence family of $\mathbf{A}$ including X and closed under the operations in N.

Finally, let $X \in \mathrm{SenSys}(\mathbf{A})$. Suppose $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\phi \in \nu^{\mathbf{A}}_\Sigma(X)$. Then, there exists σ in N and $\vec{\phi} \in X_\Sigma$, such that $\phi = \sigma_\Sigma(\vec{\phi})$. We now get

$$\begin{array}{rcll}\mathrm{SEN}(f)(\phi) & = & \mathrm{SEN}(f)(\sigma_\Sigma(\vec{\phi})) & \\ & = & \sigma_{\Sigma'}(\mathrm{SEN}(f)(\vec{\phi})) & (\sigma \text{ in } N) \\ & \in & \nu^{\mathbf{A}}_{\Sigma'}(X). & (\sigma \text{ in } N,\ \phi \in X_\Sigma,\ X \in \mathrm{SenSys}(\mathbf{A}))\end{array}$$

Therefore $\nu^{\mathbf{A}}(X) \in \mathrm{SenSys}(\mathbf{A})$. ■

Theorem 83 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *be an algebraic system and consider* $X \in \mathrm{SenFam}(\mathbf{A})$. *Then* $\nu^{\mathbf{A}}(\overrightarrow{X})$ *is the least universe of* $\mathbf{A}$ *including* X.

Proof: By Proposition 2, $\overrightarrow{X} \in \mathrm{SenSys}(\mathbf{A})$. Therefore, by Proposition 82, $\nu^{\mathbf{A}}(\overrightarrow{X}) \in \mathrm{Unv}(\mathbf{A})$. Suppose, that $U \in \mathrm{Unv}(\mathbf{A})$, such that $X \leq U$. Since U is a universe, it is a sentence system. Thus, by Proposition 2, $\overrightarrow{X} \leq U$. Moreover, since U is a universe, it is closed under the operations in N, whence, by Proposition 82, $\nu^{\mathbf{A}}(\overrightarrow{X}) \leq U$. We conclude that $\nu^{\mathbf{A}}(\overrightarrow{X})$ is the least universe of $\mathbf{A}$ containing X. ■

Based on Theorem 83, given $X \in \mathrm{SenFam}(\mathbf{A})$, we call $\nu^{\mathbf{A}}(\overrightarrow{X})$ the **universe of A generated by** X and sometimes denote it by

$$\langle X \rangle = \{\langle X \rangle_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}.$$

We adopt many simplifying notations such as writing $\langle \Phi \rangle$, $\Phi \subseteq \mathrm{SEN}(\Sigma)$, for the universe $\langle T \rangle$, generated by $T \in \mathrm{SenFam}(\mathbf{A})$, with

$$T_{\Sigma'} = \begin{cases} \Phi, & \text{if } \Sigma' = \Sigma \\ \varnothing, & \text{if } \Sigma' \neq \Sigma \end{cases}$$

and $\langle\phi,\psi\rangle$ for $\langle\{\phi,\psi\}\rangle$, $\phi,\psi \in \mathrm{SEN}(\Sigma)$, if such overloading is unlikely to result into major mayhem.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system. Consider an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ and let $\mathbf{A}' = \langle \mathbf{Sign}, \mathrm{SEN}', N'\rangle$ be an algebraic subsystem of $\mathbf{A}$. Define $\alpha^{-1}(\mathrm{SEN}') = \{\alpha_\Sigma^{-1}(\mathrm{SEN}')\}_{\Sigma\in|\mathbf{Sign}^\flat|}$ by letting $\alpha_\Sigma^{-1}(\mathrm{SEN}')$ be given, for all $\Sigma \in |\mathbf{Sign}^\flat|$, by

$$\alpha_\Sigma^{-1}(\mathrm{SEN}') = \alpha_\Sigma^{-1}(\mathrm{SEN}'(F(\Sigma))).$$

Lemma 84 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, be an $\mathbf{F}$-algebraic system. If $\mathbf{A}' = \langle \mathbf{Sign}, \mathrm{SEN}', N'\rangle \le \mathbf{A}$ is an algebraic subsystem of $\mathbf{A}$, then $\alpha^{-1}(\mathrm{SEN}')$ is a universe of $\mathbf{F}$.*

Proof: Since SEN' is a sentence system of $\mathcal{A}$, by Lemma 6, we get that $\alpha^{-1}(\mathrm{SEN}')$ is a sentence system of $\mathbf{F}$. So it suffices to show that $\alpha^{-1}(\mathrm{SEN}')$ is closed under the operations in $N^\flat$. To this end, let $\sigma^\flat \in N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi} \in \alpha^{-1}(\mathrm{SEN}'(F(\Sigma))$. Then we have

$$\begin{array}{rcl}\alpha_\Sigma(\sigma^\flat_\Sigma(\vec{\phi})) & = & \sigma^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi}))\\ & \in & \mathrm{SEN}'(F(\Sigma)),\end{array}$$

since $\alpha_\Sigma(\vec{\phi}) \in \mathrm{SEN}'(F(\Sigma))$, by hypothesis, and $\mathrm{SEN}'(F(\Sigma))$ is a universe of $\mathbf{A}$. Thus $\alpha^{-1}(\mathrm{SEN}')$ is indeed a universe of $\mathbf{F}$. ■

We define the triple $\alpha^{-1}(\mathbf{A}') = \langle \mathbf{Sign}^\flat, \mathrm{SEN}'^\flat, N'^\flat\rangle$ as the algebraic subsystem of $\mathbf{F}$ determined by the universe $\alpha^{-1}(\mathrm{SEN}')$ of $\mathbf{F}$.

Let, again, $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ an $\mathbf{F}$-algebraic system, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, and $\mathbf{A}' = \langle \mathbf{Sign}, \mathrm{SEN}', N'\rangle$ be an algebraic subsystem of $\mathbf{A}$. We define the pair $\langle F,\alpha'\rangle : \alpha^{-1}(\mathbf{A}') \to \mathbf{A}'$ by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \alpha_\Sigma^{-1}(\mathrm{SEN}'(F(\Sigma))$,

$$\alpha'_\Sigma(\phi) = \alpha_\Sigma(\phi).$$

Then $\langle F,\alpha'\rangle$ turns out to be a surjective morphism from $\alpha^{-1}(\mathbf{A}')$ to $\mathbf{A}'$.

Lemma 85 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, be an $\mathbf{F}$-algebraic system. If $\mathbf{A}' = \langle \mathbf{Sign}, \mathrm{SEN}', N'\rangle \le \mathbf{A}$ is an algebraic subsystem of $\mathbf{A}$, then $\langle F,\alpha'\rangle : \alpha^{-1}(\mathbf{A}') \to \mathbf{A}'$ is a surjective morphism.*

Proof: Since $F : \mathbf{Sign}^\flat \to \mathbf{Sign}$ is surjective and full, by hypothesis, it suffices to show that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\alpha'_\Sigma : \alpha^{-1}(\mathrm{SEN}'(F(\Sigma))) \to \mathrm{SEN}'(F(\Sigma))$ is also surjective. But this follows by the definition of $\alpha^{-1}(\mathrm{SEN}')$ and the surjectivity of $\langle F,\alpha\rangle : \mathbf{F} \to \mathbf{A}$. ■

Lemma 85 shows that $\mathcal{A}' = \langle \mathbf{A}', \langle F, \alpha' \rangle \rangle$ may be viewed as an $\alpha^{-1}(\mathbf{A}')$-algebraic system.

Consider now an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ and a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ based on $\mathbf{F}$. Given an algebraic subsystem $\mathbf{F}' = \langle \mathbf{Sign}^\flat, \mathrm{SEN}'^\flat, N'^\flat \rangle$ of $\mathbf{F}$, we define the **π-subinstitution induced by**, or **associated with $\mathbf{F}'$**, to be the pair $\mathcal{I}' = \langle \mathbf{F}', C' \rangle$, where $C' : \mathcal{P}\mathrm{SEN}'^\flat \to \mathcal{P}\mathrm{SEN}'^\flat$ is defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \mathrm{SEN}'^\flat(\Sigma)$, by

$$C'_\Sigma(\Phi) = C_\Sigma(\Phi) \cap \mathrm{SEN}'^\flat(\Sigma).$$

Proposition 86 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution, based on* $\mathbf{F}$*, and* $\mathbf{F}' = \langle \mathbf{Sign}^\flat, \mathrm{SEN}'^\flat, N'^\flat \rangle$ *an algebraic subsystem of* $\mathbf{F}$*. Then* $\mathcal{I}' = \langle \mathbf{F}', C' \rangle$ *is a* π*-institution.*

Proof: We must show that $C' : \mathcal{P}\mathrm{SEN}'^\flat \to \mathcal{P}\mathrm{SEN}'^\flat$ is a closure system on $\mathbf{F}'$. The inflation property is clear, since, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \mathrm{SEN}'^\flat(\Sigma)$,

$$\Phi \subseteq C_\Sigma(\Phi) \cap \mathrm{SEN}'^\flat(\Sigma) = C'_\Sigma(\Phi).$$

Monotonicity is also clear, since, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi, \Psi \subseteq \mathrm{SEN}'^\flat(\Sigma)$, such that $\Phi \subseteq \Psi$,

$$C'_\Sigma(\Phi) = C_\Sigma(\Phi) \cap \mathrm{SEN}'^\flat(\Sigma) \subseteq C_\Sigma(\Psi) \cap \mathrm{SEN}'^\flat(\Sigma) = C'_\Sigma(\Psi).$$

For idempotency, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}'^\flat(\Sigma)$, such that $\phi \in C'_\Sigma(C'_\Sigma(\Phi))$. Then we have

$$\begin{array}{rcl}
\phi & \in & C_\Sigma(C_\Sigma(\Phi) \cap \mathrm{SEN}'^\flat(\Sigma)) \cap \mathrm{SEN}'^\flat(\Sigma) \\
& \subseteq & C_\Sigma(C_\Sigma(\Phi)) \cap \mathrm{SEN}'^\flat(\Sigma) \\
& = & C_\Sigma(\Phi) \cap \mathrm{SEN}'^\flat(\Sigma) \\
& = & C'_\Sigma(\Phi).
\end{array}$$

It now only remains to show that C' is also structural. Let $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\Phi \subseteq \mathrm{SEN}'^\flat(\Sigma)$. Then, we have

$$\begin{array}{rcl}
\mathrm{SEN}'^\flat(f)(C'_\Sigma(\Phi)) & = & \mathrm{SEN}'^\flat(f)(C_\Sigma(\Phi) \cap \mathrm{SEN}'^\flat(\Sigma)) \\
& \subseteq & \mathrm{SEN}^\flat(f)(C_\Sigma(\Phi)) \cap \mathrm{SEN}'^\flat(\Sigma') \\
& \subseteq & C_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \cap \mathrm{SEN}'^\flat(\Sigma') \\
& = & C_{\Sigma'}(\mathrm{SEN}'^\flat(f)(\Phi)) \cap \mathrm{SEN}'^\flat(\Sigma') \\
& = & C'_{\Sigma'}(\mathrm{SEN}'^\flat(f)(\Phi)).
\end{array}$$

We conclude that C' is a closure system on $\mathbf{F}'$ and, therefore, $\mathcal{I}'$ is a π-institution. ■

We also give a characterization of the theory families and the theory systems of the induced subinstitution in terms of those of its parent.

Proposition 87 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution, based on* $\mathbf{F}$*, and* $\mathbf{F}' = \langle \mathbf{Sign}^\flat, \mathrm{SEN}'^\flat, N'^\flat \rangle$ *an algebraic subsystem of* $\mathbf{F}$*. Then*

$$\mathrm{ThFam}(\mathcal{I}') = \{T \cap \mathrm{SEN}'^\flat : T \in \mathrm{ThFam}(\mathcal{I})\} \quad \textit{and} \quad \mathrm{ThSys}(\mathcal{I}') = \{T \cap \mathrm{SEN}'^\flat : T \in \mathrm{ThSys}(\mathcal{I})\}.$$

Proof: We show the first equality. The second may be proved similarly.

Suppose, first, that $T' \in \mathrm{ThFam}(\mathcal{I}')$. Then we have $C'(T') = T'$. By definition, $C'(T') = C(T') \cap \mathrm{SEN}'^\flat$. Thus, we get $T' = C(T') \cap \mathrm{SEN}'^\flat$. Since $C(T') \in \mathrm{ThFam}(\mathcal{I})$, we get that $\mathrm{ThFam}(\mathcal{I}') \subseteq \{T \cap \mathrm{SEN}'^\flat : T \in \mathrm{ThFam}(\mathcal{I})\}$.

Suppose, conversely, that $T \in \mathrm{ThFam}(\mathcal{I})$. Then, we have

$$\begin{array}{rcl} C'(T \cap \mathrm{SEN}'^\flat) & = & C(T \cap \mathrm{SEN}'^\flat) \cap \mathrm{SEN}'^\flat \\ & \subseteq & C(T) \cap \mathrm{SEN}'^\flat \\ & = & T \cap \mathrm{SEN}'^\flat. \end{array}$$

So $T \cap \mathrm{SEN}'^\flat \in \mathrm{ThFam}(\mathcal{I}')$ and we conclude that

$$\{T \cap \mathrm{SEN}'^\flat : T \in \mathrm{ThFam}(\mathcal{I})\} \subseteq \mathrm{ThFam}(\mathcal{I}').$$

Equality now follows. ∎

Proposition 87 implies that the property of all theory families being theory systems (which shall be used in the next chapter as the defining property of a *systemic* π*-institution*) is inherited by all π-subinstitutions of a π-institution:

Corollary 88 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution, based on* $\mathbf{F}$*, and* $\mathbf{F}' = \langle \mathbf{Sign}^\flat, \mathrm{SEN}'^\flat, N'^\flat \rangle$ *an algebraic subsystem of* $\mathbf{F}$*. If* $\mathcal{I}$ *is such that* $\mathrm{ThFam}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$*, then* $\mathcal{I}' = \langle \mathbf{F}', C' \rangle$ *satisfies the same property.*

Proof: If $T' \in \mathrm{ThFam}(\mathcal{I}')$, then, by Proposition 87, there exists a theory family $T \in \mathrm{ThFam}(\mathcal{I})$, such that $T' = T \cap \mathrm{SEN}'^\flat$. By hypothesis, we have $T \in \mathrm{ThSys}(\mathcal{I})$, whence $T' = T \cap \mathrm{SEN}'^\flat \in \mathrm{ThSys}(\mathcal{I}')$. It follows that $\mathrm{ThFam}(\mathcal{I}') = \mathrm{ThSys}(\mathcal{I}')$. ∎

We now look at a relationship between Leibniz congruence systems of theory families in institutions and of Leibniz congruence systems of corresponding theory families in subinstitutions associated with given universes.

Proposition 89 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution, based on* $\mathbf{F}$*, and* $\mathcal{I}' = \langle \mathbf{F}', C' \rangle$ *a* π*-subinstitution of* $\mathcal{I}$*, associated with* $\mathbf{F}' = \langle \mathbf{Sign}, \mathrm{SEN}'^\flat, N'^\flat \rangle \leq \mathbf{F}$*. Then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\Omega^{\mathbf{F}}(T) \cap (\mathrm{SEN}'^\flat)^2 \leq \Omega^{\mathbf{F}'}(T \cap \mathrm{SEN}'^\flat).$$

Proof: By the maximality property of $\Omega^{\mathbf{F}'}(T \cap \mathrm{SEN}'^{\flat})$, it suffices to show that $\Omega^{\mathbf{F}}(T) \cap (\mathrm{SEN}'^{\flat})^2$ is a congruence system on $\mathbf{F}'$ that is compatible with the theory family $T \cap \mathrm{SEN}'^{\flat}$.

The reflexivity, symmetry, transitivity and congruence properties of

$$\Omega^{\mathbf{F}}(T) \cap (\mathrm{SEN}'^{\flat})^2$$

are inherited by those of $\Omega^{\mathbf{F}}(T)$. Moreover, we have, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^{\flat}|$ and all $f \in \mathbf{Sign}^{\flat}(\Sigma, \Sigma')$,

$$\begin{aligned} \mathrm{SEN}'^{\flat}(f)(\Omega^{\mathbf{F}}_{\Sigma}(T) \cap \mathrm{SEN}'^{\flat}(\Sigma)^2) &\subseteq \mathrm{SEN}^{\flat}(f)(\Omega^{\mathbf{F}}_{\Sigma}(T)) \cap \mathrm{SEN}'^{\flat}(\Sigma')^2 \\ &\subseteq \Omega^{\mathbf{F}}_{\Sigma'}(T) \cap \mathrm{SEN}'^{\flat}(\Sigma')^2. \end{aligned}$$

So $\Omega^{\mathbf{F}}(T) \cap (\mathrm{SEN}'^{\flat})^2$ is indeed a congruence system on $\mathbf{F}'$. Finally, assume that $\Sigma \in |\mathbf{Sign}^{\flat}|$ and $\phi, \psi \in \mathrm{SEN}'^{\flat}(\Sigma)$, such that

$$\langle \phi, \psi \rangle \in \Omega^{\mathbf{F}}_{\Sigma}(T) \cap \mathrm{SEN}'^{\flat}(\Sigma)^2 \quad \text{and} \quad \phi \in T_{\Sigma} \cap \mathrm{SEN}'^{\flat}(\Sigma).$$

Then, by the compatibility of $\Omega^{\mathbf{F}}(T)$ with T, we get that $\psi \in T_{\Sigma} \cap \mathrm{SEN}'^{\flat}(\Sigma)$. We conclude that $\Omega^{\mathbf{F}}(T) \cap (\mathrm{SEN}'^{\flat})^2$ is indeed compatible with $T \cap \mathrm{SEN}'^{\flat}$ and, therefore, $\Omega^{\mathbf{F}}(T) \cap (\mathrm{SEN}'^{\flat})^2 \leq \Omega^{\mathbf{F}'}(T \cap \mathrm{SEN}'^{\flat})$. ∎

In particular, we have the following, where, recall that $\langle \phi, \psi \rangle$ denotes the universe of $\mathbf{F}$ generated by $\{\phi, \psi\} \subseteq \mathrm{SEN}^{\flat}(\Sigma)$.

Corollary 90 *Let $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat} \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. Then, for all $\Sigma \in |\mathbf{Sign}^{\flat}|$ and all $\phi, \psi \in \mathrm{SEN}^{\flat}(\Sigma)$,*

$$\langle \phi, \psi \rangle \in \Omega_{\Sigma}(T) \quad \textit{implies} \quad \langle \phi, \psi \rangle \in \Omega^{\langle \phi, \psi \rangle}_{\Sigma}(T \cap \langle \phi, \psi \rangle).$$

Proof: This follows directly by Proposition 89 by considering the universe $\langle \phi, \psi \rangle$ of $\mathbf{F}$ generated by the sentence family T, with $T_{\Sigma} = \{\phi, \psi\}$ and $T_{\Sigma'} = \varnothing$, for all $\Sigma' \neq \Sigma$. ∎

We turn now to the examination of the relation between π-institutions and their models, on the one hand, and π-subinstitutions and their models, on the other.

We show first that, for every π-institution $\mathcal{I}$, every $\mathcal{I}$-filter family on an $\mathbf{F}$-algebraic system $\mathcal{A}$ gives rise naturally to an $\mathcal{I}'$-filter family on an $\mathbf{F}'$-algebraic subsystem of $\mathcal{A}$.

Proposition 91 *Let $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat} \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Let, also $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle \leq \mathbf{A}$ be an algebraic subsystem of $\mathbf{A}$. Then $T \cap \mathrm{SEN}' \in \mathrm{FiFam}^{\mathcal{I}'}(\langle \mathbf{A}', \langle F, \alpha' \rangle \rangle)$, where $\mathcal{I}' = \langle \alpha^{-1}(\mathbf{A}'), C' \rangle$ is the π-subinstitution of $\mathcal{I}$ induced by $\alpha^{-1}(\mathbf{A}')$.*

Proof: By Lemma 84, $\alpha^{-1}(\mathbf{A}')$ is an algebraic subsystem of $\mathbf{F}$. Therefore, the pair $\mathcal{I}' = \langle \alpha^{-1}(\mathbf{A}'), C' \rangle$ is a well defined π-subinstitution of $\mathcal{I}$. So it suffices to show, by Lemma 51, that $\alpha^{-1}(T \cap \mathrm{SEN}') \in \mathrm{ThFam}(\mathcal{I}')$. But this is easy, since we have

$$\alpha^{-1}(T \cap \mathrm{SEN}') = \alpha^{-1}(T) \cap \alpha^{-1}(\mathrm{SEN}') \in \mathrm{ThFam}(\mathcal{I}'),$$

membership following by Lemma 51 and Proposition 87. ∎

As a corollary, we obtain the fact that inverse images of Leibniz congruence systems of filter families on algebraic subsystems equal Leibniz congruence systems of the corresponding theory families of π-subinstitutions.

Corollary 92 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. Let, also* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle \leq \mathbf{A}$ *be an algebraic subsystem of* $\mathbf{A}$*. Then*

$$\alpha^{-1}(\Omega^{\mathbf{A}'}(T \cap \mathrm{SEN}')) = \Omega^{\alpha^{-1}(\mathbf{A}')}(\alpha^{-1}(T) \cap \alpha^{-1}(\mathrm{SEN}')).$$

Proof: This follows by Proposition 91 and Proposition 24. ∎

2.13 Syntax

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an algebraic system and consider a set $E \subseteq N$ of natural transformations in N. All natural transformations in E are, therefore, finitary. Since, however, there may be an infinite number of them, they may be collectively of unbounded arity. As a consequence, we write, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^\omega$,

$$E_\Sigma(\vec{\phi}) = \{\sigma_\Sigma(\phi_0, \dots, \phi_{k-1}) : \sigma \in E\}$$

to denote the values of E on the tuple $\vec{\phi}$, where, for each $\sigma \in E$ k-ary, only the first k components of $\vec{\phi}$ are actually used.

In certain contexts, we will view the first k positions of each natural transformation in E as **distinguished**, while treating all remaining positions as **parametric**. In that case we have to exercise meticulous care when we employ the following notation. Given $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$, we write

$$E_\Sigma[\vec{\phi}] = \{E_{\Sigma,\Sigma'}[\vec{\phi}]\}_{\Sigma' \in |\mathbf{Sign}|},$$

where, for all $\Sigma' \in |\mathbf{Sign}|$, we define

$$E_{\Sigma,\Sigma'}[\vec{\phi}] \quad = \quad \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\vec{\phi}), \vec{\chi}) : \sigma \in E, f \in \mathbf{Sign}(\Sigma, \Sigma'), \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')\}.$$

Let, again, $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system and $E \subseteq N$. For $T \in \mathrm{SenFam}(\mathbf{A})$, we set

$$\overleftarrow{E}(T) = \{\overleftarrow{E}_\Sigma(T)\}_{\Sigma\in|\mathbf{Sign}|},$$

where, for all $\Sigma \in |\mathbf{Sign}|$,

$$\overleftarrow{E}_\Sigma(T) = \{\vec{\phi} \in \mathrm{SEN}(\Sigma) : E_\Sigma[\vec{\phi}] \leq T\}.$$

We show that $\overleftarrow{E}(T)$ is a relation system on $\mathbf{A}$.

Lemma 93 *Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system, $E \subseteq N$ and $T \in \mathrm{SenFam}(\mathbf{A})$. Then $\overleftarrow{E}(T)$ is a relation system on $\mathbf{A}$.*

Proof: Let $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi} \in \mathrm{SEN}(\Sigma)$, such that $\vec{\phi} \in \overleftarrow{E}_\Sigma(T)$. Our goal is to show that, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}(f)(\vec{\phi}) \in \overleftarrow{E}_{\Sigma'}(T).$$

So we fix $\Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma, \Sigma')$. By hypothesis, we have that, $E_\Sigma[\vec{\phi}] \leq T$. Thus, for all $\Sigma'' \in |\mathbf{Sign}|$, $g \in \mathbf{Sign}(\Sigma', \Sigma'')$ and $\vec{\chi} \in \mathrm{SEN}(\Sigma'')$,

$$\Sigma \xrightarrow{f} \Sigma' \xrightarrow{g} \Sigma''$$

$$E_{\Sigma''}(\mathrm{SEN}(gf)(\vec{\phi}), \vec{\chi}) \subseteq T_{\Sigma''},$$

or, equivalently,

$$E_{\Sigma''}(\mathrm{SEN}(g)(\mathrm{SEN}(f)(\vec{\phi})), \vec{\chi}) \subseteq T_{\Sigma''}.$$

By definition, this means that $E_{\Sigma'}[\mathrm{SEN}(f)(\vec{\phi})] \leq T$, i.e., that $\mathrm{SEN}(f)(\vec{\phi}) \in \overleftarrow{E}_{\Sigma'}(T)$. Therefore $\overleftarrow{E}(T)$ is a relation system. ■

We show, next, that $\overleftarrow{E}$ is a monotone operator on $\mathrm{SenFam}(\mathbf{A})$.

Lemma 94 *Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system and $E \subseteq N$. Then, for all $T, T' \in \mathrm{SenFam}(\mathbf{A})$,*

$$T \leq T' \quad \textit{implies} \quad \overleftarrow{E}(T) \leq \overleftarrow{E}(T').$$

Proof: Suppose that $T, T' \in \mathrm{SenFam}(\mathcal{I})$, with $T \leq T'$. Then, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)$, we have

$$\begin{array}{rcl} \vec{\phi} \in \overleftarrow{E}_\Sigma(T) & \text{iff} & E_\Sigma[\vec{\phi}] \leq T \\ & \text{implies} & E_\Sigma[\vec{\phi}] \leq T' \\ & \text{iff} & \vec{\phi} \in \overleftarrow{E}_\Sigma(T'). \end{array}$$

So $\overleftarrow{E}(T) \leq \overleftarrow{E}(T')$. ■

A very useful property of the $\overleftarrow{E}$ operator on sentence families is that it commutes with inverse surjective morphisms.

Lemma 95 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ *be* $N^\flat$*-algebraic systems and* $\langle F, \alpha \rangle : \mathbf{A} \to \mathbf{A}'$ *be a surjective morphism. Then, for all* $E \subseteq N^\flat$*, we have*

$$\alpha^{-1}(\overleftarrow{E^{\mathbf{A}'}}(T)) = \overleftarrow{E^{\mathbf{A}}}(\alpha^{-1}(T)), \quad \text{for all } T \in \mathrm{SenFam}(\mathbf{A}').$$

Proof: Let $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi} \in \mathrm{SEN}(\Sigma)$. Then we have $\vec{\phi} \in \alpha_\Sigma^{-1}(\overleftarrow{E^{\mathbf{A}'}_{F(\Sigma)}}(T))$ iff $\alpha_\Sigma(\vec{\phi}) \in \overleftarrow{E^{\mathbf{A}'}_{F(\Sigma)}}(T)$ iff $E^{\mathbf{A}'}_{F(\Sigma)}[\alpha_\Sigma(\vec{\phi})] \leq T$ iff, by surjectivity, for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

$$E^{\mathbf{A}'}_{F(\Sigma')}(\mathrm{SEN}'(F(f))(\alpha_\Sigma(\vec{\phi})), \alpha_{\Sigma'}(\vec{\chi})) \subseteq T_{F(\Sigma')}$$

iff for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

$$E^{\mathbf{A}'}_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}(f)(\vec{\phi})), \alpha_{\Sigma'}(\vec{\chi})) \subseteq T_{F(\Sigma')}$$

iff for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

$$\alpha_{\Sigma'}(E^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\vec{\phi}), \vec{\chi})) \subseteq T_{F(\Sigma')}$$

iff for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

$$E^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\vec{\phi}), \vec{\chi}) \subseteq \alpha_{\Sigma'}^{-1}(T_{F(\Sigma')})$$

iff $E^{\mathbf{A}}_\Sigma[\vec{\phi}] \leq \alpha^{-1}(T)$ iff $\vec{\phi} \in \overleftarrow{E^{\mathbf{A}}_\Sigma}(\alpha^{-1}(T))$. ■

On the other hand, there is also a relationship between the operator $\overleftarrow{E}$ and images under morphisms.

Lemma 96 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ *be* $N^\flat$*-algebraic systems,* $\langle F, \alpha \rangle : \mathbf{A} \to \mathbf{A}'$ *be a morphism and* $E \subseteq N$*. Then, for all* $\Sigma \in |\mathbf{Sign}|$*, all* $\vec{\phi} \in \mathrm{SEN}(\Sigma)$ *and all* $\Sigma' \in |\mathbf{Sign}|$*, we have*

$$\alpha_{\Sigma'}(E_{\Sigma,\Sigma'}[\vec{\phi}]) \subseteq E'_{F(\Sigma),F(\Sigma')}[\alpha_\Sigma(\vec{\phi})],$$

with equality holding in case $\langle F, \alpha \rangle$ *is surjective.*

Proof: Let $\varepsilon \in E$ and $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\vec{\chi} \in \mathrm{SEN}(\Sigma')$. Then, we have

$$\begin{aligned}
&\alpha_{\Sigma'}(\varepsilon_{\Sigma'}(\mathrm{SEN}(f)(\vec{\phi}), \vec{\chi})) \\
&\quad = \varepsilon'_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}(f)(\vec{\phi})), \alpha_{\Sigma'}(\vec{\chi})) \\
&\quad = \varepsilon'_{F(\Sigma')}(\mathrm{SEN}'(F(f))(\alpha_\Sigma(\vec{\phi})), \alpha_{\Sigma'}(\vec{\chi})) \\
&\quad \in E'_{F(\Sigma),F(\Sigma')}[\alpha_\Sigma(\vec{\phi})].
\end{aligned}$$

If $\langle F,\alpha\rangle$ is surjective, then every element in $E'_{F(\Sigma),F(\Sigma')}[\alpha_\Sigma(\vec{\phi})]$ is of the form $\varepsilon'_{F(\Sigma')}(\mathrm{SEN}'(F(f))(\alpha_\Sigma(\vec{\phi})),\alpha_{\Sigma'}(\vec{\chi}))$, for some $\varepsilon\in E$, $\Sigma'\in|\mathbf{Sign}|$, $f\in\mathbf{Sign}(\Sigma,\Sigma')$ and $\vec{\chi}\in\mathrm{SEN}(\Sigma')$. Thus, by following the preceding equalities bottom-up, we get the reverse inclusion. ∎

Finally, we prove a close relationship between $\overleftarrow{E^{\mathbf{A}}}$, where E is a collection of natural transformations, with two distinguished arguments, and the Leibniz operator on the algebraic system $\mathbf{A}$.

Proposition 97 *Let $\mathbf{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$ be an algebraic system, $E\subseteq N$, with two distinguished arguments, and $T\in\mathrm{SenFam}(\mathbf{A})$. If $\overleftarrow{E}(T)$ is a reflexive relation system on $\mathbf{A}$, then*

$$\Omega^{\mathbf{A}}(T)\leq\overleftarrow{E}(T).$$

Proof: Suppose that $\Sigma\in|\mathbf{Sign}|$ and $\phi,\psi\in\mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle\in\Omega^{\mathbf{A}}_\Sigma(T)$. Since $\Omega^{\mathbf{A}}(T)$ is a congruence system, we have, for all $\sigma\in E\subseteq N$ and all $\Sigma'\in|\mathbf{Sign}|$, $f\in\mathbf{Sign}(\Sigma,\Sigma')$, $\vec{\chi}\in\mathrm{SEN}(\Sigma')$,

$$\begin{aligned}&\langle\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi),\mathrm{SEN}(f)(\psi),\vec{\chi}),\\&\qquad\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi),\mathrm{SEN}(f)(\phi),\vec{\chi})\rangle\in\Omega^{\mathbf{A}}_{\Sigma'}(T).\end{aligned}$$

By the assumption of reflexivity, we get that, for all $\sigma\in E$, all $\Sigma'\in|\mathbf{Sign}|$, $f\in\mathbf{Sign}(\Sigma,\Sigma')$ and $\vec{\chi}\in\mathrm{SEN}(\Sigma')$, $\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi),\mathrm{SEN}(f)(\phi),\vec{\chi})\in T_{\Sigma'}$. Therefore, by the compatibility of $\Omega^{\mathbf{A}}(T)$ with T, we conclude that, for all $\sigma\in E$, all $\Sigma'\in|\mathbf{Sign}|$, $f\in\mathbf{Sign}(\Sigma,\Sigma')$ and $\vec{\chi}\in\mathrm{SEN}(\Sigma')$,

$$\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi),\mathrm{SEN}(f)(\psi),\vec{\chi})\in T_{\Sigma'}.$$

This means that $E_\Sigma[\phi,\psi]\leq T$, or equivalently, $\langle\phi,\psi\rangle\in\overleftarrow{E_\Sigma}(T)$. Therefore, $\Omega^{\mathbf{A}}(T)\leq\overleftarrow{E}(T)$. ∎

Proposition 97 allows us to conclude that in cases where $\overleftarrow{E}(T)$ is actually a congruence system compatible with the sentence family T, it coincides with the Leibniz congruence system of T on $\mathbf{A}$.

Corollary 98 *Let $\mathbf{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$ be an algebraic system, $E\subseteq N$, with two distinguished arguments, and $T\in\mathrm{SenFam}(\mathbf{A})$. If $\overleftarrow{E}(T)$ is a congruence system on $\mathbf{A}$ compatible with T, then*

$$\overleftarrow{E}(T)=\Omega^{\mathbf{A}}(T).$$

Proof: Since, by hypothesis, $\overleftarrow{E}(T)$ is a congruence system on $\mathbf{A}$, it is reflexive. So, by Proposition 97, we have $\Omega^{\mathbf{A}}(T)\leq\overleftarrow{E}(T)$. On the other hand, since it is a congruence system on $\mathbf{A}$ compatible with T and, by definition, $\Omega^{\mathbf{A}}(T)$ is the largest such, we get that $\overleftarrow{E}(T)\leq\Omega^{\mathbf{A}}(T)$. We conclude that $\overleftarrow{E}(T)=\Omega^{\mathbf{A}}(T)$. ∎

2.14 Global versus Local Membership

We turn now to exploring some syntactic conditions with respect to morphisms, parameters and theory families in a π-institution. We consider the following setting: Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $E \subseteq N^\flat$ a collection of natural transformations in $N^\flat$, with k distinguished arguments, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^k$.

- We say $\vec{\phi}$ is E-**locally in** T if, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

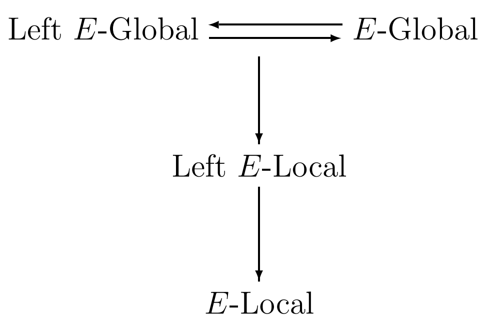
$$E_\Sigma(\vec{\phi}, \vec{\chi}) \subseteq T_\Sigma.$$

- We say that $\vec{\phi}$ is E-**globally in** T if

$$E_\Sigma[\vec{\phi}] \leq T.$$

- We say $\vec{\phi}$ is **left** E-**locally in** T if it is E-locally in $\overleftarrow{T}$.
- Similarly, $\vec{\phi}$ is **left** E-**globally in** T if it is E-globally in $\overleftarrow{T}$.

We show next that these properties satisfy the following diagram, where arrows are implications pointing from the stronger to the weaker property. After the lemma proving this result, we construct some examples showing that all implications are proper (i.e., none of them are equivalences in general for arbitrary π-institutions).



Proposition 99 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $E \subseteq N^\flat$, with k distinguished variables, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^k$.*

(a) $\vec{\phi}$ is left E-globally in T if and only if it is E-globally in T.

(b) If $\vec{\phi}$ is E-globally in T, then it is left E-locally in T. The implication becomes an equivalence if all arguments in E are distinguished (i.e., there are no parameters).

(c) *If $\vec{\phi}$ is left E-locally in T, then it is E-locally in T. The implication becomes an equivalence if $T \in \mathrm{ThSys}(\mathcal{I})$.*

Proof:

(a) If $\vec{\phi}$ is left E-globally in T, then $E_\Sigma[\vec{\phi}] \leq \overleftarrow{T}$. But, by Proposition 2, $\overleftarrow{T} \leq T$, whence $E_\Sigma[\vec{\phi}] \leq T$. Thus, $\vec{\phi}$ is E-globally in T.

Suppose, conversely, that $\vec{\phi}$ is E-globally in T. Then, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$E_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}), \vec{\chi}) \subseteq T_{\Sigma'}.$$

As a special case, we get that, for all $\Sigma', \Sigma'' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, $g \in \mathbf{Sign}^\flat(\Sigma', \Sigma'')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\begin{array}{ccccc} \Sigma & \xrightarrow{\quad f \quad} & \Sigma' & \xrightarrow{\quad g \quad} & \Sigma'' \\ \vec{\phi} & \longmapsto & \mathrm{SEN}^\flat(f)(\vec{\phi}) & \longmapsto & \mathrm{SEN}^\flat(g \circ f)(\vec{\phi}) \\ & & \vec{\chi} & \longmapsto & \mathrm{SEN}^\flat(g)(\vec{\chi}) \end{array}$$

$$E_{\Sigma''}(\mathrm{SEN}^\flat(g \circ f)(\vec{\phi}), \mathrm{SEN}^\flat(g)(\vec{\chi})) \subseteq T_{\Sigma''}.$$

So $\mathrm{SEN}^\flat(g)(E_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}), \vec{\chi})) \subseteq T_{\Sigma''}$. Since this holds for all $\Sigma'' \in |\mathbf{Sign}^\flat|$ and all $g \in \mathbf{Sign}^\flat(\Sigma', \Sigma'')$, we get $E_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}), \vec{\chi}) \subseteq \overleftarrow{T}_{\Sigma'}$. Since this holds for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, we get $E_\Sigma[\vec{\phi}] \leq \overleftarrow{T}$. We now conclude that $\vec{\phi}$ is left E-globally in T.

(b) Suppose $\vec{\phi}$ is E-globally in T. Then, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$E_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}), \vec{\chi}) \subseteq T_{\Sigma'}.$$

Thus, in particular, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $E_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}), \mathrm{SEN}^\flat(f)(\vec{\chi})) \subseteq T_{\Sigma'}$. Therefore,

$$\mathrm{SEN}^\flat(f)(E_\Sigma(\vec{\phi}, \vec{\chi})) \subseteq T_{\Sigma'},$$

which shows that $E_\Sigma(\vec{\phi}, \vec{\chi}) \subseteq \overleftarrow{T}_\Sigma$. So $\vec{\phi}$ is left E-locally in T.

Finally, assume all arguments in E are distinguished. Then, if $\vec{\phi}$ is left E-locally in T, we have $E_\Sigma(\vec{\phi}) \subseteq \overleftarrow{T}_\Sigma$, whence, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$E_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi})) = \mathrm{SEN}^\flat(f)(E_\Sigma(\vec{\phi})) \subseteq T_{\Sigma'}.$$

Hence $E_\Sigma[\vec{\phi}] \leq T$ and $\vec{\phi}$ is E-globally in T.

(c) The implication holds, exactly as the left to right implication in Part (a), because $\overleftarrow{T} \leq T$, for all $T \in \mathrm{ThFam}(\mathcal{I})$. The equivalence statement holds because, by Proposition 2, $\overleftarrow{T} = T$, whenever $T \in \mathrm{ThSys}(\mathcal{I})$. ■

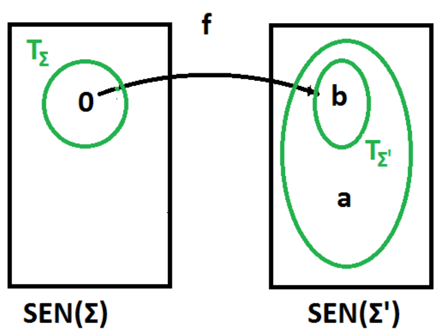
We provide examples to show that the implications in Proposition 99 are proper in general, i.e., they are not equivalences for arbitrary π-institutions, arbitrary sets of natural transformations E and arbitrary theory families T.

Example 100 *Define the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *as follows:*

- $\mathbf{Sign}^\flat$ *is the category with two objects* Σ, Σ' *and a single non-identity morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is determined by* $\mathrm{SEN}^\flat(\Sigma) = \{0\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ *and* $\mathrm{SEN}^\flat(f) : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SEN}^\flat(\Sigma')$*, given by* $\mathrm{SEN}^\flat(f)(0) = b$*;*
- $N^\flat$ *is the category of natural transformations generated by the binary transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$*, determined by the following tables:*

$\sigma^\flat_\Sigma$	
0	0

$\sigma^\flat_{\Sigma'}$	a	b
a	b	b
b	a	b



Consider the closure system C *on* $\mathbf{F}$ *defined by setting*

$$\mathcal{C}_\Sigma = \{\{0\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\}, \{a, b\}\}$$

and let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the associated* π*-institution.*

Finally, take $T = \{T_\Sigma, T_{\Sigma'}\} \in \mathrm{ThFam}(\mathcal{I})$ *to be the theory family specified by*

$$T_\Sigma = \{0\} \quad \textit{and} \quad T_{\Sigma'} = \{b\}$$

and consider $E = \{\sigma^\flat\} \subseteq N^\flat$*, with one distinguished argument. Notice that* $\overleftarrow{T} = T$*.*

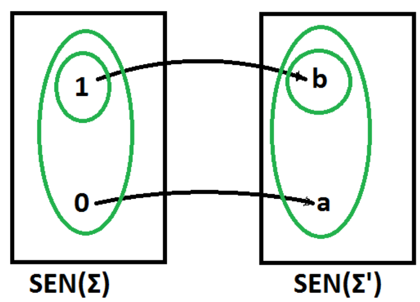
We now have $\sigma^\flat_\Sigma(0, 0) \in T_\Sigma = \overleftarrow{T}_\Sigma$*. Thus, 0 is* E*-left locally in* T*. On the other hand* $\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(0), a) = \sigma^\flat_{\Sigma'}(b, a) = a \notin T_{\Sigma'}$*. Therefore 0 is not* E*-globally in* T*.*

Example 101 *Define the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *as follows:*

- $\mathbf{Sign}^\flat$ *is the category with two objects* Σ, Σ' *and a single non-identity morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is determined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f) : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SEN}^\flat(\Sigma')$*, given by* $\mathrm{SEN}^\flat(f)(0) = a$ *and* $\mathrm{SEN}^\flat(f)(1) = b$*;*
- $N^\flat$ *is the category of natural transformations generated by the binary transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$*, determined by the following tables:*

$\sigma^\flat_\Sigma$	0	1
0	1	1
1	0	1

$\sigma^\flat_{\Sigma'}$	a	b
a	b	b
b	a	b



Consider the closure system C *on* $\mathbf{F}$ *defined by setting*

$$\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\},\{a,b\}\}$$

and let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the associated* π*-institution.*

Finally, take $T = \{T_\Sigma, T_{\Sigma'}\} \in \mathrm{ThFam}(\mathcal{I})$ *to be the theory family specified by*

$$T_\Sigma = \{0,1\} \quad \text{and} \quad T_{\Sigma'} = \{b\}$$

and consider $E = \{\sigma^\flat\} \subseteq N^\flat$*, with one distinguished argument. Notice that we have* $\overleftarrow{T} = \{\{1\},\{b\}\}$*.*

Since $\sigma^\flat_\Sigma(1,0) = 0$ *and* $\sigma^\flat_\Sigma(1,1) = 1$ *are both in* T_Σ*, we conclude that* 1 *is* E*-locally in* T*. On the other hand,* $\sigma^\flat_\Sigma(1,0) = 0 \notin \overleftarrow{T}_\Sigma$*. Thus* 1 *is not left* E*-locally in* T*.*

Let again $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $E \subseteq N^\flat$ a set of natural transformations, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $T \in \mathrm{ThFam}(\mathcal{I})$. Quantifying over all signatures and all sentences, we get the following definitions:

- We say E is **locally in** T if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,
$$E_\Sigma(\vec{\phi}, \vec{\chi}) \subseteq T_\Sigma.$$
- We say E is **left locally in** T if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,
$$E_\Sigma(\vec{\phi}, \vec{\chi}) \subseteq \overleftarrow{T}_\Sigma.$$
- We say E is **globally in** T if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,
$$E_\Sigma[\vec{\phi}] \leq T.$$

Of course, we have, taking into account Proposition 99:

Corollary 102 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $E \subseteq N^\flat$, *with* k *distinguished arguments,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution, based on* $\mathbf{F}$, *and* $T \in \mathrm{ThFam}(\mathcal{I})$.

(a) If E *is globally in* T, *then it is left locally in* T;

(b) If E *is left locally in* T, *then it is locally in* T.

Proof: Directly by Proposition 99. ∎

But Corollary 102 gives only half the true story. It turns out all three universal properties are equivalent.

Proposition 103 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $E \subseteq N^\flat$, *with* k *distinguished arguments,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution, based on* $\mathbf{F}$, *and* $T \in \mathrm{ThFam}(\mathcal{I})$. E *is globally in* T *if and only if it is locally in* T.

Proof: By Corollary 102, it suffices to show that, if E is locally in T, then it is also globally in T. To this end, suppose E is locally in T, i.e., that for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,
$$E_\Sigma(\vec{\phi}, \vec{\psi}) \subseteq T_\Sigma.$$
Thus, in particular, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,
$$\begin{array}{ccc} \Sigma & \xrightarrow{\quad f \quad} & \Sigma' \\ \vec{\phi} & \longmapsto & \mathrm{SEN}^\flat(f)(\vec{\phi}) \\ & & \vec{\chi} \end{array}$$
$$E_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}), \vec{\chi}) \subseteq T_{\Sigma'}.$$
But this is equivalent to $E_\Sigma[\vec{\phi}] \leq T$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$. Thus, E is globally in T. ∎

2.15 Global Properties and Parameters

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. Every $\sigma^\flat \in N^\flat$ has finite arity, but, when the exact arity is unimportant, we will write

$$\sigma^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell.$$

As already mentioned at the beginning of Section 2.13, this is also convenient in case we are dealing with a set $S^\flat \subseteq N^\flat$. In that case the set of arities of the natural transformations in $S^\flat$ may be unbounded and we write

$$S^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell,$$

even though, again, the arity of each member of $S^\flat$ is finite. Finally, we denote

$$p^k := \langle p^{k,0}, p^{k,1}, \ldots, p^{k,k-1} \rangle : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^k$$

the identity natural transformation, being a tuple of the appropriate k-ary projections.

For all $\sigma^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$ in $N^\flat$, with k distinguished arguments, we denote by

$$\dot{\sigma}^\flat : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^\ell$$

the collection of k-ary natural transformations in $N^\flat$, defined by

$$\dot{\sigma}^\flat = \{\sigma^\flat \circ \langle p^k, \tau^\flat \rangle : \tau^\flat : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^\omega \in N^\flat\}.$$

More generally, given a collection $S^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$ in $N^\flat$, with k distinguished arguments, we denote by

$$\dot{S}^\flat : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^\ell$$

the collection of all k-ary natural transformations in $N^\flat$ defined by

$$\dot{S}^\flat = \bigcup \{\dot{\sigma}^\flat : \sigma^\flat \in S^\flat\}.$$

Concerning these definitions, we adopt the following conventions:

1. If $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^\ell$ is k-ary, with k distinguished arguments, i.e., is thought of as parameter free, then $\dot{\sigma}^\flat = \{\sigma^\flat\}$. In this case, we identify the singleton $\dot{\sigma}^\flat$ with $\sigma^\flat$, the unique element that it contains. Similarly, for a parameterless collection $S^\flat : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^\ell$ in $N^\flat$, we identify $\dot{S}^\flat$ with $S^\flat$.

2. If $\sigma^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$ has 2 distinguished arguments, we write $\ddot{\sigma}^\flat : (\mathrm{SEN}^\flat)^2 \to (\mathrm{SEN}^\flat)^\ell$ for the collection $\dot{\sigma}^\flat$ to emphasize the binary character of $\ddot{\sigma}^\flat$. More generally, $\ddot{S}^\flat : (\mathrm{SEN}^\flat)^2 \to (\mathrm{SEN}^\flat)^\ell$ stands for the collection $\dot{S}^\flat$, when $S^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$ has 2 distinguished arguments.

We have the following relation concerning global membership based on a set of natural transformations and membership based on the corresponding parameter free counterpart.

Lemma 104 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and consider a collection* $S^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$ *of natural transformations in* $N^\flat$*, with* k *distinguished arguments. Then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^k$*,*

$$\dot{S}^\flat_\Sigma[\vec{\phi}] \leq S^\flat_\Sigma[\vec{\phi}].$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^k$. Then, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, we have

$$\begin{array}{rcl} \dot{S}^\flat_{\Sigma,\Sigma'}[\vec{\phi}] & = & \bigcup_{f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')} \{\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}), \tau^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}))) : \\ & & \qquad \sigma^\flat \in S^\flat, \tau^\flat \in N^\flat\} \quad \text{(by definition)} \\ & \subseteq & \bigcup_{f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')} \{\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}), \vec{\chi}) : \sigma^\flat \in S^\flat, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')\} \\ & & \text{(set theoretic)} \\ & = & S^\flat_{\Sigma,\Sigma'}[\vec{\phi}]. \quad \text{(by definition)} \end{array}$$

Since $\Sigma' \in |\mathbf{Sign}^\flat|$ was arbitrary, we conclude that $\dot{S}^\flat_\Sigma[\vec{\phi}] \leq S^\flat_\Sigma[\vec{\phi}]$. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $S^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$ a collection of natural transformations in $N^\flat$, with k distinguished arguments, and $T \in \mathrm{SenFam}(\mathbf{F}^\ell)$. Recall that by $\overleftarrow{S^\flat}(T)$ is denoted the k-ary relation system $\overleftarrow{S^\flat}(T) = \{\overleftarrow{S^\flat_\Sigma}(T)\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ on $\mathbf{F}$, given, for all $\Sigma \in |\mathbf{Sign}^\flat|$, by

$$\overleftarrow{S^\flat_\Sigma}(T) = \{\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^k : S^\flat_\Sigma[\vec{\phi}] \leq T\}.$$

Then we obtain

Corollary 105 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and consider a collection* $S^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$ *of natural transformations in* $N^\flat$*, with* k *distinguished arguments. Then, for all* $T \in \mathrm{SenFam}(\mathbf{F}^\ell)$*,*

$$\overleftarrow{S^\flat}(T) \leq \overleftarrow{\dot{S}^\flat}(T).$$

Proof: We have, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\begin{array}{rcl} \overleftarrow{S^\flat_\Sigma}(T) & = & \{\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^k : S^\flat_\Sigma[\vec{\phi}] \leq T\} \quad \text{(definition)} \\ & \subseteq & \{\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^k : \dot{S}^\flat_\Sigma[\vec{\phi}] \leq T\} \quad \text{(Lemma 104)} \\ & = & \overleftarrow{\dot{S}^\flat_\Sigma}(T). \quad \text{(definition)} \end{array}$$

We conclude that $\overleftarrow{S^\flat}(T) \leq \overleftarrow{\dot{S}^\flat}(T)$. ∎

We now turn to collections of natural transformations satisfying certain properties globally.

For fixed k, we assume P is a **(antimonotone) global property** of natural transformations $\sigma^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$ in $N^\flat$, with k distinguished arguments. That is:

(a) For $\sigma^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$ in $N^\flat$, with k distinguished arguments, $\sigma^\flat$ either does or does not satisfy P;

(b) For every $\sigma^\flat, \tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$ in $N^\flat$, with k distinguished arguments, if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^k$, $\sigma^\flat_\Sigma[\vec{\phi}] \leq \tau^\flat_\Sigma[\vec{\phi}]$, then, if $\tau^\flat$ satisfies P, then $\sigma^\flat$ also satisfies P.

For instance, given $T \in \mathrm{SenFam}(\mathbf{F}^\ell)$,

$$P^T(\sigma): \quad \text{for all } \Sigma \in |\mathbf{Sign}^\flat| \text{ and all } \vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^k, \quad \sigma_\Sigma[\vec{\phi}] \leq T$$

is a global property of natural transformations in $N^\flat$, with k distinguished arguments.

Given such a global property P, we denote by $P^\flat \subseteq N^\flat$ the collection of all $\sigma^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$ in $N^\flat$, with k distinguished arguments, that satisfy property P:

$$P^\flat = \{\sigma^\flat \in N^\flat : P(\sigma^\flat)\}.$$

We call $P^\flat$ the P-**core** of $N^\flat$.

As an example, for the property P^T introduced above, based on a fixed sentence family $T \in \mathrm{SenFam}(\mathbf{F}^\ell)$, we have

$$P^{T\flat} = \{\sigma^\flat \in N^\flat : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^k)(\sigma^\flat_\Sigma[\vec{\phi}] \leq T)\}.$$

Along the same lines, given a global property P of natural transformations in $N^\flat$, with k distinguished arguments, we may consider the restriction $\hat{P}$ of P to the collection of parameter free k-ary natural transformations in $N^\flat$:

$$\hat{P}: \quad \sigma : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^\ell \in N^\flat \text{ and } P(\sigma).$$

Then we define

$$\hat{P}^\flat = \{\sigma^\flat \in N^\flat : \hat{P}(\sigma^\flat)\}.$$

We call $\hat{P}^\flat$ the k-**ary** P-**core** of $N^\flat$ or the **parameter free** P-**core** of $N^\flat$. The following inclusion is straightforward:

Lemma 106 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and consider a global property* P *of natural transformations* $\sigma^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$ *in* $N^\flat$*, with* k *distinguished arguments. Then*

$$\hat{P}^\flat \subseteq P^\flat.$$

Proof: Straightforward by the definition of $\hat{P}$, since any parameter free k-ary natural transformation is a natural transformation with k distinguished arguments and no parameters. ■

Let P be a global property of natural transformations in $N^\flat$, with k distinguished arguments. We have now defined two sets of k-ary natural transformations in $N^\flat$ associated with P:

- The first set is $\dot{P}^\flat$, obtained by $P^\flat$ by applying the dot operator;
- The second is the set $\hat{P}^\flat$ obtained by restricting the property P on the subfamily of parameter free k-ary natural transformations in $N^\flat$.

In the main theorem of the section we show that, for any global property P of natural transformations in $N^\flat$, with k distinguished arguments, these two sets are identical, i.e., $\hat{P}^\flat = \dot{P}^\flat$.

Theorem 107 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and consider a global property* P *of natural transformations* $\sigma^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$ *in* $N^\flat$*, with* k *distinguished arguments. Then*

$$\hat{P}^\flat = \dot{P}^\flat.$$

Proof: Suppose, first, that $\sigma^\flat \in \hat{P}^\flat$. Then, by definition, $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^\ell$ is parameter free and satisfies P. Thus, we have $\sigma^\flat \in P^\flat$ and $\sigma^\flat = \dot{\sigma}^\flat \in \dot{P}^\flat$. Therefore, $\hat{P}^\flat \subseteq \dot{P}^\flat$.

Suppose, conversely, that $\rho^\flat \in \dot{P}^\flat$. Then, by definition, there exists $\sigma^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^\ell$, with k distinguished arguments, in $P^\flat$ and $\tau^\flat : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^\omega$ in $N^\flat$, such that

$$\rho^\flat = \sigma^\flat \circ \langle p^k, \tau^\flat \rangle : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^\ell.$$

Noting that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$, $\rho^\flat_\Sigma[\vec{\phi}] \le \sigma^\flat_\Sigma[\vec{\phi}]$, and taking into account that P is global and that $\sigma^\flat \in P^\flat$, we obtain that $\rho^\flat \in P^\flat$. But $\rho^\flat : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^\ell$ is also parameter free. Therefore $\rho^\flat \in \hat{P}^\flat$. We conclude that $\dot{P}^\flat \subseteq \hat{P}^\flat$. ■

2.16 Finitarity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

$\mathcal{I}$ is **finitary** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$, if $\phi \in C_\Sigma(\Phi)$, then, there exists finite $\Psi \subseteq \Phi$, such that $\phi \in C_\Sigma(\Psi)$.

Equivalently, $\mathcal{I}$ is finitary if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$C_\Sigma(\Phi) = \bigcup \{ C_\Sigma(\Psi) : \Psi \subseteq_\omega \Phi \},$$

where $\Psi \subseteq_\omega \Phi$ denotes the finite subset relation.

Yet another well-known equivalent characterization of finitarity asserts that $\mathcal{I}$ is finitary if and only if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and every upward directed collection $\{T^i_\Sigma : i \in I\}$ of Σ-theories, i.e., a collection, such that:

- $C_\Sigma(T^i_\Sigma) = T^i_\Sigma$, for all $i \in I$;
- for every $i, j \in I$, there exists $k \in I$, such that $T^i_\Sigma, T^j_\Sigma \subseteq T^k_\Sigma$,

the union $\bigcup_{i \in I} T^i_\Sigma$ is also a Σ-theory.

We formulate next some versions of these properties with reference to theory families.

Let **Sign** be a category and SEN : **Sign** → **Set** a functor. A sentence family $X \in \mathrm{SenFam}(\mathrm{SEN})$ is called **locally finite** if, for all $\Sigma \in |\mathbf{Sign}|$, X_Σ is finite. In this case we write $|X| <_l \omega$. Given sentence families $X, Y \in \mathrm{SenFam}(\mathrm{SEN})$, we use the notation $X \leq_{lf} Y$ to suggest that X is a locally finite subfamily of Y.

We say that a collection $\{X^i : i \in I\} \subseteq \mathrm{SenFam}(\mathrm{SEN})$ is:

- **locally directed** if, for all $\Sigma \in |\mathbf{Sign}|$ and all finite $J \subseteq I$, there exists $k \in I$, such that $X^j_\Sigma \leq X^k_\Sigma$, for all $j \in J$;
- **directed** if, for all finite $J \subseteq I$, there exists $k \in I$, such that $X^j \leq X^k$, for all $j \in J$.

Directedness is a stronger property than local directedness.

Lemma 108 *Let* **Sign** *be a category,* SEN : **Sign** → **Set** *a sentence functor and* $\{T^i : i \in I\} \subseteq \mathrm{SenFam}(\mathrm{SEN})$*. If* $\{T^i\}$ *is directed, then it is locally directed.*

Proof: Suppose $\{T^i\}$ is directed. Let $\Sigma \in |\mathbf{Sign}|$ and $i, j \in I$. Since $\{T^i\}$ is directed, there exists a $k \in I$, such that $T^i, T^j \leq T^k$. In particular, $T^i_\Sigma, T^j_\Sigma \subseteq T^k_\Sigma$. Therefore, $\{T^i\}$ is also locally directed. ∎

The opposite implication patently fails, i.e., in general, local directedness does not imply directedness.

Example 109 *Let* **Sign** *be the discrete category with objects* Σ *and* Σ'*. Let* $\mathrm{SEN}^\flat : \mathbf{Sign} \to \mathbf{Set}$ *be defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$ *and* $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$*.*

Consider the sentence families $T = \{\{1\}, \{a, b\}\}$ *and* $T' = \{\{0, 1\}, \{b\}\}$ *and the collection* $\mathcal{T} = \{T, T'\}$*.*

$\mathcal{T}$ *is locally directed, since* $T_\Sigma \subseteq T'_\Sigma$ *and* $T'_{\Sigma'} \subseteq T_{\Sigma'}$*.*

On the other hand, $\mathcal{T}$ *is not directed since there does not exist* $X \in \mathcal{T}$*, such that* $T, T' \leq X$*.*

Given an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ and a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, based on $\mathbf{F}$, we say that $\mathcal{I}$ is:

- **locally continuous** if, for every locally directed family $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$,
$$\bigcup_{i\in I} T^i \in \mathrm{ThFam}(\mathcal{I});$$
- **continuous** if, for every directed family $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$,
$$\bigcup_{i\in I} T^i \in \mathrm{ThFam}(\mathcal{I}).$$

Since, by Lemma 108, directedness implies local directedness, we get the following straightforward relationship between local continuity and continuity.

Corollary 110 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is locally continuous, then it is continuous.*

Proof: Assume $\mathcal{I}$ is locally continuous and let $\mathcal{T} \subseteq \mathrm{ThFam}(\mathcal{I})$ be directed. By Lemma 108, $\mathcal{T}$ is locally directed. Thus, by local continuity, $\bigcup \mathcal{T} \in \mathrm{ThFam}(\mathcal{I})$. Hence, $\mathcal{I}$ is continuous. ■

However, more is true. In fact, continuity turns out to be equivalent to the seemingly stronger notion of local continuity.

Theorem 111 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is locally continuous if and only if it is continuous.*

Proof: The "only if" is by Corollary 110. Suppose, conversely, that $\mathcal{I}$ is continuous. Let $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$ be locally directed. We construct the collection

$$\mathcal{T} = \{T' \in \mathrm{ThFam}(\mathcal{I}) : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\exists i \in I)(T'_\Sigma = T^i_\Sigma)\}.$$

First, note that $\mathcal{T}$ is directed. In fact, let $T, T' \in \mathcal{T}$ and $\Sigma \in |\mathbf{Sign}^\flat|$. By the definition of $\mathcal{T}$, there exist $i(\Sigma), j(\Sigma) \in I$, such that $T_\Sigma = T^{i(\Sigma)}_\Sigma$ and $T'_\Sigma = T^{j(\Sigma)}_\Sigma$. Since $\{T^i : i \in I\}$ is locally directed, there exists $k(\Sigma) \in I$, such that $T^{i(\Sigma)}_\Sigma, T^{j(\Sigma)}_\Sigma \subseteq T^{k(\Sigma)}_\Sigma$. Consider $T'' = \{T''_\Sigma\}_{\Sigma\in|\mathbf{Sign}^\flat|}$, where, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$T''_\Sigma = T^{k(\Sigma)}_\Sigma.$$

Then $T'' \in \mathcal{T}$ and, moreover, $T, T' \leq T''$. Thus, $\mathcal{T}$ is indeed directed. Second, notice that $\bigcup \mathcal{T} = \bigcup_{i\in I} T^i$. Thus, taking into account the continuity of $\mathcal{I}$, we get

$$\bigcup_{i\in I} T^i = \bigcup \mathcal{T} \in \mathrm{ThFam}(\mathcal{I}).$$

Therefore, $\mathcal{I}$ is locally continuous. ■

Now we get the following characterizations of finitarity.

Proposition 112 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then the following conditions are equivalent:*

(i) $\mathcal{I}$ *is finitary;*

(ii) For every $X \in \mathrm{SenFam}(\mathbf{F})$*,*

$$C(X) = \bigcup \{C(Y) : Y \leq_{lf} X\};$$

(iii) $\mathcal{I}$ *is locally continuous.*

(iv) $\mathcal{I}$ *is continuous.*

Proof:

(i)⇒(ii) Suppose $\mathcal{I}$ is finitary and let $X \in \mathrm{SenFam}(\mathbf{F})$. Clearly, by the monotonicity of C, $\bigcup\{C(Y) : Y \leq_{lf} X\} \leq C(X)$. To prove the converse, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(X_\Sigma)$. By finitarity, there exists $Y_\Sigma \subseteq_f X_\Sigma$, such that $\phi \in C_\Sigma(Y_\Sigma)$. Now set $Y = \{Y_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}^\flat|}$, where, for all $\Sigma' \in |\mathbf{Sign}^\flat|$,

$$Y_{\Sigma'} = \begin{cases} Y_\Sigma, & \text{if } \Sigma' = \Sigma \\ \varnothing, & \text{if } \Sigma' \neq \Sigma \end{cases}$$

Clearly, $Y \leq_{lf} X$ and, moreover, $\phi \in C_\Sigma(Y)$. Thus, we get $C(X) \leq \bigcup\{C(Y) : Y \leq_{lf} X\}$.

(ii)⇒(iii) Suppose that, for every $X \in \mathrm{SenFam}(\mathbf{F})$, $C(X) = \bigcup\{C(Y) : Y \leq_{lf} X\}$ and let $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$ be locally directed. Consider $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\bigcup_{i \in I} T^i_\Sigma)$. By hypothesis, there exists locally finite $Y \leq \bigcup_{i \in I} T^i$, such that $\phi \in C_\Sigma(Y_\Sigma)$. Since $\{T^i : i \in I\}$ is locally directed, there exists $i \in I$, such $Y_\Sigma \subseteq T^i_\Sigma$. Now we get $\phi \in C_\Sigma(T^i_\Sigma) = T^i_\Sigma \subseteq \bigcup_{i \in I} T^i_\Sigma$. We conclude that $\bigcup_{i \in I} T^i$ is a theory family and, therefore, $\mathcal{I}$ is locally continuous.

(iii)⇒(i) Assume that $\mathcal{I}$ is locally continuous and let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$. We define a collection of theory families of $\mathcal{I}$ as follows: For every finite subset $\Psi \subseteq_f \Phi$, let $T^\Psi = \{T^\Psi_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}^\flat|}$ be given, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, by setting

$$T^\Psi_{\Sigma'} = \begin{cases} C_\Sigma(\Psi), & \text{if } \Sigma' = \Sigma \\ C_{\Sigma'}(\varnothing), & \text{if } \Sigma' \neq \Sigma \end{cases}$$

Clearly, $\{T^\Psi : \Psi \subseteq_f \Phi\}$ is a locally directed. Therefore, by hypothesis

$$C(\bigcup\{T^\Psi : \Psi \subseteq_f \Phi\}) = \bigcup\{T^\Psi : \Psi \subseteq_f \Phi\}.$$

Now we get
$$\phi \in C_\Sigma(\Phi) = C_\Sigma(\bigcup_{\Psi \subseteq_f \Phi} T_\Sigma^\Psi) = \bigcup_{\Psi \subseteq_f \Phi} T_\Sigma^\Psi,$$
whence $\phi \in T_\Sigma^\Psi = C_\Sigma(\Psi)$, for some $\Psi \subseteq_f \Phi$. We conclude that $\mathcal{I}$ is finitary.

(iii)⇔(iv) This is the content of Theorem 111.

■

We now prove a lemma concerning $\mathcal{I}$-filter generation to the effect that, for a finitary π-institution, the $\mathcal{I}$-filter generated by a certain sentence family can be built inductively by "closing under consequences" in a structured way.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a finitary π-institution based on $\mathbf{F}$. Then, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $X \in \mathrm{SenFam}(\mathcal{A})$, we define
$$\Xi^{\mathcal{I},\mathcal{A},n}(X) = \{\Xi_\Sigma^{\mathcal{I},\mathcal{A},n}(X)\}_{\Sigma \in |\mathbf{Sign}|}, \quad n < \omega,$$
by induction on n, as follows:

- If $n = 0$, $\Xi^{\mathcal{I},\mathcal{A},0}(X) = X$;
- Assume $\Xi^{\mathcal{I},\mathcal{A},i}(X)$ has been defined, for all $i < n$. We define
$$\Xi^{\mathcal{I},\mathcal{A},n}(X) = \{\Xi_\Sigma^{\mathcal{I},\mathcal{A},n}(X)\}_{\Sigma \in |\mathbf{Sign}|},$$
by setting, for all $\Sigma' \in |\mathbf{Sign}|$,
$$\begin{aligned}\Xi_{\Sigma'}^{\mathcal{I},\mathcal{A},n}(X) \quad = \quad & \{\alpha_\Sigma(\phi) : \Sigma \in |\mathbf{Sign}^\flat|, \text{ such that } F(\Sigma) = \Sigma', \\ & \text{and } \Phi \cup \{\phi\} \subseteq_\omega \mathrm{SEN}^\flat(\Sigma), \text{ such that} \\ & \phi \in C_\Sigma(\Phi) \text{ and } \alpha_\Sigma(\Phi) \subseteq \Xi_{\Sigma'}^{\mathcal{I},\mathcal{A},n-1}(X)\}.\end{aligned}$$

We may write the latter set more concisely as
$$\Xi_{\Sigma'}^{\mathcal{I},\mathcal{A},n}(X) = \bigcup_{\Sigma : F(\Sigma) = \Sigma'} \{\alpha_\Sigma(\phi) : \phi \in C_\Sigma(\Phi), \alpha_\Sigma(\Phi) \subseteq \Xi_{\Sigma'}^{\mathcal{I},\mathcal{A},n-1}(X)\}.$$

We prove some basic properties of this set.

Lemma 113 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a finitary* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $X \in \mathrm{SenFam}(\mathcal{A})$.

(a) For all $n < \omega$, $\Xi^{\mathcal{I},\mathcal{A},n}(X) \leq \Xi^{\mathcal{I},\mathcal{A},n+1}(X)$;

(b) $\bigcup_{n<\omega} \Xi^{\mathcal{I},\mathcal{A},n}(X) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$;

(c) $\bigcup_{n<\omega} \Xi^{\mathcal{I},\mathcal{A},n}(X) \leq C^{\mathcal{I},\mathcal{A}}(X)$.

Proof:

(a) Let $\Sigma' \in |\mathbf{Sign}|$, $\phi' \in \mathrm{SEN}(\Sigma)$, such that $\phi' \in \Xi^{\mathcal{I},\mathcal{A},n}_{\Sigma'}(X)$. By surjectivity of $\langle F,\alpha\rangle$, there exists $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $F(\Sigma) = \Sigma'$ and $\alpha_\Sigma(\phi) = \phi'$. But $\phi \in C_\Sigma(\phi)$ and $\alpha_\Sigma(\phi) = \phi' \in \Xi^{\mathcal{I},\mathcal{A},n}_{\Sigma'}(X)$ imply that $\phi' \in \alpha_\Sigma(\phi) \in \Xi^{\mathcal{I},\mathcal{A},n+1}_{\Sigma'}(X)$. So, we get $\Xi^{\mathcal{I},\mathcal{A},n}(X) \le \Xi^{\mathcal{I},\mathcal{A},n+1}(X)$.

(b) Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq_\omega \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$ and assume that $\alpha_\Sigma(\Phi) \subseteq \bigcup_{n<\omega} \Xi^{\mathcal{I},\mathcal{A},n}_{F(\Sigma)}(X)$. Then, since $\Phi \subseteq_\omega \mathrm{SEN}^\flat(\Sigma)$, there exists $n < \omega$, such that $\alpha_\Sigma(\Phi) \subseteq \Xi^{\mathcal{I},\mathcal{A},n}_{F(\Sigma)}(X)$. Since $\phi \in C_\Sigma(\Phi)$, we get, by the definition of $\Xi^{\mathcal{I},\mathcal{A},n+1}(X)$,

$$\alpha_\Sigma(\phi) \in \Xi^{\mathcal{I},\mathcal{A},n+1}_{F(\Sigma)}(X) \subseteq \bigcup_{n<\omega} \Xi^{\mathcal{I},\mathcal{A},n}_{F(\Sigma')}(X).$$

We conclude that $\bigcup_{n<\omega} \Xi^{\mathcal{I},\mathcal{A},n}(X) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

(c) We prove this by induction on $n < \omega$.

For $n = 0$, $\Xi^{\mathcal{I},\mathcal{A},0}(X) = X \le C^{\mathcal{I},\mathcal{A}}(X)$.

Suppose that $\Xi^{\mathcal{I},\mathcal{A},i}(X) \le C^{\mathcal{I},\mathcal{A}}(X)$, for all $i < n$.

Let $\Sigma' \in |\mathbf{Sign}|$, $\phi' \in \Xi^{\mathcal{I},\mathcal{A},n}_{\Sigma'}(X)$. Thus, there exists $\Sigma \in |\mathbf{Sign}^\flat|$, such that $F(\Sigma) = \Sigma'$, and $\Phi \cup \{\phi\} \subseteq_\omega \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$, $\alpha_\Sigma(\Phi) \subseteq \Xi^{\mathcal{I},\mathcal{A},n-1}_{\Sigma'}(X)$ and $\alpha_\Sigma(\phi) = \phi'$. By the induction hypothesis, $\alpha_\Sigma(\Phi) \subseteq C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(X)$. Hence, since $\phi \in C_\Sigma(\Phi)$ and $C^{\mathcal{I},\mathcal{A}}(X) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, it follows that $\phi' = \alpha_\Sigma(\phi) \in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(X)$. We conclude that $\Xi^{\mathcal{I},\mathcal{A},n}_{\Sigma'}(X) \subseteq C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(X)$.

It now follows that $\bigcup_{n<\omega} \Xi^{\mathcal{I},\mathcal{A},n}(X) \le C^{\mathcal{I},\mathcal{A}}(X)$.

■

We set

$$\Xi^{\mathcal{I},\mathcal{A}}(X) := \bigcup_{n<\omega} \Xi^{\mathcal{I},\mathcal{A},n}(X).$$

Proposition 114 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a finitary* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $X \in \mathrm{SenFam}(\mathcal{A})$*. Then*

$$C^{\mathcal{I},\mathcal{A}}(X) = \Xi^{\mathcal{I},\mathcal{A}}(X).$$

Proof: Since by Lemma 113, $\Xi^{\mathcal{I},\mathcal{A}}(X) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $X \le \Xi^{\mathcal{I},\mathcal{A}}(X)$, we get, by the minimality of $C^{\mathcal{I},\mathcal{A}}(X)$, that $C^{\mathcal{I},\mathcal{A}}(X) \le \Xi^{\mathcal{I},\mathcal{A}}(X)$. On the other hand, by Lemma 113, $\Xi^{\mathcal{I},\mathcal{A}}(X) \le C^{\mathcal{I},\mathcal{A}}(X)$. Thus, we conclude that $C^{\mathcal{I},\mathcal{A}}(X) = \Xi^{\mathcal{I},\mathcal{A}}(X)$. ■

2.17 Equational π-Institutions

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and K be a class of $\mathbf{F}$-algebraic systems. Denote by $\mathrm{Eq}(\mathbf{F}) = \{\mathrm{Eq}_\Sigma(\mathbf{F})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ the family of **$\mathbf{F}$-equations**, i.e., for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{Eq}_\Sigma(\mathbf{F}) = \mathrm{SEN}^\flat(\Sigma)^2.$$

The **equational consequence relative to K** or **K-equational consequence** is the closure family $D^{\mathsf{K}} : \mathcal{P}\mathrm{Eq}(\mathbf{F}) \to \mathcal{P}\mathrm{Eq}(\mathbf{F})$, defined by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$D^{\mathsf{K}}_\Sigma : \mathcal{P}(\mathrm{Eq}_\Sigma(\mathbf{F})) \to \mathcal{P}(\mathrm{Eq}_\Sigma(\mathbf{F}))$$

be given, for all $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$, by

$$\begin{array}{lll}\phi \approx \psi \in D^{\mathsf{K}}_\Sigma(E) & \text{iff} & \text{for all } \mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle \in \mathsf{K}, \\ & & \alpha_\Sigma(E) \subseteq \Delta^{\mathcal{A}}_{F(\Sigma)} \text{ implies } \alpha_\Sigma(\phi) = \alpha_\Sigma(\psi).\end{array}$$

This closure operator appeared, for the first time, in Section 2.4 as a means to characterize the relative congruence system generated by a family of equations, with respect to the class K of $\mathbf{F}$-algebraic systems. In Proposition 29, it was shown that, given an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ and a class K of $\mathbf{F}$-algebraic systems, $D^{\mathsf{K}} : \mathcal{P}\mathrm{Eq}(\mathbf{F}) \to \mathcal{P}\mathrm{Eq}(\mathbf{F})$ is a (not necessarily structural) closure family on $\mathrm{Eq}(\mathbf{F})$.

Moreover, it turns out that the closure family D^{K} satisfies the properties of reflexivity, symmetry, transitivity, congruence and invariance, detailed in the following

Proposition 115 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and let K be a class of $\mathbf{F}$-algebraic systems. For all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, all $\sigma^\flat$ in $N^\flat$, all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$, all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,*

(Reflexivity) $\phi \approx \phi \in D^{\mathsf{K}}_\Sigma(\varnothing)$;

(Symmetry) $\psi \approx \phi \in D^{\mathsf{K}}_\Sigma(\phi \approx \psi)$;

(Transitivity) $\phi \approx \chi \in D^{\mathsf{K}}_\Sigma(\phi \approx \psi, \psi \approx \chi)$;

(Congruence) $\sigma^\flat_\Sigma(\vec{\phi}) \approx \sigma^\flat_\Sigma(\vec{\psi}) \in D^{\mathsf{K}}_\Sigma(\{\phi_i \approx \psi_i : i \in I\})$;

(Invariance) $\mathrm{SEN}^\flat(f)(\phi) \approx \mathrm{SEN}^\flat(f)(\psi) \in D^{\mathsf{K}}_{\Sigma'}(\phi \approx \psi)$.

Proof: All properties follow directly by applying Proposition 30. ■

Corollary 116 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and K a class of $\mathbf{F}$-algebraic systems. Then $\mathrm{ThFam}(D^{\mathsf{K}}) = \mathrm{ThSys}(D^{\mathsf{K}}) = \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$.*

Proof: The first equality is a direct consequence of Invariance, given in Proposition 115, while the second follows directly from Theorem 32. ■

Assume, next, that $Q = \{Q_\Sigma\}_{\Sigma\in|\mathbf{Sign}^\flat|} \leq \mathrm{Eq}(\mathbf{F})$ is an $\mathbf{F}$-equation system, i.e., a family of $\mathbf{F}$-equations that is invariant under signature morphisms in the sense that, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\mathrm{SEN}^\flat(f)(Q_\Sigma) \subseteq Q_{\Sigma'}.$$

Let, also $E = \{E_\Sigma\}_{\Sigma\in|\mathbf{Sign}^\flat|} \leq \mathrm{Eq}(\mathbf{F})$ be an $\mathbf{F}$-equation family (not necessarily invariant under signature morphisms). We define, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $n < \omega$,

$$\Xi^{Q,n}_\Sigma(E) : \mathcal{P}(\mathrm{Eq}_\Sigma(\mathbf{F})) \to \mathcal{P}(\mathrm{Eq}_\Sigma(\mathbf{F})),$$

by induction on $n < \omega$, as follows:

- $\Xi^{Q,0}_\Sigma(E) = \{\phi \approx \phi : \phi \in \mathrm{SEN}^\flat(\Sigma)\} \cup Q_\Sigma \cup E_\Sigma$;
- Assuming that $\Xi^{Q,n}_\Sigma(E)$ has been defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$, we set

$$\begin{array}{rcl}\Xi^{Q,n+1}_\Sigma(E) & = & \{\psi \approx \phi : \phi \approx \psi \in \Xi^{Q,n}_\Sigma(E)\}\\ & & \cup\{\phi \approx \chi : \phi \approx \psi, \psi \approx \chi \in \Xi^{Q,n}_\Sigma(E)\}\\ & & \cup\{\sigma^\flat_\Sigma(\vec{\phi}) \approx \sigma^\flat_\Sigma(\vec{\psi}) : \phi_i \approx \psi_i \in \Xi^{Q,n}_\Sigma(E), i < k\}\\ & & \cup\{\mathrm{SEN}^\flat(f)(\phi \approx \psi) : \phi \approx \psi \in \Xi^{Q,n}_{\Sigma'}(E),\\ & & \qquad \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma', \Sigma)\}.\end{array}$$

Now, set, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\Xi^Q_\Sigma(E) = \bigcup_{n<\omega} \Xi^{Q,n}_\Sigma(E)$$

and, finally,

$$\Xi^Q(E) = \{\Xi^Q_\Sigma(E)\}_{\Sigma\in|\mathbf{Sign}^\flat|}.$$

We show that $\Xi^Q : \mathcal{P}(\mathrm{Eq}(\mathbf{F})) \to \mathcal{P}(\mathrm{Eq}(\mathbf{F}))$ is a closure family on $\mathrm{Eq}(\mathbf{F})$ that satisfies Reflexivity, Symmetry, Transitivity, Congruence and Invariance.

Proposition 117 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $Q \leq \mathrm{Eq}(\mathbf{F})$ an $\mathbf{F}$-equation system. Then $\Xi^Q : \mathcal{P}\mathrm{Eq}(\mathbf{F}) \to \mathcal{P}\mathrm{Eq}(\mathbf{F})$ is a closure family on $\mathrm{Eq}(\mathbf{F})$, that satisfies Reflexivity, Symmetry, Transitivity, Congruence and Invariance.*

Proof: We start by showing that Ξ^Q is a closure family.

- Let $\Sigma \in |\mathbf{Sign}^\flat|$, $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$, such that $\phi \approx \psi \in E$. Then, by definition, $\phi \approx \psi \in \Xi^{Q,0}_\Sigma(E) \subseteq \Xi^Q_\Sigma(E)$. So Ξ^Q is inflationary.

- Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $E \cup F \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$, such that $\phi \approx \psi \in \Xi^Q_\Sigma(E)$ and $E \subseteq F$. Then, for some $n < \omega$, $\phi \approx \psi \in \Xi^{Q,n}_\Sigma(E)$. We show by induction on $n < \omega$ that, for all $n < \omega$,

$$\phi \approx \psi \in \Xi^{Q,n}_\Sigma(E) \quad \text{implies} \quad \phi \approx \psi \in \Xi^{Q,n}_\Sigma(F).$$

  - For $n = 0$, we have $\phi = \psi$ or $\phi \approx \psi \in Q_\Sigma$ or $\phi \approx \psi \in E$. In the first two cases, $\phi \approx \psi \in \Xi^{Q,0}_\Sigma(F)$ by definition, and, in the last case, $\phi \approx \psi \in \Xi^{Q,0}_\Sigma(F)$, since $E \subseteq F$, by hypothesis.
  - Assume, next, that the statement holds for $n > 0$. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \approx \psi \in \Xi^{Q,n+1}_\Sigma(E)$.

    If $\psi \approx \phi \in \Xi^{Q,n}_\Sigma(E)$, then, by the induction hypothesis, $\psi \approx \phi \in \Xi^{Q,n}_\Sigma(F)$, whence, by definition, $\phi \approx \psi \in \Xi^{Q,n+1}_\Sigma(F)$.

    If $\phi \approx \chi, \chi \approx \psi \in \Xi^{Q,n}_\Sigma(E)$, then, by the induction hypothesis, $\phi \approx \chi, \chi \approx \psi \in \Xi^{Q,n}_\Sigma(F)$, whence, by definition, $\phi \approx \psi \in \Xi^{Q,n+1}_\Sigma(F)$.

    If $\phi \approx \psi$ is of the form $\sigma^\flat_\Sigma(\vec{\phi}) \approx \sigma^\flat_\Sigma(\vec{\psi})$, with $\phi_i \approx \psi_i \in \Xi^{Q,n}_\Sigma(E)$, $i < k$, then, by the induction hypothesis, $\phi_i \approx \psi_i \in \Xi^{Q,n}_\Sigma(F)$, whence, by definition, $\sigma^\flat_\Sigma(\vec{\phi}) \approx \sigma^\flat_\Sigma(\vec{\psi}) \in \Xi^{Q,n+1}_\Sigma(F)$.

    Finally, if $\phi \approx \psi$ is of form $\mathrm{SEN}^\flat(f)(\phi' \approx \psi')$, with $\phi' \approx \psi' \in \Xi^{Q,n}_{\Sigma'}(E)$, then, by the induction hypothesis, $\phi' \approx \psi' \in \Xi^{Q,n}_{\Sigma'}(F)$, and, therefore, by definition, $\mathrm{SEN}^\flat(f)(\phi' \approx \psi') \in \Xi^{Q,n+1}_\Sigma(F)$.

  Thus, if $\phi \approx \psi \in \Xi^Q_\Sigma(E)$, then $\phi \approx \psi \in \Xi^Q_\Sigma(F)$ and Ξ^Q is monotone.

- Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$, such that $\phi \approx \psi \in \Xi^Q_\Sigma(\Xi^Q_\Sigma(E))$. Then, for some $n < \omega$, $\phi \approx \psi \in \Xi^{Q,n}_\Sigma(\Xi^Q_\Sigma(E))$. We show by induction on $n < \omega$ that, for all $n < \omega$,

$$\phi \approx \psi \in \Xi^{Q,n}_\Sigma(\Xi^Q_\Sigma(E)) \quad \text{implies} \quad \phi \approx \psi \in \Xi^Q_\Sigma(E).$$

  - For $n = 0$, $\phi = \psi$ or $\phi \approx \psi \in Q_\Sigma$ or $\phi \approx \psi \in \Xi^Q_\Sigma(E)$. In the first two cases $\phi \approx \psi \in \Xi^{Q,0}_\Sigma(E) \subseteq \Xi^Q_\Sigma(E)$, by definition, and in the last the implication is trivial.
  - Suppose that the statement holds for $n > 0$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \approx \psi \in \Xi^{Q,n+1}_\Sigma(\Xi^Q_\Sigma(E))$.

    If $\psi \approx \phi \in \Xi^{Q,n}_\Sigma(\Xi^Q_\Sigma(E))$, then, by the induction hypothesis, $\psi \approx \phi \in \Xi^Q_\Sigma(E)$, i.e., $\psi \approx \phi \in \Xi^{Q,m}_\Sigma(E)$, for some $m < \omega$. Thus, by definition, $\phi \approx \psi \in \Xi^{Q,m+1}_\Sigma(E) \subseteq \Xi^Q_\Sigma(E)$.

    If $\phi \approx \chi, \chi \approx \psi \in \Xi^{Q,n}_\Sigma(\Xi^Q_\Sigma(E))$, then, by the induction hypothesis, $\phi \approx \chi, \chi \approx \psi \in \Xi^Q_\Sigma(E)$, i.e., for some $m < \omega$, $\phi \approx \chi, \chi \approx \psi \in \Xi^{Q,m}_\Sigma(E)$. Thus, by definition, $\phi \approx \psi \in \Xi^{Q,m+1}_\Sigma(E) \subseteq \Xi^Q_\Sigma(E)$.

If $\phi \approx \psi$ is of the form $\sigma^\flat_\Sigma(\vec{\phi}) \approx \sigma^\flat_\Sigma(\vec{\psi})$, with $\phi_i \approx \psi_i \in \Xi^{Q,n}_\Sigma(\Xi^Q_\Sigma(E))$, $i < k$, then, by the induction hypothesis, $\phi_i \approx \psi_i \in \Xi^Q_\Sigma(E)$, for all $i < k$. Thus, there exists $m < \omega$, such that $\phi_i \approx \psi_i \in \Xi^{Q,m}_\Sigma(E)$, for all $i < k$, and, consequently, by definition, $\sigma^\flat_\Sigma(\vec{\phi}) \approx \sigma^\flat_\Sigma(\vec{\psi}) \in \Xi^{Q,m+1}_\Sigma(E) \subseteq \Xi^Q_\Sigma(E)$.

Finally, suppose that $\phi \approx \psi$ is of the form $\mathrm{SEN}^\flat(f)(\phi' \approx \psi')$, where $\phi' \approx \psi' \in \Xi^{Q,n}_{\Sigma'}(E)$. Then, by the induction hypothesis, $\phi' \approx \psi' \in \Xi^Q_{\Sigma'}(E)$, whence, there exists $m < \omega$, such that $\phi' \approx \psi' \in \Xi^{Q,m}_{\Sigma'}(E)$. But, then, by definition, $\mathrm{SEN}^\flat(f)(\phi' \approx \psi') \in \Xi^{Q,m+1}_\Sigma(E) \subseteq \Xi^Q_\Sigma(E)$.

So Ξ^Q is a closure family. Finally, we show that it satisfies the five extra rules.

- For Reflexivity, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, by definition, $\phi \approx \phi \in \Xi^{Q,0}_\Sigma(\varnothing) \subseteq \Xi^Q_\Sigma(\varnothing)$, whence Ξ^Q is Reflexive.

- For Symmetry, let $E \le \mathrm{Eq}(\mathbf{F})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \approx \psi \in \Xi^Q_\Sigma(E)$. Then, there exists $n < \omega$, such that $\phi \approx \psi \in \Xi^{Q,n}_\Sigma(E)$. By definition, $\psi \approx \phi \in \Xi^{Q,n+1}_\Sigma(E) \subseteq \Xi^Q_\Sigma(E)$. We conclude that $\psi \approx \phi \in \Xi^Q_\Sigma(\phi \approx \psi)$ and, hence Ξ^Q is Symmetric.

- For Transitivity, let $E \le \mathrm{Eq}(\mathbf{F})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \approx \psi, \psi \approx \chi \in \Xi^Q_\Sigma(E)$. Then, there exists $n < \omega$, such that $\phi \approx \psi, \psi \approx \chi \in \Xi^{Q,n}_\Sigma(E)$. By definition, $\phi \approx \chi \in \Xi^{Q,n+1}_\Sigma(E) \subseteq \Xi^Q_\Sigma(E)$. So Ξ^Q is Transitive.

- For Congruence, let $E \le \mathrm{Eq}(\mathbf{F})$, $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi_i, \psi_i \in \mathrm{SEN}^\flat(\Sigma)$, $i < k$, such that $\phi_i \approx \psi_i \in \Xi^Q_\Sigma(E)$. Then, there exists $n < \omega$, such that $\phi_i \approx \psi_i \in \Xi^{Q,n}_\Sigma(E)$, for all $i < k$, whence, by definition, $\sigma^\flat_\Sigma(\vec{\phi}) \approx \sigma^\flat_\Sigma(\vec{\psi}) \in \Xi^{Q,n+1}_\Sigma(E) \subseteq \Xi^Q_\Sigma(E)$. Thus Ξ^Q satisfies Congruence.

- Finally, for Invariance, let $E \le \mathrm{Eq}(\mathbf{F})$, $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \approx \psi \in \Xi^Q_\Sigma(E)$. Then, there exists $n < \omega$, such that $\phi \approx \psi \in \Xi^{Q,n}_\Sigma(E)$, and, hence, by definition,

  $$\mathrm{SEN}^\flat(f)(\phi \approx \psi) \in \Xi^{Q,n+1}_{\Sigma'}(E) \subseteq \Xi^Q_{\Sigma'}(E).$$

  We conclude that Ξ^Q satisfies Invariance as well.

This shows that $\Xi^Q : \mathcal{P}\mathrm{Eq}(\mathbf{F}) \to \mathcal{P}\mathrm{Eq}(\mathbf{F})$ is a closure family that satisfies Reflexivity, Symmetry, Transitivity, Congruence and Invariance. ■

We show that, given a semantic variety, i.e., a class K of $\mathbf{F}$-algebraic systems, such that $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$, we have $D^{\mathsf{K}} = \Xi^{\mathrm{Ker}(\mathsf{K})}$.

We prove first a lemma.

Lemma 118 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a semantic variety of* $\mathbf{F}$*-algebraic systems, i.e., a class of* $\mathbf{F}$*-algebraic systems, such that* $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$*. Then, for all* $E \subseteq \mathrm{Eq}(\mathbf{F})$*,* $\Xi^{\mathrm{Ker}(\mathsf{K})}(E) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$*.*

Proof: By Proposition 117, $\Xi^{\mathrm{Ker}(\mathsf{K})}(E)$ is a congruence system on $\mathbf{F}$. Moreover, by definition, $\mathrm{Ker}(\mathsf{K}) \leq \Xi^{\mathrm{Ker}(\mathsf{K})}(E)$. But, note that

$$\mathrm{Ker}(\mathcal{F}/\Xi^{\mathrm{Ker}(\mathsf{K})}(E)) = \Xi^{\mathrm{Ker}(\mathsf{K})}(E).$$

Thus, we have $\mathrm{Ker}(\mathsf{K}) \leq \mathrm{Ker}(\mathcal{F}/\Xi^{\mathrm{Ker}(\mathsf{K})}(E))$. Thus, by definition and the hypothesis,

$$\mathcal{F}/\Xi^{\mathrm{Ker}(\mathsf{K})}(E) \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}.$$

We conclude that $\Xi^{\mathrm{Ker}(\mathsf{K})}(E) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$. ■

Theorem 119 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems, such that* $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$*. Then* $D^{\mathsf{K}} = \Xi^Q$*, where* $Q = \mathrm{Ker}(\mathsf{K})$*.*

Proof: Assume, first, that $\Sigma \in |\mathbf{Sign}^\flat|$, $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$, such that $\phi \approx \psi \in D^{\mathsf{K}}_\Sigma(E)$. Thus, by definition, for all $\mathcal{A} \in \mathsf{K}$,

$$E \subseteq \mathrm{Ker}_\Sigma(\mathcal{A}) \quad \text{implies} \quad \langle \phi, \psi \rangle \in \mathrm{Ker}_\Sigma(\mathcal{A}).$$

In particular, by Lemma 118,

$$E \subseteq \mathrm{Ker}_\Sigma(\mathcal{F}/\Xi^Q(E)) \quad \text{implies} \quad \langle \phi, \psi \rangle \in \mathrm{Ker}_\Sigma(\mathcal{F}/\Xi^Q(E)).$$

Equivalently, we have $E \subseteq \Xi^Q_\Sigma(E)$ implies $\phi \approx \psi \in \Xi^Q_\Sigma(E)$. Since the first inclusion holds by the definition of Ξ^Q, we have $\phi \approx \psi \in \Xi^Q_\Sigma(E)$. We conclude that $D^{\mathsf{K}} \leq \Xi^Q$.

Assume, conversely, that $\Sigma \in |\mathbf{Sign}^\flat|$, $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$, such that $\phi \approx \psi \in \Xi^Q_\Sigma(E)$. Then, by definition, there exists an $n < \omega$, such that $\phi \approx \psi \in \Xi^{Q,n}_\Sigma(E)$. We show, by induction on $n < \omega$, that, for all $n < \omega$,

$$\phi \approx \psi \in \Xi^{Q,n}_\Sigma(E) \quad \text{implies} \quad \phi \approx \psi \in D^{\mathsf{K}}_\Sigma(E).$$

- If $n = 0$, then $\phi = \psi$ or $\phi \approx \psi \in \mathrm{Ker}_\Sigma(\mathsf{K})$ or $\phi \approx \psi \in E$.

  In the first case, the conclusion follows by Proposition 115, and in the last, by Proposition 29.

  In the second case, we have, for all $\mathcal{A} \in \mathsf{K}$, $\mathrm{Ker}(\mathsf{K}) \leq \mathrm{Ker}(\mathcal{A})$, whence $\phi \approx \psi \in \mathrm{Ker}_\Sigma(\mathcal{A})$. So $\phi \approx \psi \in D^{\mathsf{K}}_\Sigma(\varnothing) \subseteq D^{\mathsf{K}}_\Sigma(E)$.

- Assume, now, that the conclusion holds for $n > 0$. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$, such that $\phi \approx \psi \in \Xi_\Sigma^{Q,n+1}(E)$.

  If $\psi \approx \phi \in \Xi_\Sigma^{Q,n}(E)$, then, by the induction hypothesis, $\psi \approx \phi \in D_\Sigma^{\mathsf{K}}(E)$, whence, by Proposition 115, $\phi \approx \psi \in D_\Sigma^{\mathsf{K}}(E)$.

  If $\phi \approx \chi, \chi \approx \psi \in \Xi_\Sigma^{Q,n}(E)$, then , by the induction hypothesis, $\phi \approx \chi, \chi \approx \psi \in D_\Sigma^{\mathsf{K}}(E)$. So, by Proposition 115, we have $\phi \approx \psi \in D_\Sigma^{\mathsf{K}}(E)$.

  If $\phi \approx \psi$ is of the form $\sigma_\Sigma^\flat(\vec{\phi}) \approx \sigma_\Sigma^\flat(\vec{\psi})$, for some $\sigma^\flat$ in $N^\flat$ and $\phi_i \approx \psi_i \in \Xi_\Sigma^{Q,n}(E)$, $i < k$, then, by the induction hypothesis, $\phi_i \approx \psi_i \in D_\Sigma^{\mathsf{K}}(E)$, for all $i < k$, whence, once more by Proposition 115, $\sigma_\Sigma^\flat(\vec{\phi}) \approx \sigma_\Sigma^\flat(\vec{\psi}) \in D_\Sigma^{\mathsf{K}}(E)$.

  Finally, if $\phi \approx \psi$ is of the form $\mathrm{SEN}^\flat(f)(\phi' \approx \psi')$, for some $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma', \Sigma)$ and $\phi' \approx \psi' \in \Xi_{\Sigma'}^{Q,n}(E)$, then, by the induction hypothesis, $\phi' \approx \psi' \in D_{\Sigma'}^{\mathsf{K}}(E)$, whence, by Proposition 115, $\mathrm{SEN}^\flat(f)(\phi' \approx \psi') \in D_\Sigma^{\mathsf{K}}(E)$.

Thus, we get $\Xi^Q \leq D^{\mathsf{K}}$ and, therefore, $D^{\mathsf{K}} = \Xi^Q$. ■

2.18 Categorical Universal Algebra

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. We define **F**-equations, **F**-quasiequations and **F**-guasiequations (standing for generalized **F**-quasiequations). Recall that **F**-equations have already been introduced in Section 2.17, but the definition is repeated here for the sake of completeness.

- The family $\mathrm{Eq}(\mathbf{F}) = \{\mathrm{Eq}_\Sigma(\mathbf{F})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ of **F-equations** is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,
$$\mathrm{Eq}_\Sigma(\mathbf{F}) = \mathrm{SEN}^\flat(\Sigma)^2 = \{\phi \approx \psi : \phi, \psi \in \mathrm{SEN}^\flat(\Sigma)\};$$

- The family $\mathrm{QEq}(\mathbf{F}) = \{\mathrm{QEq}_\Sigma(\mathbf{F})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ of **F-quasiequations** is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,
$$\mathrm{QEq}_\Sigma(\mathbf{F}) = \{\langle \{\phi_i \approx \psi_i : i < k\}, \phi \approx \psi \rangle : k \in \omega, \vec{\phi}, \vec{\psi}, \phi, \psi \in \mathrm{SEN}^\flat(\Sigma)\};$$

- The family $\mathrm{GEq}(\mathbf{F}) = \{\mathrm{GEq}_\Sigma(\mathbf{F})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ of **F-guasiequations** is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,
$$\mathrm{GEq}_\Sigma(\mathbf{F}) = \{\langle \{\phi_i \approx \psi_i : i \in I\}, \phi \approx \psi \rangle : \vec{\phi}, \vec{\psi}, \phi, \psi \in \mathrm{SEN}^\flat(\Sigma)\}.$$

Sometimes we write $\langle \phi, \psi \rangle$ in place of $\phi \approx \psi$. Moreover, we use the notation
$$\vec{\phi} \approx \vec{\psi} := \{\phi_i \approx \psi_i : i \in I\}.$$

Thus, the **F**-guasiequation $\langle\{\phi_i \approx \psi_i : i \in I\}, \phi \approx \psi\rangle$ may be written more compactly $\langle\vec{\phi} \approx \vec{\psi}, \phi \approx \psi\rangle$ and, sometimes, also $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi$. Note that

$$\mathrm{Eq}(\mathbf{F}) \leq \mathrm{QEq}(\mathbf{F}) \leq \mathrm{GEq}(\mathbf{F}).$$

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\Sigma \in |\mathbf{Sign}^\flat|$, $\langle\vec{\phi} \approx \vec{\psi}, \phi \approx \psi\rangle \in \mathrm{GEq}_\Sigma(\mathbf{F})$ and $\mathcal{A} = \langle\mathbf{A}, \langle F, \alpha\rangle\rangle$ an **F**-algebraic system. We say that $\mathcal{A}$ **satisfies** $\langle\vec{\phi} \approx \vec{\psi}, \phi \approx \psi\rangle$ or that the guasiequation $\langle\vec{\phi} \approx \vec{\psi}, \phi \approx \psi\rangle$ **is true in**, or **is satisfied in**, or **holds in** $\mathcal{A}$, written

$$\mathcal{A} \vDash_\Sigma \langle\vec{\phi} \approx \vec{\psi}, \phi \approx \psi\rangle,$$

if

$$\alpha_\Sigma(\phi_i) = \alpha_\Sigma(\psi_i),\ i \in I,\ \text{imply}\ \alpha_\Sigma(\phi) = \alpha_\Sigma(\psi).$$

Since **F**-quasiequations and **F**-equations are special cases of **F**-guasiequations, the definition covers them as well. Thus, we have

- $\mathcal{A}$ satisfies the **F**-quasiequation $\langle\vec{\phi} \approx \vec{\psi}, \phi \approx \psi\rangle$ if $\alpha_\Sigma(\phi_i) = \alpha_\Sigma(\psi_i)$, for all $i < k$, imply $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$;
- $\mathcal{A}$ satisfies the **F**-equation $\phi \approx \psi$ if $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$.

Now we define the following families:

- The family $\mathrm{Eq}(\mathcal{A}) = \{\mathrm{Eq}_\Sigma(\mathcal{A})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ of **F**-equations satisfied by $\mathcal{A}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,
$$\mathrm{Eq}_\Sigma(\mathcal{A}) = \{e \in \mathrm{Eq}_\Sigma(\mathbf{F}) : \mathcal{A} \vDash_\Sigma e\};$$
- The family $\mathrm{QEq}(\mathcal{A}) = \{\mathrm{QEq}_\Sigma(\mathcal{A})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ of **F**-quasiequations satisfied by $\mathcal{A}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,
$$\mathrm{QEq}_\Sigma(\mathcal{A}) = \{q \in \mathrm{QEq}_\Sigma(\mathbf{F}) : \mathcal{A} \vDash_\Sigma q\};$$
- The family $\mathrm{GEq}(\mathcal{A}) = \{\mathrm{GEq}_\Sigma(\mathcal{A})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ of **F**-guasiequations satisfied by $\mathcal{A}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,
$$\mathrm{GEq}_\Sigma(\mathcal{A}) = \{g \in \mathrm{GEq}_\Sigma(\mathbf{F}) : \mathcal{A} \vDash_\Sigma g\}.$$

Finally, given a class K of **F**-algebraic systems, we define:

- $\mathrm{Eq}(\mathsf{K}) = \bigcap\{\mathrm{Eq}(\mathcal{A}) : \mathcal{A} \in \mathsf{K}\}$;
- $\mathrm{QEq}(\mathsf{K}) = \bigcap\{\mathrm{QEq}(\mathcal{A}) : \mathcal{A} \in \mathsf{K}\}$;
- $\mathrm{GEq}(\mathsf{K}) = \bigcap\{\mathrm{GEq}(\mathcal{A}) : \mathcal{A} \in \mathsf{K}\}$.

Note, again, that

$$\mathrm{Eq}(\mathcal{A}) \leq \mathrm{QEq}(\mathcal{A}) \leq \mathrm{GEq}(\mathcal{A}) \quad \text{and} \quad \mathrm{Eq}(\mathsf{K}) \leq \mathrm{QEq}(\mathsf{K}) \leq \mathrm{GEq}(\mathsf{K}).$$

Given a class $G \leq \mathrm{GEq}(\mathbf{F})$ of $\mathbf{F}$-guasiequations (which includes the case of quasiequations or equations), we define $\mathrm{AlgSys}(G)$ or, sometimes, $\mathrm{Mod}(G)$, to be the collection of all $\mathbf{F}$-algebraic systems that satisfy all the $\mathbf{F}$-guasiequations in G:

$$\mathrm{AlgSys}(G) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : \mathcal{A} \vDash G\}.$$

As is well-known, based on an underlying Galois connection, we get the following, for all $G, G' \leq \mathrm{GEq}(\mathbf{F})$ and all $\mathsf{K}, \mathsf{K}' \subseteq \mathrm{AlgSys}(\mathbf{F})$,

- If $\mathsf{K} \subseteq \mathsf{K}'$, then $\mathrm{GEq}(\mathsf{K}') \leq \mathrm{GEq}(\mathsf{K})$;
- If $G \leq G'$, then $\mathrm{AlgSys}(G') \subseteq \mathrm{AlgSys}(G)$;
- $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathrm{GEq}(\mathsf{K}))$ and $\mathrm{GEq}(\mathsf{K}) = \mathrm{GEq}(\mathrm{AlgSys}(\mathrm{GEq}(\mathsf{K})))$;
- $G \subseteq \mathrm{GEq}(\mathrm{AlgSys}(G))$ and $\mathrm{AlgSys}(G) = \mathrm{AlgSys}(\mathrm{GEq}(\mathrm{AlgSys}(G)))$.

Similar relations hold with the GEq operator replaced by either the Eq or the QEq operator. We may apply some of these either without providing explicit justification or, simply, by saying "by the Galois connection".

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and K a class of $\mathbf{F}$-algebraic systems.

- K is called an **equational class** if there exists $E \leq \mathrm{Eq}(\mathbf{F})$, such that $\mathsf{K} = \mathrm{AlgSys}(E)$;
- K is called a **quasiequational class** if there exists $Q \leq \mathrm{QEq}(\mathbf{F})$, such that $\mathsf{K} = \mathrm{AlgSys}(Q)$;
- K is called a **guasiequational class** if there exists $G \leq \mathrm{GEq}(\mathbf{F})$, such that $\mathsf{K} = \mathrm{AlgSys}(G)$.

Clearly, by definition, if K is an equational class, then it is a quasiequational class and, if it is a quasiequational class, then it is a guasiequational class.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and K a class of $\mathbf{F}$-algebraic systems. We define:

- The **semantic variety generated by K**

$$\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : \mathrm{Eq}(\mathsf{K}) \leq \mathrm{Eq}(\mathcal{A})\};$$

- The **semantic quasivariety generated by K**

$$\mathbb{Q}^{\mathsf{Sem}}(\mathsf{K}) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : \mathrm{QEq}(\mathsf{K}) \leq \mathrm{QEq}(\mathcal{A})\};$$

- The **semantic guasivariety generated by** K

$$\mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : \mathrm{GEq}(\mathsf{K}) \leq \mathrm{GEq}(\mathcal{A})\}.$$

We have the following straightforward relationships between these classes.

Lemma 120 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems. Then*

$$\mathsf{K} \subseteq \mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{Q}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{V}^{\mathsf{Sem}}(\mathsf{K}).$$

Proof: The essential observation we use, which has been discussed before, is that

$$\mathrm{Eq}(\mathsf{K}) \leq \mathrm{QEq}(\mathsf{K}) \leq \mathrm{GEq}(\mathsf{K}).$$

Thus, we get

$$\begin{array}{l}\{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : (\forall g \in \mathrm{GEq}(\mathsf{K}))(\mathcal{A} \vDash g)\}\\ \qquad \subseteq \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : (\forall q \in \mathrm{QEq}(\mathsf{K}))(\mathcal{A} \vDash q)\}\\ \qquad \subseteq \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : (\forall e \in \mathrm{Eq}(\mathsf{K}))(\mathcal{A} \vDash e)\}.\end{array}$$

In other words, $\mathsf{K} \subseteq \mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{Q}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. ■

Given a class K of $\mathbf{F}$-algebraic systems

- K is a **semantic variety** if $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$;
- K is a **semantic quasivariety** if $\mathbb{Q}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$;
- K is a **semantic guasivariety** if $\mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$.

We have the following result identifying equational classes with semantic varieties, quasiequational classes with semantic quasivarieties and guasiequational classes with semantic guasivarieties.

Proposition 121 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems.*

(a) K *is an equational class iff it is a semantic variety;*

(b) K *is a quasiequational class iff it is a semantic quasivariety;*

(c) K *is a guasiequational class iff it is a semantic guasivariety.*

Proof:

(a) Suppose, first, that K is an equational class. Then, there exists $E \leq \mathrm{Eq}(\mathbf{F})$, such that $\mathsf{K} = \mathrm{AlgSys}(E)$. Let $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, such that $\mathrm{Eq}(\mathsf{K}) \leq \mathrm{Eq}(\mathcal{A})$. Then we have

$$\begin{array}{rcl} \mathcal{A} & \in & \mathrm{AlgSys}(\mathrm{Eq}(\mathcal{A})) \\ & \subseteq & \mathrm{AlgSys}(\mathrm{Eq}(\mathsf{K})) \\ & = & \mathrm{AlgSys}(\mathrm{Eq}(\mathrm{AlgSys}(E))) \\ & = & \mathrm{AlgSys}(E) = \mathsf{K}. \end{array}$$

Therefore, K is a semantic variety.

Suppose, conversely, that K is a semantic variety. Set $E = \mathrm{Eq}(\mathsf{K})$. Then $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathrm{Eq}(\mathsf{K})) = \mathrm{AlgSys}(E)$. On the other hand, if $\mathcal{A} \in \mathrm{AlgSys}(E)$, then

$$\mathrm{Eq}(\mathsf{K}) = \mathrm{Eq}(\mathrm{AlgSys}(\mathrm{Eq}(\mathsf{K}))) = \mathrm{Eq}(\mathrm{AlgSys}(E)) \leq \mathrm{Eq}(\mathcal{A}),$$

whence, by hypothesis, $\mathcal{A} \in \mathsf{K}$. Therefore, $\mathsf{K} = \mathrm{AlgSys}(E)$ and K is an equational class.

(b) Suppose, first, that K is a quasiequational class. Then, there exists $Q \leq \mathrm{QEq}(\mathbf{F})$, such that $\mathsf{K} = \mathrm{AlgSys}(Q)$. Let $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, such that $\mathrm{QEq}(\mathsf{K}) \leq \mathrm{QEq}(\mathcal{A})$. Then we have

$$\begin{array}{rcl} \mathcal{A} & \in & \mathrm{AlgSys}(\mathrm{QEq}(\mathcal{A})) \\ & \subseteq & \mathrm{AlgSys}(\mathrm{QEq}(\mathsf{K})) \\ & = & \mathrm{AlgSys}(\mathrm{QEq}(\mathrm{AlgSys}(Q))) \\ & = & \mathrm{AlgSys}(Q) = \mathsf{K}. \end{array}$$

Therefore, K is a semantic quasivariety.

Suppose, conversely, that K is a semantic quasivariety. Set $Q = \mathrm{QEq}(\mathsf{K})$. Then $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathrm{QEq}(\mathsf{K})) = \mathrm{AlgSys}(Q)$. On the other hand, if $\mathcal{A} \in \mathrm{AlgSys}(Q)$, then

$$\mathrm{QEq}(\mathsf{K}) = \mathrm{QEq}(\mathrm{AlgSys}(\mathrm{QEq}(\mathsf{K}))) = \mathrm{QEq}(\mathrm{AlgSys}(Q)) \leq \mathrm{QEq}(\mathcal{A}),$$

whence, by hypothesis, $\mathcal{A} \in \mathsf{K}$. Therefore, $\mathsf{K} = \mathrm{AlgSys}(Q)$ and K is a quasiequational class.

(c) Very similar to Part (b).

■

We define or revisit, next, some operators on classes of $\mathbf{F}$-algebraic systems that will serve to provide different characterizations to the equational, quasi-equational and guasiequational classes of $\mathbf{F}$-algebraic systems.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, K a class of $\mathbf{F}$-algebraic systems and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system.

- Given $\Sigma \in |\mathbf{Sign}^\flat|$, we say that $\mathcal{A}$ is Σ-K-**certified** if there exists $\mathcal{A}^\Sigma \in \mathsf{K}$, such that $\mathrm{Eq}_\Sigma(\mathcal{A}) = \mathrm{Eq}_\Sigma(\mathcal{A}^\Sigma)$. In this case $\mathcal{A}^\Sigma$ is called the Σ-K-**certificate** of $\mathcal{A}$.

- We say that $\mathcal{A}$ is K-**certified** if it is Σ-K-certified, for all $\Sigma \in |\mathbf{Sign}^\flat|$. This, of course, means that

$$(\forall \Sigma \in |\mathbf{Sign}^\flat|)(\exists \mathcal{A}^\Sigma \in \mathsf{K})(\mathrm{Eq}_\Sigma(\mathcal{A}) = \mathrm{Eq}_\Sigma(\mathcal{A}^\Sigma)).$$

We write $\mathbb{C}(\mathsf{K})$ for the class of all $\mathbf{F}$-algebraic systems that are K-certified. We say that K is an **abstract class** whenever every K-certified $\mathbf{F}$-algebraic system belongs to K, i.e., when $\mathbb{C}(\mathsf{K}) = \mathsf{K}$.

It is not difficult to show that $\mathbb{C}$ is a closure operator on classes of $\mathbf{F}$-algebraic systems.

Proposition 122 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system. Then the operator* $\mathbb{C}$ *on classes of* $\mathbf{F}$*-algebraic systems is a closure operator.*

Proof: Suppose K is a class of $\mathbf{F}$-algebraic systems.

- Let $\mathcal{A} \in \mathsf{K}$. Then, since for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathcal{A}^\Sigma = \mathcal{A} \in \mathsf{K}$ is a Σ-K-certificate for $\mathcal{A}$, we get that $\mathcal{A} \in \mathbb{C}(\mathsf{K})$. Thus, $\mathsf{K} \subseteq \mathbb{C}(\mathsf{K})$ and $\mathbb{C}$ is inflationary.

- If $\mathsf{K} \subseteq \mathsf{K}'$ and $\mathcal{A} \in \mathbb{C}(\mathsf{K})$, then, by definition, for every $\Sigma \in |\mathbf{Sign}^\flat|$, there exists a Σ-K-certificate $\mathcal{A}^\Sigma$. Since $\mathsf{K} \subseteq \mathsf{K}'$, $\mathcal{A}^\Sigma \in \mathsf{K}'$ is also a Σ-K'-certificate. Thus, $\mathcal{A} \in \mathbb{C}(\mathsf{K}')$ and $\mathbb{C}$ is also monotone.

- Finally, suppose that $\mathcal{A} \in \mathbb{C}(\mathbb{C}(\mathsf{K}))$. Then, there exists, for all $\Sigma \in |\mathbf{Sign}^\flat|$, a Σ-$\mathbb{C}(\mathsf{K})$-certificate $\mathcal{A}^\Sigma$ for $\mathcal{A}$. Therefore, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Sigma' \in |\mathbf{Sign}^\flat|$, there exists a Σ'-K-certificate $\mathcal{A}^{\langle \Sigma, \Sigma' \rangle}$ for $\mathcal{A}^\Sigma$. But, then, for every $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{Eq}_\Sigma(\mathcal{A}) = \mathrm{Eq}_\Sigma(\mathcal{A}^\Sigma) = \mathrm{Eq}_\Sigma(\mathcal{A}^{\langle \Sigma, \Sigma \rangle}).$$

  Thus, for every $\Sigma \in |\mathbf{Sign}^\flat|$, there exists a Σ-K-certificate $\mathcal{A}^{\langle \Sigma, \Sigma \rangle}$ for $\mathcal{A}$, i.e., $\mathcal{A} \in \mathbb{C}(\mathsf{K})$ and $\mathbb{C}$ is also idempotent.

Thus $\mathbb{C}$ is a closure operator on classes of $\mathbf{F}$-algebraic systems. ∎

The importance of abstract classes of $\mathbf{F}$-algebraic systems here, and the reason why they will be our exclusive focus in this section, rests on the following observation to the effect that the validity of a guasiequation transfers from K-certificates of an $\mathbf{F}$-algebraic system to the $\mathbf{F}$-algebraic system itself.

Lemma 123 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* K *a class of* $\mathbf{F}$*-algebraic systems and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system. If* $\mathcal{A} \in \mathbb{C}(\mathsf{K})$*, then* $\mathrm{GEq}(\mathsf{K}) \leq \mathrm{GEq}(\mathcal{A})$.

Proof: Suppose $\mathcal{A} \in \mathbb{C}(\mathsf{K})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\langle \vec{\phi} \approx \vec{\psi}, \phi \approx \psi \rangle \in \mathrm{GEq}_\Sigma(\mathsf{K})$, such that $\vec{\phi} \approx \vec{\psi} \subseteq \mathrm{Eq}_\Sigma(\mathcal{A})$. Let $\mathcal{A}^\Sigma \in \mathsf{K}$ be a Σ-K-certificate for $\mathcal{A}$. Then, by definition $\vec{\phi} \approx \vec{\psi} \subseteq \mathrm{Eq}_\Sigma(\mathcal{A}^\Sigma)$. Since $\mathcal{A}^\Sigma \in \mathsf{K}$ and $\langle \vec{\phi} \approx \vec{\psi}, \phi \approx \psi \rangle \in \mathrm{GEq}_\Sigma(\mathsf{K})$, we get $\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathcal{A}^\Sigma) = \mathrm{Eq}_\Sigma(\mathcal{A})$. Therefore, $\langle \vec{\phi} \approx \vec{\psi}, \phi \approx \psi \rangle \in \mathrm{GEq}_\Sigma(\mathcal{A})$. We conclude that $\mathrm{GEq}(\mathsf{K}) \leq \mathrm{GEq}(\mathcal{A})$. ■

Using Lemma 123, we get the following corollary to the effect that all semantically defined classes of algebraic systems are abstract.

Corollary 124 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems. If* K *is a guasiequational class (and, hence, a fortiori, if it is a quasiequational class or an equational class), then it is abstract.*

Proof: Suppose K is a guasiequational class defined by the family of $\mathbf{F}$-guasiequations $G \leq \mathrm{GEq}(\mathbf{F})$ and let $\mathcal{A} \in \mathbb{C}(\mathsf{K})$. Then, by Lemma 123, $\mathrm{GEq}(\mathsf{K}) \leq \mathrm{GEq}(\mathcal{A})$, whence

$$\begin{array}{rcl}
\mathcal{A} & \in & \mathrm{AlgSys}(\mathrm{GEq}(\mathcal{A})) \\
 & \subseteq & \mathrm{AlgSys}(\mathrm{GEq}(\mathsf{K})) \\
 & = & \mathrm{AlgSys}(\mathrm{GEq}(\mathrm{AlgSys}(G))) \\
 & = & \mathrm{AlgSys}(G) \\
 & = & \mathsf{K}.
\end{array}$$

Thus, K is an abstract class. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, K a class of $\mathbf{F}$-algebraic systems and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system.

- Given $\Sigma \in |\mathbf{Sign}^\flat|$, we say that $\mathcal{A}$ is **directedly** Σ-K-**certified** if there exists a collection of $\mathbf{F}$-algebraic systems $\{\mathcal{A}^{\Sigma,i} : i \in I\} \subseteq \mathsf{K}$, such that:
  - $\bigcup_{i \in I} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i})$ is directed, where, for all $i \in I$, $\mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i})$ denotes the collection of all finite subsets of $\mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,i})$, and
  - $\mathrm{Ker}_\Sigma(\mathcal{A}) = \bigcup_{i \in I} \mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,i})$.

  We call $\{\mathcal{A}^{\Sigma,i} : i \in I\}$ the **directed** Σ-K-**certificate** of $\mathcal{A}$.

- We say that $\mathcal{A}$ is **directedly** K-**certified** if it is directedly Σ-K-certified, for all $\Sigma \in |\mathbf{Sign}^\flat|$.

We write $\mathbb{C}^*(\mathsf{K})$ for the class of all $\mathbf{F}$-algebraic systems that are directedly K-certified. We say that K is a **directedly abstract class** whenever every directedly K-certified $\mathbf{F}$-algebraic system belongs to K, i.e., when $\mathbb{C}^*(\mathsf{K}) = \mathsf{K}$.

We show that, like $\mathbb{C}$, $\mathbb{C}^*$ is a closure operator on classes of $\mathbf{F}$-algebraic systems.

Proposition 125 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system. Then the operator* $\mathbb{C}^*$ *on classes of* $\mathbf{F}$*-algebraic systems is a closure operator.*

Proof: Suppose K is a class of $\mathbf{F}$-algebraic systems.

- Let $\mathcal{A} \in \mathsf{K}$. Then, since, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\{\mathcal{A}\} \subseteq \mathsf{K}$ is a directed Σ-K-certificate for $\mathcal{A}$, we get that $\mathcal{A} \in \mathbb{C}^*(\mathsf{K})$. Thus, $\mathsf{K} \subseteq \mathbb{C}^*(\mathsf{K})$ and $\mathbb{C}^*$ is inflationary.

- If $\mathsf{K} \subseteq \mathsf{K}'$ and $\mathcal{A} \in \mathbb{C}^*(\mathsf{K})$, then, by definition, for every $\Sigma \in |\mathbf{Sign}^\flat|$, there exists a directed Σ-K-certificate $\{\mathcal{A}^{\Sigma,i} : i \in I_\Sigma\} \subseteq \mathsf{K}$. Since $\mathsf{K} \subseteq \mathsf{K}'$, $\{\mathcal{A}^{\Sigma,i} : i \in I_\Sigma\} \subseteq \mathsf{K}'$ is also a directed Σ-K'-certificate. Thus, $\mathcal{A} \in \mathbb{C}^*(\mathsf{K}')$ and $\mathbb{C}^*$ is also monotone.

- Finally, suppose that $\mathcal{A} \in \mathbb{C}^*(\mathbb{C}^*(\mathsf{K}))$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$, there exists $\{\mathcal{A}^{\Sigma,i} : i \in I_\Sigma\} \subseteq \mathbb{C}^*(\mathsf{K})$, such that $\bigcup_{i \in I_\Sigma} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i})$ is directed and $\mathrm{Ker}_\Sigma(\mathcal{A}) = \bigcup_{i \in I_\Sigma} \mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,i})$. Thus, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$ and all $i \in I_\Sigma$, there exists $\{\mathcal{A}^{\Sigma,i,\Sigma',j} : j \in J^{\Sigma,i}_{\Sigma'}\} \subseteq \mathsf{K}$, such that $\bigcup_{j \in J^{\Sigma,i}_{\Sigma'}} \mathrm{Eq}^\omega_{\Sigma'}(\mathcal{A}^{\Sigma,i,\Sigma',j})$ is directed and, moreover, $\mathrm{Ker}_{\Sigma'}(\mathcal{A}^{\Sigma,i}) = \bigcup_{j \in J^{\Sigma,i}_{\Sigma'}} \mathrm{Ker}_{\Sigma'}(\mathcal{A}^{\Sigma,i,\Sigma',j})$. Now notice that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, the collection
$$\{\mathcal{A}^{\Sigma,i,\Sigma,j} : i \in I_\Sigma, j \in J^{\Sigma,i}_\Sigma\} \subseteq \mathsf{K}$$
satisfies
$$\mathrm{Ker}_\Sigma(\mathcal{A}) = \bigcup_{i \in I_\Sigma} \mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,i}) = \bigcup_{i \in I_\Sigma} \bigcup_{j \in J^{\Sigma,i}_\Sigma} \mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,i,\Sigma,j}).$$
Thus, to see that $\mathcal{A} \in \mathbb{C}^*(\mathsf{K})$, it suffices to show that the collection
$$\bigcup_{i \in I_\Sigma} \bigcup_{j \in J^{\Sigma,i}_\Sigma} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i,\Sigma,j})$$
is directed. Consider $X \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i,\Sigma,j})$ and $X' \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i',\Sigma,j'})$. Then, as
$$\begin{array}{rcl}\mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,i}) & = & \bigcup_{j \in J^{\Sigma,i}_\Sigma} \mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,i,\Sigma,j}),\\ \mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,i'}) & = & \bigcup_{j \in J^{\Sigma,i'}_\Sigma} \mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,i',\Sigma,j}),\end{array}$$
we get that $X \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i})$ and $X' \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i'})$. As $\bigcup_{i \in I_\Sigma} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i})$ is directed, there exists $k \in I_\Sigma$ and $Y \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,k})$, such that $X, X' \subseteq Y$. Now, from $\mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,k}) = \bigcup_{j \in J^{\Sigma,k}_\Sigma} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,k,\Sigma,j})$, the finiteness of Y and the fact that the union is directed, there must exist $\ell \in J^{\Sigma,k}_\Sigma$, such that $Y \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,k,\Sigma,\ell})$. This establishes the directedness of the collection $\bigcup_{i \in I_\Sigma} \bigcup_{j \in J^{\Sigma,i}_\Sigma} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i,\Sigma,j})$.

Thus $\mathbb{C}^*$ is a closure operator on classes of $\mathbf{F}$-algebraic systems. ∎

The importance of directedly abstract classes of $\mathbf{F}$-algebraic systems stems from the fact that the validity of a quasiequation transfers from directed K-certificates of an $\mathbf{F}$-algebraic system to the $\mathbf{F}$-algebraic system itself.

Lemma 126 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* K *a class of* $\mathbf{F}$*-algebraic systems and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system. If* $\mathcal{A} \in \mathbb{C}^*(\mathsf{K})$*, then* $\mathrm{QEq}(\mathsf{K}) \leq \mathrm{QEq}(\mathcal{A})$.

Proof: Suppose $\mathcal{A} \in \mathbb{C}^*(\mathsf{K})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\langle \vec{\phi} \approx \vec{\psi}, \phi \approx \psi \rangle \in \mathrm{QEq}_\Sigma(\mathsf{K})$, such that $\vec{\phi} \approx \vec{\psi} \subseteq \mathrm{Eq}_\Sigma(\mathcal{A})$. Let $\{\mathcal{A}^{\Sigma,i} : i \in I\} \subseteq \mathsf{K}$ be a directed Σ-K-certificate for $\mathcal{A}$. Then, by definition $\vec{\phi} \approx \vec{\psi} \subseteq \bigcup_{i \in I} \mathrm{Eq}_\Sigma^\omega(\mathcal{A}^{\Sigma,i})$. Since $\vec{\phi} \approx \vec{\psi}$ is finite and the union is directed, there exists $i \in I$, such that $\vec{\phi} \approx \vec{\psi} \subseteq \mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,i})$. But $\mathcal{A}^{\Sigma,i} \in \mathsf{K}$ and $\langle \vec{\phi} \approx \vec{\psi}, \phi \approx \psi \rangle \in \mathrm{QEq}_\Sigma(\mathsf{K})$, whence

$$\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,i}) \subseteq \bigcup_{i \in I} \mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,i}) = \mathrm{Eq}_\Sigma(\mathcal{A}).$$

Therefore, $\langle \vec{\phi} \approx \vec{\psi}, \phi \approx \psi \rangle \in \mathrm{QEq}_\Sigma(\mathcal{A})$ and $\mathrm{QEq}(\mathsf{K}) \leq \mathrm{QEq}(\mathcal{A})$. ∎

Using Lemma 126, we get that all semantic quasivarieties of algebraic systems are directedly abstract.

Corollary 127 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems. If* K *is a quasiequational class (and, hence, a fortiori, if it is an equational class), then it is directedly abstract.*

Proof: Suppose K is a quasiequational class defined by the family of $\mathbf{F}$-quasiequations $Q \leq \mathrm{QEq}(\mathbf{F})$ and let $\mathcal{A} \in \mathbb{C}^*(\mathsf{K})$. Then, by Lemma 126, $\mathrm{QEq}(\mathsf{K}) \leq \mathrm{QEq}(\mathcal{A})$, whence

$$\begin{array}{rcl}\mathcal{A} & \in & \mathrm{AlgSys}(\mathrm{QEq}(\mathcal{A})) \\ & \subseteq & \mathrm{AlgSys}(\mathrm{QEq}(\mathsf{K})) \\ & = & \mathrm{AlgSys}(\mathrm{QEq}(\mathrm{AlgSys}(Q))) \\ & = & \mathrm{AlgSys}(Q) \\ & = & \mathsf{K}.\end{array}$$

Thus, K is a directedly abstract class. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{A}^i = \langle \mathbf{A}^i, \langle F^i, \alpha^i \rangle \rangle$, $i \in I$, $\mathbf{F}$-algebraic systems and $\langle H^i, \gamma^i \rangle : \mathcal{A} \to \mathcal{A}^i$, $i \in I$, surjective morphisms. Recall from Section 2.4 that we say that the collection

$$\langle H^i, \gamma^i \rangle : \mathcal{A} \to \mathcal{A}^i, \quad i \in I,$$

is a **subdirect intersection** if

$$\bigcap_{i \in I} \mathrm{Ker}(\langle H^i, \gamma^i \rangle) = \Delta^{\mathcal{A}}.$$

Given a class K of $\mathbf{F}$-algebraic systems, we write $\mathcal{A} \in \overset{\triangleleft}{\mathbb{IP}}(\mathsf{K})$ in case there exists a subdirect intersection $\{\langle H^i, \gamma^i\rangle : \mathcal{A} \to \mathcal{A}^i, i \in I\}$, with $\mathcal{A}^i \in \mathsf{K}$, for all $i \in I$. If $\overset{\triangleleft}{\mathbb{IP}}(\mathsf{K}) = \mathsf{K}$, we say that K is **closed under subdirect intersections**.

The following lemma provides an alternative characterization of the concept of subdirect intersection.

Lemma 128 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*,* $\mathcal{A}^i = \langle \mathbf{A}^i, \langle F^i, \alpha^i\rangle\rangle$*,* $i \in I$*,* $\mathbf{F}$*-algebraic systems and* $\{\langle H^i, \gamma^i\rangle : \mathcal{A} \to \mathcal{A}^i : i \in I\}$ *a collection of morphisms. The collection* $\{\langle H^i, \gamma^i\rangle : i \in I\}$ *is a subdirect intersection if and only if* $\mathrm{Ker}(\langle F, \alpha\rangle) = \bigcap_{i\in I} \mathrm{Ker}(\langle F^i, \alpha^i\rangle)$.

Proof: Suppose, first, that $\{\langle H^i, \gamma^i\rangle : i \in I\}$ is a subdirect intersection and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{lll}
\langle \phi, \psi\rangle \in \mathrm{Ker}_\Sigma(\langle F, \alpha\rangle) & \text{iff} & \alpha_\Sigma(\phi) = \alpha_\Sigma(\psi) \\
& \text{iff} & \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in \Delta^{\mathcal{A}}_{F(\Sigma)} \\
& \text{iff} & \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in \bigcap_{i\in I} \mathrm{Ker}_\Sigma(\langle H^i, \gamma^i\rangle) \\
& \text{iff} & \gamma^i_{F(\Sigma)}(\alpha_\Sigma(\phi)) = \gamma^i_{F(\Sigma)}(\alpha_\Sigma(\psi)), i \in I \\
& \text{iff} & \alpha^i_\Sigma(\phi) = \alpha^i_\Sigma(\psi), i \in I \\
& \text{iff} & \langle \phi, \psi\rangle \in \bigcap_{i\in I} \mathrm{Ker}_\Sigma(\langle F^i, \alpha^i\rangle).
\end{array}$$

The reverse relies on the surjectivity of $\langle F, \alpha\rangle$. Suppose $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then we get

$$\begin{array}{lll}
\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in \Delta^{\mathcal{A}}_{F(\Sigma)} & \text{iff} & \langle \phi, \psi\rangle \in \mathrm{Ker}_\Sigma(\langle F, \alpha\rangle) \\
& \text{iff} & \langle \phi, \psi\rangle \in \bigcap_{i\in I} \mathrm{Ker}_\Sigma(\langle F^i, \alpha^i\rangle) \\
& \text{iff} & \alpha^i_\Sigma(\phi) = \alpha^i_\Sigma(\psi), i \in I \\
& \text{iff} & \gamma^i_{F(\Sigma)}(\alpha_\Sigma(\phi)) = \gamma^i_{F(\Sigma)}(\alpha_\Sigma(\psi)), i \in I \\
& \text{iff} & \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in \bigcap_{i\in I} \mathrm{Ker}_{F(\Sigma)}(\langle H^i, \gamma^i\rangle).
\end{array}$$

Thus, by the surjectivity of $\langle F, \alpha\rangle$ we get that $\Delta^{\mathcal{A}} = \bigcap_{i\in I} \mathrm{Ker}(\langle H^i, \gamma^i\rangle)$. ■

It is not difficult to verify that the subdirect intersection operator is also a closure operator on classes of $\mathbf{F}$-algebraic systems.

Proposition 129 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system. Then the operator* $\overset{\triangleleft}{\mathbb{IP}}$ *on classes of* $\mathbf{F}$*-algebraic systems is a closure operator.*

Proof: Suppose K is a class of $\mathbf{F}$-algebraic systems.

- If $\mathcal{A} \in \mathsf{K}$, then $\{\langle I, \iota\rangle : \mathcal{A} \to \mathcal{A}\}$, where $\langle I, \iota\rangle : \mathcal{A} \to \mathcal{A}$ is the identity morphism, is a subdirect intersection family. Thus, we get that $\mathcal{A} \in \overset{\triangleleft}{\mathbb{IP}}(\mathsf{K})$. Hence $\mathsf{K} \subseteq \overset{\triangleleft}{\mathbb{IP}}(\mathsf{K})$ and $\overset{\triangleleft}{\mathbb{IP}}$ is inflationary;

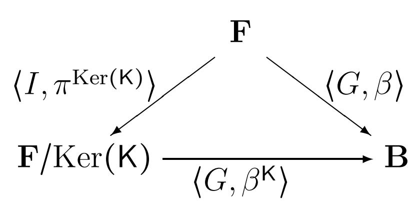
- It is obvious that $\overset{\triangleleft}{\mathbb{I}\Pi}$ is monotonic;
- Suppose that $\mathcal{A} \in \overset{\triangleleft}{\mathbb{I}\Pi}(\overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K}))$. Then, there exists a subdirect intersection family $\{\langle H^i, \gamma^i\rangle : \mathcal{A} \to \mathcal{A}^i, i \in I\}$, with $\mathcal{A}^i \in \overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K})$, for all $i \in I$. Therefore, for each $i \in I$, there exists a sibdirect intersection family $\{\langle H^{ij}, \gamma^{ij}\rangle : \mathcal{A}^i \to \mathcal{A}^{ij}, j \in J_i\}$, with $\mathcal{A}^{ij} \in \mathsf{K}$, for all $i \in I$ and all $j \in J_i$. Consider
$$\{\langle H^{ij}, \gamma^{ij}\rangle \circ \langle H^i, \gamma^i\rangle : \mathcal{A} \to \mathcal{A}^{ij}, i \in I, j \in J_i\}.$$
It is a subdirect intersection family, since
$$\begin{aligned}\textstyle\bigcap_{i\in I, j\in J_i} \mathrm{Ker}(\langle H^{ij}, \gamma^{ij}\rangle \circ \langle H^i, \gamma^i\rangle) & \\ &= \textstyle\bigcap_{i\in I, j\in J_i} (\gamma^{ij} \circ \gamma^i)^{-1}(\Delta^{\mathcal{A}^{ij}}) \\ &= \textstyle\bigcap_{i\in I, j\in J_i} (\gamma^i)^{-1}((\gamma^{ij})^{-1}(\Delta^{\mathcal{A}^{ij}})) \\ &= \textstyle\bigcap_{i\in I} (\gamma^i)^{-1}(\bigcap_{j\in J_i} (\gamma^{ij})^{-1}(\Delta^{\mathcal{A}^{ij}})) \\ &= \textstyle\bigcap_{i\in I} (\gamma^i)^{-1}(\Delta^{\mathcal{A}^i}) \\ &= \Delta^{\mathcal{A}}.\end{aligned}$$
Since $\mathcal{A}^{ij} \in \mathsf{K}$, for all $i \in I$, $j \in J_i$, we get that $\overset{\triangleleft}{\mathbb{I}\Pi}(\overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K})) \subseteq \overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K})$ and $\overset{\triangleleft}{\mathbb{I}\Pi}$ is idempotent.

Thus, $\overset{\triangleleft}{\mathbb{I}\Pi}$ is a closure operator. ∎

A key property concerning subdirect intersections, which is very useful in applying the concept, is given in the following

Lemma 130 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and consider a class* $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$. *The class of morphisms*
$$\langle G, \beta^{\mathsf{K}}\rangle : \mathcal{F}/\bigcap_{\mathcal{B}\in\mathsf{K}} \mathrm{Ker}(\langle G, \beta\rangle) \to \mathcal{B}, \quad \mathcal{B} = \langle \mathbf{B}, \langle G, \beta\rangle\rangle \in \mathsf{K},$$
where, for all $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\beta^{\mathsf{K}}_\Sigma(\phi/\bigcap_{\mathcal{B}\in\mathsf{K}} \mathrm{Ker}_\Sigma(\langle G, \beta\rangle)) = \beta_\Sigma(\phi),$$
forms a subdirect intersection.

Proof: It is not difficult to see that β^{K} is well defined and forms a natural transformation. Moreover, $\langle G, \beta^{\mathsf{K}}\rangle$ is an $\mathbf{F}$-morphism. Letting $\mathrm{Ker}(\mathsf{K}) = \bigcap_{\mathcal{B}\in\mathsf{K}} \mathrm{Ker}(\langle G, \beta\rangle)$, we have, by definition, the following commutative triangle.



To show that the displayed family forms a subdirect intersection, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then, we get

$$\begin{array}{ll}\langle \phi/\mathrm{Ker}_\Sigma(\mathsf{K}), \psi/\mathrm{Ker}_\Sigma(\mathsf{K})\rangle \in \bigcap_{\mathcal{B}\in\mathsf{K}} \mathrm{Ker}_\Sigma(\langle G, \beta^{\mathsf{K}}\rangle) & \\ \quad \text{iff} \quad \beta^{\mathsf{K}}_\Sigma(\phi/\mathrm{Ker}_\Sigma(\mathsf{K})) = \beta^{\mathsf{K}}_\Sigma(\psi/\mathrm{Ker}_\Sigma(\mathsf{K})), \quad \mathcal{B} \in \mathsf{K}, & \\ \quad \text{iff} \quad \beta_\Sigma(\phi) = \beta_\Sigma(\psi), \quad \mathcal{B} \in \mathsf{K}, & \\ \quad \text{iff} \quad \phi/\mathrm{Ker}_\Sigma(\mathsf{K}) = \psi/\mathrm{Ker}_\Sigma(\mathsf{K}). & \end{array}$$

Thus, $\bigcap_{\mathcal{B}\in\mathsf{K}} \mathrm{Ker}(\langle G, \beta^{\mathsf{K}}\rangle) = \Delta^{\mathcal{F}/\mathrm{Ker}(\mathsf{K})}$, showing that

$$\langle G, \beta^{\mathsf{K}}\rangle : \mathcal{F}/\bigcap_{\mathcal{B}\in\mathsf{K}} \mathrm{Ker}(\langle G, \beta\rangle) \to \mathcal{B}, \quad \mathcal{B} = \langle \mathbf{B}, \langle G, \beta\rangle\rangle \in \mathsf{K},$$

constitutes indeed a subdirect intersection. ■

Finally, we show that every semantic guasivariety of $\mathbf{F}$-algebraic systems is closed under subdirect intersections.

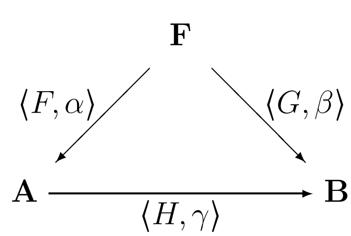
Proposition 131 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$. *If* $\mathsf{K} = \mathbb{G}^{\mathsf{Sem}}(\mathsf{K})$, *then* $\overset{\triangleleft}{\mathbb{\Pi}}(\mathsf{K}) \subseteq \mathsf{K}$.

Proof: Assume that $\mathsf{K} = \mathbb{G}^{\mathsf{Sem}}(\mathsf{K})$. Let $X = \mathrm{GEq}(\mathsf{K})$. Assume that $\mathcal{A} \in \overset{\triangleleft}{\mathbb{\Pi}}(\mathsf{K})$ and $\Sigma \in |\mathbf{Sign}^\flat|$, $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in X_\Sigma$, such that $\mathcal{A} \vDash_\Sigma \vec{\phi} \approx \vec{\psi}$, i.e., $\vec{\phi} \approx \vec{\psi} \subseteq \mathrm{Eq}_\Sigma(\mathcal{A})$. Since $\mathcal{A} \in \overset{\triangleleft}{\mathbb{\Pi}}(\mathsf{K})$, there exists a subdirect intersection

$$\langle H^i, \gamma^i\rangle : \mathcal{A} \to \mathcal{A}^i, \quad i \in I,$$

such that $\mathcal{A}^i \in \mathsf{K}$, for all $i \in I$. Hence, we get $\vec{\phi} \approx \vec{\psi} \subseteq \mathrm{Eq}_\Sigma(\mathcal{A}^i)$, $i \in I$. Now, since $\mathcal{A}^i \in \mathsf{K}$ and $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in X_\Sigma = \mathrm{GEq}_\Sigma(\mathsf{K})$, we conclude that $\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathcal{A}^i)$, for all $i \in I$. Therefore, $\phi \approx \psi \in \bigcap_{i\in I} \mathrm{Eq}_\Sigma(\mathcal{A}^i) = \mathrm{Eq}_\Sigma(\mathcal{A})$, the latter by the definition of subdirect intersection and Lemma 128. Therefore, $\mathcal{A} \vDash_\Sigma \vec{\phi} \approx \vec{\psi} \to \phi \approx \psi$. This shows that $\mathcal{A} \in \mathrm{AlgSys}(X) = \mathrm{AlgSys}(\mathrm{GEq}(\mathsf{K})) = \mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$. We conclude that $\overset{\triangleleft}{\mathbb{\Pi}}(\mathsf{K}) \subseteq \mathsf{K}$, i.e., K is closed under subdirect intersections. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta\rangle\rangle$ $\mathbf{F}$-algebraic systems and $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism.



In this case we say $\mathcal{B}$ is a **morphic image** of $\mathcal{A}$. Given a class K of $\mathbf{F}$-algebraic systems, we write $\mathcal{B} \in \mathbb{H}(\mathsf{K})$ in case there exists a surjective morphism $\langle H, \gamma\rangle$:

$\mathcal{A} \to \mathcal{B}$, with $\mathcal{A} \in \mathsf{K}$. If $\mathbb{H}(\mathsf{K}) = \mathsf{K}$, we say that K is **closed under morphic images**.

It is straightforward to verify that $\mathbb{H}$ is a closure operator on classes of $\mathbf{F}$-algebraic systems.

Proposition 132 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system. Then the operator* $\mathbb{H}$ *on classes of* $\mathbf{F}$*-algebraic systems is a closure operator.*

Proof: Let K be a class of $\mathbf{F}$-algebraic systems. If $\mathcal{A} \in \mathsf{K}$, then, using again the identity $\langle I, \iota \rangle : \mathcal{A} \to \mathcal{A}$, we see that $\mathcal{A} \in \mathbb{H}(\mathsf{K})$, and, hence, $\mathbb{H}$ is inflationary. It is again obvious that it is monotonic. Finally, if $\mathcal{A} \in \mathbb{H}(\mathbb{H}(\mathsf{K}))$, then, there exists a surjective morphism $\langle G, \beta \rangle : \mathcal{A}' \to \mathcal{A}$, with $\mathcal{A}' \in \mathbb{H}(\mathsf{K})$, whence, there also exists a surjective morphism $\langle H, \gamma \rangle : \mathcal{A}'' \to \mathcal{A}'$, with $\mathcal{A}'' \in \mathsf{K}$. Now the surjective morphism $\langle G, \beta \rangle \circ \langle H, \gamma \rangle : \mathcal{A}'' \to \mathcal{A}$ witnesses the fact that $\mathcal{A} \in \mathbb{H}(\mathsf{K})$. Therefore, $\mathbb{H}(\mathbb{H}(\mathsf{K})) \subseteq \mathbb{H}(\mathsf{K})$, and $\mathbb{H}$ is idempotent. Thus, $\mathbb{H}$ is a closure operator. ■

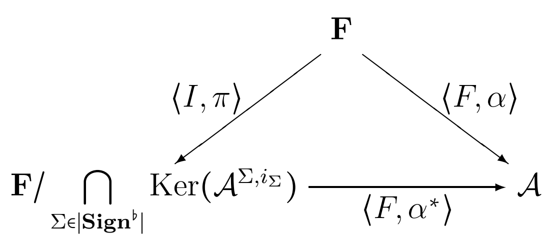
We show, next, that, if a class K of $\mathbf{F}$-algebraic systems is closed under subdirect intersections and morphic images, then it is also closed under directed K-certifications.

Proposition 133 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *be a class of* $\mathbf{F}$*-algebraic systems. Then* $\mathbb{C}^*(\mathsf{K}) \subseteq \mathbb{H}(\overset{\triangleleft}{\mathbb{II}}(\mathsf{K}))$.

Proof: Suppose $\mathcal{A} \in \mathbb{C}^*(\mathsf{K})$. Then, by definition, for all $\Sigma \in |\mathbf{Sign}^\flat|$, there exists a collection $\{\mathcal{A}^{\Sigma,i} : i \in I_\Sigma\} \subseteq \mathsf{K}$, such that $\bigcup_{i \in I_\Sigma} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i})$ is directed and $\mathrm{Ker}_\Sigma(\mathcal{A}) = \bigcup_{i \in I_\Sigma} \mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,i})$. Fix, for every $\Sigma \in |\mathbf{Sign}^\flat|$, an $i_\Sigma \in I_\Sigma$ and consider the family of morphisms

$$\langle H^{\Sigma,i_\Sigma}, \gamma^{\Sigma,i_\Sigma} \rangle : \mathcal{F}/\bigcap_{\Sigma \in |\mathbf{Sign}^\flat|} \mathrm{Ker}(\mathcal{A}^{\Sigma,i_\Sigma}) \to \mathcal{A}^{\Sigma,i_\Sigma}, \quad \Sigma \in |\mathbf{Sign}^\flat|.$$

By Lemma 130, it constitutes a subdirect intersection, whence, since $\mathcal{A}^{\Sigma,i_\Sigma} \in \mathsf{K}$, for all $\Sigma \in |\mathbf{Sign}^\flat|$, we get $\mathcal{F}/\bigcap_{\Sigma \in |\mathbf{Sign}^\flat|} \mathrm{Ker}(\mathcal{A}^{\Sigma,i_\Sigma}) \in \overset{\triangleleft}{\mathbb{II}}(\mathsf{K})$. Now it is not difficult to see that there exists a morphism $\langle F, \alpha^* \rangle : \mathcal{F}/\bigcap_{\Sigma \in |\mathbf{Sign}^\flat|} \mathrm{Ker}(\mathcal{A}^{\Sigma,i_\Sigma}) \to \mathcal{A}$, such that the following diagram commutes



The natural transformation α^* is defined, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma')$, by

$$\alpha^*_{\Sigma'}(\phi/\bigcap_{\Sigma \in |\mathbf{Sign}^\flat|} \mathrm{Ker}_{\Sigma'}(\mathcal{A}^{\Sigma,i_\Sigma})) = \alpha_{\Sigma'}(\phi).$$

It is well-defined, since, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma')$, we have

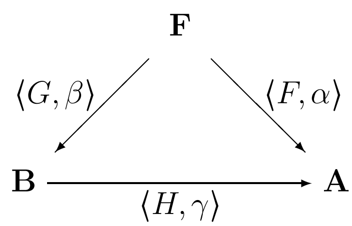
$$\begin{array}{lll}\langle \phi, \psi \rangle \in \bigcap_{\Sigma \in |\mathbf{Sign}^\flat|} \mathrm{Ker}_{\Sigma'}(\mathcal{A}^{\Sigma, i_\Sigma}) & \text{implies} & \langle \phi, \psi \rangle \in \mathrm{Ker}_{\Sigma'}(\mathcal{A}^{\Sigma', i_{\Sigma'}}) \\ & \text{implies} & \langle \phi, \psi \rangle \in \bigcup_{i \in I_{\Sigma'}} \mathrm{Ker}_{\Sigma'}(\mathcal{A}^{\Sigma', i}) \\ & \text{implies} & \langle \phi, \psi \rangle \in \mathrm{Ker}_{\Sigma'}(\mathcal{A}).\end{array}$$

Taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that $\mathcal{A} \in \mathbb{H}(\overset{\triangleleft}{\mathbb{P}}(\mathsf{K}))$. Therefore, $\mathbb{C}^*(\mathsf{K}) \subseteq \mathbb{H}(\overset{\triangleleft}{\mathbb{P}}(\mathsf{K}))$. ■

Finally, it is not difficult to see that semantic varieties are closed under morphic images.

Proposition 134 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$. *If* $\mathsf{K} = \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$, *then* $\mathbb{H}(\mathsf{K}) \subseteq \mathsf{K}$.

Proof: Assume that $\mathsf{K} = \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. Let $X = \mathrm{Eq}(\mathsf{K})$. Assume that $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle \in \mathbb{H}(\mathsf{K})$ and $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \approx \psi \in X_\Sigma$. Since $\mathcal{A} \in \mathbb{H}(\mathsf{K})$, there exists $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle \in K$ and $\langle H, \gamma \rangle : \mathcal{B} \to \mathcal{A}$:



Since $\mathcal{B} \in \mathsf{K}$ and $\phi \approx \psi \in X_\Sigma = \mathrm{Eq}_\Sigma(\mathsf{K})$, we conclude that $\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathcal{B})$. Therefore, $\beta_\Sigma(\phi) = \beta_\Sigma(\psi)$. But this gives $\gamma_{G(\Sigma)}(\beta_\Sigma(\phi)) = \gamma_{G(\Sigma)}(\beta_\Sigma(\psi))$ or, equivalently, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. Therefore, $\mathcal{A} \vDash_\Sigma \phi \approx \psi$. This shows that $\mathcal{A} \in \mathrm{AlgSys}(X) = \mathrm{AlgSys}(\mathrm{Eq}(\mathsf{K})) = \mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$. We conclude that $\mathbb{H}(\mathsf{K}) \subseteq \mathsf{K}$, i.e., K is closed under morphic images. ■

We are now ready to provide alternative characterizations of equational, quasiequational and guasiequational classes of $\mathbf{F}$-algebraic systems. Namely, we show that a class of $\mathbf{F}$-algebraic systems is:

- a guasiequational class if and only if it is abstract and closed under subdirect intersections;
- a quasiequational class if and only if it is directedly abstract and closed under subdirect intersections;
- an equational class if and only if it is closed under subdirect intersections and morphic images.

Theorem 135 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems.* K *is a guasiequational class if and only if it is abstract and closed under subdirect intersections.*

Proof: If K is a guasiequational class, then it is abstract by Corollary 124 and it is closed under subdirect intersections by Proposition 131.

Assume, conversely, that K is abstract and closed under subdirect intersections and set $G = \mathrm{GEq}(\mathsf{K})$. Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle \in \mathrm{AlgSys}(\mathbf{F})$, such that $G \leq \mathrm{GEq}(\mathcal{A})$. Let $\Sigma \in |\mathbf{Sign}^\flat|$, and $\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathbf{F})$, such that $\phi \approx \psi \notin \mathrm{Eq}_\Sigma(\mathcal{A})$, i.e., such that $\alpha_\Sigma(\phi) \neq \alpha_\Sigma(\psi)$. Thus, by definition, the guasiequation

$$\langle \mathrm{Eq}_\Sigma(\mathcal{A}), \phi \approx \psi\rangle \notin \mathrm{GEq}_\Sigma(\mathcal{A}).$$

Therefore, since $G \leq \mathrm{GEq}(\mathcal{A})$, $\langle \mathrm{Eq}_\Sigma(\mathcal{A}), \phi \approx \psi\rangle \notin \mathrm{GEq}_\Sigma(\mathsf{K})$. Hence, for every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \approx \psi \notin \mathrm{Eq}_\Sigma(\mathcal{A})$, there exists $\mathcal{K}^{\langle\Sigma,\phi\approx\psi\rangle} \in \mathsf{K}$, such that $\mathrm{Eq}_\Sigma(\mathcal{A}) \subseteq \mathrm{Eq}_\Sigma(\mathcal{K}^{\langle\Sigma,\phi\approx\psi\rangle})$, but $\phi \approx \psi \notin \mathrm{Eq}_\Sigma(\mathcal{K}^{\langle\Sigma,\phi\approx\psi\rangle})$. We conclude that

$$\mathrm{Eq}_\Sigma(\mathcal{A}) = \bigcap\{\mathrm{Eq}_\Sigma(\mathcal{K}^{\langle\Sigma,\phi\approx\psi\rangle}) : \phi \approx \psi \notin \mathrm{Eq}_\Sigma(\mathcal{A})\}.$$

Let, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathsf{K}^\Sigma = \{\mathcal{K}^{\langle\Sigma,\phi\approx\psi\rangle} : \phi \approx \psi \notin \mathrm{Eq}_\Sigma(\mathcal{A})\}.$$

- Since, by hypothesis, K is closed under subdirect intersections, and, by Lemma 130,
$$\{\langle F^\mathcal{K}, \alpha^\mathcal{K}\rangle : \mathcal{F}/\mathrm{Ker}(\mathsf{K}^\Sigma) \to \mathcal{K}, \mathcal{K} \in \mathsf{K}^\Sigma\}$$
is a subdirect intersection, we get that $\mathcal{F}/\mathrm{Ker}(\mathsf{K}^\Sigma) \in \mathsf{K}$.
- Since, for all $\Sigma \in |\mathbf{Sign}^\flat|$,
$$\mathrm{Ker}_\Sigma(\mathcal{A}) = \mathrm{Ker}_\Sigma(\mathsf{K}^\Sigma) = \mathrm{Ker}_\Sigma(\mathcal{F}/\mathrm{Ker}(\mathsf{K}^\Sigma))$$
and $\mathcal{F}/\mathrm{Ker}(\mathsf{K}^\Sigma) \in \mathsf{K}$, $\mathcal{A} \in \mathbb{C}(\mathsf{K})$. Since K is abstract, we conclude that $\mathcal{A} \in \mathsf{K}$.

Therefore, K is a guasiequational class of $\mathbf{F}$-algebraic systems. ∎

Theorem 136 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems.* K *is a quasiequational class if and only if it is directedly abstract and closed under subdirect intersections.*

Proof: If K is a quasiequational class, then it is directedly abstract by Corollary 127 and it is closed under subdirect intersections by Proposition 131.

Conversely, suppose that $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$, such that $\mathbb{C}^*(\mathsf{K}) \subseteq \mathsf{K}$ and $\overset{\triangleleft}{\mathbb{IP}}(\mathsf{K}) \subseteq \mathsf{K}$. It suffices to show that $\mathsf{K} = \mathrm{AlgSys}(\mathrm{QEq}(\mathsf{K}))$. The left to right inclusion always holds. For the converse, consider $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle \in \mathrm{AlgSys}(\mathrm{QEq}(\mathsf{K}))$. For all $\Sigma \in |\mathbf{Sign}^\flat|$, all $X \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A})$ and all $\phi \approx \psi \notin \mathrm{Eq}_\Sigma(\mathcal{A})$, we consider the $\mathbf{F}$-quasiequation

$$q^{\Sigma,X,\phi\approx\psi} := X \to \phi \approx \psi.$$

Since $\mathcal{A} \vDash_\Sigma \mathrm{Eq}_\Sigma(\mathcal{A})$ and $\mathcal{A} \nvDash_\Sigma \phi \approx \psi$, we get that $q^{\Sigma,X,\phi\approx\psi} \notin \mathrm{QEq}_\Sigma(\mathcal{A})$. Thus, since $\mathcal{A} \in \mathrm{AlgSys}(\mathrm{QEq}(\mathsf{K}))$, we infer that $q^{\Sigma,X,\phi\approx\psi} \notin \mathrm{QEq}_\Sigma(\mathsf{K})$. Therefore, there exists $\mathcal{A}^{\Sigma,X,\phi\approx\psi} \in \mathsf{K}$, such that $\mathcal{A}^{\Sigma,X,\phi\approx\psi} \nvDash_\Sigma q^{\Sigma,X,\phi\approx\psi}$, i.e.,

$$\mathcal{A}^{\Sigma,X,\phi\approx\psi} \vDash_\Sigma X \quad \text{and} \quad \mathcal{A}^{\Sigma,X,\phi\approx\psi} \nvDash_\Sigma \phi \approx \psi.$$

Let, for all $X \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A})$,

$$\mathsf{A}^{\Sigma,X} = \{\mathcal{A}^{\Sigma,X,\phi\approx\psi} : \phi \approx \psi \notin \mathrm{Eq}_\Sigma(\mathcal{A})\}.$$

Define, for all $X \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A})$,

$$\mathcal{A}^{\Sigma,X} := \mathcal{F}/\mathrm{Ker}(\mathsf{A}^{\Sigma,X}) = \mathcal{F}/\bigcap_{\phi\approx\psi\notin\mathrm{Eq}_\Sigma(\mathcal{A})} \mathrm{Ker}(\mathcal{A}^{\Sigma,X,\phi\approx\psi}).$$

By Proposition 130, for all $X \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A})$, $\mathcal{A}^{\Sigma,X} \in \overset{\triangleleft}{\mathbb{P}}(\mathsf{K}) = \mathsf{K}$. Consequently, it suffices to show the following:

- $\bigcup_{X\in\mathrm{Eq}^\omega_\Sigma(\mathcal{A})} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,X})$ is directed;
- $\mathrm{Ker}_\Sigma(\mathcal{A}) = \bigcup_{X\in\mathrm{Eq}^\omega_\Sigma(\mathcal{A})} \mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,X})$.

Suppose, first, that $E \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,X})$ and $E' \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,X'})$, for some $X, X' \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A})$. Then, by construction of $\mathcal{A}^{\Sigma,X}$ and $\mathcal{A}^{\Sigma,X'}$, we get that $E, E' \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A})$. Therefore, $E \cup E' \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,E\cup E'})$ and, hence, $\bigcup_{X\in\mathrm{Eq}^\omega_\Sigma(\mathcal{A})} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,X})$ is indeed directed.

Finally, note that, by construction, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{Ker}_\Sigma(\mathcal{A}) = \bigcup_{X\in\mathrm{Eq}^\omega_\Sigma(\mathcal{A})} \mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,X}).$$

Indeed, for all $\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathbf{F})$,

- if $\phi \approx \psi \in \mathrm{Ker}_\Sigma(\mathcal{A})$, then, $\phi \approx \psi \in \mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,\{\phi\approx\psi\}})$, whence $\phi \approx \psi \in \bigcup_{X\in\mathrm{Eq}^\omega_\Sigma(\mathcal{A})} \mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,X})$.
- if $\phi \approx \psi \notin \mathrm{Ker}_\Sigma(\mathcal{A})$, then, by construction, for all $X \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A})$, $\phi \approx \psi \notin \mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,X})$. Therefore, $\phi \approx \psi \notin \bigcup_{X\in\mathrm{Eq}^\omega_\Sigma(\mathcal{A})} \mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,X})$.

Since, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $X \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A})$, $\mathcal{A}^{\Sigma,X} \in \mathsf{K}$, we get, by the definition of $\mathbb{C}^*$ and the two properties just proven, that $\mathcal{A} \in \mathbb{C}^*(\mathsf{K}) = \mathsf{K}$. Thus, K is a quasiequational class of $\mathbf{F}$-algebraic systems. ∎

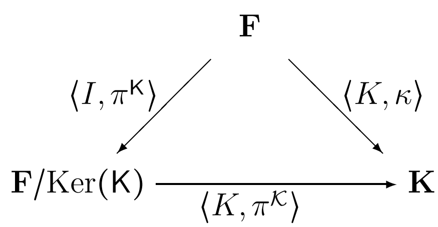
Theorem 137 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems.* K *is an equational class if and only if it is closed under subdirect intersections and morphic images.*

Proof: If K is an equational class, then it is closed under subdirect intersections by Proposition 131 and under morphic images by Proposition 134.

Suppose, conversely, that K is a class of $\mathbf{F}$-algebraic systems that is closed under subdirect intersections and morphic images. Set $E = \mathrm{Eq}(\mathsf{K})$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system, such that $E \leq \mathrm{Eq}(\mathcal{A})$. Consider, for all $\mathcal{K} = \langle \mathbf{K}, \langle K, \kappa \rangle \rangle \in \mathsf{K}$, the mapping

$$\langle K, \pi^{\mathcal{K}} \rangle : \mathbf{F}/\mathrm{Ker}(\mathsf{K}) \to \mathbf{K}$$

defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,



$$\pi^{\mathcal{K}}_\Sigma(\phi/\mathrm{Ker}_\Sigma(\mathsf{K})) = \kappa_\Sigma(\phi).$$

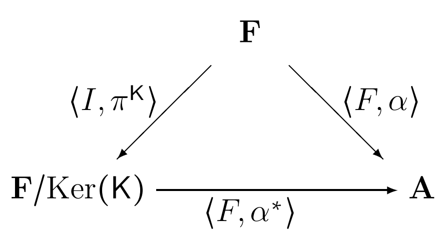
By Lemma 130, the collection

$$\{\langle K, \pi^{\mathcal{K}} \rangle : \mathcal{F}/\mathrm{Ker}(\mathsf{K}) \to \mathcal{K}, \mathcal{K} = \langle \mathbf{K}, \langle K, \kappa \rangle \rangle \in \mathsf{K}\}$$

forms a subdirect intersection. Since all codomains are in K and K is closed under subdirect intersections, we get $\mathcal{F}/\mathrm{Ker}(\mathsf{K}) \in \mathsf{K}$. Now consider the morphism

$$\langle F, \alpha^* \rangle : \mathcal{F}/\mathrm{Ker}(\mathsf{K}) \to \mathcal{A},$$

given, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, by



$$\alpha^*_\Sigma(\phi/\mathrm{Ker}_\Sigma(\mathsf{K})) = \alpha_\Sigma(\phi).$$

It is well defined, since, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, if $\langle \phi, \psi \rangle \in \mathrm{Ker}_\Sigma(\mathsf{K})$, then, by hypothesis, $\langle \phi, \psi \rangle \in \mathrm{Eq}_\Sigma(\mathcal{A})$ and, hence, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. Moreover, since $\langle F, \alpha \rangle$ is surjective, so is $\langle F, \alpha^* \rangle$. Since $\mathcal{F}/\mathrm{Ker}(\mathsf{K}) \in \mathsf{K}$ and K is closed under morphic images, we conclude that $\mathcal{A} \in \mathsf{K}$. Therefore, K is an equational class of $\mathbf{F}$-algebraic systems. ■

We prove, next, the following result to the effect that, for any guasiequational class K of $\mathbf{F}$-algebraic systems, the theory families of the equational structure $\mathcal{Q}^{\mathsf{K}} = \langle \mathbf{F}, D^{\mathsf{K}} \rangle$ coincide with the K-congruence systems on $\mathcal{F}$.

Corollary 138 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems. If* K *is a guasiequational class, then*

$$\mathrm{ThFam}(\mathcal{Q}^{\mathsf{K}}) = \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F}).$$

Proof: We can rely on preceding results, but we also give a direct proof.

Since K is a guasiequational class, by Theorem 135, it is abstract and closed under subdirect intersections. Since any guasiequational class also contains a trivial $\mathbf{F}$-algebraic system, we conclude, by Theorem 32, that $\mathrm{ThFam}(\mathcal{Q}^{\mathsf{K}}) = \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$.

Next, we provide a direct proof of the same result. Suppose K is a guasiequational class of $\mathbf{F}$-algebraic systems.

Let $\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ and $\phi \approx \psi \in D^{\mathsf{K}}_\Sigma(\theta_\Sigma)$. Then $\langle \theta_\Sigma, \phi \approx \psi \rangle \in \mathrm{GEq}_\Sigma(\mathsf{K})$. Thus, since, by hypothesis, $\mathcal{F}/\theta \in \mathsf{K}$, $\mathcal{F}/\theta \vDash_\Sigma \langle \theta_\Sigma, \phi \approx \psi \rangle$. But, obviously, $\mathcal{F}/\theta \vDash_\Sigma \theta_\Sigma$. Therefore, we get $\mathcal{F}/\theta \vDash_\Sigma \phi \approx \psi$, or, equivalently, $\langle \phi, \psi \rangle \in \theta_\Sigma$. We conclude that $D^{\mathsf{K}}(\theta) = \theta$ and, hence, $\theta \in \mathrm{ThFam}(\mathcal{Q}^{\mathsf{K}})$.

Assume, conversely, that $\theta \in \mathrm{ThFam}(\mathcal{Q}^{\mathsf{K}})$ and consider $\Sigma \in |\mathbf{Sign}^\flat|$, $\vec{\phi}$, $\vec{\psi}$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \vec{\phi} \approx \vec{\psi}, \phi \approx \psi \rangle \in \mathrm{GEq}_\Sigma(\mathsf{K})$ and $\mathcal{F}/\theta \vDash_\Sigma \vec{\phi} \approx \vec{\psi}$. Then, $\langle \phi_i, \psi_i \rangle \in \theta_\Sigma$, for all $i \in I$. Since $\langle \vec{\phi} \approx \vec{\psi}, \phi \approx \psi \rangle \in \mathrm{GEq}_\Sigma(\mathsf{K})$ and $\theta \in \mathrm{ThFam}(\mathcal{Q}^{\mathsf{K}})$, we get $\langle \phi, \psi \rangle \in \theta_\Sigma$. Hence, $\mathcal{F}/\theta \vDash_\Sigma \phi \approx \psi$. We conclude, taking into account the fact that K is a guasiequational class, that $\mathcal{F}/\theta \in \mathrm{AlgSys}(\mathrm{GEq}(\mathsf{K})) = \mathsf{K}$. Thus, $\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$. ■

We obtain, as a corollary, that, if the relative equational consequences of two semantic guasivarieties are identical, then the two guasivarieties coincide.

Proposition 139 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *and* K' *semantic guasivarieties of* $\mathbf{F}$*-algebraic systems, such that* $D^{\mathsf{K}} = D^{\mathsf{K}'}$. *Then* $\mathsf{K} = \mathsf{K}'$.

Proof: Let $\mathcal{A} \in \mathsf{K}$ and consider $\Sigma \in |\mathbf{Sign}^\flat|$, $\vec{\phi}, \vec{\psi}, \phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$\mathsf{K}' \vDash_\Sigma \langle \vec{\phi} \approx \vec{\psi}, \phi \approx \psi \rangle.$$

This is equivalent to $\phi \approx \psi \in D^{\mathsf{K}'}_\Sigma(\vec{\phi} \approx \vec{\psi})$. By hypothesis, we get $\phi \approx \psi \in D^{\mathsf{K}}_\Sigma(\vec{\phi} \approx \vec{\psi})$, i.e., $\mathsf{K} \vDash_\Sigma \langle \vec{\phi} \approx \vec{\psi}, \phi \approx \psi \rangle$. Since $\mathcal{A} \in \mathsf{K}$, $\mathcal{A} \vDash_\Sigma \langle \vec{\phi} \approx \vec{\psi}, \phi \approx \psi \rangle$. This shows that $\mathrm{GEq}(\mathsf{K}') \leq \mathrm{GEq}(\mathcal{A})$ and, hence $\mathcal{A} \in \mathbb{G}^{\mathsf{Sem}}(\mathsf{K}') = \mathsf{K}'$, the latter equation by the assumption that K' is a semantic guasivariety. We conclude that $\mathsf{K} \subseteq \mathsf{K}'$. By symmetry, we get $\mathsf{K} = \mathsf{K}'$. ■

These results allow us to obtain another round of different characterizations of equational, quasiequational and guasiequational classes of $\mathbf{F}$-algebraic systems.

To provide the characterization of guasiequational classes, we need, first, some technical lemmas.

Lemma 140 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* K *an abstract class of* $\mathbf{F}$*-algebraic systems.* K *is closed under subdirect intersections if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ *is closed under intersection.*

Proof: Let K be an abstract class of $\mathbf{F}$-algebraic systems. Suppose, first, that K is closed under subdirect intersections and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $\{\theta^i : i \in I\} \subseteq \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$. Then, by definition, $\mathcal{A}/\theta^i \in \mathsf{K}$, for all $i \in I$. Let, for all $i \in I$,

$$\langle I, \rho^i \rangle : \mathcal{A}/\bigcap_{i \in I} \theta^i \to \mathcal{A}/\theta^i$$

be defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, by

$$\begin{array}{ccc} & \mathbf{F} & \\ \langle F, \pi \circ \alpha \rangle \swarrow & & \searrow \langle F, \pi^i \circ \alpha \rangle \\ \mathcal{A}/\bigcap_{i \in I} \theta^i & \xrightarrow[\langle I, \rho^i \rangle]{} & \mathcal{A}/\theta^i \end{array}$$

$$\rho^i_\Sigma(\phi/\bigcap_{i \in I} \theta^i_\Sigma) = \phi/\theta^i_\Sigma.$$

Then, we have $\bigcap_{i \in I} \mathrm{Ker}(\langle I, \rho^i \rangle) = \Delta^{\mathcal{A}/\bigcap_{i \in I} \theta^i}$. Hence, the family $\{\langle I, \rho^i \rangle : i \in I\}$ forms a subdirect intersection. Thus, by hypothesis, since $\mathcal{A}/\theta^i \in \mathsf{K}$, for all $i \in I$, $\mathcal{A}/\bigcap_{i \in I} \theta^i \in \mathsf{K}$ and, therefore, $\bigcap_{i \in I} \theta^i \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$. We conclude that $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ is closed under intersections.

Suppose, conversely, that, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ is closed under intersection and let

$$\langle H^i, \gamma^i \rangle : \mathcal{A} \to \mathcal{A}^i, \quad i \in I,$$

be a subdirect intersection, such that $\mathcal{A}^i \in \mathsf{K}$, for all $i \in I$. For every $i \in I$, consider the morphism

$$\langle H^i, \delta^i \rangle : \mathcal{A}/\mathrm{Ker}(\langle H^i, \gamma^i \rangle) \to \mathcal{A}^i,$$

defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, by

$$\begin{array}{ccc} & \mathbf{F} & \\ \langle F, \pi^i \circ \alpha \rangle \swarrow & & \searrow \langle F^i, \alpha^i \rangle \\ \mathbf{A}/\mathrm{Ker}(\langle H^i, \gamma^i \rangle) & \xrightarrow[\langle H^i, \delta^i \rangle]{} & \mathbf{A}^i \end{array}$$

$$\delta^i_\Sigma(\phi/\mathrm{Ker}_\Sigma(\langle H^i, \gamma^i \rangle)) = \gamma^i_\Sigma(\phi).$$

It is clearly well-defined and, moreover, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rll}\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathcal{A}/\mathrm{Ker}(\langle H^i, \gamma^i\rangle)) & \text{iff} & \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in \mathrm{Ker}_{F(\Sigma)}(\langle H^i, \gamma^i\rangle)\\ & \text{iff} & \gamma^i_{F(\Sigma)}(\alpha_\Sigma(\phi)) = \gamma^i_{F(\Sigma)}(\alpha_\Sigma(\psi))\\ & \text{iff} & \alpha^i_\Sigma(\phi) = \alpha^i_\Sigma(\psi)\\ & \text{iff} & \phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathcal{A}^i).\end{array}$$

Thus, $\mathcal{A}^i$ is a Σ-K-certificate for $\mathcal{A}/\mathrm{Ker}(\langle H^i, \gamma^i\rangle)$, for all $\Sigma \in |\mathbf{Sign}^\flat|$. Since K is abstract, we get that $\mathcal{A}/\mathrm{Ker}(\langle H^i, \gamma^i\rangle) \in \mathsf{K}$ and, hence, $\mathrm{Ker}(\langle H^i, \gamma^i\rangle) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$. Thus, by hypothesis, $\Delta^{\mathcal{A}} = \bigcap_{i\in I}\mathrm{Ker}(\langle H^i, \gamma^i\rangle) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$, showing that $\mathcal{A} \in \mathsf{K}$. We conclude that K is closed under subdirect intersections. ∎

Lemma 141 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* K *be an abstract class of* $\mathbf{F}$*-algebraic systems,* $\mathcal{A}$ *an* $\mathbf{F}$*-algebraic system and* $\theta \in \mathrm{ConSys}(\mathcal{A})$*. Then* $\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ *if and only if* $\mathrm{Ker}(\langle F, \alpha^\theta\rangle) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$*,*

$$\mathbf{F} \xrightarrow{\langle F, \alpha\rangle} \mathbf{A} \xrightarrow{\langle I, \pi^\theta\rangle} \mathbf{A}/\theta$$

where $\langle F, \alpha^\theta\rangle = \langle I, \pi^\theta\rangle \circ \langle F, \alpha\rangle$.

Proof: Consider the diagram,

$$\begin{array}{ccc} & \mathbf{F} & \\ \langle I,\pi\rangle \swarrow & & \searrow \langle F,\alpha^\theta\rangle \\ \mathbf{F}/\mathrm{Ker}(\langle F,\alpha^\theta\rangle) & \xrightarrow[\langle F,\rho\rangle]{} & \mathcal{A}/\theta\end{array}$$

where $\langle F, \rho\rangle : \mathbf{F}/\mathrm{Ker}(\langle F, \alpha^\theta\rangle) \to \mathcal{A}/\theta$ is defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, by

$$\rho_\Sigma(\phi/\mathrm{Ker}_\Sigma(\langle F, \alpha^\theta\rangle)) = \alpha_\Sigma(\phi)/\theta_{F(\Sigma)}.$$

This is well-defined, since, if $\langle \phi, \psi\rangle \in \mathrm{Ker}_\Sigma(\langle F, \alpha^\theta\rangle)$, then $\alpha^\theta_\Sigma(\phi) = \alpha^\theta_\Sigma(\psi)$, i.e., by definition, $\alpha_\Sigma(\phi)/\theta_{F(\Sigma)} = \alpha_\Sigma(\psi)/\theta_{F(\Sigma)}$. Moreover, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, we have

$$\begin{array}{rll}\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathcal{F}/\mathrm{Ker}(\langle F, \alpha^\theta\rangle)) & \text{iff} & \langle \phi, \psi\rangle \in \mathrm{Ker}_\Sigma(\langle I, \pi\rangle)\\ & \text{iff} & \langle \phi, \psi\rangle \in \mathrm{Ker}_\Sigma(\langle F, \alpha^\theta\rangle)\\ & \text{iff} & \phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathcal{A}/\theta).\end{array}$$

Thus, $\mathrm{Eq}(\mathcal{F}/\mathrm{Ker}(\langle F, \alpha^\theta\rangle)) = \mathrm{Eq}(\mathcal{A}/\theta)$. Since K is abstract, we conclude that $\mathbf{F}/\mathrm{Ker}(\langle F, \alpha^\theta\rangle) \in \mathsf{K}$ if and only if $\mathcal{A}/\theta \in \mathsf{K}$. Therefore, $\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ if and only if $\mathrm{Ker}(\langle F, \alpha^\theta\rangle) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$. ∎

Lemma 142 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system and* K *an abstract class of* $\mathbf{F}$*-algebraic systems.* $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ *is closed under intersection if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ *is closed under intersection.*

Proof: The "if" direction is obvious. For the only if, suppose $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ is closed under intersection and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $\{\theta^i : i \in I\} \subseteq \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$. Note that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rll}\langle \phi, \psi\rangle \in \mathrm{Ker}_\Sigma(\langle F, \alpha^{\bigcap_{i\in I}\theta^i}\rangle) & \text{iff} & \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in \bigcap_{i\in I}\theta^i_{F(\Sigma)}\\ & \text{iff} & \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in \theta^i_{F(\Sigma)},\ \text{all } i \in I,\\ & \text{iff} & \langle \phi, \psi\rangle \in \bigcap_{i\in I}\mathrm{Ker}_\Sigma(\langle F, \alpha^{\theta^i}\rangle).\end{array}$$

Thus, $\mathrm{Ker}(\langle F, \alpha^{\bigcap_{i\in I}\theta^i}\rangle) = \bigcap_{i\in I}\mathrm{Ker}(\langle F, \alpha^{\theta^i}\rangle)$. Using Lemma 141, we now get

$$\begin{array}{rcl}\theta^i \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A}), i \in I, & \text{iff} & \mathrm{Ker}(\langle F, \alpha^{\theta^i}\rangle) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F}), i \in I,\\ & \text{implies} & \bigcap_{i\in I}\mathrm{Ker}(\langle F, \alpha^{\theta^i}\rangle) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})\\ & \text{iff} & \mathrm{Ker}(\langle F, \alpha^{\bigcap_{i\in I}\theta^i}\rangle) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})\\ & \text{iff} & \bigcap_{i\in I}\theta^i \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A}).\end{array}$$

Therefore, $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ is closed under intersection. ■

Now we formulate our first characterization theorem.

Theorem 143 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems.* K *is a guasiequational class if and only if it is abstract and* $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ *is closed under intersection.*

Proof: We have K is a guasiequational class if and only if, by Theorem 135, it is abstract and closed under subdirect intersections, if and only if, by Lemma 140, it is abstract and, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ is closed under intersection, if and only if, by Lemma 142, it is abstract and $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ is closed under intersection. ■

A similar characterization can be obtained for quasiequational classes.

Theorem 144 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems.* K *is a quasiequational class if and only if it is directedly abstract and* $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ *is closed under intersection.*

Proof: We have K is a quasiequational class if and only if, by Theorem 136, it is directedly abstract and closed under subdirect intersections, if and only if, by Lemma 140 (taking into account that directed abstraction implies abstraction), it is directedly abstract and, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ is closed under intersections, if and only if, by Lemma 142, it is directedly abstract and $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ is closed under intersection. ■

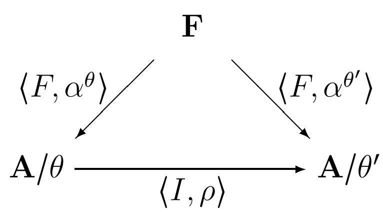
Finally, we work with equational classes. Again, to provide an analogous characterization, we go through a couple of technical lemmas.

The first is an analog of Lemma 140, but instead of addressing subdirect intersections and intersections of relative congruence systems, it addresses morphic images and shows that closure of an abstract class under morphic images amounts to the collection of all relative congruence systems on every algebraic system being an up-set in the lattice of congruence systems.

Lemma 145 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* K *an abstract class of* $\mathbf{F}$*-algebraic systems.* K *is closed under morphic images if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ *is an up-set in* $\mathrm{ConSys}(\mathcal{A})$*.*

Proof: Let K be an abstract class of $\mathbf{F}$-algebraic systems.

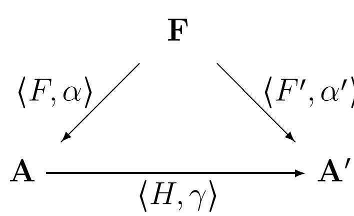
Assume, first, that K is closed under morphic images and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $\theta, \theta' \in \mathrm{ConSys}(\mathcal{A})$, such that $\theta \leq \theta'$ and $\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$. We consider the morphism $\langle I, \rho \rangle : \mathcal{A}/\theta \to \mathcal{A}/\theta'$, given, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, by



$$\rho_\Sigma(\phi/\theta_\Sigma) = \phi/\theta'_\Sigma.$$

It is clearly, well-defined, since $\theta \leq \theta'$. Since $\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$, $\mathcal{A}/\theta \in \mathsf{K}$, whence, since K is closed under morphic images, $\mathcal{A}/\theta' \in \mathsf{K}$, giving $\theta' \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$. Therefore, $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ is an up-set in $\mathrm{ConSys}(\mathcal{A})$.

Suppose, conversely, that $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ is an up-set in $\mathrm{ConSys}(\mathcal{A})$, for every $\mathbf{F}$-algebraic system $\mathcal{A}$. Consider $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$ and a surjective morphism $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{A}'$



and assume that $\mathcal{A} \in \mathsf{K}$. Then, we have $\Delta^{\mathcal{A}} \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ iff, by Lemma 141, $\mathrm{Ker}(\langle F, \alpha \rangle) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ implies, by the hypothesis and the commutativity of the triangle, $\mathrm{Ker}(\langle F', \alpha' \rangle) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ iff, again by Lemma 141, $\Delta^{\mathcal{A}'} \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A}')$ iff $\mathcal{A}' \in \mathsf{K}$. Therefore, K is closed under morphic images. ■

The second is an analog of Lemma 142, but instead of addressing closure of the collections of relative congruence systems under intersection, it deals with their upward closure under the signature-wise ordering in the lattices of congruence systems.

Lemma 146 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* K *an abstract class of* $\mathbf{F}$*-algebraic systems.* $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ *is an upset in* $\mathrm{ConSys}(\mathcal{F})$ *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ *is an upset in* $\mathrm{ConSys}(\mathcal{A})$*.*

Proof: The "if" direction is obvious.

For the "only if" assume that $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ is an up-set in $\mathrm{ConSys}(\mathcal{F})$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $\theta, \theta' \in \mathrm{ConSys}(\mathcal{A})$, such that $\theta \leq \theta'$ and $\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$. Then, taking into account the fact that $\mathrm{Ker}(\langle F, \alpha^\theta \rangle) \leq \mathrm{Ker}(\langle F, \alpha^{\theta'} \rangle)$ and that $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ is an upset and using Lemma 141, we have

$$\begin{array}{lcl} \theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A}) & \text{iff} & \mathrm{Ker}(\langle F, \alpha^\theta \rangle) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F}) \\ & \text{implies} & \mathrm{Ker}(\langle F, \alpha^{\theta'} \rangle) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F}) \\ & \text{iff} & \theta' \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A}). \end{array}$$

Therefore, $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ is an up-set in $\mathrm{ConSys}(\mathcal{A})$. ■

Now we get the following theorem characterizing equational classes of $\mathbf{F}$-algebraic systems.

Theorem 147 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems.* K *is an equational class if and only if it is abstract and* $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ *is an upset in* $\mathrm{ConSys}(\mathcal{F})$*, closed under intersections.*

Proof: We have K is an equational class if and only if, by Theorem 137, it is closed under subdirect intersections and morphic images, if and only if, by Proposition 133 and Lemmas 140 and 145, it is abstract and, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$ is an upset in $\mathrm{ConSys}(\mathcal{A})$, closed under intersections, if and only if, by Lemmas 142 and 146, it is abstract and $\mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ is an upset in $\mathrm{ConSys}(\mathcal{F})$, closed under intersections. ■

Chapter 3

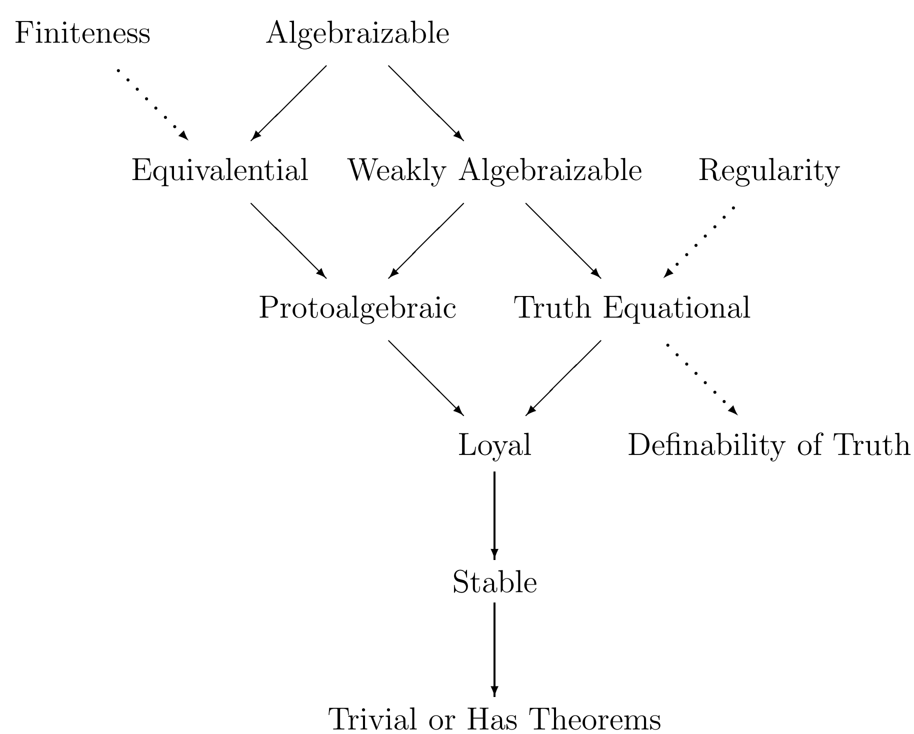
The Semantic Leibniz Hierarchy: Bottom Half

3.1 Introduction

In this chapter we give the main definitions and some key properties of the main classes of the *semantic Leibniz hierarchy*. The term *Leibniz hierarchy* refers to the classification of logical systems according to the strength of their relation to classes of algebraic systems. The term *semantic* refers to the definition of those classes by means of the *Leibniz operator* as applied to the theory families/systems of the corresponding π-institution or, via available so called *transfer theorems*, to the filter families/systems of their Leibniz reduced matrix systems.

As we will see later, there is also a *syntactic Leibniz hierarchy* whose classes do not coincide with those of the semantic hierarchy in general. However, we will show that under certain conditions, there is a correspondence between the classes in the two hierarchies.

A *very rough idea* of the main classes of the semantic Leibniz hierarchy, with the inclusion relations between them, is given in the following diagram. The hierarchy parallels that of sentential logics, established in the classical theory (see, e.g., Figure 9 on Page 316 of [86]). However, as we will see in this and subsequent chapters, in the case of logics formalized as π-institutions, various refinements of these classes are possible. One of the main goals of the monograph is to study those refinements and their interrelations. In the diagram we also give an idea of how this hierarchy is extended with other classes that are "attached" via dotted links to the main classes. Some of these extensions will also be studied later.



In Section 3.2, we introduce three fundamental classes of π-institutions, namely *systemic*, *stable* and *loyal* π-institutions. *Systemicity* and *stability* play a very important role throughout the monograph and facilitate discussions about the refinements of the various classes alluded to previously. *Loyalty* does not play a comparable role, but it constitutes at the same time a relaxation of *order preservation* and of *order reflectivity* and, as such, defines an important class close to the bottom of the hierarchy.

A π-institution is called *systemic* if all of its theory families are theory systems, i.e., invariant under the action of signature morphisms. In the context of π-institutions, the importance of systemicity was brought to the fore in the study of protoalgebraicity [105, 104]. Generally speaking, however, preservation of relations or, more concretely, of distinguished sets, is an important property in the model theory of first-order logics (see, e.g., page 71 of [17], page 5 of [43] or page 8 of [66]) and, hence, also in the theory of logical matrices serving as models of sentential logics (see, e.g., page 31 of [64] or page 200 of [86]). Systemicity is characterized by asserting that the closure family generated by any sentence of the given π-institution includes all translates of that sentence via signature morphisms. Systemicity also affords the chance to introduce the first of a host of so-called *transfer theorems*. This term refers to a property holding on the lattice of theory families of a π-institution *transferring* to the lattice of its filter families on arbitrary algebraic systems. This paradigm follows an oft-encountered situation in the theory of sentential logics (see, e.g., Section 3.6 of [86]). In this specific instance, it is shown that a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is systemic if and only if every filter family on every $\mathbf{F}$-algebraic system is actually a filter system.

Recall from Chapter 2 that, given a π-institution $\mathcal{I}$ and a theory family T of $\mathcal{I}$, $\overleftarrow{T}$ is the largest theory system of $\mathcal{I}$ included in T. In Section 2.3 it was shown (Proposition 20) that the Leibniz congruence system associated with T is included in the one associated with $\overleftarrow{T}$. We call the π-institution $\mathcal{I}$ *stable* if, for all theory families T of $\mathcal{I}$, these two Leibniz congruence systems coincide. Since systemicity directly yields that, for all T, $\overleftarrow{T} = T$, it clearly implies stability. Moreover, this implication is proper. As was the case with systemicity, stability also transfers.

The last concept introduced in Section 3.2 is that of *loyalty*. There are four possible flavors, analogs of which also arise and are pursued for many other properties considered in the monograph. The general idea of the property is, perhaps, best conveyed by *family loyalty*. The property holds for a π-institution $\mathcal{I}$ if, for no pair T and T' of theory families of $\mathcal{I}$ is it the case that $T < T'$ and $\Omega(T) > \Omega(T')$, i.e., $\mathcal{I}$ is *family loyal* if proper inclusion between theory families is never reversed when passing to corresponding Leibniz congruence systems. As in most other properties that we study, the other three versions are obtained from the family version as follows:

- *left loyalty* by replacing on the theory family side T and T' by their arrow counterparts $\overleftarrow{T}$, $\overleftarrow{T'}$, respectively;

- *right loyalty* by replacing on the congruence system side T and T' by $\overleftarrow{T}$ and $\overleftarrow{T'}$, respectively, i.e., by considering the inequality $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$ in place of $\Omega(T) \leq \Omega(T')$;

- *system loyalty* by applying the defining condition only on the collection of theory systems of $\mathcal{I}$, instead of considering arbitrary pairs of theory families.

It turns out that family loyalty implies stability. Moreover, family loyalty is the strongest of the four properties, followed by left loyalty, which, in turn, implies system loyalty, which is equivalent to right loyalty. Both implications are proper. Another feature of family loyalty is that, apart from trivial π-institutions, all family loyal ones must possess theorems. In closing the section, it is shown that all flavors of loyalty also transfer from theory families/systems to filter families/systems over arbitrary algebraic systems.

In Section 3.3, we introduce versions of the *monotonicity property*. This property is very important historically, since one of the first major classes of sentential logics to be studied in detail in the context of abstract algebraic logic was that of protoalgebraic logics [28] (see, also, Chapter 1 of [64] and Section 6.2 of [86]). They are characterized by the monotonicity of the Leibniz operator on their theory lattices. In the context of π-institutions, *family monotonicity* asserts that, for every pair T, T' of theory families, if T is included in T', then the Leibniz congruence system of T is also dominated by that of T'. *Left monotonicity* results by replacing, on the theory family side (hypothesis), T and T' by $\overleftarrow{T}$ and $\overleftarrow{T'}$, respectively. Similarly, *right monotonicity* ensues when the same is done on the congruence system side (conclusion). Finally, *system monotonicity* is monotonicity restricted to the collection of theory systems. It is shown that family and left monotonicity coincide, as do right and system monotonicity. In agreement with the terminology inherited by the sentential framework, we call a π-institution satisfying family monotonicity *protoalgebraic*, whereas one satisfying system monotonicity is termed *prealgebraic* [105, 104]. Since prealgebraicity is defined by the same monotonicity condition as protoalgebraicity, but restricted to theory systems, protoalgebraic π-institutions form a subclass of the class of prealgebraic ones. Moreover, it turns out that a π-institution is protoalgebraic if and only if it is prealgebraic and stable. Protoalgebraicity actually implies family loyalty, a condition stronger than stability, and similarly, prealgebraicity implies system loyalty. Finally, it is shown that both monotonicity properties transfer.

In Sections 3.4 and 3.5, we undertake the study of properties that may be referred to, collectively, as *complete monotonicity* properties. The reason for studying these properties can be traced back to the work of Raftery

[77] in an indirect way, but they are also loosely related, especially in the study of finitary deductive systems, to the property of continuity, whose importance was already apparent in [35]. Raftery used a property termed complete order reflectivity to characterize truth equationality of sentential logics. The property asserts that, given a sentential logic $\mathcal{S}$, for all collections $\mathcal{T} \cup \{T'\}$ of theories of $\mathcal{S}$, $\bigcap_{T\in\mathcal{T}} \Omega(T) \subseteq \Omega(T')$ implies $\bigcap \mathcal{T} \subseteq T'$. Noting that in both the lattice of theories and the lattice of congruences on the formula algebra intersection coincides with meet, this property may be rewritten as $\bigwedge_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$ implies $\bigwedge \mathcal{T} \leq T'$. Since, however, join and union of both theories and congruences differ, depending on whether one adopts a set-theoretic or a lattice-theoretic point of view, the dual property of complete order reflectivity may take one of two possible forms. The first asserts that, for all $\mathcal{T} \cup \{T'\}$, $T' \subseteq \bigcup \mathcal{T}$ implies $\Omega(T') \subseteq \bigcup_{T\in\mathcal{T}} \Omega(T)$. The second stipulates that, for all $\mathcal{T} \cup \{T'\}$, $T' \leq \bigvee \mathcal{T}$ implies $\Omega(T') \leq \bigvee_{T\in\mathcal{T}} \Omega(T)$. The translation of complete order reflectivity in the categorical context was first introduced in [107]. Various flavors of it are studied in detail in Section 3.8. In Sections 3.4 and 3.5, we study the properties corresponding to the two aforementioned duals.

In Section 3.4, we look at the various flavors of *complete* $\bigcup$*-monotonicity*. Again, the simplest one is *family complete* $\bigcup$*-monotonicity*. A π-institution is *family completely* $\bigcup$*-monotone* if, for all collections $\mathcal{T} \cup \{T'\}$ of theory families, $T' \leq \bigcup_{T\in\mathcal{T}} T$ implies $\Omega(T') \leq \bigcup_{T\in\mathcal{T}} \Omega(T)$. *Left complete* $\bigcup$*-monotonicity* results by replacing all theory families appearing in the hypothesis by their arrow counterparts. Similarly, *right complete* $\bigcup$*-monotonicity* arises by performing the same replacement in the conclusion. Finally, *system complete* $\bigcup$*-monotonicity* is the property resulting by applying the same condition defining the family version to collections of theory systems only. We use the abbreviation *c*$^\cup$*-monotonicity* to refer to complete $\bigcup$-monotonicity. Moreover, when we drop the $\bigcup$ (or $^\cup$) from the notation, it is to this version of complete monotonicity that we refer to. Family or left c$^\cup$-monotonicity are strong enough to imply stability. Moreover, family c$^\cup$-monotonicity is equivalent to possessing both left and right c$^\cup$-monotonicity. Either left or right c$^\cup$-monotonicity on its own implies system c$^\cup$-monotonicity. For these four properties, it is also the case that they transfer from theory families/systems to filter families/systems on arbitrary algebraic systems. In closing, it is established that left c$^\cup$-monotonicity implies protoalgebraicity, whereas system c$^\cup$-monotonicity is sufficient for prealgebraicity.

In Section 3.5, we undertake a similar study of the complete monotonicity properties involving the join instead of the union operation. We say that a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is *family completely* $\bigvee$*-monotone*, abbreviated *family c*$^\vee$*-monotone*, if, for every collection $\mathcal{T} \cup \{T'\}$ of theory families, $T' \leq \bigvee^{\mathcal{I}} \mathcal{T}$ implies $\Omega(T') \leq \bigvee^{\mathbf{F}}_{T\in\mathcal{T}} \Omega(T)$, where $\bigvee^{\mathcal{I}}$ denotes the join in the complete lattice of theory families of $\mathcal{I}$ and $\bigvee^{\mathbf{F}}$ the join in the complete lattice of congruence systems on $\mathbf{F}$. As in Section 3.4, *left c*$^\vee$*-monotonicity* results by replacing in the hypothesis every theory family by its arrow counterpart, *right* $c^\vee$*-monotonicity* by doing the same in the conclusion and *system* $c^\vee$*-monotonicity* by restricting the defining implication to all collections of theory systems, instead of insisting that it hold for arbitrary collections of theory families. Working with join instead of union leaves most of our conclusions intact. It is still the case that family $c^\vee$-monotonicity and left $c^\vee$-monotonicity are each sufficient for stability. Family $c^\vee$ monotonicity is equivalent to the combination of left and right $c^\vee$-monotonicities and each of the latter implies system $c^\vee$-monotonicity. And it is still the case that left $c^\vee$-monotonicity implies protoalgebraicity and system $c^\vee$-monotonicity implies prealgebraicity. There are, however, some differences between $c^\vee$-monotonicity properties and their counterparts using union. One example is that $c^\vee$-monotonicity properties, unlike $c^\cup$-monotonicity properties, do not transfer in general. This is due to the fact that, unlike union, the join operation does not commute with inverse surjective morphisms between algebraic systems. Another difference, which may also be viewed as a partial justification for considering both properties, is that the corresponding classes in the two hierarchies are incomparable. For instance, there exists a family $c^\cup$-monotone π-institution which is not family $c^\vee$-monotone and vice-versa.

In Section 3.6 we switch from the study of monotonicity properties to the study of *injectivity properties*. The importance of injectivity in the context of sentential logics was already apparent in the work of Blok and Pigozzi [35], but it was brought more in focus following its generalizations, first by Herrmann [43] and, ultimately, with the work of Czelakowski and Jansana [62] on weakly algebraizable logics. A π-institution is *family injective* if, for all theory families T and T', $\Omega(T) = \Omega(T')$ implies $T = T'$, i.e., when the Leibniz operator on theory families is injective. *Left injectivity* is obtained by replacing T and T' on the theory family side (conclusion) by $\overleftarrow{T}$ and $\overleftarrow{T'}$, respectively, while *right injectivity* by doing the same on the congruence system side (hypothesis). Finally, *system injectivity* imposes injectivity of the Leibniz operator on theory systems only. Here, right injectivity turns out to be the most potent of the four properties and it implies systemicity. Then comes family injectivity, followed by left injectivity, which, in turn, implies system injectivity. Right injectivity is equivalent to system injectivity coupled with systemicity, whereas, if system injectivity is combined with stability, they imply left injectivity. All injectivity properties transfer.

In the last two sections of the chapter, Sections 3.7 and 3.8, we study *reflectivity properties*, which are dual to the monotonicity properties delved into in Sections 3.3, 3.4 and 3.5.

In Section 3.7, we study the various flavors of *reflectivity*. In the context of algebraizable sentential logics the importance of this property was at least implicit, if not apparent, in the work of Blok and Pigozzi [35]. And, as was the case with injectivity, it kept its central role in the generalizations to infinitary algebraizable [43] and weakly algebraizable logics [62]. A π-institution $\mathcal{I}$ is *family reflective* if, for all theory families T and T', $\Omega(T) \leq \Omega(T')$ implies $T \leq T'$, i.e., if the Leibniz operator is order reflecting on the theory families of $\mathcal{I}$. *Left reflectivity* replaces T and T' on the theory family side (conclusion) by their arrow counterparts, while *right reflectivity* applies the same replacement on the congruence system side (hypothesis). Finally, *system reflectivity* imposes order reflectivity of the Leibniz operator on theory systems only. Each of family and right reflectivity implies systemicity. This allows proving that these two versions of reflectivity are actually equivalent. They imply left reflectivity, which dominates system reflectivity. System reflectivity together with stability imply left reflectivity. System reflectivity, coupled with systemicity, is equivalent to family reflectivity. All these properties transfer. Section 3.7 ends by establishing some relations between reflectivity and properties introduced in preceding sections. More precisely, it is shown that family, left and system reflectivity imply, respectively, right, left and system injectivity. Additionally, family, left and system reflectivity imply, respectively, family, left and system loyalty.

In Section 3.8, we study versions of *complete reflectivity*. As was mentioned previously, in the context of sentential logics, the property was introduced by Raftery in [77], where it was used to characterize truth equationality of sentential logics. A logic is truth equational if the filters of its Leibniz reduced matrix models are equationally definable. This is equivalent to the assertion that the filters of arbitrary matrix models are definable using equations via the corresponding Leibniz congruences. Raftery showed that truth equationality is equivalent to the complete order reflectivity of the Leibniz operator on the theories of the logic, i.e., the property that, for every collection $\mathcal{T} \cup \{T'\}$ of theories, $\bigcap_{T\in\mathcal{T}} \Omega(T) \subseteq \Omega(T')$ implies $\bigcap \mathcal{T} \subseteq T'$ (see, e.g., pages 371-382 of [86]; in particular, Theorem 6.101). Given a π-institution $\mathcal{I}$, *family complete reflectivity* asserts that, for every collection $\mathcal{T} \cup \{T'\}$ of theory families of $\mathcal{I}$, $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$ implies $\bigcap \mathcal{T} \leq T'$. *Left complete reflectivity* is obtained by replacing on the theory family side (conclusion) each theory appearing by its arrow version. Similarly, *right complete reflectivity* results by performing the same replacement on the congruence system side (hypothesis). *System complete reflectivity* is the restriction of the condition defining family complete reflectivity on collections of theory systems. We abbreviate complete reflectivity by *c-reflectivity*. On their own, family and right c-reflectivity each implies systemicity, and this allows showing that they are equivalent properties. They imply left c-reflectivity, which, in turn, implies system c-reflectivity. System c-reflectivity, coupled with stability implies left c-reflectivity. Moreover, together with systemicity, it turns out to be equivalent to family c-reflectivity. All three different properties transfer and it is fairly obvious that they generalize the corresponding reflectivity properties, since the latter are special cases of the former in which the collection $\mathcal{T}$ is a singleton.

The properties of systemicity, stability, loyalty, monotonicity, $c^{\cup}$-monotonicity, $c^{\vee}$-monotonicity, injectivity, reflectivity and c-reflectivity constitute the building blocks of the hierarchies of π-institutions that will be presented and studied in subsequent chapters of the monograph.

3.2 Systemicity, Stability and Loyalty

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system. Recall that, by Proposition 42, for every theory family $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T}$ is the largest theory system included in T. Moreover, recall that, by Proposition 20, $\Omega(T) \leq \Omega(\overleftarrow{T})$.

Definition 148 (Systemicity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is called* **systemic** *if* $\mathrm{ThFam}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$*, or, equivalently, if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\overleftarrow{T} = T.$$

Another interesting characterization is the following. Recall that, given an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$, a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, based on $\mathbf{F}$, and a sentence family $X \in \mathrm{SenFam}(\mathbf{F})$, we denote by

$$C(X) = \{C_\Sigma(X)\}_{\Sigma \in |\mathbf{Sign}^\flat|}$$

the least theory family of $\mathcal{I}$ including X. Moreover if $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, we use $C(\Phi)$ and $C(\phi) \coloneqq C(\{\phi\})$ to denote $C(X)$, where $X = \{X_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}^\flat|}$ is such that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, $X_{\Sigma'} = \begin{cases} \Phi, & \text{if } \Sigma' = \Sigma \\ \varnothing, & \text{if } \Sigma' \neq \Sigma \end{cases}$.

Proposition 149 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is systemic if and only if, for all* $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$*, all* $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\mathrm{SEN}^\flat(f)(\phi) \in C_{\Sigma'}(\phi).$$

Proof: Suppose, first, that $\mathcal{I}$ is systemic. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Since, by hypothesis, for all $T \in \mathrm{ThFam}(\mathcal{I})$, such that $\phi \in T_\Sigma$, we have, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, $\mathrm{SEN}^\flat(f)(\phi) \in T_{\Sigma'}$, we conclude that for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, $\mathrm{SEN}^\flat(f)(\phi) \in C_{\Sigma'}(\phi)$.

Assume, conversely, that the displayed condition in the statement holds and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$. Consider $\Sigma' \in |\mathbf{Sign}^\flat|$ and $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$. Then, by hypothesis,

$$\begin{array}{rcl}\mathrm{SEN}^\flat(f)(\phi) & \in & C_{\Sigma'}(\phi) \\ & = & \bigcap\{T'_{\Sigma'} : T' \in \mathrm{ThFam}(\mathcal{I}), \phi \in T'_\Sigma\} \\ & \subseteq & T_{\Sigma'}.\end{array}$$

Thus, $T \in \mathrm{ThSys}(\mathcal{I})$ and $\mathcal{I}$ is systemic. ∎

The following is one of many typical transfer theorems that we will encounter for various properties regarding π-institutions.

Theorem 150 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is systemic if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$.*

Proof: The right to left implication follows by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$, where $\langle I, \iota \rangle : \mathbf{F} \to \mathbf{F}$ is the identity morphism, and taking into account the fact that, by Lemma 51, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}) = \mathrm{ThFam}(\mathcal{I})$ and $\mathrm{FiSys}^{\mathcal{I}}(\mathcal{F}) = \mathrm{ThSys}(\mathcal{I})$.

For the left to right implication, suppose that $\mathcal{I}$ is systemic and assume that $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ is an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$. Thus, by hypothesis, $\alpha^{-1}(T) \in \mathrm{ThSys}(\mathcal{I})$. Hence, using again Lemma 51, $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$. Therefore $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$. ∎

Now we introduce another important class of π-institutions in the semantic Leibniz hierarchy.

Definition 151 (Stability) *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is called* **stable** *if, for all $T \in \mathrm{ThFam}(\mathcal{I})$,*

$$\Omega(\overleftarrow{T}) = \Omega(T).$$

Since, by Proposition 20, it always holds that $\Omega(T) \leq \Omega(\overleftarrow{T})$, we have that

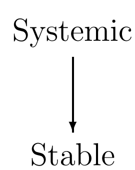
$$\mathcal{I} \text{ is stable if and only if, for all } T \in \mathrm{ThFam}(\mathcal{I}),\ \Omega(\overleftarrow{T}) \leq \Omega(T).$$

The following obvious relation holds between these two classes.

Proposition 152 *Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution. If $\mathcal{I}$ is systemic, then it is stable.*

Proof: If $\mathcal{I}$ is systemic, then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} = T$, whence $\Omega(\overleftarrow{T}) = \Omega(T)$. Thus, $\mathcal{I}$ is stable. ∎

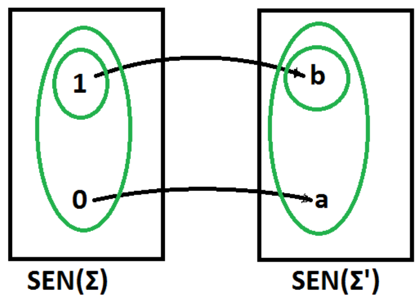
We denote this relation by the following diagram, where the arrow represents inclusion.



We show that this is a proper inclusion, i.e., there are stable π-institutions that are not systemic.

Example 153 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is a category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$*,* $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$*,* $\mathrm{SEN}^\flat(f)(1) = b$*;*
- $N^\flat$ *is the trivial clone (consisting only of the projection natural transformations).*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{1\}, \{0, 1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\}, \{a, b\}\}.$$

There is a single theory family which is not a theory system, namely $T = \{\{0, 1\}, \{b\}\}$*. So* $\mathcal{I}$ *is not systemic. On the other hand, we have* $\Omega(\overleftarrow{T}) = \Omega(\{\{1\}, \{b\}\}) = \Delta^{\mathbf{F}} = \Omega(T)$*. Therefore,* $\mathcal{I}$ *is stable.*

The stability property transfers from the theory families of a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ to all $\mathcal{I}$-filter families on all $\mathbf{F}$-algebraic systems.

Theorem 154 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is stable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $\Omega^{\mathcal{A}}(\overleftarrow{T}) = \Omega^{\mathcal{A}}(T)$*.*

Proof: The "if" part is trivial, since stability is defined by the given condition on all theory families of the π-institution, which, by Lemma 51, are exactly the $\mathcal{I}$-filter families on $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$.

For the "only if" assume that $\mathcal{I}$ is stable and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then we have:

$$\begin{array}{rcl}
\alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T})) & = & \Omega(\alpha^{-1}(\overleftarrow{T})) \quad \text{(by Proposition 24)} \\
& = & \Omega(\overleftarrow{\alpha^{-1}(T)}) \quad \text{(by Lemma 6)} \\
& = & \Omega(\alpha^{-1}(T)) \\
& & \text{(by Lemma 51, Proposition 42 and stability)} \\
& = & \alpha^{-1}(\Omega^{\mathcal{A}}(T)) \quad \text{(by Proposition 24).}
\end{array}$$

By surjectivity of $\langle F,\alpha\rangle$, we get that $\Omega^{\mathcal{A}}(\overleftarrow{T}) = \Omega^{\mathcal{A}}(T)$. ■

Next, we introduce various versions of the loyalty property and the corresponding classes in the loyalty hierarchy of π-institutions.

Definition 155 (Loyalty) *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.*

- *$\mathcal{I}$ is called* **family loyal** *if, for all $T,T' \in \mathrm{ThFam}(\mathcal{I})$,*
$$T \nleq T' \quad or \quad \Omega(T) \ngeq \Omega(T').$$
- *$\mathcal{I}$ is called* **left loyal** *if, for all $T,T' \in \mathrm{ThFam}(\mathcal{I})$,*
$$\overleftarrow{T} \nleq \overleftarrow{T'} \quad or \quad \Omega(T) \ngeq \Omega(T').$$
- *$\mathcal{I}$ is called* **right loyal** *if, for all $T,T' \in \mathrm{ThFam}(\mathcal{I})$,*
$$T \nleq T' \quad or \quad \Omega(\overleftarrow{T}) \ngeq \Omega(\overleftarrow{T'}).$$
- *$\mathcal{I}$ is called* **system loyal** *if, for all $T,T' \in \mathrm{ThSys}(\mathcal{I})$,*
$$T \nleq T' \quad or \quad \Omega(T) \ngeq \Omega(T').$$

We establish relationships between these properties that lead to a loyalty hierarchy of π-institutions.

We show, first, that family loyalty implies stability.

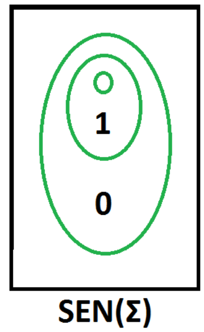
Lemma 156 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is family loyal, then it is stable.*

Proof: Suppose $\mathcal{I}$ is family loyal and let $T \in \mathrm{ThFam}(\mathcal{I})$. We must show that $\Omega(\overleftarrow{T}) = \Omega(T)$. If $\overleftarrow{T} = T$, then the conclusion is obvious. So suppose $\overleftarrow{T} \neq T$. Then, by Proposition 42, we have $\overleftarrow{T} < T$. Using family loyalty, we get $\Omega(\overleftarrow{T}) \ngeq \Omega(T)$. Hence, by Proposition 20, we have $\Omega(\overleftarrow{T}) = \Omega(T)$. We conclude that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(\overleftarrow{T}) = \Omega(T)$, whence $\mathcal{I}$ is stable. ■

There are π-institutions that are stable but not family loyal. The following example shows that family loyal π-institutions form a proper subclass of the class of stable π-institutions.

Example 157 *Consider the algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category, with object Σ;*

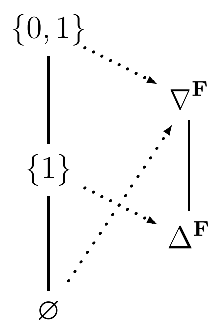


- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by setting* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$;
- $N^\flat$ *is the trivial clone (consisting only of the projections).*

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by setting*

$$C_\Sigma = \{\varnothing, \{1\}, \{1,0\}\}.$$

The lattice of theory families (which are all systems) and the corresponding Leibniz congruence systems are given in the diagram.



Since all theory families are theory systems, $\mathcal{I}$ *is clearly stable. On the other hand, letting* $T = \{\varnothing\}$ *and* $T' = \{\{1\}\}$*, we have* $T < T'$ *and* $\Omega(T) > \Omega(T')$*. Therefore,* $\mathcal{I}$ *is not family loyal.*

Now we can establish the following relationships between the four loyalty properties.

Proposition 158 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is family loyal then it is left loyal;*

(b) If $\mathcal{I}$ *is left loyal, then it is system loyal;*

(c) $\mathcal{I}$ *is system loyal if and only if it is right loyal;*

(b) $\mathcal{I}$ *is family loyal if and only if it is system loyal and stable.*

Proof:

(a) Suppose $\mathcal{I}$ is family loyal. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\overleftarrow{T} < \overleftarrow{T'}$. Then, by family loyalty $\Omega(\overleftarrow{T}) \not> \Omega(\overleftarrow{T'})$. But, by Lemma 156, $\mathcal{I}$ is stable. So we get $\Omega(T) \not> \Omega(T')$. We conclude that $\mathcal{I}$ is left loyal.

(b) Suppose that $\mathcal{I}$ is left loyal and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\Omega(T) > \Omega(T')$. Then, by left loyalty, $\overleftarrow{T} \not< \overleftarrow{T'}$. But, since T, T' are theory systems, $\overleftarrow{T} = T$ and $\overleftarrow{T'} = T'$. Hence $T \not< T'$. Therefore $\mathcal{I}$ is system loyal.

(c) Suppose, now, that $\mathcal{I}$ is right loyal and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $T < T'$. Then, by right loyalty, $\Omega(\overleftarrow{T}) \not> \Omega(\overleftarrow{T'})$. Thus, since T, T' are theory systems, $\Omega(T) \not> \Omega(T')$. We conclude that $\mathcal{I}$ is system loyal.

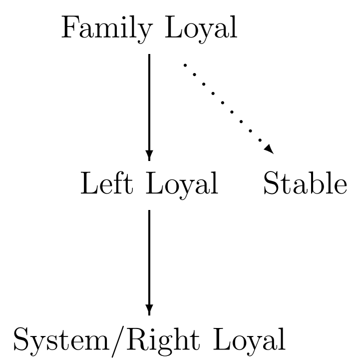
Suppose, conversely, that $\mathcal{I}$ is system loyal and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T < T'$. Then, by Lemma 1, $\overleftarrow{T} \leq \overleftarrow{T'}$. If $\overleftarrow{T} = \overleftarrow{T'}$, then $\Omega(\overleftarrow{T}) \not> \Omega(\overleftarrow{T'})$. On the other hand, if $\overleftarrow{T} < \overleftarrow{T'}$, then, by system loyalty, $\Omega(\overleftarrow{T}) \not> \Omega(\overleftarrow{T'})$. We conclude that $\mathcal{I}$ is right loyal.

(d) Suppose, first, that $\mathcal{I}$ is family loyal. Then, by Lemma 156, it is stable and it is, a fortiori, system loyal.

Suppose, conversely, that $\mathcal{I}$ is system loyal and stable. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) > \Omega(T')$. By stability, $\Omega(\overleftarrow{T}) > \Omega(\overleftarrow{T'})$. Therefore, by system loyalty, $\overleftarrow{T} \not< \overleftarrow{T'}$. Since $\Omega(\overleftarrow{T}) \neq \Omega(\overleftarrow{T'})$, we also have, $\overleftarrow{T} \not\leq \overleftarrow{T'}$. Therefore, by Lemma 1, $T \not< T'$. We conclude that $\mathcal{I}$ is family loyal.

■

By Proposition 158, the following **loyalty hierarchy** arises.



Example 157 brings to the fore another interesting point, namely, that family loyal π-institutions must have theorems, unless they are trivial.

Proposition 159 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ is trivial, then it is family loyal.

(b) If $\mathcal{I}$ is family loyal and non-trivial, then it has theorems.

Proof:

(a) If $\mathcal{I}$ is trivial, then the only Leibniz congruence system is $\nabla^{\mathbf{F}}$. So $\mathcal{I}$ is family loyal.

(b) Suppose $\mathcal{I}$ is family loyal and non-trivial. By non-triviality, it has a theory family T, such that, for some $\Sigma \in |\mathbf{Sign}^\flat|$, $T_\Sigma \neq \varnothing$ and $T_\Sigma \neq \mathrm{SEN}(\Sigma)$. Therefore, we have $\overline{\varnothing} < T$. So, by loyalty, $\Omega(\overline{\varnothing}) \not> \Omega(T)$. But $\Omega(\overline{\varnothing}) = \nabla^{\mathbf{F}}$. So $\nabla^{\mathbf{F}} \not> \Omega(T)$. This shows that $\Omega(T) = \nabla^{\mathbf{F}}$, which is a contradiction, since $\nabla^{\mathbf{F}}$ cannot be compatible with any theory family T, with a component $T_\Sigma \neq \varnothing, \mathrm{SEN}(\Sigma)$.

■

We also have the following straightforward relationship.

Proposition 160 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is systemic and system loyal, then it is family loyal.*

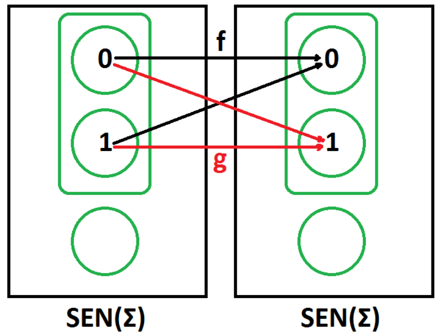
Proof: If $\mathcal{I}$ is systemic, then $\mathrm{ThFam}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$ and, as a result, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} = T$. Thus, under this hypothesis, all loyalty properties coincide. ■

The next example serves many purposes:

- It shows a π-institution that is left loyal, but not family loyal.
- It shows a π-institution that is system loyal, but not stable, and, hence, by Proposition 158, not family loyal.
- It shows an example of a nontrivial π-institution without theorems that is system loyal, but not family loyal, illustrating Proposition 159.

Example 161 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object Σ and two non-identity morphisms $f, g : \Sigma \to \Sigma$, such that $f \circ f = f$, $g \circ g = g$, $g \circ f = g$ and $f \circ g = f$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$, $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = 0$, $\mathrm{SEN}^\flat(g)(0) = \mathrm{SEN}^\flat(g)(1) = 1$;
- $N^\flat$ *is the trivial clone.*



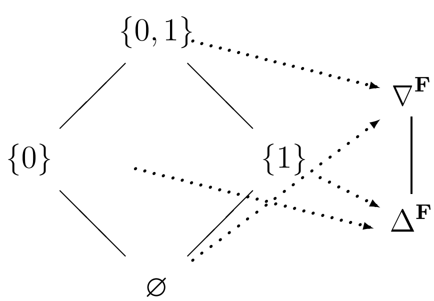
Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by setting

$$\mathcal{C}_\Sigma = \{\varnothing, \{0\}, \{1\}, \{0,1\}\}.$$

The following table gives the theory families and the theory systems of the π-institution $\mathcal{I}$:

T	$\overleftarrow{T}$
$\varnothing$	$\varnothing$
$\{0\}$	$\varnothing$
$\{1\}$	$\varnothing$
$\{0,1\}$	$\{0,1\}$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.

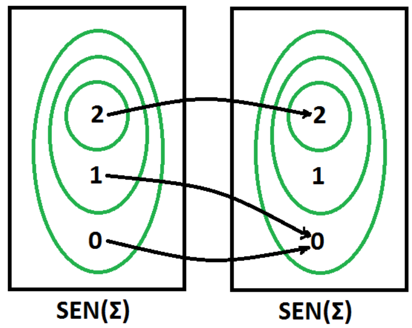


Note that, since $\overleftarrow{\{\{0\}\}} = \{\varnothing\}$ and these theory families map to different congruence systems, $\mathcal{I}$ is not stable. $\mathcal{I}$ is not family loyal, since $\{\varnothing\} < \{\{0\}\}$ and $\Omega(\{\varnothing\}) = \nabla^{\mathbf{F}} > \Delta^{\mathbf{F}} = \Omega(\{\{0\}\})$. However, $\mathcal{I}$ is left loyal, since, if $\overleftarrow{T} < \overleftarrow{T'}$, then $T' = \{\{0,1\}\}$ and, therefore, since $\Omega(T') = \nabla^{\mathbf{F}}$, $\Omega(T) \not> \Omega(T')$.

Now we provide a variety of additional examples, all showcasing π-institutions that are left loyal (and, hence, also system loyal), but fail to be family loyal.

Example 162 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial clone.*

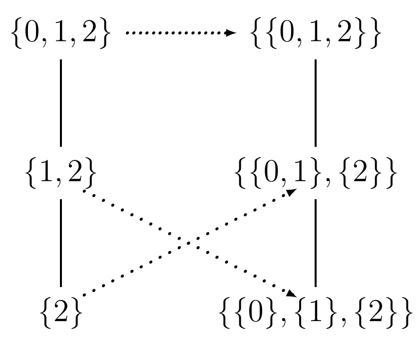


Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}$*.*

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$*:*

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



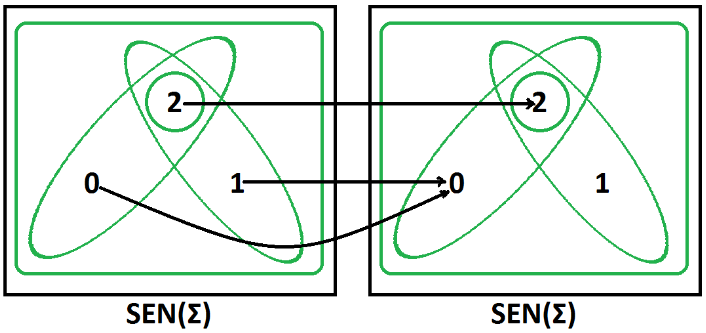
Taking into account the fact that $\{\{1,2\}\}$ *is a theory family that is not a theory system, it is easy to see that this* π*-institution is left loyal (and, hence, system loyal), but fails to be family loyal.*

Example 163 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*

- $N^\flat$ *is the trivial clone.*

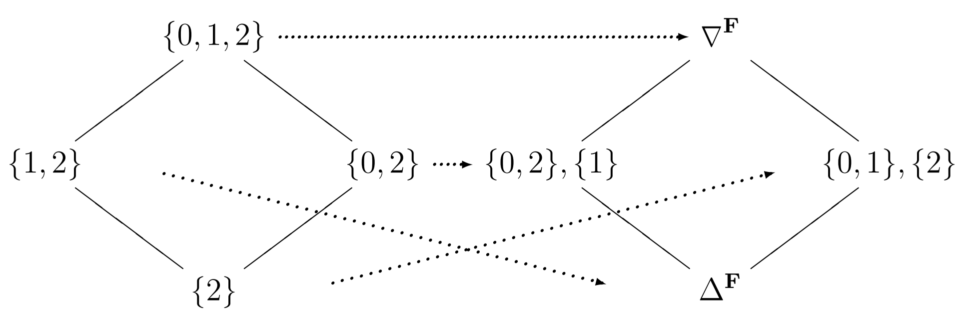


Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by setting*

$$\mathcal{C}_\Sigma = \{\{2\},\{0,2\},\{1,2\},\{0,1,2\}\}.$$

Note that $\{\{1,2\}\}$ *is the only theory family that is not a theory system.*

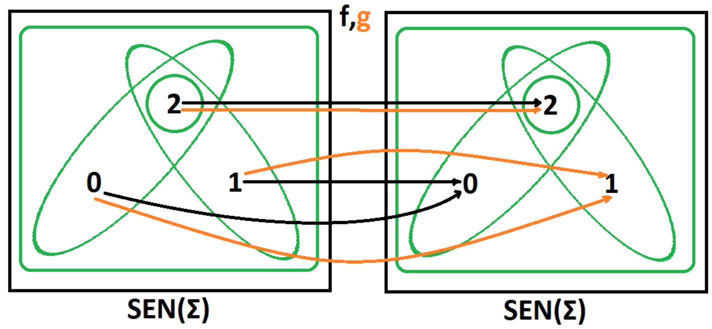
The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



Again it is not difficult to see that $\mathcal{I}$ *is left and right loyal, but fails to be family loyal.*

Example 164 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and two non-identity morphisms* $f, g : \Sigma \to \Sigma$*, such that* $f \circ f = f$*,* $g \circ g = g$*,* $g \circ f = g$ *and* $f \circ g = f$*;*



- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$, $\mathrm{SEN}^\flat(f)(2) = 2$ *and* $\mathrm{SEN}^\flat(g)(0) = 1$, $\mathrm{SEN}^\flat(g)(1) = 1$, $\mathrm{SEN}^\flat(g)(2) = 2$;
- $N^\flat$ *is the trivial clone.*

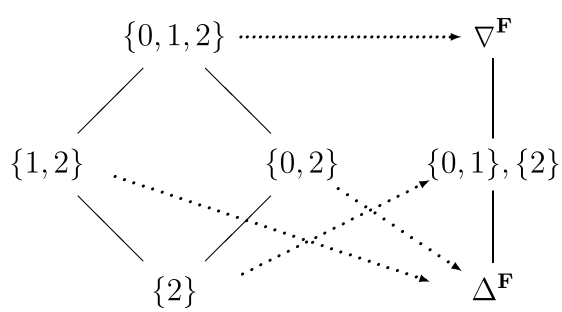
Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ by setting

$$\mathcal{C}_\Sigma = \{\{2\},\{0,2\},\{1,2\},\{0,1,2\}\}.$$

The following table gives the theory families and the theory systems of the π-institution $\mathcal{I}$:

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{0,2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



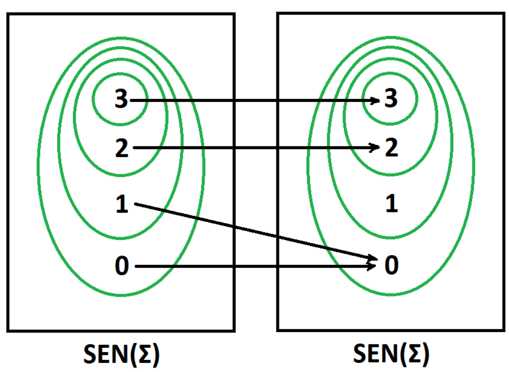
Again it is easy to check, keeping in mind that $\{\{2\}\}$ and $\{\{0,1,2\}\}$ are the only theory systems, that $\mathcal{I}$ is left loyal (and, hence, system loyal), but not family loyal.

Finally, an example of a system loyal π-institution that is not left loyal.

Example 165 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2, 3\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$, $\mathrm{SEN}^\flat(f)(2) = 2$ *and* $\mathrm{SEN}^\flat(f)(3) = 3$;
- $N^\flat$ *is the clone generated by the following two unary natural transformations* $\sigma^\flat, \tau^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$:

x	$\sigma^\flat_\Sigma(x)$	$\tau^\flat_\Sigma(x)$
0	0	0
1	1	1
2	0	3
3	3	3



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

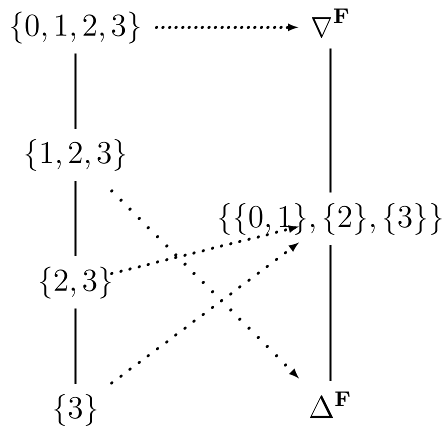
$$\mathcal{C}_\Sigma = \{\{3\}, \{2, 3\}, \{1, 2, 3\}, \{0, 1, 2, 3\}\}.$$

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$:

T	$\overleftarrow{T}$
$\{3\}$	$\{3\}$
$\{2, 3\}$	$\{2, 3\}$
$\{1, 2, 3\}$	$\{2, 3\}$
$\{0, 1, 2, 3\}$	$\{0, 1, 2, 3\}$

The lattice of theory families and the corresponding Leibniz congruence sys-

tems are shown in the diagram.



$\mathcal{I}$ has three theory systems, $\mathrm{Thm}(\mathcal{I}) = \{\{3\}\}$, $T = \{\{2,3\}\}$ *and* $\mathrm{SEN} = \{\{0,1,2,3\}\}$. *An inspection of the diagram shows that $\mathcal{I}$ is system loyal. On the other hand, setting $T' = \{\{1,2,3\}\}$, we get that*

$$\overleftarrow{T'} = \overleftarrow{\{\{1,2,3\}\}} = \{\{2,3\}\} > \{\{3\}\} = \overleftarrow{\{\{3\}\}} = \overleftarrow{\mathrm{Thm}(\mathcal{I})},$$

whereas

$$\Omega(T') = \Delta^{\mathbf{F}} < \{\{0,1\},\{2\},\{3\}\} = \Omega(\mathrm{Thm}(\mathcal{I})).$$

Therefore, $\mathcal{I}$ is not left loyal.

For all loyalty properties, we have transfer theorems, detailed in the various parts of the following result.

Theorem 166 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$.*

(a) *$\mathcal{I}$ is family loyal if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ and all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,*

$$T \nless T' \quad \text{or} \quad \Omega^{\mathcal{A}}(T) \ngtr \Omega^{\mathcal{A}}(T');$$

(b) *$\mathcal{I}$ is left loyal if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ and all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,*

$$\overleftarrow{T} \nless \overleftarrow{T'} \quad \text{or} \quad \Omega^{\mathcal{A}}(T) \ngtr \Omega^{\mathcal{A}}(T');$$

(c) *$\mathcal{I}$ is system loyal if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ and all $T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$,*

$$T \nless T' \quad \text{or} \quad \Omega^{\mathcal{A}}(T) \ngtr \Omega^{\mathcal{A}}(T').$$

Proof:

(a) For the "if", suppose that the loyalty condition holds for the $\mathcal{I}$-filter families of every $\mathbf{F}$-algebraic system. Then it holds, in particular, for the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$, where $\langle I, \iota \rangle : \mathbf{F} \to \mathbf{F}$ is the identity morphism. The fact that, by Lemma 51, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}) = \mathrm{ThFam}(\mathcal{I})$, concludes the proof.

Suppose, conversely, that $\mathcal{I}$ is family loyal. Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T < T'$. We must show that $\Omega^{\mathcal{A}}(T) \ngtr \Omega^{\mathcal{A}}(T')$. Since $T < T'$, we must have $\alpha^{-1}(T) \leq \alpha^{-1}(T')$. However, by surjectivity of $\langle F, \alpha \rangle$, if $\alpha^{-1}(T) = \alpha^{-1}(T')$, we get $T = T'$. Thus, we must have $\alpha^{-1}(T) < \alpha^{-1}(T')$. By Lemma 51, $\alpha^{-1}(T), \alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$. Thus, by loyalty, we get $\Omega(\alpha^{-1}(T)) \ngtr \Omega(\alpha^{-1}(T'))$. Thus, by Proposition 24,

$$\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \ngtr \alpha^{-1}(\Omega^{\mathcal{A}}(T')).$$

The following claim now completes the proof:

Claim: $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \ngtr \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$ implies $\Omega^{\mathcal{A}}(T) \ngtr \Omega^{\mathcal{A}}(T')$.

We work by contraposition. Assume $\Omega^{\mathcal{A}}(T) > \Omega^{\mathcal{A}}(T')$. Then, clearly, $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \geq \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. Moreover, by surjectivity, $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \neq \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. Thus, we conclude that $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) > \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$.

(b) If the left loyalty condition holds for the $\mathcal{I}$-filter families of every $\mathbf{F}$-algebraic system, it holds, in particular, for the $\mathbf{F}$-algebraic system $\mathcal{F}$. Since, by Lemma 51, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}) = \mathrm{ThFam}(\mathcal{I})$, $\mathcal{I}$ is left loyal.

Suppose, conversely, that $\mathcal{I}$ is left loyal. Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\overleftarrow{T} < \overleftarrow{T'}$. We must show that $\Omega^{\mathcal{A}}(T) \ngtr \Omega^{\mathcal{A}}(T')$. Since $\overleftarrow{T} < \overleftarrow{T'}$, we must have $\alpha^{-1}(\overleftarrow{T}) \leq \alpha^{-1}(\overleftarrow{T'})$. However, by surjectivity of $\langle F, \alpha \rangle$, if $\alpha^{-1}(\overleftarrow{T}) = \alpha^{-1}(\overleftarrow{T'})$, we get $\overleftarrow{T} = \overleftarrow{T'}$. Thus, we must have $\alpha^{-1}(\overleftarrow{T}) < \alpha^{-1}(\overleftarrow{T'})$. By Lemma 51, $\alpha^{-1}(T), \alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$. By Lemma 6, $\overleftarrow{\alpha^{-1}(T)} < \overleftarrow{\alpha^{-1}(T')}$. Thus, by left loyalty, we get $\Omega(\alpha^{-1}(T)) \ngtr \Omega(\alpha^{-1}(T'))$. Thus, by Proposition 24, $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \ngtr \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. The claim used in Part (a) is used once more to complete the proof of Part (b).

(c) The proof follows along the lines of that of Part (a). ∎

As a concluding remark, we rephrase the definitions of family and of system loyalty in terms of mappings between partially ordered sets.

Given two posets $\mathbf{P} = \langle P, \leq \rangle$ and $\mathbf{Q} = \langle Q, \preccurlyeq \rangle$, we call a mapping $f : P \to Q$ **loyal** if, for all $p_1, p_2 \in P$,

$$p_1 < p_2 \quad \text{implies} \quad f(p_1) \not\succ f(p_2).$$

Then we have the following easy consequence (or, rather, reformulation) of the definition, combined with Theorem 166.

For a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ and an $\mathbf{F}$-algebraic system $\mathcal{A}$, we define

$$\begin{array}{rcl}\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}) & := & \mathrm{ConSys}^{\mathrm{AlgSys}^*(\mathcal{I})}(\mathcal{A});\\ \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}) & := & \mathrm{ConSys}^{\mathrm{AlgSys}(\mathcal{I})}(\mathcal{A}).\end{array}$$

Moreover, we set

$$\begin{array}{rcl}\mathrm{ConSys}^*(\mathcal{I}) & := & \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{F}) = \mathrm{ConSys}^{\mathrm{AlgSys}^*(\mathcal{I})}(\mathcal{F});\\ \mathrm{ConSys}(\mathcal{I}) & := & \mathrm{ConSys}^{\mathcal{I}}(\mathcal{F}) = \mathrm{ConSys}^{\mathrm{AlgSys}(\mathcal{I})}(\mathcal{F}).\end{array}$$

Proposition 167 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is family loyal;*

(b) $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is loyal;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is loyal, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

Similarly, we get

Proposition 168 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is system loyal;*

(b) $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is loyal;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is loyal, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

3.3 Monotonicity

In this section we define and study classes of π-institutions that are defined using monotonicity properties of the Leibniz operator.

Definition 169 (Monotonicity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is called* **family monotone** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$*,*

$$T \leq T' \quad \textit{implies} \quad \Omega(T) \leq \Omega(T').$$

- $\mathcal{I}$ *is called* **left monotone** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$\overleftarrow{T} \leq \overleftarrow{T'} \quad \textit{implies} \quad \Omega(T) \leq \Omega(T').$$

- $\mathcal{I}$ *is called* **right monotone** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$T \leq T' \quad \textit{implies} \quad \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'}).$$

- $\mathcal{I}$ *is called* **system monotone** *if, for all* $T, T' \in \mathrm{ThSys}(\mathcal{I})$,

$$T \leq T' \quad \textit{implies} \quad \Omega(T) \leq \Omega(T').$$

First, we show a very useful lemma to the effect that family monotonicity implies stability.

Lemma 170 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is family monotone, then* $\mathcal{I}$ *is stable.*

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$. Then we have, by Proposition 42, that $T, \overleftarrow{T} \in \mathrm{ThFam}(\mathcal{I})$, such that $\overleftarrow{T} \leq T$. Therefore, by family monotonicity, $\Omega(\overleftarrow{T}) \leq \Omega(T)$. However, by Proposition 20, $\Omega(T) \leq \Omega(\overleftarrow{T})$. Therefore, we get that $\Omega(\overleftarrow{T}) = \Omega(T)$. So $\mathcal{I}$ is stable. ■

Using Lemma 170, we can now show that family and left monotonicity are equivalent properties.

Proposition 171 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is family monotone if and only if it is left monotone.*

Proof: Suppose, first, that $\mathcal{I}$ is left monotone. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$. Then, by Lemma 1, $\overleftarrow{T} \leq \overleftarrow{T'}$. Therefore, by hypothesis, $\Omega(T) \leq \Omega(T')$. Hence $\mathcal{I}$ is family monotone.

Suppose, conversely, that $\mathcal{I}$ is family monotone. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\overleftarrow{T} \leq \overleftarrow{T'}$. Then, by hypothesis, $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. But, by Lemma 170, $\mathcal{I}$ is stable, whence $\Omega(\overleftarrow{T}) = \Omega(T)$ and $\Omega(\overleftarrow{T'}) = \Omega(T')$. Thus, we conclude that $\Omega(T) \leq \Omega(T')$. Therefore, $\mathcal{I}$ is left monotone. ■

An interesting observation is that system monotonicity may also be defined by using arbitrary theory families, but modifying the application of monotonicy to that of "arrow monotonicity". Formally speaking, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, we say that $\mathcal{I}$ is **arrow monotone** if, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$\overleftarrow{T} \leq \overleftarrow{T'} \quad \text{implies} \quad \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'}).$$

Lemma 172 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is system monotone if and only if it is arrow monotone.*

Proof: Suppose, first, that $\mathcal{I}$ is system monotone and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\overleftarrow{T} \le \overleftarrow{T'}$. Since, by Proposition 42, $\overleftarrow{T}, \overleftarrow{T'} \in \mathrm{ThSys}(\mathcal{I})$, we get, by system monotonicity, that $\Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'})$. Thus, $\mathcal{I}$ is arrow monotone.

Suppose, conversely, that $\mathcal{I}$ is arrow monotone and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $T \le T'$. Then, again by Proposition 42, we get that $\overleftarrow{T} = T \le T' = \overleftarrow{T'}$. Therefore, by arrow monotonicity, $\Omega(T) = \Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'}) = \Omega(T')$. So $\mathcal{I}$ is system monotone. ■

Next, we show that the two properties of right monotonicity and system monotonicity also coincide.

Proposition 173 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is system monotone if and only if it is right monotone.*

Proof: Suppose, first, that $\mathcal{I}$ is right monotone and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $T \le T'$. Then, by right monotonicity, $\Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'})$. But, since T, T' are theory systems, we have $\overleftarrow{T} = T$ and $\overleftarrow{T'} = T'$. Hence, $\Omega(T) \le \Omega(T')$. Therefore, $\mathcal{I}$ is system monotone.

Suppose, conversely, that $\mathcal{I}$ is system monotone and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \le T'$. Then by Lemma 1, $\overleftarrow{T} \le \overleftarrow{T'}$. Since $\overleftarrow{T}, \overleftarrow{T'} \in \mathrm{ThSys}(\mathcal{I})$, we can apply system monotonicity to get $\Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'})$. Thus, $\mathcal{I}$ is right monotone. ■

Because of Propositions 171 and 173, we make the following definitions:

Definition 174 (Pre- and Protoalgebraicity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is called* **protoalgebraic** *if it is family monotone;*
- $\mathcal{I}$ *is called* **prealgebraic** *if it is system monotone.*

We show now that stability is exactly the separating property between prealgebraicity and protoalgebraicity.

Theorem 175 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is protoalgebraic if and only if it is prealgebraic and stable.*

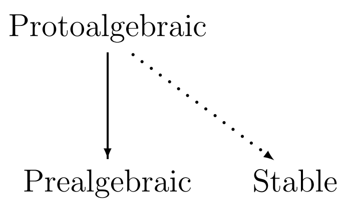
Proof: Suppose, first, that $\mathcal{I}$ is protoalgebraic. Then it is clearly prealgebraic and, by Lemma 170, it is stable.

Suppose, conversely, that $\mathcal{I}$ is stable and prealgebraic and consider $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$. Then, using stability, Proposition 42 and prealgebraicity, we get

$$\begin{array}{rcll} \Omega(T) & = & \Omega(\overleftarrow{T}) & \text{(stability)} \\ & \leq & \Omega(\overleftarrow{T'}) & \text{(Proposition 42 and prealgebraicity)} \\ & = & \Omega(T') & \text{(stability).} \end{array}$$

So Ω is monotone on theory families and $\mathcal{I}$ is protoalgebraic. ∎

In terms of monotonicity, we have established the following **monotonicity hierarchy**:



Now we give examples of π-institutions to show that the two inclusions depicted in this diagram are proper. Moreover, we show that there are π-institutions that are neither prealgebraic nor stable. In other words, we show the following

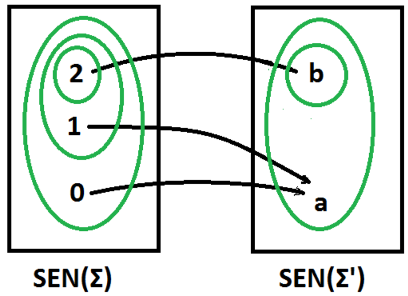
- There exist π-institutions that are neither prealgebraic nor stable.
- There exist π-institutions that are prealgebraic but not stable and, hence, not protoalgebraic.
- There exist π-institutions that are stable but not prealgebraic and, hence, not protoalgebraic.

Example 176 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$*,* $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$*,* $\mathrm{SEN}^\flat(f)(1) = a$ *and* $\mathrm{SEN}^\flat(f)(2) = b$*;*
- $N^\flat$ *is the trivial clone.*

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\},\{a,b\}\}.$$



It is easy to see that $\mathcal{I}$ is not stable: Consider the theory family $T = \{\{1,2\},\{b\}\}$. *Then we have* $\overleftarrow{T} = \{\{2\},\{b\}\}$ *and*

$$\Omega(\overleftarrow{T}) = \{\{\{0,1\},\{2\}\},\{\{a\},\{b\}\}\} \neq \Delta^{\mathbf{F}} = \Omega(T).$$

As a consequence, we get that $\mathcal{I}$ is not protoalgebraic.

We now show that it is not prealgebraic either. We use the two theory systems

$$T = \{\{2\},\{a,b\}\} \leq \{\{1,2\},\{a,b\}\} = T'.$$

We have

$$\Omega_\Sigma(T) = \{\{0,1\},\{2\}\}, \quad \Omega_{\Sigma'}(T) = \{\{a,b\}\};$$
$$\Omega_\Sigma(T') = \{\{0\},\{1,2\}\}, \quad \Omega_{\Sigma'}(T') = \{\{a,b\}\}.$$

Since $T \leq T'$ *but* $\Omega(T) \nleq \Omega(T')$, *we conclude that $\mathcal{I}$ is not prealgebraic.*

Example 177 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

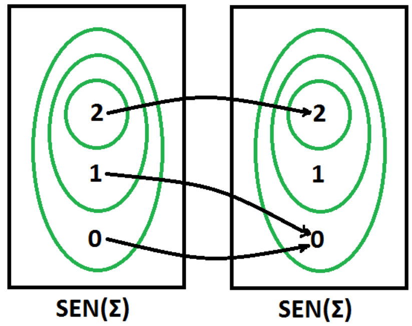
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$;
- $N^\flat$ *is the trivial clone.*

Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}$.

The following table gives the theory families and the theory systems of the π-institution $\mathcal{I}$:

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

First, observe that the only two theory systems are $T = \{\{2\}\}$ *and* $T' = \{\{0,1,2\}\}$. *Further, we have* $\Omega_\Sigma(T) = \{\{0,1\},\{2\}\}$ *and* $\Omega_\Sigma(T') = \{\{0,1,2\}\}$. *So $\mathcal{I}$ is prealgebraic.*



On the other hand, for $T'' = \{\{1,2\}\} \in \mathrm{ThFam}(\mathcal{I})$, *we have* $\overleftarrow{T''} = \{\{2\}\}$. *Moreover* $\Omega_\Sigma(T'') = \{\{0\},\{1\},\{2\}\} \not\subseteq \{\{0,1\},\{2\}\} = \Omega_\Sigma(\overleftarrow{T''})$. *Therefore, we conclude that* $\mathcal{I}$ *is not stable. As a consequence, it is not protoalgebraic either.*

Example 178 *Take any non-protoalgebraic deductive system* $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ *or, in closure system notation,* $\mathcal{S} = \langle \mathcal{L}, C_{\mathcal{S}} \rangle$. *Consider the discrete* π*-institution* $\mathcal{I}^{\mathcal{S}} = \langle \mathbf{F}^{\mathcal{L}}, C^{\mathcal{S}} \rangle$ *corresponding to the deductive system* $\mathcal{S}$ *(see Section 1.1 for details). This* π*-institution is not protoalgebraic (since the deductive system is not), but it is certainly stable (since its only signature morphism is the identity). Therefore, we conclude that it is not prealgebraic either.*

The monotonicity property transfers from the theory families/systems of a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ to all $\mathcal{I}$-filter families/systems on an arbitrary $\mathbf{F}$-algebraic system.

Theorem 179 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is prealgebraic if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$,

$$T \leq T' \quad \text{implies} \quad \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T');$$

(b) $\mathcal{I}$ *is protoalgebraic if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$T \leq T' \quad \text{implies} \quad \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T').$$

Proof:

(a) The "if" follows by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account that $\mathrm{ThSys}(\mathcal{I}) = \mathrm{FiSys}^{\mathcal{I}}(\mathcal{F})$, by Lemma 51.

For the "only if", suppose that $\mathcal{I}$ is prealgebraic and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, such that $T \leq T'$.

Then, clearly, $\alpha^{-1}(T) \le \alpha^{-1}(T')$. Since, by Lemma 51, we have $\alpha^{-1}(T)$, $\alpha^{-1}(T') \in \mathrm{ThSys}(\mathcal{I})$, we get, by applying prealgebraicity, $\Omega(\alpha^{-1}(T)) \le \Omega(\alpha^{-1}(T'))$. But, then, by Proposition 24, we get that $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \le \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. Finally, surjectivity yields that $\Omega^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')$.

(b) The "if" is obtained by considering the **F**-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$ and taking into account that $\mathrm{ThFam}(\mathcal{I}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F})$, by Lemma 51.

For the "only if", assume $\mathcal{I}$ is protoalgebraic and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an **F**-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \le T'$. Then, clearly, $\alpha^{-1}(T) \le \alpha^{-1}(T')$. Since, by Lemma 51, $\alpha^{-1}(T), \alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$, we get, by protoalgebraicity, $\Omega(\alpha^{-1}(T)) \le \Omega(\alpha^{-1}(T'))$. By Proposition 24, $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \le \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$ and, hence, using surjectivity of $\langle F, \alpha\rangle$, $\Omega^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')$.

■

As we did for loyalty, we may recast the two monotonicity classes in terms of the monotonicity of mappings from posets of theory or filter families/systems into posets of congruence systems.

Given two posets $\mathbf{P} = \langle P, \le\rangle$ and $\mathbf{Q} = \langle Q, \preccurlyeq\rangle$, we call a mapping $f : P \to Q$ **monotone** or **order preserving** if, for all $p_1, p_2 \in P$,

$$p_1 \le p_2 \quad \text{implies} \quad f(p_1) \preccurlyeq f(p_2).$$

Proposition 180 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is protoalgebraic;*

(b) $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is monotone;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is monotone, for every* **F***-algebraic system* $\mathcal{A}$*.*

Similarly, for prealgebraicity, we get

Proposition 181 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is prealgebraic;*

(b) $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is monotone;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is monotone, for every* **F***-algebraic system* $\mathcal{A}$*.*

Now we turn into exploring some of the relationships that hold between protoalgebraicity and prealgebraicity, on the one hand, and the various loyalty properties, on the other. Namely, we show that protoalgebraicity implies family loyalty and that prealgebraicity implies system loyalty. Note that, since family loyalty implies stability, the first part of the following theorem is a strengthening of Lemma 170.

Theorem 182 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is protoalgebraic, then it is family loyal;*

(b) If $\mathcal{I}$ *is prealgebraic, then it is system loyal.*

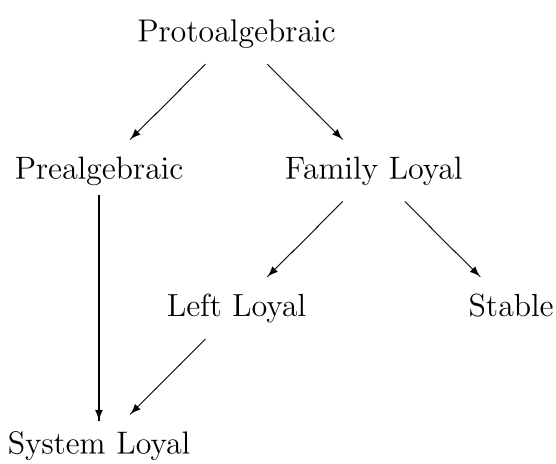
Proof:

(a) Suppose $\mathcal{I}$ is protoalgebraic and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T < T'$. Then, we have $T \leq T'$, whence, by protoalgebraicity, $\Omega(T) \leq \Omega(T')$. But this implies that $\Omega(T) \not\geq \Omega(T')$. We conclude that $\mathcal{I}$ is family loyal.

(b) Suppose that $\mathcal{I}$ is prealgebraic and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $T < T'$. Then, by prealgebraicity, $\Omega(T) \leq \Omega(T')$. This implies that $\Omega(T) \not\geq \Omega(T')$. We conclude that $\mathcal{I}$ is system loyal.

■

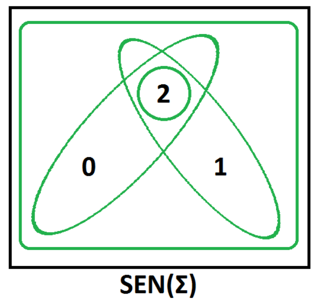
We have now established the following hierarchies:



Finally, we provide an example to show that the loyalty classes are proper subclasses of the classes defined using monotonicity.

Example 183 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be defined as follows:*

- $\mathbf{Sign}^\flat$ *is a trivial one object category, with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by setting* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*

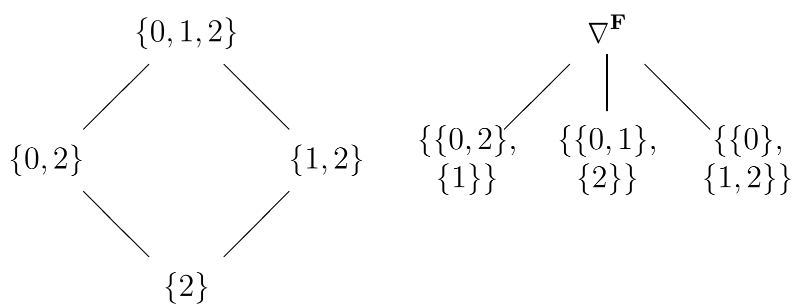


- *$N^\flat$ is the trivial clone, consisting of the projections only.*

Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by setting

$$\mathcal{C}_\Sigma = \{\{2\}, \{0,2\}, \{1,2\}, \{\{0,1,2\}\}.$$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



First, note that, since the category $\mathbf{Sign}^\flat$ is trivial, $\mathcal{I}$ is systemic, i.e., every theory family is also a theory system.

By considering, for instance, $T = \{\{2\}\}$ and $T' = \{\{0,2\}\}$, we see that $T \leq T'$, but $\Omega(T) \nleq \Omega(T')$. Thus, $\mathcal{I}$ is not prealgebraic.

On the other hand, it is clear that there do not exist $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T < T'$ and $\Omega(T) > \Omega(T')$. Hence $\mathcal{I}$ is family loyal.

3.4 Complete $\bigcup$-Monotonicity

We now define classes of π-institutions that are based on various versions of a property called *complete monotonicity*. These properties are strengthened versions of the monotonicity properties and the purpose for introducing them is that they are, in some sense, the dual properties of complete order reflectivity, which strengthens order reflectivity, which, in turn, is, in this same sense, the property dual to monotonicity. This property is somehow related to a property known as *continuity* in the context of sentential logics.

In the case of sentential logics, the property of complete reflectivity asserts that, given a sentential logic $\mathcal{S}$ and a collection of theories $\mathcal{T} \cup \{T'\}$ of $\mathcal{S}$,

$$\bigcap_{T\in\mathcal{T}} \Omega(T) \subseteq \Omega(T') \quad \text{implies} \quad \bigcap \mathcal{T} \subseteq T.$$

Note that meet and intersection coincide both in the lattice of theories of $\mathcal{S}$ and in the lattice of congruences on the formula algebra. Since, however, join and union differ, depending on the point of view, either lattice- or set-theoretic, one may perceive two different properties as dual properties of complete reflectivity. One, which we refer to as complete $\cup$-monotonicity, asserts that, for every collection $\mathcal{T} \cup \{T'\}$ of theories,

$$T' \subseteq \bigcup \mathcal{T} \quad \text{implies} \quad \Omega(T') \subseteq \bigcup_{T\in\mathcal{T}} \Omega(T).$$

The other, which may be termed complete $\vee$-monotonicity, asserts that, for every collection $\mathcal{T} \cup \{T'\}$ of theories,

$$T' \subseteq \bigvee \mathcal{T} \quad \text{implies} \quad \Omega(T') \subseteq \bigvee_{T\in\mathcal{T}} \Omega(T).$$

In this section, we deal with an analog of the first property for π-institutions. In the next section, we look at the second property.

Definition 184 (Complete $\bigcup$-Monotonicity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is* **family completely $\bigcup$-monotone** *or, simply,* **family completely monotone** *if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,
$$T' \leq \bigcup_{T\in\mathcal{T}} T \quad \textit{implies} \quad \Omega(T') \leq \bigcup_{T\in\mathcal{T}} \Omega(T).$$
- $\mathcal{I}$ *is* **left completely $\bigcup$-monotone** *or, simply,* **left completely monotone** *if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,
$$\overleftarrow{T'} \leq \bigcup_{T\in\mathcal{T}} \overleftarrow{T} \quad \textit{implies} \quad \Omega(T') \leq \bigcup_{T\in\mathcal{T}} \Omega(T).$$
- $\mathcal{I}$ *is* **right completely $\bigcup$-monotone** *or, simply,* **right completely monotone** *if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,
$$T' \leq \bigcup_{T\in\mathcal{T}} T \quad \textit{implies} \quad \Omega(\overleftarrow{T'}) \leq \bigcup_{T\in\mathcal{T}} \Omega(\overleftarrow{T}).$$
- $\mathcal{I}$ *is* **system completely $\bigcup$-monotone** *or, simply,* **system completely monotone** *if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$,
$$T' \leq \bigcup_{T\in\mathcal{T}} T \quad \textit{implies} \quad \Omega(T') \leq \bigcup_{T\in\mathcal{T}} \Omega(T).$$

Sometimes we will use the abbreviated form **c$^\cup$-monotonicity** or **c-monotonicity** to refer to complete $\bigcup$-monotonicity.

We have seen in Lemma 170 that family monotonicity (protoalgebraicity) implies stability. Since family complete monotonicity is a stronger property than family monotonicity, we get Part (a) of the following:

Lemma 185 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is family completely monotone, then it is stable.*

(b) If $\mathcal{I}$ *is left completely monotone, it is stable.*

Proof:

(a) If $\mathcal{I}$ is family completely monotone, then it is, a fortiori, family monotone. Thus, the result follows from Lemma 170.

(b) Suppose that $\mathcal{I}$ is left c-monotone and let $T \in \mathrm{ThFam}(\mathcal{I})$. By Proposition 42, $\overleftarrow{\overleftarrow{T}} = \overleftarrow{T}$. Applying left c-monotonicity, we get that $\Omega(\overleftarrow{T}) = \Omega(T)$. Hence $\mathcal{I}$ is stable.

■

Family completely monotone π-institutions are both left and right completely monotone. And, conversely, if a π-institution is both left and right c-monotone, then it is family c-monotone.

Proposition 186 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is family completely monotone if and only if it is both left and right completely monotone.*

Proof: Suppose, first, that $\mathcal{I}$ is family completely monotone.

- Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\overleftarrow{T'} \leq \bigcup_{T \in \mathcal{T}} \overleftarrow{T}$. Applying family c-monotonicity, we get $\Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$. However, by Lemma 185, $\mathcal{I}$ is stable. Hence we get $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. We conclude that $\mathcal{I}$ is left completely monotone.
- Next, let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $T' \leq \bigcup_{T \in \mathcal{T}} T$. Applying family c-monotonicity, we get $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. Once more, by Lemma 185, $\mathcal{I}$ is stable. Hence we get $\Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$. We conclude that $\mathcal{I}$ is right completely monotone.

Suppose, conversely, that $\mathcal{I}$ is both left and right completely monotone and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $T' \leq \bigcup_{T \in \mathcal{T}} T$. Then, by right c-monotonicity, we get that $\Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$. But since $\mathcal{I}$ is left completely monotone, by Lemma 185, it is stable, whence we get $\Omega(T') \leq \bigcup_{T\in\mathcal{T}}\Omega(T)$. Therefore, $\mathcal{I}$ is family completely monotone. ■

If a π-institution $\mathcal{I}$ is left or right completely monotone, then it is also system completely monotone.

Proposition 187 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is left c-monotone, then it is system c-monotone;*

(b) If $\mathcal{I}$ *is right c-monotone, then it is system c-monotone.*

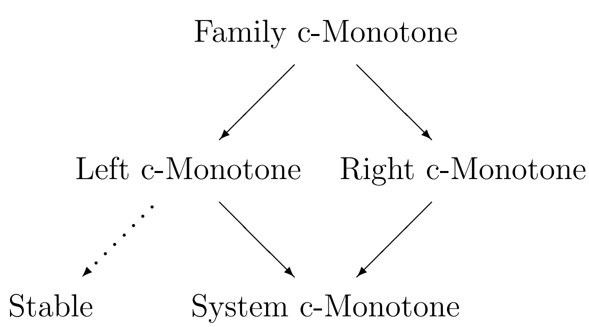
Proof:

(a) Suppose $\mathcal{I}$ is left c-monotone and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $T' \leq \bigcup_{T\in\mathcal{T}} T$. Since $\mathcal{T}\cup\{T'\}$ is a collection of theory systems, we get $\overleftarrow{T'} \leq \bigcup_{T\in\mathcal{T}} \overleftarrow{T}$. Hence, applying left c-monotonicity, we get $\Omega(T') \leq \bigcup_{T\in\mathcal{T}}\Omega(T)$. Thus, $\mathcal{I}$ is system c-monotone.

(b) Suppose $\mathcal{I}$ is right c-monotone and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $T' \leq \bigcup_{T\in\mathcal{T}} T$. Applying right c-monotonicity, we get $\Omega(\overleftarrow{T'}) \leq \bigcup_{T\in\mathcal{T}}\Omega(\overleftarrow{T})$. Since $\mathcal{T}\cup\{T'\}$ is a collection of theory systems, we now get $\Omega(T') \leq \bigcup_{T\in\mathcal{T}}\Omega(T)$. Thus, $\mathcal{I}$ is system c-monotone.

■

In terms of complete monotonicity, we have established the following hierarchy:



Now we give examples of π-institution to show that the inclusions depicted in this diagram are proper. We first give an example of a π-institution that is left c-monotone but not right c-monotone. This shows that:

- The class of family c-monotone π-institutions is properly contained in the class of all left c-monotone π-institutions;

- The class of all system c-monotone π-institutions properly includes the class of all right c-monotone π-institutions.

Example 188 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

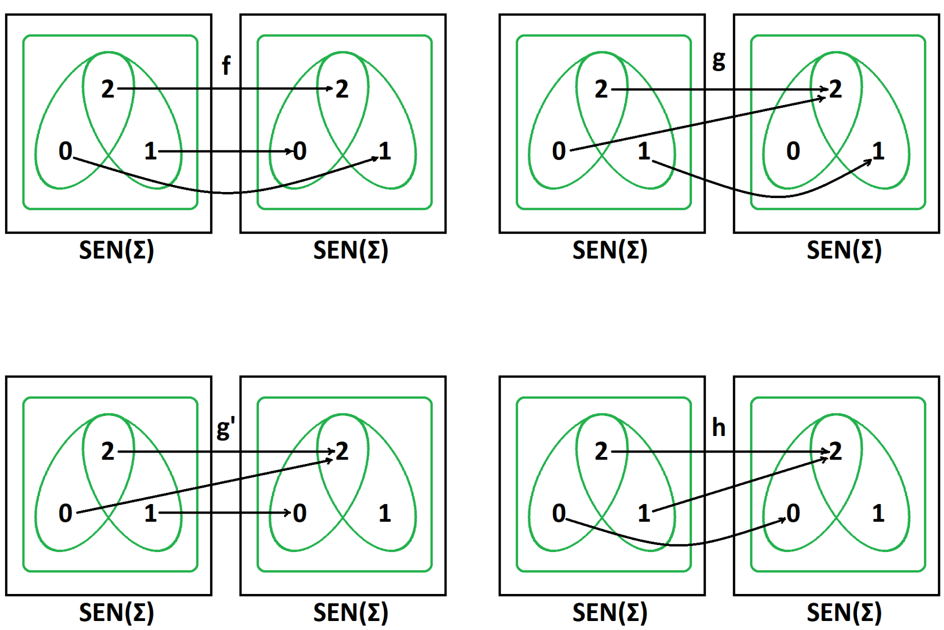
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and six non-identity morphisms* $f, g, g', h, h', t : \Sigma \to \Sigma$*, in which composition is defined by the following table, whose entry in row* k *and column* ℓ *is the result of the composition* $\ell \circ k$*:*

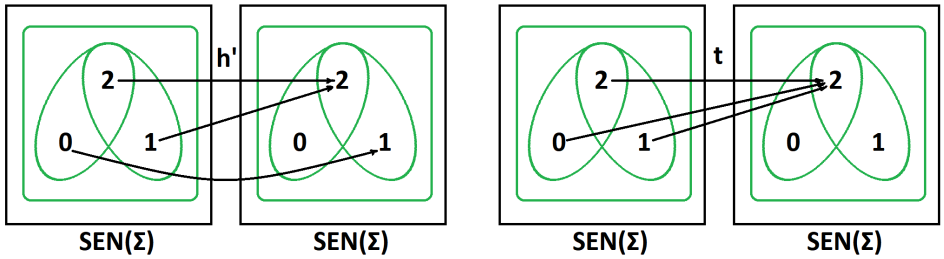
$\circ$	f	g	g'	h	h'	t
f	f	h'	h	g'	g	t
g	g'	g	g'	t	t	t
g'	g	t	t	g'	g	t
h	h'	t	t	h	h'	t
h'	h	h'	h	t	t	t
t	t	t	t	t	t	t

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given, on objects, by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and, on morphisms, by the following table, whose entries in column* k *give the values of the function* $\mathrm{SEN}^\flat(k) : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SEN}^\flat(\Sigma)$*:*

x	f	g	g'	h	h'	t
0	1	2	2	0	1	2
1	0	1	0	2	2	2
2	2	2	2	2	2	2

- $N^\flat$ *is the trivial clone.*





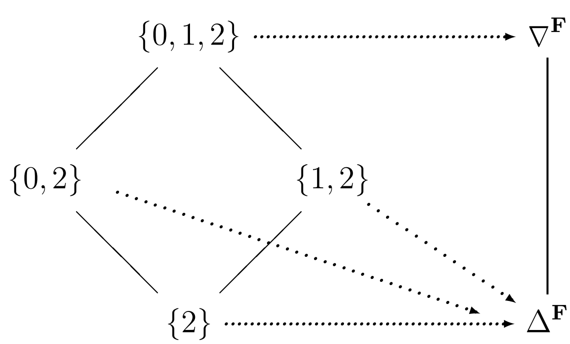
Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ by setting

$$\mathcal{C}_\Sigma = \{\{2\},\{0,2\},\{1,2\},\{0,1,2\}\}.$$

The following table gives the theory families and the theory systems of the π-institution $\mathcal{I}$:

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{0,2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



$\mathcal{I}$ has only two theory systems, $\mathrm{Thm}(\mathcal{I}) = \{\{2\}\}$*, and* $\mathrm{SEN} = \{\{0,1,2\}\}$.

To show that $\mathcal{I}$ is left completely monotone, assume that, for some $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T'} \leq \bigcup_{T\in\mathcal{T}} \overleftarrow{T}$.

- *If $\bigcup_{T\in\mathcal{T}} \overleftarrow{T} = \{\{0,1,2\}\}$, then $\{\{0,1,2\}\} \in \mathcal{T}$ and, hence,*

$$\Omega(T') \leq \nabla^{\mathbf{F}} = \Omega(\{\{0,1,2\}\}) \leq \bigcup_{T\in\mathcal{T}} \Omega(T);$$

- *If $\bigcup_{T\in\mathcal{T}} \overleftarrow{T} = \{\{2\}\}$, then $T' \neq \{\{0,1,2\}\}$, whence*

$$\Omega(T') = \Delta^{\mathbf{F}} \leq \bigcup_{T\in\mathcal{T}} \Omega(T).$$

Thus, in any case, $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$ *and* $\mathcal{I}$ *is left completely monotone.*

On the other hand, we have

$$\{\{0,1,2\}\} \leq \{\{0,2\}\} \cup \{\{1,2\}\},$$

whereas

$$\begin{array}{rcl}\Omega(\overleftarrow{\{\{0,1,2\}\}}) & = & \Omega(\{\{0,1,2\}\}) = \nabla^{\mathbf{F}} \\ & \nleq & \Delta^{\mathbf{F}} \\ & = & \Omega(\{\{2\}\}) \cup \Omega(\{\{2\}\}) \\ & = & \Omega(\overleftarrow{\{\{0,2\}\}}) \cup \Omega(\overleftarrow{\{\{1,2\}\}}).\end{array}$$

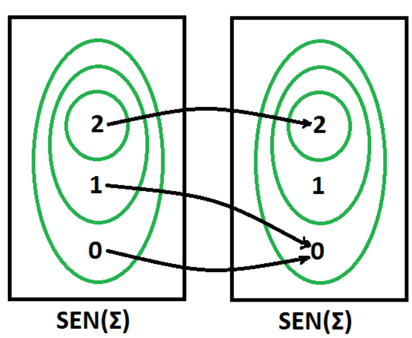
Therefore, $\mathcal{I}$ *is not right completely monotone.*

We now give an example of a right c-monotone π-institution that fails to be left c-monotone. This will show that:

- The class of family c-monotone π-institutions is properly contained in the class of right c-monotone π-institutions;
- The class of left c-monotone π-institutions is a proper subclass of the class of system c-monotone π-institutions.

Example 189 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial clone.*

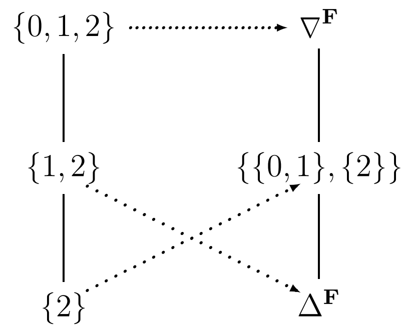


Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}$.

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$*:*

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

The structure of the lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



Taking into account that $\overleftarrow{\{\{1,2\}\}} = \{\{2\}\}$*, we can see that* $\mathcal{I}$ *is right c-monotone.*

On the other hand, for $T = \{\{1,2\}\}$ *and* $T' = \{\{2\}\}$*, we have* $\overleftarrow{T'} = \{\{2\}\} \leq \overleftarrow{T}$*, but* $\Omega(T') = \{\{\{0,1\},\{2\}\}\} \nleq \Delta^{\mathbf{F}} = \Omega(T)$*. Hence* $\mathcal{I}$ *is not left c-monotone.*

As we saw in Theorem 179 for the various monotonicity properties, all versions of complete monotonicity transfer from the theory families/systems of a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ to the $\mathcal{I}$-filter families/systems on arbitrary $\mathbf{F}$-algebraic systems.

Theorem 190 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is family c-monotone if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $T' \leq \bigcup_{T\in\mathcal{T}} T$ *implies* $\Omega^{\mathcal{A}}(T') \leq \bigcup_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(T)$*.*

(b) $\mathcal{I}$ *is left c-monotone if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $\mathcal{T}\cup\{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $\overleftarrow{T'} \leq \bigcup_{T\in\mathcal{T}} \overleftarrow{T}$ *implies* $\Omega^{\mathcal{A}}(T') \leq \bigcup_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(T)$*.*

(c) $\mathcal{I}$ *is right c-monotone if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $\mathcal{T}\cup\{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $T' \leq \bigcup_{T\in\mathcal{T}} T$ *implies* $\Omega^{\mathcal{A}}(\overleftarrow{T'}) \leq \bigcup_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(\overleftarrow{T})$*.*

(d) $\mathcal{I}$ *is system c-monotone if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$*,* $T' \leq \bigcup_{T\in\mathcal{T}} T$ *implies* $\Omega^{\mathcal{A}}(T') \leq \bigcup_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(T)$*.*

Proof: We shall prove Parts (b) and (c). Parts (a) and (d) follow along the same lines and are slightly easier.

(b) The "if" follows by considering the **F**-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account that $\mathrm{ThFam}(\mathcal{I}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F})$, by Lemma 51.

For the "only if", suppose that $\mathcal{I}$ is left c-monotone and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an **F**-algebraic system and $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\overleftarrow{T'} \le \bigcup_{T \in \mathcal{T}} \overleftarrow{T}$. Apply the inverse morphism $\langle F, \alpha \rangle$ to get $\alpha^{-1}(\overleftarrow{T'}) \le \alpha^{-1}(\bigcup_{T \in \mathcal{T}} \overleftarrow{T})$, or, equivalently, $\alpha^{-1}(\overleftarrow{T'}) \le \bigcup_{T \in \mathcal{T}} \alpha^{-1}(\overleftarrow{T})$. Now apply Lemma 6 to get $\overleftarrow{\alpha^{-1}(T')} \le \bigcup_{T \in \mathcal{T}} \overleftarrow{\alpha^{-1}(T)}$. But, by Lemma 51, $\{\alpha^{-1}(T) : T \in \mathcal{T}\} \cup \{\alpha^{-1}(T')\} \subseteq \mathrm{ThFam}(\mathcal{I})$. Therefore, applying left c-monotonicity, we get $\Omega(\alpha^{-1}(T')) \le \bigcup_{T \in \mathcal{T}} \Omega(\alpha^{-1}(T))$. Hence, by Proposition 24, $\alpha^{-1}(\Omega^{\mathcal{A}}(T')) \le \bigcup_{T \in \mathcal{T}} \alpha^{-1}(\Omega^{\mathcal{A}}(T))$, i.e.,

$$\alpha^{-1}(\Omega^{\mathcal{A}}(T')) \le \alpha^{-1}(\bigcup_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)).$$

Finally, taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that $\Omega^{\mathcal{A}}(T') \le \bigcup_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)$.

(c) The "if" is obtained by considering the **F**-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account that $\mathrm{ThFam}(\mathcal{I}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F})$, by Lemma 51.

For the "only if", suppose that $\mathcal{I}$ is right c-monotone and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an **F**-algebraic system and $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T' \le \bigcup_{T \in \mathcal{T}} T$. Apply the inverse morphism $\langle F, \alpha \rangle$ to get $\alpha^{-1}(T') \le \alpha^{-1}(\bigcup_{T \in \mathcal{T}} T)$, or, equivalently, $\alpha^{-1}(T') \le \bigcup_{T \in \mathcal{T}} \alpha^{-1}(T)$. By Lemma 51, $\{\alpha^{-1}(T) : T \in \mathcal{T}\} \cup \{\alpha^{-1}(T')\} \subseteq \mathrm{ThFam}(\mathcal{I})$. Therefore, applying right c-monotonicity, we get $\Omega(\overleftarrow{\alpha^{-1}(T')}) \le \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{\alpha^{-1}(T)})$. By Lemma 6, $\Omega(\alpha^{-1}(\overleftarrow{T'})) \le \bigcup_{T \in \mathcal{T}} \Omega(\alpha^{-1}(\overleftarrow{T}))$. Hence, by Proposition 24, $\alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'})) \le \bigcup_{T \in \mathcal{T}} \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T}))$. This is equivalent to $\alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'})) \le \alpha^{-1}(\bigcup_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(\overleftarrow{T}))$. Finally, taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that $\Omega^{\mathcal{A}}(\overleftarrow{T'}) \le \bigcup_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(\overleftarrow{T})$.

■

Next we look at the relationships that hold between protoalgebraicity and prealgebraicity, on the one hand, and the various c-monotonicity properties, on the other. More precisely, we show that left complete monotonicity implies protoalgebraicity and that system complete monotonicity implies prealgebraicity.

Theorem 191 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* **F**.

(a) If $\mathcal{I}$ *is left c-monotone, then it is protoalgebraic;*

(b) If $\mathcal{I}$ is system c-monotone, then it is prealgebraic.

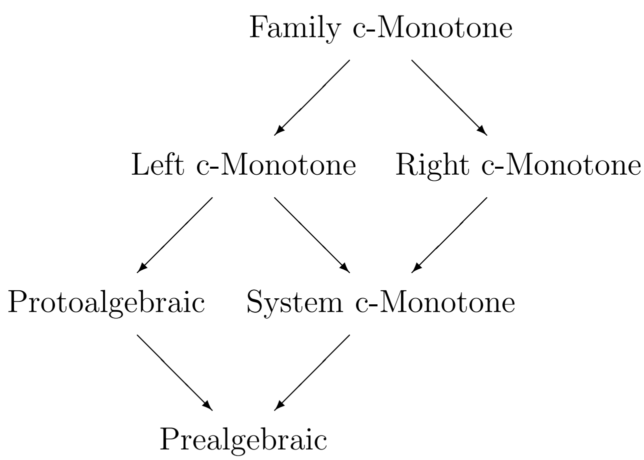
Proof:

(a) Suppose $\mathcal{I}$ is left c-monotone and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$. Then, by Lemma 1, we get $\overleftarrow{T} \leq \overleftarrow{T'}$. Thus, by left c-monotonicity, $\Omega(T) \leq \Omega(T')$. Hence $\mathcal{I}$ is protoalgebraic.

(b) Suppose that $\mathcal{I}$ is system c-monotone and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $T \leq T'$. Then we get right away from system c-monotonicity that $\Omega(T) \leq \Omega(T')$ and, therefore, $\mathcal{I}$ is prealgebraic.

■

We have now established the following hierarchy of **monotonicity** and **complete monotonicity** properties:

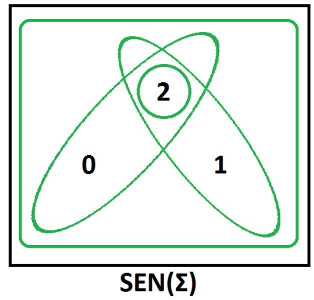


Finally, we provide an example to show that the c-monotonicity classes are proper subclasses of the monotonicity classes. Namely, we construct a protoalgebraic π-institution that fails to be system c-monotone.

Example 192 *Define the algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ as follows:*

- $\mathbf{Sign}^\flat$ *is a trivial category with object Σ;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the clone generated by the unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, given by*

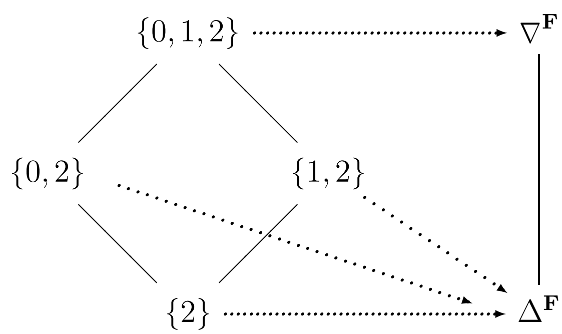
$x \in \mathrm{SEN}^\flat(\Sigma)$	$\sigma^\flat_\Sigma(x)$
0	1
1	2
2	0



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by setting

$$\mathcal{C}_\Sigma = \{\{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}.$$

It is easy to see that the lattices of theory families and corresponding Leibniz congruence systems are as given in the diagram.



From the diagram one can verify immediately that $\mathcal{I}$ is protoalgebraic, On the other hand, we have $\{\{0,1,2\}\} \leq \{\{0,2\}\} \cup \{\{1,2\}\}$, but, obviously, $\Omega(\{\{0,1,2\}\}) \nleq \Omega(\{\{0,2\}\}) \cup \Omega(\{\{1,2\}\})$. Taking into account that $\mathcal{I}$ is systemic, we conclude that $\mathcal{I}$ fails to be system c-monotone.

3.5 Complete $\bigvee$-Monotonicity

We now define classes of π-institutions that are based on the corresponding versions of the property of *complete monotonicity* using the join operation. These properties are also strengthened versions of the monotonicity properties.

To define these complete monotonicity properties, let us introduce the following notation. Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Denote by $\bigvee^{\mathcal{I}} \mathcal{T} = \bigvee^{\mathcal{I}}_{T \in \mathcal{T}} T$ the join of a collection $\mathcal{T}$ of theory families of $\mathcal{I}$ in the complete lattice $\mathbf{ThFam}(\mathcal{I})$ of theory families of $\mathcal{I}$. Analogously, denote by $\bigvee^{\mathbf{F}} \Theta = \bigvee^{\mathbf{F}}_{\theta \in \Theta} \theta$ the join of a collection Θ of congruence systems on $\mathbf{F}$ in the complete lattice $\mathbf{ConSys}(\mathbf{F})$ of congruence systems on $\mathbf{F}$.

Definition 193 (Complete $\bigvee$-Monotonicity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is* **family completely $\bigvee$-monotone** *if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$*,*

$$T' \leq \bigvee_{T \in \mathcal{T}}^{\mathcal{I}} T \quad \textit{implies} \quad \Omega(T') \leq \bigvee_{T \in \mathcal{T}}^{\mathbf{F}} \Omega(T).$$

- $\mathcal{I}$ *is* **left completely $\bigvee$-monotone** *if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$*,*

$$\overleftarrow{T'} \leq \bigvee_{T \in \mathcal{T}}^{\mathcal{I}} \overleftarrow{T} \quad \textit{implies} \quad \Omega(T') \leq \bigvee_{T \in \mathcal{T}}^{\mathbf{F}} \Omega(T).$$

- $\mathcal{I}$ *is* **right completely $\bigvee$-monotone** *if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$*,*

$$T' \leq \bigvee_{T \in \mathcal{T}}^{\mathcal{I}} T \quad \textit{implies} \quad \Omega(\overleftarrow{T'}) \leq \bigvee_{T \in \mathcal{T}}^{\mathbf{F}} \Omega(\overleftarrow{T}).$$

- $\mathcal{I}$ *is* **system completely $\bigvee$-monotone** *if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$*,*

$$T' \leq \bigvee_{T \in \mathcal{T}}^{\mathcal{I}} T \quad \textit{implies} \quad \Omega(T') \leq \bigvee_{T \in \mathcal{T}}^{\mathbf{F}} \Omega(T).$$

Sometimes we will use the abbreviated form **c$^\vee$-monotonicity** to refer to complete $\bigvee$-monotonicity.

We have seen in Lemma 170 that family monotonicity (protoalgebraicity) implies stability. Since family complete monotonicity is a stronger property than family monotonicity, we get Part (a) of the following:

Lemma 194 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is family completely* $\bigvee$*-monotone, then it is stable.*

(b) If $\mathcal{I}$ *is left completely* $\bigvee$*-monotone, it is stable.*

Proof:

(a) If $\mathcal{I}$ is family completely $\bigvee$-monotone, then it is, a fortiori, family monotone. Thus, the result follows from Lemma 170.

(b) Suppose that $\mathcal{I}$ is left c$^\vee$-monotone and let $T \in \mathrm{ThFam}(\mathcal{I})$. By Proposition 42, $\overleftarrow{\overleftarrow{T}} = \overleftarrow{T}$. Applying left c$^\vee$-monotonicity, we get that $\Omega(\overleftarrow{T}) = \Omega(T)$. Hence $\mathcal{I}$ is stable.

■

Family completely $\bigvee$-monotone π-institutions are both left and right completely $\bigvee$-monotone. And, conversely, if a π-institution is both left and right $c^\vee$-monotone, then it is family $c^\vee$-monotone. This parallels Proposition 186, which concerned the case of $c^\cup$-monotonicity properties.

Proposition 195 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is family completely* $\bigvee$*-monotone if and only if it is both left and right completely* $\bigvee$*-monotone.*

Proof: Suppose, first, that $\mathcal{I}$ is family completely $\bigvee$-monotone.

- Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\overleftarrow{T'} \leq \bigvee^{\mathcal{I}}_{T \in \mathcal{T}} \overleftarrow{T}$. Applying family $c^\vee$-monotonicity, we get $\Omega(\overleftarrow{T'}) \leq \bigvee^{\mathbf{F}}_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$. However, by Lemma 185, $\mathcal{I}$ is stable. Hence we get $\Omega(T') \leq \bigvee^{\mathbf{F}}_{T \in \mathcal{T}} \Omega(T)$. We conclude that $\mathcal{I}$ is left completely $\bigvee$-monotone.
- Next, let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $T' \leq \bigvee^{\mathcal{I}}_{T \in \mathcal{T}} T$. Applying family $c^\vee$-monotonicity, we get $\Omega(T') \leq \bigvee^{\mathbf{F}}_{T \in \mathcal{T}} \Omega(T)$. Once more, by Lemma 185, $\mathcal{I}$ is stable. Hence we get $\Omega(\overleftarrow{T'}) \leq \bigvee^{\mathbf{F}}_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$. We conclude that $\mathcal{I}$ is right completely $\bigvee$-monotone.

Suppose, conversely, that $\mathcal{I}$ is both left and right completely $\bigvee$-monotone and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $T' \leq \bigvee^{\mathcal{I}}_{T \in \mathcal{T}} T$. Then, by right $c^\vee$-monotonicity, we get that $\Omega(\overleftarrow{T'}) \leq \bigvee^{\mathbf{F}}_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$. But since $\mathcal{I}$ is left completely $\bigvee$-monotone, by Lemma 185, it is stable, whence we get $\Omega(T') \leq \bigvee^{\mathbf{F}}_{T \in \mathcal{T}} \Omega(T)$. Therefore, $\mathcal{I}$ is family completely $\bigvee$-monotone. ■

If a π-institution $\mathcal{I}$ is left or right completely $\bigvee$-monotone, then it is also system completely $\bigvee$-monotone. This is an analog of Proposition 187, which addressed the case of $c^\cup$-monotonicity.

Proposition 196 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is left* $c^\vee$*-monotone, then it is system* $c^\vee$*-monotone;*

(b) If $\mathcal{I}$ *is right* $c^\vee$*-monotone, then it is system* $c^\vee$*-monotone.*

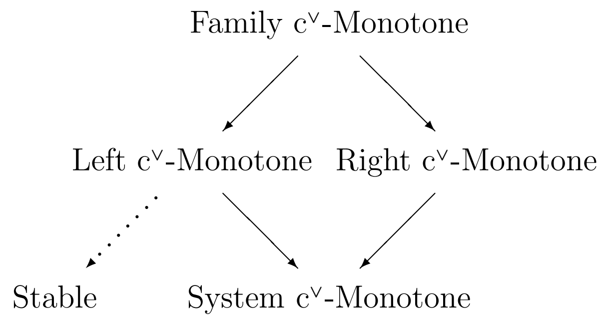
Proof:

(a) Suppose $\mathcal{I}$ is left $c^\vee$-monotone and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $T' \leq \bigvee^{\mathcal{I}}_{T \in \mathcal{T}} T$. Since $\mathcal{T} \cup \{T'\}$ is a collection of theory systems, we get $\overleftarrow{T'} \leq \bigvee^{\mathcal{I}}_{T \in \mathcal{T}} \overleftarrow{T}$. Hence, applying left $c^\vee$-monotonicity, we get $\Omega(T') \leq \bigvee^{\mathbf{F}}_{T \in \mathcal{T}} \Omega(T)$. Thus, $\mathcal{I}$ is system $c^\vee$-monotone.

(b) Suppose $\mathcal{I}$ is right $c^\vee$-monotone and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $T' \leq \bigvee^{\mathcal{I}}_{T\in\mathcal{T}} T$. Applying right $c^\vee$-monotonicity, we get $\Omega(\overleftarrow{T'}) \leq \bigvee^{\mathbf{F}}_{T\in\mathcal{T}} \Omega(\overleftarrow{T})$. Since $\mathcal{T} \cup \{T'\}$ is a collection of theory systems, we now get $\Omega(T') \leq \bigvee^{\mathbf{F}}_{T\in\mathcal{T}} \Omega(T)$. Thus, $\mathcal{I}$ is system $c^\vee$-monotone.

■

In terms of complete $\bigvee$-monotonicity, we have established the following hierarchy, which exactly mirrors the hierarchy of $c^\cup$-monotonicity classes:



Now we give examples of π-institutions to show that the inclusions depicted in this diagram are proper. We first give an example of a π-institution that is left $c^\vee$-monotone but not right $c^\vee$-monotone. This shows that:

- The class of family $c^\vee$-monotone π-institutions is properly contained in the class of all left $c^\vee$-monotone π-institutions;

- The class of all system $c^\vee$-monotone π-institutions properly includes the class of all right $c^\vee$-monotone π-institutions.

Example 197 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

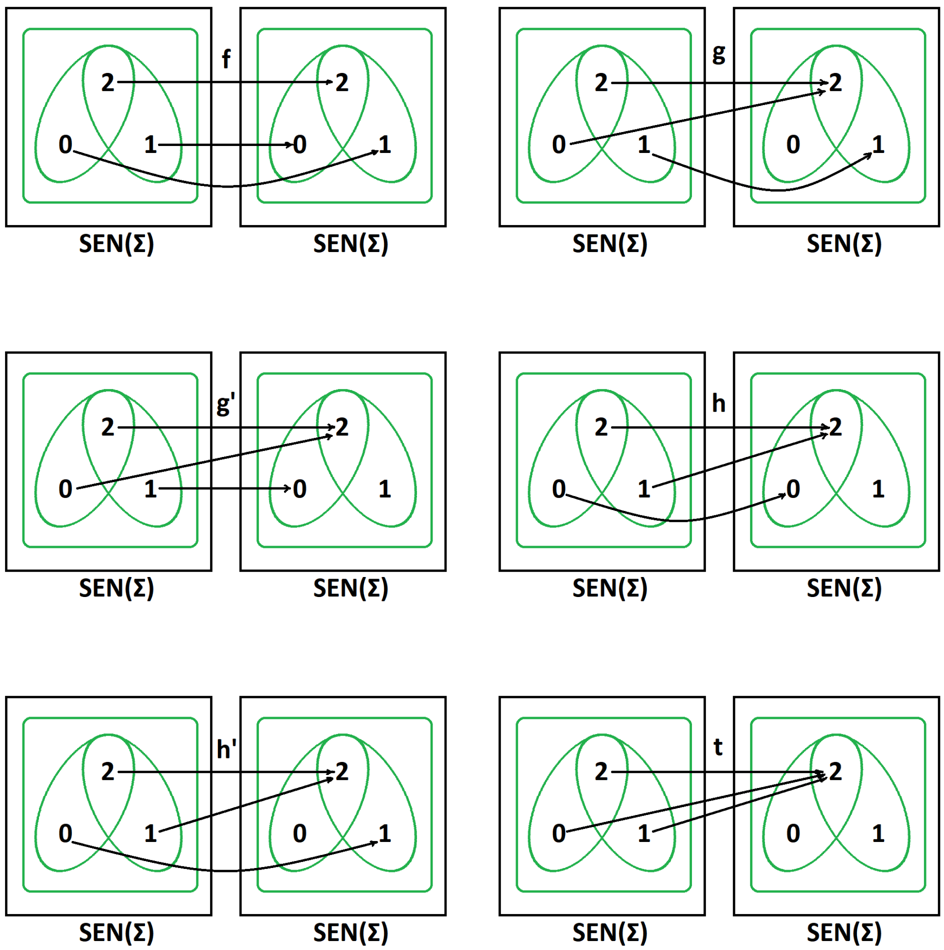
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and six non-identity morphisms* $f, g, g', h, h', t : \Sigma \to \Sigma$, *in which composition is defined by the following table, whose entry in row* k *and column* ℓ *is the result of the composition* $\ell \circ k$*:*

$\circ$	f	g	g'	h	h'	t
f	f	h'	h	g'	g	t
g	g'	g	g'	t	t	t
g'	g	t	t	g'	g	t
h	h'	t	t	h	h'	t
h'	h	h'	h	t	t	t
t	t	t	t	t	t	t

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given, on objects, by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and, on morphisms, by the following table, whose entries in column* k *give* *the values of the function* $\mathrm{SEN}^\flat(k) : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SEN}^\flat(\Sigma)$*:*

x	f	g	g'	h	h'	t
0	1	2	2	0	1	2
1	0	1	0	2	2	2
2	2	2	2	2	2	2

- $N^\flat$ *is the trivial clone.*



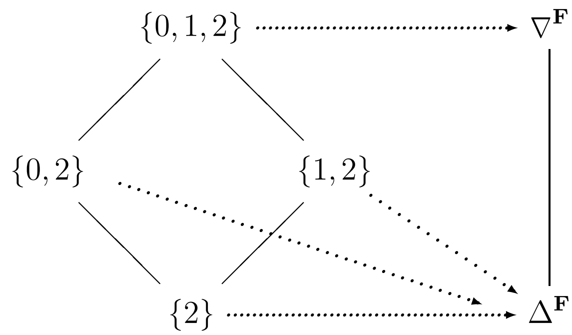
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

$$\mathcal{C}_\Sigma = \{\{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}.$$

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$*:*

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{0,2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



$\mathcal{I}$ has only two theory systems, $\mathrm{Thm}(\mathcal{I}) = \{\{2\}\}$, *and* $\mathrm{SEN} = \{\{0,1,2\}\}$.

To show that $\mathcal{I}$ is left completely monotone, assume that, for some $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T'} \leq \bigvee^{\mathcal{I}}_{T \in \mathcal{T}} \overleftarrow{T}$.

- *If $\bigvee^{\mathcal{I}}_{T \in \mathcal{T}} \overleftarrow{T} = \{\{0,1,2\}\}$, then $\{\{0,1,2\}\} \in \mathcal{T}$ and, hence,*

$$\Omega(T') \leq \nabla^{\mathbf{F}} = \Omega(\{\{0,1,2\}\}) \leq \bigvee_{T \in \mathcal{T}}^{\mathbf{F}} \Omega(T);$$

- *If $\bigvee^{\mathcal{I}}_{T \in \mathcal{T}} \overleftarrow{T} = \{\{2\}\}$, then $T' \neq \{\{0,1,2\}\}$, whence*

$$\Omega(T') = \Delta^{\mathbf{F}} \leq \bigvee_{T \in \mathcal{T}}^{\mathbf{F}} \Omega(T).$$

Thus, in any case, $\Omega(T') \leq \bigvee^{\mathbf{F}}_{T \in \mathcal{T}} \Omega(T)$ and $\mathcal{I}$ is left completely $\bigvee$-monotone.

On the other hand, we have

$$\{\{0,1,2\}\} \leq \{\{0,2\}\} \vee^{\mathcal{I}} \{\{1,2\}\},$$

whereas

$$\begin{array}{rcl}\Omega(\overleftarrow{\{\{0,1,2\}\}}) & = & \Omega(\{\{0,1,2\}\}) = \nabla^{\mathbf{F}} \\ & \nleq & \Delta^{\mathbf{F}} \\ & = & \Omega(\{\{2\}\}) \vee^{\mathbf{F}} \Omega(\{\{2\}\}) \\ & = & \Omega(\overleftarrow{\{\{0,2\}\}}) \vee^{\mathbf{F}} \Omega(\overleftarrow{\{\{1,2\}\}}).\end{array}$$

Therefore, $\mathcal{I}$ is not right completely $\bigvee$-monotone.

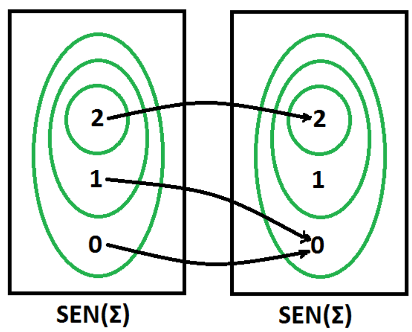
We now give an example of a right $\mathrm{c}^{\vee}$-monotone π-institution that fails to be left $\mathrm{c}^{\vee}$-monotone. This will show that:

- The class of family $\mathrm{c}^{\vee}$-monotone π-institutions is properly contained in the class of right $\mathrm{c}^{\vee}$-monotone π-institutions;

- The class of left $c^{\vee}$-monotone π-institutions is a proper subclass of the class of system $c^{\vee}$-monotone π-institutions.

Example 198 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial clone.*

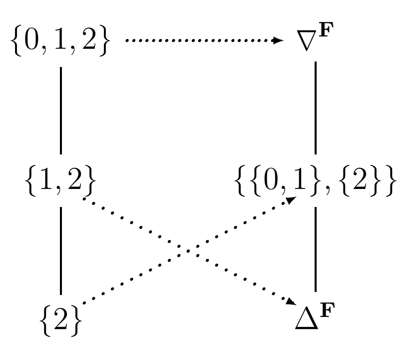


Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}$*.*

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$*:*

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

The structure of the lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



Taking into account that $\overleftarrow{\{\{1,2\}\}} = \{\{2\}\}$*, we can see that* $\mathcal{I}$ *is right* $c^{\vee}$*-monotone.*

On the other hand, for $T = \{\{1,2\}\}$ *and* $T' = \{\{2\}\}$, *we have* $\overleftarrow{T'} = \{\{2\}\} \leq \overleftarrow{T}$, *but* $\Omega(T') = \{\{\{0,1\},\{2\}\}\} \nleq \Delta^{\mathbf{F}} = \Omega(T)$. *Hence* $\mathcal{I}$ *is not left* $c^{\vee}$*-monotone.*

As we saw in Theorem 190, the various complete $\bigcup$-monotonicity properties defined based on the union operation transfer from the theory families/systems of a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ to the $\mathcal{I}$-filter families/systems on arbitrary $\mathbf{F}$-algebraic systems. On the other hand, the $\bigvee$-monotonicity versions introduced in this section do not transfer in general. The problem appears to be that the commutativity of unions with inverse surjective morphisms, i.e., $\bigcup_{T\in\mathcal{T}} \alpha^{-1}(T) = \alpha^{-1}(\bigcup_{T\in\mathcal{T}} T)$, ceases to hold when unions are replaced by joins. In that case, one has, in general, a proper inclusion instead of an equality. To describe it, let us again introduce some notation. Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Let, also $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system. Denote by $\bigvee^{\mathcal{I},\mathcal{A}} \mathcal{T} = \bigvee^{\mathcal{I},\mathcal{A}}_{T\in\mathcal{T}} T$ the join of a collection $\mathcal{T}$ of $\mathcal{I}$-filter families of $\mathcal{A}$ in the complete lattice $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})$ of $\mathcal{I}$-filter families of $\mathcal{A}$. Analogously, denote by $\bigvee^{\mathcal{A}} \Theta = \bigvee^{\mathcal{A}}_{\theta\in\Theta} \theta$ the join of a collection Θ of congruence systems on $\mathcal{A}$ in the complete lattice $\mathbf{ConSys}(\mathcal{A})$ of congruence systems on $\mathcal{A}$. According to this notation, we get, in general, that

$$\bigvee_{T\in\mathcal{T}}^{\mathcal{I}} \alpha^{-1}(T) \subsetneqq \alpha^{-1}\left(\bigvee_{T\in\mathcal{T}}^{\mathcal{I},\mathcal{A}} T\right).$$

The following example showcases this proper inclusion.

Example 199 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

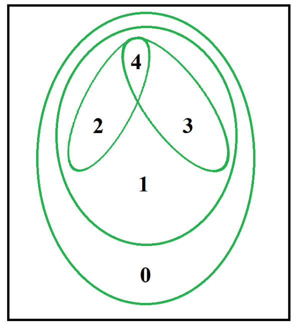
- $\mathbf{Sign}^\flat$ *is the trivial category, with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3,4\}$*;*
- $N^\flat$ *is the trivial clone.*

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by setting*

$$\mathcal{C}_\Sigma = \{\{4\},\{2,4\},\{3,4\},\{1,2,3,4\},\{0,1,2,3,4\}\}.$$

Next, consider the $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, *with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, *defined as follows:*

- *The algebraic system* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *is specified by the following data:*
  - $\mathbf{Sign}$ *is the trivial category, with object* Σ'*;*

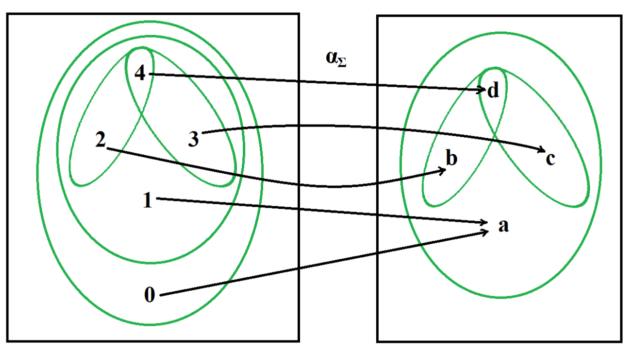


SEN(Σ)

- $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *is given by* $\mathrm{SEN}(\Sigma') = \{a, b, c, d\}$*;*
- N *is the trivial clone.*

- $F : \mathbf{Sign}^\flat \to \mathbf{Sign}$ *is the trivial functor taking* Σ *to* Σ'*;*
- $\alpha : \mathrm{SEN}^\flat \to \mathrm{SEN} \circ F$ *is determined by letting* $\alpha_\Sigma : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SEN}(\Sigma')$ *be given by*

$x \in \mathrm{SEN}^\flat(\Sigma)$	$\alpha_\Sigma(x)$
0	a
1	a
2	b
3	c
4	d



$\mathrm{SEN}^\flat(\Sigma)$ SEN(Σ')

Based on Lemma 51, we get that

$$\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \{\{\{d\}\}, \{\{b, d\}\}, \{\{c, d\}\}, \{\{a, b, c, d\}\}\}.$$

Now we can easily verify that

$$\alpha^{-1}(\{\{b, d\}\}) \vee^{\mathcal{I}} \alpha^{-1}(\{\{c, d\}\}) = \{\{2, 4\}\} \vee^{\mathcal{I}} \{\{3, 4\}\} = \{\{1, 2, 3, 4\}\},$$

whereas

$$\alpha^{-1}(\{\{b,d\}\} \vee^{\mathcal{I},\mathcal{A}} \{\{c,d\}\}) = \alpha^{-1}(\{\{a,b,c,d\}\}) = \{\{0,1,2,3,4\}\}.$$

Thus, for $\mathcal{T} = \{\{\{b,d\}\},\{\{c,d\}\}\}$, *we get* $\bigvee^{\mathcal{I}}_{T\in\mathcal{T}} \alpha^{-1}(T) \not\geq \alpha^{-1}(\bigvee^{\mathcal{I},\mathcal{A}}_{T\in\mathcal{T}} T)$.

We may characterize family and system c$^\vee$-monotonicity in terms of the complete order preservation of mappings from posets of theory families/ systems into posets of congruence systems. Given complete lattices $\mathbf{P} = \langle P, \leq\rangle$ and $\mathbf{Q} = \langle Q, \preccurlyeq\rangle$, call a mapping $f : P \to Q$ **completely order preserving** if, for all $\{x\} \cup Y \subseteq P$,

$$x \leq \bigvee_{y\in Y}^{\mathbf{P}} y \quad \text{implies} \quad f(x) \preccurlyeq \bigvee_{y\in Y}^{\mathbf{Q}} f(y).$$

Proposition 200 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is family completely* $\bigvee$*-monotone;*

(b) $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}(\mathcal{I})$ *is completely order preserving.*

Similarly, for system c-monotonicity, we have

Proposition 201 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is system completely* $\bigvee$*-monotone;*

(b) $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}(\mathcal{I})$ *is completely order preserving.*

Next we look at the relationships that hold between protoalgebraicity and prealgebraicity, on the one hand, and the various c$^\vee$-monotonicity properties, on the other. More precisely, we show that left complete $\bigvee$-monotonicity implies protoalgebraicity and that system complete $\bigvee$-monotonicity implies prealgebraicity. Once more, this theorem parallels Theorem 191, which dealt with $\bigcup$-monotonicity.

Theorem 202 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is left* c$^\vee$*-monotone, then it is protoalgebraic;*

(b) If $\mathcal{I}$ *is system* c$^\vee$*-monotone, then it is prealgebraic.*

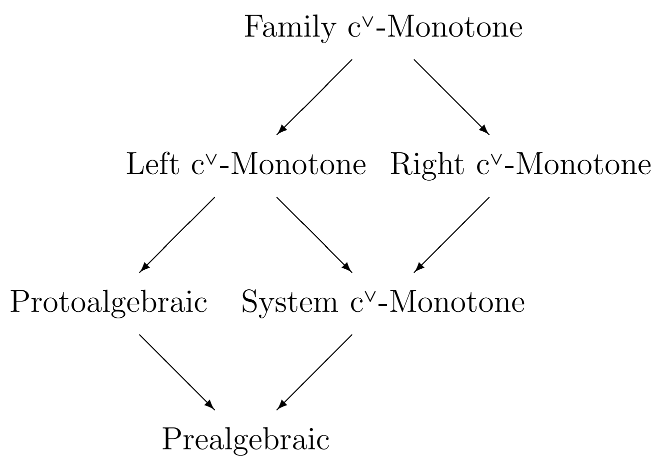
Proof:

(a) Suppose $\mathcal{I}$ is left c$^\vee$-monotone and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$. Then, by Lemma 1, we get $\overleftarrow{T} \leq \overleftarrow{T'}$. Thus, by left c$^\vee$-monotonicity, $\Omega(T) \leq \Omega(T')$. Hence $\mathcal{I}$ is protoalgebraic.

(b) Suppose that $\mathcal{I}$ is system c$^\vee$-monotone and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $T \leq T'$. Then we get right away from system c$^\vee$-monotonicity that $\Omega(T) \leq \Omega(T')$ and, therefore, $\mathcal{I}$ is prealgebraic.

■

We have now established the following hierarchy of **monotonicity** and **complete $\bigvee$-monotonicity** properties, which also mirrors the combined hierarchy of monotonicity and complete $\bigcup$-monotonicity properties:



We now provide an example to show that the c$^\vee$-monotonicity classes are proper subclasses of the monotonicity classes. Namely, we construct a protoalgebraic π-institution that fails to be system c$^\vee$-monotone.

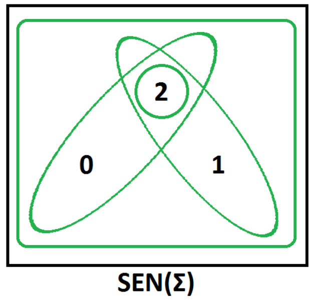
Example 203 *Define the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *as follows:*

- $\mathbf{Sign}^\flat$ *is a trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the clone generated by the unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, given by*

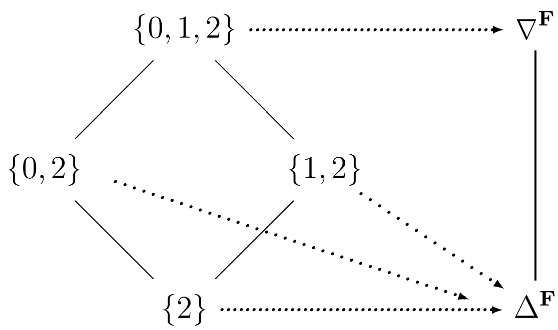
$x \in \mathrm{SEN}^\flat(\Sigma)$	$\sigma^\flat_\Sigma(x)$
0	1
1	2
2	0

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

$$\mathcal{C}_\Sigma = \{\{2\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}.$$



It is easy to see that the lattices of theory families and corresponding Leibniz congruence systems are as given in the diagram.



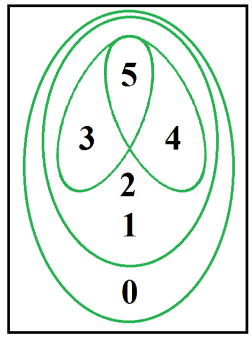
From the diagram one can verify immediately that $\mathcal{I}$ *is protoalgebraic, On the other hand, we have* $\{\{0,1,2\}\} \leq \{\{0,2\}\} \vee^{\mathcal{I}} \{\{1,2\}\}$, *but, obviously,* $\Omega(\{\{0,1,2\}\}) \nleq \Omega(\{\{0,2\}\}) \vee^{\mathbf{F}} \Omega(\{\{1,2\}\})$. *Taking into account that* $\mathcal{I}$ *is systemic, we conclude that* $\mathcal{I}$ *fails to be system* $c^{\vee}$*-monotone.*

We conclude this section with two examples showing that the classes of $c^{\cup}$-monotone and $c^{\vee}$-monotone π-institutions are incomparable. The first example shows that there exists a $c^{\cup}$-monotone π-institution which is not $c^{\vee}$-monotone.

Example 204 *Define the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *as follows:*

- $\mathbf{Sign}^\flat$ *is a trivial category with object* Σ;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3,4,5\}$;
- $N^\flat$ *is the clone generated by the unary natural transformations* $\rho^\flat, \sigma^\flat, \tau^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$, *given by*

$x \in \mathrm{SEN}^\flat(\Sigma)$	$\rho^\flat_\Sigma(x)$	$\sigma^\flat_\Sigma(x)$	$\tau^\flat_\Sigma(x)$
0	5	0	0
1	1	1	1
2	2	2	2
3	3	5	3
4	4	4	5
5	5	5	5

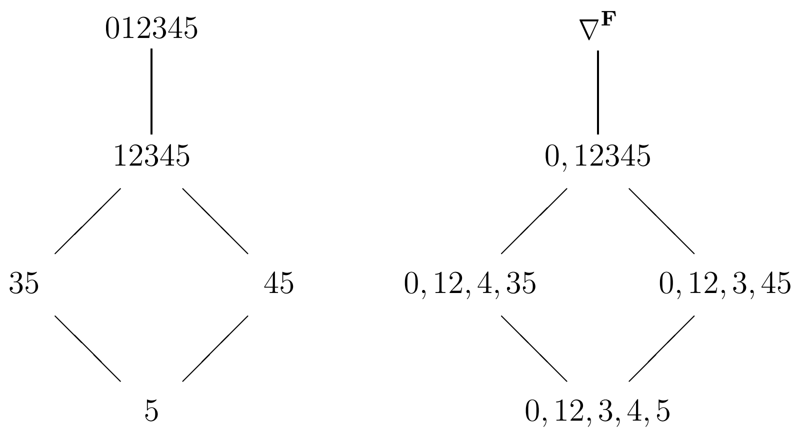


SEN(Σ)

Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by setting

$$\mathcal{C}_\Sigma = \{\{5\}, \{3,5\}, \{4,5\}, \{1,2,3,4,5\}, \{0,1,2,3,4,5\}\}.$$

The lattice of theory families and the corresponding Leibniz congruence systems (in block form) are given in the following diagram.



From the diagram one can verify that, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $T' \leq \bigcup_{T \in \mathcal{T}} T$ implies $T' \leq T$, for some $T \in \mathcal{T}$. Therefore, we get $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$ and, hence, $\mathcal{I}$ is $c^\cup$-monotone. On the other hand,

$$\{\{1,2,,3,4,5\}\} \leq \{\{3,5\}\} \vee^{\mathcal{I}} \{\{4,5\}\},$$

whereas

$$\begin{array}{rcl}\Omega(\{\{1,2,3,4,5\}\}) & = & \{\{1,12345\}\} \\ & \nleq & \{\{0,12,345\}\} \\ & = & \{\{0,12,4,35\}\} \vee^{\mathbf{F}} \{\{0,12,3,45\}\} \\ & = & \Omega(\{\{3,5\}\}) \vee^{\mathbf{F}} \Omega(\{\{4,5\}\}).\end{array}$$

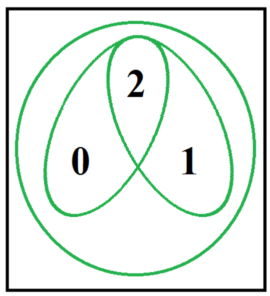
Thus, $\mathcal{I}$ is not $c^\vee$-monotone.

The second example exhibits a $c^\vee$-monotone π-institution which is not $c^\cup$-monotone.

Example 205 *Define the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *as follows:*

- $\mathbf{Sign}^\flat$ *is a trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the clone generated by the unary natural transformations* $\sigma^\flat, \tau^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, given by*

$x \in \mathrm{SEN}^\flat(\Sigma)$	$\sigma^\flat_\Sigma(x)$	$\tau^\flat_\Sigma(x)$
0	2	0
1	1	2
2	2	2

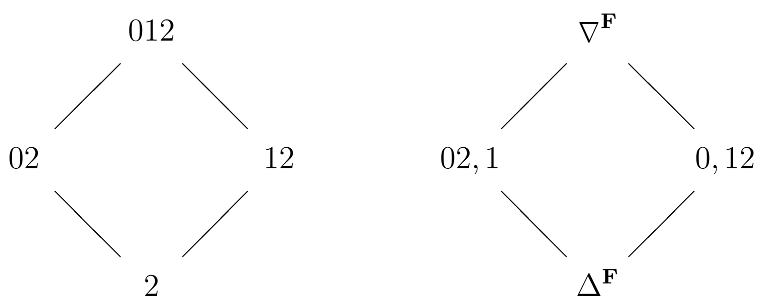


SEN(Σ)

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

$$\mathcal{C}_\Sigma = \{\{2\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

The lattice of theory families and the corresponding Leibniz congruence systems (in block form) are given in the following diagram.



Note that

$$\{\{0, 1, 2\}\} = \{\{0, 2\}\} \cup \{\{1, 2\}\} = \{\{0, 2\}\} \vee^{\mathcal{I}} \{\{1, 2\}\}.$$

But, even though

$$\begin{array}{rcl}\Omega(\{\{0, 1, 2\}\}) & = & \nabla^{\mathbf{F}} \\ & = & \{\{02, 1\}\} \vee^{\mathbf{F}} \{\{0, 12\}\} \\ & = & \Omega(\{\{0, 2\}\}) \vee^{\mathbf{F}} \Omega(\{\{1, 2\}\}),\end{array}$$

we have

$$\begin{array}{rcl}\Omega(\{\{0,1,2\}\}) & = & \nabla^{\mathbf{F}} \\ & \not\leq & \{\{02,1\}\} \cup \{\{0,12\}\} \\ & = & \Omega(\{\{0,2\}\}) \cup \Omega(\{\{1,2\}\}),\end{array}$$

since $\langle 0,1\rangle \in \nabla^{\mathbf{F}}_{\Sigma}$, *whereas* $\langle 0,1\rangle \notin \{\{02,1\}\} \cup \{\{0,12\}\}$. *Thus,* $\mathcal{I}$ *is* $c^{\vee}$*-monotone but not* $c^{\cup}$*-monotone.*

3.6 Injectivity

In this section we study classes of π-institutions defined using injectivity properties of the Leibniz operator.

Definition 206 (Injectivity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is called* **family injective** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,
$$\Omega(T) = \Omega(T') \quad \textit{implies} \quad T = T'.$$
- $\mathcal{I}$ *is called* **left injective** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,
$$\Omega(T) = \Omega(T') \quad \textit{implies} \quad \overleftarrow{T} = \overleftarrow{T'}.$$
- $\mathcal{I}$ *is called* **right injective** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,
$$\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'}) \quad \textit{implies} \quad T = T'.$$
- $\mathcal{I}$ *is called* **system injective** *if, for all* $T, T' \in \mathrm{ThSys}(\mathcal{I})$,
$$\Omega(T) = \Omega(T') \quad \textit{implies} \quad T = T'.$$

First, we show that right injectivity is so strong that it implies systemicity and, hence, a fortiori, stability.

Lemma 207 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is right injective, then it is systemic.*

Proof: Suppose that $\mathcal{I}$ is right injective and let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, we have, by Proposition 42, that $\overleftarrow{\overleftarrow{T}} = \overleftarrow{T}$. Therefore, we get $\Omega(\overleftarrow{\overleftarrow{T}}) = \Omega(\overleftarrow{T})$. Hence, by right injectivity, we get that $\overleftarrow{T} = T$. Thus, $T \in \mathrm{ThSys}(\mathcal{I})$. This proves that $\mathrm{ThFam}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$, whence $\mathcal{I}$ is systemic. ∎

Next we look into establishing the *injectivity hierarchy* of π-institutions. The following relationships can be established between the four injectivity classes.

Proposition 208 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is right injective, then it is family injective;*

(b) If $\mathcal{I}$ *is family injective, then it is left injective;*

(c) If $\mathcal{I}$ *is left injective, then it is system injective.*

Proof:

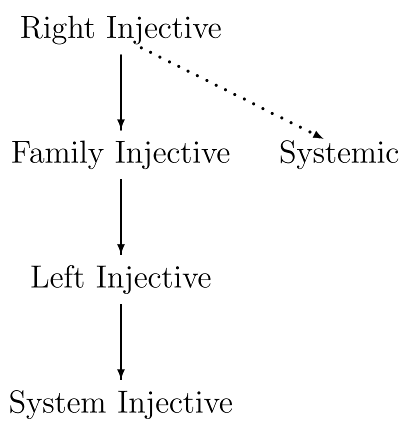
(a) Suppose that $\mathcal{I}$ is right injective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. By Lemma 207, $\mathcal{I}$ is systemic, whence $\overleftarrow{T} = T$ and $\overleftarrow{T'} = T'$. Thus, we get $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$. Now applying right injectivity, we get $T = T'$. This proves that $\mathcal{I}$ is family injective.

(b) Suppose that $\mathcal{I}$ is family injective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, by family injectivity, we get $T = T'$, whence $\overleftarrow{T} = \overleftarrow{T'}$. Therefore $\mathcal{I}$ is left injective.

(c) Suppose that $\mathcal{I}$ is left injective and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. By left injectivity, we conclude that $\overleftarrow{T} = \overleftarrow{T'}$. However, since T, T' are theory systems, we have $\overleftarrow{T} = T$ and $\overleftarrow{T'} = T'$. Hence we get $T = T'$ and $\mathcal{I}$ is system injective.

■

We have now established the following **injectivity hierarchy** of π-institutions.



There are two additional properties that can be formulated concerning the relationships between these classes. First, it turns out that the separating property between system injectivity and right injectivity is exactly systemicity.

Proposition 209 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is right injective if and only if it is system injective and systemic.*

Proof: Suppose, first, that $\mathcal{I}$ is right injective. Then, by Lemma 207, it is systemic and by Proposition 208 it is system injective.

Suppose conversely, that $\mathcal{I}$ is system injective and systemic and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$. By system injectivity we get $\overleftarrow{T} = \overleftarrow{T'}$. Hence, by systemicity, $T = T'$. Thus, $\mathcal{I}$ is right injective. ∎

Second, we show that system injectivity reinforced with stability implies left injectivity.

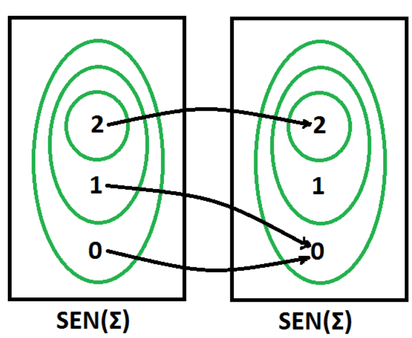
Proposition 210 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is system injective and stable, then it is left injective.*

Proof: Suppose that $\mathcal{I}$ is system injective and stable and consider $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, by stability $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$. Hence, since $\overleftarrow{T}, \overleftarrow{T'} \in \mathrm{ThSys}(\mathcal{I})$, by system injectivity, $\overleftarrow{T} = \overleftarrow{T'}$. This shows that $\mathcal{I}$ is left injective. ∎

We now present three examples to show that all inclusions established between injectivity classes and depicted in the diagram above are proper inclusions. The first example will show that the class of right injective π-institutions is a proper subclass of the class of family injective π-institutions.

Example 211 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial clone.*



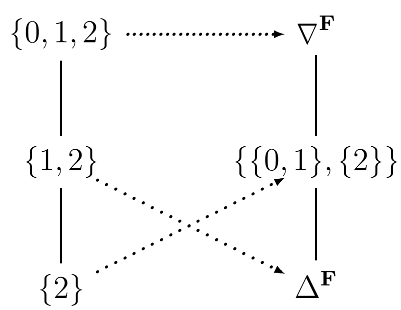
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}$.

The following table gives the theory families and the theory systems of the π-institution $\mathcal{I}$:

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

Since $\{\{1,2\}\}$ is a theory family that is not a theory system, $\mathcal{I}$ is not systemic.

The lattice of theory families and that of the corresponding Leibniz congruence systems are depicted below:



It is obvious from the diagram that $\mathcal{I}$ is family injective, since each of the three theory families has a different Leibniz congruence system.

On the other hand, $\mathcal{I}$ is not right injective. This can be seen either by applying Proposition 209 or directly. Take $T = \{\{2\}\}$ and $T' = \{\{1,2\}\}$. Then, we have $\overleftarrow{T} = \overleftarrow{T'} = T$, whence $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$, whereas, obviously, $T \neq T'$.

The second example shows that the class of family injective π-institutions is properly included in the class of left injective π-institutions.

Example 212 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

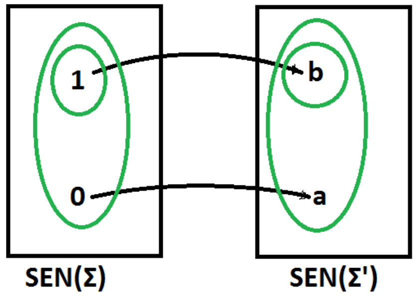
- $\mathbf{Sign}^\flat$ *is the category with objects Σ and Σ' and a unique (non-identity) morphism $f : \Sigma \to \Sigma'$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$*;*
- $N^\flat$ *is the trivial clone.*

Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

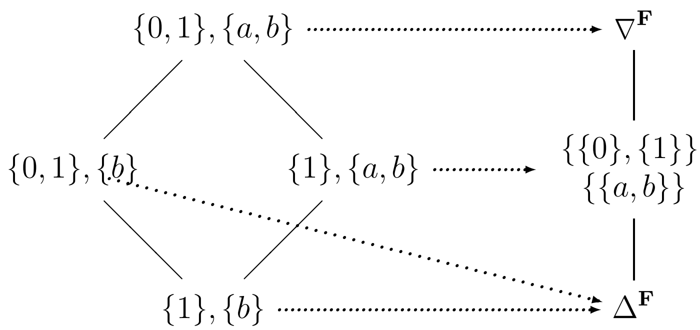
$$\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\},\{a,b\}\}.$$

The following table shows the action of $\overleftarrow{\ }$ on theory families, where rows correspond to T_Σ and columns to $T_{\Sigma'}$ and each entry is written as $\overleftarrow{T}_\Sigma, \overleftarrow{T}_{\Sigma'}$.

$\leftarrow$	$\{b\}$	$\{a,b\}$
$\{1\}$	$\{1\},\{b\}$	$\{1\},\{a,b\}$
$\{0,1\}$	$\{1\},\{b\}$	$\{0,1\},\{a,b\}$



The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right:



Since the only two theory families that have the same Leibniz congruence system are $\{\{0,1\},\{b\}\}$ *and* $\{\{1\},\{b\}\}$ *and it holds that*

$$\overleftarrow{\{\{0,1\},\{b\}\}} = \overleftarrow{\{\{1\},\{b\}\}} = \{\{1\},\{b\}\},$$

we conclude that $\mathcal{I}$ *is left injective.*

From the diagram, it is also clear that $\mathcal{I}$ *is not family injective, since the two theory families* $\{\{0,1\},\{b\}\}$ *and* $\{\{1\},\{b\}\}$ *have the same Leibniz congruence system.*

In conclusion we have shown that $\mathcal{I}$ *is left injective but not family injective.*

The next example shows that the class of left injective π-institutions is properly included in the class of all system injective π-institutions.

Example 213 *Define the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *as follows:*

- $\mathbf{Sign}^\flat$ *is the two object category with objects* Σ, Σ' *and two (non-identity) morphisms*

$$\Sigma \underset{g}{\overset{f}{\rightleftarrows}} \Sigma'$$

  such that $g \circ f = i_\Sigma$ *and* $f \circ g = i_{\Sigma'}$*;*

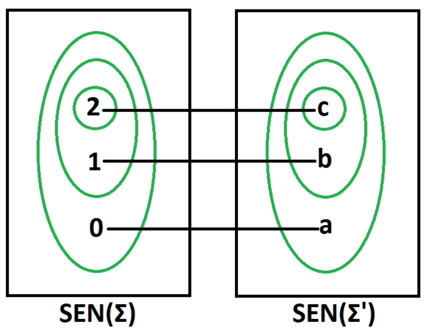
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b,c\}$ *and*

$$\begin{array}{lll}\mathrm{SEN}^\flat(f)(0) = a, & \mathrm{SEN}^\flat(f)(1) = b, & \mathrm{SEN}^\flat(f)(2) = c;\\ \mathrm{SEN}^\flat(g)(a) = 0, & \mathrm{SEN}^\flat(g)(b) = 1, & \mathrm{SEN}^\flat(g)(c) = 2;\end{array}$$

- $N^\flat$ *is the clone on* $\mathrm{SEN}^\flat$ *generated by the natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *specified by*

$x \in \mathrm{SEN}^\flat(\Sigma)$	$\sigma^\flat_\Sigma(x)$
0	0
1	1
2	0

$y \in \mathrm{SEN}^\flat(\Sigma')$	$\sigma^\flat_{\Sigma'}(y)$
a	a
b	b
c	a



Next, define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by setting*

$$\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\{c\},\{b,c\},\{a,b,c\}\}.$$

The table giving the action of $\overleftarrow{}$ *on theory families is shown below:*

$\leftarrow$	$\{c\}$	$\{b,c\}$	$\{a,b,c\}$
$\{2\}$	$\{2\},\{c\}$	$\{2\},\{c\}$	$\{2\},\{c\}$
$\{1,2\}$	$\{2\},\{c\}$	$\{1,2\},\{b,c\}$	$\{1,2\},\{b,c\}$
$\{0,1,2\}$	$\{2\},\{c\}$	$\{1,2\},\{b,c\}$	$\{0,1,2\},\{a,b,c\}$

The following table gives the Leibniz congruence systems associated with each of the nine theory families of $\mathcal{I}$*, where we have denoted by* θ *the Leibniz congruence system with* $\theta_\Sigma = \{\{0,1\},\{2\}\}$ *and* $\theta_{\Sigma'} = \{\{a,b\},\{c\}\}$*:*

$\Omega(\{T_\Sigma, T_{\Sigma'}\})$	$\{c\}$	$\{b,c\}$	$\{a,b,c\}$
$\{2\}$	θ	$\Delta^{\mathbf{F}}$	θ
$\{1,2\}$	$\Delta^{\mathbf{F}}$	$\Delta^{\mathbf{F}}$	$\Delta^{\mathbf{F}}$
$\{0,1,2\}$	θ	$\Delta^{\mathbf{F}}$	$\nabla^{\mathbf{F}}$

Observe, first, that there are only three theory systems $\{\{2\},\{c\}\}$, $\{\{1,2\},\{b,c\}\}$ *and* $\{\{0,1,2\},\{a,b,c\}\}$. *To each of these corresponds a different Leibniz congruence system. It follows that* $\mathcal{I}$ *is system injective.*

On the other hand, for $T = \{\{2\},\{b,c\}\}$ *and* $T' = \{\{1,2\},\{b,c\}\}$, *we have* $\Omega(T) = \Omega(T') = \Delta^{\mathbf{F}}$, *but* $\overleftarrow{T} = \{\{2\},\{c\}\} \neq \{\{1,2\},\{b,c\}\} = \overleftarrow{T'}$. *Therefore* $\mathcal{I}$ *is not left injective.*

The injectivity properties transfer from the theory families/systems of a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ to the collections of all $\mathcal{I}$-filter families/systems on arbitrary $\mathbf{F}$-algebraic systems.

Theorem 214 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$.

- *(a)* $\mathcal{I}$ *is family injective if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$ *implies* $T = T'$.
- *(b)* $\mathcal{I}$ *is left injective if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$ *implies* $\overleftarrow{T} = \overleftarrow{T'}$.
- *(c)* $\mathcal{I}$ *is right injective if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(\overleftarrow{T}) = \Omega^{\mathcal{A}}(\overleftarrow{T'})$ *implies* $T = T'$.
- *(d)* $\mathcal{I}$ *is system injective if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$ *implies* $T = T'$.

Proof: We will prove Parts (a) and (b) to establish the method.

- (a) The "if" follows by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I,\iota\rangle\rangle$ and taking into account that $\mathrm{ThFam}(\mathcal{I}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F})$, by Lemma 51.

  For the "only if", suppose that $\mathcal{I}$ is family injective and let $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$. Then $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) = \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. So, by Proposition 24, $\Omega(\alpha^{-1}(T)) = \Omega(\alpha^{-1}(T'))$. Since, by Lemma 51, we have that $\alpha^{-1}(T), \alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$, we get, by applying family injectivity, $\alpha^{-1}(T) = \alpha^{-1}(T')$. Finally, the surjectivity of $\langle F,\alpha\rangle$ yields that $T = T'$.

- (b) The "if" follows as in Part (a).

  For the "only if", suppose that $\mathcal{I}$ is left injective and let $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$. Then $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) = \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. So, by Proposition 24, $\Omega(\alpha^{-1}(T)) = \Omega(\alpha^{-1}(T'))$. Since, by Lemma 51, we have that $\alpha^{-1}(T), \alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$, we get, by applying left injectivity, $\overleftarrow{\alpha^{-1}(T)} = \overleftarrow{\alpha^{-1}(T')}$. Thus, by Lemma 6, $\alpha^{-1}(\overleftarrow{T}) = \alpha^{-1}(\overleftarrow{T'})$. Finally, the surjectivity of $\langle F,\alpha\rangle$ yields that $\overleftarrow{T} = \overleftarrow{T'}$.

■

Finally, we may recast the injectivity classes in terms of the injectivity of mappings from posets of theory or filter families/systems into posets of congruence systems.

Given two posets $\mathbf{P} = \langle P, \leq \rangle$ and $\mathbf{Q} = \langle Q, \preccurlyeq \rangle$, we call a mapping $f : P \to Q$ **injective** if it is injective as a set map, i.e., if, for all $p_1, p_2 \in P$,

$$f(p_1) = f(p_2) \quad \text{implies} \quad p_1 = p_2.$$

Proposition 215 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is family injective;*

(b) $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is injective;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is injective, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

Similarly, for system injectivity, we have

Proposition 216 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is system injective;*

(b) $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is injective;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is injective, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

3.7 Reflectivity

In this section we look at classes of π-institutions defined using the order reflectivity of the Leibniz operator.

Definition 217 (Reflectivity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is* **family reflective** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\Omega(T) \leq \Omega(T') \quad \text{implies} \quad T \leq T'.$$

- $\mathcal{I}$ *is* **left reflective** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\Omega(T) \leq \Omega(T') \quad \text{implies} \quad \overleftarrow{T} \leq \overleftarrow{T'}.$$

- $\mathcal{I}$ *is* **right reflective** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'}) \quad \textit{implies} \quad T \le T'.$$

- $\mathcal{I}$ *is* **system reflective** *if, for all* $T, T' \in \mathrm{ThSys}(\mathcal{I})$*,*

$$\Omega(T) \le \Omega(T') \quad \textit{implies} \quad T \le T'.$$

We first establish the fact that both family and right reflectivity imply systemicity.

Lemma 218 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is family reflective, then it is systemic;*

(b) If $\mathcal{I}$ *is right reflective, then it is systemic.*

Proof:

(a) Suppose $\mathcal{I}$ is family reflective and let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, we have, by Proposition 20, $\Omega(T) \le \Omega(\overleftarrow{T})$. Applying family reflectivity, we get $T \le \overleftarrow{T}$. Since, by Proposition 42, it always holds that $\overleftarrow{T} \le T$, we get $\overleftarrow{T} = T$. This shows that $\mathrm{ThFam}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$ and, thus, $\mathcal{I}$ is systemic.

(b) Suppose $\mathcal{I}$ is right reflective and let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, we have, by Proposition 42, $\overleftarrow{\overleftarrow{T}} = \overleftarrow{T}$. Thus, we get $\Omega(\overleftarrow{\overleftarrow{T}}) = \Omega(\overleftarrow{T})$ and, hence $\Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{\overleftarrow{T}})$. Applying right reflectivity, we get $T \le \overleftarrow{T}$. Since, by Proposition 42, it always holds that $\overleftarrow{T} \le T$, we get $\overleftarrow{T} = T$. This shows that $\mathrm{ThFam}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$ and, thus, $\mathcal{I}$ is systemic.

■

Lemma 218 enables us to prove that family reflectivity and right reflectivity are actually equivalent properties.

Proposition 219 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is family reflective if and only if it is right reflective.*

Proof: Suppose, first, that $\mathcal{I}$ is family reflective. Then, by Lemma 218, it is systemic, and, by Proposition 152, it is stable. To see that it is right reflective, let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'})$. Then, by stability, $\Omega(T) \le \Omega(T')$. Hence, by family reflectivity, $T \le T'$. Thus, $\mathcal{I}$ is right reflective.

Suppose, conversely, that $\mathcal{I}$ is right reflective. Then, by Lemma 218, it is systemic, and, by Proposition 152, it is stable. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then by stability, we get $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Now we use right reflectivity to get $T \leq T'$. Thus, $\mathcal{I}$ is family reflective. ■

Now we establish several relationships among the three reflectivity classes. First, we show that, if a π-institution is family reflective, then it is left reflective.

Proposition 220 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is family reflective, then it is left reflective.*

Proof: Suppose $\mathcal{I}$ if family reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. By family reflectivity, $T \leq T'$. But, by Lemma 218, $\mathcal{I}$ is systemic, whence we get $\overleftarrow{T} \leq \overleftarrow{T'}$ and, hence, $\mathcal{I}$ is left reflective. ■

Next, we show that left reflectivity implies system reflectivity and, moreover system reflectivity supplied with stability implies left reflectivity.

Proposition 221 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is left reflective, then it is system reflective;*

(b) If $\mathcal{I}$ *is system reflective and stable, then it is left reflective.*

Proof:

(a) Suppose that $\mathcal{I}$ is left reflective and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then, by left reflectivity, $\overleftarrow{T} \leq \overleftarrow{T'}$. But, as T, T' are theory systems, we have $\overleftarrow{T} = T$ and $\overleftarrow{T'} = T'$, whence $T \leq T'$ and $\mathcal{I}$ is system reflective.

(b) Suppose that $\mathcal{I}$ is system reflective and stable. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then, by stability, $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Since $\overleftarrow{T}, \overleftarrow{T'} \in \mathrm{ThSys}(\mathcal{I})$, we apply system reflectivity to get $\overleftarrow{T} \leq \overleftarrow{T'}$. Thus, $\mathcal{I}$ is left reflective.

■

Finally, we show that systemicity is exactly the separating property between family reflectivity and system reflectivity.

Proposition 222 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is family reflective if and only if it is system reflective and systemic.*

Proof: Suppose, first, that $\mathcal{I}$ is family reflective. Then, by Lemma 218, it is systemic. Moreover, by hypothesis, for all $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$, we get, $T \leq T'$. So $\mathcal{I}$ is also system reflective.

Suppose, conversely, that $\mathcal{I}$ is system reflective and systemic and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. By systemicity, $T, T' \in \mathrm{ThSys}(\mathcal{I})$, whence, by hypothesis, we get $T \leq T'$. Thus, $\mathcal{I}$ is family reflective. ∎

We have now established the following **reflectivity hierarchy**:

Family/Right Reflective

$\downarrow$

Left Reflective

$\downarrow$

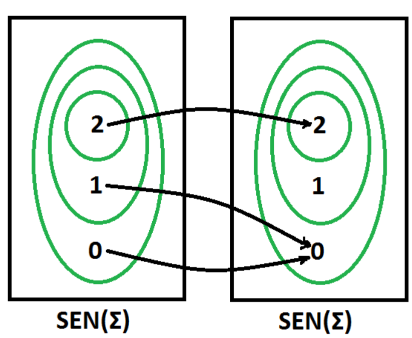
System Reflective

We present now some examples to show that the inclusions shown in the diagram are indeed proper inclusions.

The first example showcases a π-institution which is left reflective, but not systemic, and, hence, according to Proposition 220, not family reflective.

Example 223 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial clone.*



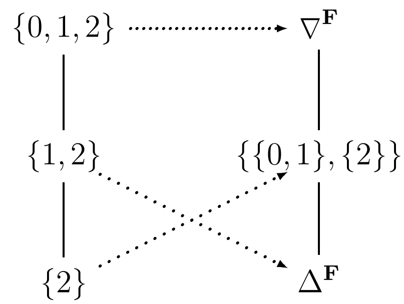
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}$.

The following table gives the theory families and the theory systems of the π-institution $\mathcal{I}$:

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

Since $\{\{1,2\}\}$ is a theory family that is not a theory system, $\mathcal{I}$ is not systemic.

The lattice of theory families and the corresponding Leibniz congruence systems are depicted below:



By the diagram, keeping in mind that $\overleftarrow{\{\{1,2\}\}} = \{\{2\}\}$, one can see that $\mathcal{I}$ is left reflective. But, clearly, it is not family reflective, as $\Omega(\{\{1,2\}\}) \leq \Omega(\{\{2\}\})$, whereas, obviously, $\{\{1,2\}\} \nleq \{\{2\}\}$.

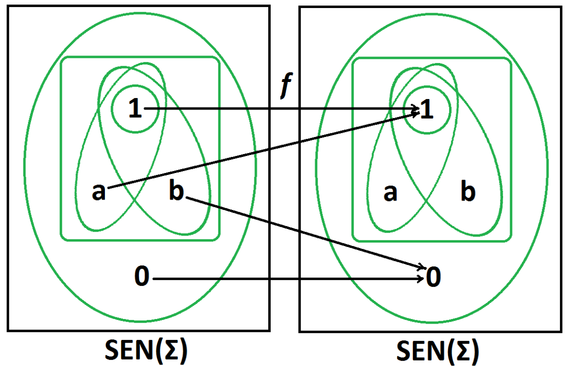
The second example presents a π-institution which is system reflective, but fails to be left reflective and, hence, by Proposition 221, is not stable.

Example 224 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object Σ and a single non-identity morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, a, b, 1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(a) = 1$, $\mathrm{SEN}^\flat(f)(b) = 0$ *and* $\mathrm{SEN}^\flat(f)(1) = 1$*;*
- $N^\flat$ *is the category of natural transformations generated by the two binary natural transformations* $\wedge, \vee : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by the following tables:*

$\wedge_\Sigma$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee_\Sigma$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1



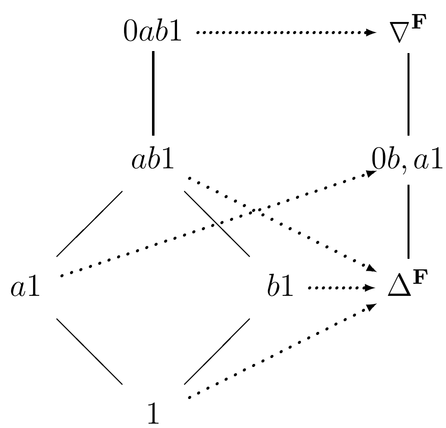
Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution defined by setting*

$$\mathcal{C}_\Sigma = \{\{1\}, \{a,1\}, \{b,1\}, \{a,b,1\}, \{0,a,b,1\}\}.$$

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$*:*

T	$\overleftarrow{T}$
$\{1\}$	$\{1\}$
$\{a,1\}$	$\{a,1\}$
$\{b,1\}$	$\{1\}$
$\{a,b,1\}$	$\{a,1\}$
$\{0,a,b,1\}$	$\{0,a,b,1\}$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



Since $\Omega(\overleftarrow{\{\{a,b,1\}\}}) = \Omega(\{\{a,1\}\}) = \{\{0,b\},\{a,1\}\} \neq \Delta^{\mathbf{F}} = \Omega(\{\{a,b,1\}\})$*, we conclude that* $\mathcal{I}$ *is not stable.*

Since $\{\{1\}\}$*,* $\{\{a,1\}\}$ *and* $\mathrm{SEN}^\flat$ *are the only theory systems of* $\mathcal{I}$*, it is clear from the diagram above that* $\mathcal{I}$ *is system reflective. On the other hand, we have*

$$\Omega(\{\{a,b,1\}\}) = \Delta^{\mathbf{F}} = \Omega(\{\{b,1\}\}),$$

but

$$\overleftarrow{\{\{a,b,1\}\}} = \{\{a,1\}\} \nleq \{\{1\}\} = \overleftarrow{\{\{b,1\}\}},$$

whence, $\mathcal{I}$ is not left reflective.

We turn now to transfer theorems regarding the reflectivity properties studied in this section.

Theorem 225 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is family reflective if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ implies $T \leq T'$.

(b) $\mathcal{I}$ is left reflective if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T, T' \in \mathrm{FiFam}^{\mathcal{A}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ implies $\overleftarrow{T} \leq \overleftarrow{T'}$.

(c) $\mathcal{I}$ is system reflective if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T, T' \in \mathrm{FiSys}^{\mathcal{A}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ implies $T \leq T'$.

Proof: We prove Part (b).

The "if" is obtained by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account that $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}) = \mathrm{ThFam}(\mathcal{I})$, by Lemma 51.

For the "only if" suppose that $\mathcal{I}$ is left reflective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. Apply the inverse of $\langle F, \alpha \rangle$ to get $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. Thus, by Proposition 24, we get $\Omega(\alpha^{-1}(T)) \leq \Omega(\alpha^{-1}(T'))$. Take into account the fact that, by Lemma 51, $\alpha^{-1}(T), \alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$ and apply left reflectivity to get $\overleftarrow{\alpha^{-1}(T)} \leq \overleftarrow{\alpha^{-1}(T')}$. Hence, by Lemma 6, we get $\alpha^{-1}(\overleftarrow{T}) \leq \alpha^{-1}(\overleftarrow{T'})$. Finally, by the surjectivity of $\langle F, \alpha \rangle$, we conclude that $\overleftarrow{T} \leq \overleftarrow{T'}$. ■

Turning to characterizations in terms of mappings between posets, we get the following characterizations of family and system reflectivity in terms of the order reflectivity of mappings from posets of theory or filter families/systems into posets of congruence systems. Given posets $\mathbf{P} = \langle P, \leq \rangle$ and $\mathbf{Q} = \langle Q, \preccurlyeq \rangle$, call a mapping $f : P \to Q$ **order reflecting** if, for all $p_1, p_2 \in P$,

$$f(p_1) \preccurlyeq f(p_2) \quad \text{implies} \quad p_1 \leq p_2.$$

Proposition 226 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The following statements are equivalent:*

(a) $\mathcal{I}$ is family reflective;

(b) $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is order reflecting;

(c) $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is order reflecting, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

Similarly, for system reflectivity, we have

Proposition 227 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is system reflective;*

(b) $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is order reflecting;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is order reflecting, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

We continue our studies of reflectivity properties by looking at the relationships governing classes defined using reflectivity with corresponding classes defined using injectivity. We have the following result:

Proposition 228 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is family/right reflective, then it is right injective;*

(b) If $\mathcal{I}$ *is left reflective, then it is left injective;*

(c) If $\mathcal{I}$ *is system reflective, then it is system injective.*

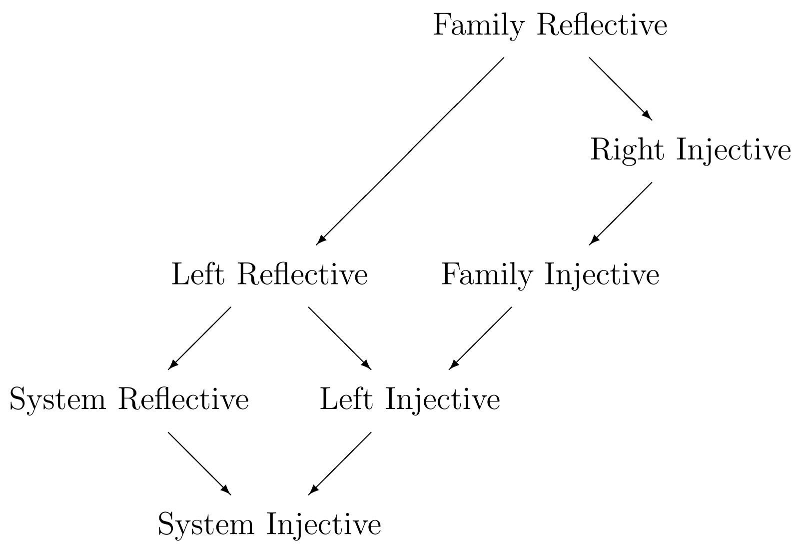
Proof:

(a) Suppose $\mathcal{I}$ is right reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$. Then, a fortiori, $\Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'})$ and $\Omega(\overleftarrow{T'}) \le \Omega(\overleftarrow{T})$. Thus, by right reflectivity, $T \le T'$ and $T' \le T$. It follows that $T = T'$. Therefore, $\mathcal{I}$ is right injective.

(b),(c) Very similar to Part (a).

■

Proposition 228 has established the following combined hierarchy of injectivity and reflectivity properties.

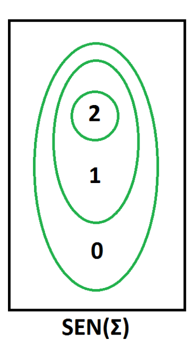


Now we turn to an example that will show that all three fresh inclusions depicted in the diagram, i.e., those established in Proposition 228, are actually proper inclusions.

Example 229 *Define the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor determined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the clone generated by the natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, defined by the following table:*

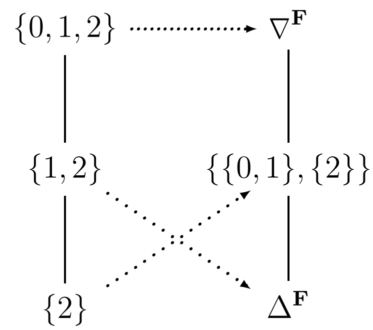
$x \in \mathrm{SEN}^\flat(\Sigma)$	$\sigma^\flat_\Sigma(x)$
0	0
1	0
2	2



The π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is defined by setting

$$\mathcal{C}_\Sigma = \{\{2\}, \{1,2\}, \{0,1,2\}\}.$$

Note that $\mathcal{I}$ is systemic, since the category $\mathbf{Sign}^\flat$ is trivial. The lattice of theory families and that of the corresponding Leibniz congruence systems are shown in the diagram.



Since all three theory families/systems have distinct Leibniz congruence systems, $\mathcal{I}$ is right injective.

On the other hand, $\Omega(\{\{1,2\}\}) \leq \Omega(\{\{2\}\})$, whereas, obviously, $\{\{1,2\}\} \nleq \{\{2\}\}$, whence $\mathcal{I}$ is not system reflective.

We close this section on reflectivity by looking at the relationships between the various reflectivity classes and the classes in the loyalty hierarchy.

Proposition 230 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) If $\mathcal{I}$ is family reflective, then it is family loyal;

(b) If $\mathcal{I}$ is left reflective, then it is left loyal;

(c) If $\mathcal{I}$ is system reflective, then it is system loyal.

Proof:

(a) Suppose that $\mathcal{I}$ is family reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) > \Omega(T')$. Then, a fortiori, $\Omega(T) \geq \Omega(T')$. Therefore, by family reflectivity, $T \geq T'$ and, hence $T \nless T'$. We conclude that $\mathcal{I}$ is family loyal.

(b) Suppose that $\mathcal{I}$ is left reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) > \Omega(T')$. Then, a fortiori, $\Omega(T) \geq \Omega(T')$. Hence, by left reflectivity, $\overleftarrow{T} \geq \overleftarrow{T'}$. But this implies that $\overleftarrow{T} \nless \overleftarrow{T'}$. Therefore, $\mathcal{I}$ is left loyal.

(c) Very similar to Parts (a) and (b).

■

Proposition 230 establishes the following combined hierarchy of reflectivity and loyalty classes of π-institutions.

Family/Right Reflective

Family Loyal Left Reflective

Left Loyal System Reflective

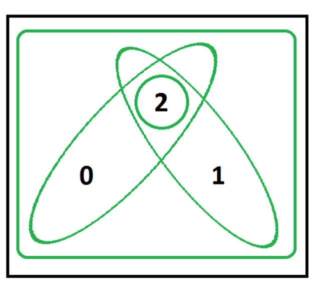
System/Right Loyal

We reinforce the picture by constructing a π-institution that is family loyal but fails to be system reflective. This demonstrates that all three inclusions established in Proposition 230 are proper.

Example 231 *Define the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category, with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor determined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the clone of natural transformations generated by the three unary natural transformations* $\rho^\flat, \sigma^\flat, \tau^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, defined by the following table:*

$x \in \mathrm{SEN}^\flat(\Sigma)$	$\rho^\flat_\Sigma(x)$	$\sigma^\flat_\Sigma(x)$	$\tau^\flat_\Sigma(x)$
0	0	0	1
1	2	0	1
2	2	2	2



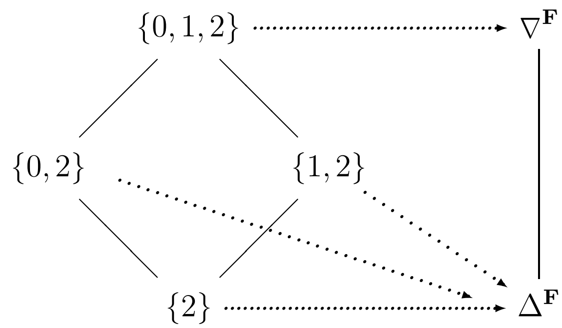
SEN(Σ)

The π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is defined by setting

$$\mathcal{C}_\Sigma = \{\{2\},\{0,2\},\{1,2\},\{0,1,2\}\}.$$

Note that, since $\mathbf{Sign}^\flat$ is trivial, all theory families are theory systems.

The lattice of theory families and the corresponding Leibniz congruence systems are given in the following diagram.



From the diagram it is clear that there are no theory families T, T', such that $T < T'$ and $\Omega(T) > \Omega(T')$. Thus, $\mathcal{I}$ is family loyal.

On the other hand, setting $T = \{\{0,2\}\}$ and $T' = \{\{1,2\}\}$, we have $\Omega(T) \leq \Omega(T')$, whereas, obviously, $T \nleq T'$. Therefore $\mathcal{I}$ is not system reflective.

3.8 Complete Reflectivity

In this section we define the classes arising by imposing the various versions of the property of complete order reflectivity.

Definition 232 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.*

- *$\mathcal{I}$ is* **family completely reflective** *if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,*

$$\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T') \quad \textit{implies} \quad \bigcap_{T\in\mathcal{T}} T \leq T'.$$

- *$\mathcal{I}$ is* **left completely reflective** *if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,*

$$\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T') \quad \textit{implies} \quad \bigcap_{T\in\mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}.$$

- *$\mathcal{I}$ is* **right completely reflective** *if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,*

$$\bigcap_{T\in\mathcal{T}} \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'}) \quad \textit{implies} \quad \bigcap_{T\in\mathcal{T}} T \leq T'.$$

- $\mathcal{I}$ *is* **system completely reflective** *if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$,

$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \quad \textit{implies} \quad \bigcap_{T \in \mathcal{T}} T \leq T'.$$

For the sake of brevity, we sometimes shorten "complete reflectivity" to **c-reflectivity**.

Lemma 218 allows us to obtain easily the fact that both family c-reflectivity and right c-reflectivity imply systemicity.

Lemma 233 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ is family completely reflective, then it is systemic;

(b) If $\mathcal{I}$ is right completely reflective, then it is systemic.

Proof: Note that, if $\mathcal{I}$ is family completely reflective, then it is family reflective and that, if $\mathcal{I}$ is right completely reflective, then it is right reflective. Therefore, the conclusion is obtained by applying Lemma 218. ■

Now we show in an analog of Proposition 219 for complete reflectivity that family complete reflectivity and right complete reflectivity are equivalent properties.

Proposition 234 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *$\mathcal{I}$ is family completely reflective if and only if it is right completely reflective.*

Proof: Suppose, first, that $\mathcal{I}$ is family completely reflective and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. By Lemma 233, $\mathcal{I}$ is systemic, whence we get $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. This allows us to apply family c-reflectivity to obtain $\bigcap \mathcal{T} \leq T'$. Hence $\mathcal{I}$ is right completely reflective.

Suppose, conversely, that $\mathcal{I}$ is right completely reflective and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. By Lemma 233, $\mathcal{I}$ is systemic, whence, we get $\bigcap_{T \in \mathcal{T}} \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Applying right c-reflectivity, we get that $\bigcap \mathcal{T} \leq T'$. Therefore, $\mathcal{I}$ is family completely reflective. ■

We now look at the relationships that govern the three complete reflectivity classes, which also parallel the ones established in Propositions 220, 221 and 222 for the various classes defined using reflectivity.

We show, first, that family complete reflectivity implies left complete reflectivity.

Proposition 235 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *If $\mathcal{I}$ is family completely reflective then it is left completely reflective.*

Proof: Suppose that $\mathcal{I}$ is family completely reflective and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. By family complete reflectivity, we get $\bigcap_{T\in\mathcal{T}} T \leq T'$, whence, by Lemma 233, $\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$. Thus, $\mathcal{I}$ is left completely reflective. ■

Next, we show that left complete reflectivity implies system complete reflectivity and, moreover, system complete reflectivity, combined with stability, implies left complete reflectivity.

Proposition 236 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is left completely reflective, then it is system completely reflective;*

(b) If $\mathcal{I}$ *is system completely reflective and stable, then it is left completely reflective.*

Proof:

(a) Suppose that $\mathcal{I}$ is left c-reflective and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. Then, by left c-reflectivity, $\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$. But, as $\mathcal{T} \cup \{T'\}$ consists of theory systems, we get $\bigcap \mathcal{T} \leq T'$. So $\mathcal{I}$ is system c-reflective.

(b) Suppose that $\mathcal{I}$ is system c-reflective and stable. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. Then, by stability, we get $\bigcap_{T\in\mathcal{T}} \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Since $\{\overleftarrow{T} : T \in \mathcal{T}\} \cup \{\overleftarrow{T'}\} \in \mathrm{ThSys}(\mathcal{I})$, we apply system c-reflectivity to get $\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$. Thus, $\mathcal{I}$ is left c-reflective.

■

Finally, it is not difficult to see that family complete reflectivity is tantamount to system complete reflectivity plus systemicity.

Proposition 237 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is family completely reflective if and only if it is system completely reflective and systemic.*

Proof: Suppose, first, that $\mathcal{I}$ is family completely reflective. Then, by Lemma 233, it is systemic. Moreover, by Propositions 235 and 236, it is system completely reflective.

Suppose, conversely, that $\mathcal{I}$ is system completely reflective and systemic and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. Since, by systemicity, we get $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, we get, by system complete reflectivity, $\bigcap_{T\in\mathcal{T}} T \leq T'$. Thus, $\mathcal{I}$ is family completely reflective. ■

We have now established the following **complete reflectivity hierarchy**:

Family/Right c-Reflective

↓

Left c-Reflective

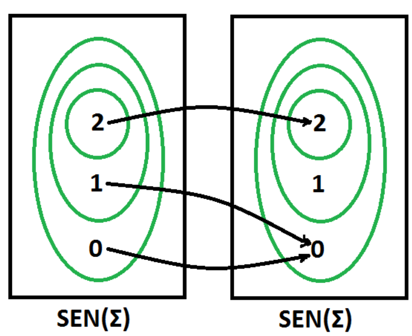
↓

System c-Reflective

We now provide two examples to show that the inclusions depicted in the diagram between the three complete reflectivity classes introduced in this section are proper. The first example presents a π-institution which is left completely reflective, but not systemic, and, hence, according to Proposition 237, not family completely reflective.

Example 238 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial clone.*



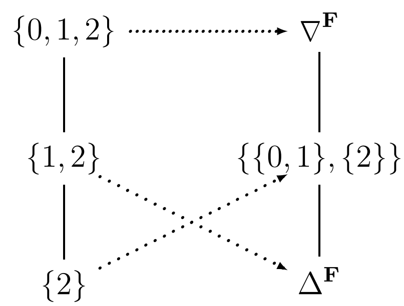
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}$.

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$*:*

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1, 2\}$	$\{2\}$
$\{0, 1, 2\}$	$\{0, 1, 2\}$

Since $\{\{1,2\}\}$ is a theory family that is not a theory system, $\mathcal{I}$ is not systemic.

The lattice of theory families and that of the corresponding Leibniz congruence systems are depicted below:



By the diagram, keeping in mind that $\overleftarrow{\{\{1,2\}\}} = \{\{2\}\}$, one can see that $\mathcal{I}$ is left c-reflective. But, clearly, it is not family c-reflective, as $\Omega(\{\{1,2\}\}) \le \Omega(\{\{2\}\})$, whereas, obviously, $\{\{1,2\}\} \nleq \{\{2\}\}$, giving, as we have already seen in Example 223, that $\mathcal{I}$ is not even family reflective.

The second example presents a π-institution which is system c-reflective, but fails to be left c-reflective and, hence, by Proposition 236, is not stable.

Example 239 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

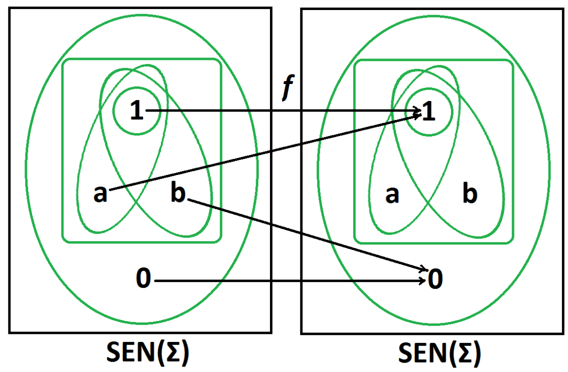
- *$\mathbf{Sign}^\flat$ is the category with a single object Σ and a single non-identity morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- *$\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ is the functor specified by $\mathrm{SEN}^\flat(\Sigma) = \{0, a, b, 1\}$ and $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(a) = 1$, $\mathrm{SEN}^\flat(f)(b) = 0$ and $\mathrm{SEN}^\flat(f)(1) = 1$;*
- *$N^\flat$ is the category of natural transformations generated by the two binary natural transformations $\wedge, \vee : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ defined by the following tables:*

$\wedge_\Sigma$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee_\Sigma$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be the π-institution defined by setting

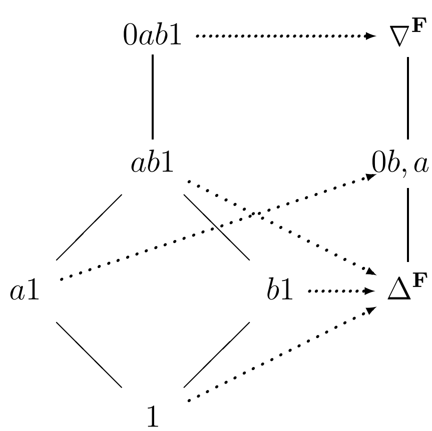
$$\mathcal{C}_\Sigma = \{\{1\}, \{a,1\}, \{b,1\}, \{a,b,1\}, \{0,a,b,1\}\}.$$



The following table gives the theory families and the theory systems of the π-institution $\mathcal{I}$:

T	$\overleftarrow{T}$
$\{1\}$	$\{1\}$
$\{a,1\}$	$\{a,1\}$
$\{b,1\}$	$\{1\}$
$\{a,b,1\}$	$\{a,1\}$
$\{0,a,b,1\}$	$\{0,a,b,1\}$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



Since $\Omega(\overleftarrow{\{\{a,b,1\}\}}) = \Omega(\{\{a,1\}\}) = \{\{0,b\},\{a,1\}\} \neq \Delta^{\mathbf{F}} = \Omega(\{\{a,b,1\}\})$, *we conclude that $\mathcal{I}$ is not stable.*

Note that, since $\{\{1\}\}$, $\{\{a,1\}\}$ *and* $\mathrm{SEN}^\flat$ *are the only theory systems of* $\mathcal{I}$, *the Leibniz operator* $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is an order isomorphism. Hence, $\mathcal{I}$ is system completely reflective. On the other hand, we have*

$$\Omega(\{\{a,b,1\}\}) = \Delta^{\mathbf{F}} = \Omega(\{\{b,1\}\}),$$

but

$$\overleftarrow{\{\{a,b,1\}\}} = \{\{a,1\}\} \nleq \{\{1\}\} = \overleftarrow{\{\{b,1\}\}},$$

whence, $\mathcal{I}$ is not left reflective and, a fortiori, it is not left completely reflective either.

We turn now to transfer theorems regarding the reflectivity properties studied in this section.

Theorem 240 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is family completely reflective if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ implies $\bigcap \mathcal{T} \leq T'$.

(b) $\mathcal{I}$ is left completely reflective if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{A}}(\mathcal{A})$, $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ implies $\bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$.

(c) $\mathcal{I}$ is system completely reflective if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiSys}^{\mathcal{A}}(\mathcal{A})$, $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ implies $\bigcap_{T \in \mathcal{T}} T \leq T'$.

Proof: We prove Part (b).

The "if" is obtained by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account that $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}) = \mathrm{ThFam}(\mathcal{I})$, by Lemma 51.

For the "only if" suppose that $\mathcal{I}$ is left c-reflective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. Apply the inverse of $\langle F, \alpha \rangle$ to get $\alpha^{-1}(\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. This yields that $\bigcap_{T \in \mathcal{T}} \alpha^{-1}(\Omega^{\mathcal{A}}(T)) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. Thus, by Proposition 24, we get $\bigcap_{T \in \mathcal{T}} \Omega(\alpha^{-1}(T)) \leq \Omega(\alpha^{-1}(T'))$. Take into account the fact that, by Lemma 51, $\{\alpha^{-1}(T) : T \in \mathcal{T}\} \cup \{\alpha^{-1}(T')\} \subseteq \mathrm{ThFam}(\mathcal{I})$ and apply left c-reflectivity to get $\bigcap_{T \in \mathcal{T}} \overleftarrow{\alpha^{-1}(T)} \leq \overleftarrow{\alpha^{-1}(T')}$. Hence, by Lemma 6, we get $\bigcap_{T \in \mathcal{T}} \alpha^{-1}(\overleftarrow{T}) \leq \alpha^{-1}(\overleftarrow{T'})$. Hence, $\alpha^{-1}(\bigcap_{T \in \mathcal{T}} \overleftarrow{T}) \leq \alpha^{-1}(\overleftarrow{T'})$. Finally, by the surjectivity of $\langle F, \alpha \rangle$, we conclude that $\bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$. ■

Next, we obtain characterizations of family and system c-reflectivity in terms of the complete order reflectivity of mappings from posets of theory or filter families/systems into posets of congruence systems. Given complete lattices $\mathbf{P} = \langle P, \leq \rangle$ and $\mathbf{Q} = \langle Q, \preccurlyeq \rangle$, call a mapping $f : P \to Q$ **completely order reflecting** if, for all $X \cup \{y\} \subseteq P$,

$$\bigwedge_{x \in X}^{\mathbf{Q}} f(x) \preccurlyeq f(y) \quad \text{implies} \quad \bigwedge_{x \in X}^{\mathbf{P}} x \leq y.$$

Proposition 241 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The following statements are equivalent:*

(a) *$\mathcal{I}$ is family completely reflective;*

(b) $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}(\mathcal{I})$ *is completely order reflecting;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}(\mathcal{A})$ *is completely order reflecting, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

Similarly, for system c-reflectivity, we have

Proposition 242 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) *$\mathcal{I}$ is system completely reflective;*

(b) $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}(\mathcal{I})$ *is completely order reflecting;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}(\mathcal{A})$ *is completely order reflecting, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

We look now at the relationships governing classes defined using complete reflectivity with corresponding classes defined using reflectivity. We have referred to this straightforward relationships already in the proof of Lemma 233.

Proposition 243 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

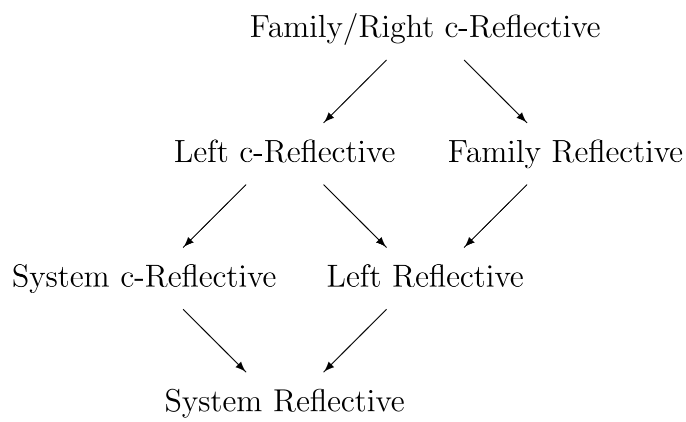
(a) *If $\mathcal{I}$ is family/right completely reflective, then it is family/right reflective;*

(b) *If $\mathcal{I}$ is left completely reflective, then it is left reflective;*

(c) *If $\mathcal{I}$ is system completely reflective, then it is system reflective.*

Proof: All three parts are based on the observation that the reflectivity conditions are specializations of the corresponding complete reflectivity conditions to the special case of singleton collections of theory families/systems on the left hand sides of the relevant inequalities. ∎

Proposition 243 has established the following combined hierarchy of injectivity and reflectivity properties.



Now we turn to an example that will show that all three inclusions established in Proposition 243 and depicted in the diagram are proper. Namely, we construct a family reflective π-institution that is not system completely reflective.

Example 244 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category, with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2, 3, 4\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the two unary natural transformations* $\sigma^\flat, \tau^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined by the following table:*

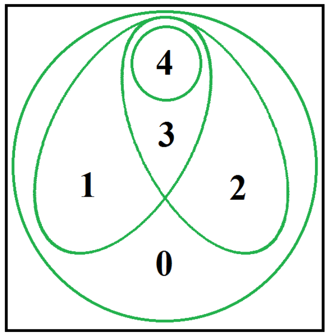
$x \in \mathrm{SEN}^\flat(\Sigma)$	$\sigma^\flat_\Sigma(x)$	$\tau^\flat_\Sigma(x)$
0	2	0
1	2	0
2	2	0
3	1	2
4	2	0

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

$$\mathcal{C}_\Sigma = \{\{4\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{0, 1, 2, 3, 4\}\}.$$

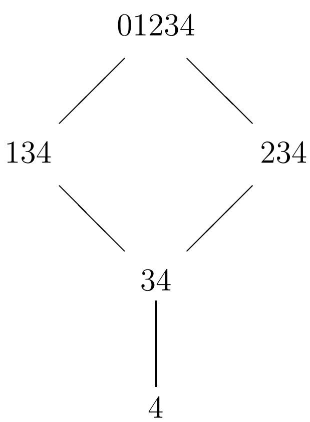
Clearly, since $\mathbf{Sign}^\flat$ *is trivial,* $\mathcal{I}$ *is systemic.*

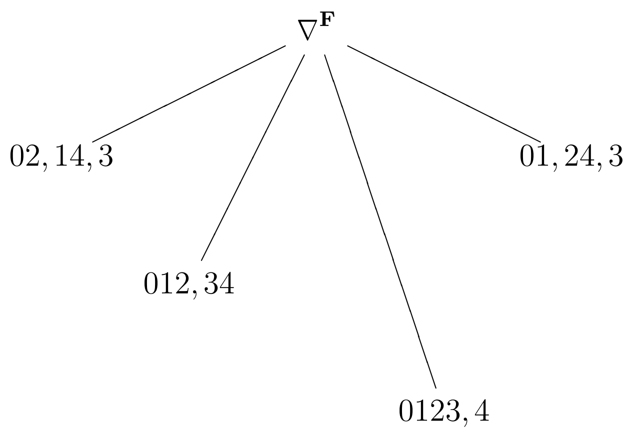
The lattice of theory families and the corresponding Leibniz congruence



SEN(Σ)

systems are shown in the diagram.





From the diagram, it is clear that, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) \leq \Omega(T')$ *implies* $T \leq T'$, *i.e.,* $\mathcal{I}$ *is family reflective. On the other hand, setting* $T^1 = \{\{1,3,4\}\}$, $T^2 = \{\{2,3,4\}\}$ *and* $T' = \{\{4\}\}$, *we get*

$$\begin{array}{rcl}\Omega(T^1) \cap \Omega(T^2) & = & \{\{02,14,3\}\} \cap \{\{01,24,3\}\}\\ & = & \Delta^{\mathbf{F}}\\ & \leq & \{\{0123,4\}\} = \Omega(T'),\end{array}$$

whereas $T^1 \cap T^2 = \{\{3,4\}\} \nleq \{\{4\}\} = T'$. *Hence,* $\mathcal{I}$ *is not system completely reflective.*

Chapter 4

The Semantic Leibniz Hierarchy: Top Half

4.1 Introduction

In this chapter, we study the classes of π-institutions that result when combining monotonicity properties of the Leibniz operator with injectivity, reflectivity or complete reflectivity. As such, all those classes correspond, in the categorical framework, to the class of weakly algebraizable sentential logics [62], which is obtained by combining protoalgebraicity [28] (see, also, [26]) with truth equationality [77] (see, also, Section 6.4 of [86]). It should also be mentioned that algebraizable logics, as introduced in [35] and generalized in [43, 53, 54], form subclasses of weakly algebraizable ones obtained by strengthening protoalgebraicity to equivalentiality [19, 23, 24]. The analogs of equivalentiality for π-institutions and the corresponding subclasses of algebraizable π-institutions will be considered in Chapters 5 and 5.

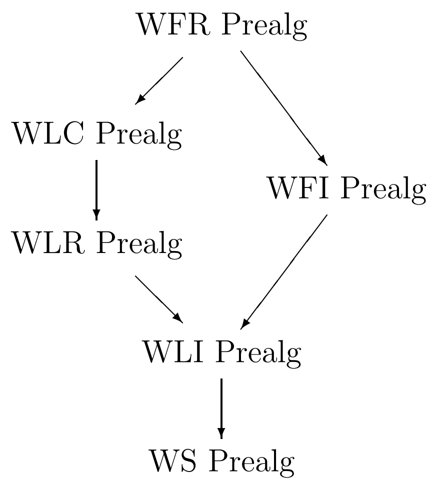
In Section 4.2, we study the hierarchy that results when combining prealgebraicity, i.e., monotonicity of the Leibniz operator on theory systems (Section 3.3) with each of the various flavors of injectivity (Section 3.6), reflectivity (Section 3.7) or complete reflectivity (Section 3.8). Since there are four different flavors of injectivity, three of reflectivity and three of complete reflectivity, we get, a priori, ten classes of *weakly prealgebraizable π-institutions*. The qualifier "weakly" suggests the use of monotonicity rather than equivalentiality, and the prefix "pre" in prealgebraizable that prealgebraicity, i.e., system monotonicity, rather than protoalgebraicity, i.e., family monotonicity, is used in the definition of these ten classes. Since prealgebraicity is common to all ten, the differentiating factor is the type of injectivity, reflectivity or c-reflectivity being imposed. Accordingly, the following ten classes are obtained, all named "weakly X prealgebraizable", or "WX Prealgebraizable" for short, where the string X stands for one of the following:

- SI for system injective, LI for left injective, FI for family injective, RI for right injective; or
- SR for system reflective, LR for left reflective, FR for family reflective; or
- SC for system c-reflective, LC for left c-reflective, FC for family c-reflective.

A fundamental result is that, under prealgebraicity, all three system properties (SI, SR and SC) coincide. Thus, WSI, WSR and WSC prealgebraizability are identical properties. We call π-institutions belonging to this class *WS prealgebraizable*. It is shown that WS prealgebraizability transfers. Moreover, WS prealgebraizable π-institutions $\mathcal{I} = \langle \mathbf{F}, C \rangle$ are characterized by the property that $\Omega^{\mathcal{A}}$ on $\mathcal{I}$-filter systems is an order embedding, for every $\mathbf{F}$-algebraic system $\mathcal{A}$. As prealgebraicity identifies also family reflectivity with family c-reflectivity, the classes of WFR prealgebraizable and WFC prealgebraizable

π-institutions coincide. Finally, both WFR and WRI prealgebraizability turn out to be equivalent, as they are both equivalent to WFI prealgebraizability plus systemicity. Hence, at the top of the weak prealgebraizability hierarchy, only two of the four classes are potentially different. We refer to them as *WFR* and *WFI prealgebraizability*. Both properties transfer. Moreover, both have characterizations in terms of the Leibniz operator viewed as a mapping between ordered sets. Namely, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is WFR prealgebraizable iff $\Omega^{\mathcal{A}}$ is an order isomorphism and it is WFI prealgebraizable iff $\Omega^{\mathcal{A}}$ is a bijection on $\mathcal{I}$-filter families, which restricts to an order embedding on $\mathcal{I}$-filter systems, for every $\mathbf{F}$-algebraic system $\mathcal{A}$.

As no further identifications seem possible, one obtains the hierarchy

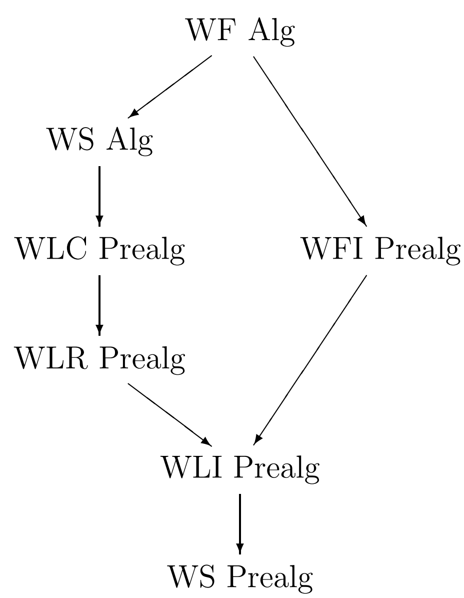


Some specialized results reduce the hierarchy further under additional provisos. First, under systemicity, the entire hierarchy collapses to a single class. Second, it is shown that, under stability, the two family properties coincide, as do all four remaining properties. Thus, under stability, the hierarchy reduces to only two distinct classes.

The section focuses, next, to the three left properties. More precisely, it is shown that all three of WLI, WLR and WLC prealgebraizability versions transfer and that each is characterized via theorems perceiving the Leibniz operator as a mapping from filter families to congruence systems over arbitrary algebraic systems. Briefly, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, it turns out that $\mathcal{I}$ is WLI (WLR, WLC, respectively) prealgebraizable iff, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}(\mathcal{A})$ is a left injective (left order reflecting, left completely order reflecting, respectively) surjection, which restricts to an order embedding on filter systems.

In Section 4.3, we study those classes that are formed by combining protoalgebraicity (family monotonicity) with each of the ten versions of injectivity, reflectivity or complete reflectivity properties. So, once more, a priori, before any detailed study, one obtains ten potentially different classes of weakly algebraizable π-institutions. However, since protoalgebraicity is a stronger condition than prealgebraicity, one obtains immediately at least those identifications that apply to the weak prealgebraizability hierarchy. So, e.g., we get that WFI, WRI, WFR and WFC algebraizable π-institutions coincide. We term the corresponding property *WF algebraizability*. It turns out to be equivalent to the conjunction of WS prealgebraizability and systemicity. WF algebraizability transfers and, moreover, it can be characterized by $\Omega^{\mathcal{A}}$ being an order isomorphism on every algebraic system $\mathcal{A}$. It follows that this class of π-institutions is actually identical to the class of WFR prealgebraizable ones, i.e., those belonging to the top class in the weak prealgebraizability hierarchy. What is a massive collapse, however, results from showing that the lowest class in the weak algebraizability hierarchy, WSI algebraizability, can be characterized as the conjunction of stability with $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ being an order isomorphism. This allows showing that all classes of WS, WLI, WLR, WLC and WFI algebraizable π-institutions are identical. We term the corresponding property *WS algebraizability*. It is shown that WS algebraizability also transfers.

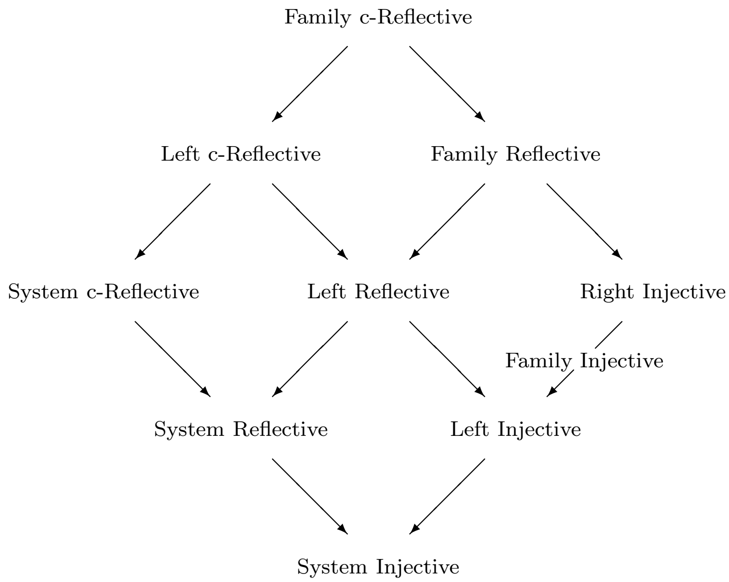
Having reduced the weak algebraizability hierarchy down to two classes, we conclude Section 4.3 (and Chapter 4) by merging it with the weak prealgebraizability hierarchy to obtain the following refinement of the classes that correspond, in the categorical framework, to the class of weakly algebraizable deductive systems.



4.2 Weak PreAlgebraizability

We now shift attention to classes of π-institutions that are defined as a result of interactions between the various kinds of injectivity, reflectivity and complete reflectivity, on the one hand, and prealgebraicity and protoalgebraicity, on the other.

Recall that the hierarchy that was established in the preceding chapter as regards the various versions of injectivity, reflectivity and complete reflectivity has the following form:



Thus, a priori, based on the preceding hierarchy, and combining with prealgebraicity, we obtain a mimicking hierarchy of ten classes which are defined, and whose hierarchy is shown, below.

Definition 245 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is* **weakly system injective prealgebraizable** *or* **WSI Prealgebraizable***, for short, if it is system injective and prealgebraic.*
- $\mathcal{I}$ *is* **weakly left injective prealgebraizable** *or* **WLI Prealgebraizable***, for short, if it is left injective and prealgebraic.*
- $\mathcal{I}$ *is* **weakly family injective prealgebraizable** *or* **WFI Prealgebraizable***, for short, if it is family injective and prealgebraic.*
- $\mathcal{I}$ *is* **weakly right injective prealgebraizable** *or* **WRI Prealgebraizable***, for short, if it is right injective and prealgebraic.*

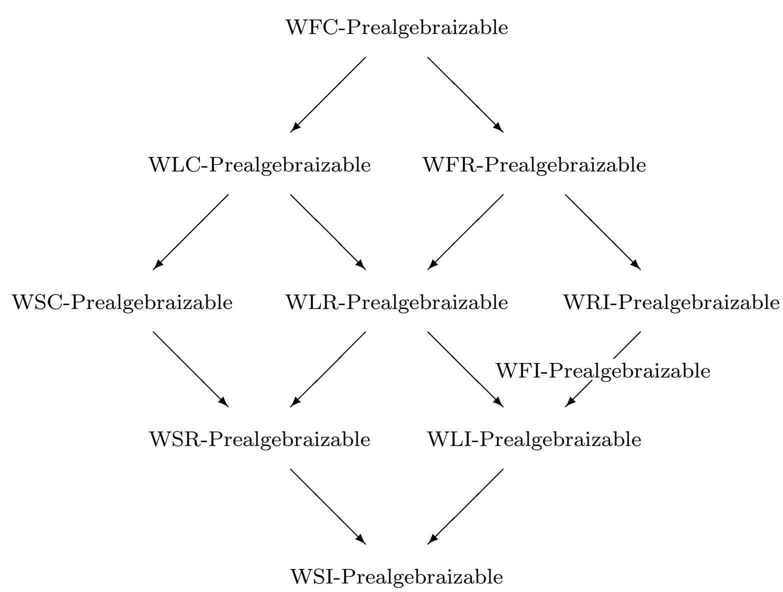
Definition 246 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is* **weakly system reflective prealgebraizable** *or* **WSR Prealgebraizable***, for short, if it is system reflective and prealgebraic.*

- $\mathcal{I}$ *is* **weakly left reflective prealgebraizable** *or* **WLR Prealgebraizable**, *for short, if it is left reflective and prealgebraic.*
- $\mathcal{I}$ *is* **weakly family reflective prealgebraizable** *or* **WFR Prealgebraizable**, *for short, if it is family reflective and prealgebraic.*

Definition 247 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is* **weakly system completely reflective prealgebraizable** *or* **WSC Prealgebraizable**, *for short, if it is system completely reflective and prealgebraic.*
- $\mathcal{I}$ *is* **weakly left completely reflective prealgebraizable** *or* **WLC Prealgebraizable**, *for short, if it is left completely reflective and prealgebraic.*
- $\mathcal{I}$ *is* **weakly family completely reflective prealgebraizable** *or* **WFC Prealgebraizable**, *for short, if it is family completely reflective and prealgebraic.*



A few words in the nomenclature used in this diagram are in order.

- W stands for "weakly" which refers to the fact that these classes are defined using forms of monotonicity of the Leibniz operator without any stipulation as to commutativity of the Leibniz operator with inverse special endomorphisms (to be studied later in the chapter). If one adds that condition (using essentially (pre)equivalentiality instead of pre- or protoalgebraicity), then the letter is dropped.

- The letters S for "system", L for "left", R for "right" and F for "family" have obvious meanings referring to which of the four versions (family, left, right or system) of injectivity (I), reflectivity (R) or complete reflectivity (C) conditions are used in the definition.
- Finally, the term "prealgebraizable" is associated with application of monotonicity to theory systems only (as in "prealgebraic"), as opposed to the term "algebraizable", which stipulates monotonicity for all theory families.

We start by proving that under prealgebraicity, system injectivity, system reflectivity and system complete reflectivity turn out to be equivalent properties.

Theorem 248 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is prealgebraic, then the following statements are equivalent:*

(a) $\mathcal{I}$ *is system injective;*

(b) $\mathcal{I}$ *is system reflective;*

(c) $\mathcal{I}$ *is system completely reflective.*

Proof:

(a)⇒(b) Suppose that $\mathcal{I}$ is system injective. Let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\Omega(T) \le \Omega(T')$. Then we get $\Omega(T) = \Omega(T) \cap \Omega(T')$. Moreover, by Lemma 23, $\Omega(T) \cap \Omega(T') \le \Omega(T \cap T')$. On the other hand, by prealgebraicity, we have $\Omega(T \cap T') \le \Omega(T)$ and $\Omega(T \cap T') \le \Omega(T')$, whence $\Omega(T \cap T') \le \Omega(T) \cap \Omega(T')$. We conclude that

$$\Omega(T) = \Omega(T) \cap \Omega(T') = \Omega(T \cap T').$$

Now we use system injectivity to get $T = T \cap T'$. Therefore, $T \le T'$. So $\mathcal{I}$ is also system reflective.

(b)⇒(c) Suppose, next, that $\mathcal{I}$ is system reflective. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \le \Omega(T')$. Since $\mathcal{I}$ is prealgebraic, i.e., Ω is monotone on theory systems, we have, for all $T \in \mathcal{T}$, $\Omega(\bigcap \mathcal{T}) \le \Omega(T)$. Therefore, we get

$$\Omega(\bigcap_{T \in \mathcal{T}} T) \le \bigcap_{T \in \mathcal{T}} \Omega(T) \le \Omega(T').$$

Since, by hypothesis, $\mathcal{I}$ is system reflective, we get $\bigcap_{T \in \mathcal{T}} T \le T'$. Thus, $\mathcal{I}$ is system completely reflective.

(c)⇒(a) Suppose, finally, that $\mathcal{I}$ is system completely reflective. By Proposition 243, it is system reflective, and, then, by Proposition 228, it is system injective.

■

Theorem 248 shows that three of the classes in the previous diagram coincide.

Corollary 249 *The classes of WSI prealgebraizable, WSR prealgebraizable, and WSC prealgebraizable π-institutions coincide.*

Taking advantage of Corollary 249 we define:

Definition 250 (WS Prealgebraizable) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is called* **weakly system prealgebraizable** *(or* **WS prealgebraizable** *for short) if it is prealgebraic and system injective, i.e., if the Leibniz operator is monotone and injective on theory systems: For all* $T, T' \in \mathrm{ThSys}(\mathcal{I})$,

$$T \le T' \quad \textit{implies} \quad \Omega(T) \le \Omega(T');$$
$$\Omega(T) = \Omega(T') \quad \textit{implies} \quad T = T'.$$

We present two examples of WS prealgebraizable π-institutions. They are crafted to provide a sneak preview of the state of affairs in the case of systemic and non-systemic π-institutions with regards to weak prealgebraizability. The reader will notice that, in both examples, there is an order isomorphism between the lattice of theory systems of the π-institution and that of the associated Leibniz congruence systems. On the other hand, for this isomorphism to extend to an isomorphism between the lattice of all theory families and the corresponding Leibniz congruence systems, the condition of systemicity on the π-institution under consideration seems to be required (and is, as we shall see later).

Example 251 *Define the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *as follows:*

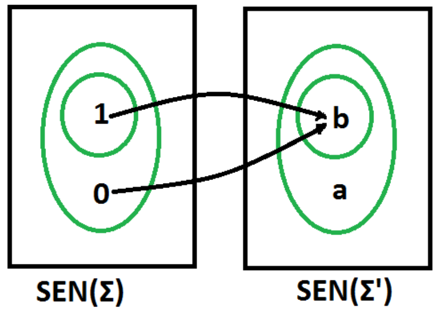
- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = b = \mathrm{SEN}^\flat(f)(1)$;
- $N^\flat$ *is the trivial clone.*

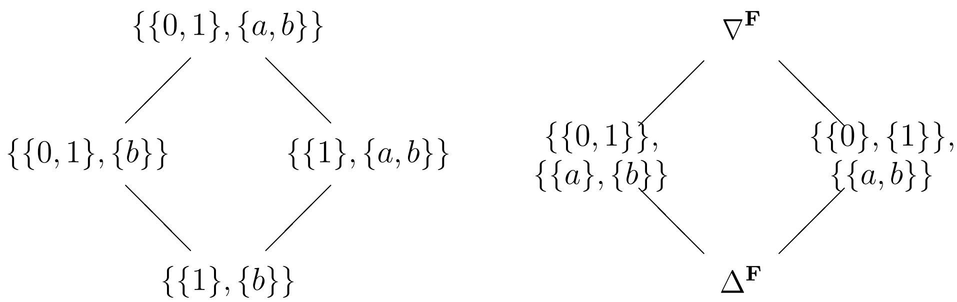
Specify the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

$$\mathcal{C}_\Sigma = \{\{1\}, \{0, 1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\}, \{a, b\}\}.$$

Notice that every theory family is a theory system, whence $\mathcal{I}$ *is systemic.*

The following diagrams show the lattices of theory families and of the corresponding Leibniz congruence systems:

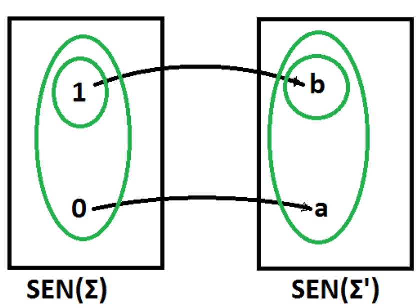




Note that the π-institution $\mathcal{I}$ is WS prealgebraizable and that the two lattices are clearly isomorphic.

Example 252 *Define the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *as follows:*

- $\mathbf{Sign}^\flat$ *is the category with two objects* Σ, Σ' *and a single (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by setting* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*,* $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$*,* $\mathrm{SEN}^\flat(f)(0) = a$ *and* $\mathrm{SEN}^\flat(f)(1) = b$*;*
- $N^\flat$ *is the trivial clone.*



Specify the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

$$\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\},\{a,b\}\}.$$

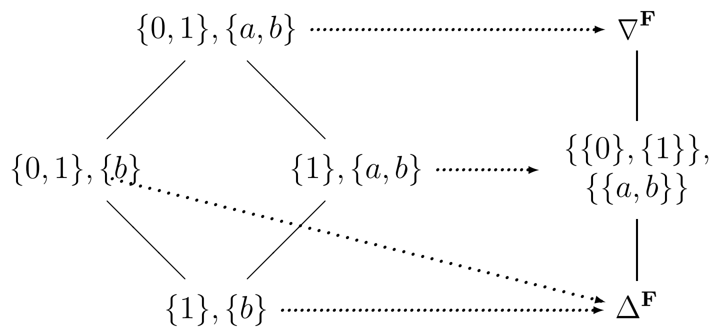
Notice that the theory family $T = \{\{0,1\},\{b\}\}$ *is not a theory system, whence* $\mathcal{I}$ *is not systemic. In fact,* $\overleftarrow{} : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ThSys}(\mathcal{I})$ *is given by the following table:*

	$\{b\}$	$\{a,b\}$
$\{1\}$	$\{1\},\{b\}$	$\{1\},\{a,b\}$
$\{0,1\}$	$\{1\},\{b\}$	$\{0,1\},\{a,b\}$

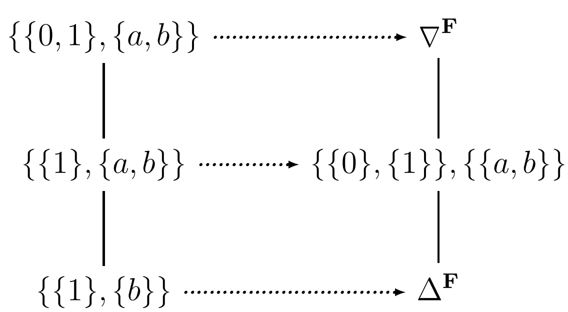
The next table gives the theory families and the associated Leibniz congruence systems:

T	$\Omega(T)$
$\{\{1\},\{b\}\}$	$\{\{0\},\{1\}\},\{\{a\},\{b\}\}$
$\{\{0,1\},\{b\}\}$	$\{\{0\},\{1\}\},\{\{a\},\{b\}\}$
$\{\{1\},\{a,b\}\}$	$\{\{0\},\{1\}\},\{\{a,b\}\}$
$\{\{0,1\},\{a,b\}\}$	$\{\{0,1\}\},\{\{a,b\}\}$

So, even though the lattices of theory families and of the corresponding Leibniz congruence systems are not isomorphic,



the lattices of theory systems and of the corresponding Leibniz congruence systems are indeed isomorphic:



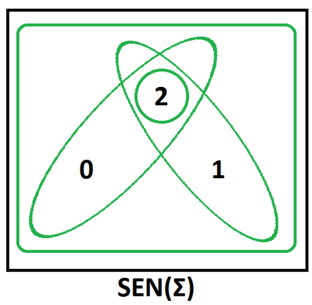
This π*-institution is also WS prealgebraizable.*

We present next examples to show that the class of weakly system prealgebraizable π-institutions is properly included in both the class of prealgebraic and that of system completely reflective π-institutions.

Example 253 *Consider the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$*;*
- $N^\flat$ *is the clone of natural transformations on* $\mathrm{SEN}^\flat$ *generated by the the two unary natural transformations* $\sigma^\flat, \tau^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, given by the following table:*

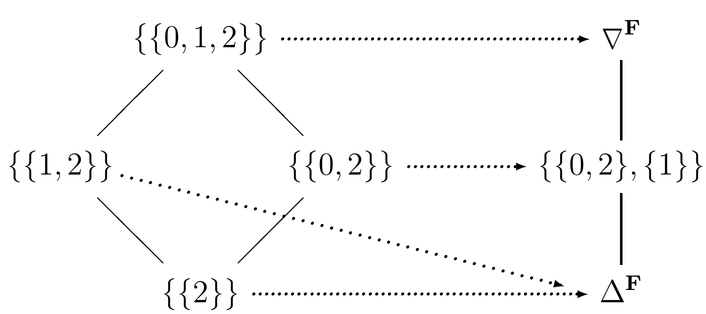
$x \in \mathrm{SEN}^\flat(\Sigma)$	$\sigma^\flat_\Sigma(x)$	$\tau^\flat_\Sigma(x)$
0	0	0
1	1	2
2	0	2



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$*, by setting*

$$C_\Sigma = \{\{2\},\{0,2\},\{1,2\},\{0,1,2\}\}.$$

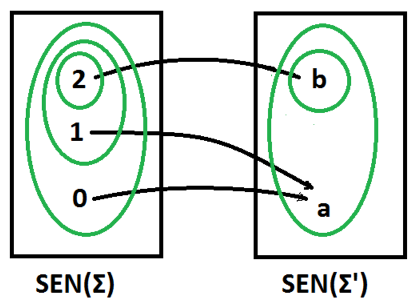
The lattice of the theory systems of $\mathcal{I}$ *and that of the associated Leibniz congruence systems are shown in the following diagrams*



It is clear that the Leibniz operator is monotone. On the other hand, the Leibniz operator is not injective on theory systems. Therefore, we conclude that $\mathcal{I}$ *is prealgebraic but that it fails to be WS prealgebraizable.*

Example 254 *Define the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *as follows:*

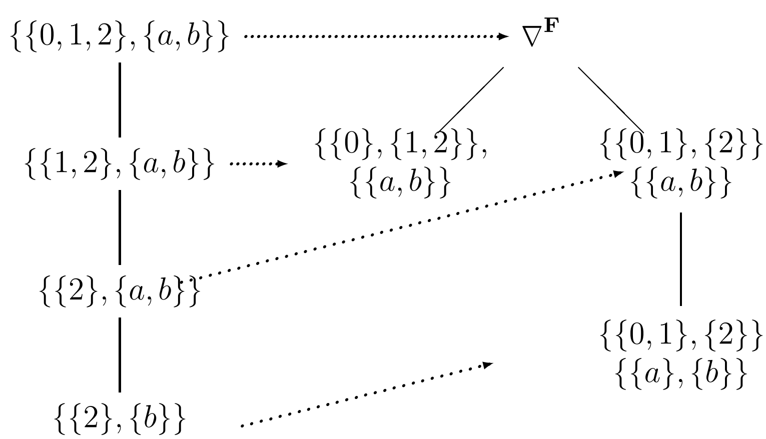
- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = a$ *and* $\mathrm{SEN}^\flat(f)(2) = b$*;*
- $N^\flat$ *is the trivial clone.*



We consider the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *defined by*

$$\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\}, \{a, b\}\}.$$

This π*-institution has six theory families, but only four theory systems. The lattice of theory systems and the associated congruence systems are shown below.*



It is clear from these that the Leibniz operator is completely order reflecting on the theory systems of $\mathcal{I}$*, but it is not monotonic. It follows that* $\mathcal{I}$ *is system c-reflective but not prealgebraic. Therefore, it is system c-reflective, but fails to be weakly system prealgebraizable.*

The defining properties of weak system prealgebraizability transfer from theory systems to filter systems over arbitrary algebraic systems. This result follows naturally from corresponding constituent pieces that have already been put in place when studying monotonicity and injectivity.

Theorem 255 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is WS prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, the Leibniz operator on* $\mathcal{A}$ *is monotone and injective on the* $\mathcal{I}$*-filter systems of* $\mathcal{A}$*, i.e., for all* $T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$*,*

$$\begin{array}{rcl} T \le T' & \textit{implies} & \Omega^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T'); \\ \Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T') & \textit{implies} & T = T'. \end{array}$$

Proof: Suppose, first, that the displayed implications hold for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $\mathcal{I}$-filter systems T, T' on $\mathcal{A}$. By taking $\mathcal{A} = \mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and keeping in mind Lemma 51, we conclude that the Leibniz operator is monotone and injective on all theory systems of $\mathcal{I}$. Thus, by definition, $\mathcal{I}$ is WS prealgebraizable.

Suppose, conversely, that $\mathcal{I}$ is WS prealgebraizable. Then, by definition, it is prealgebraic and system injective. Thus, by Theorems 179 and 214, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, the Leibniz operator $\Omega^{\mathcal{A}}$ is monotone and injective on the $\mathcal{I}$-filter systems of $\mathcal{A}$. ■

We finally establish the result that we alluded to before presenting Examples 251 and 252. Namely, we show that WS prealgebraizability can be equivalently characterized by the fact that the Leibniz operator $\Omega^{\mathcal{A}}$ over an arbitrary $\mathbf{F}$-algebraic system $\mathcal{A}$ establishes an order embedding from the lattice of filter systems on $\mathcal{A}$ into the poset of all relative congruence systems on $\mathcal{A}$ with respect to the class $\mathrm{AlgSys}^*(\mathcal{I})$.

Consider an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ and a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ based on $\mathbf{F}$. Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and let $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, observe that the $\mathcal{I}$-matrix family $\langle \mathcal{A}^{\Omega^{\mathcal{A}}(T)}, T/\Omega^{\mathcal{A}}(T) \rangle$ is Leibniz reduced. Hence, the $\mathbf{F}$-algebraic system $\mathcal{A}^{\Omega^{\mathcal{A}}(T)}$ is in $\mathrm{AlgSys}^*(\mathcal{I})$. Equivalently, we have that $\Omega^{\mathcal{A}}(T) \in \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. Thus, the Leibniz operator is always a well defined function

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

In particular, it restricts to a well-defined function

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Additionally, by definition of $\mathrm{AlgSys}^\bullet(\mathcal{I})$, this may be perceived also as a function

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A}),$$

where, we set

$$\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A}) = \{\theta \in \mathrm{ConSys}(\mathcal{A}) : \mathcal{A}/\theta \in \mathrm{AlgSys}^{\bullet}(\mathcal{I})\}.$$

We keep these remarks in mind in the formulation of several of the following results.

Theorem 256 *Let* $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat}\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is WS prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order embedding.

Proof: Suppose, first, that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, $\Omega^{\mathcal{A}}$ is an order embedding. In particular, because of Lemma 51,

$$\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^{*}(\mathcal{I})$$

is an order embedding. This implies that the Leibniz operator is monotone and injective on theory systems. Thus $\mathcal{I}$ is WS prealgebraizable.

Suppose, conversely, that $\mathcal{I}$ is WS prealgebraizable. Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system. Consider the mapping

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

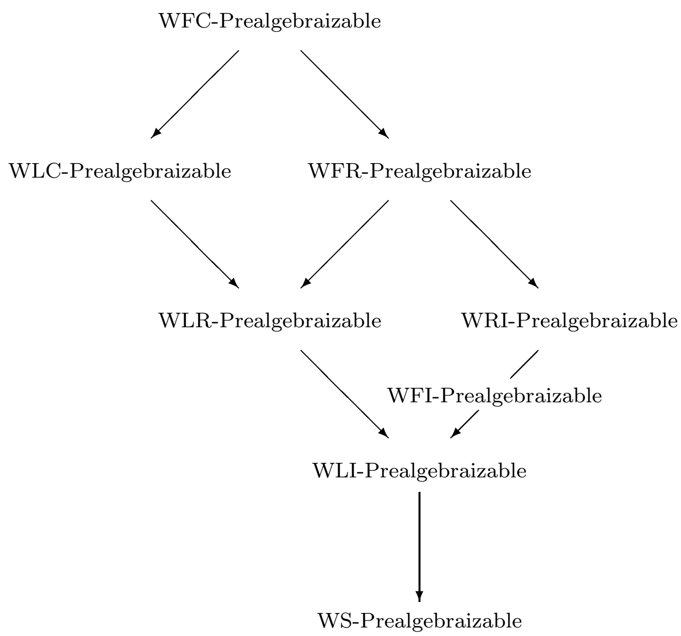
By Theorem 255, this mapping is monotone and injective. To show that it is an order embedding, we must show that it is also order reflecting. By Theorem 225, it suffices to show that $\mathcal{I}$ is system reflective. But this was accomplished in Theorem 248. ∎

Corollary 257 *Let* $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat}\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is WS prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$$

is an order isomorphism.

At this point in our studies we have the following hierarchy, which results from the preceding one by the identification established in Corollary 249.



We continue our study by showing that all three upper diagonal classes, namely those of WFC, WFR and WRI prealgebraizable π-institutions also coincide. To accomplish this for WFC and WFR prealgebraizability, we prove a partial analog of Theorem 248 that under prealgebraicity, family reflectivity and family complete reflectivity turn out to be equivalent properties. The crucial observation is that, as shown in Lemma 218, family reflectivity implies systemicity.

Theorem 258 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is prealgebraic and family reflective, then it is family completely reflective.*

Proof: Since $\mathcal{I}$ is family reflective, by Lemma 218, it is systemic. Since it is prealgebraic and system reflective, by Theorem 248, it is also system completely reflective. Hence, by systemicity, it is also family completely reflective. ■

Theorem 258 shows that two of the top classes in the previous diagram coincide.

Corollary 259 *The classes of WFR prealgebraizable and WFC prealgebraizable* π*-institutions coincide.*

Next we show that the classes of WFR and WRI prealgebraizable π-institutions coincide. We do this indirectly by providing identical characterizations of both classes involving WFI prealgebraizability and systemicity.

First, we need a result which will also prove useful later in our investigations. Namely, we look at an interesting and useful connection between family injectivity and family reflectivity, by means of protoalgebraicity, that forms a partial analog of Theorem 248, which related system injectivity with system reflectivity in the presence of prealgebraicity.

Proposition 260 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is protoalgebraic and family injective, then it is family reflective.*

Proof: Suppose that $\mathcal{I}$ is protoalgebraic and family injective. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then we get $\Omega(T) = \Omega(T) \cap \Omega(T')$. Moreover, by Lemma 23, $\Omega(T) \cap \Omega(T') \leq \Omega(T \cap T')$. On the other hand, by protoalgebraicity, we have $\Omega(T \cap T') \leq \Omega(T)$ and $\Omega(T \cap T') \leq \Omega(T')$, whence $\Omega(T \cap T') \leq \Omega(T) \cap \Omega(T')$. We conclude that

$$\Omega(T) = \Omega(T) \cap \Omega(T') = \Omega(T \cap T').$$

Now we use family injectivity to get $T = T \cap T'$. Therefore, $T \leq T'$. So $\mathcal{I}$ is also family reflective. ∎

Now we characterize WFR prealgebraizability.

Theorem 261 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. Then* $\mathcal{I}$ *is WFR prealgebraizable if and only if it is WFI prealgebraizable and systemic.*

Proof: Suppose, first, that $\mathcal{I}$ is WFR prealgebraizable. Then it is, by definition, prealgebraic, it is, by Lemma 218, systemic and, by Proposition 228, it is family injective. Thus, it is WFI prealgebraizable and systemic.

Suppose, conversely, that $\mathcal{I}$ is WFI prealgebraizable and systemic. Then, it is, by definition, prealgebraic and family injective, which, by systemicity, imply that it is protoalgebraic and family injective. Thus, by Proposition 260, it is protoalgebraic and family reflective. Hence, it is, a fortiori, WFR prealgebraizable. ∎

But it is easy to show also that WRI prealgebraizability has exactly the same characterization as WFR prealgebraizability.

Theorem 262 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. Then* $\mathcal{I}$ *is WRI prealgebraizable if and only if it is WFI prealgebraizable and systemic.*

Proof: This follows directly from Proposition 209. ∎

Corollary 263 *The classes of WFR prealgebraizable and WRI prealgebraizable* π*-institutions coincide.*

Proof: The conclusion follows from Theorems 261 and 262. ■

Corollaries 259 and 263 show that, among the top four classes of the hierarchy in the preceding diagram, only two may be (and are, as we show in the following example) different. We keep the names WFR prealgebraizable and WFI prealgebraizable for the π-institutions in each of these classes. Thus, given an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ and a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ based on $\mathbf{F}$:

- $\mathcal{I}$ is *WFR prealgebraizable* if it prealgebraic and family reflective (or, equivalently, family c-reflective or right injective), i.e., if

$$\begin{array}{rcl} T \leq T' & \text{implies} & \Omega(T) \leq \Omega(T'), \text{ for all } T, T' \in \mathrm{ThSys}(\mathcal{I}); \\ \Omega(T) \leq \Omega(T') & \text{implies} & T \leq T', \text{ for all } T, T' \in \mathrm{ThFam}(\mathcal{I}); \end{array}$$

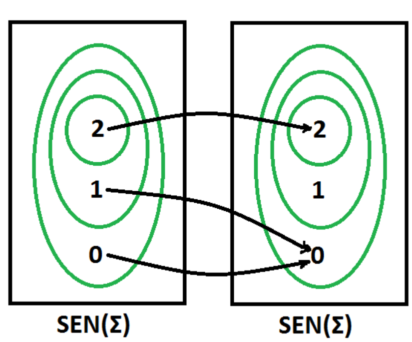
- $\mathcal{I}$ is *WFI prealgebraizable* if it is prealgebraic and family injective, i.e., if

$$\begin{array}{rcl} T \leq T' & \text{implies} & \Omega(T) \leq \Omega(T'), \text{ for all } T, T' \in \mathrm{ThSys}(\mathcal{I}); \\ \Omega(T) = \Omega(T') & \text{implies} & T = T', \text{ for all } T, T' \in \mathrm{ThFam}(\mathcal{I}). \end{array}$$

We provide an example to show that these two classes of π-institutions are indeed different, i.e., we exhibit a WFI prealgebraizable π-institution which is not WFR prealgebraizable.

Example 264 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial clone.*

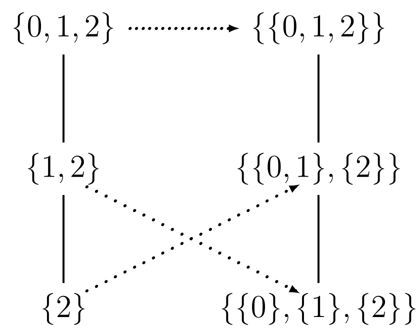


Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by setting $\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}$.

The table giving the action of $\overleftarrow{\ }$ on theory families is shown below. It is clear that $\mathcal{I}$ is not systemic.

T_Σ	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$
$\overleftarrow{T}_\Sigma$	$\{2\}$	$\{2\}$	$\{0,1,2\}$

The following diagram gives the lattice of theory families and the corresponding Leibniz congruence systems.



We can see that $\mathcal{I}$ is prealgebraic and family injective. Since it is not systemic, by Theorem 261, it follows that it is not family reflective, a fact that can also be directly verified by the diagram. We conclude that $\mathcal{I}$ is a WFI prealgebraizable π-institution, which is not WFR prealgebraizable.

We now provide a theorem to the effect that both classes are characterized by theorems asserting that their properties transfer from theory systems/families to filter systems/families on arbitrary algebraic systems.

Theorem 265 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is WFI prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, the Leibniz operator on $\mathcal{A}$ is monotone on $\mathcal{I}$-filter systems and injective on $\mathcal{I}$-filter families, i.e.,*

$$T \leq T' \text{ implies } \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T'), \text{ for all } T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A});$$
$$\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T') \text{ implies } T = T', \text{ for all } T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}).$$

Proof: The "if" direction follows by specializing to $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account Lemma 51.

For the "only if" suppose that $\mathcal{I}$ is WFI prealgebraizable and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system. By definition, $\mathcal{I}$ is prealgebraic and family injective. Thus, by Theorem 179, the Leibniz operator on the $\mathcal{I}$-filter systems of $\mathcal{A}$ is monotone and, by Theorem 214, the Leibniz operator on the $\mathcal{I}$-filter families of $\mathcal{A}$ is injective. ∎

Theorem 266 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is WFR prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, the Leibniz operator on* $\mathcal{A}$ *is monotone on* $\mathcal{I}$*-filter systems and order reflecting on* $\mathcal{I}$*-filter families, i.e.,*

$$T \leq T' \text{ implies } \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T'), \text{ for all } T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A});$$
$$\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T') \text{ implies } T \leq T', \text{ for all } T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}).$$

Proof: The "if" direction follows by specializing to $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account Lemma 51.

For the "only if" suppose that $\mathcal{I}$ is WFR prealgebraizable and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system. By definition, $\mathcal{I}$ is prealgebraic and family reflective. Thus, by Theorem 179, the Leibniz operator on the $\mathcal{I}$-filter systems of $\mathcal{A}$ is monotone and, by Theorem 225, the Leibniz operator on the $\mathcal{I}$-filter families of $\mathcal{A}$ is injective. ■

Next we give two important results, along the lines of the characterization Theorem 256 for WS prealgebraizability, characterizing the classes of WFI prealgebraizable and WFR prealgebraizable π-institutions.

Theorem 267 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is WFI prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a bijection which restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: Suppose, first, that $\mathcal{I}$ is WFI prealgebraizable. Then, it is a fortiori WS prealgebraizable. Thus, by Theorem 256, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order embedding. So it suffices to show that it extends to a bijection from $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ onto $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. Since $\mathcal{I}$ is family injective, this mapping is injective. It is also surjective: Given $\theta \in \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$, we have by definition, $\mathcal{A}^\theta \in \mathrm{AlgSys}^*(\mathcal{I})$. Thus, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta)$, such that $\Omega^{\mathcal{A}^\theta}(T) = \Delta^{\mathcal{A}^\theta}$. Now applying the inverse of the canonical quotient morphism $\langle I, \pi^\theta \rangle : \mathcal{A} \to \mathcal{A}^\theta$, we get $\pi^{\theta^{-1}}(\Omega^{\mathcal{A}^\theta}(T)) = \pi^{\theta^{-1}}(\Delta^{\mathcal{A}^\theta})$, whence, by Proposition 24, $\Omega^{\mathcal{A}}(\pi^{\theta^{-1}}(T)) = \theta$. Since, by Corollary 55, $\pi^{\theta^{-1}}(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get that the Leibniz operator is also surjective.

Suppose, conversely, that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a bijection which restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Then, by Theorem 256, $\mathcal{I}$ is WS prealgebraizable. Thus, in particular, it is prealgebraic. The fact that $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is a bijection ensures that the Leibniz operator on $\mathrm{ThFam}(\mathcal{I})$ is injective. Thus $\mathcal{I}$ is also family injective and, therefore, it is WFI prealgebraizable. ∎

And now an analogous characterization for WFR prealgebraizability.

Theorem 268 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is WFR prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism.

Proof: Suppose, first, that $\mathcal{I}$ is WFR prealgebraizable. Then, it is a fortiori WFI prealgebraizable. So by Theorem 267

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order embedding which extends to a bijection

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

But, by Theorem 261, $\mathcal{I}$ is systemic. Therefore, we get an order isomorphism

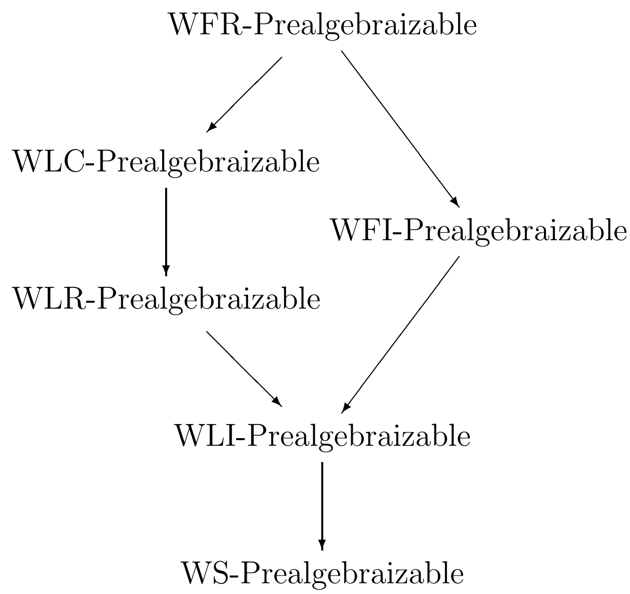
$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Suppose, conversely, that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism. In particular, $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{F})$ is an order isomorphism. This ensures that the Leibniz operator is monotone on theory families, hence on theory systems, and, moreover, that it is reflective on theory families. Thus, $\mathcal{I}$ is prealgebraic and family reflective, i.e., it is a WFR prealgebraizable π-institution. ∎

We take a break again to draw the hierarchy incorporating the information that we have currently available.



Recall again the formal definitions of the three classes that have not yet been at the focus of our investigations, namely those of WLC, WLR and WLI prealgebraizable π-institutions:

- $\mathcal{I}$ is *WLI Prealgebraizable* if it is prealgebraic and left injective, i.e., if

$$T \leq T' \text{ implies } \Omega(T) \leq \Omega(T'), \text{ for all } T, T' \in \mathrm{ThSys}(\mathcal{I});$$
$$\Omega(T) = \Omega(T') \text{ implies } \overleftarrow{T} = \overleftarrow{T'}, \text{ for all } T, T' \in \mathrm{ThFam}(\mathcal{I});$$

- $\mathcal{I}$ is *WLR Prealgebraizable* if it is prealgebraic and left reflective, i.e., if

$$T \leq T' \text{ implies } \Omega(T) \leq \Omega(T'), \text{ for all } T, T' \in \mathrm{ThSys}(\mathcal{I});$$
$$\Omega(T) \leq \Omega(T') \text{ implies } \overleftarrow{T} \leq \overleftarrow{T'}, \text{ for all } T, T' \in \mathrm{ThFam}(\mathcal{I});$$

- $\mathcal{I}$ is *WLC Prealgebraizable* if it is prealgebraic and left completely reflective, i.e., if

$$T \leq T' \text{ implies } \Omega(T) \leq \Omega(T'), \text{ for all } T, T' \in \mathrm{ThSys}(\mathcal{I});$$
$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \text{ implies } \bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}, \text{ for all } \mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I}).$$

We showed in Example 264 that the top right arrow in the preceding diagram represents a proper inclusion. Moreover, we showed in Theorem 261 that the two classes are separated by systemicity. Now we study the remaining five inclusions to reveal relationships between them and to verify that they are also proper.

We look, first, at the top left arrow, i.e., at the inclusion of the class of WFR prealgebraizable into that of WLC prealgebraizable π-institutions. We have the following extension of Theorem 261, which shows that systemicity is actually the property that separates the top class from every other class in this hierarchy.

Theorem 269 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is WFR prealgebraizable if and only if it is WLC, WLR, WFI, WLI or WS prealgebraizable and systemic.*

Proof: Suppose that $\mathcal{I}$ is WFR prealgebraizable. We showed in Theorem 261 that it is systemic. Moreover, it belongs, a fortiori, to all other classes in the hierarchy, since the conditions defining them are weaker than prealgebraicity and family complete reflectivity (which was showed to be equivalent to family reflectivity under prealgebraicity in Theorem 261).

Suppose, conversely, that $\mathcal{I}$ is WS prealgebraizable and systemic. This implies, by definition, that it is prealgebraic and system completely reflective. Thus, by systemicity, it is also family completely reflective. Therefore, since it is prealgebraic and family completely reflective, it is, by definition, WFR prealgebraizable. ∎

A more interesting, perhaps, view is the status of this hierarchy under the milder assumption of stability. Even though systemicity leads to a total collapse of the hierarchy into a single class, it turns out that stability allows for a two-class hierarchy.

Proposition 270 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is WFI prealgebraizable and stable, then it is systemic.*

Proof: Suppose that $\mathcal{I}$ is WFI prealgebraizable and stable and let $T \in \mathrm{ThFam}(\mathcal{I})$. Since $\mathcal{I}$ is stable, we have $\Omega(T) = \Omega(\overleftarrow{T})$. Thus, using family injectivity, we get $T = \overleftarrow{T}$. It follows that $T \in \mathrm{ThSys}(\mathcal{I})$. We now conclude that $\mathrm{ThFam}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$ and, therefore, $\mathcal{I}$ is systemic. ∎

Theorem 271 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is WFR prealgebraizable if and only if it is WFI prealgebraizable and stable.*

Proof: If $\mathcal{I}$ is WFR prealgebraizable, then it is, a fortiori, WFI prealgebraizable and, by Theorem 261, systemic and, therefore, stable. On the other hand, if $\mathcal{I}$ is WFI prealgebraizable and stable, then, by Proposition 270, it is systemic and, hence, by Theorem 269, it is WFR prealgebraizable. ∎

Proposition 272 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is WS prealgebraizable and stable, then it is WLC prealgebraizable.*

Proof: Suppose that $\mathcal{I}$ is WS prealgebraizable and stable and consider $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. By stability, we get that $\bigcap_{T\in\mathcal{T}} \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Since $\{\overleftarrow{T} : T \in \mathcal{T}\} \cup \{\overleftarrow{T'}\} \subseteq \mathrm{ThSys}(\mathcal{I})$, we get, by WS prealgebraizability, that $\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$. This proves that $\mathcal{I}$ is left c-reflective and, hence, that it is WLC prealgebraizable. ■

Theorem 273 *For stable π-institutions the weak prealgebraizability hierarchy collapses to the classes of weakly family prealgebraizable and weakly system/left prealgebraizable classes that are related as follows*

WF Prealgebraizable

↓

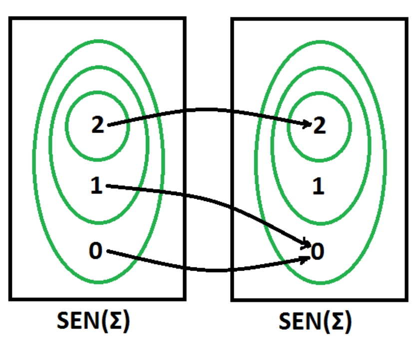
WS Prealgebraizable

Proof: This follows by Theorem 271 and Proposition 272. ■

Now we look at an example to verify that WFR prealgebraizable π-institutions form a proper subclass of WLC prealgberaizable π-institutions.

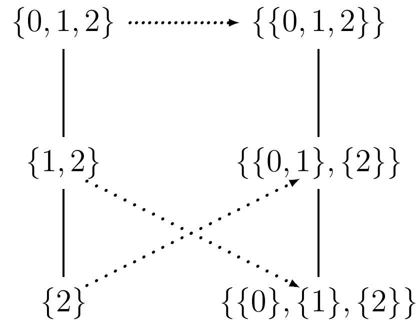
Example 274 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object Σ and a single non-identity morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial clone.*



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by setting $\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}$.

It has three theory families $T :=$ Thm$(\mathcal{I})$, T' and $T'' :=$ SEN, with $T_\Sigma = \{2\}$, $T'_\Sigma = \{1,2\}$ and $T''_\Sigma = \{0,1,2\}$, but only two theory systems T and T'', *since* $\overleftarrow{T'} = T$. *A look at the lattice of theory families and the corresponding Leibniz congruence systems shows that it is prealgebraic and left completely reflective.*



On the other hand, it is not family reflective, since $\Omega(T') \le \Omega(T)$, *but* $T' \nleq T$. *So* $\mathcal{I}$ *it is WLC prealgebraizable, but not WFR prealgebraizable.*

For WLC prealgebraizable π-institutions we have the following transfer theorem.

Theorem 275 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is WLC prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, the Leibniz operator on* $\mathcal{A}$ *is monotone on* $\mathcal{I}$*-filter systems and left completely order reflecting on* $\mathcal{I}$*-filter families, i.e.,*

$$T \le T' \text{ implies } \Omega^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T'), \text{ for all } T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A});$$
$$\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T') \text{ implies } \bigcap_{T \in \mathcal{T}} \overleftarrow{T} \le \overleftarrow{T'}, \text{ for all } \mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}).$$

Proof: The "if" direction follows by specializing to $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account Lemma 51.

For the "only if" suppose that $\mathcal{I}$ is WLC prealgebraizable and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system. By definition, $\mathcal{I}$ is prealgebraic and left completely reflective. Thus, by Theorem 179, the Leibniz operator on the $\mathcal{I}$-filter systems of $\mathcal{A}$ is monotone and, by Theorem 240, the Leibniz operator on the $\mathcal{I}$-filter families of $\mathcal{A}$ is left completely order reflecting. ∎

Moreover, we obtain the following characterization theorem:

Theorem 276 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is WLC prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left completely order reflecting surjection that restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: Suppose, first, that $\mathcal{I}$ is WLC prealgebraizable. Then, it is a fortiori WS prealgebraizable. Thus, by Theorem 256, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order embedding. So it suffices to show that it extends to a left completely order reflecting surjection from $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ onto $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. Since $\mathcal{I}$ is left completely order reflective, by Theorem 275, this mapping is left completely order reflecting. That it is also surjective may be seen by the same argument used in the proof of Theorem 267.

Suppose, conversely, that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left completely order reflecting surjection that restricts to an order embedding

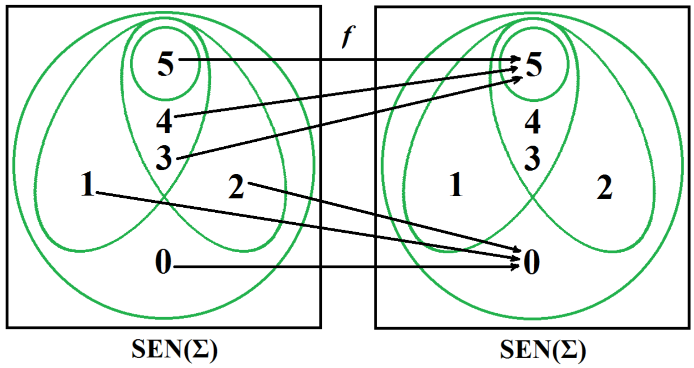
$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Then, by Theorem 256, $\mathcal{I}$ is WS prealgebraizable. Thus, in particular, it is prealgebraic. The fact that $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{F})$ is left completely order reflecting ensures that the Leibniz operator on $\mathrm{ThFam}(\mathcal{I})$ is left completely order reflecting. Thus $\mathcal{I}$ is also completely order reflective and, therefore, it is WLC prealgebraizable. ■

We switch to the left vertical arrow in the diagram. We present an example to verify that WLC prealgebraizable π-institutions form a proper subclass of WLR prealgebraizable π-institutions.

Example 277 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3,4,5\}$ *and*
$$\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = \mathrm{SEN}^\flat(f)(2) = 0,$$
$$\mathrm{SEN}^\flat(f)(3) = \mathrm{SEN}^\flat(f)(4) = \mathrm{SEN}^\flat(f)(5) = 5;$$
- $N^\flat$ *is the category of natural transformations generated by the two unary natural transformations* $\sigma^\flat, \tau^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$, *with*
$$\sigma^\flat_\Sigma, \tau^\flat_\Sigma : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SEN}^\flat(\Sigma)$$
defined by
  - $\sigma^\flat_\Sigma(3) = 1$ *and* $\sigma^\flat_\Sigma(x) = 0$, *for all* $x \in \{0,1,2,4,5\}$;
  - $\sigma^\flat_\Sigma(4) = 2$ *and* $\sigma^\flat_\Sigma(x) = 0$, *for all* $x \in \{0,1,2,3,5\}$.



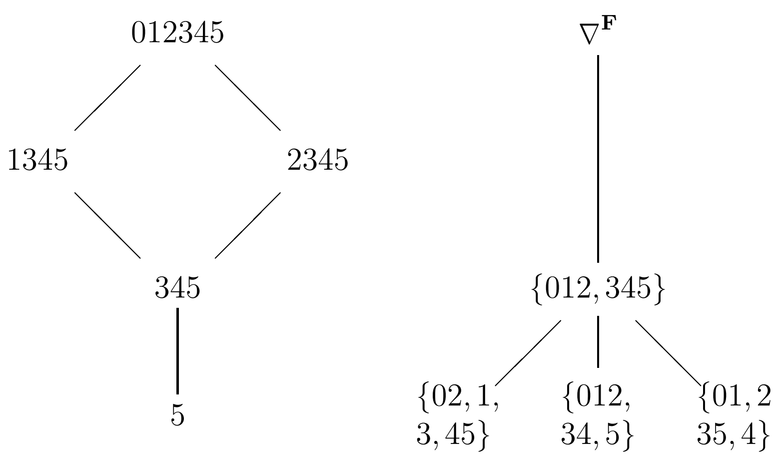
Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by setting

$$\mathcal{C}_\Sigma = \{\{5\}, \{3,4,5\}, \{1,3,4,5\}, \{2,3,4,5\}, \{0,1,2,3,4,5\}\}.$$

$\mathcal{I}$ has five theory families but only three theory systems. The action of $\overleftarrow{}$ on theory families is given by the following table.

T	$\overleftarrow{T}$
$\{5\}$	$\{5\}$
$\{3,4,5\}$	$\{3,4,5\}$
$\{1,3,4,5\}$	$\{3,4,5\}$
$\{2,3,4,5\}$	$\{3,4,5\}$
$\{0,1,2,3,4,5\}$	$\{0,1,2,3,4,5\}$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



From the diagram, it is clear that $\mathcal{I}$ is prealgebraic, i.e., that, for all $T, T' \in \mathrm{ThSys}(\mathcal{I})$, $T \le T'$ implies $\Omega(T) \le \Omega(T')$. Moreover, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$, if $\Omega(T) \le \Omega(T')$, then $\overleftarrow{T} \le \overleftarrow{T'}$, i.e., $\mathcal{I}$ is left reflective. Therefore,

$\mathcal{I}$ if WLR prealgebraizable. On the other hand, setting, $T^1 = \{\{1,3,4,5\}\}$, $T^2 = \{\{2,3,4,5\}\}$ and $T' = \{\{5\}\}$, we get

$$\begin{array}{rcl}\Omega(T^1) \cap \Omega(T^2) & = & \{\{02,1,3,45\}\} \cap \{\{01,2,35,4\}\} \\ & = & \Delta^{\mathbf{F}} \\ & \leq & \{\{012,34,5\}\} = \Omega(T'),\end{array}$$

whereas

$$\overleftarrow{T^1} \cap \overleftarrow{T^2} = \{\{3,4,5\}\} \cap \{\{3,4,5\}\} = \{\{3,4,5\}\} \nleq \{\{5\}\} = \overleftarrow{T'}.$$

Hence, $\mathcal{I}$ is not left completely reflective and, thus, a fortiori, it fails to be WLC prealgebraizable.

For WLR prealgebraizable π-institutions we have the following transfer theorem.

Theorem 278 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is WLR prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, the Leibniz operator on $\mathcal{A}$ is monotone on $\mathcal{I}$-filter systems and left order reflecting on $\mathcal{I}$-filter families, i.e.,*

$$\begin{array}{c} T \leq T' \text{ implies } \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T'), \text{ for all } T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}); \\ \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T') \text{ implies } \overleftarrow{T} \leq \overleftarrow{T'}, \text{ for all } T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}). \end{array}$$

Proof: The "if" direction follows by specializing to $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account Lemma 51.

For the "only if" suppose that $\mathcal{I}$ is WLR prealgebraizable and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system. By definition, $\mathcal{I}$ is prealgebraic and left reflective. Thus, by Theorem 179, the Leibniz operator on the $\mathcal{I}$-filter systems of $\mathcal{A}$ is monotone and, by Theorem 225, the Leibniz operator on the $\mathcal{I}$-filter families of $\mathcal{A}$ is left order reflecting. ■

Moreover, we obtain the following characterization theorem:

Theorem 279 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is WLR prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left order reflecting surjection that restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: Suppose, first, that $\mathcal{I}$ is WLR prealgebraizable. Then, it is, a fortiori, WS prealgebraizable. Thus, by Theorem 256, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order embedding. So it suffices to show that it extends to a left order reflecting surjection from $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ onto $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. Since $\mathcal{I}$ is left reflective, by Theorem 278, this mapping is left order reflecting. That it is also surjective may be seen by the same argument used in the proof of Theorem 267.

Suppose, conversely, that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left order reflecting surjection that restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Then, by Theorem 256, $\mathcal{I}$ is WS prealgebraizable. So, it is prealgebraic. The fact that $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is left order reflecting ensures that the Leibniz operator on $\mathrm{ThFam}(\mathcal{I})$ is left order reflecting. Thus $\mathcal{I}$ is also left reflective and, hence, WLR prealgebraizable. ∎

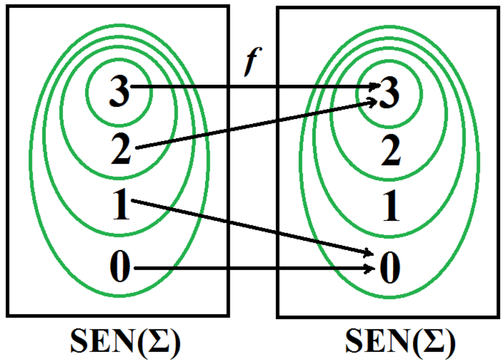
We switch to the bottom left arrow in the diagram. We present an example to verify that WLR prealgebraizable π-institutions form a proper subclass of WLI prealgebraizable π-institutions.

Example 280 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3\}$ *and*
$$\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = 0,$$
$$\mathrm{SEN}^\flat(f)(2) = \mathrm{SEN}^\flat(f)(3) = 3;$$
- $N^\flat$ *is the category of natural transformations generated by the single unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined by* $\sigma^\flat_\Sigma(0) = \sigma^\flat_\Sigma(1) = \sigma^\flat_\Sigma(3) = 0$ *and* $\sigma^\flat_\Sigma(2) = 1$.

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

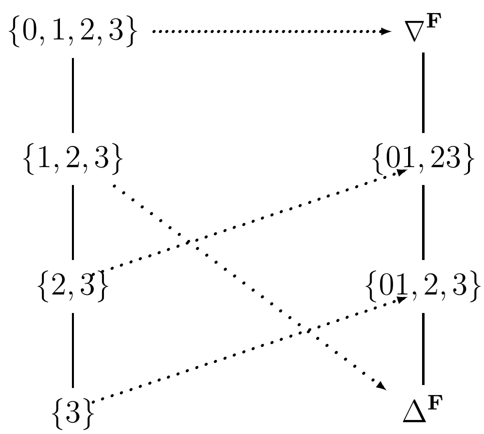
$$\mathcal{C}_\Sigma = \{\{3\}, \{2,3\}, \{1,2,3\}, \{0,1,2,3\}\}.$$



$\mathcal{I}$ has four theory families but only three theory systems. The action of $\leftarrow$ on theory families is given by the following table.

T	$\overleftarrow{T}$
$\{3\}$	$\{3\}$
$\{2,3\}$	$\{2,3\}$
$\{1,2,3\}$	$\{2,3\}$
$\{0,1,2,3\}$	$\{0,1,2,3\}$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



From the diagram, it is clear that $\mathcal{I}$ is prealgebraic, i.e., that, for all $T,T' \in \mathrm{ThSys}(\mathcal{I})$, $T \leq T'$ implies $\Omega(T) \leq \Omega(T')$. Moreover, for all $T,T' \in \mathrm{ThFam}(\mathcal{I})$, the implication $\Omega(T) = \Omega(T')$ implies $\overleftarrow{T} = \overleftarrow{T'}$ holds trivially, since no two different theory families share a common Leibniz congruence system. Hence, $\mathcal{I}$ is left injective. We conclude that $\mathcal{I}$ if WLI prealgebraizable. On the other hand, setting, $T = \{\{1,2,3\}\}$ and $T' = \{\{3\}\}$, we get

$$\Omega(T) = \Delta^{\mathbf{F}} \leq \{\{01,2,3\}\} = \Omega(T'),$$

whereas

$$\overleftarrow{T} = \{\{2,3\}\} \nleq \{\{3\}\} = \overleftarrow{T'}.$$

Hence, $\mathcal{I}$ is not left reflective and, therefore, a fortiori, it is not WLR prealgebraizable.

For WLI prealgebraizable π-institutions we have the following transfer theorem.

Theorem 281 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is WLI prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, the Leibniz operator on $\mathcal{A}$ is monotone on $\mathcal{I}$-filter systems and left injective on $\mathcal{I}$-filter families, i.e.,*

$$T \leq T' \text{ implies } \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T'), \text{ for all } T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A});$$
$$\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T') \text{ implies } \overleftarrow{T} = \overleftarrow{T'}, \text{ for all } T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}).$$

Proof: The "if" direction follows by specializing to $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account Lemma 51.

For the "only if" suppose that $\mathcal{I}$ is WLI prealgebraizable and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system. By definition, $\mathcal{I}$ is prealgebraic and left injective. Thus, by Theorem 179, the Leibniz operator on the $\mathcal{I}$-filter systems of $\mathcal{A}$ is monotone and, by Theorem 214, the Leibniz operator on the $\mathcal{I}$-filter families of $\mathcal{A}$ is left injective. ■

For WLI prealgebraizable π-institutions, we obtain the following characterization theorem:

Theorem 282 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is WLI prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left injective surjection that restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: Suppose, first, that $\mathcal{I}$ is WLI prealgebraizable. Then, it is, a fortiori, WS prealgebraizable. Thus, by Theorem 256, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a lattice embedding. So it suffices to show that it extends to a left injective surjection from $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ onto $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. Since $\mathcal{I}$ is left injective, by Theorem 281, this mapping is left injective. That it is also surjective may be seen by the same argument used in the proof of Theorem 267.

Suppose, conversely, that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left injective surjection that restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

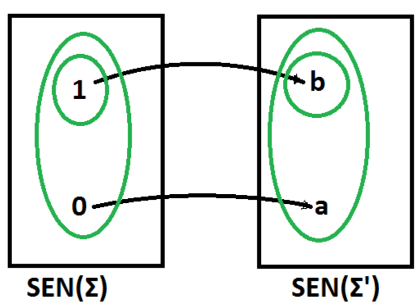
Then, by Theorem 256, $\mathcal{I}$ is WS prealgebraizable. Thus, in particular, it is prealgebraic. The fact that $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is left injective ensures that the Leibniz operator on $\mathrm{ThFam}(\mathcal{I})$ is left injective. Thus $\mathcal{I}$ is also left injective and, hence, WLI prealgebraizable. ■

We turn next to the bottom right arrow in the diagram.

We know by Proposition 208 that WFI π-institutions form a subclass of the class of WLI π-institutions. Moreover, we know by Theorem 269 that, if $\mathcal{I}$ is WLI prealgebraizable and systemic, then it is WFI prealgebraizable. We give now an example showing that the inclusion of the class of WFI prealgebraizable π-institutions into the class of WLI prealgebraizable π-institutions is proper.

Example 283 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*,* $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$*,* $\mathrm{SEN}^\flat(f)(1) = b$*;*
- $N^\flat$ *is the trivial clone.*



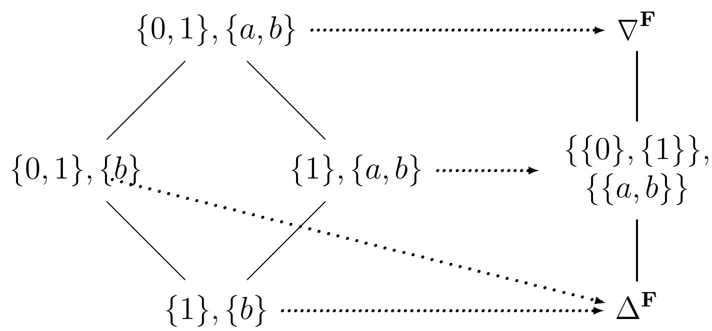
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\},\{a,b\}\}.$$

The table yielding the action of $\overleftarrow{\ }$ *on theory families is shown below.*

$\leftarrow$	$\{b\}$	$\{a,b\}$
$\{1\}$	$\{1\},\{b\}$	$\{1\},\{a,b\}$
$\{0,1\}$	$\{1\},\{b\}$	$\{0,1\},\{a,b\}$

The accompanying diagram gives the structure of the lattice of theory families and the corresponding Leibniz congruence systems.



From the diagram one can check that the Leibniz operator is monotone on theory systems and left injective on theory families. Thus, the π-institution is prealgebraic and left injective, i.e., WLI prealgebraizable.

On the other hand, letting $T = \{\{1\},\{b\}\}$ and $T' = \{\{0,1\},\{b\}\}$, we have $\Omega(T) = \Omega(T')$, but $T \neq T'$, whence $\mathcal{I}$ is not family injective and, therefore, it is not WFI prealgebraizable.

We look now at the very bottom arrow of the diagram. By Theorem 273, if $\mathcal{I}$ is a WS prealgebraizble and stable π-institution, then it is WLI prealgebraizable. We provide, next, an example to show that these two classes are different, i.e., the class of WLI prealgebraizable π-institutions is properly included in that of WS prealgebraizble π-institutions.

Example 284 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

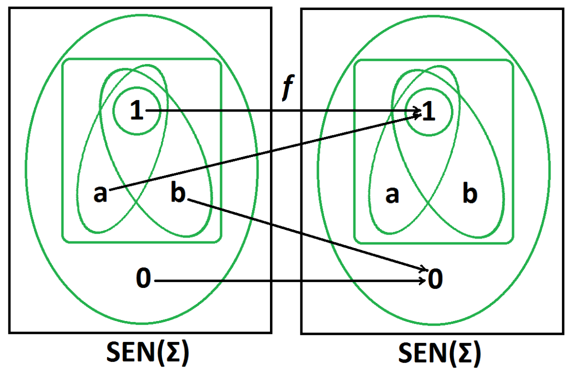
- $\mathbf{Sign}^\flat$ *is the category with a single object Σ and a single non-identity morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, a, b, 1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(a) = 1$, $\mathrm{SEN}^\flat(f)(b) = 0$ *and* $\mathrm{SEN}^\flat(f)(1) = 1$;
- $N^\flat$ *is the category of natural transformations generated by the two binary natural transformations $\wedge, \vee : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ defined by the following tables:*

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be the π-institution, defined by setting

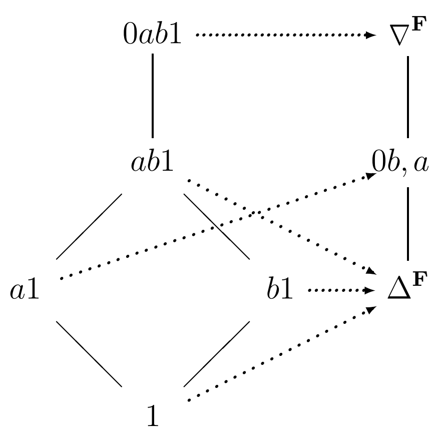
$$C_\Sigma = \{\{1\},\{a,1\},\{b,1\},\{a,b,1\},\{0,a,b,1\}\}.$$



The following table gives the theory families and the theory systems of the π-institution $\mathcal{I}$:

T	$\overleftarrow{T}$
$\{1\}$	$\{1\}$
$\{a,1\}$	$\{a,1\}$
$\{b,1\}$	$\{1\}$
$\{a,b,1\}$	$\{a,1\}$
$\{0,a,b,1\}$	$\{0,a,b,1\}$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



Since $\Omega(\overleftarrow{\{\{a,b,1\}\}}) = \Omega(\{\{a,1\}\}) = \{\{0,b\},\{a,1\}\} \neq \Delta^{\mathbf{F}} = \Omega(\{\{a,b,1\}\})$, *we conclude that* $\mathcal{I}$ *is not stable.*

Note that, since $\{\{1\}\}$, $\{\{a,1\}\}$ *and* $\mathrm{SEN}^\flat$ *are the only theory systems of* $\mathcal{I}$, *the Leibniz operator* $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is an order isomorphism. Hence,* $\mathcal{I}$ *is both prealgebraic and system injective, i.e., it is WS prealgebraizable. On the other hand, we have*

$$\Omega(\{\{a,b,1\}\}) = \Delta^{\mathbf{F}} = \Omega(\{\{b,1\}\}),$$

but

$$\overleftarrow{\{\{a,b,1\}\}} = \{\{a,1\}\} \neq \{\{1\}\} = \overleftarrow{\{\{b,1\}\}},$$

whence, $\mathcal{I}$ is not left injective and, hence, it fails to be WLI prealgebraizable.

4.3 Weak Algebraizability

We now shift attention to classes of π-institutions that are defined as a result of interactions between the various kinds of injectivity, reflectivity and complete reflectivity, on the one hand, and protoalgebraicity on the other. A priori, based on the ordering of the various injectivity, reflectivity and complete reflectivity properties, we have ten classes, which are defined below and whose hierarchy is shown in the accompanying diagram.

Definition 285 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

- *$\mathcal{I}$ is* **weakly system injective algebraizable** *or* **WSI Algebraizable***, for short, if it is system injective and protoalgebraic.*
- *$\mathcal{I}$ is* **weakly left injective algebraizable** *or* **WLI Algebraizable***, for short, if it is left injective and protoalgebraic.*
- *$\mathcal{I}$ is* **weakly family injective algebraizable** *or* **WFI Algebraizable***, for short, if it is family injective and protoalgebraic.*
- *$\mathcal{I}$ is* **weakly right injective algebraizable** *or* **WRI Algebraizable***, for short, if it is right injective and protoalgebraic.*

Definition 286 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

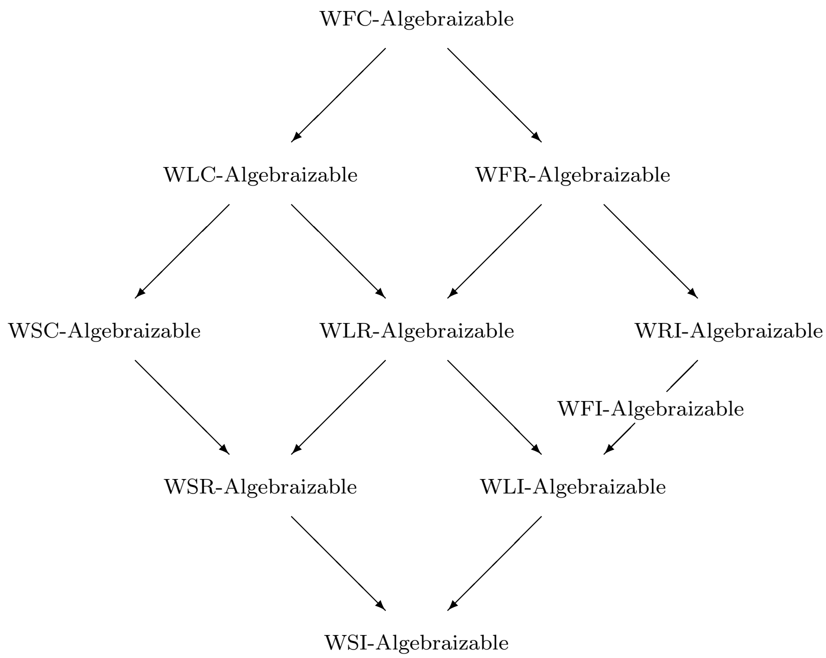
- *$\mathcal{I}$ is* **weakly system reflective algebraizable** *or* **WSR Algebraizable***, for short, if it is system reflective and protoalgebraic.*
- *$\mathcal{I}$ is* **weakly left reflective algebraizable** *or* **WLR Algebraizable***, for short, if it is left reflective and protoalgebraic.*
- *$\mathcal{I}$ is* **weakly family reflective algebraizable** *or* **WFR Algebraizable***, for short, if it is family reflective and protoalgebraic.*

Definition 287 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

- *$\mathcal{I}$ is* **weakly system completely reflective algebraizable** *or* **WSC Algebraizable***, for short, if it is system completely reflective and protoalgebraic.*

- *$\mathcal{I}$ is* **weakly left completely reflective algebraizable** *or* **WLC Algebraizable**, *for short, if it is left completely reflective and protoalgebraic.*

- *$\mathcal{I}$ is* **weakly family completely reflective algebraizable** *or* **WFC Algebraizable**, *for short, if it is family completely reflective and protoalgebraic.*



In view of the remarks made about terminology at the beginning of Section 4.2, the naming conventions here should be fairly obvious. The only difference is that the term "prealgebraizable" has been replaced by the term "algebraizable" to reflect the fact that the condition that the π-institution be prealgebraic is being replaced in the definitions by that of being protoalgebraic.

Recall from Theorem 248 that, under prealgebraicity, the properties of being system injective, system reflective and system completely reflective coincide. A similar result holds for the properties of family injectivity, right injectivity, family reflectivity and family complete reflectivity under the assumption of protoalgebraicity.

Theorem 288 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *If $\mathcal{I}$ is protoalgebraic, then the following statements are equivalent:*

(a) $\mathcal{I}$ is family injective;

(b) $\mathcal{I}$ is family reflective;

(c) $\mathcal{I}$ is family completely reflective;

(d) $\mathcal{I}$ is right injective.

Proof:

(a)⇒(b) Suppose that $\mathcal{I}$ is fanily injective. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then we get $\Omega(T) = \Omega(T) \cap \Omega(T')$. Moreover, by Lemma 23, $\Omega(T) \cap \Omega(T') \leq \Omega(T \cap T')$. On the other hand, by protoalgebraicity, we have $\Omega(T \cap T') \leq \Omega(T)$ and $\Omega(T \cap T') \leq \Omega(T')$, whence $\Omega(T \cap T') \leq \Omega(T) \cap \Omega(T')$. We conclude that

$$\Omega(T) = \Omega(T) \cap \Omega(T') = \Omega(T \cap T').$$

Now we use family injectivity to get $T = T \cap T'$. Therefore, $T \leq T'$. So $\mathcal{I}$ is also family reflective.

(b)⇒(c) Suppose, next, that $\mathcal{I}$ is family reflective. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. Since $\mathcal{I}$ is protoalgebraic, i.e., Ω is monotone on theory families, we have, for all $T \in \mathcal{T}$, $\Omega(\bigcap \mathcal{T}) \leq \Omega(T)$. Therefore, we get

$$\Omega(\bigcap_{T \in \mathcal{T}} T) \leq \bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T').$$

Since, by hypothesis, $\mathcal{I}$ is family reflective, we get $\bigcap_{T \in \mathcal{T}} T \leq T'$. Thus, $\mathcal{I}$ is family completely reflective.

(c)⇒(d) Suppose that $\mathcal{I}$ is family completely reflective. By Proposition 243, it is family reflective, and, then, by Proposition 228, it is right injective.

(d)⇒(a) Suppose, finally, that $\mathcal{I}$ is right injective. Then, by Proposition 208, it is family injective.

■

Theorem 288 shows that four of the classes in the previous diagram coincide.

Corollary 289 *The classes of WFI, WRI, WFR and WFC algebraizable π-institutions coincide.*

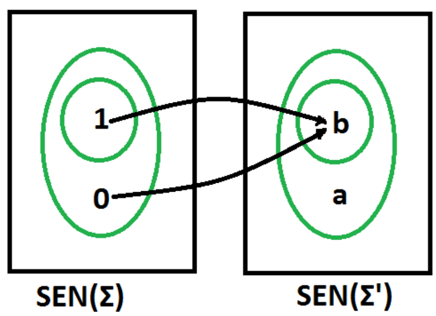
Given an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ and a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ based on $\mathbf{F}$, we use the term **weakly family algebraizable** (ot **WF algebraizable** for short) for $\mathcal{I}$ if it is protoalgebraic and family injective (or, equivalently, right injective or family reflective or family completely reflective), i.e., if the Leibniz operator is monotone and injective on theory families: For all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$\begin{array}{rcl} T \leq T' & \text{implies} & \Omega(T) \leq \Omega(T'); \\ \Omega(T) = \Omega(T') & \text{implies} & T = T'. \end{array}$$

We revisit a previously constructed example to give a WF algebraizable π-institution. Note that the π-institution in question is systemic. As we will see in Theorem 291, this is no coincidence!

Example 290 *Define the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$*,* $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = b = \mathrm{SEN}^\flat(f)(1)$*;*
- $N^\flat$ *is the trivial clone.*

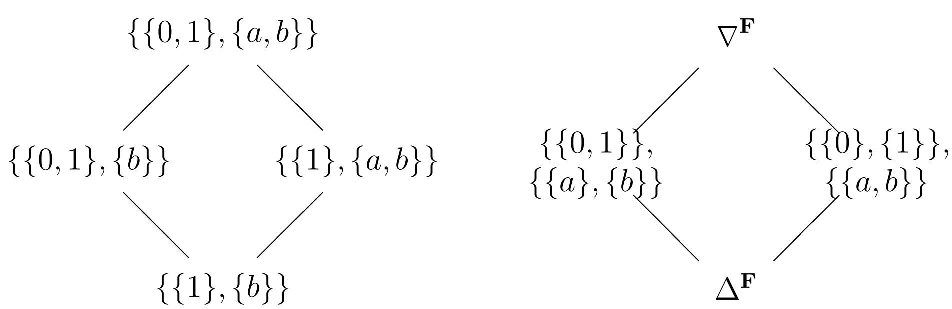


Specify the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

$$\mathcal{C}_\Sigma = \{\{1\}, \{0, 1\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\}, \{a, b\}\}.$$

Notice that every theory family is a theory system, whence $\mathcal{I}$ *is systemic.*

The following diagrams show the lattices of theory families and of the corresponding Leibniz congruence systems:



$\mathcal{I}$ *is protoalgebraic and family injective. Therefore, it is a WF algebraizable* π*-institution.*

We show that a π-institution that is WF algebraizable is necessarily systemic.

Theorem 291 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is WF algebraizable, then it is systemic.*

Proof: Suppose that $\mathcal{I}$ is WF algebraizable. Let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, by Proposition 42, $T, \overleftarrow{T} \in \mathrm{ThFam}(\mathcal{I})$, such that $\overleftarrow{T} \leq T$. Thus, by protoalgebraicity, we get $\Omega(\overleftarrow{T}) \leq \Omega(T)$. But, by Proposition 20, it is always the case that $\Omega(T) \leq \Omega(\overleftarrow{T})$. Therefore, we have $\Omega(\overleftarrow{T}) = \Omega(T)$. Thus, by family injectivity, we conclude that $\overleftarrow{T} = T$. Therefore $T \in \mathrm{ThSys}(\mathcal{I})$. We conclude that $\mathrm{ThFam}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$ and, hence, $\mathcal{I}$ is systemic. ■

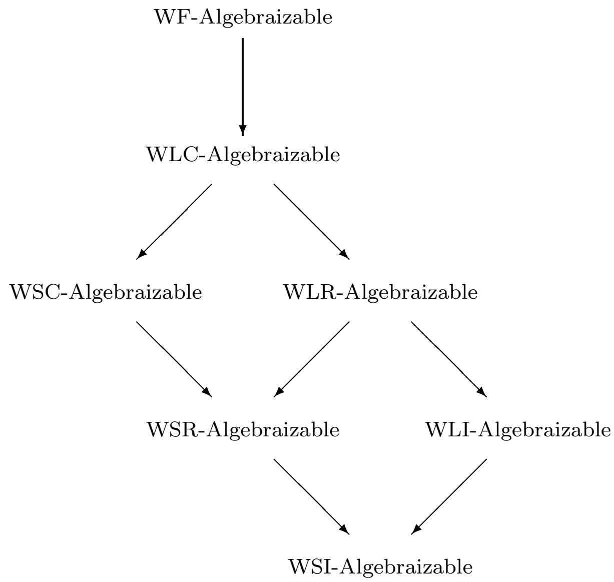
An interesting consequence of Theorem 291 is an exact characterization of those WS prealgebraizable π-institutions that are WF algebraizable.

Corollary 292 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then* $\mathcal{I}$ *is WF algebraizable if and only if it is WS prealgebraizable and systemic.*

Proof: Suppose $\mathcal{I}$ is WF algebraizable. Then, by Theorem 291, it is systemic. Moreover, by definition, its Leibniz operator is monotone and injective on theory families. Thus, it is also monotone and injective on theory systems. So $\mathcal{I}$ is WS prealgebraizable.

Suppose conversely, that $\mathcal{I}$ is WS prealgebraizable and systemic. Then, by definition, its Leibniz operator is monotone and injective on theory systems. But, by systemicity, the collection of theory systems coincides with the collection of theory families. Therefore, the Leibniz operator is monotone and injective on theory families. It follows, by definition, that $\mathcal{I}$ is WF algebraizable. ■

We pause to give an updated version of the hierarchical diagram regarding weak algebraizability classes:



We present examples to show that the class of weakly family algebraizable π-institutions is properly included in both the class of protoalgebraic and that of family completely reflective π-institutions.

Example 293 *Consider the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *defined as follows:*

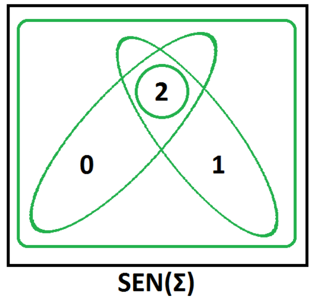
- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the clone of natural transformations on* $\mathrm{SEN}^\flat$ *generated by the the two unary natural transformations* $\sigma^\flat, \tau^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *given by the following table:*

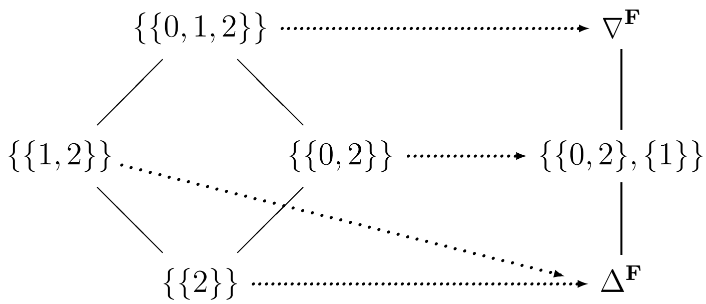
$x \in \mathrm{SEN}^\flat(\Sigma)$	$\sigma^\flat_\Sigma(x)$	$\tau^\flat_\Sigma(x)$
0	0	0
1	1	2
2	0	2

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$*, by setting*

$$\mathcal{C}_\Sigma = \{\{2\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

The lattice of theory families of $\mathcal{I}$ *and the associated Leibniz congruence systems are shown in the diagram.*

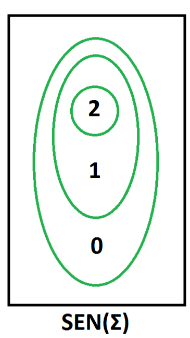




The Leibniz operator is monotone, but not injective on theory families. Therefore, we conclude that $\mathcal{I}$ is protoalgebraic but that it fails to be WF algebraizable.

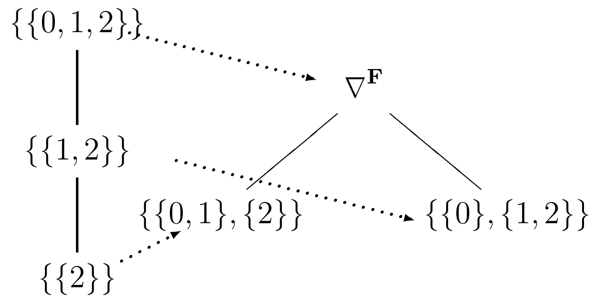
Example 294 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by setting* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}$.

The lattice of theory families and the associated Leibniz congruence systems (in block form) are shown in the diagram.



It is clear from these that the Leibniz operator is completely order reflecting on the theory families of $\mathcal{I}$, but it is not monotone. It follows that $\mathcal{I}$ is family completely reflective but not protoalgebraic. Therefore, $\mathcal{I}$ is family completely reflective, but fails to be WF algebraizable.

The properties defining weak family algebraizability transfer from theory families to filter families over arbitrary algebraic systems.

Theorem 295 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is WF algebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, the Leibniz operator on* $\mathcal{A}$ *is monotone and injective on* $\mathcal{I}$*-filter families, i.e., for all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$\begin{array}{rcl} T \leq T' & \textit{implies} & \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T'); \\ \Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T') & \textit{implies} & T = T'. \end{array}$$

Proof: Suppose, first, that the displayed implications hold for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $\mathcal{I}$-filter families T, T' on $\mathcal{A}$. By taking $\mathcal{A} = \mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and keeping in mind Lemma 51, we conclude that the Leibniz operator is monotone and injective on all theory families of $\mathcal{I}$. Thus, by definition, $\mathcal{I}$ is WF algebraizable.

Suppose, conversely, that $\mathcal{I}$ is WF algebraizable. Then, by Theorem 288, it is protoalgebraic and family completely reflective. Thus, by Theorems 179 and 240, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, the Leibniz operator $\Omega^{\mathcal{A}}$ is monotone and completely order reflecting on the $\mathcal{I}$-filter families of $\mathcal{A}$. Thus, by Propositions 243 and 228, the Leibniz operator is monotone and injective on the $\mathcal{I}$-filter families of $\mathcal{A}$. ■

We showed in Theorem 256 that WS prealgebraizability is equivalent to the Leibniz operator $\Omega^{\mathcal{A}}$ over an arbitrary $\mathbf{F}$-algebraic system $\mathcal{A}$ establishing an order embedding from the lattice of $\mathcal{I}$-filter systems on $\mathcal{A}$ into the poset of all $\mathrm{AlgSys}^*(\mathcal{I})$-congruence systems on $\mathcal{A}$. We show, next, that WF algebraizability has a similar characterization. Namely, it can be characterized by the fact that the Leibniz operator $\Omega^{\mathcal{A}}$ over an arbitrary **F**-algebraic system $\mathcal{A}$ establishes an order isomorphism from the lattice of $\mathcal{I}$-filter families of $\mathcal{A}$ into the lattice of all $\mathrm{AlgSys}^*(\mathcal{I})$-congruence systems on $\mathcal{A}$.

Recall the function

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

that we have introduced before Theorem 256, that restricts to a well-defined function from $\mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ into $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$.

Theorem 296 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is WF algebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism.

Proof: Suppose, first, that, for every **F**-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ is an order isomorphism. Then, by Theorem 268, $\mathcal{I}$ is WFR prealgebraizable. To show that it is WF algebraizable, it suffices, by Corollary 292, to show that it is systemic. But, by Theorem 261, every WFR prealgebraizable π-institution is systemic.

Suppose, conversely, that $\mathcal{I}$ is WF algebraizable. Then it is, a fortiori, WFR prealgebraizable. Therefore, by Theorem 268, for every **F**-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism. ∎

We now have the following corollary to the effect that the classes of WF algebraizable π-institutions and of WFR prealgebraizable π-institutions coincide.

Corollary 297 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is WF algebraizable if and only if it is WFR prealgebraizable.*

Proof: By Theorems 268 and 296, membership in each of these two classes is characterized by $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ being an order isomorphism, for every **F**-algebraic system $\mathcal{A}$. ∎

In light of Corollary 297, we shall call both the class of WF algebraizable π-institutions and the class of WFR prealgebraizable π-institutions by the term **weakly family algebraizable** or **WF algebraizable**, for short.

We now work towards a sweeping contraction of the classes appearing in the weak algebraizability hierarchy. To accomplish this, we provide, first, a characterization of the class of WSI algebraizable π-institutions. Namely, we show that a π-institution is WSI algebraizable if and only if it is stable and $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism, for all **F**-algebraic systems $\mathcal{A}$.

Theorem 298 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is WSI algebraizable if and only if* $\mathcal{I}$ *is stable and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism.

Proof: Suppose, first, that $\mathcal{I}$ is WSI algebraizable. Then it is, by definition, protoalgebraic and, hence, by Lemma 170, it is stable. Also, it is, a fortiori, WS prealgebraizable. Thus, by Theorem 256, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order embedding. So it suffices to show that $\Omega^{\mathcal{A}}$ on $\mathcal{I}$-filter systems on $\mathcal{A}$ is surjective. To this end, consider $\theta \in \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. Then $\mathcal{A}^\theta \in \mathrm{AlgSys}^*(\mathcal{I})$. Thus, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta)$, such that $\Omega^{\mathcal{A}^\theta}(T) = \Delta^{\mathcal{A}^\theta}$. Applying the inverse of $\langle I, \pi^\theta \rangle : \mathcal{A} \to \mathcal{A}^\theta$, we get $\pi^{\theta^{-1}}(\Omega^{\mathcal{A}^\theta}(T)) = \pi^{\theta^{-1}}(\Delta^{\mathcal{A}^\theta})$. So, by Proposition 24, $\Omega^{\mathcal{A}}(\pi^{\theta^{-1}}(T)) = \theta$. By stability and Theorem 154, we get that $\Omega^{\mathcal{A}}(\overleftarrow{\pi^{\theta^{-1}}(T)}) = \theta$. Hence, by Lemma 6, $\Omega^{\mathcal{A}}(\pi^{\theta^{-1}}(\overleftarrow{T})) = \theta$. Now, by Proposition 53 and Lemma 51, $\pi^{\theta^{-1}}(\overleftarrow{T}) \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$. Therefore, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is surjective, as was to be shown.

Suppose, conversely, that $\mathcal{I}$ is stable and for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism. In particular, we have that $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is an order isomorphism. This yields immediately that the Leibniz operator is injective on theory systems and, hence $\mathcal{I}$ is system injective. The isomorphism also yields that the Leibniz operator is monotone on theory systems, i.e., that $\mathcal{I}$ is prealgebraic. So it suffices to show that it is monotone on all theory families. To this end, let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$. Then, by Lemma 1, $\overleftarrow{T} \leq \overleftarrow{T'}$. Thus, taking into account Proposition 42, by prealgebraicity, $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Now using the postulated stability of $\mathcal{I}$, we get $\Omega(T) \leq \Omega(T')$. Therefore, $\mathcal{I}$ is protoalgebraic. ■

Using this characterization of WSI algebraizable π-institutions, we now show that the class of WSI algebraizable π-institutions and that of WLC algebraizable π-institutions coincide. This causes a collapse of both squares of the diagram describing the weak algebraizability hierarchy (i.e., of all six bottom classes) into a single class.

Theorem 299 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is WSI algebraizable, then it is WLC algebraizable.*

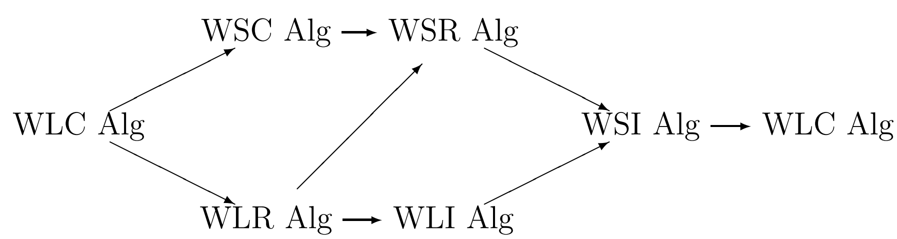
Proof: Suppose $\mathcal{I}$ is WSI algebraizable. Then it is, by definition, protoalgebraic. Moreover, by Theorem 298, it is stable and, for every **F**-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism. To see that it is WLC algebraizable, it suffices to show that it is left completely reflective. So consider $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. By protoalgebraicity, we get $\bigcap_{T \in \mathcal{T}} \Omega(T) = \Omega(\bigcap_{T \in \mathcal{T}} T)$. Thus, $\Omega(\bigcap_{T \in \mathcal{T}} T) \leq \Omega(T')$. By stability, $\Omega(\overleftarrow{\bigcap_{T \in \mathcal{T}} T}) \leq \Omega(\overleftarrow{T'})$. By Proposition 42 and the hypothesis, $\overleftarrow{\bigcap_{T \in \mathcal{T}} T} \leq \overleftarrow{T'}$. By Lemma 3, $\bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$. We conclude that the Leibniz operator is left completely order reflecting on theory families and, therefore, $\mathcal{I}$ is WLC algebraizable. ∎

Corollary 300 *The classes of WLC, WSC, WLR, WSR, WLI and WSI algebraizable π-institutions coincide.*

Proof: According to Theorem 299 and because of the hierarchy of the defining properties, we get the following diagram, where the arrows denote inclusions.



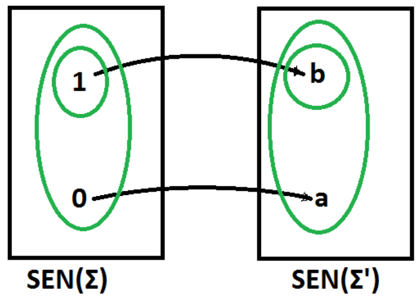
The conclusion readily follows. ∎

Because of Corollary 300, we shall call a π-institution belonging to any of these six classes **weakly** (**system**) **algebraizable**, or **W algebraizable** (sometimes **WS algebraizable**) for short.

We revisit an example showing that the inclusion of the class of WF algebraizable π-institutions into the class of WS algebraizable π-institutions is proper.

Example 301 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$*;*
- $N^\flat$ *is the trivial clone.*



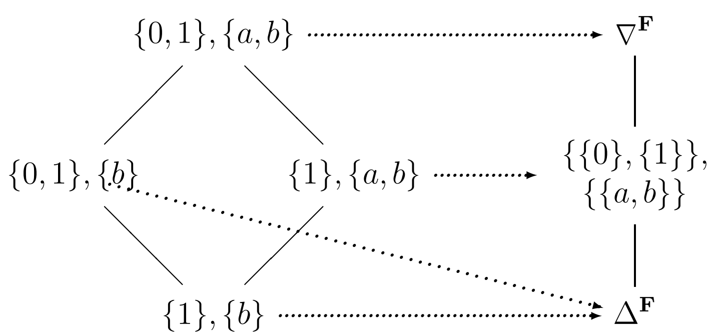
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{1\}, \{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\}, \{a,b\}\}.$$

The table yielding the action of $\overleftarrow{\ }$ *on theory families is shown below.*

$\leftarrow$	$\{b\}$	$\{a,b\}$
$\{1\}$	$\{1\},\{b\}$	$\{1\},\{a,b\}$
$\{0,1\}$	$\{1\},\{b\}$	$\{0,1\},\{a,b\}$

The accompanying diagram gives the structure of the lattice of theory families and the corresponding Leibniz congruence systems.



From the diagram one can check that the Leibniz operator is monotone on theory families and left injective on theory families (or injective on theory systems). Thus, the π*-institution is protoalgebraic and system injective, i.e., WS algebraizable. On the other hand,* $\mathcal{I}$ *is, obviously, not family injective and, therefore, it is not WF algebraizable.*

As with other classes in the hierarchy, we have a number of transfer theorems for weakly algebraizable π-institutions. We choose here to formalize the result by providing the two most powerful implications:

Theorem 302 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then the following statements are equivalent:*

(a) *$\mathcal{I}$ is weakly system algebraizable;*

(b) *For every $\mathbf{F}$-algebraic system $\mathcal{A}$, the Leibniz operator on $\mathcal{A}$ is monotone and left completely order reflecting on $\mathcal{I}$-filter families, i.e., for all $\mathcal{T} \cup \{T, T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,*

$$\begin{array}{rcl} T \leq T' & \textit{implies} & \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T'); \\ \bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T') & \textit{implies} & \bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}. \end{array}$$

(c) *For every $\mathbf{F}$-algebraic system $\mathcal{A}$, the Leibniz operator on $\mathcal{A}$ is monotone on $\mathcal{I}$-filter families and injective on $\mathcal{I}$-filter systems.*

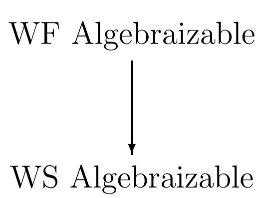
Proof:

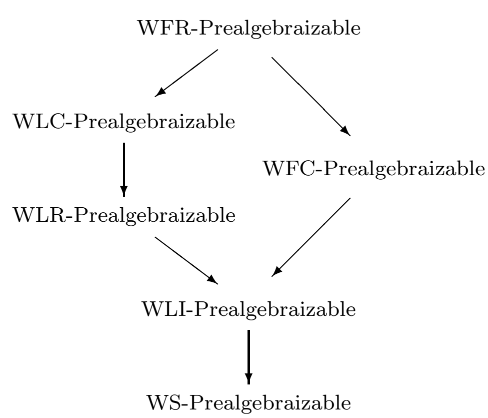
(a)⇒(b) Suppose that $\mathcal{I}$ is weakly system algebraizable. Then, by Theorem 300, it is protoalgebraic and family completely reflective. Thus, by Theorems 179 and 240, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, the Leibniz operator $\Omega^{\mathcal{A}}$ is monotone and left completely order reflecting on $\mathcal{I}$-filter families.

(b)⇒(c) Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system. By hypothesis, the Leibniz operator is monotone and left completely order reflecting on the $\mathcal{I}$-filter families of $\mathcal{A}$. By Propositions 243 and 228, the Leibniz operator is monotone on the $\mathcal{I}$-filter families and injective on the $\mathcal{I}$-filter systems of $\mathcal{A}$.

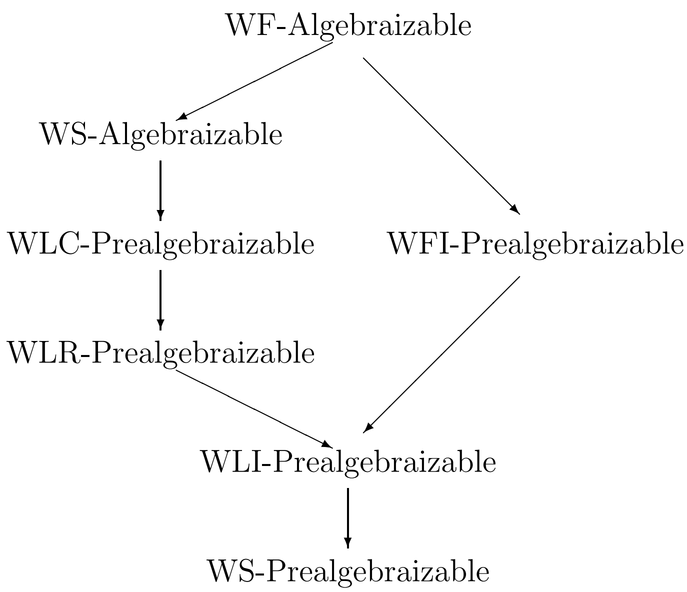
(c)⇒(a) Suppose that, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, the Leibniz operator is monotone on the $\mathcal{I}$-filter families and injective on the $\mathcal{I}$-filter systems of $\mathcal{A}$. By taking $\mathcal{A} = \mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and keeping in mind Lemma 51, we conclude that the Leibniz operator is monotone on all theory families and injective on all theory systems of $\mathcal{I}$. Thus, by definition, $\mathcal{I}$ is weakly system algebraizable. ∎

We are left now with a weak algebraizability hierarchy consisting of only two classes as shown on the left below. On the right is reprinted the weak prealgebraizability hierarchy, as revealed in the previous section.





Recalling that, by Corollary 297, the classes of WF algebraizable π-institutions and WFR prealgebraizable π-institutions coincide and noting that the class of weakly algebraizable π-institutions (coinciding with WLC algebraizable π-institutions) is included in the class of WLC prealgebraizable π-institutions, we get the following complete picture of weak (pre)algebraizability.



We close with an example that shows that the class of weakly system algebraizable π-institutions is properly contained in the class of WLC prealgebraizable π-institutions.

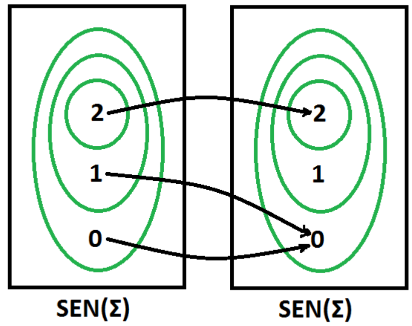
Example 303 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial clone.*

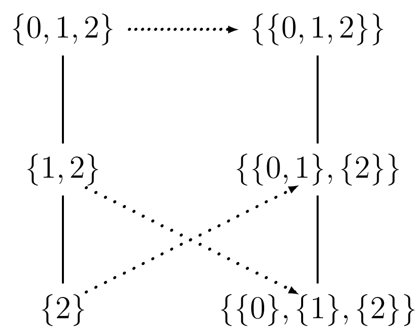
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}$.

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$*:*

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$



The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



$\mathcal{I}$ is prealgebraic, but not protoalgebraic. Moreover, it is left completely reflective. Thus, it is WLC prealgebraizable but not WS algebraizable.

Chapter 5

The Semantic Leibniz Hierarchy: Extensionality

5.1 Introduction

Protoalgebraic sentential logics were introduced by Czelakowski in [26, 29] and studied by Blok and Pigozzi [28]. Perhaps the best known among several existing characterizations of protoalgebraicity is the property of monotonicity of the Leibniz operator on the filters of a logic over arbitrary algebras (of the same algebraic type). Equivalential logics were introduced by Prucnal and Wroński [19] and studied by Czelakowski [22, 24]. A Leibniz characterization asserts that a sentential logic is equivalential iff, for every algebra, the Leibniz operator on its filters is monotone and commutes with inverse endomorphisms. More details may be found in Section 3.4 of [69], Sections 6.1-6.3 of [86] and Chapters 1-3 of [64]. In addition, whereas protoalgebraicity, in conjunction with injectivity of the Leibniz operator, is used to define weakly algebraizable logics [62], the stronger condition of equivalentiality, coupled with injectivity of the Leibniz operator, is used to define algebraizable logics [35, 54]. Section 3.4 of [69], Sections 6.4 and 6.5 of [86] and Chapter 4 of [64] provide detailed information about these classes of sentential logics.

In Section 3.3, we studied classes of π-institutions defined using monotonicity properties of the Leibniz operator. In Chapter 4, we used monotonicity to define the weak algebraizability hierarchy of π-institutions. The present chapter introduces analogs of the property of equivalentiality for π-institutions, strengthening monotonicity. Further, by replacing monotonicity by equivalentiality, one obtains from the weak algebraizability hierarchy the hierarchy of algebraizable π-institutions.

Strengthening protoalgebraicity to equivalentiality involves adding, on top of monotonicity properties, some property that emulates (or forms an analog of) the property of commutativity of the Leibniz operator with inverse endomorphisms. This desideratum informs the structure of the current chapter. In Sections 5.2 and 5.3, properties that can be used as analogs of commutativity with inverse endomorphisms in the framework of π-institutions are discussed and some of their interrelationships are explored. These are combined with monotonicity in Section 5.4 to define equivalentiality. Finally, in Sections 5.5 and 5.6, we obtain the (pre)algebraizability hierarchy of π-institutions, based on the weak (pre)algebraizability hierarchy, studied in Chapter 4. More details, by section, follow.

In Section 5.2, we study *extensionality*. Recall that, given an algebraic system $\mathbf{F}$ and a sentence family X of $\mathbf{F}$, one may determine the subsystem $\langle X\rangle$ of $\mathbf{F}$ generated by X. Moreover, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ and a subsystem $\mathbf{F}'$ of $\mathbf{F}$, $\mathbf{F}'$ determines a π-subinstitution $\mathcal{I}' = \langle \mathbf{F}', C'\rangle$ of $\mathcal{I}$ which is obtained by restricting the action of C on $\mathbf{F}'$. For details on these constructions, see Section 2.12. A π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is said to be *family extensional* if, roughly speaking, the action of the Leibniz operator on theory families of subinstitutions can be obtained as the restriction of the Leibniz operator of $\mathcal{I}$ on the universe corresponding to the subinstitution. More precisely, $\mathcal{I}$ is *family extensional* if, for every sentence family X of $\mathbf{F}$ and every theory family T of $\mathcal{I}$, $\Omega^{\langle X\rangle}(T \cap \langle X\rangle) = \Omega(T) \cap \langle X\rangle^2$. *System extensionality* is defined similarly, except that T is allowed to range over theory systems only, instead of over arbitrary theory families. By definition, family extensionality implies system extensionality. Further, system extensionality, combined with stability, implies family extensionality. The significance of extensionality stems, in part, from allowing important properties of a π-institution to be inherited by its subinstitutions. Indicative of this phenomenon are the facts that, under system extensionality, stability in inherited and, under family (system, respectively) extensionality, prealgebraicity (protoalgebraicity, respectively) is also inherited. Both versions of extensionality transfer. A seemingly weaker version of extensionality is *2-extensionality*. Roughly speaking, 2-extensionality is extensionality restricted to universes generated by two sentences over the same signature. More precisely, a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is *family 2-extensional* if, for every signature Σ, all Σ-sentences ϕ, ψ and every theory family T, $\langle \phi, \psi\rangle \in \Omega_\Sigma(T)$ if and only if $\langle \phi, \psi\rangle \in \Omega_\Sigma^{\langle \phi,\psi\rangle}(T \cap \langle \phi, \psi\rangle)$. If T is quantified over theory systems, *system 2-extensionality* is obtained instead. Despite its apparent weakness in comparison to extensionality, it turns out that a π-institution is family/system extensional if and only if it is family/system 2-extensional, respectively. Extensionality is one manifestation of the property that is used as an analog of commutativity with inverse endomorphisms, employed in the sentential framework to define equivalentiality. An alternative formalization, closer in spirit to commutativity, is introduced in Section 5.3.

In Section 5.3, we study *Leibniz commutativity* or, simply, *commutativity*, a property closer in spirit to the original property used in the sentential context to characterize equivalentiality. Let $\mathbf{F}$ be an algebraic system and X a sentence family of $\mathbf{F}$. Recall the subsystem $\langle X\rangle$ of $\mathbf{F}$ generated by X. A morphism of the form $\langle I, \alpha\rangle : \langle X\rangle \to \mathbf{F}$, where I is the identity functor on signatures, is called an *extension*. Recall also that, if $\mathbf{F}$ happens to be the base algebraic system of a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$, then X induces a π-subinstitution $\mathcal{I}^{\langle X\rangle} = \langle \langle X\rangle, C^{\langle X\rangle}\rangle$ of $\mathcal{I}$ based on $\langle X\rangle$, whose closure system is essentially C restricted on $\langle X\rangle$ and whose theory families are obtained by the theory families of $\mathcal{I}$ via restriction on $\langle X\rangle$. In this enriched context, an extension $\langle I, \alpha\rangle : \langle X\rangle \to \mathbf{F}$ is called *logical*, denoted by $\langle I, \alpha\rangle : \mathcal{I}^{\langle X\rangle} \to \mathcal{I}$, if it preserves the closure structure in the sense that, for all signatures Σ and all $\Phi \subseteq \langle X\rangle_\Sigma$, $\alpha_\Sigma(C_\Sigma^{\langle X\rangle}(\Phi)) \subseteq C_\Sigma(\alpha_\Sigma(\Phi))$. This condition is tantamount to preservation of theory families under α^{-1}, i.e., to $\alpha^{-1}(T)$ being a theory family of $\mathcal{I}^{\langle X\rangle}$, for every theory family T of $\mathcal{I}$. Logical extensions lay the groundwork for building the notion of (*Leibniz*) *commutativity*. We say that a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is *family commuting* if, for all sentence families X of $\mathbf{F}$, all logical extensions $\langle I, \alpha\rangle : \mathcal{I}^{\langle X\rangle} \to \mathcal{I}$ and all theory families T' of $\mathcal{I}^{\langle X\rangle}$, $\alpha(\Omega^{\langle X\rangle}(T')) \leq \Omega(C(\alpha(T')))$. *System commutativity* applies the same condition on theory systems only. A similar, but not identical in general, property is *inverse* (*Leibniz*) *commutativity*. $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is *family inverse commuting* if, for every sentence family X, all logical extensions $\langle I, \alpha \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$ and all theory families T of $\mathcal{I}$, $\alpha^{-1}(\Omega(T)) = \Omega^{\langle X \rangle}(\alpha^{-1}(T))$. *System inverse commutativity* results by quantifying T over theory systems instead. It is elementary to check, based on the definition of $\mathcal{I}^{\langle X \rangle}$, that injection morphisms $\langle I, j \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$ qualify as logical extensions. This permits establishing that family (system, repectively) inverse commutativity implies family (system, respectively) extensionality. Also, since theory systems form a subclass of theory families, it is obvious that family inverse commutativity is stronger than the system version. In addition, it is shown that the system version, coupled with stability, implies the family version. The last results of Section 5.3 are critical for our further investigations.

Since commutativity and inverse commutativity are used mainly in conjunction with monotonicity properties to obtain equivalentiality, it is important that, under system (family) monotonicity (i.e., pre- and protoalgebraicity, respectively), system (family, respectively) commutativity and system (family, respectively) inverse commutativity coincide. Further, in a result that allows us to switch between commutativity properties and the extensionality properties of Section 5.2, and which strengthens a previously mentioned implication, it is shown that system (family) inverse commutativity is equivalent to system (family, respectively) extensionality. Based on these equivalences and a transfer theorem from Section 5.2, it is also shown that both versions of inverse commutativity transfer. Summarizing, the corresponding (system or family) versions of extensionality and 2-extensionality and of inverse commutativity are equivalent without proviso. On the other hand, for these three to be equivalent to the corresponding commutativity version, a sufficient condition is that the corresponding version of monotonicity holds.

In Section 5.4, we define versions of *equivalentiality*, resulting by combining monotonicity and extensionality properties. Since both come in two flavors, we get, a priori, four potentially different equivalentiality classes. A π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is (*family*) *equivalential* if it is protoalgebraic and family extensional. Weakening protoalgebraicity to prealgebraicity we get *family preequivalentiality*. On the other hand, weakening family to system extensionality, we get *system equivalentiality*. Finally, if both properties are weakened in tandem, we get (*system*) *preequivalentiality*. Equivalentiality, as opposed to preequivalentiality, incorporates protoalgebraicity, which implies stability. But, under stability, the two versions of extensionality coincide. This reasoning shows that family and system equivalentiality are identical properties. So when referring to this property, we use the term *equivalentiality*, without qualification. It turns out to be equivalent to preequivalentiality plus stability. All three distinct versions transfer. We also obtain characterizations of both equivalentiality and preequivalentiality in terms of the Leibniz operator seeing as a mapping between lattices of filter families (systems) and congruence systems over arbitrary algebraic systems.

In Section 5.5, we explore the hierarchy of *prealgebraizable π-institutions*. Prealgebraizability results from weak prealgebraizability when prealgebraicity is strengthened to either family or system preequivalentiality, i.e., when either family or system extensionality is added into the mix. Accordingly, two parallel hierarchies mimicking that of weakly prealgebraizable π-institutions, detailed in Chapter 4, are formed depending on the version of preequivalentiality used. If family preequivalentiality is postulated, we get the five classes of *XF prealgebraizable π-institutions*, whereas, if (system) preequivalentiality is dictated, we get five corresponding *X prealgebraizability* classes, where X is a string reflecting which injectivity, reflectivity or complete reflectivity condition is coupled with preequivalentiality, i.e., X can be one of:

- LC for left complete reflectivity;
- LR for left reflectivity;
- FI for family injectivity;
- LI for left injectivity; and
- S for system (injectivity, reflectivity and complete reflectivity all being equivalent under preequivalentiality).

Systemicity leads to a total collapse of the ten classes into a single class. Stability results to FIF and FI prealgebraizable π-institutions being identified and to a collapse of all remaining eight classes into a single class. Thus, it yields a 2-class hierarchy. After showing that all ten prealgebraizability properties transfer, the section is dedicated to obtaining characterization theorems for each of the classes in terms of the Leibniz operator on arbitrary algebraic systems perceived as a mapping between ordered sets. The ten characterizations can be divided into five pairs, each pair addressing XF and X prealgebraizability for the same X in {LC, LR, FI, LI, S}. Making a somewhat arbitrary choice here, we look at the cases of LR and S to provide a flavor of these results. The interested reader is, of course, referred to the main text for further details on all ten properties. A π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is LRF prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is a left order reflecting surjection commuting with inverse logical extensions, which restricts to an order embedding on filter systems. A subtle, but important, change occurs in most pairs in passing from the XF to the X sibling. $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is LR prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is a left order reflecting surjection, which restricts to an order embedding commuting with inverse logical extensions on filter systems. Along similar lines, we get that $\mathcal{I}$ is SF prealgebraizable iff, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ commutes with inverse logical extensions and restricts to an order embedding on filter systems, whereas $\mathcal{I}$ is S prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order embedding commuting with inverse logical extensions.

In Section 5.6, we examine *algebraizability*. This hierarchy results from weak algebraizability when protoalgebraicity is replaced by equivalentiality. Equivalently, it ensues from prealgebraizability when, instead of imposing family or system preequivalentiality, we insist on the stronger condition of equivalentiality. Exactly due to this strengthening, only two classes may be distinguished here, *family algebraizability*, combining equivalentiality with family injectivity, and (*system*) *algebraizability*, coupling equivalentiality with system injectivity. The family version is equivalent to the system version augmented by systemicity. Both flavors transfer. Finally, a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is family algebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism commuting with inverse logical extensions, whereas it is system algebraizable if and only if it is stable and, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism commuting with inverse logical extensions.

5.2 Extensionality

The first two important ingredients in classifying π-institutions according to their algebraic character were:

- the monotonicity properties of the Leibniz operator, which gave rise to the classes of prealgebraic and protoalgebraic π-institutions, as well as the various classes defined using versions of complete monotonicity;
- the various properties involving injectivity and reflectivity, varying from the weakest, system injectivity, to the strongest, family complete reflectivity.

Two additional important properties are the extensionality of the Leibniz operator and the commutativity of the Leibniz operator, which we now introduce and study. The variants studied here will give rise to classes in the equivalential hierarchy of π-institutions and, based on these, in the semantic hierarchy of algebraizable π-institutions (as opposed to *weak* (pre)algebraizability, studied in Chapter 4).

We first define two versions of the extensionality property and two corresponding versions of 2-extensionality, which is an apparently relaxed version of extensionality, but will be shown to be equivalent to extensionality.

Recall from Section 2.12 that, given an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ and a sentence family $X \in \mathrm{SenFam}(\mathbf{F})$, we denote by $\langle X\rangle = \{\langle X\rangle_\Sigma\}_{\Sigma\in|\mathbf{Sign}^\flat|}$ the universe of $\mathbf{F}$ generated by X, i.e., $\langle X\rangle = \nu(\overrightarrow{X})$.

Definition 304 (Extensionality) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is called* **family extensional** *if, for all* $X \in \mathrm{SenFam}(\mathcal{I})$ *and all* $T \in \mathrm{ThFam}(\mathcal{I})$,
$$\Omega(T) \cap \langle X \rangle^2 = \Omega^{\langle X \rangle}(T \cap \langle X \rangle);$$
- $\mathcal{I}$ *is called* **system extensional** *if, for all* $X \in \mathrm{SenFam}(\mathcal{I})$ *and all* $T \in \mathrm{ThSys}(\mathcal{I})$,
$$\Omega(T) \cap \langle X \rangle^2 = \Omega^{\langle X \rangle}(T \cap \langle X \rangle).$$

Taking into account Proposition 89, one obtains the following equivalent formulations.

Lemma 305 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is family (system) extensional if and only if, for all* $X \in \mathrm{SenFam}(\mathcal{I})$ *and all* $T \in \mathrm{ThFam}(\mathcal{I})$ *(*$T \in \mathrm{ThSys}(\mathcal{I})$*, respectively),*
$$\Omega^{\langle X \rangle}(T \cap \langle X \rangle) \leq \Omega(T) \cap \langle X \rangle^2.$$

Proof: Since, by Proposition 89, for all $X \in \mathrm{SenFam}(\mathcal{I})$ and $T \in \mathrm{ThFam}(\mathcal{I})$, the inclusion
$$\Omega(T) \cap \langle X \rangle^2 \leq \Omega^{\langle X \rangle}(T \cap \langle X \rangle)$$
always holds, we get the statement using the definition. ∎

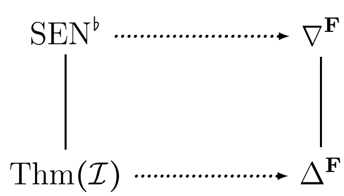
Here is a simple example of a family extensional π-institution.

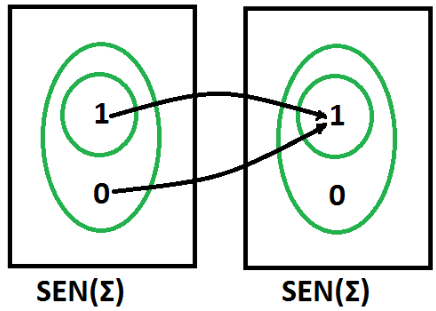
Example 306 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 1$, $\mathrm{SEN}^\flat(f)(1) = 1$*;*
- $N^\flat$ *is the trivial category of natural transformations, consisting of the projections only.*

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution, defined by setting* $\mathcal{C}_\Sigma = \{\{1\}, \{0, 1\}\}$.

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.





Note that the only universes are $\mathrm{Thm}(\mathcal{I}) = \{\{1\}\}$ *and* $\mathrm{SEN}^\flat$. *For the second one,* $\Omega(T) \cap \langle X\rangle^2 = \Omega^{\langle X\rangle}(T \cap \langle X\rangle)$ *holds trivially for all* $T \in \mathrm{ThFam}(\mathcal{I})$, *since both sides boil down to* $\Omega(T)$. *For the first, we have*

$$\Omega^{\mathrm{Thm}(\mathcal{I})}(\mathrm{Thm}(\mathcal{I}) \cap \mathrm{Thm}(\mathcal{I})) = \mathrm{Thm}(\mathcal{I})^2 = \Omega(\mathrm{Thm}(\mathcal{I})) \cap \mathrm{Thm}(\mathcal{I})^2;$$
$$\Omega^{\mathrm{Thm}(\mathcal{I})}(\mathrm{SEN}^\flat \cap \mathrm{Thm}(\mathcal{I})) = \mathrm{Thm}(\mathcal{I})^2 = \Omega(\mathrm{SEN}^\flat) \cap \mathrm{Thm}(\mathcal{I})^2.$$

So $\mathcal{I}$ *is family extensional, that is, for all* $X \in \mathrm{SenFam}(\mathcal{I})$ *and all* $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) \cap \langle X\rangle^2 = \Omega^{\langle X\rangle}(T \cap \langle X\rangle)$.

We present, now, two examples of π-institutions that are not system extensional.

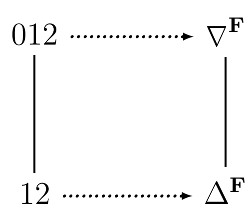
Example 307 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

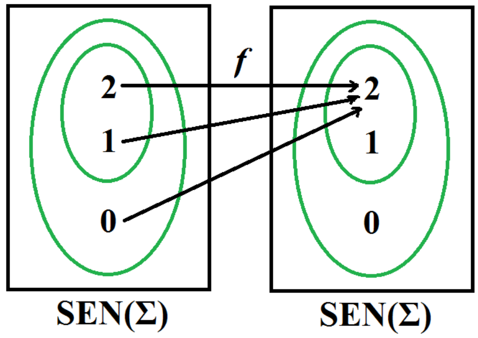
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f)(x) = 2$, *for all* $x \in \{0, 1, 2\}$;
- $N^\flat$ *is the category of natural transformations generated by the binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by the following table:*

$\sigma^\flat_\Sigma$	0	1	2
0	0	0	2
1	0	1	1
2	2	1	2

Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution defined by setting* $C_\Sigma = \{\{1, 2\}, \{0, 1, 2\}\}$.

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.





For the universe $\mathbf{X} = \{\{1,2\}\}$ *and the theory system* $T = \{\{1,2\}\}$*, we get*

$$\Omega(T) \cap \mathbf{X}^2 = \{\{1\},\{2\}\} \subsetneq \{\{1,2\}\} = \Omega^{\mathbf{X}}(T \cap \mathbf{X}).$$

Therefore, $\mathcal{I}$ *is not system extensional.*

And here is a second example of a non-system extensional π-institution.

Example 308 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with the single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, a, b, 1\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the two binary natural transformations* $\wedge : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *and* $\vee : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by the following tables:*

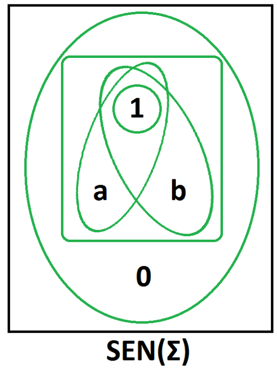
$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

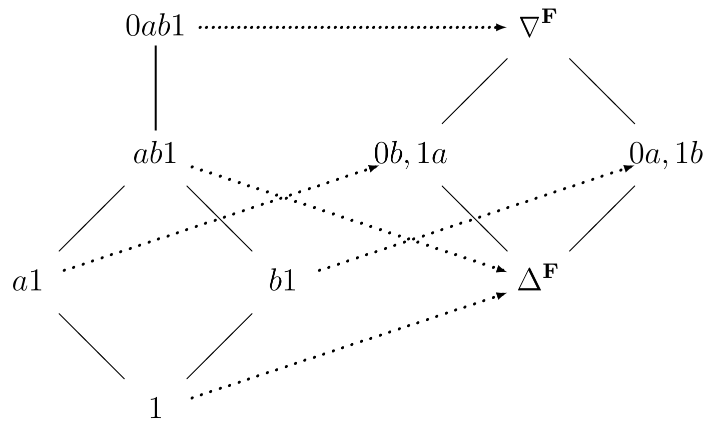
Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution defined by setting*

$$\mathcal{C}_\Sigma = \{\{1\},\{a,1\},\{b,1\},\{a,b,1\},\{0,a,b,1\}\}.$$

The lattice of theory families and the corresponding Leibniz congruence



systems are shown in the diagram.



For the universe $\mathbf{X} = \{\{0, a, 1\}\}$ *and the theory system* $T = \{\{a, b, 1\}\}$, *we get*

$$\Omega(T) \cap \mathbf{X}^2 = \{\{0\}, \{a\}, \{1\}\} \subsetneqq \{\{0\}, \{a, 1\}\} = \Omega^{\mathbf{X}}(T \cap \mathbf{X}).$$

Therefore, $\mathcal{I}$ *is not system extensional and, a fortiori, not family extensional either.*

The following clarifies the relation between family and system extensionality.

Proposition 309 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) *If* $\mathcal{I}$ *is family extensional, then it is system extensional.*

(b) *If* $\mathcal{I}$ *is system extensional and stable, then it is family extensional.*

Proof:

(a) Since all theory systems are also theory families, it follows that every family extensional π-institution is also system extensional.

(b) Suppose that $\mathcal{I}$ is system extensional and stable. Let $X \in \mathrm{SenFam}(\mathcal{I})$ and $T \in \mathrm{ThFam}(\mathcal{I})$. Then we have

$$\begin{array}{rcll}\Omega^{\langle X\rangle}(T \cap \langle X\rangle) & \leq & \Omega^{\langle X\rangle}(\overleftarrow{T \cap \langle X\rangle}) & \text{(by Proposition 20)}\\ & = & \Omega^{\langle X\rangle}(\overleftarrow{T} \cap \langle X\rangle) & \text{(by Lemma 3)}\\ & = & \Omega(\overleftarrow{T}) \cap \langle X\rangle^2 & \text{(by system extensionality)}\\ & = & \Omega(T) \cap \langle X\rangle^2. & \text{(by stability)}\end{array}$$

By Lemma 305, $\mathcal{I}$ is family extensional.

■

According to Proposition 309 we have the following **extensionality hierarchy**:

Family Extensional

↓

System Extensional

The reverse, however, does not hold in general, as the following example, exhibiting a π-institution which is system but not family extensional, shows.

Example 310 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

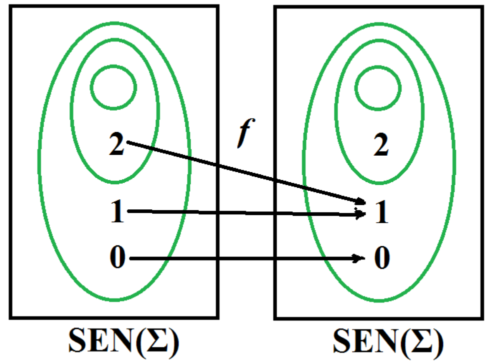
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 1$ *and* $\mathrm{SEN}^\flat(f)(2) = 1$;
- $N^\flat$ *is the category of natural transformations generated by the binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by the following table:*

$\sigma^\flat_\Sigma$	0	1	2
0	1	1	2
1	1	1	1
2	2	1	2

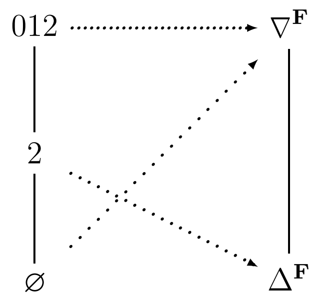
Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution defined by setting*

$$\mathcal{C}_\Sigma = \{\varnothing, \{2\}, \{0,1,2\}\}.$$

$\mathcal{I}$ *has three theory families,* $\overline{\varnothing}$, $T = \{\{2\}\}$ *and* $\mathrm{SEN}^\flat$, *but only* $\overline{\varnothing}$ *and* $\mathrm{SEN}^\flat$ *are theory systems. The lattice of theory families and the corresponding*



Leibniz congruence systems are shown in the diagram.



Moreover, $\mathbf{F}$ *has five universes* $\{\{0\}\}$, $\{\{1\}\}$, $\{\{0,1\}\}$, $\{\{1,2\}\}$ *and* $\{\{0,1,2\}\}$. *Since the only theory systems of* $\mathcal{I}$ *are* $\overline{\varnothing}$ *and* $\mathrm{SEN}^\flat$, *it is trivial to check that* $\mathcal{I}$ *is system extensional.*

For the universe $\mathbf{X} = \{\{0,1\}\}$ *and the theory family* $T = \{\{2\}\}$, *we get*

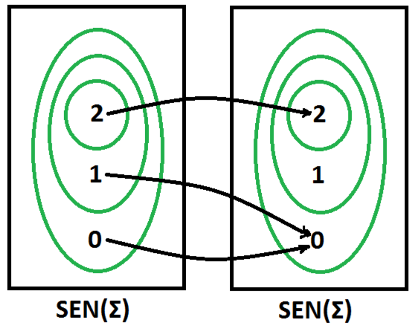
$$\Omega(T) \cap \mathbf{X}^2 = \{\{0\},\{1\}\} \subsetneq \{\{0,1\}\} = \Omega^{\mathbf{X}}(T \cap \mathbf{X}).$$

Therefore, $\mathcal{I}$ *is not family extensional.*

Moreover, as the following example shows, the converse of Part (b) of Proposition 309 does not hold in general, i.e., stability is not necessary for family extensionality.

Example 311 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

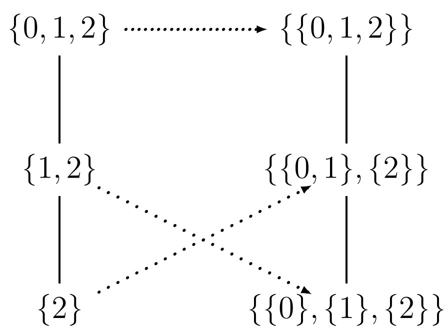
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$;
- $N^\flat$ *is the trivial clone.*



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by setting $\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}$.

$\mathcal{I}$ has three theory families, $\mathrm{Thm}(\mathcal{I})$*,* $T = \{\{1,2\}\}$ *and* $\mathrm{SEN}^\flat$*, but only* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$ *are theory systems.*

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



It is not difficult to check that $\mathcal{I}$ is family extensional, that is, for all $X \in$ $\mathrm{SenFam}(\mathcal{I})$ *and all $T \in$* $\mathrm{ThFam}(\mathcal{I})$*,*

$$\Omega(T) \cap \langle X \rangle^2 = \Omega^{\langle X \rangle}(T \cap \langle X \rangle).$$

In fact, $\mathbf{F}$ has five universes $\{\{0\}\}$, $\{\{2\}\}$, $\{\{0,1\}\}$, $\{\{0,2\}\}$ and $\{\{0,1,2\}\}$, only two of which are proper and non-singletons. $\mathcal{I}$ has three theory families, two of which are different from $\mathrm{SEN}^\flat$*. Thus, there are only four cases to check, shown below, adopting, for brevity, an obvious shorthand notation.*

$$\begin{aligned}
&\Omega(2) \cap (01)^2 = \{\{0,1\}\} = \Omega^{01}(2 \cap 01),\\
&\Omega(12) \cap (01)^2 = \{\{0\},\{1\}\} = \Omega^{01}(12 \cap 01),\\
&\Omega(2) \cap (02)^2 = \{\{0\},\{2\}\} = \Omega^{02}(2 \cap 02),\\
&\Omega(12) \cap (02)^2 = \{\{0\},\{2\}\} = \Omega^{02}(12 \cap 02).
\end{aligned}$$

Clearly, since for $T \in$ $\mathrm{ThFam}(\mathcal{I}) \backslash \mathrm{ThSys}(\mathcal{I})$*,*

$$\Omega(T) = \Delta^{\mathbf{F}} \neq \{\{0,1\},\{2\}\} = \Omega(\mathrm{Thm}(\mathcal{I})) = \Omega(\overleftarrow{T}),$$

$\mathcal{I}$ is not stable.

A related result is that, under system extensionality, stability is inherited by π-subinstitutions of a given π-institution.

Proposition 312 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a system extensional* π*-institution based on* $\mathbf{F}$ *and* $\mathbf{F}' = \langle \mathbf{Sign}^\flat, \mathrm{SEN}'^\flat, N'^\flat\rangle \leq \mathbf{F}$ *an algebraic subsystem of* $\mathbf{F}$*. If* $\mathcal{I}$ *is stable, then* $\mathcal{I}' = \langle \mathbf{F}', C'\rangle$ *is also stable.*

Proof: Suppose that $\mathcal{I}$ is system extensional and stable. Let $T \in \mathrm{ThFam}(\mathcal{I})$. Then we have

$$\begin{array}{rcll}\Omega^{\mathbf{F}'}(\overleftarrow{T \cap \mathrm{SEN}'^\flat}) & = & \Omega^{\mathbf{F}'}(\overleftarrow{T} \cap \mathrm{SEN}'^\flat) & \text{(by Lemma 3)}\\ & = & \Omega^{\mathbf{F}}(\overleftarrow{T}) \cap (\mathrm{SEN}'^\flat)^2 & \text{(by system extensionality)}\\ & = & \Omega^{\mathbf{F}}(T) \cap (\mathrm{SEN}'^\flat)^2 & \text{(by stability)}\\ & \leq & \Omega^{\mathbf{F}'}(T \cap \mathrm{SEN}'^\flat). & \text{(by Proposition 89)}\end{array}$$

Since, by Proposition 20, the reverse inclusion always holds, we conclude that $\mathcal{I}'$ is also stable. ∎

A similar preservation result, under extensionality, may also be proven with regards to pre- and protoalgebraicity. More precisely, we show that if a π-institution is family extensional and protoalgebraic, then all its π-subinstitutions are also protoalgebraic. Analogously, if a π-institution is system extensional and prealgebraic, then prealgebraicity is inherited by all its π-subinstitutions.

Proposition 313 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is family extensional and protoalgebraic, then, for all* $\mathbf{F}' = \langle \mathbf{Sign}^\flat, \mathrm{SEN}'^\flat, N'^\flat\rangle \leq \mathbf{F}$*,* $\mathcal{I}' = \langle \mathbf{F}', C'\rangle$ *is also protoalgebraic;*

(b) If $\mathcal{I}$ *is system extensional and prealgebraic, then, for all* $\mathbf{F}' = \langle \mathbf{Sign}^\flat, \mathrm{SEN}'^\flat, N'^\flat\rangle \leq \mathbf{F}$*,* $\mathcal{I}' = \langle \mathbf{F}', C'\rangle$ *is also prealgebraic.*

Proof: We only prove Part (a). Part (b) may be proven similarly. Suppose that $\mathcal{I}$ is family extensional and protoalgebraic and let $\mathbf{F}' \leq \mathbf{F}$. If $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$, then, by protoalgebraicity, $\Omega^{\mathbf{F}}(T) \leq \Omega^{\mathbf{F}}(T')$. Thus, $\Omega^{\mathbf{F}}(T) \cap (\mathrm{SEN}'^\flat)^2 \leq \Omega^{\mathbf{F}}(T') \cap (\mathrm{SEN}'^\flat)^2$. Therefore, by family extensionality, $\Omega^{\mathbf{F}'}(T \cap \mathrm{SEN}'^\flat) \leq \Omega^{\mathbf{F}'}(T' \cap \mathrm{SEN}'^\flat)$. By Proposition 87, we conclude that $\mathcal{I}'$ is also protoalgebraic. ∎

There are transfer theorems that hold for both system and family extensionality.

Theorem 314 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is family (system) extensional if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, all* $Y \in \mathrm{SenFam}(\mathcal{A})$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *(*$T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$*, respectively)*

$$\Omega^{\mathcal{A}}(T) \cap \langle Y \rangle^2 = \Omega^{\langle Y \rangle}(T \cap \langle Y \rangle).$$

Proof: We present the proof for theory families. The case of theory systems is similar.

The "if" direction follows by taking $\mathcal{A} = \mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and observing that, in that case, the displayed condition reduces to the definition of family extensionality.

For the "only if", assume that $\mathcal{I}$ is family extensional and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system, $Y \in \mathrm{SenFam}(\mathcal{A})$, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \Omega^{\langle Y \rangle}_{F(\Sigma)}(T \cap \langle Y \rangle).$$

Then we have

$$\begin{array}{lll}
\langle \phi, \psi \rangle & \in & \alpha_\Sigma^{-1}(\Omega^{\langle Y \rangle}_{F(\Sigma)}(T \cap \langle Y \rangle)) \quad \text{(set theory)} \\
& = & \Omega_\Sigma^{\alpha^{-1}(\langle Y \rangle)}(\alpha^{-1}(T) \cap \alpha^{-1}(\langle Y \rangle)) \quad \text{(Corollary 92)} \\
& = & \Omega_\Sigma(\alpha^{-1}(T)) \cap \alpha_\Sigma^{-1}(\langle Y \rangle_{F(\Sigma)})^2 \quad \text{(hypothesis)} \\
& = & \alpha_\Sigma^{-1}(\Omega^{\mathcal{A}}_{F(\Sigma)}(T)) \cap \alpha_\Sigma^{-1}(\langle Y \rangle_{F(\Sigma)})^2 \quad \text{(Proposition 24)} \\
& = & \alpha_\Sigma^{-1}(\Omega^{\mathcal{A}}_{F(\Sigma)}(T) \cap \langle Y \rangle^2_{F(\Sigma)}). \quad \text{(set theory)}
\end{array}$$

Therefore $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \Omega^{\mathcal{A}}_{F(\Sigma)}(T) \cap \langle Y \rangle^2_{F(\Sigma)}$. Since, by Proposition 89, the opposite inclusion always holds, we get, taking into account the surjectivity of $\langle F, \alpha \rangle$, that

$$\Omega^{\mathcal{A}}(T) \cap \langle Y \rangle^2 = \Omega^{\langle Y \rangle}(T \cap \langle Y \rangle).$$

The conclusion now follows. ∎

We define, next, the second property, a seemingly relaxed version of extensionality that we call 2-extensionality.

Definition 315 (2-Extensionality) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is called* **family 2-extensional** *if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(T) \quad \textit{iff} \quad \langle \phi, \psi \rangle \in \Omega_\Sigma^{\langle \phi, \psi \rangle}(T \cap \langle \phi, \psi \rangle);$$

- $\mathcal{I}$ *is called* **system 2-extensional** *if, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(T) \quad \textit{iff} \quad \langle \phi, \psi \rangle \in \Omega_\Sigma^{\langle \phi, \psi \rangle}(T \cap \langle \phi, \psi \rangle).$$

Taking into account Proposition 89, one obtains the following equivalent formulations.

Lemma 316 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is family (system) 2-extensional if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$ *(*$T \in \mathrm{ThSys}(\mathcal{I})$*, respectively), all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\langle \phi, \psi \rangle \in \Omega_\Sigma^{\langle \phi, \psi \rangle}(T \cap \langle \phi, \psi \rangle) \quad \textit{implies} \quad \langle \phi, \psi \rangle \in \Omega_\Sigma(T).$$

Proof: By Proposition 89, for all $T \in \mathrm{ThFam}(\mathcal{I})$, the inclusion

$$\Omega(T) \cap \langle \phi, \psi \rangle^2 \leq \Omega^{\langle \phi, \psi \rangle}(T \cap \langle \phi, \psi \rangle)$$

always holds. Since $\phi, \psi \in \langle \phi, \psi \rangle_\Sigma$, if $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$, then $\langle \phi, \psi \rangle \in \Omega_\Sigma^{\langle \phi, \psi \rangle}(T \cap \langle \phi, \psi \rangle)$.

Thus, 2-extensionality is, by definition, equivalent to

$$\langle \phi, \psi \rangle \in \Omega_\Sigma^{\langle \phi, \psi \rangle}(T \cap \langle \phi, \psi \rangle) \quad \text{implies} \quad \langle \phi, \psi \rangle \in \Omega_\Sigma(T).$$

■

It turns out that the corresponding versions of extensionality and 2-extensionality are equivalent. That extensionality implies 2-extensionality is fairly straightforward.

Proposition 317 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is family (system) extensional, then it is family (system, respectively) 2-extensional.*

Proof: We present the proof for theory families. The case of theory systems is similar. Suppose $\mathcal{I}$ is family extensional and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in \Omega_\Sigma^{\langle \phi, \psi \rangle}(T \cap \langle \phi, \psi \rangle)$. Then, by family extensionality, we get that $\langle \phi, \psi \rangle \in \Omega_\Sigma(T) \cap \langle \phi, \psi \rangle_\Sigma^2$, which implies that $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. Thus, by Lemma 316, $\mathcal{I}$ is family 2-extensional. ■

The full equivalence is given in the following

Theorem 318 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is family (system) extensional if and only if it is family (system, respectively) 2-extensional.*

Proof: Again we prove only the equivalence of the family versions of the two properties, since the system versions can be proven similarly.

The "only if" was the content of Proposition 317. For the "if", suppose that $\mathcal{I}$ is family 2-extensional and let $X \in \mathrm{SenFam}(\mathcal{I})$, $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \notin \Omega_\Sigma(T) \cap \langle X \rangle_\Sigma^2$.

If $\langle \phi, \psi \rangle \notin \langle X \rangle_\Sigma^2$, then, a fortiori, $\langle \phi, \psi \rangle \notin \Omega_\Sigma^{\langle X \rangle}(T \cap \langle X \rangle)$, and we are done.

If, on the other hand, $\langle\phi,\psi\rangle\in\langle X\rangle_\Sigma^2$, then, we have $\langle\phi,\psi\rangle\notin\Omega_\Sigma(T)$. Thus, by hypothesis, $\langle\phi,\psi\rangle\notin\Omega_\Sigma^{\langle\phi,\psi\rangle}(T\cap\langle\phi,\psi\rangle)$. So, by Theorem 19, there exist $\sigma^\flat\in N^\flat$, $\Sigma'\in|\mathbf{Sign}^\flat|$, $f\in\mathbf{Sign}^\flat(\Sigma,\Sigma')$ and $\vec{\chi}\in\langle\phi,\psi\rangle_{\Sigma'}$, such that (without loss of generality)

$$\begin{array}{l}\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi})\in T_{\Sigma'}\cap\langle\phi,\psi\rangle_{\Sigma'}\\ \qquad\text{but}\quad\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi})\notin T_{\Sigma'}\cap\langle\phi,\psi\rangle_{\Sigma'}.\end{array}$$

Since $\phi,\psi\in\langle X\rangle_\Sigma$ and $\vec{\chi}\in\langle\phi,\psi\rangle_{\Sigma'}$, we get that

$$\begin{array}{l}\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi})\in T_{\Sigma'}\cap\langle X\rangle_{\Sigma'}\\ \qquad\text{but}\quad\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi})\notin T_{\Sigma'}\cap\langle X\rangle_{\Sigma'}.\end{array}$$

Thus, again by Theorem 19, we get $\langle\phi,\psi\rangle\notin\Omega_\Sigma^{\langle X\rangle}(T\cap\langle X\rangle)$. Hence, $\Omega^{\langle X\rangle}(T\cap\langle X\rangle)\le\Omega(T)\cap\langle X\rangle^2$. We now conclude, using Lemma 305, that $\mathcal{I}$ is family extensional. ■

5.3 Leibniz Commutativity

Another important property is that of commutativity with a special type of logical morphism, which we now introduce and study.

Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and $X\in\mathrm{SenFam}(\mathbf{F})$. An algebraic system morphism of the form $\langle I,\alpha\rangle:\langle X\rangle\to\mathbf{F}$, where $I:\mathbf{Sign}^\flat\to\mathbf{Sign}^\flat$ is the identity functor, will be called an **extension**.

Further, given a π-institution $\mathcal{I}=\langle\mathbf{F},C\rangle$ based on $\mathbf{F}$, an extension $\langle I,\alpha\rangle:\langle X\rangle\to\mathbf{F}$ is said to be **logical** if it is a logical morphism $\langle I,\alpha\rangle:\mathcal{I}^{\langle X\rangle}\to\mathcal{I}$, where $\mathcal{I}^{\langle X\rangle}=\langle\langle X\rangle,C^{\langle X\rangle}\rangle$ is the π-subinstitution of $\mathcal{I}$ induced by $\langle X\rangle$. In other words $\langle I,\alpha\rangle:\langle X\rangle\to\mathbf{F}$ is a logical extension if, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\Phi\subseteq\langle X\rangle_\Sigma$,

$$\alpha_\Sigma(C_\Sigma^{\langle X\rangle}(\Phi))\subseteq C_\Sigma(\alpha_\Sigma(\Phi)).$$

This is abbreviated to $\alpha(C^{\langle X\rangle}(\Phi))\le C(\alpha(\Phi))$.

Using Lemma 47, we get the following characterization:

Corollary 319 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a π-institution based on* $\mathbf{F}$*,* $X\in\mathrm{SenFam}(\mathbf{F})$ *and* $\langle I,\alpha\rangle:\langle X\rangle\to\mathbf{F}$ *an extension.* $\langle I,\alpha\rangle:\mathcal{I}^{\langle X\rangle}\to\mathcal{I}$ *is a logical extension if and only if*

$$\alpha^{-1}(T)\in\mathrm{ThFam}(\mathcal{I}^{\langle X\rangle}),\ \textit{for all } T\in\mathrm{ThFam}(\mathcal{I}).$$

Proof: Immediate by Lemma 47. ■

We now define the two notions of Leibniz commutativity that we wish to study.

Definition 320 (Leibniz Commutativity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is called* **family (Leibniz) commuting** *if the Leibniz operator on theory families commutes with logical extensions, i.e., if, for every* $X \in \mathrm{SenFam}(\mathcal{I})$, *all logical extensions* $\langle I, \alpha \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$ *and all* $T' \in \mathrm{ThFam}(\mathcal{I}^{\langle X \rangle})$,
$$\alpha(\Omega^{\langle X \rangle}(T')) \leq \Omega(C(\alpha(T')));$$
- $\mathcal{I}$ *is called* **system (Leibniz) commuting** *if the Leibniz operator on theory systems commutes with logical extensions, i.e., if, for every* $X \in \mathrm{SenFam}(\mathcal{I})$, *all logical extensions* $\langle I, \alpha \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$ *and all* $T' \in \mathrm{ThSys}(\mathcal{I}^{\langle X \rangle})$,
$$\alpha(\Omega^{\langle X \rangle}(T')) \leq \Omega(C(\alpha(T'))).$$

We now give a useful characterization of those two properties, in the case of protoalgebraic and of prealgebraic π-institutions, respectively. To do this, however, we need some preliminary work. First, we note that injection morphisms are logical extensions.

Lemma 321 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *For all* $X \in \mathrm{SenFam}(\mathcal{I})$, $\langle I, j \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$ *is a logical extension, where* $\langle I, j \rangle : \langle X \rangle \to \mathbf{F}$ *is the injection morphism.*

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, we have
$$j^{-1}(T) = T \cap \langle X \rangle \in \mathrm{ThFam}(\mathcal{I}^{\langle X \rangle}),$$
where the membership follows by Proposition 87. Therefore, by Corollary 319, $\langle I, j \rangle$ is a logical extension. ■

Next we define two alternative versions of Leibniz commutativity, which we term inverse Leibniz commutativity.

Definition 322 (Inverse (Leibniz) Commutativity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is* **family inverse (Leibniz) commuting** *if, for all* $X \in \mathrm{SenFam}(\mathcal{I})$, *all logical extensions* $\langle I, \alpha \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$ *and all* $T \in \mathrm{ThFam}(\mathcal{I})$,
$$\alpha^{-1}(\Omega(T)) = \Omega^{\langle X \rangle}(\alpha^{-1}(T));.$$
- $\mathcal{I}$ *is* **system inverse (Leibniz) commuting** *if, for all* $X \in \mathrm{SenFam}(\mathcal{I})$, *all logical extensions* $\langle I, \alpha \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$ *and all* $T \in \mathrm{ThSys}(\mathcal{I})$,
$$\alpha^{-1}(\Omega(T)) = \Omega^{\langle X \rangle}(\alpha^{-1}(T)).$$

We now show that inverse commutativity implies extensionality. Naturally enough, we have two versions of this implication.

Proposition 323 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is family (system) inverse commuting, then it is family (system, respectively) extensional.*

Proof: We show the family version. The system version is similar.

Assume that $\mathcal{I}$ is family inverse commuting and let $X \in \mathrm{SenFam}(\mathcal{I})$, $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$\langle \phi, \psi \rangle \in \Omega^{\langle X \rangle}_\Sigma (T \cap \langle X \rangle).$$

Considering the injection morphism $\langle I, j \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$, which is a logical extension by Lemma 321, the hypothesis can be rewritten as $\langle \phi, \psi \rangle \in \Omega^{\langle X \rangle}_\Sigma (j^{-1}(T))$. Thus, by inverse family commutativity, $\langle \phi, \psi \rangle \in j^{-1}_\Sigma (\Omega_\Sigma(T))$. But this is equivalent to $\langle \phi, \psi \rangle \in \Omega_\Sigma(T) \cap \langle X \rangle^2_\Sigma$. We conclude, using Lemma 305, that $\mathcal{I}$ is family extensional. ■

It is clear that family inverse commutativity implies system inverse commutativity. We show, next, that under stability, the system and the family versions of inverse commutativity coincide.

Proposition 324 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is family inverse commuting, then it is system inverse commuting;*

(b) If $\mathcal{I}$ *is system inverse commuting and stable, then it is also family inverse commuting.*

Proof: Family inverse commutativity always implies system inverse commutativity. Conversely, assume that $\mathcal{I}$ is stable and system inverse commuting and let $X \in \mathrm{SenFam}(\mathcal{I})$, $\langle I, \alpha \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$ a logical extension and $T \in \mathrm{ThFam}(\mathcal{I})$. Then we have

$$\begin{array}{rcll} \alpha^{-1}(\Omega(T)) & = & \alpha^{-1}(\Omega(\overleftarrow{T})) & \text{(stability)} \\ & = & \Omega^{\langle X \rangle}(\alpha^{-1}(\overleftarrow{T})) & \text{(system inverse commutativity)} \\ & = & \Omega^{\langle X \rangle}(\overleftarrow{\alpha^{-1}(T)}) & \text{(Lemma 6)} \\ & = & \Omega^{\langle X \rangle}(\alpha^{-1}(T)). & \text{(Propositions 323 and 312)} \end{array}$$

Thus, $\mathcal{I}$ is family inverse commuting. ■

Finally, the promised characterization that relates family (system) commutativity with family (system) inverse commutativity under the hypothesis of proto(pre)algebraicity. We present the two results separately for the sake of clarity.

Theorem 325 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a protoalgebraic* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is family commuting if and only if it is family inverse commuting.*

Proof: Note, first, that, for all $X \in \mathrm{SenFam}(\mathcal{I})$, all logical extensions $\langle F, \alpha \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$ and all $T \in \mathrm{ThFam}(\mathcal{I})$, $\alpha^{-1}(\Omega(T))$ is a congruence system on $\langle X \rangle$ that is compatible with $\alpha^{-1}(T)$. Thus, by the maximality property of the Leibniz congruence system, we have, regardless of commutativity, that, for all $X \in \mathrm{SenFam}(\mathcal{I})$, all $\langle I, \alpha \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$ and all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\alpha^{-1}(\Omega(T)) \le \Omega^{\langle X \rangle}(\alpha^{-1}(T)).$$

Therefore, it suffices to show that $\mathcal{I}$ is family commuting if and only if, for all $X \in \mathrm{SenFam}(\mathcal{I})$, all $\langle I, \alpha \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$ and all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\Omega^{\langle X \rangle}(\alpha^{-1}(T)) \le \alpha^{-1}(\Omega(T)).$$

For the "only if" direction, assume that $\mathcal{I}$ is family commuting and let $X \in \mathrm{SenFam}(\mathcal{I})$, $\langle I, \alpha \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$, $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in \Omega^{\langle X \rangle}_\Sigma(\alpha^{-1}(T))$. Then we have

$$\begin{array}{rcll}
\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle & \in & \alpha_\Sigma(\Omega^{\langle X \rangle}_\Sigma(\alpha^{-1}(T))) & \\
& \subseteq & \Omega_\Sigma(C(\alpha(\alpha^{-1}(T)))) & \text{(commutativity)} \\
& \subseteq & \Omega_\Sigma(C(T)) & \text{(protoalgebraicity)} \\
& = & \Omega_\Sigma(T). &
\end{array}$$

We conclude that $\langle \phi, \psi \rangle \in \alpha^{-1}_\Sigma(\Omega_\Sigma(T))$. Therefore, $\mathcal{I}$ is family inverse commuting.

For the "if" direction, assume $\mathcal{I}$ is family inverse commuting and let $X \in \mathrm{SenFam}(\mathcal{I})$, $\langle I, \alpha \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$ and $T' \in \mathrm{ThFam}(\mathcal{I}^{\langle X \rangle})$. Then we have

$$\begin{array}{rcl}
\alpha(\Omega^{\langle X \rangle}(T')) & \le & \alpha(\Omega^{\langle X \rangle}(\alpha^{-1}(C(\alpha(T'))))) \\
& & \qquad \text{(Propositions 323 and 313)} \\
& = & \alpha(\alpha^{-1}(\Omega(C(\alpha(T'))))) \\
& & \qquad \text{(inverse commutativity)} \\
& \le & \Omega(C(\alpha(T'))). \quad \text{(set theory)}
\end{array}$$

Thus, $\mathcal{I}$ is family commuting. ■

Similarly, we may obtain the following analog for the system versions of the corresponding properties.

Theorem 326 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a prealgebraic* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is system commuting if and only if it is system inverse commuting.*

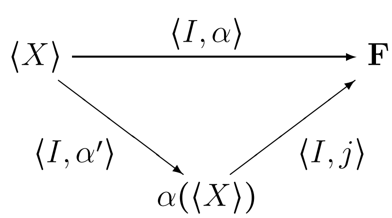
Proof: Along the lines of the proof of Theorem 325. ∎

In Proposition 323 we saw that inverse commutativity implies extensionality. We now show that extensionality is in fact equivalent to inverse commutativity.

Theorem 327 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is family (system) inverse commuting if and only if it is family (system, respectively) extensional.*

Proof: By Proposition 323, family inverse commutativity implies family extensionality.

Suppose, conversely, that $\mathcal{I}$ is family extensional and let $X \in \mathrm{SenFam}(\mathcal{I})$, $\langle I, \alpha \rangle : \mathcal{I}^{\langle X \rangle} \to \mathcal{I}$ be a logical extension and $T \in \mathrm{ThFam}(\mathcal{I})$. We exploit the epi-mono factorization of $\langle I, \alpha \rangle$ provided in Proposition 81:



We have

$$\begin{array}{rcl}
\Omega^{\langle X \rangle}(\alpha^{-1}(T)) & = & \Omega^{\langle X \rangle}(\alpha'^{-1}(j^{-1}(T))) \quad (\langle I, \alpha \rangle = \langle I, j \rangle \circ \langle I, \alpha' \rangle) \\
& = & \Omega^{\langle X \rangle}(\alpha'^{-1}(T \cap \mathrm{SEN}^{\flat\alpha})) \quad (\text{definition of } \langle I, j \rangle) \\
& = & \alpha'^{-1}(\Omega^{\alpha(\langle X \rangle)}(T \cap \mathrm{SEN}^{\flat\alpha})) \quad (\text{Proposition 24}) \\
& = & \alpha'^{-1}(\Omega(T) \cap (\mathrm{SEN}^{\flat\alpha})^2) \quad (\text{extensionality}) \\
& = & \alpha'^{-1}(j^{-1}(\Omega(T))) \quad (\text{definition of } \langle I, j \rangle) \\
& = & \alpha^{-1}(\Omega(T)). \quad (\langle I, \alpha \rangle = \langle I, j \rangle \circ \langle I, \alpha' \rangle)
\end{array}$$

Therefore, $\mathcal{I}$ is family inverse commuting.

The system version can be proven analogously. ∎

Finally, we have the following transfer theorem for inverse commutativity.

Theorem 328 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is family (system) inverse commuting if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, the* π*-institution* $\langle \mathbf{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ *is family (system, respectively) inverse commuting.*

Proof: This follows by combining Theorem 327 with Theorem 314. ∎

5.4 Equivalential π-Institutions

By combining prealgebraicity or protoalgebraicity, on the one hand, with system or family extensionality, on the other, we obtain another hierarchy, the hierarchy of *equivalential π-institutions*. The terminology is built by abiding to the following guidelines:

- The qualification "system" or "family" refers to the version of extensionality employed;

- "preequivalential" or "equivalential" is used depending on whether prealgebraicity or protoalgebraicity is assumed.

According to this nomenclature, we may define four classes of π-institutions as follows:

Definition 329 ((Pre)Equivalentiality) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$.

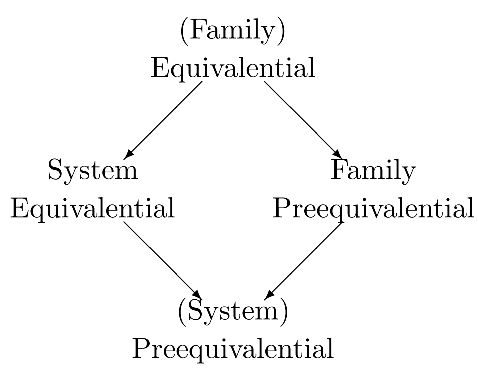
- $\mathcal{I}$ *is* **(family) equivalential** *if it is protoalgebraic and family extensional;*

- $\mathcal{I}$ *is* **system equivalential** *if it is protoalgebraic and system extensional;*

- $\mathcal{I}$ *is* **family preequivalential** *if it is prealgebraic and family extensional;*

- $\mathcal{I}$ *is* **(system) preequivalential** *if it is prealgebraic and system extensional.*

A priori, these four classes form the hierarchy depicted in the diagram.



However, it is easy to show that family and system equivalentiality are equivalent properties.

Proposition 330 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is family equivalential if and only if it is system equivalential.*

Proof: First, if $\mathcal{I}$ is family equivalential, then it is also system equivalential, since family extensionality implies system extensionality.

Suppose, conversely, that $\mathcal{I}$ is system equivalential. Then, by Theorem 175, it is stable and, by definition, it is system extensional, whence, by Proposition 309, it is also family extensional. Since it is protoalgebraic and family extensional, it is family equivalential. ■

Taking into account Proposition 330, we call $\mathcal{I}$ **equivalential** if it is protoalgebraic and (family or system) extensional.

Using this terminology, the hierarchy depicted in the preceding diagram reduces to the following linear **equivalentiality hierarchy**:

Equivalential

↓

Family Preequivalential

↓

(System) Preequivalential

It is easy to see that the separating property of the top level from the bottom level in the hierarchy is exactly stability.

Proposition 331 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is equivalential if and only if it is system preequivalential and stable.*

Proof: If $\mathcal{I}$ is equivalential, then it is trivially system preequivalential. Moreover, it is protoalgebraic and, therefore, by Theorem 175, it is stable.

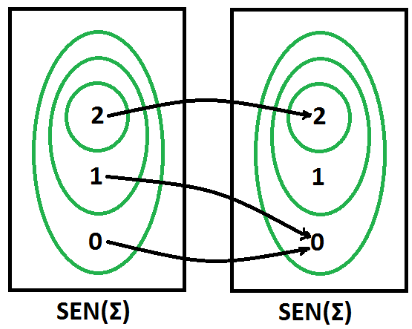
Suppose, conversely, that $\mathcal{I}$ is system prequivalential and stable. Then, by definition, it is system extensional, prealgebraic and stable. Thus, again by Theorem 175, it is system extensional and protoalgebraic and, hence, by Proposition 330, equivalential. ■

Examples are in order to show that the inclusions between the three classes of the equivalentiality hierarchy are proper.

We revisit, first, a familiar example of a π-institution that turns out to be family preequivalential but not equivalential.

Example 332 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

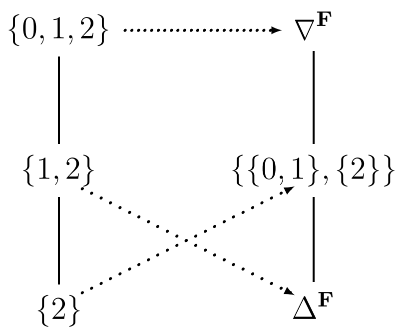
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}$*.*

The theory family $\{\{1, 2\}\}$ *is not a theory system.*

The structure of the lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



It is clear from the diagram that $\mathcal{I}$ *is prealgebraic but not protoalgebraic. So to see that it is family preequivalential but not equivalential, it suffices to show that* $\mathcal{I}$ *is family extensional. We can easily see that* $\mathbf{F}$ *has two nontrivial proper universes and three theory families:*

Universes	$\mathbf{F}_{01} = \{\{0, 1\}\}$	$\mathbf{F}_{02} = \{\{0, 2\}\}$	
Theory Families	$\mathrm{Thm}(\mathcal{I})$	$T = \{\{1, 2\}\}$	$\mathrm{SEN}^\flat$

For verification we perform the following calculations, since the case of $\mathrm{SEN}^\flat$ *is trivial:*

$$\begin{array}{rcccl}
\Omega^{\mathbf{F}_{01}}(\mathrm{Thm}(\mathcal{I}) \cap \mathbf{F}_{01}) & = & \mathbf{F}_{01}^2 & = & \Omega(\mathrm{Thm}(\mathcal{I})) \cap \mathbf{F}_{01}^2;\\
\Omega^{\mathbf{F}_{02}}(\mathrm{Thm}(\mathcal{I}) \cap \mathbf{F}_{02}) & = & \Delta^{\mathbf{F}_{02}} & = & \Omega(\mathrm{Thm}(\mathcal{I})) \cap \mathbf{F}_{02}^2;\\
\Omega^{\mathbf{F}_{01}}(T \cap \mathbf{F}_{01}) & = & \Delta^{\mathbf{F}_{01}} & = & \Omega(T) \cap \mathbf{F}_{01}^2;\\
\Omega^{\mathbf{F}_{02}}(T \cap \mathbf{F}_{02}) & = & \Delta^{\mathbf{F}_{02}} & = & \Omega(T) \cap \mathbf{F}_{02}^2.
\end{array}$$

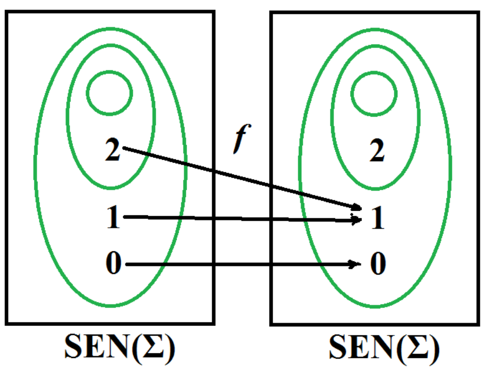
We conclude that $\mathcal{I}$ is indeed family extensional. Thus, $\mathcal{I}$ is an example of a family preequivalential π-institution, which is not equivalential.

Next we give an example of a π-institution that is system preequivalential but not family preequivalential.

Example 333 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 1$ *and* $\mathrm{SEN}^\flat(f)(2) = 1$*;*
- $N^\flat$ *is the category of natural transformations generated by the binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by the following table:*

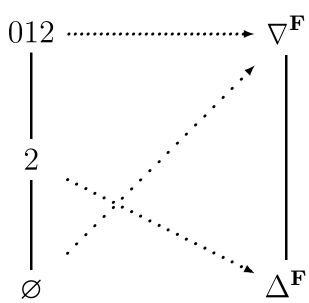
$\sigma^\flat_\Sigma$	0	1	2
0	1	1	2
1	1	1	1
2	2	1	2



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution defined by setting*

$$\mathcal{C}_\Sigma = \{\varnothing, \{2\}, \{0, 1, 2\}\}.$$

$\mathcal{I}$ *has three theory families,* $\overline{\varnothing}$*,* $T = \{\{2\}\}$ *and* $\mathrm{SEN}^\flat$*, but only* $\overline{\varnothing}$ *and* $\mathrm{SEN}^\flat$ *are theory systems. The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.*



From the diagram and the fact that $T \notin \mathrm{ThSys}(\mathcal{I})$ it follows that $\mathcal{I}$ is prealgebraic.

$\mathbf{F}$ has five universes $\{\{0\}\}$, $\{\{1\}\}$, $\{\{0,1\}\}$, $\{\{1,2\}\}$ and $\{\{0,1,2\}\}$. Since the only theory systems of $\mathcal{I}$ are $\overline{\varnothing}$ and $\mathrm{SEN}^\flat$, it is trivial to check that $\mathcal{I}$ is system extensional. Hence, being prealgebraic and system extensional, $\mathcal{I}$ is preequivalential.

For the universe $\mathbf{X} = \{\{0,1\}\}$ and the theory family $T = \{\{2\}\}$, we get

$$\Omega(T) \cap \mathbf{X}^2 = \{\{0\},\{1\}\} \subsetneq \{\{0,1\}\} = \Omega^{\mathbf{X}}(T \cap \mathbf{X}).$$

This shows that $\mathcal{I}$ is not family extensional and, hence, $\mathcal{I}$ is not family preequivalential.

Theorems 179 and 314 allow us to formulate transfer results for (pre)equivalential π-institutions.

Theorem 334 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is equivalential if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ is monotone on $\mathcal{I}$-filter families and, for all $Y \in \mathrm{SenFam}(\mathcal{A})$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,*

$$\Omega^{\mathcal{A}}(T) \cap \langle Y \rangle^2 = \Omega^{\langle Y \rangle}(T \cap \langle Y \rangle).$$

Proof: This follows from Theorems 179 and 314. ∎

Similarly, we have the following versions for the preequivalentiality properties:

Theorem 335 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is family (system) preequivalential if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ is monotone on $\mathcal{I}$-filter systems and, for all $Y \in \mathrm{SenFam}(\mathcal{A})$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ ($T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, respectively),*

$$\Omega^{\mathcal{A}}(T) \cap \langle Y \rangle^2 = \Omega^{\langle Y \rangle}(T \cap \langle Y \rangle).$$

Proof: This follows from Theorems 179 and 314. ∎

The definitions of equivalentiality and of system preequivalentiality may be recast in terms of properties of mappings between the lattice of theory families/systems and congruence systems. We have the following

Theorem 336 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is equivalential if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$:*

- *The mapping $\Omega : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is monotone;*

- *The following diagram commutes, for every* $Y \in \mathrm{SenFam}(\mathcal{A})$*.*

$$\begin{array}{ccc}
\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) & \xrightarrow{\Omega} & \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}) \\
{\scriptstyle - \cap \langle Y\rangle}\downarrow & & \downarrow{\scriptstyle - \cap \langle Y\rangle^2} \\
\mathrm{FiFam}^{\mathcal{I}^{\alpha^{-1}(\langle Y\rangle)}}(\langle Y\rangle) & \xrightarrow[\Omega^{\langle Y\rangle}]{} & \mathrm{ConSys}^{\mathcal{I}^{\alpha^{-1}(\langle Y\rangle)}}(\langle Y\rangle)
\end{array}$$

Proof: The "only if" follows from Theorem 334. The "if" follows by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$. ■

The version for system preequivalentiality has the following form.

Theorem 337 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is preequivalential if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*:*

- *The mapping* $\Omega : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ *is monotone;*
- *The following diagram commutes, for every* $Y \in \mathrm{SenFam}(\mathcal{A})$*.*

$$\begin{array}{ccc}
\mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) & \xrightarrow{\Omega} & \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}) \\
{\scriptstyle - \cap \langle Y\rangle}\downarrow & & \downarrow{\scriptstyle - \cap \langle Y\rangle^2} \\
\mathrm{FiSys}^{\mathcal{I}^{\alpha^{-1}(\langle Y\rangle)}}(\langle Y\rangle) & \xrightarrow[\Omega^{\langle Y\rangle}]{} & \mathrm{ConSys}^{\mathcal{I}^{\alpha^{-1}(\langle Y\rangle)}}(\langle Y\rangle)
\end{array}$$

Proof: Along the lines of Theorem 336, using Theorem 335. ■

We now state formally some straightforward relationships between the classes in the equivalential hierarchy and those in the monotonicity hierarchy.

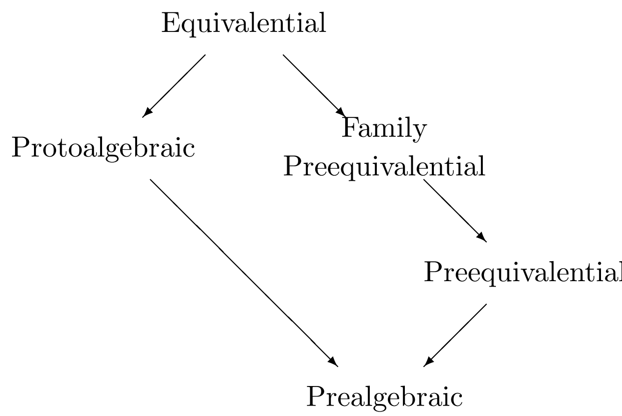
Proposition 338 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is equivalential, then it is protoalgebraic;*

(b) If $\mathcal{I}$ *is system preequivalential, then it is prealgebraic.*

Proof: Both statements follow directly from the definitions of equivalentiality and system preequivalentiality. ■

Proposition 338 establishes the following hierarchy:



The next example shows that the two inclusions from the classes in the equivalential hierarchy to the monotonicity hierarchy are proper inclusions. More precisely, a π-institution is constructed that is protoalgebraic but fails to be (system) preequivalential.

Example 339 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, a, b, 1\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the two binary natural transformations* $\wedge, \vee : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by the following tables:*

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

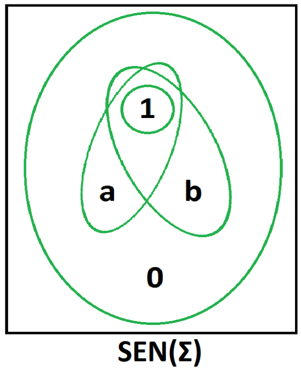
$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

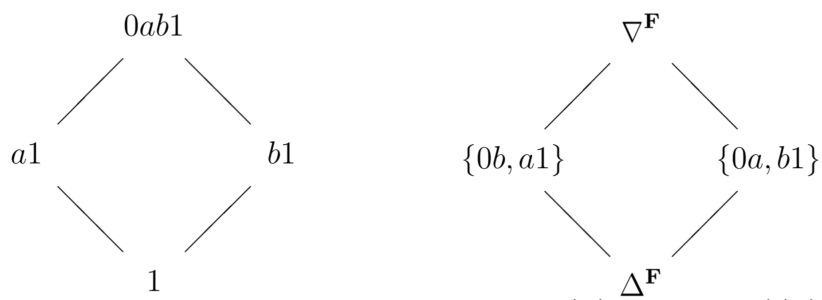
Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution defined by setting*

$$\mathcal{C}_\Sigma = \{\{1\}, \{a, 1\}, \{b, 1\}, \{0, a, b, 1\}\}.$$

$\mathcal{I}$ *has four theory families, all of which are also theory systems.*

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.





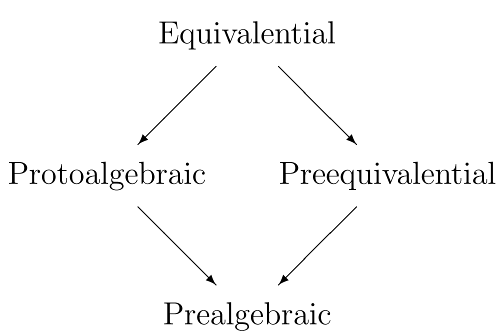
From the diagram, we can see that $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is an order isomorphism, whence,* $\mathcal{I}$ *is, in particular, protoalgebraic.*

On the other hand, for the universe $\mathbf{X} = \{\{0,a,1\}\}$ *and the theory system* $T = \{\{1\}\}$*, we get*

$$\Omega(T) \cap \mathbf{X}^2 = \{\{0\},\{a\},\{1\}\} \subsetneq \{\{0,a\},\{1\}\} = \Omega^{\mathbf{X}}(T \cap \mathbf{X}).$$

Thus, $\mathcal{I}$ *is not system extensional and, therefore, it fails to be (system) preequivalential.*

In our future work we will deal mostly with equivalential and system preequivalential π-institutions, referring to them as *equivalential* and *preequivalential*, respectively (as has already been suggested). So we focus mostly on the following part of the hierarchy:



Whenever the need to refer to family preequivalential π-institutions arises, the "family" qualification shall not be omitted to avoid confusion.

5.5 PreAlgebraizability

We study now the hierarchy that results by taking the various classes of weakly prealgebraizable π-institutions and adding to them family or system extensionality. Equivalently, we may replace prealgebraicity by either family or system preequivalentiality. Since, for every weak prealgebraizability class, we have two strengthening (or replacement) options, we get a sort of a double (or parallel) hierarchy whose classes are defined formally as follows and which is depicted in the accompanying diagram.

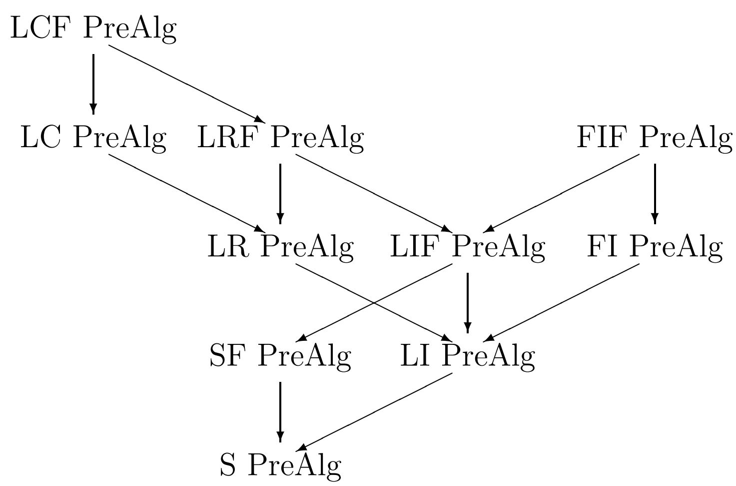
Definition 340 (Family PreAlgebraizability) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is* **left completely reflective family prealgebraizable***, or* **LCF prealgebraizable** *for short, if it is family preequivalential and left completely reflective, i.e., if it is system monotone, family extensional and left completely reflective;*
- $\mathcal{I}$ *is* **left reflective family prealgebraizable***, or* **LRF prealgebraizable** *for short, if it is family preequivalential and left reflective, i.e., if it is system monotone, family extensional and left reflective;*
- $\mathcal{I}$ *is* **family injective family prealgebraizable***, or* **FIF prealgebraizable** *for short, if it is family preequivalential and family injective, i.e., if it is system monotone, family extensional and family injective;*
- $\mathcal{I}$ *is* **left injective family prealgebraizable***, or* **LIF prealgebraizable** *for short, if it is family preequivalential and left injective, i.e., if it is system monotone, family extensional and left injective;*
- $\mathcal{I}$ *is* **system family prealgebraizable***, or* **SF prealgebraizable** *for short, if it is family preequivalential and system injective, i.e., if it is system monotone, family extensional and system injective.*

Definition 341 (PreAlgebraizability) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is* **left completely reflective prealgebraizable***, or* **LC prealgebraizable** *for short, if it is preequivalential and left completely reflective, i.e., if it is system monotone, system extensional and left completely reflective;*
- $\mathcal{I}$ *is* **left reflective prealgebraizable***, or* **LR prealgebraizable** *for short, if it is preequivalential and left reflective, i.e., if it is system monotone, system extensional and left reflective;*

- $\mathcal{I}$ *is* **family injective prealgebraizable**, *or* **FI prealgebraizable** *for short, if it is preequivalential and family injective, i.e., if it is system monotone, system extensional and family injective;*
- $\mathcal{I}$ *is* **left injective prealgebraizable**, *or* **LI prealgebraizable** *for short, if it is preequivalential and left injective, i.e., if it is system monotone, system extensional and left injective;*
- $\mathcal{I}$ *is* **system prealgebraizable**, *or* **S prealgebraizable** *for short, if it is preequivalential and system injective, i.e., if it is system monotone, system extensional and system injective.*



The nomenclature here uses the term "prealgebraizable" to suggest that we are applying prealgebraicity. The first two qualifying capitals reflect the kind of injectivity, reflectivity or c-reflectivity that is applied and, finally, the addition or omission of "F" conveys whether family or system extensionality is applied, i.e., (together with prealgebraicity) whether family or system preequivalentiality is postulated. For instance, a π-institution is *LRF prealgebraizable* if it is

- prealgebraic;
- left reflective;
- family extensional,

i.e., if it is family preequivalential and left reflective or, equivalently, if it is weakly LR prealgebraizable and family extensional.

Directly from corresponding theorems pertaining to weakly prealgebraizable π-institutions, we obtain the following results that clarify the status of this hierarchy under systemicity, on the one hand, and under the weaker condition of stability, on the other.

Theorem 342 *For systemic π-institutions, all ten classes shown in the hierarchical diagram coincide.*

Proof: First, if $\mathcal{I}$ is systemic, then it is, a fortiori, stable. Therefore, by Proposition 309, the properties of family and system extensionality coincide. Thus, the two parallel hierarchies of the diagram collapse into one. Finally, by Theorem 269, all these five classes coincide. Therefore, restricted to systemic π-institutions, the entire hierarchy of the diagram collapses into a single class. ■

Theorem 343 *For stable π-institutions, the ten-class prealgebraizability hierarchy shown in the diagram collapses to only two different classes, as shown in the diagram below*

F Prealgebraizable

↓

S Prealgebraizable

where F Prealgebraizable encompasses the classes of FIF and FI Prealgebraizable π-institutions and S Prealgebraizable encompasses the remaining eight classes in the original hierarchy.

Proof: Indeed, by Proposition 309, the properties of family and system extensionality coincide under stability. Therefore, the five pairs of parallel classes of the original hierarchy coincide, giving a 5-class hierarchy. But, according to Theorem 273, under stability, these five classes reduce to only two, as shown in the diagram of the statement. ■

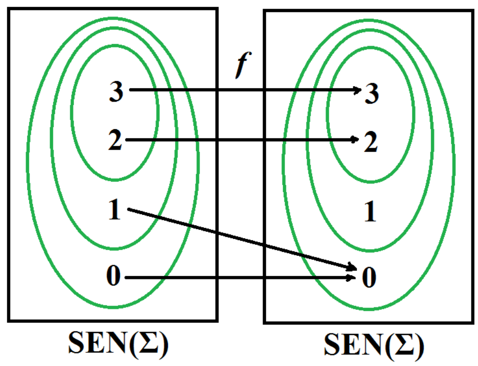
A few examples are now in order to separate the various classes of this prealgebraizability hierarchy. The first example serves in separating each pair of the two parallel hierarchies shown in the diagram. Namely, a π-institution is constructed which is LC prealgebraizable and FI prealgebraizable and, hence, belongs to all five levels of the lower hierarchy, but fails to be SF prealgebraizable and, as a consequence, belongs to none of the five upper levels of the hierarchy.

Example 344 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2, 3\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$*,* $\mathrm{SEN}^\flat(f)(2) = 2$ *and* $\mathrm{SEN}^\flat(f)(3) = 3$*;*

- $N^\flat$ *is the category of natural transformations generated by the binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by the following table:*

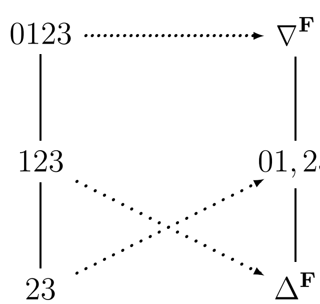
$\sigma^\flat_\Sigma$	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	2	2
3	0	1	2	3



Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution defined by setting*

$$\mathcal{C}_\Sigma = \{\{2,3\},\{1,2,3\},\{0,1,2,3\}\}.$$

$\mathcal{I}$ *has three theory families, but only two theory systems. The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.*



From the diagram, we can see that $\mathcal{I}$ *is prealgebraic, i.e., that* Ω *is monotone on* $\mathrm{ThSys}(\mathcal{I})$*, and, also, left c-reflective and family injective.*

To see that $\mathcal{I}$ *is system extensional, note that* $\mathbf{F}$ *has eleven universes,* $\{\{0\}\}$, $\{\{2\}\}$, $\{\{3\}\}$, $\{\{0,1\}\}$, $\{\{0,2\}\}$, $\{\{0,3\}\}$, $\{\{2,3\}\}$, $\{\{0,1,2\}\}$, $\{\{0,1,3\}\}$, $\{\{0,2,3\}\}$ *and* $\{\{0,1,2,3\}\}$*, seven of which are proper and non-singletons. Moreover,* $\mathcal{I}$ *has two theory systems, only one of which is proper. Thus, we* *have seven cases to check, shown below adopting obvious shorthand notation:*

$$\begin{aligned}
&\Omega(23)\cap\{01\}^2=\{01\}=\Omega^{01}(\varnothing)=\Omega^{01}(23\cap 01);\\
&\Omega(23)\cap\{02\}^2=\{0,2\}=\Omega^{02}(2)=\Omega^{01}(23\cap 02);\\
&\Omega(23)\cap\{03\}^2=\{0,3\}=\Omega^{03}(3)=\Omega^{01}(23\cap 03);\\
&\Omega(23)\cap\{23\}^2=\{23\}=\Omega^{23}(23)=\Omega^{01}(23\cap 23);\\
&\Omega(23)\cap\{012\}^2=\{01,2\}=\Omega^{012}(2)=\Omega^{01}(23\cap 012);\\
&\Omega(23)\cap\{013\}^2=\{01,3\}=\Omega^{013}(3)=\Omega^{01}(23\cap 013);\\
&\Omega(23)\cap\{023\}^2=\{0,23\}=\Omega^{023}(23)=\Omega^{01}(23\cap 023).
\end{aligned}$$

On the other hand, for the universe $\mathbf{X}=\{\{0,2,3\}\}$ *and the theory system* $T=\{\{1,2,3\}\}$*, we get*

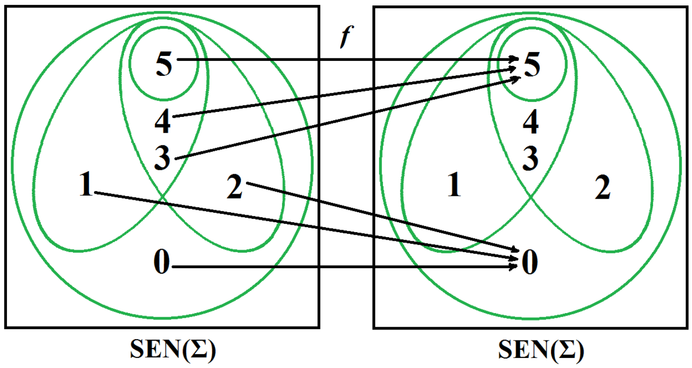
$$\Omega(T)\cap\mathbf{X}^2=\{\{0\},\{2\},\{3\}\}\subsetneqq\{\{0\},\{2,3\}\}=\Omega^{\mathbf{X}}(T\cap\mathbf{X}).$$

Thus, $\mathcal{I}$ *is not family extensional and, therefore, it fails to be SF prealgebraizable.*

We now present examples that separate each parallel step from the one immediately below it. The first is an example of an LRF prealgebraizable π-institution that fails to be LC prealgebraizable. This shows that LCF prealgebraizable π-institutions form a proper subclass of the class of LRF prealgebraizable ones and that the class of LC prealgebraizable π-institutions is a proper subclass of the class of LR prealgebraizable π-institutions.

Example 345 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single non-identity morphism* $f:\Sigma\to\Sigma$*, such that* $f\circ f=f$*;*
- $\mathrm{SEN}^\flat:\mathbf{Sign}^\flat\to\mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma)=\{0,1,2,3,4,5\}$ *and*
  $$\begin{aligned}&\mathrm{SEN}^\flat(f)(0)=\mathrm{SEN}^\flat(f)(1)=\mathrm{SEN}^\flat(f)(2)=0,\\ &\mathrm{SEN}^\flat(f)(3)=\mathrm{SEN}^\flat(f)(4)=\mathrm{SEN}^\flat(f)(5)=5;\end{aligned}$$
- $N^\flat$ *is the category of natural transformations generated by the two unary natural transformations* $\sigma^\flat,\tau^\flat:\mathrm{SEN}^\flat\to\mathrm{SEN}^\flat$*, with*
  $$\sigma^\flat_\Sigma,\tau^\flat_\Sigma:\mathrm{SEN}^\flat(\Sigma)\to\mathrm{SEN}^\flat(\Sigma)$$
  defined by
  - $\sigma^\flat_\Sigma(3)=1$ *and* $\sigma^\flat_\Sigma(x)=0$*, for all* $x\in\{0,1,2,4,5\}$*;*
  - $\sigma^\flat_\Sigma(4)=2$ *and* $\sigma^\flat_\Sigma(x)=0$*, for all* $x\in\{0,1,2,3,5\}$*.*



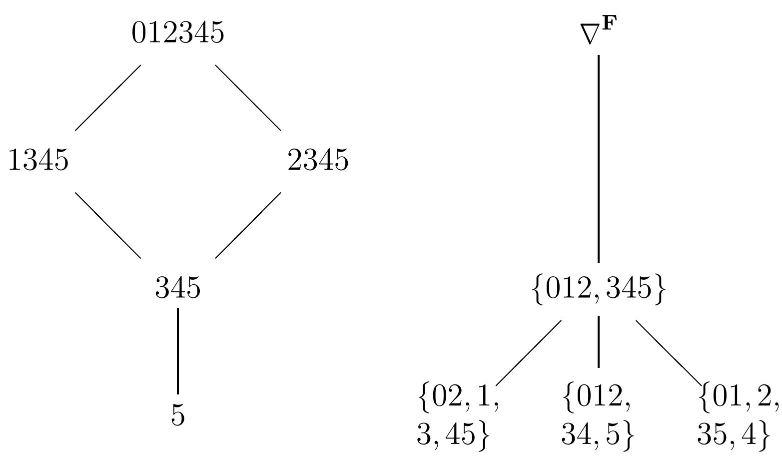
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

$$\mathcal{C}_\Sigma = \{\{5\}, \{3,4,5\}, \{1,3,4,5\}, \{2,3,4,5\}, \{0,1,2,3,4,5\}\}.$$

$\mathcal{I}$ *has five theory families but only three theory systems. The action of* $\overleftarrow{\ }$ *on theory families is given by the following table.*

T	$\overleftarrow{T}$
$\{5\}$	$\{5\}$
$\{3,4,5\}$	$\{3,4,5\}$
$\{1,3,4,5\}$	$\{3,4,5\}$
$\{2,3,4,5\}$	$\{3,4,5\}$
$\{0,1,2,3,4,5\}$	$\{0,1,2,3,4,5\}$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



From the diagram, it is clear that $\mathcal{I}$ *is prealgebraic, i.e., that, for all* $T, T' \in \mathrm{ThSys}(\mathcal{I})$, $T \leq T'$ *implies* $\Omega(T) \leq \Omega(T')$. *Moreover, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$, *if* $\Omega(T) \leq \Omega(T')$, *then* $\overleftarrow{T} \leq \overleftarrow{T'}$, *i.e.,* $\mathcal{I}$ *is left reflective. On the* *other hand, setting,* $T^1 = \{\{1,3,4,5\}\}$, $T^2 = \{\{2,3,4,5\}\}$ *and* $T' = \{\{5\}\}$, *we get*

$$\begin{array}{rcl}\Omega(T^1)\cap\Omega(T^2) & = & \{\{02,1,3,45\}\}\cap\{\{01,2,35,4\}\}\\ & = & \Delta^{\mathbf{F}}\\ & \leq & \{\{012,34,5\}\} = \Omega(T'),\end{array}$$

whereas

$$\overleftarrow{T^1}\cap\overleftarrow{T^2} = \{\{3,4,5\}\}\cap\{\{3,4,5\}\} = \{\{3,4,5\}\} \not\leq \{\{5\}\} = \overleftarrow{T'}.$$

Hence, $\mathcal{I}$ *is not left completely reflective. Hence to see that* $\mathcal{I}$ *is LRF prealgebraizable but not LC prealgebraizable, it suffices to show that it is family extensional. The verification is routine, but rather tedious. Note that* $\mathbf{F}$ *has eleven proper and non-trivial universes, namely* $\{\{0,1\}\}$, $\{\{0,2\}\}$, $\{\{0,5\}\}$, $\{\{0,1,2\}\}$, $\{\{0,1,5\}\}$, $\{\{0,2,5\}\}$, $\{\{0,1,2,5\}\}$, $\{\{0,1,3,5\}\}$, $\{\{0,2,4,5\}\}$, $\{\{0,1,2,3,5\}\}$ *and* $\{\{0,1,2,4,5\}\}$. *Moreover, it has four proper theory families,* $T^1 = \{\{5\}\}$, $T^2 = \{\{3,4,5\}\}$, $T^3 = \{\{1,3,4,5\}\}$ *and* $T^4 = \{\{2,3,4,5\}\}$. *So, one has to check forty-four cases in total which are summarized in the following table, where each entry in the column labeled by universe* $\mathbf{F}'$ *and the row labeled by theory family* T *shows the congruence system* $\Omega(T)\cap\mathbf{F}'^2 = \Omega^{\mathbf{F}'}(T\cap\mathbf{F}')$ *in shorthand block notation.*

	01	02	05	012	015	025	0125
5	01	02	0,5	012	01,5	02,5	012,5
345	01	02	0,5	012	01,5	02,5	012,5
1345	0,1	02	0,5	02,1	0,1,5	02,5	02,1,5
2345	01	0,2	0,5	01,2	01,5	0,2,5	01,2,5

	0135	0245	01235	01245
5	01,3,5	02,4,5	012,3,5	012,4,5
345	01,35	02,45	012,35	012,45
1345	0,1,3,5	02,45	02,1,3,5	02,1,45
2345	01,35	0,2,4,5	01,2,35	01,2,4,5

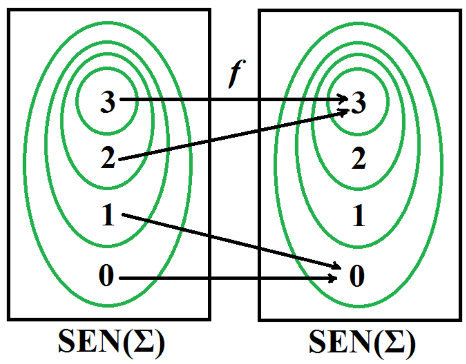
The second example is an example of an LIF prealgebraizable π-institution that fails to be LR prealgebraizable. This shows, on the one hand, that the class of LRF prealgebraizable π-institutions is a proper subclass of the class of LIF prealgebraizable π-institutions and, on the other, that the class of LR prealgebraizable π-institutions is a proper subclass of the class of LI prealgebraizable π-institutions.

Example 346 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f:\Sigma\to\Sigma$, *such that* $f\circ f = f$;

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$, $\mathrm{SEN}^\flat(f)(2) = 3$ *and* $\mathrm{SEN}^\flat(f)(3) = 3$*;*
- $N^\flat$ *is the category of natural transformations generated by the unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined by the following table:*

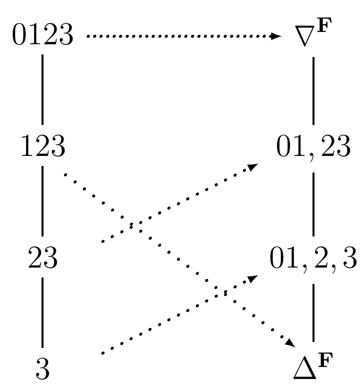
x	0	1	2	3
$\sigma^\flat_\Sigma(x)$	0	1	1	0



Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution defined by setting*

$$\mathcal{C}_\Sigma = \{\{3\},\{2,3\},\{1,2,3\},\{0,1,2,3\}\}.$$

$\mathcal{I}$ *has four theory families, but only three theory systems. The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.*



From the diagram, we can see that $\mathcal{I}$ *is prealgebraic, i.e., that* Ω *is monotone on* $\mathrm{ThSys}(\mathcal{I})$ *and, also left injective. But* $\mathcal{I}$ *is not left reflective, since* $\Omega(\{1,2,3\}) \le \Omega(\{3\})$*, whereas* $\overleftarrow{\{1,2,3\}} = \{2,3\} \nleq \{3\} = \overleftarrow{\{3\}}$*. Therefore, to see that* $\mathcal{I}$ *is LIF prealgebraizable but not LR prealgebraizable, it suffices to show that it is family extensional.*

Note that $\mathbf{F}$ *has three proper and non-singleton universes,* $\{\{0,1\}\}$, $\{\{0,3\}\}$ *and* $\{\{0,1,3\}\}$. *Moreover,* $\mathcal{I}$ *has three proper theory families. Thus, we only have nine cases to check, shown in the following array, which in the row labeled by theory family* T *and the column labeled by universe* $\mathbf{F}'$ *shows the congruence system* $\Omega(T)\cap\mathbf{F}'^2 = \Omega^{\mathbf{F}'}(T\cap\mathbf{F}')$ *in an obvious shorthand notation in terms of blocks.*

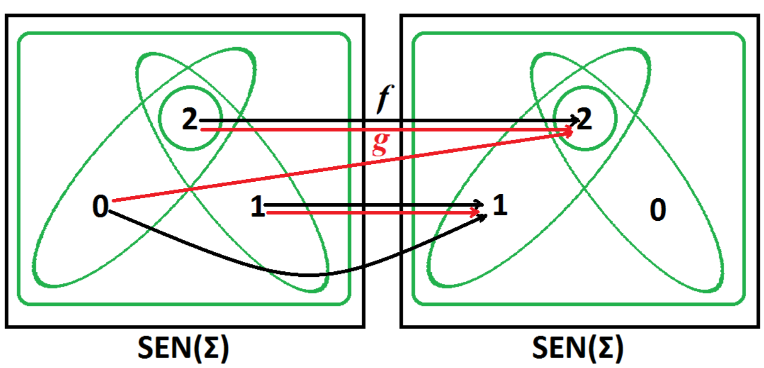
	01	03	013
3	01	0,3	01,3
23	01	0,3	01,3
123	0,1	0,3	0,1,3

We conclude that $\mathcal{I}$ *is LIF prealgebraizable but not LR prealgebraizable.*

The third example is an example of an LIF prealgebraizable π-institution that fails to be FI prealgebraizable. This shows that the class of FIF prealgebraizable π-institutions is a proper subclass of the class of LIF prealgebraizable π-institutions and that the class of FI prealgebraizable π-institutions is a proper subclass of the class of LI prealgebraizable π-institutions.

Example 347 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

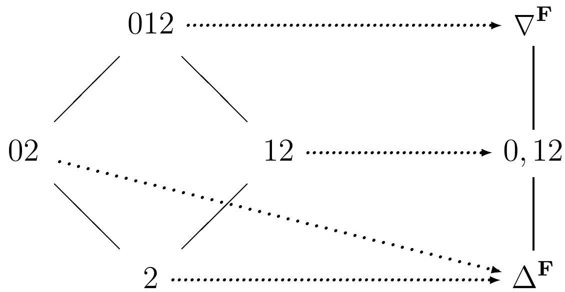
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and two (non-identity) morphisms* $f,g:\Sigma\to\Sigma$, *such that* $f\circ f = f$, $f\circ g = g$, $g\circ f = f$ *and* $g\circ g = g$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and, on morphisms* $\mathrm{SEN}^\flat(f)(0) = 1$, $\mathrm{SEN}^\flat(f)(1) = 1$, $\mathrm{SEN}^\flat(f)(2) = 2$ *and* $\mathrm{SEN}^\flat(g)(0) = 2$, $\mathrm{SEN}^\flat(g)(1) = 1$ *and* $\mathrm{SEN}^\flat(g)(2) = 2$;
- $N^\flat$ *is the trivial category of natural transformations.*



Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution defined by setting*

$$\mathcal{C}_\Sigma = \{\{2\},\{0,2\},\{1,2\},\{0,1,2\}\}.$$

$\mathcal{I}$ has four theory families, but only three theory systems. The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



From the diagram, we can see that $\mathcal{I}$ is prealgebraic and left injective. But $\mathcal{I}$ is clearly not family injective, since the theory families $\{\{2\}\}$ and $\{\{0,2\}\}$ map to the same congruence system. Therefore, to see that $\mathcal{I}$ is LIF prealgebraizable but not FI prealgebraizable, it suffices to show that it is family extensional.

Note that $\mathbf{F}$ has only one proper and non-singleton universe, $\{\{1,2\}\}$, and three proper theory families $\{\{2\}\}$, $\{\{0,2\}\}$ and $\{\{1,2\}\}$. Thus, we only have three cases to check, shown below in a shorthand notation.

$$\begin{aligned}\Omega(2)\cap(12)^2 &= \{1,2\} = \Omega^{12}(2) = \Omega^{12}(2\cap 12);\\ \Omega(02)\cap(12)^2 &= \{1,2\} = \Omega^{12}(2) = \Omega^{12}(02\cap 12);\\ \Omega(12)\cap(12)^2 &= \{12\} = \Omega^{12}(12) = \Omega^{12}(12\cap 12).\end{aligned}$$

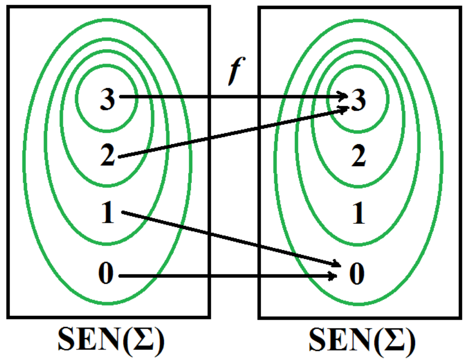
We conclude that $\mathcal{I}$ is LIF prealgebraizable but not FI prealgebraizable.

The last example in this series is an example of an SF prealgebraizable π-institution that fails to be LI prealgebraizable. This shows that LIF prealgebraizable π-institutions form a proper subclass of the class of SF prealgebraizable ones and that the class of LI prealgebraizable π-institutions is a proper subclass of the class of S prealgebraizable ones.

Example 348 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object Σ and a single (non-identity) morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$, $\mathrm{SEN}^\flat(f)(2) = 3$ *and* $\mathrm{SEN}^\flat(f)(3) = 3$*;*
- $N^\flat$ *is the category of natural transformations generated by the unary natural transformation $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ defined by the following table:*

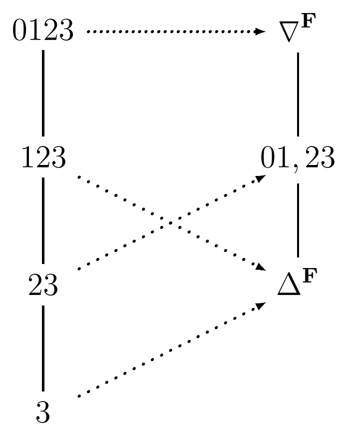
x	0	1	2	3
$\sigma^\flat_\Sigma(x)$	3	2	1	0



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be the π-institution defined by setting

$$\mathcal{C}_\Sigma = \{\{3\}, \{2,3\}, \{1,2,3\}, \{0,1,2,3\}\}.$$

$\mathcal{I}$ has four theory families, but only three theory systems. The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



From the diagram, we can see that $\mathcal{I}$ is prealgebraic, i.e., that Ω is monotone on ThSys($\mathcal{I}$) *and, also system injective, i.e., Ω is injective on theory systems. But $\mathcal{I}$ is not left injective, since $\Omega(\{1,2,3\}) = \Omega(\{3\})$, whereas $\overleftarrow{\{1,2,3\}} = \{2,3\} \neq \{3\} = \overleftarrow{\{3\}}$. Therefore, to see that $\mathcal{I}$ is SF prealgebraizable but not LI prealgebraizable, it suffices to show that it is family extensional.*

Note that $\mathbf{F}$ has only one proper and non-singleton universe, $\{\{0,3\}\}$. Moreover, $\mathcal{I}$ has three proper theory families. Thus, we have only three cases to check, shown below in shorthand notation:

$$\begin{array}{l}\Omega(3) \cap \{03\}^2 = \{0,3\} = \Omega^{03}(3) = \Omega^{03}(3 \cap 03);\\ \Omega(23) \cap \{03\}^2 = \{0,3\} = \Omega^{03}(3) = \Omega^{03}(23 \cap 03);\\ \Omega(123) \cap \{03\}^2 = \{0,3\} = \Omega^{03}(3) = \Omega^{03}(123 \cap 03).\end{array}$$

We conclude that $\mathcal{I}$ is SF prealgebraizable but not LI prealgebraizable.

We now turn to establishing transfer properties for the π-institutions belonging to the various classes of the preceding hierarchy. We do this by formulating a comprehensive result encompassing the transference of all ten properties of the above hierarchy. It is hoped that, despite its all-encompassing character, the formulation will be sufficiently clear.

Theorem 349 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *belongs to one of the ten prealgebraizability classes in the prealgebraizability hierarchy if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, the Leibniz operator on* $\mathcal{A}$*, relative to* $\mathcal{I}$*, satisfies the properties defining the corresponding class.*

For example, $\mathcal{I}$ is FIF prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, the Leibniz operator on $\mathcal{A}$ is monotone on $\mathcal{I}$-filter systems, injective on $\mathcal{I}$-filter families and family extensional, i.e.,

- for all $T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, $T \leq T'$ implies $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$;
- for all $T, T' \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$ implies $T = T'$;
- for all $Y \in \mathrm{SenFam}(\mathcal{A})$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$\Omega^{\mathcal{A}}(T) \cap \langle Y \rangle^2 = \Omega^{\langle Y \rangle}(T \cap \langle Y \rangle).$$

Proof: First, observe that the "if" is trivially satisfied, since, if the postulated conditions hold for every $\mathbf{F}$-algebraic system, then they hold, in particular, for $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and this ensures that, by definition, $\mathcal{I}$ belongs to the corresponding prealgebraizability class.

So we turn to the "only if". First, in all cases $\mathcal{I}$ is prealgebraic, i.e., system monotone, and this property transfers to all $\mathbf{F}$-algebraic systems and $\mathcal{I}$-filter systems by Theorem 179. Then, depending on whether $\mathcal{I}$ belongs to one of the classes in the upper or the lower hierarchy of the two parallel hierarchies, it is family or system extensional, respectively. But, by Theorem 314, both of these properties transfer. Finally, depending on the class $\mathcal{I}$ is postulated to belong to, it satisfies one of the properties of system injectivity, family injectivity, left injectivity, left reflectivity or left c-reflectivity. The first three properties transfer by Theorem 214, the fourth transfers by Theorem 225 and the last transfers by Theorem 240. Therefore, the conclusion holds for each of the ten prealgebraizability classes in the prealgebraizability hierarchy. ■

Finally, we turn to characterizations of the classes in the hierarchy in the form of isomorphism theorems between lattices of theory families/systems and lattices of congruence systems. We start, first with FIF prealgebraizability.

Theorem 350 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is FIF prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a bijection which commutes with inverse logical extensions and which restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: The proof is based on Theorem 267, characterizing weak FI prealgebraizbility. We have that $\mathcal{I}$ is FIF prealgebraizable if and only if, by definition, it is weakly FI prealgebraizable and family extensional if and only if, by Theorem 267 and Theorem 327, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is a bijection which commutes with inverse logical extensions and which restricts to an order embedding $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. ■

FI prealgebraizability is characterized in a similar way, the difference being that commutativity with inverse logical extensions is restricted to the application of the Leibniz operator on $\mathcal{I}$-filter systems only, rather than being valid for its operation on the entire collection of $\mathcal{I}$-filter families.

Theorem 351 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is FI prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a bijection which restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

that commutes with inverse logical extensions.

Proof: Similar to the proof of Theorem 350. ■

We turn now to a similar characterization of SF prealgebraizability.

Theorem 352 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is SF prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *commutes with inverse logical extensions and restricts to an order embedding*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: The proof is based on Theorem 256, characterizing weak system prealgebraizability. We have that $\mathcal{I}$ is SF prealgebraizable if and only if, by definition, it is weakly system prealgebraizable and family extensional if and only if, by Theorem 256 and Theorem 327, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$ commutes with inverse logical extensions and restricts to an order embedding $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$. ■

S prealgebraizability is characterized in a similar way, the difference being that commutativity with inverse logical extensions is restricted to the application of the Leibniz operator on $\mathcal{I}$-filter systems only, rather than to its operation on the entire collection of $\mathcal{I}$-filter families.

Theorem 353 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is S prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$$

is an order embedding which commutes with inverse logical extensions.

Proof: Similar to the proof of Theorem 352. ■

We continue with LCF prealgebraizability.

Theorem 354 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is LCF prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$$

is a left completely order reflecting surjection, which commutes with inverse logical extensions and which restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A}).$$

Proof: The proof is based on Theorem 276, characterizing weak LC prealgebraizbility. We have that $\mathcal{I}$ is LCF prealgebraizable if and only if, by definition, it is weakly LC prealgebraizable and family extensional, if and only if, by Theorem 276 and Theorem 327, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$ is a left completely order reflecting surjection, which commutes with inverse logical extensions and which restricts to an order embedding $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$. ■

LC prealgebraizability is characterized in a similar way, the difference being that commutativity with inverse logical extensions is restricted to the application of the Leibniz operator on $\mathcal{I}$-filter systems only, rather than to its operation on the entire collection of $\mathcal{I}$-filter families.

Theorem 355 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is LC prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left completely order reflecting surjection that restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

that commutes with inverse logical extensions.

Proof: Similar to the proof of Theorem 354. ∎

A characterization of LRF prealgebraizability in the same spirit follows.

Theorem 356 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is LRF prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left order reflecting surjection which commutes with inverse logical extensions and which restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: The proof is based on Theorem 279, characterizing weak LR prealgebraizability. We have that $\mathcal{I}$ is LRF prealgebraizable if and only if, by definition, it is weakly LR prealgebraizable and family extensional, if and only if, by Theorem 279 and Theorem 327, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is a left order reflecting surjection which commutes with inverse logical extensions and which restricts to an order embedding $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. ∎

LR prealgebraizability is characterized in a similar way, the difference being that commutativity with inverse logical extensions is restricted to the application of the Leibniz operator on $\mathcal{I}$-filter systems only, rather than to its operation on the entire collection of $\mathcal{I}$-filter families.

Theorem 357 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is LR prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left order reflecting surjection that restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

that commutes with inverse logical extensions.

Proof: Similar to the proof of Theorem 356. ∎

Finally, along the same lines we obtain a characterization of LIF prealgebraizability.

Theorem 358 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is LIF prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left injective surjection which commutes with inverse logical extensions and which restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: The proof is based on Theorem 282, characterizing weak LI prealgebraizability. We have that $\mathcal{I}$ is LIF prealgebraizable if and only if, by definition, it is weakly LI prealgebraizable and family extensional, if and only if, by Theorem 282 and Theorem 327, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is a left injective surjection which commutes with inverse logical extensions and which restricts to an order embedding $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. ∎

And, of course, LI prealgebraizability is characterized in a similar way, the difference being that commutativity with inverse logical extensions is restricted to the application of the Leibniz operator on $\mathcal{I}$-filter systems only, rather than to its operation on the entire collection of $\mathcal{I}$-filter families.

Theorem 359 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is LI prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left injective surjection that restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

that commutes with inverse logical extensions.

Proof: Similar to the proof of Theorem 358. ∎

5.6 Algebraizability

Since equivalentiality implies protoalgebraicity, the hierarchy of algebraizable π-institutions, which results from the hierarchy of weakly algebraizable π-institutions by replacing protoalgebraicity by equivalentiality, is simpler, reflecting the simplicity of the weak algebraizability hierarchy.

Definition 360 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is* **family algebraizable***, or* **F Algebraizable** *for short, if it is equivalential and family injective, i.e., if it is protoalgebraic, family extensional and family injective;*
- $\mathcal{I}$ *is (***system***)* **algebraizable** *if it is equivalential and system injective, i.e., if it is protoalgebraic, family extensional and system injective.*

These two classes form the following **algebraizability hierarchy**:

Family Algebraizable

↓

(System) Algebraizable

It is clear that these two classes are separated exactly by systemicity, as is shown in the following proposition:

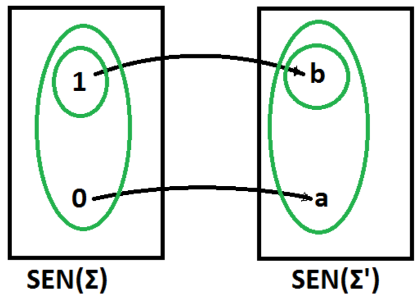
Proposition 361 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I}$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is family algebraizable if and only if it is algebraizable and systemic.*

Proof: We have that $\mathcal{I}$ is family algebraizable if and only if, by definition, it is equivalential and family injective if and only if, by Theorem 291 it is equivalential, systemic and system injective if and only if it is, by definition, algebraizable and systemic. ■

We next present an example to show that these two classes are different. It consists of an algebraizable π-institution, which fails to be family algebraizable.

Example 362 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*



- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$*;*

- $N^\flat$ *is the trivial clone.*

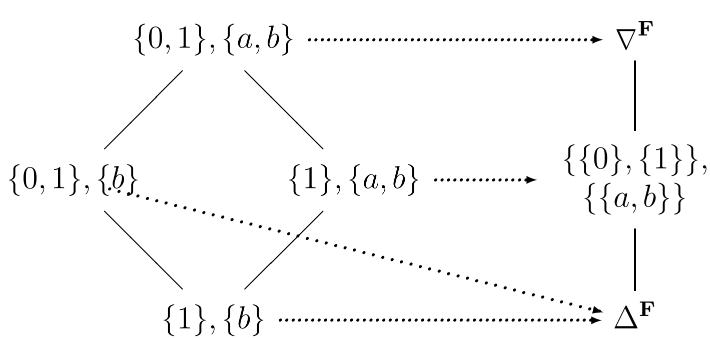
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\} \quad and \quad \mathcal{C}_{\Sigma'} = \{\{b\},\{a,b\}\}.$$

The table yielding the action of $\overleftarrow{\ }$ *on theory families is shown below.*

$\leftarrow$	$\{b\}$	$\{a,b\}$
$\{1\}$	$\{1\},\{b\}$	$\{1\},\{a,b\}$
$\{0,1\}$	$\{1\},\{b\}$	$\{0,1\},\{a,b\}$

The accompanying diagram gives the structure of the lattice of theory families and the corresponding Leibniz congruence systems.



From the diagram one can check that the Leibniz operator is monotone on theory families and injective on theory systems. Thus, the π*-institution is protoalgebraic and system injective. Moreover, as is shown in the following table, which summarizes the congruence systems of the form* $\Omega(T) \cap \langle X\rangle^2 = \Omega^{\langle X\rangle}(T \cap \langle X\rangle)$ *for the various combinations of nonempty universes and theory* *families, $\mathcal{I}$ is family extensional.*

$\langle X\rangle\backslash T$	$1\ b$	$01\ b$	$1\ ab$	$01\ ab$
$0\ a$	$0\ a$	$0\ a$	$0\ a$	$0\ a$
$0\ ab$	$0\ a,b$	$0\ a,b$	$0\ ab$	$0\ ab$
$1\ b$	$1\ b$	$1\ b$	$1\ b$	$1\ b$
$1\ ab$	$1\ a,b$	$1\ a,b$	$1\ ab$	$1\ ab$
$01\ ab$	$0,1\ a,b$	$0,1\ a,b$	$0,1\ ab$	$01\ ab$

Therefore, $\mathcal{I}$ is clearly equivalential and system injective, i.e., it is algebraizable.

On the other hand, letting $T = \{\{1\},\{b\}\}$ and $T' = \{\{0,1\},\{b\}\}$, we have $\Omega(T) = \Omega(T')$, but $T \neq T'$, whence $\mathcal{I}$ is not family injective and, therefore, it is not family algebraizable.

It is not difficult to show, based on preceding work, that both properties transfer.

Theorem 363 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is algebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, the Leibniz operator on $\mathcal{A}$ is monotone on $\mathcal{I}$-filter families, injective on $\mathcal{I}$-filter systems and family extensional, i.e.,*

- *for all $T,T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T \leq T'$ implies $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$;*
- *for all $T,T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$ implies $T = T'$;*
- *for all $Y \in \mathrm{SenFam}(\mathcal{A})$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,*

$$\Omega^{\mathcal{A}}(T) \cap \langle Y\rangle^2 = \Omega^{\langle Y\rangle}(T \cap \langle Y\rangle).$$

Proof: Suppose, first, that the three conditions hold. Consider the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I,\iota\rangle\rangle$, where $\langle I,\iota\rangle : \mathbf{F} \to \mathbf{F}$ is the identity morphism. By hypothesis, Ω is monotone on theory families and family extensional. Thus, $\mathcal{I}$ is equivalential. Also by hypothesis, Ω is injective on theory systems. Therefore, by definition, $\mathcal{I}$ is algebraizable.

Assume, conversely, that $\mathcal{I}$ is algebraizable. Thus, it is equivalential and system injective, i.e., its Leibniz operator is monotone on theory families, injective on theory systems and family extensional. Now we use Theorems 179, 214 and 314, which guarantee that monotonicity, injectivity and extensionality, respectively, transfer, to conclude that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, the Leibniz operator of $\mathcal{A}$ is monotone on $\mathcal{I}$-filter families, injective on $\mathcal{I}$-filter systems and family extensional. ∎

And, similarly, for family algebraizability, we obtain

Theorem 364 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is family algebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, the Leibniz operator on* $\mathcal{A}$ *is monotone and injective on* $\mathcal{I}$*-filter families and family extensional, i.e.,*

- *for all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $T \le T'$ *implies* $\Omega^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')$*;*
- *for all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$ *implies* $T = T'$*;*
- *for all* $Y \in \mathrm{SenFam}(\mathcal{A})$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$\Omega^{\mathcal{A}}(T) \cap \langle Y \rangle^2 = \Omega^{\langle Y \rangle}(T \cap \langle Y \rangle).$$

Proof: The proof is similar to that given for Theorem 363. It suffices to observe that family injectivity, like system injectivity, also transfers from the theory families of a π-institution $\mathcal{I}$ to all $\mathcal{I}$-filter families on an arbitrary $\mathbf{F}$-algebraic system. ■

We turn now to characterizations of the classes in the algebraizability hierarchy in terms of order isomorphisms between lattices of filter families/systems and lattices of congruence systems that satisfy additional properties. For algebraizability we have the following characterization.

Theorem 365 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is algebraizable if and only if* $\mathcal{I}$ *is stable and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism that commutes with inverse logical extensions.

Proof: Suppose, first, that $\mathcal{I}$ is algebraizable. Then it is weakly algebraizable and family extensional. Thus, by Theorem 298, $\mathcal{I}$ is stable and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a lattice isomorphism. Commutativity with inverse logical extensions follows by family extensionality and Theorems 327 and 328.

Assume, conversely, that $\mathcal{I}$ is stable and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism that commutes with inverse logical extensions. Then, again by Theorem 298, we get that $\mathcal{I}$ is weakly algebraizable and, by Theorems 328 and 327, that $\mathcal{I}$ is family extensional. It follows, by definition, that $\mathcal{I}$ is algebraizable. ■

For family algebraizability, we get an analogous characterization.

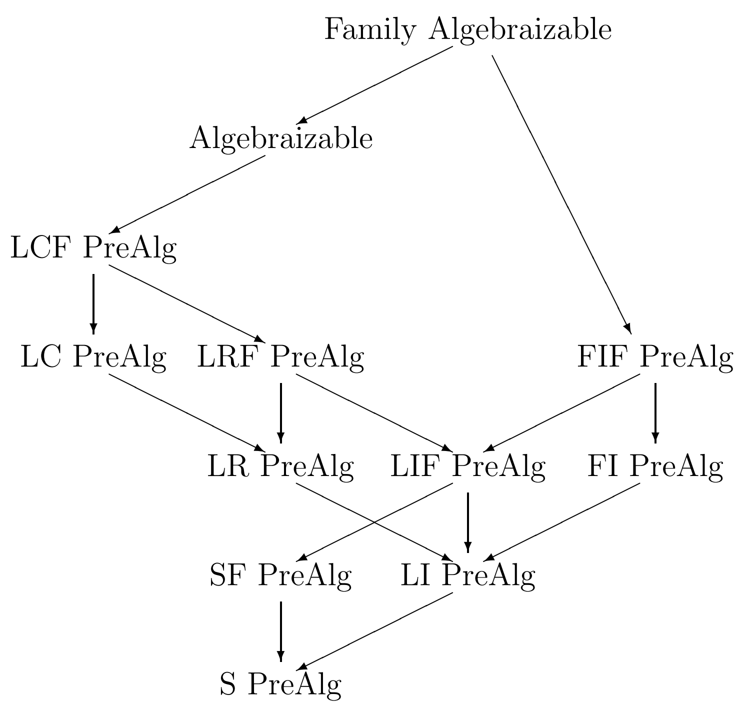
Theorem 366 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is family algebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism that commutes with inverse logical extensions.

Proof: The proof follows along lines similar to the proof of Theorem 365, except references to Theorem 298, characterizing weak algebraizability, must be replaced by referring instead to Theorem 296, which provides a corresponding characterization for weak family algebraizability. ∎

Finally, we note that the two classes sit on top of the prealgebraizability hierarchy that was studied in the preceding section. Namely, we have the hierarchy pictured below:

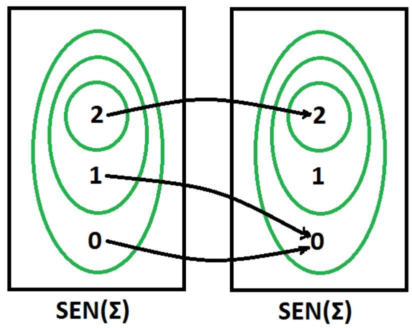


To separate the classes of the algebraizability from those of the prealgebraizability hierarchy, we provide an additional example. It is an example of an LCF and FIF prealgebraizable π-institution which is not algebraizable and, hence, a fortiori, not family algebraizable either.

Example 367 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*

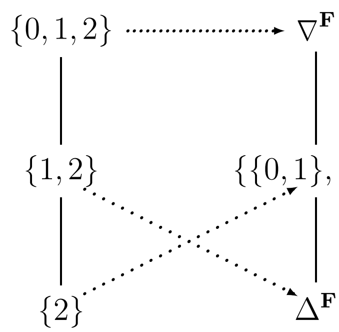
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}$.

The theory family $\{\{1,2\}\}$ *is not a theory system.*

The structure of the lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



Since $\mathcal{I}$ *is not protoalgebraic, it is clear that* $\mathcal{I}$ *is not algebraizable and, a fortiori, it is not family algebraizable either. On the other hand,* $\mathcal{I}$ *is prealgebraic and both left c-reflective and family injective. So, to see that it is both LCF and FIF prealgebraizable, it suffices to show that it is also family extensional. This is done by computing, for all* $T \in \mathrm{ThFam}(\mathcal{I})$ *and all* $X \in \mathrm{SenFam}(\mathcal{I})$ *the congruence systems* $\Omega(T) \cap \langle X\rangle^2$ *and* $\Omega^{\langle X\rangle}(T \cap \langle X\rangle)$ *and verifying that they are identical. This is detailed in the table below:*

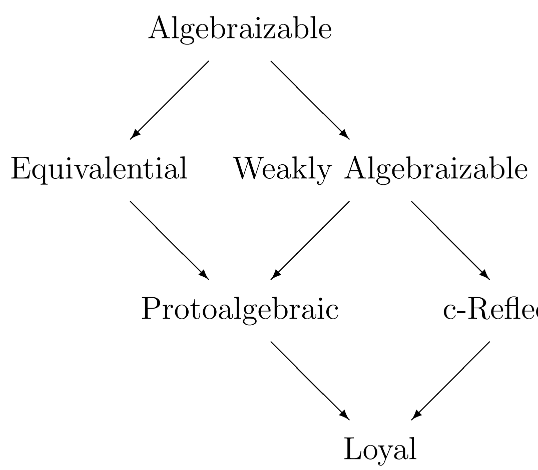
$\langle X\rangle\backslash T$	2	12	012
0	0	0	0
2	2	2	2
01	01	0,1	01
02	0,2	0,2	02
012	01,2	0,1,2	012

We conclude that $\mathcal{I}$ *is family extensional and, therefore, it is, indeed, both LCF and FIF prealgebraizable.*

The last example shows that the hierarchy depicted in the preceding diagram consists of pairwise distinct classes of π-institutions.

5.7 The Semantic Systemic Hierarchy

It is worth stopping momentarily to take a look at the semantic hierarchy that we have studied so far. It has been the case invariably that at each level studied, all classes were identical if restricted to systemic π-institutions. Therefore, considering only systemic π-institutions, one can construct a "skeleton" of the entire hierarchy that is depicted in the accompanying diagram:



It is, therefore, clear that, when restricted to systemic π-institutions, one recovers the fundamental classes and the shape of the well-known Leibniz hierarchy of propositional logics. We view this as a favorable omen that adds credibility to our institutional hierarchical investigations and the hierarchies established through them.

Chapter 6

The Semantic Leibniz Hierarchy: Under the Bottom I

6.1 Introduction

The study of some of the lowest classes in the Leibniz hierarchy of abstract algebraic logic presupposes in a certain sense that the logics studied have theorems. This occurs because the defining conditions of those classes do not hold for nontrivial logics without theorems. Realizing this shortcoming, Moraschini, in Chapter 3 of his Doctoral Dissertation [87] (see, also, [89]) introduced and studied weaker versions accommodating logics without theorems. Our investigations in this chapter have their origins in Moraschini's work, but are suitably adapted to cover logics formalized as π-institutions. At the π-institution level, an injective, and, hence, a fortiori, a reflective or completely reflective, π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ must have theorems. Otherwise, both $\mathrm{SEN}^\flat$ and $\overline{\varnothing}$ are theory families, with $\Omega(\mathrm{SEN}^\flat) = \nabla^{\mathbf{F}} = \Omega(\overline{\varnothing})$ and this contradicts injectivity. So, if one wishes to allow, in a context where injectivity is enforced, π-institutions without theorems, the condition of injectivity must be weakened to either exclude, or bypass in some other way, theory families with empty components. In this chapter we present two such attempts. The first is based on the notion of rough equivalence, under which two theory families are identified if, at those signatures Σ where they differ, one has an empty and the other a $\mathrm{SEN}^\flat(\Sigma)$ component. The second, more straightforward, approach disregards all theory families with at least one empty component. The collection of theory families all of whose components are nonempty is denoted by $\mathrm{ThFam}^{\not\flat}(\mathcal{I})$ and, similarly, $\mathrm{ThSys}^{\not\flat}(\mathcal{I})$ stands for the collection of all theory systems all of whose components are nonempty.

In Section 6.2, we introduce the notion of rough equivalence between theory families of a π-institution. Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Given a theory family T of $\mathcal{I}$, we define its *rough companion* or *associate* $\widetilde{T}$ to be the theory family resulting from T by replacing each empty Σ-component by $\mathrm{SEN}^\flat(\Sigma)$. Then we say that two theory families T, T' are *roughly equivalent*, written $T \sim T'$, if they have the same rough companion, i.e., if $\widetilde{T} = \widetilde{T'}$. Rough equivalence is an equivalence relation on theory families. The equivalence class of T is denoted by $\widetilde{[T]}$ and the collection of all rough equivalence classes by $\widetilde{\mathrm{ThFam}}(\mathcal{I})$. When restricted to theory systems, it is still an equivalence relation and the equivalence class of a theory system T is denoted $\widetilde{\lfloor T \rfloor}$, whereas the corresponding collection of rough equivalence classes by $\widetilde{\mathrm{ThSys}}(\mathcal{I})$. The key observation making rough equivalence appropriate as a vehicle for defining classes at the bottom of the Leibniz hierarchy is that, if two theory families are roughly equivalent, then they have identical associated Leibniz congruence systems. That is, the Leibniz operator is constant on rough equivalence classes and, hence, may be viewed as an operator on $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ or on $\widetilde{\mathrm{ThSys}}(\mathcal{I})$, depending on the context. The remainder of Section 6.2 deals with several technical issues concerning rough equivalence. First, by definition, $\widetilde{T}$ is the largest theory family in the class $\widetilde{[T]}$. On the other hand, even if T is a theory system, $\widetilde{T}$ may not be one. Nevertheless, $\widetilde{[T]}$ still has a largest element, which, in that case, is clearly different from $\widetilde{T}$. Another drawback is that, even when T and T' are roughly equivalent, it may not be the case that $\overleftarrow{T}$ and $\overleftarrow{T'}$ are roughly equivalent. This introduces some unexpected complications when studying the Leibniz hierarchies based on roughness and narrowness. On the positive side, given an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and an $\mathcal{I}$-filter family $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we have $\widetilde{\alpha^{-1}(T)} = \alpha^{-1}(\widetilde{T})$. Form this it follows that, for all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, if T and T' are roughly equivalent, then so are $\alpha^{-1}(T)$ and $\alpha^{-1}(T')$.

In Section 6.3, we introduce some weakened versions of systemicity adapted to the study of roughness and narrowness. A π-institution is *roughly systemic* if, for every theory family T, $\overleftarrow{T} \sim T$. It is called *narrowly systemic* if, for every theory family T, with all components nonempty, i.e., such that $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $\overleftarrow{T} = T$. Finally, it is called *exclusively systemic* if, for all $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, such that $\overleftarrow{T} \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, we have $\overleftarrow{T} = T$. Systemicity implies both rough and narrow systemicity, and each of these two implies exclusive systemicity. On the other hand, for π-institutions having theorems all four properties become identical.

In Section 6.4, we turn to the study of rough injectivity properties. These are obtained by combining injectivity with rough equivalence. A π-institution $\mathcal{I}$ is *roughly family injective* if, for all theory families T and T', $\Omega(T) = \Omega(T')$ implies $T \sim T'$. $\mathcal{I}$ is *roughly left injective* if the same condition holds, but in the conclusion T, T' are replaced by $\overleftarrow{T}$, $\overleftarrow{T'}$, respectively. It is *roughly right injective* if, similarly, the same condition holds, with T, T' in the hypothesis replaced by $\overleftarrow{T}$, $\overleftarrow{T'}$, respectively. Finally, $\mathcal{I}$ is *roughly system injective* if the implication defining rough family injectivity holds, but with T, T' allowed to range over theory systems only, instead of over arbitrary theory families. Rough right injectivity is strong enough to imply rough systemicity. It also implies rough family injectivity, which implies rough system injectivity. Rough left injectivity also implies rough system injectivity. Moreover, rough right injectivity is equivalent to rough system injectivity and rough systemicity, whereas rough system injectivity, coupled with stability, implies rough left injectivity. All four rough injectivity properties are equivalent to the corresponding injectivity properties under availability of theorems. In addition, all four rough injectivity properties transfer. Section 6.4 concludes with characterizations of the family and system versions in terms of the Leibniz operator Ω, viewed as a mapping from $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ and $\widetilde{\mathrm{ThSys}}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

In Section 6.5, we study narrow injectivity properties. These are defined like the injectivity properties of Section 3.6, but only theory families with all components nonempty are taken into account. Accordingly, a π-institution

$\mathcal{I}$ is *narrowly family injective* if, for all theory families $T, T' \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, $\Omega(T) = \Omega(T')$ implies $T = T'$. In the *left version* T and T' are replaced in the conclusion by $\overleftarrow{T}$ and $\overleftarrow{T'}$, respectively. In the *right version* the same replacement occurs in the hypothesis, whereas the *system version* results by imposing the same condition as in the family version, but T, T' are allowed to range only over $\mathrm{ThSys}^{\nleq}(\mathcal{I})$. Narrow right injectivity implies exclusive systemicity, but does not imply either rough or narrow systemicity. The narrow injectivity hierarchy recovers the linearity of the injectivity hierarchy, which was established in Section 3.6. Narrow right injectivity implies the family version, which implies the left version, which, in turn, implies the system version. The latter, coupled with narrow systemicity, implies narrow right injectivity. A comparison is made between corresponding narrow and rough injectivity properties. The family versions are identical. The left versions are incomparable. For both right and system versions, the rough properties imply the corresponding narrow properties. Each of the narrow injectivity properties is identical to the corresponding injectivity property in the presence of theorems. In addition, all four of them transfer. The section concludes by formulating characterization theorems for the family and system versions in terms of the Leibniz operator seen as a mapping from $\mathrm{ThFam}^{\nleq}(\mathcal{I})$ and $\mathrm{ThSys}^{\nleq}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

In Sections 6.6 and 6.7, we undertake the study of rough and narrow reflectivity properties, respectively, following the format of the study of rough and narrow injectivity from Sections 6.4 and 6.5. Subsequently, Sections 6.8 and 6.9, still following the same paradigm, present an analogous study of rough and narrow complete reflectivity properties. A π-institution $\mathcal{I}$ is *roughly family reflective* if, for all theory families T, T', $\Omega(T) \le \Omega(T')$ implies $\widetilde{T} \le \widetilde{T'}$. *Rough left* and *rough right reflectivity* result by replacing T and T' in the conclusion and in the hypothesis, respectively, by $\overleftarrow{T}$ and $\overleftarrow{T'}$. *Rough system reflectivity* imposes the same condition as the family version, but applies it only to theory systems. Rough right reflectivity implies rough systemicity. Moreover, it implies rough family reflectivity, which implies rough system reflectivity. The left version also implies the system version. Rough right reflectivity is equivalent to the system version plus rough systemicity and, furthermore, the system version, augmented by stability, implies rough left reflectivity. Comparing with previously studied properties, it is fairly obvious that each version of rough reflectivity implies the corresponding rough injectivity version. In addition, each rough reflectivity version is equivalent to the corresponding reflectivity version under the existence of theorems. All four rough reflectivity properties transfer. Finally, characterizations are provided of rough family and rough system reflectivity in terms of the Leibniz operator, perceived as a mapping from $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ and $\widetilde{\mathrm{ThSys}}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

In Section 6.7, we turn to narrow reflectivity. A π-institution $\mathcal{I}$ is *nar-*

rowly family reflective if, for all theory families T, T', with all components nonempty, $\Omega(T) \leq \Omega(T')$ implies $T \leq T'$. As before, in the *left version* T and T' are replaced in the conclusion by $\overleftarrow{T}$ and $\overleftarrow{T'}$, respectively, and, in the *right version* the same change is applied in the hypothesis instead. *Narrow system reflectivity* stipulates that the same condition as in the family version hold, but applied only to theory systems with all components nonempty. Narrow family reflectivity implies exclusive systemicity. In terms of the narrow reflectivity hierarchy, the right version is the strongest, followed by the family, then the left and, finally, the system version. Narrow system reflectivity and narrow systemicity imply narrow right reflectivity. Comparisons between the rough reflectivity and the narrow reflectivity classes lead to conclusions similar to those obtained in the injectivity case. The two family versions are equivalent, the left versions are incomparable, whereas rough right and rough system reflectivity imply, respectively, narrow right and narrow system reflectivity. Each narrow reflectivity property implies in a straightforward way the corresponding narrow injectivity property and, moreover, gets identified with the corresponding reflectivity property in the presence of theorems. All four narrow reflectivity properties transfer. Finally, the family and system versions may be characterized in terms of the Leibniz operator, viewed as a mapping from $\mathrm{ThFam}^{\nmid}(\mathcal{I})$ and $\mathrm{ThSys}^{\nmid}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

Section 6.8 starts the study of complete reflectivity with the rough versions, continued in Section 6.9 with the narrow versions. A π-institution $\mathcal{I}$ is *roughly family c-reflective* if, for every collection $\mathcal{T} \cup \{T'\}$ of theory families, $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$ implies $\bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T'}$. In the *left version* all theory families in the conclusion appear in their arrow versions and, in the *right version* the same happens in the hypothesis instead. Finally, the *system version* stipulates that the same condition as in the family version hold, by $\mathcal{T} \cup \{T'\}$ ranges over collections of theory systems only. The hierarchy established here mimics the one of rough reflectivity properties. Rough right c-reflectivity implies the family version, which implies the system version, which is also implied by rough left c-reflectivity. Rough system c-reflectivity and rough systemicity together are equivalent to rough right c-reflectivity. Moreover, rough system c-reflectivity, coupled with stability, implies the left version. It is clear that each rough c-reflectivity property implies the corresponding rough reflectivity property and, further, each rough c-reflectivity property is equivalent to the corresponding c-reflectivity property in the presence of theorems. All four rough c-reflectivity properties transfer and, as previously, one may formulate characterizations of rough family and rough system c-reflectivity in terms of Ω, seen as a mapping from $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ and $\widetilde{\mathrm{ThSys}}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

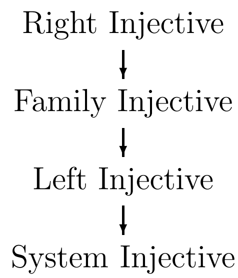
Section 6.9 continues the study of complete reflectivity by looking at narrow c-reflectivity properties. A π-institution $\mathcal{I}$ is *narrowly family c-reflective* if, for every collection $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$ implies

$\bigcap \mathcal{T} \leq T'$. The *left* and *right versions* are obtained as before by replacing all theory families in the conclusion and in the hypothesis, respectively, by their arrow versions, whereas the *system version* imposes the condition above for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^{\nleq}(\mathcal{I})$. Narrow family c-reflectivity implies exclusive systemicity. The narrow c-reflectivity hierarchy reflects the structure of the narrow reflectivity hierarchy. The right version is the strongest, followed by the family version, then by the left version, while the system version is the weakest of the four. Narrow system c-reflectivity and narrow systemicity imply narrow right c-reflectivity. Comparisons between the rough and narrow versions also follow along lines similar to those between rough and narrow reflectivity properties. The family versions are equivalent, the left versions are incomparable, whereas for both the right and the system versions, rough c-reflectivity implies the corresponding narrow c-reflectivity version. Clearly, each of the four narrow c-reflectivity properties implies the corresponding narrow reflectivity property. As was the case in the rough setting, each narrow c-reflectivity property is identified with the corresponding c-reflectivity property in the presence of theorems. The section concludes with transfer theorems and with characterizations of narrow family and narrow system c-reflectivity via Ω, perceived as a mapping from $\mathrm{ThFam}^{\nleq}(\mathcal{I})$ and $\mathrm{ThSys}^{\nleq}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

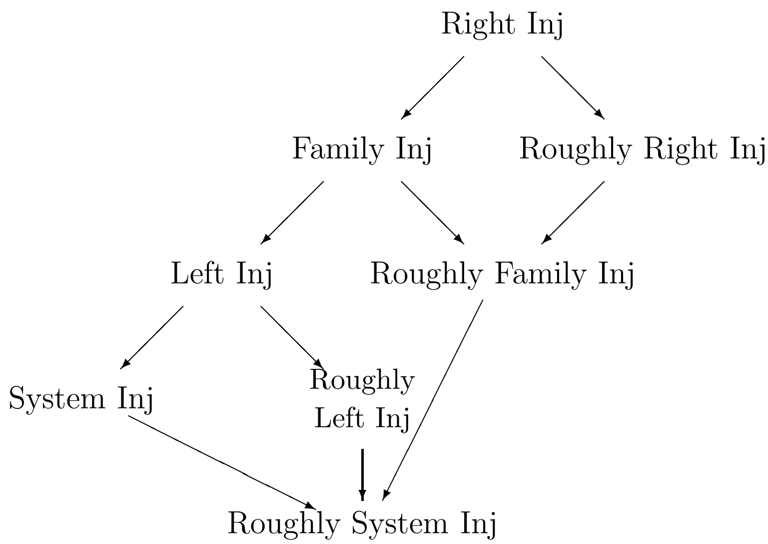
As is clear from all features described, if one considers π-institutions with theorems, the rough and narrow properties become identical to the corresponding properties studied in Chapter 3. Consequently, presence or absence of theorems is a critical characteristic underlying the considerations and hierarchies established in Sections 6.2-6.9. In Section 6.10, the concluding section of the chapter, we turn to some conditions characterizing the existence of theorems via the Frege equivalence family and the Lindenbaum equivalence family operators, introduced in Section 2.11. More precisely, we show that a π-institution $\mathcal{I}$ has theorems if and only if the Frege operator $\lambda : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{EqvFam}(\mathbf{F})$ is injective. Other equivalent conditions to the availability of theorems are the injectivity of the Lindenbaum operator $\widetilde{\lambda}^{\mathcal{I}} : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{EqvFam}(\mathbf{F})$ or, alternatively, its complete reflectivity. Finally, a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ has theorems if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, the induced π-institution $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ has theorems. This constitutes a sort of transfer theorem for the property of possessing theorems.

6.2 Rough Equivalence

Recall from Chapter 3 the injectivity hierarchy, depicted in the following diagram, lying close to the bottom of the semantic Leibniz hierarchy.



Our goal in this section is to add new classes to the semantic Leibniz hierarchy that lie below those injectivity classes. We will eventually build the following hierarchy:



Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. Given a theory family $T \in \mathrm{ThFam}(\mathcal{I})$, the **rough companion** or **rough associate** or **rough representative of** T, denoted $\widetilde{T}$, is the theory family of $\mathcal{I}$ that results from T after replacing every empty Σ-component by the corresponding universe $\mathrm{SEN}^\flat(\Sigma)$. More formally, we set

$$\widetilde{T} = \{\widetilde{T}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|},$$

where, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\widetilde{T}_\Sigma = \begin{cases} T_\Sigma, & \text{if } T_\Sigma \neq \varnothing \\ \mathrm{SEN}^\flat(\Sigma), & \text{if } T_\Sigma = \varnothing \end{cases}.$$

The operator $\widetilde{\ }: \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$ is clearly idempotent:

Lemma 368 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. Then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\widetilde{T}} = \widetilde{T}$.

Proof: We have, by construction, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\widetilde{T}_\Sigma \neq \varnothing$, whence, we get, by definition, $\widetilde{\widetilde{T}}_\Sigma = \widetilde{T}_\Sigma$. ∎

Define on $\mathrm{ThFam}(\mathcal{I})$ the relation $\sim \subseteq \mathrm{ThFam}(\mathcal{I})^2$ of **rough equivalence** by setting, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$T \sim T' \quad \text{iff} \quad \widetilde{T} = \widetilde{T'}.$$

It is not difficult to see that rough equivalence is indeed an equivalence relation on the collection of theory families of $\mathcal{I}$, since it is the relational kernel of the mapping $\widetilde{\ } : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$. We call two theories $T, T' \in \mathrm{ThFam}(\mathcal{I})$ **roughly equivalent** if $T \sim T'$. We denote by $\widetilde{[T]}$ the rough equivalence class of a theory family T and let $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ be the collection of all rough equivalence classes of theory families of $\mathcal{I}$.

Since the collection of theory systems of $\mathcal{I}$ is a subcollection of the collection of theory families of $\mathcal{I}$, the rough equivalence relation restricts to an equivalence relation, which we also term **rough equivalence**, on the collection $\mathrm{ThSys}(\mathcal{I})$. We denote by $\widetilde{[T]}$ the rough equivalence class of a theory system T and let $\widetilde{\mathrm{ThSys}}(\mathcal{I})$ be the collection of all rough equivalence classes of theory systems of $\mathcal{I}$.

We now introduce a notation that will prove very handy in subsequent considerations, especially in contexts where the π-institutions under scrutiny may not have theorems. Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We define:

- $\mathrm{ThFam}^{\not\emptyset}(\mathcal{I})$ to be the collection of all theory families of $\mathcal{I}$ with all components nonempty.

  Note that

  $$\begin{array}{rcl}\mathrm{ThFam}^{\not\emptyset}(\mathcal{I}) & = & \{T \in \mathrm{ThFam}(\mathcal{I}) : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(T_\Sigma \neq \varnothing)\} \\ & = & \{T \in \mathrm{ThFam}(\mathcal{I}) : \widetilde{T} = T\}.\end{array}$$

  Note, also, that, in case $\mathcal{I}$ has theorems, $\mathrm{ThFam}^{\not\emptyset}(\mathcal{I}) = \mathrm{ThFam}(\mathcal{I})$.

- $\mathrm{ThSys}^{\not\emptyset}(\mathcal{I})$ to be the collection of all theory systems of $\mathcal{I}$ with all components nonempty.

  Note that

  $$\begin{array}{rcl}\mathrm{ThSys}^{\not\emptyset}(\mathcal{I}) & = & \{T \in \mathrm{ThSys}(\mathcal{I}) : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(T_\Sigma \neq \varnothing)\} \\ & = & \{T \in \mathrm{ThSys}(\mathcal{I}) : \widetilde{T} = T\}.\end{array}$$

  Note, again, that, in case $\mathcal{I}$ has theorems, $\mathrm{ThSys}^{\not\emptyset}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$.

A key result in our use of the rough equivalence relation to define the semantic hierarchy classes "down under" is the realization that two roughly equivalent theory families have the same Leibniz congruence family and, as a result, the Leibniz operator may be unambiguously applied on rough equivalence classes of theory families. This follows from the fact that, for every theory family T, T and $\widetilde{T}$ share the same Leibniz congruence system.

Proposition 369 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\Omega(T) = \Omega(\widetilde{T}).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$ and $\phi \in \widetilde{T}_\Sigma$.

- If $T_\Sigma = \varnothing$, then $\widetilde{T}_\Sigma = \mathrm{SEN}^\flat(\Sigma)$ and, hence, $\psi \in \widetilde{T}_\Sigma$;
- If $T_\Sigma \neq \varnothing$, then $\widetilde{T}_\Sigma = T_\Sigma$ and, hence, by the compatibility of $\Omega(T)$ with T, we get $\psi \in T_\Sigma = \widetilde{T}_\Sigma$.

We conclude that $\Omega(T)$ is compatible with $\widetilde{T}$ and, hence, $\Omega(T) \leq \Omega(\widetilde{T})$.

Suppose, conversely, that $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in \Omega_\Sigma(\widetilde{T})$ and $\phi \in T_\Sigma$. Then $T_\Sigma \neq \varnothing$, whence $\widetilde{T}_\Sigma = T_\Sigma$. Thus, $\phi \in \widetilde{T}_\Sigma$ and, by the compatibility of $\Omega(\widetilde{T})$ with $\widetilde{T}$, we get that $\psi \in \widetilde{T}_\Sigma = T_\Sigma$. Thus, $\Omega(\widetilde{T})$ is compatible with T and we get $\Omega(\widetilde{T}) \leq \Omega(T)$.

We conclude that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(\widetilde{T}) = \Omega(T)$. ■

Theorem 370 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$*,*

$$T \sim T' \quad \textit{implies} \quad \Omega(T) = \Omega(T').$$

Proof: Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \sim T'$. Then, by definition, $\widetilde{T} = \widetilde{T'}$. Thus, we get, by Proposition 369,

$$\Omega(T) = \Omega(\widetilde{T}) = \Omega(\widetilde{T'}) = \Omega(T')$$

and T and T' have, indeed, the same Leibniz congruence system. ■

We define, next, an ordering relation on the rough equivalence classes of theory families of a π-institution $\mathcal{I}$. But we start by looking at maximal elements.

Proposition 371 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For every* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\widetilde{T} = \max \widetilde{[T]}.$$

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$ and consider $T' \in \lceil\widetilde{T}\rceil$. Then, clearly, $T' \leq \widetilde{T'} = \widetilde{T}$. Therefore, $\widetilde{T}$ is a maximum element in $\lceil\widetilde{T}\rceil$. ∎

What is, perhaps, more surprising is that each rough equivalence class in $\mathrm{ThSys}(\mathcal{I})$ also has a maximum element. First, we show that it has maximal elements and then prove that there cannot exist two different maximal elements and, hence, that it has a maximum element.

Proposition 372 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *Then, for all* $T \in \mathrm{ThSys}(\mathcal{I})$, $\lceil\widetilde{T}\rceil$ *has a maximal element.*

Proof: Let $T \in \mathrm{ThSys}(\mathcal{I})$. We show that every chain in $\lceil\widetilde{T}\rceil$ has an upper bound in $\lceil\widetilde{T}\rceil$. Then the conclusion follows by applying Zorn's Lemma. Assume that $\{T^i : i \in I\}$ is a chain in $\lceil\widetilde{T}\rceil$. We consider $\bigcup_{i\in I} T^i$.

- $\bigcup_{i\in I} T^i \in \mathrm{ThFam}(\mathcal{I})$: Let $\Sigma \in |\mathbf{Sign}^\flat|$. If, for some $j \in I$, $T^j_\Sigma \neq \varnothing$ and $T^j_\Sigma \neq \mathrm{SEN}^\flat(\Sigma)$, then, since all members of $\{T^i : i \in I\}$ are roughly equivalent, we have $\bigcup_{i\in I} T^i_\Sigma = T^j_\Sigma$ is a Σ-theory. If, on the other hand, $T^i_\Sigma = \varnothing$, for all $i \in I$, then $\bigcup_{i\in I} T^i_\Sigma = \varnothing = T^i_\Sigma$, which is again a Σ-theory. Finally, if, for some $i \in I$, $T^i = \mathrm{SEN}^\flat(\Sigma)$, then $\bigcup_{i\in I} T^i_\Sigma = \mathrm{SEN}^\flat(\Sigma)$, which is again a Σ-theory. Therefore, we conclude that $\bigcup_{i\in I} T^i \in \mathrm{ThFam}(\mathcal{I})$.

- $\bigcup_{i\in I} T^i \in \mathrm{ThSys}(\mathcal{I})$: Let $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \bigcup_{i\in I} T^i_\Sigma$. Then, for some $i \in I$, $\phi \in T^i_\Sigma$. Since $T^i \in \mathrm{ThSys}(\mathcal{I})$, we have $\mathrm{SEN}^\flat(f)(\phi) \in T^i_{\Sigma'}$ and, therefore, $\mathrm{SEN}^\flat(f)(\phi) \in \bigcup_{i\in I} T^i_{\Sigma'}$. We conclude that $\bigcup_{i\in I} T^i \in \mathrm{ThSys}(\mathcal{I})$.

- $\bigcup_{i\in I} T^i \sim T$: Let $\Sigma \in |\mathbf{Sign}^\flat|$. If $\bigcup_{i\in I} T^i_\Sigma = \varnothing$, then $T^i_\Sigma = \varnothing$, for all $i \in I$, and hence, $\widetilde{T^i}_\Sigma = \mathrm{SEN}^\flat(\Sigma)$. Therefore, $\widetilde{\bigcup_{i\in I} T^i}_\Sigma = \mathrm{SEN}^\flat(\Sigma) = \widetilde{T^i}_\Sigma$.

  Suppose, next, that $\bigcup_{i\in I} T^i_\Sigma \neq \varnothing$. Thus, there exists $j \in I$, such that $T^j_\Sigma \neq \varnothing$. If there exists $i \in I$, such that $T^i_\Sigma = \mathrm{SEN}^\flat(\Sigma)$, then $\bigcup_{i\in I} T^i_\Sigma = \mathrm{SEN}^\flat(\Sigma)$, whence

  $$(\widetilde{\bigcup_{i\in I} T^i})_\Sigma = \mathrm{SEN}^\flat(\Sigma) = \widetilde{T^i}_\Sigma.$$

  So assume that $T^i_\Sigma \neq \mathrm{SEN}^\flat(\Sigma)$, for all $i \in I$. Then, we conclude that $T^j_\Sigma \neq \varnothing, \mathrm{SEN}^\flat(\Sigma)$ and, therefore, since all T^i's are roughly equivalent, $T^i_\Sigma = T^j_\Sigma$, for all $i \in I$. Then $\bigcup_{i\in I} T^i_\Sigma = T^j_\Sigma$ and, therefore,

  $$(\widetilde{\bigcup_{i\in I} T^i})_\Sigma = T^j_\Sigma = \widetilde{T^j}_\Sigma.$$

  Thus, $\widetilde{\bigcup_{i\in I} T^i} = \widetilde{T}$ and we conclude that $\bigcup_{i\in I} T^i \in \lceil\widetilde{T}\rceil$.

Therefore, $\bigcup_{i\in I} T^i$ is clearly an upper bound of $\{T^i : i \in I\}$ in $\widetilde{[T]}$. By Zorn's Lemma, we conclude that $\widetilde{[T]}$ has a maximal element. ∎

Now we show that in a rough equivalence class in $\mathrm{ThSys}(\mathcal{I})$, there cannot exist two different maximal elements and, therefore, that every rough equivalence class in $\mathrm{ThSys}(\mathcal{I})$ has a maximum element.

Theorem 373 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *Then, for all* $T \in \mathrm{ThSys}(\mathcal{I})$, $\widetilde{[T]}$ *has a maximum element.*

Proof: The proof is quite similar to the proof of Proposition 372. The key is to show that, if $T', T'' \in \mathrm{ThSys}(\mathcal{I})$, such that $T' \sim T''$, then $T' \cup T'' \in \mathrm{ThSys}(\mathcal{I})$, such that $T' \cup T'' \sim T$. So, unless $T' = T''$, not both can be maximal in $\widetilde{[T]}$.

- $T' \cup T'' \in \mathrm{ThFam}(\mathcal{I})$: Let $\Sigma \in |\mathbf{Sign}^\flat|$. If $T'_\Sigma \neq \varnothing, \mathrm{SEN}^\flat(\Sigma)$ and $T''_\Sigma \neq \varnothing, \mathrm{SEN}^\flat(\Sigma)$, then, since $T' \sim T''$, $T'_\Sigma = T''_\Sigma$. Thus, $T'_\Sigma \cup T''_\Sigma = T'_\Sigma$ and, hence, it is a Σ-theory. If $T'_\Sigma = T''_\Sigma = \varnothing$, then $T'_\Sigma \cup T''_\Sigma = \varnothing$, which is again a Σ-theory. Otherwise, $T'_\Sigma \cup T''_\Sigma = \mathrm{SEN}^\flat(\Sigma)$, which is also a Σ-theory.

- $T' \cup T'' \in \mathrm{ThSys}(\mathcal{I})$: Let $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T'_\Sigma \cup T''_\Sigma$. Then $\phi \in T'_\Sigma$ or $\phi \in T''_\Sigma$. In the first case, $\mathrm{SEN}^\flat(f)(\phi) \in T'_{\Sigma'}$ and, in the second, $\mathrm{SEN}^\flat(f)(\phi) \in T''_{\Sigma'}$. In either case, $\mathrm{SEN}^\flat(f)(\phi) \in T'_{\Sigma'} \cup T''_{\Sigma'}$. Thus, $T' \cup T'' \in \mathrm{ThSys}(\mathcal{I})$.

- $T' \cup T'' \sim T$: Let $\Sigma \in |\mathbf{Sign}^\flat|$. If $T'_\Sigma \cup T''_\Sigma = \varnothing$, then $T'_\Sigma = T''_\Sigma = \varnothing$. So $\widetilde{T' \cup T''}_\Sigma = \mathrm{SEN}^\flat(\Sigma) = \widetilde{T'}_\Sigma = \widetilde{T}_\Sigma$.

  If $T'_\Sigma \cup T''_\Sigma \neq \varnothing$, then $T'_\Sigma \neq \varnothing$ or $T''_\Sigma \neq \varnothing$, say, without loss of generality, $T'_\Sigma \neq \varnothing$. If $T'_\Sigma \neq \mathrm{SEN}^\flat(\Sigma)$, then, since $T' \sim T''$, $T'_\Sigma = T''_\Sigma$ and, hence $T'_\Sigma \cup T''_\Sigma = T'_\Sigma$ and we have $\widetilde{T' \cup T''}_\Sigma = T'_\Sigma = \widetilde{T'}_\Sigma = \widetilde{T}_\Sigma$. If, on the other hand, $T'_\Sigma = \mathrm{SEN}^\flat(\Sigma)$, then $T'_\Sigma \cup T''_\Sigma = \mathrm{SEN}^\flat(\Sigma)$, whence $\widetilde{T' \cup T''}_\Sigma = \mathrm{SEN}^\flat(\Sigma) = \widetilde{T'}_\Sigma = \widetilde{T}_\Sigma$.

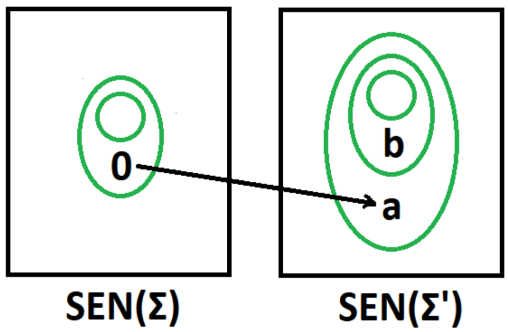
Therefore $T' \cup T'' \in \mathrm{ThSys}(\mathcal{I})$ and $T' \cup T'' \sim T$. We conclude that all maximal elements in $\widetilde{[T]}$ must be equal, i.e., $\widetilde{[T]}$ has a maximum element. ∎

It is worth noting, however, that the maximum element of a class $\widetilde{[T]}$ may not be $\widetilde{T} = \max\widetilde{[T]}$, since this theory family may not be a theory system.

Example 374 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$;

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$;
- $N^\flat$ *is the trivial clone.*



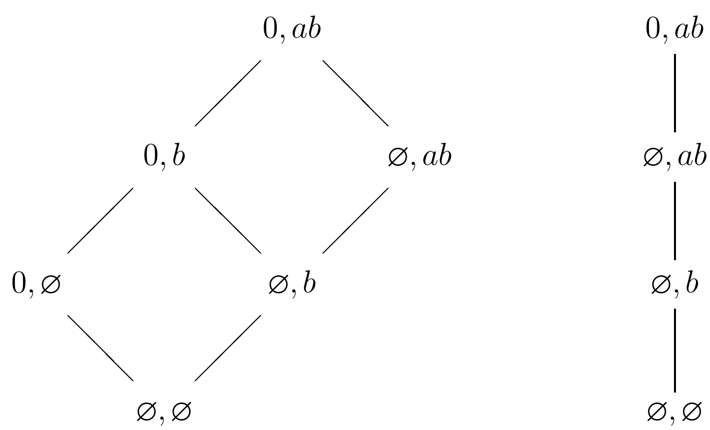
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\emptyset, \{0\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\emptyset, \{b\}, \{a, b\}\}.$$

There are six theory families, but only four theory systems. The action of $\overleftarrow{\ }$ *on theory families is given in the table below.*

T	$\overleftarrow{T}$
$\emptyset, \emptyset$	$\emptyset, \emptyset$
$0, \emptyset$	$\emptyset, \emptyset$
$\emptyset, b$	$\emptyset, b$
$0, b$	$\emptyset, b$
$\emptyset, ab$	$\emptyset, ab$
$0, ab$	$0, ab$

The complete lattice of theory families is shown on the left:



That of the theory systems is shown on the right. Now note that

$$\max\lfloor \widetilde{\{\emptyset, \{b\}\}} \rfloor = \{\emptyset, \{b\}\},$$

whereas $\widetilde{\{\emptyset, \{b\}\}} = \{\{0\}, \{b\}\} \notin \mathrm{ThSys}(\mathcal{I})$.

For what follows, we also need to point out the fact that, roughly speaking, the $\sim$ operator does not interact smoothly with the $\leftarrow$ operator. More precisely, for arbitrary π-institutions, and theory families T, T',

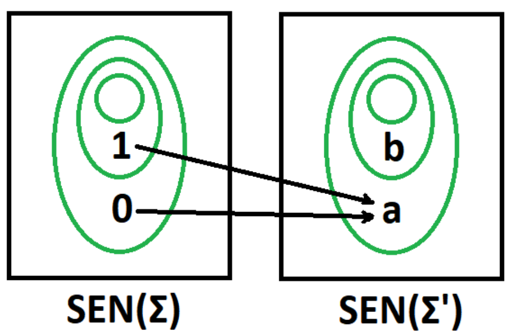
$$\text{the relation } T \sim T' \text{ does not imply, in general, that } \overleftarrow{T} \sim \overleftarrow{T'}.$$

We showcase the potential failure by giving a *counterexample to the statement*, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$T \sim T' \quad \text{implies} \quad \overleftarrow{T} \sim \overleftarrow{T'}.$$

Example 375 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = a$*;*
- $N^\flat$ *is the trivial clone.*



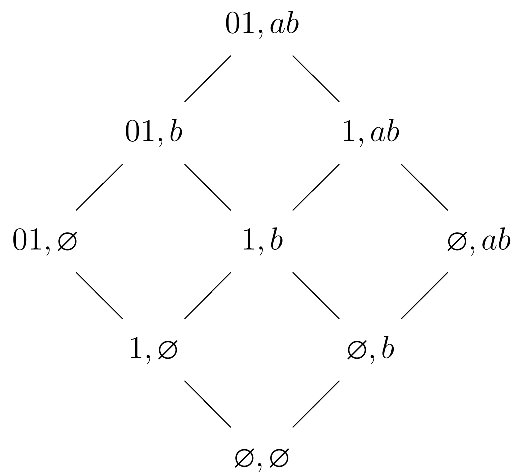
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0, 1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a, b\}\}.$$

There are nine theory families, but only five theory systems. The action of $\leftarrow$ *on theory families is given in the table below.*

T	$\overleftarrow{T}$	T	$\overleftarrow{T}$
$\varnothing, \varnothing$	$\varnothing, \varnothing$	$\varnothing, ab$	$\varnothing, ab$
$1, \varnothing$	$\varnothing, \varnothing$	$01, b$	$\varnothing, b$
$\varnothing, b$	$\varnothing, b$	$1, ab$	$1, ab$
$01, \varnothing$	$\varnothing, \varnothing$	$01, ab$	$01, ab$
$1, b$	$\varnothing, b$		

The lattice of theory families of $\mathcal{I}$ is shown in the diagram.



Consider $T = \{\{1\},\{a,b\}\}$ *and* $T' = \{\{1\},\varnothing\}$. *We clearly have* $\widetilde{T} = \widetilde{T'} = T$, *whence* $T \sim T'$. *On the other hand,*

$$\overleftarrow{T} = T \nsim \{\varnothing,\varnothing\} = \overleftarrow{T'}.$$

Therefore, even though $T \sim T'$, *it is not the case that* $\overleftarrow{T} \sim \overleftarrow{T'}$.

To establish some transfer theorems for the classes to be introduced shortly, we need a few results pertaining to the interaction of rough equivalence with inverse images. Key to these considerations is the following technical lemma to the effect that a filter family has an empty component if and only if its inverse theory family has a corresponding empty component. This is a relatively easy consequence of the surjectivity of interpretation morphisms.

Lemma 376 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. *Then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$,

$$T_{F(\Sigma)} = \varnothing \quad \textit{iff} \quad \alpha_\Sigma^{-1}(T_{F(\Sigma)}) = \varnothing.$$

Proof: Let $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $\Sigma \in |\mathbf{Sign}^\flat|$. If $T_{F(\Sigma)} = \varnothing$, then, obviously, $\alpha^{-1}(T_{F(\Sigma)}) = \varnothing$. If, conversely, $\alpha_\Sigma^{-1}(T_{F(\Sigma)}) = \varnothing$, then, by surjectivity of $\langle F, \alpha \rangle$, $T_{F(\Sigma)} = \alpha_\Sigma(\alpha_\Sigma^{-1}(T_{F(\Sigma)})) = \alpha_\Sigma(\varnothing) = \varnothing$. ■

We can now show that the maximum $\widetilde{\alpha^{-1}(T)}$ of the rough equivalence class of the theory family $\alpha^{-1}(T)$ in the π-institution $\mathcal{I}$ coincides with the inverse image $\alpha^{-1}(\widetilde{T})$ of the maximum $\widetilde{T}$ of the rough equivalence class of the $\mathcal{I}$-filter family T of $\mathcal{A}$ in $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

Theorem 377 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. Then*

$$\widetilde{\alpha^{-1}(T)} = \alpha^{-1}(\widetilde{T}).$$

Proof: Let $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $\Sigma \in |\mathbf{Sign}^\flat|$. We separate two cases depending on whether or not $\alpha_\Sigma^{-1}(T_{F(\Sigma)}) = \varnothing$.

- Suppose $\alpha_\Sigma^{-1}(T_{F(\Sigma)}) = \varnothing$. Then, by Lemma 376, we get $T_{F(\Sigma)} = \varnothing$. Thus, we get

$$\widetilde{\alpha^{-1}(T)}_\Sigma = \mathrm{SEN}^\flat(\Sigma) = \alpha_\Sigma^{-1}(\mathrm{SEN}(F(\Sigma))) = \alpha_\Sigma^{-1}(\widetilde{T}_{F(\Sigma)}).$$

- Suppose $\alpha_\Sigma^{-1}(T_{F(\Sigma)}) \neq \varnothing$. Then, by Lemma 376, we get $T_{F(\Sigma)} \neq \varnothing$. Thus, we get

$$\widetilde{\alpha^{-1}(T)}_\Sigma = \alpha_\Sigma^{-1}(T_{F(\Sigma)}) = \alpha_\Sigma^{-1}(\widetilde{T}_{F(\Sigma)}).$$

In either case, we have $\widetilde{\alpha^{-1}(T)}_\Sigma = \alpha_\Sigma^{-1}(\widetilde{T}_{F(\Sigma)})$. Therefore, we get $\widetilde{\alpha^{-1}(T)} = \alpha^{-1}(\widetilde{T})$. ■

This implies that rough equivalence interacts smoothly with inverse images. More precisely, given two $\mathcal{I}$-filter families $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T \sim T'$ in $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ if and only if $\alpha^{-1}(T) \sim \alpha^{-1}(T')$ in $\mathrm{ThFam}(\mathcal{I})$. Contrast this with the rather rocky interaction between rough equivalence and the $\overleftarrow{\ }$ operator, as detailed before (and in) Example 375.

Corollary 378 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. Then*

$$T \sim T' \quad \textit{iff} \quad \alpha^{-1}(T) \sim \alpha^{-1}(T').$$

Proof: Let $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. We get

$$\begin{array}{lll} \alpha^{-1}(T) \sim \alpha^{-1}(T') & \text{iff} & \widetilde{\alpha^{-1}(T)} = \widetilde{\alpha^{-1}(T')} \quad (\text{Definition of } \sim) \\ & \text{iff} & \alpha^{-1}(\widetilde{T}) = \alpha^{-1}(\widetilde{T}') \quad (\text{Theorem 377}) \\ & \text{iff} & \widetilde{T} = \widetilde{T}' \quad (\text{Surjectivity of } \langle F, \alpha \rangle) \\ & \text{iff} & T \sim T'. \quad (\text{Definition of } \sim) \end{array}$$

This establishes the conclusion. ■

6.3 Roughness and Systemicity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. Recall that $\mathcal{I}$ was called **systemic** if all its theory families are theory systems. This can be expressed in symbols by writing $\mathrm{ThFam}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$ or, alternatively, by the assertion that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} = T$.

We now introduce three other *systemicity* properties that are inspired by the original, but avoid in some way the consideration of theory families with empty components or take into account the rough equivalence relation between theory families.

- We say that $\mathcal{I}$ is **roughly systemic** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} \sim T$;
- We say that $\mathcal{I}$ is **narrowly systemic** if, for all $T \in \mathrm{ThFam}^{\not\flat}(\mathcal{I})$, $\overleftarrow{T} = T$;
- We say that $\mathcal{I}$ is **exclusively systemic** if, for all $T \in \mathrm{ThFam}^{\not\flat}(\mathcal{I})$, such that $\overleftarrow{T} \in \mathrm{ThSys}^{\not\flat}(\mathcal{I})$, $\overleftarrow{T} = T$.

The inclusions between these four classes are straightforward and recounted in the following

Proposition 379 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is systemic, then it is both roughly and narrowly systemic;*

(b) If $\mathcal{I}$ *is roughly systemic, then it is exclusively systemic;*

(c) If $\mathcal{I}$ *is narrowly systemic, then it is exclusively systemic.*

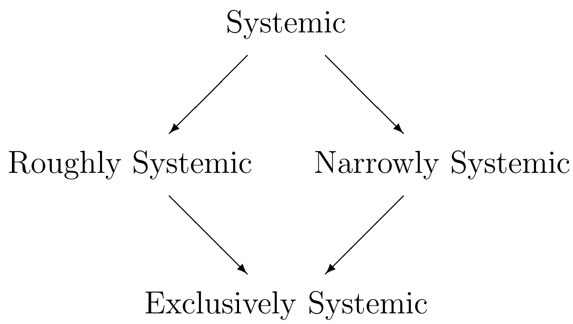
Proof:

(a) Suppose that $\mathcal{I}$ is systemic. If $T \in \mathrm{ThFam}(\mathcal{I})$, then $T = \overleftarrow{T}$. Thus, $\widetilde{T} = \widetilde{\overleftarrow{T}}$, i.e., $T \sim \overleftarrow{T}$ and, hence, $\mathcal{I}$ is roughly systemic. On the other hand, if $T \in \mathrm{ThFam}^{\not\flat}(\mathcal{I})$, then, since $\mathrm{ThFam}^{\not\flat}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I})$, we get, by hypothesis, $\overleftarrow{T} = T$. Thus, $\mathcal{I}$ is also narrowly systemic.

(b) Suppose that $\mathcal{I}$ is roughly systemic. Let $T \in \mathrm{ThFam}^{\not\flat}(\mathcal{I})$, such that $\overleftarrow{T} \in \mathrm{ThSys}^{\not\flat}(\mathcal{I})$. Since $\mathrm{ThFam}^{\not\flat}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I})$, we get, by hypothesis, $\overleftarrow{T} \sim T$, i.e., $\widetilde{\overleftarrow{T}} = \widetilde{T}$. However, since $T \in \mathrm{ThFam}^{\not\flat}(\mathcal{I})$ and $\overleftarrow{T} \in \mathrm{ThSys}^{\not\flat}(\mathcal{I})$, we conclude that $\overleftarrow{T} = \widetilde{\overleftarrow{T}} = \widetilde{T} = T$. Thus, $\mathcal{I}$ is exclusively systemic.

(c) Suppose $\mathcal{I}$ is narrowly systemic. Then, by hypothesis, for all $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$ and, therefore, a fortiori, for all such T, such that $\overleftarrow{T} \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$, we get that $\overleftarrow{T} = T$. Hence, $\mathcal{I}$ is exclusively systemic. ■

Proposition 379 establishes the following *hierarchy of roughness and systemicity properties*:



A related result, which partially explains the introduction of the roughness and systemicity classes and which, in fact, forms the undercurrent of much of the ideas underlying developments in the entire chapter, assures that all three bottom properties actually coincide with systemicity itself, in case the π-institution under consideration has theorems.

Proposition 380 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is exclusively systemic and has theorems, then it is systemic.*

Proof: Suppose $\mathcal{I}$ has theorems. Then $\mathrm{ThFam}^{\lightning}(\mathcal{I}) = \mathrm{ThFam}(\mathcal{I})$, and $\mathrm{ThSys}^{\lightning}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$. Moreover, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} \in \mathrm{ThSys}(\mathcal{I}) = \mathrm{ThSys}^{\lightning}(\mathcal{I})$. Therefore, the defining condition of exclusive systemicity is equivalent to asserting that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} = T$, i.e., it is equivalent to systemicity. ■

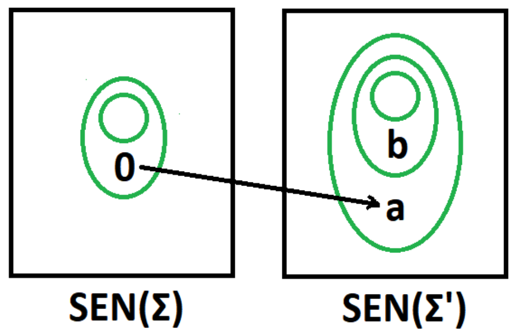
We present two examples that will show that all four classes in the roughness and systemicity hierarchy depicted above are indeed different. The first example shows that the southwest arrows represent proper inclusions, i.e.,

- Systemic π-institutions form a proper subclass of roughly systemic π-institutions;
- Exclusively systemic π-institutions form a proper subclass of narrowly systemic π-institutions.

This is accomplished by presenting a π-institution which is roughly systemic but fails to be narrowly systemic.

Example 381 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0\}$*,* $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$*;*
- $N^\flat$ *is the trivial clone.*



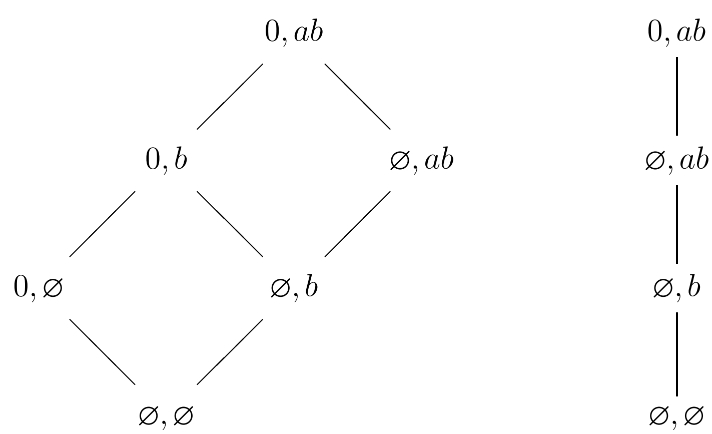
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{0\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a, b\}\}.$$

There are six theory families, but only four theory systems. The action of $\overleftarrow{\ }$ *on theory families is given in the table below.*

T	$\overleftarrow{T}$
$\varnothing, \varnothing$	$\varnothing, \varnothing$
$0, \varnothing$	$\varnothing, \varnothing$
$\varnothing, b$	$\varnothing, b$
$0, b$	$\varnothing, b$
$\varnothing, ab$	$\varnothing, ab$
$0, ab$	$0, ab$

The complete lattice of theory families is shown on the left, whereas that of the theory systems is shown on the right.



To check that this π-institution is roughly systemic, it is only necessary to focus on theory families T for which $\overleftarrow{T} \neq T$. There are two such, namely $T = \{\{0\}, \varnothing\}$ and $T = \{\{0\}, \{b\}\}$. We have (using obvious shorthand):

$$\overleftarrow{\widetilde{0,\varnothing}} = \widetilde{\varnothing,\varnothing} = 0, ab = \widetilde{0,\varnothing};$$
$$\overleftarrow{\widetilde{0,b}} = \widetilde{\varnothing,b} = 0, b = \widetilde{0,b}.$$

Thus, $\mathcal{I}$ is indeed roughly systemic. On the other hand, for the theory $T = \{\{0\}, \{b\}\}$ above, we have $T \in \mathrm{ThFam}^{\not\sharp}(\mathcal{I})$ and, moreover, $\overleftarrow{T} = \{\varnothing, \{b\}\} \neq T$. Hence, $\mathcal{I}$ fails to be narrowly systemic.

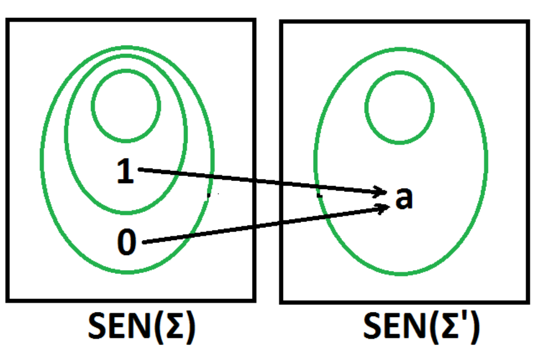
The second example shows that the southeast arrows represent proper inclusions, i.e.,

- The class of systemic π-institutions is a proper subclass of that of narrowly systemic π-institutions;
- The class of roughly systemic π-institutions forms a proper subclass of that of exclusively systemic π-institutions.

It exhibits a π-institution which is narrowly systemic but not roughly systemic.

Example 382 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

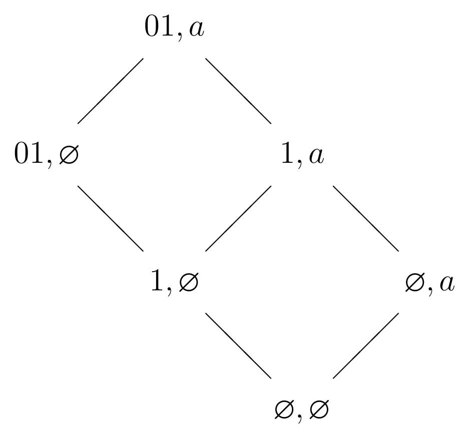
- $\mathbf{Sign}^\flat$ *is the category with objects Σ and Σ' and a unique (non-identity) morphism $f : \Sigma \to \Sigma'$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$*,* $\mathrm{SEN}^\flat(\Sigma') = \{a\}$ *and* $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = a$*;*
- $N^\flat$ *is the trivial clone.*



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{a\}\}.$$

Clearly, there are six theory families in $\mathrm{ThFam}(\mathcal{I})$*, only four of which are theory systems, and only two of which are in* $\mathrm{ThFam}^{\nsim}(\mathcal{I})$*. The lattice of theory families is shown in the diagram:*



Since $\mathrm{ThFam}^{\nsim}(\mathcal{I}) = \{\{1,a\},\{01,a\}\}$ *and* $\overleftarrow{1,a} = 1,a$ *and* $\overleftarrow{01,a} = 01,a$*, we get that $\mathcal{I}$ is narrowly systemic. On the other hand, consider $T = \{\{1\},\varnothing\}$. We have*

$$\widetilde{\overleftarrow{1,\varnothing}} = \widetilde{\varnothing,\varnothing} = 01,a \neq 1,a = \widetilde{1,\varnothing},$$

whence, $\overleftarrow{1,\varnothing} \nsim 1,\varnothing$ *and, hence, $\mathcal{I}$ is not roughly systemic.*

Finally, it is not difficult to show that rough systemicity implies stability.

Lemma 383 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a π-institution based on* $\mathbf{F}$*. If $\mathcal{I}$ is roughly systemic, then it is stable.*

Proof: Suppose $\mathcal{I}$ is roughly systemic and let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, by rough systemicity, $\overleftarrow{T} \sim T$, i.e., $\widetilde{\overleftarrow{T}} = \widetilde{T}$. Therefore, using Proposition 369, we get $\Omega(\overleftarrow{T}) = \Omega(\widetilde{\overleftarrow{T}}) = \Omega(\widetilde{T}) = \Omega(T)$. This shows that $\mathcal{I}$ is stable. ∎

6.4 Rough Injectivity

In this section we study classes of π-institutions defined using injectivity properties of the Leibniz operator applied on rough equivalence classes.

Definition 384 (Rough Injectivity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a π-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is called* **roughly family injective** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$\Omega(T) = \Omega(T') \quad \textit{implies} \quad T \sim T';$$

- $\mathcal{I}$ *is called* **roughly left injective** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$\Omega(T) = \Omega(T') \quad \textit{implies} \quad \overleftarrow{T} \sim \overleftarrow{T'}.$$

- $\mathcal{I}$ *is called* **roughly right injective** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'}) \quad \textit{implies} \quad T \sim T'.$$

- $\mathcal{I}$ *is called* **roughly system injective** *if, for all* $T, T' \in \mathrm{ThSys}(\mathcal{I})$,

$$\Omega(T) = \Omega(T') \quad \textit{implies} \quad T \sim T'.$$

Recall that, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$, based on an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$, we say that $\mathcal{I}$ is *roughly systemic* if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} \sim T$. In an analog of Lemma 207, we show that rough right injectivity implies rough systemicity and, hence, by Theorem 370, stability.

Lemma 385 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is roughly right injective, then it is roughly systemic.*

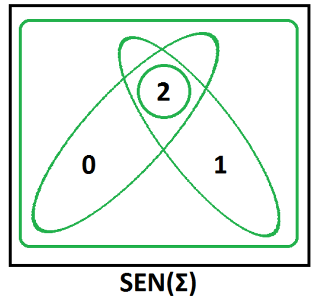
Proof: Suppose that $\mathcal{I}$ is roughly right injective and let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, we have, by Proposition 42, that $\overleftarrow{\overleftarrow{T}} = \overleftarrow{T}$. Therefore, we get $\Omega(\overleftarrow{\overleftarrow{T}}) = \Omega(\overleftarrow{T})$. Hence, by rough right injectivity, we get that $\overleftarrow{T} \sim T$. Hence $\mathcal{I}$ is roughly systemic. ∎

We give another example to show that the converse of Lemma 385 does not hold in general. That is, that there exists a roughly systemic π-institution that is not roughly right injective.

Example 386 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with the single object* Σ;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$;
- $N^\flat$ *is the clone generated by the three unary natural transformations* $\sigma^\flat, \tau^\flat, \rho^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *given by the following table:*

x	$\sigma^\flat_\Sigma(x)$	$\tau^\flat_\Sigma(x)$	$\rho^\flat_\Sigma(x)$
0	0	0	0
1	2	1	0
2	2	1	2

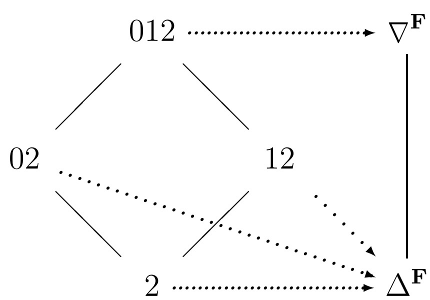


Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}.$$

$\mathcal{I}$ has theorems and, therefore, rough equivalence coincides with the identity relation on $\mathrm{ThFam}(\mathcal{I})$. *Moreover, since* $\mathbf{Sign}^\flat$ *is trivial, $\mathcal{I}$ is systemic. These observations imply that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T \sim \overleftarrow{T}$ and, hence, $\mathcal{I}$ is roughly systemic.*

On the other hand, the lattice of theory families of $\mathcal{I}$ and the corresponding Leibniz congruence systems are given in the diagram.



Since

$$\Omega(\overleftarrow{\{\{0,2\}\}}) = \Omega(\{\{0,2\}\}) = \Delta^{\mathbf{F}} = \Omega(\{\{1,2\}\}) = \Omega(\overleftarrow{\{\{1,2\}\}}),$$

whereas $\{\{0,2\}\} \nsim \{\{1,2\}\}$, we get that $\mathcal{I}$ is not roughly right injective.

Next we look into establishing the *rough injectivity hierarchy* of π-institutions. The following relationships can be established between the four rough injectivity classes.

Proposition 387 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) If $\mathcal{I}$ is roughly right injective, then it is roughly family injective;

(b) If $\mathcal{I}$ is roughly family injective, then it is roughly system injective;

(c) *If $\mathcal{I}$ is roughly left injective, then it is roughly system injective.*

Proof:

(a) Suppose that $\mathcal{I}$ is roughly right injective and let $T, T' \in \text{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. By Lemma 385, $\mathcal{I}$ is roughly systemic, whence $\overleftarrow{T} \sim T$ and $\overleftarrow{T'} \sim T'$. Thus, by Theorem 370, we get

$$\Omega(\overleftarrow{T}) = \Omega(T) = \Omega(T') = \Omega(\overleftarrow{T'}).$$

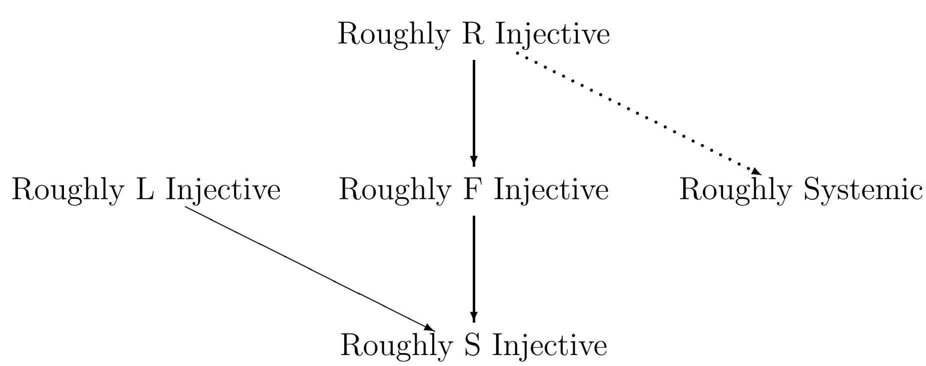
Now applying rough right injectivity gives $T \sim T'$. Hence, $\mathcal{I}$ is roughly family injective.

(b) Suppose that $\mathcal{I}$ is roughly family injective and let $T, T' \in \text{ThSys}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, by rough family injectivity, we get $T \sim T'$. Therefore, $\mathcal{I}$ is roughly system injective.

(c) Suppose that $\mathcal{I}$ is roughly left injective and let $T, T' \in \text{ThSys}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. By rough left injectivity, we conclude that $\overleftarrow{T} \sim \overleftarrow{T'}$. However, since T, T' are theory systems, we have $\overleftarrow{T} = T$ and $\overleftarrow{T'} = T'$. Hence we get $T \sim T'$ and $\mathcal{I}$ is roughly system injective.

■

We have now established the following **rough injectivity hierarchy** of π-institutions.



We formulate, next, two additional properties concerning the relationships between rough injectivity classes. First, it turns out that the separating property between rough right injectivity and rough system injectivity is exactly rough systemicity.

Proposition 388 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *$\mathcal{I}$ is roughly right injective if and only if it is roughly system injective and roughly systemic.*

Proof: Suppose, first, that $\mathcal{I}$ is roughly right injective. Then, by Lemma 385, it is roughly systemic and by Proposition 387 it is roughly system injective.

Suppose conversely, that $\mathcal{I}$ is roughly system injective and roughly systemic and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$. By rough system injectivity and Proposition 42, we get $\overleftarrow{T} \sim \overleftarrow{T'}$. Hence, by rough systemicity, $T \sim \overleftarrow{T} \sim \overleftarrow{T'} \sim T'$. Thus, $\mathcal{I}$ is roughly right injective. ■

Second, we show that rough system injectivity together with stability imply rough left injectivity.

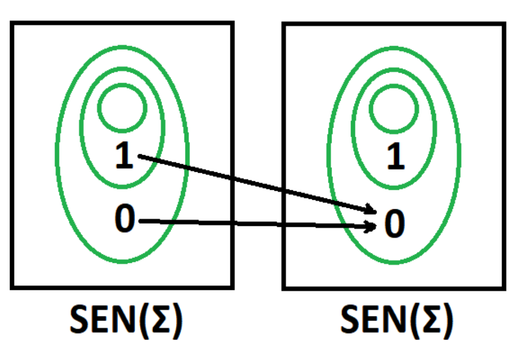
Proposition 389 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is roughly system injective and stable, then it is roughly left injective.*

Proof: Suppose that $\mathcal{I}$ is roughly system injective and stable and consider $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, by stability, $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$. Hence, since $\overleftarrow{T}, \overleftarrow{T'} \in \mathrm{ThSys}(\mathcal{I})$, by rough system injectivity, $\overleftarrow{T} \sim \overleftarrow{T'}$. This shows that $\mathcal{I}$ is roughly left injective. ■

Even though rough left injectivity does imply rough system injectivity, as was shown in Proposition 387, rough left injectivity does not imply stability in general, as is shown in the following example, and, hence, the converse of Proposition 389 fails in general.

Example 390 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

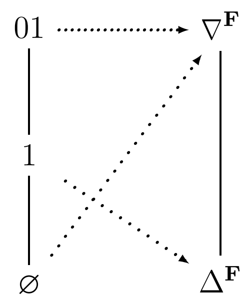
- *$\mathbf{Sign}^\flat$ is the category with the single object Σ and a single (non-identity) morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- *$\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ is defined by $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$ and $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$;*
- *$N^\flat$ is the trivial clone, consisting of the projections only.*



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\}.$$

$\mathcal{I}$ has three theory families $\{\varnothing\}$, $\{\{1\}\}$ and $\{\{0,1\}\}$, but only two theory systems, $\{\varnothing\}$ and $\{\{0,1\}\}$. The lattice of theory families of $\mathcal{I}$ and the corresponding Leibniz congruence systems are given in the diagram.



The only theory families for which $\Omega(T) = \Omega(T')$ are $T = \{\{0,1\}\}$ and $T' = \{\varnothing\}$. For those, we get $\overleftarrow{T} = T \sim T' = \overleftarrow{T'}$. Therefore, $\mathcal{I}$ is roughly left injective. On the other hand, we get $\Omega(\overleftarrow{\{\{1\}\}}) = \Omega(\{\varnothing\}) = \nabla^{\mathbf{F}} \neq \Delta^{\mathbf{F}} = \Omega(\{\{1\}\})$. Therefore, $\mathcal{I}$ is not a stable π-institution.

We now present three examples to show that all inclusions established between rough injectivity classes and depicted in the diagram above are proper inclusions. The first example will show that the class of roughly right injective π-institutions is a proper subclass of the class of roughly family injective π-institutions.

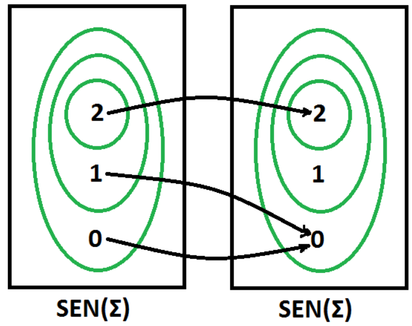
Example 391 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- *$\mathbf{Sign}^\flat$ is the category with a single object Σ and a single non-identity morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- *$\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ is given by $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ and $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$ and $\mathrm{SEN}^\flat(f)(2) = 2$;*
- *$N^\flat$ is the trivial clone.*

Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by setting $\mathcal{C}_\Sigma = \{\{2\}, \{1,2\}, \{0,1,2\}\}$. Since $\mathcal{I}$ has theorems, rough equivalence on $\mathrm{ThFam}(\mathcal{I})$ coincides with the identity relation.

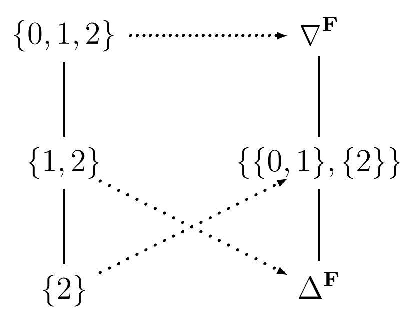
The following table gives the theory families and the theory systems of the π-institution $\mathcal{I}$:

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$



Since $\{\{1,2\}\}$ is a theory family that is not a theory system, $\mathcal{I}$ is not systemic. Thus, rough equivalence being the identity, $\mathcal{I}$ is not roughly systemic and, hence, by Lemma 385, $\mathcal{I}$ is not roughly right injective.

The lattice of theory families and the corresponding Leibniz congruence systems are depicted below:



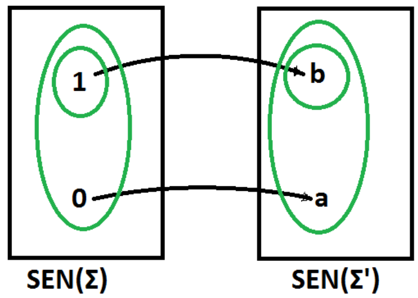
It is obvious from the diagram that $\mathcal{I}$ is family injective, and therefore, rough equivalence being the identity, it is also roughly family injective.

Returning more explicitly to right rough injectivity, note that for $T = \{\{2\}\}$ and $T' = \{\{1,2\}\}$, we have $\overleftarrow{T} = T = \overleftarrow{T'}$, whence $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$, whereas, obviously, $T \neq T'$ and, hence, $T \not\approx T'$.

The second example shows that there exists a roughly left injective π-institution that is not roughly family injective.

Example 392 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- *$\mathbf{Sign}^\flat$ is the category with objects Σ and Σ' and a unique (non-identity) morphism $f : \Sigma \to \Sigma'$;*
- *$\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ is defined by $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ and $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$;*
- *$N^\flat$ is the trivial clone.*



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

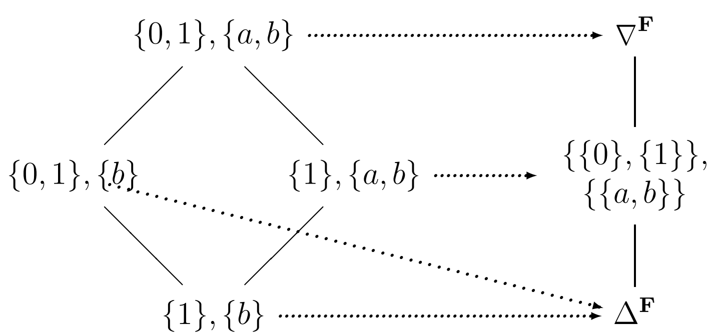
$$\mathcal{C}_\Sigma = \{\{1\}, \{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\}, \{a,b\}\}.$$

Again, since $\mathcal{I}$ has theorems, rough equivalence coincides with the identity relation on $\mathrm{ThFam}(\mathcal{I})$.

The following table shows the action of $\overleftarrow{\ }$ *on theory families, where rows correspond to T_Σ and columns to $T_{\Sigma'}$ and each entry is written as $\overleftarrow{T}_\Sigma, \overleftarrow{T}_{\Sigma'}$.*

$\leftarrow$	$\{b\}$	$\{a,b\}$
$\{1\}$	$\{1\}, \{b\}$	$\{1\}, \{a,b\}$
$\{0,1\}$	$\{1\}, \{b\}$	$\{0,1\}, \{a,b\}$

The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right:



Since the only two theory families that have the same Leibniz congruence system are $\{\{0,1\},\{b\}\}$ and $\{\{1\},\{b\}\}$ and it holds that

$$\overleftarrow{\{\{0,1\},\{b\}\}} = \overleftarrow{\{\{1\},\{b\}\}} = \{\{1\},\{b\}\},$$

we conclude that $\mathcal{I}$ is left injective. Moreover, since rough equivalence coincides with the identity, $\mathcal{I}$ is also roughly left injective.

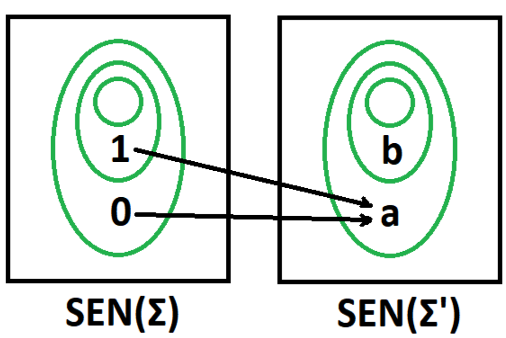
From the diagram, it is also clear that $\mathcal{I}$ is not family injective, since the two theory families $\{\{0,1\},\{b\}\}$ and $\{\{1\},\{b\}\}$ have the same Leibniz *congruence system. The same counterexample, keeping in mind the fact that rough equivalence coincides with the identity, showcases that $\mathcal{I}$ is not roughly family injective either.*

The third example shows that there exists a roughly family injective π-institution that is not roughly left injective. Combined with the preceding example, it has the effect of establishing the following facts:

- The classes of roughly family injective and roughly left injective π-institutions are incomparable. Contrast this with the case of injectivity, where family injectivity implies left injectivity.
- The class of roughly family injective π-institutions is properly contained in the class of roughly system injective π-institutions.
- Similarly, the class of roughly left injective π-institutions is a proper subclass of the class of roughly system injective π-institutions.

Example 393 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = a$;
- $N^\flat$ *is the trivial clone.*



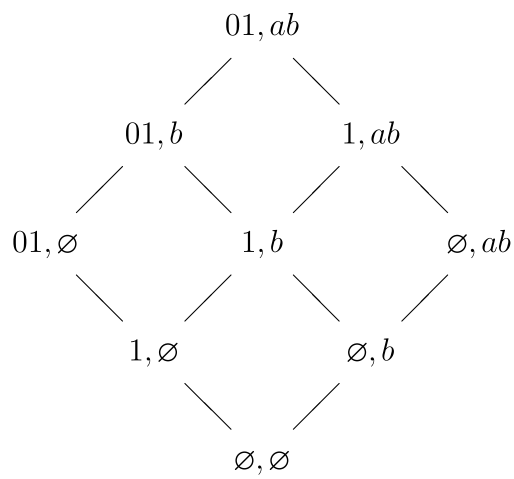
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a,b\}\}.$$

There are nine theory families, but only five theory systems. The action of $\overleftarrow{\ }$ *on theory families is given in the table below.*

T	$\overleftarrow{T}$	T	$\overleftarrow{T}$
$\varnothing,\varnothing$	$\varnothing,\varnothing$	$\varnothing,ab$	$\varnothing,ab$
$1,\varnothing$	$\varnothing,\varnothing$	$01,b$	$\varnothing,b$
$\varnothing,b$	$\varnothing,b$	$1,ab$	$1,ab$
$01,\varnothing$	$\varnothing,\varnothing$	$01,ab$	$01,ab$
$1,b$	$\varnothing,b$		

The lattice of theory families of $\mathcal{I}$ *is shown in the diagram.*



We show that $\mathcal{I}$ *is roughly family injective. The following table summarizes the theory families together with their associated Leibniz congruence systems.*

T	$\Omega(T)$
$\{\varnothing,\varnothing\},\{01,\varnothing\},\{\varnothing,ab\},\{01,ab\}$	$\nabla^{\mathbf{F}}$
$\{\varnothing,b\},\{01,b\}$	$\{\nabla^{\mathbf{F}}_{\Sigma},\Delta^{\mathbf{F}}_{\Sigma'}\}$
$\{1,\varnothing\},\{1,ab\}$	$\{\Delta^{\mathbf{F}}_{\Sigma},\nabla^{\mathbf{F}}_{\Sigma'}\}$
$\{1,b\}$	$\Delta^{\mathbf{F}}$

Since, every row on the left column of this table contains roughly equivalent theory families, we conclude that $\mathcal{I}$ *is roughly family injective.*

On the other hand, consider $T=\{1,ab\}$ *and* $T'=\{1,\varnothing\}$. *We have*

$$\overleftarrow{T}=\{1,ab\}\not\sim\{\varnothing,\varnothing\}=\overleftarrow{T'},$$

but

$$\Omega(T)=\Omega(\{1,ab\})=\{\Delta^{\mathbf{F}}_{\Sigma},\nabla^{\mathbf{F}}_{\Sigma'}\}=\Omega(\{1,\varnothing\})=\Omega(T').$$

We conclude that $\mathcal{I}$ *is not roughly left injective.*

We now clarify the connections between rough injectivity and injectivity classes. It turns out that membership in an injectivity class implies membership in the corresponding rough injectivity class and, also, possession of theorems. Conversely, membership in a rough injectivity class plus possession of theorems entails membership in the corresponding injectivity class.

Theorem 394 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is right injective if and only if it is roughly right injective and has theorems;*

(b) $\mathcal{I}$ *is family injective if and only if it is roughly family injective and has theorems;*

(c) $\mathcal{I}$ *is left injective if and only if it is roughly left injective and has theorems;*

(d) $\mathcal{I}$ *is system injective if and only if it is roughly system injective and has theorems.*

Proof:

(a) Suppose that $\mathcal{I}$ is right injective. First, note that $\Omega(\overleftarrow{\mathrm{SEN}^\flat}) = \Omega(\mathrm{SEN}^\flat) = \nabla^{\mathbf{F}} = \Omega(\varnothing) = \Omega(\overleftarrow{\varnothing})$. Thus, if $\mathcal{I}$ does not have theorems, $\mathrm{SEN}^\flat = \varnothing$, a contradiction. Therefore, $\mathcal{I}$ has theorems. Second, if $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$, then, by right injectivity, $T = T'$ and, hence, $T \sim T'$. Thus, $\mathcal{I}$ is roughly right injective.

Assume, conversely, that $\mathcal{I}$ is roughly right injective and has theorems. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$. Then, by rough right injectivity, we get $T \sim T'$. On the other hand, since $\mathcal{I}$ has theorems, rough equivalence collapses to the identity relation, whence $T = T'$. Therefore, $\mathcal{I}$ is right injective.

(b) Suppose that $\mathcal{I}$ is family injective. First, note that $\Omega(\mathrm{SEN}^\flat) = \nabla^{\mathbf{F}} = \Omega(\varnothing)$. Thus, if $\mathcal{I}$ does not have theorems, $\mathrm{SEN}^\flat = \varnothing$, a contradiction. Therefore, $\mathcal{I}$ has theorems. Second, if $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$, then, by family injectivity, $T = T'$ and, hence, $T \sim T'$. Thus, $\mathcal{I}$ is roughly family injective.

Assume, conversely, that $\mathcal{I}$ is roughly family injective and has theorems. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, by rough family injectivity, we get $T \sim T'$. On the other hand, since $\mathcal{I}$ has theorems, rough equivalence collapses to the identity relation, whence $T = T'$. Therefore, $\mathcal{I}$ is family injective.

(c) Suppose that $\mathcal{I}$ is left injective. First, note that $\Omega(\mathrm{SEN}^\flat) = \nabla^{\mathbf{F}} = \Omega(\varnothing)$. Thus, if $\mathcal{I}$ does not have theorems, $\mathrm{SEN}^\flat = \overleftarrow{\mathrm{SEN}^\flat} = \overleftarrow{\varnothing} = \varnothing$, a contradiction. Therefore, $\mathcal{I}$ has theorems. Second, if $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$, then, by left injectivity, $\overleftarrow{T} = \overleftarrow{T'}$ and, hence, $\overleftarrow{T} \sim \overleftarrow{T'}$. Thus, $\mathcal{I}$ is roughly left injective.

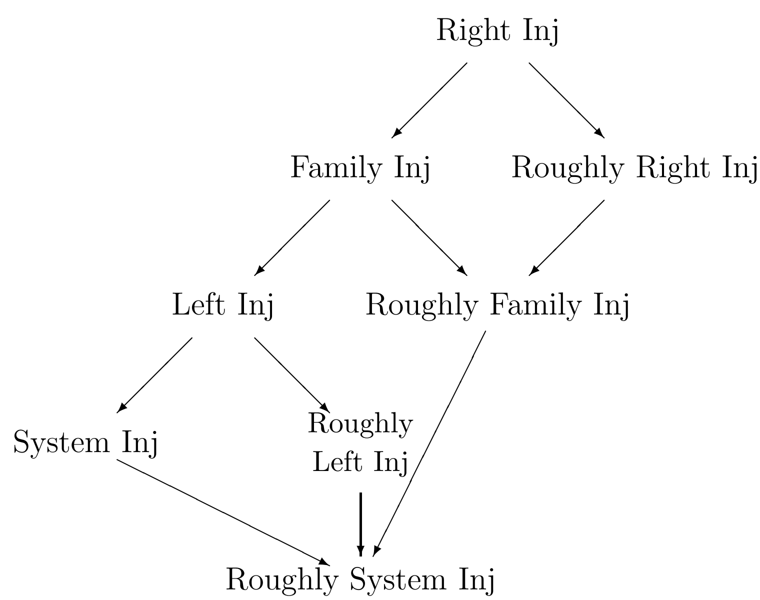
Assume, conversely, that $\mathcal{I}$ is roughly left injective and has theorems. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, by rough left injectivity, we get $\overleftarrow{T} \sim \overleftarrow{T'}$. On the other hand, since $\mathcal{I}$ has theorems, rough equivalence collapses to the identity relation, whence $\overleftarrow{T} = \overleftarrow{T'}$. Therefore, $\mathcal{I}$ is left injective.

(d) Suppose that $\mathcal{I}$ is system injective. First, note that $\Omega(\mathrm{SEN}^\flat) = \nabla^{\mathbf{F}} = \Omega(\varnothing)$. Thus, if $\mathcal{I}$ does not have theorems, $\mathrm{SEN}^\flat = \varnothing$, a contradiction. Therefore, $\mathcal{I}$ has theorems. Second, if $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$, then, by system injectivity, $T = T'$ and, hence, $T \sim T'$. Thus, $\mathcal{I}$ is roughly system injective.

Assume, conversely, that $\mathcal{I}$ is roughly system injective and has theorems. Let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, by rough system injectivity, we get $T \sim T'$. On the other hand, since $\mathcal{I}$ has theorems, rough equivalence collapses to the identity relation, whence $T = T'$. Therefore, $\mathcal{I}$ is system injective.

■

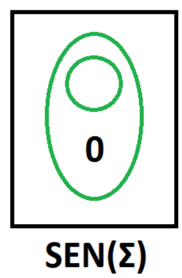
The work in Section 3.6, together with the work done in the present section and Theorem 394, reveal the following hierarchy of injectivity and rough injectivity classes, which was previewed at the beginning of Section 6.2.



To complete the demonstration that all classes in the depicted hierarchy are distinct we provide an example of a π-institution which belongs to all steps in the rough injectivity hierarchy but possesses none of the four (gentle) injectivity properties.

Example 395 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

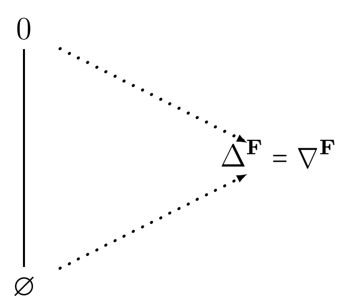
- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0\}$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{0\}\}.$$

$\mathcal{I}$ *is systemic and its lattice of theory families and corresponding Leibniz congruence systems are shown in the diagram.*



$\mathcal{I}$ *is roughly system injective, since* $\Omega(T) = \Omega(T')$ *implies* $T \sim T'$*. Since* $\mathcal{I}$ *is also systemic, it is, a fortiori, roughly systemic and stable. Now, by either direct calculation or based on Propositions 388 and 389, we get that* $\mathcal{I}$ *is also roughly right injective (and, hence, roughly family injective) and roughly left injective, respectively.*

On the other hand, since $\varnothing \neq \{0\}$ *but* $\Omega(\varnothing) = \nabla^{\mathbf{F}} = \Omega(\{0\})$*,* $\mathcal{I}$ *is not system injective and, hence, a fortiori,* $\mathcal{I}$ *has none of the four injectivity properties.*

The rough injectivity properties transfer from the theory families/systems of a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ to all $\mathcal{I}$-filter families/systems on arbitrary $\mathbf{F}$-algebraic systems.

Theorem 396 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is roughly right injective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $\Omega^{\mathcal{A}}(\overleftarrow{T}) = \Omega^{\mathcal{A}}(\overleftarrow{T'})$ *implies* $T \sim T'$*;*

(b) $\mathcal{I}$ *is roughly family injective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$ *implies* $T \sim T'$*;*

(c) $\mathcal{I}$ *is roughly left injective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$ *implies* $\overleftarrow{T} \sim \overleftarrow{T'}$*;*

(d) $\mathcal{I}$ *is roughly system injective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$*,* $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$ *implies* $T \sim T'$*.*

Proof:

(a) The "if" follows by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$ and taking into account that, by Lemma 51, $\mathrm{ThFam}(\mathcal{I}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F})$.

For the "only if", suppose that $\mathcal{I}$ is roughly right injective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(\overleftarrow{T}) = \Omega^{\mathcal{A}}(\overleftarrow{T'})$. Then $\alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T})) = \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'}))$. So, by Proposition 24, $\Omega(\alpha^{-1}(\overleftarrow{T})) = \Omega(\alpha^{-1}(\overleftarrow{T'}))$. Hence, by Lemma 6, $\Omega(\overleftarrow{\alpha^{-1}(T)}) = \Omega(\overleftarrow{\alpha^{-1}(T')})$. Since, by Lemma 51, $\alpha^{-1}(T), \alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$, we get, by applying rough right injectivity, $\alpha^{-1}(T) \sim \alpha^{-1}(T')$. Thus, by Corollary 378, $T \sim T'$.

(b) The "if" follows as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is roughly family injective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$. Then $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) = \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. So, by Proposition 24, $\Omega(\alpha^{-1}(T)) = \Omega(\alpha^{-1}(T'))$. Since, by Lemma 51, $\alpha^{-1}(T), \alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$, we get, by applying rough family injectivity, $\alpha^{-1}(T) \sim \alpha^{-1}(T')$. Thus, by Corollary 378, $T \sim T'$.

(c) The "if" follows as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is roughly left injective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$. Then $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) = \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. So, by Proposition 24, $\Omega(\alpha^{-1}(T)) = \Omega(\alpha^{-1}(T'))$. Since, by Lemma 51, $\alpha^{-1}(T), \alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$, we get, by applying rough left injectivity, $\overleftarrow{\alpha^{-1}(T)} \sim \overleftarrow{\alpha^{-1}(T')}$. Thus, by Lemma 6, $\alpha^{-1}(\overleftarrow{T}) \sim \alpha^{-1}(\overleftarrow{T'})$. Hence, by Corollary 378, $\overleftarrow{T} \sim \overleftarrow{T'}$.

(d) Similar to Part (b).

■

Finally, we may recast the rough injectivity classes in terms of the injectivity of mappings from posets of classes of theory or filter families/systems into posets of congruence systems.

Proposition 397 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is roughly family injective;*

(b) $\Omega : \widetilde{\mathrm{ThFam}}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is injective;*

(c) $\Omega^{\mathcal{A}} : \widetilde{\mathrm{FiFam}}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is injective, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

Similarly, for system injectivity, we have

Proposition 398 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is roughly system injective;*

(b) $\Omega : \widetilde{\mathrm{ThSys}}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is injective;*

(c) $\Omega^{\mathcal{A}} : \widetilde{\mathrm{FiSys}}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is injective, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

6.5 Narrow Injectivity

In this section we study classes of π-institutions defined using injectivity properties of the Leibniz operator restricted to $\mathrm{ThFam}^{\natural}(\mathcal{I})$. We call those *narrow injectivity* properties in analogy with the terminology adopted in Section 6.3, differentiating rough systemicity and narrow systemicity, the two strongest properties combining systemicity with rough equivalence.

Definition 399 (Narrow Injectivity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is called* **narrowly family injective** *if, for all* $T, T' \in \mathrm{ThFam}^{\not\wr}(\mathcal{I})$,
$$\Omega(T) = \Omega(T') \quad \textit{implies} \quad T = T';$$
- $\mathcal{I}$ *is called* **narrowly left injective** *if, for all* $T, T' \in \mathrm{ThFam}^{\not\wr}(\mathcal{I})$,
$$\Omega(T) = \Omega(T') \quad \textit{implies} \quad \overleftarrow{T} = \overleftarrow{T'}.$$
- $\mathcal{I}$ *is called* **narrowly right injective** *if, for all* $T, T' \in \mathrm{ThFam}^{\not\wr}(\mathcal{I})$,
$$\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'}) \quad \textit{implies} \quad T = T'.$$
- $\mathcal{I}$ *is called* **narrowly system injective** *if, for all* $T, T' \in \mathrm{ThSys}^{\not\wr}(\mathcal{I})$,
$$\Omega(T) = \Omega(T') \quad \textit{implies} \quad T = T'.$$

These narrow injectivity properties have the following useful characterizations.

Proposition 400 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is narrowly family injective if and only if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) = \Omega(T')$ *implies* $T \sim T'$;

(b) $\mathcal{I}$ *is narrowly left injective if and only if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) = \Omega(T')$ *implies* $\overleftarrow{\widetilde{T}} = \overleftarrow{\widetilde{T'}}$;

(c) $\mathcal{I}$ *is narrowly right injective if and only if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(\overleftarrow{\widetilde{T}}) = \Omega(\overleftarrow{\widetilde{T'}})$ *implies* $T \sim T'$;

(d) $\mathcal{I}$ *is narrowly system injective if and only if, for all* $T, T' \in \mathrm{ThSys}(\mathcal{I})$, *such that* $\widetilde{T}, \widetilde{T'} \in \mathrm{ThSys}(\mathcal{I})$, $\Omega(T) = \Omega(T')$ *implies* $T \sim T'$.

Proof:

(a) Suppose that $\mathcal{I}$ is narrowly family injective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Consider $\widetilde{T}, \widetilde{T'} \in \mathrm{ThFam}^{\not\wr}(\mathcal{I})$. By Proposition 369, $\Omega(\widetilde{T}) = \Omega(T) = \Omega(T') = \Omega(\widetilde{T'})$. Thus, by hypothesis, $\widetilde{T} = \widetilde{T'}$, i.e., $T \sim T'$. Therefore, the asserted condition holds.

Assume, conversely, that the asserted condition holds and let $T, T' \in \mathrm{ThFam}^{\not\wr}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, since $\mathrm{ThFam}^{\not\wr}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I})$, we get, by hypothesis, $T \sim T'$, i.e., $\widetilde{T} = \widetilde{T'}$. Since, however, $T, T' \in \mathrm{ThFam}^{\not\wr}(\mathcal{I})$, we get $T = \widetilde{T} = \widetilde{T'} = T'$. Thus, $\mathcal{I}$ is narrowly family injective.

(b) Suppose that $\mathcal{I}$ is narrowly left injective and let $T,T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then $\widetilde{T},\widetilde{T'} \in \mathrm{ThFam}^{\not\leq}(\mathcal{I})$ and, by Proposition 369, $\Omega(\widetilde{T}) = \Omega(\widetilde{T'})$. Thus, by hypothesis, $\overleftarrow{\widetilde{T}} = \overleftarrow{\widetilde{T'}}$.

Assume, conversely, that the asserted condition holds and let $T,T' \in \mathrm{ThFam}^{\not\leq}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, by hypothesis, $\overleftarrow{\widetilde{T}} = \overleftarrow{\widetilde{T'}}$. Since, however, $T,T' \in \mathrm{ThFam}^{\not\leq}(\mathcal{I})$, we get $\overleftarrow{T} = \overleftarrow{\widetilde{T}} = \overleftarrow{\widetilde{T'}} = \overleftarrow{T'}$. Therefore, $\mathcal{I}$ is narrowly left injective.

(c) Suppose that $\mathcal{I}$ is narrowly right injective and let $T,T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(\overleftarrow{\widetilde{T}}) = \Omega(\overleftarrow{\widetilde{T'}})$. Since $\widetilde{T},\widetilde{T'} \in \mathrm{ThFam}^{\not\leq}(\mathcal{I})$, we get, by hypothesis, $\widetilde{T} = \widetilde{T'}$, i.e., $T \sim T'$.

Assume, conversely, that the asserted condition holds and let $T,T' \in \mathrm{ThFam}^{\not\leq}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$. Then, since $T,T' \in \mathrm{ThFam}^{\not\leq}(\mathcal{I})$, we get $\Omega(\overleftarrow{\widetilde{T}}) = \Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'}) = \Omega(\overleftarrow{\widetilde{T'}})$. Now, by hypothesis, $T \sim T'$, i.e., $\widetilde{T} = \widetilde{T'}$ and, therefore, $T = T'$. We conclude that $\mathcal{I}$ is narrowly right injective.

(d) Suppose $\mathcal{I}$ is narrowly system injective and let $T,T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T},\widetilde{T'} \in \mathrm{ThSys}(\mathcal{I})$ and $\Omega(T) = \Omega(T')$. Then $\widetilde{T},\widetilde{T'} \in \mathrm{ThSys}^{\not\leq}(\mathcal{I})$ and, by Proposition 369, $\Omega(\widetilde{T}) = \Omega(T) = \Omega(T') = \Omega(\widetilde{T'})$. Thus, by hypothesis, $\widetilde{T} = \widetilde{T'}$, i.e., $T \sim T'$.

Assume, conversely, that the asserted condition holds and let $T,T' \in \mathrm{ThSys}^{\not\leq}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, since $T,T' \in \mathrm{ThSys}^{\not\leq}(\mathcal{I})$, we get $\widetilde{T} = T, \widetilde{T'} = T' \in \mathrm{ThSys}(\mathcal{I})$ and, therefore, by hypothesis, $T \sim T'$, i.e., $\widetilde{T} = \widetilde{T'}$. But this gives $T = \widetilde{T} = \widetilde{T'} = T'$. Thus, $\mathcal{I}$ is narrowly system injective.

■

It will be shown, next, in an analog of Lemma 385, that narrow right injectivity implies exclusive systemicity. Recall that, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$, based on an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$, we say that $\mathcal{I}$ is *exclusively systemic* if, for all $T \in \mathrm{ThFam}^{\not\leq}(\mathcal{I})$, such that $\overleftarrow{T} \in \mathrm{ThSys}^{\not\leq}(\mathcal{I})$, $\overleftarrow{T} = T$.

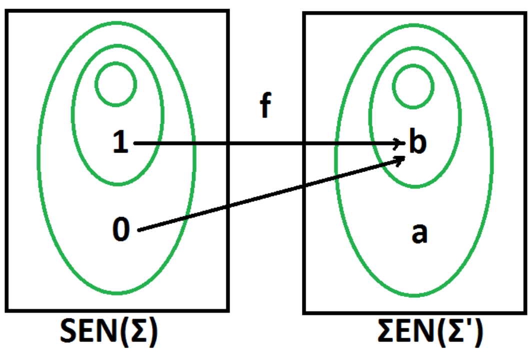
Lemma 401 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is narrowly right injective, then it is exclusively systemic.*

Proof: Assume $\mathcal{I}$ is narrowly right injective and let $T \in \mathrm{ThFam}^{\not\leq}(\mathcal{I})$, such that $\overleftarrow{T} \in \mathrm{ThSys}^{\not\leq}(\mathcal{I})$. Then, since, by Proposition 2, $\overleftarrow{\overleftarrow{T}} = \overleftarrow{T}$, we have $\Omega(\overleftarrow{\overleftarrow{T}}) = \Omega(\overleftarrow{T})$ and, hence, by narrow right injectivity, $\overleftarrow{T} = T$. Therefore, $\mathcal{I}$ is exclusively systemic. ■

However, as opposed to rough right injectivity, as the next examples demonstrate, narrow right injectivity implies neither rough nor narrow systemicity, in general. The first example showcases a π-institution which is narrowly right injective, but fails to be roughly systemic.

Example 402 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

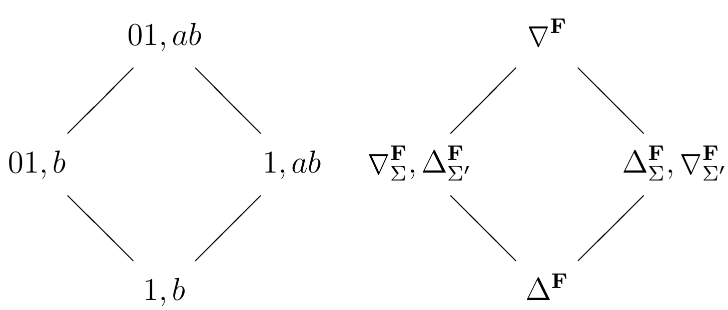
- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = b$, $\mathrm{SEN}^\flat(f)(1) = b$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a,b\}\}.$$

Clearly, there are only four theory families in $\mathrm{ThFam}^\sharp(\mathcal{I})$*, all of which are theory systems. Their lattice together with the associated Leibniz congruence systems are shown in the diagram:*



From this diagram and the fact that all theory families depicted are theory systems, we can see that, for all $T, T' \in \mathrm{ThFam}^\sharp(\mathcal{I})$,

$$\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'}) \quad \textit{implies} \quad T = T'.$$

Therefore, $\mathcal{I}$ is indeed narrowly right injective.

On the other hand, consider $T = \{\{1\}, \varnothing\}$. Then we have

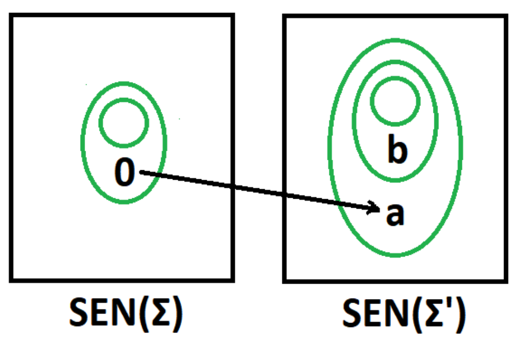
$$\widetilde{\overleftarrow{T}} = \widetilde{\widetilde{\varnothing}} = \mathrm{SEN}^\flat \neq \{\{1\}, \{a,b\}\} = \widetilde{T}.$$

This shows that $\overleftarrow{T} \nsim T$ and, therefore, $\mathcal{I}$ is not roughly systemic.

The next example exhibits a π-institution which is also narrowly right injective, but fails to be narrowly systemic.

Example 403 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects Σ and Σ' and a unique morphism $f : \Sigma \to \Sigma'$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by $\mathrm{SEN}^\flat(\Sigma) = \{0\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ and $\mathrm{SEN}^\flat(f)(0) = a$;*
- $N^\flat$ *is the trivial clone.*



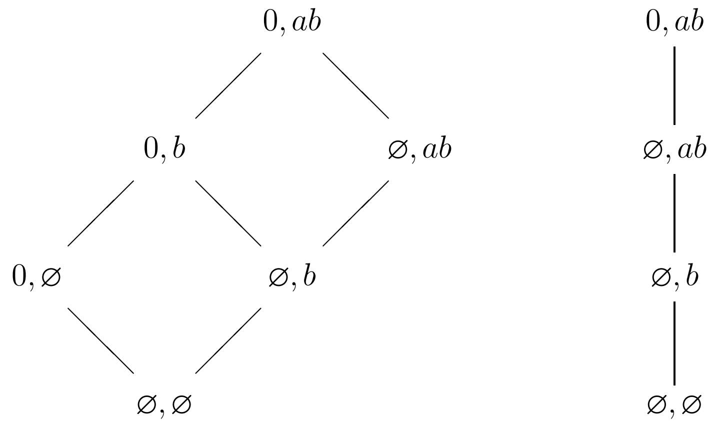
Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\varnothing, \{0\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a,b\}\}.$$

There are six theory families, but only four theory systems. The action of $\overleftarrow{\ }$ on theory families is given in the table below.

T	$\overleftarrow{T}$
$\varnothing, \varnothing$	$\varnothing, \varnothing$
$0, \varnothing$	$\varnothing, \varnothing$
$\varnothing, b$	$\varnothing, b$
$0, b$	$\varnothing, b$
$\varnothing, ab$	$\varnothing, ab$
$0, ab$	$0, ab$

The complete lattice of theory families is shown on the left.



That of the theory systems is shown on the right.

The only theory families in $\mathrm{ThFam}^{\not\sim}(\mathcal{I})$ *are* $T = \{\{0\},\{b\}\}$ *and* $\mathrm{SEN}^\flat$. *Since*

$$\Omega(\overleftarrow{T}) = \Omega(\varnothing,b) = \nabla^{\mathbf{F}}_{\Sigma},\Delta^{\mathbf{F}}_{\Sigma} \neq \nabla^{\mathbf{F}} = \Omega(\mathrm{SEN}^\flat) = \Omega(\overleftarrow{\mathrm{SEN}^\flat}),$$

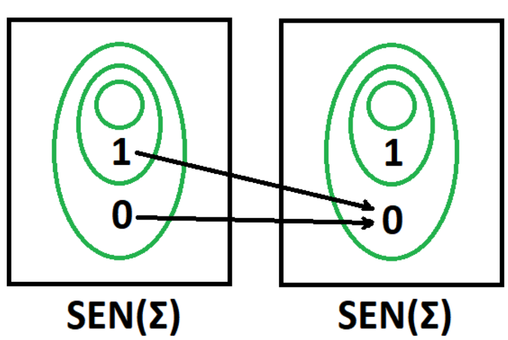
we conclude that $\mathcal{I}$ *is narrowly right injective.*

On the other hand, since $\overleftarrow{T} = \{\varnothing,\{b\}\} \neq T$, $\mathcal{I}$ *is not narrowly systemic.*

The converse of Lemma 401 fails in general. That is, there exists a π-institution which is exclusively systemic but is not narrowly right injective.

Example 404 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

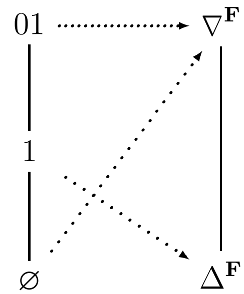
- $\mathbf{Sign}^\flat$ *is the category with the single object* Σ *and a single (non-identity) morphism* $f:\Sigma\to\Sigma$, *such that* $f\circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$;
- $N^\flat$ *is the trivial clone, consisting of the projections only.*



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\}.$$

$\mathcal{I}$ has three theory families, $\varnothing$, $\{\{1\}\}$ and $\{\{0,1\}\}$, but only two theory systems, $\varnothing$ and $\{\{0,1\}\}$. The lattice of theory families of $\mathcal{I}$ and the corresponding Leibniz congruence systems are given in the diagram.



Since there exists only one theory family T in $\mathrm{ThFam}^\natural(\mathcal{I})$, such that $\overleftarrow{T} \in \mathrm{ThSys}^\natural(\mathcal{I})$, namely $T = \mathrm{SEN}^\flat$, and $\overleftarrow{\mathrm{SEN}^\flat} = \mathrm{SEN}^\flat$, $\mathcal{I}$ is exclusively systemic. On the other hand, $\{\{1\}\}, \mathrm{SEN}^\flat \in \mathrm{ThFam}^\natural(\mathcal{I})$ and

$$\Omega(\overleftarrow{1}) = \Omega(\varnothing) = \nabla^{\mathbf{F}} = \Omega(\mathrm{SEN}^\flat) = \Omega(\overleftarrow{\mathrm{SEN}^\flat}),$$

but $\{\{1\}\} \neq \mathrm{SEN}^\flat$. Therefore, $\mathcal{I}$ fails to be narrowly right injective.

Following a similar vein, we establish a weakened analog of Lemma 207 for narrow right injectivity. This will play a key role in some of the classifications obtained in this and in subsequent sections.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that $\mathcal{I}$ is **narrowly stable** if, for all $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, $\Omega(\overleftarrow{T}) = \Omega(T)$. We return to this notion and study it in more detail in Section 7.2.

Lemma 405 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is narrowly right injective, then it is narrowly stable.*

Proof: Suppose that $\mathcal{I}$ is narrowly right injective. Let $T \in \mathrm{ThFam}^\natural(\mathcal{I})$. If $T \neq \overleftarrow{T} \in \mathrm{ThFam}^\natural(\mathcal{I})$, then, since $\overleftarrow{\overleftarrow{T}} = \overleftarrow{T}$, we would get $\Omega(\overleftarrow{\overleftarrow{T}}) = \Omega(\overleftarrow{T})$ and, hence, by narrow right injectivity, $\overleftarrow{T} = T$, a contradiction. Thus, we get that, for all $T \in \mathrm{ThFam}^\natural(\mathcal{I})$,

$$\overleftarrow{T} = T \quad \text{or} \quad \overleftarrow{T} \notin \mathrm{ThFam}^\natural(\mathcal{I}).$$

If $T \in \mathrm{ThFam}^{\not\vdash}(\mathcal{I})$ is such that $\overleftarrow{T} = \overline{\varnothing}$, then we would have $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{\mathrm{SEN}^\flat}) = \nabla^{\mathbf{F}}$, whence, by narrow right injectivity, $T = \mathrm{SEN}^\flat$, giving $\overline{\varnothing} = \overleftarrow{T} = \overleftarrow{\mathrm{SEN}^\flat} = \mathrm{SEN}^\flat$, a contradiction. Thus, $\overleftarrow{T} \neq \overline{\varnothing}$. So, there exists $P \in |\mathbf{Sign}^\flat|$, such that $\overleftarrow{T}_P \neq \varnothing$. If for such P, $\overleftarrow{T}_P \neq T_P$, then, setting $T' = \{T'_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, with $T'_\Sigma = \begin{cases} T_\Sigma, & \text{if } \Sigma \neq P \\ \overleftarrow{T}_P, & \text{if } \Sigma = P \end{cases}$, we would get $T, T' \in \mathrm{ThFam}^{\not\vdash}(\mathcal{I})$, with $\overleftarrow{T'} = \overleftarrow{T}$ and $T' \neq T$, contradicting narrow right injectivity. Therefore, for all $T \in \mathrm{ThFam}^{\not\vdash}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\overleftarrow{T}_\Sigma = T_\Sigma \quad \text{or} \quad \overleftarrow{T}_\Sigma = \varnothing.$$

Based on this fact, given $T \in \mathrm{ThFam}^{\not\vdash}(\mathcal{I})$, we partition the signatures into Part I, consisting of $\Sigma \in |\mathbf{Sign}^\flat|$, such that $\overleftarrow{T}_\Sigma = T_\Sigma$, and Part II, consisting of $\Sigma \in |\mathbf{Sign}^\flat|$, such that $\overleftarrow{T}_\Sigma = \varnothing$. Note that no morphism can have a domain of Type I and a codomain of Type II. Thus, letting $T' = \{T'_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, with

$$T'_\Sigma = \begin{cases} T_\Sigma, & \text{if } \Sigma \text{ is of Type I} \\ \mathrm{SEN}^\flat(\Sigma), & \text{if } \Sigma \text{ is of Type II} \end{cases},$$

we get $\overleftarrow{T'}_\Sigma = \overleftarrow{T}_\Sigma = T_\Sigma$, if Σ is of Type I, and, by the displayed condition above, $\overleftarrow{T'}_\Sigma = \varnothing$ or $\mathrm{SEN}^\flat(\Sigma)$, if Σ is of Type II. In either case, it follows by Theorem 370 that $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$, whence, by narrow right injectivity, $T = T'$. We finally conclude that

$$\begin{array}{rcll} \Omega(\overleftarrow{T}) & = & \Omega(\overleftarrow{T'}) & (T = T') \\ & = & \Omega(T') & (\text{Theorem } 370) \\ & = & \Omega(T). & (T = T') \end{array}$$

Therefore, $\mathcal{I}$ is narrowly stable. ■

We establish, next, the narrow injectivity hierarchy. The following proposition forms an analog of Proposition 387, which established the rough injectivity hierarchy.

Proposition 406 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) If $\mathcal{I}$ is narrowly right injective, then it is narrowly family injective;

(b) If $\mathcal{I}$ is narrowly family injective, then it is narrowly left injective;

(c) If $\mathcal{I}$ is narrowly left injective, then it is narrowly system injective.

Proof:

(a) Suppose that $\mathcal{I}$ is narrowly right injective and let $T, T' \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. By hypothesis and Lemma 405, $\Omega(\overleftarrow{T}) = \Omega(T) = \Omega(T') = \Omega(\overleftarrow{T'})$. By narrow right injectivity, we conclude that $T = T'$. Hence, $\mathcal{I}$ is narrowly family injective.

(b) Suppose that $\mathcal{I}$ is narrowly family injective and let $T, T' \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, by hypothesis, $T = T'$, whence, $\overleftarrow{T} = \overleftarrow{T'}$. Thus, $\mathcal{I}$ is narrowly left injective.

(c) Suppose that $\mathcal{I}$ is narrowly left injective and let $T, T' \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, by hypothesis, we get $\overleftarrow{T} = \overleftarrow{T'}$. Therefore, since T, T' are theory systems, $T = T'$ and, hence, $\mathcal{I}$ is narrowly system injective.

■

We have now established the following **narrow injectivity hierarchy** of π-institutions.

Narrowly R Injective
↓
Narrowly F Injective
↓
Narrowly L Injective
↓
Narrowly S Injective

We give some additional relations governing the hierarchy of narrow injectivity. The following proposition may be viewed as an analog of Propositions 388 and 389.

Proposition 407 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is narrowly system injective and narrowly systemic, then it is narrowly right injective.*

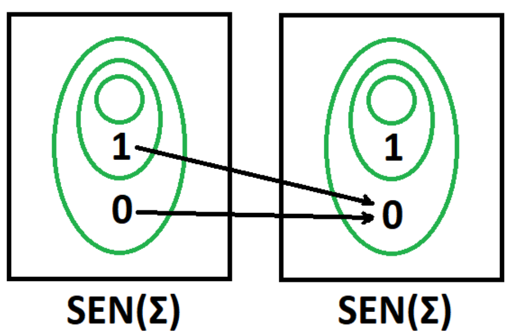
Proof: Suppose $\mathcal{I}$ is narrowly system injective and narrowly systemic. Let $T, T' \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$. By narrow systemicity, $T = \overleftarrow{T}$ and $T' = \overleftarrow{T'}$. Hence, on the one hand, $\Omega(T) = \Omega(T')$ and, on the other, $T, T' \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$. Thus, by narrow system injectivity, $T = T'$. Thus, $\mathcal{I}$ is narrowly right injective. ■

It was shown in Example 403 that narrow right injectivity does not imply, in general, narrow systemicity. Thus, the converse of Proposition 407 does not hold in general.

We present three examples to show that all inclusions established between the narrow injectivity classes and shown in the preceding diagram are indeed proper inclusions. The first example depicts a π-institution which is narrowly family injective but not narrowly right injective. This shows that the class of narrowly right injective π-institutions constitutes a proper subclass of the class of narrowly family injective π-institutions.

Example 408 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

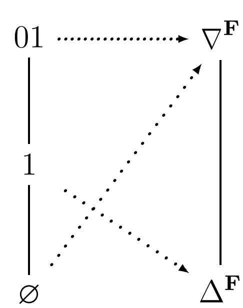
- $\mathbf{Sign}^\flat$ *is the category with the single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$;
- $N^\flat$ *is the trivial clone, consisting of the projections only.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\}.$$

$\mathcal{I}$ *has three theory families,* $\varnothing$, $\{\{1\}\}$ *and* $\{\{0,1\}\}$, *but only two theory systems,* $\varnothing$ *and* $\{\{0,1\}\}$. *The lattice of theory families of* $\mathcal{I}$ *and the corresponding Leibniz congruence systems are given in the diagram.*



Since there exists only two theory families in $\mathrm{ThFam}^{\nparallel}(\mathcal{I})$, $\{\{0,1\}\}$ *and* $\{\{1\}\}$, *and* $\Omega(\{\{0,1\}\}) \neq \Omega(\{\{1\}\})$, $\mathcal{I}$ *is trivially narrowly family injective. On the other hand,* $\{\{1\}\},\{\{0,1\}\} \in \mathrm{ThFam}^{\nparallel}(\mathcal{I})$ *and*

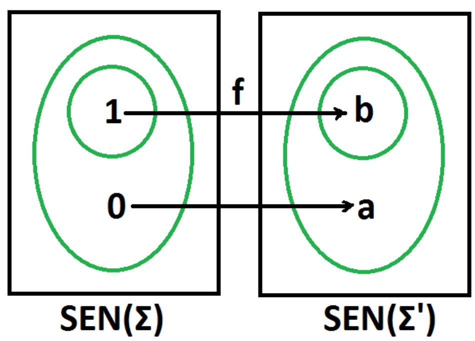
$$\Omega(\overleftarrow{\{\{1\}\}}) = \Omega(\{\varnothing\}) = \nabla^{\mathbf{F}} = \Omega(\{\{0,1\}\}) = \Omega(\overleftarrow{\{\{0,1\}\}}),$$

but $\{\{1\}\} \neq \{\{0,1\}\}$. *Therefore,* $\mathcal{I}$ *fails to be narrowly right injective.*

The next example depicts a π-institution which is narrowly left injective but not narrowly family injective. This shows that the class of narrowly family injective π-institutions constitutes a proper subclass of the class of narrowly left injective π-institutions.

Example 409 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$;
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

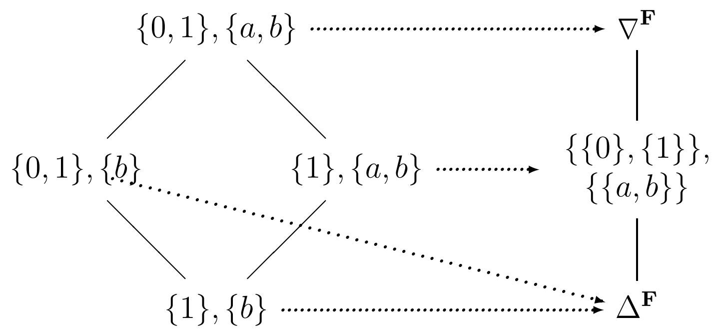
$$\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\},\{a,b\}\}.$$

Since $\mathcal{I}$ *has theorems, we get* $\mathrm{ThFam}^{\nparallel}(\mathcal{I}) = \mathrm{ThFam}(\mathcal{I})$. *Hence, both narrow family injectivity and narrow left injectivity coincide with family injectivity and left injectivity, respectively.*

The following table shows the action of $^{\leftarrow}$ *on theory families, where rows correspond to* T_Σ *and columns to* $T_{\Sigma'}$ *and each entry is written as* $\overleftarrow{T}_\Sigma, \overleftarrow{T}_{\Sigma'}$.

$\leftarrow$	$\{b\}$	$\{a,b\}$
$\{1\}$	$\{1\},\{b\}$	$\{1\},\{a,b\}$
$\{0,1\}$	$\{1\},\{b\}$	$\{0,1\},\{a,b\}$

The diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right.



Since the only two theory families that have the same Leibniz congruence system are $\{\{0,1\},\{b\}\}$ *and* $\{\{1\},\{b\}\}$ *and it holds that*

$$\overleftarrow{\{\{0,1\},\{b\}\}} = \overleftarrow{\{\{1\},\{b\}\}} = \{\{1\},\{b\}\},$$

we conclude that $\mathcal{I}$ *is left injective. Therefore, taking into account the remark above, we get that* $\mathcal{I}$ *is also narrowly left injective.*

From the diagram, it is also clear that $\mathcal{I}$ *is not family injective, since the two theory families* $\{\{0,1\},\{b\}\}$ *and* $\{\{1\},\{b\}\}$ *have the same Leibniz congruence system. The same counterexample shows that* $\mathcal{I}$ *is not narrowly family injective either.*

We finish the sequence of examples by presenting a narrowly system injective π-institution which, however, fails to be narrowly left injective. This example shows that narrowly left injective π-institutions form a proper subclass of the class of narrowly system injective π-institutions.

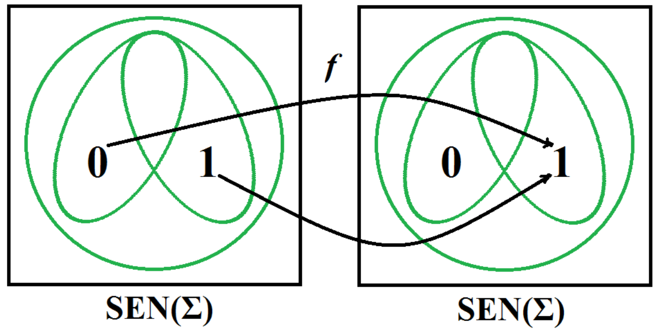
Example 410 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 1$ *and* $\mathrm{SEN}^\flat(f)(1) = 1$*;*
- $N^\flat$ *is the trivial clone.*

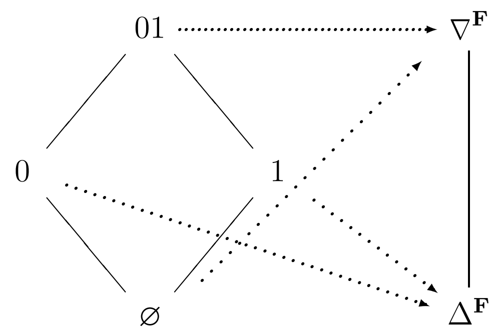
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\varnothing, \{0\}, \{1\}, \{0,1\}\}$*.*

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$*.*

T	$\overleftarrow{T}$
$\varnothing$	$\varnothing$
$\{0\}$	$\varnothing$
$\{1\}$	$\{1\}$
$\{0,1\}$	$\{0,1\}$



The lattice of theory families and the corresponding Leibniz congruence systems are depicted below.



It is obvious from the diagram that for no $T, T' \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$*, such that* $T \neq T'$ *is it the case that* $\Omega(T) = \Omega(T')$*. Therefore,* $\mathcal{I}$ *is trivially narrowly system injective. On the other hand, for* $T = \{\{0\}\}$*,* $T' = \{\{1\}\}$*, both members of* $\mathrm{ThFam}^{\nmid}(\mathcal{I})$*, we have* $\Omega(T) = \Omega(T') = \Delta^{\mathbf{F}}$*, whereas* $\overleftarrow{T} = \{\varnothing\} \neq \{\{1\}\} = \overleftarrow{T'}$*. Therefore,* $\mathcal{I}$ *fails to be narrowly left injective.*

We turn now to the relationships between corresponding classes of the rough injectivity and the narrow injectivity hierarchies.

First, it is easy to see, using the characterization in Part (a) of Proposition 400 that the two types of family injectivity involved coincide.

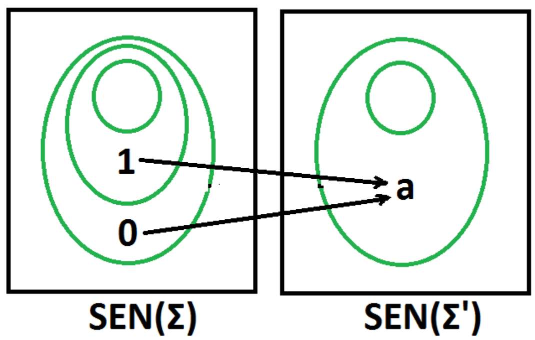
Corollary 411 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is roughly family injective if and only if it is narrowly family injective.*

Proof: Part (a) of Proposition 400. ∎

Unfortunately, the relationship between the remaining classes are not so straightforward, due to the necessity of investigating the mode of interaction between rough equivalence and the $\leftarrow$ operator. We look, next, at the two classes of left injective π-institutions. We start by showing that the class of narrow left injective π-institutions is not included in the class of roughly left injective π-institutions. The next example exhibits a π-institution which is narrowly left injective but not roughly left injective.

Example 412 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

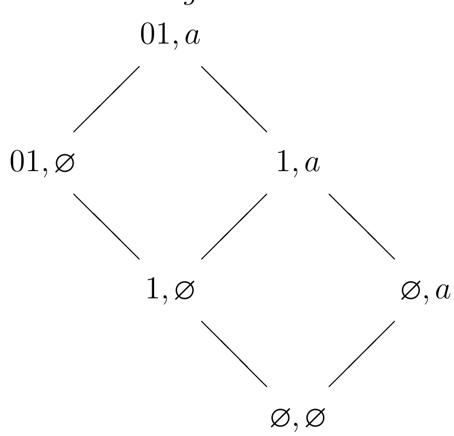
- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a\}$ *and* $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = a$;
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{a\}\}.$$

Clearly, there are six theory families in $\mathrm{ThFam}(\mathcal{I})$, *only four of which are theory systems, and only two of which are in* $\mathrm{ThFam}^{\natural}(\mathcal{I})$. *The lattice of theory families is shown in the diagram:*



Since $\mathrm{ThFam}^{\natural}(\mathcal{I}) = \{\{1,a\},\{01,a\}\}$ *and* $\Omega(1,a) = \{\Delta^{\mathbf{F}}_\Sigma, \nabla^{\mathbf{F}}_{\Sigma'}\} \neq \nabla^{\mathbf{F}} = \Omega(01,a)$, *it follows that* $\mathcal{I}$ *is trivially narrowly left injective.*

On the other hand, consider $T = \{1,\varnothing\}$ *and* $T' = \{1,a\}$. *We have* $\Omega(1,\varnothing) = \{\Delta^{\mathbf{F}}_\Sigma, \nabla^{\mathbf{F}}_{\Sigma'}\} = \Omega(1,a)$, *but*

$$\overleftarrow{\widetilde{1,\varnothing}} = \widetilde{\varnothing,\varnothing} = 01,a \neq 1,a = \widetilde{1,a} = \overleftarrow{\widetilde{1,a}}.$$

This proves that $\mathcal{I}$ *is not roughly left injective.*

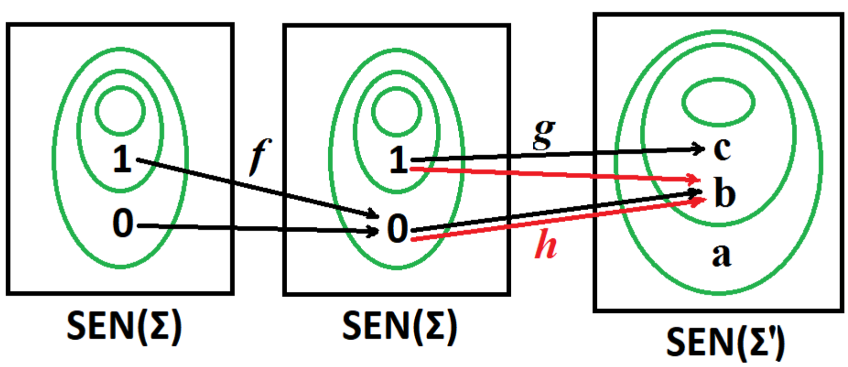
We now look at a π-institution that is roughly left injective, while it fails to be narrowly left injective. Combined with Example 412, this will show that the two left injectivity classes, rough and narrow, are incomparable from the point of view of inclusion.

Example 413 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and three nonidentity morphisms* $f : \Sigma \to \Sigma$ *and* $g, h : \Sigma \to \Sigma'$*, such that* $f \circ f = f$*,* $g \circ f = h$ *and* $h \circ f = h$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$*,* $\mathrm{SEN}^\flat(\Sigma') = \{a, b, c\}$*,* $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = 0$*,* $\mathrm{SEN}^\flat(g)(0) = b$*,* $\mathrm{SEN}^\flat(g)(1) = c$ *and* $\mathrm{SEN}^\flat(h)(0) = \mathrm{SEN}^\flat(h)(1) = b$*;*
- $N^\flat$ *is the clone generated by a single binarry natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$*, whose components are defined by the following tables:*

$\sigma^\flat_\Sigma$	0	1
0	0	1
1	1	1

$\sigma^\flat_{\Sigma'}$	a	b	c
a	a	a	c
b	a	b	c
c	c	c	c



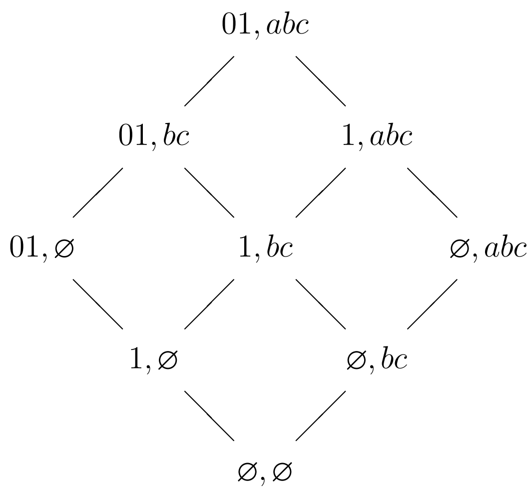
It is not difficult, albeit slightly tedious, to check that this is a well-defined natural transformation. We summarize the checking in the accompanying table.

(x, y)	$f(\sigma^\flat_\Sigma(x, y)) = \sigma^\flat_\Sigma(f(x), f(y))$	$g(\sigma^\flat_\Sigma(x, y)) = \sigma^\flat_{\Sigma'}(g(x), g(y))$	$h(\sigma^\flat_\Sigma(x, y)) = \sigma^\flat_{\Sigma'}(h(x), h(y))$
$(0, 0)$	$0 = 0$	$b = b$	$b = b$
$(0, 1)$	$0 = 0$	$c = c$	$b = b$
$(1, 0)$	$0 = 0$	$c = c$	$b = b$
$(1, 1)$	$0 = 0$	$c = c$	$b = b$

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b,c\}, \{a,b,c\}\}.$$

Clearly, there are nine theory families in $\mathrm{ThFam}(\mathcal{I})$*, five of which are theory systems, and four of which are in* $\mathrm{ThFam}^{\nmid}(\mathcal{I})$*. The lattice of theory families is shown in the diagram:*



The action of $\overleftarrow{\ }$ *on theory families is given in the following table.*

T	$\overleftarrow{T}$	T	$\overleftarrow{T}$
$01, abc$	$01, abc$	$\varnothing, abc$	$\varnothing, abc$
$01, bc$	$01, bc$	$1, \varnothing$	$\varnothing, \varnothing$
$1, abc$	$\varnothing, abc$	$\varnothing, bc$	$\varnothing, bc$
$01, \varnothing$	$\varnothing, \varnothing$	$\varnothing, \varnothing$	$\varnothing, \varnothing$
$1, bc$	$\varnothing, bc$		

The table below provides the Leibniz congruence systems associated with the theory families of $\mathcal{I}$*.*

T	$\Omega(T)$
$\{01, abc\}, \{01, \varnothing\}, \{\varnothing, abc\}, \{\varnothing, \varnothing\}$	$\nabla^{\mathbf{F}}$
$\{1, abc\}, \{1, \varnothing\}$	$\{\Delta^{\mathbf{F}}_\Sigma, \nabla^{\mathbf{F}}_{\Sigma'}\}$
$\{01, bc\}, \{1, bc\}, \{\varnothing, bc\}$	$\Delta^{\mathbf{F}}$

To see that $\mathcal{I}$ *is roughly left injective, note that all elements in a single row of the table have associated theory systems that are roughly equivalent.*

$$\widetilde{\overleftarrow{\{01, abc\}}} = \widetilde{\overleftarrow{\{01, \varnothing\}}} = \widetilde{\overleftarrow{\{\varnothing, abc\}}} = \widetilde{\overleftarrow{\{\varnothing, \varnothing\}}} = \{01, abc\};$$
$$\widetilde{\overleftarrow{\{1, abc\}}} = \widetilde{\overleftarrow{\{1, \varnothing\}}} = \{01, abc\};$$
$$\widetilde{\overleftarrow{\{01, bc\}}} = \widetilde{\overleftarrow{\{1, bc\}}} = \widetilde{\overleftarrow{\{\varnothing, bc\}}} = \{01, bc\}.$$

But $\mathcal{I}$ *is not narrowly left injective. In fact, setting* $T = \{1, bc\}$ *and* $T' = \{01, bc\}$*, we get* $\Omega(T) = \Omega(T') = \Delta^{\mathbf{F}}$*, whereas* $\overleftarrow{T} = \{\varnothing, bc\} \neq \{01, bc\} = \overleftarrow{T'}$*.*

We turn, next to the relationship between the two kinds of right injectivity. We show, first, that rough right injectivity implies narrow right injectivity.

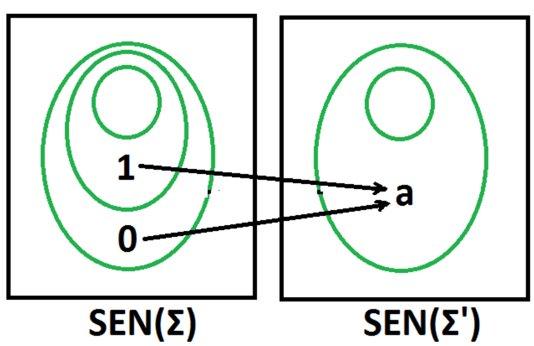
Proposition 414 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly right injective, then it is narrowly right injective.*

Proof: Suppose $\mathcal{I}$ is roughly right injective and let $T, T' \in \mathrm{ThFam}^{\not\,}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$. By rough right injectivity, we get that $T \sim T'$, i.e., that $\widetilde{T} = \widetilde{T'}$. Since, however, $T, T' \in \mathrm{ThFam}^{\not\,}(\mathcal{I})$, we get $T = \widetilde{T} = \widetilde{T'} = T'$. Therefore, $\mathcal{I}$ is narrowly right injective. ■

The converse, on the other hand, does not hold in general, as the following example demonstrates.

Example 415 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

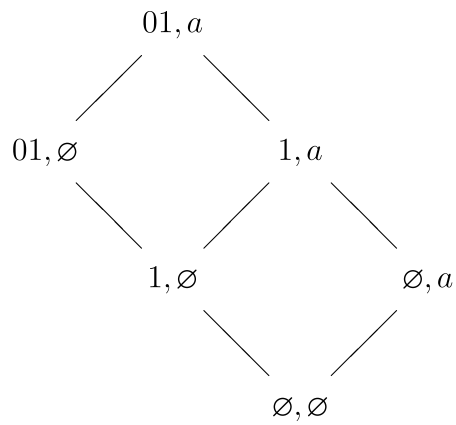
- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*,* $\mathrm{SEN}^\flat(\Sigma') = \{a\}$ *and* $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = a$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{a\}\}.$$

Clearly, there are six theory families in $\mathrm{ThFam}(\mathcal{I})$*, only four of which are theory systems and only two of which are in* $\mathrm{ThFam}^{\not\,}(\mathcal{I})$*. The lattice of* *theory families is shown in the diagram.*



The only theory families in $\mathrm{ThFam}^{\natural}(\mathcal{I})$ *are* $\{1, a\}$ *and* $\{01, a\}$*. Moreover,*

$$\Omega(\overleftarrow{\{1, a\}}) = \Omega(\{1, a\}) = \{\Delta^{\mathbf{F}}_{\Sigma}, \nabla^{\mathbf{F}}_{\Sigma}\} \neq \nabla^{\mathbf{F}} = \Omega(\{01, a\}) = \Omega(\overleftarrow{\{01, a\}}).$$

Thus, $\mathcal{I}$ *is trivially narrowly right injective.*

On the other hand, letting $T = \{1, \varnothing\}$ *and* $T' = \{01, \varnothing\}$*, we get*

$$\Omega(\overleftarrow{T}) = \Omega(\{\varnothing, \varnothing\}) = \nabla^{\mathbf{F}} = \Omega(\{\varnothing, \varnothing\}) = \Omega(\overleftarrow{T'}),$$

but, clearly, $\widetilde{T} = \{1, a\} \neq \{01, a\} = \widetilde{T'}$*, i.e.,* $T \nsim T'$*. Therefore,* $\mathcal{I}$ *is not roughly right injective.*

Finally, we look at system injectivity. Again, it turns out that rough system injectivity implies narrow system injectivity. However, the converse fails in general.

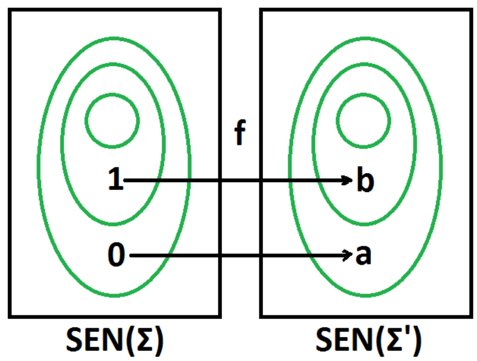
Proposition 416 *Let* $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat} \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly system injective, then it is narrowly system injective.*

Proof: Suppose $\mathcal{I}$ is roughly system injective and let $T, T' \in \mathrm{ThSys}^{\natural}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, by rough system injectivity, $T \sim T'$, i.e., $\widetilde{T} = \widetilde{T'}$. However, since $T, T' \in \mathrm{ThSys}^{\natural}(\mathcal{I})$, we get $T = \widetilde{T} = \widetilde{T'} = T'$. Therefore, $\mathcal{I}$ is narrowly system injective. ∎

And now we present an example of a π-institution that is narrowly system injective but not roughly system injective. This, combined with Proposition 416, shows that the class of narrowly system injective π-institutions properly contains the class of roughly system injective π-institutions.

Example 417 *Let* $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat} \rangle$ *be the algebraic system defined as follows:*

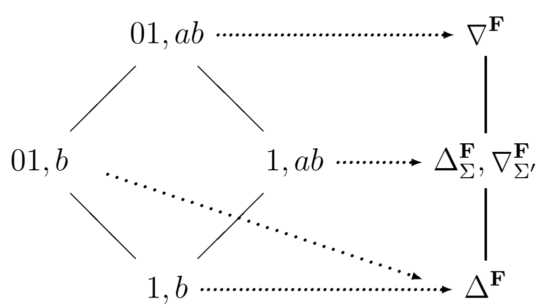
- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique morphism* $f : \Sigma \to \Sigma'$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$;
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a,b\}\}.$$

There are only four theory families in $\mathrm{ThFam}^{\natural}(\mathcal{I})$, *all of which except for* $\{01, b\}$ *are theory systems. Their lattice together with the associated Leibniz congruence systems are shown in the diagram:*

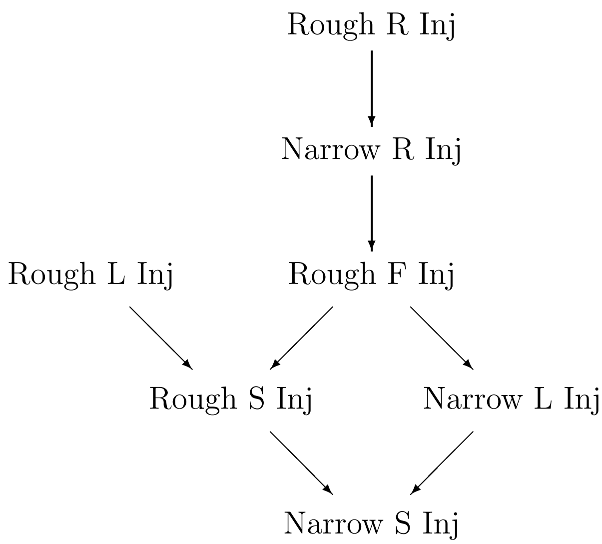


From this diagram we see that for no $T, T' \in \mathrm{ThSys}^{\natural}(\mathcal{I})$, *with* $T \neq T'$ *is it the case that* $\Omega(T) = \Omega(T')$. *Therefore,* $\mathcal{I}$ *is trivially narrowly system injective.*

On the other hand, consider $T = \{1, b\}$, $T' = \{\varnothing, b\} \in \mathrm{ThSys}(\mathcal{I})$. *Even though* $T \not\leftrightarrow T'$, *we have* $\Omega(T) = \Delta^{\mathbf{F}} = \Omega(T')$. *Hence,* $\mathcal{I}$ *is not roughly system injective.*

The results obtained and the counterexamples presented, thus far, reveal the following mixed hierarchy of rough and narrow injectivity classes of π-

institutions.



A theorem, analogous to Theorem 394 asserts that ordinary injectivity is equivalent to narrow injectivity in the presence of theorems. This holds for all four injectivity classes.

Theorem 418 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is right injective if and only if it is narrowly right injective and has theorems;*

(b) $\mathcal{I}$ *is family injective if and only if it is narrowly family injective and has theorems;*

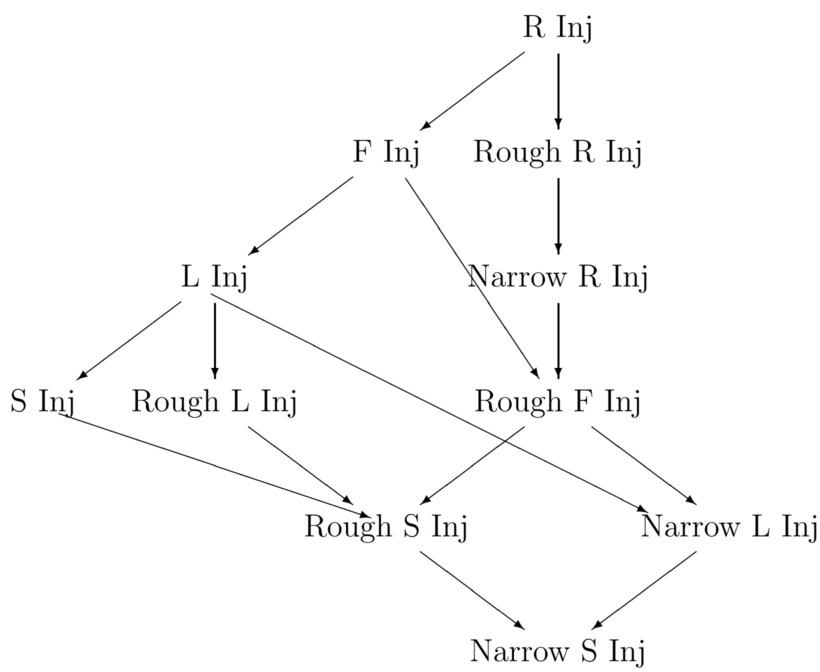
(c) $\mathcal{I}$ *is left injective if and only if it is narrowly left injective and has theorems;*

(d) $\mathcal{I}$ *is system injective if and only if it is narrowly system injective and has theorems.*

Proof: By Theorem 394, if $\mathcal{I}$ has one of the four injectivity properties, then it has theorems. Moreover, by the same theorem, an injectivity property implies the corresponding rough injectivity property and, by Corollary 411, Proposition 414 and Proposition 416, each implies the corresponding narrow injectivity property except in the case of left injectivity, where (as actually in all other cases, as well) one can easily see directly, that left injectivity implies narrow left injectivity, since the defining condition of the latter is a specialization of that of the former.

All converses are also easily verified, since, in the presence of theorems, $\mathrm{ThFam}^{\lightning}(\mathcal{I}) = \mathrm{ThFam}(\mathcal{I})$ and $\mathrm{ThSys}^{\lightning}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$, which makes the four defining conditions for the narrow classes identical with the corresponding conditions for the ordinary injectivity classes. ■

We now have the following hierarchy.



The narrow injectivity properties transfer from the theory families/systems of a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ to all $\mathcal{I}$-filter families/systems on arbitrary $\mathbf{F}$-algebraic systems. This result forms an analog of Theorem 396, which applied to rough injectivity classes.

Theorem 419 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is narrowly right injective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}\natural}(\mathcal{A})$*,* $\Omega^{\mathcal{A}}(\overleftarrow{T}) = \Omega^{\mathcal{A}}(\overleftarrow{T'})$ *implies* $T = T'$*;*

(b) $\mathcal{I}$ *is narrowly family injective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}\natural}(\mathcal{A})$*,* $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$ *implies* $T = T'$*;*

(c) $\mathcal{I}$ *is narrowly left injective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}\natural}(\mathcal{A})$*,* $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$ *implies* $\overleftarrow{T} = \overleftarrow{T'}$*;*

(d) $\mathcal{I}$ *is narrowly system injective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T, T' \in \mathrm{FiSys}^{\mathcal{I}\natural}(\mathcal{A})$*,* $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$ *implies* $T = T'$*.*

Proof: The proof follows the steps of the proofs of the various parts of Theorem 214, but, in addition, it takes into account Lemma 376. We do Part (a) in detail to give a flavor of what is involved.

The "if" follows by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account that $\mathrm{ThFam}^{\lightning}(\mathcal{I}) = \mathrm{FiFam}^{\mathcal{I}\lightning}(\mathcal{F})$, by Lemmas 51 and 376.

For the "only if", suppose that $\mathcal{I}$ is narrowly right injective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}\lightning}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(\overleftarrow{T}) = \Omega^{\mathcal{A}}(\overleftarrow{T'})$. Then $\alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T})) = \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'}))$. So, by Proposition 24, $\Omega(\alpha^{-1}(\overleftarrow{T})) = \Omega(\alpha^{-1}(\overleftarrow{T'}))$. Hence, by Lemma 6, $\Omega(\overleftarrow{\alpha^{-1}(T)}) = \Omega(\overleftarrow{\alpha^{-1}(T')})$. Since, by Lemmas 51 and 376, $\alpha^{-1}(T), \alpha^{-1}(T') \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, we get, by applying narrow right injectivity, $\alpha^{-1}(T) = \alpha^{-1}(T')$. This yields, taking into account the surjectivity of $\langle F, \alpha \rangle$, $T = T'$. ■

We finally recast narrow injectivity in terms of the injectivity of mappings from posets of theory or filter families/systems into posets of congruence systems. The following results form, roughly, analogs of Propositions 397 and 398, respectively, except that special attention must be paid to the fact that neither $\mathrm{ThFam}^{\lightning}(\mathcal{I})$ nor $\mathrm{FiFam}^{\mathcal{I}\lightning}(\mathcal{A})$ is necessarily a lattice.

Proposition 420 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is narrowly family injective;*

(b) $\Omega : \mathrm{ThFam}^{\lightning}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is injective;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}\lightning}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is injective, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

Similarly, for system injectivity, we have

Proposition 421 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is narrowly system injective;*

(b) $\Omega : \mathrm{ThSys}^{\lightning}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is injective;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}\lightning}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is injective, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

6.6 Rough Reflectivity

In this section we study classes of π-institutions defined using reflectivity properties of the Leibniz operator applied on rough equivalence classes.

Definition 422 (Rough Reflectivity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is called* **roughly family reflective** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,
$$\Omega(T) \leq \Omega(T') \quad \textit{implies} \quad \widetilde{T} \leq \widetilde{T'}.$$
- $\mathcal{I}$ *is called* **roughly left reflective** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,
$$\Omega(T) \leq \Omega(T') \quad \textit{implies} \quad \widetilde{\overleftarrow{T}} \leq \widetilde{\overleftarrow{T'}}.$$
- $\mathcal{I}$ *is called* **roughly right reflective** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,
$$\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'}) \quad \textit{implies} \quad \widetilde{T} \leq \widetilde{T'}.$$
- $\mathcal{I}$ *is called* **roughly system reflective** *if, for all* $T, T' \in \mathrm{ThSys}(\mathcal{I})$,
$$\Omega(T) \leq \Omega(T') \quad \textit{implies} \quad \widetilde{T} \leq \widetilde{T'}.$$

In a partial analog of Lemma 218, we show that rough right reflectivity implies rough systemicity and, hence, by Theorem 370, stability.

Lemma 423 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly right reflective, then it is roughly systemic.*

Proof: Suppose that $\mathcal{I}$ is roughly right reflective and let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, we have, by Proposition 42, $\overleftarrow{\overleftarrow{T}} = \overleftarrow{T}$. Therefore, we get $\Omega(\overleftarrow{\overleftarrow{T}}) = \Omega(\overleftarrow{T})$. Hence, by rough right reflectivity, we get that $\widetilde{\overleftarrow{T}} = \widetilde{T}$, i.e., $\overleftarrow{T} \sim T$. Hence $\mathcal{I}$ is roughly systemic. ■

Next we look into establishing the *rough reflectivity hierarchy* of π-institutions. The following relationships can be established between the four rough reflectivity classes.

Proposition 424 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is roughly right reflective, then it is roughly family reflective;*

(b) If $\mathcal{I}$ *is roughly family reflective, then it is roughly system reflective;*

(c) If $\mathcal{I}$ *is roughly left reflective, then it is roughly system reflective.*

Proof:

(a) Suppose that $\mathcal{I}$ is roughly right reflective and let $T,T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \le \Omega(T')$. By Lemma 423, $\mathcal{I}$ is roughly systemic, whence $\overleftarrow{T} \sim T$ and $\overleftarrow{T'} \sim T'$. Thus, by Theorem 370, we get

$$\Omega(\overleftarrow{T}) = \Omega(T) \le \Omega(T') = \Omega(\overleftarrow{T'}).$$

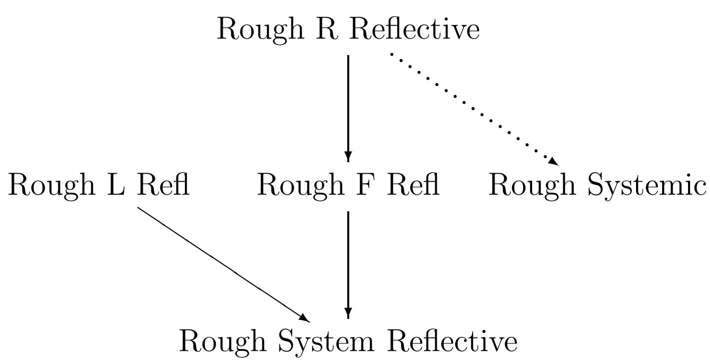
Now applying rough right reflectivity, we get $\widetilde{T} \le \widetilde{T'}$. This proves that $\mathcal{I}$ is roughly family reflective.

(b) Suppose that $\mathcal{I}$ is roughly family reflective and let $T,T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\Omega(T) \le \Omega(T')$. Then, by rough family reflectivity, we get $\widetilde{T} \le \widetilde{T'}$, whence, $\mathcal{I}$ is roughly system reflective.

(c) Suppose that $\mathcal{I}$ is roughly left reflective and let $T,T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\Omega(T) \le \Omega(T')$. By rough left reflectivity, we conclude that $\widetilde{\overleftarrow{T}} \le \widetilde{\overleftarrow{T'}}$. However, since T,T' are theory systems, we have $\overleftarrow{T} = T$ and $\overleftarrow{T'} = T'$. Hence we get $\widetilde{T} \le \widetilde{T'}$ and $\mathcal{I}$ is roughly system reflective.

■

We have now established the following **rough reflectivity hierarchy** of π-institutions.



We formulate two additional properties concerning the relationships between rough reflectivity classes. First, rough right reflectivity turns out to be equivalent to rough system reflectivity combined with rough systemicity.

Proposition 425 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is roughly right reflective if and only if it is roughly system reflective and roughly systemic.*

Proof: Suppose, first, that $\mathcal{I}$ is roughly right reflective. Then, by Lemma 423, it is roughly systemic and by Proposition 424 it is roughly system reflective.

Suppose, conversely, that $\mathcal{I}$ is roughly system reflective and roughly systemic and let $T,T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'})$. By rough system reflectivity and Proposition 42, we get $\widetilde{\overleftarrow{T}} \leq \widetilde{\overleftarrow{T'}}$. Hence, by rough systemicity, $\widetilde{T} = \widetilde{\overleftarrow{T}} \leq \widetilde{\overleftarrow{T'}} = \widetilde{T'}$. Thus, $\mathcal{I}$ is roughly right reflective. ■

Second, we show that rough system reflectivity together with stability imply rough left reflectivity.

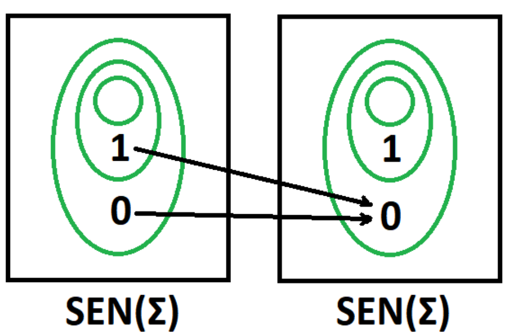
Proposition 426 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly system reflective and stable, then it is roughly left reflective.*

Proof: Suppose that $\mathcal{I}$ is roughly system reflective and stable and consider $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then, by stability $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Hence, since $\overleftarrow{T}, \overleftarrow{T'} \in \mathrm{ThSys}(\mathcal{I})$, by rough system reflectivity, $\widetilde{\overleftarrow{T}} \leq \widetilde{\overleftarrow{T'}}$. This shows that $\mathcal{I}$ is roughly left reflective. ■

We present three examples to show that all inclusions established between rough reflectivity classes and depicted in the diagram above are proper inclusions. The first example will show that the class of roughly right reflective π-institutions is a proper subclass of the class of roughly family reflective π-institutions.

Example 427 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

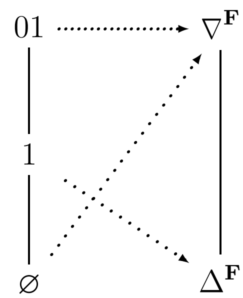
- $\mathbf{Sign}^\flat$ *is the category with the single object* Σ *and a single (no-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0, 1\}\}.$$

$\mathcal{I}$ has three theory families $\{\varnothing\}$, $\{\{1\}\}$ and $\{\{0,1\}\}$, but only two theory systems, $\{\varnothing\}$ and $\{\{0,1\}\}$. The lattice of theory families of $\mathcal{I}$ and the corresponding Leibniz congruence systems are given in the diagram.



It is easy to see that $\mathcal{I}$ is roughly family reflective. Suppose that for $T, T' \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) \leq \Omega(T')$.

- *If $\Omega(T') = \Delta^{\mathbf{F}}$, then $\Omega(T) = \Delta^{\mathbf{F}}$, whence $T' = T = \{\{1\}\}$. Thus, $\widetilde{T} \leq \widetilde{T'}$.*
- *If $\Omega(T') = \nabla^{\mathbf{F}}$, then $T' = \{\varnothing\}$ or $T' = \{\{0,1\}\}$. In either case, $\widetilde{T} \leq \{\{0,1\}\} = \widetilde{T'}$.*

On the other hand, for $T = \{\{1\}\}$, we get $\widetilde{T} = \{\{1\}\} \neq \{\{0,1\}\} = \widetilde{\{\varnothing\}} = \overleftarrow{\widetilde{T}}$, whence $T \not\approx \overleftarrow{T}$ and, hence, $\mathcal{I}$ is not roughly systemic. Therefore, by Lemma 423, $\mathcal{I}$ is not roughly right reflective.

The second example shows that there exists a roughly left reflective π-institution that is not roughly family reflective.

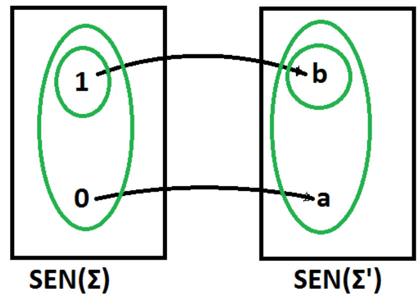
Example 428 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- *$\mathbf{Sign}^\flat$ is the category with objects Σ and Σ' and a unique (non-identity) morphism $f : \Sigma \to \Sigma'$;*
- *$\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ is defined by $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ and $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$;*
- *$N^\flat$ is the trivial clone.*

Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\},\{a,b\}\}.$$

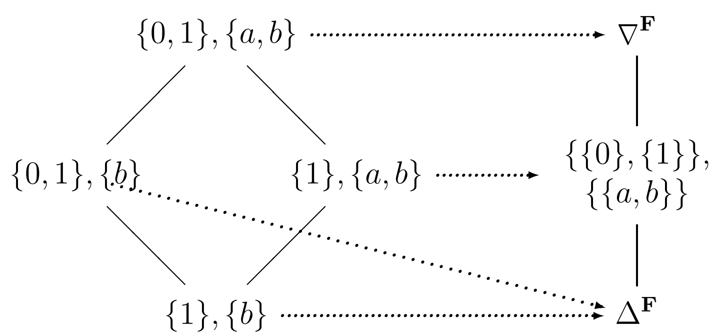
Again, since $\mathcal{I}$ has theorems, rough equivalence coincides with the identity relation on $\mathrm{ThFam}(\mathcal{I})$.



The following table shows the action of $\leftarrow$ *on theory families, where rows correspond to* T_Σ *and columns to* $T_{\Sigma'}$ *and each entry is written as* $\overleftarrow{T}_\Sigma, \overleftarrow{T}_{\Sigma'}$.

$\leftarrow$	$\{b\}$	$\{a,b\}$
$\{1\}$	$\{1\},\{b\}$	$\{1\},\{a,b\}$
$\{0,1\}$	$\{1\},\{b\}$	$\{0,1\},\{a,b\}$

The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right:



To see that $\mathcal{I}$ *is roughly left reflective, suppose* $T,T' \in \mathrm{ThFam}(\mathcal{I})$, *such that* $\Omega(T) \leq \Omega(T')$.

- *If* $\Omega(T') = \nabla^{\mathbf{F}}$, *then* $T' = \{\{0,1\},\{a,b\}\}$, *whence* $\overleftarrow{T} \leq \{\{0,1\},\{a,b\}\} = \overleftarrow{T'}$ *and, hence,* $\overleftarrow{\overleftarrow{T}} \leq \overleftarrow{\overleftarrow{T'}}$.
- *If* $\Omega(T') = \{\{\{0\},\{1\}\},\{\{a,b\}\}\}$, *then* $T' = \{\{1\},\{a,b\}\}$ *and* $T = \{\{0,1\},\{b\}\}$ *or* $T = \{\{1\},\{b\}\}$. *In either case* $\overleftarrow{T} = \{\{1\},\{b\}\} \leq T' = \overleftarrow{T'}$ *and, hence,* $\overleftarrow{\overleftarrow{T}} \leq \overleftarrow{\overleftarrow{T'}}$.
- *If* $\Omega(T') = \Delta^{\mathbf{F}}$, *then both* T *and* T' *have to be either* $\{\{0,1\},\{b\}\}$ *or* $\{\{1\},\{b\}\}$. *Thus, we get* $\overleftarrow{T} = \{\{1\},\{b\}\} = \overleftarrow{T'}$ *and, hence,* $\overleftarrow{\overleftarrow{T}} \leq \overleftarrow{\overleftarrow{T'}}$.

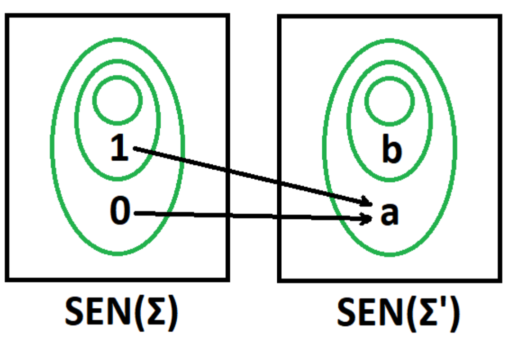
On the other hand, we have $\Omega(\{\{0,1\},\{b\}\}) \leq \Omega(\{\{1\},\{b\}\})$*, but, clearly,* $\{\{0,1\},\{b\}\} \nleq \{\{1\},\{b\}\}$*. Thus, since rough equivalence is the identity on* $\mathrm{ThFam}(\mathcal{I})$*, we conclude that* $\mathcal{I}$ *is not roughly family reflective.*

The third example shows that there exists a roughly family reflective π-institution that is not roughly left reflective. Combined with the preceding example, it has the effect of establishing the following facts:

- The classes of roughly family reflective and roughly left reflective π-institutions are incomparable.
- The class of roughly family reflective π-institutions is properly contained in the class of roughly system reflective π-institutions.
- Similarly, the class of roughly left reflective π-institutions is a proper subclass of the class of roughly system reflective π-institutions.

Example 429 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*,* $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$*,* $\mathrm{SEN}^\flat(f)(1) = a$*;*
- $N^\flat$ *is the trivial clone.*



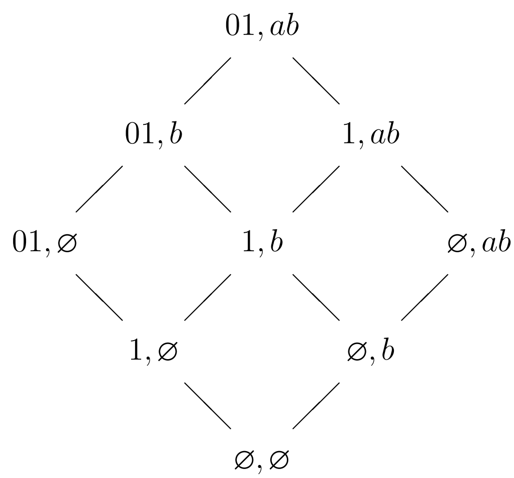
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a,b\}\}.$$

There are nine theory families, but only five theory systems. The action of $\overleftarrow{}$ *on theory families is given in the table below.*

T	$\overleftarrow{T}$	T	$\overleftarrow{T}$
$\varnothing,\varnothing$	$\varnothing,\varnothing$	$\varnothing,ab$	$\varnothing,ab$
$1,\varnothing$	$\varnothing,\varnothing$	$01,b$	$\varnothing,b$
$\varnothing,b$	$\varnothing,b$	$1,ab$	$1,ab$
$01,\varnothing$	$\varnothing,\varnothing$	$01,ab$	$01,ab$
$1,b$	$\varnothing,b$		

The lattice of theory families of $\mathcal{I}$ *is shown in the diagram.*



We show that $\mathcal{I}$ *is roughly family reflective. The following table summarizes the theory families together with their associated Leibniz congruence systems.*

T	$\Omega(T)$
$\{\varnothing,\varnothing\},\{01,\varnothing\},\{\varnothing,ab\},\{01,ab\}$	$\nabla^{\mathbf{F}}$
$\{\varnothing,b\},\{01,b\}$	$\{\nabla^{\mathbf{F}}_{\Sigma},\Delta^{\mathbf{F}}_{\Sigma'}\}$
$\{1,\varnothing\},\{1,ab\}$	$\{\Delta^{\mathbf{F}}_{\Sigma},\nabla^{\mathbf{F}}_{\Sigma'}\}$
$\{1,b\}$	$\Delta^{\mathbf{F}}$

Let $T,T' \in \mathrm{ThFam}(\mathcal{I})$*, such that* $\Omega(T) \le \Omega(T')$*.*

- *If* $\Omega(T') = \nabla^{\mathbf{F}}$*, then* $\widetilde{T} \le \{01,ab\} = \widetilde{T'}$*.*
- *If* $\Omega(T') = \{\Delta^{\mathbf{F}}_{\Sigma},\nabla^{\mathbf{F}}_{\Sigma'}\}$*, then* $\widetilde{T} = \{1,ab\}$ *or* $\widetilde{T} = \{1,b\}$ *and, hence,* $\widetilde{T} \le \{1,ab\} = \widetilde{T'}$*.*
- *If* $\Omega(T') = \{\nabla^{\mathbf{F}}_{\Sigma},\Delta^{\mathbf{F}}_{\Sigma'}\}$*, then* $\widetilde{T} = \{01,b\}$ *or* $\widetilde{T} = \{1,b\}$ *and, hence,* $\widetilde{T} \le \{01,b\} = \widetilde{T'}$*.*
- *If* $\Omega(T') = \Delta^{\mathbf{F}}$*, then* $\widetilde{T} = \{1,b\} = \widetilde{T'}$*.*

On the other hand, consider $T = \{1, \varnothing\}$ *and* $T' = \{1, ab\}$. *We have*

$$\Omega(T) = \Omega(\{1, \varnothing\}) = \{\Delta^{\mathbf{F}}_{\Sigma}, \nabla^{\mathbf{F}}_{\Sigma'}\} = \Omega(\{1, ab\}) = \Omega(T'),$$

whereas

$$\overleftarrow{\widetilde{T}} = \widetilde{\{\varnothing, \varnothing\}} = \{01, ab\} \nleq T' = \widetilde{T'} = \overleftarrow{\widetilde{T'}}.$$

Hence, $\mathcal{I}$ *is not roughly left reflective.*

We look, next, at the connections between rough reflectivity and rough injectivity classes. It turns out that membership in a rough reflectivity class implies membership in the corresponding rough injectivity class. We have the following straightforward inclusions.

Theorem 430 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is roughly right reflective, then it is roughly right injective;*

(b) If $\mathcal{I}$ *is roughly family reflective, then it is roughly family injective;*

(c) If $\mathcal{I}$ *is roughly left reflective, then it is roughly left injective;*

(d) If $\mathcal{I}$ *is roughly system reflective, then it is roughly system injective.*

Proof:

(a) Suppose that $\mathcal{I}$ is roughly right reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$. Then, by hypothesis, $\widetilde{T} \leq \widetilde{T'}$ and $\widetilde{T'} \leq \widetilde{T}$, whence $\widetilde{T} = \widetilde{T'}$, i.e., $T \sim T'$. Therefore, $\mathcal{I}$ is roughly right injective.

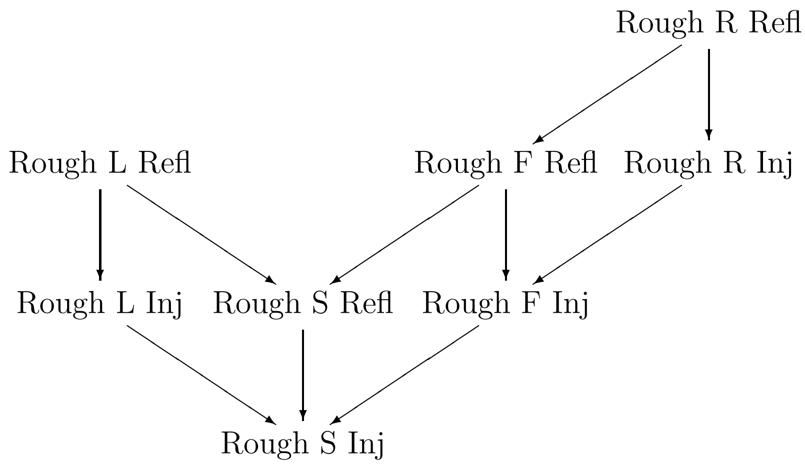
(b) Suppose that $\mathcal{I}$ is roughly family reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, by hypothesis, $\widetilde{T} \leq \widetilde{T'}$ and $\widetilde{T'} \leq \widetilde{T}$, whence $\widetilde{T} = \widetilde{T'}$, i.e., $T \sim T'$. Therefore, $\mathcal{I}$ is roughly family injective.

(c) Suppose that $\mathcal{I}$ is roughly left reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, by hypothesis, $\overleftarrow{\widetilde{T}} \leq \overleftarrow{\widetilde{T'}}$ and $\overleftarrow{\widetilde{T'}} \leq \overleftarrow{\widetilde{T}}$, whence $\overleftarrow{\widetilde{T}} = \overleftarrow{\widetilde{T'}}$, i.e., $\overleftarrow{T} \sim \overleftarrow{T'}$. Therefore, $\mathcal{I}$ is roughly left injective.

(d) Similar to Part (b).

■

Theorem 430 establishes the mixed rough hierarchy depicted in the diagram.

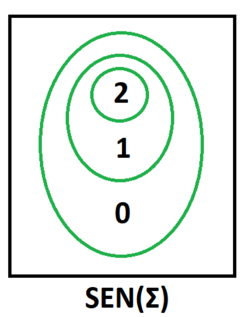


To see that all classes in the hierarchy are different, we give an example of a π-institution satisfying all four rough injectivity properties, which is not, however, roughly system reflective and, therefore, a fortiori, belongs to none of the four rough reflectivity classes.

Example 431 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with the single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$*;*
- $N^\flat$ *is the clone generated by the single unary operation* $\sigma : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *determined by the following table:*

x	0	1	2
$\sigma^\flat_\Sigma(x)$	0	0	2

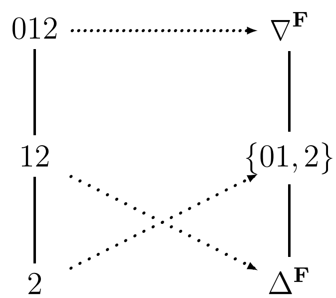


Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{1\}, \{0,1\}, \{0,1,2\}\}.$$

$\mathcal{I}$ has three theory families $\{\{2\}\}$ and $\{\{1,2\}\}$ and $\{\{0,1,2\}\}$, all of which are theory systems. Moreover, $\mathcal{I}$ has theorems. It follows that the action of $\overleftarrow{\ }$ is trivial and that rough equivalence in $\mathcal{I}$ coincides with the identity relation on $\mathrm{ThFam}(\mathcal{I})$.

The lattice of theory families of $\mathcal{I}$ and the corresponding Leibniz congruence systems are given in the diagram.



We show that $\mathcal{I}$ is both roughly right and roughly left injective and, hence, belongs to all four classes in the rough injectivity hierarchy.

- *Suppose $\Omega(\overleftarrow{T}) = \Omega(\overleftarrow{T'})$. Then $\Omega(T) = \Omega(T')$ and, hence, $T = T'$, i.e., $T \sim T'$. Thus, $\mathcal{I}$ is rough right injective.*

- *Suppose $\Omega(T) = \Omega(T')$. Then $T = T'$. This gives $\overleftarrow{T} = \overleftarrow{T'}$, which, in turn, implies $\overleftarrow{T} \sim \overleftarrow{T'}$. Thus, $\mathcal{I}$ is roughly left injective.*

On the other hand, we have $\Omega(\{\{1,2\}\}) = \Delta^{\mathbf{F}} \leq \{\{0,1\},\{2\}\} = \Omega(\{2\})$, but $\{\{1,2\}\} \nleq \{\{2\}\}$, whence $\mathcal{I}$ is not roughly system reflective.

We now clarify the connections between rough reflectivity and reflectivity classes. It turns out that membership in a reflectivity class implies membership in the corresponding rough reflectivity class and, also, possession of theorems and, conversely, that membership in a rough reflectivity class plus possession of theorems entails membership in the corresponding reflectivity class.

Theorem 432 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is right/family reflective if and only if it is roughly right reflective and has theorems;

(b) $\mathcal{I}$ is right/family reflective if and only if it is roughly family reflective and has theorems;

(c) $\mathcal{I}$ is left reflective if and only if it is roughly left reflective and has theorems;

(d) *$\mathcal{I}$ is system reflective if and only if it is roughly system reflective and has theorems.*

Proof:

(a) Suppose that $\mathcal{I}$ is right reflective. Then, by Proposition 228, it is right injective. Hence, by Theorem 394, it has theorems. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Then, by right reflectivity, $T \leq T'$. Since $\mathcal{I}$ has theorems, $\widetilde{T} = T$ and $\widetilde{T'} = T'$. Therefore, $\widetilde{T} \leq \widetilde{T'}$ and $\mathcal{I}$ is roughly right reflective.

Assume, conversely, that $\mathcal{I}$ is roughly right reflective and has theorems. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Then, by rough right reflectivity, we get $\widetilde{T} \leq \widetilde{T'}$. On the other hand, since $\mathcal{I}$ has theorems, $\widetilde{T} = T$ and $\widetilde{T'} = T'$. Therefore, $T \leq T'$ and $\mathcal{I}$ is right reflective.

(b) Suppose that $\mathcal{I}$ is family reflective. Then, by Proposition 228, it is family injective. Hence, by Theorem 394, it has theorems. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then, by family reflectivity, $T \leq T'$. Since $\mathcal{I}$ has theorems, $\widetilde{T} = T$ and $\widetilde{T'} = T'$. Therefore, $\widetilde{T} \leq \widetilde{T'}$ and $\mathcal{I}$ is roughly family reflective.

Assume, conversely, that $\mathcal{I}$ is roughly family reflective and has theorems. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then, by rough family reflectivity, we get $\widetilde{T} \leq \widetilde{T'}$. On the other hand, since $\mathcal{I}$ has theorems, $\widetilde{T} = T$ and $\widetilde{T'} = T'$. Therefore, $T \leq T'$ and $\mathcal{I}$ is family reflective.

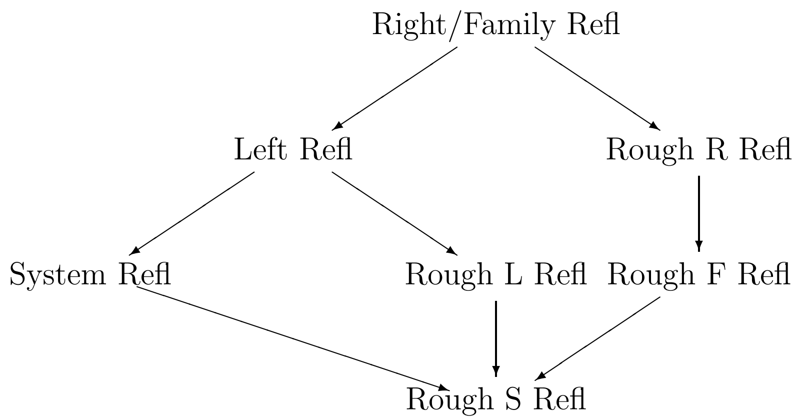
(c) Suppose that $\mathcal{I}$ is left reflective. Then, by Proposition 228, it is left injective. Hence, by Theorem 394, it has theorems. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then, by left reflectivity, $\overleftarrow{T} \leq \overleftarrow{T'}$. Since $\mathcal{I}$ has theorems, $\widetilde{\overleftarrow{T}} = \overleftarrow{T}$ and $\widetilde{\overleftarrow{T'}} = \overleftarrow{T'}$. Therefore, $\widetilde{\overleftarrow{T}} \leq \widetilde{\overleftarrow{T'}}$ and $\mathcal{I}$ is roughly left reflective.

Assume, conversely, that $\mathcal{I}$ is roughly left reflective and has theorems. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then, by rough left reflectivity, we get $\widetilde{\overleftarrow{T}} \leq \widetilde{\overleftarrow{T'}}$. On the other hand, since $\mathcal{I}$ has theorems, $\widetilde{\overleftarrow{T}} = \overleftarrow{T}$ and $\widetilde{\overleftarrow{T'}} = \overleftarrow{T'}$. Therefore, $\overleftarrow{T} \leq \overleftarrow{T'}$ and $\mathcal{I}$ is left reflective.

(d) Similar to Part (b).

■

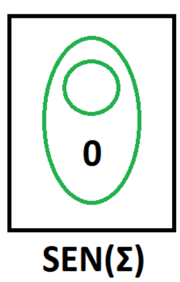
The work in Chapter 3, together with the work done in the present section and Theorem 432, reveal a hierarchy of reflectivity and rough reflectivity classes shown in the accompanying diagram.



To complete the demonstration that all classes in the depicted hierarchy are distinct we provide an example of a π-institution which belongs to all steps in the rough reflectivity hierarchy but possesses none of the four (gentle) reflectivity properties.

Example 433 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

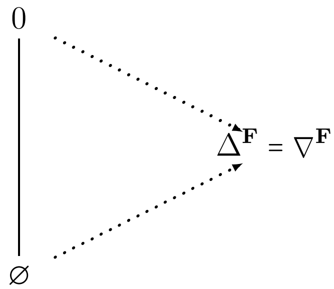
- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0\}$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{0\}\}.$$

$\mathcal{I}$ *is systemic and its lattice of theory families and corresponding Leibniz* *congruence systems are shown in the diagram.*



Note that $\widetilde{\{\{0\}\}} = \widetilde{\{\varnothing\}} = \{\{0\}\}$, *whence, trivially,* $\mathcal{I}$ *is both roughly right and roughly left reflective.*

On the other hand, since $\Omega(\{\{0\}\}) = \nabla^{\mathbf{F}} = \Omega(\{\varnothing\})$, *whereas* $\{\{0\}\} \nleq \{\varnothing\}$, $\mathcal{I}$ *is not system reflective and, hence, a fortiori,* $\mathcal{I}$ *has none of the four reflectivity properties.*

The rough reflectivity properties transfer from the theory families/systems of a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ to all $\mathcal{I}$-filter families/systems on arbitrary $\mathbf{F}$-algebraic systems.

Theorem 434 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is roughly right reflective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(\overleftarrow{T}) \leq \Omega^{\mathcal{A}}(\overleftarrow{T'})$ *implies* $\widetilde{T} \leq \widetilde{T'}$;

(b) $\mathcal{I}$ *is roughly family reflective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ *implies* $\widetilde{T} \leq \widetilde{T'}$;

(c) $\mathcal{I}$ *is roughly left reflective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ *implies* $\widetilde{\overleftarrow{T}} \leq \widetilde{\overleftarrow{T'}}$;

(d) $\mathcal{I}$ *is roughly system reflective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ *implies* $\widetilde{T} \leq \widetilde{T'}$.

Proof:

(a) The "if" follows by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$ and taking into account that, by Lemma 51, $\mathrm{ThFam}(\mathcal{I}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F})$.

For the "only if", suppose that $\mathcal{I}$ is roughly right reflective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that

$\Omega^{\mathcal{A}}(\overleftarrow{T}) \leq \Omega^{\mathcal{A}}(\overleftarrow{T'})$. Then $\alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T})) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'}))$. So, by Proposition 24, $\Omega(\alpha^{-1}(\overleftarrow{T})) \leq \Omega(\alpha^{-1}(\overleftarrow{T'}))$. Hence, by Lemma 6, $\Omega(\overleftarrow{\alpha^{-1}(T)}) \leq \Omega(\overleftarrow{\alpha^{-1}(T')})$. Since, by Lemma 51, $\alpha^{-1}(T), \alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$, we get, by applying rough right reflectivity, $\widetilde{\alpha^{-1}(T)} \leq \widetilde{\alpha^{-1}(T')}$. Thus, by Theorem 377, $\alpha^{-1}(\widetilde{T}) \leq \alpha^{-1}(\widetilde{T'})$. Therefore, taking into account the surjectivity of $\langle F, \alpha\rangle$, we conclude that $\widetilde{T} \leq \widetilde{T'}$.

(b) The "if" follows as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is roughly family reflective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. Then $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. So, by Proposition 24, $\Omega(\alpha^{-1}(T)) \leq \Omega(\alpha^{-1}(T'))$. Since, by Lemma 51, $\alpha^{-1}(T), \alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$, we get, by applying rough family reflectivity, $\widetilde{\alpha^{-1}(T)} \leq \widetilde{\alpha^{-1}(T')}$. Thus, by Theorem 377, $\alpha^{-1}(\widetilde{T}) \leq \alpha^{-1}(\widetilde{T'})$. Therefore, taking into account the surjectivity of $\langle F, \alpha\rangle$, we conclude that $\widetilde{T} \leq \widetilde{T'}$.

(c) The "if" follows as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is roughly left reflective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. Then $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. So, by Proposition 24, $\Omega(\alpha^{-1}(T)) \leq \Omega(\alpha^{-1}(T'))$. Since, by Lemma 51, $\alpha^{-1}(T), \alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$, we get, by applying rough left reflectivity, $\widetilde{\overleftarrow{\alpha^{-1}(T)}} \leq \widetilde{\overleftarrow{\alpha^{-1}(T')}}$. Thus, by Lemma 6, $\widetilde{\alpha^{-1}(\overleftarrow{T})} \leq \widetilde{\alpha^{-1}(\overleftarrow{T'})}$. Hence, by Theorem 377, $\alpha^{-1}(\widetilde{\overleftarrow{T}}) \leq \alpha^{-1}(\widetilde{\overleftarrow{T'}})$. Therefore, taking into account the surjectivity of $\langle F, \alpha\rangle$, we conclude that $\widetilde{\overleftarrow{T}} \leq \widetilde{\overleftarrow{T'}}$.

(d) Similar to Part (b).

■

Finally, we may recast the rough reflectivity classes in terms of the order reflectivity of mappings from posets of classes of theory or filter families/systems into posets of congruence systems.

Note for the following, that the collections $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ and $\widetilde{\mathrm{ThSys}}(\mathcal{I})$ may be ordered by setting, respectively, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$\widetilde{[T]} \leq \widetilde{[T']} \quad \text{iff} \quad \widetilde{T} \leq \widetilde{T'}$$

and, for all $T, T' \in \mathrm{ThSys}(\mathcal{I})$

$$\widetilde{\lfloor T\rfloor} \leq \widetilde{\lfloor T'\rfloor} \quad \text{iff} \quad \widetilde{T} \leq \widetilde{T'}.$$

We denote by $\widetilde{\mathbf{ThFam}}(\mathcal{I}) = \langle \widetilde{\mathrm{ThFam}}(\mathcal{I}), \leq\rangle$ and $\widetilde{\mathbf{ThSys}}(\mathcal{I}) = \langle \widetilde{\mathrm{ThSys}}(\mathcal{I}), \leq\rangle$, respectively, the corresponding ordered sets.

Proposition 435 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is roughly family reflective;*

(b) $\Omega : \widetilde{\mathbf{ThFam}}(\mathcal{I}) \to \mathbf{ConSys}^*(\mathcal{I})$ *is order reflecting;*

(c) $\Omega^{\mathcal{A}} : \widetilde{\mathbf{FiFam}}^{\mathcal{I}}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is order reflecting, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

Similarly, for system reflectivity, we have

Proposition 436 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is roughly system reflective;*

(b) $\Omega : \widetilde{\mathbf{ThSys}}(\mathcal{I}) \to \mathbf{ConSys}^*(\mathcal{I})$ *is order reflecting;*

(c) $\Omega^{\mathcal{A}} : \widetilde{\mathbf{FiSys}}^{\mathcal{I}}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is order reflecting, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

6.7 Narrow Reflectivity

In this section we study classes of π-institutions defined using reflectivity properties of the Leibniz operator restricted to $\mathrm{ThFam}^{\nleq}(\mathcal{I})$. We call those *narrow reflectivity* properties in analogy with the terminology adopted when differentiating rough injectivity and narrow injectivity classes.

Definition 437 (Narrow Reflectivity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is called* **narrowly family reflective** *if, for all* $T, T' \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$*,*
$$\Omega(T) \leq \Omega(T') \quad \textit{implies} \quad T \leq T';$$
- $\mathcal{I}$ *is called* **narrowly left reflective** *if, for all* $T, T' \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$*,*
$$\Omega(T) \leq \Omega(T') \quad \textit{implies} \quad \overleftarrow{T} \leq \overleftarrow{T'}.$$
- $\mathcal{I}$ *is called* **narrowly right reflective** *if, for all* $T, T' \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$*,*
$$\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'}) \quad \textit{implies} \quad T \leq T'.$$
- $\mathcal{I}$ *is called* **narrowly system reflective** *if, for all* $T, T' \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$*,*
$$\Omega(T) \leq \Omega(T') \quad \textit{implies} \quad T \leq T'.$$

The narrow reflectivity properties have the following characterizations, paralleling those given for the narrow injectivity classes in Proposition 400.

Proposition 438 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is narrowly family reflective if and only if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) \leq \Omega(T')$ *implies* $\widetilde{T} \leq \widetilde{T'}$*;*

(b) $\mathcal{I}$ *is narrowly left reflective if and only if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) \leq \Omega(T')$ *implies* $\overleftarrow{\widetilde{T}} \leq \overleftarrow{\widetilde{T'}}$*;*

(c) $\mathcal{I}$ *is narrowly right reflective if and only if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(\overleftarrow{\widetilde{T}}) \leq \Omega(\overleftarrow{\widetilde{T'}})$ *implies* $\widetilde{T} \leq \widetilde{T'}$*;*

(d) $\mathcal{I}$ *is narrowly system reflective if and only if, for all* $T, T' \in \mathrm{ThSys}(\mathcal{I})$, *such that* $\widetilde{T}, \widetilde{T'} \in \mathrm{ThSys}(\mathcal{I})$, $\Omega(T) \leq \Omega(T')$ *implies* $\widetilde{T} \leq \widetilde{T'}$.

Proof:

(a) Suppose that $\mathcal{I}$ is narrowly family reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Consider $\widetilde{T}, \widetilde{T'} \in \mathrm{ThFam}^\natural(\mathcal{I})$. By Proposition 369, $\Omega(\widetilde{T}) = \Omega(T) \leq \Omega(T') = \Omega(\widetilde{T'})$. Thus, by hypothesis, $\widetilde{T} \leq \widetilde{T'}$. Therefore, the asserted condition holds.

Assume, conversely, that the asserted condition holds and let $T, T' \in \mathrm{ThFam}^\natural(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then, since $\mathrm{ThFam}^\natural(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I})$, we get, by hypothesis, $\widetilde{T} \leq \widetilde{T'}$. Since, however, $T, T' \in \mathrm{ThFam}^\natural(\mathcal{I})$, we get $T = \widetilde{T} \leq \widetilde{T'} = T'$. Thus, $\mathcal{I}$ is narrowly family reflective.

(b) Suppose that $\mathcal{I}$ is narrowly left reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then $\widetilde{T}, \widetilde{T'} \in \mathrm{ThFam}^\natural(\mathcal{I})$ and, by Proposition 369, $\Omega(\widetilde{T}) \leq \Omega(\widetilde{T'})$. Thus, by hypothesis, $\overleftarrow{\widetilde{T}} \leq \overleftarrow{\widetilde{T'}}$.

Assume, conversely, that the asserted condition holds and let $T, T' \in \mathrm{ThFam}^\natural(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then, by hypothesis, $\overleftarrow{\widetilde{T}} \leq \overleftarrow{\widetilde{T'}}$. Since, however, $T, T' \in \mathrm{ThFam}^\natural(\mathcal{I})$, we get $\overleftarrow{T} = \overleftarrow{\widetilde{T}} \leq \overleftarrow{\widetilde{T'}} = \overleftarrow{T'}$. Therefore, $\mathcal{I}$ is narrowly left reflective.

(c) Suppose that $\mathcal{I}$ is narrowly right reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(\overleftarrow{\widetilde{T}}) \leq \Omega(\overleftarrow{\widetilde{T'}})$. Since $\widetilde{T}, \widetilde{T'} \in \mathrm{ThFam}^\natural(\mathcal{I})$, we get, by hypothesis, $\widetilde{T} \leq \widetilde{T'}$.

Assume, conversely, that the asserted condition holds and let $T, T' \in \mathrm{ThFam}^\natural(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Then, since $T, T' \in \mathrm{ThFam}^\natural(\mathcal{I})$, we get $\Omega(\overleftarrow{\widetilde{T}}) = \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'}) = \Omega(\overleftarrow{\widetilde{T'}})$. Now, by hypothesis, $\widetilde{T} \leq \widetilde{T'}$ and, therefore, $T \leq T'$. We conclude that $\mathcal{I}$ is narrowly right reflective.

(d) Suppose $\mathcal{I}$ is narrowly system reflective and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T}, \widetilde{T'} \in \mathrm{ThSys}(\mathcal{I})$ and $\Omega(T) \leq \Omega(T')$. Then $\widetilde{T}, \widetilde{T'} \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$ and, by Proposition 369, $\Omega(\widetilde{T}) = \Omega(T) \leq \Omega(T') = \Omega(\widetilde{T'})$. Thus, by hypothesis, $\widetilde{T} \leq \widetilde{T'}$.

Assume, conversely, that the asserted condition holds and let $T, T' \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then, since $T, T' \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, we get $\widetilde{T} = T, \widetilde{T'} = T' \in \mathrm{ThSys}(\mathcal{I})$ and, therefore, by hypothesis, $\widetilde{T} \leq \widetilde{T'}$. But this gives $T = \widetilde{T} \leq \widetilde{T'} = T'$. Thus, $\mathcal{I}$ is narrowly system reflective. ∎

As was shown in Lemma 401, narrow right injectivity implies exclusive systemicity. In the next lemma, we show that narrow family reflectivity also implies exclusive systemicity.

Lemma 439 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is narrowly family reflective, then it is exclusively systemic.*

Proof: Assume $\mathcal{I}$ is narrowly family reflective and let $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, such that $\overleftarrow{T} \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$. By Proposition 20, $\Omega(T) \leq \Omega(\overleftarrow{T})$. Thus, by hypothesis, $T \leq \overleftarrow{T}$. Since, by Proposition 2, the reverse inclusion always holds, we get $\overleftarrow{T} = T$. Thus, $\mathcal{I}$ is exclusively systemic. ∎

Lemma 405 also has the following direct consequence.

Corollary 440 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is narrowly right reflective, then it is narrowly stable.*

Proof: Since narrow right reflectivity implies narrow right injectivity, this follows directly from Lemma 405. ∎

We establish, next the *narrow reflectivity hierarchy*. The following proposition forms an analog of Proposition 406, which established the narrow injectivity hierarchy.

Proposition 441 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is narrowly right reflective, then it is narrowly family reflective;*

(b) If $\mathcal{I}$ *is narrowly family reflective, then it is narrowly left reflective;*

(c) If $\mathcal{I}$ *is narrowly left reflective, then it is narrowly system reflective.*

Proof:

(a) Suppose that $\mathcal{I}$ is narrowly right reflective and let $T, T' \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. By Corollary 440, $\mathcal{I}$ is narrowly stable. Now we obtain $\Omega(\overleftarrow{T}) = \Omega(T) \leq \Omega(T') = \Omega(\overleftarrow{T'})$. Hence, by narrow right reflectivity, $T \leq T'$. Hence, $\mathcal{I}$ is narrowly family reflective.

(b) Suppose that $\mathcal{I}$ is narrowly family reflective and let $T, T' \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then, by hypothesis, $T \leq T'$, whence, by Lemma 1, $\overleftarrow{T} \leq \overleftarrow{T'}$. Thus, $\mathcal{I}$ is narrowly left reflective.

(c) Suppose that $\mathcal{I}$ is narrowly left reflective and let $T, T' \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then, by hypothesis, we get $\overleftarrow{T} \leq \overleftarrow{T'}$. Therefore, since T, T' are theory systems, $T \leq T'$ and, hence, $\mathcal{I}$ is narrowly system reflective.

■

We have now established the following **narrow reflectivity hierarchy** of π-institutions.

Narrowly R Reflective

$\downarrow$

Narrowly F Reflective

$\downarrow$

Narrowly L Reflective

$\downarrow$

Narrowly S Reflective

We give an additional result pertaining to the hierarchy of narrow reflectivity properties depicted in the diagram. The following proposition may be viewed as an analog of Proposition 407.

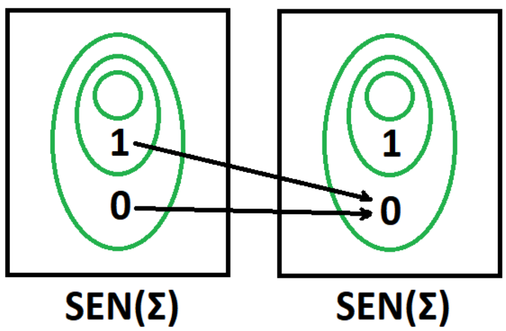
Proposition 442 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is narrowly system reflective and narrowly systemic, then it is narrowly right reflective.*

Proof: Suppose $\mathcal{I}$ is narrowly system reflective and narrowly systemic. Let $T, T' \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. By narrow systemicity, $T = \overleftarrow{T}$ and $T' = \overleftarrow{T'}$. Hence, on the one hand $\Omega(T) \leq \Omega(T')$ and, on the other, $T, T' \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$. Thus, by narrow system reflectivity, $T \leq T'$. Thus, $\mathcal{I}$ is narrowly right reflective. ■

We present three examples to show that all inclusions established between the narrow reflectivity classes and shown in the preceding diagram are indeed proper inclusions. The first example depicts a π-institution which is narrowly family reflective but not narrowly right reflective. This shows that the class of narrowly right reflective π-institutions constitutes a proper subclass of the class of narrowly family reflective π-institutions.

Example 443 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

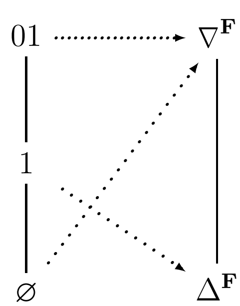
- $\mathbf{Sign}^\flat$ *is the category with the single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$*;*
- $N^\flat$ *is the trivial clone, consisting of the projections only.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\}.$$

$\mathcal{I}$ *has three theory families,* $\varnothing$*,* $\{\{1\}\}$ *and* $\{\{0,1\}\}$*, but only two theory systems,* $\varnothing$ *and* $\{\{0,1\}\}$*. The lattice of theory families of* $\mathcal{I}$ *and the corresponding Leibniz congruence systems are given in the diagram.*



Since the Leibniz operator is an isomorphism on $\mathrm{ThFam}^{\lightning}(\mathcal{I})$*,* $\mathcal{I}$ *is narrowly family reflective. On the other hand,* $\{\{1\}\}, \{\{0,1\}\} \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$ *and*

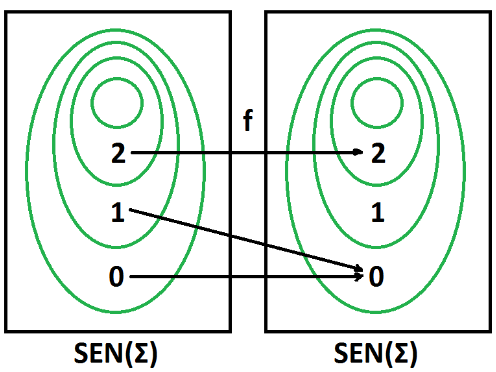
$$\Omega(\overleftarrow{\{\{1\}\}}) = \Omega(\{\varnothing\}) = \nabla^{\mathbf{F}} = \Omega(\{\{0,1\}\}) = \Omega(\overleftarrow{\{\{0,1\}\}}),$$

but $\{\{1\}\} \neq \{\{0,1\}\}$. *Therefore,* $\mathcal{I}$ *is not narrowly right injective and, a fortiori, it fails to be narrowly right reflective.*

The next example depicts a π-institution which is narrowly left reflective but not narrowly family reflective. This shows that the class of narrowly family reflective π-institutions is a proper subclass of the class of narrowly left reflective π-institutions.

Example 444 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

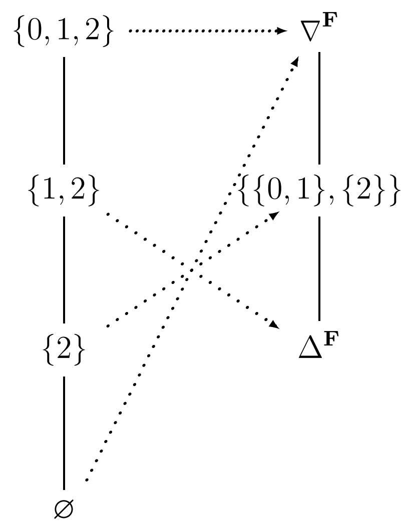
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$, $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$;
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{2\}, \{1,2\}, \{0,1,2\}\}.$$

$\mathcal{I}$ *has four theory families, but only three theory systems, namely* $\varnothing$, $\{2\}$ *and* $\{0,1,2\}$. *Moreover, clearly,* $\mathrm{ThFam}^{\nmid}(\mathcal{I}) = \{\{2\},\{1,2\},\{0,1,2\}\}$. *The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right:*



There are three pairs (T, T'), *with* $T, T' \in \mathrm{ThFam}^{\natural}(\mathcal{I})$ *and* $T \neq T'$, *such that* $\Omega(T) \leq \Omega(T')$, *namely,*

$$(\{1,2\},\{0,1,2\}), \quad (\{2\},\{0,1,2\}), \quad (\{1,2\},\{2\}).$$

For all three, we get $\overleftarrow{T} \leq \overleftarrow{T'}$. *Thus,* $\mathcal{I}$ *is narrowly left reflective. On the other hand, for* $T = \{1,2\}$ *and* $T' = \{2\}$, *even though* $\Omega(T) \leq \Omega(T')$, *we get* $T \nleq T'$, *whence* $\mathcal{I}$ *fails to be narrowly family reflective.*

We finish the sequence of examples by presenting a narrowly system reflective π-institution which fails to be narrowly left reflective. This example shows that narrowly left reflective π-institutions form a proper subclass of the class of narrowly system reflective π-institutions.

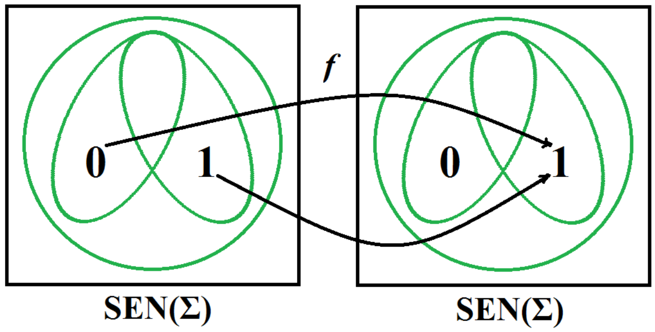
Example 445 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 1$ *and* $\mathrm{SEN}^\flat(f)(1) = 1$;
- $N^\flat$ *is the trivial clone.*

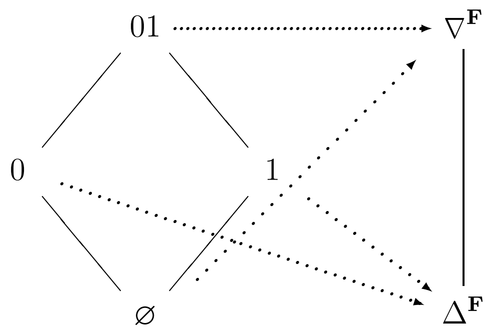
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\varnothing, \{0\}, \{1\}, \{0,1\}\}$.

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$.

T	$\overleftarrow{T}$
$\varnothing$	$\varnothing$
$\{0\}$	$\varnothing$
$\{1\}$	$\{1\}$
$\{0,1\}$	$\{0,1\}$



The lattice of theory families and the corresponding Leibniz congruence systems are depicted below.



It is obvious from the diagram that the Leibniz operator is an isomorphism on $\mathrm{ThSys}^{\nmid}(\mathcal{I})$. *Therefore,* $\mathcal{I}$ *is narrowly system reflective. On the other hand, for* $T = \{\{0\}\}$, $T' = \{\{1\}\}$, *both members of* $\mathrm{ThFam}^{\nmid}(\mathcal{I})$, *we have* $\Omega(T) = \Omega(T') = \Delta^{\mathbf{F}}$, *whereas* $\overleftarrow{T} = \{\varnothing\} \neq \{\{1\}\} = \overleftarrow{T'}$. *Therefore,* $\mathcal{I}$ *is not narrowly left injective and, a fortiori, it fails to be narrowly left reflective.*

We turn now to the relationships between corresponding classes of the rough reflectivity and the narrow reflectivity hierarchies. These parallel the ones already established between the rough injectivity and narrow injectivity classes in Section 6.5.

Using the characterization in Part (a) of Proposition 438, we can immediately see that the two types of family reflectivity coincide.

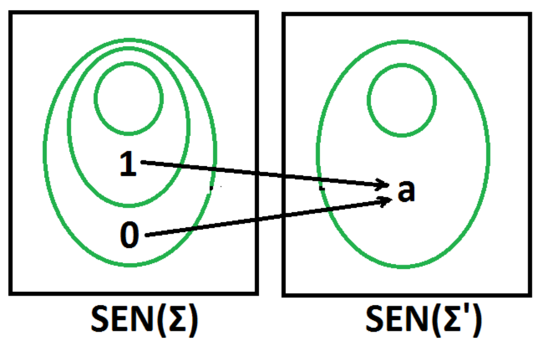
Corollary 446 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is roughly family reflective if and only if it is narrowly family reflective.*

Proof: Part (a) of Proposition 438. ∎

As was the case with rough and narrow injectivity properties, the relationships between the remaining classes are not so straightforward, due to the necessity of investigating the mode of interaction between rough equivalence and the $\leftarrow$ operator. Starting with the two left reflectivity classes, we show that the class of narrow left reflective π-institutions is not included in the class of roughly left reflective π-institutions. This is accomplished by constructing a π-institution which is narrowly left reflective but not roughly left reflective.

Example 447 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

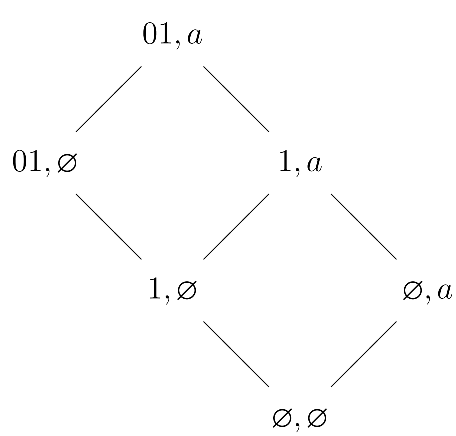
- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*,* $\mathrm{SEN}^\flat(\Sigma') = \{a\}$ *and* $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = a$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{a\}\}.$$

Clearly, there are six theory families in $\mathrm{ThFam}(\mathcal{I})$*, only four of which are theory systems, and only two of which are in* $\mathrm{ThFam}^{\lightning}(\mathcal{I})$*. The lattice of theory families is shown in the diagram:*



The only pair (T,T'), *with* $T,T' \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, $T \neq T'$ *and* $\Omega(T) \leq \Omega(T')$ *is* $(\{1,a\},\ \{01,a\})$, *Since,* $\overleftarrow{\{1,a\}} = \{1,a\} \leq \{01,a\} = \overleftarrow{\{01,a\}}$, *it follows that* $\mathcal{I}$ *is narrowly left reflective.*

On the other hand, consider $T = \{1,\varnothing\}$ *and* $T' = \{1,a\}$. *We have* $\Omega(1,\varnothing) = \{\Delta^{\mathbf{F}}_{\Sigma}, \nabla^{\mathbf{F}}_{\Sigma'}\} = \Omega(1,a)$, *but*

$$\overleftarrow{\widetilde{1,\varnothing}} = \widetilde{\varnothing,\varnothing} = \{01,a\} \nleq \{1,a\} = \widetilde{1,a} = \overleftarrow{\widetilde{1,a}}.$$

This proves that $\mathcal{I}$ *is not roughly left reflective.*

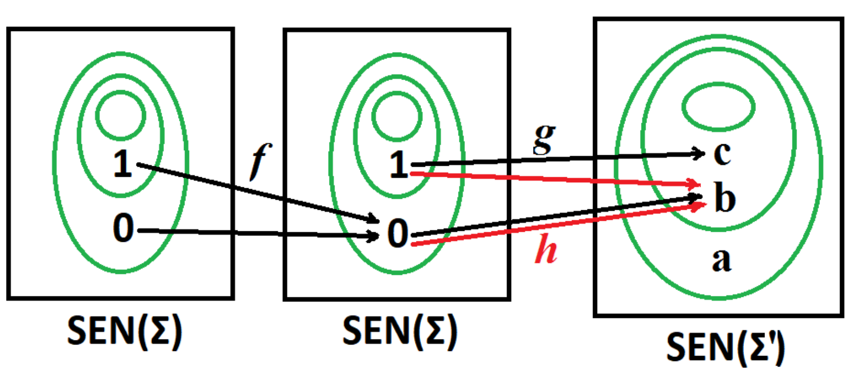
We exhibit, next a π-institution that is roughly left reflective, while it fails to be narrowly left reflective. Combined with Example 447, this will show that the two left reflectivity classes, rough and narrow, are incomparable from the point of view of inclusion.

Example 448 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and three nonidentity morphisms* $f : \Sigma \to \Sigma$ *and* $g, h : \Sigma \to \Sigma'$, *such that* $f \circ f = f$, $g \circ f = h$ *and* $h \circ f = h$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b,c\}$, $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = 0$, $\mathrm{SEN}^\flat(g)(0) = b$, $\mathrm{SEN}^\flat(g)(1) = c$ *and* $\mathrm{SEN}^\flat(h)(0) = \mathrm{SEN}^\flat(h)(1) = b$;
- $N^\flat$ *is the clone generated by a single binarry natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$, *whose components are defined by the following tables:*

$\sigma^\flat_\Sigma$	0	1
0	0	1
1	1	1

$\sigma^\flat_{\Sigma'}$	a	b	c
a	a	a	c
b	a	b	c
c	c	c	c



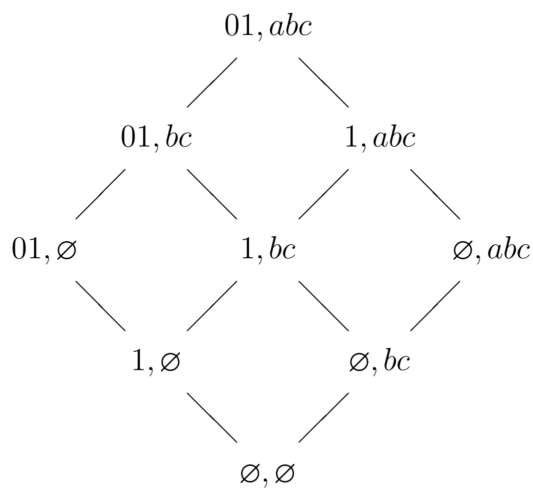
It is not difficult, albeit slightly tedious, to check that this is a well-defined natural transformation. We summarize the checking in the accompanying table.

(x,y)	$f(\sigma^\flat_\Sigma(x,y)) = \sigma^\flat_\Sigma(f(x),f(y))$	$g(\sigma^\flat_\Sigma(x,y)) = \sigma^\flat_{\Sigma'}(g(x),g(y))$	$h(\sigma^\flat_\Sigma(x,y)) = \sigma^\flat_{\Sigma'}(h(x),h(y))$
$(0,0)$	$0=0$	$b=b$	$b=b$
$(0,1)$	$0=0$	$c=c$	$b=b$
$(1,0)$	$0=0$	$c=c$	$b=b$
$(1,1)$	$0=0$	$c=c$	$b=b$

Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\varnothing,\{1\},\{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing,\{b,c\},\{a,b,c\}\}.$$

Clearly, there are nine theory families in $\mathrm{ThFam}(\mathcal{I})$*, five of which are theory systems, and four of which are in* $\mathrm{ThFam}^{\lightning}(\mathcal{I})$*. The lattice of theory families is shown in the diagram:*



The action of $\overleftarrow{\ }$ *on theory families is given in the following table.*

T	$\overleftarrow{T}$	T	$\overleftarrow{T}$
$01, abc$	$01, abc$	$\varnothing, abc$	$\varnothing, abc$
$01, bc$	$01, bc$	$1, \varnothing$	$\varnothing, \varnothing$
$1, abc$	$\varnothing, abc$	$\varnothing, bc$	$\varnothing, bc$
$01, \varnothing$	$\varnothing, \varnothing$	$\varnothing, \varnothing$	$\varnothing, \varnothing$
$1, bc$	$\varnothing, bc$		

The table below provides the Leibniz congruence systems associated with the theory families of $\mathcal{I}$.

T	$\Omega(T)$
$\{01,abc\},\{01,\varnothing\},\{\varnothing,abc\},\{\varnothing,\varnothing\}$	$\nabla^{\mathbf{F}}$
$\{1,abc\},\{1,\varnothing\}$	$\{\Delta^{\mathbf{F}}_\Sigma, \nabla^{\mathbf{F}}_{\Sigma'}\}$
$\{01,bc\},\{1,bc\},\{\varnothing,bc\}$	$\Delta^{\mathbf{F}}$

To see that $\mathcal{I}$ is roughly left reflective, suppose that $\Omega(T) \le \Omega(T')$. We separate cases depending on $\Omega(T')$.

- *If $\Omega(T') = \Delta^{\mathbf{F}}$, then $T, T' \in \{\{01, bc\}, \{1, bc\}, \{\varnothing, b\}\}$, whence $\widetilde{\overleftarrow{T}} = \{01, bc\} = \widetilde{\overleftarrow{T'}}$;*
- *If $\Omega(T') = \{\Delta^{\mathbf{F}}_{\Sigma}, \nabla^{\mathbf{F}}_{\Sigma'}\}$ or $\Omega(T') = \nabla^{\mathbf{F}}$, then $\widetilde{\overleftarrow{T}} \le \{01, abc\} = \widetilde{\overleftarrow{T'}}$.*

On the other hand, for $T = \{01, bc\}$ and $T' = \{1, bc\}$, we get $\Omega(T) = \Delta^{\mathbf{F}} = \Omega(T')$, whereas $\overleftarrow{T} = \{01, bc\} \nleq \{\varnothing, bc\} = \overleftarrow{T'}$. Therefore, $\mathcal{I}$ is not narrowly left reflective.

We turn, next to the relationship between the two kinds of right reflectivity. We show, first, that rough right reflectivity implies narrow right reflectivity.

Proposition 449 *Let $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat} \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is roughly right reflective, then it is narrowly right reflective.*

Proof: Suppose $\mathcal{I}$ is roughly right reflective and let $T, T' \in \mathrm{ThFam}^{\natural}(\mathcal{I})$, such that $\Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'})$. By rough right reflectivity, we get that $\widetilde{T} \le \widetilde{T'}$. Since, however, $T, T' \in \mathrm{ThFam}^{\natural}(\mathcal{I})$, we get $T = \widetilde{T} \le \widetilde{T'} = T'$. Therefore, $\mathcal{I}$ is narrowly right reflective. ∎

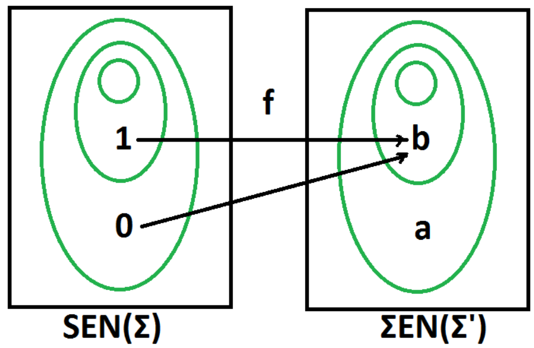
The converse, on the other hand, does not hold in general, as the following example demonstrates.

Example 450 *Let $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat} \rangle$ be the algebraic system defined as follows:*

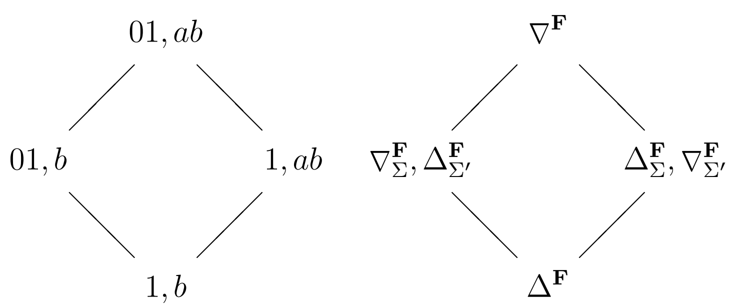
- $\mathbf{Sign}^{\flat}$ *is the category with objects Σ and Σ' and a unique morphism $f : \Sigma \to \Sigma'$;*
- $\mathrm{SEN}^{\flat} : \mathbf{Sign}^{\flat} \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^{\flat}(\Sigma) = \{0, 1\}$, $\mathrm{SEN}^{\flat}(\Sigma') = \{a, b\}$ *and* $\mathrm{SEN}^{\flat}(f)(0) = b$, $\mathrm{SEN}^{\flat}(f)(1) = b$;
- *$N^{\flat}$ is the trivial clone.*

Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_{\Sigma} = \{\varnothing, \{1\}, \{0, 1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a, b\}\}.$$



Clearly, there are only four theory families in $\mathrm{ThFam}^{\sharp}(\mathcal{I})$*, all of which are theory systems. Their lattice together with the associated Leibniz congruence systems are shown in the diagram:*



From this diagram and the fact that all theory families in $\mathrm{ThFam}^{\sharp}(\mathcal{I})$ *are theory systems, we see that, for all* $T, T' \in \mathrm{ThFam}^{\sharp}(\mathcal{I})$*,*

$$\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'}) \quad \textit{iff} \quad \Omega(T) \leq \Omega(T') \quad \textit{iff} \quad T \leq T'.$$

Therefore, $\mathcal{I}$ *is indeed narrowly right reflective.*

On the other hand, consider $T = \{01, ab\}$ *and* $T' = \{1, \varnothing\}$*. Then we have*

$$\Omega(\overleftarrow{T}) = \Omega(01, ab) = \nabla^{\mathbf{F}} = \Omega(\overline{\varnothing}) = \Omega(\overleftarrow{T'}),$$

whereas $\widetilde{01, ab} = \{01, ab\} \nleq \{1, ab\} = \widetilde{1, \varnothing}$*. This shows that* $\mathcal{I}$ *is not roughly right reflective.*

Finally, we look at system reflectivity. We show that rough system reflectivity implies narrow system reflectivity, but that the converse implication fails in general.

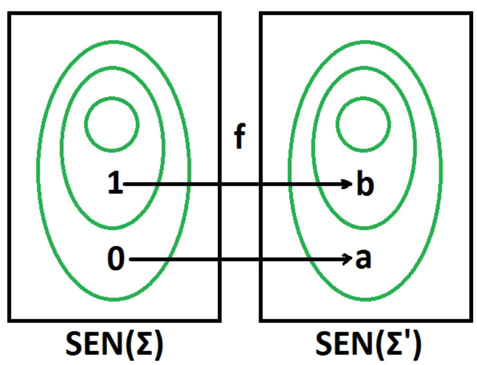
Proposition 451 *Let* $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat} \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly system reflective, then it is narrowly system reflective.*

Proof: Suppose $\mathcal{I}$ is roughly system reflective and let $T, T' \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$, such that $\Omega(T) \leq \Omega(T')$. Then, by rough system reflectivity, $\widetilde{T} \leq \widetilde{T'}$. However, since $T, T' \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$, we get $T = \widetilde{T} \leq \widetilde{T'} = T'$. Therefore, $\mathcal{I}$ is narrowly system reflective. ■

And now we present an example of a π-institution that is narrowly system reflective but not roughly system reflective. This, combined with Proposition 451, shows that the class of narrowly system reflective π-institutions properly contains the class of roughly system reflective π-institutions.

Example 452 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

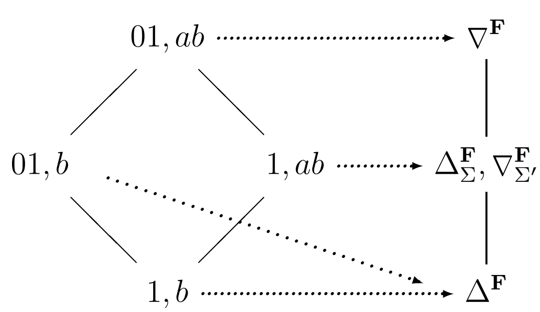
- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0, 1\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a, b\}\}.$$

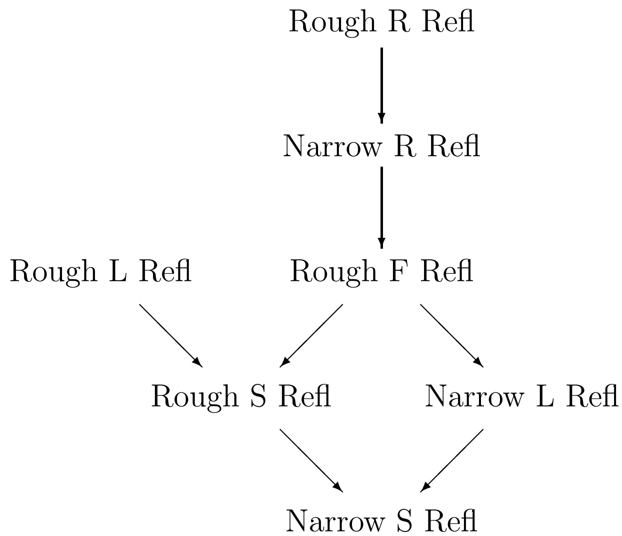
There are only four theory families in $\mathrm{ThFam}^{\lightning}(\mathcal{I})$*, all of which except for* $\{01, b\}$ *are theory systems. Their lattice together with the associated Leibniz congruence systems are shown in the diagram:*



From this diagram we see that for all $T, T' \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$*, we get* $\Omega(T) \leq \Omega(T')$ *if and only if* $T \leq T'$*. Therefore,* $\mathcal{I}$ *is narrowly system reflective.*

On the other hand, consider $T = \{\varnothing, b\}$*,* $T' = \{1, b\} \in \mathrm{ThSys}(\mathcal{I})$*. Even though* $\widetilde{T} = \{01, b\} \nleq \{1, b\} = \widetilde{T}'$*, we have* $\Omega(T) = \Delta^{\mathbf{F}} = \Omega(T')$*. Hence,* $\mathcal{I}$ *is not roughly system reflective.*

The results obtained and the counterexamples presented, thus far, reveal the following mixed hierarchy of rough and narrow reflectivity classes of π-institutions, paralleling the one presented for rough and narrow injectivity properties.



As far as narrow injectivity versus narrow reflectivity properties, it is easy to show that a narrow reflectivity property implies the corresponding narrow injectivity property. (In fact, this observation, formalized in Proposition 453, has already been used before, e.g., in the proof of Part (a) of Proposition 441.)

Proposition 453 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is narrowly family reflective, then it is narrowly family injective;*

(b) If $\mathcal{I}$ *is narrowly left reflective, then it is narrowly left injective;*

(c) If $\mathcal{I}$ *is narrowly right reflective, then it is narrowly right injective;*

(d) If $\mathcal{I}$ *is narrowly system reflective, then it is narrowly system injective.*

Proof: We only deal with the family case, since the other three implications are equally straightforward to prove.

Assume that $\mathcal{I}$ is narrowly family reflective and let $T, T' \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Since this implies that $\Omega(T) \leq \Omega(T')$ and that

$\Omega(T') \leq \Omega(T)$, we get, by applying narrow family reflectivity, that $T \leq T'$ and $T' \leq T$. Therefore, $T = T'$ and, hence, $\mathcal{I}$ is narrowly family injective. ■

Turning to the relationships between narrow reflectivity classes and corresponding reflectivity classes, we prove a theorem, analogous to Theorem 418, asserting that ordinary reflectivity is equivalent to narrow reflectivity in the presence of theorems.

Theorem 454 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is family reflective if and only if it is narrowly family reflective and has theorems;*

(b) $\mathcal{I}$ *is left reflective if and only if it is narrowly left reflective and has theorems;*

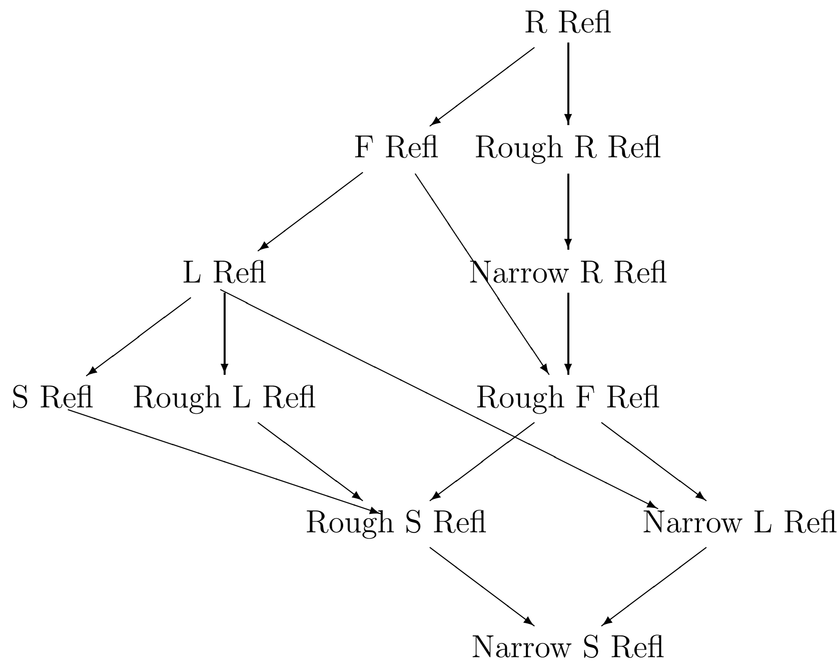
(c) $\mathcal{I}$ *is right reflective if and only if it is narrowly right reflective and has theorems;*

(d) $\mathcal{I}$ *is system reflective if and only if it is narrowly system reflective and has theorems.*

Proof: By Theorem 432, if $\mathcal{I}$ has one of the four reflectivity properties, then it has theorems. Moreover, by the same theorem, a reflectivity property implies the corresponding rough reflectivity property and, by Corollary 446, Proposition 449 and Proposition 451, each implies the corresponding narrow reflectivity property except in the case of left reflectivity, where (as actually in all other cases, as well) one can easily see directly that left reflectivity implies narrow left reflectivity, since the defining condition of the latter is a special case of that of the former.

All converses are also easily verified, since, in the presence of theorems, $\mathrm{ThFam}^{\natural}(\mathcal{I}) = \mathrm{ThFam}(\mathcal{I})$ and $\mathrm{ThSys}^{\natural}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$, which makes the four defining conditions for the narrow classes identical with the corresponding conditions for the ordinary reflectivity classes. ■

We now have the following hierarchy.



The narrow reflectivity properties transfer from the theory families/systems of a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ to all $\mathcal{I}$-filter families/systems on arbitrary $\mathbf{F}$-algebraic systems. This result forms an analog of Theorem 419, which applied to narrow injectivity classes.

Theorem 455 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is narrowly right reflective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}\nmid}(\mathcal{A})$, $\Omega^{\mathcal{A}}(\overleftarrow{T}) \leq \Omega^{\mathcal{A}}(\overleftarrow{T'})$ *implies* $T \leq T'$*;*

(b) $\mathcal{I}$ *is narrowly family reflective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}\nmid}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ *implies* $T \leq T'$*;*

(c) $\mathcal{I}$ *is narrowly left reflective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}\nmid}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ *implies* $\overleftarrow{T} \leq \overleftarrow{T'}$*;*

(d) $\mathcal{I}$ *is narrowly system reflective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T, T' \in \mathrm{FiSys}^{\mathcal{I}\nmid}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ *implies* $T \leq T'$*.*

Proof: The proof follows the steps of the proofs of the various parts of Theorem 419. We do Part (a) in detail to give a flavor of what is involved.

The "if" follows by considering the **F**-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account that $\mathrm{ThFam}^{\nmid}(\mathcal{I}) = \mathrm{FiFam}^{\mathcal{I}\nmid}(\mathcal{F})$, by Lemmas 51 and 376.

For the "only if", suppose that $\mathcal{I}$ is narrowly right reflective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an **F**-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}\nmid}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(\overleftarrow{T}) \leq \Omega^{\mathcal{A}}(\overleftarrow{T'})$. Then $\alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T})) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'}))$. So, by Proposition 24, $\Omega(\alpha^{-1}(\overleftarrow{T})) \leq \Omega(\alpha^{-1}(\overleftarrow{T'}))$. Hence, by Lemma 6, $\Omega(\overleftarrow{\alpha^{-1}(T)}) \leq \Omega(\overleftarrow{\alpha^{-1}(T')})$. Since, by Lemmas 51 and 376, $\alpha^{-1}(T), \alpha^{-1}(T') \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, we get, by applying narrow right reflectivity, $\alpha^{-1}(T) \leq \alpha^{-1}(T')$. This yields, taking into account the surjectivity of $\langle F, \alpha \rangle$, $T \leq T'$. ■

We finally recast narrow reflectivity in terms of the order reflectivity of mappings from posets of theory or filter families/systems into posets of congruence systems. The following results form analogs of Propositions 420 and 421, respectively, addressing reflectivity instead of injectivity properties.

Proposition 456 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* **F**. *The following statements are equivalent:*

(a) $\mathcal{I}$ *is narrowly family reflective;*

(b) $\Omega : \mathrm{ThFam}^{\nmid}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is order reflecting;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}\nmid}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is order reflecting, for every* **F**-*algebraic system* $\mathcal{A}$.

Similarly, for system reflectivity, we have

Proposition 457 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* **F**. *The following statements are equivalent:*

(a) $\mathcal{I}$ *is narrowly system reflective;*

(b) $\Omega : \mathrm{ThSys}^{\nmid}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is order reflecting;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}\nmid}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is order reflecting, for every* **F**-*algebraic system* $\mathcal{A}$.

6.8 Rough Complete Reflectivity

In this section we study classes of π-institutions defined using complete reflectivity properties of the Leibniz operator applied on rough equivalence classes.

Definition 458 (Rough c-Reflectivity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* **F**.

- $\mathcal{I}$ *is called* **roughly family completely reflective**, *or* **roughly family c-reflective** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,

$$\bigcap_{T \in \mathcal{T}} \Omega(T) \le \Omega(T') \quad \textit{implies} \quad \bigcap_{T \in \mathcal{T}} \widetilde{T} \le \widetilde{T'}.$$

- $\mathcal{I}$ *is called* **roughly left completely reflective**, *or* **roughly left c-reflective** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,

$$\bigcap_{T \in \mathcal{T}} \Omega(T) \le \Omega(T') \quad \textit{implies} \quad \bigcap_{T \in \mathcal{T}} \overleftarrow{\widetilde{T}} \le \overleftarrow{\widetilde{T'}}.$$

- $\mathcal{I}$ *is called* **roughly right completely reflective**, *or* **roughly right c-reflective** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,

$$\bigcap_{T \in \mathcal{T}} \Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'}) \quad \textit{implies} \quad \bigcap_{T \in \mathcal{T}} \widetilde{T} \le \widetilde{T'}.$$

- $\mathcal{I}$ *is called* **roughly system completely reflective**, *or* **roughly system c-reflective** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$,

$$\bigcap_{T \in \mathcal{T}} \Omega(T) \le \Omega(T') \quad \textit{implies} \quad \bigcap_{T \in \mathcal{T}} \widetilde{T} \le \widetilde{T'}.$$

As was shown to be the case with rough right reflectivity in Lemma 423, we show that rough right c-reflectivity implies rough systemicity and, hence, by Theorem 370, stability.

Lemma 459 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly right completely reflective, then it is roughly systemic.*

Proof: This is a consequence of Lemma 423, since rough right c-reflectivity implies trivially rough right reflectivity. ∎

Next we establish the *rough c-reflectivity hierarchy* of π-institutions.

Proposition 460 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) *If* $\mathcal{I}$ *is roughly right c-reflective, then it is roughly family c-reflective;*

(b) *If* $\mathcal{I}$ *is roughly family c-reflective, then it is roughly system c-reflective;*

(c) *If* $\mathcal{I}$ *is roughly left c-reflective, then it is roughly system c-reflective.*

Proof:

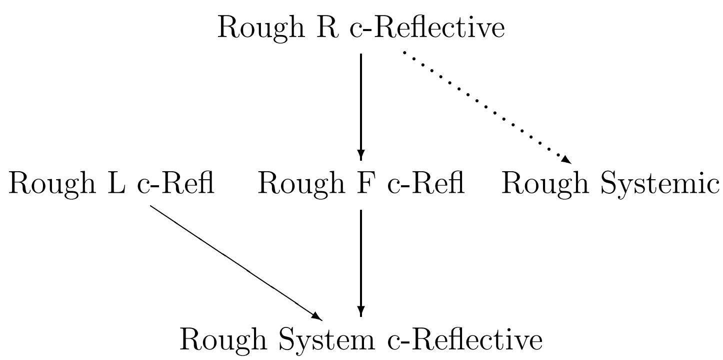
(a) Suppose $\mathcal{I}$ is roughly right c-reflective and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. By Lemma 459, $\mathcal{I}$ is roughly systemic, whence $\overleftarrow{T} \sim T$, for all $T \in \mathcal{T}$, and $\overleftarrow{T'} \sim T'$. Thus, by Theorem 370, we get
$$\bigcap_{T\in\mathcal{T}} \Omega(\overleftarrow{T}) = \bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T') = \Omega(\overleftarrow{T'}).$$
Now applying rough right c-reflectivity, we get $\bigcap_{T\in\mathcal{T}} \widetilde{T} \leq \widetilde{T'}$. This proves that $\mathcal{I}$ is roughly family c-reflective.

(b) Suppose $\mathcal{I}$ is roughly family c-reflective and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. Then, by rough family c-reflectivity, we get $\bigcap_{T\in\mathcal{T}} \widetilde{T} \leq \widetilde{T'}$, whence, $\mathcal{I}$ is roughly system c-reflective.

(c) Suppose $\mathcal{I}$ is roughly left c-reflective and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. By rough left c-reflectivity, we conclude that $\bigcap_{T\in\mathcal{T}} \overleftarrow{\widetilde{T}} \leq \overleftarrow{\widetilde{T'}}$. However, since $\mathcal{T} \cup \{T'\}$ consists of theory systems, we have $\overleftarrow{T} = T$, for all $T \in \mathcal{T}$, and $\overleftarrow{T'} = T'$. Hence we get $\bigcap_{T\in\mathcal{T}} \widetilde{T} \leq \widetilde{T'}$ and, hence, $\mathcal{I}$ is roughly system reflective.

■

We have now established the following **rough complete reflectivity hierarchy** of π-institutions.



We formulate two additional properties concerning the relationships between rough c-reflectivity classes. First, rough right c-reflectivity turns out to be equivalent to rough system c-reflectivity combined with rough systemicity.

Proposition 461 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is roughly right c-reflective if and only if it is roughly system c-reflective and roughly systemic.*

Proof: Suppose, first, that $\mathcal{I}$ is roughly right c-reflective. Then, by Lemma 459, it is roughly systemic and by Proposition 460 it is roughly system c-reflective.

Suppose, conversely, that $\mathcal{I}$ is roughly system c-reflective and roughly systemic and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. By rough system c-reflectivity and Proposition 42, we get $\bigcap_{T\in\mathcal{T}}\widetilde{\overleftarrow{T}} \leq \widetilde{\overleftarrow{T'}}$. Hence, by rough systemicity, $\bigcap_{T\in\mathcal{T}}\widetilde{T} = \bigcap_{T\in\mathcal{T}}\widetilde{\overleftarrow{T}} \leq \widetilde{\overleftarrow{T'}} = \widetilde{T'}$. Thus, $\mathcal{I}$ is roughly right c-reflective. ■

Second, we show that rough system c-reflectivity together with stability imply rough left c-reflectivity.

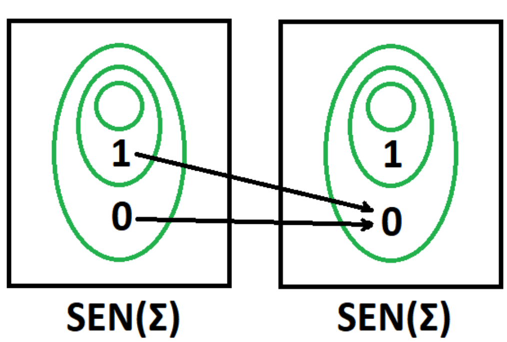
Proposition 462 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is roughly system c-reflective and stable, then it is roughly left c-reflective.*

Proof: Suppose that $\mathcal{I}$ is roughly system c-reflective and stable and consider $\mathcal{T}\cup\{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}}\Omega(T) \leq \Omega(T')$. Then, by stability $\bigcap_{T\in\mathcal{T}}\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Hence, since $\{\overleftarrow{T} : T\in\mathcal{T}\}\cup\{\overleftarrow{T'}\} \subseteq \mathrm{ThSys}(\mathcal{I})$, by rough system c-reflectivity, $\bigcap_{T\in\mathcal{T}}\widetilde{\overleftarrow{T}} \leq \widetilde{\overleftarrow{T'}}$. This shows that $\mathcal{I}$ is roughly left c-reflective. ■

We present three examples to show that all inclusions established between rough c-reflectivity classes and depicted in the diagram above are proper inclusions. The first example will show that the class of roughly right c-reflective π-institutions is a proper subclass of the class of roughly family c-reflective π-institutions.

Example 463 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

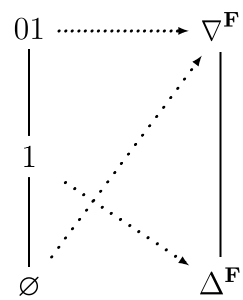
- $\mathbf{Sign}^\flat$ *is the category with the single object* Σ *and a single (non-identity) morphism* $f:\Sigma\to\Sigma$, *such that* $f\circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$;
- $N^\flat$ *is the trivial clone.*



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\}.$$

$\mathcal{I}$ has three theory families $\{\varnothing\}$, $\{\{1\}\}$ and $\{\{0,1\}\}$, but only two theory systems, $\{\varnothing\}$ and $\{\{0,1\}\}$. The lattice of theory families of $\mathcal{I}$ and the corresponding Leibniz congruence systems are given in the diagram.



It is easy to see that $\mathcal{I}$ is roughly family c-reflective. Suppose that for $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\bigcap_{T \in \mathcal{T}} \Omega(T) \le \Omega(T')$.

- *If $\Omega(T') = \Delta^{\mathbf{F}}$, then $\bigcap_{T \in \mathcal{T}} \Omega(T) = \Delta^{\mathbf{F}}$, whence $T' = \{\{1\}\}$ and $\{\{1\}\} \in \mathcal{T}$. Thus, $\bigcap_{T \in \mathcal{T}} \widetilde{T} \le \{\{1\}\} = \widetilde{T'}$.*

- *If $\Omega(T') = \nabla^{\mathbf{F}}$, then $T' = \{\varnothing\}$ or $T' = \{\{0,1\}\}$. In either case, $\bigcap_{T \in \mathcal{T}} \widetilde{T} \le \{\{0,1\}\} = \widetilde{T'}$.*

On the other hand, for $T = \{\{1\}\}$, we get $\widetilde{T} = \{\{1\}\} \ne \{\{0,1\}\} = \widetilde{\{\varnothing\}} = \widetilde{\overleftarrow{T}}$, whence $T \not\approx \overleftarrow{T}$ and, hence, $\mathcal{I}$ is not roughly systemic. Therefore, by Lemma 459, $\mathcal{I}$ is not roughly right c-reflective.

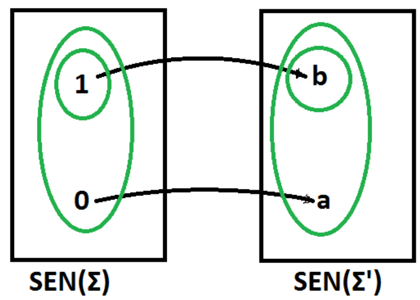
The second example shows that there exists a roughly left c-reflective π-institution that is not roughly family c-reflective.

Example 464 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- *$\mathbf{Sign}^\flat$ is the category with objects Σ and Σ' and a unique (non-identity) morphism $f : \Sigma \to \Sigma'$;*

- *$\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ is defined by $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ and $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$;*

- *$N^\flat$ is the trivial clone.*



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

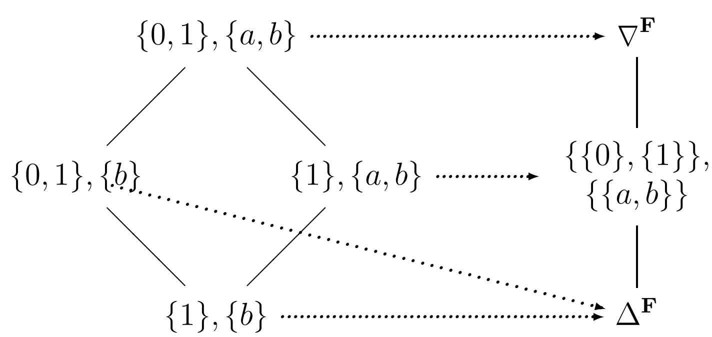
$$\mathcal{C}_\Sigma = \{\{1\}, \{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\}, \{a,b\}\}.$$

Again, since $\mathcal{I}$ has theorems, rough equivalence coincides with the identity relation on ThFam$(\mathcal{I})$*.*

The following table shows the action of $\overleftarrow{\ }$ on theory families.

T	$\{1,b\}$	$\{01,b\}$	$\{1,ab\}$	$\{01,ab\}$
$\overleftarrow{T}$	$\{1,b\}$	$\{1,b\}$	$\{1,ab\}$	$\{01,ab\}$

The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right:



We show, first, that $\mathcal{I}$ is roughly left c-reflective. Suppose $\mathcal{T} \cup \{T'\} \subseteq$ ThFam$(\mathcal{I})$*, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$.*

- *If $\Omega(T') = \nabla^{\mathbf{F}}$, then $T' = \{\{0,1\},\{a,b\}\}$, whence*

$$\bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \{\{0,1\},\{a,b\}\} = \overleftarrow{T'}$$

and, hence, $\bigcap_{T \in \mathcal{T}} \overleftarrow{\widetilde{T}} \leq \overleftarrow{\widetilde{T'}}$.

- *If* $\Omega(T') = \{\{\{0\},\{1\}\},\{\{a,b\}\}\}$*, then* $T' = \{\{1\},\{a,b\}\}$ *and one of* $\{\{0,1\},\{b\}\}$ *or* $\{\{1\},\{a,b\}\}$ *or* $\{\{1\},\{b\}\}$ *must belong to* $\mathcal{T}$*. In either case*
$$\bigcap_{T\in\mathcal{T}}\overleftarrow{T} \le \{\{1\},\{a,b\}\} = \overleftarrow{T'}$$
and, hence, $\bigcap_{T\in\mathcal{T}}\widetilde{\overleftarrow{T}} \le \widetilde{\overleftarrow{T'}}$.

- *If* $\Omega(T') = \Delta^{\mathbf{F}}$*, then* T' *must be either* $\{\{0,1\},\{b\}\}$ *or* $\{\{1\},\{b\}\}$ *and, moreover,* $\{\{0,1\},\{b\}\}$ *or* $\{\{1\},\{b\}\}$ *is in* $\mathcal{T}$*. Thus, we get*
$$\bigcap_{T\in\mathcal{T}}\overleftarrow{T} \le \{\{1\},\{b\}\} = \overleftarrow{T'}$$
and, hence, $\bigcap_{T\in\mathcal{T}}\widetilde{\overleftarrow{T}} \le \widetilde{\overleftarrow{T'}}$.

On the other hand, we have $\Omega(\{\{0,1\},\{b\}\}) \le \Omega(\{\{1\},\{b\}\})$*, but, clearly,* $\{\{0,1\},\{b\}\} \not\le \{\{1\},\{b\}\}$*. Thus, since rough equivalence is the identity on* $\mathrm{ThFam}(\mathcal{I})$*, we conclude that* $\mathcal{I}$ *is not roughly family c-reflective.*

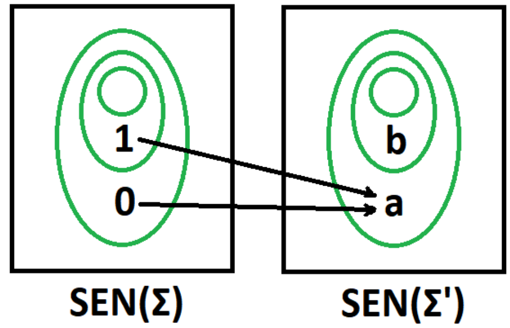
The third example shows that there exists a roughly family c-reflective π-institution that is not roughly left c-reflective. Combined with the preceding example, it has the effect of establishing the following facts:

- The classes of roughly family c-reflective and roughly left c-reflective π-institutions are incomparable.
- The class of roughly family c-reflective π-institutions is properly contained in the class of roughly system c-reflective π-institutions.
- Similarly, the class of roughly left c-reflective π-institutions is a proper subclass of the class of roughly system c-reflective π-institutions.

Example 465 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = a$*;*
- $N^\flat$ *is the trivial clone.*

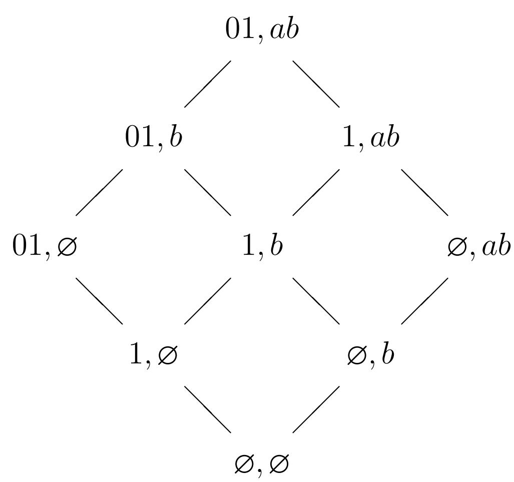
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by stipulating that*
$$\mathcal{C}_\Sigma = \{\varnothing,\{1\},\{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing,\{b\},\{a,b\}\}.$$



There are nine theory families, but only five theory systems. The action of $\overleftarrow{\ }$ *on theory families is given in the table below.*

T	$\overleftarrow{T}$	T	$\overleftarrow{T}$
$\varnothing, \varnothing$	$\varnothing, \varnothing$	$\varnothing, ab$	$\varnothing, ab$
$1, \varnothing$	$\varnothing, \varnothing$	$01, b$	$\varnothing, b$
$\varnothing, b$	$\varnothing, b$	$1, ab$	$1, ab$
$01, \varnothing$	$\varnothing, \varnothing$	$01, ab$	$01, ab$
$1, b$	$\varnothing, b$		

The lattice of theory families of $\mathcal{I}$ *is shown in the diagram.*



We show that $\mathcal{I}$ *is roughly family c-reflective. The following table summarizes the theory families together with their associated Leibniz congruence systems.*

T	$\Omega(T)$
$\{\varnothing, \varnothing\}, \{01, \varnothing\}, \{\varnothing, ab\}, \{01, ab\}$	$\nabla^{\mathbf{F}}$
$\{\varnothing, b\}, \{01, b\}$	$\{\nabla^{\mathbf{F}}_{\Sigma}, \Delta^{\mathbf{F}}_{\Sigma'}\}$
$\{1, \varnothing\}, \{1, ab\}$	$\{\Delta^{\mathbf{F}}_{\Sigma}, \nabla^{\mathbf{F}}_{\Sigma'}\}$
$\{1, b\}$	$\Delta^{\mathbf{F}}$

Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$*, such that* $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$*.*

- *If* $\Omega(T') = \nabla^{\mathbf{F}}$*, then* $\bigcap_{T\in\mathcal{T}} \widetilde{T} \leq \{01, ab\} = \widetilde{T}'$.
- *If* $\Omega(T') = \{\Delta^{\mathbf{F}}_{\Sigma}, \nabla^{\mathbf{F}}_{\Sigma'}\}$*, then* $\mathcal{T}$ *must include one of the theory families* $\{1, \varnothing\}$, $\{1, ab\}$, $\{1, b\}$. *Hence,* $\bigcap_{T\in\mathcal{T}} \widetilde{T} \leq \{1, ab\} = \widetilde{T}'$.
- *If* $\Omega(T') = \{\nabla^{\mathbf{F}}_{\Sigma}, \Delta^{\mathbf{F}}_{\Sigma'}\}$*, then* $\mathcal{T}$ *must include one of the theory families* $\{\varnothing, b\}$, $\{01, b\}$, $\{1, b\}$. *Hence,* $\bigcap_{T\in\mathcal{T}} \widetilde{T} \leq \{01, b\} = \widetilde{T}'$.
- *If* $\Omega(T') = \Delta^{\mathbf{F}}$*, then* $\bigcap_{T\in\mathcal{T}} \Omega(T) = \Delta^{\mathbf{F}}$ *and* $\widetilde{T}' = \widetilde{\{1, b\}} = \{1, b\}$.
  - *If* $\{1, b\} \in \mathcal{T}$*, then* $\bigcap_{T\in\mathcal{T}} \widetilde{T} \leq \{1, b\} = \widetilde{T}'$;
  - *If* $\{1, b\} \notin \mathcal{T}$*, then* $\mathcal{T}$ *must include at least one member of each of the pairs*
  $$\{\varnothing, b\},\ \{01, b\} \quad \text{and} \quad \{1, \varnothing\},\ \{1, ab\}.$$
  Thus, $\bigcap_{T\in\mathcal{T}} \widetilde{T} \leq \{01, b\} \cap \{1, ab\} = \{1, b\} = \widetilde{T}'$.

On the other hand, consider $T = \{1, \varnothing\}$ *and* $T' = \{1, ab\}$. *We have*

$$\Omega(T) = \Omega(\{1, \varnothing\}) = \{\Delta^{\mathbf{F}}_{\Sigma}, \nabla^{\mathbf{F}}_{\Sigma'}\} = \Omega(\{1, ab\}) = \Omega(T'),$$

whereas

$$\overleftarrow{\widetilde{T}} = \widetilde{\{\varnothing, \varnothing\}} = \{01, ab\} \nleq T' = \widetilde{T}' = \overleftarrow{\widetilde{T}'}.$$

hence, $\mathcal{I}$ *is not roughly left c-reflective.*

We look, next, at the connections between rough c-reflectivity and rough reflectivity classes. Membership in a rough c-reflectivity class implies, in a straightforward way, membership in the corresponding rough reflectivity class.

Theorem 466 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is roughly right c-reflective, then it is roughly right reflective;*

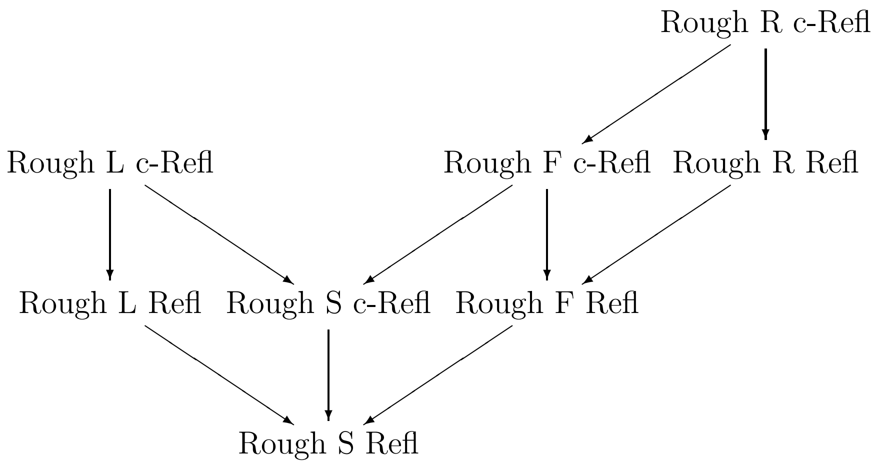
(b) If $\mathcal{I}$ *is roughly family c-reflective, then it is roughly family reflective;*

(c) If $\mathcal{I}$ *is roughly left c-reflective, then it is roughly left reflective;*

(d) If $\mathcal{I}$ *is roughly system c-reflective, then it is roughly system reflective.*

Proof: The property of being, e.g., roughly right reflective is a specialization of the property of being roughly right c-reflective, where one replaces the class $\mathcal{T}$ of theory families by the singleton $\{T\}$. The same holds for the remaining three types of rough reflectivity and rough c-reflectivity, respectively. ■

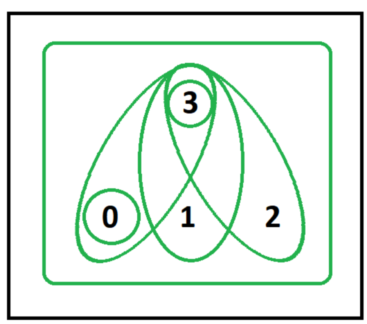
Theorem 466 establishes the mixed rough hierarchy depicted in the diagram.



To see that all classes in the hierarchy are different, we give an example of a π-institution satisfying all four rough reflectivity properties, which is not, however, roughly system c-reflective and, therefore, a fortiori, belongs to none of the four rough c-reflectivity classes.

Example 467 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with the single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3\}$*;*
- $N^\flat$ *is the trivial clone.*



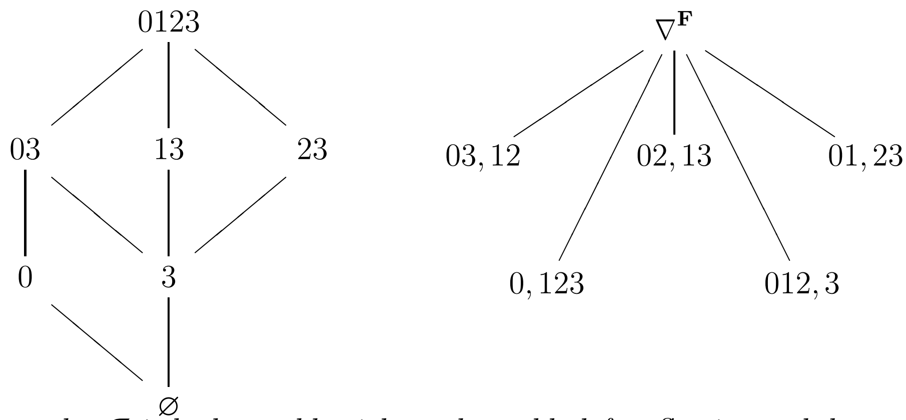
SEN(Σ)

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{0\}, \{3\}, \{0,3\}, \{1,3\}, \{2,3\}, \{0,1,2,3\}\}.$$

$\mathcal{I}$ *has seven theory families all of which are theory systems. It follows that the action of* $\overleftarrow{\ }$ *is trivial. Moreover, the only non-singleton rough equivalence class is the one consisting of* $\{\varnothing\}$ *and* $\{\{0,1,2,3\}\}$*.*

The lattice of theory families of $\mathcal{I}$ and the corresponding Leibniz congruence systems are given in the diagram.



We show that $\mathcal{I}$ is both roughly right and roughly left reflective and, hence, belongs to all four classes in the rough reflectivity hierarchy. Note that, since $\mathcal{I}$ is systemic, both rough reflectivity properties boil down to showing that, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$\Omega(T) \leq \Omega(T') \quad \textit{implies} \quad \widetilde{T} \leq \widetilde{T'}.$$

- *If $\Omega(T') = \nabla^{\mathbf{F}}$, then $T' = \{\varnothing\}$ or $T' = \{\{0,1,2,3\}\}$. Therefore, $\widetilde{T} \leq \{\{0,1,2,3\}\} = \widetilde{T'}$;*
- *If $\Omega(T') \neq \nabla^{\mathbf{F}}$, then, since $\Omega(T) \leq \Omega(T')$, we must have $T = T'$ and, hence, $\widetilde{T} \leq \widetilde{T'}$.*

On the other hand, we have

$$\Omega(\{03\}) \cap \Omega(\{3\}) = \{03, 12\} \cap \{012, 3\} = \{0, 12, 3\} \leq \{0, 123\} = \Omega(\{\{0\}\}),$$

whereas

$$\widetilde{\{03\}} \cap \widetilde{\{3\}} = \{03\} \cap \{3\} = \{3\} \nleq \{0\} = \widetilde{\{0\}}.$$

Hence, $\mathcal{I}$ is not roughly system c-reflective and, therefore, it belongs to none of the four rough c-reflectivity classes.

We explore, next, the connections between rough c-reflectivity and c-reflectivity classes. By analogy with the case of reflectivity and rough reflectivity (Theorem 432), we get that membership in a c-reflectivity class implies membership in the corresponding rough c-reflectivity class and, also, possession of theorems. Conversely, membership in a rough c-reflectivity class plus possession of theorems entails membership in the corresponding c-reflectivity class.

Theorem 468 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) *$\mathcal{I}$ is right/family c-reflective if and only if it is roughly right c-reflective and has theorems;*

(b) *$\mathcal{I}$ is right/family c-reflective if and only if it is roughly family c-reflective and has theorems;*

(c) *$\mathcal{I}$ is left c-reflective if and only if it is roughly left c-reflective and has theorems;*

(d) *$\mathcal{I}$ is system c-reflective if and only if it is roughly system c-reflective and has theorems.*

Proof:

(a) Suppose that $\mathcal{I}$ is right c-reflective. Then, by Proposition 243, it is right reflective. Hence, by Theorem 432, it has theorems. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Then, by right c-reflectivity, $\bigcap_{T\in\mathcal{T}} T \leq T'$. Since $\mathcal{I}$ has theorems, $\widetilde{T} = T$, for all $T \in \mathcal{T}$, and $\widetilde{T'} = T'$. Therefore, $\bigcap_{T\in\mathcal{T}} \widetilde{T} \leq \widetilde{T'}$ and $\mathcal{I}$ is roughly right c-reflective.

Assume, conversely, that $\mathcal{I}$ is roughly right c-reflective and has theorems. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Then, by rough right c-reflectivity, we get $\bigcap_{T\in\mathcal{T}} \widetilde{T} \leq \widetilde{T'}$. On the other hand, since $\mathcal{I}$ has theorems, $\widetilde{T} = T$, for all $T \in \mathcal{T}$, and $\widetilde{T'} = T'$. Therefore, $\bigcap_{T\in\mathcal{T}} T \leq T'$ and $\mathcal{I}$ is right c-reflective.

(b) Suppose that $\mathcal{I}$ is family c-reflective. Then, by Proposition 243, it is family reflective. Hence, by Theorem 432, it has theorems. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. Then, by family c-reflectivity, $\bigcap_{T\in\mathcal{T}} T \leq T'$. Since $\mathcal{I}$ has theorems, $\widetilde{T} = T$, for all $T \in \mathcal{T}$, and $\widetilde{T'} = T'$. Therefore, $\bigcap_{T\in\mathcal{T}} \widetilde{T} \leq \widetilde{T'}$ and $\mathcal{I}$ is roughly family c-reflective.

Assume, conversely, that $\mathcal{I}$ is roughly family c-reflective and has theorems. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. Then, by rough family c-reflectivity, we get $\bigcap_{T\in\mathcal{T}} \widetilde{T} \leq \widetilde{T'}$. On the other hand, since $\mathcal{I}$ has theorems, $\widetilde{T} = T$, for all $T \in \mathcal{T}$, and $\widetilde{T'} = T'$. Therefore, $\bigcap_{T\in\mathcal{T}} T \leq T'$ and $\mathcal{I}$ is family c-reflective.

(c) Suppose that $\mathcal{I}$ is left c-reflective. Then, by Proposition 243, it is left reflective. Hence, by Theorem 432, it has theorems. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. Then, by left reflectivity, $\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$. Since $\mathcal{I}$ has theorems, $\overleftarrow{\widetilde{T}} = \overleftarrow{T}$, for all $T \in \mathcal{T}$, and $\overleftarrow{\widetilde{T'}} = \overleftarrow{T'}$. Therefore, $\bigcap_{T\in\mathcal{T}} \overleftarrow{\widetilde{T}} \leq \overleftarrow{\widetilde{T'}}$ and $\mathcal{I}$ is roughly left c-reflective.

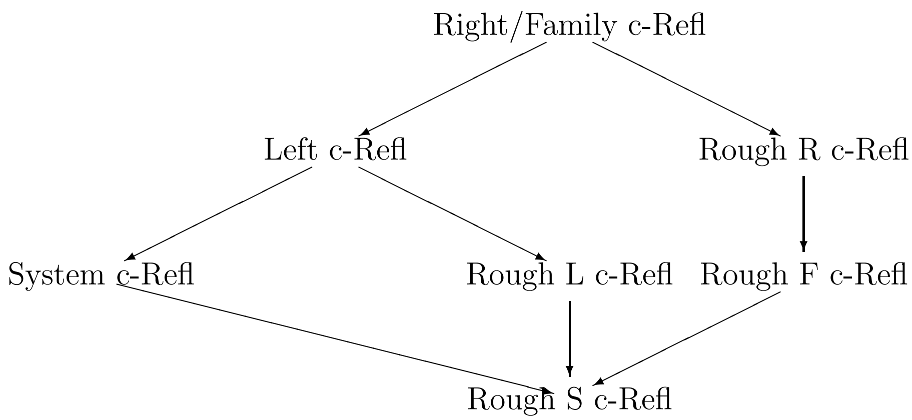
Assume, conversely, that $\mathcal{I}$ is roughly left c-reflective and has theorems. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. Then, by rough left c-reflectivity, we get $\bigcap_{T\in\mathcal{T}} \overleftarrow{\widetilde{T}} \leq \overleftarrow{\widetilde{T'}}$. On the other hand, since

$\mathcal{I}$ has theorems, $\widetilde{\overleftarrow{T}} = \overleftarrow{T}$, for all $T \in \mathcal{T}$, and $\widetilde{\overleftarrow{T'}} = \overleftarrow{T'}$. Therefore, $\bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$ and $\mathcal{I}$ is left c-reflective.

(d) Similar to Part (b).

■

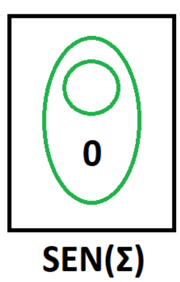
The work in Chapter 3, together with the work done in the present section and Theorem 468, reveal a hierarchy of c-reflectivity and rough c-reflectivity classes shown in the accompanying diagram.



To complete the demonstration that all classes in the depicted hierarchy are distinct we provide an example of a π-institution which belongs to all steps in the rough c-reflectivity hierarchy but possesses none of the four (gentle) c-reflectivity properties.

Example 469 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

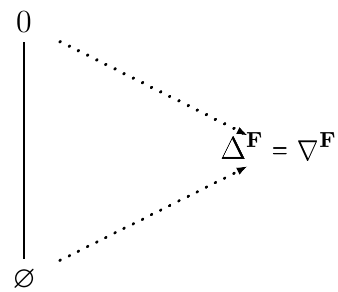
- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0\}$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{0\}\}.$$

$\mathcal{I}$ *is systemic and its lattice of theory families and corresponding Leibniz congruence systems are shown in the diagram.*



Note that $\widetilde{\{\{0\}\}} = \widetilde{\{\varnothing\}} = \{\{0\}\}$*, whence, trivially,* $\mathcal{I}$ *is both roughly right and roughly left c-reflective.*

On the other hand, since $\Omega(\{\{0\}\}) = \nabla^{\mathbf{F}} = \Omega(\{\varnothing\})$*, whereas* $\{\{0\}\} \nleq \{\varnothing\}$*,* $\mathcal{I}$ *is not system c-reflective and, hence, a fortiori,* $\mathcal{I}$ *has none of the four c-reflectivity properties.*

As was shown to be the case with the rough reflectivity properties in Theorem 434, the rough c-reflectivity properties transfer from the theory families/systems of a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ to all $\mathcal{I}$-filter families/systems on arbitrary $\mathbf{F}$-algebraic systems.

Theorem 470 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is roughly right c-reflective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $\bigcap_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(\overleftarrow{T}) \leq \Omega^{\mathcal{A}}(\overleftarrow{T'})$ *implies* $\bigcap_{T\in\mathcal{T}} \widetilde{T} \leq \widetilde{T'}$*;*

(b) $\mathcal{I}$ *is roughly family c-reflective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $\bigcap_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ *implies* $\bigcap_{T\in\mathcal{T}} \widetilde{T} \leq \widetilde{T'}$*;*

(c) $\mathcal{I}$ *is roughly left c-reflective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $\bigcap_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ *implies* $\bigcap_{T\in\mathcal{T}} \overleftarrow{\widetilde{T}} \leq \overleftarrow{\widetilde{T'}}$*;*

(d) $\mathcal{I}$ *is roughly system c-reflective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $\mathcal{T} \cup \{T'\} \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$*,* $\bigcap_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ *implies* $\bigcap_{T\in\mathcal{T}} \widetilde{T} \leq \widetilde{T'}$*.*

Proof:

(a) The "if" follows by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account that, by Lemma 51, $\mathrm{ThFam}(\mathcal{I}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F})$.

For the "only if", suppose that $\mathcal{I}$ is roughly right c-reflective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(\overleftarrow{T}) \le \Omega^{\mathcal{A}}(\overleftarrow{T'})$. Then $\alpha^{-1}(\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(\overleftarrow{T})) \le \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'}))$, whence, $\bigcap_{T \in \mathcal{T}} \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T})) \le \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'}))$. So, by Proposition 24, $\bigcap_{T \in \mathcal{T}} \Omega(\alpha^{-1}(\overleftarrow{T})) \le \Omega(\alpha^{-1}(\overleftarrow{T'}))$. By Lemma 6,

$$\bigcap_{T \in \mathcal{T}} \Omega(\overleftarrow{\alpha^{-1}(T)}) \le \Omega(\overleftarrow{\alpha^{-1}(T')}).$$

Since, by Lemma 51, $\{\alpha^{-1}(T) : T \in \mathcal{T}\} \cup \{\alpha^{-1}(T')\} \subseteq \mathrm{ThFam}(\mathcal{I})$, we get, by applying rough right c-reflectivity,

$$\bigcap_{T \in \mathcal{T}} \widetilde{\alpha^{-1}(T)} \le \widetilde{\alpha^{-1}(T')}.$$

Thus, by Theorem 377, $\bigcap_{T \in \mathcal{T}} \alpha^{-1}(\widetilde{T}) \le \alpha^{-1}(\widetilde{T'})$, i.e., $\alpha^{-1}(\bigcap_{T \in \mathcal{T}} \widetilde{T}) \le \alpha^{-1}(\widetilde{T'})$. Therefore, taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that $\bigcap_{T \in \mathcal{T}} \widetilde{T} \le \widetilde{T'}$.

(b) The "if" follows as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is roughly family c-reflective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')$. Then we get $\alpha^{-1}(\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)) \le \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$, whence, $\bigcap_{T \in \mathcal{T}} \alpha^{-1}(\Omega^{\mathcal{A}}(T)) \le \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. So, by Proposition 24, $\bigcap_{T \in \mathcal{T}} \Omega(\alpha^{-1}(T)) \le \Omega(\alpha^{-1}(T'))$. Since, by Lemma 51,

$$\{\alpha^{-1}(T) : T \in \mathcal{T}\} \cup \{\alpha^{-1}(T')\} \subseteq \mathrm{ThFam}(\mathcal{I}),$$

we get, by applying rough family c-reflectivity, $\bigcap_{T \in \mathcal{T}} \widetilde{\alpha^{-1}(T)} \le \widetilde{\alpha^{-1}(T')}$. Thus, by Theorem 377, $\bigcap_{T \in \mathcal{T}} \alpha^{-1}(\widetilde{T}) \le \alpha^{-1}(\widetilde{T'})$, i.e., $\alpha^{-1}(\bigcap_{T \in \mathcal{T}} \widetilde{T}) \le \alpha^{-1}(\widetilde{T'})$. Therefore, taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that $\bigcap_{T \in \mathcal{T}} \widetilde{T} \le \widetilde{T'}$.

(c) The "if" follows as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is roughly left c-reflective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')$. Then $\alpha^{-1}(\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)) \le \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$, whence, $\bigcap_{T \in \mathcal{T}} \alpha^{-1}(\Omega^{\mathcal{A}}(T)) \le \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. So, by Proposition 24, $\bigcap_{T \in \mathcal{T}} \Omega(\alpha^{-1}(T)) \le \Omega(\alpha^{-1}(T'))$. Since, by Lemma 51,

$$\{\alpha^{-1}(T) : T \in \mathcal{T}\} \cup \{\alpha^{-1}(T')\} \subseteq \mathrm{ThFam}(\mathcal{I}),$$

we get, by applying rough left c-reflectivity, $\bigcap_{T \in \mathcal{T}} \overleftarrow{\widetilde{\alpha^{-1}(T)}} \le \overleftarrow{\widetilde{\alpha^{-1}(T')}}$. Thus, by Lemma 6, $\bigcap_{T \in \mathcal{T}} \widetilde{\alpha^{-1}(\overleftarrow{T})} \le \widetilde{\alpha^{-1}(\overleftarrow{T'})}$. Hence, by Theorem 377,

$\bigcap_{T\in\mathcal{T}}\alpha^{-1}(\widetilde{\overleftarrow{T}}) \le \alpha^{-1}(\widetilde{\overleftarrow{T'}})$, i.e., $\alpha^{-1}(\bigcap_{T\in\mathcal{T}}\widetilde{\overleftarrow{T}}) \le \alpha^{-1}(\widetilde{\overleftarrow{T'}})$. Therefore, taking into account the surjectivity of $\langle F,\alpha\rangle$, we conclude that $\bigcap_{T\in\mathcal{T}}\widetilde{\overleftarrow{T}} \le \widetilde{\overleftarrow{T'}}$.

(d) Similar to Part (b).

■

Finally, we may recast the rough c-reflectivity classes in terms of complete order reflectivity of mappings from posets of classes of theory or filter families/systems into posets of congruence systems.

Recall that the collections $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ and $\widetilde{\mathrm{ThSys}}(\mathcal{I})$ may be ordered by setting, respectively, for all $T,T' \in \mathrm{ThFam}(\mathcal{I})$,

$$\widetilde{\lceil T\rceil} \le \widetilde{\lceil T'\rceil} \quad \text{iff} \quad \widetilde{T} \le \widetilde{T'}$$

and, for all $T,T' \in \mathrm{ThSys}(\mathcal{I})$,

$$\widetilde{\lfloor T\rfloor} \le \widetilde{\lfloor T'\rfloor} \quad \text{iff} \quad \widetilde{T} \le \widetilde{T'},$$

and that the corresponding ordered sets are denoted by $\widetilde{\mathbf{ThFam}}(\mathcal{I})$ and $\widetilde{\mathbf{ThSys}}(\mathcal{I})$, respectively.

Proposition 471 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. The following statements are equivalent:*

(a) $\mathcal{I}$ is roughly family c-reflective;

(b) $\Omega : \widetilde{\mathbf{ThFam}}(\mathcal{I}) \to \mathbf{ConSys}^*(\mathcal{I})$ is completely order reflecting;

(c) $\Omega^{\mathcal{A}} : \widetilde{\mathbf{FiFam}}^{\mathcal{I}}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is completely order reflecting, for every $\mathbf{F}$-algebraic system $\mathcal{A}$.

Similarly, for system c-reflectivity, we have

Proposition 472 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. The following statements are equivalent:*

(a) $\mathcal{I}$ is roughly system c-reflective;

(b) $\Omega : \widetilde{\mathbf{ThSys}}(\mathcal{I}) \to \mathbf{ConSys}^*(\mathcal{I})$ is completely order reflecting;

(c) $\Omega^{\mathcal{A}} : \widetilde{\mathbf{FiSys}}^{\mathcal{I}}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is completely order reflecting, for every $\mathbf{F}$-algebraic system $\mathcal{A}$.

6.9 Narrow Complete Reflectivity

In this section we study classes of π-institutions defined using complete reflectivity properties of the Leibniz operator restricted to $\mathrm{ThFam}^{\nleq}(\mathcal{I})$. We call those *narrow complete reflectivity* properties in analogy with the terminology adopted when differentiating rough reflectivity and narrow reflectivity classes.

Definition 473 (Narrow c-Reflectivity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is called* **narrowly family completely reflective***, or* **narrowly family c-reflective** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nleq}(\mathcal{I})$,
$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \quad \textit{implies} \quad \bigcap_{T \in \mathcal{T}} T \leq T';$$
- $\mathcal{I}$ *is called* **narrowly left completely reflective***, or* **narrowly left c-reflective** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nleq}(\mathcal{I})$,
$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \quad \textit{implies} \quad \bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'};$$
- $\mathcal{I}$ *is called* **narrowly right completely reflective***, or* **narrowly right c-reflective** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nleq}(\mathcal{I})$,
$$\bigcap_{T \in \mathcal{T}} \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'}) \quad \textit{implies} \quad \bigcap_{T \in \mathcal{T}} T \leq T';$$
- $\mathcal{I}$ *is called* **narrowly system completely reflective***, or* **narrowly system c-reflective** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^{\nleq}(\mathcal{I})$,
$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \quad \textit{implies} \quad \bigcap_{T \in \mathcal{T}} T \leq T'.$$

The narrow complete reflectivity properties have the following characterizations, paralleling those given for the narrow reflectivity classes, given in Proposition 438.

Proposition 474 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is narrowly family c-reflective if and only if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$ *implies* $\bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T'}$;

(b) $\mathcal{I}$ *is narrowly left c-reflective if and only if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$ *implies* $\bigcap_{T \in \mathcal{T}} \overleftarrow{\widetilde{T}} \leq \overleftarrow{\widetilde{T'}}$;

(c) *$\mathcal{I}$ is narrowly right c-reflective if and only if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\bigcap_{T \in \mathcal{T}} \Omega(\overleftarrow{\widetilde{T}}) \leq \Omega(\overleftarrow{\widetilde{T'}})$ implies $\bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T'}$;*

(d) *$\mathcal{I}$ is narrowly system c-reflective if and only if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $\{\widetilde{T} : T \in \mathcal{T}\} \cup \{\widetilde{T'}\} \subseteq \mathrm{ThSys}(\mathcal{I})$, $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$ implies $\bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T'}$.*

Proof: The proofs of the various parts mimic those of the corresponding parts for the narrow reflectivity properties, presented in detail in Proposition 438. Thus, we only do Part (b) in detail here, trusting that the reader may easily reproduce the other proofs.

Suppose that $\mathcal{I}$ is narrowly left c-reflective and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. Then $\{\widetilde{T} : T \in \mathcal{T}\} \cup \{\widetilde{T'}\} \subseteq \mathrm{ThFam}^{\lightning}(\mathcal{I})$ and, by Proposition 369, $\bigcap_{T \in \mathcal{T}} \Omega(\widetilde{T}) \leq \Omega(\widetilde{T'})$. Thus, by hypothesis, $\bigcap_{T \in \mathcal{T}} \overleftarrow{\widetilde{\widetilde{T}}} \leq \overleftarrow{\widetilde{\widetilde{T'}}}$.

Assume, conversely, that the asserted condition holds and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\lightning}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. Then, by hypothesis, $\bigcap_{T \in \mathcal{T}} \overleftarrow{\widetilde{\widetilde{T}}} \leq \overleftarrow{\widetilde{\widetilde{T'}}}$. Since, however, $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\lightning}(\mathcal{I})$, we get

$$\bigcap_{T \in \mathcal{T}} \overleftarrow{T} = \bigcap_{T \in \mathcal{T}} \overleftarrow{\widetilde{\widetilde{T}}} \leq \overleftarrow{\widetilde{\widetilde{T'}}} = \overleftarrow{T'}.$$

Therefore, $\mathcal{I}$ is narrowly left c-reflective. ∎

As was shown in Lemma 439, narrow family reflectivity implies exclusive systemicity. Since narrow family c-reflectivity implies narrow family reflectivity, it follows that it also implies exclusive systemicity.

Corollary 475 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is narrowly family c-reflective, then it is exclusively systemic.*

Proof: If $\mathcal{I}$ is narrowly family c-reflective, then it is, a fortiori, narrow family reflective, whence, by Lemma, 439, it is exclusively systemic. ∎

Similarly, the fact that narrow right c-reflectivity strengthens narrow right reflectivity, implies immediately the following

Corollary 476 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is narrowly right c-reflective, then it is narrowly stable.*

Proof: Since narrow right c-reflectivity implies narrow right reflectivity, this follows from Corollary 440. ∎

We establish, next the *narrow c-reflectivity hierarchy*. The following proposition forms an analog of Proposition 441, which dealt with the narrow reflectivity hierarchy.

Proposition 477 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(c) If $\mathcal{I}$ *is narrowly right c-reflective, then it is narrowly family c-reflective;*

(b) If $\mathcal{I}$ *is narrowly family c-reflective, then it is narrowly left c-reflective;*

(c) If $\mathcal{I}$ *is narrowly left c-reflective, then it is narrowly system c-reflective.*

Proof:

(a) Suppose that $\mathcal{I}$ is narrowly right c-reflective and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nsim}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. By Corollary 476, $\mathcal{I}$ is narrowly stable. Now we obtain $\bigcap_{T \in \mathcal{T}} \Omega(\overleftarrow{T}) = \bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') = \Omega(\overleftarrow{T'})$. Hence, by narrow right c-reflectivity, $\bigcap \mathcal{T} \leq T'$. Hence, $\mathcal{I}$ is narrowly family c-reflective.

(b) Suppose that $\mathcal{I}$ is narrowly family c-reflective and consider $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nsim}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. Then, by hypothesis, $\bigcap_{T \in \mathcal{T}} T \leq T'$, whence, by Lemma 1, $\overleftarrow{\bigcap_{T \in \mathcal{T}} T} \leq \overleftarrow{T'}$. Thus, by Lemma 3, $\bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$. Thus, $\mathcal{I}$ is narrowly left c-reflective.

(c) Suppose that $\mathcal{I}$ is narrowly left c-reflective and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^{\nsim}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. Then, by hypothesis, we get $\bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$. Therefore, since $\mathcal{T} \cup \{T'\}$ is a collection of theory systems, $\bigcap_{T \in \mathcal{T}} T \leq T'$ and, hence, $\mathcal{I}$ is narrowly system c-reflective.

■

We have now established the following **narrow complete reflectivity hierarchy** of π-institutions.

Narrowly R c-Reflective

↓

Narrowly F c-Reflective

↓

Narrowly L c-Reflective

↓

Narrowly S c-Reflective

We give an additional result pertaining to the hierarchy of narrow complete reflectivity properties depicted in the diagram. It forms an analog of Proposition 442, establishing a similar result for the narrow reflectivity hierarchy.

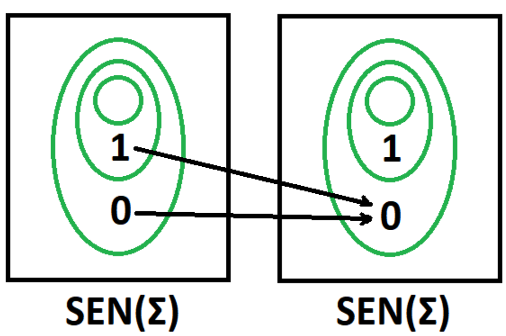
Proposition 478 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is narrowly system c-reflective and narrowly systemic, then it is narrowly right c-reflective.*

Proof: Suppose $\mathcal{I}$ is narrowly system c-reflective and narrowly systemic. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nmid}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. By narrow systemicity, $T = \overleftarrow{T}$, for all $T \in \mathcal{T}$, and $T' = \overleftarrow{T'}$. Hence, on the one hand, $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$ and, on the other, $\{T : T \in \mathcal{T}\} \cup \{T'\} \subseteq \mathrm{ThSys}^{\nmid}(\mathcal{I})$. Thus, by narrow system c-reflectivity, $\bigcap_{T \in \mathcal{T}} T \leq T'$. Thus, $\mathcal{I}$ is narrowly right c-reflective. ■

We present three examples to show that all inclusions established between the narrow c-reflectivity classes and shown in the preceding diagram are indeed proper inclusions. The first example depicts a π-institution which is narrowly family c-reflective but not narrowly right c-reflective. This shows that the class of narrowly right c-reflective π-institutions constitutes a proper subclass of the class of narrowly family c-reflective π-institutions.

Example 479 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

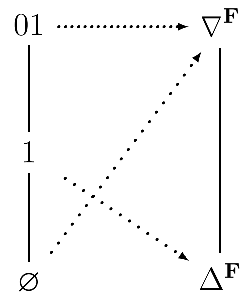
- $\mathbf{Sign}^\flat$ *is the category with the single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$*;*
- $N^\flat$ *is the trivial clone, consisting of the projections only.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0, 1\}\}.$$

$\mathcal{I}$ *has three theory families,* $\varnothing$*,* $\{\{1\}\}$ *and* $\{\{0, 1\}\}$*, but only two theory systems,* $\varnothing$ *and* $\{\{0, 1\}\}$*. The lattice of theory families of* $\mathcal{I}$ *and the corresponding Leibniz congruence systems are given in the diagram.*



Since the Leibniz operator is an isomorphism on $\mathrm{ThFam}^{\nmid}(\mathcal{I})$, $\mathcal{I}$ *is narrowly family c-reflective. On the other hand,* $\{\{1\}\},\{\{0,1\}\} \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$ *and*

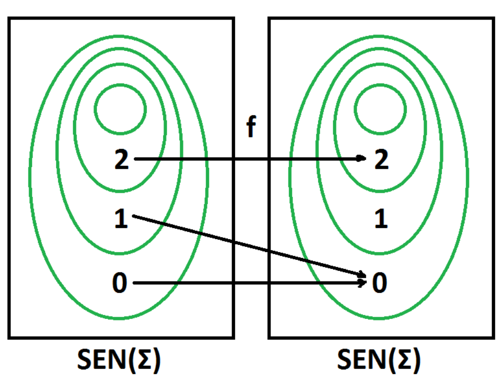
$$\Omega(\overleftarrow{\{\{1\}\}}) = \Omega(\{\varnothing\}) = \nabla^{\mathbf{F}} = \Omega(\{\{0,1\}\}) = \Omega(\overleftarrow{\{\{0,1\}\}}),$$

but $\{\{1\}\} \neq \{\{0,1\}\}$. *Therefore,* $\mathcal{I}$ *is not narrowly right injective and, a fortiori, it fails to be narrowly right c-reflective.*

The next example depicts a π-institution which is narrowly left c-reflective but not narrowly family c-reflective. This shows that the class of narrowly family c-reflective π-institutions is a proper subclass of the class of narrowly left c-reflective π-institutions.

Example 480 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

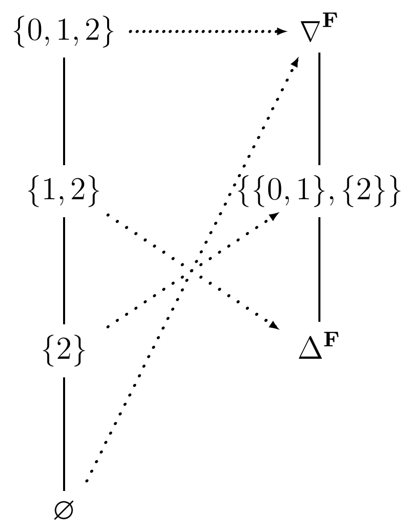
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$, $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$;
- $N^\flat$ *is the trivial clone.*



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\varnothing, \{2\}, \{1,2\}, \{0,1,2\}\}.$$

$\mathcal{I}$ has four theory families, but only three theory systems, namely $\varnothing$, $\{2\}$ and $\{0,1,2\}$. Moreover, clearly, $\mathrm{ThFam}^{\not\natural}(\mathcal{I}) = \{\{2\},\{1,2\},\{0,1,2\}\}$. The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right:



To show that $\mathcal{I}$ is narrowly left c-reflective, let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\not\natural}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \le \Omega(T')$. We distinguish the following cases, based on the value of $\Omega(T')$:

- *If $\Omega(T') = \nabla^{\mathbf{F}}$, then $T' = \mathrm{SEN}^\flat$. Hence, we get*
$$\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \le \mathrm{SEN}^\flat = \overleftarrow{\mathrm{SEN}^\flat} = \overleftarrow{T'};$$
- *If $\Omega(T') = \{\{0,1\},\{2\}\}$, then $T' = \{2\}$. Then, by hypothesis, $\{2\} \in \mathcal{T}$ or $\{1,2\} \in \mathcal{T}$. Hence, in either case, $\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \le \{2\} = T' = \overleftarrow{T'}$;*
- *If $\Omega(T') = \Delta^{\mathbf{F}}$, then $T' = \{1,2\}$ and, by hypothesis, $\{1,2\} \in \mathcal{T}$. Hence, in this case as well, $\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \le \overleftarrow{\{1,2\}} = \overleftarrow{T'}$.*

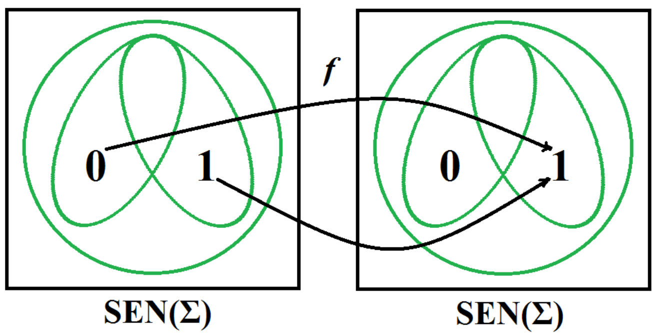
Thus, $\mathcal{I}$ is narrowly left c-reflective.

On the other hand, for $T = \{1,2\}$ and $T' = \{2\}$, even though $\Omega(T) \le \Omega(T')$, we get $T \nleq T'$, whence $\mathcal{I}$ fails to be narrowly family reflective and, hence, a fortiori, it is not narrow family c-reflective.

We finish the sequence of examples by presenting a narrowly system c-reflective π-institution which fails to be narrowly left c-reflective. This example shows that narrowly left c-reflective π-institutions form a proper subclass of the class of narrowly system c-reflective π-institutions.

Example 481 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 1$ *and* $\mathrm{SEN}^\flat(f)(1) = 1$;
- $N^\flat$ *is the trivial clone.*

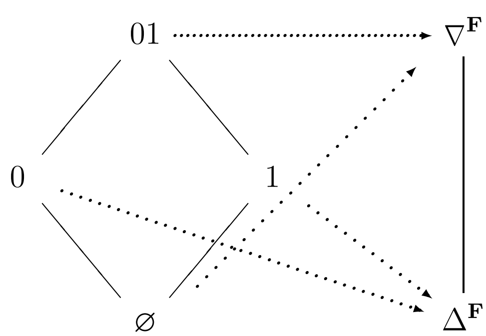


Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\varnothing, \{0\}, \{1\}, \{0,1\}\}$.

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$.

T	$\overleftarrow{T}$
$\varnothing$	$\varnothing$
$\{0\}$	$\varnothing$
$\{1\}$	$\{1\}$
$\{0,1\}$	$\{0,1\}$

The lattice of theory families and the corresponding Leibniz congruence systems are depicted below.



It is obvious from the diagram that the Leibniz operator is an isomorphism on $\mathrm{ThSys}^{\nmid}(\mathcal{I})$*. Therefore,* $\mathcal{I}$ *is narrowly system c-reflective. On the other hand, for* $T = \{\{0\}\}$*,* $T' = \{\{1\}\}$*, both members of* $\mathrm{ThFam}^{\nmid}(\mathcal{I})$*, we have* $\Omega(T) = \Omega(T') = \Delta^{\mathbf{F}}$*, whereas* $\overleftarrow{T} = \{\varnothing\} \neq \{\{1\}\} = \overleftarrow{T'}$*. Therefore,* $\mathcal{I}$ *is not narrowly left injective and, a fortiori, it fails to be narrowly left c-reflective.*

We turn now to the relationships between corresponding classes of the rough complete reflectivity and the narrow complete reflectivity hierarchies. These parallel the ones already established between the rough reflectivity and narrow reflectivity classes.

Using the characterization in Part (a) of Proposition 474, we can immediately see that the two types of family complete reflectivity coincide.

Corollary 482 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is roughly family c-reflective if and only if it is narrowly family c-reflective.*

Proof: Part (a) of Proposition 474. ■

As was the case with rough and narrow reflectivity properties, the relationships between the remaining classes are more involved. Starting with the two left complete reflectivity classes, we show that the class of narrowly left c-reflective π-institutions is not included in the class of roughly left c-reflective π-institutions. This is accomplished by constructing a π-institution which is narrowly left c-reflective but not roughly left c-reflective.

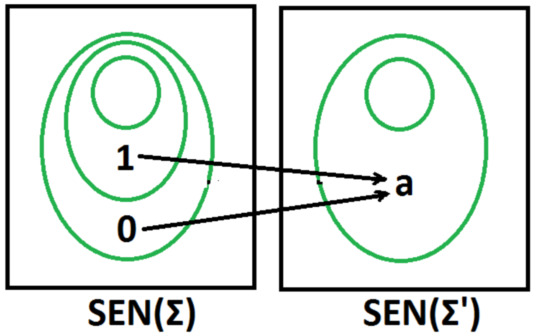
Example 483 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$*,* $\mathrm{SEN}^\flat(\Sigma') = \{a\}$ *and* $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = a$*;*
- $N^\flat$ *is the trivial clone.*

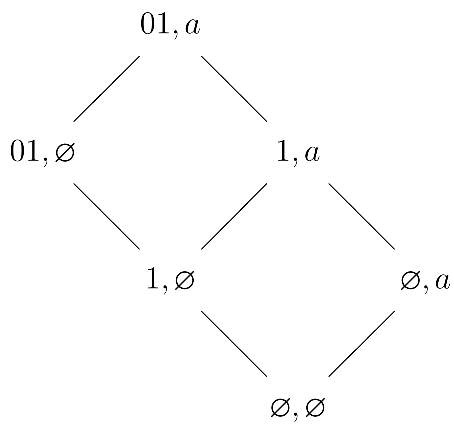
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0, 1\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{a\}\}.$$

Clearly, there are six theory families in $\mathrm{ThFam}(\mathcal{I})$*, only four of which are theory systems, and only two of which are in* $\mathrm{ThFam}^{\nmid}(\mathcal{I})$*. The lattice of*



theory families is shown in the diagram:



To see that $\mathcal{I}$ is narrowly left c-reflective, let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\not\,}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. We reasons by cases, depending on the value of $\Omega(T')$:

- *If $\Omega(T') = \nabla^{\mathbf{F}}$, then $T' = \mathrm{SEN}^\flat$. So we get $\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \leq \mathrm{SEN}^\flat = \overleftarrow{\mathrm{SEN}^\flat} = \overleftarrow{T'}$;*
- *If $\Omega(T') = \Delta^{\mathbf{F}}$, then $T' = \{1, a\}$ and, by hypothesis, we must have $T' \in \mathcal{T}$. Thus, $\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$.*

We conclude that $\mathcal{I}$ is narrowly left c-reflective.

On the other hand, consider $T = \{1, \varnothing\}$ and $T' = \{1, a\}$. We have $\Omega(1, \varnothing) = \{\Delta^{\mathbf{F}}_{\Sigma}, \nabla^{\mathbf{F}}_{\Sigma'}\} = \Omega(1, a)$, but

$$\overleftarrow{\widetilde{1,\varnothing}} = \widetilde{\varnothing,\varnothing} = \{01, a\} \not\leq \{1, a\} = \widetilde{1,a} = \overleftarrow{\widetilde{1,a}}.$$

This proves that $\mathcal{I}$ is not roughly left reflective and, hence, a fortiori, it fails to be roughly left c-reflective.

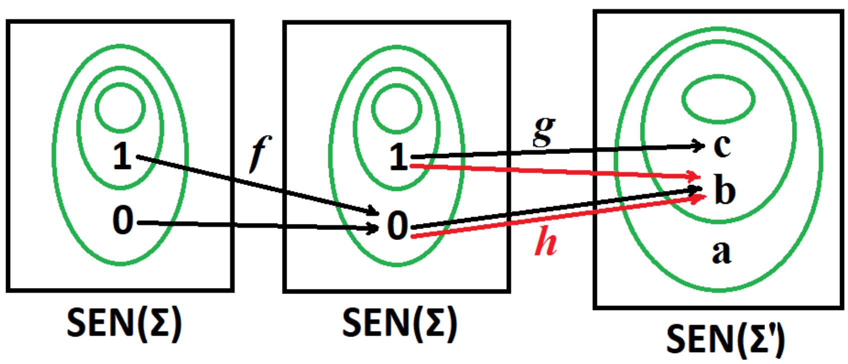
We exhibit, next a π-institution that is roughly left c-reflective, while it fails to be narrowly left c-reflective. Combined with Example 483, this will show that the two left complete reflectivity classes, rough and narrow, are incomparable from the point of view of inclusion.

Example 484 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and three nonidentity morphisms* $f : \Sigma \to \Sigma$ *and* $g, h : \Sigma \to \Sigma'$, *such that* $f \circ f = f$, $g \circ f = h$ *and* $h \circ f = h$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a, b, c\}$, $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = 0$, $\mathrm{SEN}^\flat(g)(0) = b$, $\mathrm{SEN}^\flat(g)(1) = c$ *and* $\mathrm{SEN}^\flat(h)(0) = \mathrm{SEN}^\flat(h)(1) = b$;
- $N^\flat$ *is the clone generated by a single binarry natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$, *whose components are defined by the following tables:*

$\sigma^\flat_\Sigma$	0	1
0	0	1
1	1	1

$\sigma^\flat_{\Sigma'}$	a	b	c
a	a	a	c
b	a	b	c
c	c	c	c



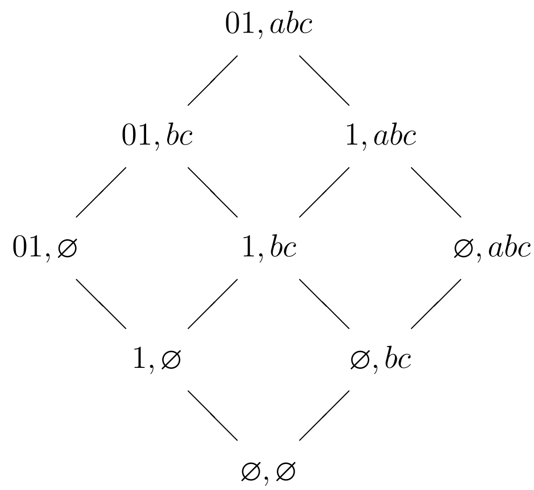
It is not difficult, albeit slightly tedious, to check that this is a well-defined natural transformation. We summarize the checking in the accompanying table.

(x, y)	$f(\sigma^\flat_\Sigma(x,y)) = \sigma^\flat_\Sigma(f(x), f(y))$	$g(\sigma^\flat_\Sigma(x,y)) = \sigma^\flat_{\Sigma'}(g(x), g(y))$	$h(\sigma^\flat_\Sigma(x,y)) = \sigma^\flat_{\Sigma'}(h(x), h(y))$
$(0,0)$	$0 = 0$	$b = b$	$b = b$
$(0,1)$	$0 = 0$	$c = c$	$b = b$
$(1,0)$	$0 = 0$	$c = c$	$b = b$
$(1,1)$	$0 = 0$	$c = c$	$b = b$

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b, c\}, \{a, b, c\}\}.$$

Clearly, there are nine theory families in $\mathrm{ThFam}(\mathcal{I})$*, five of which are theory systems, and four of which are in* $\mathrm{ThFam}^{\nmid}(\mathcal{I})$*. The lattice of theory families is shown in the diagram:*



The action of $\overleftarrow{\ }$ *on theory families is given in the following table.*

T	$\overleftarrow{T}$	T	$\overleftarrow{T}$
$01, abc$	$01, abc$	$\varnothing, abc$	$\varnothing, abc$
$01, bc$	$01, bc$	$1, \varnothing$	$\varnothing, \varnothing$
$1, abc$	$\varnothing, abc$	$\varnothing, bc$	$\varnothing, bc$
$01, \varnothing$	$\varnothing, \varnothing$	$\varnothing, \varnothing$	$\varnothing, \varnothing$
$1, bc$	$\varnothing, bc$		

The table below provides the Leibniz congruence systems associated with the theory families of $\mathcal{I}$*.*

T	$\Omega(T)$
$\{01, abc\}, \{01, \varnothing\}, \{\varnothing, abc\}, \{\varnothing, \varnothing\}$	$\nabla^{\mathbf{F}}$
$\{1, abc\}, \{1, \varnothing\}$	$\{\Delta^{\mathbf{F}}_{\Sigma}, \nabla^{\mathbf{F}}_{\Sigma'}\}$
$\{01, bc\}, \{1, bc\}, \{\varnothing, bc\}$	$\Delta^{\mathbf{F}}$

To see that $\mathcal{I}$ *is roughly left c-reflective, suppose that* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$*, such that* $\bigcap_{T \in \mathcal{T}} \Omega(T) \le \Omega(T')$*. We separate cases depending on* $\Omega(T')$*.*

- *If* $\Omega(T') = \Delta^{\mathbf{F}}$*, then, by hypothesis, at least one among* $\{01, bc\}$*,* $\{1, bc\}$*,* $\{\varnothing, b\}$ *must be in* $\mathcal{T}$*. But, then, we get* $\bigcap_{T \in \mathcal{T}} \widetilde{\overleftarrow{T}} \le \{01, bc\} = \widetilde{\overleftarrow{T'}}$*;*

- *If* $\Omega(T') = \{\Delta^{\mathbf{F}}_{\Sigma}, \nabla^{\mathbf{F}}_{\Sigma'}\}$ *or* $\Omega(T') = \nabla^{\mathbf{F}}$*, then* $\bigcap_{T \in \mathcal{T}} \widetilde{\overleftarrow{T}} \le \{01, abc\} = \widetilde{\overleftarrow{T'}}$*.*

On the other hand, for $T = \{01, bc\}$ *and* $T' = \{1, bc\}$*, we get* $\Omega(T) = \Delta^{\mathbf{F}} = \Omega(T')$*, whereas* $\overleftarrow{T} = \{01, bc\} \nleq \{\varnothing, bc\} = \overleftarrow{T'}$*. Therefore,* $\mathcal{I}$ *is not narrowly left c-reflective.*

We turn, next to the relationship between the two kinds of right c-reflectivity. We show, first, that rough right c-reflectivity implies narrow right c-reflectivity, a direct analog of Proposition 449, which established the corresponding result for the two right reflectivity classes.

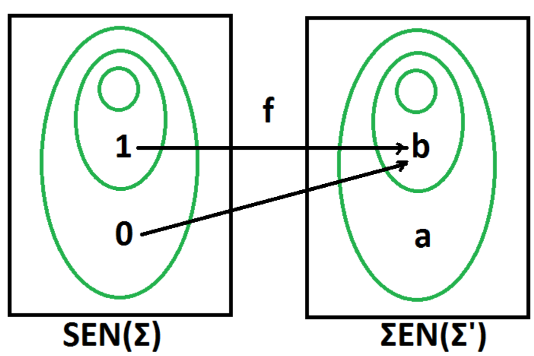
Proposition 485 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly right c-reflective, then it is narrowly right c-reflective.*

Proof: Suppose $\mathcal{I}$ is roughly right reflective and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nmid}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. By rough right reflectivity, we get that $\bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T}'$. Since, however, $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nmid}(\mathcal{I})$, we get $\bigcap_{T \in \mathcal{T}} T = \bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T}' = T'$. Therefore, $\mathcal{I}$ is narrowly right c-reflective. ■

The converse, on the other hand, does not hold in general, as the following example demonstrates.

Example 486 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

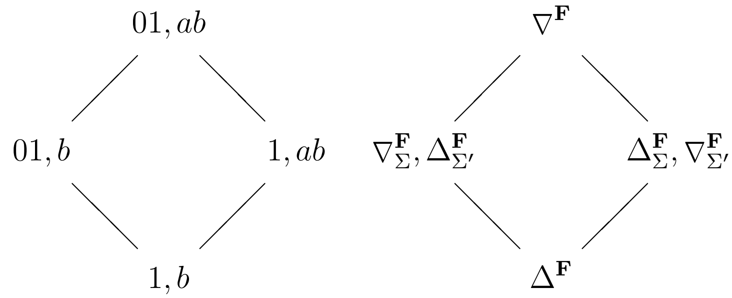
- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$*,* $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = b$*,* $\mathrm{SEN}^\flat(f)(1) = b$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0, 1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a, b\}\}.$$

Clearly, there are only four theory families in $\mathrm{ThFam}^{\nmid}(\mathcal{I})$*, all of which are theory systems. Their lattice together with the associated Leibniz congruence* *systems are shown in the diagram:*



From this diagram and the fact that all theory families in $\mathrm{ThFam}^{\nmid}(\mathcal{I})$ *are theory systems, we see that, for all* $T, T' \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$,

$$\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'}) \quad \textit{iff} \quad \Omega(T) \leq \Omega(T') \quad \textit{iff} \quad T \leq T'.$$

Therefore, $\mathcal{I}$ *is indeed narrowly right c-reflective.*

On the other hand, consider $T = \{01, ab\}$ *and* $T' = \{1, \varnothing\}$*. Then we have*

$$\Omega(\overleftarrow{T}) = \Omega(01, ab) = \nabla^{\mathbf{F}} = \Omega(\overline{\varnothing}) = \Omega(\overleftarrow{T'}),$$

whereas $\widetilde{01, ab} = \{01, ab\} \nleq \{1, ab\} = \widetilde{1, \varnothing}$. *This shows that* $\mathcal{I}$ *is not roughly right reflective and, hence, a fortiori, it fails to be roughly right c-reflective.*

Finally, we look at system complete reflectivity. We show that rough system c-reflectivity implies narrow system c-reflectivity, but that the converse implication fails in general.

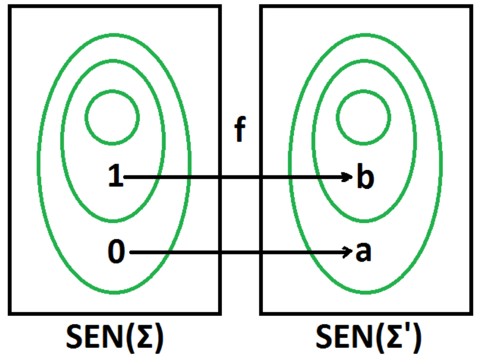
Proposition 487 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is roughly system c-reflective, then it is narrowly system c-reflective.*

Proof: Suppose $\mathcal{I}$ is roughly system c-reflective and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^{\nmid}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. Then, by rough system c-reflectivity, $\bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T}'$. However, since $\mathcal{T} \cup \{T'\} \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, we get $\bigcap_{T \in \mathcal{T}} T = \bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T}' = T'$. Therefore, $\mathcal{I}$ is narrowly system c-reflective. ■

We present an example of a π-institution that is narrowly system c-reflective but not roughly system c-reflective. This, combined with Proposition 487, shows that the class of narrowly system c-reflective π-institutions properly contains the class of roughly system c-reflective π-institutions.

Example 488 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique morphism* $f : \Sigma \to \Sigma'$;

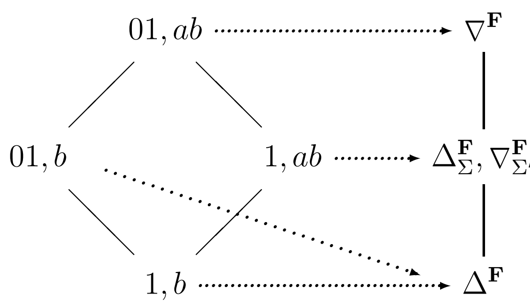


- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$;
- $N^\flat$ *is the trivial clone.*

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a,b\}\}.$$

There are only four theory families in $\mathrm{ThFam}^{\nleq}(\mathcal{I})$*, all of which except for* $\{01,b\}$ *are theory systems. Their lattice together with the associated Leibniz congruence systems are shown in the diagram:*

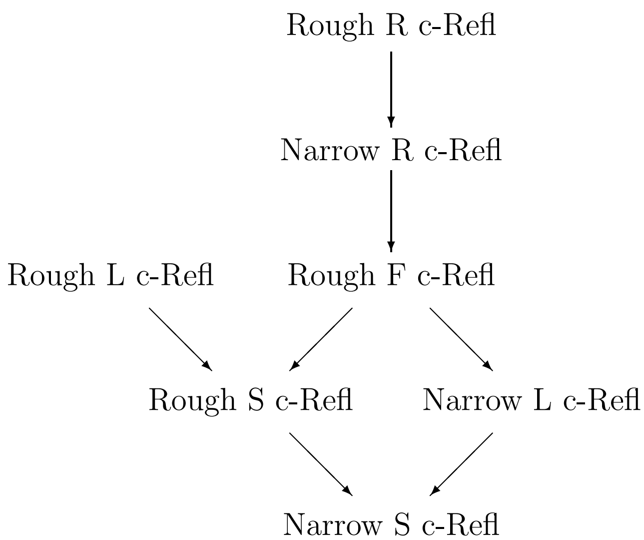


To see that $\mathcal{I}$ *is narrowly system c-reflective, let* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^{\nleq}(\mathcal{I})$*, such that* $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$*. We distinguish three cases, depending on the value of* $\Omega(T')$*:*

- *If* $\Omega(T') = \nabla^{\mathbf{F}}$*, then* $T' = \mathrm{SEN}^\flat$*. Hence,* $\bigcap_{T\in\mathcal{T}} T \leq \mathrm{SEN}^\flat = T'$*;*
- *If* $\Omega(T') = \{\Delta^{\mathbf{F}}_\Sigma, \nabla^{\mathbf{F}}_{\Sigma'}\}$*, then* $T' = \{1, ab\}$*, whence, by hypothesis,* $T' \in \mathcal{T}$ *or* $\{1,b\} \in \mathcal{T}$*. In either case,* $\bigcap_{T\in\mathcal{T}} T \leq \{1,ab\} = T'$*;*
- *If* $\Omega(T') = \Delta^{\mathbf{F}}$*, then* $T' = \{1,b\}$ *and, hence, by hypothesis,* $T' \in \mathcal{T}$*, which shows that* $\bigcap_{T\in\mathcal{T}} T \leq T'$*.*

On the other hand, consider $T = \{\varnothing, b\}$*,* $T' = \{1,b\} \in \mathrm{ThSys}(\mathcal{I})$*. Even though* $\widetilde{T} = \{01,b\} \nleq \{1,b\} = \widetilde{T}'$*, we have* $\Omega(T) = \Delta^{\mathbf{F}} = \Omega(T')$*. Hence,* $\mathcal{I}$ *is not roughly system reflective and, hence, a fortiori, it is not roughly system c-reflective.*

The results obtained and the counterexamples presented, thus far, reveal the following mixed hierarchy of rough and narrow c-reflectivity classes of π-institutions, paralleling the one presented for rough and narrow reflectivity properties.



We have already used in the context of the preceding examples the fact that a narrow c-reflectivity property implies the corresponding narrow reflectivity property, since the latter is a special case of the former in which $\mathcal{T}$ is taken to be a singleton. These observations are formalized in the following

Proposition 489 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is narrowly family c-reflective, then it is narrowly family reflective;*

(b) If $\mathcal{I}$ *is narrowly left c-reflective, then it is narrowly left reflective;*

(c) If $\mathcal{I}$ *is narrowly right c-reflective, then it is narrowly right reflective;*

(d) If $\mathcal{I}$ *is narrowly system c-reflective, then it is narrowly system reflective.*

Proof: All four reflectivity properties are special cases of the corresponding c-reflectivity properties, in which $\mathcal{T}$ is taken to be a singleton collection of theory families. ∎

Turning to the relationships between narrow c-reflectivity classes and corresponding c-reflectivity classes, we prove a theorem, analogous to Theorem 454, asserting that ordinary c-reflectivity is equivalent to narrow c-reflectivity in the presence of theorems.

Theorem 490 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is family c-reflective if and only if it is narrowly family c-reflective and has theorems;*

(b) $\mathcal{I}$ *is left c-reflective if and only if it is narrowly left c-reflective and has theorems;*

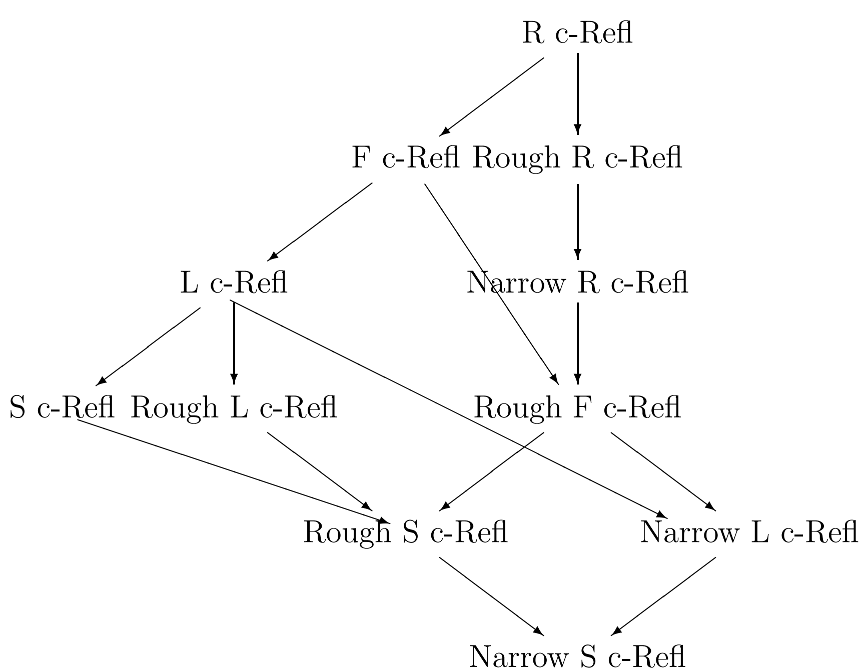
(c) $\mathcal{I}$ *is right c-reflective if and only if it is narrowly right c-reflective and has theorems;*

(d) $\mathcal{I}$ *is system c-reflective if and only if it is narrowly system c-reflective and has theorems.*

Proof: By Theorem 468, if $\mathcal{I}$ has one of the four complete reflectivity properties, then it has theorems. Moreover, by the same theorem, a complete reflectivity property implies the corresponding rough complete reflectivity property and, by Corollary 482, Proposition 485 and Proposition 487, each implies the corresponding narrow complete reflectivity property except in the case of left complete reflectivity, where (as actually in all other cases, as well) one can easily see directly, that left c-reflectivity implies narrow left c-reflectivity, since the defining condition of the latter is a special case of that of the former.

All converses are also easily verified, since, in the presence of theorems, $\mathrm{ThFam}^{\nmid}(\mathcal{I}) = \mathrm{ThFam}(\mathcal{I})$ and $\mathrm{ThSys}^{\nmid}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$, which makes the four defining conditions for the narrow c-reflectivity classes identical with the corresponding conditions for the ordinary c-reflectivity classes. ■

We now have the following hierarchy, paralleling the mixed reflectivity and narrow reflectivity hierarchy, given previously.



The narrow complete reflectivity properties transfer from the theory families/systems of a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ to all $\mathcal{I}$-filter families/systems on arbitrary $\mathbf{F}$-algebraic systems. This result forms an analog of Theorem 455, which applied to narrow reflectivity classes.

Theorem 491 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is narrowly right c-reflective if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ and all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}\nmid}(\mathcal{A})$, $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(\overleftarrow{T}) \leq \Omega^{\mathcal{A}}(\overleftarrow{T'})$ implies $\bigcap_{T\in\mathcal{T}} T \leq T'$;

(b) $\mathcal{I}$ is narrowly family c-reflective if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ and all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}\nmid}(\mathcal{A})$, $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ implies $\bigcap_{T\in\mathcal{T}} T \leq T'$;

(c) $\mathcal{I}$ is narrowly left c-reflective if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ and all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}\nmid}(\mathcal{A})$, $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ implies $\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$;

(d) $\mathcal{I}$ is narrowly system c-reflective if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ and all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiSys}^{\mathcal{I}\nmid}(\mathcal{A})$, $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ implies $\bigcap_{T\in\mathcal{T}} T \leq T'$.

Proof: The proof follows the steps of the proofs of the various parts of Theorem 455. We do Part (a) in detail to give a flavor of what is involved.

The "if" follows by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$ and taking into account that $\mathrm{ThFam}^{\nmid}(\mathcal{I}) = \mathrm{FiFam}^{\mathcal{I}\nmid}(\mathcal{F})$, by Lemmas 51 and 376.

For the "only if", suppose that $\mathcal{I}$ is narrowly right c-reflective and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}\nmid}(\mathcal{A})$, such that $\bigcap_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(\overleftarrow{T}) \leq \Omega^{\mathcal{A}}(\overleftarrow{T'})$. Then $\alpha^{-1}(\bigcap_{T\in\mathcal{T}} \Omega^{\mathcal{A}}(\overleftarrow{T})) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'}))$. Thus, $\bigcap_{T\in\mathcal{T}} \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T})) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'}))$. So, by Proposition 24, $\bigcap_{T\in\mathcal{T}} \Omega(\alpha^{-1}(\overleftarrow{T})) \leq \Omega(\alpha^{-1}(\overleftarrow{T'}))$. Hence, by Lemma 6, $\bigcap_{T\in\mathcal{T}} \Omega(\overleftarrow{\alpha^{-1}(T)}) \leq \Omega(\overleftarrow{\alpha^{-1}(T')})$. Since, by Lemmas 51 and 376, $\{\alpha^{-1}(T) : T \in \mathcal{T}\} \cup \{\alpha^{-1}(T')\} \subseteq \mathrm{ThFam}^{\nmid}(\mathcal{I})$, we get, by applying narrow right c-reflectivity, $\bigcap_{T\in\mathcal{T}} \alpha^{-1}(T) \leq \alpha^{-1}(T')$ or, equivalently, $\alpha^{-1}(\bigcap_{T\in\mathcal{T}} T) \leq \alpha^{-1}(T')$. This yields, taking into account the surjectivity of $\langle F, \alpha\rangle$, $\bigcap_{T\in\mathcal{T}} T \leq T'$. ∎

We finally recast narrow complete reflectivity in terms of the complete order reflectivity of mappings from posets of theory or filter families/systems into posets of congruence systems. The following results form analogs of Propositions 456 and 457, respectively, addressing complete reflectivity instead of reflectivity properties.

Proposition 492 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. The following statements are equivalent:*

(a) $\mathcal{I}$ *is narrowly family c-reflective;*

(b) $\Omega : \mathrm{ThFam}^{\nleq}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is completely order reflecting;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}\nleq}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is completely order reflecting, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$.

Similarly, for system c-reflectivity, we have

Proposition 493 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is narrowly system c-reflective;*

(b) $\Omega : \mathrm{ThSys}^{\nleq}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is completely order reflecting;*

(c) $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}\nleq}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is completely order reflecting, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$.

6.10 Availability of Theorems

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall that, by convention, if $\mathcal{I}$ has theorems, then, for every $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathcal{I}$ has a Σ-theorem, i.e., there exists $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\varnothing)$.

Recall, also, from our work in the present chapter, that all levels of the injectivity, reflectivity and complete reflectivity hierarchies imply the existence of theorems and that, moreover, any rough injectivity, rough reflectivity or rough complete reflectivity property, complemented with the existence of theorems, implies the corresponding (gentle) injectivity, reflectivity or complete reflectivity property, respectively. In other words, insisting on existence of theorems causes all pairs of rough and gentle properties to collapse to a single class.

In this section, due to the importance of the property of "having theorems", we give a few more results characterizing that property.

It turns out that existence of theorems is tantamount to the injectivity of the local Frege operator λ.

Theorem 494 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *has theorems if and only if* λ *is injective.*

Proof: Suppose, first, that $\mathcal{I}$ has theorems. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\lambda(T) = \lambda(T')$. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$. Since $\mathcal{I}$ has theorems, there exists $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$. Then $t \in T_\Sigma$ and, therefore, $\langle \phi, t \rangle \in \lambda_\Sigma(T)$. By hypothesis, $\langle \phi, t \rangle \in \lambda_\Sigma(T')$. But, clearly, $t \in T'_\Sigma$. Hence

$\phi \in T'_\Sigma$. We conclude that $T \leq T'$ and, by symmetry, $T = T'$. Thus, λ is injective.

Assume, conversely, that $\mathcal{I}$ does not have theorems. Then, we have $\varnothing, \mathrm{SEN}^\flat \in \mathrm{ThFam}(\mathcal{I})$, with $\varnothing \neq \mathrm{SEN}^\flat$, whereas $\lambda(\varnothing) = \lambda(\mathrm{SEN}^\flat) = \nabla^{\mathbf{F}}$. Therefore, λ is not injective. ■

It turns out that existence of theorems is also equivalent to both the injectivity and the c-reflectivity of the local Lindenbaum operator $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}$ on all $\mathbf{F}$-algebraic systems.

Theorem 495 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following conditions are equivalent:*

(i) $\mathcal{I}$ *has theorems;*

(ii) $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}$ *is injective, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*;*

(iii) $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}$ *is completely reflective, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

Proof:

(i)⇒(iii) Assume that $\mathcal{I}$ has theorems and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, be an $\mathbf{F}$-algebraic system, $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that

$$\bigcap_{T \in \mathcal{T}} \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \leq \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T').$$

Let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in \bigcap_{T \in \mathcal{T}} T_\Sigma$. By hypothesis and the surjectivity of $\langle F, \alpha \rangle$, there exists $t \in C^{\mathcal{I},\mathcal{A}}_\Sigma(\varnothing)$. Then, we have

$$C^{\mathcal{I},\mathcal{A}}_\Sigma(\bigcap_{T \in \mathcal{T}} T_\Sigma, \phi) = C^{\mathcal{I},\mathcal{A}}_\Sigma(\bigcap_{T \in \mathcal{T}} T_\Sigma) = C^{\mathcal{I},\mathcal{A}}_\Sigma(\bigcap_{T \in \mathcal{T}} T_\Sigma, t).$$

Thus, we get

$$\langle \phi, t \rangle \in \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(\bigcap_{T \in \mathcal{T}} T) \leq \bigcap_{T \in \mathcal{T}} \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T) \leq \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T').$$

We conclude that

$$\begin{array}{lll} \phi & \in & C^{\mathcal{I},\mathcal{A}}_\Sigma(T'_\Sigma, \phi) \quad \text{(inflationarity)} \\ & = & C^{\mathcal{I},\mathcal{A}}_\Sigma(T'_\Sigma, t) \quad (\langle \phi, t \rangle \in \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T')) \\ & = & T'_\Sigma. \quad (t \in C^{\mathcal{I},\mathcal{A}}_\Sigma(\varnothing)) \end{array}$$

Therefore, $\bigcap_{T \in \mathcal{T}} T \leq T'$ and $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}$ is completely reflective.

(iii)⇒(ii) Complete reflectivity implies injectivity.

(ii)⇒(i) Finally, suppose that $\mathcal{I}$ does not have theorems. We let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be the trivial $\mathbf{F}$-algebraic system, with single signature object $*$ and singleton $\mathrm{SEN}(*) = \{0\}$. Since $\mathcal{I}$ does not have theorems, both $\varnothing$ and SEN are $\mathcal{I}$-filter families of $\mathcal{A}$. Now we have

$$\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\varnothing) = \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\mathrm{SEN}) = \nabla^{\mathcal{A}}.$$

Hence, the Leibniz operator $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}$ is not injective.

■

The property of having theorems clearly transfers from a π-institution to all its gmatrix families.

Theorem 496 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *has theorems if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T \neq \overline{\varnothing}$.

Proof: The right-to-left inclusion follows by considering the algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. For the converse, assume $\mathcal{I}$ has theorems and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Let $\Sigma \in |\mathbf{Sign}^\flat|$. Then, there exists $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$. By definition, $\alpha_\Sigma(t) \in T_{F(\Sigma)}$. Hence, $T \neq \overline{\varnothing}$. ■

Note that an alterrnative way of expressing the assertion of Theorem 496 is to say that $\mathcal{I}$ has theorems if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, the π-institution $\langle \mathbf{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ has theorems.

Chapter 7

The Semantic Leibniz Hierarchy: Under the Bottom II

7.1 Introduction

In this chapter we continue the study of properties lying below properties in the bottom half of the classical Leibniz hierarchy [64, 86]. The underlying motivation is identical to that presented in the Introduction to, and governing the studies presented in, Chapter 6. Briefly, we note that, when one studies protoalgebraicity, no π-institution that is not almost inconsistent and does not have theorems can be considered. This is because such a π-institution has a theory family $T \in \mathrm{ThFam}^{\not\emptyset}(\mathcal{I})$, i.e., with all its components nonempty, for which $\overline{\varnothing} \leq T$, whereas $\Omega(T) \leq \nabla^{\mathbf{F}} = \Omega(\overline{\varnothing})$. Consequently, to incorporate nontrivial π-institutions without theorems in studies involving monotonicity properties of the Leibniz operator, one would have to devise ways to bypass, or otherwise suitably handle, theory families with one or more empty components. For properties involving reflectivity, which were handled in Chapter 6, this was done in the context of sentential logics in [87] (see, also, [89]). Here, we undertake a study similar to that presented in Chapter 6, but, instead of injectivity, reflectivity and complete reflectivity properties, we focus on monotonicity and complete monotonicity (c-monotonicity) properties.

In Section 7.2, we introduce some weakened versions of stability which serve in formalizing some of the properties studied later in the chapter. Recall from Section 3.2 that a π-institution $\mathcal{I}$ is *stable* if, for every theory family T of $\mathcal{I}$, $\Omega(\overleftarrow{T}) = \Omega(T)$. A first weakening is obtained by restricting the scope of the quantifier to theory families with all components nonempty. The ensuing property is termed *narrow stability*. A further weakening applies the condition only to those theory families T with all components nonempty which, in addition, satisfy that $\overleftarrow{T}$ has all its components nonempty. The resulting concept is termed *exclusive stability*. By definition, stability implies narrow stability, which implies exclusive stability and, as it turns out, both implications are actually strict.

In Section 7.3, we study *rough monotonicity properties*. These are the product of combining monotonicity properties with rough equivalence, introduced in Section 6.2. Rough equivalence formalizes an attempt at overcoming the hurdle imposed by theory families with empty components. Recall that two theory families are *roughly equivalent* if, whenever they differ at some signature Σ, one has Σ-component $\varnothing$ and the other $\mathrm{SEN}^\flat(\Sigma)$. Recall, also, that, given a theory family T, $\widetilde{T}$ denotes its *rough companion*, which results from T by replacing each of its empty Σ-components by $\mathrm{SEN}^\flat(\Sigma)$. Clearly $\widetilde{T}$ is roughly equivalent to T and, moreover, it is the largest theory family in the rough equivalence class $\widetilde{[T]}$ of T. All roughly equivalent theory families have identical Leibniz congruence systems. A π-institution $\mathcal{I}$ is called *roughly family monotone* if, for all theory families $T, T' \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{T} \leq \widetilde{T'}$ implies $\Omega(T) \leq \Omega(T')$. *Rough left monotonicity* results by replacing T and T' in the hypothesis by $\overleftarrow{T}$ and $\overleftarrow{T'}$, respectively. *Rough right monotonicity* is the result of the same replacement performed in the conclusion instead. *Rough system monotonicity* stipulates that $\widetilde{T} \leq \widetilde{T'}$ implies $\Omega(T) \leq \Omega(T')$ hold for all theory systems T and T'. Rough left monotonicity implies both rough family and rough right monotonicity, and each of the latter two implies the system version. Additionally, rough left monotonicity is equivalent to the conjunction of rough system monotonicity and stability. Protoalgebraicity (which, recall from Section 3.3, names the equivalent notions of left and family monotonicity) implies rough left monotonicity. Prealgebraicity (naming the equivalent notions of right and system monotonicity), on the other hand, implies rough right monotonicity. But these interrelationships may be tied further, subject to some additional mild hypotheses. Namely, for non-almost inconsistent π-institutions, protoalgebraicity is equivalent to rough left or rough family monotonicity, coupled with availability of theorems. Moreover, for π-institutions possessing a theory family $T \neq \mathrm{SEN}^\flat$, with $\overleftarrow{T} \neq \overline{\varnothing}$, prealgebraicity is equivalent to rough right or rough system monotonicity, couple with availability of theorems. All four rough monotonicity properties transfer. E.g., a π-institution $\mathcal{I}$ is roughly right monotone if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $\mathcal{I}$-filter families $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{T} \leq \widetilde{T'}$ implies $\Omega^{\mathcal{A}}(\overleftarrow{T}) \leq \Omega^{\mathcal{A}}(\overleftarrow{T'})$. Finally, it is possible to recast rough family and rough system monotonicity in terms of the Leibniz operator viewed as a mapping from $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ and $\widetilde{\mathrm{ThSys}}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$. The property one imposes is monotonicity, where, for rough equivalence classes $\widetilde{[T]}$, $\widetilde{[T']}$ in $\widetilde{\mathrm{ThFam}}(\mathcal{I})$, e.g., the order $\widetilde{[T]} \leq \widetilde{[T']}$ is the one induced by comparing the maximum elements $\widetilde{T} \leq \widetilde{T'}$ in the complete lattice of theory families of $\mathcal{I}$.

In Section 7.4, we look at *narrow monotonicity properties*. Narrowness is an alternative approach to roughness in dealing with theory families having one or more empty components. It literally bypasses theory families with empty components by altogether ignoring them and applying the relevant monotonicity conditions on the collections $\mathrm{ThFam}^{\nleq}(\mathcal{I})$ and $\mathrm{ThSys}^{\nleq}(\mathcal{I})$ of theory families and systems, respectively, all of whose components are nonempty. Accordingly, we say that a π-institution $\mathcal{I}$ is *narrowly family monotone* if, for all $T, T' \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, $T \leq T'$ implies $\Omega(T) \leq \Omega(T')$. In *narrow left monotonicity* T, T' in the hypothesis, are replaced by $\overleftarrow{T}$, $\overleftarrow{T'}$, respectively, and the same substitution is applied in the conclusion, instead, for *narrow right monotonicity*. *Narrow system monotonicity* imposes the same condition as the family version, but restricts its scope to $T, T' \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$. Narrow left monotonicity implies narrow family monotonicity, which, in turn, implies narrow system monotonicity. The latter is also a consequence of narrow right monotonicity. The left version also implies exclusive stability, whereas the weakest version, i.e., narrow system monotonicity, supplemented by narrow systemicity, introduced in Section 6.3, implies both the left and right versions. Protoalgebraicity implies narrow left monotonicity and prealgebraicity implies narrow right monotonicity. As in the case of rough monotonicity properties, these connections may be strengthened under some fairly mild hypotheses. More precisely, for non almost inconsistent π-institutions, protoalgebraicity is equivalent to narrow left or narrow family monotonicity, augmented by existence of theorems. Similarly, for π-institutions possessing a theory system different from $\overline{\varnothing}$ and $\mathrm{SEN}^\flat$, prealgebraicity is equivalent to narrow right or narrow system monotonicity, coupled with existence of theorems. Of course, having introduced two seemingly different approaches to handling empty theory family components, it is of central importance to investigate the relations between rough monotonicity and narrow monotonicity classes. Narrow family monotonicity turns out to be equivalent to rough family monotonicity, whereas, with regards to the three remaining versions, each of the rough properties implies the corresponding narrow property. All four narrow monotonicity properties transfer. The section concludes with characterizations of narrow family and narrow system monotonicity in terms of the Leibniz operator viewed as a mapping from $\mathrm{ThFam}^{\nleq}(\mathcal{I})$ and $\mathrm{ThSys}^{\nleq}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

In Section 7.5, we look at *rough complete monotonicity* (c-monotonicity) properties. These concepts, in analogy with the extension of monotonicity to the c-monotonicity properties of Section 3.4, extend rough monotonicity properties by allowing arbitrary unions on the right-hand side of the relevant inequalities. A π-institution $\mathcal{I}$ is called *roughly family c-monotone* if, for every collection $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T}$ implies $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. In *rough left c-monotonicity* the hypothesis is replaced by $\overleftarrow{\widetilde{T'}} \leq \bigcup_{T \in \mathcal{T}} \overleftarrow{\widetilde{T}}$ and, in *rough right c-monotonicity*, the conclusion is replaced by $\Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$. The *system version* imposes the same condition as the family version, but restricts it on collections $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$. Here, the only inclusions are those establishing that each of the rough left, family and right c-monotonicity classes form a subclass of the class of roughly system c-monotone π-institutions. Rough left c-monotonicity is equivalent to rough system c-monotonicity plus stability. Under stability, rough family c-monotonicity and rough right c-monotonicity are equivalent and, furthermore, under rough systemicity, the entire hierarchy collapses to a single class. From the definitions, it is obvious that each version of rough c-monotonicity implies the corresponding version of rough monotonicity, since, the definition of the latter specializes that of the former. Moreover, each version of c-monotonicity implies the corresponding version of rough c-monotonicity. As far as closer ties, analogous to those detailed for rough monotonicity classes in Section 7.3, for non almost inconsistent π-institutions, $\mathcal{I}$ is family (left, respectively) c-monotone if and only if it is rough;y family (left, respectively) c-monotone and has theorems. Along similar lines, for $\mathcal{I}$ having a theory family $T \neq \mathrm{SEN}^\flat$, such that $\overleftarrow{T} \neq \overline{\varnothing}$, $\mathcal{I}$ is system (right, respectively) c-monotone if and only if it is roughly system (right, respectively) c-monotone and has theorems. All four rough c-monotonicity properties transfer and, as was the case with rough monotonicity, the family and system versions have characterizations in terms of Ω seen as a mapping from $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ and $\widetilde{\mathrm{ThSys}}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

The same extension that led from rough monotonicity to rough c-monotonicity properties may be applied to narrow monotonicity properties and leads to *narrow c-monotonicity* properties, which constitute the objects of study in Section 7.6. A π-institution $\mathcal{I}$ is called *narrowly family c-monotone* if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nleq}(\mathcal{I})$, $T' \leq \bigcup_{T \in \mathcal{T}} T$ implies $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. Once more, the *left version* results by replacing in the hypothesis all theory families by their arrow counterparts, and, similarly for the *right version*, except that the replacement is applied in the conclusion of the implication instead. The *system version* applies the same condition as the family version, but restricts its scope on collections of theory systems in $\mathrm{ThSys}^{\nleq}(\mathcal{I})$. As was the case with rough c-monotonicity in Section 7.5, the only three implications assert that each of the narrow left, family and right c-monotonicity properties implies narrow system c-monotonicity. Each version of c-monotonicity implies its narrow c-monotonicity counterpart. It turns out that rough family c-monotonicity is equivalent to narrow family c-monotonicity. On the other hand, for the remaining three versions, each rough c-monotonicity variant implies the corresponding narrow c-monotonicity variant. Of course, due to the specializations in the relevant definitions, each narrow c-monotonicity property implies the corresponding narrow monotonicity property. All four narrow c-monotonicity properties transfer. Finally, it is the case here as well, that the family and the system versions can be characterized in terms of the Leibniz operator viewed as a mapping from $\mathrm{ThFam}^{\nleq}(\mathcal{I})$ and $\mathrm{ThSys}^{\nleq}(\mathcal{I})$, respectively, to $\mathrm{ConSys}^*(\mathcal{I})$.

7.2 Narrow and Exclusive Stability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall that $\mathcal{I}$ is called *stable* if, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\Omega(\overleftarrow{T}) = \Omega(T).$$

Recall, also, that, in Section 6.5, we defined *narrow stability*, a concept that proved handy in demonstrating that the narrow right properties studied there implied the corresponding narrow family properties. We recall that definition and look at an additional concept weakening stability. These two notions aim at bypassing theory families with empty components.

Definition 497 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is* **narrowly stable** *if, for all* $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$,

$$\Omega(\overleftarrow{T}) = \Omega(T);$$

- $\mathcal{I}$ *is* **exclusively stable** *if, for all* $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$*, such that* $\overleftarrow{T} \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$*,*
$$\Omega(\overleftarrow{T}) = \Omega(T).$$

It is clear that stability is the strongest of the three properties followed by narrow stability and exclusive stability, which is the weakest of the three.

Proposition 498 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is stable, then it is narrowly stable;*

(b) If $\mathcal{I}$ *is narrowly stable, then it is exclusively stable.*

Proof: It suffices to note that each property is a specialization of the one immediately dominating it in strength. ■

Thus, the following linear **stability hierarchy** is established.

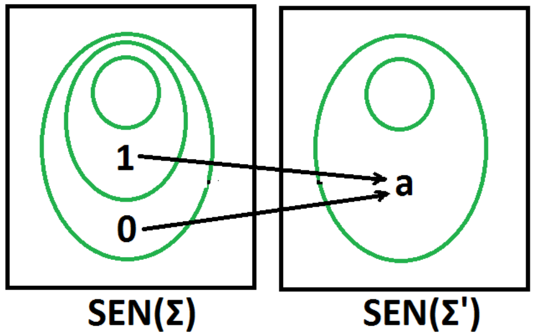
$$\begin{array}{c} \text{Stable} \\ \downarrow \\ \text{Narrowly Stable} \\ \downarrow \\ \text{Exclusively Stable} \end{array}$$

It is not difficult to see that all three classes are different. The following example provides a π-institution that is narrowly stable but not stable, showing that stable π-institutions form a proper subclass of the class consisting of the narrowly stable ones.

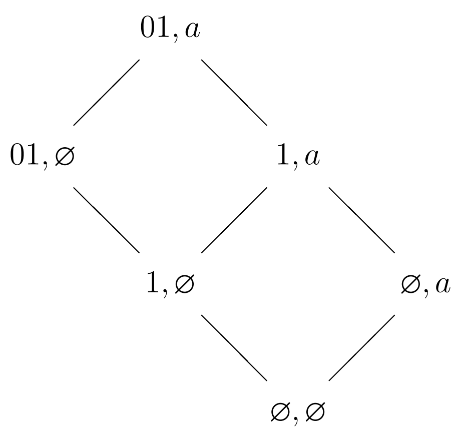
Example 499 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a\}$ *and* $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = a$*;*
- $N^\flat$ *is the trivial clone.*

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*
$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{a\}\}.$$



Clearly, there are six theory families in $\mathrm{ThFam}(\mathcal{I})$*, only four of which are theory systems, and only two of which are in* $\mathrm{ThFam}^{\nmid}(\mathcal{I})$*. The lattice of theory families is shown in the diagram:*



Since $\mathrm{ThFam}^{\nmid}(\mathcal{I}) = \{\{1, a\}, \{01, a\}\}$ *and* $\overleftarrow{\{1, a\}} = \{1, a\}$ *and* $\overleftarrow{\{01, a\}} = \{01, a\}$*, we get that* $\mathcal{I}$ *is narrowly systemic and, hence, a fortiori, also narrowly stable. On the other hand, consider* $T = \{\{1\}, \varnothing\}$*. We have*

$$\Omega(\overleftarrow{\{1, \varnothing\}}) = \Omega(\overline{\varnothing}) = \nabla^{\mathbf{F}} \neq \{\Delta^{\mathbf{F}}_{\Sigma}, \nabla^{\mathbf{F}}_{\Sigma'}\} = \Omega(\{1, \varnothing\}),$$

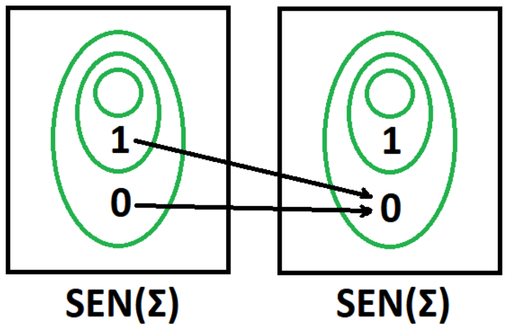
whence $\mathcal{I}$ *is not stable.*

Finally, we give an example of an exclusively stable π-institution which, however, fails to be narrowly stable. This shows that the inclusion of the class of narrowly stable π-institutions into the class of exclusively stable ones is also proper.

Example 500 *Let* $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat} \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^{\flat}$ *is the category with the single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*

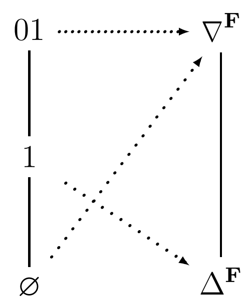
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$*;*
- $N^\flat$ *is the trivial clone, consisting of the projections only.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\}.$$

$\mathcal{I}$ *has three theory families* $\{\varnothing\}$, $\{\{1\}\}$ *and* $\{\{0,1\}\}$, *but only two theory systems,* $\{\varnothing\}$ *and* $\{\{0,1\}\}$. *The lattice of theory families of* $\mathcal{I}$ *and the corresponding Leibniz congruence systems are given in the diagram.*



The only theory family $T \in \mathrm{ThFam}^{\sharp}(\mathcal{I})$, *such that* $\overleftarrow{T} \in \mathrm{ThSys}^{\sharp}(\mathcal{I})$ *is* $\{\{0,1\}\}$. *Moreover,* $\overleftarrow{\{0,1\}} = \{0,1\}$, *whence we get that* $\mathcal{I}$ *is exclusively stable. On the other hand, for* $\{\{1\}\} \in \mathrm{ThFam}^{\sharp}(\mathcal{I})$, *we get*

$$\Omega(\overleftarrow{\{1\}}) = \Omega(\overline{\varnothing}) = \nabla^{\mathbf{F}} \neq \Delta^{\mathbf{F}} = \Omega(\{1\}).$$

Therefore, $\mathcal{I}$ *is not narrowly stable.*

7.3 Rough Monotonicity

In this section we exploit the notion of rough equivalence, which was studied in some detail in Section 6.2, to introduce and study classes of π-institutions defined using monotonicity properties of the Leibniz operator applied on rough equivalence classes.

Definition 501 (Rough Monotonicity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is called* **roughly family monotone** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$*,*
$$\widetilde{T} \leq \widetilde{T'} \quad \textit{implies} \quad \Omega(T) \leq \Omega(T').$$
- $\mathcal{I}$ *is called* **roughly left monotone** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$*,*
$$\widetilde{\overleftarrow{T}} \leq \widetilde{\overleftarrow{T'}} \quad \textit{implies} \quad \Omega(T) \leq \Omega(T').$$
- $\mathcal{I}$ *is called* **roughly right monotone** *if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$*,*
$$\widetilde{T} \leq \widetilde{T'} \quad \textit{implies} \quad \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'}).$$
- $\mathcal{I}$ *is called* **roughly system monotone** *if, for all* $T, T' \in \mathrm{ThSys}(\mathcal{I})$*,*
$$\widetilde{T} \leq \widetilde{T'} \quad \textit{implies} \quad \Omega(T) \leq \Omega(T').$$

Next we look into establishing the *rough monotonicity hierarchy* of π-institutions. We show, first, that rough left monotonicity implies stability.

Lemma 502 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly left monotone, then it is stable.*

Proof: Suppose $\mathcal{I}$ is roughly left monotone and let $T \in \mathrm{ThFam}(\mathcal{I})$. Since $\overleftarrow{\overleftarrow{T}} = \overleftarrow{T}$, we get that $\widetilde{\overleftarrow{\overleftarrow{T}}} = \widetilde{\overleftarrow{T}}$. Thus, by rough left monotonicity, $\Omega(\overleftarrow{T}) = \Omega(T)$. Hence, $\mathcal{I}$ is stable. ■

Lemma 502 leads to the conclusion that, under rough left monotonicity, the properties of rough family monotonicity and rough right monotonicity are equivalent.

Corollary 503 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly left monotone, then it is roughly family monotone if and only if it is roughly right monotone.*

Proof: Suppose $\mathcal{I}$ is roughly left monotone. Then, by Lemma 502, it is stable. Now note that rough family monotonicity is equivalent to the condition that, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,
$$\widetilde{T} \leq \widetilde{T'} \quad \text{implies} \quad \Omega(T) \leq \Omega(T'),$$
which, by stability, is equivalent to, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,
$$\widetilde{T} \leq \widetilde{T'} \quad \text{implies} \quad \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'}),$$
and this is equivalent, by definition, to rough right monotonicity. ■

Next we show that rough left monotonicity implies rough family monotonicity.

Proposition 504 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly left monotone, then it is roughly family monotone.*

Proof: Suppose $\mathcal{I}$ is roughly left monotone, i.e., for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{\widetilde{T}} \leq \overleftarrow{\widetilde{T'}}$ implies $\Omega(T) \leq \Omega(T')$. Let $X, Y \in \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{X} \leq \widetilde{Y}$. If $\overleftarrow{\widetilde{X}} \leq \overleftarrow{\widetilde{Y}}$, then, by rough left monotonicity, $\Omega(X) \leq \Omega(Y)$. So, assume that $\overleftarrow{\widetilde{X}} \nleq \overleftarrow{\widetilde{Y}}$, that is, that there exists $P \in |\mathbf{Sign}^\flat|$, such that $\overleftarrow{\widetilde{X}}_P \nsubseteq \overleftarrow{\widetilde{Y}}_P$. At the same time, since $\widetilde{X} \leq \widetilde{Y}$, we have that $\widetilde{X}_P \subseteq \widetilde{Y}_P$. This implies that $X_P \subseteq Y_P$ or $Y_P = \varnothing$. However, if $Y_P = \varnothing$, then, we would also have $\overleftarrow{Y}_P = \varnothing$, whence $\overleftarrow{\widetilde{X}}_P \subseteq \mathrm{SEN}^\flat(P) = \overleftarrow{\widetilde{Y}}_P$, contradicting our assumption. Hence, we conclude that $X_P \subseteq Y_P$. Now, based on $\overleftarrow{\widetilde{X}}_P \nsubseteq \overleftarrow{\widetilde{Y}}_P$, we distinguish two possibilities, $\overleftarrow{X}_P \nsubseteq \overleftarrow{Y}_P$ or $\overleftarrow{X}_P = \varnothing$.

- Suppose $X_P \subseteq Y_P$ and $\overleftarrow{X}_P \nsubseteq \overleftarrow{Y}_P$. Then, there exists $Q \in |\mathbf{Sign}^\flat|$ and $P \stackrel{f}{\to} Q$, such that $X_Q \nsubseteq Y_Q$. Since, however, $\widetilde{X}_Q \subseteq \widetilde{Y}_Q$, we would have $Y_Q = \varnothing$. This, combined with the fact that $\overleftarrow{\widetilde{X}}_P \nsubseteq \overleftarrow{\widetilde{Y}}_P$ implies that $\overleftarrow{Y}_P \neq \varnothing$, yield that there cannot exist $f : P \to Q$, a contradiction.

- So it must be the case that $X_P \subseteq Y_P$ and $\overleftarrow{X}_P = \varnothing$. Since $\overleftarrow{\widetilde{X}}_P \nsubseteq \overleftarrow{\widetilde{Y}}_P$, we must have $\overleftarrow{Y}_P \neq \varnothing$ and $\overleftarrow{Y}_P \neq \mathrm{SEN}^\flat(P)$. Note that it is not possible to have both $X_P = \overleftarrow{X}_P$ and $Y_P = \overleftarrow{Y}_P$. If that had been the case, we would have $X_P = \varnothing$ and $Y_P \neq \varnothing$ or $\mathrm{SEN}^\flat(P)$, whence $\widetilde{X}_P \nsubseteq \widetilde{Y}_P$, which contradicts the hypothesis. So, we must have $\varnothing = \overleftarrow{X}_P \subsetneq X_P$ or $\overleftarrow{Y}_P \subsetneq Y_P$.

  - Assume, first, that $\varnothing = \overleftarrow{X}_P \subsetneq X_P \subseteq Y_P \neq \mathrm{SEN}^\flat(P)$. Define $Z = \{Z_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

  $$Z_\Sigma = \begin{cases} \varnothing, & \text{if } \Sigma \neq P \\ X_P, & \text{if } \Sigma = P \end{cases}.$$

    Then, we have $Z \leq X$, whence $\overleftarrow{Z} \leq \overleftarrow{X}$ and, hence, $\overleftarrow{Z} = \overline{\varnothing} = \overleftarrow{\overline{\varnothing}}$. So, whereas $\overleftarrow{\widetilde{Z}} = \overleftarrow{\widetilde{\overline{\varnothing}}}$, $\Omega(Z) \neq \nabla^{\mathbf{F}} = \Omega(\overline{\varnothing})$. This contradicts rough right monotonicity.

  - Suppose, next, that $\varnothing = \overleftarrow{X}_P = X_P$ and $\overleftarrow{Y}_P \subsetneq Y_P$. We already know that $\overleftarrow{Y}_P \neq \varnothing$. Moreover, since $\widetilde{X}_P \subseteq \widetilde{Y}_P$, we must have $Y_P = \mathrm{SEN}^\flat(P)$. Now we define $Z = \{Z_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by setting

  $$Z_\Sigma = \begin{cases} \varnothing, & \text{if } \Sigma \neq P \\ \overleftarrow{Y}_P, & \text{if } \Sigma = P \end{cases}.$$

If there had been no morphism of the form $P \xrightarrow{f} Q$, with $Q \neq P$, in $\mathbf{Sign}^\flat$, then, since $Y_P = \mathrm{SEN}^\flat(P)$, we would have $\overleftarrow{Y}_P = \mathrm{SEN}^\flat(P)$, contradicting our assumption. The existence of such a morphism implies that $\overleftarrow{\overline{Z}} = \overline{\varnothing} = \overleftarrow{\overline{\varnothing}}$. However, $\Omega(Z) \neq \nabla^{\mathbf{F}} = \Omega(\overline{\varnothing})$, which contradicts rough left monotonicity.

We conclude that $\mathcal{I}$ must be roughly family monotone. ∎

We now have a picture of the rough monotonicity hierarchy.

Proposition 505 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is roughly left monotone, then it is both roughly family and roughly right monotone;*

(b) If $\mathcal{I}$ *is roughly family or roughly right monotone, then it is roughly system monotone.*

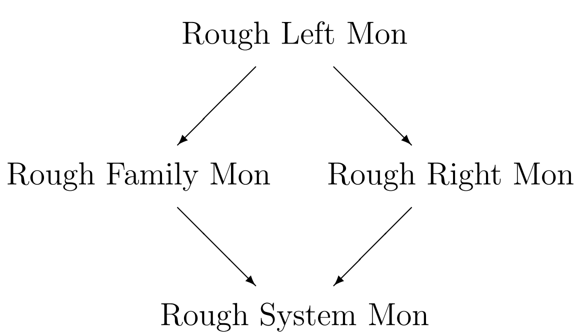
Proof:

(a) Suppose $\mathcal{I}$ is roughly left monotone. By Proposition 504, $\mathcal{I}$ is roughly family monotone. Therefore, by Corollary 503, it is also roughly right monotone.

(b) If $\mathcal{I}$ is roughly family monotone, then it is, a fortiori, roughly system monotone, since the condition defining the latter notion is a specialization of that defining the former. So, suppose $\mathcal{I}$ is roughly right monotone and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T} \leq \widetilde{T'}$. Then, by rough right monotonicity, $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Since T, T' are theory systems, $\overleftarrow{T} = T$ and $\overleftarrow{T'} = T'$, whence $\Omega(T) \leq \Omega(T')$ and, hence, $\mathcal{I}$ is roughly system monotone.

∎

We have now established the following **rough monotonicity hierarchy** of π-institutions.



It is not difficult to see that being roughly left monotone is equivalent to being roughly system monotone and stable.

Proposition 506 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is roughly left monotone if and only if it is roughly system monotone and stable.*

Proof: Suppose, first, that $\mathcal{I}$ is roughly left monotone. Then, by Proposition 505, it is roughly system monotone. Moreover, by Lemma 502, $\mathcal{I}$ is stable.

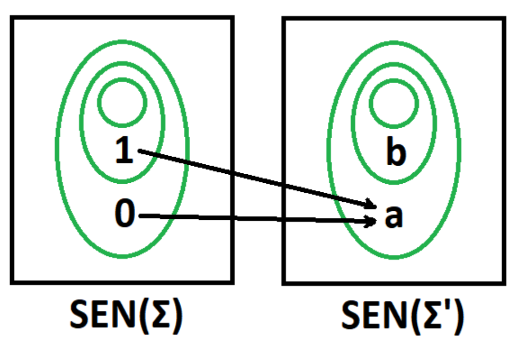
Assume, conversely, that $\mathcal{I}$ is stable and roughly system monotone. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\overleftarrow{\overleftarrow{T}} \leq \overleftarrow{\overleftarrow{T'}}$. Then, since $\overleftarrow{T}, \overleftarrow{T'} \in \mathrm{ThSys}(\mathcal{I})$, we get, by rough system monotonicity, $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Thus, by stability, $\Omega(T) \leq \Omega(T')$. We conclude that $\mathcal{I}$ is roughly left monotone. ■

By Proposition 506, under stability, the rough monotonicity hierarchy collapses to a single class. Moreover, by Lemma 383, the same happens, a fortiori, under rough systemicity.

We present two examples to show that all four rough monotonicity classes depicted in the diagram above are different. The first example gives a roughly family monotone π-institution that is not roughly right monotone. It shows that the inclusions represented in the diagram by the two southwest pointing arrows are proper inclusions.

Example 507 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = a$*;*
- $N^\flat$ *is the trivial clone.*



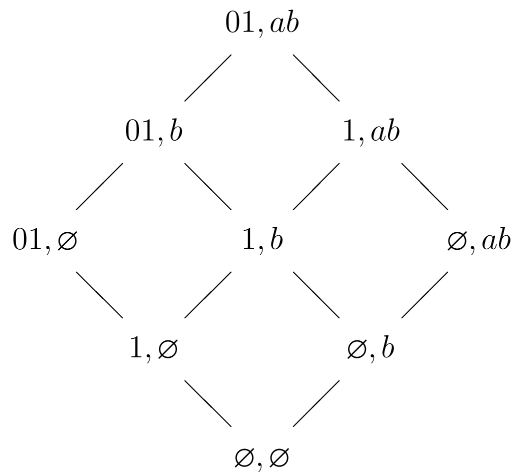
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0, 1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a, b\}\}.$$

There are nine theory families, but only five theory systems. The action of $\overleftarrow{\ }$ *on theory families is given in the table below.*

T	$\overleftarrow{T}$	T	$\overleftarrow{T}$
$\varnothing,\varnothing$	$\varnothing,\varnothing$	$\varnothing,ab$	$\varnothing,ab$
$1,\varnothing$	$\varnothing,\varnothing$	$01,b$	$\varnothing,b$
$\varnothing,b$	$\varnothing,b$	$1,ab$	$1,ab$
$01,\varnothing$	$\varnothing,\varnothing$	$01,ab$	$01,ab$
$1,b$	$\varnothing,b$		

The lattice of theory families of $\mathcal{I}$ *is shown in the diagram.*



We show that $\mathcal{I}$ *is roughly family monotone. To this end, suppose* $\widetilde{T} \le \widetilde{T'}$.

- *If* $\widetilde{T'} = \{01,ab\}$, *then* $T' = \{\varnothing,\varnothing\}$ *or* $\{01,\varnothing\}$ *or* $\{\varnothing,ab\}$ *or* $\{01,ab\}$. *In all cases* $\Omega(T) \le \nabla^{\mathbf{F}} = \Omega(T')$;
- *If* $\widetilde{T'} = \{01,b\}$, *then* $T' = \{\varnothing,b\}$ *or* $\{01,b\}$ *and* $\widetilde{T} = \widetilde{T'}$ *or* $\widetilde{T} = \{1,b\} = T$, *whence* $\Omega(T) \le \{\nabla^{\mathbf{F}}_\Sigma, \Delta^{\mathbf{F}}_{\Sigma'}\} = \Omega(T')$;
- *If* $\widetilde{T'} = \{1,ab\}$, *then* $T' = \{1,\varnothing\}$ *or* $\{1,ab\}$ *and* $\widetilde{T} = \widetilde{T'}$ *or* $\widetilde{T} = \{1,b\} = T$, *whence* $\Omega(T) \le \{\Delta^{\mathbf{F}}_\Sigma, \nabla^{\mathbf{F}}_{\Sigma'}\} = \Omega(T')$;
- *If* $\widetilde{T'} = \{1,b\}$, *then* $\widetilde{T} = \{1,b\}$, *whence* $T = T' = \{1,b\}$ *and* $\Omega(T) = \Omega(T')$.

Therefore, $\mathcal{I}$ *is indeed roughly family monotone.*

On the other hand, we have $\widetilde{\{1,b\}} = \{1,b\} \le \{1,ab\} = \widetilde{\{1,ab\}}$, *whereas*

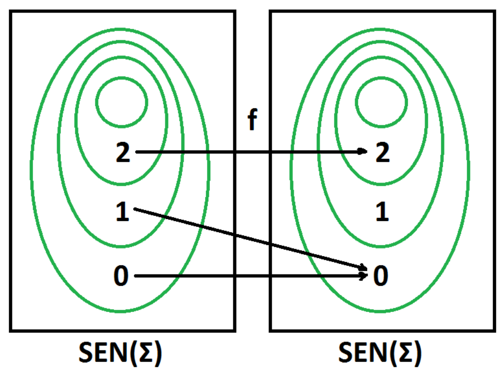
$$\Omega(\overleftarrow{\{1,b\}}) = \Omega(\{\varnothing,b\}) = \{\nabla^{\mathbf{F}}_\Sigma, \Delta^{\mathbf{F}}_{\Sigma'}\} \nleq \{\Delta^{\mathbf{F}}_\Sigma, \nabla^{\mathbf{F}}_{\Sigma'}\} = \Omega(\{1,ab\}) = \Omega(\overleftarrow{\{1,ab\}}).$$

Therefore, $\mathcal{I}$ *is not roughly right monotone.*

The second example shows that there exists a roughly right monotone π-institution that is not roughly family monotone. This has the effect of establishing that the inclusions represented by the two southeast arrows in the hierarchy diagram are also proper inclusions.

Example 508 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

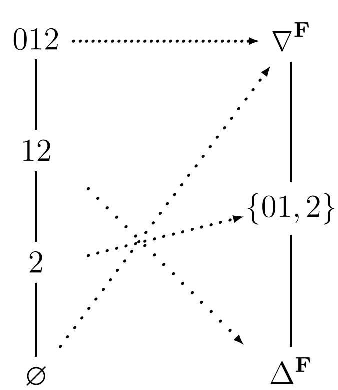
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$, $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$;
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{2\}, \{1,2\}, \{0,1,2\}\}.$$

$\mathcal{I}$ *has four theory families, but only three theory systems, namely* $\overline{\varnothing}$, $\{\{2\}\}$ *and* $\{\{0,1,2\}\}$. *The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right:*



We show that $\mathcal{I}$ *is roughly right monotone. Suppose* $T, T' \in \mathrm{ThFam}(\mathcal{I})$, *such that* $\widetilde{T} \leq \widetilde{T'}$.

- *If $T' = \{\varnothing\}$ or $T' = \{\{0,1,2\}\}$, i.e., if $\widetilde{T'} = \{\{0,1,2\}\}$, $\Omega(\overleftarrow{T}) \le \nabla^{\mathbf{F}} = \Omega(\overleftarrow{T'})$;*
- *If $T' = \{\{1,2\}\}$, then $T = \{\{2\}\}$ or $T = \{\{1,2\}\}$. So $\Omega(\overleftarrow{T}) = \Omega(\{\{2\}\}) = \Omega(\overleftarrow{T'})$;*
- *If $T' = \{\{2\}\}$, i.e., if $\widetilde{T'} = \{1\}$, then $T = \{\{2\}\}$, and the conclusion is trivial.*

Thus, $\mathcal{I}$ is indeed roughly right monotone.

On the other hand, setting $T = \{\{2\}\}$ and $T' = \{\{1,2\}\}$, we get $\widetilde{T} \le \widetilde{T'}$, but $\Omega(T) = \{\{0,1\},\{2\}\} \nleq \Delta^{\mathbf{F}} = \Omega(T')$. Therefore, $\mathcal{I}$ is not roughly family monotone.

We conclude, after these two examples, that the structure of the rough monotonicity hierarchy is, in fact, exactly as depicted in the diagram and no two classes are identical.

We look, next, at the connections between rough monotonicity and monotonicity classes. It turns out that protoalgebraicity (i.e., family/left monotonicity, by Proposition 171) is strong enough to ensure membership in all classes of the rough monotonicity hierarchy, whereas prealgebraicity (i.e., system/right monotonicity, by Proposition 173) is only sufficiently strong to yield corresponding rough monotonicity properties, i.e., implies rough right monotonicity and, a fortiori, rough system monotonicity.

Theorem 509 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.*

(a) If $\mathcal{I}$ is protoalgebraic, then it is roughly left monotone;

(b) If $\mathcal{I}$ is prealgebraic, then it is roughly right monotone.

Proof:

(a) Suppose that $\mathcal{I}$ is protoalgebraic. By Lemma 170, this implies that $\mathcal{I}$ is stable. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\overleftarrow{\overleftarrow{T}} \le \overleftarrow{\overleftarrow{T'}}$. Then, by protoalgebraicity, $\Omega(\overleftarrow{\overleftarrow{T}}) \le \Omega(\overleftarrow{\overleftarrow{T'}})$. Hence, by Proposition 369, $\Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'})$. Thus, by stability, $\Omega(T) \le \Omega(T')$. Therefore, $\mathcal{I}$ is roughly left monotone.

(b) Suppose that $\mathcal{I}$ is prealgebraic. If $\mathrm{ThSys}(\mathcal{I})$ consists of a single rough equivalence class, then $\mathcal{I}$ is trivially roughly right monotone. Otherwise, since $\mathcal{I}$ is prealgebraic and $\Omega(\overline{\varnothing}) = \Omega(\mathrm{SEN}^\flat) = \nabla^{\mathbf{F}}$, $\mathcal{I}$ must have theorems. Therefore, rough equivalence is the identity relation on $\mathrm{ThFam}(\mathcal{I})$. Thus, for $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{T} \le \widetilde{T'}$, we get $T \le T'$, whence, by Lemma 1, $\overleftarrow{T} \le \overleftarrow{T'}$. Thus, by prealgebraicity, $\Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'})$, showing that $\mathcal{I}$ is roughly right monotone.

■

Moreover, the following additional relations hold.

Theorem 510 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a non-almost inconsistent* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is protoalgebraic if and only if it has theorems and is roughly family or roughly left monotone.*

Proof: Suppose $\mathcal{I}$ is protoalgebraic. Since, by hypothesis, it is not almost inconsistent, it must have theorems. Moreover, by Theorem 509 and Proposition 505, it is both roughly left and roughly family monotone.

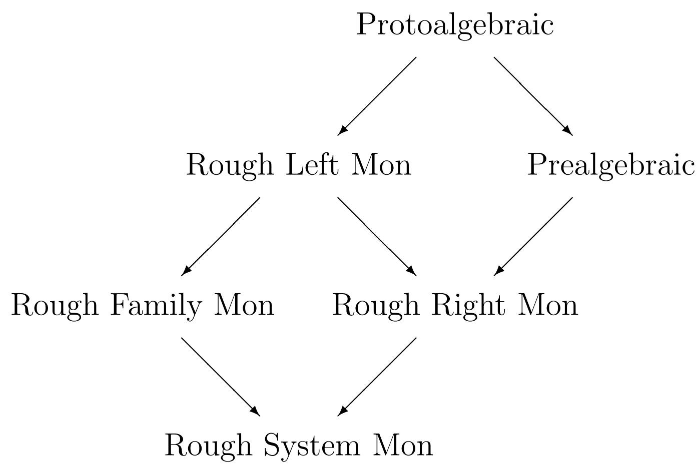
Assume, conversely, that $\mathcal{I}$ is roughly family or roughly left monotone and has theorems. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$. Then, by Lemma 1, we get $\overleftarrow{T} \leq \overleftarrow{T'}$. Since $\mathcal{I}$ has theorems, rough equivalence coincides with the identity relation on $\mathrm{ThFam}(\mathcal{I})$, whence, we get both $\widetilde{T} \leq \widetilde{T'}$ and $\overleftarrow{\widetilde{T}} \leq \overleftarrow{\widetilde{T'}}$. Using either rough family or rough left monotonicity, as the case requires, we obtain $\Omega(T) \leq \Omega(T')$. Therefore, $\mathcal{I}$ is protoalgebraic. ■

Theorem 511 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, that has a theory family* $T \neq \mathrm{SEN}^\flat$ *such that* $\overleftarrow{T} \neq \overline{\varnothing}$. $\mathcal{I}$ *is prealgebraic if and only if it has theorems and it is roughly right or roughly system monotone.*

Proof: Suppose $\mathcal{I}$ is prealgebraic. Since, by hypothesis, it has a theory system $\overleftarrow{T} \neq \mathrm{SEN}^\flat, \overline{\varnothing}$, it must have theorems. Moreover, by Theorem 509, it is roughly right monotone and, hence, by Proposition 505, it is roughly system monotone.

Assume, conversely, that $\mathcal{I}$ is roughly right or roughly system monotone and has theorems. By Proposition 505, it is roughly system monotone and has theorems. Let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $T \leq T'$. Since $\mathcal{I}$ has theorems, rough equivalence coincides with the identity relation on $\mathrm{ThFam}(\mathcal{I})$, whence, we get $\widetilde{T} \leq \widetilde{T'}$. By rough system monotonicity, we obtain $\Omega(T) \leq \Omega(T')$. Therefore, $\mathcal{I}$ is prealgebraic. ■

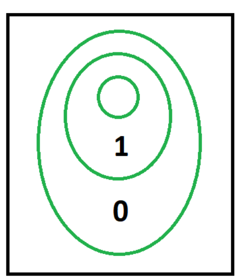
Theorem 509, together with Theorem 175 and Proposition 505, establish the mixed monotonicity and rough monotonicity hierarchy depicted in the diagram.



To see that all classes in the hierarchy are different, we give an example of a π-institution satisfying all four rough monotonicity properties, which is not, however, prealgebraic and, therefore, a fortiori, it is not protoalgebraic either.

Example 512 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with the single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$*;*
- $N^\flat$ *is the trivial clone.*



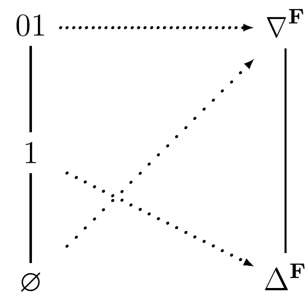
SEN(Σ)

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0, 1\}\}.$$

$\mathcal{I}$ has three theory families $\{\varnothing\}$ *and* $\{\{1\}\}$ *and* $\{\{0, 1\}\}$*, all of which are theory systems.*

The lattice of theory families of $\mathcal{I}$ and the corresponding Leibniz congruence systems are given in the diagram.



$\mathcal{I}$ belongs to all four classes of the rough monotonicity hierarchy. Indeed, since it is systemic, all four rough monotonicity conditions boil down to checking that, for all $T,T' \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{T} \leq \widetilde{T'}$ implies $\Omega(T) \leq \Omega(T')$.

- *If $\widetilde{T'} = \{\{0,1\}\}$, then $T' = \{\varnothing\}$ or $T' = \{\{0,1\}\}$, whence $\Omega(T) \leq \nabla^{\mathbf{F}} = \Omega(T')$;*
- *If $\widetilde{T'} = \{\{1\}\}$, then $\widetilde{T} = \{\{1\}\}$ and, hence, $T = T' = \{\{1\}\}$. Thus, the implication holds trivially.*

On the other hand, we have $\{\varnothing\} \leq \{\{1\}\}$, whereas $\Omega(\{\varnothing\}) \nleq \Omega(\{\{1\}\})$, whence $\mathcal{I}$ is not prealgebraic.

The rough monotonicity properties transfer from the theory families/ systems of a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ to all $\mathcal{I}$-filter families/systems on arbitrary $\mathbf{F}$-algebraic systems.

Theorem 513 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is roughly family monotone if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ and all $T,T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{T} \leq \widetilde{T'}$ implies $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$;

(b) $\mathcal{I}$ is roughly left monotone if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ and all $T,T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\overset{\leftleftarrows}{T} \leq \overset{\leftleftarrows}{T'}$ implies $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$;

(c) $\mathcal{I}$ is roughly right monotone if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ and all $T,T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{T} \leq \widetilde{T'}$ implies $\Omega^{\mathcal{A}}(\overleftarrow{T}) \leq \Omega^{\mathcal{A}}(\overleftarrow{T'})$;

(d) $\mathcal{I}$ is roughly system monotone if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ and all $T,T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{T} \leq \widetilde{T'}$ implies $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$.

Proof:

(a) The "if" follows by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account that, by Lemma 51, $\mathrm{ThFam}(\mathcal{I}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F})$.

For the "only if", suppose that $\mathcal{I}$ is roughly family monotone and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{T} \leq \widetilde{T'}$. Then $\alpha^{-1}(\widetilde{T}) \leq \alpha^{-1}(\widetilde{T'})$. By Theorem 377, $\widetilde{\alpha^{-1}(T)} \leq \widetilde{\alpha^{-1}(T')}$. Since, by Lemma 51, both $\alpha^{-1}(T)$ and $\alpha^{-1}(T')$ are theory families of $\mathcal{I}$, we get, by rough family monotonicity, $\Omega(\alpha^{-1}(T)) \leq \Omega(\alpha^{-1}(T'))$. Hence, by Proposition 24, $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. Taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$.

(b) The "if" follows as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is roughly left monotone and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\overleftarrow{T}} \leq \widetilde{\overleftarrow{T'}}$. Then $\alpha^{-1}(\widetilde{\overleftarrow{T}}) \leq \alpha^{-1}(\widetilde{\overleftarrow{T'}})$. By Theorem 377, $\widetilde{\alpha^{-1}(\overleftarrow{T})} \leq \widetilde{\alpha^{-1}(\overleftarrow{T'})}$. Hence, by Lemma 6, $\widetilde{\overleftarrow{\alpha^{-1}(T)}} \leq \widetilde{\overleftarrow{\alpha^{-1}(T')}}$. Since, by Lemma 51, $\alpha^{-1}(T)$ and $\alpha^{-1}(T')$ are theory families, we get, by rough left monotonicity, $\Omega(\alpha^{-1}(T)) \leq \Omega(\alpha^{-1}(T'))$, whence, by Proposition 24, $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. Thus, by the surjectivity of $\langle F, \alpha \rangle$, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$.

(c) The "if" follows as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is roughly right monotone and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{T} \leq \widetilde{T'}$. Then $\alpha^{-1}(\widetilde{T}) \leq \alpha^{-1}(\widetilde{T'})$ and, hence, by Theorem 377, $\widetilde{\alpha^{-1}(T)} \leq \widetilde{\alpha^{-1}(T')}$. Since, by Lemma 51, $\alpha^{-1}(T)$ and $\alpha^{-1}(T')$ are theory families, we get, by rough right monotonicity, $\Omega(\overleftarrow{\alpha^{-1}(T)}) \leq \Omega(\overleftarrow{\alpha^{-1}(T')})$. Thus, by Lemma 6, $\Omega(\alpha^{-1}(\overleftarrow{T})) \leq \Omega(\alpha^{-1}(\overleftarrow{T'}))$. Now, by Proposition 24, we get $\alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T})) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'}))$, whence, by the surjectivity of $\langle F, \alpha \rangle$, $\Omega^{\mathcal{A}}(\overleftarrow{T}) \leq \Omega^{\mathcal{A}}(\overleftarrow{T'})$.

(d) Similar to Part (a).

■

Finally, we may recast the rough monotonicity classes in terms of the monotonicity of mappings from posets of classes of theory or filter families/systems into posets of congruence systems.

Recall the orderings of the collections $\widetilde{\mathrm{ThFam}}(\mathcal{I})$ and $\widetilde{\mathrm{ThSys}}(\mathcal{I})$: For all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$\widetilde{[T]} \leq \widetilde{[T']} \quad \text{iff} \quad \widetilde{T} \leq \widetilde{T'}$$

and, for all $T, T' \in \mathrm{ThSys}(\mathcal{I})$,

$$\widetilde{[T]} \leq \widetilde{[T']} \quad \text{iff} \quad \widetilde{T} \leq \widetilde{T'}$$

and the notation $\widetilde{\mathbf{ThFam}}(\mathcal{I}) = \langle \widetilde{\mathrm{ThFam}}(\mathcal{I}), \leq \rangle$ and $\widetilde{\mathbf{ThSys}}(\mathcal{I}) = \langle \widetilde{\mathrm{ThSys}}(\mathcal{I}), \leq \rangle$ for the corresponding ordered sets.

Proposition 514 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is roughly family monotone;*

(b) $\Omega : \widetilde{\mathbf{ThFam}}(\mathcal{I}) \to \mathbf{ConSys}^*(\mathcal{I})$ *is monotone;*

(c) $\Omega^\mathcal{A} : \widetilde{\mathbf{FiFam}}^\mathcal{I}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is monotone, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

Similarly, for rough system monotonicity, we have

Proposition 515 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is roughly system monotone;*

(b) $\Omega : \widetilde{\mathbf{ThSys}}(\mathcal{I}) \to \mathbf{ConSys}^*(\mathcal{I})$ *is monotone;*

(c) $\Omega^\mathcal{A} : \widetilde{\mathbf{FiSys}}^\mathcal{I}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is monotone, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

7.4 Narrow Monotonicity

We now introduce and study classes of π-institutions defined using, once more, monotonicity properties of the Leibniz operator, but applied only on theory families with all components nonempty. This is one of the ways used already in Chapter 6 to bypass theory families with empty components that may cause lack of monotonicity.

Definition 516 (Narrow Monotonicity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is called* **narrowly family monotone** *if, for all* $T, T' \in \mathrm{ThFam}^{\not\emptyset}(\mathcal{I})$*,*
$$T \leq T' \quad \textit{implies} \quad \Omega(T) \leq \Omega(T').$$
- $\mathcal{I}$ *is called* **narrowly left monotone** *if, for all* $T, T' \in \mathrm{ThFam}^{\not\emptyset}(\mathcal{I})$*,*
$$\overleftarrow{T} \leq \overleftarrow{T'} \quad \textit{implies} \quad \Omega(T) \leq \Omega(T').$$
- $\mathcal{I}$ *is called* **narrowly right monotone** *if, for all* $T, T' \in \mathrm{ThFam}^{\not\emptyset}(\mathcal{I})$*,*
$$T \leq T' \quad \textit{implies} \quad \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'}).$$

- *$\mathcal{I}$ is called* **narrowly system monotone** *if, for all $T, T' \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$,*

$$T \leq T' \quad \textit{implies} \quad \Omega(T) \leq \Omega(T').$$

We establish now the *narrow monotonicity hierarchy* of π-institutions.

Proposition 517 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) If $\mathcal{I}$ is narrowly left monotone, then it is narrowly family monotone;

(b) If $\mathcal{I}$ is narrowly family monotone, then it is narrowly system monotone;

(c) If $\mathcal{I}$ is narrowly right monotone, then it is narrowly system monotone.

Proof:

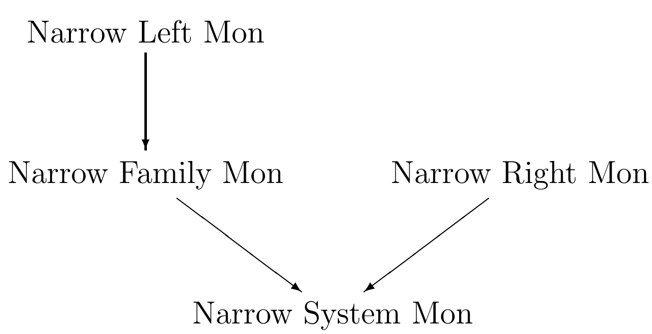
(a) Suppose that $\mathcal{I}$ is narrowly left monotone and let $T, T' \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, such that $T \leq T'$. Then, by Lemma 1, $\overleftarrow{T} \leq \overleftarrow{T'}$, whence, by narrow left monotonicity, $\Omega(T) \leq \Omega(T')$. Hence $\mathcal{I}$ is narrow family monotone.

(b) Suppose $\mathcal{I}$ is narrow family monotone. Then it is a fortiori narrow system monotone, since the condition defining the latter is a specialization of the one defining the former.

(c) Suppose $\mathcal{I}$ is narrowly right monotone and let $T, T' \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$, such that $T \leq T'$. Then, by narrow right monotonicity, $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$. Since T, T' are theory systems, $\overleftarrow{T} = T$ and $\overleftarrow{T'} = T'$, whence $\Omega(T) \leq \Omega(T')$ and, hence, $\mathcal{I}$ is narrowly system monotone.

■

We have now established the following **narrow monotonicity hierarchy** of π-institutions.



Some additional relationships may be established between the narrow monotonicity classes. More precisely, we show that narrow left monotonicity implies exclusive stability, whereas narrow system monotonicity together with narrow systemicity, yield both narrow left and narrow right monotonicity.

Proposition 518 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is narrowly left monotone, then it is exclusively stable.*

Proof: Suppose that $\mathcal{I}$ is narrowly left monotone and let $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, such that $\overleftarrow{T} \in \mathrm{ThSys}^\natural(\mathcal{I})$. Since $\overleftarrow{\overleftarrow{T}} = \overleftarrow{T}$ and $T, \overleftarrow{T} \in \mathrm{ThFam}^\natural(\mathcal{I})$, we get, by narrow left monotonicity, $\Omega(\overleftarrow{T}) = \Omega(T)$. Thus, $\mathcal{I}$ is exclusively stable. ■

Proposition 519 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is narrowly system monotone and narrowly systemic, then it is both narrowly left and narrowly right monotone.*

Proof: Suppose that $\mathcal{I}$ is narrowly system monotone and narrowly systemic and let $T, T' \in \mathrm{ThFam}^\natural(\mathcal{I})$.

- Assume that $\overleftarrow{T} \leq \overleftarrow{T'}$. By narrow systemicity, $\overleftarrow{T} = T$ and $\overleftarrow{T'} = T'$, whence $T, T' \in \mathrm{ThSys}^\natural(\mathcal{I})$. Thus, by narrow system monotonicity, $\Omega(T) \leq \Omega(T')$ and, therefore, $\mathcal{I}$ is narrowly left monotone.
- Assume that $T \leq T'$. Again, by narrow systemicity, $\overleftarrow{T} = T$ and $\overleftarrow{T'} = T'$, which yields that $\overleftarrow{T}, \overleftarrow{T'} \in \mathrm{ThSys}^\natural(\mathcal{I})$. Hence, by narrow system monotonicity, $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$, showing that $\mathcal{I}$ is narrowly right monotone.

■

By Propositions 517 and 519, under narrow systemicity, the narrow monotonicity hierarchy collapses to a single class.

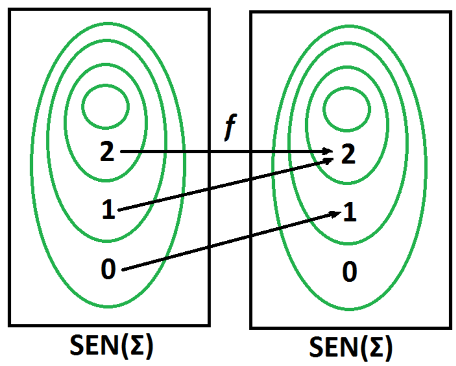
We present three examples to show that all four narrow monotonicity classes depicted in the diagram above are different. The first example gives a narrowly family monotone π-institution which is not narrowly left monotone. Thus, it shows that the class of narrowly left monotone π-institutions is properly contained in the class of narrowly family monotone π-institutions.

Example 520 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with the single object* Σ *and four non-identity morphisms* $f, z, o, t : \Sigma \to \Sigma$*, whose composition table is the following:*

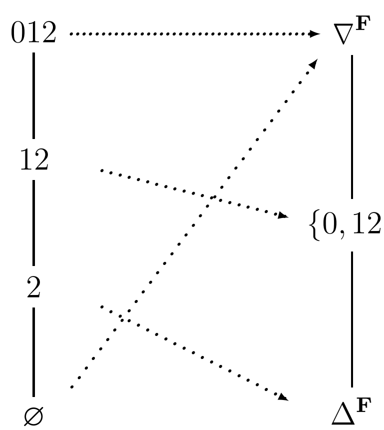
$\circ$	f	z	o	t
f	t	o	t	t
z	z	z	z	z
o	o	o	o	o
t	t	t	t	t

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$, *with*
$$\mathrm{SEN}^\flat(f)(0) = 1, \quad \mathrm{SEN}^\flat(f)(1) = 2, \quad \mathrm{SEN}^\flat(f)(2) = 2,$$
whereas $\mathrm{SEN}^\flat(z)(x) = 0$, $\mathrm{SEN}^\flat(o)(x) = 1$ *and* $\mathrm{SEN}^\flat(t)(x) = 2$, *for all* $x \in \mathrm{SEN}^\flat(\Sigma)$;
- $N^\flat$ *is the trivial clone, consisting of the projections only.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by stipulating that*
$$C_\Sigma = \{\varnothing, \{2\}, \{1,2\}, \{0,1,2\}\}.$$

$\mathcal{I}$ *has four theory families* $\varnothing$, $\{\{2\}\}$, $\{\{1,2\}\}$ *and* $\{\{0,1,2\}\}$, *but only two theory systems,* $\varnothing$ *and* $\{\{0,1,2\}\}$. *The lattice of theory families of* $\mathcal{I}$ *and the corresponding Leibniz congruence systems are given in the diagram.*



Since $\mathrm{ThFam}^{\sharp}(\mathcal{I}) = \{\{\{2\}\}, \{\{1,2\}\}, \mathrm{SEN}^\flat\}$, *it is clear that* $\mathcal{I}$ *is narrowly family monotone.*

On the other hand, for $T = \{\{1,2\}\}$ *and* $T' = \{\{2\}\}$, *we get* $\overleftarrow{T} = \overline{\varnothing} = \overleftarrow{T'}$, *whereas* $\Omega(T) = \{0,12\} \nleq \Delta^{\mathbf{F}} = \Omega(T')$. *Therefore,* $\mathcal{I}$ *is not narrowly left monotone.*

The second example shows that there exists a narrowly family monotone π-institution that is not narrowly right monotone, thus showing, on the one hand, that the class of narrowly right monotone π institutions is properly included in the class of narrowly system monotone π-institutions and, on the other, that narrowly family monotone π-institutions do not form a subclass of narrowly right monotone π-institutions.

Example 521 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with the single object* Σ *and four non-identity morphisms* $f, g, o, t : \Sigma \to \Sigma$*, whose composition table is the following:*

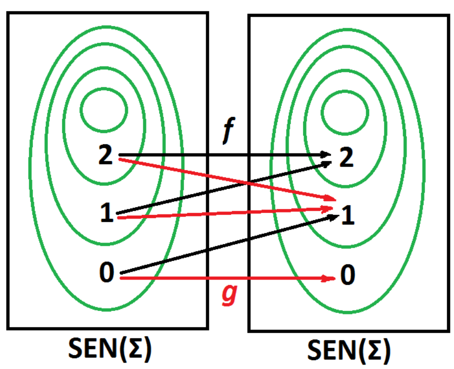
$\circ$	f	g	o	t
f	t	f	t	t
g	o	g	o	o
o	o	o	o	o
t	t	t	t	t

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*, with*

$$\begin{array}{lll} \mathrm{SEN}^\flat(f)(0) = 1, & \mathrm{SEN}^\flat(f)(1) = 2, & \mathrm{SEN}^\flat(f)(2) = 2; \\ \mathrm{SEN}^\flat(g)(0) = 0, & \mathrm{SEN}^\flat(g)(1) = 1, & \mathrm{SEN}^\flat(g)(2) = 1, \end{array}$$

whereas $\mathrm{SEN}^\flat(o)(x) = 1$ *and* $\mathrm{SEN}^\flat(t)(x) = 2$*, for all* $x \in \mathrm{SEN}^\flat(\Sigma)$*;*

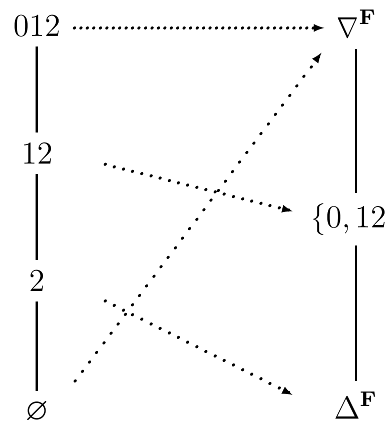
- $N^\flat$ *is the trivial clone, consisting of the projections only.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

$\mathcal{I}$ *has four theory families* $\varnothing$, $\{\{2\}\}$, $\{\{1, 2\}\}$ *and* $\{\{0, 1, 2\}\}$*, but only three theory systems,* $\varnothing$, $\{\{1, 2\}\}$ *and* $\{\{0, 1, 2\}\}$*. The lattice of theory families of* $\mathcal{I}$ *and the corresponding Leibniz congruence systems are given in the diagram.*



Since $\mathrm{ThFam}^{\nmid}(\mathcal{I}) = \{\{\{2\}\}, \{\{1,2\}\}, \mathrm{SEN}^\flat\}$*, it is clear that* $\mathcal{I}$ *is narrowly family monotone.*

On the other hand, for $T = \{\{2\}\}$ *and* $T' = \{\{1,2\}\}$*, we get* $T \le T'$*, whereas* $\Omega(\overleftarrow{T}) = \Omega(\overline{\varnothing}) = \nabla^{\mathbf{F}} \not\le \{0,12\} = \Omega(T') = \Omega(\overleftarrow{T'})$*. Therefore,* $\mathcal{I}$ *is not narrowly right monotone.*

The third example shows that there exists a narrowly right monotone π-institution that is not narrowly family monotone. Combined with the preceding examples, it has the effect of establishing the following facts:

- The classes of narrowly family monotone and narrowly right monotone π-institutions are pairwise incomparable.
- The class of narrowly family monotone π-institutions is properly contained in the class of narrowly system monotone π-institutions.

Example 522 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

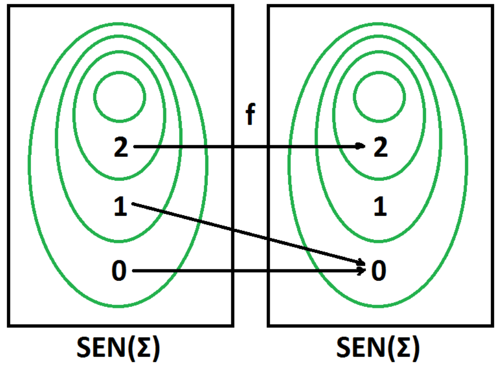
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$*,* $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial clone.*

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

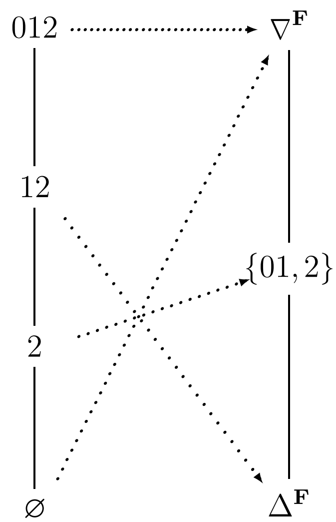
$$\mathcal{C}_\Sigma = \{\varnothing, \{2\}, \{1,2\}, \{0,1,2\}\}.$$

$\mathcal{I}$ *has four theory families, but only three theory systems, namely* $\overline{\varnothing}$*,* $\{\{2\}\}$ *and* $\{\{0,1,2\}\}$*. Moreover, clearly,*

$$\mathrm{ThFam}^{\nmid}(\mathcal{I}) = \{\{\{2\}\}, \{\{1,2\}\}, \{\{0,1,2\}\}\}.$$



The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right:



We have

$$\begin{array}{rcl}\Omega(\overleftarrow{2}) & = & \Omega(2) = \{01,2\};\\ \Omega(\overleftarrow{12}) & = & \Omega(2) = \{01,2\};\\ \Omega(\overleftarrow{012}) & = & \Omega(012) = \nabla^{\mathbf{F}}.\end{array}$$

Thus, we get $\Omega(\overleftarrow{2}) \leq \Omega(\overleftarrow{12}) \leq \Omega(\overleftarrow{012})$ *and, therefore,* $\mathcal{I}$ *is narrowly right monotone.*

On the other hand, for $T = \{\{2\}\}$ *and* $T' = \{\{1,2\}\}$*, we get* $T \leq T'$*, whereas* $\Omega(T) = \{01,2\} \nleq \Delta^{\mathbf{F}} = \Omega(T')$*. Thus,* $\mathcal{I}$ *is not narrowly family monotone.*

We conclude that the structure of the narrow monotonicity hierarchy is, in fact, exactly as depicted in the diagram and no two classes are identical.

We look, next, at the connections between narrow monotonicity and monotonicity classes. Once more, as was the case with monotonicity and rough monotonicity in Section 7.3, protoalgebraicity (i.e., family/left monotonicity, by Proposition 171) is strong enough to ensure membership in all classes of the narrow monotonicity hierarchy, whereas prealgebraicity (i.e., system/right monotonicity, by Proposition 173) is only sufficiently strong to yield corresponding narrow monotonicity properties, i.e., implies narrow right monotonicity and, a fortiori, narrow system monotonicity.

Theorem 523 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is protoalgebraic, then it is narrowly left monotone;*

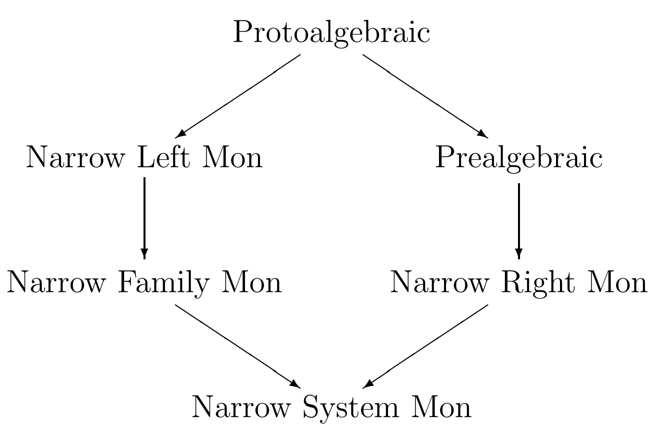
(b) If $\mathcal{I}$ *is prealgebraic, then it is narrowly right monotone.*

Proof:

(a) Suppose that $\mathcal{I}$ is protoalgebraic. By Proposition 171, it is left monotone, whence, it is, a fortiori, narrowly left monotone, since the condition defining the latter is a specialization of that defining the former.

(b) Suppose that $\mathcal{I}$ is prealgebraic. By Proposition 173, it is right monotone, whence, it is, a fortiori, narrowly right monotone, since the condition defining the latter is a specialization of that defining the former. ■

Thus, the following mixed monotonicity and narrow monotonicity hierarchy emerges.



We also have the following additional relations, paralleling the ones established between monotonicity and rough monotonicity classes in Theorems 510 and 511.

Theorem 524 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a non-almost inconsistent* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is protoalgebraic if and only if it has theorems and is narrowly left or narrowly family monotone.*

Proof: Suppose $\mathcal{I}$ is protoalgebraic. Since, by hypothesis, it is not almost inconsistent, it must have theorems. Moreover, by Theorem 523 and Proposition 517, it is both narrowly left and narrowly family monotone.

Assume, conversely, that $\mathcal{I}$ is narrowly left or narrowly family monotone and has theorems. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$. Then, since $\mathcal{I}$ has theorems, $T, T' \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$ and, moreover, by Lemma 1, we get $\overleftarrow{T} \leq \overleftarrow{T'}$. Using either narrow family or narrow left monotonicity, as the case requires, we obtain $\Omega(T) \leq \Omega(T')$. Therefore, $\mathcal{I}$ is protoalgebraic. ■

Theorem 525 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, that has a theory system* $T \neq \overline{\varnothing}, \mathrm{SEN}^\flat$*.* $\mathcal{I}$ *is prealgebraic if and only if it has theorems and it is narrowly right or narrowly system monotone.*

Proof: Suppose $\mathcal{I}$ is prealgebraic. Since, by hypothesis, it has a theory system $T \neq \overline{\varnothing}, \mathrm{SEN}^\flat$, it must have theorems. Moreover, by Theorem 523, it is narrowly right monotone and, hence, by Proposition 517, it is narrowly system monotone.

Assume, conversely, that $\mathcal{I}$ is narrowly right or narrowly system monotone and has theorems. By Proposition 517, it is narrowly system monotone and has theorems. Let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $T \leq T'$. Since $\mathcal{I}$ has theorems, $T, T' \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$. By narrow system monotonicity, we obtain $\Omega(T) \leq \Omega(T')$. Therefore, $\mathcal{I}$ is prealgebraic. ■

To see that all classes in the hierarchy are different, we give an example of a π-institution satisfying all four narrow monotonicity properties, which is not, however, prealgebraic and, therefore, a fortiori, it is not protoalgebraic either.

Example 526 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

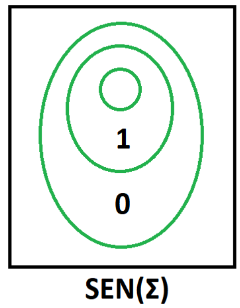
- $\mathbf{Sign}^\flat$ *is the trivial category with the single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$*;*
- $N^\flat$ *is the trivial clone.*

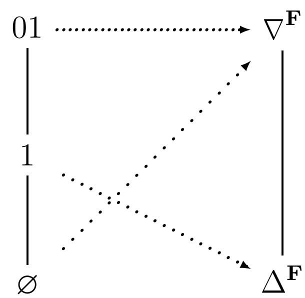
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0, 1\}\}.$$

$\mathcal{I}$ *has three theory families* $\{\varnothing\}$ *and* $\{\{1\}\}$ *and* $\{\{0, 1\}\}$*, all of which are theory systems.*

The lattice of theory families of $\mathcal{I}$ *and the corresponding Leibniz congruence systems are given in the diagram.*





$\mathcal{I}$ belongs to all four classes of the narrow monotonicity hierarchy. Indeed, since it is systemic, all four narrow monotonicity conditions boil down to checking that, for all $T,T' \in \mathrm{ThFam}^{\not\downarrow}(\mathcal{I})$, $T \leq T'$ implies $\Omega(T) \leq \Omega(T')$. This is obvious, since the only $T,T' \in \mathrm{ThFam}^{\not\downarrow}(\mathcal{I})$, with $T \lneq T'$, are $T = \{\{1\}\}$ and $T' = \{\{0,1\}\}$ and $\Omega(T) = \Delta^{\mathbf{F}} \leq \nabla^{\mathbf{F}} = \Omega(T')$.

On the other hand, we have $\{\varnothing\} \leq \{\{1\}\}$, whereas $\Omega(\{\varnothing\}) \nleq \Omega(\{\{1\}\})$, whence $\mathcal{I}$ is not prealgebraic.

We look, next, at relationships between narrow monotonicity and rough monotonicity classes. We show that the two family versions coincide and that, for the remaining three properties, each of the rough versions implies the corresponding narrow version.

Theorem 527 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ if roughly family monotone iff it is narrowly family monotone;

(b) If $\mathcal{I}$ is roughly left monotone, then it is narrowly left monotone;

(c) If $\mathcal{I}$ is roughly right monotone, then it is narrowly right monotone;

(d) If $\mathcal{I}$ is roughly system monotone, then it is narrowly system monotone.

Proof:

(a) Suppose that $\mathcal{I}$ is roughly family monotone and let $T,T' \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, such that $T \leq T'$. Since $T,T' \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, $\widetilde{T} = T$ and $\widetilde{T'} = T'$, whence, by hypothesis, $\widetilde{T} \leq \widetilde{T'}$. Thus, by rough family monotonicity, $\Omega(T) \leq \Omega(T')$ and, therefore, $\mathcal{I}$ is narrowly family monotone. Suppose, conversely, that $\mathcal{I}$ is narrowly family monotone and let $T,T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{T} \leq \widetilde{T'}$. Since $\widetilde{T},\widetilde{T'} \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, we get, by narrow family monotonicity, $\Omega(\widetilde{T}) \leq \Omega(\widetilde{T'})$. Therefore, by Proposition 369, $\Omega(T) \leq \Omega(T')$ and, hence, $\mathcal{I}$ is roughly family monotone.

(b) Suppose that $\mathcal{I}$ is roughly left monotone, i.e., that, for all $T,T' \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\overleftarrow{T}} \leq \widetilde{\overleftarrow{T'}}$ implies $\Omega(T) \leq \Omega(T')$. Assume, for the sake of obtaining a contradiction, that $\mathcal{I}$ is not narrowly left monotone. Then, there exist $X,Y \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, such that $\overleftarrow{X} \leq \overleftarrow{Y}$ and $\Omega(X) \nleq \Omega(Y)$.

First, observe that, if there existed $Z \in \mathrm{ThFam}(\mathcal{I})$ and $P \in |\mathbf{Sign}^\flat|$, such that $Z_P \neq \varnothing$ and $\overleftarrow{Z}_P = \varnothing$, then, setting $Z' = \{Z_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, with

$$Z'_\Sigma = \begin{cases} \varnothing, & \text{if } \Sigma \neq P \\ Z_P, & \text{if } \Sigma = P \end{cases},$$

we would have $\widetilde{\overleftarrow{Z'}} = \widetilde{\overleftarrow{\overline{\varnothing}}}$, but $\Omega(Z') \neq \Omega(\overline{\varnothing})$, which contradicts rough left monotonicity. Thus, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$, $T_\Sigma \neq \varnothing$ implies $\overleftarrow{T}_\Sigma \neq \varnothing$.

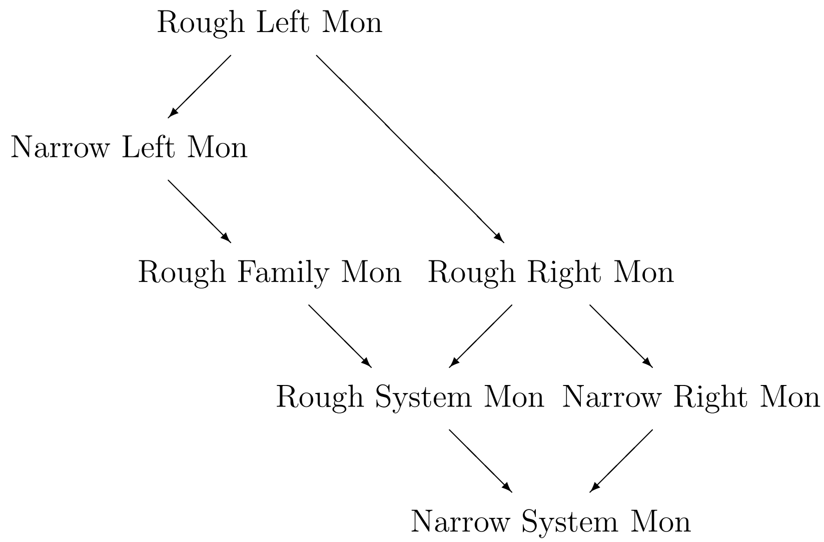
Continuing with the proof, by hypothesis, $\overleftarrow{X} \leq \overleftarrow{Y}$ and $\Omega(X) \nleq \Omega(Y)$. Hence, by rough left monotonicity, $\widetilde{\overleftarrow{X}} \nleq \widetilde{\overleftarrow{Y}}$. Thus, there exists $P \in |\mathbf{Sign}^\flat|$, such that $\widetilde{\overleftarrow{X}}_P \nsubseteq \widetilde{\overleftarrow{Y}}_P$, whereas $\overleftarrow{X}_P \subseteq \overleftarrow{Y}_P$. But this gives $\overleftarrow{X}_P = \varnothing$, whence, by the preceding observation, $X_P = \varnothing$, which contradicts $X \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$. Therefore, $\mathcal{I}$ must be narrowly left monotone.

(c) Suppose that $\mathcal{I}$ is roughly right monotone and let $T,T' \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, such that $T \leq T'$. Since $T,T' \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, we get $\widetilde{T} = T$ and $\widetilde{T'} = T'$, whence, by hypothesis, $\widetilde{T} \leq \widetilde{T'}$. By rough right monotonicity, $\Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'})$, whence $\mathcal{I}$ is narrowly right monotone.

(d) Suppose that $\mathcal{I}$ is roughly system monotone and let $T,T' \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$, such that $T \leq T'$. Since $T,T' \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$, $\widetilde{T} = T$ and $\widetilde{T'} = T'$, whence, by hypothesis, $\widetilde{T} \leq \widetilde{T'}$. Thus, by rough system monotonicity, $\Omega(T) \leq \Omega(T')$ and, therefore, $\mathcal{I}$ is narrowly system monotone.

∎

Thus, the following mixed rough monotonicity and narrow monotonicity hierarchy emerges.



To see that all classes in the hierarchy are different, we must find examples that separate the class of rough monotone from the class of narrow monotone π-institutions for each of the three allegedly distinct types, subject to the inclusions established in Theorem 527.

First, we provide an example of a narrowly left monotone π-institution that is not roughly left monotone. This proves that the class of roughly left monotone π-institutions is a proper subclass of the class of narrowly left monotone ones.

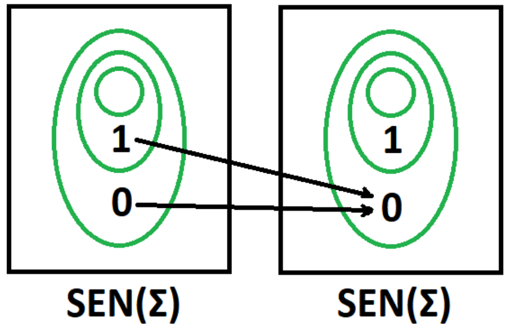
Example 528 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

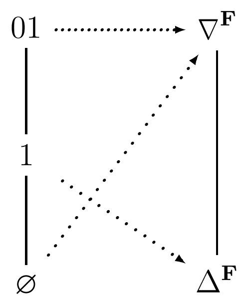
- $\mathbf{Sign}^\flat$ *is the category with the single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$*;*
- $N^\flat$ *is the trivial clone, consisting of the projections only.*

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\}.$$

$\mathcal{I}$ *has three theory families* $\{\varnothing\}$, $\{\{1\}\}$ *and* $\{\{0,1\}\}$*, but only two theory systems,* $\{\varnothing\}$ *and* $\{\{0,1\}\}$*. The lattice of theory families of* $\mathcal{I}$ *and the corresponding Leibniz congruence systems are given in the diagram.*





To see that $\mathcal{I}$ is narrowly left monotone, note that the only two different theory families in $\mathrm{ThFam}^{\not\, \sharp}(\mathcal{I})$ *are* $\{\{1\}\}$ *and* $\{\{0,1\}\}$ *and we have*

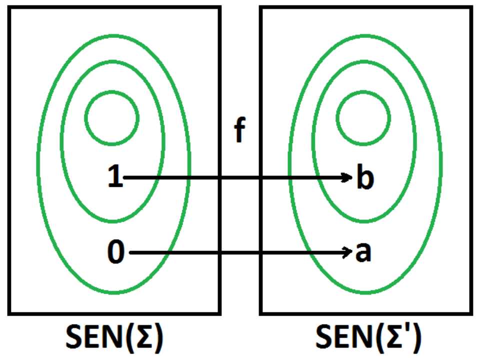
$$\overleftarrow{\{\{1\}\}} = \{\varnothing\} \le \{\{0,1\}\} = \overleftarrow{\{\{0,1\}\}} \quad \text{and} \quad \Omega(\{\{1\}\}) = \Delta^{\mathbf{F}} \le \nabla^{\mathbf{F}} = \Omega(\{\{0,1\}\}).$$

On the other hand, $\mathcal{I}$ is not roughly left monotone, since $\widetilde{\overleftarrow{\{\varnothing\}}} = \{\{0,1\}\} = \widetilde{\overleftarrow{\{\{1\}\}}}$*, but* $\Omega(\{\varnothing\}) \not\le \Omega(\{\{1\}\})$*.*

Next we exhibit a narrowly right monotone but not roughly right monotone π-institution, showing that the class of roughly right monotone π-institutions is a proper subclass of that of the narrowly right monotone ones.

Example 529 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

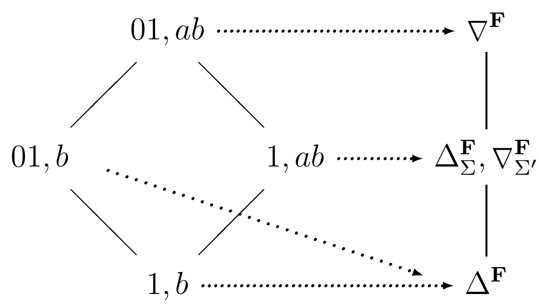
- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a,b\}\}.$$

There are only four theory families in $\mathrm{ThFam}^{\natural}(\mathcal{I})$*, all of which except for* $\{01, b\}$ *are theory systems. Their lattice together with the associated Leibniz congruence systems are shown in the diagram:*



To see that $\mathcal{I}$ *is narrowly right monotone, we check all cases comparing theory families in* $\mathrm{ThFam}^{\natural}(\mathcal{I})$*:*

$$\begin{array}{ll}
\{1,b\} \leq \{01,b\}, & \Omega(\overleftarrow{\{1,b\}}) = \Omega(\{1,b\}) = \Omega(\overleftarrow{\{01,b\}});\\
\{1,b\} \leq \{1,ab\}, & \Omega(\overleftarrow{\{1,b\}}) = \Delta^{\mathbf{F}} \leq \Omega(\overleftarrow{\{1,ab\}});\\
\{01,b\} \leq \{01,ab\}, & \Omega(\overleftarrow{\{01,b\}}) \leq \nabla^{\mathbf{F}} = \Omega(\overleftarrow{\{01,ab\}});\\
\{1,ab\} \leq \{01,ab\}, & \Omega(\overleftarrow{\{1,ab\}}) \leq \nabla^{\mathbf{F}} = \Omega(\overleftarrow{\{01,ab\}}).
\end{array}$$

On the other hand, since $\widetilde{\{1,\varnothing\}} \leq \widetilde{\{1,ab\}}$*, but*

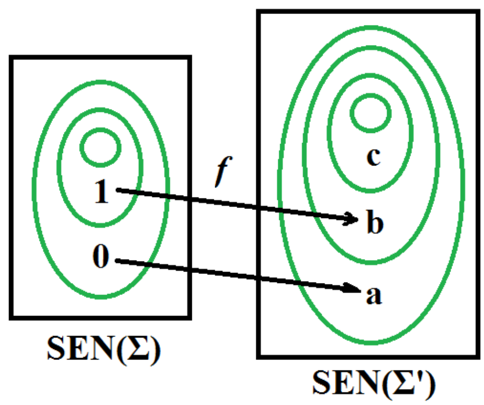
$$\Omega(\overleftarrow{\{1,\varnothing\}}) = \Omega(\{\varnothing,\varnothing\}) \nleq \Omega(\{1,ab\}) = \Omega(\overleftarrow{\{1,ab\}}),$$

$\mathcal{I}$ *is not roughly right monotone.*

Finally, we present an example of a narrowly system monotone π-institution which fails to be roughly system monotone, thereby establishing that the class of roughly system monotone π-institutions is properly contained in the class of narrowly system monotone ones.

Example 530 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with two object* Σ, Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b,c\}$, *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$;
- $N^\flat$ *is the trivial clone.*



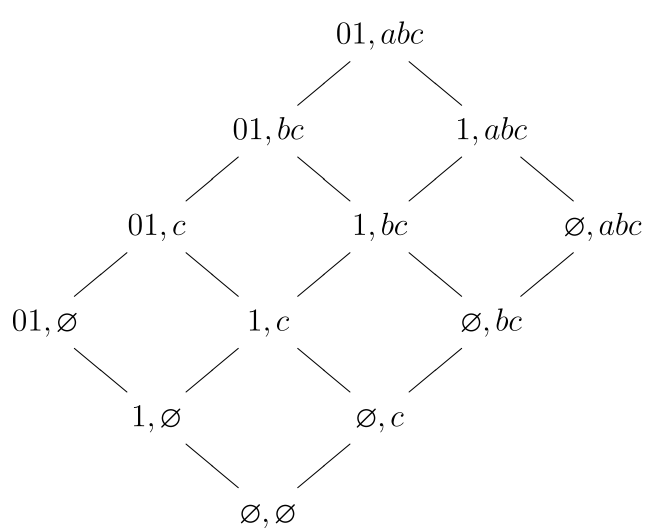
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{c\}, \{b,c\}, \{a,b,c\}\}.$$

$\mathcal{I}$ *has twelve theory families, but only seven theory systems. These are*

$$\overline{\varnothing}, \{\varnothing, c\}, \{\varnothing, bc\}, \{\varnothing, abc\}, \{1, bc\}, \{1, abc\}, \{01, abc\}.$$

The following diagram shows the structure of the lattice of theory families.



To see that $\mathcal{I}$ is narrow system monotone, note that there are only three theory systems in $\mathrm{ThSys}^{\nleq}(\mathcal{I})$*, namely,* $\{1, bc\}$*,* $\{1, abc\}$ *and* $\{01, abc\}$ *and we have* $\{1, bc\} \leq \{1, abc\} \leq \{01, abc\}$ *and, also,*

$$\begin{aligned}\Omega(\{1, bc\}) = \{\Delta^{\mathbf{F}}_{\Sigma}, \{a, bc\}\} \\ \leq \Omega(\{1, abc\}) = \{\Delta^{\mathbf{F}}_{\Sigma}, \nabla^{\mathbf{F}}_{\Sigma'}\} \\ \leq \Omega(\{01, abc\}) = \nabla^{\mathbf{F}}.\end{aligned}$$

On the other hand, setting $T = \{\varnothing, c\}$ *and* $T' = \{\varnothing, bc\}$*, which are both theory systems, we get*

$$\widetilde{T} = \{01, c\} \leq \{01, bc\} = \widetilde{T}',$$

whereas

$$\Omega(T) = \{\nabla^{\mathbf{F}}_{\Sigma}, \{ab, c\}\} \nleq \{\Delta^{\mathbf{F}}_{\Sigma}, \{a, bc\}\} = \Omega(T').$$

Therefore, $\mathcal{I}$ is not roughly system monotone.

The narrow monotonicity properties transfer from the theory families/systems of a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ to all $\mathcal{I}$-filter families/systems on arbitrary $\mathbf{F}$-algebraic systems. Recall the notations $\mathrm{FiFam}^{\mathcal{I}\nleq}(\mathcal{A})$ and $\mathrm{FiSys}^{\mathcal{I}\nleq}(\mathcal{A})$ for the collections of $\mathcal{I}$-filter families and $\mathcal{I}$-filter systems, respectively, of an $\mathbf{F}$-algebraic system $\mathcal{A}$ all of whose components are nonempty.

Theorem 531 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is narrowly family monotone if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}\nleq}(\mathcal{A})$*,* $T \leq T'$ *implies* $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$*;*

(b) $\mathcal{I}$ is narrowly left monotone if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}\nleq}(\mathcal{A})$*,* $\overleftarrow{T} \leq \overleftarrow{T'}$ *implies* $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$*;*

(c) $\mathcal{I}$ is narrowly right monotone if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}\nleq}(\mathcal{A})$*,* $T \leq T'$ *implies* $\Omega^{\mathcal{A}}(\overleftarrow{T}) \leq \Omega^{\mathcal{A}}(\overleftarrow{T'})$*;*

(d) $\mathcal{I}$ is narrowly system monotone if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $T, T' \in \mathrm{FiSys}^{\mathcal{I}\nleq}(\mathcal{A})$*,* $T \leq T'$ *implies* $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$*.*

Proof:

(a) The "if" follows by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and taking into account that, by Lemma 51, $\mathrm{ThFam}(\mathcal{I}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F})$.

For the "only if", suppose that $\mathcal{I}$ is narrowly family monotone and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}\lightning}(\mathcal{A})$, such that $T \leq T'$. Then $\alpha^{-1}(T) \leq \alpha^{-1}(T')$. Since, by Lemmas 51 and 376, both $\alpha^{-1}(T)$ and $\alpha^{-1}(T')$ are in $\mathrm{ThFam}^{\lightning}(\mathcal{I})$, we get, by narrow family monotonicity, $\Omega(\alpha^{-1}(T)) \leq \Omega(\alpha^{-1}(T'))$. Hence, by Proposition 24, $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. Taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$.

(b) The "if" follows as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is narrowly left monotone and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}\lightning}(\mathcal{A})$, such that $\overleftarrow{T} \leq \overleftarrow{T'}$. Then $\alpha^{-1}(\overleftarrow{T}) \leq \alpha^{-1}(\overleftarrow{T'})$. By Lemma 6, $\overleftarrow{\alpha^{-1}(T)} \leq \overleftarrow{\alpha^{-1}(T')}$. Since, by Lemmas 51 and 376, $\alpha^{-1}(T)$ and $\alpha^{-1}(T')$ are in $\mathrm{ThFam}^{\lightning}(\mathcal{I})$, we get, by narrow left monotonicity, $\Omega(\alpha^{-1}(T)) \leq \Omega(\alpha^{-1}(T'))$, whence, by Proposition 24, $\alpha^{-1}(\Omega^{\mathcal{A}}(T)) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T'))$. Thus, by the surjectivity of $\langle F, \alpha \rangle$, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$.

Parts (c) and (d) may be proved similarly. ■

Finally, in analogs of Propositions 514 and 515, we recast the narrow monotonicity properties in terms of the monotonicity of mappings from posets of theory or filter families/systems into posets of congruence systems.

Proposition 532 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *The following statements are equivalent:*

(a) $\mathcal{I}$ *is narrowly family monotone;*

(b) $\Omega : \mathbf{ThFam}^{\lightning}(\mathcal{I}) \to \mathbf{ConSys}^*(\mathcal{I})$ *is monotone;*

(c) $\Omega^{\mathcal{A}} : \mathbf{FiFam}^{\mathcal{I}\lightning}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is monotone, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$.

Similarly, for narrow system monotonicity, we have

Proposition 533 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *The following statements are equivalent:*

(a) $\mathcal{I}$ *is narrowly system monotone;*

(b) $\Omega : \mathbf{ThSys}^{\lightning}(\mathcal{I}) \to \mathbf{ConSys}^*(\mathcal{I})$ *is monotone;*

(c) $\Omega^{\mathcal{A}} : \mathbf{FiSys}^{\mathcal{I}\lightning}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is monotone, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$.

7.5 Rough Complete Monotonicity

In this section we study classes of π-institutions defined using complete monotonicity properties of the Leibniz operator applied on rough equivalence classes.

Definition 534 (Rough c-Monotonicity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is called* **roughly family completely monotone**, *or* **roughly family c-monotone** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,
$$\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T} \quad \textit{implies} \quad \Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T).$$
- $\mathcal{I}$ *is called* **roughly left completely monotone**, *or* **roughly left c-monotone** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,
$$\overleftarrow{\widetilde{T'}} \leq \bigcup_{T \in \mathcal{T}} \overleftarrow{\widetilde{T}} \quad \textit{implies} \quad \Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T).$$
- $\mathcal{I}$ *is called* **roughly right completely monotone**, *or* **roughly right c-monotone** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,
$$\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T} \quad \textit{implies} \quad \Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T}).$$
- $\mathcal{I}$ *is called* **roughly system completely monotone**, *or* **roughly system c-monotone** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$,
$$\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T} \quad \textit{implies} \quad \Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T).$$

We start with an analog of Corollary 503 applying to rough complete monotonicity properties.

Corollary 535 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is roughly left monotone, then it is roughly family c-monotone if and only if it is roughly right c-monotone.*

Proof: Suppose $\mathcal{I}$ is roughly left monotone. Then, by Lemma 502, it is stable. Now note that rough family c-monotonicity is equivalent to the condition that, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,
$$\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T} \quad \text{implies} \quad \Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T),$$

which, by stability, is equivalent to, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,

$$\widetilde{T'} \le \bigcup_{T \in \mathcal{T}} \widetilde{T} \quad \text{implies} \quad \Omega(\overleftarrow{T'}) \le \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T}),$$

and this is equivalent, by definition, to rough right c-monotonicity. ∎

We establish a rough complete monotonicity hierarchy analogous to the one obtained in Proposition 505 for rough monotonicity.

Proposition 536 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is roughly family c-monotone, then it is roughly system c-monotone;*

(b) If $\mathcal{I}$ *is roughly left c-monotone, then it is roughly system c-monotone;*

(c) If $\mathcal{I}$ *is roughly right c-monotone, then it is roughly system c-monotone.*

Proof:

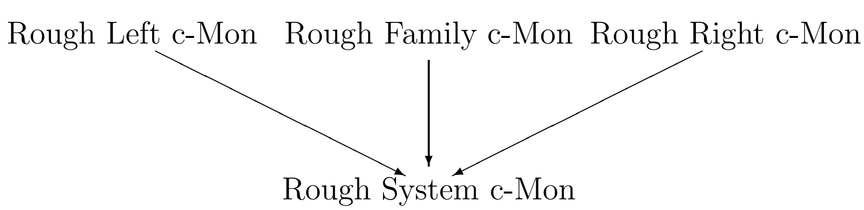
(a) The definition of rough system c-monotonicity is a specialization of that of rough family c-monotonicity, in which the universal quantification is restricted over theory systems.

(b) Suppose $\mathcal{I}$ is roughly left c-monotone and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T'} \le \bigcup_{T \in \mathcal{T}} \widetilde{T'}$. Since $\mathcal{T} \cup \{T'\}$ consists of theory systems, $\overleftarrow{T} = T$, for all $T \in \mathcal{T}$, and $\overleftarrow{T'} = T'$. Hence, we get $\widetilde{\overleftarrow{T'}} \le \bigcup_{T \in \mathcal{T}} \widetilde{\overleftarrow{T}}$. Thus, by rough left monotonicity, we get $\Omega(T') \le \bigcup_{T \in \mathcal{T}} \Omega(T)$. Therefore, $\mathcal{I}$ is roughly system c-monotone.

(c) Suppose $\mathcal{I}$ is roughly right monotone and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T'} \le \bigcup_{T \in \mathcal{T}} \widetilde{T}$. Then, by rough right monotonicity, $\Omega(\overleftarrow{T'}) \le \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$. Since $\mathcal{T} \cup \{T'\}$ consists of theory systems, $\overleftarrow{T} = T$, for all $T \in \mathcal{T}$, and $\overleftarrow{T'} = T'$, whence $\Omega(T') \le \bigcup_{T \in \mathcal{T}} \Omega(T)$ and, hence, $\mathcal{I}$ is roughly system c-monotone.

∎

We have now established the following **rough c-monotonicity hierarchy** of π-institutions.



In an analog of Proposition 506, it is shown that being roughly left c-monotone is equivalent to being roughly system c-monotone and stable.

Proposition 537 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is roughly left c-monotone if and only if it is roughly system c-monotone and stable.*

Proof: Suppose, first, that $\mathcal{I}$ is roughly left c-monotone. Then, by Proposition 536, it is roughly system c-monotone. Moreover, it is, a fortiori, roughly left monotone and, hence, by Lemma 502, it is stable.

Assume, conversely, that $\mathcal{I}$ is stable and roughly system c-monotone. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\overleftarrow{\overleftarrow{T'}} \leq \bigcup_{T \in \mathcal{T}} \overleftarrow{\overleftarrow{T}}$. Then, since $\{\overleftarrow{T} : T \in \mathcal{T}\} \cup \{\overleftarrow{T'}\} \subseteq \mathrm{ThSys}(\mathcal{I})$, we get, by rough system c-monotonicity, $\Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$. Thus, by stability, $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. We conclude that $\mathcal{I}$ is roughly left c-monotone. ∎

We show, next, that the rough complete monotonicity hierarchy collapses to two classes under stability and to a single class under rough systemicity.

Proposition 538 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is stable and roughly system c-monotone, then it is roughly left c-monotone.*

(b) If $\mathcal{I}$ *is stable, then it is roughly family c-monotone if and only if it is roughly right c-monotone.*

Proof:

(a) By Proposition 537.

(b) Suppose that $\mathcal{I}$ is stable. Then, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\Omega(\overleftarrow{T}) = \Omega(T)$, $T \in \mathcal{T}$, and $\Omega(\overleftarrow{T'}) = \Omega(T')$, whence the conditions defining rough family c-monotonicity and rough right c-monotonicity become identical. Therefore, under stability, roughly family and roughly right c-monotone π-institutions coincide.

∎

By Proposition 538, under stability, we get the reduced hierarchy

Rough Family/Right c-Monotonicity

↓

Rough System/Left c-Monotonicity

We also get that rough systemicity causes the further collapse of the entire hierarchy into a single class.

Proposition 539 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For roughly systemic* π*-institutions, all four rough c-monotonicity classes coincide.*

Proof: Since rough systemicity implies stability, by Proposition 538, it suffices to show that, if $\mathcal{I}$ is roughly system c-monotone and roughly systemic, then it is roughly family c-monotone. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T}$. By rough systemicity, $\overleftarrow{\widetilde{T}} = \widetilde{T}$, for all $T \in \mathcal{T}$, and $\overleftarrow{\widetilde{T'}} = \widetilde{T'}$. Therefore, $\overleftarrow{\widetilde{T'}} \leq \bigcup_{T \in \mathcal{T}} \overleftarrow{\widetilde{T}}$. Thus, since $\{\overleftarrow{T} : T \in \mathcal{T}\} \cup \{\overleftarrow{T'}\}$ consists of theory systems, by rough system c-monotonicity, $\Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$. Since rough systemicity implies stability, $\Omega(\overleftarrow{T}) = \Omega(T)$, for all $T \in \mathcal{T}$, and $\Omega(\overleftarrow{T'}) = \Omega(T')$. Thus, $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. We conclude that $\mathcal{I}$ is roughly family c-monotone. ■

We present several examples to show that all four rough complete monotonicity classes depicted in the diagram above are different and that the hierarchy is exactly as shown.

The first example gives a roughly left c-monotone π-institution which is not roughly family c-monotone. Thus, it shows that the class of roughly family c-monotone π-institutions is properly contained in the class of roughly system c-monotone π-institutions and that, moreover, the class of roughly left c-monotone π-institutions is not a subclass of the class of roughly family c-monotone π-institutions.

Example 540 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

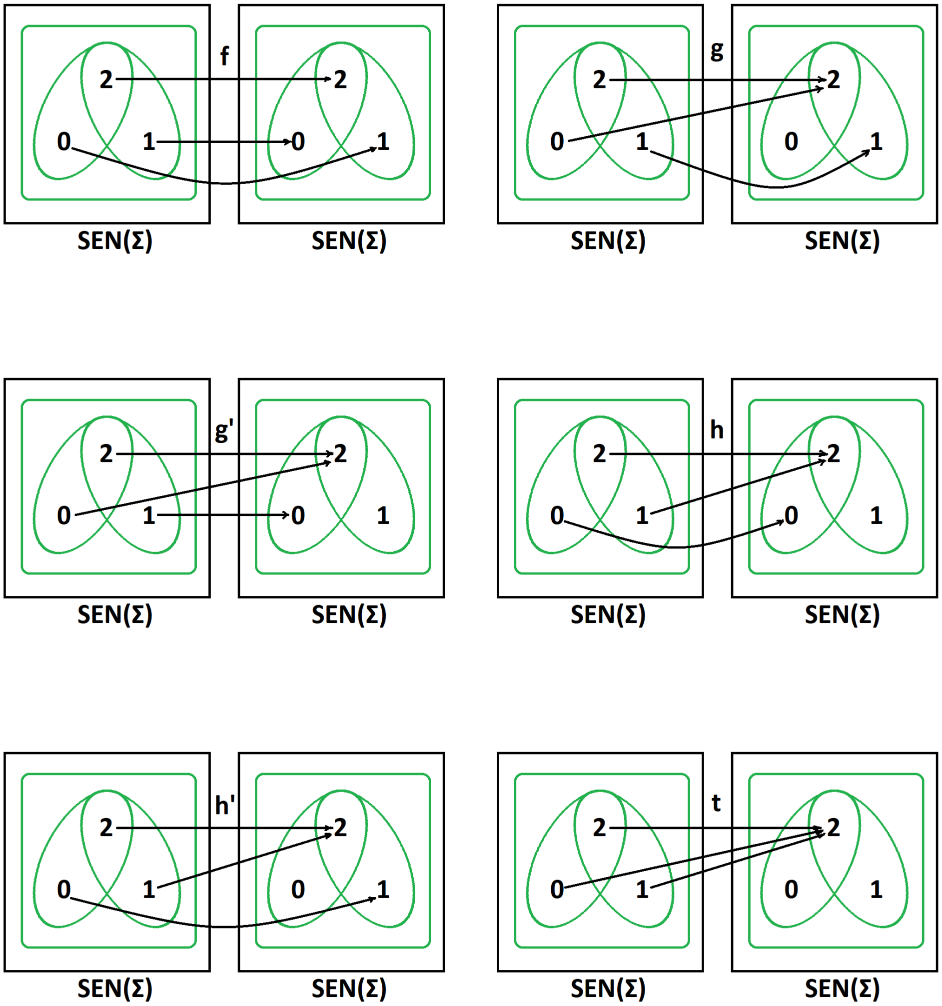
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and six non-identity morphisms* $f, g, g', h, h', t : \Sigma \to \Sigma$*, in which composition is defined by the following table, whose entry in row* k *and column* ℓ *is the result of the composition* $\ell \circ k$*:*

$\circ$	f	g	g'	h	h'	t
f	f	h'	h	g'	g	t
g	g'	g	g'	t	t	t
g'	g	t	t	g'	g	t
h	h'	t	t	h	h'	t
h'	h	h'	h	t	t	t
t	t	t	t	t	t	t

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given, on objects, by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and, on morphisms, by the following table, whose entries in column* k *give the values of the function* $\mathrm{SEN}^\flat(k) : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SEN}^\flat(\Sigma)$*:*

x	f	g	g'	h	h'	t
0	1	2	2	0	1	2
1	0	1	0	2	2	2
2	2	2	2	2	2	2

- *$N^\flat$ is the trivial clone.*



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by setting

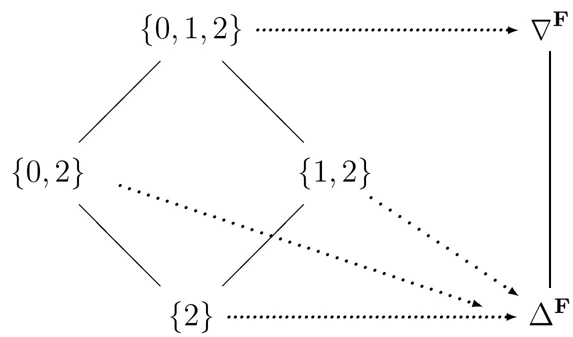
$$\mathcal{C}_\Sigma = \{\{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}.$$

Note that since $\mathcal{I}$ has theorems, rough equivalence coincides with the identity relation on ThFam$(\mathcal{I})$*.*

The following table gives the theory families and the theory systems of the π-institution $\mathcal{I}$:

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{0,2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



$\mathcal{I}$ has only two theory systems, $\mathrm{Thm}(\mathcal{I}) = \{\{2\}\}$*, and* $\mathrm{SEN} = \{\{0,1,2\}\}$.

To show that $\mathcal{I}$ is (roughly) left c-monotone, assume that, for some $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T'} \leq \bigcup_{T\in\mathcal{T}} \overleftarrow{T}$.

- *If $\bigcup_{T\in\mathcal{T}} \overleftarrow{T} = \{\{0,1,2\}\}$, then $\{\{0,1,2\}\} \in \mathcal{T}$ and, hence,*
$$\Omega(T') \leq \nabla^{\mathbf{F}} = \Omega(\{\{0,1,2\}\}) \leq \bigcup_{T\in\mathcal{T}} \Omega(T);$$
- *If $\bigcup_{T\in\mathcal{T}} \overleftarrow{T} = \{\{2\}\}$, then $T' \neq \{\{0,1,2\}\}$, whence*
$$\Omega(T') = \Delta^{\mathbf{F}} \leq \bigcup_{T\in\mathcal{T}} \Omega(T).$$

Thus, in any case, $\Omega(T') \leq \bigcup_{T\in\mathcal{T}} \Omega(T)$ and $\mathcal{I}$ is roughly left c-monotone.

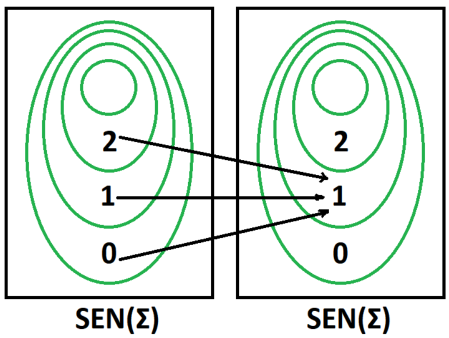
On the other hand, we have
$$\{\{0,1,2\}\} \leq \{\{0,2\}\} \cup \{\{1,2\}\},$$
whereas
$$\Omega(\{\{0,1,2\}\}) = \nabla^{\mathbf{F}} \nleq \Delta^{\mathbf{F}} = \Omega(\{\{0,2\}\}) \cup \Omega(\{\{1,2\}\}).$$
Therefore, $\mathcal{I}$ is not roughly family c-monotone.

An additional conclusion obtained from Example 540, combined with the statement of Corollary 535, is that the class of roughly left c-monotone π-institutions is not a subclass of the class of roughly right c-monotone π-institutions either, since that inclusion would imply that the former is also a subclass of the class of roughly family c-monotone π-institutions, contradicting Example 540.

The second example shows that there exists a roughly family c-monotone π-institution that is not roughly right c-monotone, thus showing, on the one hand, that the class of roughly right c-monotone π-institutions is properly included in the class of roughly system c-monotone π-institutions and, on the other, that roughly family c-monotone π-institutions do not form a subclass of roughly right c-monotone π-institutions.

Example 541 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = \mathrm{SEN}^\flat(f)(2) = 1$;
- $N^\flat$ *is the clone generated by the unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *specified by* $\sigma^\flat_\Sigma(0) = 2$, $\sigma^\flat_\Sigma(1) = 1$ *and* $\sigma^\flat_\Sigma(2) = 2$.



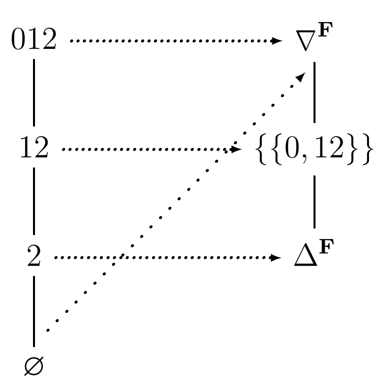
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

There are four theory families, but only three theory systems. The action of $\overleftarrow{\ }$ *on theory families is given in the table below.*

T	$\overleftarrow{T}$
$\{\varnothing\}$	$\{\varnothing\}$
$\{2\}$	$\{\varnothing\}$
$\{12\}$	$\{12\}$
$\{012\}$	$\{012\}$

The lattice of theory families of $\mathcal{I}$ *together with the Leibniz congruence systems are shown in the diagram.*



We show that $\mathcal{I}$ is roughly family c-monotone. To this end, suppose $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{T'} \leq \bigcup_{T\in\mathcal{T}} \widetilde{T}$.

- *If $\widetilde{T'} = \{012\}$, then we must have $\{\varnothing\} \in \mathcal{T}$ or $\{012\} \in \mathcal{T}$. Hence, we get $\Omega(T') = \nabla^{\mathbf{F}} = \bigcup_{T\in\mathcal{T}} \Omega(T)$;*
- *If $\widetilde{T'} = \{12\}$, then $\mathcal{T}$ must include one of $\{12\}$, $\{\varnothing\}$ or $\{012\}$. Hence, we get $\Omega(T') = \{\{0,12\}\} \leq \bigcup_{T\in\mathcal{T}} \Omega(T)$;*
- *If $\widetilde{T'} = \{2\}$, then $T' = \{2\}$ and, hence, $\Omega(T') = \Delta^{\mathbf{F}} \leq \bigcup_{T\in\mathcal{T}} \Omega(T)$.*

Therefore, $\mathcal{I}$ is indeed roughly family c-monotone.

On the other hand, we have $\widetilde{\{2\}} = \{2\} \leq \{12\} = \widetilde{\{12\}}$, whereas

$$\Omega(\overleftarrow{\{2\}}) = \Omega(\{\varnothing\}) = \nabla^{\mathbf{F}} \nleq \{\{0,12\}\} = \Omega(\{12\}) = \Omega(\overleftarrow{\{12\}}).$$

Therefore, $\mathcal{I}$ is not roughly right c-monotone.

An additional conclusion obtained from Example 541, combined with the statement of Corollary 535, is that the class of roughly family c-monotone π-institutions is not a subclass of the class of roughly left c-monotone π-institutions. Otherwise, by Corollary 535, rough family c-monotonicity would imply rough right c-monotonicity, contradicting Example 541.

The third example shows that there exists a roughly right c-monotone π-institution that is not roughly left c-monotone. It establishes that the class of roughly left c-monotone π-institutions is properly contained in the class of roughly system c-monotone π-institutions and, also, that the class of roughly right c-monotone π-institutions does not form a subclass of the class consisting of the roughly left c-monotone ones.

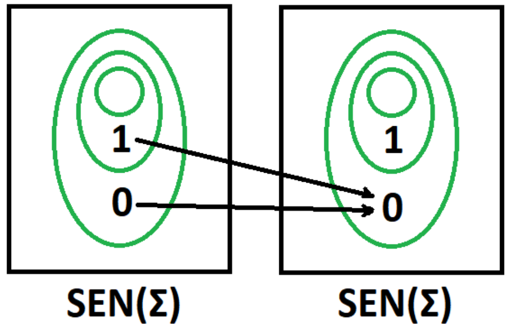
Example 542 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be the algebraic system defined as follows:*

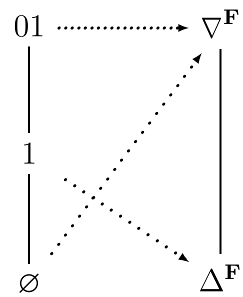
- *$\mathbf{Sign}^\flat$ is the category with the single object Σ and a single (non-identity) morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- *$\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ is defined by $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$ and $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$;*
- *$N^\flat$ is the trivial clone, consisting of the projections only.*

Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\}.$$

$\mathcal{I}$ has three theory families $\varnothing$, $\{\{1\}\}$ and $\{\{0,1\}\}$, but only two theory systems, $\varnothing$ and $\{\{0,1\}\}$. The lattice of theory families of $\mathcal{I}$ and the corresponding Leibniz congruence systems are given in the diagram.





We show that $\mathcal{I}$ is roughly right c-monotone. Suppose $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T}$. Since, for all $T \in \mathcal{T}$, we have $\overleftarrow{T} = \{\varnothing\}$ or $\overleftarrow{T} = \{01\}$, we have, trivially,

$$\Omega(\overleftarrow{T'}) \leq \nabla^{\mathbf{F}} = \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T}).$$

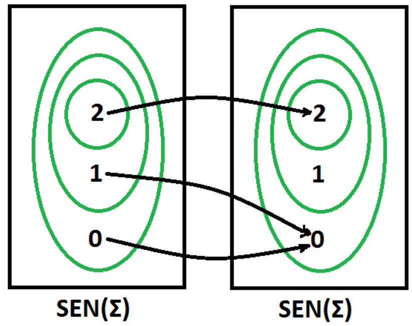
Thus, $\mathcal{I}$ is indeed roughly right monotone.

On the other hand, we have $\widetilde{\overleftarrow{\{\varnothing\}}} = \{01\} = \widetilde{\{\varnothing\}} = \widetilde{\overleftarrow{\{1\}}}$, but $\Omega(\{\varnothing\}) \nleq \Omega(\{1\})$. Therefore, $\mathcal{I}$ is not roughly left c-monotone.

The last example in this series depicts a roughly right c-monotone π-institution, which is not roughly family c-monotone. This shows that the class of roughly right c-monotone π-institutions does not form a subclass of the class of roughly family c-monotone ones.

Example 543 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object Σ and a single non-identity morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial clone.*

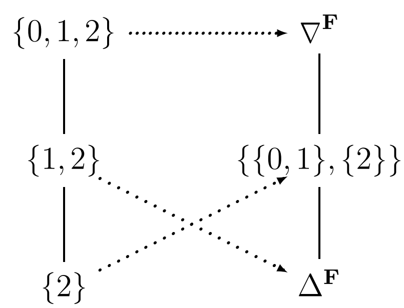


Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by setting $\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}$. Since $\mathcal{I}$ has theorems, rough equivalence on ThFam$(\mathcal{I})$ *coincides with the identity relation.*

The following table gives the theory families and the theory systems of the π-institution $\mathcal{I}$:

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

The lattice of theory families and the corresponding Leibniz congruence systems are depicted below:



We show that $\mathcal{I}$ is roughly right c-monotone. Suppose $\mathcal{T} \cup \{T'\} \subseteq$ ThFam$(\mathcal{I})$*, such that $\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T}$.*

- *If $T' = \{012\}$, then $T' \in \mathcal{T}$, whence we get $\Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$;*
- *If $T' = \{12\}$, then $\{12\} \in \mathcal{T}$ or $\{012\} \in \mathcal{T}$. In either case $\Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$;*
- *If $T' = \{2\}$, then $\mathcal{T} \neq \varnothing$ and, since $\overleftarrow{\{12\}} = \{2\}$, we get $\Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$.*

Therefore, $\mathcal{I}$ is roughly right c-monotone. On the other hand, since $\{2\} \leq \{12\}$, but $\Omega(\{2\}) = \{\{01,2\}\} \nleq \Delta^{\mathbf{F}} = \Omega(\{12\})$, $\mathcal{I}$ is not roughly family c-monotone.

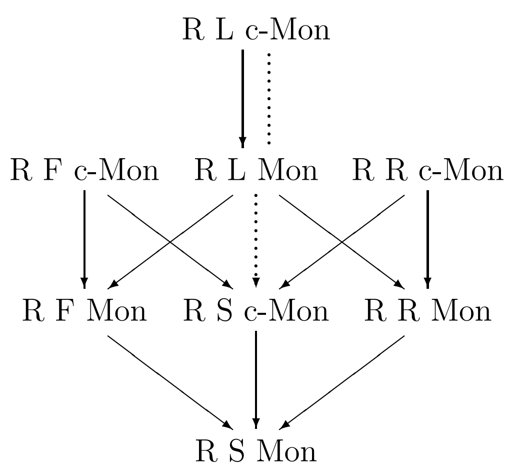
We conclude, after these four examples, that the structure of the rough complete monotonicity hierarchy is exactly as depicted in the diagram and no two classes are identical.

We look, next at the connections between the classes in the rough monotonicity and rough complete monotonicity hierarchies. We have a straightforward

Proposition 544 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly family (respectively, left, right, system) c-monotone, then it is roughly family (respectively, left, right, system) monotone.*

Proof: The condition defining a rough monotonicity class is a special case of the condition defining the respective rough c-monotonicity class, where the collection $\mathcal{T}$, in that definition, is taken to be a singleton. ■

Proposition 544, in view of Propositions 505 and 536, establishes the hierarchy depicted in the diagram (the dotted line and arrow represent jointly a single arrow signifying the inclusion of the class of roughly left c-monotone into the class of roughly system c-monotone π-institutions).



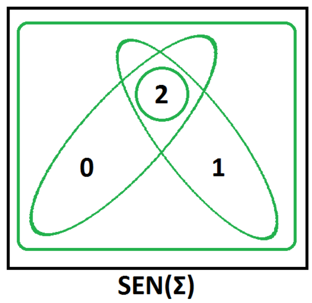
We present an example to show that the two hierarchies are separated. It shows a π-institution, which belongs to all steps of the rough monotonicity hierarchy but to none of the four rough complete monotonicity classes.

Example 545 *Define the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *as follows:*

- $\mathbf{Sign}^\flat$ *is a trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*

- *$N^\flat$ is the clone generated by the unary natural transformation $\sigma^\flat$: $\mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$, given by*

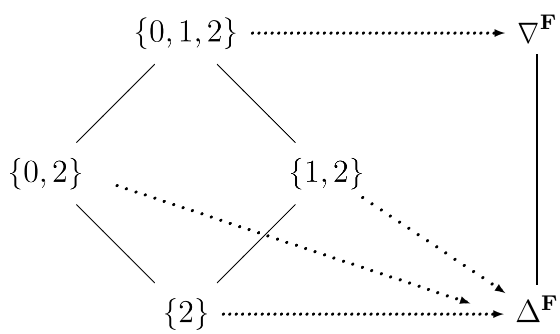
$x \in \mathrm{SEN}^\flat(\Sigma)$	$\sigma^\flat_\Sigma(x)$
0	1
1	2
2	0



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ by setting

$$\mathcal{C}_\Sigma = \{\{2\},\{0,2\},\{1,2\},\{0,1,2\}\}.$$

It is easy to see that the lattices of theory families and corresponding Leibniz congruence systems are as given in the diagram.



Since $\mathbf{Sign}^\flat$ *is trivial, $\mathcal{I}$ is systemic and, since $\mathcal{I}$ has theorems, rough equivalence is the identity relation on* $\mathrm{FiFam}(\mathcal{I})$. *We conclude that all four rough monotonicity properties for $\mathcal{I}$ coincide and, moreover, they are identical with both monotonicity properties, which they also coincide, due to systemicity. The same holds for c-monotonicity. All four rough c-monotonicity properties coincide and they, in turn, are identical with all c-monotonicity conditions.*

From the diagram one can verify immediately that $\mathcal{I}$ is (roughly left, right and family) monotone, On the other hand, we have $\{\{0,1,2\}\} \le \{\{0,2\}\} \cup \{\{1,2\}\}$, but, obviously, $\Omega(\{\{0,1,2\}\}) \nleq \Omega(\{\{0,2\}\}) \cup \Omega(\{\{1,2\}\})$. Taking into account that $\mathcal{I}$ is systemic, we conclude that $\mathcal{I}$ fails to be roughly system c-monotone.

We turn, next, our attention to the relations between the classes in the rough c-monotonicity hierarchy and those in the c-monotonicity hierarchy. We start by showing that possessing any type of c-monotonicity forces a π-institution to either have theorems or, else, to have only one theory system rough equivalence class.

Proposition 546 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is left c-monotone without theorems, then* $|\widetilde{\mathrm{ThFam}}(\mathcal{I})| = 1$*.*

(b) If $\mathcal{I}$ *is system c-monotone without theorems, then* $|\widetilde{\mathrm{ThSys}}(\mathcal{I})| = 1$*.*

Proof:

(a) Suppose that $\mathcal{I}$ is left c-monotone and does not have theorems. If $|\widetilde{\mathrm{ThFam}}(\mathcal{I})| > 1$, then there exists $T \in \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{T} \neq \mathrm{SEN}^\flat$. Thus, we get

$$\overleftarrow{\overline{\varnothing}} = \overline{\varnothing} \leq \overleftarrow{T} \quad \text{and} \quad \Omega(\overline{\varnothing}) \nleq \Omega(T).$$

Therefore, $\mathcal{I}$ is not left c-monotone, a contradiction. Thus, we must have $|\widetilde{\mathrm{ThFam}}(\mathcal{I})| = 1$, as claimed.

(b) Suppose that $\mathcal{I}$ is system c-monotone and does not have theorems. If $|\widetilde{\mathrm{ThSys}}(\mathcal{I})| > 1$, then there exists $T \in \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T} \neq \mathrm{SEN}^\flat$. Thus, we get

$$\overline{\varnothing} < T \quad \text{and} \quad \Omega(\overline{\varnothing}) \nleq \Omega(T).$$

Therefore, $\mathcal{I}$ is not system c-monotone, a contradiction. Thus, we must have $|\widetilde{\mathrm{ThSys}}(\mathcal{I})| = 1$, as claimed.

■

We can establish the following relations between c-monotonicity and rough c-monotonicity classes.

Proposition 547 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is left c-monotone, then it is roughly left c-monotone;*

(b) If $\mathcal{I}$ *is family c-monotone, then it is roughly family c-monotone;*

(c) If $\mathcal{I}$ *is right c-monotone, then it is roughly right c-monotone;*

(d) If $\mathcal{I}$ *is system c-monotone, then it is roughly system c-monotone.*

Proof:

(a) Suppose that $\mathcal{I}$ is left c-monotone. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\overleftarrow{\overleftarrow{T'}} \leq \bigcup_{T \in \mathcal{T}} \overleftarrow{\overleftarrow{T}}$. If $\mathcal{I}$ has theorems, then $\overleftarrow{\overleftarrow{T}} = \overleftarrow{T}$, for all $T \in \mathcal{T}$, and $\overleftarrow{\overleftarrow{T'}} = \overleftarrow{T'}$, whence $\overleftarrow{T'} \leq \bigcup_{t \in \mathcal{T}} \overleftarrow{T}$. Thus, by left c-monotonicity, $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. On the other hand, if $\mathcal{I}$ does not have theorems, then, by Proposition 546, $|\widetilde{\mathrm{ThFam}}(\mathcal{I})| = 1$, whence, by Theorem 370, $\Omega(T') = \bigcup_{T \in \mathcal{T}} \Omega(T)$.

(b) Suppose that $\mathcal{I}$ is family c-monotone. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T}$. If $\mathcal{I}$ has theorems, then we get $\widetilde{T} = T$, for all $T \in \mathcal{T}$, and $\widetilde{T'} = T'$. Thus, $T' \leq \bigcup_{T \in \mathcal{T}} T$. By family c-monotonicity, we now get $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. On the other hand, if $\mathcal{I}$ does not have theorems, then, by Propositions 186 and 546, $|\widetilde{\mathrm{ThFam}}(\mathcal{I})| = 1$, whence, by Theorem 370, $\Omega(T') = \bigcup_{T \in \mathcal{T}} \Omega(T)$.

(c) Suppose that $\mathcal{I}$ is right c-monotone. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T}$. If $\mathcal{I}$ has theorems, then we get $\widetilde{T} = T$, for all $T \in \mathcal{T}$, and $\widetilde{T'} = T'$. Thus, $T' \leq \bigcup_{T \in \mathcal{T}} T$. By right c-monotonicity, we now get $\Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$. On the other hand, if $\mathcal{I}$ does not have theorems, then, by Propositions 187 and 546, $|\widetilde{\mathrm{ThSys}}(\mathcal{I})| = 1$, whence, by Theorem 370, $\Omega(\overleftarrow{T'}) = \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$.

(d) Suppose that $\mathcal{I}$ is system c-monotone. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T}$. If $\mathcal{I}$ has theorems, then we get $\widetilde{T} = T$, for all $T \in \mathcal{T}$, and $\widetilde{T'} = T'$. Thus, $T' \leq \bigcup_{T \in \mathcal{T}} T$. By system c-monotonicity, we now get $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. On the other hand, if $\mathcal{I}$ does not have theorems, then, by Proposition 546, $|\widetilde{\mathrm{ThSys}}(\mathcal{I})| = 1$, whence, by Theorem 370, $\Omega(T') = \bigcup_{T \in \mathcal{T}} \Omega(T)$.

■

We can now prove the following additional, and more precise, relations.

Theorem 548 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a non-almost inconsistent* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is family (left, respectively) c-monotone if and only if it is roughly family (left, respectively) c-monotone and has theorems.*

Proof: Suppose $\mathcal{I}$ is family or left c-monotone. Since, by hypothesis, it is not almost inconsistent, $|\widetilde{\mathrm{ThFam}}(\mathcal{I})| > 1$. Thus, by Proposition 546, $\mathcal{I}$ has theorems. Moreover, by Theorem 547, it is roughly family or left c-monotone, respectively.

Assume, conversely, that $\mathcal{I}$ is roughly family (or left c-monotone) and has theorems. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $T' \leq \bigcup_{T \in \mathcal{T}} T$ (or $\overleftarrow{T'} \leq \bigcup_{T \in \mathcal{T}} \overleftarrow{T}$, respectively). Since $\mathcal{I}$ has theorems, rough equivalence coincides with the identity relation on $\mathrm{ThFam}(\mathcal{I})$, whence, we get $\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T}$ (or $\overleftarrow{\overleftarrow{T'}} \leq \bigcup_{T \in \mathcal{T}} \overleftarrow{\overleftarrow{T}}$). Using rough family (or left, respectively) c-monotonicity, we obtain $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. Therefore, $\mathcal{I}$ is family (or left) c-monotone. ■

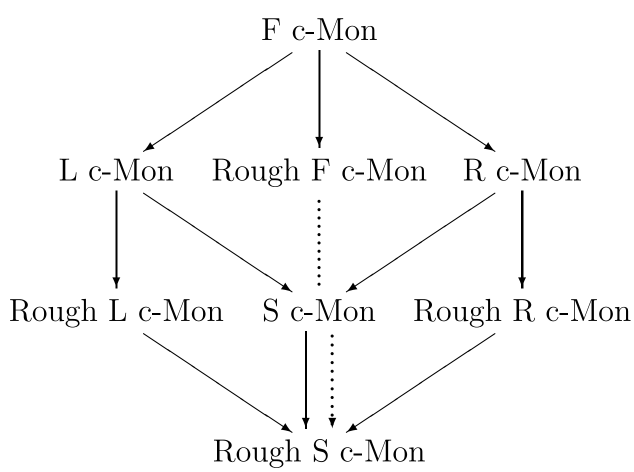
Analogously, for the cases of system and right c-monotonicity, we get the following

Theorem 549 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, that has a theory family* $T \neq \mathrm{SEN}^\flat$ *such that* $\overleftarrow{T} \neq \overline{\varnothing}$*.* $\mathcal{I}$ *is system (right, respectively) c-monotone if and only if it roughly system (right, respectively) c-monotone and has theorems.*

Proof: Suppose $\mathcal{I}$ is system or right c-monotone. Since, by hypothesis, it has a theory system $\overleftarrow{T} \neq \mathrm{SEN}^\flat, \overline{\varnothing}$, we get $|\widetilde{\mathrm{ThSys}}(\mathcal{I})| > 1$. Thus, by Proposition 546, $\mathcal{I}$ must have theorems. Moreover, by Theorem 547, it is roughly system or right c-monotone, respectively.

Assume, conversely, that $\mathcal{I}$ is roughly system (or right) c-monotone and has theorems. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$ (or $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, respectively), such that $T' \leq \bigcup_{T \in \mathcal{T}} T$. Since $\mathcal{I}$ has theorems, rough equivalence coincides with the identity relation on $\mathrm{ThFam}(\mathcal{I})$, whence, we get $\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T}$. By rough system (or right, respectively) c-monotonicity, we obtain $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$ (or $\Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$, respectively). Therefore, $\mathcal{I}$ is system (or right) c-monotone. ∎

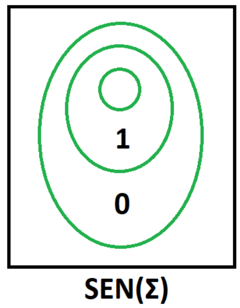
The preceding propositions allow us to draw the following hierarchical diagram concerning the complete monotonicity and the rough complete monotonicity classes.



To see that the rough c-monotonicity classes are separated from the c-monotonicity classes, we give an example. A π-institution is presented which belongs to all four rough c-monotonicity classes, but fails to be system c-monotone and, therefore, belongs to none of the four c-monotonicity classes. The secret lies, of course, in the absence of theorems.

Example 550 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with the single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$*;*
- $N^\flat$ *is the trivial clone.*

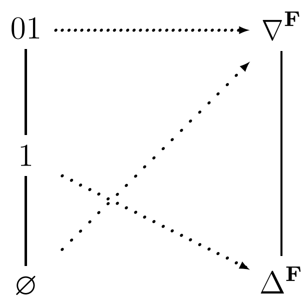


Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0, 1\}\}.$$

$\mathcal{I}$ *has three theory families* $\{\varnothing\}$ *and* $\{\{1\}\}$ *and* $\{\{0, 1\}\}$*, all of which are theory systems.*

The lattice of theory families of $\mathcal{I}$ *and the corresponding Leibniz congruence systems are given in the diagram.*



$\mathcal{I}$ *belongs to all four classes of the rough c-monotonicity hierarchy. Indeed, since it is systemic, all four rough monotonicity conditions boil down to checking that, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$*,* $\widetilde{T'} \le \bigcup_{T \in \mathcal{T}} \widetilde{T}$ *implies* $\Omega(T') \le \bigcup_{T \in \mathcal{T}} \Omega(T)$*.*

- *If* $\bigcup_{T \in \mathcal{T}} \widetilde{T} = \{01\}$*, then* $\mathcal{T}$ *must include* $\{\varnothing\}$ *or* $\{01\}$*, whence* $\Omega(T') \le \nabla^{\mathbf{F}} = \bigcup_{T \in \mathcal{T}} \Omega(T)$*;*
- *If* $\bigcup_{T \in \mathcal{T}} \widetilde{T} = \{1\}$*, then* $\widetilde{T'} = \{1\}$ *and, hence,* $\mathcal{T} = \{\{1\}\}$ *and* $T' = \{1\}$*. Thus, the implication holds trivially.*

Since $\bigcup_{T \in \mathcal{T}} \widetilde{T} = \{\varnothing\}$ *cannot occur, we get that* $\mathcal{I}$ *is roughly family c-monotone.*

On the other hand, we have $\{\varnothing\} \le \{1\}$*, whereas* $\Omega(\{\varnothing\}) \nleq \Omega(\{1\})$*, whence* $\mathcal{I}$ *is not system c-monotone.*

Next, we turn to transfer theorems for the rough c-monotonicity classes.

Theorem 551 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is roughly family c-monotone if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T}$ *implies* $\Omega^{\mathcal{A}}(T') \leq \bigcup_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)$;

(b) $\mathcal{I}$ *is roughly left c-monotone if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\overleftarrow{\widetilde{T'}} \leq \bigcup_{T \in \mathcal{T}} \overleftarrow{\widetilde{T}}$ *implies* $\Omega^{\mathcal{A}}(T') \leq \bigcup_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)$;

(c) $\mathcal{I}$ *is roughly right c-monotone if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T}$ *implies* $\Omega^{\mathcal{A}}(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(\overleftarrow{T})$;

(d) $\mathcal{I}$ *is roughly system c-monotone if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T}$ *implies* $\Omega^{\mathcal{A}}(T') \leq \bigcup_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)$.

Proof:

(a) The "if" results by applying the hypothesis to the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$.

For the "only if", suppose that $\mathcal{I}$ is roughly family c-monotone and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T}$. Then we get $\alpha^{-1}(\widetilde{T'}) \leq \alpha^{-1}(\bigcup_{T \in \mathcal{T}} \widetilde{T})$, whence $\alpha^{-1}(\widetilde{T'}) \leq \bigcup_{T \in \mathcal{T}} \alpha^{-1}(\widetilde{T})$. Thus, by Theorem 377, $\widetilde{\alpha^{-1}(T')} \leq \bigcup_{T \in \mathcal{T}} \widetilde{\alpha^{-1}(T)}$. Since, by Lemma 51, $\{\alpha^{-1}(T) : T \in \mathcal{T}\} \cup \{\alpha^{-1}(T')\} \subseteq \mathrm{ThFam}(\mathcal{I})$, we get, by rough family c-monotonicity, $\Omega(\alpha^{-1}(T')) \leq \bigcup_{T \in \mathcal{T}} \Omega(\alpha^{-1}(T))$. Hence, by Proposition 24, we get $\alpha^{-1}(\Omega^{\mathcal{A}}(T')) \leq \bigcup_{T \in \mathcal{T}} \alpha^{-1}(\Omega^{\mathcal{A}}(T))$, i.e., $\alpha^{-1}(\Omega^{\mathcal{A}}(T')) \leq \alpha^{-1}(\bigcup_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T))$. Taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that $\Omega^{\mathcal{A}}(T') \leq \bigcup_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)$.

(b) The "if" is obtained as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is roughly left c-monotone and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\overleftarrow{\widetilde{T'}} \leq \bigcup_{T \in \mathcal{T}} \overleftarrow{\widetilde{T}}$. Then we get $\alpha^{-1}(\overleftarrow{\widetilde{T'}}) \leq \alpha^{-1}(\bigcup_{T \in \mathcal{T}} \overleftarrow{\widetilde{T}})$, whence $\alpha^{-1}(\overleftarrow{\widetilde{T'}}) \leq \bigcup_{T \in \mathcal{T}} \alpha^{-1}(\overleftarrow{\widetilde{T}})$. Thus, by Theorem 377, $\widetilde{\alpha^{-1}(\overleftarrow{T'})} \leq \bigcup_{T \in \mathcal{T}} \widetilde{\alpha^{-1}(\overleftarrow{T})}$. Hence, by Lemma 6, we get $\overleftarrow{\widetilde{\alpha^{-1}(T')}} \leq \bigcup_{T \in \mathcal{T}} \overleftarrow{\widetilde{\alpha^{-1}(T)}}$. Since, by Lemma 51, $\{\alpha^{-1}(T) : T \in \mathcal{T}\} \cup \{\alpha^{-1}(T')\} \subseteq \mathrm{ThFam}(\mathcal{I})$, we get, by rough left c-monotonicity, $\Omega(\alpha^{-1}(T')) \leq \bigcup_{T \in \mathcal{T}} \Omega(\alpha^{-1}(T))$. Hence, by Proposition 24, we get $\alpha^{-1}(\Omega^{\mathcal{A}}(T')) \leq \bigcup_{T \in \mathcal{T}} \alpha^{-1}(\Omega^{\mathcal{A}}(T))$, i.e., $\alpha^{-1}(\Omega^{\mathcal{A}}(T')) \leq$

$\alpha^{-1}(\bigcup_{T\in\mathcal{T}}\Omega^{\mathcal{A}}(T))$. Taking into account the surjectivity of $\langle F,\alpha\rangle$, we conclude that $\Omega^{\mathcal{A}}(T') \leq \bigcup_{T\in\mathcal{T}}\Omega^{\mathcal{A}}(T)$.

(c) The "if" is obtained as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is roughly right c-monotone and let $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $\mathcal{T}\cup\{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{T'} \leq \bigcup_{T\in\mathcal{T}}\widetilde{T}$. Then we get $\alpha^{-1}(\widetilde{T'}) \leq \alpha^{-1}(\bigcup_{T\in\mathcal{T}}\widetilde{T})$, whence $\alpha^{-1}(\widetilde{T'}) \leq \bigcup_{T\in\mathcal{T}}\alpha^{-1}(\widetilde{T})$. Thus, by Theorem 377, $\widetilde{\alpha^{-1}(T')} \leq \bigcup_{T\in\mathcal{T}}\widetilde{\alpha^{-1}(T)}$. Since, by Lemma 51, $\{\alpha^{-1}(T) : T \in \mathcal{T}\}\cup\{\alpha^{-1}(T')\} \subseteq \mathrm{ThFam}(\mathcal{I})$, we get, by rough right c-monotonicity, $\Omega(\overleftarrow{\alpha^{-1}(T')}) \leq \bigcup_{T\in\mathcal{T}}\Omega(\overleftarrow{\alpha^{-1}(T)})$. Thus, by Lemma 6, $\Omega(\alpha^{-1}(\overleftarrow{T'})) \leq \bigcup_{T\in\mathcal{T}}\Omega(\alpha^{-1}(\overleftarrow{T}))$. Hence, by Proposition 24, we get $\alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'})) \leq \bigcup_{T\in\mathcal{T}}\alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T}))$, i.e., $\alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'})) \leq \alpha^{-1}(\bigcup_{T\in\mathcal{T}}\Omega^{\mathcal{A}}(\overleftarrow{T}))$. Taking into account the surjectivity of $\langle F,\alpha\rangle$, we conclude that $\Omega^{\mathcal{A}}(\overleftarrow{T'}) \leq \bigcup_{T\in\mathcal{T}}\Omega^{\mathcal{A}}(\overleftarrow{T})$.

(d) Similar to Part (a).

■

We close this section by giving two characterizations concerning the rough family and rough system c-monotonicity classes, based on mappings between posets satisfying the complete monotonicity property.

Proposition 552 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is roughly family c-monotone;*

(b) $\Omega : \widetilde{\mathbf{ThFam}}(\mathcal{I}) \to \mathbf{ConSys}^*(\mathcal{I})$ *is completely monotone;*

(c) $\Omega^{\mathcal{A}} : \widetilde{\mathbf{FiFam}}^{\mathcal{I}}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is completely monotone, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

Proposition 553 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(a) $\mathcal{I}$ *is roughly system c-monotone;*

(b) $\Omega : \widetilde{\mathbf{ThSys}}(\mathcal{I}) \to \mathbf{ConSys}^*(\mathcal{I})$ *is completely monotone;*

(c) $\Omega^{\mathcal{A}} : \widetilde{\mathbf{FiSys}}^{\mathcal{I}}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is completely monotone, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

7.6 Narrow Complete Monotonicity

In this section we revisit classes of π-institutions defined using complete monotonicity properties of the Leibniz operator. However, complete monotonicity is only applied on theory systems/families all of whose components are nonempty.

Definition 554 (Narrow c-Monotonicity) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is called* **narrowly family completely monotone**, *or* **narrowly family c-monotone** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\lightning}(\mathcal{I})$,
$$T' \leq \bigcup_{T \in \mathcal{T}} T \quad \textit{implies} \quad \Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T).$$
- $\mathcal{I}$ *is called* **narrowly left completely monotone**, *or* **narrowly left c-monotone** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\lightning}(\mathcal{I})$,
$$\overleftarrow{T'} \leq \bigcup_{T \in \mathcal{T}} \overleftarrow{T} \quad \textit{implies} \quad \Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T).$$
- $\mathcal{I}$ *is called* **narrowly right completely monotone**, *or* **narrowly right c-monotone** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\lightning}(\mathcal{I})$,
$$T' \leq \bigcup_{T \in \mathcal{T}} T \quad \textit{implies} \quad \Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T}).$$
- $\mathcal{I}$ *is called* **narrowly system completely monotone**, *or* **narrowly system c-monotone** *for short, if, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^{\lightning}(\mathcal{I})$,
$$T' \leq \bigcup_{T \in \mathcal{T}} T \quad \textit{implies} \quad \Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T).$$

We establish a narrow complete monotonicity hierarchy analogous to the one obtained in Proposition 536 for rough complete monotonicity.

Proposition 555 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

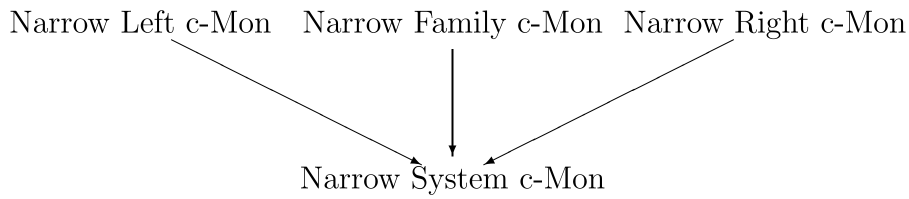
(a) If $\mathcal{I}$ *is narrowly left c-monotone, then it is narrowly system c-monotone;*

(b) If $\mathcal{I}$ *is narrowly family c-monotone, then it is narrowly system c-monotone;*

(c) If $\mathcal{I}$ *is narrowly right c-monotone, then it is narrowly system c-monotone.*

Proof: We sketch a proof that works for all three cases. Suppose that $\mathcal{I}$ is narrowly left (family or right) c-monotone and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^\natural(\mathcal{I})$, such that $T' \leq \bigcup_{T\in\mathcal{T}} T$. Since $\mathcal{T} \cup \{T'\}$ consists of theory systems, we have $\overleftarrow{T} = T$, for all $T \in \mathcal{T}$, and $\overleftarrow{T'} = T'$. Thus, by hypothesis, $\overleftarrow{T'} \leq \bigcup_{T\in\mathcal{T}} \overleftarrow{T}$. Now we apply narrow left (narrow family or narrow right) c-monotonicity to get $\Omega(T') \leq \bigcup_{T\in\mathcal{T}} \Omega(T)$ ($\Omega(T') \leq \bigcup_{T\in\mathcal{T}} \Omega(T)$ or $\Omega(\overleftarrow{T'}) \leq \bigcup_{T\in\mathcal{T}} \Omega(\overleftarrow{T})$). However, in all three cases, we conclude that $\Omega(T') \leq \bigcup_{T\in\mathcal{T}} \Omega(T)$. Therefore, $\mathcal{I}$ is narrowly system c-monotone. ■

We have now established the following **narrow c-monotonicity hierarchy** of π-institutions.



We may establish some additional relationships between those classes once various types of stability and monotonicity are allowed into the mix. First, we show that narrow left c-monotonicity implies exclusive stability and that, under narrow stability, narrow family c-monotonicity and narrow right c-monotonicity coincide.

Proposition 556 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.*

(a) If $\mathcal{I}$ is narrowly left c-monotone, then it is exclusively stable.

(b) If $\mathcal{I}$ is narrowly stable, then it is narrowly family c-monotone if and only if it is narrowly right c-monotone.

Proof:

(a) Suppose $\mathcal{I}$ is narrowly left c-monotone. Let $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, such that $\overleftarrow{T} \in \mathrm{ThSys}^\natural(\mathcal{I})$. Then, since $\overleftarrow{\overleftarrow{T}} = \overleftarrow{T}$, we get, by applying narrow left c-monotonicity, $\Omega(\overleftarrow{T}) = \Omega(T)$. Thus, $\mathcal{I}$ is exclusively stable.

(b) Suppose $\mathcal{I}$ is narrowly stable. Then, for all $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, we have $\Omega(\overleftarrow{T}) = \Omega(T)$. Thus, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^\natural(\mathcal{I})$, the condition $\Omega(T') \leq \bigcup_{T\in\mathcal{T}} \Omega(T)$ is equivalent to the condition $\Omega(\overleftarrow{T'}) \leq \bigcup_{T\in\mathcal{T}} \Omega(\overleftarrow{T})$. Therefore, the condition defining narrow family c-monotonicity is identical to that defining narrow right c-monotonicity.

■

Finally, under narrow systemicity, all four narrow complete monotonicity classes collapse into a single class.

Proposition 557 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is narrowly systemic and narrowly system c-monotone, then it has all four narrow c-monotonicity properties.*

Proof: Suppose that $\mathcal{I}$ is narrowly systemic and narrowly system c-monotone.

- Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nleq}(\mathcal{I})$, such that $\overleftarrow{T'} \leq \bigcup_{T \in \mathcal{T}} \overleftarrow{T}$. By narrow systemicity, $\overleftarrow{T} = T$, for all $T \in \mathcal{T}$, and $\overleftarrow{T'} = T'$, whence $T' \leq \bigcup_{T \in \mathcal{T}} T$ and, moreover, $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^{\nleq}(\mathcal{I})$. Thus, by narrow system c-monotonicity, $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. This proves that $\mathcal{I}$ is narrowly left c-monotone.

- Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nleq}(\mathcal{I})$, such that $T' \leq \bigcup_{T \in \mathcal{T}} T$. By narrow systemicity, $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^{\nleq}(\mathcal{I})$. Thus, by narrow system c-monotonicity, $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. This proves that $\mathcal{I}$ is narrowly family c-monotone.

- Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nleq}(\mathcal{I})$, such that $T' \leq \bigcup_{T \in \mathcal{T}} T$. By narrow systemicity, $\overleftarrow{T} = T$, for all $T \in \mathcal{T}$, and $\overleftarrow{T'} = T'$, and, thus, $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^{\nleq}(\mathcal{I})$. Thus, by narrow system c-monotonicity, $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. But, then, we get $\Omega(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{T})$, whence $\mathcal{I}$ is also narrowly right c-monotone. ■

We reuse some examples to show that all four rough complete monotonicity classes depicted in the diagram above are different and that no inclusions hold among the three top classes.

The first example gives a narrowly left c-monotone π-institution which is neither narrowly family nor narrowly right c-monotone. Thus, it shows that the classes of narrowly family and of narrowly right c-monotone π-institutions are properly contained in the class of narrowly system c-monotone π-institutions and that, moreover, the class of narrowly left c-monotone π-institutions is not a subclass of either the class of narrowly family or the class of narrowly right c-monotone π-institutions.

Example 558 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

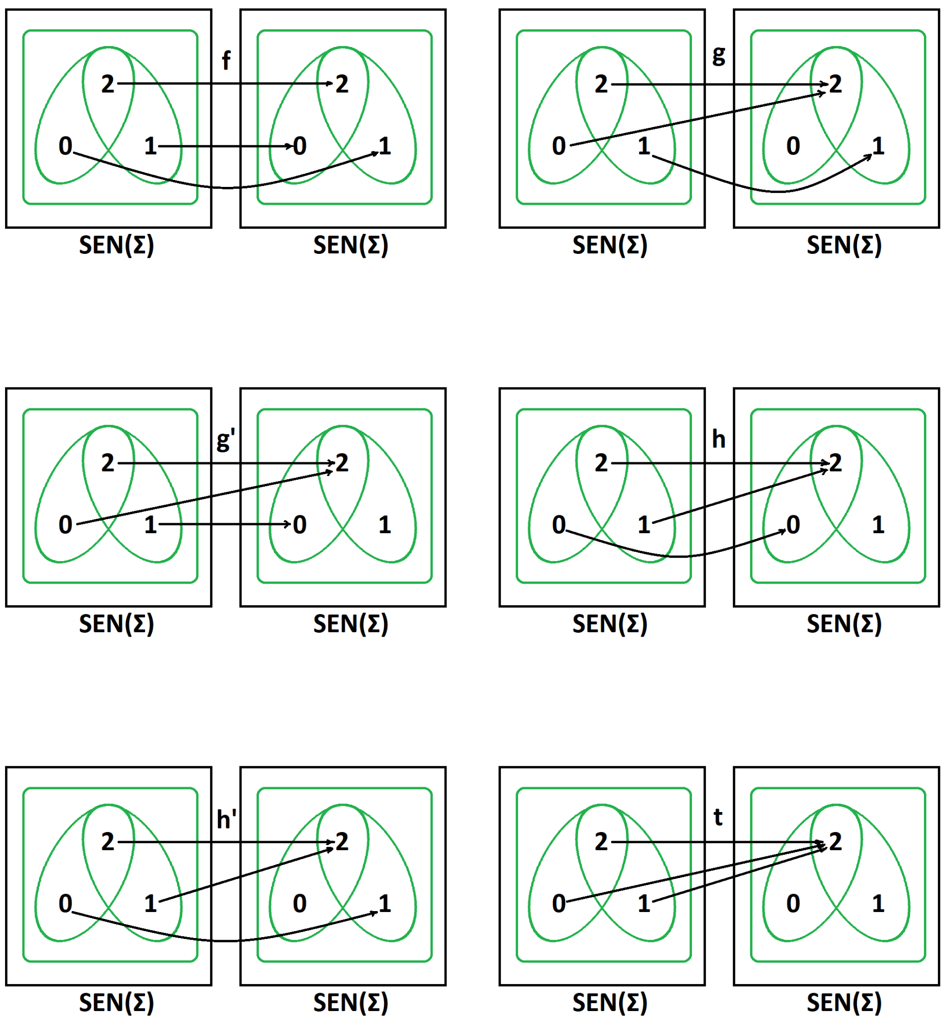
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and six non-identity morphisms* $f, g, g', h, h', t : \Sigma \to \Sigma$, *in which composition is defined by the following table, whose entry in row* k *and column* ℓ *is the result of the composition* $\ell \circ k$:

$\circ$	f	g	g'	h	h'	t
f	f	h'	h	g'	g	t
g	g'	g	g'	t	t	t
g'	g	t	t	g'	g	t
h	h'	t	t	h	h'	t
h'	h	h'	h	t	t	t
t	t	t	t	t	t	t

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given, on objects, by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and, on morphisms, by the following table, whose entries in column* k *give the values of the function* $\mathrm{SEN}^\flat(k) : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SEN}^\flat(\Sigma)$*:*

x	f	g	g'	h	h'	t
0	1	2	2	0	1	2
1	0	1	0	2	2	2
2	2	2	2	2	2	2

- $N^\flat$ *is the trivial clone.*



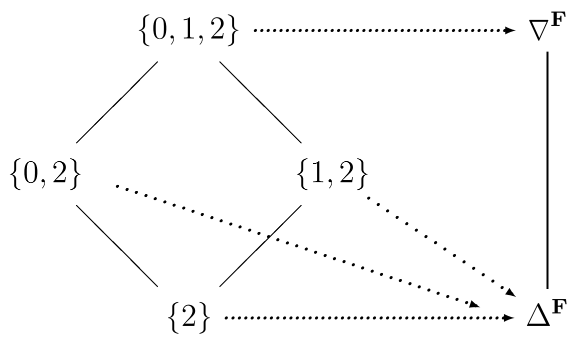
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

$$\mathcal{C}_\Sigma = \{\{2\},\{0,2\},\{1,2\},\{0,1,2\}\}.$$

The following table gives the theory families and the theory systems of the π-institution $\mathcal{I}$:

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{0,2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



$\mathcal{I}$ *has only two theory systems,* $\mathrm{Thm}(\mathcal{I}) = \{\{2\}\}$, *and* $\mathrm{SEN} = \{\{0,1,2\}\}$.

Since $\mathcal{I}$ has theorems, narrow left c-monotonicity coincides with left c-monotonicity. To show that $\mathcal{I}$ is left c-monotone, assume that, for some $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T'} \leq \bigcup_{T\in\mathcal{T}} \overleftarrow{T}$.

- *If $\bigcup_{T\in\mathcal{T}} \overleftarrow{T} = \{\{0,1,2\}\}$, then $\{\{0,1,2\}\} \in \mathcal{T}$ and, hence,*

$$\Omega(T') \leq \nabla^{\mathbf{F}} = \Omega(\{\{0,1,2\}\}) \leq \bigcup_{T\in\mathcal{T}} \Omega(T);$$

- *If $\bigcup_{T\in\mathcal{T}} \overleftarrow{T} = \{\{2\}\}$, then $T' \neq \{\{0,1,2\}\}$, whence*

$$\Omega(T') = \Delta^{\mathbf{F}} \leq \bigcup_{T\in\mathcal{T}} \Omega(T).$$

Thus, in any case, $\Omega(T') \leq \bigcup_{T\in\mathcal{T}} \Omega(T)$ and $\mathcal{I}$ is left completely monotone.

On the other hand, we have

$$\{\{0,1,2\}\} \leq \{\{0,2\}\} \cup \{\{1,2\}\},$$

whereas

$$\begin{aligned} \Omega(\overleftarrow{\{\{0,1,2\}\}}) &= \Omega(\{\{0,1,2\}\}) = \nabla^{\mathbf{F}} \\ &\nleq \Delta^{\mathbf{F}} \\ &= \Omega(\{\{2\}\}) \cup \Omega(\{\{2\}\}) \\ &= \Omega(\overleftarrow{\{\{0,2\}\}}) \cup \Omega(\overleftarrow{\{\{1,2\}\}}). \end{aligned}$$

Therefore, $\mathcal{I}$ is not (narrowly) right c-monotone. Using the same theory families, we also get $\{\{0,1,2\}\} \leq \{\{0,2\}\} \cup \{\{1,2\}\}$, whereas $\Omega(\{\{0,1,2\}\}) = \nabla^{\mathbf{F}} \nleq \Delta^{\mathbf{F}} = \Omega(\{\{0,2\}\}) \cup \Omega(\{\{1,2\}\})$, whence $\mathcal{I}$ is not (narrowly) family c-monotone either.

The second example shows that there exists a narrowly family c-monotone π-institution that is not narrowly right c-monotone, thus showing that narrowly family c-monotone π-institutions do not form a subclass of the class of narrowly right c-monotone π-institutions.

Example 559 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- *$\mathbf{Sign}^\flat$ is the category with the single object Σ and four non-identity morphisms $f, g, o, t : \Sigma \to \Sigma$, whose composition table is the following:*

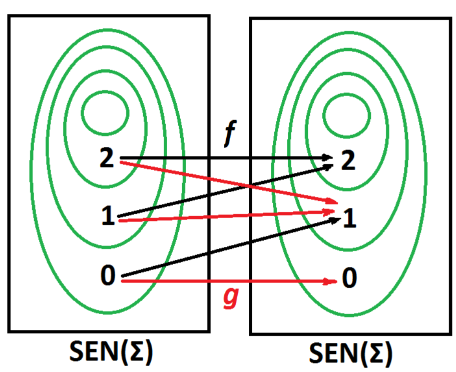
$\circ$	f	g	o	t
f	t	f	t	t
g	o	g	o	o
o	o	o	o	o
t	t	t	t	t

- *$\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ is defined by $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$, with*

$$\begin{array}{lll} \mathrm{SEN}^\flat(f)(0) = 1, & \mathrm{SEN}^\flat(f)(1) = 2, & \mathrm{SEN}^\flat(f)(2) = 2; \\ \mathrm{SEN}^\flat(g)(0) = 0, & \mathrm{SEN}^\flat(g)(1) = 1, & \mathrm{SEN}^\flat(g)(2) = 1, \end{array}$$

  whereas $\mathrm{SEN}^\flat(o)(x) = 1$ and $\mathrm{SEN}^\flat(t)(x) = 2$, for all $x \in \mathrm{SEN}^\flat(\Sigma)$;

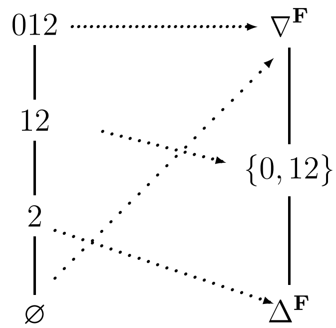
- *$N^\flat$ is the trivial clone, consisting of the projections only.*



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\varnothing, \{2\}, \{1,2\}, \{0,1,2\}\}.$$

$\mathcal{I}$ has four theory families $\varnothing$, $\{\{2\}\}$, $\{\{1,2\}\}$ and $\{\{0,1,2\}\}$, but only three theory systems, $\varnothing$, $\{\{1,2\}\}$ and $\{\{0,1,2\}\}$. The lattice of theory families of $\mathcal{I}$ and the corresponding Leibniz congruence systems are given in the diagram.



Since, as shown in the diagram, $\Omega : \mathrm{ThFam}^{\nleq}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is an order isomorphism, $\mathcal{I}$ is narrowly family c-monotone.

On the other hand, for $T = \{\{2\}\}$ and $T' = \{\{1,2\}\}$, we get $T \le T'$, whereas $\Omega(\overleftarrow{T}) = \Omega(\overline{\varnothing}) = \nabla^{\mathbf{F}} \nleq \{0,12\} = \Omega(T') = \Omega(\overleftarrow{T'})$. Therefore, $\mathcal{I}$ is not narrowly right c-monotone.

The third example gives a narrowly right c-monotone π-institution which is neither narrowly family nor narrowly left c-monotone. Thus, it shows that the classes of narrowly family and of narrowly left c-monotone π-institutions are properly contained in the class of narrowly system c-monotone π-institutions and that, moreover, the class of narrowly right c-monotone π-institutions is not a subclass of either the class of narrowly family or the class of narrowly left c-monotone π-institutions.

Example 560 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

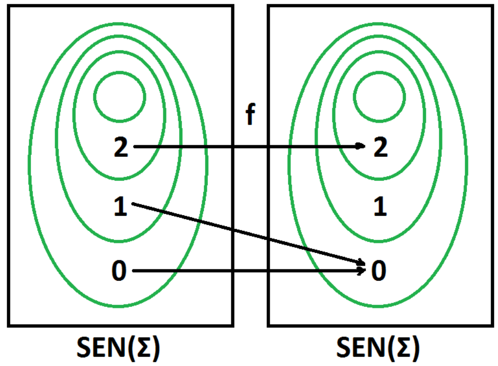
- $\mathbf{Sign}^\flat$ *is the category with a single object Σ and a unique (non-identity) morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$, $\mathrm{SEN}^\flat(f)(0) = \mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial clone.*

Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

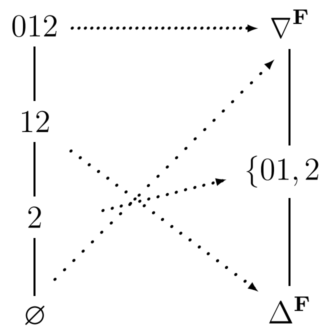
$$\mathcal{C}_\Sigma = \{\varnothing, \{2\}, \{1,2\}, \{0,1,2\}\}.$$

$\mathcal{I}$ has four theory families, but only three theory systems, namely $\overline{\varnothing}$, $\{\{2\}\}$ and $\{\{0,1,2\}\}$. Moreover, clearly,

$$\mathrm{ThFam}^{\nleq}(\mathcal{I}) = \{\{\{2\}\}, \{\{1,2\}\}, \{\{0,1,2\}\}\}.$$



The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right:



To see that $\mathcal{I}$ is narrowly right c-monotone, suppose $\mathcal{T}\cup\{T'\}\subseteq \mathrm{ThFam}^{\natural}(\mathcal{I})$, such that $T' \leq \bigcup_{T\in\mathcal{T}} T$. Then,

- *if $T' = \{\{0,1,2\}\}$, we must have $\{\{0,1,2\}\} \in \mathcal{T}$, whence $\Omega(\overleftarrow{T'}) = \nabla^{\mathbf{F}} = \bigcup_{T\in\mathcal{T}} \Omega(\overleftarrow{T})$;*
- *if $T' \neq \{\{0,1,2\}\}$, then $\Omega(\overleftarrow{T'}) \leq \{\{01,2\}\} \leq \bigcup_{T\in\mathcal{T}} \Omega(\overleftarrow{T})$.*

Therefore, $\mathcal{I}$ is narrowly right c-monotone.

On the other hand, for $T = \{\{2\}\}$ and $T' = \{\{1,2\}\}$, we get $T \leq T'$, whereas $\Omega(T) = \{01,2\} \nleq \Delta^{\mathbf{F}} = \Omega(T')$. Thus, $\mathcal{I}$ is not narrowly family c-monotone. Moreover, for the same theory families, $\overleftarrow{T} = \{\{2\}\} = \overleftarrow{T'}$, whereas $\Omega(T) = \{01,2\} \nleq \Delta^{\mathbf{F}} = \Omega(T')$. Thus, $\mathcal{I}$ is not narrowly left c-monotone.

The last example shows that there exists a narrowly family c-monotone π-institution that is not narrowly left c-monotone, thus showing that narrowly family c-monotone π-institutions do not form a subclass of the class of narrowly left c-monotone ones.

Example 561 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with the single object* Σ *and four non-identity morphisms* $f, z, o, t : \Sigma \to \Sigma$, *whose composition table is the following:*

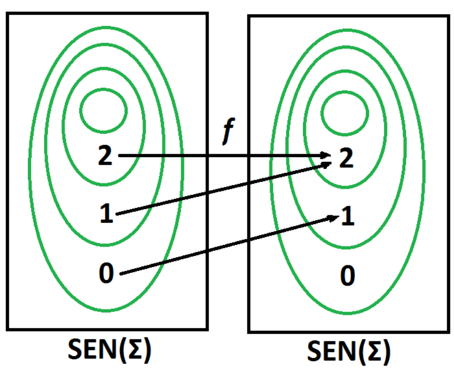
$\circ$	f	z	o	t
f	t	o	t	t
z	z	z	z	z
o	o	o	o	o
t	t	t	t	t

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$, *with*

$$\mathrm{SEN}^\flat(f)(0) = 1, \quad \mathrm{SEN}^\flat(f)(1) = 2, \quad \mathrm{SEN}^\flat(f)(2) = 2,$$

whereas $\mathrm{SEN}^\flat(z)(x) = 0$, $\mathrm{SEN}^\flat(o)(x) = 1$ *and* $\mathrm{SEN}^\flat(t)(x) = 2$, *for all* $x \in \mathrm{SEN}^\flat(\Sigma)$;

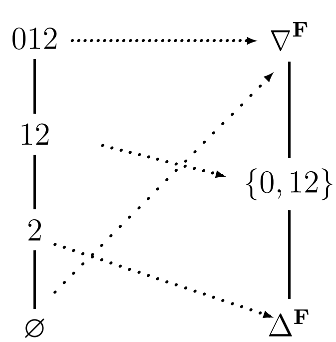
- $N^\flat$ *is the trivial clone, consisting of the projections only.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{2\}, \{1,2\}, \{0,1,2\}\}.$$

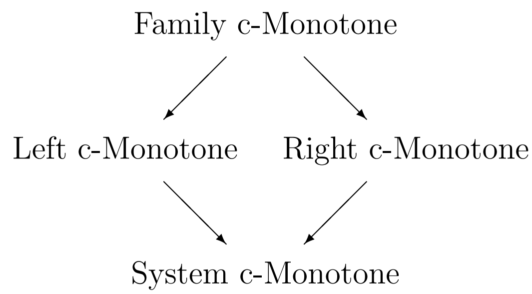
$\mathcal{I}$ *has four theory families* $\varnothing$, $\{\{2\}\}$, $\{\{1,2\}\}$ *and* $\{\{0,1,2\}\}$, *but only two theory systems,* $\varnothing$ *and* $\{\{0,1,2\}\}$. *The lattice of theory families of* $\mathcal{I}$ *and the corresponding Leibniz congruence systems are given in the diagram.*



Since $\Omega : \mathrm{ThFam}^{\lightning}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is an order isomorphism,* $\mathcal{I}$ *is narrowly family c-monotone. On the other hand, for* $T = \{\{1,2\}\}$ *and* $T' = \{\{2\}\}$*, we get* $\overleftarrow{T} = \overline{\varnothing} = \overleftarrow{T'}$*, whereas* $\Omega(T) = \{0,12\} \nleq \Delta^{\mathbf{F}} = \Omega(T')$*. Therefore,* $\mathcal{I}$ *is not narrowly left c-monotone.*

We conclude, after these examples, that the structure of the narrow complete monotonicity hierarchy is, in fact, exactly as depicted in the diagram and no two classes are identical.

Recall from Chapter 3 that we have the following complete monotonicity hierarchy of π-institutions.



We establish now a combined c-monotonicity and narrow c-monotonicity hierarchy. It is not difficult to see that a c-monotonicity property implies the corresponding narrow c-monotonicity property. Alternatively, these relations can be derived by the relationships governing rough and narrow c-monotonicity classes, on the one hand, and the ones governing rough c-monotonicity and c-monotonicity classes on the other.

Proposition 562 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is family (left, right, system, respectively) c-monotone, then it is narrowly family (narrowly left, narrow right, narrowly system, respectively) c-monotone.*

Proof: If $\mathcal{I}$ has a certain type of c-monotonicity, then it has, a fortiori, the same type of narrow c-monotonicity, since the condition defining the latter is a specialization of that defining the former, in which $\mathcal{T} \cup \{T'\}$ are only allowed to range over theory families or systems, as the case may be, in $\mathrm{ThFam}^{\lightning}(\mathcal{I})$ or $\mathrm{ThSys}^{\lightning}(\mathcal{I})$, respectively. (An alternative way is to combine Proposition 547 with Theorem 566 that follows.) ■

Analogously to Theorems 548 and 549, we also get more precise relationships between c-monotonicity and narrow c-monotonicity classes.

Theorem 563 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a non-almost inconsistent* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is family (left, respectively) c-monotone if and only if it is narrowly family (left, respectively) c-monotone and has theorems.*

Proof: Suppose $\mathcal{I}$ is family or left c-monotone. Since, by hypothesis, it is not almost inconsistent, $|\widetilde{\mathrm{ThFam}}(\mathcal{I})| > 1$. Thus, by Proposition 546, $\mathcal{I}$ has theorems. Moreover, by Proposition 562, it is narrowly family or left c-monotone, respectively.

Assume, conversely, that $\mathcal{I}$ is narrowly family (or left c-monotone) and has theorems. Then, since $\mathrm{ThFam}^{\natural}(\mathcal{I}) = \mathrm{ThFam}(\mathcal{I})$, the condition defining narrow family (left) c-monotonicity coincides with the one defining family (left, respectively) c-monotonicity. ∎

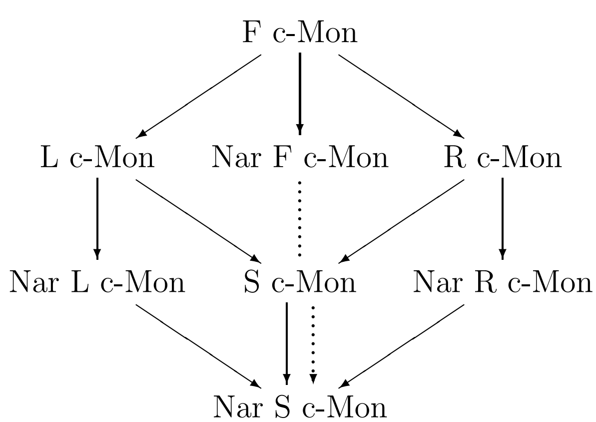
Analogously, for the cases of system and right c-monotonicity, we get the following

Theorem 564 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, that has a theory family* $T \neq \mathrm{SEN}^\flat$ *such that* $\overleftarrow{T} \neq \overline{\varnothing}$*.* $\mathcal{I}$ *is system (right, respectively) c-monotone if and only if it roughly system (right, respectively) c-monotone and has theorems.*

Proof: Suppose $\mathcal{I}$ is system or right c-monotone. Since, by hypothesis, it has a theory system $\overleftarrow{T} \neq \mathrm{SEN}^\flat, \overline{\varnothing}$, we get $|\widetilde{\mathrm{ThSys}}(\mathcal{I})| > 1$. Thus, by Proposition 546, $\mathcal{I}$ must have theorems. Moreover, by Proposition 562, it is narrowly system or right c-monotone, respectively.

Assume, conversely, that $\mathcal{I}$ is narrowly system (or right) c-monotone and has theorems. Then, since $\mathrm{ThFam}^{\natural}(\mathcal{I}) = \mathrm{ThFam}(\mathcal{I})$ and $\mathrm{ThSys}^{\natural}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$, the condition defining narrow right (system) c-monotonicity coincides with the one defining right (system, respectively) c-monotonicity. ∎

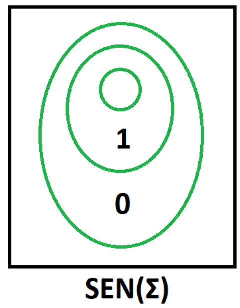
Thus, we have the following mixed hierarchy of c-monotonicity and narrow c-monotonicity properties.



We provide an example of a π-institution which has all four types of narrow c-monotonicity but fails to be system c-monotone and, as a consequence, does not belong to any of the four c-monotonicity classes.

Example 565 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with the single object* Σ;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$;
- $N^\flat$ *is the trivial clone.*

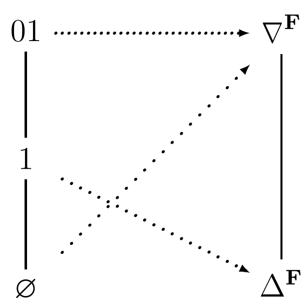


Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\}.$$

$\mathcal{I}$ *has three theory families* $\{\varnothing\}$ *and* $\{\{1\}\}$ *and* $\{\{0,1\}\}$, *all of which are theory systems.*

The lattice of theory families of $\mathcal{I}$ *and the corresponding Leibniz congruence systems are given in the diagram.*



$\mathcal{I}$ *belongs to all four classes of the narrow c-monotonicity hierarchy. Indeed, since it is systemic, all four narrow c-monotonicity conditions boil down to checking that, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^\sharp(\mathcal{I})$, $T' \le \bigcup_{T\in\mathcal{T}} T$ *implies* $\Omega(T') \le \bigcup_{T\in\mathcal{T}} \Omega(T)$.

- *If* $T' = \{\{1\}\}$, *then* $\Omega(T') = \Delta^{\mathbf{F}} \le \bigcup_{T\in\mathcal{T}} \Omega(T)$;
- *If* $T' = \mathrm{SEN}^\flat$, *then* $\mathrm{SEN}^\flat \in \mathcal{T}$, *whence* $\Omega(T') \le \nabla^{\mathbf{F}} = \bigcup_{T\in\mathcal{T}} \Omega(T)$.

On the other hand, we have $\{\varnothing\} \le \{\{1\}\}$, *whereas* $\Omega(\{\varnothing\}) \nleq \Omega(\{\{1\}\})$, *whence* $\mathcal{I}$ *is not system c-monotone.*

As far as connections between the rough c-monotonicity and narrow c-monotonicity classes are concerned, we get the following analog of Theorem 527.

Theorem 566 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is roughly family c-monotone if and only if it is narrowly family c-monotone;*

(b) $\mathcal{I}$ *is roughly left c-monotone, then it is narrowly left c-monotone;*

(c) If $\mathcal{I}$ *is roughly right c-monotone, then it is narrowly right c-monotone;*

(d) If $\mathcal{I}$ *is roughly system c-monotone, then it is narrowly system c-monotone.*

Proof:

(a) Suppose $\mathcal{I}$ is roughly family c-monotone and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nsim}(\mathcal{I})$, such that $T' \leq \bigcup_{T \in \mathcal{T}} T$. Then, by hypothesis, $\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T}$, whence, by rough family c-monotonicity, $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$. Thus $\mathcal{I}$ is narrowly family c-monotone.

Assume, conversely, that $\mathcal{I}$ is narrowly family c-monotone and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{T'} \leq \bigcup_{T \in \mathcal{T}} \widetilde{T}$. Since $\{\widetilde{T} : T \in \mathcal{T}\} \cup \{\widetilde{T'}\} \subseteq \mathrm{ThFam}^{\nsim}(\mathcal{I})$, we get, by narrow family c-monotonicity, $\Omega(\widetilde{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\widetilde{T})$. Thus, by Proposition 369, $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T)$, showing that $\mathcal{I}$ is roughly family c-monotone.

(b) Suppose that $\mathcal{I}$ is roughly left c-monotone, i.e., that, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{\widetilde{T'}} \leq \bigcup_{T \in \mathcal{T}} \overleftarrow{\widetilde{T}}$ implies $\Omega(T') \leq \bigcup_{T \in \mathcal{T}} \Omega(T')$. Assume, for the sake of obtaining a contradiction, that $\mathcal{I}$ is not narrowly left c-monotone. Then, there exist $\mathcal{X} \cup \{Y\} \subseteq \mathrm{ThFam}^{\nsim}(\mathcal{I})$, such that $\overleftarrow{Y} \leq \bigcup_{X \in \mathcal{X}} \overleftarrow{X}$ and $\Omega(Y) \nleq \bigcup_{X \in \mathcal{X}} \Omega(X)$.

First, observe that, if there existed $Z \in \mathrm{ThFam}(\mathcal{I})$ and $P \in |\mathbf{Sign}^\flat|$, such that $Z_P \neq \varnothing$ and $\overleftarrow{Z}_P = \varnothing$, then, setting $Z' = \{Z_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, with

$$Z'_\Sigma = \begin{cases} \varnothing, & \text{if } \Sigma \neq P \\ Z_P, & \text{if } \Sigma = P \end{cases},$$

we would have $\overleftarrow{\widetilde{Z'}} = \overleftarrow{\widetilde{\varnothing}}$, but $\Omega(Z') \neq \Omega(\overline{\varnothing})$, which contradicts rough left c-monotonicity. Thus, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$, $T_\Sigma \neq \varnothing$ implies $\overleftarrow{T}_\Sigma \neq \varnothing$.

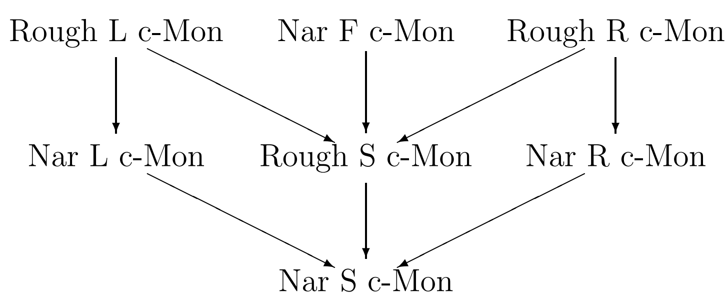
Continuing with the proof, by hypothesis, $\overleftarrow{Y} \leq \bigcup_{X\in\mathcal{X}} \overleftarrow{X}$ and $\Omega(Y) \nleq \bigcup_{X\in\mathcal{X}} \Omega(X)$. Hence, by rough left c-monotonicity, $\overleftarrow{\overleftarrow{Y}} \nleq \bigcup_{X\in\mathcal{X}} \overleftarrow{\overleftarrow{X}}$. Thus, there exists $P \in |\mathbf{Sign}^\flat|$, such that $\overleftarrow{\overleftarrow{Y}}_P \nsubseteq \bigcup_{X\in\mathcal{X}} \overleftarrow{\overleftarrow{X}}_P$, whereas $\overleftarrow{Y}_P \subseteq \bigcup_{X\in\mathcal{X}} \overleftarrow{X}_P$. But this gives $\overleftarrow{Y}_P = \varnothing$, whence, by the preceding observation, $Y_P = \varnothing$, which contradicts $Y \in \mathrm{ThFam}^\nmid(\mathcal{I})$. Therefore, $\mathcal{I}$ must be narrowly left c-monotone.

(c) Suppose $\mathcal{I}$ is roughly right c-monotone and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^\nmid(\mathcal{I})$, such that $T' \leq \bigcup_{T\in\mathcal{T}} T$. By hypothesis, $\widetilde{T'} \leq \bigcup_{T\in\mathcal{T}} \widetilde{T}$. Thus, by rough right c-monotonicity, $\Omega(\overleftarrow{T'}) \leq \bigcup_{T\in\mathcal{T}} \Omega(\overleftarrow{T})$. Thus, $\mathcal{I}$ is narrowly right c-monotone.

(d) Suppose $\mathcal{I}$ is roughly system c-monotone and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^\nmid(\mathcal{I})$, such that $T' \leq \bigcup_{T\in\mathcal{T}} T$. Then, by hypothesis, $\widetilde{T'} \leq \bigcup_{T\in\mathcal{T}} \widetilde{T}$, whence, by rough system c-monotonicity, $\Omega(T') \leq \bigcup_{T\in\mathcal{T}} \Omega(T)$. Thus $\mathcal{I}$ is narrowly system c-monotone.

■

Theorem 566 gives rise to the following mixed rough and narrow c-monotonicity hierarchy.

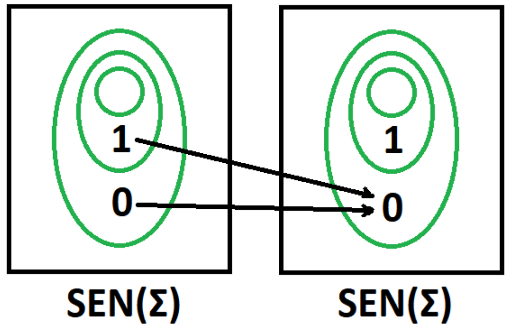


We insert, again, some examples to show that each of the three rough c-monotonicity classes is different from its narrow counterpart.

The first example gives a narrowly left c-monotone π-institution which is not roughly left c-monotone.

Example 567 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

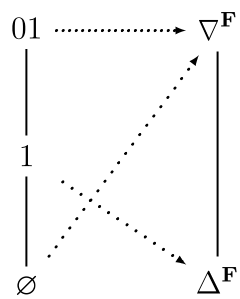
- $\mathbf{Sign}^\flat$ *is the category with the single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$;
- $N^\flat$ *is the trivial clone, consisting of the projections only.*



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\}.$$

$\mathcal{I}$ has three theory families $\{\varnothing\}$, $\{\{1\}\}$ and $\{\{0,1\}\}$, but only two theory systems, $\{\varnothing\}$ and $\{\{0,1\}\}$. The lattice of theory families of $\mathcal{I}$ and the corresponding Leibniz congruence systems are given in the diagram.



To see that $\mathcal{I}$ is narrowly left c-monotone, note that the only two different theory families in $\mathrm{ThFam}^{\nsubseteq}(\mathcal{I})$ are $\{\{1\}\}$ and $\{\{0,1\}\}$ and we have

$$\overleftarrow{\{\{1\}\}} = \{\varnothing\} \leq \{\{0,1\}\} = \overleftarrow{\{\{0,1\}\}}$$
$$\text{and} \quad \Omega(\{\{1\}\}) = \Delta^{\mathbf{F}} \leq \nabla^{\mathbf{F}} = \Omega(\{\{0,1\}\}).$$

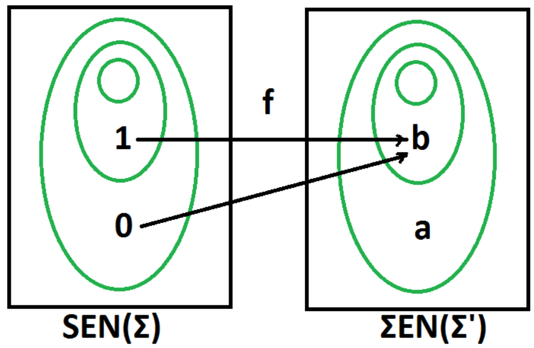
On the other hand, $\mathcal{I}$ is not roughly left c-monotone, since $\overleftarrow{\widetilde{\{\varnothing\}}} = \{\{0,1\}\} = \overleftarrow{\widetilde{\{\{1\}\}}}$, but $\Omega(\{\varnothing\}) \nleq \Omega(\{\{1\}\})$.

The second example shows that there exists a narrowly right c-monotone π-institution that is not roughly right c-monotone.

Example 568 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects Σ and Σ' and a unique morphism $f : \Sigma \to \Sigma'$;*

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = b$, $\mathrm{SEN}^\flat(f)(1) = b$*;*
- $N^\flat$ *is the trivial clone.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{b\}, \{a,b\}\}.$$

Clearly, there are only four theory families in $\mathrm{ThFam}^{\nleq}(\mathcal{I})$*, all of which are theory systems. Their lattice together with the associated Leibniz congruence systems are shown in the diagram:*

$$\begin{array}{ccc} & 01, ab & \\ 01, b & & 1, ab \\ & 1, b & \end{array} \qquad \begin{array}{ccc} & \nabla^{\mathbf{F}} & \\ \nabla^{\mathbf{F}}_\Sigma, \Delta^{\mathbf{F}}_{\Sigma'} & & \Delta^{\mathbf{F}}_\Sigma, \nabla^{\mathbf{F}}_{\Sigma'} \\ & \Delta^{\mathbf{F}} & \end{array}$$

From this diagram and the fact that all theory families depicted are theory systems, we can see that, for all $T, T' \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$*,*

$$T \le T' \quad \textit{iff} \quad \Omega(\overleftarrow{T}) \le \Omega(\overleftarrow{T'}).$$

Therefore, $\mathcal{I}$ *is indeed narrowly right c-monotone.*

On the other hand, consider $T = \{1, \varnothing\}$ *and* $T' = \{1, ab\}$*. Then we have* $\widetilde{T} = \{1, ab\} = \widetilde{T}'$*, whereas*

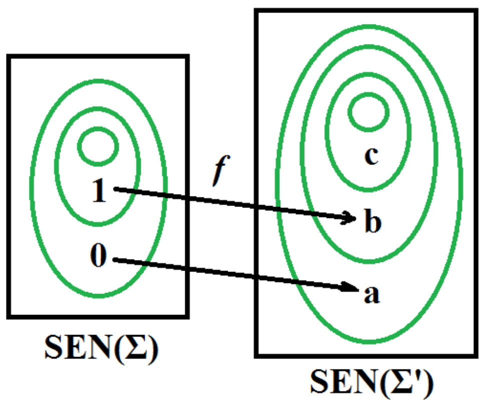
$$\Omega(\overleftarrow{T}) = \Omega(\overline{\varnothing}) = \nabla^{\mathbf{F}} \nleq \{\Delta^{\mathbf{F}}_\Sigma, \nabla^{\mathbf{F}}_{\Sigma'}\} = \Omega(\{1, ab\}) = \Omega(\overleftarrow{T'}).$$

This shows that $\mathcal{I}$ *is not roughly right c-monotone.*

The last example gives a narrowly system c-monotone π-institution which is not roughly system c-monotone.

Example 569 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with two object* Σ, Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b,c\}$, *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$*;*
- $N^\flat$ *is the trivial clone.*



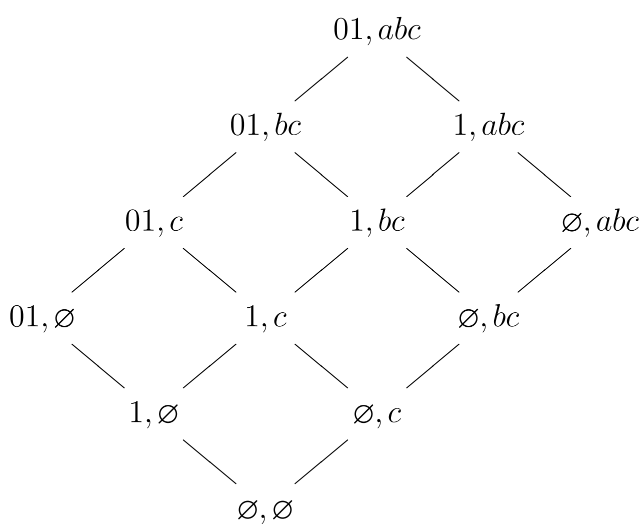
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0,1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\varnothing, \{c\}, \{b,c\}, \{a,b,c\}\}.$$

$\mathcal{I}$ *has twelve theory families, but only seven theory systems. These are*

$$\overline{\varnothing}, \{\varnothing, c\}, \{\varnothing, bc\}, \{\varnothing, abc\}, \{1, bc\}, \{1, abc\}, \{01, abc\}.$$

The following diagram shows the structure of the lattice of theory families.



To see that $\mathcal{I}$ is narrow system c-monotone, note that there are only three theory systems in $\mathrm{ThSys}^{\nleq}(\mathcal{I})$*, namely,* $\{1,bc\}$, $\{1,abc\}$ *and* $\{01,abc\}$ *and we have* $\{1,bc\} \le \{1,abc\} \le \{01,abc\}$ *and, also,*

$$\begin{aligned}\Omega(\{1,bc\}) &= \{\Delta^{\mathbf{F}}_{\Sigma}, \{a,bc\}\}\\ &\le \Omega(\{1,abc\}) = \{\Delta^{\mathbf{F}}_{\Sigma}, \nabla^{\mathbf{F}}_{\Sigma'}\}\\ &\le \Omega(\{01,abc\}) = \nabla^{\mathbf{F}}.\end{aligned}$$

On the other hand, setting $T = \{\varnothing, c\}$ *and* $T' = \{\varnothing, bc\}$*, which are both theory systems, we get*

$$\widetilde{T} = \{01,c\} \le \{01,bc\} = \widetilde{T'},$$

whereas

$$\Omega(T) = \{\nabla^{\mathbf{F}}_{\Sigma}, \{ab,c\}\} \nleq \{\Delta^{\mathbf{F}}_{\Sigma}, \{a,bc\}\} = \Omega(T').$$

Therefore, $\mathcal{I}$ is not roughly system c-monotone.

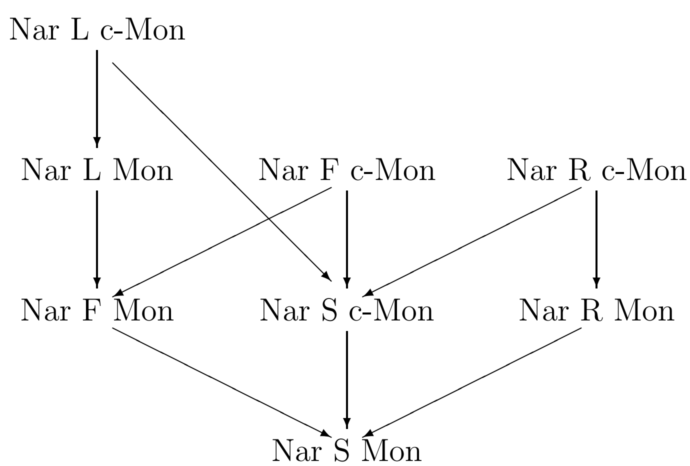
We conclude, after these examples, that the structure of the joint rough and narrow c-monotonicity hierarchy is as depicted in the diagram following Theorem 566, with no two classes being identical.

Finally, we look at some straightforward connections between the classes in the narrow monotonicity and narrow complete monotonicity hierarchies. These follow directly by the definitions involved.

Proposition 570 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$*. If $\mathcal{I}$ is narrowly family (respectively, left, right, system) c-monotone, then it is narrowly family (respectively, left, right, system) monotone.*

Proof: The condition defining a narrow monotonicity class is a special case of the condition defining the corresponding narrow c-monotonicity class, where the collection $\mathcal{T}$, in that definition, is taken to be a singleton. ■

Proposition 570, in view of Propositions 517 and 555, establishes the hierarchy depicted in the diagram.

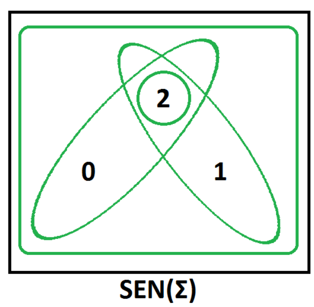


We present an example to show that the two hierarchies are separated. The showcased π-institution belongs to all steps of the narrow monotonicity hierarchy but to none of the four narrow c-monotonicity classes.

Example 571 *Define the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *as follows:*

- $\mathbf{Sign}^\flat$ *is a trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$*;*
- $N^\flat$ *is the clone generated by the unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, given by*

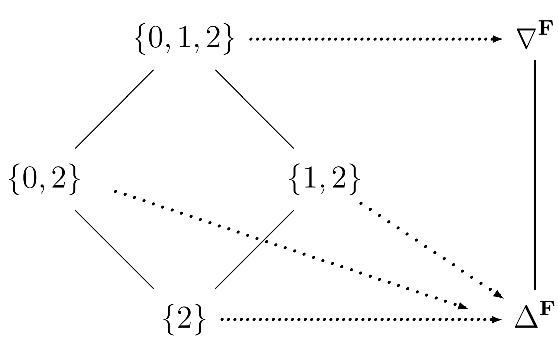
$x \in \mathrm{SEN}^\flat(\Sigma)$	$\sigma^\flat_\Sigma(x)$
0	1
1	2
2	0



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting*

$$\mathcal{C}_\Sigma = \{\{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}.$$

It is easy to see that the lattices of theory families and corresponding Leibniz congruence systems are as given in the diagram.



Since $\mathbf{Sign}^\flat$ *is trivial,* $\mathcal{I}$ *is systemic and, since* $\mathcal{I}$ *has theorems,* $\mathrm{FiFam}^\sharp(\mathcal{I}) = \mathrm{FiFam}(\mathcal{I})$*. We conclude that all four narrow monotonicity properties for*

$\mathcal{I}$ coincide and, moreover, they are identical with both monotonicity properties, which they also coincide, due to systemicity. The same holds for c-monotonicity. All four narrow c-monotonicity properties coincide and they, in turn, are identical with all c-monotonicity conditions.

From the diagram one can verify immediately that $\mathcal{I}$ is (narrowly left, right and family) monotone, On the other hand, we have

$$\{\{0,1,2\}\} \le \{\{0,2\}\} \cup \{\{1,2\}\},$$

but, obviously, $\Omega(\{\{0,1,2\}\}) \not\le \Omega(\{\{0,2\}\}) \cup \Omega(\{\{1,2\}\})$. Taking into account that $\mathcal{I}$ is systemic, we conclude that $\mathcal{I}$ fails to be narrowly system c-monotone.

Next, we turn to transfer theorems for the various narrow c-monotonicity properties.

Theorem 572 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

- *(a) $\mathcal{I}$ is narrowly family c-monotone if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}\nleq}(\mathcal{A})$, $T' \le \bigcup_{T \in \mathcal{T}} T$ implies $\Omega^{\mathcal{A}}(T') \le \bigcup_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)$;*
- *(b) $\mathcal{I}$ is narrowly left c-monotone if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}\nleq}(\mathcal{A})$, $\overleftarrow{T'} \le \bigcup_{T \in \mathcal{T}} \overleftarrow{T}$ implies $\Omega^{\mathcal{A}}(T') \le \bigcup_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)$;*
- *(c) $\mathcal{I}$ is narrowly right c-monotone if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}\nleq}(\mathcal{A})$, $T' \le \bigcup_{T \in \mathcal{T}} T$ implies $\Omega^{\mathcal{A}}(\overleftarrow{T'}) \le \bigcup_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(\overleftarrow{T})$;*
- *(d) $\mathcal{I}$ is narrowly system c-monotone if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiSys}^{\mathcal{I}\nleq}(\mathcal{A})$, $T' \le \bigcup_{T \in \mathcal{T}} T$ implies $\Omega^{\mathcal{A}}(T') \le \bigcup_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)$.*

Proof:

- (a) The "if" results by applying the hypothesis to the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$.

  For the "only if", suppose that $\mathcal{I}$ is narrowly family c-monotone and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\nleq}(\mathcal{I})$, such that $T' \le \bigcup_{T \in \mathcal{T}} T$. Then we get $\alpha^{-1}(T') \le \alpha^{-1}(\bigcup_{T \in \mathcal{T}} T)$, whence $\alpha^{-1}(T') \le \bigcup_{T \in \mathcal{T}} \alpha^{-1}(T)$. Since, by Lemmas 51 and 376, $\{\alpha^{-1}(T) : T \in \mathcal{T}\} \cup \{\alpha^{-1}(T')\} \subseteq \mathrm{ThFam}^{\nleq}(\mathcal{I})$, we get, by narrow family c-monotonicity, $\Omega(\alpha^{-1}(T')) \le \bigcup_{T \in \mathcal{T}} \Omega(\alpha^{-1}(T))$. Hence, by Proposition 24, $\alpha^{-1}(\Omega^{\mathcal{A}}(T')) \le \bigcup_{T \in \mathcal{T}} \alpha^{-1}(\Omega^{\mathcal{A}}(T))$, i.e., $\alpha^{-1}(\Omega^{\mathcal{A}}(T')) \le \alpha^{-1}(\bigcup_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T))$. Taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that $\Omega^{\mathcal{A}}(T') \le \bigcup_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)$.

(b) The "if" is obtained as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is narrowly left c-monotone and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\lightning}(\mathcal{I})$, such that $\overleftarrow{T'} \leq \bigcup_{T \in \mathcal{T}} \overleftarrow{T}$. Then we get $\alpha^{-1}(\overleftarrow{T'}) \leq \alpha^{-1}(\bigcup_{T \in \mathcal{T}} \overleftarrow{T})$, whence $\alpha^{-1}(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \alpha^{-1}(\overleftarrow{T})$. By Lemma 6, we get $\overleftarrow{\alpha^{-1}(T')} \leq \bigcup_{T \in \mathcal{T}} \overleftarrow{\alpha^{-1}(T)}$. Since, by Lemmas 51 and 376, it holds $\{\alpha^{-1}(T) : T \in \mathcal{T}\} \cup \{\alpha^{-1}(T')\} \subseteq \mathrm{ThFam}^{\lightning}(\mathcal{I})$, we get, by narrow left c-monotonicity, that $\Omega(\alpha^{-1}(T')) \leq \bigcup_{T \in \mathcal{T}} \Omega(\alpha^{-1}(T))$. Thus, by Proposition 24, we now get $\alpha^{-1}(\Omega^{\mathcal{A}}(T')) \leq \bigcup_{T \in \mathcal{T}} \alpha^{-1}(\Omega^{\mathcal{A}}(T))$, i.e., $\alpha^{-1}(\Omega^{\mathcal{A}}(T')) \leq \alpha^{-1}(\bigcup_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T))$. Taking into account the surjectivity of $\langle F, \alpha \rangle$, we obtain $\Omega^{\mathcal{A}}(T') \leq \bigcup_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T)$.

(c) The "if" is obtained as in Part (a).

For the "only if", suppose that $\mathcal{I}$ is narrowly right c-monotone and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\lightning}(\mathcal{I})$, such that $T' \leq \bigcup_{T \in \mathcal{T}} T$. Then we get $\alpha^{-1}(T') \leq \alpha^{-1}(\bigcup_{T \in \mathcal{T}} T)$, whence $\alpha^{-1}(T') \leq \bigcup_{T \in \mathcal{T}} \alpha^{-1}(T)$. Since, by Lemmas 51 and 376, $\{\alpha^{-1}(T) : T \in \mathcal{T}\} \cup \{\alpha^{-1}(T')\} \subseteq \mathrm{ThFam}^{\lightning}(\mathcal{I})$, we get, by narrow right c-monotonicity, $\Omega(\overleftarrow{\alpha^{-1}(T')}) \leq \bigcup_{T \in \mathcal{T}} \Omega(\overleftarrow{\alpha^{-1}(T)})$. Thus, by Lemma 6, $\Omega(\alpha^{-1}(\overleftarrow{T'})) \leq \bigcup_{T \in \mathcal{T}} \Omega(\alpha^{-1}(\overleftarrow{T}))$. Hence, by Proposition 24, we get $\alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'})) \leq \bigcup_{T \in \mathcal{T}} \alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T}))$, i.e., $\alpha^{-1}(\Omega^{\mathcal{A}}(\overleftarrow{T'})) \leq \alpha^{-1}(\bigcup_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(\overleftarrow{T}))$. Taking into account the surjectivity of $\langle F, \alpha \rangle$, we obtain $\Omega^{\mathcal{A}}(\overleftarrow{T'}) \leq \bigcup_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(\overleftarrow{T})$.

(d) Similar to Part (a). ■

We close this section by giving two characterizations concerning the narrow family and narrow system c-monotonicity classes, based on mappings between posets satisfying the complete monotonicity property.

Proposition 573 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *The following statements are equivalent:*

(a) $\mathcal{I}$ *is narrowly family c-monotone;*

(b) $\Omega : \mathbf{ThFam}^{\lightning}(\mathcal{I}) \to \mathbf{ConSys}^*(\mathcal{I})$ *is completely monotone;*

(c) $\Omega^{\mathcal{A}} : \mathbf{FiFam}^{\mathcal{I}\lightning}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is completely monotone, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$.

Proposition 574 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *The following statements are equivalent:*

(a) $\mathcal{I}$ *is narrowly system c-monotone;*

(b) $\Omega : \mathbf{ThSys}^{\lightning}(\mathcal{I}) \to \mathbf{ConSys}^*(\mathcal{I})$ *is completely monotone;*

(c) $\Omega^{\mathcal{A}} : \mathbf{FiSys}^{\mathcal{I}\lightning}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is completely monotone, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$.

Chapter 8

The Semantic Leibniz Hierarchy: Over the Top I

8.1 Introduction

The prototypical example of an algebraizable deductive system, classical propositional calculus, has the additional distinctive feature of being *1-algebraizable* or *regularly algebraizable* (see, e.g., Chapter 5 of [64], Chapter 3 (p. 66) of [52] and Section 3.4 of [86]). This means that any two theorems are equivalent or, more generally, that any two sentences belonging to a theory T are equivalent relative to T. In this chapter, we undertake the study of regularity, a property that, when added to algebraizability, yields regular algebraizability, featured among the topmost classes in the entire semantic hierarchy discussed in this monograph.

In Section 8.2, we introduce and study the basic *regularity properties*, which form the basis for developing the regular algebraizability classes in Sections 8.4-8.7. A π-institution $\mathcal{I}$ is *family regular* if, for every theory family T of $\mathcal{I}$, all signatures Σ and all Σ-sentences ϕ and ψ, $\phi,\psi \in T_\Sigma$ implies $\langle\phi,\psi\rangle \in \Omega_\Sigma(T)$. It is *left regular* if it satisfies the same condition, but with T in the hypothesis replaced by $\overleftarrow{T}$, and *right regular* if T is replaced by $\overleftarrow{T}$ in the conclusion instead. Finally, $\mathcal{I}$ is *system regular* if, in the implication defining family regularity, T is restricted to range only over theory systems, instead of being allowed to range over arbitrary theory families. These four properties form a linear hierarchy, with family regularity being the strongest, followed by right regularity, then by left regularity, with the system version being the weakest of the four. Stability causes the collapse of this hierarchy into two levels, since, under stability, system regularity implies left regularity and right regularity implies family regularity. More transparently, systemicity causes a total collapse of the hierarchy into a single class. The family, left and system versions have characterizations involving the Suszko operator and one of its variants. For a sneak preview, $\mathcal{I}$ is system regular if and only if for every signature Σ and all Σ-sentences ϕ and ψ, $\langle\phi,\psi\rangle \in \widehat{\Omega}^{\mathcal{I}}_\Sigma(\overrightarrow{C}(\phi,\psi))$, where $\overrightarrow{C}(\phi,\psi)$ is the least theory system of $\mathcal{I}$ containing ϕ and ψ and $\widehat{\Omega}^{\mathcal{I}}$: $\mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}(\mathcal{I})$ gives, for a given theory system T of $\mathcal{I}$, the largest congruence system $\widehat{\Omega}^{\mathcal{I}}(T)$ compatible with every theory system including T. All four regularity properties transfer, e.g., looking at right regularity, it holds for a π-institution $\mathcal{I}$ if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, all $\mathcal{I}$-filter families T of $\mathcal{A}$, all signatures Σ of $\mathcal{A}$ and all Σ-sentences ϕ, ψ, $\phi,\psi \in T_\Sigma$ implies $\langle\phi,\psi\rangle \in \Omega^{\mathcal{A}}_\Sigma(\overleftarrow{T})$. Finally, the family and system versions have natural characterizations in terms of the form of the filter families/systems, respectively, of the reduced matric families/systems of $\mathcal{I}$. The condition here is that, if $\langle\mathcal{A},T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$, then T is at most a singleton, i.e., each of its components T_Σ has at most one element.

In Section 8.3, we look at *assertionality*, which is the property ensuing from regularity when existence of theorems is also postulated. Thus, a π-institution $\mathcal{I}$ is *family*, *right*, *left* or *system assertional* if it is family, right, left or system regular, respectively, and has theorems. The hierarchy of regularity properties established in Section 8.2 immediately yields an a priori linear assertionality hierarchy with four classes, the family version implying the right, which, in turn, implies the left version, with the system being the weakest of the four versions. However, it turns out that right assertionality is strong enough to imply systemicity and, as a consequence, the family and right versions are equivalent. Thus, the hierarchy consists of only three distinct classes. The weakest property, system assertionality, coupled with systemicity, is equivalent to the strongest, family assertionality. It is straightforward by the definitions that each asertionality property implies its regularity counterpart. More interestingly, each assertionality property implies the corresponding complete reflectivity (c-reflectivity) property (see Section 3.8). All three versions of assertionality transfer. This follows from the fact that both regularity and existence of theorems transfer. Additionally, based on the characterizations of family and system regularity in terms of reduced matrix families/systems, one may obtain similar characterizations of family/system assertionality. Again, for the sake of preview, the condition characterizing family assertionality is that, for every reduced $\mathcal{I}$-matrix family $\langle\mathcal{A},T\rangle$, T is a singleton, i.e., $|T_\Sigma| = 1$, for all signatures Σ of $\mathcal{A}$.

Having discussed, to some extent, the foundations in Sections 8.2 and 8.3, we embark, in Section 8.4, on the study of algebraizability properties, starting with *regular weak prealgebraizability*. The three classes defined here reflect the type of asssertionality combined with prealgebraicity. Accordingly, a π-institution $\mathcal{I}$ is *regularly weakly family* (RWF) *prealgebraizable* if it is prealgebraic and family assertional. It is *regularly weakly left* (RWL) *prealgebraizable* if it is prealgebraic and left assertional, and it is *regularly weakly system* (RWS) *prealgebraizable* if it is prealgebraic and system assertional. The hierarchy of assertionality properties of Section 8.3 yields that RWF prealgebraizability implies RWL prealgebraizability, which, in turn, implies RWS prealgebraizability. By definition, RWF/L/S prealgebraizability implies, respectively, family/left/system assertionality. More noteworthy, however, is the fact that, since each version of assertionality implies the corresponding c-reflectivity version, RWF/L/S prealgebraizability implies, respectively, WF/L/SC prealgebraizability (see Section 4.2). All three regular weak prealgebraizability properties transfer. This property stems from the transferability of both prealgebraicity and assertionality. It is possible to formulate characterizations of the regular weak prealgebraizability properties in terms of the Leibniz operator viewed as a mapping between ordered sets. E.g., a π-institution $\mathcal{I}$ is RWF prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism, such that, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T/\Omega^{\mathcal{A}}(T)$ is a singleton. The other two characterizations assume similar forms.

In Section 8.5, we switch from regular weak prealgebraizability to *regular weak algebraizability* properties. The former involve prealgebraicity, which, when strengthened to protoalgebraicity, yield the latter. In accordance, a π-institution $\mathcal{I}$ is *regularly weakly family* (RWF) *algebraizable* if it is protoalgebraic and family assertional. It is *regularly weakly left* (RWL) *algebraizable* if it is protoalgebraic and left assertional, and it is *regularly weakly system* (RWS) *algebraizable* if it is protoalgebraic and system assertional. The strengthening of prealgebraicity to protoalgebraicity results in the identification of the left and system versions. Thus, the regular weak algebraizability hierarchy consists of only two distinct classes, that of regularly weakly family algebraizable π-institutions and its proper superclass of regularly weakly system algebraizable π-institutions. Further, in comparing regular weak algebraizability with regular weak prealgebraizability properties, it is revealed that the strongest versions of each, i.e., RWF algebraizability and RWF prealgebraizability, are actually equivalent. In contrast, RWS algebraizability strictly implies RWL prealgebraizability. Again, based on the fact that assertionality implies c-reflectivity, one infers that each regular weak algebraizability property implies the corresponding weak algebraizability property (see Section 4.3). Both regular weak algebraizability properties transfer and both can be characterized in terms of the Leibniz operator seen as a mapping between ordered sets. Clearly, since RWF algebraizability coincides with RWF prealgebraizability, the characterization, given previously, regarding the latter applies to the former as well.

In Section 8.6, we turn to *regular prealgebraizability* properties, which are obtained from the regular weak prealgebraizability properties of Section 8.4, not by strengthening prealgebraicity to protoalgebraicity, as was done in Section 8.5, but, by adding, instead, system extensionality, i.e., by replacing prealgebraicity by preequivalentiality. Consequently, a π-institution $\mathcal{I}$ is *regularly family* (RF) *prealgebraizable* if it is preequivalential (prealgebraic and system extensional) and family assertional. It is *regularly left* (RL) *prealgebraizable* if it is preequivalential and left assertional, and it is *regularly system* (RS) *prealgebraizable* if it is preequivalential and system assertional. RF prealgebraizability implies RL prealgebraizability, which implies RS prealgebraizability, based on the asertionality hierarchy of Section 8.3. Since preequivalentiality implies prealgebraicity, each of the three regular prealgebraizability properties implies the corresponding regular weak prealgebraizability property. Furthermore, since assertionality implies c-reflectivity, each of the regular prealgebraizability properties implies its prealgebraizability counterpart (see Section 5.5). All three regular prealgebraizability properties transfer. In addition, each can be characterized via the use of the Leibniz operator perceived as a mapping between ordered sets. Roughly speaking, these characterizations mimic the ones used in Section 8.4 for regular weak prealgebraizability properties, while adding some form of commutativity with inverse logical extensions, which, by Theorem 327, captures extensionality.

In Section 8.7, the last section of the chapter, we look at *regular algebraizability* properties, which are obtained from the regular prealgebraizability properties of Section 8.6 by strengthening preequivalentiality to equivalentiality or, alternatively, from the regular weak algebraizability properties of Section 8.5 by strengthening protoalgebraicity to equivalentiality. Either point of view leads to defining a π-institution $\mathcal{I}$ being *regularly family* (RF) *algebraizable* if it is equivalential and family assertional, *regularly left* (RL) *algebraizable* if it is equivalential and left assertional, and *regularly system* (RS) *algebraizable* if it is equivalential and system assertional. As transpired with regular weak algebraizability in Section 8.5, the left and system versions are equivalent, and this results in a two-class hierarchy, with RF algebraizability at the top, dominating RS algebraizability. The reasoning naturally leading to the establishment of these classes, permits us to conclude, on the one hand, that each regular algebraizability property implies the corresponding regular prealgebraizability property and, on the other, that each regular algebraizability property implies its regular weak counterpart. But, in addition, in establishing the relations between regular algebraizability and regular prealgebraizability properties, it is seen that the two family versions coincide. A final comparison is made between regular algebraizability and algebraizability (see Section 5.6). Since assertionality implies c-reflectivity, one obtains that each of the two distinct regular algebraizability versions implies the corresponding algebraizability version. Both regular algebraizability properties transfer. Finally, each possesses a characterization via the Leibniz operator, viewed as a mapping between ordered sets, satisfying some additional properties.

8.2 Semantic Regularity

In this chapter, we deal with π-institutions that have theorems and that, in addition, satisfy some form of the *semantic regularity property*, which is detailed in the following

Definition 575 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is* **family regular** *if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*
$$\phi, \psi \in T_\Sigma \quad \text{implies} \quad \langle \phi, \psi \rangle \in \Omega_\Sigma(T);$$
- $\mathcal{I}$ *is* **left regular** *if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*
$$\phi, \psi \in \overleftarrow{T}_\Sigma \quad \text{implies} \quad \langle \phi, \psi \rangle \in \Omega_\Sigma(T);$$

- $\mathcal{I}$ *is* **right regular** *if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\phi, \psi \in T_\Sigma \quad \textit{implies} \quad \langle \phi, \psi \rangle \in \Omega_\Sigma(\overleftarrow{T});$$

- $\mathcal{I}$ *is* **system regular** *if, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\phi, \psi \in T_\Sigma \quad \textit{implies} \quad \langle \phi, \psi \rangle \in \Omega_\Sigma(T).$$

We establish a hierarchy of regularity properties by looking at the relationships that hold between the properties introduced in Definition 575.

Proposition 576 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is family regular, then it is left regular;*

(b) If $\mathcal{I}$ *is family regular, then it is right regular;*

(c) If $\mathcal{I}$ *is left regular, then it is system regular;*

(d) If $\mathcal{I}$ *is right regular, then it is system regular.*

Proof:

(a) Suppose that $\mathcal{I}$ is family regular and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in \overleftarrow{T}_\Sigma$. Then, by Proposition 42, $\phi, \psi \in T_\Sigma$. Thus, by family regularity, $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. Therefore, $\mathcal{I}$ is left regular.

(b) Suppose that $\mathcal{I}$ is family regular and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Then, by family regularity, $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. Therefore, by Proposition 20, $\langle \phi, \psi \rangle \in \Omega_\Sigma(\overleftarrow{T})$. Thus, $\mathcal{I}$ is right regular.

(c) Suppose that $\mathcal{I}$ is left regular and let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Since T is a theory system, $\overleftarrow{T} = T$, whence, $\phi, \psi \in \overleftarrow{T}_\Sigma$. Hence, by left regularity, $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. Therefore, $\mathcal{I}$ is system regular.

(d) Suppose that $\mathcal{I}$ is right regular and let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Then, by right regularity, $\langle \phi, \psi \rangle \in \Omega_\Sigma(\overleftarrow{T})$. But T is a theory system, i.e., $\overleftarrow{T} = T$, whence $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. Thus, $\mathcal{I}$ is system regular.

■

We now show that, in fact, right regularity implies left regularity. This is a more challenging result that requires a technical lemma.

Lemma 577 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is right regular and not systemic, then, for all* $T \in \mathrm{ThFam}(\mathcal{I}) \backslash \mathrm{ThSys}(\mathcal{I})$ *and all* $\Sigma \in |\mathbf{Sign}^\flat|$*, such that* $\overleftarrow{T}_\Sigma \subsetneq T_\Sigma$*,* $\overleftarrow{T}_\Sigma = \varnothing$*.*

Proof: Suppose that $\mathcal{I}$ is right regular and not systemic and consider $T \in \mathrm{ThFam}(\mathcal{I}) \backslash \mathrm{ThSys}(\mathcal{I})$ and $\Sigma \in |\mathbf{Sign}^\flat|$, such that $\overleftarrow{T}_\Sigma \subsetneq T_\Sigma$ and $\overleftarrow{T}_\Sigma \neq \varnothing$. Then, on the one hand, there exists $\phi \in T_\Sigma$, such that $\phi \notin \overleftarrow{T}_\Sigma$ and, on the other, there exists $\psi \in \overleftarrow{T}_\Sigma$. Thus, by the compatibility of $\Omega(\overleftarrow{T})$ with $\overleftarrow{T}$, we get that $\langle \phi, \psi \rangle \notin \Omega_\Sigma(\overleftarrow{T})$, whereas, since $\overleftarrow{T} \leq T$, $\phi, \psi \in T_\Sigma$. Therefore, $\mathcal{I}$ is not right regular, a contradiction. We conclude that $\overleftarrow{T}_\Sigma = \varnothing$. ■

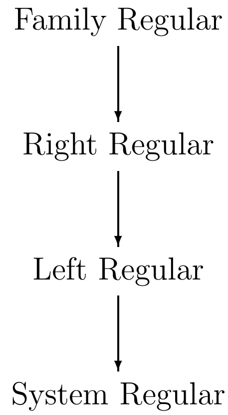
Theorem 578 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is right regular, then it is left regular.*

Proof: Suppose $\mathcal{I}$ is right regular. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in \overleftarrow{T}_\Sigma$. Then, also, $\phi, \psi \in T_\Sigma$.

- If $\mathcal{I}$ is systemic, then, by right regularity, $\langle \phi, \psi \rangle \in \Omega_\Sigma(\overleftarrow{T}) = \Omega_\Sigma(T)$, whence $\mathcal{I}$ is left regular.
- Suppose, now, that $\mathcal{I}$ is not systemic, whence Lemma 577 applies. Since $\phi, \psi \in \overleftarrow{T}_\Sigma$, by Lemma 577, we must have $\overleftarrow{T}_\Sigma = T_\Sigma$. But then, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ such that $\mathbf{Sign}^\flat(\Sigma, \Sigma') \neq \varnothing$, we get $\overleftarrow{T}_{\Sigma'} \neq \varnothing$, whence $\overleftarrow{T}_{\Sigma'} = T_{\Sigma'}$. Thus, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$, the condition
$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \in \overleftarrow{T}_{\Sigma'} \quad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}) \in \overleftarrow{T}_{\Sigma'}$$
is equivalent to the condition
$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \in T_{\Sigma'} \quad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}) \in T_{\Sigma'}.$$
Hence, $\langle \phi, \psi \rangle \in \Omega_\Sigma(\overleftarrow{T}) = \Omega_\Sigma(T)$.

We conclude that $\mathcal{I}$ is left regular. ■

Proposition 576 and Theorem 578 establish the hierarchy depicted in the diagram.



We show, next, that, adding stability to system regularity and to right regularity takes us, respectively, into the classes of left regular and family regular π-institutions.

Proposition 579 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is system regular and stable, then it is left regular;*

(b) If $\mathcal{I}$ *is right regular and stable, then it is family regular.*

Proof:

(a) Suppose $\mathcal{I}$ is system regular and stable. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in \overleftarrow{T}_\Sigma$. Since, by Proposition 42, $\overleftarrow{T} \in \mathrm{ThSys}(\mathcal{I})$, we may apply system regularity to conclude that $\langle \phi, \psi \rangle \in \Omega_\Sigma(\overleftarrow{T})$. Therefore, by stability, $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. Thus, $\mathcal{I}$ is left regular.

(b) Suppose $\mathcal{I}$ is right regular and stable. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. By right regularity, we get that $\langle \phi, \psi \rangle \in \Omega_\Sigma(\overleftarrow{T})$. Therefore, by stability, $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. Thus, $\mathcal{I}$ is family regular.

■

Of course, if systemicity is assumed, then all four classes in the regularity hierarchy collapse into a single class.

Proposition 580 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is system regular and systemic, then it is family regular.*

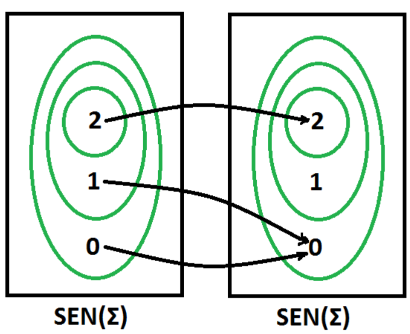
Proof: Under systemicity, all theory families are also theory systems. Hence the conditions defining family and system regularity are identical. ■

To show that all four classes in the hierarchy above are different, we must present some examples that separate them. The first example provides an unstable π-institution which is left regular but not right regular. This accomplishes two goals:

- It shows that the class of right regular π-institutions is a proper subclass of the class of left regular ones;
- It shows that the converse of Part (a) of Proposition 579 does not hold in general, as the π-institution constructed is left regular but fails to be stable.

Example 581 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial category of natural transformations.*

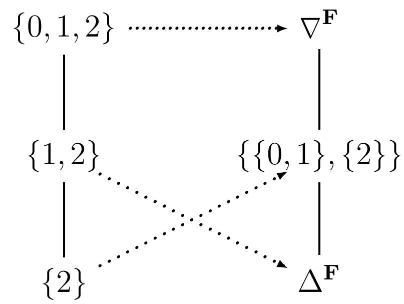


Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}$*.*

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$*:*

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

The lattice of theory families and the corresponding Leibniz congruence systems are depicted below:



Since

$$\Omega(\overleftarrow{\{\{1,2\}\}}) = \Omega(\{\{2\}\}) = \{\{\{0,1\},\{2\}\}\} \neq \Delta^{\mathbf{F}} = \Omega(\{\{1,2\}\}),$$

$\mathcal{I}$ is not stable.

We show that $\mathcal{I}$ is left regular, i.e., that, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, if $\phi, \psi \in \overleftarrow{T}_\Sigma$, then $\langle\phi,\psi\rangle \in \Omega_\Sigma(T)$.

- *If $T = \{\{0,1,2\}\}$, then, for all ϕ, ψ, $\langle\phi,\psi\rangle \in \nabla^{\mathbf{F}}_\Sigma = \Omega_\Sigma(\{\{0,1,2\}\})$;*
- *If $T \neq \{\{0,1,2\}\}$, then $\phi, \psi \in \overleftarrow{T}_\Sigma$ implies $\phi = \psi = 2$, whence, $\langle\phi,\psi\rangle \in \Delta^{\mathbf{F}}_\Sigma \subseteq \Omega_\Sigma(T)$.*

On the other hand, for $T = \{\{1,2\}\}$, we have $1, 2 \in T_\Sigma$, but

$$\langle 1,2\rangle \notin \{\{\{0,1\},\{2\}\}\} = \Omega_\Sigma(\{\{2\}\}) = \Omega_\Sigma(\overleftarrow{T}).$$

Therefore, $\mathcal{I}$ is not right regular.

The second example presents a π-institution which is right regular, but fails to be family regular.

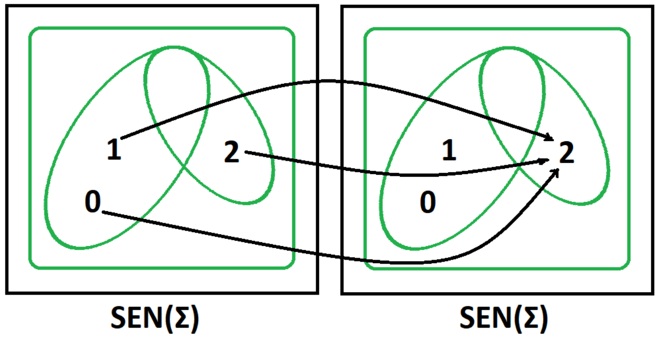
Example 582 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be the algebraic system defined as follows:*

- *$\mathbf{Sign}^\flat$ is the category with a single object Σ and a unique (non-identity) morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- *$\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ is defined by $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ and*

  $$\mathrm{SEN}^\flat(f)(0) = 2,\ \mathrm{SEN}^\flat(f)(1) = 2,\ \mathrm{SEN}^\flat(f)(2) = 2;$$

- *$N^\flat$ is the category of natural transformations generated by the unary natural transformation $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ determined by*

  $$\sigma^\flat_\Sigma(0) = 0,\ \sigma^\flat_\Sigma(1) = 2,\ \sigma^\flat_\Sigma(2) = 2.$$



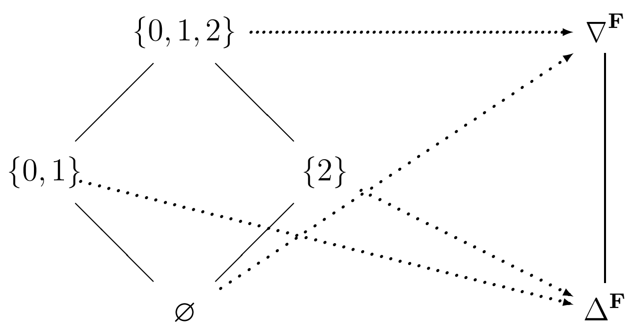
Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\varnothing, \{0,1\}, \{2\}, \{0,1,2\}\}.$$

The following table shows the action of $\overleftarrow{}$ *on theory families.*

T	$\varnothing$	$\{0,1\}$	$\{2\}$	$\{0,1,2\}$
$\overleftarrow{T}$	$\varnothing$	$\varnothing$	$\{2\}$	$\{0,1,2\}$

The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right:



We show, first, that $\mathcal{I}$ is right regular, i.e., that it satisfies, for all $T \in$ ThFam$(\mathcal{I})$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi,\psi \in T_\Sigma$ implies $\langle\phi,\psi\rangle \in \Omega_\Sigma(\overleftarrow{T})$.

- *If $T = \{\varnothing\}$, then the conclusion is vacuously true;*
- *If $T = \{\{0,1\}\}$, then, since $\Omega(\overleftarrow{T}) = \Omega(\{\varnothing\}) = \nabla^{\mathbf{F}}$, the conclusion is trivial;*
- *If $T = \{\{2\}\}$, then $\phi,\psi \in T_\Sigma$ implies $\phi = \psi = 2$, whence $\langle\phi,\psi\rangle \in \Delta^{\mathbf{F}}_\Sigma \subseteq \Omega_\Sigma(\overleftarrow{T})$;*
- *If $T = \{\{0,1,2\}\}$, then, since $\Omega(\overleftarrow{T}) = \Omega(T) = \nabla^{\mathbf{F}}$, the conclusion is trivial.*

On the other hand, for $T = \{\{0,1\}\}$, *we have* $0,1 \in T_\Sigma$, *whereas* $\langle 0,1\rangle \notin \Delta^{\mathbf{F}}_\Sigma = \Omega_\Sigma(T)$. *We conclude that* $\mathcal{I}$ *is not family regular.*

The last example shows a system regular π-institution which fails to be left regular.

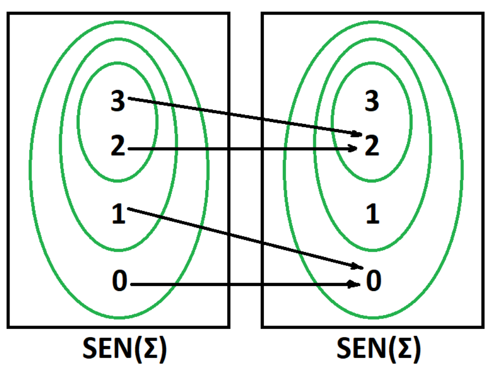
Example 583 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3\}$ *and*

  $$\mathrm{SEN}^\flat(f)(0) = 0,\ \mathrm{SEN}^\flat(f)(1) = 0,\ \mathrm{SEN}^\flat(f)(2) = 2,\ \mathrm{SEN}^\flat(f)(3) = 2;$$

- $N^\flat$ *is the category of natural transformations generated by the unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *determined by*

x	0	1	2	3
$\sigma^\flat_\Sigma(x)$	0	1	0	1



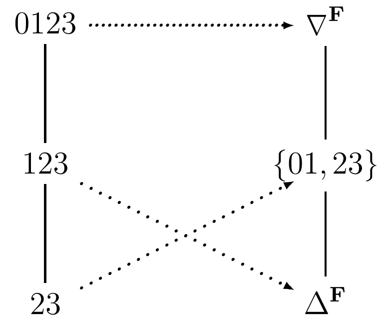
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{2,3\},\{1,2,3\},\{0,1,2,3\}\}.$$

The following table shows the action of $\overleftarrow{\ }$ *on theory families.*

T	$\{2,3\}$	$\{1,2,3\}$	$\{0,1,2,3\}$
$\overleftarrow{T}$	$\{2,3\}$	$\{2,3\}$	$\{0,1,2,3\}$

The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in *terms of blocks) on the right:*



We show, first, that $\mathcal{I}$ *is system regular, i.e., that it satisfies, for all* $T \in \mathrm{ThSys}(\mathcal{I})$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi, \psi \in T_\Sigma$ *implies* $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$.

- *If* $T = \{\{2, 3\}\}$, *then,* $\phi = \psi$ *or* $\{\phi, \psi\} = \{2, 3\}$. *In either case* $\langle \phi, \psi \rangle \in \{\{0, 1\}, \{2, 3\}\} = \Omega_\Sigma(T)$;
- *If* $T = \{\{0, 1, 2, 3\}\}$, *then, since* $\Omega(T) = \nabla^{\mathbf{F}}$, *the conclusion is trivial.*

On the other hand, for $T = \{\{1, 2, 3\}\}$, *we have* $2, 3 \in \{2, 3\} = \overleftarrow{T}_\Sigma$, *whereas* $\langle 2, 3 \rangle \notin \Delta^{\mathbf{F}}_\Sigma = \Omega_\Sigma(T)$. *We conclude that* $\mathcal{I}$ *is not left regular.*

We provide, next, characterizations of three of the four regularity classes in terms of the Suszko operator acting on the theory families of a π-institution.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Given a theory system $T \in \mathrm{ThSys}(\mathcal{I})$, we set

$$\widehat{\Omega}^{\mathcal{I}}(T) = \bigcap \{\Omega(T') : T \leq T' \in \mathrm{ThSys}(\mathcal{I})\},$$

a system version of the Suszko operator on $\mathcal{I}$.

Theorem 584 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is family regular if and only if, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\langle \phi, \psi \rangle \in \widetilde{\Omega}^{\mathcal{I}}_\Sigma(C(\phi, \psi));$$

(b) $\mathcal{I}$ *is left regular if and only if, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\langle \phi, \psi \rangle \in \widetilde{\Omega}^{\mathcal{I}}_\Sigma(\overrightarrow{C}(\phi, \psi));$$

(c) $\mathcal{I}$ *is system regular if and only if, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\langle \phi, \psi \rangle \in \widehat{\Omega}^{\mathcal{I}}_\Sigma(\overrightarrow{C}(\phi, \psi)).$$

Proof:

(a) Suppose $\mathcal{I}$ is family regular. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then, we have, by family regularity, $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$, for all $T \in \mathrm{ThFam}(\mathcal{I})$, such that $\phi, \psi \in T_\Sigma$. Therefore, by the definition of $\widetilde{\Omega}^{\mathcal{I}}$,

$$\begin{array}{lll}\langle \phi, \psi \rangle & \in & \bigcap\{\Omega_\Sigma(T) : T \in \mathrm{ThFam}(\mathcal{I}), \phi, \psi \in T_\Sigma\} \\ & = & \widetilde{\Omega}^{\mathcal{I}}_\Sigma(C(\phi, \psi)).\end{array}$$

Assume, conversely, that the displayed condition holds. To show family regularity, let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Then, $C(\phi, \psi) \leq T$, whence, by the hypothesis and the monotonicity of $\widetilde{\Omega}^{\mathcal{I}}$,

$$\langle \phi, \psi \rangle \in \widetilde{\Omega}^{\mathcal{I}}_\Sigma(C(\phi, \psi)) \subseteq \widetilde{\Omega}^{\mathcal{I}}_\Sigma(T) \subseteq \Omega_\Sigma(T).$$

Hence, $\mathcal{I}$ is family regular.

(b) Suppose $\mathcal{I}$ is left regular. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then, we have, by left regularity, $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$, for all $T \in \mathrm{ThFam}(\mathcal{I})$, such that $\phi, \psi \in \overleftarrow{T}_\Sigma$. Therefore, by the definition of $\widetilde{\Omega}^{\mathcal{I}}$,

$$\begin{array}{lll}\langle \phi, \psi \rangle & \in & \bigcap\{\Omega_\Sigma(T) : T \in \mathrm{ThFam}(\mathcal{I}), \phi, \psi \in \overleftarrow{T}_\Sigma\} \\ & = & \bigcap\{\Omega_\Sigma(T) : T \in \mathrm{ThFam}(\mathcal{I}), \overrightarrow{\{\phi, \psi\}} \leq T\} \\ & = & \widetilde{\Omega}^{\mathcal{I}}_\Sigma(\overrightarrow{C}(\phi, \psi)).\end{array}$$

Assume, conversely, that the displayed condition holds. To show left regularity, let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in \overleftarrow{T}_\Sigma$. Then, $\overrightarrow{\{\phi, \psi\}} \leq T$, whence, by the hypothesis and the monotonicity of $\widetilde{\Omega}^{\mathcal{I}}$,

$$\langle \phi, \psi \rangle \in \widetilde{\Omega}^{\mathcal{I}}_\Sigma(\overrightarrow{C}(\phi, \psi)) \subseteq \widetilde{\Omega}^{\mathcal{I}}_\Sigma(T) \subseteq \Omega_\Sigma(T).$$

Hence, $\mathcal{I}$ is left regular.

(c) Suppose $\mathcal{I}$ is system regular. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then, we have, by system regularity, $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$, for all $T \in \mathrm{ThSys}(\mathcal{I})$, such that $\phi, \psi \in T_\Sigma$. Therefore, by the definition of $\widehat{\Omega}^{\mathcal{I}}$,

$$\begin{array}{lll}\langle \phi, \psi \rangle & \in & \bigcap\{\Omega_\Sigma(T) : T \in \mathrm{ThSys}(\mathcal{I}), \phi, \psi \in T_\Sigma\} \\ & = & \bigcap\{\Omega_\Sigma(T) : T \in \mathrm{ThSys}(\mathcal{I}), \overrightarrow{\{\phi, \psi\}} \leq T\} \\ & = & \widehat{\Omega}^{\mathcal{I}}_\Sigma(\overrightarrow{C}(\phi, \psi)).\end{array}$$

Assume, conversely, that the displayed condition holds. To show system regularity, let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Then, $\overrightarrow{\{\phi, \psi\}} \leq T$, whence, by the hypothesis and the monotonicity of $\widehat{\Omega}^{\mathcal{I}}$,

$$\langle \phi, \psi \rangle \in \widehat{\Omega}^{\mathcal{I}}_\Sigma(\overrightarrow{C}(\phi, \psi)) \subseteq \widehat{\Omega}^{\mathcal{I}}_\Sigma(T) \subseteq \Omega_\Sigma(T).$$

Hence, $\mathcal{I}$ is system regular.

■

We show, next, that all four regularity properties transfer from theory families/systems to $\mathcal{I}$-filter families/systems over arbitrary $\mathbf{F}$-algebraic systems.

Theorem 585 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is family regular if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,*

$$\phi, \psi \in T_\Sigma \quad \textit{implies} \quad \langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_\Sigma(T);$$

(b) $\mathcal{I}$ *is right regular if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,*

$$\phi, \psi \in T_\Sigma \quad \textit{implies} \quad \langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_\Sigma(\overleftarrow{T});$$

(c) $\mathcal{I}$ *is left regular if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,*

$$\phi, \psi \in \overleftarrow{T}_\Sigma \quad \textit{implies} \quad \langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_\Sigma(T);$$

(d) $\mathcal{I}$ *is system regular if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, all* $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$*, all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,*

$$\phi, \psi \in T_\Sigma \quad \textit{implies} \quad \langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_\Sigma(T).$$

Proof:

(a) The "if" follows easily by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and recalling from Lemma 51 that $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}) = \mathrm{ThFam}(\mathcal{I})$.

Assume, conversely, that $\mathcal{I}$ is family regular and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \in T_{F(\Sigma)}$. Then, we get $\phi, \psi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})$. By Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, whence, by family regularity, we get that $\langle \phi, \psi \rangle \in \Omega_\Sigma(\alpha^{-1}(T))$. Thus, by Proposition 24, we get $\langle \phi, \psi \rangle \in \alpha_\Sigma^{-1}(\Omega^{\mathcal{A}}_{F(\Sigma)}(T))$. Hence, $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \Omega^{\mathcal{A}}_{F(\Sigma)}(T)$. Taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, if $\phi, \psi \in T_\Sigma$, then $\langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_\Sigma(T)$.

(b) The "if" follows as in Part (a).

Assume, conversely, that $\mathcal{I}$ is right regular and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\in T_{F(\Sigma)}$. Then, we get $\phi,\psi\in\alpha_\Sigma^{-1}(T_{F(\Sigma)})$. By Lemma 51, $\alpha^{-1}(T)\in\mathrm{ThFam}(\mathcal{I})$, whence, by right regularity, we get that $\langle\phi,\psi\rangle\in\Omega_\Sigma(\overleftarrow{\alpha^{-1}(T)})$. By Lemma 6, we get $\langle\phi,\psi\rangle\in\Omega_\Sigma(\alpha^{-1}(\overleftarrow{T}))$. Thus, by Proposition 24, we get $\langle\phi,\psi\rangle\in\alpha_\Sigma^{-1}(\Omega^{\mathcal{A}}_{F(\Sigma)}(\overleftarrow{T}))$. Hence, $\langle\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\rangle\in\Omega^{\mathcal{A}}_{F(\Sigma)}(\overleftarrow{T})$. Taking into account the surjectivity of $\langle F,\alpha\rangle$, we conclude that, for all $\Sigma\in|\mathbf{Sign}|$ and all $\phi,\psi\in\mathrm{SEN}(\Sigma)$, if $\phi,\psi\in T_\Sigma$, then $\langle\phi,\psi\rangle\in\Omega^{\mathcal{A}}_\Sigma(\overleftarrow{T})$.

(c) The "if" follows as in Part (a).

Assume, conversely, that $\mathcal{I}$ is left regular and let $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system, $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma\in|\mathbf{Sign}^\flat|$ and $\phi,\psi\in\mathrm{SEN}^\flat(\Sigma)$, such that $\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\in\overleftarrow{T}_{F(\Sigma)}$. Then, we get $\phi,\psi\in\alpha_\Sigma^{-1}(\overleftarrow{T}_{F(\Sigma)})$, i.e., by Lemma 6, $\phi,\psi\in\overleftarrow{\alpha^{-1}(T)}_\Sigma$. By Lemma 51, $\alpha^{-1}(T)\in\mathrm{ThFam}(\mathcal{I})$, whence, by left regularity, we get that $\langle\phi,\psi\rangle\in\Omega_\Sigma(\alpha^{-1}(T))$. Thus, by Proposition 24, we get $\langle\phi,\psi\rangle\in\alpha_\Sigma^{-1}(\Omega^{\mathcal{A}}_{F(\Sigma)}(T))$. Hence, $\langle\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\rangle\in\Omega^{\mathcal{A}}_{F(\Sigma)}(T)$. Taking into account the surjectivity of $\langle F,\alpha\rangle$, we conclude that, for all $\Sigma\in|\mathbf{Sign}|$ and all $\phi,\psi\in\mathrm{SEN}(\Sigma)$, if $\phi,\psi\in\overleftarrow{T}_\Sigma$, then $\langle\phi,\psi\rangle\in\Omega^{\mathcal{A}}_\Sigma(T)$.

(d) Similar to Part (a). ■

We also have the following characterizations in terms of reduced $\mathcal{I}$-matrix families and $\mathcal{I}$-matrix systems.

Theorem 586 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is family regular if and only if, for every* $\langle\mathcal{A},T\rangle\in\mathrm{MatFam}^*(\mathcal{I})$*, with* $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$, $\mathbf{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$*, and all* $\Sigma\in|\mathbf{Sign}|$, $|T_\Sigma|\le1$*;*

(b) $\mathcal{I}$ *is system regular if and only if, for every* $\langle\mathcal{A},T\rangle\in\mathrm{MatSys}^*(\mathcal{I})$*, with* $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$, $\mathbf{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$*, and all* $\Sigma\in|\mathbf{Sign}|$, $|T_\Sigma|\le1$.

Proof:

(a) Suppose, first, that $\mathcal{I}$ is family regular. Let $\langle\mathcal{A},T\rangle\in\mathrm{MatFam}^*(\mathcal{I})$, $\Sigma\in|\mathbf{Sign}|$ and $\phi,\psi\in\mathrm{SEN}(\Sigma)$, such that $\phi,\psi\in T_\Sigma$. Then, we have, using Theorem 585 and the fact that $\langle\mathcal{A},T\rangle$ is reduced,

$$\langle\phi,\psi\rangle\in\Omega^{\mathcal{A}}_\Sigma(T)=\Delta^{\mathcal{A}}_\Sigma,$$

whence $\phi=\psi$. Therefore, $|T_\Sigma|\le1$.

Suppose, conversely, that the given condition holds. Let $T\in\mathrm{ThFam}(\mathcal{I})$, $\Sigma\in|\mathbf{Sign}^\flat|$ and $\phi,\psi\in\mathrm{SEN}^\flat(\Sigma)$, such that $\phi,\psi\in T_\Sigma$. Then, $\langle\mathcal{F}/\Omega(T),$

$T/\Omega(T)\rangle$ is reduced and, moreover, $\phi/\Omega_\Sigma(T), \psi/\Omega_\Sigma(T) \in T_\Sigma/\Omega_\Sigma(T)$. Hence, by hypothesis, $\phi/\Omega_\Sigma(T) = \psi/\Omega_\Sigma(T)$, i.e., $\langle\phi,\psi\rangle \in \Omega_\Sigma(T)$. We conclude that $\mathcal{I}$ is family regular.

(b) Suppose, first, that $\mathcal{I}$ is system regular. Let $\langle\mathcal{A},T\rangle \in \mathrm{MatSys}^*(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\phi,\psi \in T_\Sigma$. Then, we have, using Theorem 585 and the fact that $\langle\mathcal{A},T\rangle$ is reduced,

$$\langle\phi,\psi\rangle \in \Omega^{\mathcal{A}}_\Sigma(T) = \Delta^{\mathcal{A}}_\Sigma,$$

whence $\phi = \psi$. Therefore, $|T_\Sigma| \le 1$.

Suppose, conversely, that the given condition holds. Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi,\psi \in T_\Sigma$. Then, $\langle\mathcal{F}/\Omega(T), T/\Omega(T)\rangle$ is a reduced $\mathcal{I}$-matrix system. Moreover, we have $\phi/\Omega_\Sigma(T)$, $\psi/\Omega_\Sigma(T) \in T_\Sigma/\Omega_\Sigma(T)$. Hence, by hypothesis, $\phi/\Omega_\Sigma(T) = \psi/\Omega_\Sigma(T)$, i.e., $\langle\phi,\psi\rangle \in \Omega_\Sigma(T)$. We conclude that $\mathcal{I}$ is system regular.

■

8.3 Assertionality

In this section, we introduce the *assertionality hierarchy* of π-institutions. The properties defining this hierarchy are obtained simply by adding to the various properties defining the regularity hierarchy the stipulation that $\mathcal{I}$ have theorems.

Definition 587 (Assertionality) *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is* **family assertional** *if it is family regular and has theorems;*
- $\mathcal{I}$ *is* **left assertional** *if it is left regular and has theorems;*
- $\mathcal{I}$ *is* **right assertional** *if it is right regular and has theorems;*
- $\mathcal{I}$ *is* **system assertional** *if it is system regular and has theorems.*

Definition 587 and Proposition 576 allow us to obtain the following a priori assertionality hierarchy of π-institutions.

Family Assertional

↓

Right Assertional

↓

Left Assertional

↓

System Assertional

However, using the characterizing properties included in the following proposition, we shall see that right assertionality implies systemicity and, hence, the classes of family assertional and right assertional π-institutions coincide.

Proposition 588 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is family assertional if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$ *and all* $\Sigma \in |\mathbf{Sign}^\flat|$, $T_\Sigma = t_\Sigma / \Omega_\Sigma(T)$, *for some* $t_\Sigma \in \mathrm{Thm}_\Sigma(\mathcal{I})$;

(b) $\mathcal{I}$ *is right assertional if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$ *and all* $\Sigma \in |\mathbf{Sign}^\flat|$, $T_\Sigma = t_\Sigma / \Omega_\Sigma(\overleftarrow{T})$, *for some* $t_\Sigma \in \mathrm{Thm}_\Sigma(\mathcal{I})$;

(c) $\mathcal{I}$ *is left assertional if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$ *and all* $\Sigma \in |\mathbf{Sign}^\flat|$, $\overleftarrow{T}_\Sigma = t_\Sigma / \Omega_\Sigma(T)$, *for some* $t_\Sigma \in \mathrm{Thm}_\Sigma(\mathcal{I})$;

(d) $\mathcal{I}$ *is system assertional if and only if, for all* $T \in \mathrm{ThSys}(\mathcal{I})$ *and all* $\Sigma \in |\mathbf{Sign}^\flat|$, $T_\Sigma = t_\Sigma / \Omega_\Sigma(T)$, *for some* $t_\Sigma \in \mathrm{Thm}_\Sigma(\mathcal{I})$.

Proof: If, in a certain context, a π-institution $\mathcal{I}$ has theorems, we shall use t_Σ to denote an arbitrary Σ-theorem of $\mathcal{I}$, $\Sigma \in |\mathbf{Sign}^\flat|$.

(a) Suppose that $\mathcal{I}$ is family assertional and let $T \in \mathrm{ThFam}(\mathcal{I})$ and $\Sigma \in |\mathbf{Sign}^\flat|$. If $\phi \in T_\Sigma$, then $\phi, t_\Sigma \in T_\Sigma$, whence, by family regularity, $\langle \phi, t_\Sigma \rangle \in \Omega_\Sigma(T)$, i.e., $\phi \in t_\Sigma / \Omega_\Sigma(T)$. On the other hand, if $\langle \phi, t_\Sigma \rangle \in \Omega_\Sigma(T)$, then, since $t_\Sigma \in T_\Sigma$, we get, by the compatibility of $\Omega(T)$ with T, $\phi \in T_\Sigma$.

Suppose, conversely, that, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$, $T_\Sigma = t_\Sigma / \Omega_\Sigma(T)$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Then, by hypothesis

$$\phi \ \Omega_\Sigma(T) \ t_\Sigma \ \Omega_\Sigma(T) \ \psi,$$

whence $\mathcal{I}$ is family regular.

(b) Suppose that $\mathcal{I}$ is right assertional and let $T \in \mathrm{ThFam}(\mathcal{I})$ and $\Sigma \in |\mathbf{Sign}^\flat|$. If $\phi \in T_\Sigma$, then $\phi, t_\Sigma \in T_\Sigma$, whence, by right regularity, $\langle\phi, t_\Sigma\rangle \in \Omega_\Sigma(\overleftarrow{T})$, i.e., $\phi \in t_\Sigma/\Omega_\Sigma(\overleftarrow{T})$. On the other hand, if $\langle\phi, t_\Sigma\rangle \in \Omega_\Sigma(\overleftarrow{T})$, then, since $t_\Sigma \in \overleftarrow{T}_\Sigma$, we get, by the compatibility of $\Omega(\overleftarrow{T})$ with $\overleftarrow{T}$, $\phi \in \overleftarrow{T}_\Sigma \subseteq T_\Sigma$.

Suppose, conversely, that, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$, $T_\Sigma = t_\Sigma/\Omega_\Sigma(\overleftarrow{T})$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Then, by hypothesis

$$\phi \ \Omega_\Sigma(\overleftarrow{T}) \ t_\Sigma \ \Omega_\Sigma(\overleftarrow{T}) \ \psi,$$

whence $\mathcal{I}$ is right regular.

(c) Suppose that $\mathcal{I}$ is left assertional and let $T \in \mathrm{ThFam}(\mathcal{I})$ and $\Sigma \in |\mathbf{Sign}^\flat|$. If $\phi \in \overleftarrow{T}_\Sigma$, then $\phi, t_\Sigma \in \overleftarrow{T}_\Sigma$, whence, by left regularity, $\langle\phi, t_\Sigma\rangle \in \Omega_\Sigma(T)$, i.e., $\phi \in t_\Sigma/\Omega_\Sigma(T)$. On the other hand, if $\langle\phi, t_\Sigma\rangle \in \Omega_\Sigma(T)$, then, since $\Omega(T) \le \Omega(\overleftarrow{T})$, we get $\langle\phi, t_\Sigma\rangle \in \Omega_\Sigma(\overleftarrow{T})$. But $t_\Sigma \in \overleftarrow{T}_\Sigma$, whence, by the compatibility of $\Omega(\overleftarrow{T})$ with $\overleftarrow{T}$, $\phi \in \overleftarrow{T}_\Sigma$.

Suppose, conversely, that, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$, $\overleftarrow{T}_\Sigma = t_\Sigma/\Omega_\Sigma(T)$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in \overleftarrow{T}_\Sigma$. Then, by hypothesis

$$\phi \ \Omega_\Sigma(T) \ t_\Sigma \ \Omega_\Sigma(T) \ \psi,$$

whence $\mathcal{I}$ is left regular.

(d) Similar to Part (a).

∎

Using the characterizations in Proposition 588, we can show that right assertionality implies systemicity.

Proposition 589 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is right assertional, then* $\mathcal{I}$ *is systemic.*

Proof: Suppose that $\mathcal{I}$ is right assertional. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$. Then, by right assertionality and Proposition 588, $\langle\phi, t_\Sigma\rangle \in \Omega_\Sigma(\overleftarrow{T})$, for some $t_\Sigma \in \mathrm{Thm}_\Sigma(\mathcal{I})$. But $t_\Sigma \in \overleftarrow{T}_\Sigma$, whence, by compatibility of $\Omega(\overleftarrow{T})$ with $\overleftarrow{T}$, $\phi \in \overleftarrow{T}_\Sigma$. Therefore, $T \le \overleftarrow{T}$ and, hence, $T \in \mathrm{ThSys}(\mathcal{I})$. Thus, $\mathcal{I}$ is systemic. ∎

Proposition 590 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is right assertional if and only if* $\mathcal{I}$ *is family assertional.*

Proof: If $\mathcal{I}$ is family assertional, then, by definition, it is family regular and has theorems, whence, by Proposition 576, it is right regular and has theorems and, therefore, by definition, it is right assertional.

Suppose, conversely, that $\mathcal{I}$ is right assertional. Then, by Proposition 589, it is systemic and, hence, a fortiori, stable. Therefore, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, if $\phi, \psi \in T_\Sigma$, then, by right assertionality, $\langle \phi, \psi \rangle \in \Omega_\Sigma(\overleftarrow{T})$ and, hence, by stability, $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. Therefore, $\mathcal{I}$ is family regular and, hence, family assertional. ■

We can also show easily that, in case $\mathcal{I}$ is systemic, the entire assertionailty hierarchy collapses into a single class.

Proposition 591 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is family assertional if and only if it is system assertional and systemic.*

Proof: If $\mathcal{I}$ is systemic, the conditions defining family assertionality and system assertionality coincide.

On the other hand, if $\mathcal{I}$ is family assertional, then, by definition, it is family regular and has theorems, whence, by Proposition 576, it is right regular and has theorems. Thus, by definition, $\mathcal{I}$ is right assertional and, hence, by Proposition 589, it is systemic. Moreover, using again Proposition 576, we conclude that $\mathcal{I}$ is also system assertional. ■

Thus, we get, regarding the assertionality hierarchy the following

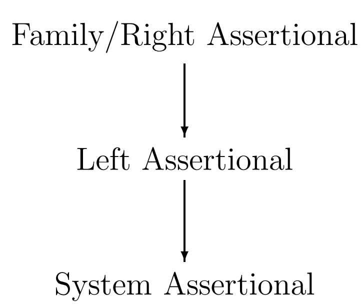
Proposition 592 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is family/right assertional, then it is left assertional;*

(b) If $\mathcal{I}$ *is left assertional, then it is system assertional.*

Proof: By Definition 587, Proposition 576 and Proposition 591. ■

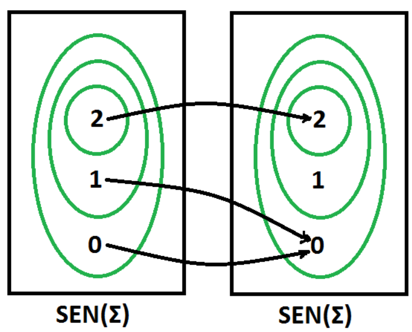
Proposition 592 establishes the **assertionality hierarchy** depicted in the accompanying diagram.



We show, next, that all three classes are different, by constructing two examples to separate them. The first is an example of a left assertional π-institution which fails to satisfy family assertionality.

Example 593 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial category of natural transformations.*



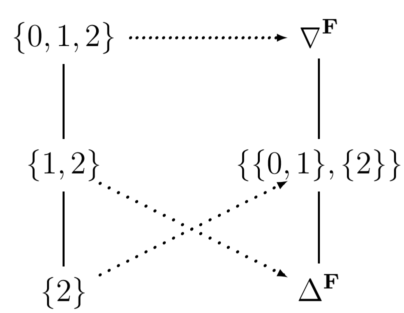
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}$*.*

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$*:*

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1, 2\}$	$\{2\}$
$\{0, 1, 2\}$	$\{0, 1, 2\}$

Since $\mathcal{I}$ *is not systemic, then, by Proposition 591, it fails to be family assertional.*

The lattice of theory families and the corresponding Leibniz congruence systems are depicted below:



Clearly, $\mathcal{I}$ *has theorems. Thus, to show that it is left assertional, it suffices to show, by Proposition 588, that, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,* $\overleftarrow{T}_\Sigma = 2/\Omega_\Sigma(T)$*.*

- $\overleftarrow{\{\{2\}\}}_\Sigma = \{2\} = 2/\Omega_\Sigma(\{\{2\}\})$;
- $\overleftarrow{\{\{1,2\}\}}_\Sigma = \{2\} = 2/\Omega_\Sigma(\{\{1,2\}\})$;
- $\overleftarrow{\{\{0,1,2\}\}}_\Sigma = \{0,1,2\} = 2/\Omega_\Sigma(\{\{0,1,2\}\})$.

The next example showcases a system assertional π-institution which is not left assertional.

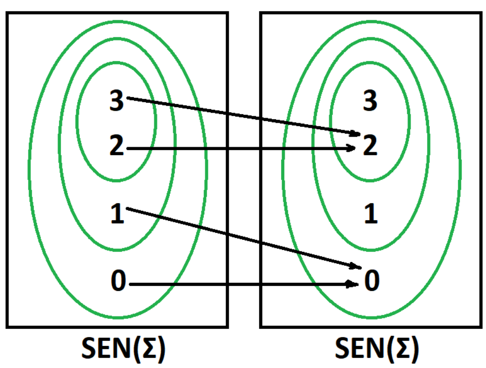
Example 594 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3\}$ *and*

  $$\mathrm{SEN}^\flat(f)(0) = 0,\ \mathrm{SEN}^\flat(f)(1) = 0,\ \mathrm{SEN}^\flat(f)(2) = 2,\ \mathrm{SEN}^\flat(f)(3) = 2;$$

- $N^\flat$ *is the category of natural transformations generated by the unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *determined by*

x	0	1	2	3
$\sigma^\flat_\Sigma(x)$	0	1	0	1



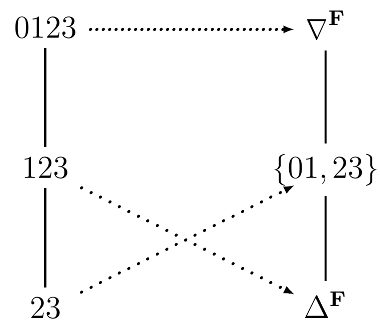
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{2,3\},\{1,2,3\},\{0,1,2,3\}\}.$$

The following table shows the action of $\overleftarrow{\ }$ *on theory families.*

T	$\{2,3\}$	$\{1,2,3\}$	$\{0,1,2,3\}$
$\overleftarrow{T}$	$\{2,3\}$	$\{2,3\}$	$\{0,1,2,3\}$

The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right:



Clearly, $\mathcal{I}$ has theorems. To see that $\mathcal{I}$ is system assertional, it suffices to show, by Proposition 588, that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $T_\Sigma = 2/\Omega_\Sigma(T)$. We do have indeed:

- $\{2,3\} = 2/\Omega_\Sigma(\{\{2,3\}\})$;
- $\{0,1,2,3\} = 2/\Omega_\Sigma(\{\{0,1,2,3\}\})$.

On the other hand, for $T = \{\{1,2,3\}\}$, we have $2,3 \in \{2,3\} = \overleftarrow{T}_\Sigma$, whereas $\langle 2,3\rangle \notin \Delta^{\mathbf{F}}_\Sigma = \Omega_\Sigma(T)$. We conclude that $\mathcal{I}$ is not left regular and, hence, a fortiori, not left assertional either.

We proceed by exploring the relationships that hold between the various classes of the assertionality hierarchy, introduced in the present section, with the classes of the regularity hierarchy, which were introduced in Section 8.2. We have the following straightforward implications, which follow directly from the definitions involved.

Proposition 595 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.*

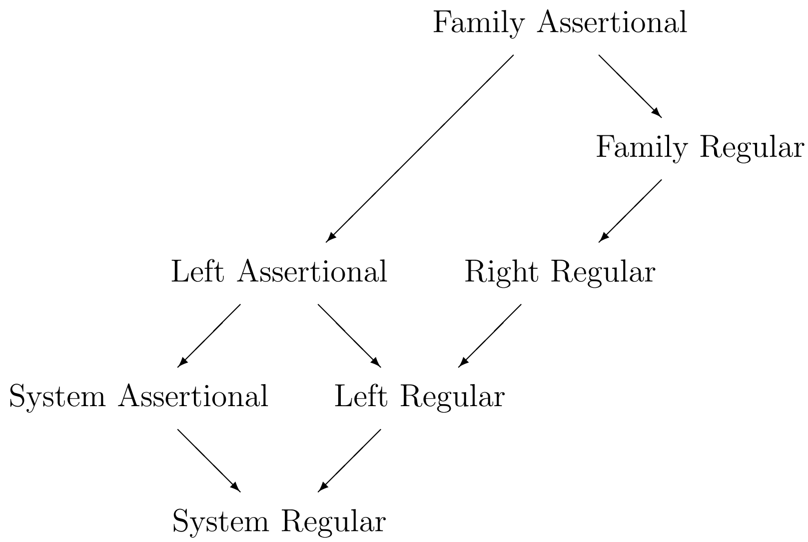
(a) If $\mathcal{I}$ is family assertional, then it is family regular;

(b) If $\mathcal{I}$ is left assertional, then it is left regular;

(c) If $\mathcal{I}$ is system assertional, then it is system regular.

Proof: Directly from Definition 587. ■

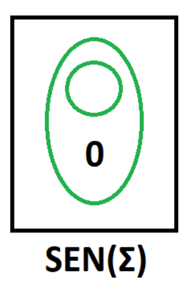
Thus, taking into account Propositions 576 and 592, we have the following mixed assertionality and regularity hierarchy.



An easy example shows that the three southeast arrows from the assertionality classes to the corresponding regularity classes correspond to proper inclusions.

Example 596 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

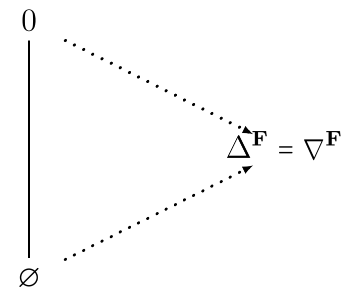
- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0\}$*;*
- $N^\flat$ *is the trivial category of natural transformations.*



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{0\}\}.$$

$\mathcal{I}$ is systemic and its lattice of theory families and corresponding Leibniz congruence systems are shown in the diagram.



$\mathcal{I}$ is family regular, since, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\langle 0,0\rangle \in \nabla^{\mathbf{F}}_{\Sigma} = \Omega_{\Sigma}(T)$.

On the other hand, since $\mathcal{I}$ does not have theorems, $\mathcal{I}$ does not belong to any of the steps in the assertionality hierarchy.

We examine next, the relationships between the classes in the assertionality hierarchy and those in the complete reflectivity hierarchy.

Theorem 597 *Let $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat}\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.*

(a) If $\mathcal{I}$ is family/right assertional, then it is family/right completely reflective;

(b) If $\mathcal{I}$ is left assertional, then it is left completely reflective;

(c) If $\mathcal{I}$ is system assertional, then it is system completely reflective.

Proof:

(a) Suppose that $\mathcal{I}$ is family assertional. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \le \Omega(T')$, $\Sigma \in |\mathbf{Sign}^{\flat}|$ and $\phi \in \bigcap_{T\in\mathcal{T}} T_{\Sigma}$. By assertionality, there exists $t_{\Sigma} \in \mathrm{Thm}_{\Sigma}(\mathcal{I})$, whence, $\phi, t_{\Sigma} \in T_{\Sigma}$, for all $T \in \mathcal{T}$. Thus, by family regularity, $\langle \phi, t_{\Sigma}\rangle \in \Omega_{\Sigma}(T)$, for all $T \in \mathcal{T}$, i.e., $\langle \phi, t_{\Sigma}\rangle \in \bigcap_{T\in\mathcal{T}} \Omega_{\Sigma}(T)$. By hypothesis, $\langle \phi, t_{\Sigma}\rangle \in \Omega_{\Sigma}(T')$. Therefore, since $t_{\Sigma} \in T'_{\Sigma}$, we get, by compatibility of $\Omega(T')$ with T', $\phi \in T'_{\Sigma}$. We conclude that $\bigcap_{T\in\mathcal{T}} T \le T'$ and, hence, that $\mathcal{I}$ is family c-reflective.

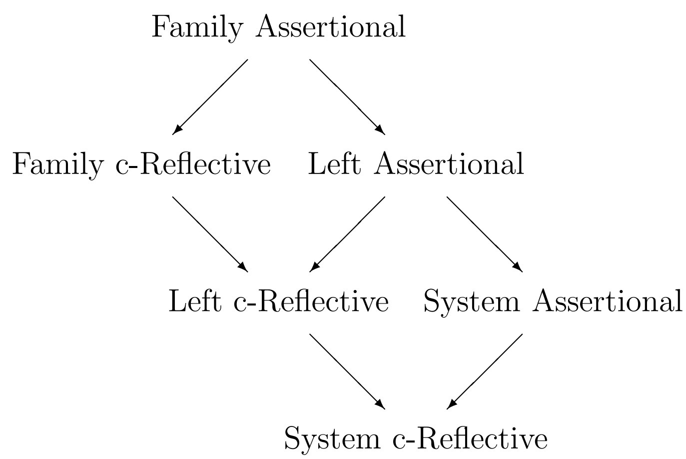
(b) Suppose that $\mathcal{I}$ is left assertional. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \le \Omega(T')$, $\Sigma \in |\mathbf{Sign}^{\flat}|$ and $\phi \in \bigcap_{T\in\mathcal{T}} \overleftarrow{T}_{\Sigma}$. By assertionality, there exists $t_{\Sigma} \in \mathrm{Thm}_{\Sigma}(\mathcal{I})$, whence, $\phi, t_{\Sigma} \in \overleftarrow{T}_{\Sigma}$, for all $T \in \mathcal{T}$. Thus, by left regularity, $\langle \phi, t_{\Sigma}\rangle \in \Omega_{\Sigma}(T)$, for all $T \in \mathcal{T}$, i.e., $\langle \phi, t_{\Sigma}\rangle \in \bigcap_{T\in\mathcal{T}} \Omega_{\Sigma}(T)$. By hypothesis, $\langle \phi, t_{\Sigma}\rangle \in \Omega_{\Sigma}(T') \subseteq \Omega_{\Sigma}(\overleftarrow{T'})$. Therefore, since $t_{\Sigma} \in \overleftarrow{T'}_{\Sigma}$, we get, by compatibility of $\Omega(\overleftarrow{T'})$ with $\overleftarrow{T'}$, $\phi \in \overleftarrow{T'}_{\Sigma}$. We conclude that $\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \le \overleftarrow{T'}$ and, hence, that $\mathcal{I}$ is left c-reflective.

(c) Similar to Part (a).

∎

Alternatively, Theorem 597 may be proven by employing the characterizations provided in Proposition 588.

Based on the complete reflectivity hierarchy, which was established in Section 3.8, on the assertionality hierarchy established in Proposition 592 and on Theorem 597, we get the hierarchy relating assertionality with complete reflectivity classes shown in the diagram.



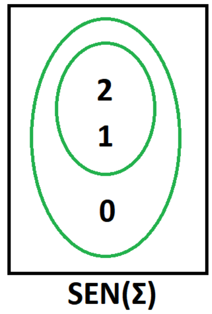
To show that all southwest inclusion arrows, connecting the various assertionality classes with the corresponding c-reflectivity classes, represent proper inclusions we construct an example of a family completely reflective π-institution which fails to be system assertional. Note that, since family c-reflectivity implies family injectivity, any π-institution fulfilling these requirements must have theorems. Therefore, the failure of assertionality must be due to failure of family regularity rather than the absence of theorems.

Example 598 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

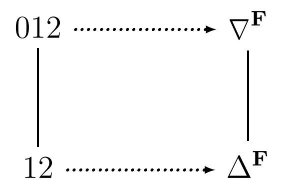
- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, specified by* $\sigma^\flat_\Sigma(0) = 0$*,* $\sigma^\flat_\Sigma(1) = 1$ *and* $\sigma^\flat_\Sigma(2) = 0$*.*

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{1,2\},\{0,1,2\}\}.$$



$\mathcal{I}$ is systemic and its lattice of theory families and corresponding Leibniz congruence systems are shown in the diagram.



Since the lattice of theory families of $\mathcal{I}$ is order isomorphic with the lattice of $\mathrm{AlgSys}^*(\mathcal{I})$*-congruence systems, $\mathcal{I}$ is family completely reflective.*

On the other hand, for $T = \{\{2,3\}\}$, we have $2,3 \in T_\Sigma$, but $\langle 2,3\rangle \notin \Delta^{\mathbf{F}}_\Sigma = \Omega_\Sigma(T)$, whence $\mathcal{I}$ is not system regular and, hence, a fortiori, belongs to none of the three classes in the assertionality hierarchy.

We show, next, that all assertionality properties transfer from theory families/systems to $\mathcal{I}$-filter families/systems over arbitrary $\mathbf{F}$-algebraic systems. This is a consequence of the facts that, by Theorem 585, all regularity properties transfer and, also, that the property of having theorems carries from the collection of all theory families to the collections of all filter systems over arbitrary algebraic systems, as seen in Lemma 376.

Theorem 599 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is family (respectively, left, system) assertional if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, $\langle \mathbf{A}, C^{\mathcal{I},\mathcal{A}}\rangle$ is family (respectively, left, system) assertional.*

Proof: Directly from Lemma 376 and Theorem 585. ■

We also have the following characterizations in terms of reduced $\mathcal{I}$-matrix families and $\mathcal{I}$-matrix systems.

Theorem 600 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is family assertional if and only if, for every $\langle \mathcal{A}, T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, and all $\Sigma \in |\mathbf{Sign}|$, $|T_\Sigma| = 1$;

(b) *$\mathcal{I}$ is system assertional if and only if, for every $\langle\mathcal{A},T\rangle \in \mathrm{MatSys}^*(\mathcal{I})$, with $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle$, $\mathbf{A} = \langle\mathbf{Sign},\mathrm{SEN},N\rangle$, and all $\Sigma \in |\mathbf{Sign}|$, $|T_\Sigma| = 1$.*

Proof:

(a) Suppose, first, that $\mathcal{I}$ is family assertional. Then, by definition, it is family regular. Thus, by Theorem 586, for all $\langle\mathcal{A},T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$, $|T_\Sigma| \leq 1$. However, by family assertionality, $\mathcal{I}$ has theorems, whence, by Lemma 376, $|T_\Sigma| = 1$.

Suppose, conversely, that the given condition holds. Then $\mathcal{I}$ has theorems and, by Lemma 586, it is family regular. Therefore, $\mathcal{I}$ is family assertional.

(b) Similar to Part (a).

■

8.4 Regular Weak Prealgebraizability

We look, next, at those classes of π-institutions that are formed by adding prealgebraicity to the various levels of assertionality.

Definition 601 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$.*

- *$\mathcal{I}$ is* **regularly weakly family prealgebraizable***, or* **RWF prealgebraizable** *for short, if it is prealgebraic and family assertional;*
- *$\mathcal{I}$ is* **regularly weakly left prealgebraizable***, or* **RWL prealgebraizable** *for short, if it is prealgebraic and left assertional;*
- *$\mathcal{I}$ is* **regularly weakly system prealgebraizable***, or* **RWS prealgebraizable** *for short, if it is prealgebraic and system assertional.*

Based on the assertionality hierarchy established in Proposition 592, we have the following

Proposition 602 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$.*

(a) *If $\mathcal{I}$ is regularly weakly family prealgebraizable, then it is regularly weakly left prealgebraizable;*

(b) *If $\mathcal{I}$ is regularly weakly left prealgebraizable, then it is regularly weakly system prealgebraizable.*

Proof: Straightforward by combining Definition 601 and Proposition 592. ■

Proposition 602 establishes the **regular weak prealgebraizability hierarchy** depicted in the following diagram.

Regular Weak Family Prealgebraizable

↓

Regular Weak Left Prealgebraizable

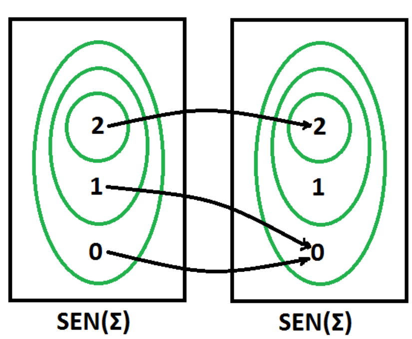
↓

Regular Weak System Prealgebraizable

We reuse two examples to show that all classes in this hierarchy are different, i.e., that the arrows in the diagram represent proper inclusions. The first describes a π-institution that is regularly weakly left prealgebraizable but fails to be regularly weakly family prealgebraizable, thus showing that the family class is properly included in the left class.

Example 603 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$*,* $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$*;*
- $N^\flat$ *is the trivial category of natural transformations.*



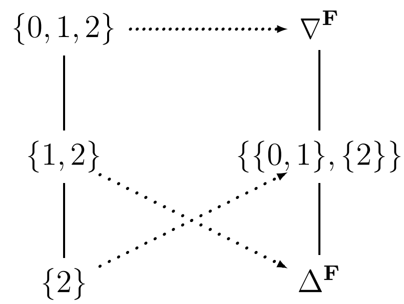
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}$.

The following table gives the theory families and the theory systems of the π-institution $\mathcal{I}$:

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

Since $\mathcal{I}$ is not systemic, by Proposition 591, it fails to be family assertional and, hence, it is not regularly weakly family prealgebraizable.

The lattice of theory families and the corresponding Leibniz congruence systems are depicted below:



Since the only theory systems of $\mathcal{I}$ are $\{\{2\}\}$ and $\{\{0,1,2\}\}$, it is clear that Ω is monotone on theory systems and, hence, $\mathcal{I}$ is prealgebraic. Clearly, $\mathcal{I}$ has theorems. Thus, to complete the proof that it is regularly weakly left prealgebraizable, it suffices to show that it is left assertional, i.e., by Proposition 588, that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T}_\Sigma = 2/\Omega_\Sigma(T)$.

- $\overleftarrow{\{\{2\}\}}_\Sigma = \{2\} = 2/\Omega_\Sigma(\{\{2\}\})$;
- $\overleftarrow{\{\{1,2\}\}}_\Sigma = \{2\} = 2/\Omega_\Sigma(\{\{1,2\}\})$;
- $\overleftarrow{\{\{0,1,2\}\}}_\Sigma = \{0,1,2\} = 2/\Omega_\Sigma(\{\{0,1,2\}\})$.

The second example presents a regularly weakly system prealgebraizable π-institution that is not regularly weakly left prealgebraizable.

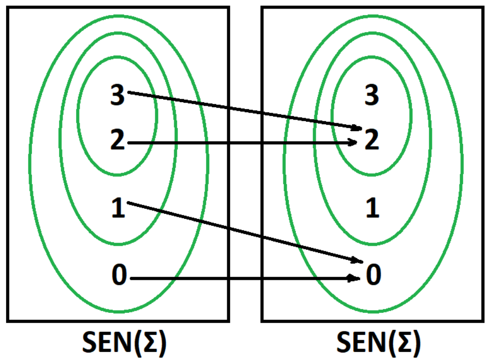
Example 604 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object Σ and a unique (non-identity) morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3\}$ *and*

  $\mathrm{SEN}^\flat(f)(0) = 0,\ \mathrm{SEN}^\flat(f)(1) = 0,\ \mathrm{SEN}^\flat(f)(2) = 2,\ \mathrm{SEN}^\flat(f)(3) = 2;$

- *$N^\flat$ is the category of natural transformations generated by the unary natural transformation $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ determined by*

x	0	1	2	3
$\sigma^\flat_\Sigma(x)$	0	1	0	1



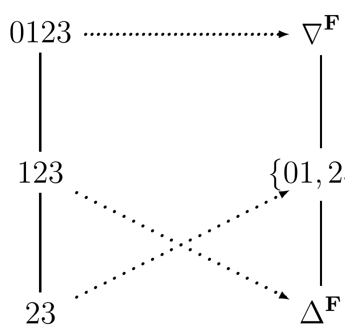
Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\{2,3\},\{1,2,3\},\{0,1,2,3\}\}.$$

The following table shows the action of $\overleftarrow{}$ on theory families.

T	$\{2,3\}$	$\{1,2,3\}$	$\{0,1,2,3\}$
$\overleftarrow{T}$	$\{2,3\}$	$\{2,3\}$	$\{0,1,2,3\}$

The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right:



Since the only theory systems of $\mathcal{I}$ are $\{\{2,3\}\}$ and $\{\{0,1,2,3\}\}$, it is obvious that Ω is monotone on theory systems and, hence, that $\mathcal{I}$ is prealgebraic. Clearly, $\mathcal{I}$ has theorems. To see that $\mathcal{I}$ is regularly weakly system prealgebraizable it suffices to show that it is system assertional, i.e., by Proposition 588, that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $T_\Sigma = 2/\Omega_\Sigma(T)$. We do have indeed:

- $\{2,3\} = 2/\Omega_\Sigma(\{\{2,3\}\})$;
- $\{0,1,2,3\} = 2/\Omega_\Sigma(\{\{0,1,2,3\}\})$.

On the other hand, for $T = \{\{1,2,3\}\}$*, we have* $2,3 \in \{2,3\} = \overleftarrow{T}_\Sigma$*, whereas* $\langle 2,3\rangle \notin \Delta^{\mathbf{F}}_\Sigma = \Omega_\Sigma(T)$*. We conclude that* $\mathcal{I}$ *is not left regular and, hence, a fortiori, it is not regularly weakly left prealgebraizable.*

We investigate, next, the relationships that hold between the various regular weak prealgebraizability classes, introduced in the present section, and the corresponding assertional classes, that were introduced in Section 8.3.

Directly from the definitions involved, we get the following

Proposition 605 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

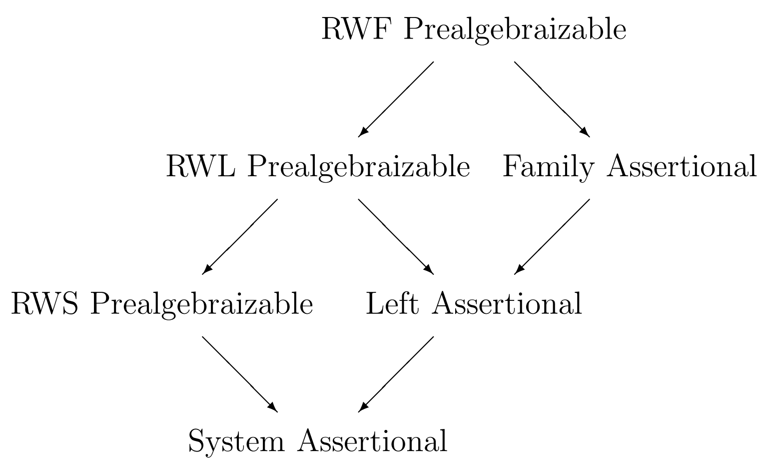
(a) If $\mathcal{I}$ *is regularly weakly family prealgebraizable, then it is family assertional;*

(b) If $\mathcal{I}$ *is regularly weakly left prealgebraizable, then it is left assertional;*

(c) If $\mathcal{I}$ *is regularly weakly system prealgebraizable, then it is system assertional.*

Proof: Directly from Definition 601. ■

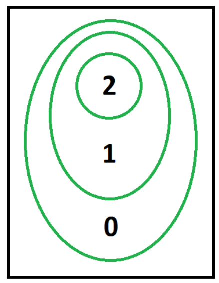
Therefore, we get the mixed regular weak prealgebraizability and assertionality hierarchy depicted in the diagram.



To show that all classes in this hierarchy are different, we provide an example of a π-institution that is family assertional, and, thus, belongs to all three assertionality classes, but fails to be regularly weakly system prealgebraizable, whence it belongs to none of three steps in the regular weak prealgebraizability hierarchy. This example shows that all three southeast arrows represent proper inclusions.

Example 606 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the trivial category of natural transformations.*

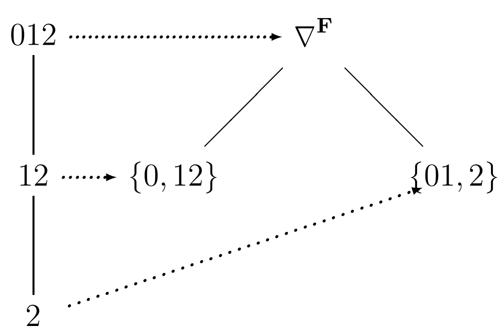


SEN(Σ)

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

$\mathcal{I}$ *is systemic and its lattice of theory families and corresponding Leibniz congruence systems are shown in the diagram.*



$\mathcal{I}$ *has theorems, whence to show that it is family assertional, it suffices to show that, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,* $T_\Sigma = 2/\Omega_\Sigma(T)$*. Indeed, we have:*

- *For* $T = \{\{2\}\}$*,* $\{2\} = 2/\Omega_\Sigma(\{\{2\}\})$*;*
- *For* $T = \{\{1, 2\}\}$*,* $\{1, 2\} = 2/\Omega_\Sigma(\{\{1, 2\}\})$*;*
- *For* $T = \{\{0, 1, 2\}\}$*,* $\{0, 1, 2\} = 2/\Omega_\Sigma(\{\{0, 1, 2\}\})$*.*

On the other hand, since $\{\{2\}\} \leq \{\{1, 2\}\}$*, but* $\Omega(\{\{2\}\}) \nleq \Omega(\{\{1, 2\}\})$*,* $\mathcal{I}$ *is not prealgebraic and, hence, fails to be regularly weakly system prealgebraizable.*

Turning now to the relationship between the regular weak prealgebraizability hierarchy and the weak prealgebraizability hierarchy, we get the following

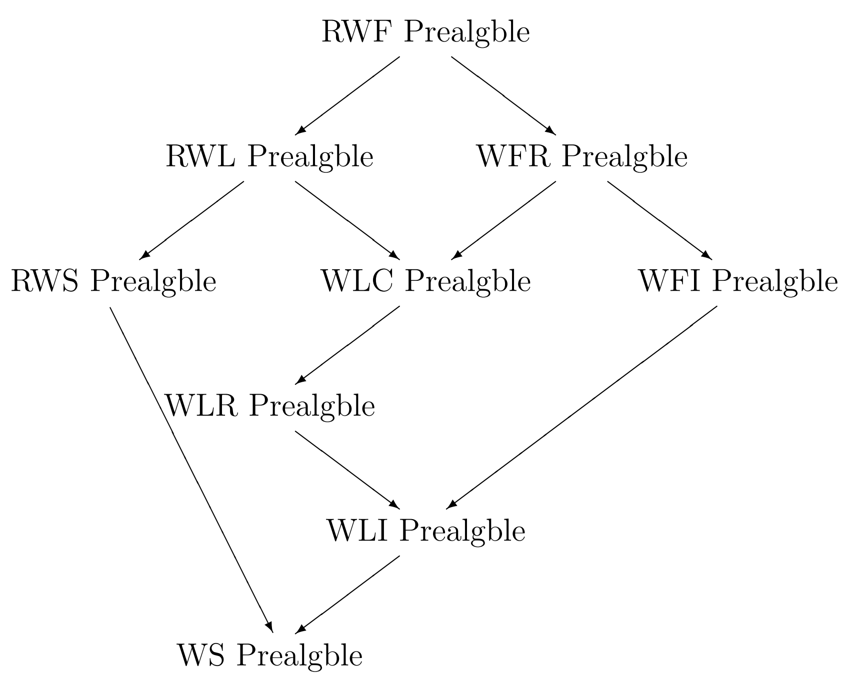
Proposition 607 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- *If* $\mathcal{I}$ *is regularly weakly family prealgebraizable, then it is weakly family (completely) reflective prealgebraizable;*
- *If* $\mathcal{I}$ *is regularly weakly left prealgebraizable, then it is weakly left completely reflective prealgebraizable;*
- *If* $\mathcal{I}$ *is regularly weakly system prealgebraizable, then it is weakly system prealgebraizable.*

Proof: We show Part (a) in detail. The remaining parts can be proved similarly.

Suppose $\mathcal{I}$ is regularly weakly family prealgebraizable. Then, by definition, it is prealgebraic and family assertional. Hence, by Theorem 597, it is prealgebraic and family completely reflective. Thus, by definition, it is weakly family prealgebraizable. ■

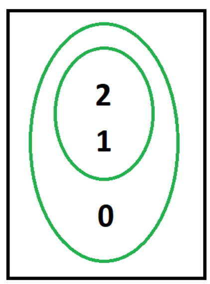
Thus, Proposition 607, together with Proposition 602 and the hierarchy established in Section 4.2, point to the following hierarchy of regularly weakly prealgebraizable and weakly prealgebraizable π-institutions.



Again it is not difficult to see that the classes in the regular weak prealgebraizability hierarchy are different from the classes of weakly prealgebraizable π-institutions. This is accomplished by constructing an example of a π-institution which is weakly family completely reflective prealgebraizable but is not regularly weakly system prealgebraizable.

Example 608 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *specified by* $\sigma^\flat_\Sigma(0) = 0$*,* $\sigma^\flat_\Sigma(1) = 1$ *and* $\sigma^\flat_\Sigma(2) = 0$*.*

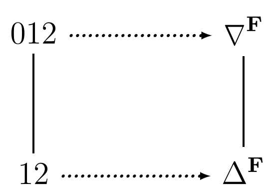


SEN(Σ)

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{1, 2\}, \{0, 1, 2\}\}.$$

$\mathcal{I}$ *is systemic and its lattice of theory families and corresponding Leibniz congruence systems are shown in the diagram.*



Since the lattice of theory families of $\mathcal{I}$ *is order isomorphic with the lattice of* $\mathrm{AlgSys}^*(\mathcal{I})$*-congruence systems,* $\mathcal{I}$ *is weakly family c-reflective prealgebraizable.*

On the other hand, for $T = \{\{2, 3\}\}$*, we have* $2, 3 \in T_\Sigma$*, but* $\langle 2, 3 \rangle \notin \Delta^{\mathbf{F}}_\Sigma = \Omega_\Sigma(T)$*, whence* $\mathcal{I}$ *is not system regular and, hence, a fortiori, it is not regularly weakly system prealgebraizable either.*

Based on existing results, we can show that all three kinds of regular weak prealgebraizability transfer from theory families/systems to filter families/systems over arbitrary **F**-algebraic systems.

Theorem 609 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) *$\mathcal{I}$ is regularly weakly family prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, all* $T', T'' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$*,*

- $T' \leq T''$ *implies* $\Omega^{\mathcal{A}}(T') \leq \Omega^{\mathcal{A}}(T'')$*;*
- $|T_\Sigma / \Omega^{\mathcal{A}}_\Sigma(T)| = 1$*;*

(b) *$\mathcal{I}$ is regularly weakly left prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, all* $T', T'' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$*,*

- $T' \leq T''$ *implies* $\Omega^{\mathcal{A}}(T') \leq \Omega^{\mathcal{A}}(T'')$*;*
- $|\overleftarrow{T}_\Sigma / \Omega^{\mathcal{A}}_\Sigma(T)| = 1$*;*

(c) *$\mathcal{I}$ is regularly weakly system prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, all* $T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$*,*

- $T \leq T'$ *implies* $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$*;*
- $|T_\Sigma / \Omega^{\mathcal{A}}_\Sigma(T)| = 1$*.*

Proof: Combine Theorem 179 with Theorem 599. ■

Finally, we may also adapt previously obtained results characterizing weak prealgebraizability to obtain similar characterizations of regular weak prealgebraizability in terms of mappings between posets of filter families/systems (including theory families/systems) and congruence systems.

Theorem 610 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. $\mathcal{I}$ is regularly weakly family prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism, such that, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$*,* $|T_\Sigma / \Omega^{\mathcal{A}}_\Sigma(T)| = 1$*.*

Proof: Suppose, first, that $\mathcal{I}$ is regularly weakly family prealgebraizable. Then it is, by definition, prealgebraic and, moreover, by definition, Proposition 605 and Theorem 597, it is family c-reflective. Therefore, it is WFR prealgebraizable. Thus, the required isomorphism is given by Theorem 268. The expression for T is obtained by applying Theorem 609.

Assume, conversely, that the postulated condition holds. Then, the hypotheses of Theorem 609, Part (a), are satisfied and, therefore, $\mathcal{I}$ is regularly weakly family prealgebraizable. ∎

Theorem 611 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is regularly weakly left prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order embedding, such that, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$, $|\overleftarrow{T}_\Sigma / \Omega^{\mathcal{A}}_\Sigma(T)| = 1$.

Proof: Suppose, first, that $\mathcal{I}$ is regularly weakly left prealgebraizable. Then it is, by definition, prealgebraic and, moreover, by definition, Proposition 605 and Theorem 597, it is left c-reflective. Therefore, it is WLC prealgebraizable. Thus, the required embedding is given by Theorem 276. The expression for T is obtained by applying Theorem 609.

Assume, conversely, that the postulated condition holds. Then, the hypotheses of Theorem 609, Part (b), are satisfied and, therefore, $\mathcal{I}$ is regularly weakly left prealgebraizable. ∎

Theorem 612 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is regularly weakly system prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order embedding, such that, for all $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$, $|T_\Sigma / \Omega^{\mathcal{A}}_\Sigma(T)| = 1$.

Proof: Suppose, first, that $\mathcal{I}$ is regularly weakly system prealgebraizable. Then it is, by definition, prealgebraic and, moreover, by definition, Proposition 605 and Theorem 597, it is system c-reflective. Therefore, it is WS prealgebraizable. Thus, the required embedding is given by Theorem 256. The expression for T is obtained by applying Theorem 609.

Assume, conversely, that the postulated condition holds. Then, the hypotheses of Theorem 609, Part (c), are satisfied and, therefore, $\mathcal{I}$ is regularly weakly system prealgebraizable. ∎

8.5 Regular Weak Algebraizability

We look, next, at those classes of π-institutions that are formed by adding protoalgebraicity to the various levels of assertionality.

Definition 613 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is* **regularly weakly family algebraizable**, *or* **RWF algebraizable** *for short, if it is protoalgebraic and family assertional;*
- $\mathcal{I}$ *is* **regularly weakly left algebraizable**, *or* **RWL algebraizable** *for short, if it is protoalgebraic and left assertional;*
- $\mathcal{I}$ *is* **regularly weakly system algebraizable**, *or* **RWS algebraizable** *for short, if it is protoalgebraic and system assertional.*

Even though there seem to be three classes in the regular weak algebraizability hierarchy, in reality there are only two, since it is easy to see that the classes of regularly weakly left and of regularly weakly system π-institutions coincide.

Proposition 614 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is regularly weakly left algebraizable if and only if it is regularly weakly system algebraizable.*

Proof: The "only if" follows directly by the definition and Proposition 592. For the "if", suppose that $\mathcal{I}$ is regularly weakly system algebraizable. Then it is, a fortiori, protoalgebraic, whence, by Lemma 170, it is stable. Therefore, since $\mathcal{I}$ is system regular and stable, by Proposition 579, it is left regular. We conclude that $\mathcal{I}$ is regularly weakly left algebraizable. ∎

The assertionality hierarchy, established in Proposition 592, and Proposition 614 allow us to establish the following regular weak algebraizability hierarchy.

Proposition 615 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is regularly weakly family algebraizable, then it is regularly weakly system algebraizable.*

Proof: Straightforward by combining Definition 601 and Proposition 592. ∎

The **regular weak algebraizability hierarchy** is depicted in the following diagram.

Regular Weak Family Algebraizable

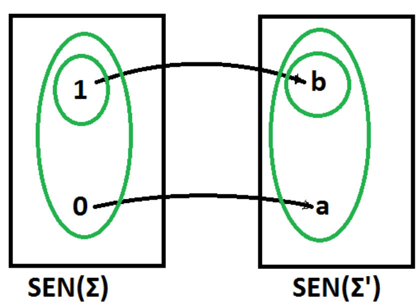
↓

Regular Weak System Algebraizable

We use an example to show that the two classes in this hierarchy are different. Namely, we construct a π-institution that is regularly weakly system algebraizable but fails to be regularly weakly family algebraizable.

Example 616 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$, $\mathrm{SEN}^\flat(f)(1) = b$*;*
- $N^\flat$ *is the trivial category of natural transformations.*



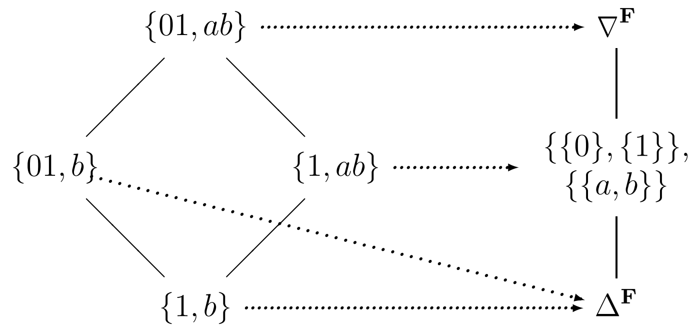
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{1\}, \{0, 1\}\} \quad \text{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\}, \{a, b\}\}.$$

The following table shows the action of $\overleftarrow{\ }$ *on theory families, where rows correspond to* T_Σ *and columns to* $T_{\Sigma'}$ *and each entry is written as* $\overleftarrow{T}_\Sigma, \overleftarrow{T}_{\Sigma'}$.

$\leftarrow$	$\{b\}$	$\{a, b\}$
$\{1\}$	$\{1\}, \{b\}$	$\{1\}, \{a, b\}$
$\{0, 1\}$	$\{1\}, \{b\}$	$\{0, 1\}, \{a, b\}$

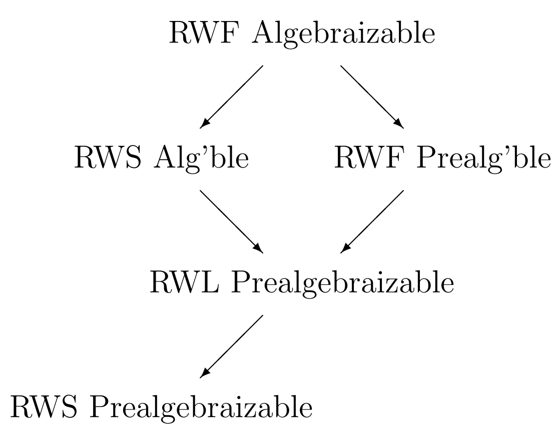
The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in *terms of blocks) on the right:*



The Leibniz operator is monotone on theory families, whence, $\mathcal{I}$ is protoalgebraic. Moreover, $\mathrm{Thm}(\mathcal{I}) = \{\{1\}, \{b\}\}$ *and, for every theory system T, $T_\Sigma = 1/\Omega_\Sigma(T)$ and $T_{\Sigma'} = b/\Omega_{\Sigma'}(T)$. Therefore, $\mathcal{I}$ is system assertional. Thus, $\mathcal{I}$ is regularly weakly system algebraizable.*

On the other hand, for $T = \{\{0, 1\}, \{b\}\} \in \mathrm{ThFam}(\mathcal{I})$, we have $0, 1 \in T_\Sigma$, but $\langle 0, 1\rangle \notin \Omega_\Sigma(T)$. Therefore, $\mathcal{I}$ fails to be family regular and, hence, a fortiori, it is not regularly weakly family algebraizable.

We investigate, next, the relationships that hold between the two regular weak algebraizability classes, introduced in the present section, and the three regular weak prealgebraizability classes, that were introduced in Section 8.4. Since, by Theorem 175, protoalgebraicity implies prealgebraicity, we get, a priori, the following mixed hierarchy.



However, we can show that the two top classes of the hierarchies coincide.

Theorem 617 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is regularly weakly family prealgebraizable if and only if it is regularly weakly family algebraizable.*

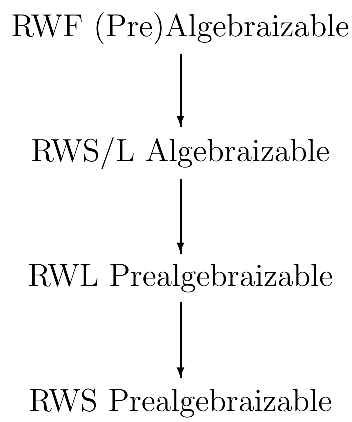
Proof: The "if" follows from the relevant definitions and the fact that, by Theorem 175, protoalgebraicity implies prealgebraicity. For the "only if", it suffices to show that, under family assertionality, prealgebraicity implies protoalgebraicity. By Theorem 175, it suffices, in turn, to show that family assertionality implies stability and, by Proposition 152, that family assertionality implies systemicity. Indeed, by Theorem 597, family assertionality implies family c-reflectivity and, by Proposition 237, we get that $\mathcal{I}$ is systemic. ■

Moreover, from the definitions involved, we get the following

Proposition 618 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is regularly weakly system algebraizable, then it is regularly weakly left prealgebraizable.*

Proof: Suppose $\mathcal{I}$ is regularly weakly system algebraizable. Equivalently, by Proposition 614, it is regularly weakly left algebraizable. Then, by definition, it is protoalgebraic and left assertional. Thus, by Theorem 175, it is prealgebraic and left assertional, i.e., by definition, it is regularly weakly left prealgebraizable. ■

Based on Theorem 617 and Proposition 618, we get the following updated version of the mixed hierarchy shown in the preceding diagram.



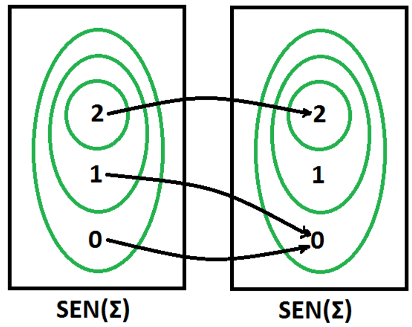
To show that all classes in this hierarchy are different, we provide an example of a π-institution that is regularly weakly left prealgebraizable, but fails to be regularly weakly system algebraizable, i.e., an example that separates the regular weak algebraizability from the regular weak prealgebraizability classes.

Example 619 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$;

- $N^\flat$ *is the trivial category of natural transformations.*

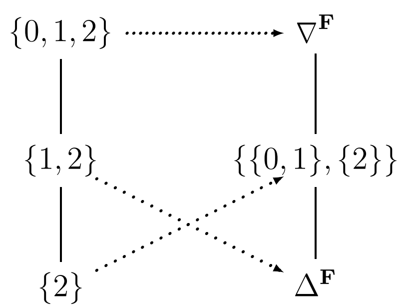


Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}$.

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$.

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

The lattice of theory families and the corresponding Leibniz congruence systems are depicted below.



Since the only theory systems of $\mathcal{I}$ *are* $\{\{2\}\}$ *and* $\{\{0,1,2\}\}$, *it is clear that* Ω *is monotone on theory systems and, hence,* $\mathcal{I}$ *is prealgebraic. Clearly,* $\mathcal{I}$ *has theorems. Thus, to complete the proof that it is regularly weakly left prealgebraizable, it suffices to show that it is left assertional, i.e., by Proposition 588, that, for all* $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T}_\Sigma = 2/\Omega_\Sigma(T)$. *Indeed, we get:*

- $\overleftarrow{\{\{2\}\}}_\Sigma = \{2\} = 2/\Omega_\Sigma(\{\{2\}\})$;

- $\overleftarrow{\{\{1,2\}\}}_\Sigma = \{2\} = 2/\Omega_\Sigma(\{\{1,2\}\})$;

- $\overleftarrow{\{\{0,1,2\}\}}_\Sigma = \{0,1,2\} = 2/\Omega_\Sigma(\{\{0,1,2\}\})$.

On the other hand, since $\{\{2\}\} \leq \{\{1,2\}\}$*, but*

$$\Omega(\{\{2\}\}) = \{\{\{0,1\},\{2\}\}\} \nleq \Delta^{\mathbf{F}} = \Omega(\{\{1,2\}\}),$$

$\mathcal{I}$ *is not protoalgebraic and, hence, it fails to be regularly weakly system algebraizable.*

Turning now to the relationship between regular weak algebraizability and weak algebraizability, we get, by definition

Proposition 620 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

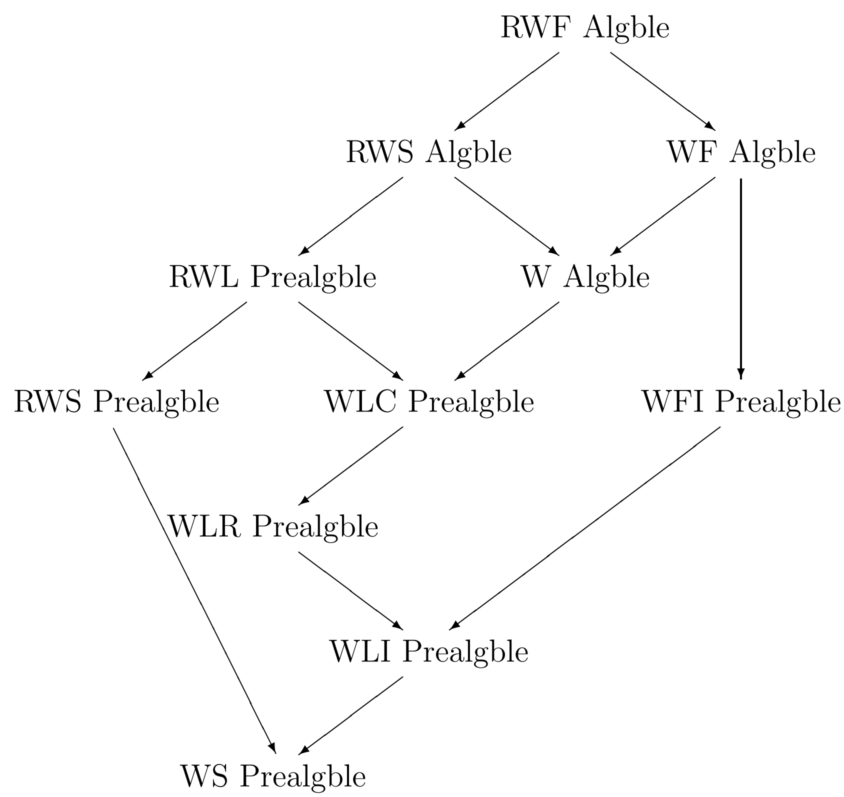
(a) If $\mathcal{I}$ *is regularly weakly family algebraizable, then it is weakly family algebraizable;*

(b) If $\mathcal{I}$ *is regularly weakly system algebraizable, then it is weakly (system/left) algebraizable.*

Proof: For Part (a) note that, by Theorem 617, regular weak family algebraizability coincides with regular weak family prealgebraizability. In turn, by Proposition 607, regular weak family prealgebraizability entails weak family prealgebraizability. But, by Corollary 297, the latter property is identical with weak family algebraizability.

For Part (b), if $\mathcal{I}$ is regularly weakly system algebraizable, then it is, by definition, protoalgebraic and system assertional, whence, by Theorem 597, it is protoalgebraic and system completely reflective. Therefore, it is, by definition, weakly (system or, equivalently, left) algebraizable. ■

Thus, Proposition 620, together with Propositions 607 and 618, point to the following hierarchy of regularly weakly (pre)algebraizable π-institutions and weakly (pre)algebraizable π-institutions.



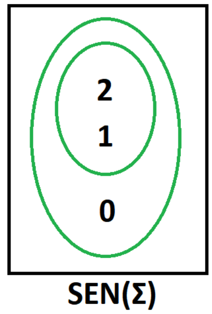
Again it is not difficult to see that the classes in the regular weak algebraizability hierarchy are different from the classes in the weak algebraizability hierarchy. This is accomplished by constructing an example of a π-institution which is weakly family algebraizable but is not regularly weakly system prealgebraizable.

Example 621 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

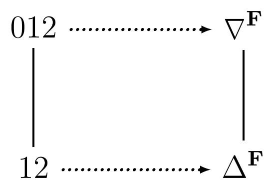
- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *specified by* $\sigma^\flat_\Sigma(0) = 0$*,* $\sigma^\flat_\Sigma(1) = 1$ *and* $\sigma^\flat_\Sigma(2) = 0$*.*

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{1, 2\}, \{0, 1, 2\}\}.$$



$\mathcal{I}$ is systemic and its lattice of theory families and corresponding Leibniz congruence systems are shown in the diagram.



Since the lattice of theory families of $\mathcal{I}$ is order isomorphic with the lattice of $\mathrm{AlgSys}^*(\mathcal{I})$*-congruence systems, $\mathcal{I}$ is weakly family algebraizable.*

On the other hand, for $T = \{\{2,3\}\}$, we have $2,3 \in T_\Sigma$, but $\langle 2,3\rangle \notin \Delta^{\mathbf{F}}_\Sigma = \Omega_\Sigma(T)$, whence $\mathcal{I}$ is not system regular. Hence, a fortiori, $\mathcal{I}$ is not regularly weakly system prealgebraizable.

As was the case with regular weak prealgebraizability, we can show that both kinds of regular weak algebraizability transfer from theory families/systems to filter families/systems over arbitrary **F**-algebraic systems.

Theorem 622 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is regularly weakly family algebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and all $\Sigma \in |\mathbf{Sign}|$,

- *$T \leq T'$ implies $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$;*
- *$|T_\Sigma / \Omega^{\mathcal{A}}_\Sigma(T)| = 1$;*

(b) $\mathcal{I}$ is regularly weakly system algebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, all $T'' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ and all $\Sigma \in |\mathbf{Sign}|$,

- *$T \leq T'$ implies $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$;*
- *$|T''_\Sigma / \Omega^{\mathcal{A}}_\Sigma(T)| = 1$.*

Proof: Combine Theorem 179 with Theorems 599 and 600. ∎

Finally, we may also adapt previously obtained results characterizing weak algebraizability to obtain similar characterizations of regular weak algebraizability in terms of mappings between posets of filter families/ systems and congruence systems.

Corollary 623 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is regularly weakly family algebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism, such that, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$*,* $|T_\Sigma / \Omega^{\mathcal{A}}_\Sigma(T)| = 1$*.*

Proof: By Theorems 617 and 610. ∎

Theorem 624 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is regularly weakly system algebraizable if and only if it is stable and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism, such that, for all $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$*,* $|T_\Sigma / \Omega^{\mathcal{A}}_\Sigma(T)| = 1$*.*

Proof: Suppose, first, that $\mathcal{I}$ is regularly weakly system algebraizable. Then it is, by definition, protoalgebraic and, thus, by Theorem 175, stable. Moreover, by Propositions 618 and 605 and Theorem 597, it is system c-reflective. Therefore, it is weakly algebraizable. Thus, the required isomorphism is given by Theorem 268. The expression for T is obtained by applying Theorem 609.

Assume, conversely, that the postulated condition holds. Consider the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. Since Ω on the collection of theory systems is an order isomorphism, it is monotone and, hence, $\mathcal{I}$ is prealgebraic. Thus, by stability and Theorem 175, $\mathcal{I}$ is protoalgebraic. Moreover, by hypothesis and Theorem 609, $\mathcal{I}$ is system assertional. Thus, by definition, $\mathcal{I}$ is regularly weakly system algebraizable. ∎

8.6 Regular Prealgebraizability

We look, next, at those classes of π-institutions that are formed by adding preequivalentiality to the various levels of assertionality.

Definition 625 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- *$\mathcal{I}$ is* **regularly family prealgebraizable***, or* **RF prealgebraizable** *for short, if it is preequivalential and family assertional;*
- *$\mathcal{I}$ is* **regularly left prealgebraizable***, or* **RL prealgebraizable** *for short, if it is preequivalential and left assertional;*
- *$\mathcal{I}$ is* **regularly system prealgebraizable***, or* **RS prealgebraizable** *for short, if it is preequivalential and system assertional.*

Based on the assertionality hierarchy established in Proposition 592, we have the following

Proposition 626 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) If $\mathcal{I}$ is regularly family prealgebraizable, then it is regularly left prealgebraizable;

(b) If $\mathcal{I}$ is regularly left prealgebraizable, then it is regularly system prealgebraizable.

Proof: Straightforward by combining Definition 625 and Proposition 592. ■

Proposition 626 establishes the **regular prealgebraizability hierarchy** depicted in the following diagram.

Regular Family Prealgebraizable

↓

Regular Left Prealgebraizable

↓

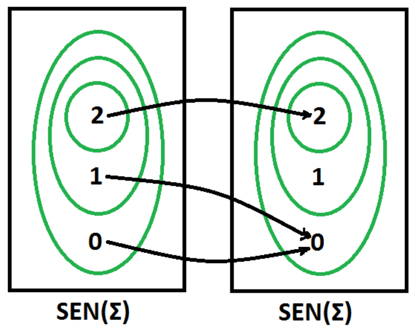
Regular System Prealgebraizable

We give again two examples to show that all classes in this hierarchy are different, i.e., that the arrows in the diagram represent proper inclusions. The first describes a π-institution that is regularly left prealgebraizable but fails to be regularly family prealgebraizable, thus showing that the top arrow stands for a proper inclusion.

Example 627 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- *$\mathbf{Sign}^\flat$ is the category with a single object Σ and a single non-identity morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$;
- $N^\flat$ *is the trivial category of natural transformations.*



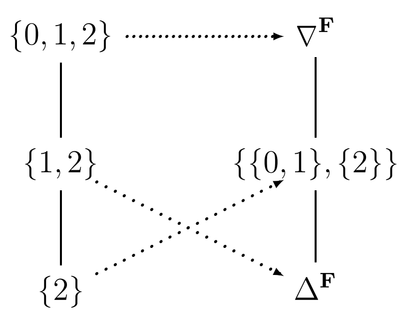
Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ by setting $\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}$.

The following table gives the theory families and the theory systems of the π-institution $\mathcal{I}$.

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1,2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$

Since $\mathcal{I}$ is not systemic, by Proposition 591, it fails to be family assertional and, hence, it is not regularly family prealgebraizable.

The lattice of theory families and the corresponding Leibniz congruence systems are depicted below:



Since the only theory systems of $\mathcal{I}$ are $\{\{2\}\}$ and $\{\{0,1,2\}\}$, it is clear that Ω is monotone on theory systems and, hence, $\mathcal{I}$ is prealgebraic. To see that it is preequivalential, we must also show that it is system extensional. To simplify the process, we note that the only non-trivial proper universes of $\mathbf{F}$ are $\mathbf{X} = \{\{0,1\}\}$ and $\mathbf{Y} = \{\{0,2\}\}$, and the only proper theory system is $\mathrm{Thm}(\mathcal{I}) = \{\{2\}\}$. *Hence, there are only two cases to check, as shown below (written, as done elsewhere, in shorthand):*

- $\Omega^{\mathbf{X}}(2 \cap \mathbf{X}) = \Omega^{\mathbf{X}}(\varnothing) = \{01\} = \{01,2\} \cap \mathbf{X}^2 = \Omega(2) \cap \mathbf{X}^2$;

- $\Omega^{\mathbf{Y}}(2 \cap \mathbf{Y}) = \Omega^{\mathbf{Y}}(2) = \{0,2\} = \{01,2\} \cap \mathbf{Y}^2 = \Omega(2) \cap \mathbf{Y}^2$.

Clearly, $\mathcal{I}$ has theorems. Thus, to complete the proof that it is regularly left prealgebraizable, it suffices to show that it is left assertional, i.e., by Proposition 588, that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T}_\Sigma = 2/\Omega_\Sigma(T)$.

- $\overleftarrow{\{\{2\}\}}_\Sigma = \{2\} = 2/\Omega_\Sigma(\{\{2\}\})$;
- $\overleftarrow{\{\{1,2\}\}}_\Sigma = \{2\} = 2/\Omega_\Sigma(\{\{1,2\}\})$;
- $\overleftarrow{\{\{0,1,2\}\}}_\Sigma = \{0,1,2\} = 2/\Omega_\Sigma(\{\{0,1,2\}\})$.

The second example gives a regularly system prealgebraizable π-institution that is not regularly left prealgebraizable.

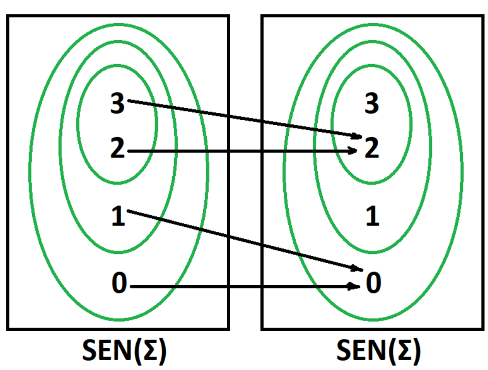
Example 628 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object Σ and a unique (non-identity) morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3\}$ *and*

  $$\mathrm{SEN}^\flat(f)(0) = 0,\ \mathrm{SEN}^\flat(f)(1) = 0,\ \mathrm{SEN}^\flat(f)(2) = 2,\ \mathrm{SEN}^\flat(f)(3) = 2;$$

- $N^\flat$ *is the category of natural transformations generated by the unary natural transformation $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ determined by*

x	0	1	2	3
$\sigma^\flat_\Sigma(x)$	0	1	0	1



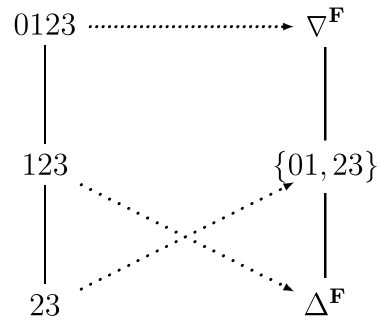
Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\{2,3\},\{1,2,3\},\{0,1,2,3\}\}.$$

The following table shows the action of $\overleftarrow{\ }$ *on theory families.*

T	$\{2,3\}$	$\{1,2,3\}$	$\{0,1,2,3\}$
$\overleftarrow{T}$	$\{2,3\}$	$\{2,3\}$	$\{0,1,2,3\}$

The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right.



Since the only theory systems of $\mathcal{I}$ *are* $\{\{2,3\}\}$ *and* $\{\{0,1,2,3\}\}$, *it is obvious that* Ω *is monotone on theory systems and, hence, that* $\mathcal{I}$ *is prealgebraic. To see that it is preequivalential, it suffices, thus, to show that it is also system extensional. To simplify the process, we note that the only non-trivial proper universes of* $\mathbf{F}$ *are* $\mathbf{X} = \{\{0,1\}\}$, $\mathbf{Y} = \{\{0,2\}\}$ *and* $\mathbf{Z} = \{\{0,1,2\}\}$, *and the only proper theory system is* $\mathrm{Thm}(\mathcal{I}) = \{\{2,3\}\}$. *Hence, there are three cases to check, as shown below (written, as done elsewhere, in shorthand):*

- $\Omega^{\mathbf{X}}(23 \cap \mathbf{X}) = \Omega^{\mathbf{X}}(\varnothing) = \{01\} = \{01,23\} \cap \mathbf{X}^2 = \Omega(23) \cap \mathbf{X}^2$;
- $\Omega^{\mathbf{Y}}(23 \cap \mathbf{Y}) = \Omega^{\mathbf{Y}}(2) = \{0,2\} = \{01,23\} \cap \mathbf{Y}^2 = \Omega(23) \cap \mathbf{Y}^2$;
- $\Omega^{\mathbf{Z}}(23 \cap \mathbf{Z}) = \Omega^{\mathbf{Z}}(2) = \{01,2\} = \{01,23\} \cap \mathbf{Z}^2 = \Omega(23) \cap \mathbf{Z}^2$.

Clearly, $\mathcal{I}$ *has theorems. To see that* $\mathcal{I}$ *is regularly system prealgebraizable it suffices to show that it is system assertional, i.e., by Proposition 588, that, for all* $T \in \mathrm{ThSys}(\mathcal{I})$, $T_\Sigma = 2/\Omega_\Sigma(T)$. *We do have indeed:*

- $\{2,3\} = 2/\Omega_\Sigma(\{\{2,3\}\})$;
- $\{0,1,2,3\} = 2/\Omega_\Sigma(\{\{0,1,2,3\}\})$.

On the other hand, for $T = \{\{1,2,3\}\}$, *we have* $2,3 \in \{2,3\} = \overleftarrow{T}_\Sigma$, *whereas* $\langle 2,3\rangle \notin \Delta^{\mathbf{F}}_\Sigma = \Omega_\Sigma(T)$. *We conclude that* $\mathcal{I}$ *is not left regular and, hence, a fortiori, it is not regularly left prealgebraizable.*

We investigate, next, the relationships that hold between the various regular prealgebraizability classes, introduced in the present section, and the corresponding regular weak prealgebraizability classes, that were introduced in Section 8.4.

Directly from the definitions involved, we get the following

Proposition 629 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

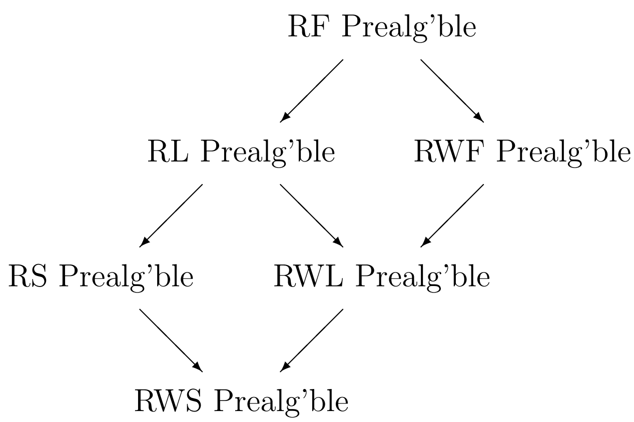
(a) If $\mathcal{I}$ *is regularly family prealgebraizable, then it is regularly weakly family prealgebraizable;*

(b) If $\mathcal{I}$ *is regularly left prealgebraizable, then it is regularly weakly left prealgebraizable;*

(c) If $\mathcal{I}$ *is regularly system prealgebraizable, then it is regularly weakly system prealgebraizable.*

Proof: If $\mathcal{I}$ is regularly family prealgebraizable, then, by definition, it is preequivalential and family assertional. Hence, by Proposition 338, it is prealgebraic and family assertional. Thus, it is, by definition, regularly weakly family prealgebraizable. Parts (b) and (c) can be proven similarly. ■

Therefore, we get the mixed regular prealgebraizability and regular weak prealgebraizability hierarchy depicted in the diagram.



To show that all classes in this hierarchy are different, we provide an example of a π-institution that is regularly weakly family prealgebraizable, and, thus, belongs to all three regular weak prealgebraizability classes, but fails to be regularly system prealgebraizable, whence it belongs to none of three steps in the regular prealgebraizability hierarchy. This example shows that all three southeast arrows represent proper inclusions.

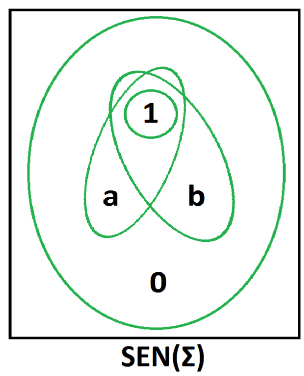
Example 630 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, a, b, 1\}$*;*

- *$N^\flat$ is the category of natural transformations generated by the two binary natural transformations $\wedge, \vee : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ defined by the following tables.*

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

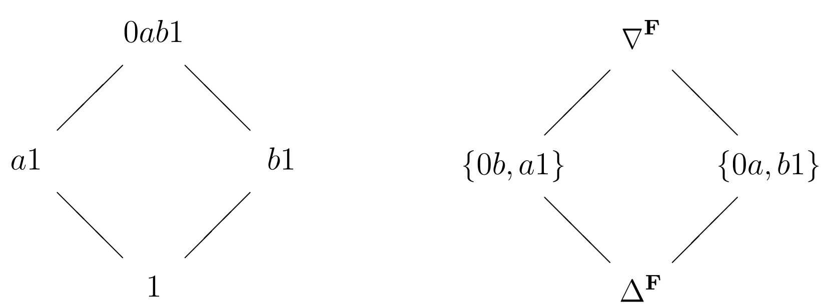


Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be the π-institution defined by setting

$$\mathcal{C}_\Sigma = \{\{1\}, \{a, 1\}, \{b, 1\}, \{0, a, b, 1\}\}.$$

$\mathcal{I}$ has four theory families, all of which are also theory systems.

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



From the diagram, we can see that $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is an order isomorphism, whence, $\mathcal{I}$ is weakly family prealgebraizable. Since it has theorems, to show that it is, also, family assertional, it suffices to show that it satisfies, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T_\Sigma = 1/\Omega_\Sigma(T)$. This is easily checked from the diagram above, giving the Leibniz congruence systems corresponding to the various theory families of $\mathcal{I}$.

On the other hand, for the universe $\mathbf{X} = \{\{0, a, 1\}\}$ *and the theory system* $T = \{\{1\}\}$*, we get*

$$\Omega(T) \cap \mathbf{X}^2 = \{\{0\}, \{a\}, \{1\}\} \subsetneq \{\{0\}, \{a, 1\}\} = \Omega^{\mathbf{X}}(T \cap \mathbf{X}).$$

Thus, $\mathcal{I}$ *is not system extensional and, therefore, it fails to be (system) preequivalential and, a fortiori, it also fails to be regularly system prealgebraizable.*

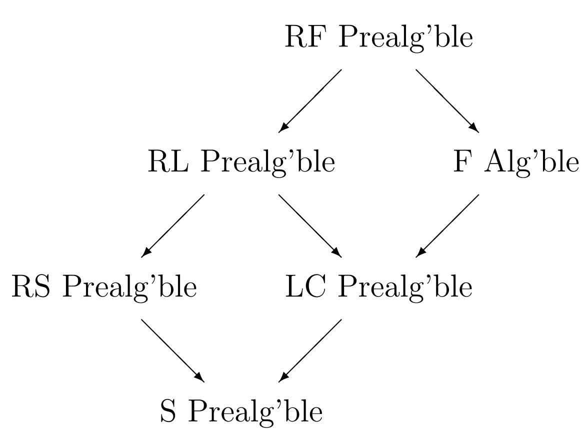
Turning now to the relationship between the regular prealgebraizability hierarchy and the prealgebraizability hierarchy, we get the following

Proposition 631 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- *If* $\mathcal{I}$ *is regularly family prealgebraizable, then it is family (pre)algebraizable;*
- *If* $\mathcal{I}$ *is regularly left prealgebraizable, then it is left completely reflective prealgebraizable;*
- *If* $\mathcal{I}$ *is regularly system prealgebraizable, then it is system prealgebraizable.*

Proof: We show Part (a) in detail. The remaining parts can be proved similarly. Suppose $\mathcal{I}$ is regularly family prealgebraizable. Then, by definition, it is preequivalential and family assertional. Hence, by Theorem 597, it is preequivalential and family completely reflective. Thus, by definition, it is weakly family (pre)algebraizable. ■

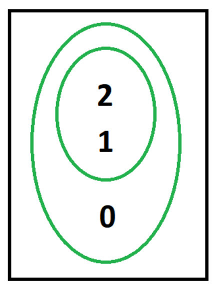
Proposition 631, together with Proposition 626 and the hierarchy established in Section 5.6, point to the following hierarchy of regularly prealgebraizable and (pre)algebraizable π-institutions. Note that the complete hierarchy is larger, but we only show those classes in the (pre)algebraizability hierarchy that are directly related to those in the regular prealgebraizability hierarchy via Theorem 631.



Again it is not difficult to see that the classes in the regular prealgebraizability hierarchy are different from the classes of prealgebraizable π-institutions. This is accomplished by constructing an example of a π-institution which is family completely reflective prealgebraizable (equivalently, family algebraizable), but is not regularly system prealgebraizable.

Example 632 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *specified by* $\sigma^\flat_\Sigma(0) = 0$*,* $\sigma^\flat_\Sigma(1) = 1$ *and* $\sigma^\flat_\Sigma(2) = 0$*.*

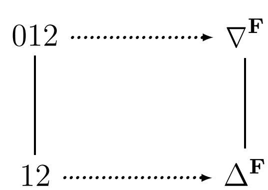


SEN(Σ)

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{1,2\},\{0,1,2\}\}.$$

$\mathcal{I}$ *is systemic and its lattice of theory families and corresponding Leibniz congruence systems are shown in the diagram.*



Since the lattice of theory families of $\mathcal{I}$ *is order isomorphic with the lattice of* $\mathrm{AlgSys}^*(\mathcal{I})$*-congruence systems,* $\mathcal{I}$ *is weakly family c-reflective prealgebraizable. To see that it is family c-reflective prealgebraizable, it suffices to show that it is also system extensional. The only nontrivial proper universes of* $\mathbf{F}$ *are* $\mathbf{X} = \{\{0,1\}\}$ *and* $\mathbf{Y} = \{\{0,2\}\}$ *and the only proper theory system is* $\{\{1,2\}\}$*. Thus, we only need to check two cases:*

- $\Omega^{\mathbf{X}}(12 \cap \mathbf{X}) = \Omega^{\mathbf{X}}(1) = \{0,1\} = \Delta^{\mathbf{F}} \cap \mathbf{X}^2 = \Omega(12) \cap \mathbf{X}^2$;
- $\Omega^{\mathbf{Y}}(12 \cap \mathbf{Y}) = \Omega^{\mathbf{Y}}(2) = \{0,2\} = \Delta^{\mathbf{F}} \cap \mathbf{Y}^2 = \Omega(12) \cap \mathbf{Y}^2$.

On the other hand, for $T = \{\{1,2\}\}$, *we have* $1,2 \in T_\Sigma$, *but* $\langle 1,2\rangle \notin \Delta^{\mathbf{F}}_\Sigma = \Omega_\Sigma(T)$, *whence* $\mathcal{I}$ *is not system regular and, hence, a fortiori, it is not regularly system prealgebraizable.*

Based on existing results, we can show that all three kinds of regular prealgebraizability transfer from theory families/systems to filter families/systems over arbritrary $\mathbf{F}$-algebraic systems.

Theorem 633 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is regularly family prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, *with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, *every universe* $\mathbf{X} \leq \mathbf{A}$, *all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, *all* $T', T'' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$,

  - $T' \leq T''$ *implies* $\Omega^{\mathcal{A}}(T') \leq \Omega^{\mathcal{A}}(T'')$;
  - $\Omega^{\mathbf{X}}(T' \cap \mathbf{X}) \leq \Omega^{\mathcal{A}}(T') \cap \mathbf{X}^2$;
  - $|T_\Sigma/\Omega^{\mathcal{A}}_\Sigma(T)| = 1$;

(b) $\mathcal{I}$ *is regularly left prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, *with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, *every universe* $\mathbf{X} \leq \mathbf{A}$, *all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, *all* $T', T'' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$,

  - $T' \leq T''$ *implies* $\Omega^{\mathcal{A}}(T') \leq \Omega^{\mathcal{A}}(T'')$;
  - $\Omega^{\mathbf{X}}(T' \cap \mathbf{X}) \leq \Omega^{\mathcal{A}}(T') \cap \mathbf{X}^2$;
  - $|\overleftarrow{T}_\Sigma/\Omega^{\mathcal{A}}_\Sigma(T)| = 1$;

(c) $\mathcal{I}$ *is regularly weakly system prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, *with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, *every universe* $\mathbf{X} \leq \mathbf{A}$, *all* $T, T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$,

  - $T \leq T'$ *implies* $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$;
  - $\Omega^{\mathbf{X}}(T \cap \mathbf{X}) \leq \Omega^{\mathcal{A}}(T) \cap \mathbf{X}^2$;
  - $|T_\Sigma/\Omega^{\mathcal{A}}_\Sigma(T)| = 1$.

Proof: Combine Theorems 179 and 314 with Theorem 599. ■

Finally, we adapt previously obtained results characterizing prealgebraizability to obtain similar characterizations of regular prealgebraizability in terms of mappings between posets of filter families/ systems (including theory families/systems) and congruence systems.

Theorem 634 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is regularly family prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism that commutes with inverse logical extensions, such that, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$*,* $|T_\Sigma / \Omega^{\mathcal{A}}_\Sigma(T)| = 1$*.*

Proof: Suppose, first, that $\mathcal{I}$ is regularly family prealgebraizable. Then it is, by definition, preequivalential and, moreover, by definition, Proposition 629, Proposition 605 and Theorem 597, it is family c-reflective. Therefore, it is F (pre)algebraizable. Thus, the required isomorphism is given by Theorem 366. The expression for T is obtained by applying Theorem 600.

Assume, conversely, that the postulated condition holds. Consider the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. Since Ω is an order isomorphism, which commutes with inverse logical extensions, $\mathcal{I}$ is preequivalential. Moreover, by hypothesis and Theorem 600, $\mathcal{I}$ is family assertional. Thus, by definition, $\mathcal{I}$ is regularly family prealgebraizable. ■

Theorem 635 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is regularly left prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order embedding that commutes with inverse logical extensions, such that, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$*,* $|\overleftarrow{T}_\Sigma / \Omega^{\mathcal{A}}_\Sigma(T)| = 1$*.*

Proof: Suppose, first, that $\mathcal{I}$ is regularly left prealgebraizable. Then it is, by definition, preequivalential and, moreover, by definition, Propositions 629 and 605 and Theorem 597, it is left c-reflective. Therefore, it is LC prealgebraizable. Thus, the required embedding is given by Theorem 355. The expression for T is obtained by applying Theorem 600.

Assume, conversely, that the postulated condition holds. Consider the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. Since Ω on the collection of theory systems is an order embedding that commutes with inverse logical extensions, $\mathcal{I}$ is preequivalential. Moreover, by hypothesis and Theorem 600, $\mathcal{I}$ is left assertional. Thus, by definition, $\mathcal{I}$ is regularly left prealgebraizable. ■

Theorem 636 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is regularly system prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order embedding that commutes with inverse logical extensions, such that, for all $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$*,* $|T_\Sigma / \Omega^{\mathcal{A}}_\Sigma(T)| = 1$*.*

Proof: Suppose, first, that $\mathcal{I}$ is regularly system prealgebraizable. Then it is, by definition, preequivalential and, moreover, by definition, Propositions 629 and 605 and Theorem 597, it is system c-reflective. Therefore, it is system prealgebraizable. Thus, the required embedding is given by Theorem 353. The expression for T is obtained by applying Theorem 600.

Assume, conversely, that the postulated condition holds. Consider the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. Since Ω on the collection of theory systems is an order embedding that commutes with inverse logical extensions, $\mathcal{I}$ is preequivalential. Moreover, by hypothesis and Theorem 600, $\mathcal{I}$ is system assertional. Thus, by definition, $\mathcal{I}$ is regularly system prealgebraizable. ■

8.7 Regular Algebraizability

We look, next, at those classes of π-institutions that are formed by adding equivalentiality to the various levels of assertionality.

Definition 637 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

- $\mathcal{I}$ *is* **regularly family algebraizable**, *or* **RF algebraizable** *for short, if it is equivalential and family assertional;*
- $\mathcal{I}$ *is* **regularly left algebraizable**, *or* **RL algebraizable** *for short, if it is equivalential and left assertional;*
- $\mathcal{I}$ *is* **regularly system algebraizable**, *or* **RS algebraizable** *for short, if it is equivalential and system assertional.*

Even though, there are apparently three classes in the regular algebraizability hierarchy, in reality there are only two, since, as was the case with regular weak algebraizability, the classes of regularly left and of regularly system π-institutions coincide.

Proposition 638 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is regularly left algebraizable if and only if it is regularly system algebraizable.*

Proof: The "only if" follows directly by the definition and Proposition 592. For the "if", suppose that $\mathcal{I}$ is regularly system algebraizable. Then it is, a fortiori, equivalential and, hence, protoalgebraic. Thus, by Lemma 170, it is stable. Therefore, since $\mathcal{I}$ is system regular and stable, by Proposition 579, it is left regular. We conclude that $\mathcal{I}$ is regularly left algebraizable. ■

The assertionality hierarchy, established in Proposition 592, and Proposition 638 allow us to establish the following regular algebraizability hierarchy.

Proposition 639 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is regularly family algebraizable, then it is regularly system algebraizable.*

Proof: Straightforward by combining Definition 637 and Proposition 592, and taking into account Proposition 638. ■

The **regular algebraizability hierarchy** is depicted in the following diagram.

Regular Family Algebraizable

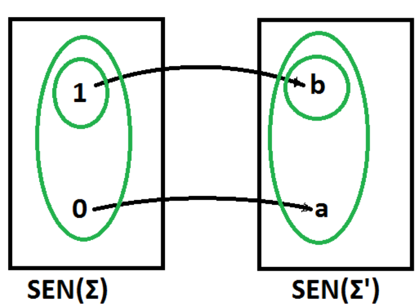
↓

Regular System Algebraizable

We use an example to show that the two classes in this hierarchy are different. Namely, we construct a π-institution that is regularly system algebraizable but fails to be regularly family algebraizable.

Example 640 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with objects* Σ *and* Σ' *and a unique (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$*,* $\mathrm{SEN}^\flat(\Sigma') = \{a, b\}$ *and* $\mathrm{SEN}^\flat(f)(0) = a$*,* $\mathrm{SEN}^\flat(f)(1) = b$*;*
- $N^\flat$ *is the trivial category of natural transformations.*



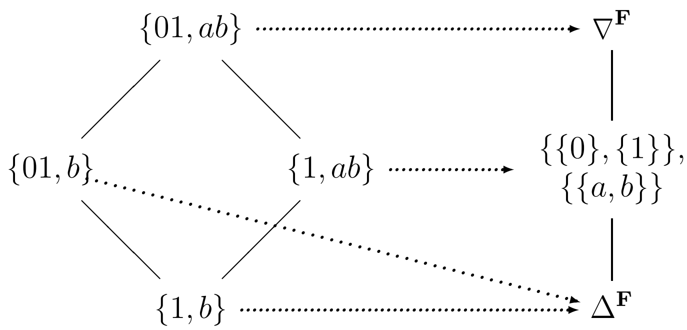
Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{1\}, \{0, 1\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{b\}, \{a, b\}\}.$$

The following table shows the action of $^{\leftarrow}$ *on theory families, where rows correspond to* T_Σ *and columns to* $T_{\Sigma'}$ *and each entry is written as* $\overleftarrow{T}_\Sigma, \overleftarrow{T}_{\Sigma'}$.

$\leftarrow$	$\{b\}$	$\{a,b\}$
$\{1\}$	$\{1\},\{b\}$	$\{1\},\{a,b\}$
$\{0,1\}$	$\{1\},\{b\}$	$\{0,1\},\{a,b\}$

The following diagram shows the structure of the lattice of theory families on the left and the structure of the corresponding Leibniz congruence systems (in terms of blocks) on the right:



The Leibniz operator is monotone on theory families, whence, $\mathcal{I}$ *is protoalgebraic. To see that it is equivalential, we must show that it is family extensional. The only non-trivial proper subuniverses of* $\mathbf{F}$ *are* $\mathbf{X} = \{\{0\},\{a,b\}\}$ *and* $\mathbf{Y} = \{\{1\},\{a,b\}\}$. *Moreover, there are only three theory families different from* $\mathrm{SEN}^\flat$. *Thus, we have six cases to examine, accomplished below:*

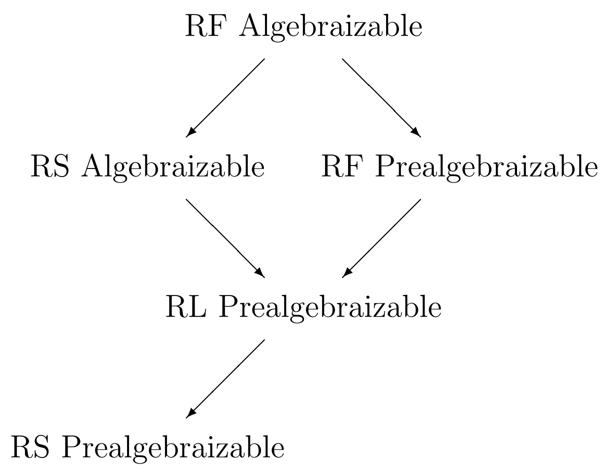
- $\Omega^{\mathbf{X}}(\{1,b\}\cap\mathbf{X}) = \Omega^{\mathbf{X}}(\{\varnothing,b\}) = \{\Delta^{\mathbf{X}}_\Sigma,\nabla^{\mathbf{X}}_{\Sigma'}\} = \Delta^{\mathbf{F}}\cap\mathbf{X}^2 = \Omega(\{1,b\})\cap\mathbf{X}^2$;
- $\Omega^{\mathbf{X}}(\{01,b\}\cap\mathbf{X}) = \Omega^{\mathbf{X}}(\{0,b\}) = \Delta^{\mathbf{X}} = \Delta^{\mathbf{F}}\cap\mathbf{X}^2 = \Omega(\{01,b\})\cap\mathbf{X}^2$;
- $\Omega^{\mathbf{X}}(\{1,ab\}\cap\mathbf{X}) = \Omega^{\mathbf{X}}(\{\varnothing,ab\}) = \nabla^{\mathbf{X}} = \{\Delta^{\mathbf{F}}_\Sigma,\nabla^{\mathbf{F}}_{\Sigma'}\}\cap\mathbf{X}^2 = \Omega(\{1,ab\})\cap\mathbf{X}^2$;
- $\Omega^{\mathbf{Y}}(\{1,b\}\cap\mathbf{Y}) = \Omega^{\mathbf{Y}}(\{1,b\}) = \Delta^{\mathbf{Y}} = \Delta^{\mathbf{F}}\cap\mathbf{Y}^2 = \Omega(\{1,b\})\cap\mathbf{Y}^2$;
- $\Omega^{\mathbf{Y}}(\{01,b\}\cap\mathbf{Y}) = \Omega^{\mathbf{Y}}(\{1,b\}) = \Delta^{\mathbf{Y}} = \Delta^{\mathbf{F}}\cap\mathbf{Y}^2 = \Omega(\{01,b\})\cap\mathbf{Y}^2$;
- $\Omega^{\mathbf{Y}}(\{1,ab\}\cap\mathbf{Y}) = \Omega^{\mathbf{Y}}(\{1,ab\}) = \nabla^{\mathbf{Y}} = \{\Delta^{\mathbf{F}}_\Sigma,\nabla^{\mathbf{F}}_{\Sigma'}\}\cap\mathbf{Y}^2 = \Omega(\{1,ab\})\cap\mathbf{Y}^2$.

We showed that $\mathcal{I}$ *is equivalential. We also have,* $\mathrm{Thm}(\mathcal{I}) = \{\{1\},\{b\}\}$ *and, for every theory system* T, $T_\Sigma = 1/\Omega_\Sigma(T)$ *and* $T_{\Sigma'} = b/\Omega_{\Sigma'}(T)$. *Therefore,* $\mathcal{I}$ *is system assertional. Thus,* $\mathcal{I}$ *is regularly system algebraizable.*

On the other hand, for $T = \{\{0,1\},\{b\}\} \in \mathrm{ThFam}(\mathcal{I})$, *we have* $0,1 \in T_\Sigma$, *but* $\langle 0,1\rangle \notin \Omega_\Sigma(T)$. *Therefore,* $\mathcal{I}$ *fails to be family regular and, hence,* $\mathcal{I}$ *is not regularly family algebraizable.*

We investigate, next, the relationships that hold between the two regular algebraizability classes, introduced in the present section, and the three regular prealgebraizability classes, that were introduced in Section 8.6. Since, by

Proposition 331, equivalentiality implies preequivalentiality, we get, a priori, the following mixed hierarchy.



As was the case with the corresponding weak classes, we can show that the top classes of the regular prealgebraizability and the regular algebraizability hierarchies coincide.

Theorem 641 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is regularly family prealgebraizable if and only if it is regularly family algebraizable.*

Proof: The "if" follows from the relevant definitions and the fact that, by Proposition 331, equivalentiality implies preequivalentiality. For the "only if", it suffices to show that, under family assertionality, preequivalentiality implies equivalentiality. By Proposition 331, it suffices, in turn, to show that family assertionality implies stability and, by Proposition 152, that family assertionality implies systemicity. Indeed, by Theorem 597, family assertionality implies family c-reflectivity and, by Proposition 237, we get that $\mathcal{I}$ is systemic. ∎

Moreover, from the definitions involved, we get the following

Proposition 642 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is regularly system algebraizable, then it is regularly left prealgebraizable.*

Proof: Suppose $\mathcal{I}$ is regularly system algebraizable. Equivalently, by Proposition 638, it is regularly left algebraizable. Then, by definition, it is equivalential and left assertional. Thus, by Proposition 331, it is preequivalential and left assertional, i.e., by definition, it is regularly left prealgebraizable. ∎

Based on Theorem 641 and Proposition 642, we get the following updated version of the mixed hierarchy shown in the preceding diagram.

RF (Pre)Algebraizable

↓

RS Algebraizable

↓

RL Prealgebraizable

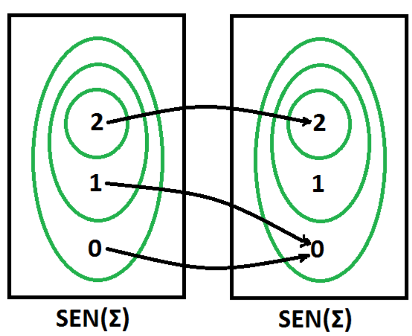
↓

RS Prealgebraizable

To show that all classes in this hierarchy are different, we provide an example of a π-institution that is regularly left prealgebraizable, but fails to be regularly system algebraizable, i.e., an example that separates the regular algebraizability from the regular prealgebraizability classes.

Example 643 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single non-identity morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is given by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$ *and* $\mathrm{SEN}^\flat(f)(2) = 2$;
- $N^\flat$ *is the trivial category of natural transformations.*

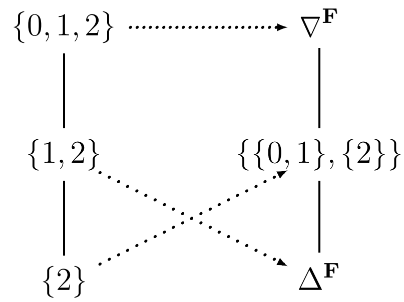


Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting* $\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}$.

The following table gives the theory families and the theory systems of the π*-institution* $\mathcal{I}$:

T	$\overleftarrow{T}$
$\{2\}$	$\{2\}$
$\{1, 2\}$	$\{2\}$
$\{0, 1, 2\}$	$\{0, 1, 2\}$

The lattice of theory families and the corresponding Leibniz congruence systems are depicted below:



Since the only theory systems of $\mathcal{I}$ *are* $\{\{2\}\}$ *and* $\{\{0,1,2\}\}$, *it is clear that* Ω *is monotone on theory systems and, hence,* $\mathcal{I}$ *is prealgebraic. To see that it is preequivalential, we must also show that it is system extensional. To simplify the process, we note that the only non-trivial proper universes of* $\mathbf{F}$ *are* $\mathbf{X} = \{\{0,1\}\}$ *and* $\mathbf{Y} = \{\{0,2\}\}$ *and the only proper theory system is* $\mathrm{Thm}(\mathcal{I}) = \{\{2\}\}$. *Hence, there are two cases to check, as shown below (written, as done elsewhere, in shorthand):*

- $\Omega^{\mathbf{X}}(2 \cap \mathbf{X}) = \Omega^{\mathbf{X}}(\varnothing) = \{01\} = \{01,2\} \cap \mathbf{X}^2 = \Omega(2) \cap \mathbf{X}^2$;
- $\Omega^{\mathbf{Y}}(2 \cap \mathbf{Y}) = \Omega^{\mathbf{Y}}(2) = \{0,2\} = \{01,2\} \cap \mathbf{Y}^2 = \Omega(2) \cap \mathbf{Y}^2$.

Clearly, $\mathcal{I}$ *has theorems. Thus, to complete the proof that it is regularly left prealgebraizable, it suffices to show that it is left assertional, i.e., by Proposition 588, that, for all* $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T}_\Sigma = 2/\Omega_\Sigma(T)$. *Indeed, we get:*

- $\overleftarrow{\{\{2\}\}}_\Sigma = \{2\} = 2/\Omega_\Sigma(\{\{2\}\})$;
- $\overleftarrow{\{\{1,2\}\}}_\Sigma = \{2\} = 2/\Omega_\Sigma(\{\{1,2\}\})$;
- $\overleftarrow{\{\{0,1,2\}\}}_\Sigma = \{0,1,2\} = 2/\Omega_\Sigma(\{\{0,1,2\}\})$.

On the other hand, since $\{\{2\}\} \leq \{\{1,2\}\}$, *but*

$$\Omega(\{\{2\}\}) = \{\{\{0,1\},\{2\}\}\} \nleq \Delta^{\mathbf{F}} = \Omega(\{\{1,2\}\}),$$

$\mathcal{I}$ *is not protoalgebraic and, hence, a fortioti, it is not equivalential. As a consequence, it fails to be regularly system algebraizable.*

Turning now to the relationship between regular (pre)algebraizability and regular weak (pre)algebraizability, we complete the picture given in Section 8.6.

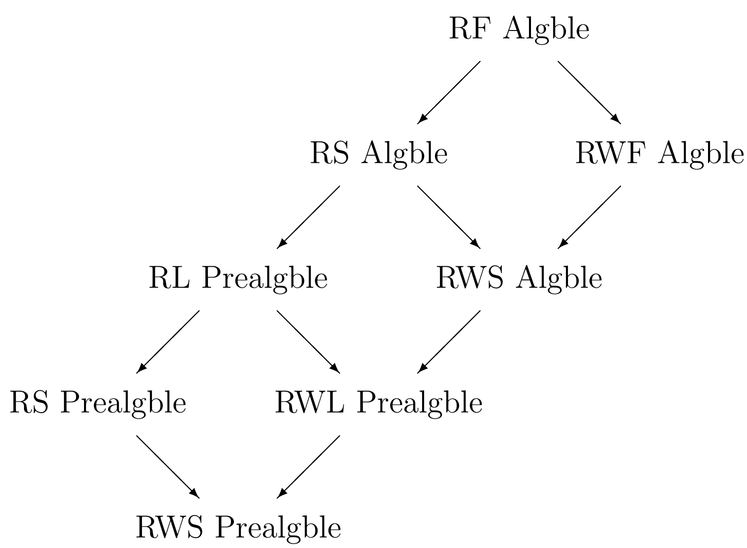
Proposition 644 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ is regularly family algebraizable, then it is regularly weakly family algebraizable;

(b) If $\mathcal{I}$ is regularly system algebraizable, then it is regularly weakly system algebraizable.

Proof: By Definition 329, equivalentiality implies protoalgebraicity. From this fact, and Definitions 637 and 613, both implications follow directly. ■

Thus, Proposition 644, together with Propositions 642 and 629, point to the following hierarchy of regularly (pre)algebraizable π-institutions and regularly weakly (pre)algebraizable π-institutions.



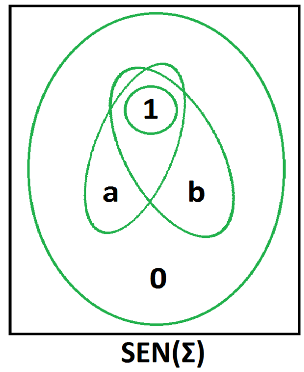
To see that all southeast arrows represent proper inclusions, we give an example of a regularly weakly family algebraizable π-institution which fails to be regularly system prealgebraizable.

Example 645 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, a, b, 1\}$*;*
- $N^\flat$ *is the category of natural transformations, generated by the two binary natural transformations* $\wedge, \vee : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$*, defined by the following tables:*

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

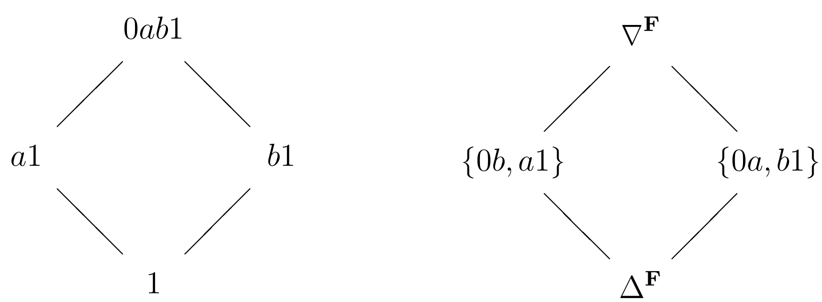


Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be the π-institution, defined by setting

$$\mathcal{C}_\Sigma = \{\{1\}, \{a, 1\}, \{b, 1\}, \{0, a, b, 1\}\}.$$

$\mathcal{I}$ has four theory families, all of which are also theory systems.

The lattice of theory families and the corresponding Leibniz congruence systems are shown in the diagram.



From the diagram, we can see that $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is an order isomorphism, whence, $\mathcal{I}$ is weakly family algebraizable. Since it has theorems, to show that it is, also, family assertional, it suffices to show that it satisfies, for all $T \in \mathrm{Thfam}(\mathcal{I})$, $T_\Sigma = 1/\Omega_\Sigma(T)$. This is easily checked from the diagram above, giving the Leibniz congruence systems corresponding to the various theory families of $\mathcal{I}$.

On the other hand, for the universe $\mathbf{X} = \{\{0, a, 1\}\}$ and the theory system $T = \{\{1\}\}$, we get

$$\Omega(T) \cap \mathbf{X}^2 = \{\{0\}, \{a\}, \{1\}\} \subsetneq \{\{0\}, \{a, 1\}\} = \Omega^{\mathbf{X}}(T \cap \mathbf{X}).$$

Thus, $\mathcal{I}$ is not system extensional and, therefore, it fails to be (system) prequivalential and, a fortiori, it also fails to be regularly system prealgebraizable.

Turning now to the relationship between regular (pre)algebraizability and (pre)algebraizability, we get, by definition,

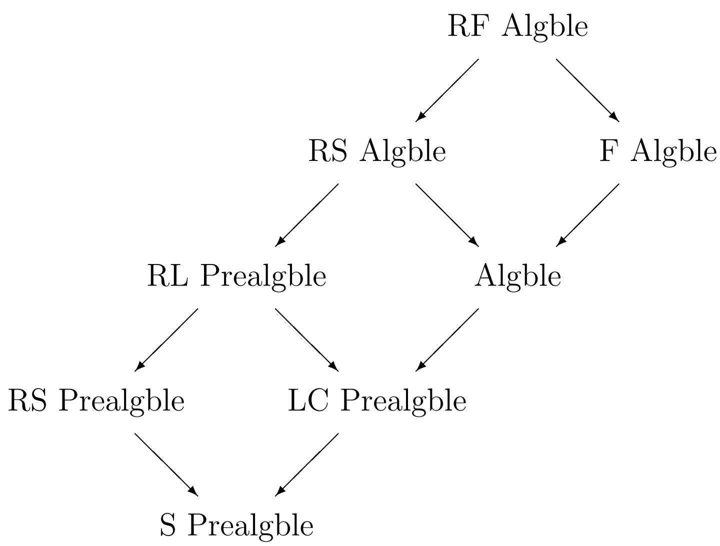
Proposition 646 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.*

(a) If $\mathcal{I}$ is regularly family algebraizable, then it is family algebraizable;

(b) If $\mathcal{I}$ is regularly system algebraizable, then it is (system) algebraizable.

Proof: For Part (a) note that, by definition, $\mathcal{I}$ is regularly family algebraizable if and only if it is equivalential and family assertional. Thus, by Theorem 597, it is equivalential and family completely reflective. Thus, by Definition 360, it is family algebraizable. Part (b) follows along similar lines. ■

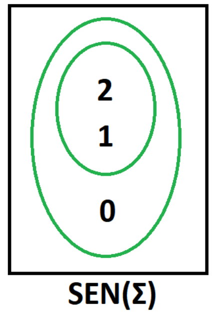
Proposition 646 completes the picture given by Proposition 631 and Proposition 642, establishing the following mixed regular (pre)algebraizability and (pre)algebraizability hierarchies, where, on the prealgebraizability side, only the classes immediately interacting with the regular prealgebraizability classes are shown.



Again it is not difficult to see that the classes in the regular (pre)algebraizability hierarchy are different from the classes in the (pre)algebraizability hierarchy. This is accomplished by constructing an example of a π-institution which is family algebraizable but is not regularly system prealgebraizable.

Example 647 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

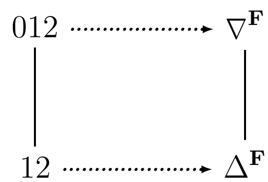
- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *specified by* $\sigma^\flat_\Sigma(0) = 0$*,* $\sigma^\flat_\Sigma(1) = 1$ *and* $\sigma^\flat_\Sigma(2) = 0$*.*



Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by stipulating that

$$\mathcal{C}_\Sigma = \{\{1,2\}, \{0,1,2\}\}.$$

$\mathcal{I}$ is systemic and its lattice of theory families and corresponding Leibniz congruence systems are shown in the diagram.



Since the lattice of theory families of $\mathcal{I}$ is order isomorphic with the lattice of $\mathrm{AlgSys}^*(\mathcal{I})$*-congruence systems, $\mathcal{I}$ is weakly family algebraizable. To see that it is family algebraizable, it suffices to show that it is also family extensional. The only nontrivial proper universes of $\mathbf{F}$ are $\mathbf{X} = \{\{0,1\}\}$ and $\mathbf{Y} = \{\{0,2\}\}$ and the only proper theory family is $\{\{1,2\}\}$. Thus, we only need to check two cases:*

- $\Omega^{\mathbf{X}}(12 \cap \mathbf{X}) = \Omega^{\mathbf{X}}(1) = \{0,1\} = \Delta^{\mathbf{F}} \cap \mathbf{X}^2 = \Omega(12) \cap \mathbf{X}^2$;
- $\Omega^{\mathbf{Y}}(12 \cap \mathbf{Y}) = \Omega^{\mathbf{Y}}(2) = \{0,2\} = \Delta^{\mathbf{F}} \cap \mathbf{Y}^2 = \Omega(12) \cap \mathbf{Y}^2$.

On the other hand, for $T = \{\{2,3\}\}$, we have $2,3 \in T_\Sigma$, but $\langle 2,3 \rangle \notin \Delta^{\mathbf{F}}_\Sigma = \Omega_\Sigma(T)$, whence $\mathcal{I}$ is not system regular and, hence, a fortiori, it is not regularly system prealgebraizable.

As was the case with regular weak algebraizability, we can show that both kinds of regular algebraizability transfer from theory families/ systems to filter families/systems over arbritrary $\mathbf{F}$-algebraic systems.

Theorem 648 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is regularly family algebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, all $\mathbf{X} \leq \mathbf{A}$, all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and all $\Sigma \in |\mathbf{Sign}|$,

– $T \leq T'$ *implies* $\Omega(T) \leq \Omega(T')$*;*
– $\Omega^{\mathbf{X}}(T \cap \mathbf{X}) = \Omega(T) \cap \mathbf{X}^2$*;*
– $|T_\Sigma/\Omega_\Sigma(T)| = 1$*;*

(b) $\mathcal{I}$ *is regularly system algebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, all* $\mathbf{X} \leq \mathbf{A}$*, all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, all* $T'' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$*,*

– $T \leq T'$ *implies* $\Omega(T) \leq \Omega(T')$*;*
– $\Omega^{\mathbf{X}}(T \cap \mathbf{X}) = \Omega(T) \cap \mathbf{X}^2$*;*
– $|T''_\Sigma/\Omega_\Sigma(T)| = 1$*.*

Proof: Combine Theorem 334 with Theorem 599. ■

Finally, we obtain characterizations of regular algebraizability in terms of mappings between posets of filter families/ systems and congruence systems.

Corollary 649 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is regularly family algebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism commuting with inverse logical extensions, such that, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$*,* $|T_\Sigma/\Omega^{\mathcal{A}}_\Sigma(T)| = 1$*.*

Proof: By Theorems 641 and 634. ■

Theorem 650 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is regularly system algebraizable if and only if it is stable and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism commuting with inverse logical extensions, such that, for all $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ *and all* $\Sigma \in |\mathbf{Sign}|$*,* $|T_\Sigma/\Omega^{\mathcal{A}}_\Sigma(T)| = 1$*.*

Proof: Suppose, first, that $\mathcal{I}$ is regularly system algebraizable. Then it is, by definition, equivalential and, thus, by Proposition 331, stable. Moreover, by Proposition 646, it is algebraizable, whence, the required isomorphism is given by Theorem 365. The expression for T is obtained by applying Theorem 600.

Assume, conversely, that the postulated condition holds. Consider the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. Since Ω is an order isomorphism that commutes with inverse logical extensions and $\mathcal{I}$ is stable, $\mathcal{I}$ is, by Theorem 365, algebraizable. Hence, $\mathcal{I}$ is, a fortiori, equivalential. Moreover, by hypothesis and Theorem 600, $\mathcal{I}$ is system assertional. Thus, by definition, $\mathcal{I}$ is regularly system algebraizable. ■

Chapter 9

The Semantic Leibniz Hierarchy: Over the Top II

9.1 Introduction

At the apex of the Leibniz hierarchy of sentential logics one finds the class of algebraizable logics [35, 43] (see, also, Chapter 4 of [64] and Sections 3.2 and 6.5 of [86]). The concept was first introduced by Blok and Pigozzi in [35] for finitary logics. It was later generalized to arbitrary sentential logics by Herrmann [43]. Roughly speaking, a sentential logic is *algebraizable* when there exist a class K of algebras, termed the *equivalent algebraic semantics*, and two translations $\delta \approx \varepsilon$ from formulas to equations, called *defining equations*, and Δ from equations to formulas, called *equivalence formulas*, which are interpretations, i.e., preserve and reflect the logical and the equational closures and vice-versa and, in addition, are inverses of one another in a specific sense. For a detailed study of this framework, apart from the original monograph by Blok and Pigozzi [35] and Herrmann's Dissertation [43], one may consult Chapter 4 of [64] and Sections 3.2 and 6.5 of [86]. Partly due to the historical progression, but also due to the intrinsic importance and ubiquity of finitarity, its key role in studies of classical logical systems and a host of advantageous properties associated with it, the finitary aspects of algebraizability have been extensively studied and tight relations between them have been established. A very illuminative and beautifully written summary of these results, as pertaining to algebraizability, appears in Section 3.4 of [86], which constitutes the inspiration and starting point of the investigations presented here.

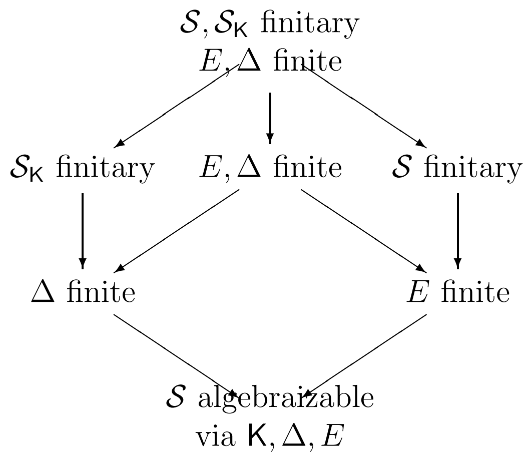
We first give a quick overview of the aforementioned work pertaining to algebraizable sentential logics. We fix an algebraizable sentential logic $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$, with equivalent algebraic semantics the generalized quasivariety K, as witnessed by a set $\delta \approx \varepsilon$ of defining equations and a set Δ of equivalence formulas. Lemma 3.36 of [86] asserts that, in case $\mathcal{S}$ is finitary and has a finite set of equivalence formulas, then every set of equivalence formulas contains a finite subset that also serves as a set of equivalence formulas. Dually, if the equational logic $\mathcal{S}_{\mathsf{K}} = \langle \mathcal{L}, \vDash_{\mathsf{K}} \rangle$ induced by the class K is finitary, i.e., if the class K happens to be a quasivariety, and $\mathcal{S}$ has a finite set of defining equations, then every set of defining equations has a finite subset that also serves in the same capacity. Besides these conditional "finitarization" results, Theorem 3.37 of [86] details some important relationships between the following four conditions: $\mathcal{S}$ finitary; $\mathcal{S}_{\mathsf{K}}$ finitary; $\delta \approx \varepsilon$ finite; and Δ finite. On the one hand, if $\mathcal{S}$ is finitary, $\delta \approx \varepsilon$ may be taken finite, and, dually, if $\mathcal{S}_{\mathsf{K}}$ is finitary, then Δ may be taken to be finite. Moreover, if $\mathcal{S}$ is finitary and has a finite set Δ of equivalence formulas, then $\mathcal{S}_{\mathsf{K}}$ is finitary also, and, dually, if $\mathcal{S}_{\mathsf{K}}$ is finitary and $\mathcal{S}$ has a finite set $\delta \approx \varepsilon$ of defining equations, then $\mathcal{S}$ is finitary also. These implications lead to Corollary 3.38 of [86], which asserts the following three conditional equivalences:

1. If $\delta \approx \varepsilon$ and Δ are both finite, then $\mathcal{S}$ is finitary iff $\mathcal{S}_{\mathsf{K}}$ is finitary.

2. If $\mathcal{S}$ is finitary, then $\mathcal{S}_{\mathsf{K}}$ is finitary iff Δ may be taken finite.

3. If $\mathcal{S}_{\mathsf{K}}$ is finitary, then $\mathcal{S}$ is finitary iff $\delta \approx \varepsilon$ may be taken finite.

These lead to the hierarchy depicted in the diagram shown in Figure 3, p. 137 of [86] and duplicated below.



After outlining the results that lead to the depicted hierarchy, Font gathers pointers to three examples of sentential logics that serve to separate the various classes in the hierarchy. Example 3.41 of [86] revisits Łukasiewicz's infinite valued logic Ł_∞, which is not finitary, has a non-finitary equivalent algebraic semantics, but is algebraized via finite sets of defining equations and equivalence formulas. As a result, it serves to separate classes related by the vertical arrows in the diagram. In Example 3.42 of [86], the so-called Logic of Last Judgement $\mathcal{L}J$, introduced by Herrmann [53], is presented. This is a finitary logic, algebraized by a non-finitary equational consequence, via a single defining equation, but a necessarily infinite set of equivalence formulas. So $\mathcal{L}J$ serves in separating the logics related by the southeast arrows in the diagram. Additional examples that can serve the same purpose were presented by Dellunde [48] and by Lewin, Mikenberg and Schwartze [55]. Finally, in Example 3.43 of [86], Font presents a logic due to Raftery [82]. Raftery's work was motivated by a question posed by Czelakowski in Note 4.5.2 (4) of [64], which was also implicit in Problem 3.18 of [43]. Raftery's logic is not finitary, but is algebraizable, with a finitary equivalent algebraic semantics which is actually a variety, via an infinite set of defining equations and a singleton set of equivalence formulas. It serves in separating the classes of sentential logics connected via the southwest arrows of the diagram. In Section 9.5, we revisit Łukasiewicz's logic and the logics of Dellunde and Raftery in much more detail.

Our own goal in this chapter is to provide analogs of the classes in the finitarity hierarchy of algebraizable sentential logics for logics formalized as π-institutions. The finitarity conditions pertaining to π-institutions remain roughly unchanged. However, keeping in the spirit of dealing with semantically defined classes (i.e., relying on properties of the Leibniz operator) in this part of the monograph, the finitarity conditions regarding $\delta \approx \varepsilon$ and Δ are modified. They are recast as continuity properties of the Leibniz operator and of its inverse. Subject to these modifications, the results obtained for semantically defined finitarity properties pertaining to weakly family algebraizable π-institutions reflect those outlined above for algebraizable sentential logics.

In Section 9.2, we introduce the concept of a π-*structure*, which abstracts that of a π-institution by eliminating the requirement of structurality. That is, a π-structure $\mathcal{I}$ consists of an algebraic system together with a collection of closure operators, one on each of its sentence components, which are not required to satisfy structurality. The *finitary companion* of a π-structure $\mathcal{I}$ is the π-structure obtained by considering the closure family induced by all finite consequences of $\mathcal{I}$. As a consequence, it constitutes the largest finitary π-structure included in $\mathcal{I}$. In addition, it can be shown that it is structural when the given π-structure satisfies structurality, i.e., when it is a π-institution. Finitary companions may also be characterized via their theory families. Namely, a sentence family of a π-institution $\mathcal{I}$ is a theory family of its finitary companion if and only if it is the union of a directed collection of locally finitely generated theory families of $\mathcal{I}$.

In Section 9.3, we focus on some of the fundamental properties that determine the classes in the semantic Leibniz hierarchy and investigate whether they are transferred from a π-institution to its finitary companion and vice-versa, and, if yes, under which conditions. In this vein, protoalgebraicity is shown to hold for a π-institution $\mathcal{I}$ if its finitary companion is protoalgebraic. A similar property holds for family reflectivity. As a consequence, a π-institution is weakly family algebraizable if its finitary companion has the same property. In the opposite direction, for properties of $\mathcal{I}$ to be inherited by its finitary companion, additional provisos are needed. We say that the Leibniz operator of a π-institution is *continuous* if, for every directed collection $\{T^i\}_{i\in I}$ of theory families, such that $\bigcup_{i\in I} T^i$ is also a theory family, $\Omega(\bigcup_{i\in I} T^i) = \bigcup_{i\in I} \Omega(T^i)$. Continuity of the Leibniz operator is a stronger property than, i.e., implies, protoalgebraicity. It turns out that it is also sufficient for the finitary companion of $\mathcal{I}$ to be protoalgebraic, subject to the category of signatures being finite. Along dual lines, we say that the inverse Leibniz operator of a weakly family algebraizable π-institution $\mathcal{I}$ is *continuous* if, for all directed collections $\{\theta^i\}_{i\in I}$ of congruence systems in $\mathrm{ConSys}^*(\mathcal{I})$, such that $\bigcup_{i\in I} \theta^i$ is also a congruence system in $\mathrm{ConSys}^*(\mathcal{I})$, $\Omega^{-1}(\bigcup_{i\in I} \theta^i) = \bigcup_{i\in I} \Omega^{-1}(\theta^i)$. This condition, when supplementing continuity of the Leibniz operator, ensures that weak family algberaizability of a π-institution $\mathcal{I}$, with a finite category of signatures, is inherited by its finitary companion.

Section 9.4 is the main section of the chapter. Here, we establish the semantic finitarity hierarchy of weakly family algebraizable π-institutions, which parallels the hierarchy of sentential logics studied in Section 3.4 of [86] and summarized both in Subsection 1.3.8 and at the beginning of this Introduction. Throughout, the object of study is a weakly family algebraizable π-institution $\mathcal{I}$. Moreover, we denote by $\mathsf{K} \coloneqq \mathrm{AlgSys}(\mathcal{I})$ and by $\mathcal{Q}^{\mathsf{K}}$ the equational π-structure induced by the class K. Note that $\mathcal{I}$ being weakly family algebraizable ensures that the Leibniz operator $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is an isomorphism, whence the inverse Ω^{-1} is well-defined. It is shown, first, that the finitarity of $\mathcal{I}$ ensures the continuity of the inverse Leibniz operator on $\mathrm{ConSys}^*(\mathcal{I})$ and that, dually, the finitarity of $\mathcal{Q}^{\mathsf{K}}$ guarantees that the Leibniz operator itself is continuous on $\mathrm{ThFam}(\mathcal{I})$. Further, if to the finitarity of $\mathcal{I}$ is added the continuity of the Leibniz operator, then the finitarity of $\mathcal{Q}^{\mathsf{K}}$ follows. Dually, if to the finitarity of $\mathcal{Q}^{\mathsf{K}}$ is added the continuity of the inverse Leibniz operator, then $\mathcal{I}$ is also finitary. These results are summarized in three statements, which parallel those governing sentential logics, stated in Corollary 3.38 of [86]. Namely, under continuity of both the Leibniz operator and its inverse, finitarity of $\mathcal{I}$ is equivalent to finitarity of $\mathcal{Q}^{\mathsf{K}}$. Under finitarity of $\mathcal{I}$, finitarity of $\mathcal{Q}^{\mathsf{K}}$ is equivalent to continuity of the Leibniz operator and, dually, under finitarity of $\mathcal{Q}^{\mathsf{K}}$, finitarity of $\mathcal{I}$ is tantamount to continuity of the inverse Leibniz operator.

In Section 9.5, we take a brief detour to present in detail three examples of sentential logics, which serve to separate the classes in the finitarity hierarchy of algebraizable sentential logics, presented in Section 3.4 of [86]. Even though our focus here is not on sentential logics, we showed in Section 1.1 how a sentential logic gives rise to a π-institution in a rather straightforward way. Accordingly, the purpose of presenting these three sentential logics is to construct, based on them, corresponding π-institutions that will serve to separate the classes in the finitarity hierarchy of weakly family algebraizable π-institutions, studied in Section 9.4. The constructions of the π-institutions, based on the sentential logics introduced here, and the separation properties they help establish will be described in some detail in Section 9.6.

The first example is Łukasiewicz's infinite valued logic (see, e.g., Example 3.41 of [86]). It is a logic over a language with three binary connectives $\wedge$, $\vee$, $\to$ and one unary connective $\neg$. It is semantically defined via a logical matrix. It is shown that it is not finitary, but that it is algebraizable via a singleton set of defining equations $E(x) = \{x \approx \top\}$, where $\top \coloneqq x \to x$, and the doubleton set of equivalence formulas $\Delta(x,y) = \{x \to y, y \to x\}$. This logic serves to separate the classes of sentential logics related by vertical arrows in the finitarity hierarchy of algebraizable sentential logics, depicted in the preceding diagram. The second example is a logic introduced by Dellunde in [48]. It is a logic defined over a language with one binary connective $\leftrightarrow$ and one unary connective $\Box$. It is defined via a Hilbert calculus and, as a result, it is finitary. It is shown that it is regularly algebraizable via the infinite set of equivalence formulas $\Delta(x,y) = \{\Box^n x \leftrightarrow \Box^n y : n \in \omega\}$. According to the general theory, regular algebraizability implies that the set of defining equations is the singleton $E(x) = \{x \approx \top\}$, where $\top := x \leftrightarrow x$ defines the unique element in the filter of any reduced matrix of the logic. What is key for our purposes is that there does not exist a finite subset $\Delta_0 \subseteq \Delta$ that also serves as a set of equivalence formulas for this logic. Consequently, this example serves in separating those classes in the hierarchy of sentential logics connected via southeast arrows in the diagram. The last example presented in Section 9.5 is a logic introduced by Raftery [82]. It is a logic defined over a language with one binary connective $\leftrightarrow$ and three unary connectives π_1, π_2 and $\Diamond$. It is semantically defined as a weakening of another logic, which, in turn, is defined using a logical matrix. The weakening, roughly speaking, is obtained by considering an entire variety of algebras to which the underlying algebra of this logical matrix belongs. Raftery shows that neither logic is finitary and that, in addition, the weaker logic, corresponding to the variety, is algebraizable via an infinite set of defining equations and a singleton set of equivalence formulas. As a result, Raftery's logic serves as an example separating the classes related by the southwest arrows in the finitarity hierarchy depicted in the preceding diagram.

In Section 9.6, we use the framework outlined in Section 1.1 to formalize the three sentential logics of Section 9.5 as π-institutions. The resulting examples enable us to separate the classes in the semantic finitarity hierarchy of weakly family algebraizable π-institutions, studied in Section 9.4, in a way that parallels the separation of the classes in the hierarchy of algebraizable sentential logics.

9.2 The Finitary Companion

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. A **π-structure** $\mathcal{I} = \langle \mathbf{F}, D \rangle$, **based on F**, is like a π-institution except that D is a **closure family** on $\mathbf{F}$ instead of a closure system, i.e., the only requirement is that

$$D_\Sigma : \mathcal{P}\mathrm{SEN}^\flat(\Sigma) \to \mathcal{P}\mathrm{SEN}^\flat(\Sigma)$$

be a closure operator on $\mathrm{SEN}^\flat(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}^\flat|$. On the other hand, D is not required to be structural. A heavier use of π-structures will be encountered in Chapter 12, where the concept will be defined anew and more details given. D is called the **closure family of the π-structure** $\mathcal{I}$. Note that π-structures generalize π-institutions.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, D \rangle$ be a π-structure based on $\mathbf{F}$. Define the family

$$D^f = \{D^f_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$$

by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$D^f_\Sigma : \mathcal{P}\mathrm{SEN}^\flat(\Sigma) \to \mathcal{P}\mathrm{SEN}^\flat(\Sigma)$$

be given, for all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$, by

$$D^f_\Sigma(\Phi) = \bigcup\{D_\Sigma(\Phi') : \Phi' \subseteq_f \Phi\},$$

where $\subseteq_f$ denotes the finite subset relation.

It is not hard to show that D^f is a finitary closure family on $\mathbf{F}$ and that, moreover, it is a closure system (i.e., structural) in case D itself happens to be structural.

Lemma 651 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, D\rangle$ *a* π*-structure based on* $\mathbf{F}$. *Then* D^f *is a finitary closure family on* $\mathbf{F}$. *Further, if* D *is structural, i.e., if* $\mathcal{I}$ *is a* π*-institution, then* D^f *is also structural.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \Phi$. Then $\phi \in D_\Sigma(\phi) \subseteq D^f_\Sigma(\Phi)$. Thus, D^f is inflationary.

Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \Psi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\Phi \subseteq \Psi$. If $\phi \in D^f_\Sigma(\Phi)$, then there exists $\Phi' \subseteq_f \Phi$, such that $\phi \in D_\Sigma(\Phi')$. But $\Phi' \subseteq_f \Phi \subseteq \Psi$, whence, $\phi \in D^f_\Sigma(\Psi)$. Thus, $D^f_\Sigma(\Phi) \subseteq D^f_\Sigma(\Psi)$ and D^f is also monotone.

Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in D^f_\Sigma(D^f_\Sigma(\Phi))$. Then, there exists $\Phi' \subseteq_f D^f_\Sigma(\Phi)$, such that $\phi \in D_\Sigma(\Phi')$. Since $\Phi' \subseteq D^f_\Sigma(\Phi)$, for all $\phi' \in \Phi'$, there exists $\Phi'^{\phi'} \subseteq_f \Phi$, such that $\phi' \in D_\Sigma(\Phi'^{\phi'})$. Hence, we get

$$\phi \in D_\Sigma(\Phi') \subseteq D_\Sigma(\bigcup_{\phi' \in \Phi'} \Phi'^{\phi'}).$$

Since $\bigcup_{\phi' \in \Phi'} \Phi'^{\phi'} \subseteq_f \Phi$, we get, by definition, $\phi \in D^f_\Sigma(\Phi)$. Thus, D^f is also idempotent.

Finally, to show finitarity, let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in D^f_\Sigma(\Phi)$. Thus, there exists $\Phi' \subseteq_f \Phi$, such that $\phi \in D_\Sigma(\Phi')$. Then, by definition, $\phi \in D^f_\Sigma(\Phi')$. Thus, $D^f_\Sigma(\Phi) = \bigcup_{\Phi' \subseteq_f \Phi} D^f_\Sigma(\Phi')$ and, hence, D^f is a finitary closure family on $\mathbf{F}$.

To prove the last statement concerning structurality, assume that D is structural. Let $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in D^f_\Sigma(\Phi)$. Then, there exists $\Phi' \subseteq_f \Phi$, such that $\phi \in D_\Sigma(\Phi')$. Since D is assumed structural, we get $\mathrm{SEN}^\flat(f)(\phi) \in D_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi'))$. But Φ' being a finite subset of Φ, $\mathrm{SEN}^\flat(f)(\Phi')$ is a finite subset of $\mathrm{SEN}^\flat(f)(\Phi)$, whence, $\mathrm{SEN}^\flat(f)(\phi) \in D^f_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))$. This shows that D^f is also structural. ∎

The following proposition provides a characterization of D^f.

Proposition 652 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, D\rangle$ *a* π*-structure based on* $\mathbf{F}$. *Then* D^f *is the largest finitary closure family on* $\mathbf{F}$ *lying below* D *in the* $\leq$ *ordering.*

Proof: By Lemma 651, D^f is a finitary closure family. By its definition and the monotonicity of D, it is clear that $D^f \leq D$. To complete the proof, suppose D' is a finitary closure family on $\mathbf{F}$, such that $D' \leq D$. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in D'_\Sigma(\Phi)$. Since, by hypothesis, D' is finitary, there exists $\Phi' \subseteq_f \Phi$, such that $\phi \in D'_\Sigma(\Phi')$. Since, also by hypothesis, $D' \leq D$, we get $\phi \in D_\Sigma(\Phi')$. Thus, since $\Phi' \subseteq_f \Phi$, we get, by definition of D^f, $\phi \in D^f_\Sigma(\Phi)$. Thus, $D' \leq D^f$ and, therefore, D^f is the largest finitary closure family below D. ∎

Corollary 653 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *Then* C^f *is the largest finitary closure system on* $\mathbf{F}$ *lying below* C *in the* $\leq$ *ordering.*

Proof: By Proposition 652, C^f is the largest finitary closure family lying below C. But, by Lemma 651, it is a closure system on $\mathbf{F}$. Hence, it is the largest finitary closure system lying below C. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, D \rangle$ be a π-structure based on $\mathbf{F}$. We call D^f the **finitary companion of** D. Moreover, we set $\mathcal{I}^f = \langle \mathbf{F}, D^f \rangle$ and call it the **finitary companion of** $\mathcal{I}$. Of course, these terms apply, in particular, to the case of π-institutions.

We would like to provide an alternative characterization of the finitary companion that is also very useful in various applications of the notion. With an eye towards this goal, we make the following definitions.

Definition 654 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$ *and* $\mathcal{T} \cup \{T\} \subseteq \mathrm{ThFam}(\mathcal{I})$.

- T *is called* **locally finitely generated** *if, for all* $\Sigma \in |\mathbf{Sign}^\flat|$, *there exists* $\Phi_\Sigma \subseteq_f T_\Sigma$, *such that* $T_\Sigma = C_\Sigma(\Phi_\Sigma)$.
- $\mathcal{T}$ *is* **locally finitely generated** *if all its theory families are locally finitely generated.*

The following proposition provides a characterization of those sentence families of a π-institution $\mathcal{I}$ that are theory families of its finitary companion.

Proposition 655 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$ *and* $T \in \mathrm{SenFam}(\mathbf{F})$. *Then* $T \in \mathrm{ThFam}(\mathcal{I}^f)$ *if and only if, there exists a directed locally finitely generated collection* $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$, *such that*

$$T = \bigcup_{i \in I} T^i.$$

Proof: Let $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$ be a directed locally finitely generated collection. Since it is locally finitely generated, we have, by definition, $T^i \in \mathrm{ThFam}(\mathcal{I}^f)$, for all $i \in I$. But, by Lemma 651, $\mathcal{I}^f$ is finitary. Thus, by Proposition 112, it is continuous. Hence $\bigcup_{i\in I} T^i \in \mathrm{ThFam}(\mathcal{I}^f)$.

Suppose, conversely, that $T \in \mathrm{ThFam}(\mathcal{I}^f)$. Set

$$\mathcal{T} = \{C(X) : X \leq_{lf} T\},$$

where $\leq_{lf}$ denotes the locally finite subfamily relation. It is clear, by its definition, that $\mathcal{T}$ is locally finitely generated. Suppose $C(X), C(Y) \in \mathcal{T}$. Then $C(X \cup Y) \in \mathcal{T}$ and, moreover, $C(X), C(Y) \leq C(X \cup Y)$. Hence, $\mathcal{T}$ is also directed. Finally, it is not difficult to see that $T = \bigcup \mathcal{T}$. Thus, the declared characterization holds. ■

9.3 π-Institutions & Companions: Hierarchy

In this section, we study how some of the properties that have been used to build hierarchies of π-institutions are inherited by the finitary companion of a π-institution from the π-institution itself and vice-versa. In some instances the inheritance is immediate, but, in others, additional conditions need to be imposed. We focus on the property of weak family algebraizability. That is the reason of selecting the few properties studied here versus some of the remaining properties introduced previously.

First, we show that protoalgebraicity is passed up to $\mathcal{I}$ from its finitary companion $\mathcal{I}^f$.

Lemma 656 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}^f = \langle \mathbf{F}, C^f \rangle$ *is protoalgebraic, then so is* $\mathcal{I}$*.*

Proof: Suppose $\mathcal{I}^f$ is protoalgebraic. Then, by definition, the Leibniz operator $\Omega : \mathrm{ThFam}(\mathcal{I}^f) \to \mathrm{ConSys}^*(\mathcal{I}^f)$ is monotone. Since $C^f \leq C$, we have $\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$. Therefore, the Leibniz operator $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is also monotone. We conclude that $\mathcal{I}$ is protoalgebraic. ■

Similarly, if $\mathcal{I}^f$ is family reflective, then so is $\mathcal{I}$.

Lemma 657 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}^f = \langle \mathbf{F}, C^f \rangle$ *is family reflective, then so is* $\mathcal{I}$*.*

Proof: Suppose $\mathcal{I}^f$ is family reflective. Then, by definition, the Leibniz operator $\Omega : \mathrm{ThFam}(\mathcal{I}^f) \to \mathrm{ConSys}^*(\mathcal{I}^f)$ is order reflecting. Since $C^f \leq C$, we have $\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$. Therefore, a fortiori, the Leibniz operator

$\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is also order reflecting. We conclude that $\mathcal{I}$ is family reflective. ∎

Combining Lemmas 656 and 657, we get that weak family algebraizability for a π-institution is obtained, provided that its finitary companion has the same property.

Proposition 658 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}^f = \langle \mathbf{F}, C^f \rangle$ is weakly family algebraizable, then so is $\mathcal{I}$.*

Proof: Suppose $\mathcal{I}^f$ is weakly family algebraizable. Then it is protoalgebraic and family reflective. Thus, by Lemmas 656 and 657, respectively, $\mathcal{I}$ is also protoalgebraic and family reflective. Hence, by definition, $\mathcal{I}$ is weakly family algebraizable. ∎

Now we turn to the question of the same properties passing down to $\mathcal{I}^f$ from $\mathcal{I}$. In this direction, additional conditions are needed to ensure inheritance.

Given a directed family $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$ it is not, in general, the case that $\bigcup_{i \in I} T^i$ is a theory family of $\mathcal{I}$. However, as we saw in Proposition 112, this is always the case when $\mathcal{I}$ is a finitary π-institution.

Motivated by this consideration, we define the following property of the Leibniz operator:

Definition 659 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The Leibniz operator*

$$\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$$

is **continuous** *if, for every directed family $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcup_{i \in I} T^i \in \mathrm{ThFam}(\mathcal{I})$,*

$$\Omega(\bigcup_{i \in I} T^i) = \bigcup_{i \in I} \Omega(T^i).$$

It is easy to see that continuity implies protoalgebraicity.

Lemma 660 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is continuous, then $\mathcal{I}$ is protoalgebraic.*

Proof: Suppose Ω is continuous and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$. Then $T' = T \cup T'$ and we get

$$\Omega(T) \cup \Omega(T') = \Omega(T \cup T') = \Omega(T').$$

Hence, $\Omega(T) \leq \Omega(T')$ and $\mathcal{I}$ is protoalgebraic. ∎

We saw in Lemma 656 that protoalgebraicity of the finitary companion $\mathcal{I}^f$ of a π-institution $\mathcal{I}$ ensures that $\mathcal{I}$ is also protoalgebraic. We now see that working over finite signature categories and imposing the stronger property of continuity of the Leibniz operator on $\mathcal{I}$ ensure that $\mathcal{I}^f$ is protoalgebraic.

Lemma 661 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is continuous, then* $\mathcal{I}^f$ *is protoalgebraic.*

Proof: Suppose that Ω is continuous on $\mathrm{ThFam}(\mathcal{I})$. Then, by Lemma 660, $\mathcal{I}$ is protoalgebraic. To show that $\mathcal{I}^f$ is protoalgebraic, let $T, T' \in \mathrm{ThFam}(\mathcal{I}^f)$, such that $T \leq T'$. By Proposition 655, there exist directed locally finitely generated collections $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$ and $\{T'^j : j \in J\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that

$$T = \bigcup_{i\in I} T^i \quad \text{and} \quad T' = \bigcup_{j\in J} T'^j.$$

Since, by hypothesis, $T \leq T'$, we get, for all $i \in I$, $T^i \leq \bigcup_{j\in J} T'^j$. Since $\mathbf{Sign}^\flat$ is finite, T^i is locally finitely generated and $\{T'^j : j \in J\}$ is directed, there exists $j_i \in J$, such that $T^i \leq T'^{j_i}$, for all $i \in I$. Now we get

$$\begin{array}{rcll}
\Omega(T) & = & \Omega(\bigcup_{i\in I} T^i) & (T = \bigcup_{i\in I} T^i)\\
& = & \bigcup_{i\in I} \Omega(T^i) & (\Omega \text{ continuous})\\
& \leq & \bigcup_{i\in I} \Omega(T'^{j_i}) & (T^i \leq T'^{j_i} \text{ and protoalgebraicity})\\
& \leq & \bigcup_{j\in J} \Omega(T'^j) & (\text{Set Theory})\\
& = & \Omega(\bigcup_{j\in J} T'^j) & (\Omega \text{ continuous})\\
& = & \Omega(T'). & (T' = \bigcup_{j\in J} T'^j)
\end{array}$$

Thus, $\mathcal{I}^f$ is protoalgebraic. ■

Lemma 661 allows us to prove the following result, giving sufficient conditions for weak family algebraizability to be inherited by the finitary companion $\mathcal{I}^f$ from a π-institution $\mathcal{I}$.

Definition 662 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a weakly family algebraizable* π*-institution based on* $\mathbf{F}$*. The inverse* $\Omega^{-1} : \mathrm{ConSys}^*(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$ *of the Leibniz operator is* **continuous** *if, for every directed family* $\{\theta^i : i \in I\} \subseteq \mathrm{ConSys}^*(\mathcal{I})$*, such that* $\bigcup_{i\in I} \theta^i \in \mathrm{ConSys}^*(\mathcal{I})$*,*

$$\Omega^{-1}(\bigcup_{i\in I} \theta^i) = \bigcup_{i\in I} \Omega^{-1}(\theta^i).$$

Theorem 663 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a weakly family algebraizable* π*-institution based on* $\mathbf{F}$*. If*

$$\mathrm{ThFam}(\mathcal{I}) \underset{\Omega^{-1}}{\overset{\Omega}{\rightleftarrows}} \mathrm{ConSys}^*(\mathcal{I})$$

are continuous, then $\mathcal{I}^f$ *is also weakly family algebraizable.*

Proof: By Lemma 661, $\mathcal{I}^f$ is protoalgebraic. Thus, it suffices to show that $\mathcal{I}^f$ is also family injective. To this end, let $T, T' \in \mathrm{ThFam}(\mathcal{I}^f)$, such that $\Omega(T) = \Omega(T')$. By Proposition 655, there exist directed locally finitely generated collections $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$ and $\{T'^j : j \in J\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that

$$T = \bigcup_{i\in I} T^i \quad \text{and} \quad T' = \bigcup_{j\in J} T'^j.$$

Now we work as follows:

$$\begin{array}{rcl}
T & = & \bigcup_{i\in I} T^i \\
 & = & \bigcup_{i\in I} \Omega^{-1}(\Omega(T^i)) \\
 & = & \Omega^{-1}(\bigcup_{i\in I} \Omega(T^i)) \\
 & & (\bigcup_{i\in I} \Omega(T^i) \in \mathrm{ConSys}^*(\mathcal{I}) \text{ and } \Omega^{-1} \text{ continuous}) \\
 & = & \Omega^{-1}(\Omega(\bigcup_{i\in I} T^i)) \\
 & & (\bigcup_{i\in I} T^i \in \mathrm{ThFam}(\mathcal{I}) \text{ and } \Omega \text{ continuous}) \\
 & = & \Omega^{-1}(\Omega(\bigcup_{j\in J} T'^j)) \quad (\Omega(T) = \Omega(T')) \\
 & = & \Omega^{-1}(\bigcup_{j\in J} \Omega(T'^j)) \\
 & & (\bigcup_{j\in J} T'^j \in \mathrm{ThFam}(\mathcal{I}) \text{ and } \Omega \text{ continuous}) \\
 & = & \bigcup_{j\in J} \Omega^{-1}(\Omega(T'^j)) \\
 & & (\bigcup_{j\in J} \Omega(T'^j) \in \mathrm{ConSys}^*(\mathcal{I}) \text{ and } \Omega^{-1} \text{ continuous}) \\
 & = & \bigcup_{j\in J} T'^j \\
 & = & T'.
\end{array}$$

Hence, $\mathcal{I}^f$ is family injective and, thus, weakly family algebraizable. ■

9.4 Finitarity and Continuity

In this section, we establish some results pertaining to the finitarity of weakly family algebraizable π-institutions. We stay with semantic notions, using the Leibniz operator, and aim at establishing relations between various aspects of finitarity.

We begin by showing that the finitarity of a weakly family algebraizable π-institution $\mathcal{I}$ entails the continuity of the inverse Leibniz operator on the $\mathcal{I}^*$-congruence systems on $\mathcal{F}$.

Proposition 664 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a weakly family algebraizable* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is finitary, then* $\Omega^{-1} : \mathrm{ConSys}^*(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$ *is continuous.*

Proof: Suppose that $\{\theta^i : i \in I\} \subseteq \mathrm{ConSys}^*(\mathcal{I})$ is directed, such that $\bigcup_{i\in I} \theta^i \in \mathrm{ConSys}^*(\mathcal{I})$. Since $\{\theta^i : i \in I\} \subseteq \mathrm{ConSys}^*(\mathcal{I})$, there exist $T^i \in \mathrm{ThFam}(\mathcal{I})$, such that $\theta^i = \Omega(T^i)$, for all $i \in I$. Note that, since, by Theorem 296, Ω is an order isomorphism, $\{T^i : i \in I\}$ is also directed. Moreover, since $\mathcal{I}$ is finitary, we have, by Proposition 112, $\bigvee_{i\in I} T^i = \bigcup_{i\in I} T^i$. Hence, we get

$$\begin{array}{rcll}\bigcup_{i\in I}\theta^i & = & \bigvee_{i\in I}\theta^i & (\bigcup_{i\in I}\theta^i \in \mathrm{ConSys}^*(\mathcal{I}))\\ & = & \bigvee_{i\in I}\Omega(T^i) & (\theta^i = \Omega(T^i))\\ & = & \Omega(\bigvee_{i\in I} T^i) & (\Omega \text{ order isomorphism})\\ & = & \Omega(\bigcup_{i\in I} T^i). & (\mathcal{I} \text{ finitary})\end{array}$$

From this, we get

$$\Omega^{-1}(\bigcup_{i\in I}\theta^i) = \bigcup_{i\in I} T^i = \bigcup_{i\in I}\Omega^{-1}(\theta^i).$$

Hence Ω^{-1} is indeed continuous. ∎

Next, we show that the finitarity of the π-structure $\mathcal{Q}^{\mathsf{K}} = \langle \mathbf{F}, D^{\mathsf{K}}\rangle$, where $\mathsf{K} = \mathrm{AlgSys}(\mathcal{I})$, for a weakly family algebraizable π-institution $\mathcal{I}$, entails the continuity of the Leibniz operator on the collection of theory families of $\mathcal{I}$.

Proposition 665 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a weakly family algebraizable π-institution based on $\mathbf{F}$. If $\mathcal{Q}^{\mathsf{K}}$ is finitary, for $\mathsf{K} = \mathrm{AlgSys}(\mathcal{I})$, then $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is continuous.*

Proof: Suppose that $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$ is directed, such that $\bigcup_{i\in I} T^i \in \mathrm{ThFam}(\mathcal{I})$. Since $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$, $\Omega(T^i) \in \mathrm{ThFam}(\mathcal{Q}^{\mathsf{K}})$, where $\mathsf{K} = \mathrm{AlgSys}(\mathcal{I})$. Moreover, since, by Theorem 296, Ω is an order isomorphism, $\{\Omega(T^i) : i \in I\}$ is also directed. Hence, since $\mathcal{Q}^{\mathsf{K}}$ is finitary, by Proposition 112, we have $\bigvee_{i\in I}\Omega(T^i) = \bigcup_{i\in I}\Omega(T^i)$. Hence, we get

$$\begin{array}{rcll}\Omega(\bigcup_{i\in I} T^i) & = & \Omega(\bigvee_{i\in I} T^i) & (\bigcup_{i\in I} T^i \in \mathrm{ThFam}(\mathcal{I}))\\ & = & \bigvee_{i\in I}\Omega(T^i) & (\Omega \text{ order isomorphism})\\ & = & \bigcup_{i\in I}\Omega(T^i). & (\mathcal{Q}^{\mathsf{K}} \text{ finitary})\end{array}$$

Hence, Ω is continuous. ∎

We saw in Proposition 664 that finitarity of a weakly family algebraizable π-institution $\mathcal{I}$ entails the continuity of the inverse Leibniz operator. If, to the finitarity of $\mathcal{I}$, we add continuity of the Leibniz operator Ω, then finitarity of $\mathcal{Q}^{\mathsf{K}}$ is ensured, where $\mathsf{K} = \mathrm{AlgSys}(\mathcal{I})$.

Proposition 666 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a weakly family algebraizable π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is finitary, and $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is continuous, then $\mathcal{Q}^{\mathsf{K}}$, where $\mathsf{K} = \mathrm{AlgSys}(\mathcal{I})$, is also finitary.*

Proof: Assume that $\mathcal{I}$ is a finitary, weakly family algebraizable π-institution and that Ω is continuous. Let $\{\theta^i : i \in I\} \subseteq \mathrm{ConSys}^*(\mathcal{I})$ be a directed family of congruence systems. Then, there exist $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that

$\Omega(T^i) = \theta^i$, for all $i \in I$. Moreover, since $\{\theta^i : i \in I\}$ is directed and Ω is, by Theorem 296, an order isomorphism, $\{T^i : i \in I\}$ is also directed. Hence, since $\mathcal{I}$ is finitary, by Proposition 112, $\bigvee_{i\in I} T^i = \bigcup_{i\in I} T^i$. Now we have

$$\begin{array}{lll}\bigvee_{i\in I}\theta^i & = & \bigvee_{i\in I}\Omega(T^i) \quad (\Omega(T^i) = \theta^i)\\ & = & \Omega(\bigvee_{i\in I} T^i) \quad (\Omega \text{ order isomorphism})\\ & = & \Omega(\bigcup_{i\in I} T^i) \quad (\mathcal{I} \text{ finitary})\\ & = & \bigcup_{i\in I}\Omega(T^i) \quad (\Omega \text{ continuous})\\ & = & \bigcup_{i\in I}\theta^i. \quad (\Omega(T^i) = \theta^i)\end{array}$$

Thus, $\mathrm{ConSys}^*(\mathcal{I})$ is closed under directed unions and, therefore, by Proposition 112, $\mathcal{Q}^{\mathsf{K}}$ is finitary. ∎

Dually, we have seen in Proposition 665 that if $\mathcal{Q}^{\mathsf{K}}$ is finitary, where $\mathsf{K} = \mathrm{AlgSys}(\mathcal{I})$, then Ω is continuous. If, to the finitarity of $\mathcal{Q}^{\mathsf{K}}$, we add the continuity of Ω^{-1}, then, finitarity of $\mathcal{I}$ is ensured.

Proposition 667 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a weakly family algebraizable* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{Q}^{\mathsf{K}}$ *is finitary, where* $\mathsf{K} = \mathrm{AlgSys}^*(\mathcal{I})$*, and* $\Omega^{-1} : \mathrm{ConSys}^*(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$ *is continuous, then* $\mathcal{I}$ *is also finitary.*

Proof: Assume that $\mathcal{I}$ is a weakly family algebraizable π-institution, such that $\mathcal{Q}^{\mathsf{K}}$, $\mathsf{K} = \mathrm{AlgSys}(\mathcal{I})$, is finitary, and that Ω^{-1} is continuous. Let $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$ be a directed collection of theory families. Then, since, by Theorem 296, Ω is an order isomorphism, $\{\Omega(T^i) : i \in I\}$ is a directed family of congruence systems. Since $\mathcal{Q}^{\mathsf{K}}$ is finitary, by Proposition 112, $\bigvee_{i\in I}\Omega(T^i) = \bigcup_{i\in I}\Omega(T^i)$. Now we have

$$\begin{array}{lll}\bigvee_{i\in I} T^i & = & \Omega^{-1}(\Omega(\bigvee_{i\in I} T^i)) \quad (\Omega \text{ isomorphism})\\ & = & \Omega^{-1}(\bigvee_{i\in I}\Omega(T^i)) \quad (\Omega \text{ order isomorphism})\\ & = & \Omega^{-1}(\bigcup_{i\in I}\Omega(T^i)) \quad (\mathcal{Q}^{\mathsf{K}} \text{ finitary})\\ & = & \bigcup_{i\in I}\Omega^{-1}(\Omega(T^i)) \quad (\Omega^{-1} \text{ continuous})\\ & = & \bigcup_{i\in I} T^i. \quad (\Omega \text{ isomorphism})\end{array}$$

Thus, $\mathrm{ThFam}(\mathcal{I})$ is closed under directed unions and, hence, by Proposition 112, $\mathcal{I}$ is finitary. ∎

Gathering together all conclusions drawn during the studies undertaken in this section, we get the following

Corollary 668 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a weakly family algebraizable* π*-institution based on* $\mathbf{F}$ *and set* $\mathsf{K} = \mathrm{AlgSys}(\mathcal{I})$*.*

(a) *If both* Ω *and* Ω^{-1} *are continuous, then* $\mathcal{I}$ *is finitary if and only if* $\mathcal{Q}^{\mathsf{K}}$ *is finitary.*

(b) *If $\mathcal{I}$ is finitary, then $\mathcal{Q}^{\mathsf{K}}$ is finitary if and only if* $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is continuous.*

(c) *If $\mathcal{Q}^{\mathsf{K}}$ is finitary, then $\mathcal{I}$ is finitary if and only if* $\Omega^{-1} : \mathrm{ConSys}^*(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$ *is continuous.*

In each of (a)-(c), if the two equivalent conditions hold, then all four finitarity conditions hold.

Proof:

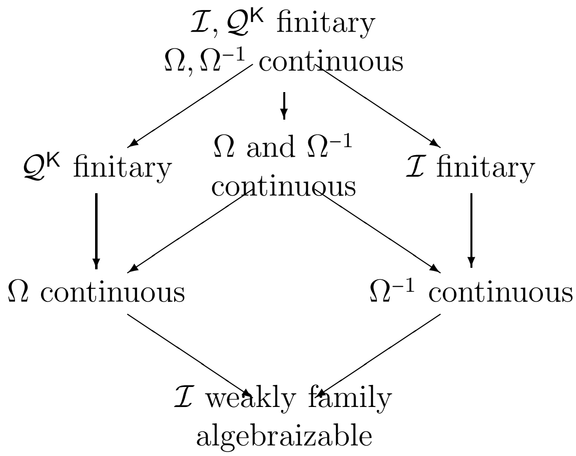
(a) Suppose Ω and Ω^{-1} are continuous. Then, if $\mathcal{I}$ is finitary, $\mathcal{Q}^{\mathsf{K}}$ is finitary, by Proposition 666, and if $\mathcal{Q}^{\mathsf{K}}$ is finitary, then $\mathcal{I}$ is finitary, by Proposition 667.

(b) Suppose that $\mathcal{I}$ is finitary. Then, by Proposition 664, Ω^{-1} is continuous. If $\mathcal{Q}^{\mathsf{K}}$ is finitary, then Ω is continuous, by Proposition 665. On the other hand, if Ω is continuous, then $\mathcal{Q}^{\mathsf{K}}$ is finitary, by Proposition 666.

(c) Suppose that $\mathcal{Q}^{\mathsf{K}}$ is finitary. Then, by Proposition 665, Ω is continuous. If $\mathcal{I}$ is finitary, then Ω^{-1} is continuous, by Proposition 664. On the other hand, if Ω^{-1} is continuous, then $\mathcal{I}$ is finitary, by Proposition 667.

We turn to the last statement. For Part (a), assume that $\mathcal{I}$ and $\mathcal{Q}^{\mathsf{K}}$ are finitary. Then, we get, by Propositions 664 and 665, that Ω and Ω^{-1} are continuous. For Part (b), if $\mathcal{I}$ is finitary, $\mathcal{Q}^{\mathsf{K}}$ is finitary and $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is continuous, then, by Proposition 664, Ω^{-1} is continuous. A similar reasoning applies to Part (c). ■

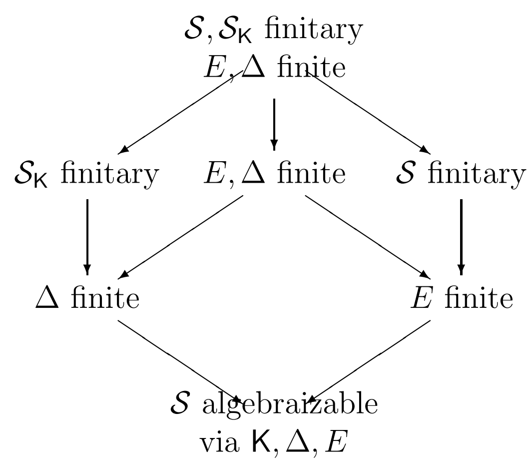
We summarize our conclusions in the accompanying diagram. At the bottom is situated the underlying assumption (holding at all levels) that $\mathcal{I}$ is weakly family algebraizable. The top consists of the situation in which all four finitarity conditions hold, i.e., both $\mathcal{I}$ and $\mathcal{Q}^{\mathsf{K}}$, for $\mathsf{K} = \mathrm{AlgSys}(\mathcal{I})$, are finitary and both Ω and Ω^{-1} are continuous. The intermediate classes constitute the various different intermediate possibilities that were detailed previously.



This diagram is a modified version of the one shown in Figure 3 of Section 3.4 of [86]. Instead of dealing with sentential logics, it concerns the more general case of π-institutions and, instead of being expressed in terms of syntactic constructs (which in the case of sentential logics turn out to be equivalent), it relies entirely on corresponding properties of the Leibniz operator and its inverse. These analogies and similarities will be exploited in the remainder of the chapter to obtain examples that separate the various classes involved in this hierarchy by adapting appropriate examples that serve an analogous purpose in the framework of sentential logics.

9.5 The Case of Sentential Logics

In the Introduction and, briefly, in concluding Section 9.4, we pointed out that the semantic finitarity hierarchy of π-institutions reflects the finitarity hierarchy presented in Section 3.4 of [86], which is duplicated below.



In [86], Font presents examples of sentential logics to separate the classes in this hierarchy. In this section, we revisit some of them in detail and, then, rely on them in Section 9.6 to separate the classes of π-institutions shown in the hierarchy of Section 9.4. We provide, now, a brief overview of the examples chosen and what each accomplishes, before describing them in full detail.

The first example we present is Łukasiewicz's infinite valued logic $Ł_\infty$. This logic is introduced in Example 1.12 of Section 1.2 of [86]. In Example 1.15, in the same section, in conjunction with Exercise 1.26 of [86], it is shown that it is not finitary. On the other hand, in Example 3.41 of Section 3.4 of [86], it is shown that it is finitely algebraizable, with defining equations $E(x) = \{x \approx \top\}$ and equivalence formulas $\Delta(x,y) = \{x \to y, y \to x\}$, both finite. Thus, $Ł_\infty$ serves in showing that the three vertical arrows in the preceding diagram represent proper inclusions. As Font points out in [86], more information about Ł$_\infty$ and similar logics may be found in specialized references, such as [61, 65, 56] and Chapters 1, 2 and 6 of [83].

The second example that Font presents in Section 3.4 of [86] is Herrmann's Last Judgement Logic $\mathcal{L}J$ [53]. This is a finitary logic which is syntactically defined and which is algebraizable with a single defining equation $E(x) = \{\neg x \approx \neg(x \to x)\}$ and an infinite set of equivalence formulas $\Delta(x,y) = \{\Box^n(x \to y) \approx \Box^n(y \to x) : n \geq 0\}$. So this logic serves to separate all classes related by southeast arrows in the diagram. The same separations may be attained by a logic introduced by Dellunde in [48]. Dellunde's logic is actually the logic we opt to present as our second example. Here, we shall name it Dellunde's Logic $\mathcal{D}$.

The last example, presented in Section 3.4 of [86] is a logic introduced by Raftery in [82]. This is a semantically defined logic which is not finitary but is algebraizable via a finitary equational consequence, with an infinite set of defining equations and a single equivalence formula. So this logic, which we shall refer to as Raftery's Logic $\mathcal{R}$, shows that all southwest arrows in the diagram represent proper inclusions. This will be the third, and last example, presented in detail in this section.

9.5.1 Łukasiewicz's Infinite Valued Logic

We begin with Łukasiewicz's infinite valued logic Ł$_\infty$. Define an algebra $\mathbf{A} = \langle A, \wedge, \vee, \to, \neg\rangle$ as follows:

- The universe A is the unit interval $A = [0,1]$.
- The operations are defined, for all $a, b \in A$, by:
  - $a \wedge b = \min\{a, b\}$;
  - $a \vee b = \max\{a, b\}$;
  - $a \to b = \min\{1, 1 - a + b\} = \begin{cases} 1, & \text{if } a \leq b \\ 1 - a + b, & \text{if } a > b \end{cases}$;
  - $\neg a = 1 - a$.

Łukasiewicz's infinite valued logic Ł$_\infty = \langle \mathcal{L}, \vdash_\infty\rangle$ is the sentential logic over the language $\mathcal{L} = \{\wedge, \vee, \to, \neg\}$ defined, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, by

$$\Gamma \vdash_\infty \varphi \quad \text{iff} \quad \begin{array}{l} \text{for every homomorphism } h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \\ h(\Gamma) \subseteq \{1\} \quad \text{implies} \quad h(\varphi) = 1. \end{array}$$

Over the same language $\mathcal{L}$, we define the derived binary connective $\oplus$ by setting

$$x \oplus y := \neg x \to y.$$

This operation satisfies some key properties.

Lemma 669 *For all $a, b \in A$,*

$$a \oplus b = \begin{cases} 1, & \textit{if } a + b \geq 1, \\ a + b, & \textit{if } a + b < 1 \end{cases}.$$

Proof: Let $a, b \in A$. We perform a straightforward calculation using the definitions of the operations in $\mathbf{A}$.

$$\begin{array}{rcl} a \oplus b & = & \neg a \to b = (1 - a) \to b \\ & = & \left\{ \begin{array}{ll} 1, & \text{if } 1 - a \leq b \\ 1 - (1 - a) + b, & \text{if } 1 - a > b \end{array} \right\} = \left\{ \begin{array}{ll} 1, & \text{if } a + b \geq 1 \\ a + b, & \text{if } a + b < 1 \end{array} \right. . \end{array}$$

■

Lemma 669 helps us establish the following

Lemma 670 *For all $a \in A$ and all $n \geq 2$,*

$$\underbrace{a \oplus \cdots \oplus a}_{n} = \begin{cases} 1, & \textit{if } a \geq \frac{1}{n} \\ na, & \textit{if } a < \frac{1}{n} \end{cases}.$$

Proof: We use induction on n. Lemma 669 guarantees that the formula holds for $n = 2$. Assume that the formula holds for some $n \geq 2$. Then, we get

$$\begin{array}{rcl} \underbrace{a \oplus \cdots \oplus a}_{n+1} & = & \underbrace{(a \oplus \cdots \oplus a)}_{n} \oplus a = \left\{ \begin{array}{ll} 1 \oplus a, & \text{if } a \geq \frac{1}{n} \\ (na) \oplus a, & \text{if } a < \frac{1}{n} \end{array} \right. \\ & = & \left\{ \begin{array}{ll} 1, & \text{if } a \geq \frac{1}{n} \\ 1, & \text{if } (n+1)a \geq 1 \\ (n+1)a, & \text{if } (n+1)a < 1 \end{array} \right\} = \left\{ \begin{array}{ll} 1, & \text{if } a \geq \frac{1}{n+1} \\ (n+1)a, & \text{if } a < \frac{1}{n+1} \end{array} \right. . \end{array}$$

■

We now show that $Ł_\infty$ is not finitary.

Theorem 671 *Łukasiewicz's infinite valued logic $Ł_\infty$ is not finitary.*

Proof: We set

$$\Phi = \{\underbrace{(x \oplus \cdots \oplus x)}_{n} \to y : n \geq 2\} \cup \{\neg x \to y\}.$$

We show that $\Phi \vdash_\infty y$, but $\Phi_0 \nvdash_\infty y$, for any finite $\Phi_0 \subseteq \Phi$.

Suppose $h : \mathbf{Fm}_\mathcal{L}(V) \to \mathbf{A}$ is such that

$$h(\neg y \to x) = 1 \quad \text{and} \quad h(\underbrace{(x \oplus \cdots \oplus x)}_{n} \to y) = 1, \text{ for all } n \geq 2.$$

By definition, these imply that $h(x) + h(y) \geq 1$ and $h(\underbrace{x \oplus \cdots \oplus x}_{n}) \leq h(y)$, for all $n \geq 2$.

- If $h(x) = 0$, then, by the first inequality, $h(y) = 1$.
- If $h(x) \neq 0$, then $h(x) \geq \frac{1}{n}$, for some $n > 0$. In this case, $h(\underbrace{x \oplus \cdots \oplus x}_{n}) = 1$, whence, by the second inequality, $h(y) = 1$.

Since, in either case, $h(y) = 1$, we get that $\Phi \vdash_\infty y$.

To refute finitarity, assume, towards obtaining a contradiction, that, for some finite $\Phi_0 \subseteq \Phi$, $\Phi_0 \vdash_\infty y$. Then, there exists $k \geq 2$, such that

$$\{(\underbrace{x \oplus \cdots \oplus x}_{n}) \to y : 2 \leq n \leq k\} \cup \{\neg x \to y\} \vdash_\infty y.$$

Consider a homomorphism $h : \mathbf{Fm}_\mathcal{L}(V) \to \mathbf{A}$, such that

$$h(x) = \frac{1}{k+1} \quad \text{and} \quad h(y) = \frac{k}{k+1}.$$

Then, we have

$$\begin{array}{l} h(\neg x \to y) = (1 - h(x)) \to y = \frac{k}{k+1} \to \frac{k}{k+1} = 1; \\ h((\underbrace{x \oplus \cdots \oplus x}_{n}) \to y) = h((\underbrace{x \oplus \cdots \oplus x}_{n})) \to h(y) \stackrel{n<k+1}{=} \frac{n}{k+1} \to \frac{k}{k+1} \stackrel{n\leq k}{=} 1. \end{array}$$

On the other hand, $h(y) = \frac{k}{k+1} \neq 1$. Therefore, $\Phi_0 \nvdash_\infty y$, contrary to hypothesis. We conclude that Ł$_\infty$ is not a finitary sentential logic. ■

Our final result pertaining to this logic is that it is algebraizable, via the class $\{\mathbf{A}\}$, with defining equation $E(x) = \{x \approx \top\}$, where $\top := x \to x$ (interpreted as 1), and equivalence formulas $\Delta(x, y) = \{x \to y, y \to x\}$. In the proof, we will rely on the general theory of algebraizable logics (see, e.g., Sections 3.2 and 6.5 of [86] or Section 4.5 of [64]).

Theorem 672 *Lukasiewicz's infinite value logic Ł$_\infty$ is algebraizable via the class $\{\mathbf{A}\}$, with defining equations $E(x) = \{x \approx \top\}$ and equivalence formulas $\Delta(x, y) = \{x \to y, y \to x\}$.*

Proof: According to the general theory of algebraizability, it suffices to show that, for all $\Gamma \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}_\mathcal{L}(V)$,

$$\begin{array}{c} \Gamma \vdash_\infty \varphi \quad \text{iff} \quad \{\gamma \approx \top : \gamma \in \Gamma\} \vDash_\mathcal{A} \varphi \approx \top, \\ \varphi \approx \psi \dashv\vDash_\mathbf{A} \{\varphi \to \psi \approx \top, \psi \to \varphi \approx \top\}. \end{array}$$

For the first, note that

$$\begin{array}{lll} \Gamma \vdash_\infty \varphi & \text{iff} & (\forall h : \mathbf{Fm}_\mathcal{L}(V) \to \mathbf{A})(h(\Gamma) \subseteq \{1\} \text{ implies } h(\varphi) = 1) \\ & \text{iff} & (\forall h : \mathbf{Fm}_\mathcal{L}(V) \to \mathbf{A})((\forall \gamma \in \Gamma)(h(\gamma) = 1) \text{ implies } h(\varphi) = 1) \\ & \text{iff} & \{\gamma \approx \top : \gamma \in \Gamma\} \vDash_\mathbf{A} \varphi \approx \top. \end{array}$$

Finally, noting that, for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$, we have $h(\varphi \to \psi) = 1$ iff $h(\varphi) \leq h(\psi)$, we get

$$h(\varphi) = h(\psi) \quad \text{iff} \quad h(\varphi \to \psi) = h(\psi \to \varphi) = 1,$$

whence $\varphi \approx \psi \mathrel{=\!|\!\models_{\mathbf{A}}} \{\varphi \to \psi \approx \top, \psi \to \varphi \approx \top\}$. The conclusion now follows. ■

9.5.2 Dellunde's Logic

We switch to the second example of the section, a logic due to Dellunde [48].

Let $\mathcal{L} = \{\leftrightarrow, \Box\}$ be the algebraic language consisting of a binary operation $\leftrightarrow$ and a unary operation $\Box$. **Dellunde's logic** $\mathcal{D} = \langle \mathcal{L}, \vdash_D \rangle$ is the logic over the language $\mathcal{L}$ defined by the following Hilbert style calculus, where x, y and x_1, y_1, x_2, y_2 denote distinct variables:

(1) $\vdash_D x \leftrightarrow x$;

(2) $x, x \leftrightarrow y \vdash_D y$;

(3) $x, y \vdash_D \Box^n x \leftrightarrow \Box^n y$, for all $n \in \omega$;

(4) $x_1 \leftrightarrow y_1, x_2 \leftrightarrow y_2 \vdash_D \Box^n(x_1 \leftrightarrow x_2) \leftrightarrow \Box^n(y_1 \leftrightarrow y_2)$, for all $n \in \omega$.

Since $\mathcal{D}$ is defined via a Hilbert calculus, it is finitary. We further define

$$\Delta(x, y) = \{\Box^n x \leftrightarrow \Box^n y : n \in \omega\}.$$

Dellunde shows that $\mathcal{D}$ is 1-equivalential, which implies that it is regularly algebraizable [53].

Theorem 673 *Dellunde's logic $\mathcal{D} = \langle \mathcal{L}, \vdash_D \rangle$ is regularly algebraizable.*

Proof: It suffices to show that, for distinct variables x, y, x_1, y_1, x_2, y_2, the following hold:

(R) $\vdash_D \Delta(x, x)$;

(MP) $x, \Delta(x, y) \vdash_D y$;

(RP) $\Delta(x, y) \vdash_D \Delta(\Box x, \Box y)$ and

$\Delta(x_1, y_1), \Delta(x_2, y_2) \vdash_D \Delta(x_1 \leftrightarrow x_2, y_1 \leftrightarrow y_2)$;

(RG) $x, y \vdash_D \Delta(x, y)$.

By (1), we have $\vdash_D x \leftrightarrow x$. By structurality, $\vdash_D \Box^n x \leftrightarrow \Box^n x$, for all $n \in \omega$. That is, $\vdash_D \Delta(x,x)$. So (R) holds.

Rule (2) assures that $x, x \leftrightarrow y \vdash_D y$. Now, note that $x \leftrightarrow y \in \Delta(x,y)$ and apply monotonicity of entailment to get $x, \Delta(x,y) \vdash_D y$. That is, (MP) holds.

The first rule in (RP) is a consequence of monotonicity, since

$$\begin{array}{rcl}\Delta(\Box x, \Box y) & = & \{\Box^n \Box x \leftrightarrow \Box^n \Box y : n \in \omega\}\\ & = & \{\Box^n x \leftrightarrow \Box^n y : n \geq 1\}\\ & \subseteq & \{\Box^n x \leftrightarrow \Box^n y : n \in \omega\}\\ & = & \Delta(x,y).\end{array}$$

For the second rule in (RP), note that (4) gives $x_1 \leftrightarrow y_1, x_2 \leftrightarrow y_2 \vdash_D \Delta(x_1 \leftrightarrow x_2, y_1 \leftrightarrow y_2)$. On the other hand, $x_1 \leftrightarrow y_1 \in \Delta(x_1,y_1)$ and $x_2 \leftrightarrow y_2 \in \Delta(x_2,y_2)$. Therefore, we conclude that $\Delta(x_1,y_1), \Delta(x_2,y_2) \vdash_D \Delta(x_1 \leftrightarrow x_2, y_1 \leftrightarrow y_2)$, whence (RP) holds.

Finally, note that, by (3), (RG) holds.

We conclude that $\mathcal{D}$ is regularly algebraizable, with a singleton set of defining equations $E(x) = \{x \approx \top\}$, where $\top := x \leftrightarrow x$ is a unary term interpreted as the unique element of the $\mathcal{D}$-filter of any reduced $\mathcal{D}$-matrix, and set of equivalence formulas $\Delta(x,y)$. ■

Finally, Dellunde shows that $\mathcal{D}$ is not finitely equivalential, i.e., that there does not exist a finite subset $\Delta_0 \subseteq \Delta$ that can also serve as a set of equivalence formulas. In relation to this, see Lemma 3.36 in Section 3.4 of [86].

Theorem 674 *Dellunde's logic $\mathcal{D} = \langle \mathcal{L}, \vdash_D \rangle$ is not finitely equivalential, i.e., there exists no finite $\Delta_0 \subseteq \Delta$ which is also a set of equivalence formulas for $\mathcal{D}$.*

Proof: Assume, towards a contradiction, that there exists finite $\Delta_0 \subseteq \Delta$, which serves as a set of equivalence formulas for $\mathcal{D}$. Then, there exists a maximum $m \in \omega$, such that $\Box^m x \leftrightarrow \Box^m y \in \Delta_0$. To obtain a contradiction, we construct a $\mathcal{D}$-matrix $\mathfrak{A} = \langle \mathbf{A}, F \rangle$ and choose elements $c, d \in A$, such that

$$\Delta_0^{\mathbf{A}}(c,d) \subseteq F \quad \text{but} \quad \langle c,d \rangle \notin \Omega_{\mathbf{A}}(F).$$

As a preparatory step in defining the $\mathcal{L}$-algebra $\mathbf{A}$, we define on $\omega \times \omega$ the following equivalence relation:

$$R = \mathrm{Id}_{\omega\times\omega} \cup \{\langle \langle i,j \rangle, \langle k,\ell \rangle \rangle : i = k, i < j, k < \ell\}.$$

The algebra $\mathbf{A} = \langle A, \leftrightarrow^{\mathbf{A}}, \Box^{\mathbf{A}} \rangle$ is defined as follows:

- $A = \omega \times \omega$;
- The operations are defined, for all $i, j \in \omega$ and all $a, b \in \omega \times \omega$,

– $\Box^{\mathbf{A}}(\langle i,j\rangle) = \langle i+1,j\rangle$;

– $\leftrightarrow^{\mathbf{A}} (a,b) = \begin{cases} \langle 1,0\rangle, & \text{if } \langle a,b\rangle \in R \\ \langle 0,0\rangle, & \text{if } \langle a,b\rangle \notin R \end{cases}$.

The filter $F = \{\langle 1,0\rangle\}$ and the elements $c,d \in A$ are chosen as $c = \langle 0,m+1\rangle$ and $d = \langle 0,m+2\rangle$, where, recall that, $m = \max\{k : \Box^k x \leftrightarrow \Box^k y \in \Delta_0\}$. It suffices, now, to show the following:

(a) $\mathfrak{A} = \langle \mathbf{A}, F\rangle$ is a $\mathcal{D}$-matrix, i.e., F is closed under all $\mathcal{D}$-rules;

(b) $\Delta_0^{\mathbf{A}}(c,d) \subseteq F$;

(c) $\langle c,d\rangle \notin \Omega_{\mathbf{A}}(F)$.

For (a), let $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$ be arbitrary. Then:

- For all $n \in \omega$, $h(\Box^n x \leftrightarrow \Box^n x) = h(\Box^n x) \leftrightarrow^{\mathbf{A}} h(\Box^n x) = \langle 1,0\rangle \in F$.
- Suppose $h(x) = \langle 1,0\rangle$ and $h(x \leftrightarrow y) = \langle 1,0\rangle$. So $h(x) = \langle 1,0\rangle$ and $h(x) \leftrightarrow^{\mathbf{A}} h(y) = \langle 1,0\rangle$. Since $h(x) = \langle 1,0\rangle$ and $1 \nleq 0$, we get $h(x) = h(y)$. So, $h(y) = \langle 1,0\rangle \in F$.
- If $h(x) = h(y) = \langle 1,0\rangle$, then $h(\Box^n x) = \Box^{\mathbf{A}^n} h(x) = \Box^{\mathbf{A}^n} h(y) = h(\Box^n y)$, whence, $h(\Box^n x \leftrightarrow \Box^n y) = h(\Box^n x) \leftrightarrow^{\mathbf{A}} h(\Box^n y) = \langle 1,0\rangle \in F$.
- If $h(x_1 \leftrightarrow y_1) = h(x_2 \leftrightarrow y_2) = \langle 1,0\rangle$, then, since R is an equivalence relation, it follows from

$$\begin{array}{ccc} h(x_1) & \text{—}R\text{—} & h(y_1) \\ \vdots & & \vdots \\ R & & R \\ \vdots & & \vdots \\ h(x_2) & \text{—}R\text{—} & h(y_2) \end{array}$$

that $\langle h(x_1),h(x_2)\rangle \in R$ iff $\langle h(y_1),h(y_2)\rangle \in R$, i.e., that

$$h(x_1 \leftrightarrow x_2) = h(y_1 \leftrightarrow y_2) = \begin{cases} \langle 1,0\rangle, & \text{if } \langle h(x_1),h(x_2)\rangle \in R \\ \langle 0,0\rangle, & \text{if } \langle h(x_1),h(x_2)\rangle \notin R \end{cases}.$$

Then, we obtain, for all $n \in \omega$, $h(\Box^n(x_1 \leftrightarrow x_2)) = h(\Box^n(y_1 \leftrightarrow y_2))$, which yields that $h(\Box^n(x_1 \leftrightarrow x_2) \leftrightarrow \Box^n(y_1 \leftrightarrow y_2)) = \langle 1,0\rangle$.

Thus, $\mathfrak{A}$ is indeed a $\mathcal{D}$-matrix.

For (b), suppose $\Box^k x \leftrightarrow \Box^k y \in \Delta_0$, i.e., $k \leq m$. Then, we have

$$\begin{aligned} \Box^{\mathbf{A}^k} c \leftrightarrow^{\mathbf{A}} \Box^{\mathbf{A}^k} d &= \Box^{\mathbf{A}^k}\langle 0,m+1\rangle \leftrightarrow^{\mathbf{A}} \Box^{\mathbf{A}^k}\langle 0,m+2\rangle \\ &= \langle k,m+1\rangle \leftrightarrow^{\mathbf{A}} \langle k,m+2\rangle \\ &= \langle 1,0\rangle. \end{aligned}$$

So, $\Delta_0^{\mathbf{A}}(c,d) \subseteq F$.

Finally, for (c), observe that

$$\begin{array}{rcl} \Box^{\mathbf{A}^{m+1}} c \leftrightarrow^{\mathbf{A}} \Box^{\mathbf{A}^{m+1}} d & = & \Box^{\mathbf{A}^{m+1}} \langle 0, m+1\rangle \leftrightarrow^{\mathbf{A}} \Box^{\mathbf{A}^{m+1}} \langle 0, m+2\rangle \\ & = & \langle m+1, m+1\rangle \leftrightarrow^{\mathbf{A}} \langle m+1, m+2\rangle \\ & = & \langle 0,0\rangle \notin F. \end{array}$$

Therefore, by Theorem 673 and the general theory of algebraizability, $\langle c,d\rangle \notin \Omega_{\mathbf{A}}(F)$.

The conjunction of assertions (a), (b) and (c) shows that Δ_0 is not a set of equivalence formulas for $\mathcal{D}$ and, consequently, taking into account the finitarity of $\mathcal{D}$ and Lemma 3.36 of Section 3.4 of [86], $\mathcal{D}$ does not possess a finite set of equivalence formulas. ■

9.5.3 Raftery's Logic

Finally, we turn to a detailed description of Raftery's logic [82].

The construction unfolds in several stages. It starts with the set

$$B = \{0,1\} \cup (\{0,1\} \times \{0,1\})^\omega$$

consisting of the bits 0 and 1 and of infinite sequences of pairs of bits. On this set B, three unary operations π_1, π_2 and $\Diamond$ are defined by setting, for all $b \in \{0,1\}$ and all $\langle\langle b_0, b_0'\rangle, \langle b_1, b_1'\rangle, \ldots\rangle \in (\{0,1\} \times \{0,1\})^\omega$,

$$\begin{array}{rl} \pi_1(b) = b, & \pi_1(\langle\langle b_0, b_0'\rangle, \langle b_1, b_1'\rangle, \ldots\rangle) = b_0; \\ \pi_2(b) = b, & \pi_2(\langle\langle b_0, b_0'\rangle, \langle b_1, b_1'\rangle, \ldots\rangle) = b_0'; \\ \Diamond b = b, & \Diamond(\langle\langle b_0, b_0'\rangle, \langle b_1, b_1'\rangle, \ldots\rangle) = \langle\langle b_1, b_1'\rangle, \langle b_2, b_2'\rangle, \ldots\rangle. \end{array}$$

In the next stage, Raftery constructs the universe A of the algebra $\mathbf{A}$ that forms the algebraic reduct of the logical matrix used to define Raftery's logic. This is accomplished by closing under the formation of ordered pairs.

$$\begin{array}{rcl} B^{[1]} & = & B; \\ B^{[n]} & = & (\bigcup_{0<m<n} B^{[m]}) \times (\bigcup_{0<m<n} B^{[m]}), \quad n > 1; \end{array}$$

and, finally,

$$A = \bigcup_{0<n\in\omega} B^{[n]}.$$

First, observe that no element of B is an ordered pair and that every element of $A - B$ is an ordered pair.

To define the algebra $\mathbf{A}$, the operations introduced previously on B are extended on A. We set, for all $\langle a, a'\rangle \in A - B$,

$$\pi_1(\langle a, a'\rangle) = a, \qquad \pi_2(\langle a, a'\rangle) = a', \qquad \Diamond\langle a, a'\rangle = \langle a, a'\rangle.$$

To complete the specification of $\mathbf{A}$, we add a "pair forming" binary operation $\leftrightarrow$, defined, for all $a, a' \in A$, by

$$a \leftrightarrow a' = \langle a, a' \rangle.$$

So the algebra used to specify Raftery's logic is $\mathbf{A} = \langle A, \leftrightarrow, \pi_1, \pi_2, \lozenge \rangle$. It is easy to check that $\mathbf{A}$ satisfies the equations

$$\begin{array}{rcl} \pi_1(x \leftrightarrow y) & \approx & x, \\ \pi_2(x \leftrightarrow y) & \approx & y, \\ \lozenge(x \leftrightarrow y) & \approx & x \leftrightarrow y. \end{array}$$

We now define two logical systems semantically. The first is defined via a logical matrix with underlying algebra $\mathbf{A}$. We define the matrix $\mathfrak{A} = \langle \mathbf{A}, D \rangle$, where D is the set of so-called "diagonal elements" of A, i.e., the elements

- 0 and 1;
- $\langle \langle b_0, b_0 \rangle, \langle b_1, b_1 \rangle, \dots \rangle$, for $b_0, b_1, \dots \in \{0, 1\}$;
- $\langle a, a \rangle$, for $a \in A$.

This matrix $\mathfrak{A}$ specifies the logic $\mathcal{S}_{\mathfrak{A}} = \langle \mathcal{L}, \vdash_{\mathfrak{A}} \rangle$ in the standard way, i.e., for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\begin{array}{lll} \Gamma \vdash_{\mathfrak{A}} \varphi & \text{iff} & \text{for every } h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \\ & & h(\Gamma) \subseteq D \quad \text{implies} \quad h(\varphi) \in D. \end{array}$$

The second logical system is defined using a variety V of $\mathcal{L}$-algebras, for $\mathcal{L} = \{\leftrightarrow, \pi_1, \pi_2, \lozenge\}$, namely the variety axiomatized by the three equations

$$\begin{array}{rcl} \pi_1(x \leftrightarrow y) & \approx & x, \\ \pi_2(x \leftrightarrow y) & \approx & y, \\ \lozenge(x \leftrightarrow y) & \approx & x \leftrightarrow y. \end{array}$$

We set $\delta_i(x) = \pi_1(\lozenge^i x)$ and $\varepsilon_i(x) = \pi_2(\lozenge^i x)$ and define **Raftery's logic** $\mathcal{R} = \langle \mathcal{L}, \vdash_R \rangle$ by setting, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Gamma \vdash_{\mathcal{R}} \varphi \quad \text{iff} \quad (\delta \approx \varepsilon)(\Gamma) \vDash_V (\delta \approx \varepsilon)(\phi),$$

i.e., $\Gamma \vdash_{\mathcal{R}} \varphi$ iff, for every $\mathbf{A} \in V$, all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$ and all $j \in \omega$,

$$\begin{array}{ll} \delta_i^{\mathbf{A}}(h(\gamma)) = \varepsilon_i^{\mathbf{A}}(h(\gamma)), & \text{for all } i \in \omega,\ \gamma \in \Gamma, \\ & \text{implies} \quad \delta_j^{\mathbf{A}}(h(\varphi)) = \varepsilon_j^{\mathbf{A}}(h(\varphi)). \end{array}$$

The first result relating the logics $\mathcal{S}_{\mathfrak{A}}$ and $\mathcal{R}$ asserts that the latter is a weakening of the former.

Lemma 675 *Raftery's logic* $\mathcal{R} = \langle \mathcal{L}, \vdash_R \rangle$ *is a weakening of* $\mathcal{S}_{\mathfrak{A}} = \langle \mathcal{L}, \vdash_{\mathfrak{A}} \rangle$.

Proof: Since, as remarked previously, $\mathbf{A}$ satisfies the three equations axiomatizing the variety V, we get that $\mathbf{A} \in V$. Consequently, it suffices to show that for all $a \in A$,

$$a \in D \quad \text{iff} \quad \delta_i^{\mathbf{A}}(a) = \epsilon_i^{\mathbf{A}}(a), \text{ for all } i \in \omega.$$

Suppose, first, that $a \in D$.

- If $a \in \{0,1\}$, then $\delta_i^{\mathbf{A}}(a) = \pi_1^{\mathbf{A}}(a)(\Diamond^{\mathbf{A}^i} a) = \pi_1^{\mathbf{A}}(a) = a = \pi_2^{\mathbf{A}}(a) = \pi_2^{\mathbf{A}}(a)(\Diamond^{\mathbf{A}^i} a) = \varepsilon_i^{\mathbf{A}}(a)$.
- If $a = \langle\langle b_0, b_0\rangle, \langle b_1, b_1\rangle, \dots\rangle$, then
$$\begin{aligned} &\delta_i^{\mathbf{A}}(a) = \pi_1^{\mathbf{A}}(\Diamond^{\mathbf{A}^i} a) = \pi_1^{\mathbf{A}}(\langle\langle b_i, b_i\rangle, \langle b_{i+1}, b_{i+1}\rangle, \dots\rangle) = b_i \\ &= \pi_2^{\mathbf{A}}(\langle\langle b_i, b_i\rangle, \langle b_{i+1}, b_{i+1}\rangle, \dots\rangle) = \pi_2^{\mathbf{A}}(\Diamond^{\mathbf{A}^i} a) = \varepsilon_i^{\mathbf{A}}(a).\end{aligned}$$
- If $a = \langle a', a'\rangle$, with $a' \in A$, then
$$\begin{aligned} &\delta_i^{\mathbf{A}}(\langle a', a'\rangle) = \pi_1^{\mathbf{A}}(\Diamond^{\mathbf{A}^i}\langle a', a'\rangle) = \pi_1^{\mathbf{A}}(\langle a', a'\rangle) = a' \\ &= \pi_2^{\mathbf{A}}(\langle a', a'\rangle) = \pi_2^{\mathbf{A}}(\Diamond^{\mathbf{A}^i}\langle a', a'\rangle) = \varepsilon_i^{\mathbf{A}}(\langle a', a'\rangle).\end{aligned}$$

Assume, conversely, that $\delta_i^{\mathbf{A}}(a) = \varepsilon_i^{\mathbf{A}}(a)$, for all $i \in \omega$. This means $\pi_1^{\mathbf{A}}(\Diamond^{\mathbf{A}^i} a) = \pi_2^{\mathbf{A}}(\Diamond^{\mathbf{A}^i} a)$, for all $i \in \omega$. if $a = 0$ or $a = 1$, there is nothing to prove. If $a = \langle\langle b_0, b_0'\rangle, \langle b_1, b_1'\rangle, \dots\rangle$, then the i-th equation gives $b_i = b_i'$. So we conclude that $a \in D$. Finally, if $a = \langle a', a''\rangle$, for some $a', a'' \in A$, then the equations ensure that $a' = a''$ and, therefore, $a = \langle a', a''\rangle \in D$. ■

To verify that Raftery's logic accomplishes its mission, one has to establish that it is not finitary, but that it is algebraizable with the variety V as its equivalent algebraic semantics. Then, by Theorem 3.37 and Corollary 3.38 of Section 3.4 of [86], it becomes clear that the algebraization of $\mathcal{R}$ is carried out by a necessarily infinite set of defining equations and a set of equivalence formulas that may be taken to be finite. We formalize the second statement first.

Theorem 676 (Fact 9 of [82]) *Raftery's logic $\mathcal{R} = \langle \mathcal{L}, \vdash_R\rangle$ is algebraizable with equivalent algebraic semantics V via the set of defining equations $\delta(x) \approx \varepsilon(x) = \{\delta_i(x) \approx \varepsilon_i(x) : i \in \omega\}$ and the equivalence formula $\Delta(x,y) = \{x \leftrightarrow y\}$.*

Proof: By the definition of $\vdash_R$, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Gamma \vdash_R \varphi \quad \text{iff} \quad (\delta \approx \varepsilon)(\Gamma) \vDash_V (\delta \approx \varepsilon)(\varphi).$$

Moreover, for all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\begin{aligned} (\delta \approx \varepsilon)(\varphi \leftrightarrow \psi) &= \pi_1(\Diamond^i(\varphi \leftrightarrow \psi)) \approx \pi_2(\Diamond^i(\varphi \leftrightarrow \psi)), i \in I \\ &\dashv\vDash_V \pi_1(\varphi \leftrightarrow \psi) \approx \pi_2(\varphi \leftrightarrow \psi) \\ &\qquad (\text{since } V \vDash \Diamond(x \leftrightarrow y) \approx x \leftrightarrow y) \\ &\dashv\vDash_V \varphi \approx \psi \\ &\qquad (\text{since } V \vDash \pi_1(x \leftrightarrow y) \approx x \\ &\qquad \text{and } V \vDash \pi_2(x \leftrightarrow y) \approx y). \end{aligned}$$

By the general theory of algebraizable logics, these two conditions suffice to guarantee the conclusion. ■

And, finally, we show that $\mathcal{R}$ is not finitary.

Theorem 677 (Fact 10 of [82]) *The logics* $\mathcal{S}_{\mathfrak{A}} = \langle \mathcal{L}, \vdash_{\mathfrak{A}} \rangle$ *and* $\mathcal{R} = \langle \mathcal{L}, \vdash_R \rangle$ *are not finitary.*

Proof: Note that the two conditions established in the proof of Theorem 676, which suffice to establish algebraizability, imply, by the general theory of algebraizability (see, e.g., Exercise 39 of Section 3.2 of [86]), that

$$\{\delta_i(x) \leftrightarrow \varepsilon_i(x) : i \in \omega\} \vdash_R x$$

also holds. In addition, since, by Lemma 675, $\mathcal{R} \leq \mathcal{S}_{\mathfrak{A}}$,

$$\{\delta_i(x) \leftrightarrow \varepsilon_i(x) : i \in \omega\} \vdash_{\mathfrak{A}} x.$$

So to prove that $\mathcal{S}_{\mathfrak{A}}$ and $\mathcal{R}$ are not finitary, it suffices to show that, for no finite $K \subseteq \omega$ is it the case that $\{\delta_k(x) \leftrightarrow \varepsilon_k(x) : k \in K\} \vdash_{\mathfrak{A}} x$.

Let $j \in \omega - K$ and consider $a = \langle \langle b_0, b'_0 \rangle, \langle b_1, b'_1 \rangle, \dots \rangle \in B - \{0, 1\}$, such that $b_k = b'_k$, for all $k \in K$, but $b_j \neq b'_j$. Now, we compute

$$\delta_i^{\mathbf{A}}(a) \leftrightarrow^{\mathbf{A}} \varepsilon_i^{\mathbf{A}}(a) = \pi_1^{\mathbf{A}}(\Diamond^{\mathbf{A}^i} a) \leftrightarrow^{\mathbf{A}} \pi_2^{\mathbf{A}}(\Diamond^{\mathbf{A}^i} a) = b_i \leftrightarrow^{\mathbf{A}} b'_i,$$

whence, $(\delta_k \leftrightarrow \varepsilon_k)^{\mathbf{A}}(a) \in D$, for all $k \in K$, whereas, since $(\delta_j \leftrightarrow \varepsilon_j)^{\mathbf{A}}(a) \notin D$, by what was proven in Lemma 675, $a \notin D$. This shows that $\{\delta_k(x) \leftrightarrow \varepsilon_k(x) : k \in K\} \nvdash_{\mathfrak{A}} x$. Hence $\mathcal{S}_{\mathfrak{A}}$ and, a fortiori, $\mathcal{R}$ are not finitary. ■

So, the logic $\mathcal{R}$ does indeed attain the goal of discovering a non-finitary logic that is elementarily algebraizable (i.e., has a finitary equivalent algebraic semantics).

9.6 Separating Classes of π-Institutions

Using the framework detailed in Section 1.1, we recast the three sentential logics introduced in Section 9.5 as π-institutions and show that they provide examples that serve to separate the classes of π-institutions appearing in the steps of the finitarity hierarchy studied in Section 9.4.

In the first example, we recast Łukasiewicz's infinite valued logic as a π-institution.

Example 678 *Consider the algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by* $\mathrm{SEN}^\flat(\Sigma) = \mathrm{Fm}_{\mathcal{L}}(V)$, *where* $\mathcal{L} = \{\wedge, \vee, \to, \neg\}$ *is the language of Łukasiewicz's infinite valued logic;*

- $N^\flat$ *is the category of natural transformations generated by the binary operations* $\wedge, \vee, \to : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *and the unary operation* $\neg : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$, *defined as usual on the absolutely free algebra of formulas.*

Now define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$, *where, for all* $\Gamma \cup \{\varphi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\varphi \in C_\Sigma(\Gamma) \quad \text{iff} \quad \Gamma \vdash_\infty \varphi.$$

By Theorem 671, $\mathcal{I}$ *is not finitary. By Theorem 672 and the general theory of algebraizable logics, for all* $T \in \mathrm{ThFam}(\mathcal{I})$ *and* $\theta = \mathrm{ConSys}^*(\mathcal{I})$,

$$\begin{array}{rcl}\Omega_\Sigma(T) & = & \{\langle\varphi,\psi\rangle \in \mathrm{Fm}^2_{\mathcal{L}}(V) : \varphi \to \psi, \psi \to \varphi \in T_\Sigma\};\\ \Omega^{-1}_\Sigma(\theta) & = & \{\varphi \in \mathrm{Fm}_{\mathcal{L}}(V) : \langle \varphi, \top\rangle \in \theta_\Sigma\}.\end{array}$$

We show that the Leibniz operator $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *and its inverse* $\Omega^{-1} : \mathrm{ConSys}^*(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$ *are continuous. Suppose* $\{T^i\}_{i\in I} \subseteq \mathrm{ThFam}(\mathcal{I})$ *is directed and that* $\bigcup_{i\in I} T^i \in \mathrm{ThFam}(\mathcal{I})$. *Then we get, for all* $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\begin{array}{rll}\langle\varphi,\psi\rangle \in \Omega_\Sigma(\bigcup_{i\in I} T^i) & \text{iff} & \varphi \to \psi, \psi \to \varphi \in \bigcup_{i\in I} T^i_\Sigma\\ & \text{iff} & \varphi \to \psi \in T^i_\Sigma, \psi \to \varphi \in T^j_\Sigma, \text{ for some } i, j \in I,\\ & \text{iff} & \varphi \to \psi, \psi \to \varphi \in T^k_\Sigma, \text{ for some } k \in I,\\ & \text{iff} & \langle\varphi,\psi\rangle \in \Omega_\Sigma(T^k), \text{ for some } k \in I,\\ & \text{iff} & \langle\varphi,\psi\rangle \in \bigcup_{i\in I} \Omega_\Sigma(T^i).\end{array}$$

The proof for Ω^{-1} *is similar.*

This π*-institution serves to separate the classes connected by the three vertical arrows in the diagram concluding Section 9.4.*

The second example revisits Dellunde's logic in a similar way.

Example 679 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by setting* $\mathrm{SEN}^\flat(\Sigma) = \mathrm{Fm}_{\mathcal{L}}(V)$, *where* $\mathcal{L} = \{\leftrightarrow, \Box\}$ *is the language of Dellunde's logic;*

- $N^\flat$ *is the category of natural transformations generated by the binary operation* $\leftrightarrow : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *and the unary operation* $\Box : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined as usual on the absolutely free algebra of formulas.*

Define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, by setting, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\varphi \in C_\Sigma(\Gamma) \quad \text{iff} \quad \Gamma \vdash_D \varphi.$$

Since, as remarked in Section 9.5, Dellunde's logic $\mathcal{D}$ is finitary, so is the π-institution $\mathcal{I}$. Moreover, by Theorem 673 and the general theory of algebraizable logics, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\theta \in \mathrm{ConSys}^*(\mathcal{I})$, we have

$$\begin{array}{lll}\Omega_\Sigma(T) & = & \{\langle \varphi, \psi \rangle \in \mathrm{Fm}^2_{\mathcal{L}}(V) : \Box^n \varphi \leftrightarrow \Box^n \psi \in T_\Sigma, \text{ for all } n \in \omega\};\\ \Omega^{-1}_\Sigma(\theta) & = & \{\varphi \in \mathrm{Fm}_{\mathcal{L}}(V) : \langle \varphi, \top \rangle \in \theta_\Sigma\}.\end{array}$$

We show that $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is not continuous. Assume to the contrary, and define, for all $i \in \omega$, $T^i = \{T^i_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by setting

$$T^i_\Sigma = C_\Sigma(\{\Box^k x \leftrightarrow \Box^k y : k \leq i\}).$$

Note the following:

(1) $\{T^i\}_{i=0}^\infty$ is directed;

(2) $\bigcup_{i=0}^\infty T^i \in \mathrm{ThFam}(\mathcal{I})$, since $\mathcal{I}$ is finitary;

(3) $\langle x, y \rangle \in \Omega_\Sigma(\bigcup_{i=0}^\infty T^i)$, since $\Box^n x \leftrightarrow \Box^n y \in \bigcup_{i=0}^\infty T^i_\Sigma$, for all $n \in \omega$.

By the hypothesized continuity of Ω, since $\langle x, y \rangle \in \bigcup_{i=0}^\infty \Omega_\Sigma(T^i)$, there exists $m \in \omega$, such that $\langle x, y \rangle \in \Omega_\Sigma(T^m)$. But this implies that, for all $n > m$,

$$\Box^n x \leftrightarrow \Box^n y \in C_\Sigma(\{\Box^k x \leftrightarrow \Box^k y : k \leq m\}),$$

which contradicts what was shown in Theorem 674.

The π-institution $\mathcal{I}$, constructed here, serves to separate the classes connected by the southeast arrows in the finitarity hierarchy of π-institutions, shown at the end of Section 9.4.

Finally, we formulate an example that employs Raftery's logic $\mathcal{R}$.

Example 680 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object Σ;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is the functor specified by $\mathrm{SEN}^\flat(\Sigma) = \mathrm{Fm}_{\mathcal{L}}(V)$, where $\mathcal{L} = \{\leftrightarrow, \pi_1, \pi_2, \Diamond\}$ is the language of Raftery's logic;*
- $N^\flat$ *is the category of natural transformations generated by the binary operation $\leftrightarrow : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ and the unary operations $\pi_1, \pi_2, \Diamond : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ defined as usual on the absolutely free algebra of formulas.*

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by setting, for all* $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\varphi \in C_\Sigma(\Gamma) \quad \textit{iff} \quad \Gamma \vdash_R \varphi.$$

By Theorem 677, $\mathcal{I}$ *is not finitary. By Theorem 676 and the general theory of algberaizable logics, for all* $T \in \mathrm{ThFam}(\mathcal{I})$ *and all* $\theta \in \mathrm{ConSys}^*(\mathcal{I})$,

$$\begin{array}{lll}\Omega_\Sigma(T) & = & \{\langle \varphi, \psi \rangle \in \mathrm{Fm}^2_{\mathcal{L}}(V) : \varphi \leftrightarrow \psi \in T_\Sigma\};\\ \Omega^{-1}_\Sigma(\theta) & = & \{\varphi \in \mathrm{Fm}_{\mathcal{L}}(V) : \langle \pi_1(\Box^i\varphi), \pi_2(\Box^i\varphi)\rangle \in \theta_\Sigma, \ \textit{for all } i \in \omega\}.\end{array}$$

We may now show that $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is continuous, but* $\Omega^{-1} : \mathrm{ConSys}^*(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$ *is not continuous.*

To show continuity of Ω*, assume* $\{T^i\}_{i \in I} \subseteq \mathrm{ThFam}(\mathcal{I})$ *is directed, such that* $\bigcup_{i \in I} T^i \in \mathrm{ThFam}(\mathcal{I})$*. Let* $\langle \varphi, \psi \rangle \in \Omega_\Sigma(\bigcup_{i \in I} T^i)$*. This holds iff* $\varphi \leftrightarrow \psi \in \bigcup_{i \in I} T^i_\Sigma$*, i.e., iff, for some* $i \in I$*,* $\varphi \leftrightarrow \psi \in T^i$*. This is equivalent to* $\langle \varphi, \psi \rangle \in \Omega_\Sigma(T^i)$*, for some* $i \in I$*, showing that* $\Omega(\bigcup_{i \in I} T^i) = \bigcup_{i \in I} \Omega(T^i)$.

To show that Ω^{-1} *is not continuous, let, for all* $i \in \omega$*,* $\theta^i = \{\theta^i_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|} \in \mathrm{ConSys}^*(\mathcal{I})$ *be defined by*

$$\theta^i_\Sigma = \{\langle \varphi, \psi \rangle \in \mathrm{Fm}^2_{\mathcal{L}}(V) : \{\delta_k(x) \approx \varepsilon_k(x) : k \leq i\} \vDash_V \varphi \approx \psi\},$$

where, as before, for all $i \in \omega$,

$$\delta_i(x) = \pi_1(\Diamond^i x) \quad \textit{and} \quad \varepsilon_i(x) = \pi_2(\Diamond^i x).$$

Note that

(1) $\{\theta^i\}^\infty_{i=0}$ *is directed;*

(2) $\bigcup^\infty_{i=0} \theta^i \in \mathrm{ConSys}^*(\mathcal{I})$*, since* $\vDash_V$ *is finitary;*

(3) $x \in \Omega^{-1}_\Sigma(\bigcup^\infty_{i=0} \theta^i)$*, since* $\delta(x) \approx \varepsilon(x) \subseteq \bigcup^\infty_{i=0} \theta^i_\Sigma$.

If Ω^{-1} *were continuous, there would exist* $m \in \omega$*, such that* $x \in \Omega^{-1}_\Sigma(\theta^m)$*. But, this would imply that*

$$\{\delta_k(x) \approx \varepsilon_k(x) : k \leq m\} \vDash_V \delta(x) \approx \varepsilon(x),$$

which yields $\{\delta_k(x) \leftrightarrow \varepsilon_k(x) : k \leq m\} \vdash_R x$*, contradicting Theorem 677.*

The π*-institution* $\mathcal{I}$*, constructed in this example, separates the classes of* π*-institutions related by the southwest arrows in the finitarity hierarchy shown at the end of Section 9.4.*

Chapter 10

Elements of Syntax

10.1 Natural Transformations and Parameters

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. Consider a set

$$I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$$

of natural transformations in $N^\flat$. Of course, by definition, each $\sigma^\flat \in I^\flat \subseteq N^\flat$ is finitary, but the arities in the collection may be unbounded, whence the notation becomes handy.

Recall that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^\omega$,

$$I^\flat_\Sigma(\vec{\phi}) = \{\sigma^\flat_\Sigma(\phi_0, \ldots, \phi_{k-1}) : \sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat \in I^\flat\}.$$

Moreover, we may view the first n of the arguments in the input sequence as **distinguished** and the remaining as **parameters** or **parametric arguments**. In that case, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} = \langle \phi_0, \ldots, \phi_{n-1} \rangle \in \mathrm{SEN}^\flat(\Sigma)$, we define

$$I^\flat_\Sigma[\vec{\phi}] = \{I^\flat_{\Sigma,\Sigma'}[\vec{\phi}]\}_{\Sigma' \in |\mathbf{Sign}^\flat|},$$

where, for all $\Sigma' \in |\mathbf{Sign}^\flat|$,

$$I^\flat_{\Sigma,\Sigma'}[\vec{\phi}] = \bigcup\{I^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}), \vec{\chi}) : f \in \mathbf{Sign}^\flat(\Sigma, \Sigma'), \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')\}.$$

The following diagram illustrates where the various sentences and components sit as we move from inputs to outputs in this construct.

$$\begin{array}{ccc}
\mathrm{SEN}^\flat(\Sigma) & \xrightarrow{\mathrm{SEN}^\flat(f)} & \mathrm{SEN}^\flat(\Sigma') \\
\vec{\phi} & \longmapsto & \mathrm{SEN}^\flat(f)(\vec{\phi}), \vec{\chi} \\
 & & \big\downarrow \\
 & & I^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}), \vec{\chi})
\end{array}$$

Suppose that in $I^\flat$ we take $n = 2$, i.e., we consider only the first two arguments as distinguished and the remaining as parameters. Then, we define for $\sigma : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat \in N^\flat$, the natural transformation $\overline{\sigma} : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$, by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\overline{\sigma}_\Sigma(\phi, \psi, \vec{\chi}) = \sigma_\Sigma(\psi, \phi, \vec{\chi}).$$

Further, we set

$$\overline{I^\flat} = \{\overline{\sigma} : \sigma \in I^\flat\}$$

and

$$\overleftrightarrow{I^\flat} = I^\flat \cup \overline{I^\flat}.$$

It is not difficult to see that, given $I^\flat \subseteq N^\flat$, the collections $\overline{I^\flat}$ and $\overleftrightarrow{I^\flat}$ both consist of natural transformations in $N^\flat$.

Lemma 681 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and let* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *be a collection of natural transformations in* $N^\flat$. *Then* $\overline{I^\flat}, \overleftrightarrow{I^\flat} \subseteq N^\flat$.

Proof: The inclusion $\overline{I^\flat} \subseteq N^\flat$ follows from Proposition 11. Then the second inclusion follows directly from the definition of $\overleftrightarrow{I^\flat}$. ∎

Finally, recall that, given an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$, a collection $I^\flat \subseteq N^\flat$, with two distinguished arguments, and $T \in \mathrm{SenFam}(\mathbf{F})$, we define $I^\flat(T) = \{I^\flat_\Sigma(T)\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in I^\flat_\Sigma(T) \quad \text{iff} \quad I^\flat_\Sigma[\phi, \psi] \leq T.$$

It was shown in Lemma 93 that $I^\flat(T)$ is a relation system on $\mathbf{F}$, i.e., invariant under signature morphisms.

In what follows we explore some properties that collections of natural transformations may or may not satisfy in π-institutions based on the algebraic systems on which they are defined.

10.2 Reflexivity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. Taking into account Proposition 103, we say that $I^\flat$ is **reflexive in** $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma(\phi, \phi, \vec{\chi}) \subseteq \mathrm{Thm}_\Sigma(\mathcal{I}) \coloneqq C_\Sigma(\varnothing).$$

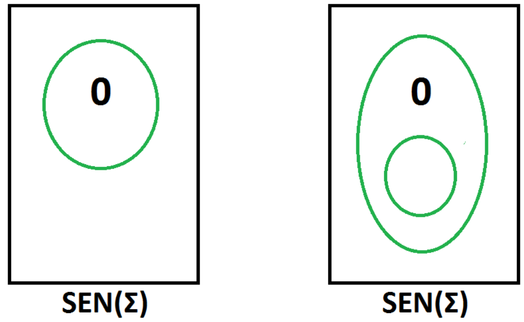
Example 682 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0\}$*;*
- $N^\flat$ *is the trivial category of natural transformations consisting only of the projections.*

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{0\}\}$ *and* $\mathcal{I}' = \langle \mathbf{F}, C' \rangle$ *be the* π*-institution determined by* $\mathcal{C}'_\Sigma = \{\varnothing, \{0\}\}$.

Consider the set $I^\flat = \{p^{2,0}\}$, *with* $p^{2,0} : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *be the 2-argument projection function projecting onto the first argument.*

In this case, it is easy to verify that $I^\flat$ *is reflexive in* $\mathcal{I}$ *but* $I^\flat$ *is not reflexive in* $\mathcal{I}'$.



As was the case with the various properties of the Leibniz operator that gave rise to the various classes of the semantic hierarchy of π-institutions, the surjectivity of the morphism components in interpreted algebraic systems affords transferring the properties that give rise to the syntactic hierarchy studied in the present chapter from the theory families of a π-institution to the filter families over arbitrary algebraic systems. The key in proving these tranfer properties is Lemma 95, which will be used repeatedly in the proofs throughout the chapter.

The first of this type of transfer properties is the transfer property for reflexivity. In formulating the property it is convenient to adopt the following terminology. We consider, as is usual in this context, a base algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ and a set $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ of natural transformations in $N^\flat$. Given a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, based on $\mathbf{F}$, and an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, we say that I is **reflexive in** $\mathcal{A}$ if the collection $I : \mathrm{SEN}^\omega \to \mathrm{SEN}$ of natural transformations in N, that are images of those in $I^\flat$, is reflexive in the π-institution $\langle \mathbf{A}, C^{\mathcal{I},\mathcal{A}} \rangle$, $C^{\mathcal{I},\mathcal{A}}$ being the closure (operator) system whose closed set families are the $\mathcal{I}$-filter families on $\mathcal{A}$.

We use similar terminology for all other properties that we study in this chapter, pertaining to subsets $I^\flat$ of $N^\flat$. In particular, such terminology will be used in all transfer results for these properties.

Proposition 683 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $I^\flat \subseteq N^\flat$ a collection of natural transformations $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $I^\flat$ is reflexive in $\mathcal{I}$ if and only if, for every algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, I is reflexive in $\mathcal{A}$.*

Proof: First, note that if reflexivity of I in $\mathcal{A}$ is assumed, for all $\mathcal{A}$, then it holds, in particular, for $\mathcal{A} = \mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. Moreover $\langle \mathbf{F}, C^{\mathcal{I},\mathcal{F}} \rangle = \mathcal{I}$. Thus, we conclude that $I^\flat$ is reflexive in $\mathcal{I}$.

Suppose, conversely, that $I^\flat$ is reflexive in $\mathcal{I}$. By the surjectivity of $\langle F, \alpha \rangle$, it suffices to show that, for all $\sigma : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat \in I^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi \in \mathrm{SEN}^\flat(\Sigma)$ and all $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\sigma_{F(\Sigma')}(\mathrm{SEN}(F(f))(\alpha_\Sigma(\phi)), \mathrm{SEN}(F(f))(\alpha_\Sigma(\phi)), \alpha_{\Sigma'}(\vec{\chi})) \in C^{\mathcal{I},\mathcal{A}}_{F(\Sigma')}(\varnothing).$$

To this end, let $\sigma : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat \in I^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$ and $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$. Since $C^{\mathcal{I},\mathcal{A}}(\varnothing)$ is, by definition, an $\mathcal{I}$-filter family on $\mathcal{A}$, by Lemma 51, $\alpha^{-1}(C^{\mathcal{I},\mathcal{A}}(\varnothing)) \in \mathrm{ThFam}(\mathcal{I})$. Hence, since $I^\flat$ is reflexive in $\mathcal{I}$, we get

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \in \alpha^{-1}_{\Sigma'}(C^{\mathcal{I},\mathcal{A}}_{F(\Sigma')}(\varnothing)).$$

This is equivalent to

$$\alpha_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\phi), \vec{\chi})) \in C^{\mathcal{I},\mathcal{A}}_{F(\Sigma')}(\varnothing),$$

which is, in turn, equivalent to

$$\sigma_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)), \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)), \alpha_{\Sigma'}(\vec{\chi})) \in C^{\mathcal{I},\mathcal{A}}_{F(\Sigma')}(\varnothing).$$

Finally, by the naturality of α, we get the conclusion. Therefore, I is indeed reflexive in $\mathcal{A}$. ■

10.3 Symmetry

We look now at various versions of the symmetry property, taking into account both the duality between local versus global membership and the difference between considering all theory families versus restricting only to theory systems.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. We say that:

- $I^\flat$ has the **local family symmetry in** $\mathcal{I}$ if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq T_\Sigma$, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$, implies that $I^\flat_\Sigma(\psi, \phi, \vec{\xi}) \subseteq T_\Sigma$, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$;
- $I^\flat$ has the **local system symmetry in** $\mathcal{I}$ if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq T_\Sigma$, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$, implies that $I^\flat_\Sigma(\psi, \phi, \vec{\xi}) \subseteq T_\Sigma$, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$;
- $I^\flat$ has the **global family symmetry in** $\mathcal{I}$ if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma[\phi, \psi] \leq T$ implies $I^\flat_\Sigma[\psi, \phi] \leq T$;
- $I^\flat$ has the **global system symmetry in** $\mathcal{I}$ if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma[\phi, \psi] \leq T$ implies $I^\flat_\Sigma[\psi, \phi] \leq T$.

The following proposition establishes a hierarchy of symmetry properties.

Proposition 684 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments.*

(a) *If $I^\flat$ has the local family symmetry, then it has the local system symmetry in $\mathcal{I}$;*

(b) *If $I^\flat$ has the local system symmetry, then it has the global family symmetry in $\mathcal{I}$;*

(c) *$I^\flat$ has the global family symmetry if and only if it has the global system symmetry in $\mathcal{I}$.*

Proof: Parts (a) and one of the implications in Part (c) follow directly from the fact that every theory system of $\mathcal{I}$ is also a theory family of $\mathcal{I}$.

For Part (b), suppose that $I^\flat$ has the local system symmetry in $\mathcal{I}$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $I^\flat_\Sigma[\phi,\psi] \leq T$. Then by Lemma 93, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$,

$$I^\flat_{\Sigma'}[\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi)] \leq T.$$

This implies, by Lemma 99, that, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$I^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), \vec{\xi}) \subseteq \overleftarrow{T}_{\Sigma'}.$$

Since $I^\flat$ has the local system symmetry and, by Proposition 42, $\overleftarrow{T} \in \mathrm{ThSys}(\mathcal{I})$, we get that $I^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \mathrm{SEN}^\flat(f)(\phi), \vec{\xi}) \subseteq \overleftarrow{T}_{\Sigma'} \subseteq T_{\Sigma'}$. Since this holds for all $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$, we conclude that $I^\flat_\Sigma[\psi,\phi] \leq T$. Therefore $I^\flat$ has the global family symmetry in $\mathcal{I}$.

Suppose, finally, that $I^\flat$ has the global system symmetry in $\mathcal{I}$ and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $I^\flat_\Sigma[\phi,\psi] \leq T$. By Lemma 99, we get that $I^\flat_\Sigma[\phi,\psi] \leq \overleftarrow{T}$. Since $I^\flat$ has the global system symmetry and, by Proposition 42, $\overleftarrow{T} \in \mathrm{ThSys}(\mathcal{I})$, we get that $I^\flat_\Sigma[\psi,\phi] \leq \overleftarrow{T}$. Using again Lemma 99, we conclude that $I^\flat_\Sigma[\psi,\phi] \leq T$. Therefore, $I^\flat$ has the global family symmetry in $\mathcal{I}$. ■

Proposition 684 has established the following hierarchy of symmetry properties:

Local Family Symmetry

↓

Local System Symmetry

↓

Global Symmetry

We look, next, at some natural sufficient conditions under which some of these three symmetry properties coincide.

Proposition 685 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$*, with two distinguished arguments.*

(a) If $\mathcal{I}$ *is systemic, then the local family and the local system symmetry coincide;*

(b) If $I^\flat$ *has only two arguments (i.e., is parameter free), then the local system symmetry and the global symmetry coincide.*

Proof: If $\mathcal{I}$ is systemic, then all theory families are theory systems and, hence, the local family and local system symmetries coincide.

Suppose, next that $I^\flat$ is parameter free and has the global system symmetry in $\mathcal{I}$. Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $I^\flat_\Sigma(\phi, \psi) \subseteq T_\Sigma$. Then, by Proposition 99, $I^\flat[\phi, \psi] \leq T$. Thus, by the global system property, $I^\flat_\Sigma[\psi, \phi] \leq T$, which implies that $I^\flat_\Sigma(\psi, \phi) \subseteq T_\Sigma$. Therefore, $I^\flat$ has the local system symmetry in $\mathcal{I}$. ■

So in the case of a systemic π-institution $\mathcal{I}$, we have the hierarchy pictured on the left, whereas in the case of a parameter-free set of natural transformations we have the hierarchy on the right.

Local Symmetry	Local Family Symmetry
↓	↓
Global Symmetry	Local System/Global Symmetry

Finally, for a systemic π-institution with a parameter-free set of natural transformations all four symmetry properties collapse to a single one.

We provide some examples to show that the implications of Proposition 684 are not equivalences in general, i.e., in the 3-class hierarchy all inclusions of classes of π-institutions with a set of natural transformations satisfying the corresponding symmetry properties are proper inclusions.

We first present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that have the local system symmetry but not the local family symmetry in $\mathcal{I}$.

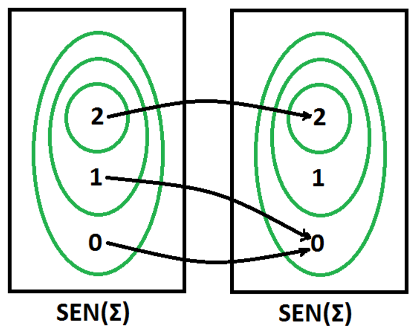
Example 686 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single objects* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0, 1, 2\} \to \{0, 1, 2\}$ *given by* $0 \mapsto 0$*,* $1 \mapsto 0$ *and* $2 \mapsto 2$*;*

- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 1, & \textit{if } (x,y) = (0,1) \\ 0, & \textit{otherwise} \end{cases}$$



Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}.$$

Note that there are three theory families, but only $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$ *are theory systems.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the local system symmetry in* $\mathcal{I}$*, but it does not have the local family symmetry in* $\mathcal{I}$*.*

For the local system symmetry note that, if $T = \mathrm{SEN}^\flat$*, then the defining implication is trivially true, whereas, if* $T = \mathrm{Thm}(\mathcal{I})$*, then, since, for all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,* $\sigma^\flat_\Sigma(\phi,\psi) \neq 2$*, the defining implication is vacuously true. So* $I^\flat$ *has the local system symmetry in* $\mathcal{I}$*.*

On the other hand, for $T = \{\{1,2\}\} \in \mathrm{ThFam}(\mathcal{I})$*, we have* $\sigma^\flat_\Sigma(0,1) = 1 \in T_\Sigma$*, but* $\sigma^\flat_\Sigma(1,0) = 0 \notin T_\Sigma$*. Therefore, the implication defining local family symmetry fails for* $T = \{\{1,2\}\}$*. So* $I^\flat$ *is not locally family symmetric in* $\mathcal{I}$*.*

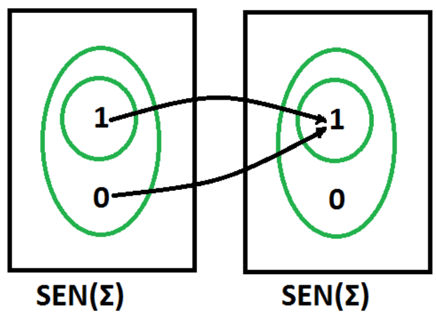
Next, we present an example to show that there is π-institution $\mathcal{I}$ with a set of natural transformations that have the global family symmetry but not the local system symmetry in $\mathcal{I}$.

Example 687 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single objects* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1\} \to \{0,1\}$ *given by* $0 \mapsto 1$ *and* $1 \mapsto 1$;
- $N^\flat$ *is the category of natural transformations generated by the single ternary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1\}^3 \to \{0,1\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y,z) = \begin{cases} 0, & \textit{if } (x,z) = (1,0) \\ 1, & \textit{otherwise} \end{cases}$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\}$. *Note that both theory families,* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$, *are also theory systems.*

Consider the set $I^\flat = \{\sigma^\flat\}$, *with* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the global family symmetry in* $\mathcal{I}$, *but it does not have the local system symmetry in* $\mathcal{I}$.

For the global family symmetry note that, if $T = \mathrm{SEN}^\flat$, *then the defining implication is trivially true, whereas, if* $T = \mathrm{Thm}(\mathcal{I})$, *then, since, for all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\sigma^\flat_\Sigma(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), 0) = \sigma^\flat_\Sigma(1,1,0) = 0,$$

the defining implication is vacuously true. So $I^\flat$ *has the global family symmetry in* $\mathcal{I}$.

On the other hand, we have $\sigma^\flat_\Sigma(0,1,\xi) = 1$, *for all* $\xi \in \mathrm{SEN}^\flat(\Sigma)$, *but* $\sigma^\flat_\Sigma(1,0,0) = 0 \notin \{1\}$. *Therefore, the implication defining local system symmetry fails for* $\mathrm{Thm}(\mathcal{I})$. *So* $I^\flat$ *is not locally system symmetric in* $\mathcal{I}$.

To close the study of symmetry properties, we prove that all three symmetry properties transfer from π-institutions to their models.

Proposition 688 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $I^\flat$ *a collection of natural transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$, *with two distinguished arguments, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $I^\flat$ *has a symmetry property in* $\mathcal{I}$ *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, *with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, I *has the corresponding symmetry property in* $\mathcal{A}$.

Proof: If I has a symmetry property in $\mathcal{A}$, for all $\mathcal{A}$, then it has the same symmetry in $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. Since $\langle \mathbf{F}, C^{\mathcal{I},\mathcal{F}} \rangle = \mathcal{I}$, we conclude that $I^\flat$ has the corresponding symmetry in $\mathcal{I}$.

Suppose, conversely, that $I^\flat$ has a symmetry in $\mathcal{I}$. We look at each of the three properties in turn.

(a) Suppose $I^\flat$ has the local family symmetry in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$I_{F(\Sigma)}(\alpha_\Sigma(\phi), \alpha_\Sigma(\psi), \alpha_\Sigma(\vec{\xi})) \subseteq T_{F(\Sigma)},$$

for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$. Since this is equivalent to $\alpha_\Sigma(I^\flat_\Sigma(\phi, \psi, \vec{\xi})) \subseteq T_{F(\Sigma)}$, we get that $I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq \alpha_\Sigma^{-1}(T_{F(\Sigma)})$, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$. But, by hypothesis, $I^\flat$ has the local family symmetry in $\mathcal{I}$ and, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$. Therefore, we get that $I^\flat_\Sigma(\psi, \phi, \vec{\xi}) \subseteq \alpha_\Sigma^{-1}(T_{F(\Sigma)})$. This now gives $\alpha_\Sigma(I^\flat_\Sigma(\psi, \phi, \vec{\xi})) \subseteq T_{F(\Sigma)}$, or, equivalently,

$$I_{F(\Sigma)}(\alpha_\Sigma(\psi), \alpha_\Sigma(\phi), \alpha_\Sigma(\vec{\xi})) \subseteq T_{F(\Sigma)}.$$

We conclude that I has the local family symmetry in $\mathcal{A}$.

(b) The case of the local system symmetry can be proven similarly, taking into account that, if $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, then $\alpha^{-1}(T) \in \mathrm{ThSys}(\mathcal{I})$.

(c) Suppose that $I^\flat$ has the global (family) symmetry in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$I_{F(\Sigma)}[\alpha_\Sigma(\phi), \alpha_\Sigma(\psi)] \le T.$$

Then, we have, by Lemma 95, $I^\flat_\Sigma[\phi, \psi] \le \alpha^{-1}(T)$. Now, since, by hypothesis, $I^\flat$ has the global family symmetry in $\mathcal{I}$ and, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, we get that $I^\flat_\Sigma[\psi, \phi] \le \alpha^{-1}(T)$, or, equivalently, by Lemma 95, $I_{F(\Sigma)}[\alpha_\Sigma(\psi), \alpha_\Sigma(\phi)] \le T$. Thus, I has the global family symmetry in $\mathcal{A}$.

■

10.4 Transitivity

We study next various versions of the transitivity property, taking into account, again, both the duality between local versus global membership and the difference between considering all theory families versus restricting only to theory systems.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. We say that:

- $I^\flat$ has the **local family transitivity in** $\mathcal{I}$ if, for all $T \in \text{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \text{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq T_\Sigma$ and $I^\flat_\Sigma(\psi, \chi, \vec{\xi}) \subseteq T_\Sigma$, for all $\vec{\xi} \in \text{SEN}^\flat(\Sigma)$, imply that $I^\flat_\Sigma(\phi, \chi, \vec{\xi}) \subseteq T_\Sigma$, for all $\vec{\xi} \in \text{SEN}^\flat(\Sigma)$;
- $I^\flat$ has the **local system transitivity in** $\mathcal{I}$ if, for all $T \in \text{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \text{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq T_\Sigma$ and $I^\flat_\Sigma(\psi, \chi, \vec{\xi}) \subseteq T_\Sigma$, for all $\vec{\xi} \in \text{SEN}^\flat(\Sigma)$, imply that $I^\flat_\Sigma(\phi, \chi, \vec{\xi}) \subseteq T_\Sigma$, for all $\vec{\xi} \in \text{SEN}^\flat(\Sigma)$;
- $I^\flat$ has the **global family transitivity in** $\mathcal{I}$ if, for all $T \in \text{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \text{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma[\phi, \psi] \leq T$ and $I^\flat_\Sigma[\psi, \chi] \leq T$ imply $I^\flat_\Sigma[\phi, \chi] \leq T$;
- $I^\flat$ has the **global system transitivity in** $\mathcal{I}$ if, for all $T \in \text{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \text{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma[\phi, \psi] \leq T$ and $I^\flat_\Sigma[\psi, \chi] \leq T$ imply $I^\flat_\Sigma[\phi, \chi] \leq T$.

The following proposition establishes the hierarchy of transitivity properties.

Proposition 689 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\text{SEN}^\flat)^\omega \to \text{SEN}^\flat$ *a set of natural transformations in* $N^\flat$*, with two distinguished arguments.*

(a) If $I^\flat$ *has the local family transitivity, then it has the local system transitivity in* $\mathcal{I}$*;*

(b) If $I^\flat$ *has the local system transitivity, then it has the global family transitivity in* $\mathcal{I}$*;*

(c) $I^\flat$ *has the global family transitivity if and only if it has the global system transitivity in* $\mathcal{I}$*.*

Proof: The statement in Part (a) as well as one of the two implications of Part (c) follow from the fact that every theory system is also a theory family of $\mathcal{I}$.

For Part (b), suppose that $I^\flat$ has the local system transitivity and let $T \in \text{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \text{SEN}^\flat(\Sigma)$, such that $I^\flat_\Sigma[\phi, \psi] \leq T$ and $I^\flat_\Sigma[\psi, \chi] \leq T$. By Lemma 93, we get that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$I^\flat_{\Sigma'}[\text{SEN}^\flat(f)(\phi), \text{SEN}^\flat(f)(\psi)] \leq T, \;\; I^\flat_{\Sigma'}[\text{SEN}^\flat(f)(\psi), \text{SEN}^\flat(f)(\chi)] \leq T.$$

So, by Proposition 99, we get, for all $\vec{\xi} \in \text{SEN}^\flat(\Sigma')$,

$$\begin{aligned} &I^\flat_{\Sigma'}(\text{SEN}^\flat(f)(\phi), \text{SEN}^\flat(f)(\psi), \vec{\xi}) \subseteq \overleftarrow{T}_{\Sigma'}, \\ &I^\flat_{\Sigma'}(\text{SEN}^\flat(f)(\psi), \text{SEN}^\flat(f)(\chi), \vec{\xi}) \subseteq \overleftarrow{T}_{\Sigma'}. \end{aligned}$$

By local system transitivity, we obtain $I^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi}) \subseteq \overleftarrow{T}_{\Sigma'}$, Since this holds for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$, we conclude that $I^\flat_{\Sigma'}[\phi, \chi] \leq T$, Therefore, $I^\flat$ has the global family transitivity in $\mathcal{I}$.

Finally, suppose that $I^\flat$ has the global system transitivity in $\mathcal{I}$ and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, such that $I^\flat_\Sigma[\phi, \psi] \leq T$ and $I^\flat_\Sigma[\psi, \chi] \leq T$. By Proposition 99, we get $I^\flat_\Sigma[\phi, \psi] \leq \overleftarrow{T}$ and $I^\flat_\Sigma[\psi, \chi] \leq \overleftarrow{T}$. Hence, by global system transitivity, $I^\flat_\Sigma[\phi, \chi] \leq \overleftarrow{T}$. Now, using Proposition 99 again, we conclude that $I^\flat_\Sigma[\phi, \chi] \leq T$. Therefore, $I^\flat$ has the global family transitivity in $\mathcal{I}$. ■

Proposition 689 has established the following hierarchy of transitivity properties:

Local Family Transitivity

$\downarrow$

Local System Transitivity

$\downarrow$

Global Transitivity

We also have the following result regarding natural sufficient conditions under which some of these three transitivity properties coincide.

Proposition 690 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$*, with two distinguished arguments.*

(a) If $\mathcal{I}$ *is systemic, then the local family and the local system transitivity coincide;*

(b) If $I^\flat$ *has only two arguments (i.e., is parameter free), then the local system transitivity and the global transitivity properties coincide.*

Proof: If $\mathcal{I}$ is systemic, then all theory families are theory systems and the local family and local system transitivity properties collapse.

Suppose that $I^\flat$ is parameter-free and that $I^\flat$ has the global (family) transitivity in $\mathcal{I}$. Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, such that $I^\flat_\Sigma(\phi, \psi) \subseteq T_\Sigma$ and $I^\flat_\Sigma(\psi, \chi) \subseteq T_\Sigma$. By Proposition 99, $I^\flat_\Sigma[\phi, \psi] \leq T$ and $I^\flat_\Sigma[\psi, \chi] \leq T$. Thus, by the global family transitivity property, $I^\flat_\Sigma[\phi, \chi] \leq T$, which implies that $I^\flat_\Sigma(\phi, \chi) \subseteq T_\Sigma$. We conclude that $I^\flat$ has the local system transitivity in $\mathcal{I}$. ■

So in the case of a systemic π-institution $\mathcal{I}$, we have the hierarchy pictured on the left, whereas in the case of a parameter-free set of natural transformations we have the hierarchy on the right.

Local Transitivity	Local Family Transitivity
$\downarrow$	$\downarrow$
Global Transitivity	Local System/Global Transitivity

Finally, for a systemic π-institution with a parameter-free set of natural transformations all four transitivity properties collapse to a single one.

We provide some examples to show that the implications of Proposition 689 are not equivalences in general, i.e., in the 3-class transitivity hierarchy all inclusions of classes of π-institutions with a set of natural transformations satisfying the corresponding transitivity properties are proper inclusions.

First, we present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that has the local system transitivity but not the local family transitivity in $\mathcal{I}$.

Example 691 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single objects* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 0$*,* $1 \mapsto 0$ *and* $2 \mapsto 2$*;*
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 1, & \textit{if } (x,y) = (0,1) \textit{ or } (1,2) \\ 0, & \textit{otherwise} \end{cases}$$

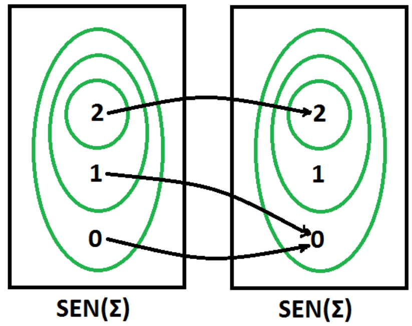
Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}.$$

Note that there are three theory families, but only $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$ *are theory systems.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the local system transitivity in* $\mathcal{I}$*, but it does not have the local family transitivity in* $\mathcal{I}$*.*

For the local system transitivity note that, if $T = \mathrm{SEN}^\flat$*, then the defining implication is trivially true, whereas, if* $T = \mathrm{Thm}(\mathcal{I})$*, then, since, for all*



$\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $\sigma^\flat_\Sigma(\phi, \psi) \neq 2$, the defining implication is vacuously true. So $I^\flat$ has the local system transitivity in $\mathcal{I}$.

On the other hand, for $T = \{\{1,2\}\} \in \mathrm{ThFam}(\mathcal{I})$, we have $\sigma^\flat_\Sigma(0,1) = \sigma^\flat_\Sigma(1,2) = 1 \in T_\Sigma$, but $\sigma^\flat_\Sigma(0,2) = 0 \notin T_\Sigma$. Therefore, the implication defining local family transitivity fails for $T = \{\{1,2\}\}$. So $I^\flat$ is not locally family transitive in $\mathcal{I}$.

We now present an example to show that there is π-institution $\mathcal{I}$, with a set of natural transformations that has the global family transitivity but not the local system transitivity in $\mathcal{I}$.

Example 692 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system determined as follows:*

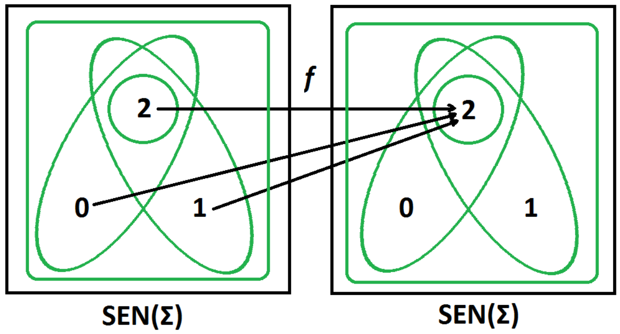
- *$\mathbf{Sign}^\flat$ is the category with a single objects Σ and a single (non-identity) morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- *$\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ is specified by $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ and $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ given by $0 \mapsto 2$, $1 \mapsto 2$ and $2 \mapsto 2$;*
- *$N^\flat$ is the category of natural transformations generated by the single ternary natural transformation $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ defined by letting $\sigma^\flat_\Sigma : \{0,1,2\}^3 \to \{0,1,2\}$ be given by*

$$\sigma^\flat_\Sigma(x,y,z) = \begin{cases} 0, & \text{if } x = y = 0 \text{ or } x = y = 1 \\ 2, & \text{if } \{x,y\} = \{0,1\} \text{ or } x = y = z = 2 \\ z, & \text{otherwise} \end{cases}$$

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be the π-institution determined by

$$\mathcal{C}_\Sigma = \{\{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}.$$

Note that all four theory families, $\mathrm{Thm}(\mathcal{I})$, $T = \{\{0,2\}\}$, $T' = \{\{1,2\}\}$ and $\mathrm{SEN}^\flat$, are also theory systems.



Consider the set $I^\flat = \{\sigma^\flat\}$, *with* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the global family transitivity in* $\mathcal{I}$, *but it does not have the local system transitivity in* $\mathcal{I}$.

For the global family transitivity note that, because, for all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $\sigma^\flat_\Sigma(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), 0) = 0$ *and* $\sigma^\flat_\Sigma(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), 1) = 1$, *the implication of the defining condition is vacuously true for* $\mathrm{Thm}(\mathcal{I})$, T *and* T' *and trivially true for* $\mathrm{SEN}^\flat$. *Therefore, we get that* $I^\flat$ *has the global family transitivity in* $\mathcal{I}$.

On the other hand, we have $\sigma^\flat_\Sigma(0,1,\xi) = \sigma^\flat_\Sigma(1,0,\xi) = 2$, *for all* $\xi \in \mathrm{SEN}^\flat(\Sigma)$, *but* $\sigma^\flat_\Sigma(0,0,0) = 0 \notin \{2\}$. *Therefore, the implication defining local system transitivity fails for* $\mathrm{Thm}(\mathcal{I})$. *So* $I^\flat$ *does not have the local system transitivity in* $\mathcal{I}$.

We close the study of transitivity by providing, again, a transfer property for transitivity.

Proposition 693 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $I^\flat$ *a collection of natural transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$, *with two distinguished arguments, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $I^\flat$ *has a transitivity property in* $\mathcal{I}$ *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, *with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, I *has the corresponding transitivity property in* $\mathcal{A}$.

Proof: If I has a transitivity property in $\mathcal{A}$, for all $\mathcal{A}$, then it has the same transitivity in $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. Since $\langle \mathbf{F}, C^{\mathcal{I},\mathcal{F}} \rangle = \mathcal{I}$, we conclude that $I^\flat$ has the corresponding transitivity in $\mathcal{I}$.

Suppose, conversely, that $I^\flat$ has a transitivity property in $\mathcal{I}$. We look at each of the three properties in turn.

(a) Suppose $I^\flat$ has the local family transitivity in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, such that, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$I_{F(\Sigma)}(\alpha_\Sigma(\phi), \alpha_\Sigma(\psi), \alpha_\Sigma(\vec{\xi})) \subseteq T_{F(\Sigma)},$$
$$I_{F(\Sigma)}(\alpha_\Sigma(\psi), \alpha_\Sigma(\chi), \alpha_\Sigma(\vec{\xi})) \subseteq T_{F(\Sigma)}.$$

These are equivalent, respectively, to

$$\alpha_\Sigma(I^\flat_\Sigma(\phi,\psi,\vec{\xi})) \subseteq T_{F(\Sigma)},\ \alpha_\Sigma(I^\flat_\Sigma(\psi,\chi,\vec{\xi})) \subseteq T_{F(\Sigma)},$$

i.e., to $I^\flat_\Sigma(\phi,\psi,\vec{\xi}) \subseteq \alpha_\Sigma^{-1}(T_{F(\Sigma)})$ and $I^\flat_\Sigma(\psi,\chi,\vec{\xi}) \subseteq \alpha_\Sigma^{-1}(T_{F(\Sigma)})$, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$. But, by hypothesis, $I^\flat$ has the local family transitivity in $\mathcal{I}$ and, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$. Therefore, we get that $I^\flat_\Sigma(\phi,\chi,\vec{\xi}) \subseteq \alpha_\Sigma^{-1}(T_{F(\Sigma)})$, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$. Thus, $\alpha_\Sigma(I^\flat_\Sigma(\phi,\chi,\vec{\xi})) \subseteq T_{F(\Sigma)}$ and, hence, $I_{F(\Sigma)}(\alpha_\Sigma(\phi),\alpha_\Sigma(\chi),\alpha_\Sigma(\vec{\xi})) \subseteq T_{F(\Sigma)}$. This, combined with the surjectivity of $\langle F,\alpha\rangle$, proves that I has the local family transitivity in $\mathcal{A}$.

(b) The case of the local system transitivity may be proven similarly, taking into account that, if $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, then $\alpha^{-1}(T) \in \mathrm{ThSys}(\mathcal{I})$.

(c) Suppose that $I^\flat$ has the global (family) transitivity in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A},\langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign},\mathrm{SEN},N\rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi,\chi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$I_{F(\Sigma)}[\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)] \leq T \quad \text{and} \quad I_{F(\Sigma)}[\alpha_\Sigma(\psi),\alpha_\Sigma(\chi)] \leq T.$$

Then, we have, by Lemma 95, $I^\flat_\Sigma[\phi,\psi] \leq \alpha^{-1}(T)$ and $I^\flat_\Sigma[\psi,\chi] \leq \alpha^{-1}(T)$. Since, by hypothesis, $I^\flat$ has the global family transitivity in $\mathcal{I}$ and, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, we get that $I^\flat_\Sigma[\phi,\chi] \leq \alpha^{-1}(T)$, or, equivalently, using again Lemma 95, $I_{F(\Sigma)}[\alpha_\Sigma(\phi),\alpha_\Sigma(\chi)] \leq T$. Thus, I has the global family transitivity in $\mathcal{A}$.

■

10.5 Equivalence

We look now at sets of natural transformations $I^\flat$, with two distinguished arguments, that define (modulo theory families) equivalence relation families on the underlying algebraic system of a π-institution $\mathcal{I}$. We assume that $I^\flat$ has the reflexivity property and study combinations of possible symmetry and transitivity properties that the set of connectives may or may not possess.

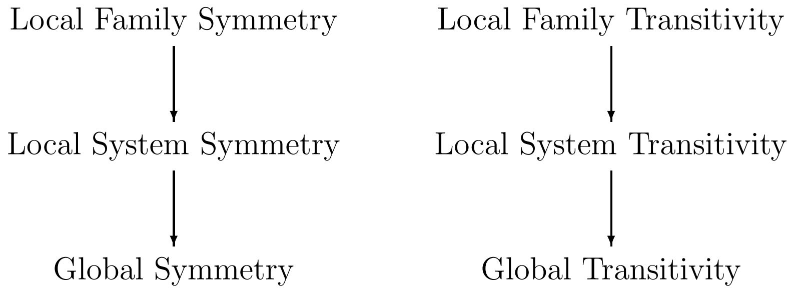
Let $\mathbf{F} = \langle \mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F},C\rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. Let $X,Y \in \{\mathrm{LF},\mathrm{LS},\mathrm{GB}\}$, where LF stands for "Local Family", LS stands for "Local System" and GB stands for "GloBal". We say that $I^\flat$ has the XY**-equivalence property in** $\mathcal{I}$ if it has

(a) reflexivity in $\mathcal{I}$;

(b) X symmetry in $\mathcal{I}$ and

(c) Y transitivity in $\mathcal{I}$.

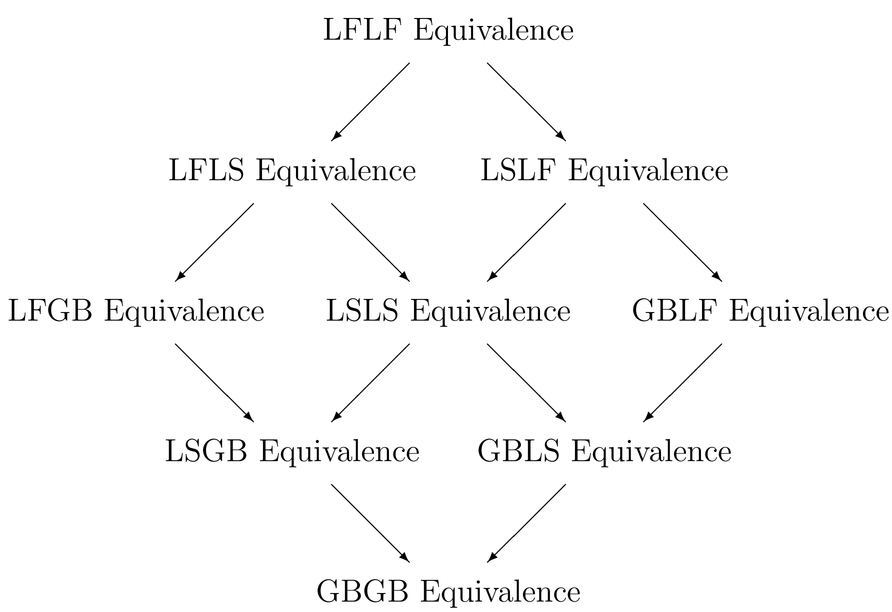
Recall the following hierarchies of symmetry and transitivity properties that we established previously:



From these, we can infer the following hierarchy of equivalence properties:

Corollary 694 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$*, with two distinguished arguments. The nine equivalence properties constitute the hierarchy depicted in the accompanying diagram.*

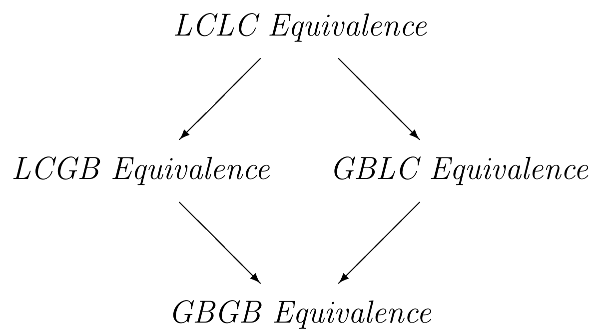
Proof: The statement is a direct consequence of Propositions 684 and 689.
■



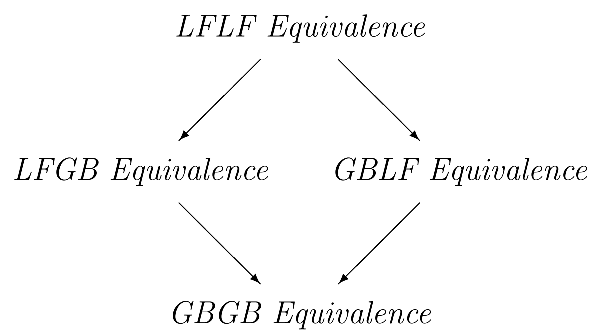
Based on the analysis performed on symmetry and transitivity, we have the following result regarding natural sufficient conditions under which some of the nine equivalence properties above coincide. We let LC stand for "LoCal" to summarize the case when the local family and the local system version of a property coincide.

Corollary 695 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$*, with two distinguished arguments.*

(a) If $\mathcal{I}$ *is systemic, then the equivalence hierarchy collapses to the one depicted below;*



(b) If $I^\flat$ *has only two arguments (i.e., is parameter free), then the equivalence hierarchy collapses to the one depicted below, where the local system versions coincide with (and, hence, are incorporated into) the global versions.*



Proof: The statement follows directly from Propositions 685 and 690. ■

For a systemic π-institution with a parameter-free set of natural transformations, there is only one equivalence property, since all versions of symmetry and all versions of transitivity collapse to a single property.

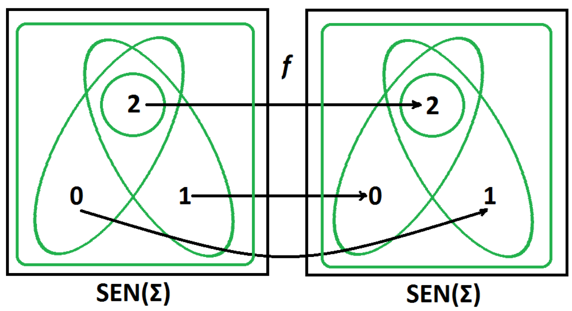
We provide some examples to show that the implications of Proposition 694 are not equivalences in general, i.e., that the nine classes of the equivalence hierarchy are all distinct.

First, we present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that has the LSLF equivalence, but not the LFGB equivalence in $\mathcal{I}$.

Example 696 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = i_\Sigma$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 1$*,* $1 \mapsto 0$ *and* $2 \mapsto 2$*;*
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given by*

$\sigma^\flat_\Sigma$	0	1	2
0	2	1	1
1	0	2	0
2	0	1	2



Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by*

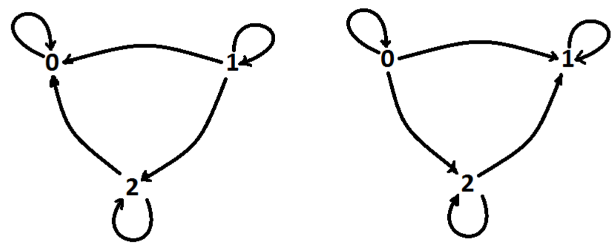
$$\mathcal{C}_\Sigma = \{\{2\},\{0,2\},\{1,2\},\{0,1,2\}\}.$$

Note that there are four theory families, but only $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$ *are theory systems.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the LSLF equivalence in* $\mathcal{I}$*, but it does not have the LFGB equivalence in* $\mathcal{I}$*.*

Note, first, that reflexivity is obvious, since, by definition $\sigma^\flat_\Sigma(x,x) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, for all* $x \in \mathrm{SEN}^\flat(\Sigma)$*. Local system symmetry is also obvious, since the only theory systems in* $\mathcal{I}$ *are* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*. Local family transitivity is a little more challenging to verify, but it suffices to observe that the pairs that are related modulo* $T = \{\{0,2\}\}$ *are as shown on the left below and the pairs that are related modulo* $T' = \{\{1,2\}\}$ *are as on the right below. We conclude that* $I^\flat$ *has the LSLF equivalence in* $\mathcal{I}$*.*

On the other hand, for $T = \{\{0,2\}\} \in \mathrm{ThFam}(\mathcal{I})$*, we have* $\sigma^\flat_\Sigma(1,0) = 0 \in T_\Sigma$*, but* $\sigma^\flat_\Sigma(0,1) = 1 \notin T_\Sigma$*. Therefore, the implication defining local family symmetry fails for* $T = \{\{0,2\}\}$*. So* $I^\flat$ *is not locally family symmetric, and, hence, a fortiori, does not have the LFGB equivalence property in* $\mathcal{I}$*.*

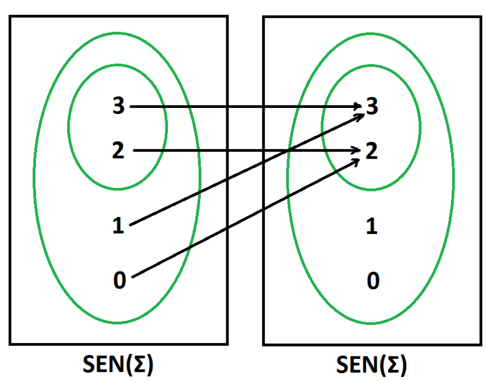


We now present an example to show that there is π-institution $\mathcal{I}$ with a set of natural transformations that has the GBLF equivalence but not the LSGB equivalence in $\mathcal{I}$.

Example 697 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, with* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2,3\} \to \{0,1,2,3\}$ *given by* $0 \mapsto 2$*,* $1 \mapsto 3$*,* $2 \mapsto 2$ *and* $3 \mapsto 3$*;*
- $N^\flat$ *is the category of natural transformations generated by the single ternary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *defined by letting:* $\sigma^\flat_\Sigma : \{0,1,2,3\}^3 \to \{0,1,2,3\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y,z) = \begin{cases} 2, & \text{if } x = y \text{ or } (x,y) = (0,1) \text{ or } z = 2 \text{ or } z = 3 \\ 0, & \text{otherwise} \end{cases}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2,3\}, \{0,1,2,3\}\}.$$

$\mathcal{I}$ *has two theory families,* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, both of which are also theory systems. So it is a systemic* π*-institution.*

Consider the set $I^\flat = \{\sigma^\flat\}$, *with* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the GBLF equivalence in* $\mathcal{I}$, *but it does not have the LSGB equivalence in* $\mathcal{I}$.

First, note that $\sigma^\flat_\Sigma(\phi,\phi,\psi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$, *for all* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$. *Thus,* $I^\flat$ *is reflexive in* $\mathcal{I}$. *For global symmetry, the case of* $T = \mathrm{SEN}^\flat$ *is trivial, whereas, for* $T = \mathrm{Thm}(\mathcal{I})$, *observe that, for no* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, *with* $\phi \neq \psi$, *is it the case that* $\sigma^\flat_\Sigma[\phi,\psi] \leq T$. *Thus, the defining condition holds trivially for* $\mathrm{Thm}(\mathcal{I})$. *So* $I^\flat$ *has the global symmetry in* $\mathcal{I}$. *For local family transitivity, the case of* $T = \mathrm{SEN}^\flat$ *is also trivial and for* $T = \mathrm{Thm}(\mathcal{I})$, *the only pair* $\langle\phi,\psi\rangle \in \mathrm{SEN}^\flat(\Sigma)$, *with* $\phi \neq \psi$, *for which* $\sigma^\flat_\Sigma(\phi,\psi,\xi) \subseteq \mathrm{Thm}_\Sigma(\mathcal{I})$, *for all* $\xi \in \mathrm{SEN}^\flat(\Sigma)$, *is the pair* $(\phi,\psi) = (0,1)$. *So the defining condition holds for* $\mathrm{Thm}(\mathcal{I})$ *also. Thus* $I^\flat$ *has the local family transitivity. We conclude that* $I^\flat$ *has the GBLF equivalence in* $\mathcal{I}$.

On the other hand, we have $\sigma^\flat_\Sigma(0,1,\xi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$, *for all* $\xi \in \mathrm{SEN}^\flat(\Sigma)$, *but* $\sigma^\flat_\Sigma(1,0,0) = 0 \notin \mathrm{Thm}_\Sigma(\mathcal{I})$. *So the implication defining local system symmetry fails for* $\mathrm{Thm}(\mathcal{I})$. *Therefore,* $I^\flat$ *does not have the local system symmetry in* $\mathcal{I}$.

Next, we present an example to show that there is π-institution $\mathcal{I}$ with a set of natural transformations that has the LFLS equivalence but not the GBLF equivalence in $\mathcal{I}$.

Example 698 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

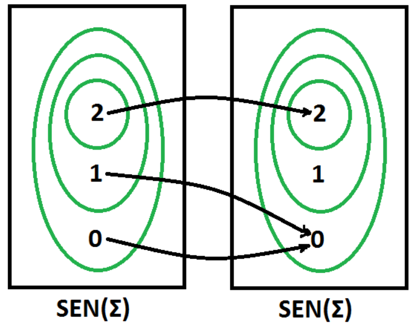
- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 0$, $1 \mapsto 0$ *and* $2 \mapsto 2$;
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given by the following table:*

$\sigma^\flat_\Sigma$	0	1	2
0	2	2	0
1	2	2	1
2	0	1	2

Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}.$$

Note that $\mathcal{I}$ *has three theory families, but only* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$ *are theory systems. So* $\mathcal{I}$ *is not systemic.*



Consider the set $I^\flat = \{\sigma^\flat\}$, *with* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the LFLS equivalence in* $\mathcal{I}$, *but it does not have the GBLF equivalence in* $\mathcal{I}$.

First, since $\sigma^\flat_\Sigma(\phi,\phi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$, *for all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $I^\flat$ *is reflexive in* $\mathcal{I}$. *Next, observe from the table that, for all* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, $\sigma^\flat_\Sigma(\phi,\psi) = \sigma^\flat_\Sigma(\psi,\phi)$. *Therefore, a fortiori, for all* $T \in \mathrm{ThFam}(\mathcal{I})$, *and all* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, *if* $\sigma^\flat_\Sigma(\phi,\psi) \in T_\Sigma$, *then* $\sigma^\flat_\Sigma(\psi,\phi) \in T_\Sigma$, *showing that* $I^\flat$ *has the local family symmetry in* $\mathcal{I}$. *For the local system transitivity, the defining implication is trivial in the case of* $\mathrm{SEN}^\flat$, *whereas in the case of* $\mathrm{Thm}(\mathcal{I})$, *it is straightforward to check based on the table defining* $\sigma^\flat_\Sigma$. *Thus,* $I^\flat$ *has indeed the LFLS equivalence in* $\mathcal{I}$.

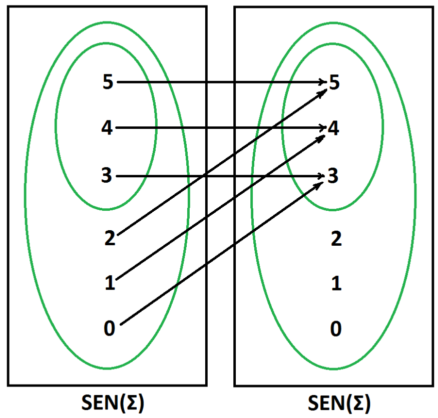
On the other hand, consider the theory family $T = \{\{1,2\}\}$. *We have* $\sigma^\flat_\Sigma(0,1) = 2$ *and* $\sigma^\flat_\Sigma(1,2) = 1$, *i.e.,* $\sigma^\flat_\Sigma(0,1), \sigma^\flat_\Sigma(1,2) \in T_\Sigma$, *whereas* $\sigma^\flat_\Sigma(0,2) = 0 \notin T_\Sigma$. *Therefore,* $I^\flat$ *does not have the local family transitivity and, hence, a fortiori, does not satisfy the GBLF equivalence property in* $\mathcal{I}$.

Finally, we present an example to show that there is π-institution $\mathcal{I}$ with a set of natural transformations that has the LFGB equivalence but not the GBLS equivalence in $\mathcal{I}$.

Example 699 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$, *with* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3,4,5\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2,3,4,5\} \to \{0,1,2,3,4,5\}$ *given by* $0 \mapsto 3$, $1 \mapsto 4$, $2 \mapsto 5$, $3 \mapsto 3$, $4 \mapsto 4$ *and* $5 \mapsto 5$;
- $N^\flat$ *is the category of natural transformations generated by the single ternary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *defined by letting:* $\sigma^\flat_\Sigma : \{0,1,2,3,4,5\}^3 \to \{0,1,2,3,4,5\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y,z) = \begin{cases} 3, & \text{if } x = y \text{ or } \{x,y\} = \{0,1\} \text{ or } \{x,y\} = \{1,2\} \\ & \quad \text{or } z = 3 \text{ or } z = 4 \text{ or } z = 5 \\ 0, & \text{otherwise} \end{cases}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be the π-institution determined by

$$\mathcal{C}_\Sigma = \{\{3,4,5\},\{0,1,2,3,4,5\}\}.$$

$\mathcal{I}$ has two theory families, $\mathrm{Thm}(\mathcal{I})$ and $\mathrm{SEN}^\flat$, both of which are also theory systems. So it is a systemic π-institution.

Consider the set $I^\flat = \{\sigma^\flat\}$, with $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ having two distinguished arguments. We show that $I^\flat$ has the LFGB equivalence in $\mathcal{I}$, but it does not have the GBLS equivalence in $\mathcal{I}$.

First, note that $\sigma^\flat_\Sigma(\phi,\phi,\psi) = 3 \in \mathrm{Thm}_\Sigma(\mathcal{I})$, for all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$. Thus, $I^\flat$ is reflexive in $\mathcal{I}$. For local family symmetry, the case of $T = \mathrm{SEN}^\flat$ is trivial, whereas, for $T = \mathrm{Thm}(\mathcal{I})$, observe that, if $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, with $\phi \neq \psi$, are such that $\sigma^\flat_\Sigma(\phi,\psi,\xi) \in \mathrm{Thm}_\Sigma(\mathcal{I})$, for all $\xi \in \mathrm{SEN}^\flat(\Sigma)$, then $\{x,y\} = \{0,1\}$ or $\{x,y\} = \{1,2\}$. Thus, $I^\flat$ is local family symmetric. For global transitivity, the case of $T = \mathrm{SEN}^\flat$ is also trivial and for $T = \mathrm{Thm}(\mathcal{I})$, there is no pair $\langle\phi,\psi\rangle \in \mathrm{SEN}^\flat(\Sigma)$, with $\phi \neq \psi$, for which $\sigma^\flat_\Sigma[\phi,\psi] \subseteq \mathrm{Thm}(\mathcal{I})$. So the defining condition holds trivially for $\mathrm{Thm}(\mathcal{I})$ also. Thus $I^\flat$ has the global transitivity. We conclude that $I^\flat$ has the LFGB equivalence in $\mathcal{I}$.

On the other hand, we have $\sigma^\flat_\Sigma(0,1,\xi) = 3 \in \mathrm{Thm}_\Sigma(\mathcal{I})$ and $\sigma^\flat_\Sigma(1,2,\xi) = 3 \in \mathrm{Thm}_\Sigma(\mathcal{I})$, for all $\xi \in \mathrm{SEN}^\flat(\Sigma)$, but $\sigma^\flat_\Sigma(0,2,0) = 0 \notin \mathrm{Thm}_\Sigma(\mathcal{I})$. So the implication defining local system transitivity fails for $\mathrm{Thm}(\mathcal{I})$. Therefore, $I^\flat$ does not have the local system transitivity in $\mathcal{I}$.

We close the study of equivalence by providing, again, a transfer property.

Corollary 700 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $I^\flat$ a collection of natural transformations $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with two distinguished arguments, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $I^\flat$ has a transitivity property in $\mathcal{I}$ if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, I has the corresponding transitivity property in $\mathcal{A}$.*

Proof: This follows directly from Propositions 688 and 693. ■

10.6 Antisymmetry

We look next at the antisymmetry property.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. We say that:

- $I^\flat$ has the **local antisymmetry in** $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq \mathrm{Thm}_\Sigma(\mathcal{I})$ and $I^\flat_\Sigma(\psi, \phi, \vec{\xi}) \subseteq \mathrm{Thm}_\Sigma(\mathcal{I})$, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$, imply $\phi = \psi$;

- $I^\flat$ has the **global antisymmetry in** $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma[\phi, \psi] \leq \mathrm{Thm}(\mathcal{I})$ and $I^\flat_\Sigma[\psi, \phi] \leq \mathrm{Thm}(\mathcal{I})$ imply $\phi = \psi$.

The antisymmetry properties stratify in the following hierarchy.

Proposition 701 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$*, with two distinguished arguments. If* $I^\flat$ *has the local antisymmetry in* $\mathcal{I}$*, then it has the global antisymmetry in* $\mathcal{I}$*.*

Proof: Suppose that $I^\flat$ has the local antisymmetry and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $I^\flat_\Sigma[\phi, \psi] \leq \mathrm{Thm}(\mathcal{I})$ and $I^\flat_\Sigma[\psi, \phi] \leq \mathrm{Thm}(\mathcal{I})$. Then we get, in particular, that, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq \mathrm{Thm}_\Sigma(\mathcal{I})$ and $I^\flat_\Sigma(\psi, \phi, \xi) \subseteq \mathrm{Thm}_\Sigma(\mathcal{I})$. Thus, by local antisymmetry, we obtain $\phi = \psi$. We conclude that $I^\flat$ has the global antisymmetry in $\mathcal{I}$. ■

Proposition 701 has established the following hierarchy of antisymmetry properties:

Local Antisymmetry

↓

Global Antisymmetry

We look, next, at a natural sufficient condition under which the antisymmetry properties coincide.

Proposition 702 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$*, with two distinguished arguments. If* $I^\flat$ *has only two arguments (i.e., is parameter free), then the local antisymmetry and the global antisymmetry properties coincide.*

Proof: Suppose that $I^\flat$ is parameter free and has the global antisymmetry in $\mathcal{I}$. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $I^\flat_\Sigma(\phi, \psi) \subseteq \mathrm{Thm}_\Sigma(\mathcal{I})$ and $I^\flat_\Sigma(\psi, \phi) \subseteq \mathrm{Thm}_\Sigma(\mathcal{I})$. Then, by Proposition 99, $I^\flat_\Sigma[\phi, \psi] \leq \mathrm{Thm}(\mathcal{I})$ and $I^\flat_\Sigma[\psi, \phi] \leq \mathrm{Thm}(\mathcal{I})$. Thus, by global antisymmetry, $\phi = \psi$. Therefore, $I^\flat$ has the local antisymmetry in $\mathcal{I}$. ■

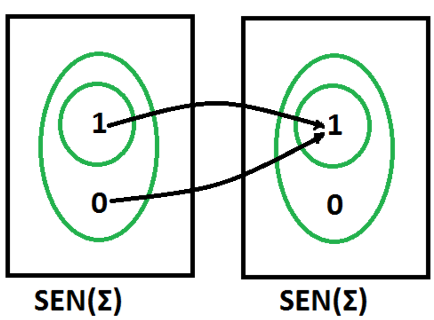
So in the case of a parameter-free set of natural transformations we have a single antisymmetry property.

We provide an example to show that the implication of Proposition 701 is not an equivalence in general. That is, we provide an example of a π-institution $\mathcal{I}$ with a set $I^\flat$ of natural transformations, with two distinguished arguments, that has the global antisymmetry but not the local antisymmetry in $\mathcal{I}$.

Example 703 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single objects* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$ *and* $\mathrm{SEN}^\flat(f) : \{0, 1\} \to \{0, 1\}$ *given by* $0 \mapsto 1$ *and* $1 \mapsto 1$*;*
- $N^\flat$ *is the category of natural transformations generated by the single ternary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0, 1\}^3 \to \{0, 1\}$ *be given by*

$$\sigma^\flat_\Sigma(x, y, z) = \begin{cases} 0, & \textit{if } (x, y, z) = (1, 1, 0) \\ 1, & \textit{otherwise} \end{cases}$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\}, \{0, 1\}\}$*. Note that there are two theory families,* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, both of which are theory systems.*

Consider the set $I^\flat = \{\sigma^\flat\}$, *with* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the global antisymmetry in* $\mathcal{I}$, *but it does not have the local antisymmetry in* $\mathcal{I}$.

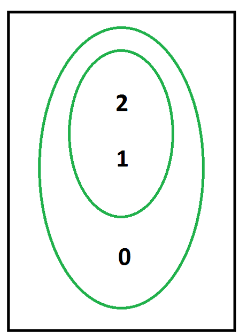
To see that $I^\flat$ *has the global antisymmetry in* $\mathcal{I}$, *it suffices to notice that, for no* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$ *is it the case that* $I^\flat_\Sigma[\phi,\psi] \le \mathrm{Thm}(\Sigma)$. *Therefore, the defining condition holds vacuously, for all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$.

On the other hand, for $0 \ne 1$, *we have* $\sigma^\flat_\Sigma(0,1,\xi) = \sigma^\flat_\Sigma(1,0,\xi) = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$, *for all* $\xi \in \{0,1\}$. *So* $I^\flat$ *is not locally antisymmetric in* $\mathcal{I}$.

To close the study of antisymmetry properties, we show that they do not transfer from π-institutions to their models. This is to be expected, since the inverse image $\alpha^{-1}(T)$ of the minimum $\mathcal{I}$ filter family of a π-institution $\mathcal{I}$ on an algebraic system $\mathcal{A}$ may not coincide with the theorem system of $\mathcal{I}$.

Example 704 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with a single object* Σ;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$;
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting:* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given by* $\sigma^\flat_\Sigma(x,y) = 0$, *for all* $x, y \in \mathrm{SEN}^\flat(\Sigma)$.



SEN(Σ)

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{1,2\},\{0,1,2\}\}.$$

Consider the set $I^\flat = \{\sigma^\flat\}$, *with* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the local antisymmetry in* $\mathcal{I}$, *but that there exists an* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, *with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, *such that* I *does not have the local antisymmetry in* $\mathcal{A}$.

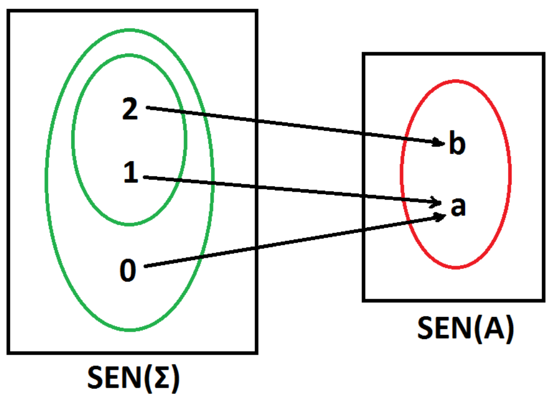
Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *be the algebraic system determined as follows:*

- **Sign** *is the trivial category with a single object* A*;*
- $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}(A) = \{a, b\}$*;*
- N *is the category of natural transformations generated by the single binary natural transformation* $\sigma : \mathrm{SEN}^2 \to \mathrm{SEN}$ *defined by letting:* $\sigma_A : \{a, b\}^2 \to \{a, b\}$ *be given by* $\sigma_A(x, y) = a$*, for all* $x, y \in \mathrm{SEN}(A)$*.*

$\mathbf{A}$ *is an* $N^\flat$*-algebraic system, as can be seen by sending* $\sigma^\flat \mapsto \sigma$ *and extending to categories by composition.*

Now let $\langle F, \alpha\rangle : \mathbf{F} \to \mathbf{A}$ *be the morphism defined as follows:*

- $F : \mathbf{Sign}^\flat \to \mathbf{Sign}$ *is the obvious functor between trivial categories;*
- $\alpha : \mathrm{SEN}^\flat \to \mathrm{SEN} \circ F$ *is defined by setting* $\alpha_\Sigma(0) = a$*,* $\alpha_\Sigma(1) = a$ *and* $\alpha_\Sigma(2) = b$*.*



Our goal is to show that $I^\flat$ *has the local antisymmetry in* $\mathcal{I}$ *but that* I *does not have the local antisymmetry in* $\mathcal{A}$*. We have, for all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,* $\sigma^\flat_\Sigma(\phi, \psi) \notin \mathrm{Thm}_\Sigma(\mathcal{I})$ *and* $\sigma^\flat_\Sigma(\psi, \phi) \notin \mathrm{Thm}_\Sigma(\mathcal{I})$*, whence the defining condition of local antisymmetry for* $I^\flat$ *is vacuously true. So* $I^\flat$ *is locally antisymmetric in* $\mathcal{I}$*.*

On the other hand, note that the least $\mathcal{I}$*-filter system on* $\mathcal{A}$ *is* SEN*. Moreover, we have* $\sigma_A(a, b) = \sigma_A(b, a) = a \in \mathrm{SEN}(A)$*, with* $a \neq b$*. Thus* $I = \{\sigma\}$ *does not have local antisymmetry in* $\mathcal{A}$*.*

10.7 Order

We look next at sets of natural transformations $I^\flat$, with two distinguished arguments, that define (modulo theory families) partial order families on the underlying algebraic system of a π-institution $\mathcal{I}$. We assume that $I^\flat$ has the reflexivity property and study combinations of possible antisymmetry and transitivity properties that the set of connectives may or may not possess.

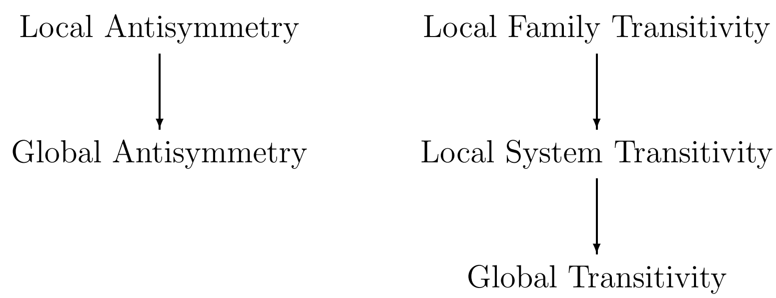
Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. Let $X \in \{LC, GB\}$, where LC and GB stand for "LoCal" and "GloBal", respectively, and let $Y \in \{LF, LS, GB\}$, where LF stands for "Local Family" and LS for "Local System". We say that $I^\flat$ has the XY **poset property in** $\mathcal{I}$ if it has

(a) reflexivity in $\mathcal{I}$;

(b) X antisymmetry in $\mathcal{I}$ and

(c) Y transitivity in $\mathcal{I}$.

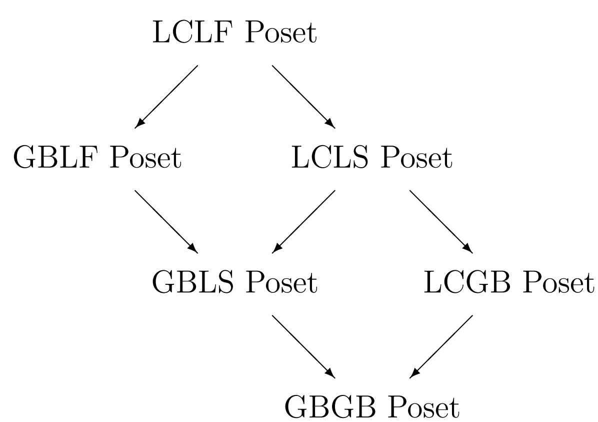
Recall, again, the following hierarchies of antisymmetry and of transitivity properties:



From these, we can infer the following hierarchy of equivalence properties:

Corollary 705 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$*, with two distinguished arguments. The six poset properties of* $I^\flat$ *satisfy the hierarchy depicted in the accompanying diagram.*

Proof: The statement is a direct consequence of Propositions 701 and 689. ■



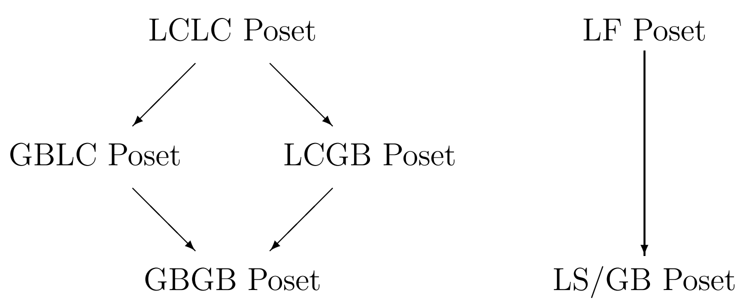
Based on the analyses performed on antisymmetry and transitivity, we have the following result regarding natural sufficient conditions under which some of these poset properties coincide.

Corollary 706 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$*, with two distinguished arguments.*

(a) If $\mathcal{I}$ *is systemic, then the equivalence hierarchy collapses to the one depicted on the left of the accompanying diagram;*

(b) If $I^\flat$ *has only two arguments (i.e., is parameter free), then the equivalence hierarchy collapses to the one depicted on the right of the diagram, where, since there is only one antisymmetry property, the qualifications refer to the type of transitivity that holds.*

Proof: The statement follows directly from Propositions 690 and 702. ∎



For a systemic π-institution with a parameter-free set of natural transformations, there is only one poset property, since the two versions of antisymmetry and all three versions of transitivity collapse, respectively, to a single property.

We provide some examples to show that the implications of Corollary 705 are not equivalences, i.e., the six classes of the poset hierarchy are all distinct in general.

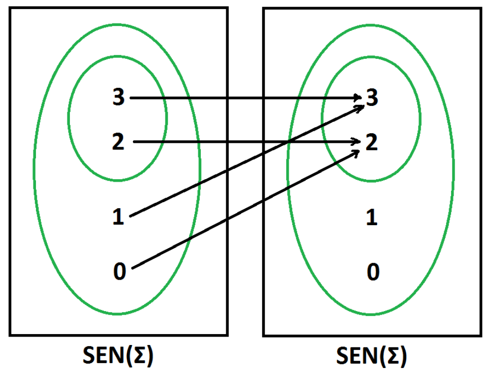
First, we present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that has the GBLF poset property, but not the LCGB poset property in $\mathcal{I}$.

Example 707 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2, 3\}$ *and* $\mathrm{SEN}^\flat(f) : \{0, 1, 2, 3\} \to \{0, 1, 2, 3\}$ *given by* $0 \mapsto 2$*,* $1 \mapsto 3$*,* $2 \mapsto 2$ *and* $3 \mapsto 3$*;*
- $N^\flat$ *is the category of natural transformations generated by the single ternary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *defined by letting*

$\sigma^\flat_\Sigma : \{0,1,2,3\}^3 \to \{0,1,2,3\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y,z) = \begin{cases} 2, & \text{if } x = y \text{ or } (x,y) = (0,1) \text{ or } (x,y) = (1,0) \\ & \quad \text{or } z = 2 \text{ or } z = 3 \\ 0, & \text{otherwise} \end{cases}$$



Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2,3\},\{0,1,2,3\}\}.$$

Note that both theory families, $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, are also theory systems. So* $\mathcal{I}$ *is a systemic* π*-institution.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the GBLF poset property in* $\mathcal{I}$*, but it does not have the LCGB poset property in* $\mathcal{I}$*.*

Note, first, that reflexivity is obvious, since, by definition, for all $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,* $\sigma^\flat_\Sigma(\phi,\phi,\xi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, for all* $\xi \in \mathrm{SEN}^\flat(\Sigma)$*. For global antisymmetry, note that if, for some* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$*,* $\sigma^\flat_\Sigma[\phi,\psi] \leq \mathrm{Thm}(\mathcal{I})$ *and* $\sigma^\flat_\Sigma[\psi,\phi] \leq \mathrm{Thm}(\mathcal{I})$*, then we must have* $\phi = \psi$*. Finally, for local family transitivity, the defining equation holds trivially for* $T = \mathrm{SEN}^\flat$*, whereas, if for some* $\phi,\psi,\chi \in \mathrm{SEN}^\flat(\Sigma)$*, with* $\phi \neq \chi$*,* $\sigma^\flat_\Sigma(\phi,\psi,\xi) \in \{2,3\}$ *and* $\sigma^\flat_\Sigma(\psi,\chi,\xi) \in \{2,3\}$*, for all* $\xi \in \mathrm{SEN}^\flat(\Sigma)$*, we must have* $\phi = \psi$ *or* $\psi = \chi$*, whence the condition is satisfied in this case as well. Thus,* $I^\flat$ *is also locally family transitive in* $\mathcal{I}$ *and, therefore, has the GBLF poset property in* $\mathcal{I}$*.*

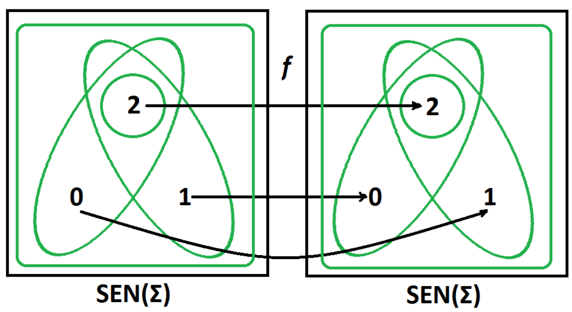
On the other hand, since $\sigma^\flat_\Sigma(0,1,\xi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$ *and* $\sigma^\flat_\Sigma(1,0,\xi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, for all* $\xi \in \mathrm{SEN}^\flat(\Sigma)$*,* $I^\flat$ *does not have the local antisymmetry property. A fortiori,* $I^\flat$ *does not have the LCGB poset property in* $\mathcal{I}$*.*

Next, we present an example to show that there is a π-institution $\mathcal{I}$ with a set of natural transformations that has the LCLS poset property but not the GBLF poset property in $\mathcal{I}$.

Example 708 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$, *with* $f \circ f = i_\Sigma$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 1$, $1 \mapsto 0$ *and* $2 \mapsto 2$;
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting:* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y,z) = \begin{cases} 0, & \text{if } (x,y) = (0,1) \text{ or } (x,y) = (1,2) \\ 1, & \text{if } (x,y) = (1,0) \text{ or } (x,y) = (0,2) \\ 2, & \text{otherwise} \end{cases}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

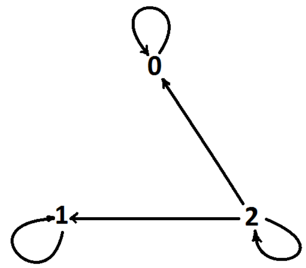
$$\mathcal{C}_\Sigma = \{\{2\},\{0,2\},\{1,2\},\{0,1,2\}\}.$$

$\mathcal{I}$ *has four theory families, but only* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$ *are theory systems. So* $\mathcal{I}$ *is not systemic.*

Consider the set $I^\flat = \{\sigma^\flat\}$, *with* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the LCLS poset property but it does not have the GBLF poset property in* $\mathcal{I}$.

First, note that $\sigma^\flat_\Sigma(\phi,\phi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$, *for all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$. *Thus,* $I^\flat$ *is reflexive in* $\mathcal{I}$. *For the local antisymmetry, note that for no* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, *with* $\phi \neq \psi$ *is it the case that both* $\sigma^\flat_\Sigma(\phi,\psi) = \sigma^\flat_\Sigma(\psi,\phi) = 2$. *Finally, for the local system transitivity, the defining condition is trivially satisfied for* $\mathrm{SEN}^\flat$, *whereas the pairs related modulo* $\mathrm{Thm}(\mathcal{I})$ *are as in the following diagram, an examination of which verifies transitivity. Therefore* $I^\flat$ *is also locally system transitive in* $\mathcal{I}$ *and, hence has the LCLS poset property in* $\mathcal{I}$.

On the other hand, for $T = \{\{0,2\}\} \in \mathrm{ThFam}(\mathcal{I})$, *we have* $\sigma^\flat_\Sigma(0,1) = \sigma^\flat_\Sigma(1,2) = 0 \in T_\Sigma$, *whereas* $\sigma^\flat_\Sigma(0,2) = 1 \notin T_\Sigma$. *So the implication defining local family transitivity fails for* T. *Therefore,* $I^\flat$ *does not have the local family transitivity and, a fortiori, does not have the GBLF poset property in* $\mathcal{I}$.

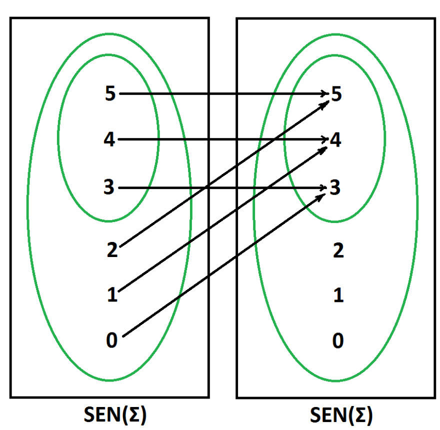


Finally, we look at an example that shows that there is π-institution $\mathcal{I}$ with a set of natural transformations that has the LCGB poset property but not the GBLS poset property in $\mathcal{I}$.

Example 709 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, with* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3,4,5\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2,3,4,5\} \to \{0,1,2,3,4,5\}$ *given by* $0 \mapsto 3$*,* $1 \mapsto 4$*,* $2 \mapsto 5$*,* $3 \mapsto 3$*,* $4 \mapsto 4$ *and* $5 \mapsto 5$*;*
- $N^\flat$ *is the category of natural transformations generated by the single ternary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *defined by letting:* $\sigma^\flat_\Sigma : \{0,1,2,3,4,5\}^3 \to \{0,1,2,3,4,5\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y,z) = \begin{cases} 3, & \text{if } x = y \text{ or } (x,y) = (0,1) \text{ or } (x,y) = (1,2) \\ & \quad \text{or } z = 3 \text{ or } z = 4 \text{ or } z = 5 \\ 0, & \text{otherwise} \end{cases}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{3,4,5\},\{0,1,2,3,4,5\}\}.$$

$\mathcal{I}$ has two theory families, $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, both of which are also theory systems. So it is a systemic π-institution.*

Consider the set $I^\flat = \{\sigma^\flat\}$, with $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ having two distinguished arguments. We show that $I^\flat$ has the LCGB poset property in $\mathcal{I}$, but not the GBLS poset property in $\mathcal{I}$.

First, note that, for all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\sigma^\flat_\Sigma(\phi,\phi,\xi) = 3 \in \mathrm{Thm}_\Sigma(\mathcal{I})$, for all $\xi \in \mathrm{SEN}^\flat(\Sigma)$. Thus, $I^\flat$ is reflexive in $\mathcal{I}$. For local antisymmetry, it suffices to observe that, for no $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, with $\phi \neq \psi$, is it the case that $\sigma^\flat_\Sigma(\phi,\psi,\xi) \in \mathrm{Thm}_\Sigma(\mathcal{I})$ and $\sigma^\flat_\Sigma(\psi,\phi,\xi) \in \mathrm{Thm}_\Sigma(\mathcal{I})$, for all $\xi \in \mathrm{SEN}^\flat(\Sigma)$. For global transitivity, the defining condition holds trivially for $T = \mathrm{SEN}^\flat$, whereas for $T = \mathrm{Thm}(\mathcal{I})$, it suffices to note that, for no $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, with $\phi \neq \psi$, is it the case that $\sigma^\flat_\Sigma[\phi,\psi] \leq \mathrm{Thm}(\mathcal{I})$. We conclude that $I^\flat$ has the LCGB poset property in $\mathcal{I}$.

On the other hand, we have $\sigma^\flat_\Sigma(0,1,\xi) = 3 \in \mathrm{Thm}_\Sigma(\mathcal{I})$ and $\sigma^\flat_\Sigma(1,2,\xi) = 3 \in \mathrm{Thm}_\Sigma(\mathcal{I})$, for all $\xi \in \mathrm{SEN}^\flat(\Sigma)$, but $\sigma^\flat_\Sigma(0,2,0) = 0 \notin \mathrm{Thm}_\Sigma(\mathcal{I})$. So the implication defining local system transitivity fails for $\mathrm{Thm}(\mathcal{I})$*. Therefore, $I^\flat$ does not have the local system transitivity in $\mathcal{I}$ and, hence, a fortiori, it does not have the GBLS poset property in $\mathcal{I}$.*

Because of the non-transference of antisymmetry, which was shown in Example 704, it is to be expected that none of the poset properties transfers from a π-institution to its models. We provide an example that showcases a π-institution $\mathcal{I}$, with a set $I^\flat$ of natural transformations having two distinguished arguments, that has the LCLF poset property, but one of whose models does not have the GBGB poset property.

Example 710 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be the algebraic system determined as follows:*

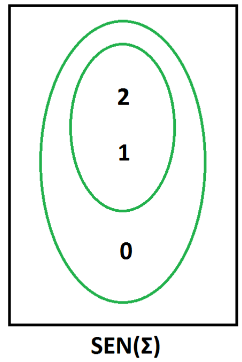
- $\mathbf{Sign}^\flat$ *is the trivial category with a single object Σ;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$*;*
- *$N^\flat$ is the category of natural transformations generated by the single binary natural transformation $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ defined by letting: $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ be given, for all $x,y \in \mathrm{SEN}^\flat(\Sigma)$, by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}.$$

Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be the π-institution determined by $\mathcal{C}_\Sigma = \{\{1,2\},\{0,1,2\}\}$.

Consider the set $I^\flat = \{\sigma^\flat\}$, with $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ having two distinguished arguments. We show that $I^\flat$ has the LCLF poset property in $\mathcal{I}$, but that there exists an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, such that I does not have the GBGB poset property in $\mathcal{A}$.

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be the algebraic system determined as follows:



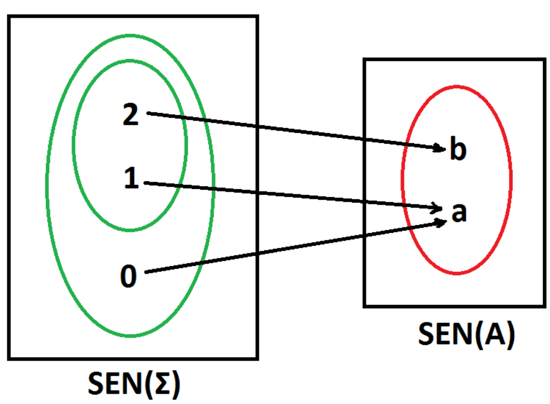
- **Sign** *is the trivial category with a single object* A*;*
- $\text{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *is specified by* $\text{SEN}(A) = \{a, b\}$*;*
- N *is the category of natural transformations generated by the single binary natural transformation* $\sigma : \text{SEN}^2 \to \text{SEN}$ *defined by letting:* $\sigma_A : \{a, b\}^2 \to \{a, b\}$ *be given, for all* $x, y \in \text{SEN}(A)$*, by*

$$\sigma_A(x, y) = a.$$

$\mathbf{A}$ *is an* $N^\flat$*-algebraic system, as can be seen by sending* $\sigma^\flat \mapsto \sigma$ *and extending to categories by composition.*

Now let $\langle F, \alpha \rangle : \mathbf{F} \to \mathbf{A}$ *be the morphism defined as follows:*

- $F : \mathbf{Sign}^\flat \to \mathbf{Sign}$ *is the obvious functor between trivial categories;*
- $\alpha : \text{SEN}^\flat \to \text{SEN} \circ F$ *is defined by setting* $\alpha_\Sigma(0) = a$*,* $\alpha_\Sigma(1) = a$ *and* $\alpha_\Sigma(2) = b$*.*



We show that $I^\flat$ *has the LCLF poset property in* $\mathcal{I}$ *but that* I *does not have the GBGB poset property in* $\mathcal{A}$*.*

First, since $\sigma^\flat_\Sigma(\phi,\phi) = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, we have that* $I^\flat$ *is reflexive in* $\mathcal{I}$*. Second, if* $\sigma^\flat_\Sigma(\phi,\psi) = 1 = \sigma^\flat_\Sigma(\psi,\phi)$*, then* $\phi = \psi$*. So* $I^\flat$ *has the local antisymmetry in* $\mathcal{I}$*. Finally, local family transitivity is obvious, since for no* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$*, with* $\phi \neq \psi$ *is it the case that* $\sigma^\flat_\Sigma(\phi,\psi) = 1$*. We conclude that* $I^\flat$ *has the LCLF poset property in* $\mathcal{I}$*.*

On the other hand, note that the least $\mathcal{I}$*-filter system on* $\mathcal{A}$ *is* SEN *and since* $\sigma_A(a,b) = \sigma_A(b,a) = a \in \mathrm{SEN}(A)$*, with* $a \neq b$*,* $I = \{\sigma\}$ *does not have the global antisymmetry in* $\mathcal{A}$*. So, a fortiori, it does not have the GBGB poset property in* $\mathcal{A}$*.*

10.8 Compatibility

We look next at various versions of the compatibility property, taking again into account both the duality between local versus global membership and the difference between considering all theory families versus restricting only to theory systems.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. We say that:

- $I^\flat$ has the **local family compatibility in** $\mathcal{I}$ if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, $\overleftrightarrow{I^\flat}_\Sigma(\phi,\psi,\vec{\xi}) \subseteq T_\Sigma$, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$, implies
$$I^\flat_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}),\vec{\xi}) \subseteq T_{\Sigma'},$$
for all $\sigma^\flat \in N^\flat$, $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and $\vec{\chi},\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$;
- $I^\flat$ has the **local system compatibility in** $\mathcal{I}$ if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, $\overleftrightarrow{I^\flat}_\Sigma(\phi,\psi,\vec{\xi}) \subseteq T_\Sigma$, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$, implies
$$I^\flat_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}),\vec{\xi}) \subseteq T_{\Sigma'},$$
for all $\sigma^\flat \in N^\flat$, $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and $\vec{\chi},\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$;
- $I^\flat$ has the **global family compatibility in** $\mathcal{I}$ if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, $\overleftrightarrow{I^\flat}_\Sigma[\phi,\psi] \leq T$ implies
$$I^\flat_{\Sigma'}[\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi})] \leq T,$$
for all $\sigma^\flat \in N^\flat$, $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$;

- $I^\flat$ has the **global system compatibility in** $\mathcal{I}$ if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi] \le T$ implies

$$I^\flat_{\Sigma'}[\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}), \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi})] \le T,$$

for all $\sigma^\flat \in N^\flat$, $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$.

The following proposition establishes a hierarchy of compatibility properties.

Proposition 711 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$*, with two distinguished arguments.*

(a) If $I^\flat$ *has the local family compatibility, then it has the local system compatibility in* $\mathcal{I}$*;*

(b) If $I^\flat$ *has the local system compatibility, then it has the global family compatibility in* $\mathcal{I}$*;*

(c) $I^\flat$ *has the global family compatibility in* $\mathcal{I}$ *if and only if it has the global system compatibility in* $\mathcal{I}$*.*

Proof: Part (a) and one of the implications in Part (c) follow directly from the fact that every theory system of $\mathcal{I}$ is also a theory family of $\mathcal{I}$.

For Part (b), suppose that $I^\flat$ has the local system compatibility in $\mathcal{I}$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi] \le T$. Then by Lemma 93, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\overleftrightarrow{I^\flat}_{\Sigma'}[\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi)] \le T.$$

This implies, by Lemma 99, that, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\overleftrightarrow{I^\flat}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), \vec{\xi}) \subseteq \overleftarrow{T}_{\Sigma'}.$$

Since $I^\flat$ has the local system compatibility and, by Proposition 42, $\overleftarrow{T} \in \mathrm{ThSys}(\mathcal{I})$, we get that, for all $\sigma^\flat \in N^\flat$, all $\Sigma'' \in |\mathbf{Sign}^\flat|$, all $g \in \mathbf{Sign}^\flat(\Sigma', \Sigma'')$

$$\Sigma \xrightarrow{\ f\ } \Sigma' \xrightarrow{\ g\ } \Sigma''$$

and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$ and $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma'')$,

$$\begin{aligned} I^\flat_{\Sigma''}(&\sigma^\flat_{\Sigma''}(\mathrm{SEN}^\flat(gf)(\phi), \mathrm{SEN}^\flat(g)(\vec{\chi})),\\ &\sigma^\flat_{\Sigma''}(\mathrm{SEN}^\flat(gf)(\psi), \mathrm{SEN}^\flat(g)(\vec{\chi})), \vec{\xi}) \subseteq T_{\Sigma''},\end{aligned}$$

or, equivalently,

$$\begin{aligned}I^\flat_{\Sigma''}(&\mathrm{SEN}^\flat(g)(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi})),\\ &\qquad\mathrm{SEN}^\flat(g)(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi})),\vec{\xi}) \subseteq T_{\Sigma''}.\end{aligned}$$

Since this holds for all $\Sigma'' \in |\mathbf{Sign}^\flat|$, $g \in \mathbf{Sign}^\flat(\Sigma',\Sigma'')$ and $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma'')$, we conclude that $I^\flat_{\Sigma'}[\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi})] \leq T$. Therefore $I^\flat$ has the global family compatibility in $\mathcal{I}$.

Suppose, finally, that $I^\flat$ has the global system compatibility in $\mathcal{I}$ and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\overleftrightarrow{I^\flat}_\Sigma[\phi,\psi] \leq T$. By Lemma 99, we get that $\overleftrightarrow{I^\flat}_\Sigma[\phi,\psi] \leq \overleftarrow{T}$. Since $I^\flat$ has the global system compatibility and, by Proposition 42, $\overleftarrow{T} \in \mathrm{ThSys}(\mathcal{I})$, we get that, for all $\sigma^\flat \in N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$, $I^\flat_{\Sigma'}[\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi})] \leq \overleftarrow{T}$. Using again Lemma 99, we conclude that $I^\flat_{\Sigma'}[\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi})] \leq T$. Therefore, $I^\flat$ has the global family compatibility in $\mathcal{I}$. ■

Proposition 711 has established the following hierarchy of compatibility properties:

Local Family Compatibility

↓

Local System Compatibility

↓

Global Compatibility

We look, next, at some natural sufficient conditions under which some of these three compatibility properties coincide.

Proposition 712 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$*, with two distinguished arguments.*

(a) If $\mathcal{I}$ *is systemic, then the local family and the local system compatibility coincide;*

(b) If $I^\flat$ *has only two arguments (i.e., is parameter free), then the local system compatibility and the global compatibility coincide.*

Proof: If $\mathcal{I}$ is systemic, then all theory families are theory systems and, hence, the local family and local system compatibility properties coincide.

Suppose, next that $I^\flat$ is parameter free and has the global system compatibility in $\mathcal{I}$. Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$\overleftrightarrow{I}^\flat{}_\Sigma(\phi,\psi) \subseteq T_\Sigma$. Then, by Proposition 99, $\overleftrightarrow{I}^\flat{}_\Sigma[\phi,\psi] \leq T$. Thus, by the global system compatibility, for all $\sigma^\flat \in N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$, $I^\flat_{\Sigma'}[\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi})] \leq T$, which implies that, for all $\sigma^\flat \in N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$, $I^\flat_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi})) \subseteq T_{\Sigma'}$. Therefore, $I^\flat$ has the local system compatibility in $\mathcal{I}$. ■

So in the case of a systemic π-institution $\mathcal{I}$, we have the hierarchy pictured on the left, whereas in the case of a parameter-free set of natural transformations we have the hierarchy on the right.

$$\begin{array}{ccc} \text{Local Compatibility} & \qquad & \text{Local Family Compatibility} \\ \downarrow & & \downarrow \\ \text{Global Compatibility} & & \text{Local System/Global Compatibility} \end{array}$$

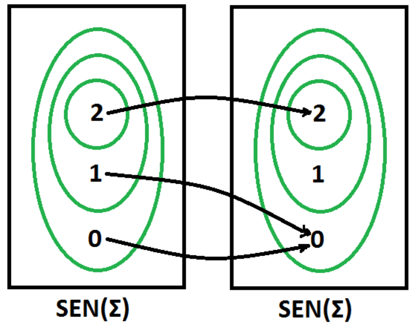
Of course, for a systemic π-institution with a parameter-free set of natural transformations all four compatibility properties coincide.

We provide some examples to show that the implications of Proposition 711 are not equivalences in general, i.e., in the 3-class hierarchy all inclusions of classes of π-institutions with a set of natural transformations satisfying the corresponding compatibility properties are proper inclusions.

We first present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that has the local system compatibility but not the local family compatibility in $\mathcal{I}$.

Example 713 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single objects* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 0$*,* $1 \mapsto 0$ *and* $2 \mapsto 2$*;*
- $N^\flat$ *is the category of natural transformations generated by two binary natural transformations:*
  - $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given by*
  $$\sigma^\flat_\Sigma(x,y) = \begin{cases} 2, & \text{if } x = 2 \text{ or } y = 2 \\ 1, & \text{if } \{x,y\} = \{0,1\} \\ 0, & \text{otherwise} \end{cases}$$
  - $\lambda^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\lambda^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given by*
  $$\lambda^\flat_\Sigma(x,y) = \begin{cases} 2, & \text{if } x = 2 \text{ or } y = 2 \\ 0, & \text{otherwise} \end{cases}$$



Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}.$$

Note that there are three theory families, but only $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$ *are theory systems.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the local system compatibility in* $\mathcal{I}$*, but it does not have the local family compatibility in* $\mathcal{I}$*.*

For the local system compatibility note that, if $T = \mathrm{SEN}^\flat$*, then the defining implication is trivially true. If, on the other hand,* $T = \mathrm{Thm}(\mathcal{I})$*, then* $\sigma^\flat_\Sigma(\phi,\psi) = 2$ *if and only if* $\phi = 2$ *or* $\psi = 2$*. But then we get, for all* $\chi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\begin{array}{l}\sigma^\flat_\Sigma(\sigma^\flat_\Sigma(\phi,\chi),\sigma^\flat_\Sigma(\psi,\chi)) = 2,\\ \sigma^\flat_\Sigma(\lambda^\flat_\Sigma(\phi,\chi),\lambda^\flat_\Sigma(\psi,\chi)) = 2.\end{array}$$

So $I^\flat$ *has the local system compatibility in* $\mathcal{I}$*.*

To see that $I^\flat$ *does not have the local family compatibility in* $\mathcal{I}$*, consider the theory family* $T = \{\{1,2\}\}$*. We have* $\sigma^\flat_\Sigma(0,1) = \sigma^\flat_\Sigma(1,0) = 1 \in T_\Sigma$*, but*

$$\sigma^\flat_\Sigma(\lambda^\flat_\Sigma(1,0),\lambda^\flat_\Sigma(0,0)) = \sigma^\flat_\Sigma(0,0) = 0 \notin T_\Sigma.$$

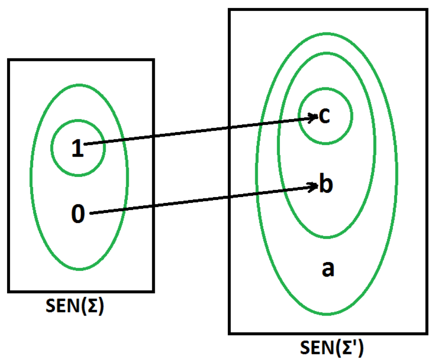
Therefore, the implication defining local family compatibility fails for $T = \{\{1,2\}\}$*. So* $I^\flat$ *does not have locally family compatibility in* $\mathcal{I}$*.*

Next, we present an example to show that there is π-institution $\mathcal{I}$ with a set of natural transformations that has the global (family) compatibility but not the local system compatibility in $\mathcal{I}$.

Example 714 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with two objects* Σ, Σ' *and a single (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*,* $\mathrm{SEN}^\flat(\Sigma') = \{a,b,c\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1\} \to \{a,b,c\}$ *given by* $0 \mapsto b$*,* $1 \mapsto c$*;*

- *$N^\flat$ is the category of natural transformations generated by one ternary natural transformation $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$, defined as follows:*
  - *$\sigma^\flat_\Sigma : \{0,1\}^3 \to \{0,1\}$ be given by*
  $$\sigma^\flat_\Sigma(x,y,z) = 0, \text{ for all } x,y,z \in \{0,1\};$$
  - *$\sigma^\flat_{\Sigma'} : \{a,b,c\}^3 \to \{a,b,c\}$ be given by*
  $$\sigma^\flat_{\Sigma'}(x,y,z) = \begin{cases} a, & \text{if } a \in \{x,y,z\} \\ b, & \text{otherise} \end{cases}$$



Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be the π-institution determined by

$$\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\}, \quad \mathcal{C}_{\Sigma'} = \{\{c\},\{b,c\},\{a,b,c\}\}.$$

$\mathrm{Thm}(\mathcal{I})$ *has six theory families all of which, except $\{\{0,1\},\{c\}\}$, are theory systems.*

Consider the set $I^\flat = \{\sigma^\flat\}$, with $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ having two distinguished arguments. We show that $I^\flat$ has the global family compatibility in $\mathcal{I}$, but it does not have the local system compatibility in $\mathcal{I}$.

For the global compatibility note that, if, for some $x,y \in \mathrm{SEN}^\flat(\Sigma)$, we have $\sigma^\flat_\Sigma[x,y] \leq T$, then $T = \mathrm{SEN}^\flat$. Similarly, if, for some $x,y \in \mathrm{SEN}^\flat(\Sigma')$, $\sigma^\flat_{\Sigma'}[x,y] \leq T$, then $T_{\Sigma'} = \{a,b,c\}$. In both cases, the conclusion of the defining implication is trivially true. So $I^\flat$ has the global compatibility in $\mathcal{I}$.

On the other hand, consider the theory system $T = \{\{0,1\},\{b,c\}\}$. Let $\phi = 0$ and $\psi = 1$. Then, we have, for all $z \in \{0,1\}$,

$$\sigma^\flat_\Sigma(0,1,z) = \sigma^\flat_\Sigma(1,0,z) = 0 \in T_\Sigma.$$

On the contrary,

$$\sigma^\flat_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(0),c,c),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(1),c,c),a) = a \notin T_{\Sigma'}.$$

Therefore, the implication defining local system compatibility fails for T. So $I^\flat$ does not have the local system compatibility in $\mathcal{I}$.

We close by proving that all three compatibility properties transfer from π-institutions to their models.

Proposition 715 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $I^\flat$ *a collection of natural transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, with two distinguished arguments, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $I^\flat$ *has a compatibility property in* $\mathcal{I}$ *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* I *has the corresponding compatibility property in* $\mathcal{A}$*.*

Proof: If I has a compatibility property in $\mathcal{A}$, for all $\mathcal{A}$, then it has the same compatibility property in $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. Since $\langle \mathbf{F}, C^{\mathcal{I},\mathcal{F}} \rangle = \mathcal{I}$, we conclude that $I^\flat$ has the corresponding compatibility in $\mathcal{I}$.

Suppose, conversely, that $I^\flat$ has a compatibility property in $\mathcal{I}$. We look at each of the three properties in turn.

(a) Suppose $I^\flat$ has the local family compatibility in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, be an $\mathbf{F}$-algebraic system, let $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$\overleftrightarrow{I}_{F(\Sigma)}(\alpha_\Sigma(\phi), \alpha_\Sigma(\psi), \alpha_\Sigma(\vec{\xi})) \subseteq T_{F(\Sigma)},$$

for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$. Since this is equivalent to $\alpha_\Sigma(\overleftrightarrow{I^\flat}_\Sigma(\phi, \psi, \vec{\xi})) \subseteq T_{F(\Sigma)}$, we get that $\overleftrightarrow{I^\flat}_\Sigma(\phi, \psi, \vec{\xi}) \subseteq \alpha_\Sigma^{-1}(T_{F(\Sigma)})$, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$. But, by hypothesis, $I^\flat$ has the local family compatibility in $\mathcal{I}$ and, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$. Therefore, we get that, for all $\lambda^\flat \in N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi}, \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$I^\flat_{\Sigma'}(\lambda^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}), \lambda^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}), \vec{\xi}) \subseteq \alpha_{\Sigma'}^{-1}(T_{F(\Sigma')}).$$

This now gives

$$\alpha_{\Sigma'}(I^\flat_{\Sigma'}(\lambda^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}), \lambda^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}), \vec{\xi})) \subseteq T_{F(\Sigma')},$$

or, equivalently,

$$\begin{aligned} &I_{F(\Sigma')}(\lambda_{F(\Sigma')}(\mathrm{SEN}(F(f))(\alpha_\Sigma(\phi)), \alpha_{\Sigma'}(\vec{\chi})), \\ &\qquad \lambda_{F(\Sigma')}(\mathrm{SEN}(F(f))(\alpha_\Sigma(\psi)), \alpha_{\Sigma'}(\vec{\chi})), \alpha_{\Sigma'}(\vec{\xi})) \subseteq T_{F(\Sigma')}. \end{aligned}$$

Taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that I has the local family compatibility in $\mathcal{A}$.

(b) The case of the local system compatibility can be proven similarly, taking into account that, if $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, then $\alpha^{-1}(T) \in \mathrm{ThSys}(\mathcal{I})$.

(c) Suppose that $I^\flat$ has the global (family) compatibility in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$\overleftrightarrow{I}_{F(\Sigma)}[\alpha_\Sigma(\phi), \alpha_\Sigma(\psi)] \leq T.$$

Then, we have, by Lemma 95, $\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi] \leq \alpha^{-1}(T)$. Now, since, by hypothesis, $I^\flat$ has the global family compatibility in $\mathcal{I}$ and, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, we get that, for all $\lambda^\flat \in N^\flat$, $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$I^\flat_{\Sigma'}[\lambda^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}), \lambda^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi})] \leq \alpha^{-1}(T),$$

or, equivalently, by Lemma 95,

$$I_{F(\Sigma')}[\alpha_{\Sigma'}(\lambda^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi})), \alpha_{\Sigma'}(\lambda^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}))] \leq T.$$

But this amounts to

$$\begin{aligned} I_{F(\Sigma')}[&\lambda_{F(\Sigma')}(\mathrm{SEN}(F(f))(\alpha_\Sigma(\phi)), \alpha_{\Sigma'}(\vec{\chi})), \\ &\lambda_{F(\Sigma')}(\mathrm{SEN}(F(f))(\alpha_\Sigma(\psi)), \alpha_{\Sigma'}(\vec{\chi}))] \leq T. \end{aligned}$$

Thus, I has the global family compatibility in $\mathcal{A}$.

■

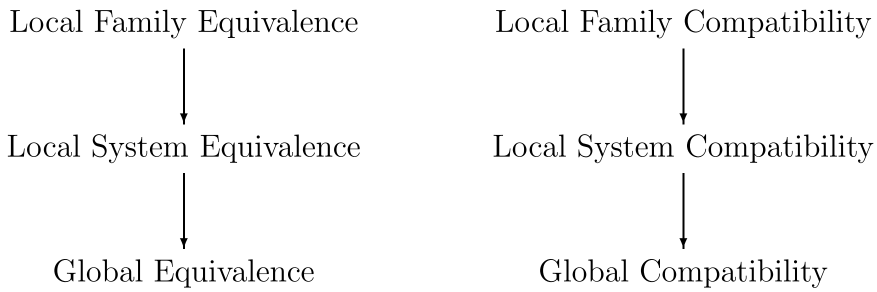
10.9 Congruence

In this section we focus on the three **uniform equivalence properties**, i.e., on LFLF equivalence, LSLS equivalence and GBGB equivalence, and we add to those versions of the compatibility property to obtain several versions of the congruence property.

To fix some terminology, we say that a set $I^\flat$ of natural transformations in a π-institution $\mathcal{I}$ has:

- the **local family equivalence in** $\mathcal{I}$ if it has the LFLF equivalence in $\mathcal{I}$;
- the **local system equivalence in** $\mathcal{I}$ if it has the LSLS equivalence in $\mathcal{I}$;
- the **global equivalence in** $\mathcal{I}$ if it has the GBGB equivalence in $\mathcal{I}$.

By previous work, we know that these three uniform equivalence properties are stratified in the linear hierarchy shown on the left below.



Moreover, by our study of the compatibility properties, we know that they also fall into a similar linear hierarchy, as shown on the right of the diagram.

By combining equivalence with compatibility properties, we obtain nine congruence properties as follows. Let $X, Y \in \{\text{LF}, \text{LS}, \text{GB}\}$, where, as before, LF stands for "Local Family", LS stands for "Local System" and GB stands for "GloBal".

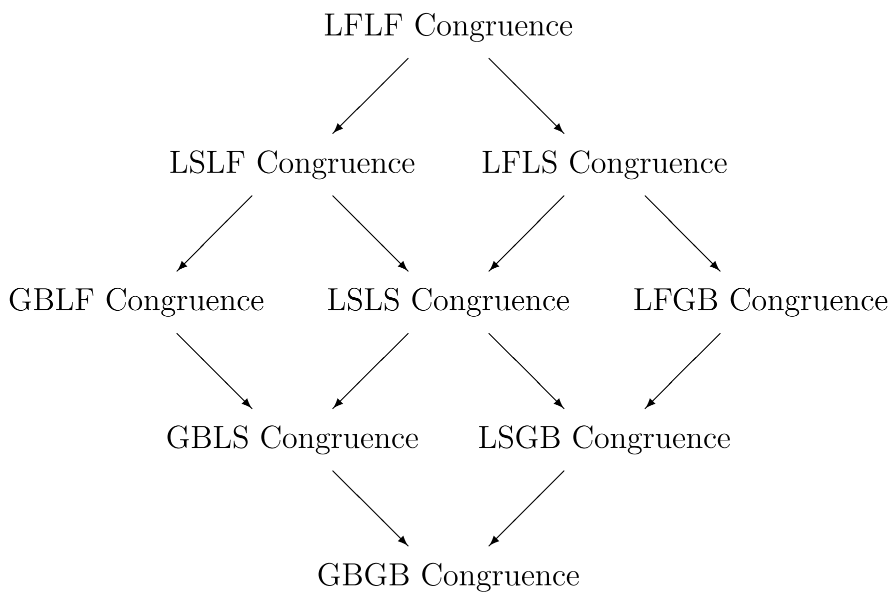
Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\text{SEN}^\flat)^\omega \to \text{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. We say that $I^\flat$ has the **XY-congruence in** $\mathcal{I}$ if it has

- the X (uniform) equivalence in $\mathcal{I}$;
- the Y compatibility in $\mathcal{I}$.

Based on the hierarchies of the equivalence and compatibility properties, we obtain the following hierarchical structure for the various flavors of the congruence property.

Corollary 716 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\text{SEN}^\flat)^\omega \to \text{SEN}^\flat$ *a set of natural transformations in* $N^\flat$*, with two distinguished arguments. The nine congruence properties form the hierarchy shown on the accompanying diagram.*

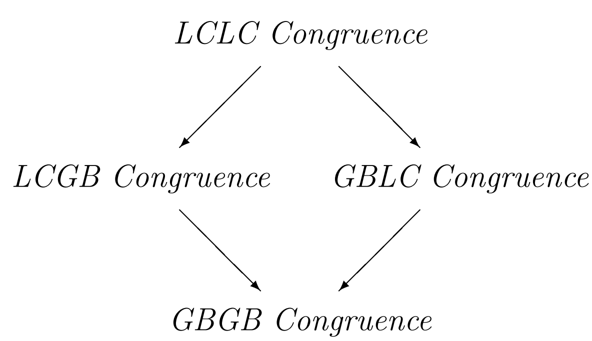
Proof: This follows directly from Corollary 694 and Proposition 711. ■



Based on the analysis performed on symmetry and transitivity, we have the following result regarding natural sufficient conditions under which some of the nine congruence properties above coincide.

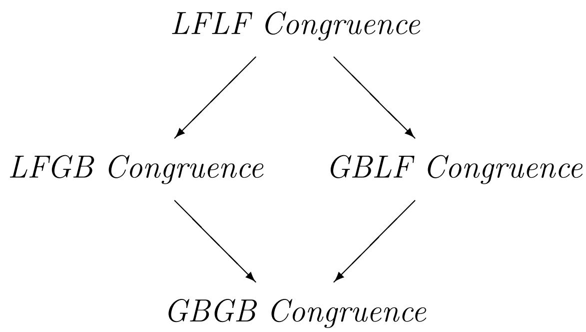
Corollary 717 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$*, with two distinguished arguments.*

(a) If $\mathcal{I}$ *is systemic, then the congruence hierarchy collapses to the one depicted below, where LC (for LoCal) is used to incorporate the LF and LS properties, which coincide;*



(b) If $I^\flat$ *has only two arguments (i.e., is parameter free), then the congruence hierarchy collapses to the one depicted below, where the Local System versions coincide with (and, thus, are incorporated into) the*

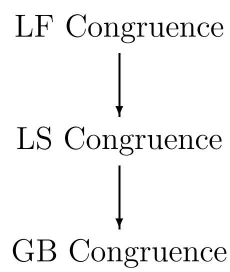
GloBal versions.



Proof: The statement follows from Corollary 695 and Proposition 712. ■

For a systemic π-institution with a parameter-free set of natural transformations, there is only one congruence property, since all versions of equivalence and all versions of compatibility collapse to a single property.

Instead of studying this entire hierarchy in detail, we refocus, once again, to the uniformly defined classes. So we define **LF congruence**, **LS congruence** and **GB congruence** to mean, respectively, LFLF congruence, LSLS congruence and GBGB congruence. These are the three diagonal classes in the original diagram that form, according to Corollary 716, the subhierarchy depicted below.



And, of course, according to Corollary 717, this reduces to the hierarchy depicted on the left below for a systemic π-institution and to the one depicted on the right below for a parameter free set of natural transformations.

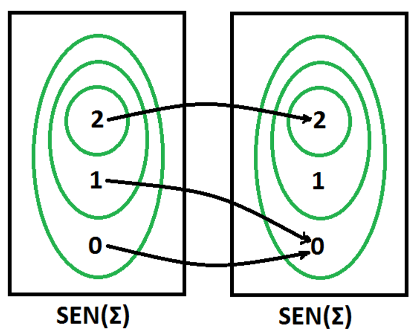
We provide examples to show that the three uniform classes of the congruence hierarchy are distinct in general.

First, we present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that has the LS congruence, but not the LF congruence property in $\mathcal{I}$.

Example 718 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 0$, $1 \mapsto 0$ *and* $2 \mapsto 2$*;*
- $N^\flat$ *is the category of natural transformations generated by a single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 2, & \text{if } x = y \text{ or } \{x,y\} = \{0,1\} \\ 1, & \text{if } (x,y) = (1,2) \\ 0, & \text{otherwise} \end{cases}$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}.$$

Note that there are three theory families, but only $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$ *are theory systems.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the local system congruence in* $\mathcal{I}$*, but it does not have the local family congruence in* $\mathcal{I}$*.*

First note that $\sigma^\flat_\Sigma(\phi,\phi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, for all* $\phi \in \{0,1,2\}$*, whence* $I^\flat$ *is reflexive in* $\mathcal{I}$*. Next note that the condition defining local system symmetry holds trivially for* $\mathrm{SEN}^\flat$*, whereas for* $\mathrm{Thm}(\mathcal{I})$*, if* $\sigma^\flat_\Sigma(\phi,\psi) \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, for some* $\phi \neq \psi$*, then* $\{\phi,\psi\} = \{0,1\}$*, whence* $\sigma^\flat_\Sigma(\psi,\phi) \in \mathrm{Thm}_\Sigma(\mathcal{I})$*. So* $I^\flat$ *is locally system symmetric in* $\mathcal{I}$*. For local system transitivity, the defining condition holds, again, trivially for* $\mathrm{SEN}^\flat$*, whereas for* $\mathrm{Thm}(\mathcal{I})$*, it holds due to the fact that* $\sigma^\flat_\Sigma(\phi,\psi) \in \mathrm{Thm}_\Sigma(\mathcal{I})$ *for no* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$*, with* $\phi \neq \psi$*, other than* $(\phi,\psi) = (0,1)$ *or* $(1,0)$*. Thus,* $I^\flat$ *is also locally system transitive in*

$\mathcal{I}$. Finally, note that the condition defining local system compatibility is also trivial for $\mathrm{SEN}^\flat$*, whereas for* $\mathrm{Thm}(\mathcal{I})$*, if* $\sigma^\flat_\Sigma(\phi,\psi) = 2$ *and* $\sigma^\flat_\Sigma(\psi,\phi) = 2$*, with* $\phi \neq \psi$*, then* $\{\phi,\psi\} = \{0,1\}$ *and, in that case,*

$$\sigma^\flat_\Sigma(\sigma^\flat_\Sigma(\mathrm{SEN}^\flat(h)(\phi),\chi),\sigma^\flat_\Sigma(\mathrm{SEN}^\flat(h)(\psi),\chi)) = 2$$

and

$$\sigma^\flat_\Sigma(\sigma^\flat_\Sigma(\chi,\mathrm{SEN}^\flat(h)(\phi)),\sigma^\flat_\Sigma(\chi,\mathrm{SEN}^\flat(h)(\psi))) = 2,$$

for all $h \in \mathbf{Sign}^\flat(\Sigma,\Sigma)$ *and all* $\chi \in \{0,1,2\}$*. Thus,* $I^\flat$ *has the local system compatibility in* $\mathcal{I}$ *and, therefore, has the local system congruence in* $\mathcal{I}$*.*

On the other hand, note that $\sigma^\flat_\Sigma(1,2) = 1 \in \{1,2\}$*, but* $\sigma^\flat_\Sigma(2,1) = 0 \notin \{1,2\}$*. Thus, the local family symmetry condition fails for* $T = \{\{1,2\}\} \in \mathrm{ThFam}(\mathcal{I})$*. Hence,* $I^\flat$ *is not locally family symmetric in* $\mathcal{I}$ *and, therefore, a fortiori, it fails to satisfy the local family congruence in* $\mathcal{I}$*.*

Finally, we present an example to show that there is a π-institution $\mathcal{I}$ with a set of natural transformations that has the GB congruence but not the LS congruence in $\mathcal{I}$.

Example 719 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2,3\} \to \{0,1,2,3\}$ *given by* $0 \mapsto 2$*,* $1 \mapsto 3$*,* $2 \mapsto 2$ *and* $3 \mapsto 3$*;*
- $N^\flat$ *is the category of natural transformations generated by the single ternary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2,3\}^3 \to \{0,1,2,3\}$ *be given by*

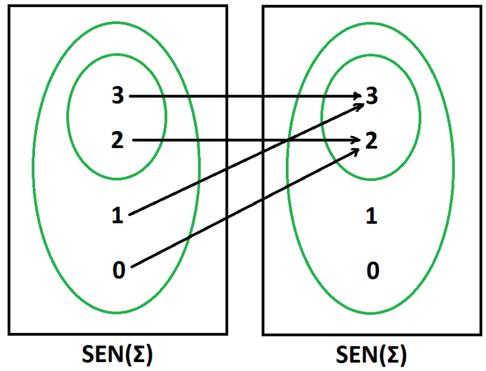
$$\sigma^\flat_\Sigma(x,y,z) = \begin{cases} 2, & \text{if } x = y \text{ or } (x,y) = (0,1) \text{ or } z = 2 \text{ or } z = 3 \\ 0, & \text{otherwise} \end{cases}.$$

Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2,3\},\{0,1,2,3\}\}.$$

Note that both theory families, $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, are also theory systems. So* $\mathcal{I}$ *is a systemic* π*-institution.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the GB congruence in* $\mathcal{I}$*, but it does not have the LS congruence in* $\mathcal{I}$*.*



Note, first, that reflexivity is obvious, since, by definition, for all $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,* $\sigma^\flat_\Sigma(\phi, \phi, \xi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, for all* $\xi \in \mathrm{SEN}^\flat(\Sigma)$*. For global symmetry, note that if, for some* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,* $\sigma^\flat_\Sigma[\phi, \psi] \le \mathrm{Thm}(\mathcal{I})$*, then we must have* $\phi = \psi$*, whence* $\sigma^\flat_\Sigma[\psi, \phi] \le \mathrm{Thm}(\mathcal{I})$ *holds. For global transitivity, note again that for no* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*, with* $\phi \ne \psi$*, is it the case that* $\sigma^\flat_\Sigma[\phi, \psi] \le \mathrm{Thm}(\mathcal{I})$*, whence the condition is satisfied in this case as well. Finally, the same observation leads to the conclusion that* $I^\flat$ *satisfies the global compatibility property in* $\mathcal{I}$*. We conclude that* $I^\flat$ *has the GB congruence in* $\mathcal{I}$*.*

On the other hand, since $\sigma^\flat_\Sigma(0, 1, \xi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, for all* $\xi \in \mathrm{SEN}^\flat(\Sigma)$*, but* $\sigma^\flat_\Sigma(1, 0, 0) = 0 \notin \mathrm{Thm}_\Sigma(\mathcal{I})$*,* $I^\flat$ *does not have the local system symmetry. A fortiori,* $I^\flat$ *does not have the LS congruence in* $\mathcal{I}$*.*

And here is a transfer property for the congruence properties that we have focused on in this section.

Corollary 720 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $I^\flat$ *a collection of natural transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, with two distinguished arguments, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $I^\flat$ *has a congruence property in* $\mathcal{I}$ *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* I *has the corresponding congrunece property in* $\mathcal{A}$*.*

Proof: This follows directly from Corollary 700 and Proposition 715. ■

10.10 Modus Ponens

We turn now to the study of various versions of the modus ponens property, taking again into account both the duality between local versus global membership and the difference between considering all theory families versus restricting only to theory systems.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. We say that:

- $I^\flat$ has the **local family modus ponens** (**local family MP**) **in** $\mathcal{I}$ if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi \in T_\Sigma \text{ and, for all } \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma),\ I^\flat_\Sigma(\phi, \psi, \vec{\chi}) \subseteq T_\Sigma \text{ imply } \psi \in T_\Sigma;$$
- $I^\flat$ has the **local system modus ponens** (**local system MP**) **in** $\mathcal{I}$ if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi \in T_\Sigma \text{ and, for all } \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma),\ I^\flat_\Sigma(\phi, \psi, \vec{\chi}) \subseteq T_\Sigma \text{ imply } \psi \in T_\Sigma;$$
- $I^\flat$ has the **global family modus ponens** (**global family MP**) **in** $\mathcal{I}$ if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi \in T_\Sigma \text{ and } I^\flat_\Sigma[\phi, \psi] \leq T \text{ imply } \psi \in T_\Sigma;$$
- $I^\flat$ has the **global system modus ponens** (**global system MP**) **in** $\mathcal{I}$ if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi \in T_\Sigma \text{ and } I^\flat_\Sigma[\phi, \psi] \leq T \text{ imply } \psi \in T_\Sigma.$$

The following proposition establishes the hierarchy of modus ponens rules.

Proposition 721 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$ having two distinguished arguments.*

(a) If $I^\flat$ has the local family MP, then it has both the global family MP in $\mathcal{I}$ and the local system MP in $\mathcal{I}$;

(b) If $I^\flat$ has the global family MP, then it has the global system MP in $\mathcal{I}$;

(c) If $I^\flat$ has the local system MP, then it has the global system MP in $\mathcal{I}$.

Proof:

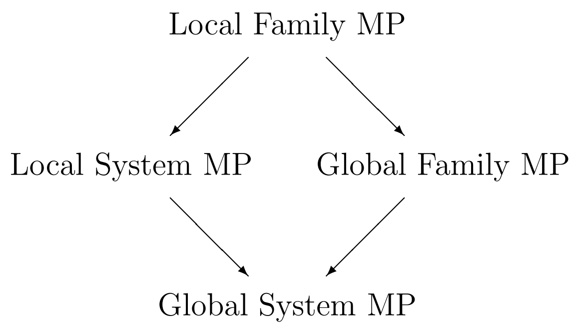
(a) Suppose that $I^\flat$ has the local family MP in $\mathcal{I}$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $I^\flat_\Sigma[\phi, \psi] \leq T$. Then, we have, in particular, that, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma(\phi, \psi, \vec{\chi}) \subseteq T_\Sigma$. But then, since $\phi \in T_\Sigma$ and, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma(\phi, \psi, \vec{\chi}) \subseteq T_\Sigma$, we get by the local family MP that $\psi \in T_\Sigma$. We conclude that $I^\flat$ has the global family MP in $\mathcal{I}$.

If $I^\flat$ has the local family MP in $\mathcal{I}$, then it has a fortiori the local system MP in $\mathcal{I}$ due to the fact that every theory system of $\mathcal{I}$ is also a theory family.

(b) This follows, similarly to the second part of (a), from the fact that every theory system of $\mathcal{I}$ is also a theory family.

(c) We repeat the argument used in the proof of the first part of (a) except reasoning exclusively in terms of theory systems rather than using arbitrary theory families. ■

Proposition 721 has established the following hierarchy of modus ponens properties, where the southwest arrows are based on the family-system duality whereas the southeast arrows on the local-global duality.



We also note the following regarding natural sufficient conditions under which some of these four classes coincide.

Proposition 722 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments.*

(a) If $\mathcal{I}$ *is systemic, then the local (global) family and the local (global) system MP coincide;*

(b) If $I^\flat$ *has only two arguments (i.e., is parameter free), then the local system MP and the global system MP coincide;*

(c) If $\mathcal{I}$ *is systemic and* $I^\flat$ *is parameter-free, then the local family MP and the global family MP also coincide.*

Proof:

(a) If $\mathcal{I}$ is systemic, then all theory families are theory systems and the family and system properties collapse.

(b) Suppose that $I^\flat$ is parameter-free and that $I^\flat$ has the global system MP in $\mathcal{I}$. Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $I^\flat_\Sigma(\phi, \psi) \subseteq T_\Sigma$. Since T is a theory system, we have, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$I^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi)) = \mathrm{SEN}^\flat(f)(I^\flat_\Sigma(\phi, \psi)) \subseteq T_{\Sigma'}.$$

Equivalently, $I^\flat_\Sigma[\phi, \psi] \leq T$. Thus, by the global system MP, we get that $\psi \in T_\Sigma$. Thus, $I^\flat$ has the local system MP.

(c) This follows from Parts (a) and (b).

■

So in the case of a systemic π-institution $\mathcal{I}$, we have the hierarchy pictured on the left, whereas in the case of a parameter-free set of natural transformations we have the hierarchy on the right.

$$\begin{array}{ccc} \text{Local MP} & \quad & \text{Local Family MP} \\ \downarrow & & \downarrow \\ \text{Global MP} & & \text{Global Family MP} \\ & & \downarrow \\ & & \text{System MP} \end{array}$$

Finally, for a systemic π-institution with a parameter-free set of natural transformations all four MP properties collapse to a single one.

We provide some examples to show that the implications of Proposition 721 are not equivalences in general, i.e., in the hierarchy shown above all inclusions of classes of π-institutions with a set of natural transformations satisfying the corresponding modus ponens properties are proper inclusions.

We first present an example to show that there is π-institution $\mathcal{I}$ with a set of natural transformations that have the global family MP but not the local system MP in $\mathcal{I}$.

Example 723 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

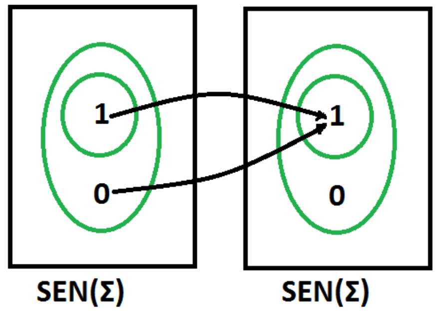
- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1\} \to \{0,1\}$ *given by* $0 \mapsto 1$ *and* $1 \mapsto 1$*;*
- $N^\flat$ *is the category of natural transformations generated by the single ternary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1\}^3 \to \{0,1\}$ *be given by*

$$\sigma^\flat_\Sigma(a,b,c) = \begin{cases} 0, & \text{if } (a,b,c) = (1,1,0) \\ 1, & \text{otherwise} \end{cases}$$

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\}$*.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *having two distinguished arguments.*

Clearly, both $\mathrm{Thm}(\mathcal{I})$ *and* SEN*, which are the only theory families, are also theory systems. We show that* $I^\flat$ *has the global family MP in* $\mathcal{I}$*, but it does not have the local system MP in* $\mathcal{I}$*.*



For the global family MP notice that we only need to check the case with $T = \mathrm{Thm}(\mathcal{I})$, $\phi = 1$ *and* $\psi = 0$. *Since* $\sigma^\flat_\Sigma(\mathrm{SEN}^\flat(f)(1), \mathrm{SEN}^\flat(f)(0), 0) = \sigma^\flat_\Sigma(1,1,0) = 0$, *we have that* $I^\flat_\Sigma[\phi,\psi] \nsubseteq T$, *whence the condition is vacuously satisfied. Therefore, we get that* $I^\flat$ *has the global family MP in* $\mathcal{I}$.

On the other hand, we have $\sigma^\flat_\Sigma(1,0,0) = \sigma^\flat_\Sigma(1,0,1) = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$ *and* $1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$, *but* $0 \notin \mathrm{Thm}_\Sigma(\mathcal{I})$, *which shows that* $I^\flat$ *does not have the local system MP in* $\mathcal{I}$.

Next we present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that have the local system MP but not the global family MP in $\mathcal{I}$.

Example 724 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

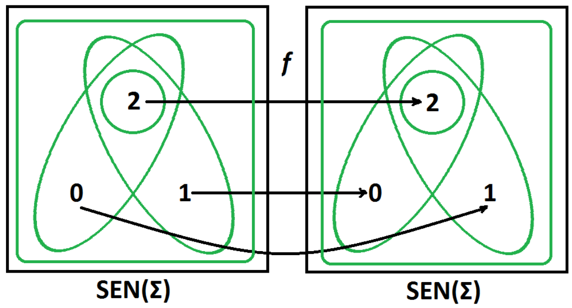
- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = i_\Sigma$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 1$, $1 \mapsto 0$ *and* $2 \mapsto 2$;
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given, for all* $a, b \in \mathrm{SEN}^\flat(\Sigma)$, *by*

$$\sigma^\flat_\Sigma(a,b) = \begin{cases} 1, & \text{if } (a,b) = (2,0) \\ 0, & \text{if } (a,b) = (2,1) \\ 2, & \text{otherwise} \end{cases}.$$

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}.$$

Consider the set $I^\flat = \{\sigma^\flat\}$.



$\mathcal{I}$ has four theory families $\mathrm{Thm}(\mathcal{I})$, $T = \{\{0,2\}\}$, $T' = \{\{1,2\}\}$ *and* $\mathrm{SEN}^\flat$, *but only two theory systems* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$. *We show that $I^\flat$ has the local system MP in $\mathcal{I}$, but it does not have the global family MP in $\mathcal{I}$.*

For the local system MP notice that we only need to check the case for $\mathrm{Thm}(\mathcal{I})$, $\phi = 2$ *and* $\psi = 0$ *or* $\psi = 1$. *Since* $\sigma^\flat_\Sigma(2,0) = 1 \notin \mathrm{Thm}_\Sigma(\mathcal{I})$ *and* $\sigma^\flat_\Sigma(2,1) = 0 \notin \mathrm{Thm}_\Sigma(\mathcal{I})$, *we conclude that $I^\flat$ has the local system MP in $\mathcal{I}$.*

On the other hand, for the theory family T and for $\phi = 0$ and $\psi = 1$, we get that $\phi = 0 \in T_\Sigma$ and $\sigma^\flat_\Sigma(\phi,\psi) = \sigma^\flat_\Sigma(0,1) = 2$ *and*

$$\sigma^\flat_\Sigma(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi)) = \sigma^\flat_\Sigma(1,0) = 2,$$

whence $\sigma^\flat_\Sigma[0,1] \leq T$. *But clearly* $1 \notin T_\Sigma$. *Therefore $I^\flat$ does not have the global family MP in $\mathcal{I}$.*

We prove next a transfer property for modus ponens.

Proposition 725 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, $I^\flat$ a collection of natural transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in $N^\flat$, with two distinguished arguments, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on $\mathbf{F}$. $I^\flat$ has a modus ponens property in $\mathcal{I}$ if and only if, for every $\mathbf{F}$-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, *with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, *$I$ has the corresponding modus ponens property in $\mathcal{A}$.*

Proof: If I has a modus ponens property in $\mathcal{A}$, for all $\mathcal{A}$, then it has the same modus ponens in $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. Since $\langle \mathbf{F}, C^{\mathcal{I},\mathcal{F}} \rangle = \mathcal{I}$, we conclude that $I^\flat$ has the corresponding modus ponens in $\mathcal{I}$.

Suppose, conversely, that $I^\flat$ has a modus ponens in $\mathcal{I}$. We look at each of the four properties in turn.

(a) Suppose $I^\flat$ has the local family MP in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$ and

$$I_{F(\Sigma)}(\alpha_\Sigma(\phi), \alpha_\Sigma(\psi), \alpha_\Sigma(\vec{\chi})) \subseteq T_{F(\Sigma)},$$

for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$. Since the latter is equivalent to $\alpha_\Sigma(I^\flat_\Sigma(\phi,\psi,\vec{\chi})) \subseteq T_{F(\Sigma)}$, we get that $\phi \in \alpha^{-1}_\Sigma(T_{F(\Sigma)})$ and $I^\flat_\Sigma(\phi,\psi,\vec{\chi}) \subseteq \alpha^{-1}_\Sigma(T_{F(\Sigma)})$, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$. But, by hypothesis, $I^\flat$ has the local family MP in $\mathcal{I}$ and, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$. Therefore, we get that $\psi \in \alpha^{-1}_\Sigma(T_{F(\Sigma)})$, or, equivalently, $\alpha_\Sigma(\psi) \in T_{F(\Sigma)}$. This proves that I has the local family MP in $\mathcal{A}$.

(b) The case of the local system MP can be proven similarly, taking into account that, if $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, then $\alpha^{-1}(T) \in \mathrm{ThSys}(\mathcal{I})$.

(c) Suppose that $I^\flat$ has the global family MP in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$\alpha_\Sigma(\phi) \quad \text{and} \quad I_{F(\Sigma)}[\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)] \leq T.$$

Then, we have $\phi \in \alpha^{-1}_\Sigma(T_{F(\Sigma)})$ and, by Lemma 95, $I^\flat_\Sigma[\phi,\psi] \leq \alpha^{-1}(T)$. Now, since, by hypothesis, $I^\flat$ has the global family MP in $\mathcal{I}$ and, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, we get that $\psi \in \alpha^{-1}_\Sigma(T_{F(\Sigma)})$, or, equivalently, $\alpha_\Sigma(\psi) \in T_{F(\Sigma)}$. Thus, I has the global family MP in $\mathcal{A}$.

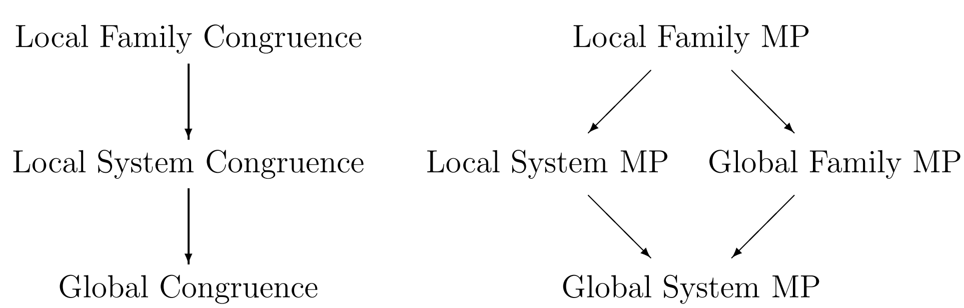
(d) Similar to (c).

■

10.11 Syntactic Protoalgebraicity

In this section we focus on the three **uniform congruence properties**, i.e., on LF congruence, LS congruence and GB congruence, and we add to those versions of the modus ponens property to obtain several versions of the syntactic protoalgebraicity property.

By previous work, we know that the three uniform congruence properties are stratified in the linear hierarchy shown on the left below.



Moreover, by our study of the modus ponens, we know that they fall into the hierarchy shown on the right of the diagram.

By combining equivalence with compatibility properties, we obtain twelve syntactic protoalgebraicity properties as follows. Let $X \in \{\mathrm{LF},\mathrm{LS},\mathrm{GB}\}$ and

$Y \in \{\mathrm{LF}, \mathrm{LS}, \mathrm{GF}, \mathrm{GS}\}$, where, as before, LF stands for "Local Family", LS stands for "Local System", GF stands for "Global Family", GS stands for "Global System" and GB stands for "GloBal".

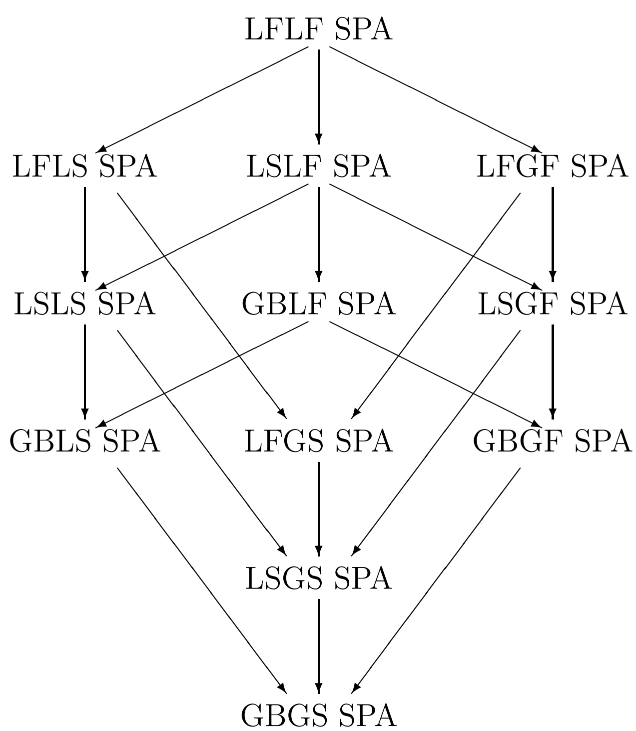
Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. We say that $I^\flat$ has the **XY syntactic protoalgebraicity in** $\mathcal{I}$ (**XY SPA in** $\mathcal{I}$) if it has

- the X congruence in $\mathcal{I}$;
- the Y modus ponens in $\mathcal{I}$.

Based on the hierarchies of the congruence and MP properties, we obtain the following hierarchical structure for the various flavors of the syntactic protoalgebraicity property.

Corollary 726 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments. The twelve syntactic protoalgebraicity properties form the hierarchy shown on the accompanying diagram.*

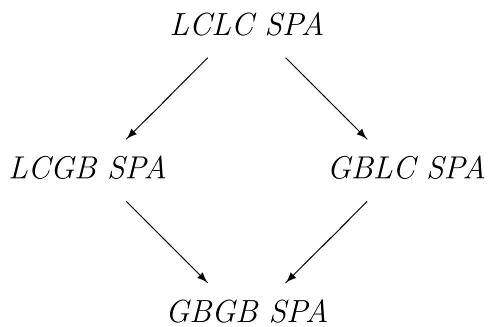
Proof: This follows directly from Corollary 716 and Proposition 721. ■



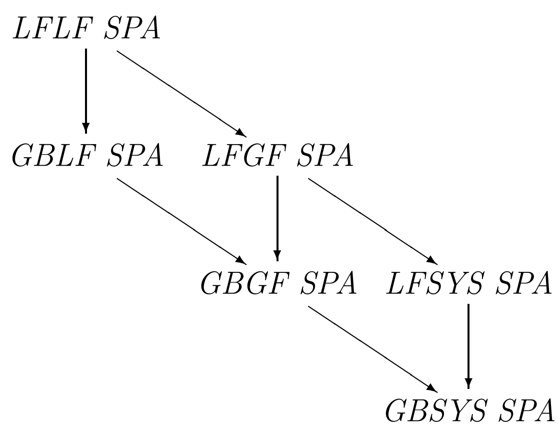
Based on the analysis performed on congruence and modus ponens, we have the following result regarding natural sufficient conditions under which some of the twelve syntactic protoalgebraicity properties above coincide.

Corollary 727 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments.*

(a) If $\mathcal{I}$ *is systemic, then the syntactic protoalgebraicity hierarchy collapses to the one depicted below;*



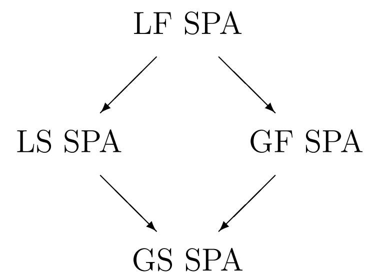
(b) If $I^\flat$ *has only two arguments (i.e., is parameter free), then the syntactic protoalgebraicity hierarchy collapses to the one depicted below, where the Local System versions of congruence coincide with (and are incorporated into) the global versions and the Local and Global System versions of MP also coincide and are denoted by SYS.*



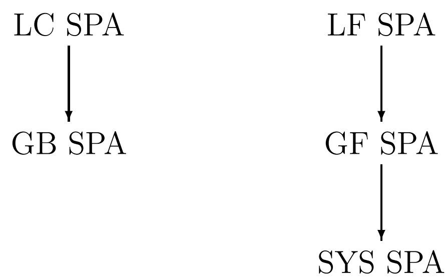
Proof: The statement follows from Corollary 717 and Proposition 722. ■

For a systemic π-institution with a parameter-free set of natural transformations, there is only one syntactic protoalgebraicity property, since all versions of congruence and all versions of modus ponens collapse to a single property.

Instead of studying this entire hierarchy in detail, we refocus, once again, on the uniformly defined classes. So we define **LF SPA**, **LS SPA**, **GF SPA** and **GS SPA** to mean, respectively, LFLF syntactic, LSLS syntactic, GFGF syntactic and GSGS syntactic protoalgebraicity. These classes form, according to the diagram above, based on Corollary 726, the sub hierarchy depicted below.



Moreover, according to Corollary 727, this reduces to the hierarchy depicted on the left below for a systemic π-institution and to the one depicted on the right below for a parameter free set of natural transformations.



We provide examples to show that the four uniform classes of the syntactic protoalgebraicity hierarchy are distinct in general.

First, we present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that has the LS syntactic protoalgebraicity, but not the GF syntactic protoalgebraicity in $\mathcal{I}$.

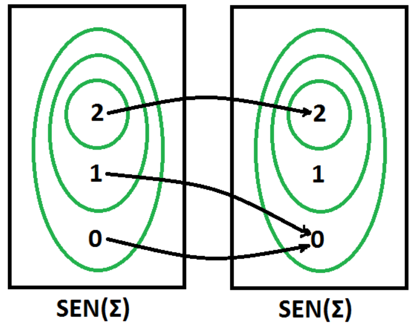
Example 728 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 0$*,* $1 \mapsto 0$ *and* $2 \mapsto 2$*;*
- $N^\flat$ *is the category of natural transformations generated by a single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 2, & \text{if } x = y \text{ or } \{x,y\} = \{0,1\} \\ 1, & \text{if } \{x,y\} = \{1,2\} \\ 0, & \text{otherwise} \end{cases}$$

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}.$$



Note that there are three theory families, but only $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$ *are theory systems.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the local system syntactic protoalgebraicity in* $\mathcal{I}$*, but it does not have the global family syntactic protoalgebraicity in* $\mathcal{I}$*.*

First note that $\sigma^\flat_\Sigma(\phi,\phi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, for all* $\phi \in \{0,1,2\}$*, whence* $I^\flat$ *is reflexive in* $\mathcal{I}$*.*

Next, note that the condition defining local system symmetry holds trivially for $\mathrm{SEN}^\flat$*, whereas for* $\mathrm{Thm}(\mathcal{I})$*, if* $\sigma^\flat_\Sigma(\phi,\psi) \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, for some* $\phi \neq \psi$*, then* $\{\phi,\psi\} = \{0,1\}$*, whence* $\sigma^\flat_\Sigma(\psi,\phi) \in \mathrm{Thm}_\Sigma(\mathcal{I})$*. So* $I^\flat$ *is local system symmetric in* $\mathcal{I}$*.*

For local system transitivity, the defining condition holds, again, trivially for $\mathrm{SEN}^\flat$*, whereas for* $\mathrm{Thm}(\mathcal{I})$*, it holds due to the fact that* $\sigma^\flat_\Sigma(\phi,\psi) \in \mathrm{Thm}_\Sigma(\mathcal{I})$ *for no* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$*, with* $\phi \neq \psi$*, other than* $(\phi,\psi) = (0,1)$ *or* $(1,0)$*. Thus,* $I^\flat$ *is also local system transitive in* $\mathcal{I}$*.*

Next, note that the condition defining local system compatibility is also trivial for $\mathrm{SEN}^\flat$*, whereas for* $\mathrm{Thm}(\mathcal{I})$*, if* $\sigma^\flat_\Sigma(\phi,\psi) = 2$ *and* $\sigma^\flat_\Sigma(\psi,\phi) = 2$*, with* $\phi \neq \psi$*, then* $\{\phi,\psi\} = \{0,1\}$ *and, in that case,*

$$\sigma^\flat_\Sigma(\sigma^\flat_\Sigma(\mathrm{SEN}^\flat(h)(\phi),\chi),\sigma^\flat_\Sigma(\mathrm{SEN}^\flat(h)(\psi),\chi)) = 2$$

and

$$\sigma^\flat_\Sigma(\sigma^\flat_\Sigma(\chi,\mathrm{SEN}^\flat(h)(\phi)),\sigma^\flat_\Sigma(\chi,\mathrm{SEN}^\flat(h)(\psi))) = 2,$$

for all $h \in \mathbf{Sign}^\flat(\Sigma,\Sigma)$ *and all* $\chi \in \{0,1,2\}$*. Thus,* $I^\flat$ *has the local system compatibility in* $\mathcal{I}$ *and, therefore, has the local system congruence in* $\mathcal{I}$*.*

To finish up, note that, since the only pairs (ϕ,ψ)*, with* $\phi \neq \psi$*, such that* $\sigma^\flat_\Sigma(\phi,\psi) \in \mathrm{Thm}_\Sigma(\mathcal{I})$ *are* $(0,1)$ *and* $(1,0)$ *and for neither of these is* $\phi \in \mathrm{Thm}_\Sigma(\mathcal{I})$*,* $I^\flat$ *has the local system modus ponens in* $\mathcal{I}$ *and, therefore, it has the local system syntactic protoalgebraicity in* $\mathcal{I}$ *as well.*

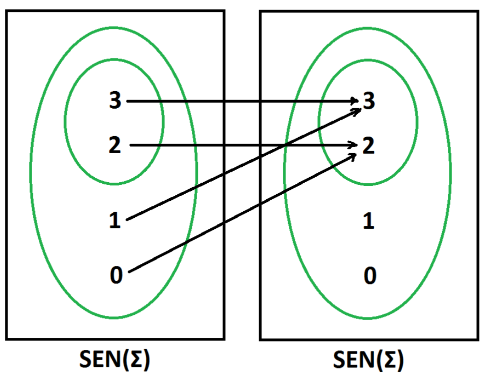
On the other hand, $1 \in \{1,2\}$ *and* $\sigma^\flat_\Sigma[1,0] \leq \{\{1,2\}\}$*, but* $0 \notin \{1,2\}$*. Therefore,* $I^\flat$ *does not have the global family modus ponens in* $\mathcal{I}$ *and, hence, a fortiori, it does not have the global family syntactic protoalgebraicity in* $\mathcal{I}$*.*

Next, we present an example to show that there is π-institution $\mathcal{I}$ with a set of natural transformations that has the GF syntactic protoalgebraicity but not the LS syntactic protoalgebraicity in $\mathcal{I}$.

Example 729 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2,3\} \to \{0,1,2,3\}$ *given by* $0 \mapsto 2$*,* $1 \mapsto 3$*,* $2 \mapsto 2$ *and* $3 \mapsto 3$*;*
- $N^\flat$ *is the category of natural transformations generated by the single ternary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2,3\}^3 \to \{0,1,2,3\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y,z) = \begin{cases} 2, & \text{if } x = y \text{ or } (x,y) = (0,1) \text{ or } z = 2 \text{ or } z = 3 \\ 0, & \text{otherwise} \end{cases}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$C_\Sigma = \{\{2,3\},\{0,1,2,3\}\}.$$

Note that both theory families, $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, are also theory systems. So* $\mathcal{I}$ *is a systemic* π*-institution.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the global family syntactic protoalgebraicity in* $\mathcal{I}$*, but it does not have the local system syntactic protoalgebraicity in* $\mathcal{I}$*.*

Note, first, that reflexivity is obvious, since, by definition, for all $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,* $\sigma^\flat_\Sigma(\phi,\phi,\xi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, for all* $\xi \in \mathrm{SEN}^\flat(\Sigma)$*. For global symmetry, note that if, for some* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$*,* $\sigma^\flat_\Sigma[\phi,\psi] \leq \mathrm{Thm}(\mathcal{I})$*, then we* *must have* $\phi = \psi$*, whence* $\sigma^\flat_\Sigma[\psi,\phi] \leq \mathrm{Thm}(\mathcal{I})$ *holds. For global transitivity, note again that for no* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$*, with* $\phi \neq \psi$*, is it the case that* $\sigma^\flat_\Sigma[\phi,\psi] \leq \mathrm{Thm}(\mathcal{I})$*, whence the condition is satisfied in this case as well. Finally, the same observation leads to the conclusion that* $I^\flat$ *satisfies both the global compatibility property in* $\mathcal{I}$ *and the global modus ponens. We conclude that* $I^\flat$ *has the global family syntactic protoalgebtaicity in* $\mathcal{I}$*.*

On the other hand, since $\sigma^\flat_\Sigma(0,1,\xi) = 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, for all* $\xi \in \mathrm{SEN}^\flat(\Sigma)$*, but* $\sigma^\flat_\Sigma(1,0,0) = 0 \notin \mathrm{Thm}_\Sigma(\mathcal{I})$*,* $I^\flat$ *does not have the local system symmetry. A fortiori,* $I^\flat$ *does not have the local system congruence and, hence, does not have the local system syntactic protoalgebraicity in* $\mathcal{I}$ *either.*

And here is a transfer property for the syntactic protoalgebraicity properties that we have focused on in this section.

Corollary 730 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $I^\flat$ *a collection of natural transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, with two distinguished arguments, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $I^\flat$ *has a (uniform) syntactic protoalgebraicity property in* $\mathcal{I}$ *if and only if, for every algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* I *has the corresponding syntactic protoalgebraicity property in* $\mathcal{A}$*.*

Proof: This follows directly from Corollary 720 and Proposition 1440. ∎

10.12 Invertibility

We study, next, various versions of the invertibility property, once again based on the local versus global and the theory family versus theory system dualities.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$ having two distinguished arguments. We say that:

- $I^\flat$ has the **local family invertibility in** $\mathcal{I}$ if there exists a set $\tau : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ of natural transformations in $N^\flat$, such that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \text{ iff } \overleftrightarrow{I^\flat}_\Sigma(\tau_\Sigma(\phi), \vec{\xi}) \subseteq T_\Sigma, \text{ for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma);$$

- $I^\flat$ has the **local system invertibility in** $\mathcal{I}$ if there exists a set $\tau : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ of natural transformations in $N^\flat$, such that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \text{ iff } \overleftrightarrow{I^\flat}_\Sigma(\tau_\Sigma(\phi), \vec{\xi}) \subseteq T_\Sigma, \text{ for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma);$$

- $I^\flat$ has the **global family invertibility in** $\mathcal{I}$ if there exists a set $\tau : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ of natural transformations in $N^\flat$, such that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi \in T_\Sigma \quad \text{iff} \quad \overleftrightarrow{I^\flat}_\Sigma[\tau_\Sigma(\phi)] \leq T;$$
- $I^\flat$ has the **global system invertibility in** $\mathcal{I}$ if there exists a set $\tau : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ of natural transformations in $N^\flat$, such that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi \in T_\Sigma \quad \text{iff} \quad \overleftrightarrow{I^\flat}_\Sigma[\tau_\Sigma(\phi)] \leq T.$$

We look at the hierarchy of invertibility properties.

Proposition 731 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments.*

(a) If $I^\flat$ *has the local (global) family invertibility, then it has the local (global) system invertibility in* $\mathcal{I}$.

(b) If $I^\flat$ *has the local system invertibility, then it has the global system invertibility in* $\mathcal{I}$.

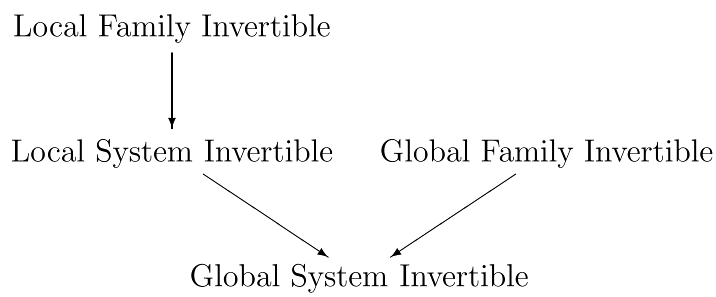
Proof: Since every theory system of $\mathcal{I}$ is a theory family, if $I^\flat$ has the local (global) family invertibility in $\mathcal{I}$, then it has, a fortiori, the local (global) system invertibility in $\mathcal{I}$, with the same witnessing set τ of natural transformations in $N^\flat$.

Suppose, next, that $I^\flat$ has the local system invertibility in $\mathcal{I}$, with witnessing set of natural transformations τ, and let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$.

- If $\phi \in T_\Sigma$, then, since $T \in \mathrm{ThSys}(\mathcal{I})$, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, $\mathrm{SEN}^\flat(f)(\phi) \in T_{\Sigma'}$. Thus, by the local family invertibility,
$$I^\flat_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)), \vec{\xi}) \subseteq T_{\Sigma'}, \text{ for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma').$$
This is equivalent to $I^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\tau_\Sigma(\phi)), \vec{\xi}) \subseteq T_{\Sigma'}$. Since $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$ were arbitrary, we conclude that $I^\flat_\Sigma[\tau_\Sigma(\phi)] \leq T$.
- Suppose, conversely, that $I^\flat_\Sigma[\tau_\Sigma(\phi)] \leq T$. This implies $I^\flat_\Sigma(\tau_\Sigma(\phi), \vec{\xi}) \subseteq T_\Sigma$, for all $\xi \in \mathrm{SEN}^\flat(\Sigma)$. Thus, by the local family invertibility, $\phi \in T_\Sigma$.

We conclude that $\phi \in T_\Sigma$ if and only if $I^\flat_\Sigma[\tau_\Sigma(\phi)] \leq T$, whence $I^\flat$ has the global system invertibility in $\mathcal{I}$. ■

Proposition 731 has established the following hierarchy of invertibility properties:



The following holds regarding natural sufficient conditions under which some of these properties coincide.

Proposition 732 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments.*

(a) If $\mathcal{I}$ *is systemic, then the local (global) family invertibility and the local (global) system invertibility properties coincide;*

(b) If $I^\flat$ *is parameter-free, then the local system invertibility and the global system invertibility properties coincide.*

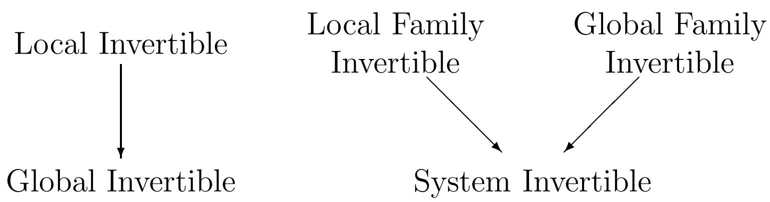
Proof: If $\mathcal{I}$ is systemic, then the local (global) system invertibility property coincides with the local (global) family invertibility property because of the fact that every theory family in $\mathcal{I}$ is also a theory system.

Suppose, next, that $I^\flat$ is parameter-free and that $I^\flat$ has the global system invertibility with witnessing set of natural transformations $\tau : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$. Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$.

- If $\phi \in T_\Sigma$, then, by the global system invertibility, $I^\flat_\Sigma[\tau_\Sigma(\phi)] \leq T$. In particular, $I^\flat_\Sigma(\tau_\Sigma(\phi)) \subseteq T_\Sigma$.
- If, conversely, $I^\flat_\Sigma(\tau_\Sigma(\phi)) \subseteq T_\Sigma$, then, since $T \in \mathrm{ThSys}(\mathcal{I})$, we get that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, $I^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\tau_\Sigma(\phi))) \subseteq T_{\Sigma'}$. Hence, $I^\flat_\Sigma[\tau_\Sigma(\phi)] \leq T$. Using the global system invertibility, we now conclude that $\phi \in T_\Sigma$.

Thus, the global system invertibility implies the local system invertibility property and, therefore that, provided $I^\flat$ is parameter-free, the local and global system invertibility properties coincide. ■

So, in the case of a systemic π-institution $\mathcal{I}$, the hierarchy of invertibility properties reduces to the one depicted on the left below, whereas in the case of a parameter-free set of natural transformations $I^\flat$, we get the hierarchy depicted on the right.



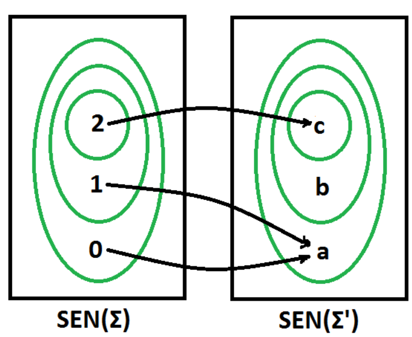
Finally, for a systemic π-institution and a parameter-free set of natural transformations, all four invertibility properties coincide.

We provide some examples to show that the implications of Proposition 731 are not equivalences in general, i.e., in the hierarchy shown above all inclusions of classes of π-institutions with a set of natural transformations satisfying the corresponding invertibility properties are proper inclusions.

We first present an example to show that there is π-institution $\mathcal{I}$ with a set of natural transformations that has the local family invertibility but not the global family invertibility in $\mathcal{I}$.

Example 733 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with two objects* Σ *and* Σ' *and a single (non-identity) morphism* $f : \Sigma \to \Sigma'$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$, $\mathrm{SEN}^\flat(\Sigma') = \{a,b,c\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{a,b,c\}$ *given by* $0 \mapsto a$, $1 \mapsto a$ *and* $2 \mapsto c$*;*
- $N^\flat$ *is the trivial category of natural transformations consisting of the projections only.*



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{c\},\{b,c\},\{a,b,c\}\}.$$

Consider the set $I^\flat = \{p^{2,0}\}$*, with* $p^{2,0} : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *being the projection binary natural transformation (onto the first coordinate), viewed as having two distinguished arguments.*

$\mathcal{I}$ *has nine theory families, but only five of those are theory systems. So it is not a systemic* π*-institution. We show that* $I^\flat$ *has the local family invertibility in* $\mathcal{I}$*, but it does not have the global family invertibility in* $\mathcal{I}$*.*

For the local family invertibility, let $\tau \equiv \{\iota \approx \iota\}$*, where* $\iota : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *is the identity (or unary first coordinate projection) natural transformation. Then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\overleftrightarrow{I^\flat}_\Sigma(\iota_\Sigma(\phi), \iota_\Sigma(\phi)) \in T_\Sigma \quad \text{iff} \quad \phi \in T_\Sigma.$$

Thus, $I^\flat$ *has the local family invertibility in* $\mathcal{I}$*.*

On the other hand, for $T = \{\{1,2\},\{b,c\}\} \in \mathrm{ThFam}(\mathcal{I})$*, we have* $1 \in T_\Sigma$*, but*

$$p^{2,0}_{\Sigma'}(\mathrm{SEN}^\flat(f)(1), \mathrm{SEN}^\flat(f)(1)) = p^{2,0}_{\Sigma'}(a,a) = a \notin T_{\Sigma'}.$$

Therefore $I^\flat$ *does not have the global family invertibility in* $\mathcal{I}$*.*

Next we present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that has the global family invertibility but not the local system invertibility in $\mathcal{I}$.

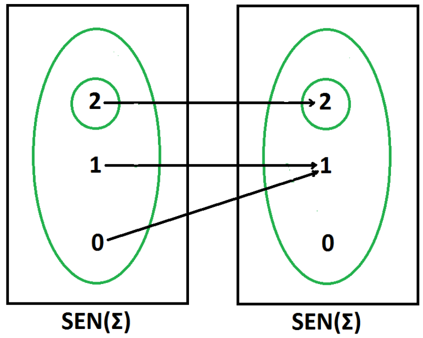
Example 734 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 1$*,* $1 \mapsto 1$ *and* $2 \mapsto 2$*;*
- $N^\flat$ *is the category of natural transformations generated by the single ternary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^3 \to \{0,1,2\}$ *be given, for all* $x,y,z \in \mathrm{SEN}^\flat(\Sigma)$*, by*

$$\sigma^\flat_\Sigma(x,y,z) = \begin{cases} 1, & \text{if } (x,y,z) = (1,1,2) \\ 2, & \text{otherwise} \end{cases}.$$

Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{2\},\{0,1,2\}\}$*. Consider the set* $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat$ *having two distinguished arguments.*

$\mathcal{I}$ *has two theory families* $\mathrm{Thm}(\mathcal{I})$*,* $\mathrm{SEN}^\flat$ *both of which are also theory systems. So* $\mathcal{I}$ *is systemic. We show that* $I^\flat$ *has the global family invertibility in* $\mathcal{I}$*, but it does not have the local system invertibility in* $\mathcal{I}$*.*



For the global family invertibility, consider $\tau = \{\iota \approx \iota\}$, *where* $\iota : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *is the identity natural transformation. The case of* $\mathrm{SEN}^\flat$ *is trivial, whereas for* $\mathrm{Thm}(\mathcal{I})$, *we have, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and* $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi = 2 \quad \textit{iff} \quad \overleftrightarrow{I^\flat}_\Sigma[\phi,\phi] \le \{\{2\}\},$$

which holds, for all $\phi \in \{0,1,2\}$, *as can be checked on a case-by-case basis.*

On the other hand, for the local system invertibility, note that $0 \notin \{2\}$, *but* $\sigma^\flat_\Sigma(\tau_\Sigma(0),\psi) = 2 \in \{2\}$, *for every set of unary natural transformations* $\tau : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$. *We conclude that* $I^\flat$ *does not have the local system invertibility in* $\mathcal{I}$.

Finally, we present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that has the local system invertibility but not the local family invertibility in $\mathcal{I}$.

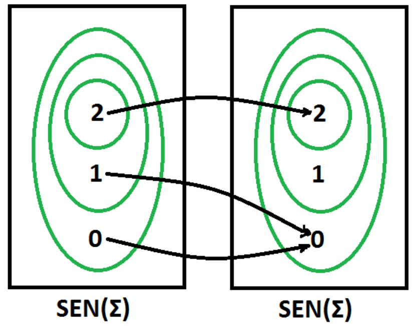
Example 735 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 0$, $1 \mapsto 0$ *and* $2 \mapsto 2$;
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given, for all* $x,y \in \mathrm{SEN}^\flat(\Sigma)$, *by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 2, & \textit{if } (x,y) = (2,2) \\ 0, & \textit{otherwise} \end{cases}.$$

Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}.$$



Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat$ *having two distinguished arguments.*

$\mathcal{I}$ *has three theory families, but only* $\mathrm{Thm}(\mathcal{I})$, $\mathrm{SEN}^\flat$ *are theory systems. So* $\mathcal{I}$ *is not systemic. We show that* $I^\flat$ *has the local system invertibility in* $\mathcal{I}$*, but it does not have the local family invertibility in* $\mathcal{I}$*.*

For the local system invertibility, consider $\tau = \{\iota \approx \iota\}$*, where* $\iota : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *is the identity natural transformation. The case of* $\mathrm{SEN}^\flat$ *is trivial, whereas for* $\mathrm{Thm}(\mathcal{I})$*, we have to verify that, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\phi = 2 \quad \textit{iff} \quad \overleftrightarrow{I^\flat}_\Sigma(\phi,\phi) \subseteq \{\{2\}\}.$$

But this obviously holds, by the definition of $I^\flat$*.*

On the other hand, for the local family invertibility, note that $1 \in \{1,2\}$*, but*

$$\sigma^\flat_\Sigma(\tau_\Sigma(1)) = 0 \notin \{1,2\},$$

for every set of unary natural transformations $\tau : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*. We conclude that* $I^\flat$ *does not have the local family invertibility in* $\mathcal{I}$*.*

We now prove a transfer property for invertibility.

Proposition 736 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $I^\flat$ *a collection of natural transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, with two distinguished arguments, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $I^\flat$ *has an invertibility property in* $\mathcal{I}$ *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*,* I *has the corresponding invertibility property in* $\mathcal{A}$*.*

Proof: If I has an invertibility property in $\mathcal{A}$, for all $\mathcal{A}$, then it has the same invertibility property in $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$. Since $\langle \mathbf{F}, C^{\mathcal{I},\mathcal{F}}\rangle = \mathcal{I}$, we conclude that $I^\flat$ has the corresponding invertibility property in $\mathcal{I}$.

Suppose, conversely, that $I^\flat$ has an invertibility property in $\mathcal{I}$, with witnessing set of natural transformations $\tau : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$. We look at each of the four properties in turn.

(a) Suppose $I^\flat$ has the global family invertibility in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. We then have

$$\begin{array}{lll}\alpha_\Sigma(\phi) \in T_{F(\Sigma)} & \text{iff} & \phi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)}) \\ & \text{iff} & \overleftrightarrow{I^\flat}_\Sigma[\tau_\Sigma(\phi)] \le \alpha^{-1}(T) \\ & \text{iff} & \overleftrightarrow{I}_{F(\Sigma)}[\alpha_\Sigma(\tau_\Sigma(\phi))] \le T \\ & \text{iff} & \overleftrightarrow{I}_{F(\Sigma)}[\tau_{F(\Sigma)}(\alpha_\Sigma(\phi))] \le T.\end{array}$$

Taking into account the surjetivity of $\langle F, \alpha \rangle$, we conclude that I has the global family invertibility in $\mathcal{A}$.

(b) The global system invertibility follows analogously, taking into account the fact that if $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, then $\alpha^{-1}(T) \in \mathrm{ThSys}(\mathcal{I})$.

(c) Suppose $I^\flat$ has the local family invertibility in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{lll}\alpha_\Sigma(\phi) \in T_{F(\Sigma)} & \text{iff} & \phi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)}) \\ & \text{iff} & \overleftrightarrow{I^\flat}_\Sigma(\tau_\Sigma(\phi), \vec{\xi}) \subseteq \alpha_\Sigma^{-1}(T_{F(\Sigma)}), \\ & & \qquad \text{for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma), \\ & \text{iff} & \alpha_\Sigma(\overleftrightarrow{I^\flat}_\Sigma(\tau_\Sigma(\phi), \vec{\xi})) \subseteq T_{F(\Sigma)}, \\ & & \qquad \text{for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma), \\ & \text{iff} & \overleftrightarrow{I}_{F(\Sigma)}(\tau_{F(\Sigma)}(\alpha_\Sigma(\phi)), \alpha_\Sigma(\vec{\xi})) \subseteq T_{F(\Sigma)}, \\ & & \qquad \text{for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma),\end{array}$$

Taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that I has the local family invertibility in $\mathcal{A}$.

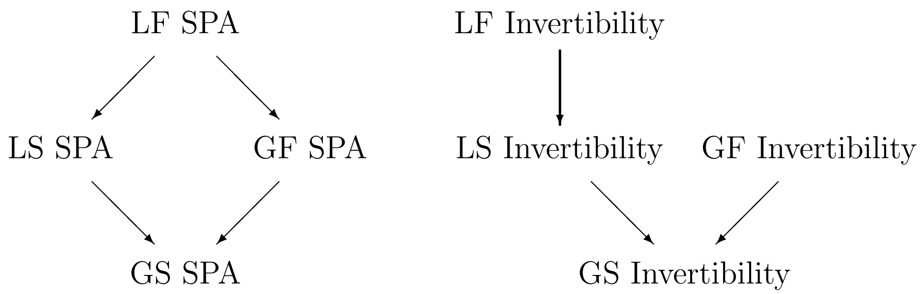
(d) The local system invertibility follows along the same lines, taking again into account the fact that if $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, then $\alpha^{-1}(T) \in \mathrm{ThSys}(\mathcal{I})$.

■

10.13 Syntactic Algebraizability

In this section we focus on the four **uniform syntactic protoalgebraicity properties**, i.e., on LF SPA, LS SPA and GF SPA and GS SPA, and we add to those versions of the invertibility property to obtain several versions of the syntactic algebraizability property.

By previous work, we know that the four uniform SPA properties constitute the hierarchy shown on the left below.



Moreover, by our study of invertibility, we know that the various versions of invertibility fall into the hierarchy shown on the right of the diagram.

By combining syntactic protoalgebraicity with invertibility properties, we obtain sixteen syntactic algebraizability properties as follows. Let $X, Y \in \{\mathrm{LF}, \mathrm{LS}, \mathrm{GF}, \mathrm{GS}\}$, where LF stands for "Local Family", LS stands for "Local System", GF stands for "Global Family" and GS stands for "Global System".

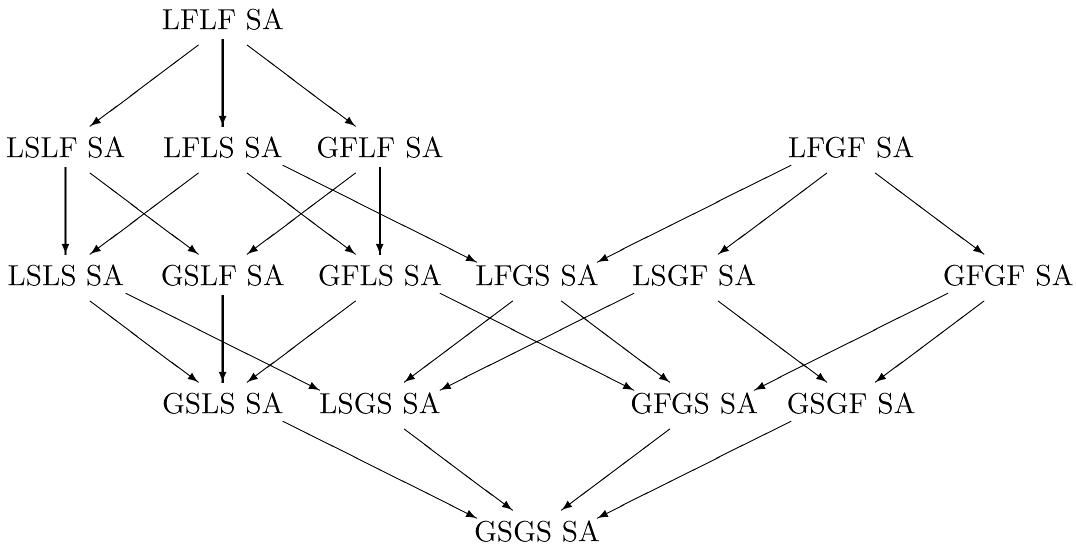
Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. We say that $I^\flat$ has the **XY syntactic algebraizability in** $\mathcal{I}$ (**XY SA in** $\mathcal{I}$) if it has

- the X syntactic protoalgebraicity in $\mathcal{I}$;
- the Y invertibility in $\mathcal{I}$.

Based on the hierarchies of the syntactic protoalgebraicity and invertibility properties, we obtain the following hierarchical structure for the various flavors of the syntactic algebraizability property.

Corollary 737 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments. The sixteen syntactic algebraizability properties form the hierarchy shown on the accompanying diagram.*

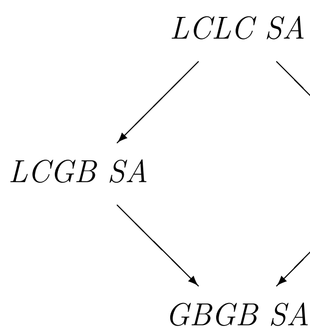
Proof: This follows directly from Corollary 726 and Proposition 731. ∎



Based on the analysis performed on SPA and Invertibility, we have the following result regarding sufficient conditions under which some of the sixteen syntactic algebraizability properties coincide.

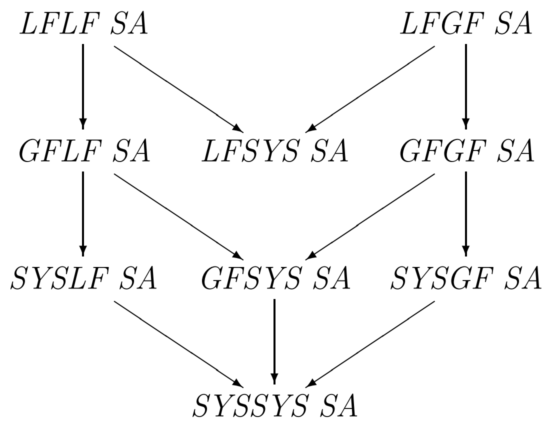
Corollary 738 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments.*

(a) If $\mathcal{I}$ *is systemic, then the syntactic algebraizability hierarchy collapses to the one depicted below;*



(b) If $I^\flat$ *has only two arguments (i.e., is parameter free), then the syntactic algebraizability hierarchy collapses to the one depicted below, where the system versions of both the SPA and the invertibility properties are*

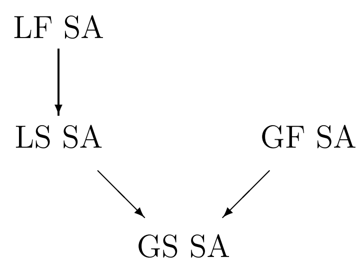
grouped together under the label SYS.



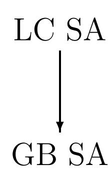
Proof: The statement follows from Corollary 727 and Proposition 732. ■

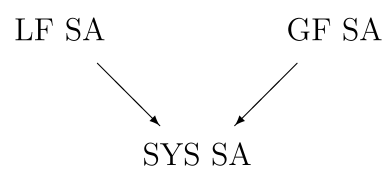
For a systemic π-institution with a parameter-free set of natural transformations, there is only one syntactic protoalgebraicity property, since all versions of syntactic protoalgebraicity and all versions of invertibility collapse to a single property.

Instead of studying this entire hierarchy in detail, we refocus, once again, on the uniformly defined classes. So we define **LF SA**, **LS SA**, **GF SA** and **GS SA** to mean, respectively, LFLF syntactic, LSLS syntactic, GFGF syntactic and GSGS syntactic algebraizability. These classes form the subhierarchy depicted below.



Moreover, according to Corollary 738, this reduces to the hierarchy depicted on the left below for a systemic π-institution and to the one depicted on the right below for a parameter free set of natural transformations.





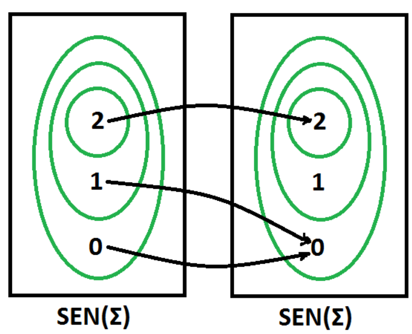
We provide examples to show that the inclusions between the four uniform classes of the syntactic algebraizability hierarchy are proper in general.

First, we present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that has the LS syntactic algebraizability, but not the LF syntactic algebraizability in $\mathcal{I}$.

Example 739 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 0$*,* $1 \mapsto 0$ *and* $2 \mapsto 2$*;*
- $N^\flat$ *is the category of natural transformations generated by*
  - *a unary natural transformation* $\lambda^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined by letting* $\lambda^\flat_\Sigma : \{0,1,2\} \to \{0,1,2\}$ *be given by* $\lambda^\flat_\Sigma(x) = 2$*, for all* $x \in \{0,1,2\}$*;*
  - *a binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 2, & \text{if } x = y \text{ or } \{x,y\} = \{0,1\} \\ 1, & \text{if } \{x,y\} = \{1,2\} \\ 0, & \text{otherwise} \end{cases}$$



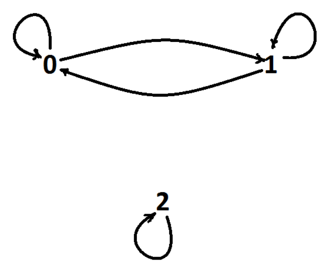
Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}.$$

Note that there are three theory families, but only $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$ *are theory systems. So* $\mathcal{I}$ *is not systemic.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the local system syntactic algebraizability in* $\mathcal{I}$*, but it does not have the local family syntactic algebraizability in* $\mathcal{I}$*.*

First, we look at local system equivalence. The defining conditions for reflexivity, symmetry and transitivity of $I^\flat$ *in* $\mathcal{I}$ *are all trivially satisfied for* $\mathrm{SEN}^\flat$*. For the theory system* $\mathrm{Thm}(\mathcal{I})$*, it suffices to observe that the elements*



of $\mathrm{SEN}^\flat(\Sigma)$ *are related in local system equivalence modulo* $\mathrm{Thm}(\mathcal{I})$ *as shown in the diagram. Therefore,* $I^\flat$ *has the local system equivalence in* $\mathcal{I}$.

Next, observe that, for all $\phi \in \mathrm{SEN}^\flat(\Sigma)$*, the pairs* $(\sigma^\flat_\Sigma(\phi,0),\sigma^\flat_\Sigma(\phi,1))$, $(\sigma^\flat_\Sigma(0,\phi),\sigma^\flat_\Sigma(1,\phi))$ *and* $(\lambda^\flat_\Sigma(0),\lambda^\flat_\Sigma(1))$ *are related via* $I^\flat$ *modulo* $\mathrm{Thm}(\mathcal{I})$. *Thus,* $I^\flat$ *has the local system congruence in* $\mathcal{I}$.

Next, note that, since the only pairs (ϕ,ψ)*, with* $\phi \neq \psi$*, such that* $\sigma^\flat_\Sigma(\phi,\psi) \in \mathrm{Thm}_\Sigma(\mathcal{I})$ *are* $(0,1)$ *and* $(1,0)$ *and for neither of these is* $\phi \in \mathrm{Thm}_\Sigma(\mathcal{I})$, $I^\flat$ *has the local system modus ponens in* $\mathcal{I}$.

Finally, consider the set $\tau : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *of natural transformations in* $N^\flat$*, given by* $\tau = \{\iota \approx \lambda^\flat\}$*, where* $\iota : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *is the identity natural transformation. Since, for every* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*, we have*

$$\phi \in \mathrm{Thm}_\Sigma(\mathcal{I}) \quad \textit{iff} \quad \overleftrightarrow{I}_\Sigma(\phi,\lambda^\flat_\Sigma(\phi)) \subseteq \mathrm{Thm}_\Sigma(\mathcal{I}),$$

we also get that $I^\flat$ *has the local system invertibility in* $\mathcal{I}$ *and, therefore, we conclude that* $I^\flat$ *has the local system algebraizability in* $\mathcal{I}$.

On the other hand, $1 \in \{1,2\}$ *and* $\sigma^\flat_\Sigma(1,0) = 2 \in \{1,2\}$*, but* $0 \notin \{1,2\}$. *Therefore,* $I^\flat$ *does not have the local family modus ponens in* $\mathcal{I}$ *and, hence, a fortiori, it does not have the local family syntactic algebraizability in* $\mathcal{I}$.

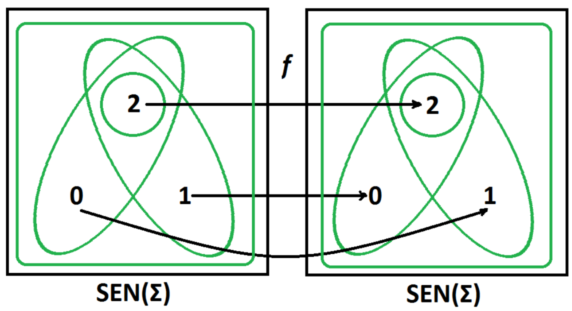
Next, we present an example to show that there is π-institution $\mathcal{I}$ with a set of natural transformations that has the GS syntactic algebraizability but not the GF syntactic algebraizability in $\mathcal{I}$.

Example 740 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = i_\Sigma$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 1$, $1 \mapsto 0$ *and* $2 \mapsto 2$*;*
- $N^\flat$ *is the category of natural transformations generated by*
  - *a unary natural transformation* $\lambda^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined by letting* $\lambda^\flat_\Sigma : \{0,1,2\} \to \{0,1,2\}$ *be given by* $\lambda^\flat_\Sigma(x) = 2$*, for all* $x \in \{0,1,2\}$*;*

– *a single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y,z) = \begin{cases} 2, & \text{if } x = y \text{ or } \{x,y\} = \{0,1\} \\ 1, & \text{if } (x,y) = (0,2) \text{ or } (x,y) = (2,0) \\ 0, & \text{if } (x,y) = (1,2) \text{ or } (x,y) = (2,1) \end{cases}.$$



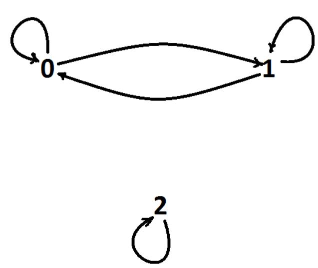
Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\},\{0,2\},\{1,2\},\{0,1,2\}\}.$$

$\mathcal{I}$ *has four theory families* $\mathrm{Thm}(\mathcal{I})$, $T = \{\{0,2\}\}$, $T' = \{\{1,2\}\}$ *and* $\mathrm{SEN}^\flat$, *but only two theory systems* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$. *In particular,* $\mathcal{I}$ *is not systemic.*

Consider the set $I^\flat = \{\sigma^\flat\}$, *with* $\sigma^\flat$ *having both arguments distinguished. We show that* $I^\flat$ *has the global system syntactic algebraizability in* $\mathcal{I}$, *but it does not have the global family syntactic algebraizability in* $\mathcal{I}$.

First, we look at global system equivalence. The defining conditions for reflexivity, symmetry and transitivity of $I^\flat$ *in* $\mathcal{I}$ *are all trivially satisfied for* $\mathrm{SEN}^\flat$. *For the theory system* $\mathrm{Thm}(\mathcal{I})$, *it suffices to observe that the elements of* $\mathrm{SEN}^\flat(\Sigma)$ *are related in global system equivalence modulo* $\mathrm{Thm}(\mathcal{I})$ *as shown in the diagram. Therefore,* $I^\flat$ *has the global system equivalence in* $\mathcal{I}$.



Next, observe that, for all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, *the pairs* $(\sigma^\flat_\Sigma(\phi,0), \sigma^\flat_\Sigma(\phi,1))$, $(\sigma^\flat_\Sigma(0,\phi), \sigma^\flat_\Sigma(1,\phi))$ *and* $(\lambda^\flat_\Sigma(0), \lambda^\flat_\Sigma(1))$ *are related via* $I^\flat$ *modulo* $\mathrm{Thm}(\mathcal{I})$. *Thus,* $I^\flat$ *has the global system congruence in* $\mathcal{I}$.

Next, note that, since the only pairs (ϕ,ψ)*, with* $\phi \neq \psi$*, such that* $\sigma^\flat_\Sigma[\phi,\psi] \leq \mathrm{Thm}(\mathcal{I})$ *are* $(0,1)$ *and* $(1,0)$ *and for neither of these is* $\phi \in \mathrm{Thm}_\Sigma(\mathcal{I})$*,* $I^\flat$ *has the global system modus ponens in* $\mathcal{I}$*.*

Finally, consider the set $\tau : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *of natural transformations in* $N^\flat$*, given by* $\tau = \{\iota \approx \lambda^\flat\}$*, where* $\iota : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *is the identity natural transformation. Since, for every* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*, we have*

$$\phi \in \mathrm{Thm}_\Sigma(\mathcal{I}) \quad \textit{iff} \quad \overleftrightarrow{I}_\Sigma[\phi, \lambda^\flat_\Sigma(\phi)] \leq \mathrm{Thm}(\mathcal{I}),$$

we also get that $I^\flat$ *has the global system invertibility in* $\mathcal{I}$ *and, therefore, we conclude that* $I^\flat$ *has the global system algebraizability in* $\mathcal{I}$*.*

On the other hand, $1 \in \{1,2\}$ *and* $\sigma^\flat_\Sigma[1,0] \leq \{\{1,2\}\}$*, but* $0 \notin \{1,2\}$*. Therefore,* $I^\flat$ *does not have the global family modus ponens in* $\mathcal{I}$ *and, hence, a fortiori, it does not have the global family syntactic algebraizability in* $\mathcal{I}$*.*

Note that the preceding example also shows that there is π-institution $\mathcal{I}$ with a set of natural transformations that has the GS syntactic algebraizability but not the LF syntactic algebraizability in $\mathcal{I}$. We present also an additional example depicting a π-institution $\mathcal{I}$ with a set of natural transformations $I^\flat$, with two distinguished arguments, that has the GS syntactic algebraizability but not the LS syntactic algebraizability in $\mathcal{I}$.

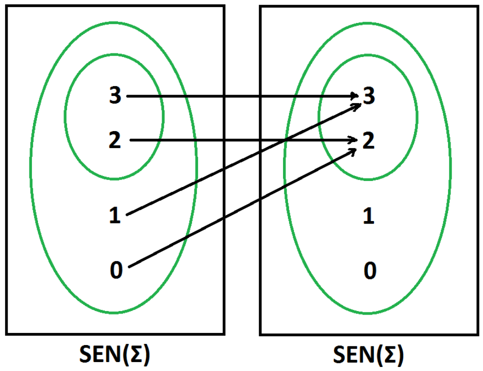
Example 741 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2,3\} \to \{0,1,2,3\}$ *given by* $0 \mapsto 2$*,* $1 \mapsto 3$*,* $2 \mapsto 2$ *and* $3 \mapsto 3$*;*
- $N^\flat$ *is the category of natural transformations generated by*
  - *a unary natural transformation* $\lambda^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined by letting* $\lambda^\flat_\Sigma : \{0,1,2,3\} \to \{0,1,2,3\}$ *be given by* $0 \mapsto 2$*,* $1 \mapsto 3$*,* $2 \mapsto 2$ *and* $3 \mapsto 3$*;*
  - *a ternary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2,3\}^3 \to \{0,1,2,3\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y,z) = \begin{cases} 2, & \textit{if } x = y \textit{ or } (x,y) = (0,1) \textit{ or } z = 2 \textit{ or } z = 3 \\ 0, & \textit{otherwise} \end{cases}.$$

Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2,3\},\{0,1,2,3\}\}.$$



Note that both theory families, $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, are also theory systems. So* $\mathcal{I}$ *is a systemic* π*-institution.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the global (family or system) syntactic algebraizability in* $\mathcal{I}$*, but it does not have the local (family or system) syntactic algebraizability in* $\mathcal{I}$*.*

Concerning global equivalence, the defining conditions for reflexivity, symmetry and transitivity of $I^\flat$ *in* $\mathcal{I}$ *are all trivially satisfied for* $\mathrm{SEN}^\flat$*. For the theory system* $\mathrm{Thm}(\mathcal{I})$*, it suffices to observe that the relation of global equivalence modulo* $\mathrm{Thm}(\mathcal{I})$ *is the identity relation. Therefore,* $I^\flat$ *has the global system equivalence in* $\mathcal{I}$*. Because of that, the global compatibility and the global modus ponens are trivially satisfied.*

Finally, consider the set $\tau : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *of natural transformations in* $N^\flat$*, given by* $\tau = \{\iota \approx \lambda^\flat\}$*, where* $\iota : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *is the identity natural transformation. Since, for every* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*, we have*

$$\phi \in \mathrm{Thm}_\Sigma(\mathcal{I}) \quad \textit{iff} \quad \overleftrightarrow{I}_\Sigma[\phi, \lambda^\flat_\Sigma(\phi)] \le \mathrm{Thm}(\mathcal{I}),$$

we also get that $I^\flat$ *has the global system invertibility in* $\mathcal{I}$ *and, therefore, we conclude that* $I^\flat$ *has the global system algebraizability in* $\mathcal{I}$*.*

On the other hand, $\sigma^\flat_\Sigma(0,1,\xi) \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, for all* $\xi \in \mathrm{SEN}^\flat(\Sigma)$*, but* $\sigma^\flat_\Sigma(1,0,0) = 0 \notin \mathrm{Thm}_\Sigma(\mathcal{I})$*, whence* $I^\flat$ *does not have the local symmetry in* $\mathcal{I}$ *and, therefore, a fortiori, fails to satisfy the local syntactic algebraizability in* $\mathcal{I}$*.*

We close with a transfer property for the syntactic algebraizability properties that we have focused on in this section.

Corollary 742 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $I^\flat$ *a collection of natural transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, with two distinguished arguments, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $I^\flat$ *has a (uniform) syntactic algebraizability property in* $\mathcal{I}$ *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* I *has the corresponding syntactic algebraizability property in* $\mathcal{A}$*.*

Proof: This follows directly from Corollary 730 and Proposition 736. ■

10.14 Regularity

We turn now to the study of various versions of the regularity property, based on the local versus global and the theory family versus theory system dualities.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$, with two distinguished arguments. We say that:

- $I^\flat$ has the **local family regularity in** $\mathcal{I}$ if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi, \psi \in T_\Sigma \text{ imply } I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq T_\Sigma, \text{ for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma);$$
- $I^\flat$ has the **local system regularity in** $\mathcal{I}$ if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi, \psi \in T_\Sigma \text{ imply } I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq T_\Sigma, \text{ for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma);$$
- $I^\flat$ has the **global family regularity in** $\mathcal{I}$ if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi, \psi \in T_\Sigma \text{ imply } I^\flat_\Sigma[\phi, \psi] \leq T;$$
- $I^\flat$ has the **global system regularity in** $\mathcal{I}$ if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi, \psi \in T_\Sigma \text{ imply } I^\flat_\Sigma[\phi, \psi] \leq T.$$

We give now the hierarchy of regularity properties.

Proposition 743 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$ having two distinguished arguments.*

(a) If $I^\flat$ has the global family regularity, then it has the local family regularity in $\mathcal{I}$;

(b) If $I^\flat$ has the local family regularity, then it has the local system regularity in $\mathcal{I}$;

(c) $I^\flat$ has the global system regularity if and only if it has the local system regularity in $\mathcal{I}$.

Proof:

(a) Suppose that $I^\flat$ has the global family regularity in $\mathcal{I}$. Consider $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Then, we have, by hypothesis, $I^\flat_\Sigma[\phi, \psi] \leq T$. But this implies that $I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq T_\Sigma$, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$. Thus, $I^\flat$ has the local family regularity in $\mathcal{I}$.

(b) The conclusion follows directly from the fact that every theory system is a theory family of $\mathcal{I}$.

(c) For the "only if" direction, we repeat the argument used in the proof of Part (a) except reasoning exclusively in terms of theory systems rather than using arbitrary theory families.

Suppose, conversely, that $I^\flat$ has the local system regularity in $\mathcal{I}$. Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Since $T \in \mathrm{ThSys}(\mathcal{I})$, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi) \in T_{\Sigma'}.$$

Thus, by the local system regularity, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$I^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), \vec{\xi}) \subseteq T_{\Sigma'}.$$

Since this holds for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$, we get that $I^\flat_\Sigma[\phi, \psi] \leq T$, Therefore, $I^\flat$ has the global system regularity in $\mathcal{I}$.

■

Proposition 743 has established the following hierarchy of regularity properties:

Global Family Regular

$\downarrow$

Local Family Regular

$\downarrow$

System Regular

We also note the following regarding natural sufficient conditions under which some of these properties coincide.

Proposition 744 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$*, with two distinguished arguments. If* $\mathcal{I}$ *is systemic, then all three regularity properties coincide.*

Proof: If $\mathcal{I}$ is systemic, then the (global) system regularity property coincides with the family regularity property and this causes the collapsing of the hierarchy. ■

So in the case of a systemic π-institution $\mathcal{I}$, there is only one possible regularity property.

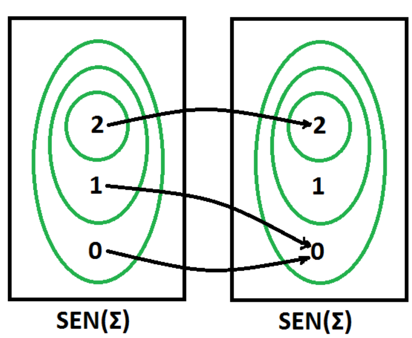
We provide some examples to show that the implications of Proposition 743 are not equivalences in general, i.e., in the hierarchy shown above all inclusions of classes of π-institutions with a set of natural transformations satisfying the corresponding regularity properties are proper inclusions.

We first present an example to show that there is π-institution $\mathcal{I}$ with a set of natural transformations that has the local family regularity but not the global family regularity in $\mathcal{I}$.

Example 745 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 0$, $1 \mapsto 0$ *and* $2 \mapsto 2$;
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1\}^2 \to \{0,1\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 2, & \text{if } x = 2 \text{ or } y = 2 \\ 1, & \text{if } (x,y) = (1,1) \\ 0, & \text{otherwise} \end{cases}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}$.

Consider the set $I^\flat = \{\sigma^\flat\}$, *with* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *having two distinguished arguments.*

$\mathcal{I}$ has three theory families, but only $\mathrm{Thm}(\mathcal{I})$ *and* SEN *are theory systems. We show that $I^\flat$ has the local family regularity in $\mathcal{I}$, but it does not have the global family regularity in $\mathcal{I}$.*

For the local family regularity note, first, that $\sigma^\flat_\Sigma(2,2) = 2$, which takes care of $\mathrm{Thm}(\mathcal{I})$ *and that the case of* $\mathrm{SEN}^\flat$ *is trivial. So we only need to check the case with $T = \{\{1,2\}\}$. Since $\sigma^\flat_\Sigma(2,2) = \sigma^\flat_\Sigma(1,2) = \sigma^\flat_\Sigma(2,1) = 2$ and $\sigma^\flat_\Sigma(1,1) = 1$, the defining condition for local family regularity is also satisfied for $T = \{\{1,2\}\}$. Therefore, $I^\flat$ has the local family regularity in $\mathcal{I}$.*

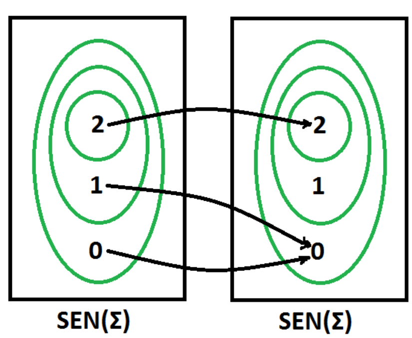
On the other hand, we have $1 \in \{1,2\}$ but $\sigma^\flat_\Sigma(\mathrm{SEN}^\flat(f)(1), \mathrm{SEN}^\flat(f)(1)) = \sigma^\flat_\Sigma(0,0) = 0 \notin \{1,2\}$. Thus, $1 \in T$, but $\sigma^\flat_\Sigma[1,1] \nleq T$, which shows that $I^\flat$ does not have the global family regularity in $\mathcal{I}$.

Next we present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that has the system regularity but not the local family regularity in $\mathcal{I}$.

Example 746 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object Σ and a single (non-identity) morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ and $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ given by $0 \mapsto 0$, $1 \mapsto 0$ and $2 \mapsto 2$;*
- *$N^\flat$ is the category of natural transformations generated by the single binary natural transformation $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ defined by letting $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ be given, for all $a, b \in \mathrm{SEN}^\flat(\Sigma)$, by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 2, & \text{if } x = 2 \text{ or } y = 2 \\ 0, & \text{otherwise} \end{cases}$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be the π-institution determined by $\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}$. Consider the set $I^\flat = \{\sigma^\flat\}$, with $\sigma^\flat$ having two distinguished arguments.

$\mathcal{I}$ has three theory families $\mathrm{Thm}(\mathcal{I})$, $T = \{\{1,2\}\}$ *and* $\mathrm{SEN}^\flat$*, but only two theory systems* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*. We show that $I^\flat$ has the (local) system regularity in $\mathcal{I}$, but it does not have the local family regularity in $\mathcal{I}$.*

For the local system regularity note that $\sigma^\flat_\Sigma(2,2) = 2$, which takes care of $\mathrm{Thm}(\mathcal{I})$*, and that the case of* $\mathrm{SEN}^\flat$ *is trivial.*

On the other hand, for the local family regularity, note that $1 \in T_\Sigma = \{1,2\}$, but $\sigma^\flat_\Sigma(1,1) = 0 \notin T_\Sigma$. Therefore $I^\flat$ does not have the local family regularity in $\mathcal{I}$.

We now prove a transfer property for regularity.

Proposition 747 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $I^\flat$ a collection of natural transformations $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with two distinguished arguments, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. $I^\flat$ has a regularity property in $\mathcal{I}$ if and only if, for every algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, I has the corresponding regularity property in $\mathcal{A}$.*

Proof: If I has a regularity property in $\mathcal{A}$, for all $\mathcal{A}$, then it has the same regularity property in $\mathcal{F} = \langle \mathbf{F}, \langle I,\iota\rangle\rangle$. Since $\langle \mathbf{F}, C^{\mathcal{F}}\rangle = \mathcal{I}$, we conclude that $I^\flat$ has the corresponding regularity property in $\mathcal{I}$.

Suppose, conversely, that $I^\flat$ has a regularity property in $\mathcal{I}$. We look at each of the three properties in turn.

(a) Suppose $I^\flat$ has the global family regularity in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$ and $\alpha_\Sigma(\psi) \in T_{F(\Sigma)}$. Then $\phi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})$ and $\psi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})$. Since, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, we get by global family regularity, $I^\flat_\Sigma[\phi,\psi] \le \alpha^{-1}(T)$. Thus, by Lemma 95, $I_{F(\Sigma)}[\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)] \le T$. We conclude that I has the global family regularity in $\mathcal{A}$.

(b) Suppose $I^\flat$ has the local family regularity in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$ and $\alpha_\Sigma(\psi) \in T_{F(\Sigma)}$. Then $\phi \in \alpha^{-1}(T_{F(\Sigma)})$ and $\psi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})$. Since $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, we get by local family regularity, that

$$I^\flat_\Sigma(\phi,\psi,\vec{\xi}) \subseteq \alpha_\Sigma^{-1}(T_{F(\Sigma)}), \text{ for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma).$$

Thus, $\alpha_\Sigma(I^\flat_\Sigma(\phi,\psi,\vec{\xi})) \subseteq T_{F(\Sigma)}$ or, equivalently,

$$I_{F(\Sigma)}(\alpha_\Sigma(\phi),\alpha_\Sigma(\psi),\alpha_\Sigma(\vec{\xi})) \subseteq T_{F(\Sigma)}, \text{ for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma).$$

It follows, taking into account the surjectivity of $\langle F,\alpha\rangle$, that I has the local family regularity in $\mathcal{A}$.

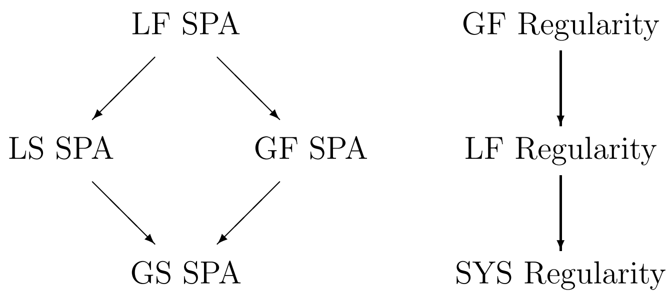
(c) The system regularity follows analogously, taking into account the fact that if $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, then $\alpha^{-1}(T) \in \mathrm{ThSys}(\mathcal{I})$.

■

10.15 Syntactic Regularity

In this section we focus on the four uniform syntactic protoalgebraicity properties, LF SPA, LS SPA, GF SPA and GS SPA, and we add to those versions of the regularity property to obtain several versions of the syntactic regularity property.

By previous work, we know that the four uniform SPA properties constitute the hierarchy shown on the left below.



Moreover, by our study of regularity, we know that the various versions of regularity fall into the linear hierarchy shown on the right of the diagram.

By combining syntactic protoalgebraicity with regularity properties, we obtain twelve syntactic regularity properties as follows. Let $X \in \{\mathrm{LF}, \mathrm{LS}, \mathrm{GF}, \mathrm{GS}\}$ and $Y \in \{\mathrm{LF}, \mathrm{GF}, \mathrm{SYS}\}$, where LF stands for "Local Family", LS stands for "Local System", GF stands for "Global Family", GS stands for "Global System" and SYS stands for "SYStem", abbreviating both the local and the global system properties, in case they are identical.

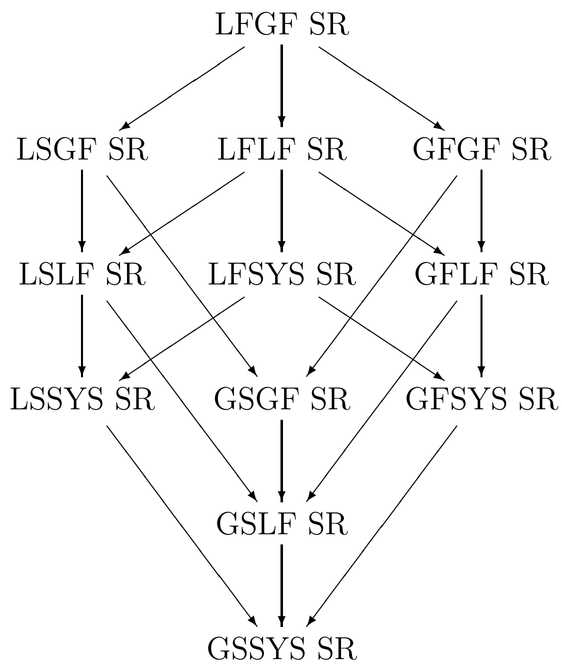
Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$ having two distinguished arguments. We say that $I^\flat$ has the **XY syntactic regularity in** $\mathcal{I}$ (**XY SR in** $\mathcal{I}$) if it has

- the X syntactic protoalgebraicity in $\mathcal{I}$;
- the Y regularity in $\mathcal{I}$.

Based on the hierarchies of the syntactic protoalgebraicity and regularity properties, we obtain the following hierarchical structure for the various flavors of the syntactic regularity property.

Corollary 748 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments. The twelve syntactic regularity properties form the hierarchy shown on the accompanying diagram.*

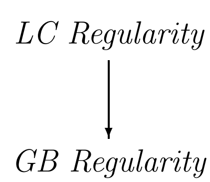
Proof: This follows directly from Corollary 726 and Proposition 743. ■



Based on the analysis performed on SPA and regularity, we have the following result regarding sufficient conditions under which some of the twelve syntactic regularity properties coincide.

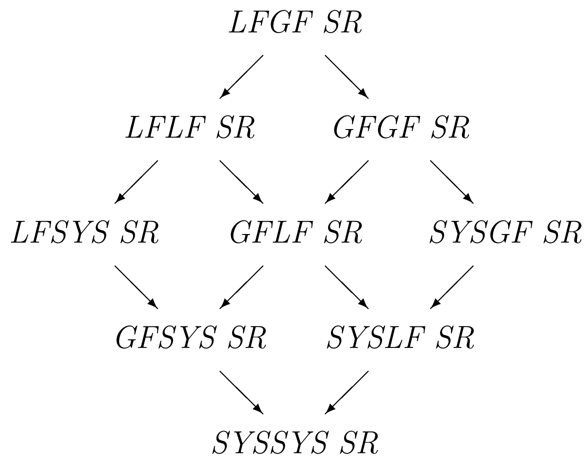
Corollary 749 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$*, with two distinguished arguments.*

(a) If $\mathcal{I}$ *is systemic, then the syntactic regularity hierarchy collapses to the one depicted below;*



(b) If $I^\flat$ *has only two arguments (i.e., is parameter free), then the syntactic regularity hierarchy collapses to the one depicted below, where the System versions of both the SPA and the invertibility properties are*

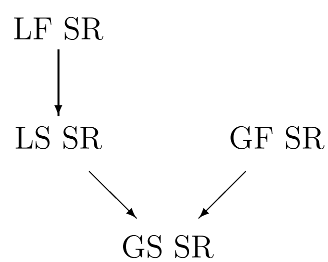
grouped together under the label SYS.



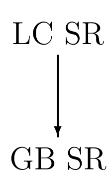
Proof: The statement follows from Corollary 727 and Proposition 743. ■

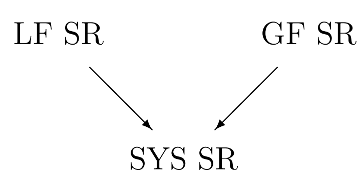
For a systemic π-institution with a parameter-free set of natural transformations, there is only one syntactic regularity property, since all versions of syntactic protoalgebraicity and all versions of regularity collapse to a single property.

Instead of studying this entire hierarchy in detail, we concentrate again on the uniformly defined classes. So we define **LF SR**, **LS SR**, **GF SR** and **GS SR** to mean, respectively, LFLF syntactic, LSLS syntactic, GFGF syntactic and GSGS syntactic regularity. These classes form, according to Corollary 748, the sub hierarchy depicted below.



Moreover, according to Corollary 749, this reduces to the hierarchy depicted on the left below for a systemic π-institution and to the one depicted on the right below for a parameter free set of natural transformations.





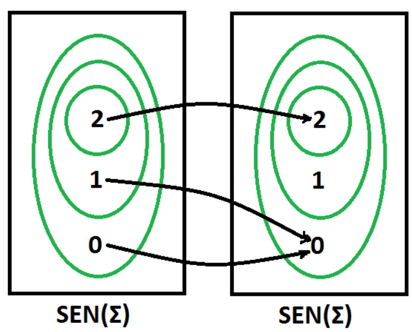
We provide examples to show that the inclusions between these four uniform classes of the syntactic regularity hierarchy are proper in general.

First, we present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that has the LS syntactic regularity, but not the LF syntactic regularity in $\mathcal{I}$.

Example 750 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 0$*,* $1 \mapsto 0$ *and* $2 \mapsto 2$*;*
- $N^\flat$ *is the category of natural transformations generated by a binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 2, & \text{if } x = y \text{ or } \{x,y\} = \{0,1\} \\ 1, & \text{if } \{x,y\} = \{1,2\} \\ 0, & \text{otherwise} \end{cases}$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

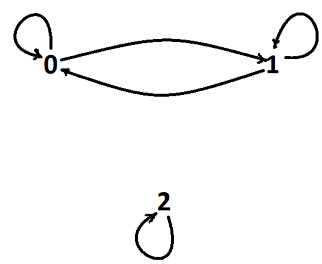
$$\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}.$$

Note that there are three theory families, but only $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$ *are theory systems. So* $\mathcal{I}$ *is not systemic.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the local system syntactic regularity in* $\mathcal{I}$*, but it does not have the local family syntactic regularity in* $\mathcal{I}$*.*

First, we look at local system equivalence. The defining conditions for reflexivity, symmetry and transitivity of $I^\flat$ *in* $\mathcal{I}$ *are all trivially satisfied for* $\mathrm{SEN}^\flat$*. For the theory system* $\mathrm{Thm}(\mathcal{I})$*, it suffices to observe that the elements of* $\mathrm{SEN}^\flat(\Sigma)$ *are related in local system equivalence modulo* $\mathrm{Thm}(\mathcal{I})$ *as shown in the diagram. Therefore,* $I^\flat$ *has the local system equivalence in* $\mathcal{I}$*.*

Next, observe that, for all $\phi \in \mathrm{SEN}^\flat(\Sigma)$*, the pairs* $(\sigma^\flat_\Sigma(\phi,0), \sigma^\flat_\Sigma(\phi,1))$ *and* $(\sigma^\flat_\Sigma(0,\phi), \sigma^\flat_\Sigma(1,\phi))$ *are related via* $I^\flat$ *modulo* $\mathrm{Thm}(\mathcal{I})$*. Thus,* $I^\flat$ *has the local system congruence in* $\mathcal{I}$*.*



Next, note that, since the only pairs (ϕ,ψ)*, with* $\phi\neq\psi$*, such that* $\sigma^\flat_\Sigma(\phi,\psi)\in \mathrm{Thm}_\Sigma(\mathcal{I})$ *are* $(0,1)$ *and* $(1,0)$ *and for neither of these is* $\phi\in\mathrm{Thm}_\Sigma(\mathcal{I})$*,* $I^\flat$ *has the local system modus ponens in* $\mathcal{I}$*.*

Finally, for local system regularity, note that the defining condition is trivially satisfied for $\mathrm{SEN}^\flat$*, whereas, for* $\mathrm{Thm}(\mathcal{I})$*, we clearly have that, if* $\phi,\psi\in\mathrm{Thm}_\Sigma(\mathcal{I})$*, then* $\phi=\psi=2$*, whence* $\sigma^\flat_\Sigma(\phi,\psi)=2\in\mathrm{Thm}_\Sigma(\mathcal{I})$*. Therefore* $I^\flat$ *has the local system regularity in* $\mathcal{I}$ *and, therefore, we conclude that* $I^\flat$ *has the local system syntactic regularity in* $\mathcal{I}$*.*

On the other hand, $1\in\{1,2\}$ *and* $\sigma^\flat_\Sigma(1,0)=2\in\{1,2\}$*, but* $0\notin\{1,2\}$*. Therefore,* $I^\flat$ *does not have the local family modus ponens in* $\mathcal{I}$ *and, hence, a fortiori, it does not have the local family syntactic regularity in* $\mathcal{I}$*.*

Next, we present an example to show that there is π-institution $\mathcal{I}$ with a set of natural transformations that has the GS syntactic regularity but not the GF syntactic regularity in $\mathcal{I}$.

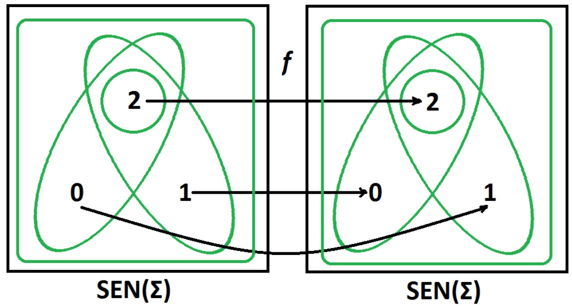
Example 751 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single (non-identity) morphism* $f:\Sigma\to\Sigma$*, such that* $f\circ f=i_\Sigma$*;*
- $\mathrm{SEN}^\flat:\mathbf{Sign}^\flat\to\mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma)=\{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f):\{0,1,2\}\to\{0,1,2\}$ *given by* $0\mapsto1$*,* $1\mapsto0$ *and* $2\mapsto2$*;*
- $N^\flat$ *is the category of natural transformations generated by a single binary natural transformation* $\sigma^\flat:(\mathrm{SEN}^\flat)^2\to\mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma:\{0,1,2\}^2\to\{0,1,2\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y)=\begin{cases}2, & \text{if } x=y \text{ or } \{x,y\}=\{0,1\}\\ 1, & \text{if } \{x,y\}=\{0,2\}\\ 0, & \text{if } \{x,y\}=\{1,2\}\end{cases}$$

Let $\mathcal{I}=\langle\mathbf{F},C\rangle$ *be the* π*-institution determined by*

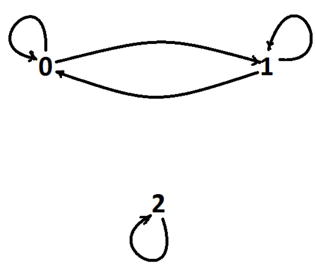
$$\mathcal{C}_\Sigma=\{\{2\},\{0,2\},\{1,2\},\{0,1,2\}\}.$$



$\mathcal{I}$ has four theory families $\mathrm{Thm}(\mathcal{I})$, $T = \{\{0,2\}\}$, $T' = \{\{1,2\}\}$ *and* $\mathrm{SEN}^\flat$, *but only* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$ *are theory systems. In particular, $\mathcal{I}$ is not systemic.*

Consider the set $I^\flat = \{\sigma^\flat\}$, with $\sigma^\flat$ having both arguments distinguished. We show that $I^\flat$ has the global system syntactic regularity in $\mathcal{I}$, but it does not have the global family syntactic regularity in $\mathcal{I}$.

First, we look at global system equivalence. The defining conditions for reflexivity, symmetry and transitivity of $I^\flat$ in $\mathcal{I}$ are all trivially satisfied for $\mathrm{SEN}^\flat$. *For the theory system* $\mathrm{Thm}(\mathcal{I})$, *it suffices to observe that the elements of* $\mathrm{SEN}^\flat(\Sigma)$ *are related in global system equivalence modulo* $\mathrm{Thm}(\mathcal{I})$ *as shown in the diagram. Therefore, $I^\flat$ has the global system equivalence in $\mathcal{I}$.*



Next, observe that, for all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, the pairs $(\sigma^\flat_\Sigma(\phi,0), \sigma^\flat_\Sigma(\phi,1))$ and $(\sigma^\flat_\Sigma(0,\phi), \sigma^\flat_\Sigma(1,\phi))$ are related via $I^\flat$ modulo $\mathrm{Thm}(\mathcal{I})$. *Thus, $I^\flat$ has the global system congruence in $\mathcal{I}$.*

Now note that, since the only pairs (ϕ,ψ), with $\phi \neq \psi$, such that $\sigma^\flat_\Sigma[\phi,\psi] \leq$ $\mathrm{Thm}(\mathcal{I})$ *are* $(0,1)$ *and* $(1,0)$ *and for neither of these is $\phi \in \mathrm{Thm}_\Sigma(\mathcal{I})$, $I^\flat$ has the global system modus ponens in $\mathcal{I}$.*

For the global system regularity, note that the defining condition is satisfied trivially for $\mathrm{SEN}^\flat$, *whereas for* $\mathrm{Thm}(\mathcal{I})$, *if $\phi,\psi \in \mathrm{Thm}_\Sigma(\mathcal{I})$, then $\phi,\psi = 2$, whence we get $\sigma^\flat_\Sigma[\phi,\phi] \leq \mathrm{Thm}(\mathcal{I})$. Therefore, we conclude that $I^\flat$ has the global system syntactic regularity in $\mathcal{I}$.*

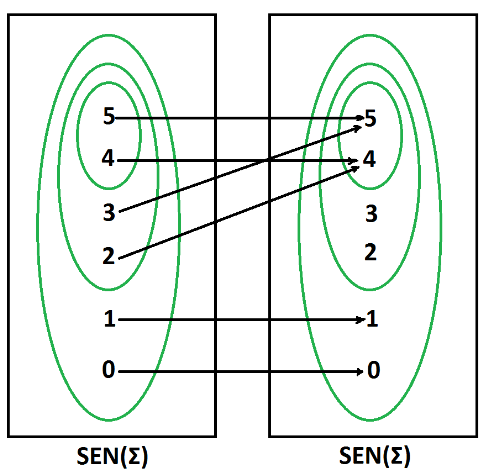
On the other hand, $1 \in \{1,2\}$ and $\sigma^\flat_\Sigma[1,0] \leq \{\{1,2\}\}$, but $0 \notin \{1,2\}$. Therefore, $I^\flat$ does not have the global family modus ponens in $\mathcal{I}$ and, hence, a fortiori, it does not have the global family syntactic regularity in $\mathcal{I}$.

Finally, we present an example of a π-institution $\mathcal{I}$ with a set of natural transformations $I^\flat$, with two distinguished arguments, that has the GS syntactic regularity but not the LS syntactic regularity in $\mathcal{I}$.

Example 752 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with a single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2,3,4,5\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2,3,4,5\} \to \{0,1,2,3,4,5\}$ *given by* $0 \mapsto 0$*,* $1 \mapsto 1$*,* $2 \mapsto 4$*,* $3 \mapsto 5$*,* $4 \mapsto 4$ *and* $5 \mapsto 5$*;*
- $N^\flat$ *is the category of natural transformations generated by a ternary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2,3,4,5\}^3 \to \{0,1,2,3,4,5\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y,z) = \begin{cases} 4, & \text{if } x = y \text{ or } \{x,y\} = \{4,5\} \\ & \quad \text{or } ((x,y) = (1,4) \text{ and } z = 0,1,4,5) \\ & \quad \text{or } ((x,y) = (1,5) \text{ and } z = 0,1,4,5) \\ 2, & \text{else if } \{x,y\} \subseteq \{2,3,4,5\} \\ & \qquad \text{or } (x,y) = (1,2) \text{ or } (x,y) = (1,3) \\ 0, & \text{otherwise} \end{cases}.$$



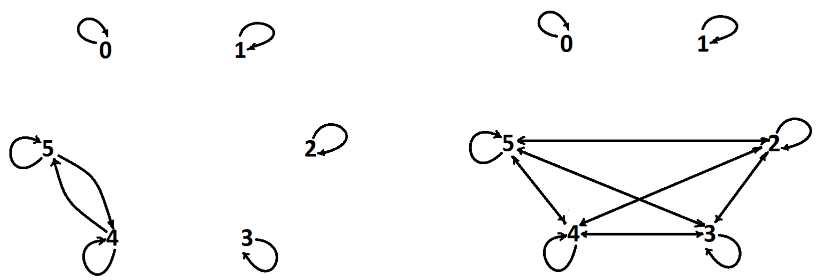
Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{4,5\},\{2,3,4,5\},\{0,1,2,3,4,5\}\}.$$

Note that all three theory families, $\mathrm{Thm}(\mathcal{I})$*,* $T = \{\{2,3,4,5\}\}$ *and* $\mathrm{SEN}^\flat$*, are also theory systems. So* $\mathcal{I}$ *is a systemic* π*-institution.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat : (\mathrm{SEN}^\flat)^3 \to \mathrm{SEN}^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has the global (family or system) syntactic regularity in* $\mathcal{I}$*, but it does not have the local (family or system) syntactic regularity in* $\mathcal{I}$*.*

Concerning global equivalence, the defining conditions for reflexivity, symmetry and transitivity of $I^\flat$ *in* $\mathcal{I}$ *are all trivially satisfied for* $\mathrm{SEN}^\flat$*. For the theory system* $\mathrm{Thm}(\mathcal{I})$*, it suffices to observe that the relation of global equivalence modulo* $\mathrm{Thm}(\mathcal{I})$ *is the binary relation on* $\mathrm{SEN}^\flat(\Sigma)$ *depicted on the left graph in the figure. Moreover, the relation of global equivalence modulo*



T *is the binary relation on* $\mathrm{SEN}^\flat(\Sigma)$ *depicted on the right graph in the figure. Therefore,* $I^\flat$ *has the global system equivalence in* $\mathcal{I}$*.*

Looking at these two graphs and taking into account the definition of $\sigma^\flat$*, we can see that the defining conditions of the global compatibility and the global modus ponens are also satisfied for all three theory systems.*

For global regularity, note again that the defining condition is trivially satisfied for $\mathrm{SEN}^\flat$*, that* $\sigma^\flat_\Sigma[\phi,\psi] \leq \mathrm{Thm}(\mathcal{I})$*, if* $\phi,\psi \in \{4,5\}$*, and that* $\sigma^\flat_\Sigma[\phi,\psi] \leq T$*, if* $\phi,\psi \in \{2,3,4,5\}$*. Thus, we conclude that* $I^\flat$ *has the global regularity in* $\mathcal{I}$ *and, therefore,* $I^\flat$ *has the global system syntactic regularity in* $\mathcal{I}$*.*

On the other hand, $\sigma^\flat_\Sigma(1,2,\xi) = 2 \in T_\Sigma$*, for all* $\xi \in \mathrm{SEN}^\flat(\Sigma)$*, whereas* $\sigma^\flat_\Sigma(2,1,0) = 0 \notin T_\Sigma$*, whence* $I^\flat$ *does not have the local system symmetry in* $\mathcal{I}$ *and, therefore, a fortiori, fails to satisfy the local system syntactic regularity in* $\mathcal{I}$*.*

We close with a transfer property for the (uniform) syntactic regularity properties that we have studied in this section.

Corollary 753 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $I^\flat$ *a collection of natural transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, with two distinguished arguments, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $I^\flat$ *has a (uniform) syntactic regularity property in* $\mathcal{I}$ *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*,* I *has the corresponding syntactic regularity property in* $\mathcal{A}$*.*

Proof: This follows directly from Corollary 730 and Proposition 747. ∎

10.16 Modus Fortis

We conclude with the study of versions of the modus fortis (also known as the Wójcicki or the Rasiowa) property. In the next setion, we call Rasiowa property the combination of syntactic protoalgebraicity with the modus fortis.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$ having two distinguished arguments. We say that:

- $I^\flat$ has the **local family modus fortis** (**local family MF**) **in** $\mathcal{I}$ if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\psi \in T_\Sigma \text{ implies } I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq T_\Sigma, \text{ for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma);$$
- $I^\flat$ has the **local system modus fortis** (**local system MF**) **in** $\mathcal{I}$ if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\psi \in T_\Sigma \text{ implies } I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq T_\Sigma, \text{ for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma);$$
- $I^\flat$ has the **global family modus fortis** (**global family MF**) **in** $\mathcal{I}$ if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\psi \in T_\Sigma \text{ implies } I^\flat_\Sigma[\phi, \psi] \le T;$$
- $I^\flat$ has the **global system modus fortis** (**global system MF**) **in** $\mathcal{I}$ if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\psi \in T_\Sigma \text{ implies } I^\flat_\Sigma[\phi, \psi] \le T.$$

We give now the hierarchy of modus fortis properties.

Proposition 754 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments.*

(a) If $I^\flat$ *has the global family MF, then it has the local family MF in* $\mathcal{I}$*;*

(b) If $I^\flat$ *has the local family MF, then it has the local system MF in* $\mathcal{I}$*;*

(c) $I^\flat$ *has the global system MF if and only if it has the local system MF in* $\mathcal{I}$*.*

Proof:

(a) Suppose that $I^\flat$ has the global family MF in $\mathcal{I}$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\psi \in T_\Sigma$. Then, we have, by hypothesis, $I^\flat_\Sigma[\phi, \psi] \leq T$. This implies that $I^\flat_\Sigma(\phi, \psi, \vec{\xi}) \subseteq T_\Sigma$, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$. Thus, $I^\flat$ has the local family MF in $\mathcal{I}$.

(b) The conclusion follows directly from the fact that every theory system is a theory family of $\mathcal{I}$.

(c) For the "only if" direction, we repeat the argument used in the proof of Part (a) except reasoning exclusively in terms of theory systems rather than using arbitrary theory families.

Suppose, conversely, that $I^\flat$ has the local system MF in $\mathcal{I}$. Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\psi \in T_\Sigma$. Since $T \in \mathrm{ThSys}(\mathcal{I})$, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, $\mathrm{SEN}^\flat(f)(\psi) \in T_{\Sigma'}$. Thus, by the local system MF, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$I^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), \vec{\xi}) \subseteq T_{\Sigma'}.$$

Since this holds, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$, we get that $I^\flat_\Sigma[\phi, \psi] \leq T$, Therefore, $I^\flat$ has the global system MF in $\mathcal{I}$.

■

Proposition 754 has established the following hierarchy of Modus Fortis properties:

Global Family MF

↓

Local Family MF

↓

System MF

We also note the following regarding natural sufficient conditions under which some of these properties coincide.

Proposition 755 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments. If* $\mathcal{I}$ *is systemic, then all three modus fortis properties coincide.*

Proof: If $\mathcal{I}$ is systemic, then the (global) system MF coincides with the family MF property and this causes the collapsing of the hierarchy. ■

So in the case of a systemic π-institution $\mathcal{I}$, there is only one possible modus fortis property.

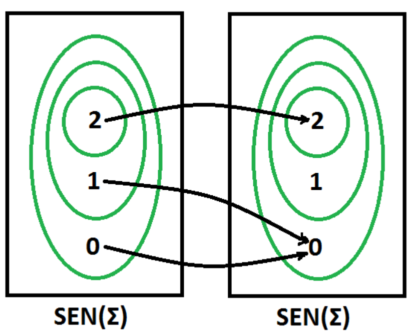
We provide some examples to show that the implications of Proposition 754 are not equivalences in general, i.e., in the hierarchy shown above all inclusions of classes of π-institutions with a set of natural transformations satisfying the corresponding modus fortis properties are proper inclusions.

We first present an example to show that there is π-institution $\mathcal{I}$ with a set of natural transformations that has the local family MF but not the global family MF in $\mathcal{I}$.

Example 756 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 0$, $1 \mapsto 0$ *and* $2 \mapsto 2$;
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1\}^2 \to \{0,1\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 2, & \text{if } x = 2 \text{ or } y = 2 \\ 1, & \text{if } x \neq 2 \text{ and } y = 1 \\ 0, & \text{otherwise} \end{cases}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$C_\Sigma = \{\{2\}, \{1,2\}, \{0,1,2\}\}.$$

Consider the set $I^\flat = \{\sigma^\flat\}$, *with* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *having two distinguished arguments.*

$\mathcal{I}$ *has three theory families, but only* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$ *are theory systems. We show that* $I^\flat$ *has the local family modus fortis, but it does not have the global family modus fortis in* $\mathcal{I}$.

For the local family MF note, first, that, for all $x \in \mathrm{SEN}^\flat(\Sigma)$, $\sigma^\flat_\Sigma(x,2) = 2$, *which takes care of* $\mathrm{Thm}(\mathcal{I})$, *and that the case of* $\mathrm{SEN}^\flat$ *is trivial. So we only need to check the case with* $T = \{\{1,2\}\}$. *Since, for all* $x \in \mathrm{SEN}^\flat(\Sigma)$, $\sigma^\flat_\Sigma(x,2) = 2$ *and, also,* $\sigma^\flat_\Sigma(0,1) = \sigma^\flat_\Sigma(1,1) = 1$ *and* $\sigma^\flat_\Sigma(2,1) = 2$, *the defining condition for local family MF is also satisfied for* $T = \{\{1,2\}\}$. *Therefore,* $I^\flat$ *has the local family MF in* $\mathcal{I}$.

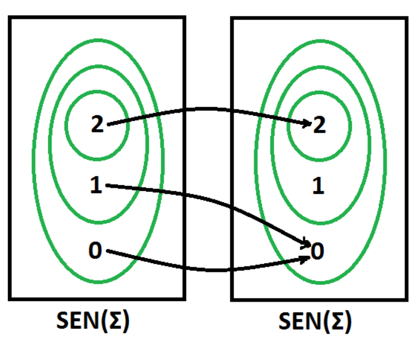
On the other hand, we have $1 \in \{1,2\}$ *but* $\sigma^\flat_\Sigma(\mathrm{SEN}^\flat(f)(0), \mathrm{SEN}^\flat(f)(1)) = \sigma^\flat_\Sigma(0,0) = 0 \notin \{1,2\}$. *Thus,* $1 \in T$, *but* $\sigma^\flat_\Sigma[0,1] \nsubseteq T$, *which shows that* $I^\flat$ *does not have the global family MF in* $\mathcal{I}$.

Next we present an example to show that there exists a π-institution $\mathcal{I}$, with a set of natural transformations that has the system modus fortis but not the local family modus fortis in $\mathcal{I}$.

Example 757 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$, *such that* $f \circ f = f$;
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 0$, $1 \mapsto 0$ *and* $2 \mapsto 2$;
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given, for all* $a, b \in \mathrm{SEN}^\flat(\Sigma)$, *by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 2, & \text{if } x = 2 \text{ or } y = 2 \\ 0, & \text{otherwise} \end{cases}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}.$$

Consider the set $I^\flat = \{\sigma^\flat\}$, *with* $\sigma^\flat$ *having two distinguished arguments.*

$\mathcal{I}$ has three theory families $\mathrm{Thm}(\mathcal{I})$, $T = \{\{1,2\}\}$ *and* $\mathrm{SEN}^\flat$, *but only two theory systems* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$. *We show that* $I^\flat$ *has the (local) system MF in* $\mathcal{I}$, *but it does not have the local family MF in* $\mathcal{I}$.

For the local system MF note that, for all $x \in \mathrm{SEN}^\flat(\Sigma)$, $\sigma^\flat_\Sigma(x,2) = 2$, *which takes care of* $\mathrm{Thm}(\mathcal{I})$, *and that the case of* $\mathrm{SEN}^\flat$ *is trivial.*

On the other hand, for the local family MF, note that $1 \in T_\Sigma = \{1,2\}$, *but* $\sigma^\flat_\Sigma(0,1) = 0 \notin T_\Sigma$. *Therefore,* $I^\flat$ *does not have the local family MF in* $\mathcal{I}$.

We finally prove a transfer property for modus fortis.

Proposition 758 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $I^\flat$ *a collection of natural transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$, *with two distinguished arguments, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $I^\flat$ *has an MF property in* $\mathcal{I}$ *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, *with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, I *has the corresponding MF property in* $\mathcal{A}$.

Proof: If I has an MF property in $\mathcal{A}$, for all $\mathcal{A}$, then it has the same property in $\mathcal{F} = \langle \mathbf{F}, \langle I,\iota\rangle\rangle$. Since $\langle \mathbf{F}, C^{\mathcal{I},\mathcal{F}}\rangle = \mathcal{I}$, we conclude that $I^\flat$ has the corresponding MF property in $\mathcal{I}$.

Suppose, conversely, that $I^\flat$ has an MF property in $\mathcal{I}$. We look at each of the three properties in turn.

(a) Suppose $I^\flat$ has the global family MF in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\alpha_\Sigma(\psi) \in T_{F(\Sigma)}$. Then $\psi \in \alpha^{-1}_\Sigma(T_{F(\Sigma)})$. Since, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, we get by global family MF, $I^\flat_\Sigma[\phi,\psi] \leq \alpha^{-1}(T)$. Thus, by Lemma 95,

$$I_{F(\Sigma)}[\alpha_\Sigma(\phi), \alpha_\Sigma(\psi)] \leq T.$$

We conclude that I has the global family MF in $\mathcal{A}$.

(b) Suppose $I^\flat$ has the local family MF in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\alpha_\Sigma(\psi) \in T_{F(\Sigma)}$. Then $\psi \in \alpha^{-1}_\Sigma(T_{F(\Sigma)})$. Since $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, we get by local family MF, that

$$I^\flat_\Sigma(\phi,\psi,\vec{\xi}) \subseteq \alpha^{-1}_\Sigma(T_{F(\Sigma)}), \text{ for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma).$$

Thus, $\alpha_\Sigma(I^\flat_\Sigma(\phi,\psi,\vec{\xi})) \subseteq T_{F(\Sigma)}$ or, equivalently,

$$I_{F(\Sigma)}(\alpha_\Sigma(\phi), \alpha_\Sigma(\psi), \alpha_\Sigma(\vec{\xi})) \subseteq T_{F(\Sigma)}, \text{ for all } \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma).$$

It follows, taking into account the surjectivity of $\langle F,\alpha\rangle$, that I has the local family MF in $\mathcal{A}$.

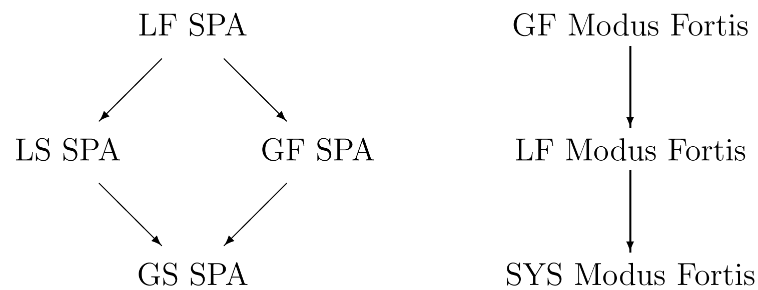
(c) The system MF follows analogously, taking into account the fact that if $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, then $\alpha^{-1}(T) \in \mathrm{ThSys}(\mathcal{I})$.

■

10.17 The Rasiowa Property

In this section we focus again on the four uniform syntactic protoalgebraicity properties, LF SPA, LS SPA, GF SPA and GS SPA, and we add to those versions of the modus fortis property to obtain several versions of the Rasiowa property.

By previous work, we know that the four uniform SPA properties constitute the hierarchy shown on the left below.



Moreover, by our study of modus fortis, we know that the various versions of modus fortis (MF) fall into the linear hierarchy shown on the right of the diagram.

By combining syntactic protoalgebraicity with MF properties, we obtain twelve Rasiowa properties as follows. Let $X \in \{\mathrm{LF}, \mathrm{LS}, \mathrm{GF}, \mathrm{GS}\}$ and $Y \in \{\mathrm{LF}, \mathrm{GF}, \mathrm{SYS}\}$, where LF stands for "Local Family", LS stands for "Local System", GF stands for "Global Family", GS stands for "Global System" and SYS stands for "SYStem", abbreviating both the local and the global system properties, when they are identical.

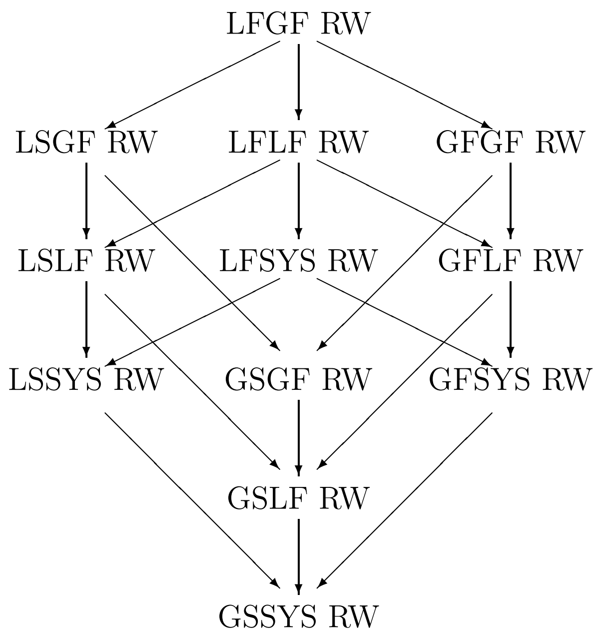
Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$ having two distinguished arguments. We say that $I^\flat$ has the **XY Rasiowa property in $\mathcal{I}$** (**XY RW in $\mathcal{I}$**), or that $I^\flat$ is **XY Rasiowan in $\mathcal{I}$**, if it has

- the X syntactic protoalgebraicity in $\mathcal{I}$;
- the Y modus fortis in $\mathcal{I}$.

Based on the hierarchies of the syntactic protoalgebraicity and MF properties, we obtain the following a priori hierarchical structure for the various flavors of the Rasiowa property.

Corollary 759 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a set of natural transformations in $N^\flat$ having two distinguished arguments. The twelve Rasiowa properties form the hierarchy shown on the accompanying diagram.*

Proof: This follows directly from Corollary 726 and Proposition 754. ∎



It turns out that all these classes collapse to a single class! Indeed, as we show next, the only π-institutions, with a set of natural transformations having two distinguished arguments, satisfying the global system syntactic protoalgebraicity and the system modus fortis are the inconsistent ones. As a consequence, they also satisfy the local family syntactic protoalgebraicity and the global family modus fortis.

Proposition 760 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments. If* $I^\flat$ *has the GSSYS Rasiowa property in* $\mathcal{I}$*, then* $\mathcal{I}$ *is inconsistent.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$. Since $I^\flat$ is reflexive, $I^\flat_\Sigma(\phi, \phi, \vec{\xi}) \subseteq \mathrm{Thm}_\Sigma(\mathcal{I})$, for all $\phi, \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$. Thus, $\mathrm{Thm}_\Sigma(\mathcal{I}) \neq \varnothing$. Fix $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$. Then, for all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, we get, using the SYS Rasiowa property, $I^\flat_\Sigma[\phi, t] \leq \mathrm{Thm}(\mathcal{I})$. Then, by GS symmetry, $I^\flat_\Sigma[t, \phi] \leq \mathrm{Thm}(\mathcal{I})$. Thus, by GS modus ponens, we get $\phi \in \mathrm{Thm}_\Sigma(\mathcal{I})$. Since this holds for all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, we conclude that $\mathrm{Thm}(\mathcal{I}) = \mathrm{SEN}^\flat$ and, therefore, $\mathcal{I}$ is inconsistent. ∎

So the hierarchy of Corollary 759 consists actually of a single property, which we call the **Rasiowa property**, and the only π-institutions satisfying that property are the inconsistent ones.

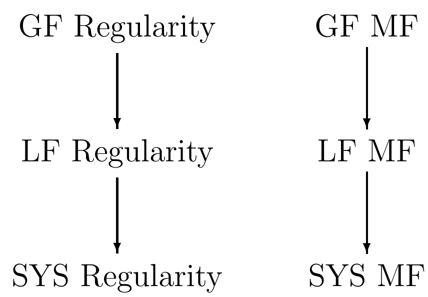
The Rasiowa property also transfers.

Corollary 761 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $I^\flat$ *a collection of natural transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, with two* *distinguished arguments, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $I^\flat$ *has the Rasiowa property in* $\mathcal{I}$ *if and only if, for every algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, *with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, I *has the Rasiowa property in* $\mathcal{A}$.

Proof: This follows directly from Corollary 730 and Proposition 758. ∎

10.18 Modus Fortis and Regularity

Recall the hierarchies of the regularity and modus fortis properties that we have introduced previously. These are depicted again below.



The various versions of these three properties are not independent. In fact the modus fortis properties imply the corresponding regularity properties. We prove these straightforward dependencies in the following two propositions.

Proposition 762 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments.*

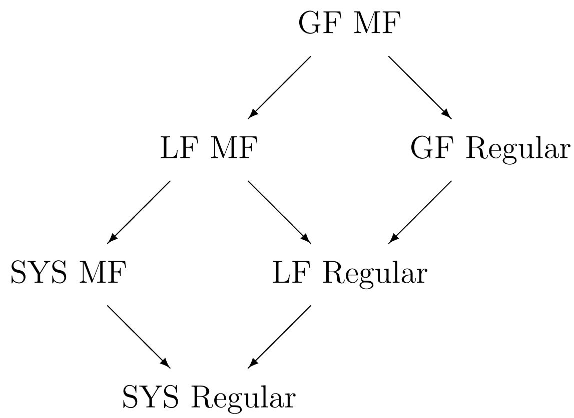
(a) If $I^\flat$ *has the global family MF, then it has the global family regularity in* $\mathcal{I}$*;*

(b) If $I^\flat$ *has the local family MF, then it has the local family regularity in* $\mathcal{I}$*;*

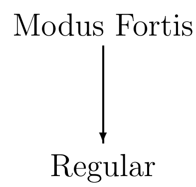
(c) If $I^\flat$ *has the system MF, then it has the system regularity in* $\mathcal{I}$.

Proof: We only provide a proof for Part (a), since Parts (b) and (c) can be proved in essentially the same way. So suppose that $I^\flat$ has the global family modus fortis in $\mathcal{I}$ and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Since $\psi \in T_\Sigma$ and $I^\flat$ has the global family modus fortis in $\mathcal{I}$, we get that $I^\flat_\Sigma[\phi, \psi] \leq T$. This show that $I^\flat$ has the global family regularity in $\mathcal{I}$. ∎

Proposition 762 together with the previously established hierarchies of regularity and modus fortis properties, establish the following combined hierarchy of these properties.



Recall, now that, if $\mathcal{I}$ is systemic, all three versions of regularity and modus fortis are identified. Therefore, in the case of a systemic π-institution $\mathcal{I}$ with a set $I^\flat$ of natural transformations having two distinguished arguments, the hierarchy above reduces to simply



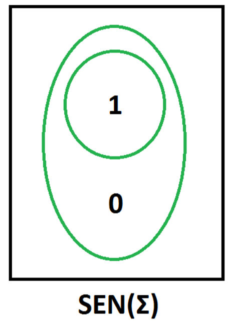
On the other hand, since the property of being parameter-free does not affect either the regularity or the Modus Fortis hierarchies, it has no effect on the mixed hierarchy either.

We present an example of a π-institution $\mathcal{I}$, with a set $I^\flat$ of natural transformations, having two distinguished variables, that has the global family regularity property in $\mathcal{I}$, but does not have the system modus fortis property in $\mathcal{I}$.

Example 763 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1\}^2 \to \{0,1\}$ *be given, for all* $x,y \in \mathrm{SEN}^\flat(\Sigma)$*, by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\}, \{0,1\}\}$. *Consider the set* $I^\flat = \{\sigma^\flat\}$, *with* $\sigma^\flat$ *having two distinguished arguments.*

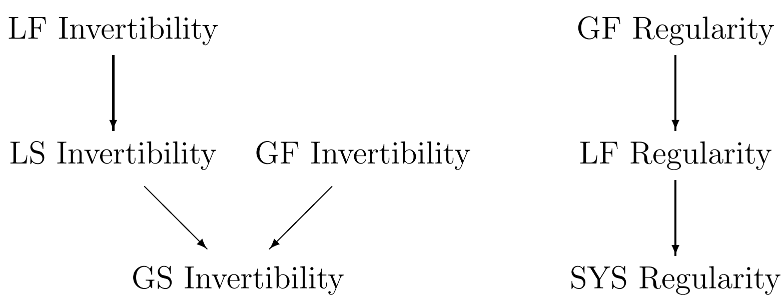
$\mathcal{I}$ *has two theory families* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$, *both of which are theory systems. So it is a systemic* π*-institution. We show that* $I^\flat$ *has the global family regularity in* $\mathcal{I}$, *but it does not have the system modus fortis in* $\mathcal{I}$.

For the global family regularity, note that the condition is trivial when $T = \mathrm{SEN}^\flat$, *whereas for* $T = \mathrm{Thm}(\mathcal{I})$, *if* $\phi = \psi = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$, *we have* $\sigma^\flat_\Sigma(1,1) = 1$, *which gives* $\sigma^\flat_\Sigma[1,1] \leq \mathrm{Thm}(\mathcal{I})$. *Thus,* $I^\flat$ *is indeed global family regular in* $\mathcal{I}$.

On the other hand, note that $1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$, *but* $\sigma^\flat_\Sigma(0,1) = 0 \notin \mathrm{Thm}_\Sigma(\mathcal{I})$. *Therefore,* $I^\flat$ *does not have the system MF in* $\mathcal{I}$.

10.19 Regularity and Invertibility

Recall the hierarchies that we have introduced previously based on invertibility and regularity. These are depicted again below.



Connecting the regularity with the invertibility conditions requires additional hypotheses. Namely, we will suppose that the π-institution under consideration has natural theorems and satisfies some form of the modus ponens property.

Proposition 764 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$, *having natural theorems, and* $I^\flat$:

$(\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments.*

(a) If $I^\flat$ *has the global family modus ponens and the global family regularity, then it has the global family invertibility in* $\mathcal{I}$*;*

(b) If $I^\flat$ *has the local family modus ponens and the local family regularity, then it has the local family invertibility in* $\mathcal{I}$*;*

(c) If $I^\flat$ *has the local system modus ponens and the system regularity, then it has the local system invertibility in* $\mathcal{I}$*;*

(d) If $I^\flat$ *has the global system modus ponens and the system regularity, then it has the global system invertibility in* $\mathcal{I}$*.*

Proof: We only provide a proof for Part (a), since Parts (b)-(d) can be proved in essentially the same way. Let $\top^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ be a natural theorem and suppose that $I^\flat$ has the global family modus ponens and the global family regularity in $\mathcal{I}$. Consider the singleton $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ of natural transformations in $N^\flat$, given by

$$\tau^\flat = \{\top^\flat \approx \iota\},$$

where $\iota : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ is the identity natural transformation. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$.

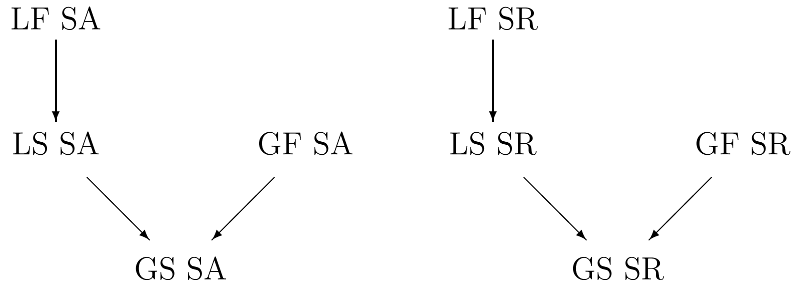
- If $\phi \in T_\Sigma$, then, since $\top^\flat_\Sigma(\phi) \in \mathrm{Thm}_\Sigma(\mathcal{I}) \subseteq T_\Sigma$, we get, by global family regularity, $I^\flat_\Sigma[\top^\flat_\Sigma(\phi), \phi] \leq T$, i.e., $I^\flat_\Sigma[\tau^\flat_\Sigma(\phi)] \leq T$.

- Suppose, conversely, that $I^\flat_\Sigma[\tau^\flat_\Sigma(\phi)] \leq T$. Then $I^\flat_\Sigma[\top^\flat_\Sigma(\phi), \phi] \leq T$. Since $\top^\flat_\Sigma(\phi) \in T_\Sigma$, we get, by global family modus ponens, $\phi \in T_\Sigma$.

We conclude that $\phi \in T_\Sigma$ if and only if $I^\flat_\Sigma[\tau^\flat_\Sigma(\phi)] \leq T$. Thus, $I^\flat$ has the global family invertibility in $\mathcal{I}$, with witnessing set of natural transformations $\tau^\flat$. ■

10.20 The Algebraic Hierarchy

Recall the three hierarchies that we have introduced previously based on uniform combinations of the syntactic protoalgebraizability properties and the invertibility, regularity and modus fortis properties. These formed the hierarchies of syntactically algebraizable (SA), syntactically regular (SR) and Rasiowa properties, respectively. The first two are depicted again below, whereas the last consists of a single property, which, as we saw in Proposition 760, is characteristic of inconsistent π-institutions.



The various versions of these three properties are not independent. Since, as was shown in Proposition 762, the modus fortis properties imply the corresponding regularity properties and, as was shown in Proposition 764, regularity properties, fortified with some form of the modus ponens, imply the corresponding invertibility properties, we obtain ensuing relationships between the Rasiowa, syntactic regularity and syntactic algebraizability properties.

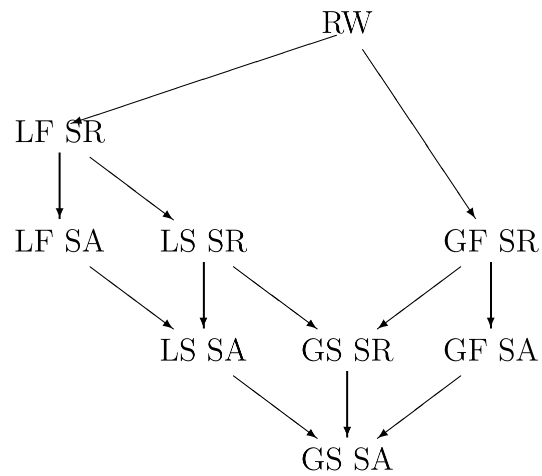
Corollary 765 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments. If* $I^\flat$ *has the Rasiowa property, then it has all four syntactic regularity properties.*

Proof: Directly from the definitions and Proposition 762. ∎

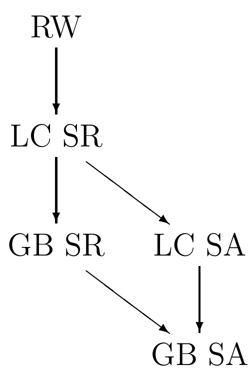
Corollary 766 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a set of natural transformations in* $N^\flat$ *having two distinguished arguments. If* $I^\flat$ *has a syntactic regularity property, then it has the corresponding syntactic algebraizability property in* $\mathcal{I}$*.*

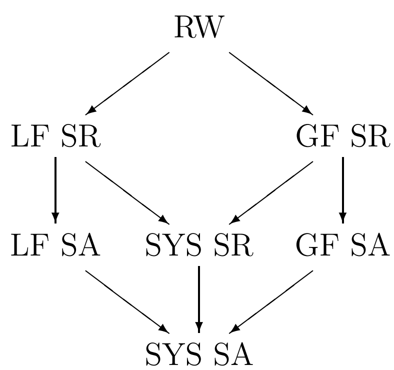
Proof: We present in detail the reasoning for the global family versions. Suppose that $I^\flat$ is a set of natural transformations, with two distinguished arguments, having the global family syntactic regularity in $\mathcal{I}$. Note that, by global family syntactic protoalgebraicity, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $I^\flat_\Sigma[\phi, \phi] \leq \mathrm{Thm}(\mathcal{I})$. Thus, $\mathcal{I}$ has natural theorems. Moreover, by the definition of global family syntactic regularity, $I^\flat$ has both the global family modus ponens and the global family regularity in $\mathcal{I}$. It follows now, by Proposition 764, that $I^\flat$ has the global family invertibility in $\mathcal{I}$. Thus, it also has the global family syntactic algebraizability in $\mathcal{I}$. ∎

Corollaries 765 and 766 together with the previously established hierarchies of syntactic algebraizability, syntactic regularity and Rasiowa properties, establish the following combined hierarchy of these properties.



Recall, now that, if $\mathcal{I}$ is systemic, then the two local versions and the two global versions of syntactic algebraizability become identified and that the same holds for syntactic regularity. Therefore, in the case of a systemic π-institution $\mathcal{I}$ with a set $I^\flat$ of natural transformations having two distinguished arguments, the hierarchy above reduces to the simpler hierarchy shown on the left below.





Furthermore, the property of being parameter-free has the effect of collapsing the two versions of system syntactic algebraizability and the two versions of system syntactic regularity properties. Thus, the hierarchy of the three properties for parameter-free sets of natural transformations $I^\flat$ in $\mathcal{I}$ is given by the diagram shown on the right above.

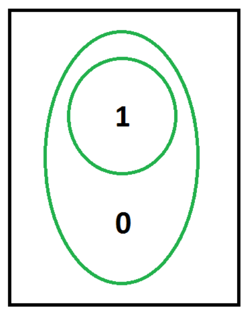
We present some examples to show that all inclusions in the diagram of the hierarchy of Rasiowa, syntactic regularity and syntactic algebraizablity properties are proper in general.

We first present an example of a π-institution $\mathcal{I}$, with a set $I^\flat$ of natural transformations, with two distinguished arguments, that has the local and global family syntactic regularity properties in $\mathcal{I}$, but does not have the Rasiowa property in $\mathcal{I}$.

Example 767 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1\}^2 \to \{0,1\}$ *be given, for all* $x, y \in \mathrm{SEN}^\flat(\Sigma)$*, by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}.$$



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Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\}, \{0,1\}\}$*.* $\mathcal{I}$ *has two theory families* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, both of which are theory systems. So it is a systemic* π*-institution.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has (all kinds of) the syntactic regularity in* $\mathcal{I}$*, but it does not have the Rasiowa property in* $\mathcal{I}$*.*

First, we look at the equivalence property. The defining conditions for reflexivity, symmetry and transitivity of $I^\flat$ *in* $\mathcal{I}$ *are all trivially satisfied for* $\mathrm{SEN}^\flat$*. For the theory system* $\mathrm{Thm}(\mathcal{I})$*, it suffices to observe that equivalence modulo* $\mathrm{Thm}(\mathcal{I})$ *coincides with the identity relation on* $\mathrm{SEN}^\flat(\Sigma)$*. Therefore,* $I^\flat$ *has the local system equivalence in* $\mathcal{I}$*.*

The fact that equivalence modulo $\mathrm{Thm}(\mathcal{I})$ *is the identity relation immediately implies that* $I^\flat$ *also has the compatibility property and the modus ponens in* $\mathcal{I}$*.*

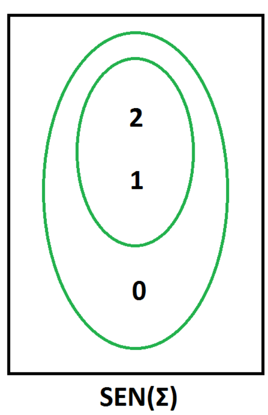
Finally, for regularity, note that the defining condition is trivially satisfied for $\mathrm{SEN}^\flat$*, whereas, for* $\mathrm{Thm}(\mathcal{I})$*, we clearly have that, if* $\phi, \psi \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, then* $\phi = \psi = 1$*, whence* $\sigma^\flat_\Sigma(\phi,\psi) = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*. Therefore* $I^\flat$ *has the regularity property in* $\mathcal{I}$ *and, therefore, we conclude that* $I^\flat$ *has the syntactic regularity in* $\mathcal{I}$*.*

On the other hand, since $\mathcal{I}$ is not an inconsistent π-institution, $I^\flat$ does not have the Rasiowa property in $\mathcal{I}$.

Next, we look at an example of a π-institution $\mathcal{I}$, with a set $I^\flat$ of natural transformations, with two distinguished arguments, that has the local and global family syntactic algebraizability properties in $\mathcal{I}$, but does not possess the global system syntactic regularity in $\mathcal{I}$.

Example 768 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$*;*
- $N^\flat$ *is the category of natural transformations generated by*
  - *a unary natural transformation* $\lambda^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined by letting* $\lambda^\flat_\Sigma : \{0,1,2\} \to \{0,1,2\}$ *be given, for all* $x \in \mathrm{SEN}^\flat(\Sigma)$*, by*
  $$\lambda^\flat_\Sigma(x) = \begin{cases} 2, & \text{if } x = 2 \\ 1, & \text{otherwise} \end{cases};$$
  - *a binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given, for all* $x, y \in \mathrm{SEN}^\flat(\Sigma)$*, by*
  $$\sigma^\flat_\Sigma(x,y) = \begin{cases} 2, & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1,2\},\{0,1,2\}\}$*.* $\mathcal{I}$ *has two theory families* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, both of which are theory systems. So it is a systemic* π*-institution.*

Consider the set $I^\flat = \{\sigma^\flat\}$*, with* $\sigma^\flat$ *having two distinguished arguments. We show that* $I^\flat$ *has (all kinds of) the syntactic algebraizability in* $\mathcal{I}$*, but it does not have (any kind of) the syntactic regularity in* $\mathcal{I}$*.*

First, we look at the equivalence property. The defining conditions for reflexivity, symmetry and transitivity of $I^\flat$ *in* $\mathcal{I}$ *are all trivially satisfied for* $\mathrm{SEN}^\flat$*. For the theory system* $\mathrm{Thm}(\mathcal{I})$*, it suffices to observe that equivalence modulo* $\mathrm{Thm}(\mathcal{I})$ *coincides with the identity relation on* $\mathrm{SEN}^\flat(\Sigma)$*. Therefore,* $I^\flat$ *has the local system equivalence in* $\mathcal{I}$*.*

The fact that equivalence modulo $\mathrm{Thm}(\mathcal{I})$ *is the identity relation immediately implies that* $I^\flat$ *also has the compatibility property and the modus ponens in* $\mathcal{I}$*.*

Finally, for invertibility, consider the set $\tau : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *of natural transformations in* $N^\flat$*, defined by* $\tau = \{\iota \approx \lambda^\flat\}$*. Note that the defining condition is trivially satisfied for* $\mathrm{SEN}^\flat$*, whereas, for* $\mathrm{Thm}(\mathcal{I})$*, we clearly have that,*

$$\phi \in \mathrm{Thm}_\Sigma(\mathcal{I}) \quad \textit{iff} \quad \sigma^\flat_\Sigma(\phi, \lambda^\flat_\Sigma(\phi)) \in \mathrm{Thm}_\Sigma(\mathcal{I}).$$

Therefore, $I^\flat$ *has the invertibility and, hence, the syntactic algebraizability property in* $\mathcal{I}$*.*

On the other hand, we have $1, 2 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, but* $\sigma^\flat_\Sigma(1,2) = 0 \notin \mathrm{Thm}_\Sigma(\mathcal{I})$*. Therefore,* $I^\flat$ *fails to have the regularity property and, hence, a fortiori, does not have the syntactic regularity property in* $\mathcal{I}$*.*

Chapter 11

The Syntactic Leibniz Hierarchy: Foundations

11.1 Syntactic Prealgebraicity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

Recall that $\mathcal{I}$ is **prealgebraic** if, for all $T, T' \in \mathrm{ThSys}(\mathcal{I})$,

$$T \leq T' \quad \text{implies} \quad \Omega(T) \leq \Omega(T').$$

We say that $\mathcal{I}$ is **syntactically prealgebraic** if there exists $I^\flat \subseteq N^\flat$, with two distinguished arguments, such that $I^\flat$ has:

- reflexivity;
- global system transitivity;
- global system compatibility; and
- global system modus ponens.

In that case, we call $I^\flat$ a **set of witnessing natural transformations**, or, more simply, **witnessing transformations** (of the syntactic prealgebraicity of $\mathcal{I}$).

It turns out that, if $\mathcal{I}$ is a syntactically prealgebraic π-institution, with witnessing transformations $I^\flat$, then $\overleftrightarrow{I^\flat}(T)$ is a congruence system on $\mathbf{F}$ compatible with T, for all $T \in \mathrm{ThSys}(\mathcal{I})$. As a consequence, using Corollary 98, we may conclude that, for all $T \in \mathrm{ThSys}(\mathcal{I})$,

$$\overleftrightarrow{I^\flat}(T) = \Omega(T).$$

Proposition 769 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is syntactically prealgebraic, with witnessing transformations $I^\flat$, then, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\overleftrightarrow{I^\flat}(T)$ is a congruence system on $\mathbf{F}$ compatible with T.*

Proof: Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$.

Since $I^\flat$ is reflexive in $\mathcal{I}$, we get that $I^\flat_\Sigma[\phi, \phi] \leq \mathrm{Thm}(\mathcal{I}) \leq T$. Therefore, $\overleftrightarrow{I^\flat}_\Sigma[\phi, \phi] \leq T$, which shows that $\langle \phi, \phi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$.

Suppose, next, that $\langle \phi, \psi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$. Thus, $\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi] \leq T$. By the definition of $\overleftrightarrow{I^\flat}$, we get $\overleftrightarrow{I^\flat}_\Sigma[\psi, \phi] \leq T$ and, hence, $\langle \psi, \phi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$.

Next, assume that $\langle \phi, \psi \rangle, \langle \psi, \chi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$. Then we get $\langle \phi, \psi \rangle$, $\langle \psi, \chi \rangle$, $\langle \psi, \phi \rangle$, $\langle \chi, \psi \rangle \in I^\flat_\Sigma(T)$. Since $I^\flat$ is globally system transitive in $\mathcal{I}$, we conclude that $\langle \phi, \chi \rangle$, $\langle \chi, \phi \rangle \in I^\flat_\Sigma(T)$ and, therefore, $\langle \phi, \chi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$.

To show the congruence property, assume that $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ is a natural transformation in $N^\flat$ and that $\langle \phi_i, \psi_i \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$, for all $i < k$. Thus, since $I^\flat$ has the global system compatibility in $\mathcal{I}$, we get that $\langle\sigma^\flat_\Sigma(\vec{\phi}),\sigma^\flat_\Sigma(\vec{\psi})\rangle \in I^\flat_\Sigma(T)$. By symmetry, we also get $\langle\sigma^\flat_\Sigma(\vec{\psi}),\sigma^\flat_\Sigma(\vec{\phi})\rangle \in I^\flat_\Sigma(T)$ and, hence, that $\langle\sigma^\flat_\Sigma(\vec{\phi}),\sigma^\flat_\Sigma(\vec{\psi})\rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$.

Finally, since by Lemma 93, $\overleftrightarrow{I^\flat}(T)$ is a relation system on $\mathbf{F}$, we conclude that $\overleftrightarrow{I^\flat}(T)$ is a congruence system on $\mathbf{F}$.

To conclude the proof, note that, if $\phi \in T_\Sigma$ and $\langle\phi,\psi\rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$, then $\psi \in T_\Sigma$ by the global system modus ponens of $I^\flat$ in $\mathcal{I}$ and the fact that $I^\flat \subseteq \overleftrightarrow{I^\flat}$. ■

Based on Proposition 769, we can conclude that $\overleftrightarrow{I^\flat}$ defines the Leibniz congruence systems of the theory systems of $\mathcal{I}$.

Corollary 770 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically prealgebraic, with witnessing transformations* $I^\flat$*, if and only if, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,*

$$\overleftrightarrow{I^\flat}(T) = \Omega(T).$$

Proof: The only if is by Proposition 769 and Corollary 98. The if is obvious, since the displayed equations immediately implies the four properties of $\overleftrightarrow{I^\flat}$ defining syntactic prealgebraicity. ■

Corollary 770 has as an immediate consequence the important fact that syntactic prealgebraicity implies (semantic) prealgebraicity.

Theorem 771 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is syntactically prealgebraic, then it is prealgebraic.*

Proof: Suppose that $\mathcal{I}$ is syntactically prealgebraic with witnessing transformations $I^\flat$. Let $T,T' \in \mathrm{ThSys}(\mathcal{I})$, such that $T \le T'$. Then

$$\begin{array}{rcll}\Omega(T) & = & \overleftrightarrow{I^\flat}(T) & \text{(by Corollary 770)}\\ & \le & \overleftrightarrow{I^\flat}(T') & \text{(by Lemma 94)}\\ & = & \Omega(T'). & \text{(by Corollary 770)}\end{array}$$

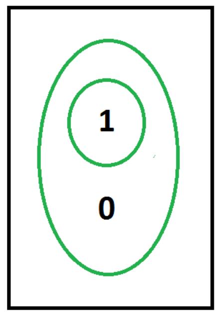
Thus, $\mathcal{I}$ is prealgebraic. ■

The following example shows that the inclusion of Theorem 771 is proper.

Example 772 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with a single object* Σ*;*

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting:* $\sigma^\flat_\Sigma : \{0,1\}^2 \to \{0,1\}$ *be given by*
$$\sigma^\flat_\Sigma(x,y) = 1, \quad \textit{for all } x,y \in \{0,1\}.$$



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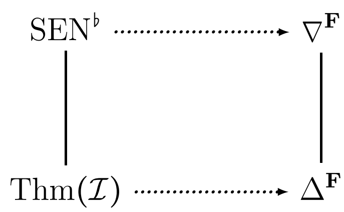
Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\}$*.*

$\mathcal{I}$ *has two theory families,* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, which are also theory systems. Clearly,* $\mathrm{Thm}(\mathcal{I}) \le \mathrm{SEN}^\flat$*. Moreover,* $\Omega(\mathrm{Thm}(\mathcal{I})) = \Delta^{\mathbf{F}}$ *and* $\Omega(\mathrm{SEN}^\flat) = \nabla^{\mathbf{F}}$*. Since* $\Omega(\mathrm{Thm}(\mathcal{I})) \le \Omega(\mathrm{SEN}^\flat)$*,* $\mathcal{I}$ *is prealgebraic.*



On the other hand, there does not exist $I^\flat \subseteq N^\flat$*, such that* $I^\flat$ *has the required properties to constitute a witnessing set of transformations in* $\mathcal{I}$*. Any set containing projections cannot satisfy reflexivity and the set consisting only of* $\sigma^\flat$ *does not satisfy the modus ponens property. We conclude that* $\mathcal{I}$ *is not syntactically prealgebraic.*

We provide, next, a characterization of syntactic prealgebraicity in terms of the global system modus ponens property of a subset of natural transformations intrinsically associated with the π-institution. Later, we use this characterization to provide an exact description of those prealgebraic π-institutions which are syntactically prealgebraic.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$. We define the **reflexive core** of $\mathcal{I}$ to be the collection
$$R^{\mathcal{I}} = \{\rho^\flat \in N^\flat : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\rho^\flat_\Sigma[\phi,\phi] \le \mathrm{Thm}(\mathcal{I}))\}.$$

Note that the defining condition is equivalent to asserting that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\rho^\flat_\Sigma(\phi, \phi, \vec{\chi}) \in \mathrm{Thm}_\Sigma(\mathcal{I}).$$

It is clear that $R^{\mathcal{I}}(T)$ is a reflexive relation system on $\mathbf{F}$, for every theory family T of $\mathcal{I}$.

Lemma 773 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,* $R^{\mathcal{I}}(T)$ *is a reflexive relation system on* $\mathbf{F}$*.*

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$. That $R^{\mathcal{I}}(T)$ is a relation system follows from Lemma 93. For reflexivity, it is required that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\langle \phi, \phi \rangle \in R^{\mathcal{I}}_\Sigma(T)$. But this is equivalent to $R^{\mathcal{I}}_\Sigma[\phi, \phi] \le T$, which certainly holds, since, by definition of $R^{\mathcal{I}}$, $R^{\mathcal{I}}_\Sigma[\phi, \phi] \le \mathrm{Thm}(\mathcal{I}) \le T$. ∎

Now, using Proposition 97, we draw a useful conclusion about the role of the reflexive core in determining the Leibniz congruence system associated with a given theory family.

Proposition 774 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\Omega(T) \le R^{\mathcal{I}}(T).$$

Proof: By Lemma 773 and Proposition 97. ∎

We next show that, for every theory family T of $\mathcal{I}$, $R^{\mathcal{I}}(T)$ is also a symmetric relation system on $\mathbf{F}$.

Lemma 775 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,* $R^{\mathcal{I}}(T)$ *is a symmetric relation system on* $\mathbf{F}$*.*

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$. That $R^{\mathcal{I}}(T)$ is a relation system follows from Lemma 93. To show that it is symmetric, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in R^{\mathcal{I}}_\Sigma(T)$. Equivalently, $R^{\mathcal{I}}_\Sigma[\phi, \psi] \le T$. Now consider any $\rho^\flat \in R^{\mathcal{I}}$. By the definition of $R^{\mathcal{I}}$, we get that $\overline{\rho^\flat} \in R^{\mathcal{I}}$. Therefore, by the hypothesis, $\overline{\rho^\flat}_\Sigma[\phi, \psi] \le T$. But this gives $\rho^\flat_\Sigma[\psi, \phi] \le T$. Since this holds for all $\rho^\flat \in R^{\mathcal{I}}$, we conclude that $R^{\mathcal{I}}_\Sigma[\psi, \phi] \le T$. Hence, $\langle \psi, \phi \rangle \in R^{\mathcal{I}}_\Sigma(T)$. Therefore, $R^{\mathcal{I}}(T)$ is a symmetric relation system on $\mathbf{F}$. ∎

We turn, next, to the congruence compatibility property. More precisely, we show that, for all theory families T of $\mathcal{I}$, $R^{\mathcal{I}}(T)$ has the compatibility property in $\mathbf{F}$.

Lemma 776 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,* $R^{\mathcal{I}}(T)$ *has the compatibility property in* $\mathbf{F}$*.*

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$. Note that, because of Corollary 12, it suffices to show that, for all $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$, and all $\phi, \psi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in R^{\mathcal{I}}_\Sigma(T) \quad \text{implies} \quad \langle \sigma^\flat_\Sigma(\phi, \vec{\chi}), \sigma^\flat_\Sigma(\psi, \vec{\chi}) \rangle \in R^{\mathcal{I}}_\Sigma(T).$$

Suppose, $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ is in $N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in R^{\mathcal{I}}_\Sigma(T)$ or, equivalently, $R^{\mathcal{I}}_\Sigma[\phi, \psi] \le T$. Let $\rho^\flat : (\mathrm{SEN}^\flat)^n \to \mathrm{SEN}^\flat$ be arbitrary in $R^{\mathcal{I}}$. We consider the natural transformation $\rho'^\flat : (\mathrm{SEN}^\flat)^{n+k} \to \mathrm{SEN}^\flat$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\zeta, \eta, \vec{\chi}, \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$, by

$$\rho'^\flat_\Sigma(\zeta, \eta, \vec{\chi}, \vec{\xi}) = \rho^\flat_\Sigma(\sigma^\flat_\Sigma(\zeta, \vec{\chi}), \sigma^\flat_\Sigma(\eta, \vec{\chi}), \vec{\xi}).$$

Now note that, since $\sigma^\flat \in N^\flat$, $\rho^\flat \in N^\flat$ and

$$\begin{aligned}\rho'^\flat \quad = \quad & \rho^\flat \circ \langle \sigma^\flat \circ \langle p^{n+k,0}, p^{n+k,2}, \dots, p^{n+k,k} \rangle, \sigma^\flat \circ \langle p^{n+k,1}, p^{n+k,2}, \dots, p^{n+k,k} \rangle, \\ & p^{n+k,k+1}, \dots, p^{n+k,n+k-1} \rangle,\end{aligned}$$

we get, by the definition of a category of natural transformations, that $\rho'^\flat \in N^\flat$.

Next, note that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\zeta, \vec{\chi}, \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{aligned}\rho'^\flat_\Sigma(\zeta, \zeta, \vec{\chi}, \vec{\xi}) \quad &= \quad \rho^\flat_\Sigma(\sigma^\flat_\Sigma(\zeta, \vec{\chi}), \sigma^\flat_\Sigma(\zeta, \vec{\chi}), \vec{\xi}) \quad \text{(by definition of } \rho'^\flat) \\ &\in \quad \mathrm{Thm}_\Sigma(\mathcal{I}). \quad \text{(since } \rho^\flat \in R^{\mathcal{I}}).\end{aligned}$$

Thus, by the definition of the reflexive core, we get that $\rho'^\flat \in R^{\mathcal{I}}$.

Now since $\rho'^\flat \in R^{\mathcal{I}}$ and, by hypothesis, $R^{\mathcal{I}}_\Sigma[\phi, \psi] \le T$, we get, in particular, that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, and $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi}, \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}), \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}), \vec{\xi}) \in T_{\Sigma'}.$$

Hence, a fortiori, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\sigma^\flat_\Sigma(\phi, \vec{\chi})), \mathrm{SEN}^\flat(f)(\sigma^\flat_\Sigma(\psi, \vec{\chi})), \vec{\xi}) \in T_{\Sigma'}.$$

This proves that

$$\rho^\flat_\Sigma[\sigma^\flat_\Sigma(\phi, \vec{\chi}), \sigma^\flat_\Sigma(\psi, \vec{\chi})] \le T.$$

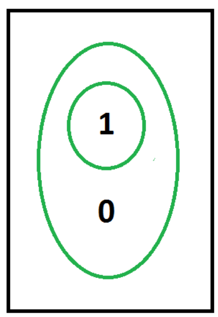
Since this holds for all $\rho^\flat \in R^{\mathcal{I}}$, we get that $R^{\mathcal{I}}_\Sigma[\sigma^\flat_\Sigma(\phi, \vec{\chi}), \sigma^\flat_\Sigma(\psi, \vec{\chi})] \le T$ or, equivalently, $\langle \sigma^\flat_\Sigma(\phi, \vec{\chi}), \sigma^\flat_\Sigma(\psi, \vec{\chi}) \rangle \in R^{\mathcal{I}}_\Sigma(T)$. Therefore, $R^{\mathcal{I}}(T)$ has the congruence compatibility property in $\mathbf{F}$. ■

It is possible, but not necessary, that the reflexive core of a π-institution has the global system modus ponens. To see this, we present two examples. In the first example, we look at a π-institution $\mathcal{I}$ whose reflexive core $R^{\mathcal{I}}$ does have the global system modus ponens in $\mathcal{I}$.

Example 777 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with a single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting:* $\sigma^\flat_\Sigma : \{0,1\}^2 \to \{0,1\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 0, & \text{if } (x,y) = (1,0) \\ 1, & \text{otherwise} \end{cases}.$$



SEN(Σ)

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\}$*. The only theory families,* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, are also theory systems.*

Note that $\sigma^\flat \in R^{\mathcal{I}}$*, since, for all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,* $\sigma^\flat_\Sigma(\phi,\phi) = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*. On the other hand, no projection natural transformation can be in the reflexive core.*

To see that $R^{\mathcal{I}}$ *satisfies the global system modus ponens in* $\mathcal{I}$*, note that it does so trivially for the theory system* $\mathrm{SEN}^\flat$*, whereas for* $\mathrm{Thm}(\mathcal{I})$*, it is possible that* $\sigma^\flat_\Sigma(\phi,\psi) = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$ *and* $\phi = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$ *only if* $\psi = 1$*. Thus,* $R^{\mathcal{I}}$ *has the global system MP in* $\mathcal{I}$*, as claimed.*

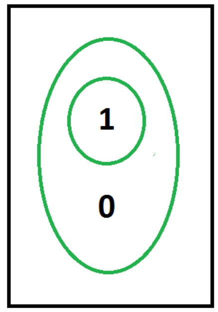
Next, we present an example of a π-institution $\mathcal{I}$ whose reflexive core $R^{\mathcal{I}}$ does not have the global system modus ponens in $\mathcal{I}$.

Example 778 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with a single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*;*

- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting:* $\sigma^\flat_\Sigma : \{0,1\}^2 \to \{0,1\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y) = 1, \quad \textit{for all } x,y \in \{0,1\}.$$



SEN(Σ)

Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\}$*. So its two theory families,* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, are also theory systems.*

Note that $\sigma^\flat \in R^{\mathcal{I}}$*, since, for all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,* $\sigma^\flat_\Sigma(\phi,\phi) = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*. On the other hand, no projection natural transformation can be in the reflexive core.*

To see that $R^{\mathcal{I}}$ *does not satisfy the global system modus ponens in* $\mathcal{I}$*, note that* $1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$ *and that* $\sigma^\flat_\Sigma(1,0) = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, but* $0 \notin \mathrm{Thm}_\Sigma(\mathcal{I})$*. Thus,* $R^{\mathcal{I}}$ *does not have the global system MP in* $\mathcal{I}$*.*

It turns out that possession of the global system modus ponens by the reflexive core intrinsically characterizes syntactic prealgebraicity. We can show, at the outset, that the reflexive core having the global system modus ponens is necessary for syntactic prealgebraicity.

Theorem 779 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically prealgebraic, then* $R^{\mathcal{I}}$ *has the global system modus ponens.*

Proof: Suppose that $\mathcal{I}$ is syntactically prealgebraic with witnessing transformations $I^\flat$. Thus, $I^\flat$ has reflexivity, global system transitivity, global system compatibility and the global system modus ponens in $\mathcal{I}$. Since $I^\flat$ is reflexive in $\mathcal{I}$, we get, by the definition of the reflexive core, that $I^\flat \subseteq R^{\mathcal{I}}$. But, then, since, by hypothesis, $I^\flat$ has the global system modus ponens, it follows that, a fortiori, $R^{\mathcal{I}}$ has the global system modus ponens in $\mathcal{I}$. ■

In proving the reverse implication, we now show that having the global system modus ponens implies the global system transitivity property of the reflexive core.

Proposition 780 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}}$ *has the global system modus ponens, then it also has the global system transitivity in* $\mathcal{I}$*.*

Proof: Suppose that $R^{\mathcal{I}}$ has the global system modus ponens in $\mathcal{I}$ and let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle, \langle \psi, \chi \rangle \in R^{\mathcal{I}}_\Sigma(T)$. This means that $R^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T$ and $R^{\mathcal{I}}_\Sigma[\psi, \chi] \leq T$. Then, by Lemma 776, we get that, for all $\rho^\flat \in R^{\mathcal{I}}$, and all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$R^{\mathcal{I}}_{\Sigma'}[\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), \vec{\xi}), \rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi})] \leq T.$$

Since $R^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T$, we get by the global system MP of $R^{\mathcal{I}}$ that, for all $\rho^\flat \in R^{\mathcal{I}}$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi}) \subseteq T_{\Sigma'}.$$

Thus, $R^{\mathcal{I}}_\Sigma[\phi, \chi] \leq T$, whence $\langle \phi, \chi \rangle \in R^{\mathcal{I}}_\Sigma(T)$. Therefore $R^{\mathcal{I}}$ has the global system transitivity in $\mathcal{I}$. ■

Proposition 780 closes a line of work that was started with the definition of a reflexive core and with Lemma 773.

Theorem 781 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}}$ *has the global system modus ponens, then* $\mathcal{I}$ *is syntactically prealgebraic, with witnessing transformations* $R^{\mathcal{I}}$*.*

Proof: By Lemma 773, $R^{\mathcal{I}}$ is reflexive in $\mathcal{I}$. By Lemma 775, $\mathcal{I}$ is globally family symmetric in $\mathcal{I}$. By hypothesis and Proposition 780, it is globally system transitive in $\mathcal{I}$. By Lemma 776 it has the global family compatibility property in $\mathcal{I}$. Finally, by hypothesis, it has the global system modus ponens in $\mathcal{I}$. We conclude that $\mathcal{I}$ is syntactically prealgebraic with witnessing transformations $R^{\mathcal{I}}$. ■

Theorems 779 and 781 provide the promised characterization of syntactic prealgebraicity in terms of the global system modus ponens of the reflexive core.

$$\mathcal{I} \text{ is Syntactically Prealgebraic} \longleftrightarrow R^{\mathcal{I}} \text{ has Global System MP.}$$

Theorem 782 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically prealgebraic if and only if* $R^{\mathcal{I}}$ *has the global system modus ponens in* $\mathcal{I}$*.*

Proof: Theorem 779 gives the "only if" and the "if" is by Theorem 781. ■

If $\mathcal{I}$ is syntactically prealgebraic, then $R^{\mathcal{I}}$ defines Leibniz congruence systems of theory systems in $\mathcal{I}$. This proposition may be viewed as a special case of Corollary 770, since $R^{\mathcal{I}}$ forms a set of witnessing transformations that, in addition, has the global family symmetry in $\mathcal{I}$.

Proposition 783 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}}$ *has the global system modus ponens, then, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,*

$$\Omega(T) = R^{\mathcal{I}}(T).$$

Proof: Let $T \in \mathrm{ThSys}(\mathcal{I})$. If $R^{\mathcal{I}}$ has the global system modus ponens, then, by Lemma 773, Lemma 775, Lemma 776, the hypothesis and Proposition 780, $R^{\mathcal{I}}(T)$ is a congruence system that is compatible with T. Therefore, by Corollary 98, we get that $\Omega(T) = R^{\mathcal{I}}(T)$. ■

We also get another related characterization of syntactic prealgebraicity.

$$\begin{array}{l}\mathcal{I} \text{ is Syntactically Prealgebraic} \\ \qquad\qquad \longleftrightarrow R^{\mathcal{I}} \text{ Defines Leibniz Congruence Systems} \\ \qquad\qquad\qquad\qquad \text{of Theory Systems in } \mathcal{I}.\end{array}$$

Theorem 784 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically prealgebraic if and only if, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,*

$$\Omega(T) = R^{\mathcal{I}}(T).$$

Proof: If $\mathcal{I}$ is syntactically prealgebraic, then, by Theorem 782, $R^{\mathcal{I}}$ has the global system modus ponens in $\mathcal{I}$. Thus, by Proposition 783, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\Omega(T) = R^{\mathcal{I}}(T)$.

Conversely, if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $R^{\mathcal{I}}(T) = \Omega(T)$, then, $R^{\mathcal{I}}$ is reflexive, globally system transitive, has the global family compatibility and the global system modus ponens. Thus, $R^{\mathcal{I}}$ is a set of witnessing transformations and $\mathcal{I}$ is syntactically prealgebraic. ■

We finally show that the property that separates prealgebraicity from syntactic prealgebraicity is exactly the Leibniz compatibility property with respect to the theory system generated by the reflexive core.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that $R^{\mathcal{I}}$ is **Leibniz** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi])).$$

We show that, if $R^{\mathcal{I}}$ has the global system modus ponens, then it is Leibniz.

Proposition 785 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}}$ *has the global system modus ponens, then it is Leibniz.*

Proof: Suppose $R^{\mathcal{I}}$ has the global system modus ponens. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. To show that $\langle \phi, \psi \rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi]))$, we use the criterion for membership given in Theorem 19. To this end, let $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ be in $N^\flat$, $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$, such that

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\xi}) \in C_{\Sigma'}(R^{\mathcal{I}}_\Sigma[\phi, \psi]).$$

By Lemma 776,

$$R^{\mathcal{I}}_{\Sigma'}[\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\xi}), \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\xi})] \leq C(R^{\mathcal{I}}_\Sigma[\phi, \psi]).$$

Since, by hypothesis, $R^{\mathcal{I}}$ has the global system modus ponens, we obtain that $\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\xi}) \in C_{\Sigma'}(R^{\mathcal{I}}_\Sigma[\phi, \psi])$. By symmetry, we now have that

$$\begin{array}{l}\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\xi}) \in C_{\Sigma'}(R^{\mathcal{I}}_\Sigma[\phi, \psi]) \\ \qquad\qquad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\xi}) \in C_{\Sigma'}(R^{\mathcal{I}}_\Sigma[\phi, \psi]).\end{array}$$

Therefore, by Theorem 19, we conclude that $\langle \phi, \psi \rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi]))$, and, hence, $R^{\mathcal{I}}$ is Leibniz. ■

Here is an example of a π-institution $\mathcal{I}$, with a Leibniz reflexive core not having the global system modus ponens.

Example 786 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

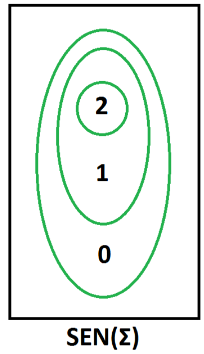
- $\mathbf{Sign}^\flat$ *is the trivial category with single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0, 1, 2\}^2 \to \{0, 1, 2\}$ *be given, for all* $x, y \in \mathrm{SEN}^\flat(\Sigma)$*, by*

$$\sigma^\flat_\Sigma(x, y) = \begin{cases} 2, & \text{if } x = y \text{ or } \{x, y\} = \{0, 1\} \\ 0, & \text{otherwise} \end{cases}.$$

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

$\mathcal{I}$ *has three theory families* $\mathrm{Thm}(\mathcal{I})$*,* $T = \{\{1, 2\}\}$ *and* $\mathrm{SEN}^\flat$*, all of which are theory systems.*



Note that $R^{\mathcal{I}} = \{\sigma^\flat\}$. *We show that* $R^{\mathcal{I}}$ *is Leibniz, but does not have the global system modus ponens.*

To verify the Leibniz property, note that, if $\phi = \psi$ *the conclusion is trivial. If* $\phi \neq \psi$*, then, if* $\{\phi,\psi\} \neq \{0,1\}$*, then* $R^{\mathcal{I}}_\Sigma[\phi,\psi] = \{\{0\}\}$*, whence* $C(R^{\mathcal{I}}_\Sigma[\phi,\psi]) = \mathrm{SEN}^\flat$ *and, therefore,*

$$\Omega(C(R^{\mathcal{I}}_\Sigma[\phi,\psi])) = \nabla^{\mathbf{F}}$$

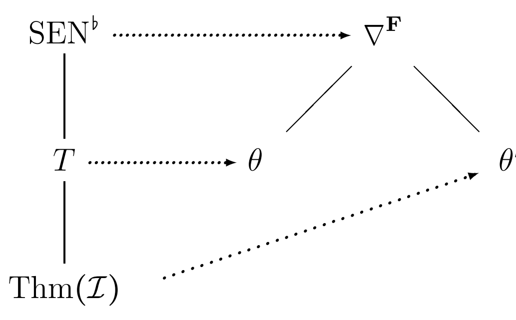
and the conclusion follows. Otherwise, if $\{\phi,\psi\} = \{0,1\}$*, then* $C(R^{\mathcal{I}}_\Sigma[\phi,\psi]) = \mathrm{Thm}(\mathcal{I})$*, whence*

$$\Omega(C(R^{\mathcal{I}}_\Sigma[\phi,\psi])) = \{\{0,1\},\{2\}\}$$

and the conclusion follows. Therefore, $R^{\mathcal{I}}$ *is Leibniz.*

On the other hand, we have $1 \in \{1,2\}$ *and* $R^{\mathcal{I}}_\Sigma[1,0] \leq \{\{1,2\}\}$*, whereas* $0 \notin \{1,2\}$*. Therefore,* $R^{\mathcal{I}}$ *fails to have the global system modus ponens in* $\mathcal{I}$*.*

We note, with a nod to what is to follow, that $\mathcal{I}$ *is not prealgebraic, since, as is clear by the poset diagrams of theory systems and associated Leibniz congruence systems, the Leibniz operator is not monotonic on theory systems (here* $\theta = \{\{0\},\{1,2\}\}$ *and* $\theta' = \{\{0,1\},\{2\}\}$*).*



In the opposite direction, and on the positive side, in a *prealgebraic* π-institution $\mathcal{I}$, if the reflexive core is Leibniz, then it does have the global system modus ponens in $\mathcal{I}$.

Proposition 787 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a prealgebraic* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}}$ *is Leibniz, then it has the global system modus ponens in* $\mathcal{I}$*.*

Proof: Suppose that $\mathcal{I}$ is prealgebraic and that $R^{\mathcal{I}}$ is Leibniz. Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $R^{\mathcal{I}}_\Sigma[\phi,\psi] \leq T$. Now we have

$$\begin{array}{lll}\langle\phi,\psi\rangle & \in & \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi,\psi])) \quad \text{(since } R^{\mathcal{I}} \text{ is Leibniz)}\\ & \subseteq & \Omega_\Sigma(T). \quad \text{(since } R^{\mathcal{I}}_\Sigma[\phi,\psi] \leq T \text{ and } \mathcal{I} \text{ is prealgebraic)}\end{array}$$

Therefore, since $\phi \in T_\Sigma$, we get, by the compatibility of $\Omega(T)$ with T, that $\psi \in T_\Sigma$. We conclude that $R^{\mathcal{I}}$ has the global system modus ponens in $\mathcal{I}$. ■

We now show that a π-institution is syntactically prealgebraic if and only if it is prealgebraic and it has a Leibniz reflexive core.

$$\begin{array}{rcl}\text{Syntactic Prealgebraicity} & = & R^{\mathcal{I}} \text{ has Global System MP}\\ & = & R^{\mathcal{I}} \text{ Defines Leibniz Congruence Systems}\\ & & \quad \text{of Theorem Systems in } \mathcal{I}\\ & = & \text{Prealgebraicity} + R^{\mathcal{I}} \text{ is Leibniz}\end{array}$$

Theorem 788 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is syntactically prealgebraic if and only if it is prealgebraic and has a Leibniz reflexive core.*

Proof: Suppose, first, that $\mathcal{I}$ is syntactically prealgebraic. Then it is prealgebraic by Theorem 771. Moreover, its reflexive core has the global family modus ponens by Theorem 782 and, hence, by Proposition 785, its reflexive core is Leibniz.

Suppose, conversely, that $\mathcal{I}$ is prealgebraic with a Leibniz reflexive core. Then, by Proposition 787, its reflexive core has the global system modus ponens and, therefore, by Theorem 782, $\mathcal{I}$ is syntactically prealgebraic. ■

It is not difficult to see that syntactic prealgebraicity transfers from a π-institution $\mathcal{I}$ to all its generalized matrix families.

Theorem 789 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is syntactically prealgebraic, with witnessing transformations $I^\flat$, if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle\mathbf{A}, \langle F, \alpha\rangle\rangle$, the generalized matrix family $\langle\mathcal{A}, C^{\mathcal{I},\mathcal{A}}\rangle$ is syntactically prealgebraic, with witnessing transformations $I^{\mathcal{A}}$.*

Proof: The "if" follows by considering the algebraic system $\mathcal{F} = \langle\mathbf{F}, \langle I, \iota\rangle\rangle$. For the "only if", assume that $\mathcal{I}$ is syntactically prealgebraic, with witnessing transformations $I^\flat$, and let $\mathcal{A} = \langle\mathbf{A}, \langle F, \alpha\rangle\rangle$, with $\mathbf{A} = \langle\mathbf{Sign}, \mathrm{SEN}, N\rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. We have

$$\begin{array}{rll}\langle\alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in T_{F(\Sigma)} & \text{iff} & \langle\phi,\psi\rangle \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})\\ & \text{iff} & I^\flat_\Sigma[\phi,\psi] \leq \alpha^{-1}(T)\\ & \text{iff} & I^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi), \alpha_\Sigma(\psi)] \leq T.\end{array}$$

Taking into account the surjectivity of $\langle F, \alpha\rangle$, we conclude that $\langle\mathcal{A}, C^{\mathcal{I},\mathcal{A}}\rangle$ is syntactically prealgebraic, with witnessing transformations $I^{\mathcal{A}}$. ■

11.2 Syntactic Protoalgebraicity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$.

Recall that $\mathcal{I}$ is **protoalgebraic** if, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$T \le T' \quad \text{implies} \quad \Omega(T) \le \Omega(T').$$

We say that $\mathcal{I}$ is **syntactically protoalgebraic** if there exists $I^\flat \subseteq N^\flat$, with two distinguished arguments, such that $I^\flat$ has:

- reflexivity;
- global family transitivity;
- global family compatibility; and
- global family modus ponens.

In that case, we call $I^\flat$ a **set of witnessing natural transformations**, or, more simply, **witnessing transformations** (of the syntactic protoalgebraicity of $\mathcal{I}$).

It turns out that, if $\mathcal{I}$ is a syntactically protoalgebraic π-institution, with witnessing transformations $I^\flat$, then $\overleftrightarrow{I^\flat}(T)$ is a congruence system on $\mathbf{F}$ compatible with T, for all $T \in \mathrm{ThFam}(\mathcal{I})$. As a consequence, using Corollary 98, we may conclude that, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\overleftrightarrow{I^\flat}(T) = \Omega(T).$$

Proposition 790 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is syntactically protoalgebraic, with witnessing transformations $I^\flat$, then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftrightarrow{I^\flat}(T)$ is a congruence system on $\mathbf{F}$ compatible with T.*

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$.

Since $I^\flat$ is reflexive in $\mathcal{I}$, we get that $I^\flat_\Sigma[\phi, \phi] \le \mathrm{Thm}(\mathcal{I}) \le T$. Therefore, $\overleftrightarrow{I^\flat}_\Sigma[\phi, \phi] \le T$, which shows that $\langle \phi, \phi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$.

Suppose, next, that $\langle \phi, \psi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$. Thus, $\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi] \le T$. By the definition of $\overleftrightarrow{I^\flat}$, we then get $\overleftrightarrow{I^\flat}_\Sigma[\psi, \phi] \le T$ and, hence, $\langle \psi, \phi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$.

Next, assume that $\langle \phi, \psi \rangle, \langle \psi, \chi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$. Then we get $\langle \phi, \psi \rangle$, $\langle \psi, \chi \rangle$, $\langle \psi, \phi \rangle$, $\langle \chi, \psi \rangle \in I^\flat_\Sigma(T)$. Since $I^\flat$ is transitive in $\mathcal{I}$, we conclude that $\langle \phi, \chi \rangle$, $\langle \chi, \phi \rangle \in I^\flat_\Sigma(T)$ and, therefore, $\langle \phi, \chi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$.

To show the congruence property, assume that $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ is a natural transformation in $N^\flat$ and that $\langle \phi_i, \psi_i \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$, for all $i < k$. Thus, since $I^\flat$ has the compatibility property in $\mathcal{I}$, we get that $\langle\sigma^\flat_\Sigma(\vec{\phi}),\sigma^\flat_\Sigma(\vec{\psi})\rangle\in I^\flat_\Sigma(T)$. By symmetry, we also get $\langle\sigma^\flat_\Sigma(\vec{\psi}),\sigma^\flat_\Sigma(\vec{\phi})\rangle\in I^\flat_\Sigma(T)$ and, hence, that $\langle\sigma^\flat_\Sigma(\vec{\phi}),\sigma^\flat_\Sigma(\vec{\psi})\rangle\in\overleftrightarrow{I^\flat}_\Sigma(T)$.

Finally, since by Lemma 93, $\overleftrightarrow{I^\flat}(T)$ is a relation system on $\mathbf{F}$, we conclude that $\overleftrightarrow{I^\flat}(T)$ is a congruence system on $\mathbf{F}$.

To conclude the proof, note that, if $\phi\in T_\Sigma$ and $\langle\phi,\psi\rangle\in\overleftrightarrow{I^\flat}_\Sigma(T)$, then $\psi\in T_\Sigma$ by the global family modus ponens of $I^\flat$ in $\mathcal{I}$ and the fact that $I^\flat\subseteq\overleftrightarrow{I^\flat}$. ■

Based on Proposition 790, we can conclude that $\overleftrightarrow{I^\flat}$ defines the Leibniz congruence systems of the theory families of $\mathcal{I}$.

Corollary 791 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically protoalgebraic, with witnessing transformations* $I^\flat$*, if and only if, for all* $T\in\mathrm{ThFam}(\mathcal{I})$*,*

$$\overleftrightarrow{I^\flat}(T)=\Omega(T).$$

Proof: The "only if" is by Proposition 790 and Corollary 98. The "if" is again obvious, as in Corollary 770. ■

Corollary 791 has as an immediate consequence the important fact that syntactic protoalgebraicity implies (semantic) protoalgebraicity.

Theorem 792 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically protoalgebraic, then it is protoalgebraic.*

Proof: Suppose that $\mathcal{I}$ is syntactically protoalgebraic with witnessing transformations $I^\flat$. Let $T,T'\in\mathrm{ThFam}(\mathcal{I})$, such that $T\le T'$. Then

$$\begin{array}{rcll}\Omega(T)&=&\overleftrightarrow{I^\flat}(T)&\text{(by Corollary 791)}\\&\le&\overleftrightarrow{I^\flat}(T')&\text{(by Lemma 94)}\\&=&\Omega(T').&\text{(by Corollary 791)}\end{array}$$

Thus, $\mathcal{I}$ is protoalgebraic. ■

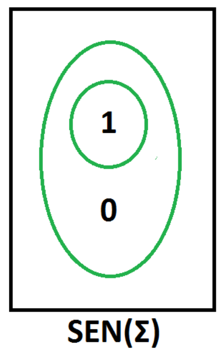
The following example shows that the inclusion of Theorem 792 is proper.

Example 793 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with a single object* Σ*;*
- $\mathrm{SEN}^\flat:\mathbf{Sign}^\flat\to\mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma)=\{0,1\}$*;*

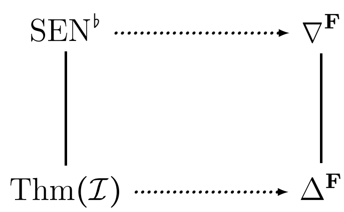
- *$N^\flat$ is the category of natural transformations generated by the single binary natural transformation $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ defined by letting: $\sigma^\flat_\Sigma : \{0,1\}^2 \to \{0,1\}$ be given by*

$$\sigma^\flat_\Sigma(x,y) = 1, \quad \textit{for all } x,y \in \{0,1\}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be the π-institution determined by $\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\}$.

$\mathcal{I}$ has two theory families, $\mathrm{Thm}(\mathcal{I})$ and $\mathrm{SEN}^\flat$, such that $\mathrm{Thm}(\mathcal{I}) \le \mathrm{SEN}^\flat$. Moreover, $\Omega(\mathrm{Thm}(\mathcal{I})) = \Delta^{\mathbf{F}}$ and $\Omega(\mathrm{SEN}^\flat) = \nabla^{\mathbf{F}}$. Since $\Omega(\mathrm{Thm}(\mathcal{I})) \le \Omega(\mathrm{SEN}^\flat)$, $\mathcal{I}$ is protoalgebraic.



On the other hand, there does not exist $I^\flat \subseteq N^\flat$, such that $I^\flat$ has the required properties to constitute a witnessing set of transformations in $\mathcal{I}$. Any set containing projections cannot satisfy reflexivity and the set consisting only of $\sigma^\flat$ does nor satisfy the modus ponens property. We conclude that $\mathcal{I}$ is not syntactically protoalgebraic.

We now work towards a dual goal. We first provide a characterization of syntactic protoalgebraicity in terms of the global family modus ponens property of the reflexive core of the π-institution. Then, we use this characterization to provide an exact description of those protoalgebraic π-institutions which are syntactically protoalgebraic.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Recall that the **reflexive core** of $\mathcal{I}$ is the collection

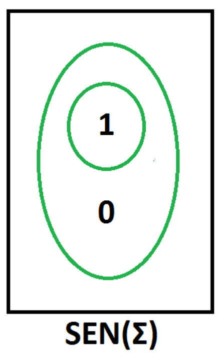
$$R^{\mathcal{I}} = \{\rho^\flat \in N^\flat : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\rho^\flat_\Sigma[\phi,\phi] \le \mathrm{Thm}(\mathcal{I}))\}.$$

It is possible, but not necessary, that the reflexive core of a π-institution has the global family modus ponens. To see this, we present two examples. In the first example, we look at a π-institution $\mathcal{I}$ whose reflexive core $R^{\mathcal{I}}$ does have the global family modus ponens in $\mathcal{I}$.

Example 794 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with a single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting:* $\sigma^\flat_\Sigma : \{0,1\}^2 \to \{0,1\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 0, & \text{if } (x,y) = (1,0) \\ 1, & \text{otherwise} \end{cases}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\}$.

Note that $\sigma^\flat \in R^{\mathcal{I}}$*, since, for all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,* $\sigma^\flat_\Sigma(\phi,\phi) = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$. *On the other hand, no projection natural transformation can be in the reflexive core.*

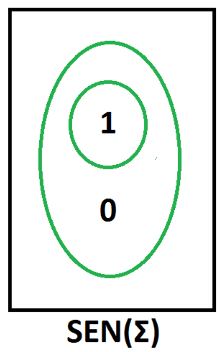
To see that $R^{\mathcal{I}}$ *satisfies the modus ponens in* $\mathcal{I}$*, note that it does so trivially for the theory family* $\mathrm{SEN}^\flat$*, whereas for* $\mathrm{Thm}(\mathcal{I})$*, it is possible that* $\sigma^\flat_\Sigma(\phi,\psi) = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$ *and* $\phi = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$ *only if* $\psi = 1$. *Thus,* $R^{\mathcal{I}}$ *has the global family MP in* $\mathcal{I}$*, as claimed.*

Next, we present an example of a π-institution $\mathcal{I}$ whose reflexive core $R^{\mathcal{I}}$ does not have the global family modus ponens in $\mathcal{I}$.

Example 795 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with a single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting:* $\sigma^\flat_\Sigma : \{0,1\}^2 \to \{0,1\}$ *be given by*

$$\sigma^\flat_\Sigma(x,y) = 1, \quad \textit{for all } x,y \in \{0,1\}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\}$.

Note that $\sigma^\flat \in R^{\mathcal{I}}$*, since, for all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,* $\sigma^\flat_\Sigma(\phi,\phi) = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*. On the other hand, no projection natural transformation can be in the reflexive core.*

To see that $R^{\mathcal{I}}$ *does not satisfy the modus ponens in* $\mathcal{I}$*, note that* $1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$ *and that* $\sigma^\flat_\Sigma(1,0) = 1 \in \mathrm{Thm}_\Sigma(\mathcal{I})$*, but* $0 \notin \mathrm{Thm}_\Sigma(\mathcal{I})$*. Thus,* $R^{\mathcal{I}}$ *does not have the global family MP in* $\mathcal{I}$*.*

It turns out that possession of the global family modus ponens by the reflexive core intrinsically characterizes syntactic protoalgebraicity. We can show, at the outset, that the reflexive core having the global family modus ponens is necessary for syntactic protoalgebraicity. Thus, there is no point in exploring syntactic protoalgebraicity unless the π-institution $\mathcal{I}$ under scrutiny is such that $R^{\mathcal{I}}$ has the global family MP.

Theorem 796 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically protoalgebraic, then* $R^{\mathcal{I}}$ *has the global family modus ponens.*

Proof: Suppose that $\mathcal{I}$ is syntactically protoalgebraic with witnessing transformations $I^\flat$. Thus, $I^\flat$ has reflexivity, global family transitivity, global family compatibility and the global family modus ponens in $\mathcal{I}$. Since $I^\flat$ is reflexive in $\mathcal{I}$, we get, by the definition of the reflexive core, that $I^\flat \subseteq R^{\mathcal{I}}$. But, then, since, by hypothesis, $I^\flat$ has the global family modus ponens, it follows that, a fortiori, $R^{\mathcal{I}}$ has the global family modus ponens in $\mathcal{I}$. ∎

To prove the reverse implication, we show, first, that having the global family modus ponens implies the global family transitivity property of the reflexive core.

Proposition 797 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $R^{\mathcal{I}}$ has the global family modus ponens, then it also has the global family transitivity in $\mathcal{I}$.*

Proof: Suppose that $R^{\mathcal{I}}$ has the global family modus ponens in $\mathcal{I}$ and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle, \langle \psi, \chi \rangle \in R^{\mathcal{I}}_\Sigma(T)$. This means that $R^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T$ and $R^{\mathcal{I}}_\Sigma[\psi, \chi] \leq T$. Then, by Lemma 776, we get that, for all $\rho^\flat \in R^{\mathcal{I}}$, and all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$R^{\mathcal{I}}_{\Sigma'}[\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), \vec{\xi}), \rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi})] \leq T.$$

Since $R^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T$, we get by the global family MP of $R^{\mathcal{I}}$ that, for all $\rho^\flat \in R^{\mathcal{I}}$, and all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi}) \subseteq T_{\Sigma'}.$$

Thus, $R^{\mathcal{I}}_\Sigma[\phi, \chi] \leq T$, whence $\langle \phi, \chi \rangle \in R^{\mathcal{I}}_\Sigma(T)$. Therefore $R^{\mathcal{I}}$ is globally family transitive in $\mathcal{I}$. ∎

Proposition 797 closes a line of work that was started with the definition of a reflexive core and goes back to Lemma 773.

Theorem 798 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $R^{\mathcal{I}}$ has the global family modus ponens, then $\mathcal{I}$ is syntactically protoalgebraic, with witnessing transformations $R^{\mathcal{I}}$.*

Proof: By Lemma 773, $R^{\mathcal{I}}$ is reflexive in $\mathcal{I}$. By Lemma 775, $\mathcal{I}$ is globally family symmetric in $\mathcal{I}$. By hypothesis and Proposition 797, it is globally family transitive in $\mathcal{I}$. By Lemma 776 it has the global family compatibility property in $\mathcal{I}$. Finally, by hypothesis, it has the global family modus ponens in $\mathcal{I}$. We conclude that $\mathcal{I}$ is syntactically protoalgebraic with witnessing transformations $R^{\mathcal{I}}$. ∎

Theorems 796 and 798 provide the promised characterization of syntactic protoalgebraicity in terms of the global family modus ponens of the reflexive core.

$\mathcal{I}$ is Syntactically Protoalgebraic $\longleftrightarrow$ $R^{\mathcal{I}}$ has Global Family MP.

Theorem 799 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically protoalgebraic if and only if* $R^{\mathcal{I}}$ *has the global family modus ponens in* $\mathcal{I}$*.*

Proof: Theorem 796 gives the "only if" and the "if" is by Theorem 798. ■

If $\mathcal{I}$ is syntactically protoalgebraic, then $R^{\mathcal{I}}$ defines Leibniz congruence systems in $\mathcal{I}$. This proposition may be viewed as a special case of Corollary 791, since $R^{\mathcal{I}}$ forms a set of witnessing transformations that, in addition, has the global family symmetry in $\mathcal{I}$.

Proposition 800 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}}$ *has the global family modus ponens, then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\Omega(T) = R^{\mathcal{I}}(T).$$

Proof: If $R^{\mathcal{I}}$ has the global family modus ponens, then, by Lemma 773, Lemma 775, Lemma 776, the hypothesis and Proposition 797, $R^{\mathcal{I}}(T)$ is a congruence system that is compatible with T. Therefore, by Corollary 98, we get that $\Omega(T) = R^{\mathcal{I}}(T)$. ■

We also get (almost) for free another related characterization of syntactic protoalgebraicity.

$$\begin{array}{l}\mathcal{I} \text{ is Syntactically Protoalgebraic} \\ \qquad\qquad \longleftrightarrow R^{\mathcal{I}} \text{ Defines Leibniz Congruence Systems.}\end{array}$$

Theorem 801 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically protoalgebraic if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\Omega(T) = R^{\mathcal{I}}(T).$$

Proof: If $\mathcal{I}$ is syntactically protoalgebraic, then, by Theorem 799, $R^{\mathcal{I}}$ has the family modus ponens in $\mathcal{I}$. Thus, by Proposition 800, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) = R^{\mathcal{I}}(T)$.

Conversely, if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $R^{\mathcal{I}}(T) = \Omega(T)$, then, $R^{\mathcal{I}}$ is reflexive, globally family transitive, has the global family compatibility and the global family modus ponens. Thus, $R^{\mathcal{I}}$ is a set of witnessing transformations and $\mathcal{I}$ is syntactically protoalgebraic. ■

We finally show that the property that separates protoalgebraicity from syntactic protoalgebraicity is exactly the Leibniz compatibility property with respect to the theory family generated by the reflexive core.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall that $R^{\mathcal{I}}$ is **Leibniz** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi])).$$

We have shown in Proposition 785 that, if $R^{\mathcal{I}}$ has the global system modus ponens, then it is Leibniz.

Corollary 802 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $R^{\mathcal{I}}$ has the global family modus ponens, then it is Leibniz.*

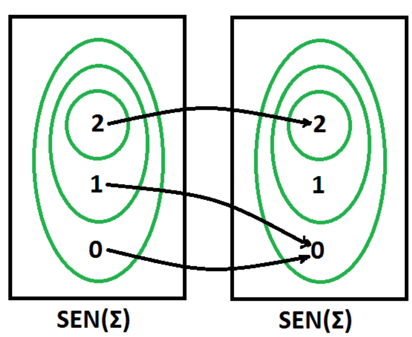
Proof: Directly from Proposition 785 ∎

Here is an example of a π-institution $\mathcal{I}$, with a Leibniz reflexive core not having the global family modus ponens.

Example 803 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object Σ and a single (non-identity) morphism $f : \Sigma \to \Sigma$, such that $f \circ f = f$;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 0$, $1 \mapsto 0$ *and* $2 \mapsto 2$;
- $N^\flat$ *is the category of natural transformations generated by the single binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\}^2 \to \{0,1,2\}$ *be given, for all* $a, b \in \mathrm{SEN}^\flat(\Sigma)$, *by*

$$\sigma^\flat_\Sigma(x,y) = \begin{cases} 2, & \text{if } x = y \text{ or } \{x,y\} = \{0,1\} \\ 0, & \text{otherwise} \end{cases}.$$



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be the π-institution determined by

$$\mathcal{C}_\Sigma = \{\{2\}, \{1,2\}, \{0,1,2\}\}.$$

$\mathcal{I}$ has three theory families $\mathrm{Thm}(\mathcal{I})$, $T = \{\{1,2\}\}$ *and* $\mathrm{SEN}^\flat$, *but only two theory systems* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$.

Note that $R^{\mathcal{I}} = \{\sigma^\flat\}$. *We show that* $R^{\mathcal{I}}$ *is Leibniz, but does not have the global family modus ponens.*

To verify the Leibniz property, note that, if $\phi = \psi$ *the conclusion is trivial and, if* $\{\phi,\psi\} \neq \{0,1\}$, *then* $C_\Sigma(R^{\mathcal{I}}_\Sigma[\phi,\psi]) = \mathrm{SEN}^\flat(\Sigma)$, *whence*

$$\Omega(C(R^{\mathcal{I}}_\Sigma[\phi,\psi])) = \nabla^{\mathbf{F}}$$

and the conclusion follows. Finally, if $\{\phi,\psi\} = \{0,1\}$, *then* $C_\Sigma(R^{\mathcal{I}}_\Sigma[\phi,\psi]) = \{2\}$, *whence* $\Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi,\psi])) = \{\{0,1\},\{2\}\}$ *and, therefore,*

$$\langle\phi,\psi\rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi,\psi])),$$

as required. We conclude that $R^{\mathcal{I}}$ *is Leibniz.*

On the other hand, we have $1 \in \{1,2\}$ *and* $R^{\mathcal{I}}_\Sigma[1,0] \leq \{\{1,2\}\}$, *whereas* $0 \notin \{1,2\}$. *Therefore,* $R^{\mathcal{I}}$ *fails to have the global family modus ponens in* $\mathcal{I}$.

In the opposite direction, and on the positive side, in a *protoalgebraic* π-institution $\mathcal{I}$, if the reflexive core is Leibniz, then it has the global family modus ponens in $\mathcal{I}$.

Proposition 804 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a protoalgebraic* π*-institution based on* $\mathbf{F}$. *If* $R^{\mathcal{I}}$ *is Leibniz, then it has the global family modus ponens in* $\mathcal{I}$.

Proof: Suppose that $\mathcal{I}$ is protoalgebraic and that $R^{\mathcal{I}}$ is Leibniz. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $R^{\mathcal{I}}_\Sigma[\phi,\psi] \leq T$. Now we have

$$\begin{array}{lll}\langle\phi,\psi\rangle & \in & \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi,\psi])) \quad \text{(since } R^{\mathcal{I}} \text{ is Leibniz)}\\ & \subseteq & \Omega_\Sigma(T). \quad \text{(since } R^{\mathcal{I}}_\Sigma[\phi,\psi] \leq T \text{ and } \mathcal{I} \text{ is protoalgebraic)}\end{array}$$

Therefore, since $\phi \in T_\Sigma$, we get, by the compatibility of $\Omega(T)$ with T, that $\psi \in T_\Sigma$. We conclude that $R^{\mathcal{I}}$ has the global family modus ponens in $\mathcal{I}$. ∎

We now show that a π-institution is syntactically protoalgebraic if and only if it is protoalgebraic and has a Leibniz reflexive core.

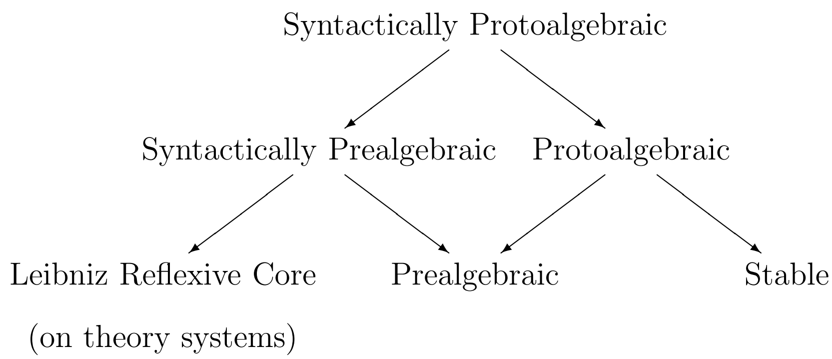
Syntactic Protoalgebraicity	=	$R^{\mathcal{I}}$ has Global Family MP
	=	$R^{\mathcal{I}}$ Defines Leibniz Congruence Systems
	=	Protoalgebraicity + $R^{\mathcal{I}}$ is Leibniz

Theorem 805 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically protoalgebraic if and only if it is protoalgebraic and has a Leibniz reflexive core.*

Proof: Suppose, first, that $\mathcal{I}$ is syntactically protoalgebraic. Then it is protoalgebraic by Theorem 792. Moreover, its reflexive core has the global family modus ponens by Theorem 799 and, hence, by Corollary 802, its reflexive core is Leibniz.

Suppose, conversely, that $\mathcal{I}$ is protoalgebraic with a Leibniz reflexive core. Then, by Proposition 804, its reflexive core has the global family modus ponens and, therefore, by Theorem 799, $\mathcal{I}$ is syntactically protoalgebraic. ∎

We have now established the following hierarchy of properties:



In fact, it turns out that many of the given characterizations of syntactic protoalgebraicity can be recast in terms of the corresponding ones concerning syntactic prealgebraicity by adding stability. The main result that allows this connection is the one corresponding to Theorem 175, but concerning syntactic protoalgebraicity and syntactic prealgebraicity rather than their respective semantic versions.

Theorem 806 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically protoalgebraic if and only if it is syntactically prealgebraic and stable.*

Proof: Suppose, first, that $\mathcal{I}$ is syntactically protoalgebraic. Then, it is, a fortiori, syntactically prealgebraic. Moreover, by Theorem 792, it is protoalgebraic. Therefore, by Theorem 175, it is stable.

Suppose, conversely, that $\mathcal{I}$ is syntactically prealgebraic and stable. Consider $T \in \mathrm{ThFam}(\mathcal{I})$. Then we have

$$\begin{array}{rcll} \Omega(T) & = & \Omega(\overleftarrow{T}) & \text{(stability)} \\ & = & R^{\mathcal{I}}(\overleftarrow{T}) & \text{(syntactic prealgebraicity and Theorem 784)} \\ & = & R^{\mathcal{I}}(T). & \text{(Proposition 99)} \end{array}$$

By Theorem 801, we conclude that $\mathcal{I}$ is syntactically protoalgebraic. ∎

Now we obtain, almost for free, the following corollaries, which contain the promised characterizations involving stability.

Corollary 807 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically protoalgebraic if and only if it is stable and* $R^{\mathcal{I}}$ *has the global system modus ponens.*

Proof: We have that $\mathcal{I}$ is syntactically protoalgebraic if and only if, by Theorem 806, it is syntactically prealgebraic and stable, if and only if, by Theorem 782, it is stable and $R^{\mathcal{I}}$ has the global system modus ponens. ■

Corollary 808 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically protoalgebraic if and only if it is stable and* $R^{\mathcal{I}}$ *defines Leibniz congruence systems of theory systems of* $\mathcal{I}$*, i.e., for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,* $\Omega(T) = R^{\mathcal{I}}(T)$*.*

Proof: We have that $\mathcal{I}$ is syntactically protoalgebraic if and only if, by Theorem 806, it is syntactically prealgebraic and stable, if and only if, by Theorem 784, it is stable and $R^{\mathcal{I}}$ defines Leibniz congruence systems of theory systems in $\mathcal{I}$. ■

Corollary 809 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically protoalgebraic if and only if it is prealgebraic, stable and* $R^{\mathcal{I}}$ *is Leibniz.*

Proof: We have that $\mathcal{I}$ is syntactically protoalgebraic if and only if, by Theorem 806, it is syntactically prealgebraic and stable, if and only if, by Theorem 788, it is prealgebraic, stable and $R^{\mathcal{I}}$ is Leibniz. ■

Finally, it is not difficult to see, in this case as well, that syntactic protoalgebraicity transfers from a π-institution $\mathcal{I}$ to all its generalized matrix families.

Theorem 810 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically protoalgebraic, with witnessing transformations* $I^\flat$*, if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, the generalized matrix family* $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ *is syntactically protoalgebraic, with witnessing transformations* $I^{\mathcal{A}}$*.*

Proof: The proof mimics the proof of Theorem 789. ■

11.3 Matrix Semantics

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall that an $\mathcal{I}$**-matrix family** is a pair $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, where $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ is an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is an $\mathcal{I}$-filter family on $\mathcal{A}$. The class of all $\mathcal{I}$-matrix families is denoted by MatFam($\mathcal{I}$). $\mathcal{I}$-matrix families on $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$, i.e., pairs of the form $\langle \mathcal{F}, T\rangle$, where $T \in \mathrm{ThFam}(\mathcal{I})$, are called **Lindenbaum $\mathcal{I}$-matrix families**. The collection of all Lindenbaum $\mathcal{I}$-matrix families is denoted by LMatFam($\mathcal{I}$).

Four subclasses of MatFam($\mathcal{I}$) are distinguished and will be of particular interest to us in the upcoming sections. These are:

- The class $\mathrm{LMatFam}^*(\mathcal{I})$ of all **reduced Lindenbaum $\mathcal{I}$-matrix families**:
$$\mathrm{LMatFam}^*(\mathcal{I}) = \{\langle \mathcal{F}/\Omega(T), T/\Omega(T)\rangle : T \in \mathrm{ThFam}(\mathcal{I})\};$$
- The class $\mathrm{LMatFam}^{Su}(\mathcal{I})$ of all **Suszko reduced Lindenbaum $\mathcal{I}$-matrix families**:
$$\mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I}) = \{\langle \mathcal{F}/\widetilde{\Omega}^{\mathcal{I}}(T), T/\widetilde{\Omega}^{\mathcal{I}}(T)\rangle : T \in \mathrm{ThFam}(\mathcal{I})\};$$
- The class $\mathrm{MatFam}^*(\mathcal{I})$ of all **reduced $\mathcal{I}$-matrix families**:
$$\mathrm{LMatFam}^*(\mathcal{I}) = \{\langle \mathcal{A}, T\rangle \in \mathrm{MatFam}(\mathcal{I}) : \Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}\};$$
- The class $\mathrm{MatFam}^{Su}(\mathcal{I})$ of all **Suszko reduced $\mathcal{I}$-matrix families**:
$$\mathrm{MatFam}^{Su}(\mathcal{I}) = \{\langle \mathcal{A}, T\rangle \in \mathrm{MatFam}(\mathcal{I}) : \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Delta^{\mathcal{A}}\}.$$

The following characterizations of the last two classes are well-known and very useful in practice.

Proposition 811 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a π-institution based on* $\mathbf{F}$. *Then the following equalities hold (where the classes are perceived as being closed under isomorphism):*

(a) $\mathrm{MatFam}^*(\mathcal{I}) = \{\langle \mathcal{A}/\Omega^{\mathcal{A}}(T), T/\Omega^{\mathcal{A}}(T)\rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}$;

(b) $\mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I}) = \{\langle \mathcal{F}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T), T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)\rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}$.

Proof: We prove Part (a). Part (b) can be proven similarly. First, if $\langle \mathcal{A}, T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$, then, since, by definition, $\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}$, we get that $\langle \mathcal{A}, T\rangle \cong \langle \mathcal{A}/\Omega^{\mathcal{A}}(T), T/\Omega^{\mathcal{A}}(T)\rangle$. For the reverse inclusion, it suffices to observe that, given an $\mathbf{F}$-algebraic system $\mathcal{A}$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we have, essentially due to the definition of the Leibniz congruence system, that
$$\Omega^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(T/\Omega^{\mathcal{A}}(T)) = \Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}.$$
Therefore, $\langle \mathcal{A}/\Omega^{\mathcal{A}}(T), T/\Omega^{\mathcal{A}}(T)\rangle \in \mathrm{MatFam}^*(\mathcal{I})$. ■

It turns out that all four classes of $\mathcal{I}$-matrix families defined above form matrix family semantics for the π-institution $\mathcal{I}$. More precisely, given an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ and a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, a class M of $\mathcal{I}$-matrix families is called a **matrix (family) semantics for $\mathcal{I}$** if $C = C^{\mathrm{M}}$, i.e., for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, $\phi \in C_\Sigma(\Phi)$ if and only if, for all $\langle \mathcal{A}, T \rangle \in \mathrm{M}$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq T_{F(\Sigma')} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \in T_{F(\Sigma')}.$$

Proposition 812 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathcal{I}$. The four classes*

$$\mathrm{LMatFam}^*(\mathcal{I}), \mathrm{LMatFam}^{\mathsf{Su}}(\mathcal{I}), \mathrm{MatFam}^*(\mathcal{I}) \text{ and } \mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})$$

are all matrix semantics for $\mathcal{I}$.

Proof: Let M be any of these four matrix family classes. Since M consists of $\mathcal{I}$-matrix families, we have that $C \leq C^{\mathrm{M}}$.

For the converse, note that the following inclusions hold:

$$\begin{array}{ccccc} & & \mathrm{LMatFam}^{\mathsf{Su}}(\mathcal{I}) & \searrow & \\ & & & & \mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I}) \\ \mathrm{LMatFam}^*(\mathcal{I}) & \longrightarrow & \mathrm{MatFam}^*(\mathcal{I}) & \nearrow & \end{array}$$

Therefore, we have, by definition, the inclusions

$$\begin{array}{ccccc} & \nearrow & C^{\mathrm{LMatFam}^{\mathsf{Su}}(\mathcal{I})} & & \\ C^{\mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})} & & & & \\ & \searrow & C^{\mathrm{MatFam}^*(\mathcal{I})} & \longrightarrow & C^{\mathrm{LMatFam}^*(\mathcal{I})} \end{array}$$

It follows that it suffices to show that the two reduced Lindenbaum matrix family classes satisfy $C^{\mathrm{LMatFam}^{\mathsf{Su}}(\mathcal{I})} \leq C$ and $C^{\mathrm{LMatFam}^*(\mathcal{I})} \leq C$. We show the first inclusion, since the second can be proven similarly.

Suppose $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that

$$\phi \in C_\Sigma^{\mathrm{LMatFam}^{\mathsf{Su}}(\mathcal{I})}(\Phi).$$

Let $T \in \mathrm{ThFam}(\mathcal{I})$, such that $\Phi \subseteq T_\Sigma$. Then, $\Phi/\widetilde{\Omega}_\Sigma^{\mathcal{I}}(T) \subseteq T_\Sigma/\widetilde{\Omega}_\Sigma^{\mathcal{I}}(T)$. Since $\langle \mathcal{F}/\widetilde{\Omega}^{\mathcal{I}}(T), T/\widetilde{\Omega}^{\mathcal{I}}(T) \rangle \in \mathrm{LMatFam}^{\mathsf{Su}}(\mathcal{I})$, we get, by hypothesis, $\phi/\widetilde{\Omega}_\Sigma^{\mathcal{I}}(T) \in T_\Sigma/\widetilde{\Omega}_\Sigma^{\mathcal{I}}(T)$. Thus, using the compatibility of $\widetilde{\Omega}^{\mathcal{I}}(T)$ with T, we get that $\phi \in T_\Sigma$. Since $T \in \mathrm{ThFam}(\mathcal{I})$ was arbitrary, we conclude that $\phi \in C_\Sigma(\Phi)$. ∎

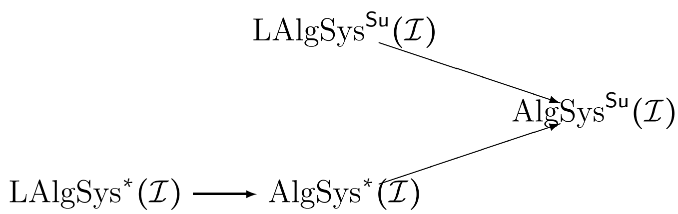
We denote the classes of the underlying $\mathbf{F}$-algebraic systems of the matrix families in $\mathrm{LMatFam}^*(\mathcal{I})$, $\mathrm{LMatFam}^{\mathsf{Su}}(\mathcal{I})$, $\mathrm{MatFam}^*(\mathcal{I})$ and $\mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})$, respectively, by

$$\mathrm{LAlgSys}^*(\mathcal{I}), \mathrm{LAlgSys}^{\mathsf{Su}}(\mathcal{I}), \mathrm{AlgSys}^*(\mathcal{I}) \text{ and } \mathrm{AlgSys}^{\mathsf{Su}}(\mathcal{I}).$$

So we have

$$\begin{array}{rcl}
\mathrm{LAlgSys}^*(\mathcal{I}) & = & \{\mathcal{F}/\Omega(T) : T \in \mathrm{ThFam}(\mathcal{I})\};\\
\mathrm{LAlgSys}^{\mathsf{Su}}(\mathcal{I}) & = & \{\mathcal{F}/\widetilde{\Omega}^{\mathcal{I}}(T) : T \in \mathrm{ThFam}(\mathcal{I})\};\\
\mathrm{AlgSys}^*(\mathcal{I}) & = & \{\mathcal{A} : (\exists T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}})\};\\
\mathrm{AlgSys}^{\mathsf{Su}}(\mathcal{I}) & = & \{\mathcal{A} : (\exists T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Delta^{\mathcal{A}})\}.
\end{array}$$

We clearly have the following inclusion relationships between those four classes of $\mathbf{F}$-algebraic systems:



11.4 Algebraic Semantics

In the study of logical systems formalized as π-institutions and, more specifically, as related to their algebraic properties, the notions of an algebraic semantics and that of equational definability of truth are paramount. We introduce and study these two notions in this section.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Consider a class K of $\mathbf{F}$-algebraic systems. We define the closure system $C^{\mathrm{K}} : \mathcal{P}(\mathrm{SEN}^\flat)^2 \to \mathcal{P}(\mathrm{SEN}^\flat)^2$, by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $C^{\mathrm{K}}_\Sigma : \mathcal{P}(\mathrm{SEN}^\flat(\Sigma)^2) \to \mathcal{P}(\mathrm{SEN}^\flat(\Sigma)^2)$ be given, for all $E \cup \{\phi \approx \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)^2$, by

$$\begin{array}{lll}
\phi \approx \psi \in C^{\mathrm{K}}_\Sigma(E) & \text{iff} & \text{for all } \mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle \in \mathrm{K},\\
& & \alpha_\Sigma(E) \subseteq \Delta^{\mathcal{A}}_{F(\Sigma)} \text{ implies } \alpha_\Sigma(\phi) = \alpha_\Sigma(\psi).
\end{array}$$

Given a set $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ of natural transformations in $N^\flat$, with a single distinguished argument, we say that the class K of $\mathbf{F}$-algebraic systems is a **$\tau^\flat$-algebraic semantics for $\mathcal{I}$**, or, more simply, a **$\tau^\flat$-semantics for $\mathcal{I}$**, if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in C_\Sigma(\Phi) \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq C^{\mathrm{K}}(\tau^\flat_\Sigma[\Phi]).$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution, based on $\mathbf{F}$, and M a class of $\mathcal{I}$-matrix families. Given a set $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ of natural transformations in $N^\flat$, we say that

truth is $\tau^\flat$-equationally definable in M, or, more simply, that **truth is $\tau^\flat$-definable in** M if, for all $\langle\mathcal{A},T\rangle \in$ M, with $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle$ and $\mathbf{A} = \langle\mathbf{Sign},\mathrm{SEN},N\rangle$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^{\mathcal{A}}_\Sigma[\phi] \leq \Delta^{\mathcal{A}}.$$

It turns out that classes of algebraic systems forming $\tau^\flat$-semantics for a π-institution and classes of matrix families in which truth is $\tau^\flat$-definable are closely interrelated. To express this connection, we first formulate a technical lemma.

Lemma 813 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and let* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *be a set of natural transformations in* $N^\flat$. *Suppose* $\mathfrak{A} = \langle\mathcal{A},T\rangle$, *with* $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle$ *and* $\mathbf{A} = \langle\mathbf{Sign},\mathrm{SEN},N\rangle$, *is an* $\mathbf{F}$*-matrix family in which truth is* $\tau^\flat$*-definable. Then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in C^{\mathfrak{A}}_\Sigma(\Phi) \quad \textit{iff} \quad \tau^\flat_\Sigma[\phi] \leq C^{\mathcal{A}}(\tau^\flat_\Sigma[\Phi]).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. Then we have the following sequence of equivalent statements:

$$\begin{array}{lll}
\phi \in C^{\mathfrak{A}}_\Sigma(\Phi) & \text{iff,} & \text{for all } \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma,\Sigma'),\\
& & \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq T_{F(\Sigma')}\\
& & \qquad\qquad \text{implies } \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \in T_{F(\Sigma')}\\
& \text{iff,} & \text{for all } \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma,\Sigma'),\\
& & \tau^{\mathcal{A}}_{F(\Sigma')}[\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))] \leq \Delta^{\mathcal{A}}\\
& & \qquad\qquad \text{implies } \tau^{\mathcal{A}}_{F(\Sigma')}[\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi))] \leq \Delta^{\mathcal{A}}\\
& \text{iff,} & \text{for all } \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma,\Sigma'),\\
& & \tau^{\mathcal{A}}_{F(\Sigma')}[\mathrm{SEN}(F(f))(\alpha_\Sigma(\Phi))] \leq \Delta^{\mathcal{A}}\\
& & \qquad\qquad \text{implies } \tau^{\mathcal{A}}_{F(\Sigma')}[\mathrm{SEN}(F(f))(\alpha_\Sigma(\phi))] \leq \Delta^{\mathcal{A}}\\
& \text{iff,} & \text{by Lemma 93,}\\
& & \tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\Phi)] \leq \Delta^{\mathcal{A}} \text{ implies } \tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi)] \leq \Delta^{\mathcal{A}}\\
& \text{iff,} & \text{by surjectivity of } \langle F,\alpha\rangle,\\
& & \alpha(\tau^\flat_\Sigma[\Phi]) \leq \Delta^{\mathcal{A}} \text{ implies } \alpha(\tau^\flat_\Sigma[\phi]) \leq \Delta^{\mathcal{A}}\\
& \text{iff} & \tau^\flat_\Sigma[\phi] \leq C^{\mathcal{A}}(\tau^\flat_\Sigma[\Phi]).
\end{array}$$

■

Now we establish the promised relationship between algebraic semantics and matrix semantics.

Theorem 814 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *a set of natural transformation in* $N^\flat$. *A class* K *of* $\mathbf{F}$*-algebraic systems is a* $\tau^\flat$*-semantics for* $\mathcal{I}$ *if and only if it is the class of underlying algebraic systems of some matrix semantics* M *for* $\mathcal{I}$ *in which truth is* $\tau^\flat$*-definable.*

Proof: Suppose, first, that M is a matrix semantics for $\mathcal{I}$ in which truth is $\tau^\flat$-definable and let K be the class of its underlying algebraic systems. Then we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}\phi \in C_\Sigma(\Phi) & \text{iff} & \phi \in C^{\mathrm{M}}_\Sigma(\Phi) \quad (\text{M a matrix semantics})\\ & \text{iff} & (\forall \mathfrak{A} \in \mathrm{M})(\phi \in C^{\mathfrak{A}}_\Sigma(\Phi)) \quad (\text{by definition})\\ & \text{iff} & (\forall \mathcal{A} \in \mathrm{K})(\tau^\flat_\Sigma[\phi] \leq C^{\mathcal{A}}(\tau^\flat_\Sigma[\Phi])) \quad (\text{by Lemma 813})\\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq C^{\mathrm{K}}(\tau^\flat_\Sigma[\Phi]). \quad (\text{by definition})\end{array}$$

Thus, K is a $\tau^\flat$-semantics for $\mathcal{I}$.

Suppose, conversely, that K is a $\tau^\flat$-semantics for $\mathcal{I}$. Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle \in$ K, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$. Define, for all $\Sigma \in |\mathbf{Sign}|$,

$$T^{\mathcal{A},\tau}_\Sigma = \{\phi \in \mathrm{SEN}(\Sigma) : \tau^{\mathcal{A}}_\Sigma[\phi] \leq \Delta^{\mathcal{A}}\},$$

and set $T^{\mathcal{A},\tau} = \{T^{\mathcal{A},\tau}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$. Then, let

$$\mathrm{M} = \{\langle \mathcal{A}, T^{\mathcal{A},\tau}\rangle : \mathcal{A} \in \mathrm{K}\}.$$

Note that K is the class of all underlying algebraic systems of the matrix systems in M and, also, that, for all $\mathcal{A} \in \mathrm{K}$, truth is $\tau^\flat$-definable in $\langle \mathcal{A}, T^{\mathcal{A},\tau}\rangle$ by the definition of $T^{\mathcal{A},\tau}$. Thus, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}\phi \in C_\Sigma(\Phi) & \text{iff} & \tau^\flat_\Sigma[\phi] \leq C^{\mathrm{K}}(\tau^\flat_\Sigma[\Phi]) \quad (\text{K a } \tau^\flat\text{-semantics})\\ & \text{iff} & (\forall \mathcal{A} \in \mathrm{K})(\tau^\flat_\Sigma[\phi] \leq C^{\mathcal{A}}(\tau^\flat_\Sigma[\Phi])) \quad (\text{by definition})\\ & \text{iff} & (\forall \mathfrak{A} \in \mathrm{M})(\phi \in C^{\mathfrak{A}}_\Sigma(\Phi)) \quad (\text{by Lemma 813})\\ & \text{iff} & \phi \in C^{\mathrm{M}}_\Sigma(\Phi). \quad (\text{by definition})\end{array}$$

We conclude that M is a matrix semantics for $\mathcal{I}$. ■

Corollary 815 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution, based on $\mathbf{F}$ and $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ a set of natural transformation in $N^\flat$. If truth is $\tau^\flat$-definable in any of the classes*

$$\mathrm{LMatFam}^*(\mathcal{I}), \mathrm{LMatFam}^{\mathsf{Su}}(\mathcal{I}), \mathrm{MatFam}^*(\mathcal{I}) \text{ or } \mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I}),$$

then, the corresponding class

$$\mathrm{LAlgSys}^*(\mathcal{I}), \mathrm{LAlgSys}^{\mathsf{Su}}(\mathcal{I}), \mathrm{AlgSys}^*(\mathcal{I}) \text{ or } \mathrm{AlgSys}^{\mathsf{Su}}(\mathcal{I})$$

is a $\tau^\flat$-semantics for $\mathcal{I}$.

Proof: This follows from Theorem 814, since the four displayed classes of $\mathcal{I}$-matrix families are matrix semantics for $\mathcal{I}$ and the four displayed classes of algebraic systems are the respective classes of their underlying algebraic systems. ■

11.5 Truth Equationality

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that $\mathcal{I}$ is:

- **Leibniz truth equational** if there exists $\tau^\flat \subseteq N^\flat$, with a single distinguished argument, such that truth is $\tau^\flat$-definable in $\mathrm{LMatFam}^*(\mathcal{I})$, i.e., such that, for all $\langle \mathcal{F}/\Omega(T), T/\Omega(T) \rangle \in \mathrm{LMatFam}^*(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi/\Omega_\Sigma(T) \in T_\Sigma/\Omega_\Sigma(T) \quad \text{iff} \quad \tau_\Sigma^{\mathcal{F}/\Omega(T)}[\phi/\Omega_\Sigma(T)] \leq \Delta^{\mathcal{F}/\Omega(T)};$$
- **Universally Leibniz truth equational** if there exists $\tau^\flat \subseteq N^\flat$, with a single distinguished argument, such that truth is $\tau^\flat$-definable in the class $\mathrm{MatFam}^*(\mathcal{I})$, i.e., such that, for all $\langle \mathcal{A}, T \rangle \in \mathrm{MatFam}^*(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,
$$\phi \in T_\Sigma \quad \text{iff} \quad \tau_\Sigma^{\mathcal{A}}[\phi] \leq \Delta^{\mathcal{A}};$$
- **Suszko truth equational** if there exists $\tau^\flat \subseteq N^\flat$, with a single distinguished argument, such that truth is $\tau^\flat$-definable in $\mathrm{LMatFam}^{\mathsf{Su}}(\mathcal{I})$, i.e., such that, for all $\langle \mathcal{F}/\widetilde{\Omega}^{\mathcal{I}}(T), T/\widetilde{\Omega}^{\mathcal{I}}(T) \rangle \in \mathrm{LMatFam}^{\mathsf{Su}}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi/\widetilde{\Omega}^{\mathcal{I}}_\Sigma(T) \in T_\Sigma/\widetilde{\Omega}^{\mathcal{I}}_\Sigma(T) \quad \text{iff} \quad \tau_\Sigma^{\mathcal{F}/\widetilde{\Omega}^{\mathcal{I}}(T)}[\phi/\widetilde{\Omega}^{\mathcal{I}}_\Sigma(T)] \leq \Delta^{\mathcal{F}/\widetilde{\Omega}^{\mathcal{I}}(T)};$$
- **Universally Suszko truth equational** if there exists $\tau^\flat \subseteq N^\flat$, with a single distinguished argument, such that truth is $\tau^\flat$-definable in the class $\mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})$, i.e., such that, for all $\langle \mathcal{A}, T \rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,
$$\phi \in T_\Sigma \quad \text{iff} \quad \tau_\Sigma^{\mathcal{A}}[\phi] \leq \Delta^{\mathcal{A}}.$$

The set $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ will be called a set of **witnessing equations** (of/for the corresponding truth equationality property).

The following proposition provides alternative conditions for testing whether a given π-institution is truth equational with respect to any of the four classes of matrix families considered above.

Proposition 816 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\tau^\flat \subseteq N^\flat$ having a single distinguished argument.*

(a) $\mathcal{I}$ is Leibniz truth equational with witnessing equations $\tau^\flat$ iff, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi \in T_\Sigma \quad \textit{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T);$$

(b) *$\mathcal{I}$ is universally Leibniz truth equational if and only if, for every* **F**-*algebraic system $\mathcal{A}$, all $T \in \mathrm{FiFam}^{\mathcal{A}}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,*

$$\phi \in T_\Sigma \quad \textit{iff} \quad \tau^{\mathcal{A}}_\Sigma[\phi] \leq \Omega^{\mathcal{A}}(T);$$

(c) *$\mathcal{I}$ is Suszko truth equational with witnessing equations $\tau^\flat$ iff, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,*

$$\phi \in T_\Sigma \quad \textit{iff} \quad \tau^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T);$$

(d) *$\mathcal{I}$ is universally Suszko truth equational if and only if, for every* **F**-*algebraic system $\mathcal{A}$, all $T \in \mathrm{FiFam}^{\mathcal{A}}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,*

$$\phi \in T_\Sigma \quad \textit{iff} \quad \tau^{\mathcal{A}}_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T).$$

Proof:

(a) Suppose, first, that $\mathcal{I}$ is Leibniz truth equational and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{lll}\phi \in T_\Sigma & \text{iff} & \phi/\Omega_\Sigma(T) \in T_\Sigma/\Omega_\Sigma(T) \quad \text{(by compatibility)}\\ & \text{iff} & \tau^{\mathcal{F}/\Omega(T)}[\phi/\Omega_\Sigma(T)] \leq \Delta^{\mathcal{F}/\Omega(T)} \quad \text{(by hypothesis)}\\ & \text{iff} & \tau^\flat_\Sigma[\phi]/\Omega(T) \leq \Delta^{\mathcal{F}/\Omega(T)} \quad \text{(by definition)}\\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(T).\end{array}$$

Assume, conversely, that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in T_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Let $\langle \mathcal{F}/\Omega(T), T/\Omega(T)\rangle \in \mathrm{LMatFam}^*(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{lll}\phi/\Omega_\Sigma(T) \in T_\Sigma/\Omega_\Sigma(T) & \text{iff} & \phi \in T_\Sigma \quad \text{(by compatibility)}\\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(T) \quad \text{(by hypothesis)}\\ & \text{iff} & \tau^\flat_\Sigma[\phi]/\Omega(T) \leq \Delta^{\mathcal{F}/\Omega(T)} \quad \text{(by definition)}\\ & \text{iff} & \tau^{\mathcal{F}/\Omega(T)}_\Sigma[\phi/\Omega_\Sigma(T)] \leq \Delta^{\mathcal{F}/\Omega(T)}.\end{array}$$

(b) Suppose, first, that $\mathcal{I}$ is universally Leibniz truth equational and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, be an **F**-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. Then

$$\begin{array}{lll}\phi \in T_\Sigma & \text{iff} & \phi/\Omega^{\mathcal{A}}_\Sigma(T) \in T_\Sigma/\Omega^{\mathcal{A}}_\Sigma(T) \quad \text{(by compatibility)}\\ & \text{iff} & \tau^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}[\phi/\Omega^{\mathcal{A}}_\Sigma(T)] \leq \Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)} \quad \text{(by hypothesis)}\\ & \text{iff} & \tau^{\mathcal{A}}_\Sigma[\phi]/\Omega^{\mathcal{A}}(T) \leq \Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)} \quad \text{(by definition)}\\ & \text{iff} & \tau^{\mathcal{A}}_\Sigma[\phi] \leq \Omega^{\mathcal{A}}(T).\end{array}$$

Assume, conversely, that, for every **F**-algebraic system $\mathcal{A}$, all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $\phi \in T_\Sigma$ if and only if $\tau^{\mathcal{A}}_\Sigma[\phi] \leq \Omega^{\mathcal{A}}(T)$. Let $\langle \mathcal{A}, T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. Then

$$\begin{array}{lll}\phi \in T_\Sigma & \text{iff} & \tau^{\mathcal{A}}_\Sigma[\phi] \leq \Omega^{\mathcal{A}}(T) \quad \text{(by hypothesis)}\\ & \text{iff} & \tau^{\mathcal{A}}_\Sigma[\phi] \leq \Delta^{\mathcal{A}}. \quad (\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}})\end{array}$$

Parts (c) and (d) follow along similar lines. ∎

We investigate next the relationships between the various types of truth equationality. Our first result is that Leibniz truth equationality and universal Leibniz truth equationality coincide.

Proposition 817 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is Leibniz truth equational if and only if it is universally Leibniz truth equational.*

Proof: First, note that, since $\mathrm{LMatFam}^*(\mathcal{I}) \subseteq \mathrm{MatFam}^*(\mathcal{I})$, universal Leibniz truth equationality trivially implies Leibniz truth equationality. Suppose, conversely, that $\mathcal{I}$ is Leibniz truth equational, with witnessing equations $\tau^\flat \subseteq N^\flat$. Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, we have

$$\begin{array}{lll} \alpha_\Sigma(\phi) \in T_{F(\Sigma)} & \text{iff} & \phi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)}) \quad \text{(set theory)} \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(\alpha^{-1}(T)) \quad \text{(Proposition 816)} \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T)) \quad \text{(Proposition 24)} \\ & \text{iff} & \alpha(\tau^\flat_\Sigma[\phi]) \leq \Omega^{\mathcal{A}}(T) \quad \text{(set theory)} \\ & \text{iff} & \tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi)] \leq \Omega^{\mathcal{A}}(T). \quad \text{(Lemma 96)} \end{array}$$

Taking into account the surjectivity of $\langle F, \alpha \rangle$ and Proposition 816, we conclude that $\mathcal{I}$ is universally Leibniz truth equational. ∎

In the next proposition, we show that (universal) Leibniz truth equationality and universal Suszko truth equationality are also identical properties.

Theorem 818 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is universally Leibniz truth equational if and only if it is universally Suszko truth equational.*

Proof: Since $\mathrm{MatFam}^*(\mathcal{I}) \subseteq \mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})$, it follows that universal Suszko truth equationality implies universal Leibniz truth equationality. Suppose, conversely, that $\mathcal{I}$ is universally Leibniz truth equational, with witnessing equations $\tau^\flat$. Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. By Proposition 816, it suffices to show that

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^{\mathcal{A}}_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T).$$

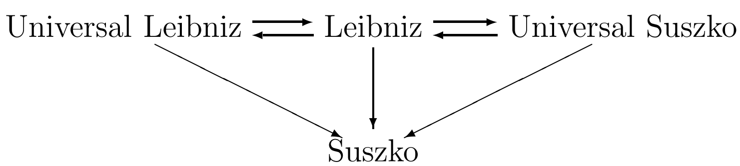
If $\phi \in T_\Sigma$, then $\phi \in T'_\Sigma$, for all $T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Thus, by hypothesis, $\tau^{\mathcal{A}}_\Sigma[\phi] \leq \Omega^{\mathcal{A}}(T')$. But then we have

$$\tau^{\mathcal{A}}_\Sigma[\phi] \leq \bigcap_{T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})} \Omega^{\mathcal{A}}(T') = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T).$$

Suppose, conversely, that $\tau^{\mathcal{A}}_{\Sigma}[\phi] \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. Then, since $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T)$, we get that $\tau^{\mathcal{A}}[\phi] \leq \Omega^{\mathcal{A}}(T)$, whence, by hypothesis, $\phi \in T_\Sigma$.

Using Proposition 816, we conclude that $\mathcal{I}$ is universally Suszko truth equational. ■

Since, for any π-institution $\mathcal{I}$, $\mathrm{LMatFam}^{\mathsf{Su}}(\mathcal{I}) \subseteq \mathrm{MatFam}^{\mathsf{Su}}(\mathcal{I})$, we have, trivially, that universal Suszko truth equationality implies Suszko truth equationality. Therefore, we get the following picture involving implications between the various truth equationality properties:



Next, we present an example showing that the top-to-bottom implication is not an equivalence in general. I.e., we construct an example of a π-institution, which is Suszko truth equational but not Leibniz truth equational.

Example 819 EXAMPLE NOT FOUND YET!

We call a π-institution that is (universally) Leibniz truth equational, or equivalently, universally Suszko truth equational, a **family truth-equational π-institution**, or more simply, a **truth equational π-institution**.

Combining these results with Corollary 815, we get the following

Corollary 820 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution and $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$. If $\mathcal{I}$ is truth equational with witnessing equations $\tau^\flat$, then the three classes $\mathrm{LAlgSys}^*(\mathcal{I})$, $\mathrm{AlgSys}^*(\mathcal{I})$ and $\mathrm{AlgSys}^{\mathsf{Su}}(\mathcal{I})$ are $\tau^\flat$-semantics for $\mathcal{I}$.*

Proof: By the definition of truth equationality and Corollary 815. ■

11.6 More on Truth Equationality

We start this section by looking closely at a property similar to the one defining truth equationality.

Lemma 821 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ a set of natural transformations in $N^\flat$. The following statements are equivalent:*

(a) For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\tau^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(C(\phi))$;

(b) *For all* $T \in \mathrm{ThFam}(\mathcal{I})$*,* $\tau^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T)$*.*

Proof: For (a)⇒(b), assume that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\tau^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(C(\phi))$, and let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in T_\Sigma$,

$$\begin{array}{rcll}\tau^\flat_\Sigma[\phi] & \leq & \widetilde{\Omega}^{\mathcal{I}}(C(\phi)) & \text{(hypothesis)}\\ & \leq & \widetilde{\Omega}^{\mathcal{I}}(T). & \text{(monotonicity of } \widetilde{\Omega}^{\mathcal{I}})\end{array}$$

Therefore, $\tau^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T)$.

For (b)⇒(a), assume that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\tau^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T)$, and let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, by hypothesis,

$$\tau^\flat_\Sigma[\phi] \leq \tau^\flat[C(\phi)] \leq \widetilde{\Omega}^{\mathcal{I}}(C(\phi)).$$

■

A very similar property holds replacing theory families by theory systems and using the arrow operators.

Lemma 822 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *a set of natural transformations in* $N^\flat$*. The following statements are equivalent:*

(a) *For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,* $\tau^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi}))$*;*

(b) *For all* $T \in \mathrm{ThSys}(\mathcal{I})$*,* $\tau^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T)$*.*

Proof: For (a)⇒(b), assume that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\tau^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi}))$, and let $T \in \mathrm{ThSys}(\mathcal{I})$. Then, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in T_\Sigma$, $C(\overrightarrow{\phi}) \leq T$ and, hence,

$$\begin{array}{rcll}\tau^\flat_\Sigma[\phi] & \leq & \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) & \text{(hypothesis)}\\ & \leq & \widetilde{\Omega}^{\mathcal{I}}(T). & \text{(monotonicity of } \widetilde{\Omega}^{\mathcal{I}})\end{array}$$

Therefore, $\tau^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T)$.

For (b)⇒(a), assume that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\tau^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T)$, and let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, by hypothesis,

$$\tau^\flat_\Sigma[\phi] \leq \tau^\flat[C(\overrightarrow{\phi})] \leq \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})).$$

■

The property studied in Lemma 821 is one that is satisfied by every π-institution possessing a $\tau^\flat$-semantics.

Proposition 823 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *a set of natural transformations in* $N^\flat$*. If* $\mathcal{I}$ *has a* $\tau^\flat$*-semantics, then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\tau^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(C(\phi)).$$

Proof: Suppose that K is a $\tau^\flat$-semantics for $\mathcal{I}$ and let $\delta^\flat \approx \epsilon^\flat$ be an arbitrary equation in $\tau^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Our goal is to show that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\langle \delta^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}), \epsilon^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi})\rangle \in \widetilde{\Omega}^{\mathcal{I}}_{\Sigma'}(C(\phi)).$$

By the characterization theorem for membership in the Suszko congruence system, and using symmetry, it suffices to show that, for all $\sigma^\flat \in N^\flat$, all $\Sigma'' \in |\mathbf{Sign}^\flat|$, all $g \in \mathbf{Sign}^\flat(\Sigma',\Sigma'')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma'')$,

$$\begin{array}{ccccc}
\Sigma & \xrightarrow{\quad f \quad} & \Sigma' & \xrightarrow{\quad g \quad} & \Sigma'' \\
\phi & \mapsto & \mathrm{SEN}^\flat(\phi) & \mapsto & \mathrm{SEN}^\flat(gf)(\phi) \\
 & & \vec{\chi} & \mapsto & \mathrm{SEN}^\flat(g)(\vec{\chi}) \\
 & & & & \vec{\xi}
\end{array}$$

$$\begin{array}{l}\sigma^\flat_{\Sigma''}(\mathrm{SEN}^\flat(g)(\epsilon^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi})),\vec{\xi}) \\ \qquad \in C_{\Sigma''}(\phi,\sigma^\flat_{\Sigma''}(\mathrm{SEN}^\flat(g)(\delta^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi})),\vec{\xi})).\end{array}$$

This is equivalent to showing

$$\begin{array}{l}\sigma^\flat_{\Sigma''}(\epsilon^\flat_{\Sigma''}(\mathrm{SEN}^\flat(gf)(\phi),\mathrm{SEN}^\flat(g)(\vec{\chi})),\vec{\xi}) \\ \qquad \in C_{\Sigma''}(\phi,\sigma^\flat_{\Sigma''}(\delta^\flat_{\Sigma''}(\mathrm{SEN}^\flat(gf)(\phi),\mathrm{SEN}^\flat(g)(\vec{\chi})),\vec{\xi})).\end{array}$$

To show this, however, it suffices to show that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\chi},\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\sigma^\flat_{\Sigma'}(\epsilon^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\vec{\xi}) \in C_{\Sigma'}(\phi,\sigma^\flat_{\Sigma'}(\delta^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\vec{\xi})).$$

This, now, follows from the fact that K is a $\tau^\flat$-semantics for $\mathcal{I}$ and that, obviously,

$$\begin{array}{l}\tau^\flat_{\Sigma'}[\sigma^\flat_{\Sigma'}(\epsilon^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\vec{\xi})] \\ \qquad \le C^{\mathrm{K}}(\tau^\flat_{\Sigma}[\phi],\tau^\flat_{\Sigma'}[\sigma^\flat_{\Sigma'}(\delta^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\vec{\xi})]).\end{array}$$

■

Now we obtain the following consequence:

Corollary 824 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *a set of natural transformations in* $N^\flat$*. If* $\mathcal{I}$ *has a* $\tau^\flat$*-semantics, then. for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\tau^\flat[T] \le \widetilde{\Omega}^{\mathcal{I}}(T).$$

Proof: By Proposition 823 and Lemma 821. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$. We say that the Suszko operator:

- is **universally family injective** if, for every **F**-algebraic system $\mathcal{A}$, and all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,
$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T') \quad \text{implies} \quad T = T';$$
- has the **universal family minimality** property if, for every **F**-algebraic system $\mathcal{A}$, and every $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$ is the least theory family of $\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$.

Universal family injectivity and universal family minimality of the Suszko operator turn out to be equivalent properties.

Theorem 825 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The Suszko operator is universally family injective if and only if it has the universal family minimality property.*

Proof: Suppose, first, that the Suszko operator has the universal family minimality property. Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an **F**-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T')$. Then both $T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$ and $T'/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T')$ are $\mathcal{I}$-filter families on the **F**-algebraic system $\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T')$. Thus, by the universal family minimality property, $T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = T'/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T')$. Since $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T')$, we get that $T = T'$. So $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ is universally family injective.

Suppose, conversely, that the Suszko operator is universally family injective. Consider an **F**-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Let T' be the least $\mathcal{I}$-filter family on $\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. Since we have $T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))$, we get, by minimality, that $T' \leq T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. But then, by the monotonicity of the Suszko operator, we get that
$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(T') \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)) = \Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}$$
and, therefore,
$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(T') = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))(= \Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}).$$
Hence, by universal family injectivity, $T' = T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$, which proves that the Suszko operator has the universal family minimality property. ■

Finally, recall that a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is called **family c-reflective** if, for every $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,
$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \quad \text{implies} \quad \bigcap \mathcal{T} \leq T'.$$
Also recall that, by the Transfer Theorem **??**, $\mathcal{I}$ is family c-reflective if and only if, for every **F**-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,
$$\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T') \quad \text{implies} \quad \bigcap \mathcal{T} \leq T'.$$

We may call this latter property **universal family complete reflectivity** or **universal family c-reflectivity**.

Our goal, in closing this section is to show that the family injectivity of the Suszko operator (and, hence, by Theorem 825, its universal family minimality) is equivalent to the (universal) family c-reflecitvity of $\mathcal{I}$.

We provide, first, an alternative characterization of universal family c-reflectivity involving both the Suszko and the Leibniz congruence systems.

Lemma 826 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is (universally) family c-reflective if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T') \quad \textit{implies} \quad T \leq T'.$$

Proof: Assume, first, that $\mathcal{I}$ is universally family c-reflective and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. Thus, by the definition of the Suszko operator,

$$\bigcap\{\Omega^{\mathcal{A}}(T'') : T \leq T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \leq \Omega^{\mathcal{A}}(T').$$

Using universal family c-reflectivity, we get that

$$\bigcap\{T'' : T \leq T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \leq T'.$$

Hence, $T \leq T'$, as required.

Suppose, conversely, that, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ implies $T \leq T'$. Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. Then we have

$$\begin{array}{rcll}\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(\bigcap_{T \in \mathcal{T}} T) & \leq & \bigcap_{T \in \mathcal{T}} \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) & (\text{monotonicity of } \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}) \\ & \leq & \bigcap_{T \in \mathcal{T}} \Omega^{\mathcal{A}}(T) & (\text{since } \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T)) \\ & \leq & \Omega^{\mathcal{A}}(T'). & (\text{by hypothesis})\end{array}$$

Using the hypothesis, we conclude that $\bigcap \mathcal{T} \leq T'$. Therefore, $\mathcal{I}$ is family c-reflective. ■

Finally, we show that family c-reflectivity is identical with the universal family injectivity of the Suszko operator.

Theorem 827 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is (universally) family c-reflective if and only if the Suszko operator is universally family injective.*

Proof: Suppose, first, that the Suszko operator is universally family injective. To show that $\mathcal{I}$ is family c-reflective, we use Lemma 826. Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$.

This implies that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$ is compatible with T'. We consider the natural transformation

$$\langle I,\gamma\rangle : \mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \to \mathcal{A}/\Omega^{\mathcal{A}}(T').$$

Since $T'/\Omega^{\mathcal{A}}(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T'))$, we get

$$\gamma^{-1}(T'/\Omega^{\mathcal{A}}(T')) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)),$$

i.e., $T'/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))$. By universal family injectivity of the Suszko operator and Theorem 825, we get that $T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \le T'/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. Taking into account the compatibility of $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$ with T', pointed out above, we get that $T \le T'$.

Assume, conversely, that $\mathcal{I}$ is (universally) family c-reflective. Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T,T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T')$. Then, we have $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')$ and $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T') \le \Omega^{\mathcal{A}}(T)$, whence, by hypothesis and Lemma 826, $T \le T'$ and $T' \le T$, showing that $T = T'$. Thus, the Suszko operator is universally family injective. ■

In a nutshell we have the following three equivalent properties, given in Theorems 825 and 827.

Suszko is Universally Family Injective

$\downarrow\uparrow$

Suszko has Universal Family Minimality

$\downarrow\uparrow$

$\mathcal{I}$ is (Universally) Family c-Reflective

11.7 Truth Equationality and c-Reflectivity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$. Recall that $\mathcal{I}$ was called *family c-reflective* if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,

$$\bigcap_{T\in\mathcal{T}} \Omega(T) \le \Omega(T') \quad \text{implies} \quad \bigcap_{T\in\mathcal{T}} T \le T'.$$

Family c-reflectivity implies family reflectivity, i.e., the property that, for all $T,T' \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) \le \Omega(T')$ implies $T \le T'$. Finally, family c-reflectivity is a property strong enough to imply systemicity. Therefore, a π-institution is family c-reflective if and only if it is system c-reflective and systemic.

Recall, also, that $\mathcal{I}$ was called (*family*) *truth equational* if there exists $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with a single distinguished argument, such that, for every $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \le \Omega(T).$$

In that case, $\tau^\flat$ is termed a **set of witnessing equations** (of/for the truth equationality of $\mathcal{I}$).

Note again that truth equationality implies systemicity. In fact, if $\mathcal{I}$ is truth equational with witnessing equations $\tau^\flat$, then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, we get

$$\begin{array}{rcl} \tau^\flat_\Sigma[\phi] \le \Omega(\overleftarrow{T}) & \text{iff} & \phi \in \overleftarrow{T}_\Sigma \\ & \text{implies} & \phi \in T_\Sigma \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \le \Omega(T) \\ & \text{implies} & \tau^\flat_\Sigma[\phi] \le \Omega(\overleftarrow{T}). \end{array}$$

So all statements above are equivalent showing that $\overleftarrow{T} = T$. Thus, $\mathcal{I}$ is systemic.

It turns out that, if $\mathcal{I}$ is a truth equational π-institution, with witnessing equations $\tau^\flat$, then $\tau^\flat(\Omega(T))$ is exactly equal to T, i.e., that the witnessing equations reflect theory families.

Proposition 828 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is truth equational, with witnessing equations $\tau^\flat$, then, for all $T \in \mathrm{ThFam}(\mathcal{I})$,*

$$\tau^\flat(\Omega(T)) = T.$$

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rcll} \phi \in \tau^\flat_\Sigma(\Omega(T)) & \text{iff} & \tau^\flat_\Sigma[\phi] \le \Omega(T) & \text{(definition)} \\ & \text{iff} & \phi \in T_\Sigma. & \text{(truth equationality)} \end{array}$$

■

Proposition 828 has as an immediate consequence the important fact that truth equationality implies family c-reflectivity.

Theorem 829 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is truth equational, then it is family c-reflective.*

Proof: Suppose that $\mathcal{I}$ is truth equational with witnessing equations $\tau^\flat$. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \le \Omega(T')$. Then

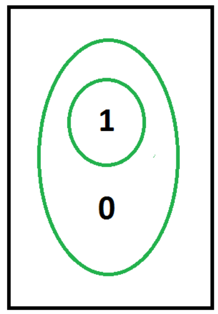
$$\begin{array}{rcll} \bigcap_{T\in\mathcal{T}} T & = & \bigcap_{T\in\mathcal{T}} \tau^\flat(\Omega(T)) & \text{(Proposition 828)} \\ & = & \tau^\flat(\bigcap_{T\in\mathcal{T}} \Omega(T)) & \text{(set theory)} \\ & \le & \tau^\flat(\Omega(T')) & \text{(hypothesis and Lemma 94)} \\ & = & T'. & \text{(Proposition 828)} \end{array}$$

Thus, $\mathcal{I}$ is family c-reflective. ■

The following example shows that the inclusion of Theorem 829 is proper.

Example 830 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$*;*
- $N^\flat$ *is the trivial category of natural transformations consisting of the projections only.*



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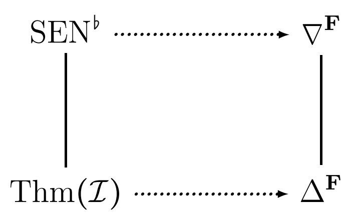
Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\}, \{0, 1\}\}$*.*

$\mathcal{I}$ *has two theory families,* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, which are also theory systems. Clearly,* $\mathrm{Thm}(\mathcal{I}) \leq \mathrm{SEN}^\flat$*. Moreover,* $\Omega(\mathrm{Thm}(\mathcal{I})) = \Delta^{\mathbf{F}}$ *and* $\Omega(\mathrm{SEN}^\flat) = \nabla^{\mathbf{F}}$*.* $\mathcal{I}$ *is clearly family c-reflective.*



On the other hand, there does not exist $\tau^\flat \subseteq N^\flat$*, such that* $I^\flat$ *has the required properties to constitute a witnessing set of equations for the truth equationality in* $\mathcal{I}$*. Any set consisting of projections only cannot satisfy the required condition since* $\tau^\flat(\Omega(T))$ *can only be* $\mathrm{SEN}^\flat$ *or* $\overline{\varnothing}$*.*

We now work towards a dual goal. We first provide a characterization of truth equationality in terms of the solubility property of the Suszko core of the π-institution. Then, we provide an exact description of those family c-reflective π-institutions which are truth equational.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. We define the **Suszko core** $S^{\mathcal{I}}$ of $\mathcal{I}$ to be the collection

$$S^{\mathcal{I}} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}(\mathcal{I}))(\sigma^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T))\}.$$

By Lemma 821, this definition is equivalent to setting

$$S^{\mathcal{I}} = \{\sigma^\flat \in N^\flat : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\sigma^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(C(\phi)))\}.$$

The Suszko core has a list of interesting properties:

Proposition 831 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$.

(a) $\iota \approx \iota \in S^{\mathcal{I}}$*, where* $\iota : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *denotes the identity;*

(b) If $\delta^\flat \approx \epsilon^\flat \in S^{\mathcal{I}}$*, then* $\epsilon^\flat \approx \delta^\flat \in S^{\mathcal{I}}$*;*

(c) If $\delta^\flat \approx \epsilon^\flat, \epsilon^\flat \approx \zeta^\flat \in S^{\mathcal{I}}$*, then* $\delta^\flat \approx \zeta^\flat \in S^{\mathcal{I}}$*;*

(d) If $\delta^\flat \approx \epsilon^\flat \in S^{\mathcal{I}}$*, then, for all* $\sigma^\flat \in N^\flat$*,*

$$\sigma^\flat \circ \langle \delta^\flat(\vec{p}), \vec{q} \rangle \approx \sigma^\flat \circ \langle \epsilon^\flat(\vec{p}), \vec{q} \rangle \in S^{\mathcal{I}},$$

where $\vec{p}, \vec{q}$ *denote vectors of projections*

$$\vec{p} = \langle p^{k+n-1,0}, \dots, p^{k+n-1,k-1} \rangle, \vec{q} = \langle p^{k+n-1,k}, \dots, p^{k+n-1,k+n-2} \rangle,$$

with k *the maximum arity between* $\delta^\flat$ *and* $\epsilon^\flat$*, and* n *the arity of* $\sigma^\flat$*.*

Proof:

(a) Since, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in T_\Sigma$,

$$(\iota \approx \iota)_\Sigma[\phi] \leq \Delta^{\mathbf{F}} \leq \widetilde{\Omega}^{\mathcal{I}}(T),$$

we get, by definition, $\iota \approx \iota \in S^{\mathcal{I}}$.

(b) Suppose that $\delta^\flat \approx \epsilon^\flat \in S^{\mathcal{I}}$. Then, by definition, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in T_\Sigma$, $(\delta^\flat \approx \epsilon^\flat)_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T)$. By the symmetry property of the Suszko congruence system $\widetilde{\Omega}^{\mathcal{I}}(T)$, we conclude that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in T_\Sigma$, $(\epsilon^\flat \approx \delta^\flat)_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T)$. Therefore, $\epsilon^\flat \approx \delta^\flat \in S^{\mathcal{I}}$.

(c) This follows along the lines of Part (b), using the transitivity of the Suszko congruence system $\widetilde{\Omega}^{\mathcal{I}}(T)$ instead of its symmetry.

(d) Suppose that $\delta^\flat \approx \epsilon^\flat \in S^{\mathcal{I}}$ and $\sigma^\flat \in N^\flat$. Then, by definition of $S^{\mathcal{I}}$, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in T_\Sigma$, $(\delta^\flat \approx \epsilon^\flat)_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T)$. Thus, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\langle \delta^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}), \epsilon^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \rangle \in \widetilde{\Omega}^{\mathcal{I}}_{\Sigma'}(T).$$

But, then, by the congruence compatibility property of $\widetilde{\Omega}^{\mathcal{I}}(T)$, we get that, for all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\langle \sigma^\flat_{\Sigma'}(\delta^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\vec{\xi}), \sigma^\flat_{\Sigma'}(\epsilon^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\vec{\xi})\rangle \in \widetilde{\Omega}^{\mathcal{I}}_{\Sigma'}(T).$$

Since $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and $\vec{\chi},\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$ were arbitrary, we get

$$(\sigma^\flat \circ \langle \delta^\flat(\vec{p}),\vec{q}\rangle \approx \sigma^\flat \circ \langle \epsilon^\flat(\vec{p}),\vec{q}\rangle)_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T).$$

Finally, since $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in T_\Sigma$ were arbitrary, we conclude that

$$\sigma^\flat \circ \langle \delta^\flat(\vec{p}),\vec{q}\rangle \approx \sigma^\flat \circ \langle \epsilon^\flat(\vec{p}),\vec{q}\rangle \in S^{\mathcal{I}}.$$

■

It is clear, by the definitions involved, that the Suszko core of a π-institution satisfies the following property:

Proposition 832 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. For every* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$T \leq S^{\mathcal{I}}(\Omega(T)).$$

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll} \phi \in T_\Sigma & \text{implies} & S^{\mathcal{I}}_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T) \quad (\text{definition of } S^{\mathcal{I}}) \\ & \text{implies} & S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T). \quad (\widetilde{\Omega}^{\mathcal{I}}(T) \leq \Omega(T)) \end{array}$$

Thus, we get that $T \leq S^{\mathcal{I}}(\Omega(T))$. ■

It is possible, but not necessary, that the Suszko core of a π-institution satisfies the reverse inclusion. We call this property solubility.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. We say that the Suszko core of $\mathcal{I}$ is **soluble** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$S^{\mathcal{I}}(\Omega(T)) \leq T.$$

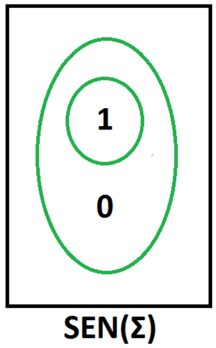
In other words, $S^{\mathcal{I}}$ is soluble if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) \quad \text{implies} \quad \phi \in T_\Sigma.$$

We present two examples to showcase the possibilities. In the first example, we look at a π-institution $\mathcal{I}$ whose Suszko core $S^{\mathcal{I}}$ is soluble.

Example 833 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

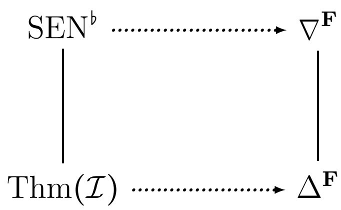
- $\mathbf{Sign}^\flat$ *is the trivial category with single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*;*



- *$N^\flat$ is the category of natural transformations generated by the single unary natural transformations $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$, specified by setting $\sigma^\flat_\Sigma(x) = 1$, for all $x \in \mathrm{SEN}^\flat(\Sigma)$.*

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be the π-institution determined by $\mathcal{C}_\Sigma = \{\{1\}, \{0,1\}\}$.

$\mathcal{I}$ has two theory families, $\mathrm{Thm}(\mathcal{I})$ and $\mathrm{SEN}^\flat$, which are also theory systems. Moreover, the structure of its posets of theory families and of their associated Leibniz congruence systems is given below.



One can see that the Suszko core of $\mathcal{I}$ is given by

$$S^{\mathcal{I}} = \{\iota \approx \iota, \iota \approx \sigma^\flat, \sigma^\flat \approx \iota, \sigma^\flat \approx \sigma^\flat\}.$$

Since the implication

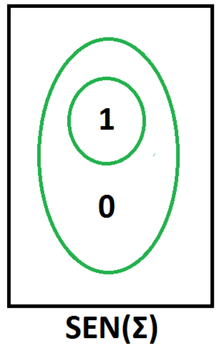
$$S^{\mathcal{I}}_\Sigma[\phi] \le T \quad \textit{implies} \quad \phi \in T_\Sigma$$

holds universally, we conclude that the Suszko core of $\mathcal{I}$ is soluble.

Next, we present an example of a π-institution $\mathcal{I}$ whose Suszko core $S^{\mathcal{I}}$ is not soluble.

Example 834 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system determined as follows:*

- *$\mathbf{Sign}^\flat$ is the trivial category with single object Σ;*
- *$\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ is specified by $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$;*



- *$N^\flat$ is the trivial category of natural transformations consisting of the projections only.*

Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be the π-institution determined by $\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\}$.

$\mathcal{I}$ has two theory families, $\mathrm{Thm}(\mathcal{I})$ and $\mathrm{SEN}^\flat$, which are also theory systems. Moreover, the structure of its posets of theory families and of their associated Leibniz congruence systems is given below.

$$\begin{array}{ccc} \mathrm{SEN}^\flat & \cdots\cdots\to & \nabla^{\mathbf{F}} \\ \big| & & \big| \\ \mathrm{Thm}(\mathcal{I}) & \cdots\cdots\to & \Delta^{\mathbf{F}} \end{array}$$

One can see that the Suszko core of $\mathcal{I}$ is given by

$$S^{\mathcal{I}} = \{\iota \approx \iota\}.$$

We, thus, have that

$$S^{\mathcal{I}}_\Sigma[0] \le \Delta^{\mathbf{F}} = \Omega(\mathrm{Thm}(\mathcal{I})) \quad \textit{but} \quad 0 \notin \mathrm{Thm}_\Sigma(\mathcal{I}).$$

Therefore $S^{\mathcal{I}}$ is not soluble.

It turns out that possession of the solubility property by the Suszko core intrinsically characterizes truth equationality. We can show, at the outset, that the Suszko core being soluble is necessary for truth equationality. Thus, there is no point in trying to discover witnessing equations unless the Suszko core of the π-institution $\mathcal{I}$ under scrutiny is soluble.

To show this, observe, first, that, in case a π-institution is truth equational, the witnessing equations form a subset of the Suszko core.

Lemma 835 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is truth equational, with witnessing equations $\tau^\flat \subseteq N^\flat$, then $\tau^\flat \subseteq S^{\mathcal{I}}$.*

Proof: By truth equationality, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

Thus, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}\phi \in T_\Sigma & \text{iff} & (\forall T \leq T' \in \mathrm{ThFam}(\mathcal{I}))(\phi \in T'_\Sigma) \\ & \text{iff} & (\forall T \leq T' \in \mathrm{ThFam}(\mathcal{I}))(\tau^\flat_\Sigma[\phi] \leq \Omega(T')) \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \bigcap\{\Omega(T') : T \leq T' \in \mathrm{ThFam}(\mathcal{I})\} \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T).\end{array}$$

We conclude, by the definition of $S^{\mathcal{I}}$, that $\tau^\flat \subseteq S^{\mathcal{I}}$. ∎

Now we prove the necessity of solubility for truth equationality.

Theorem 836 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is truth equational, then* $S^{\mathcal{I}}$ *is soluble.*

Proof: Suppose that $\mathcal{I}$ is truth equational, with witnessing equations $\tau^\flat$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{llll}S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) & \text{implies} & \tau^\flat_\Sigma[\phi] \leq \Omega(T) & \text{(Lemma 835)} \\ & \text{iff} & \phi \in T_\Sigma. & \text{(truth equationality)}\end{array}$$

Thus, $S^{\mathcal{I}}$ is soluble. ∎

The reverse implication, which also holds and completes the promised characterization of truth equationality in terms of the Suszko core, is presented in the following result.

Theorem 837 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $S^{\mathcal{I}}$ *is soluble, then* $\mathcal{I}$ *is truth equational, with witnessing equations* $S^{\mathcal{I}}$*.*

Proof: It suffices to show that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

The left-to-right implication is given in Proposition 832, whereas the converse is ensured by the postulated solubility of $S^{\mathcal{I}}$. ∎

Theorems 836 and 837 provide the promised characterization of truth equationality in terms of the solubility of the Suszko core.

$$\mathcal{I} \text{ is Truth Equational} \longleftrightarrow S^{\mathcal{I}} \text{ is Soluble.}$$

Theorem 838 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is truth equational if and only if* $S^{\mathcal{I}}$ *is soluble.*

Proof: Theorem 836 gives the "only if" and the "if" is by Theorem 837. ∎

If $\mathcal{I}$ is truth equational, then the Suszko core defines theory families in $\mathcal{I}$ in terms of their Leibniz congruence systems. This proposition may be viewed as a special case of Proposition 828, since $S^{\mathcal{I}}$ forms a maximal set of witnessing equations for the truth equationality of $\mathcal{I}$.

Proposition 839 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $S^{\mathcal{I}}$ *is soluble, then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$T = S^{\mathcal{I}}(\Omega(T)).$$

Proof: If $S^{\mathcal{I}}$ is soluble, then, by Theorem 837, $S^{\mathcal{I}}$ forms a set of witnessing equations for the truth equationality of $\mathcal{I}$. Therefore, by Proposition 828, we get that, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $T = S^{\mathcal{I}}(\Omega(T))$. ∎

In fact, this property may also be restated as another characterization of truth equationality. Let us say that $S^{\mathcal{I}}$ **defines theory families** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T = S^{\mathcal{I}}(\Omega(T))$. Then we have:

$$\mathcal{I} \text{ is Truth Equational} \longleftrightarrow S^{\mathcal{I}} \text{ Defines Theory Families.}$$

Theorem 840 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is truth equational if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$T = S^{\mathcal{I}}(\Omega(T)).$$

Proof: If $\mathcal{I}$ is truth equational, then, by Theorem 838, $S^{\mathcal{I}}$ is soluble. Thus, by Proposition 839, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T = S^{\mathcal{I}}(\Omega(T))$.

Conversely, if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T = S^{\mathcal{I}}(\Omega(T))$, then $S^{\mathcal{I}}$ is soluble. Thus, again by Theorem 838, $S^{\mathcal{I}}$ is a set of witnessing equations and $\mathcal{I}$ is truth equational. ∎

We finally show that the property that separates family complete reflectivity from truth equationality is exactly the adequacy property of the Suszko core. Roughly speaking, this property ensures that the Suszko core is rich enough to define Suszko congruence systems in terms of the Leibniz congruence systems of theory families that it selects via inclusion.

We have the following relationship connecting the Suszko core with both Leibniz and Suszko congruence systems.

Proposition 841 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)\} \le \widetilde{\Omega}^{\mathcal{I}}(C(\phi)).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\begin{array}{lll} \phi \in T_\Sigma & \text{implies} & S^{\mathcal{I}}_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T) \quad \text{(definition of the Suszko core)} \\ & \text{implies} & S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T). \quad (\widetilde{\Omega}^{\mathcal{I}}(T) \leq \Omega(T)) \end{array}$$

Therefore, we have

$$\begin{array}{rcl} \bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} & \leq & \bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T)\} \\ & \leq & \bigcap\{\Omega(T) : \phi \in T_\Sigma\} \\ & = & \widetilde{\Omega}^{\mathcal{I}}(C(\phi)). \end{array}$$

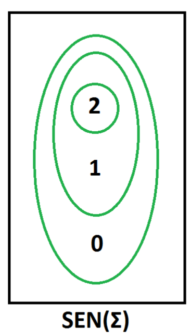
■

We provide an example, next, that shows that the inclusion proven in Proposition 841 is proper, in general. I.e., there exist π-institutions $\mathcal{I}$ in which, for some signature Σ and some Σ-sentence ϕ,

$$\bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \lneqq \widetilde{\Omega}^{\mathcal{I}}(C(\phi)).$$

Example 842 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object Σ;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$;*
- $N^\flat$ *is the trivial category of natural transformations, consisting of the projections only.*



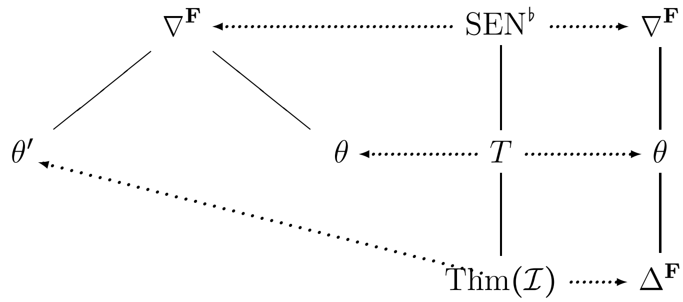
Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be the π-institution determined by

$$\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

$\mathcal{I}$ has three theory families $\mathrm{Thm}(\mathcal{I})$, $T = \{\{1, 2\}\}$ and $\mathrm{SEN}^\flat$, all of which are theory systems.

Note that $S^{\mathcal{I}} = \{\iota \approx \iota\}$. Note, also, the structure of the posets of Leibniz congruence systems and of Suszko congruence systems, that are provided in the left and right sides, respectively, of the following diagram, where

$$T = \{\{1, 2\}\}, \quad \theta = \{\{0\}, \{1, 2\}\}, \quad \theta' = \{\{0, 1\}, \{2\}\}.$$



Taking this into account, it is not difficult to see that

$$\bigcap\{\Omega(T) : S^{\mathcal{I}}_{\Sigma}[1] \leq \Omega(T)\} = \Delta^{\mathbf{F}} \lneq \theta = \widetilde{\Omega}^{\mathcal{I}}(C(1)).$$

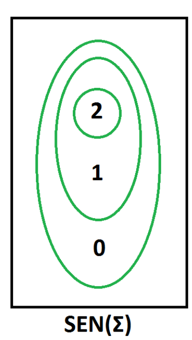
We also give an example of a π-institution $\mathcal{I}$ whose Suszko core $S^{\mathcal{I}}$ is such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) = \bigcap\{\Omega(T) : S^{\mathcal{I}}_{\Sigma}[\phi] \leq \Omega(T)\}.$$

Example 843 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the single unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0, 1, 2\} \to \{0, 1, 2\}$ *be given, for all* $x \in \mathrm{SEN}^\flat(\Sigma)$*, by*

$$\sigma^\flat_\Sigma(x) = 2.$$



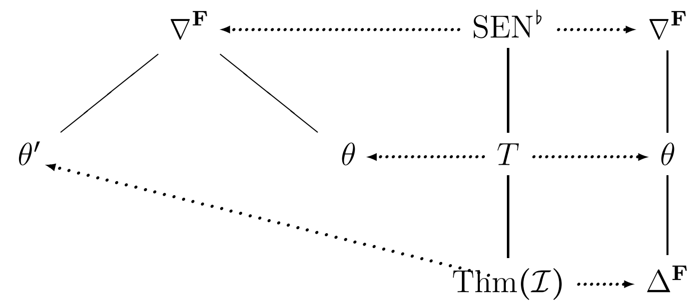
Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

$\mathcal{I}$ *has three theory families* $\mathrm{Thm}(\mathcal{I})$*,* $T = \{\{1, 2\}\}$ *and* $\mathrm{SEN}^\flat$*, all of which are theory systems.*

Note that $S^{\mathcal{I}} = \{\iota \approx \iota, \iota \approx \sigma^\flat, \sigma^\flat \approx \iota, \sigma^\flat \approx \sigma^\flat\}$. *Note, also, the structure of the posets of Leibniz congruence systems and of Suszko congruence systems, that are provided in the left and right sides, respectively, of the following diagram, where*

$$T = \{\{1,2\}\}, \quad \theta = \{\{0\},\{1,2\}\}, \quad \theta' = \{\{0,1\},\{2\}\}.$$



Now we can check:

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathcal{I}}(C(0)) & = & \nabla^{\mathbf{F}} = \Omega(\mathrm{SEN}^\flat)\\ & = & \bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[0] \le \Omega(T)\};\\ \widetilde{\Omega}^{\mathcal{I}}(C(1)) & = & \theta = \Omega(\mathrm{SEN}^\flat) \cap \Omega(T)\\ & = & \bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[1] \le \Omega(T)\};\\ \widetilde{\Omega}^{\mathcal{I}}(C(2)) & = & \Delta^{\mathbf{F}} = \Omega(\mathrm{SEN}^\flat) \cap \Omega(T) \cap \Omega(\mathrm{Thm}(\mathcal{I}))\\ & = & \bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[2] \le \Omega(T)\}.\end{array}$$

We have seen, therefore, through examples, that it is possible, but not necessary, that the Suszko core of a π-institution satisfies, for every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, the reverse inclusion of that given in Proposition 841:

$$\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) \le \bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)\}.$$

Intuitively speaking, this means that the Suszko core $S^{\mathcal{I}}$ is rich enough to allow, for every signature Σ and every Σ-sentence ϕ, the determination of those theory families whose Leibniz congruence systems form a covering of the Suszko congruence system of $C(\phi)$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$. We say that the Suszko core $S^{\mathcal{I}}$ of $\mathcal{I}$ is **adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) = \bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)\}.$$

Based on our preceding work, it is not difficult to see that, if $S^{\mathcal{I}}$ is soluble, then it is adequate.

Corollary 844 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $S^{\mathcal{I}}$ *is soluble, then it is adequate.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

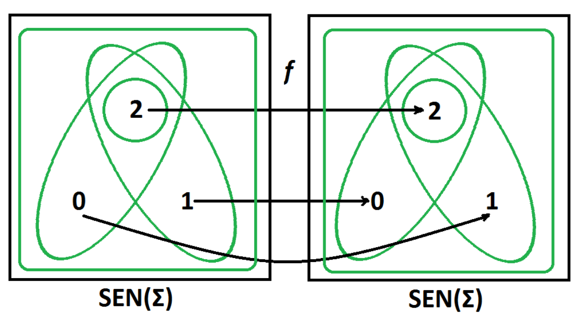
$$\begin{array}{rcl}\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) & = & \bigcap\{\Omega(T) : \phi \in T_\Sigma\} \quad (\text{definition of } \widetilde{\Omega}^{\mathcal{I}}(C(\phi)))\\ & = & \bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.\\ & & (\text{solubility of } S^{\mathcal{I}} \text{ and Proposition 839})\end{array}$$

We conclude that $S^{\mathcal{I}}$ is adequate. ■

Here is an example of a π-institution $\mathcal{I}$, with an adequate but not soluble Suszko core.

Example 845 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = i_\Sigma$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 1$*,* $1 \mapsto 0$ *and* $2 \mapsto 2$*;*
- $N^\flat$ *is the trivial category of natural transformations (consisting of the projections only).*

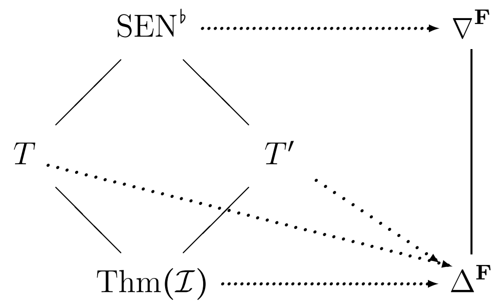


Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\},\{0,2\},\{1,2\},\{0,1,2\}\}.$$

$\mathcal{I}$ *has four theory families* $\mathrm{Thm}(\mathcal{I})$*,* $T = \{\{0,2\}\}$*,* $T' = \{\{1,2\}\}$ *and* $\mathrm{SEN}^\flat$*, but only two theory systems* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*. Therefore, being non-systemic, it can be neither family c-reflective nor truth-equational. The fact that it is not truth equational, together with Theorem 838, reveal that the Suszko core* $S^{\mathcal{I}}$ *is not soluble.*

To verify that $S^{\mathcal{I}}$ *is adequate, we look at the posets of theory families (left), Leibniz congruence systems (right) and Suszko congruence systems* *(right, identical with the Leibniz congruence systems, since the π-institution is protoalgebraic).*



Since $S^{\mathcal{I}} = \{\iota \approx \iota\}$, we verify adequacy of $S^{\mathcal{I}}$ by the following calculation, holding for all $\phi \in \mathrm{SEN}^\flat(\Sigma)$:

$$\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) = \Delta^{\mathbf{F}} = \bigcap_{T\in\mathrm{ThFam}(\mathcal{I})} \Omega(T) = \bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.$$

Thus, $S^{\mathcal{I}}$ is in fact adequate but not soluble.

In the opposite direction, and on the positive side, in a *family c-reflective* π-institution $\mathcal{I}$, if the Suszko core is adequate, then it is also soluble.

Proposition 846 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a family c-reflective π-institution based on $\mathbf{F}$. If $S^{\mathcal{I}}$ is adequate, then it is soluble.*

Proof: Suppose that $\mathcal{I}$ is family c-reflective and that $S^{\mathcal{I}}$ is adequate. We must show that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

The implication left-to-right is always satisfied by Proposition 832. For the converse, assume that $S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. Then, by the adequacy of $S^{\mathcal{I}}$, we get that $\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) \leq \Omega(T)$. Thus, by family c-reflectivity and Lemma 826, we conclude that $C(\phi) \leq T$, which gives $\phi \in T_\Sigma$. ∎

We finally show that a π-institution is truth equational if and only if it is family c-reflective and has an adequate Suszko core.

Truth Equationality	=	$S^{\mathcal{I}}$ Soluble
	=	$S^{\mathcal{I}}$ Defines Theory Families
	=	Family c-Reflectivity + $S^{\mathcal{I}}$ Adequate

Theorem 847 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is truth equational if and only if it is family c-reflective and has an adequate Suszko core.*

Proof: Suppose, first, that $\mathcal{I}$ is truth equational. Then it is family c-reflective by Theorem 829. Moreover, its Suszko core is soluble by Theorem 838 and, hence, by Corollary 844, its Suszko core is adequate.

Suppose, conversely, that $\mathcal{I}$ is family c-reflective with an adequate Suszko core. Then, by Proposition 846, its Suszko core is soluble and, therefore, by Theorem 838, $\mathcal{I}$ is truth equational. ∎

Finally, it is not difficult to see that, in some sense, truth equationality transfers from a π-institution to all $\mathcal{I}$-matrix families.

Theorem 848 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is truth equational, with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*, if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*, and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $T = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$*.*

Proof: Suppose $\mathcal{I}$ is truth equational, with witnessing transformations $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, whence, by hypothesis, $\alpha^{-1}(T) = \tau^\flat(\Omega(\alpha^{-1}(T)))$. Hence, by Proposition 24, $\alpha^{-1}(T) = \tau^\flat(\alpha^{-1}(\Omega^{\mathcal{A}}(T)))$. Therefore, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, we get

$$\begin{array}{lll} \alpha_\Sigma(\phi) \in T_{F(\Sigma)} & \text{iff} & \phi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)}) \\ & \text{iff} & \tau_\Sigma^\flat[\phi] \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T)) \\ & \text{iff} & \alpha(\tau_\Sigma^\flat[\phi]) \leq \Omega^{\mathcal{A}}(T) \\ & \text{iff} & \tau_{F(\Sigma)}^{\mathcal{A}}[\alpha_\Sigma(\phi)] \leq \Omega^{\mathcal{A}}(T). \quad (\langle F, \alpha\rangle \text{ surjective}) \end{array}$$

Taking again into account the surjectivity of $\langle F, \alpha\rangle$, we conclude that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $\phi \in T_\Sigma$ if and only if $\tau_\Sigma^{\mathcal{A}}[\phi] \leq \Omega^{\mathcal{A}}(T)$, i.e., $T = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$. ∎

11.8 Left Truth Equationality

In this section, we look at versions of truth equationality and c-reflectivity that can still be applied to general theory families but do not force the π-institutions to be systemic. In the next section we will also look at *system truth equationality*, i.e., truth equationality applied only to theory systems, and at *system c-reflectivity*. In this section we take the "leftist" approach, "left" having the meaning attributed to it in Chapter 3.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$.

Recall that $\mathcal{I}$ is **left c-reflective** if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,

$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \quad \text{implies} \quad \bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}.$$

Left c-reflectivity is not strong enough to imply systemicity. Moreover, left c-reflectivity is a property intermediate between family c-reflectivity and system c-reflectivity.

We say that the π-institution $\mathcal{I}$ is **left truth equational** if there exists $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with a single distinguished argument, such that, for every $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \le \Omega(T).$$

In that case, we call $\tau^\flat$ a **set of witnessing equations** (of/for the left truth equationality of $\mathcal{I}$).

If $\mathcal{I}$ is a left truth equational π-institution, with witnessing equations $\tau^\flat$, then $\tau^\flat(\Omega(T))$ is exactly equal to $\overleftarrow{T}$, i.e., the witnessing equations reflect theory families only "up to arrow".

Proposition 849 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is left truth equational, with witnessing equations $\tau^\flat$, then, for all $T \in \mathrm{ThFam}(\mathcal{I})$,*

$$\tau^\flat(\Omega(T)) = \overleftarrow{T}.$$

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll} \phi \in \tau^\flat_\Sigma(\Omega(T)) & \text{iff} & \tau^\flat_\Sigma[\phi] \le \Omega(T) \quad \text{(definition)} \\ & \text{iff} & \phi \in \overleftarrow{T}_\Sigma. \quad \text{(left truth equationality)} \end{array}$$

■

Proposition 849 has as an immediate consequence the important fact that left truth equationality implies left c-reflectivity.

Theorem 850 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is left truth equational, then it is left c-reflective.*

Proof: Suppose that $\mathcal{I}$ is left truth equational with witnessing equations $\tau^\flat$. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \le \Omega(T')$. Then

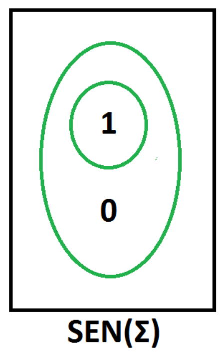
$$\begin{array}{lll} \bigcap_{T \in \mathcal{T}} \overleftarrow{T} & = & \bigcap_{T \in \mathcal{T}} \tau^\flat(\Omega(T)) \quad \text{(Proposition 849)} \\ & = & \tau^\flat(\bigcap_{T \in \mathcal{T}} \Omega(T)) \quad \text{(set theory)} \\ & \le & \tau^\flat(\Omega(T')) \quad \text{(hypothesis and Lemma 94)} \\ & = & \overleftarrow{T'}. \quad \text{(Proposition 849)} \end{array}$$

Thus, $\mathcal{I}$ is left c-reflective. ■

The following example shows that the inclusion of Theorem 850 is proper.

Example 851 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$*;*
- $N^\flat$ *is the trivial category of natural transformations consisting of the projections only.*



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\}, \{0, 1\}\}$*.*

$\mathcal{I}$ *has two theory families,* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, which are also theory systems. In other words,* $\overleftarrow{\mathrm{Thm}(\mathcal{I})} = \mathrm{Thm}(\mathcal{I})$ *and* $\overleftarrow{\mathrm{SEN}^\flat} = \mathrm{SEN}^\flat$*. Clearly,* $\mathrm{Thm}(\mathcal{I}) \leq \mathrm{SEN}^\flat$*. Moreover,* $\Omega(\mathrm{Thm}(\mathcal{I})) = \Delta^{\mathbf{F}}$ *and* $\Omega(\mathrm{SEN}^\flat) = \nabla^{\mathbf{F}}$*.* $\mathcal{I}$ *is clearly left c-reflective.*

$$\begin{array}{ccc} \mathrm{SEN}^\flat & \cdots\cdots\rightarrow & \nabla^{\mathbf{F}} \\ | & & | \\ \mathrm{Thm}(\mathcal{I}) & \cdots\cdots\rightarrow & \Delta^{\mathbf{F}} \end{array}$$

On the other hand, there does not exist $\tau^\flat \subseteq N^\flat$*, such that* $I^\flat$ *has the required properties to constitute a witnessing set of equations for the left truth equationality in* $\mathcal{I}$*. Any set consisting of projections only cannot satisfy the required condition since* $\tau^\flat(\Omega(T))$ *can only be* $\mathrm{SEN}^\flat$ *or* $\overline{\varnothing}$*.*

We provide, next, a characterization of left truth equationality in terms of the left solubility property of the left Suszko core of the π-institution. Then, we provide an exact description of those left c-reflective π-institutions which are left truth equational.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The **left Suszko core** of $\mathcal{I}$ is the collection

$$L^{\mathcal{I}} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}(\mathcal{I}))(\sigma^\flat[\overleftarrow{T}] \leq \widetilde{\Omega}^{\mathcal{I}}(T)\}.$$

There is an alternative way to define the left Suszko core of a π-institution, which may be also viewed as justifying the alternative terminology *system Suszko core* for it, which we state in the form of a property.

Proposition 852 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then*

$$L^{\mathcal{I}} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThSys}(\mathcal{I}))(\sigma^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T)\}.$$

Proof: Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution and set

$$M^{\mathcal{I}} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThSys}(\mathcal{I}))(\sigma^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T)\}.$$

Our goal is to show that $M^{\mathcal{I}} = L^{\mathcal{I}}$.

Suppose, first, that $\sigma^\flat \in L^{\mathcal{I}}$ and let $T \in \mathrm{ThSys}(\mathcal{I})$. Then, we have

$$\begin{array}{rcll} \sigma^\flat[T] & = & \sigma^\flat[\overleftarrow{T}] & (T \in \mathrm{ThSys}(\mathcal{I})) \\ & \leq & \widetilde{\Omega}^{\mathcal{I}}(T). & (\sigma^\flat \in L^{\mathcal{I}}) \end{array}$$

Therefore $\sigma^\flat \in M^{\mathcal{I}}$.

Suppose, conversely, that $\sigma^\flat \in M^{\mathcal{I}}$ and let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, we have

$$\begin{array}{rcll} \sigma^\flat[\overleftarrow{T}] & \leq & \widetilde{\Omega}^{\mathcal{I}}(\overleftarrow{T}) & (\sigma^\flat \in M^{\mathcal{I}} \text{ and } \overleftarrow{T} \in \mathrm{ThSys}(\mathcal{I})) \\ & \leq & \widetilde{\Omega}^{\mathcal{I}}(T). & (\overleftarrow{T} \leq T \text{ and monotonicity of } \widetilde{\Omega}^{\mathcal{I}}) \end{array}$$

We conclude that $\sigma^\flat \in L^{\mathcal{I}}$ and, therefore, $M^{\mathcal{I}} = L^{\mathcal{I}}$. ∎

Note that, since, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} \leq T$, we get that $S^{\mathcal{I}} \subseteq L^{\mathcal{I}}$, which implies that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $L^{\mathcal{I}}(\Omega(T)) \leq S^{\mathcal{I}}(\Omega(T))$.

Note, also, that for systemic π-institutions the left Suszko core and the Suszko core are identical.

The left Suszko core of a π-institution satisfies the following property relating to the arrow operator:

Proposition 853 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For every* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\overleftarrow{T} \leq L^{\mathcal{I}}(\Omega(T)).$$

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{rcll} \phi \in \overleftarrow{T}_\Sigma & \text{implies} & L^{\mathcal{I}}_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T) & (\text{by definition of } L^{\mathcal{I}}) \\ & \text{implies} & L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) & (\widetilde{\Omega}^{\mathcal{I}}(T) \leq \Omega(T)) \\ & \text{iff} & \phi \in L^{\mathcal{I}}(\Omega(T)). & (\text{by definition}) \end{array}$$

∎

It is possible, but not necessary, that the left Suszko core of a π-institution satisfies the reverse inclusion. We call this property left solubility.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the left Suszko core of $\mathcal{I}$ is **left soluble** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$L^{\mathcal{I}}(\Omega(T)) \leq \overleftarrow{T}.$$

In other words, $L^{\mathcal{I}}$ is left soluble if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) \quad \text{implies} \quad \phi \in \overleftarrow{T}_\Sigma.$$

We show that this property has an alternative characterization in terms of theory systems.

Proposition 854 *Let $\mathcal{I} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. The left Suszko core $L^{\mathcal{I}}$ of $\mathcal{I}$ is left soluble if and only if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $L^{\mathcal{I}}(\Omega(T)) = T$.*

Proof: For the "only if", assume that $L^{\mathcal{I}}$ is left soluble and let $T \in \mathrm{ThSys}(\mathcal{I})$. Then

$$\begin{array}{rcll} L^{\mathcal{I}}(\Omega(T)) & = & \overleftarrow{T} & (\text{Left Solubility of } L^{\mathcal{I}}) \\ & = & T. & (T \in \mathrm{ThSys}(\mathcal{I})) \end{array}$$

Conversely, assume that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $L^{\mathcal{I}}(\Omega(T)) = T$ and let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, we have

$$\begin{array}{rcll} L^{\mathcal{I}}(\Omega(T)) & \leq & L^{\mathcal{I}}(\Omega(\overleftarrow{T})) & (\text{Proposition 20 and Lemma 94}) \\ & = & \overleftarrow{T}. & (\text{by hypothesis}) \end{array}$$

Thus, $L^{\mathcal{I}}$ is left soluble. ■

Note that for systemic π-institutions, since the left Suszko core coincides with the Suszko core, left solubility of the left Suszko core coincides with the solubility of the Suszko core. These two properties are, however, different in general and, as the following proposition and example show, solubility of the Suszko core is a stronger property than left solubility of the left Suszko core.

Proposition 855 *Let $\mathcal{I} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If the Suszko core $S^{\mathcal{I}}$ of $\mathcal{I}$ is soluble, then the left Suszko core $L^{\mathcal{I}}$ of $\mathcal{I}$ is left soluble.*

Proof: Suppose that $S^{\mathcal{I}}$ is soluble, i.e., for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) \quad \text{implies} \quad \phi \in T_\Sigma.$$

Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. Then, since $S^{\mathcal{I}} \subseteq L^{\mathcal{I}}$, we get that $S^{\mathcal{I}}[\phi] \leq \Omega(T)$. Moreover, since $\Omega(T) \leq$

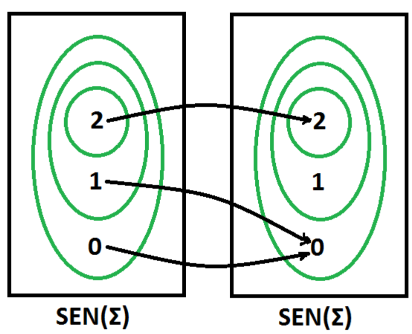
$\Omega(\overleftarrow{T})$, we get that $S^{\mathcal{I}}[\phi] \leq \Omega(\overleftarrow{T})$. Thus, by the solubility of $S^{\mathcal{I}}$, we get that $\phi \in \overleftarrow{T}_{\Sigma}$. We conclude that $L^{\mathcal{I}}$ is left soluble. ∎

The implication of Proposition 855 is proper in general.

Example 856 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$ *and* $\mathrm{SEN}^\flat(f) : \{0,1,2\} \to \{0,1,2\}$ *given by* $0 \mapsto 0$*,* $1 \mapsto 0$ *and* $2 \mapsto 2$*;*
- $N^\flat$ *is the category of natural transformations generated by the single unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\} \to \{0,1,2\}$ *be given, for all* $x \in \mathrm{SEN}^\flat(\Sigma)$*, by*

$$\sigma^\flat_\Sigma(x) = 2.$$

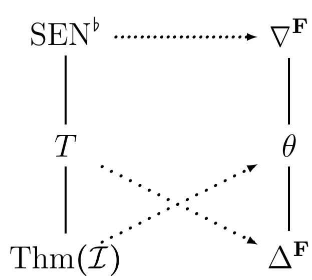


Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\}, \{1,2\}, \{0,1,2\}\}.$$

$\mathcal{I}$ *has three theory families* $\mathrm{Thm}(\mathcal{I})$*,* $T = \{\{1,2\}\}$ *and* $\mathrm{SEN}^\flat$*, but only two theory systems* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*. So it is not a systemic* π*-institution.*

The posets of theory families and associated Leibniz congruence systems are shown in the following figure (where $T \in \{\{1,2\}\}$ *and* $\theta = \{\{0,1\},\{2\}\}$*):*



Note that

$$S^{\mathcal{I}} = \{\iota \approx \iota, \sigma^\flat \approx \sigma^\flat\},$$

whereas

$$L^{\mathcal{I}} = \{\iota \approx \iota, \iota \approx \sigma^\flat, \sigma^\flat \approx \iota, \sigma^\flat \approx \sigma^\flat\}.$$

We show that $L^{\mathcal{I}}$ is left soluble, but that $S^{\mathcal{I}}$ is not soluble.

The left solubility of $L^{\mathcal{I}}$ can be seen by looking at the defining implication on a case-by-case basis. The case of the theory family $\mathrm{SEN}^\flat$ *is trivial as is the case for $\phi = 2$. For $\phi = 0$ or 1 and for the theory families T or $\mathrm{ThFam}(\mathcal{I})$, we have:*

- $L^{\mathcal{I}}_\Sigma[0] \le \Omega(T)$ *is false;*
- $L^{\mathcal{I}}_\Sigma[1] \le \Omega(T)$ *is false;*
- $L^{\mathcal{I}}_\Sigma[0] \le \Omega(\mathrm{Thm}(\mathcal{I}))$ *is false;*
- $L^{\mathcal{I}}_\Sigma[1] \le \Omega(\mathrm{Thm}(\mathcal{I}))$ *is false.*

So in every other case the defining condition is vacuously satisfied.

On the other hand, $S^{\mathcal{I}}_\Sigma[0] \le \Omega(T)$, but $0 \notin T_\Sigma$, which shows that $S^{\mathcal{I}}$ is not soluble.

It turns out that possession of left solubility by the left Suszko core intrinsically characterizes left truth equationality. We show, first, that the left Suszko core being left soluble is necessary for left truth equationality. To demonstrate this, observe, first, that, in case a π-institution is left truth equational, the witnessing equations form a subset of the left Suszko core.

Lemma 857 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is left truth equational, with witnessing equations $\tau^\flat \subseteq N^\flat$, then $\tau^\flat \subseteq L^{\mathcal{I}}$.*

Proof: By left truth equationality, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \le \Omega(T).$$

Thus, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}
\phi \in \overleftarrow{T}_\Sigma & \text{iff} & (\forall T \le T' \in \mathrm{ThFam}(\mathcal{I}))(\phi \in \overleftarrow{T'}_\Sigma) \\
& & \text{(by Lemma 1)} \\
& \text{iff} & (\forall T \le T' \in \mathrm{ThFam}(\mathcal{I}))(\tau^\flat_\Sigma[\phi] \le \Omega(T')) \\
& & \text{(left truth equationality; displayed formula above)} \\
& \text{iff} & \tau^\flat_\Sigma[\phi] \le \bigcap\{\Omega(T') : T \le T' \in \mathrm{ThFam}(\mathcal{I})\} \\
& & \text{(set theoretically)} \\
& \text{iff} & \tau^\flat_\Sigma[\phi] \le \widetilde{\Omega}^{\mathcal{I}}(T). \\
& & \text{(by definition of } \widetilde{\Omega}^{\mathcal{I}})
\end{array}$$

We conclude, by the definition of $L^{\mathcal{I}}$, that $\tau^\flat \subseteq L^{\mathcal{I}}$. ■

Now we prove the necessity of left solubility of the left Suszko core for left truth equationality.

Theorem 858 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is left truth equational, then* $L^{\mathcal{I}}$ *is left soluble.*

Proof: Suppose that $\mathcal{I}$ is left truth equational, with witnessing equations $\tau^\flat$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{llll} L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) & \text{implies} & \tau^\flat_\Sigma[\phi] \leq \Omega(T) & \text{(Lemma 857)} \\ & \text{iff} & \phi \in \overleftarrow{T}_\Sigma. & \text{(left truth equationality)} \end{array}$$

Thus, $L^{\mathcal{I}}$ is left soluble. ■

The reverse implication also holds and completes the promised characterization of left truth equationality in terms of the left Suszko core.

Theorem 859 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $L^{\mathcal{I}}$ *is left soluble, then* $\mathcal{I}$ *is left truth equational, with witnessing equations* $L^{\mathcal{I}}$*.*

Proof: It suffices to show that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{iff} \quad L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

The left-to-right implication is given in Proposition 853, whereas the converse is ensured by the postulated left solubility of $L^{\mathcal{I}}$. ■

Theorems 858 and 859 provide the promised characterization of left truth equationality in terms of the left solubility of the left Suszko core.

$$\mathcal{I} \text{ is Left Truth Equational} \longleftrightarrow L^{\mathcal{I}} \text{ is Left Soluble.}$$

Theorem 860 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is left truth equational if and only if* $L^{\mathcal{I}}$ *is left soluble.*

Proof: Theorem 858 gives the "only if" and the "if" is by Theorem 859. ■

If $\mathcal{I}$ is left truth equational, then the left Suszko core defines theory families in $\mathcal{I}$ "up to arrow" in terms of their Leibniz congruence systems. This proposition may be viewed as a special case of Proposition 849, since $L^{\mathcal{I}}$ forms a maximal set of witnessing equations of the left truth equationality of $\mathcal{I}$.

Proposition 861 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $L^{\mathcal{I}}$ *is left soluble, then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\overleftarrow{T} = L^{\mathcal{I}}(\Omega(T)).$$

Proof: If $L^{\mathcal{I}}$ is left soluble, then, by Theorem 859, $L^{\mathcal{I}}$ forms a set of witnessing equations for the left truth equationality of $\mathcal{I}$. Therefore, by Proposition 849, we get that, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} = L^{\mathcal{I}}(\Omega(T))$. ■

This property may be restated as another characterization of left truth equationality. We say that $L^{\mathcal{I}}$ **defines theory families up to arrow** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} = L^{\mathcal{I}}(\Omega(T))$. Then we have:

$\mathcal{I}$ is Left Truth Equational
$\longleftrightarrow L^{\mathcal{I}}$ Defines Theory Families Up to Arrow.

Theorem 862 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is left truth equational if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\overleftarrow{T} = L^{\mathcal{I}}(\Omega(T)).$$

Proof: If $\mathcal{I}$ is left truth equational, then, by Theorem 860, $L^{\mathcal{I}}$ is left soluble. Thus, by Proposition 861, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} = L^{\mathcal{I}}(\Omega(T))$.

Conversely, if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} = L^{\mathcal{I}}(\Omega(T))$, then, $L^{\mathcal{I}}$ is left soluble. Thus, again by Theorem 860, $L^{\mathcal{I}}$ is a set of witnessing equations and $\mathcal{I}$ is left truth equational. ■

We finally show that the property that separates left complete reflectivity from left truth equationality is a property of the left Suszko core, analogous to the adequacy property introduced previously for the Suszko core, that we call *left adequacy*. Similarly to adequacy, informally speaking, this property ensures that the left Suszko core is rich enough to define Suszko congruence systems in terms of the Leibniz congruence systems of theory families that it selects via inclusion.

The following relationship connects the left Suszko core with both Leibniz and Suszko congruence systems.

Recall that given a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, based on an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$, and a sentence family $T \in \mathrm{SenFam}(\mathcal{I})$, we denote by $\overrightarrow{T}$ the least sentence system of $\mathcal{I}$ that includes T. Because of the structurality of C, it is not difficult to see that $C(\overrightarrow{T}) = \overrightarrow{C(T)}$, for any sentence family T of $\mathcal{I}$.

Proposition 863 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)\} \le \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\begin{array}{lll}\phi \in \overleftarrow{T}_\Sigma & \text{implies} & L^{\mathcal{I}}_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T) \quad \text{(definition of the left Suszko core)}\\ & \text{implies} & L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T). \quad (\widetilde{\Omega}^{\mathcal{I}}(T) \leq \Omega(T))\end{array}$$

Therefore, we have

$$\begin{array}{lll}\bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} & \leq & \bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T)\}\\ & \leq & \bigcap\{\Omega(T) : \phi \in \overleftarrow{T}_\Sigma\}\\ & = & \bigcap\{\Omega(T) : \overrightarrow{\phi} \leq T\}\\ & = & \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})).\end{array}$$

■

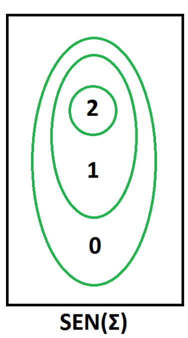
We provide an example, next, that shows that the inclusion proven in Proposition 863 is proper, in general. I.e., there exist π-institutions $\mathcal{I}$ in which, for some signature Σ and some Σ-sentence ϕ,

$$\bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \lneq \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})).$$

Of course, it is convenient that in a systemic π-institution $L^{\mathcal{I}} = S^{\mathcal{I}}$ and, moreover, for every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $C(\overrightarrow{\phi}) = \overrightarrow{C(\phi)} = C(\phi)$, whence Example 842, used to prove proper inclusion following Proposition 841, may be reused.

Example 864 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the trivial category of natural transformations, consisting of the projections only.*



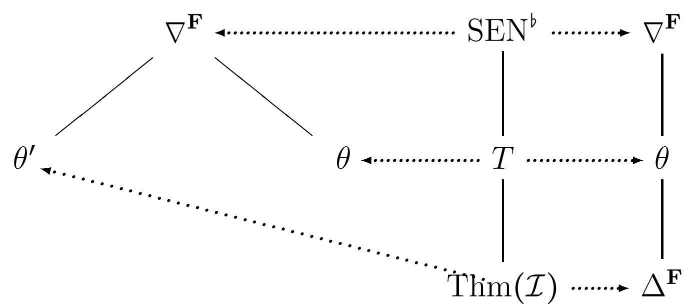
Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the* π*-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

$\mathcal{I}$ has three theory families $\mathrm{Thm}(\mathcal{I})$, $T = \{\{1,2\}\}$ *and* $\mathrm{SEN}^\flat$, *all of which are theory systems. So $\mathcal{I}$ is systemic.*

We have $L^{\mathcal{I}} = S^{\mathcal{I}} = \{\iota \approx \iota\}$. Furthermore, the structure of the posets of Leibniz congruence systems and of Suszko congruence systems are provided in the left and right sides, respectively, of the following diagram, where

$$T = \{\{1,2\}\}, \quad \theta = \{\{0\},\{1,2\}\}, \quad \theta' = \{\{0,1\},\{2\}\}.$$



Taking this into account, it is not difficult to see that

$$\bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[1] \leq \Omega(T)\} = \Delta^{\mathbf{F}} \lneq \theta = \widetilde{\Omega}^{\mathcal{I}}(C(1)) = \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{1})).$$

We also give an example of a π-institution $\mathcal{I}$ whose left Suszko core $L^{\mathcal{I}}$ is such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) = \bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.$$

This again takes after Example 843, since the π-institution used there was systemic.

Example 865 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be the algebraic system determined as follows:*

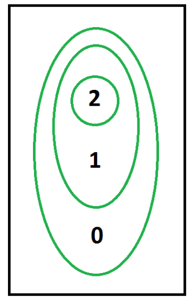
- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$*;*
- $N^\flat$ *is the category of natural transformations generated by the single unary natural transformation* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0,1,2\} \to \{0,1,2\}$ *be given, for all* $x \in \mathrm{SEN}^\flat(\Sigma)$*, by*

$$\sigma^\flat_\Sigma(x) = 2.$$

Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be the π-institution determined by*

$$\mathcal{C}_\Sigma = \{\{2\},\{1,2\},\{0,1,2\}\}.$$

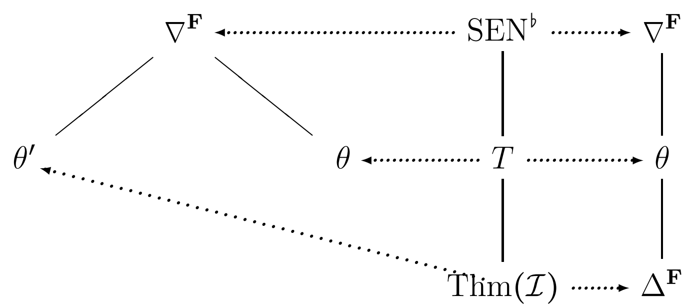
$\mathcal{I}$ has three theory families $\mathrm{Thm}(\mathcal{I})$, $T = \{\{1,2\}\}$ *and* $\mathrm{SEN}^\flat$, *all of which are theory systems. So $\mathcal{I}$ is systemic.*



SEN(Σ)

Note that $L^{\mathcal{I}} = S^{\mathcal{I}} = \{\iota \approx \iota, \iota \approx \sigma^\flat, \sigma^\flat \approx \iota, \sigma^\flat \approx \sigma^\flat\}$*. Note, also, the structure of the posets of Leibniz congruence systems and of Suszko congruence systems, that are provided in the left and right sides, respectively, of the following diagram, where*

$$T = \{\{1,2\}\}, \quad \theta = \{\{0\},\{1,2\}\}, \quad \theta' = \{\{0,1\},\{2\}\}.$$



Now we can check:

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{0})) = \widetilde{\Omega}^{\mathcal{I}}(C(0)) & = & \nabla^{\mathbf{F}} = \Omega(\mathrm{SEN}^\flat) \\ & = & \bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[0] \le \Omega(T)\}; \\ \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{1})) = \widetilde{\Omega}^{\mathcal{I}}(C(1)) & = & \theta = \Omega(\mathrm{SEN}^\flat) \cap \Omega(T) \\ & = & \bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[1] \le \Omega(T)\}; \\ \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{2})) =^{\mathcal{I}} \widetilde{\Omega}(C(2)) & = & \Delta^{\mathbf{F}} = \Omega(\mathrm{SEN}^\flat) \cap \Omega(T) \cap \Omega(\mathrm{Thm}(\mathcal{I})) \\ & = & \bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[2] \le \Omega(T)\}.\end{array}$$

We have seen, therefore, through examples, that it is possible, but not necessary, that the left Suszko core of a π-institution satisfies, for every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, the reverse inclusion of that given in Proposition 863:

$$\widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) \le \bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)\}.$$

Intuitively speaking, this means that the left Suszko core $L^{\mathcal{I}}$ is rich enough to allow, for every signature Σ and every Σ-sentence ϕ, the determination of those theory families whose Leibniz congruence systems form a covering of the Suszko congruence system of $C(\overrightarrow{\phi})$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the left Suszko core $L^\mathcal{I}$ of $\mathcal{I}$ is **left adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widetilde{\Omega}^\mathcal{I}(C(\overrightarrow{\phi})) = \bigcap\{\Omega(T) : L^\mathcal{I}_\Sigma[\phi] \leq \Omega(T)\}.$$

Based on our preceding work, it is not difficult to see that, if $L^\mathcal{I}$ is left soluble, then it is left adequate.

Corollary 866 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $L^\mathcal{I}$ *is left soluble, then it is left adequate.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

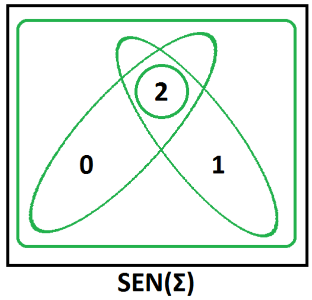
$$\begin{array}{rcll}\widetilde{\Omega}^\mathcal{I}(C(\overrightarrow{\phi})) & = & \bigcap\{\Omega(T) : \overrightarrow{\phi} \leq T\} & (\text{definition of } \widetilde{\Omega}^\mathcal{I}(C(\overrightarrow{\phi}))) \\ & = & \bigcap\{\Omega(T) : \phi \in \overleftarrow{T}_\Sigma\} & (\text{definition of } \overrightarrow{\phi} \text{ and } \overleftarrow{T}) \\ & = & \bigcap\{\Omega(T) : L^\mathcal{I}_\Sigma[\phi] \leq \Omega(T)\}. & \\ & & (\text{left solubility of } L^\mathcal{I} \text{ and Proposition 861}) & \end{array}$$

We conclude that $L^\mathcal{I}$ is left adequate. ■

Here is an example of a π-institution $\mathcal{I}$, with a left adequate but not left soluble left Suszko core.

Example 867 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the category of natural transformations generated by two unary natural transformations:*
  - $\rho^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined by letting* $\rho^\flat_\Sigma : \{0, 1, 2\} \to \{0, 1, 2\}$ *be given, for all* $x \in \mathrm{SEN}^\flat(\Sigma)$*, by*
  $$\rho^\flat_\Sigma(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x = 1 \\ 2, & \text{if } x = 2 \end{cases}.$$
  - $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *defined by letting* $\sigma^\flat_\Sigma : \{0, 1, 2\} \to \{0, 1, 2\}$ *be given, for all* $x \in \mathrm{SEN}^\flat(\Sigma)$*, by*
  $$\sigma^\flat_\Sigma(x) = \begin{cases} 2, & \text{if } x = 0 \\ 1, & \text{if } x = 1 \\ 2, & \text{if } x = 2 \end{cases}.$$

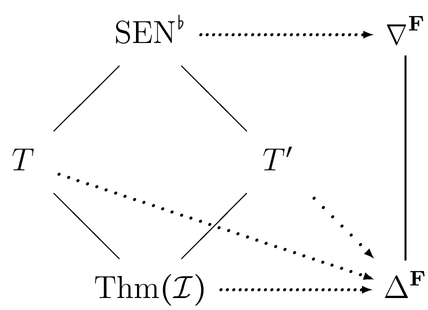


Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be the π-institution determined by

$$\mathcal{C}_\Sigma = \{\{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}.$$

$\mathcal{I}$ has four theory families $\mathrm{Thm}(\mathcal{I})$, $T = \{\{0,2\}\}$, $T' = \{\{1,2\}\}$ and $\mathrm{SEN}^\flat$, all of which are theory systems. So it is a systemic π-institution.

The posets of theory families (center), associated Leibniz congruence systems (right) and associate Suszko congruence systems (right, identical with Leibniz congruence systems, since $\mathcal{I}$ is protoalgebraic) are shown in the following figure:



Note that $L^{\mathcal{I}} = \{\iota \approx \iota\}$. We show that $L^{\mathcal{I}}$ is left adequate, but not left soluble. We are omitting arrows from the notation is the following verifications since, as $\mathcal{I}$ is based on $\mathbf{F}$ with trivial $\mathbf{Sign}^\flat$, they play no role in this context.

For left adequacy, we have

$$\begin{aligned}
\widetilde{\Omega}^{\mathcal{I}}(C(0)) &= \widetilde{\Omega}^{\mathcal{I}}(T) = \Delta^{\mathbf{F}} = \bigcap\{\Omega(T'') : T'' \in \mathrm{ThFam}(\mathcal{I})\};\\
\widetilde{\Omega}^{\mathcal{I}}(C(1)) &= \widetilde{\Omega}^{\mathcal{I}}(T') = \Delta^{\mathbf{F}} = \bigcap\{\Omega(T'') : T'' \in \mathrm{ThFam}(\mathcal{I})\};\\
\widetilde{\Omega}^{\mathcal{I}}(C(2)) &= \widetilde{\Omega}^{\mathcal{I}}(\mathrm{Thm}(\mathcal{I})) = \Delta^{\mathbf{F}} = \bigcap\{\Omega(T'') : T'' \in \mathrm{ThFam}(\mathcal{I})\}.
\end{aligned}$$

As for left solubility, note that $L^{\mathcal{I}}_\Sigma[0] \leq \Omega(T')$, but that $0 \notin T'_\Sigma$. Thus, $L^{\mathcal{I}}$ is not left soluble.

In the opposite direction, and on the positive side, in a *left c-reflective* π-institution $\mathcal{I}$, if the left Suszko core is left adequate, then it is also left soluble.

First, we note that the following variant of Lemma 826, giving an alternative characterization of left c-reflectivity in terms of both the Suszko and the Leibniz operators, holds.

Lemma 868 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is left c-reflective if and only if, for every* $T, T' \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\widetilde{\Omega}^{\mathcal{I}}(T) \leq \Omega(T') \quad \textit{implies} \quad \overleftarrow{T} \leq \overleftarrow{T'}.$$

Proof: Assume, first, that $\mathcal{I}$ is left c-reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{\Omega}^{\mathcal{I}}(T) \leq \Omega(T')$. By the definition of the Suszko operator,

$$\bigcap\{\Omega(T'') : T \leq T'' \in \mathrm{ThFam}(\mathcal{I})\} \leq \Omega(T').$$

Using left c-reflectivity, we get that

$$\bigcap\{\overleftarrow{T''} : T \leq T'' \in \mathrm{ThFam}(\mathcal{I})\} \leq \overleftarrow{T'}.$$

Hence, using Lemma 1, $\overleftarrow{T} \leq \overleftarrow{T'}$, as required.

Suppose, conversely, that, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\Omega}^{\mathcal{I}}(T) \leq \Omega(T')$ implies $\overleftarrow{T} \leq \overleftarrow{T'}$. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. Then we have

$$\begin{array}{rcll}
\widetilde{\Omega}^{\mathcal{I}}(\bigcap_{T \in \mathcal{T}} T) & \leq & \bigcap_{T \in \mathcal{T}} \widetilde{\Omega}^{\mathcal{I}}(T) & (\text{monotonicity of } \widetilde{\Omega}^{\mathcal{I}}) \\
& \leq & \bigcap_{T \in \mathcal{T}} \Omega(T) & (\text{since } \widetilde{\Omega}^{\mathcal{I}}(T) \leq \Omega(T)) \\
& \leq & \Omega(T'). & (\text{by hypothesis})
\end{array}$$

Using the hypothesis, we conclude that $\overleftarrow{\bigcap_{T \in \mathcal{T}} T} \leq \overleftarrow{T'}$. Thus, by Lemma 3, $\bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$. Therefore, $\mathcal{I}$ is left c-reflective. ■

And now for the promised result showing that in a left c-reflective π-institution $\mathcal{I}$, if the left Suszko core is left adequate, then it is also left soluble.

Proposition 869 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a left c-reflective* π*-institution based on* $\mathbf{F}$*. If* $L^{\mathcal{I}}$ *is left adequate, then it is left soluble.*

Proof: Suppose that $\mathcal{I}$ is left c-reflective and that $L^{\mathcal{I}}$ is left adequate. We must show that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{iff} \quad L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

The implication left-to-right is always satisfied by Proposition 853. For the converse, assume that $L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. Then, by the left adequacy of $L^{\mathcal{I}}$, we get that $\widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) \leq \Omega(T)$. Thus, by left c-reflectivity and Lemma 868, we conclude that $\overleftarrow{C(\overrightarrow{\phi})} \leq \overleftarrow{T}$, which gives $\phi \in \overleftarrow{T}_\Sigma$. ■

We finally show that a π-institution is left truth equational if and only if it is left c-reflective and its left Suszko core is left adequate.

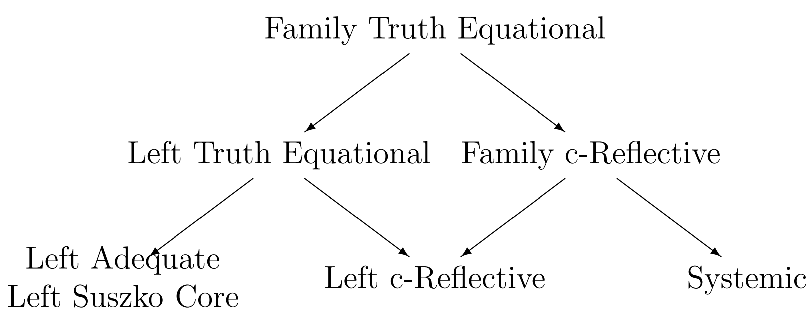
Left Truth Equationality	=	$L^{\mathcal{I}}$ Left Soluble
	=	$L^{\mathcal{I}}$ Defines Theory Families Up to Arrow
	=	Left c-Reflectivity + $L^{\mathcal{I}}$ Left Adequate

Theorem 870 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is left truth equational if and only if it is left c-reflective and has a left adequate left Suszko core.*

Proof: Suppose, first, that $\mathcal{I}$ is left truth equational. Then it is left c-reflective by Theorem 850. Moreover, its left Suszko core is left soluble by Theorem 860 and, hence, by Corollary 866, its left Suszko core is left adequate.

Suppose, conversely, that $\mathcal{I}$ is left c-reflective with a left adequate left Suszko core. Then, by Proposition 869, its left Suszko core is left soluble and, therefore, by Theorem 860, $\mathcal{I}$ is left truth equational. ■

We have now established the following hierarchy of properties:



11.9 System Truth Equationality

In this section, we look at system truth equationality and system c-reflectivity, which can also be applied to a π-institution without forcing it to be systemic. Recall that, by Proposition **??**, system c-reflectivity is a weaker property than left c-reflectivity, i.e., left c-reflectivity, which was used in the characterization of left truth equationality in the preceding section, implies system c-reflectivity.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

Recall that $\mathcal{I}$ is **system c-reflective** if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$,

$$\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T') \quad \text{implies} \quad \bigcap_{T\in\mathcal{T}} T \leq T'.$$

Since left c-reflectivity is not strong enough to imply systemicity, system c-reflectivity has, a fortiori, the same property.

We say that the π-institution $\mathcal{I}$ is **system truth equational** if there exists $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ having a single distinguished argument, such that, for every $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

In that case, we call $\tau^\flat$ a **set of witnessing equations** (of/for the system truth equationality of $\mathcal{I}$).

If $\mathcal{I}$ is a system truth equational π-institution, with witnessing equations $\tau^\flat$, then, for $T \in \mathrm{ThSys}(\mathcal{I})$, $\tau^\flat(\Omega(T))$ is exactly equal to T, i.e., the witnessing equations reflect theory systems.

Proposition 871 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is system truth equational, with witnessing equations* $\tau^\flat$*, then, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,*

$$\tau^\flat(\Omega(T)) = T.$$

Proof: Let $T \in \mathrm{ThSys}(\mathcal{I})$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{llll} \phi \in \tau^\flat_\Sigma(\Omega(T)) & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(T) & \text{(definition)} \\ & \text{iff} & \phi \in T_\Sigma. & \text{(system truth equationality)} \end{array}$$

■

Proposition 871 has as an immediate consequence the important fact that system truth equationality implies system c-reflectivity.

Theorem 872 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is system truth equational, then it is system c-reflective.*

Proof: Suppose that $\mathcal{I}$ is system truth equational with witnessing equations $\tau^\flat$. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. Then

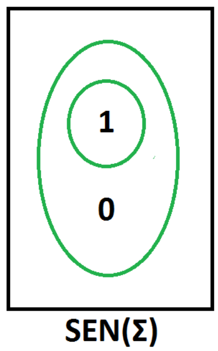
$$\begin{array}{lll} \bigcap_{T\in\mathcal{T}} T & = & \bigcap_{T\in\mathcal{T}} \tau^\flat(\Omega(T)) \quad \text{(Proposition 871)} \\ & = & \tau^\flat(\bigcap_{T\in\mathcal{T}} \Omega(T)) \quad \text{(set theory)} \\ & \leq & \tau^\flat(\Omega(T')) \quad \text{(hypothesis and Lemma 94)} \\ & = & T'. \quad \text{(Proposition 871)} \end{array}$$

Thus, $\mathcal{I}$ is system c-reflective. ■

The following example shows that the inclusion of Theorem 872 is proper.

Example 873 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$*;*
- $N^\flat$ *is the trivial category of natural transformations.*



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\}, \{0, 1\}\}$*.*

$\mathcal{I}$ *has two theory families,* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, which are also theory systems. Clearly,* $\mathrm{Thm}(\mathcal{I}) \leq \mathrm{SEN}^\flat$*. Moreover,* $\Omega(\mathrm{Thm}(\mathcal{I})) = \Delta^{\mathbf{F}}$ *and* $\Omega(\mathrm{SEN}^\flat) = \nabla^{\mathbf{F}}$*.* $\mathcal{I}$ *is clearly system c-reflective.*

$$\begin{array}{ccc}
\mathrm{SEN}^\flat & \cdots\cdots\rightarrow & \nabla^{\mathbf{F}} \\
| & & | \\
\mathrm{Thm}(\mathcal{I}) & \cdots\cdots\rightarrow & \Delta^{\mathbf{F}}
\end{array}$$

On the other hand, there does not exist $\tau^\flat \subseteq N^\flat$*, such that* $I^\flat$ *has the required properties to constitute a witnessing set of equations for the system truth equationality in* $\mathcal{I}$*. Any set consisting of projections only cannot satisfy the required condition since* $\tau^\flat(\Omega(T))$ *can only be* $\mathrm{SEN}^\flat$ *or* $\overline{\varnothing}$*.*

We provide, next, a characterization of system truth equationality in terms of the solubility property of the system core of the π-institution. Then, we provide an exact description of those system c-reflective π-institutions which are system truth equational.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. First, for $T \in \mathrm{ThSys}(\mathcal{I})$, we introduce the notation

$$\widehat{\Omega}^{\mathcal{I}}(T) = \bigcap \{\Omega(T') : T \leq T' \in \mathrm{ThSys}(\mathcal{I})\}.$$

We now define the **system core** of $\mathcal{I}$ to be the collection

$$Z^{\mathcal{I}} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThSys}(\mathcal{I}))(\sigma^\flat[T] \leq \widehat{\Omega}^{\mathcal{I}}(T)\}.$$

The following proposition clarifies the relation between the Suszko core, the left Suszko core and the system core of a π-institution $\mathcal{I}$.

Proposition 874 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $S^{\mathcal{I}} \subseteq L^{\mathcal{I}} \subseteq Z^{\mathcal{I}}$*;*

(b) For every relation family θ *on* $\mathbf{F}$*,* $Z^{\mathcal{I}}(\theta) \leq L^{\mathcal{I}}(\theta) \leq S^{\mathcal{I}}(\theta)$*.*

Proof: For Part (a), $S^{\mathcal{I}} \subseteq L^{\mathcal{I}}$ has been shown after Proposition 852. For the second inclusion, assume that $\sigma^\flat \in L^{\mathcal{I}}$ and let $T \in \mathrm{ThSys}(\mathcal{I})$. Then we have

$$\begin{array}{rcll}\sigma^\flat[T] & = & \sigma^\flat[\overleftarrow{T}] & (T \in \mathrm{ThSys}(\mathcal{I}))\\ & \leq & \widetilde{\Omega}^{\mathcal{I}}(T) & (\sigma^\flat \in L^{\mathcal{I}})\\ & \leq & \widehat{\Omega}^{\mathcal{I}}(T). & (\widetilde{\Omega}^{\mathcal{I}}(T) \leq \widehat{\Omega}^{\mathcal{I}}(T))\end{array}$$

Thus $\sigma^\flat \in Z^\flat$ and $L^{\mathcal{I}} \subseteq Z^{\mathcal{I}}$. Part (b) follows form Part (a) and the relevant definitions. ■

The system core of a π-institution satisfies the following property related to the Leibniz congruence systems of the theory systems of the π-institution:

Proposition 875 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For every* $T \in \mathrm{ThSys}(\mathcal{I})$*,*

$$T \leq Z^{\mathcal{I}}(\Omega(T)).$$

Proof: Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{rlll}\phi \in T_\Sigma & \text{implies} & Z^{\mathcal{I}}_\Sigma[\phi] \leq \widehat{\Omega}^{\mathcal{I}}(T) & (\text{by definition of } Z^{\mathcal{I}})\\ & \text{implies} & Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) & (\widehat{\Omega}^{\mathcal{I}}(T) \leq \Omega(T))\\ & \text{iff} & \phi \in Z^{\mathcal{I}}(\Omega(T)). & (\text{by definition})\end{array}$$

■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the system core $Z^{\mathcal{I}}$ of $\mathcal{I}$ is **soluble** if the converse inclusion to that proven in Proposition 875 holds, i.e., if, for all $T \in \mathrm{ThSys}(\mathcal{I})$

$$Z^{\mathcal{I}}(\Omega(T)) \leq T.$$

Equivalently, $Z^{\mathcal{I}}$ is soluble if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) \quad \text{implies} \quad \phi \in T_\Sigma.$$

We show that left solubility of the left Suszko core implies solubility of the system core of a π-institution.

Proposition 876 *Let $\mathcal{I} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If the left Suszko core $L^{\mathcal{I}}$ of $\mathcal{I}$ is left soluble, then the system core $Z^{\mathcal{I}}$ of $\mathcal{I}$ is soluble.*

Proof: Suppose that $L^{\mathcal{I}}$ is left soluble and let $T \in \mathrm{ThSys}(\mathcal{I})$. Then we have

$$\begin{array}{lll} Z^{\mathcal{I}}(\Omega(T)) & \leq & L^{\mathcal{I}}(\Omega(T)) \quad \text{(Proposition 874)} \\ & = & T. \quad \text{(hypothesis and Proposition 854)} \end{array}$$

Therefore, $Z^{\mathcal{I}}$ is soluble. ■

It turns out that the property of solubility of the system core intrinsically characterizes system truth equationality. We show, first, that the system core being soluble is necessary for system truth equationality. Observe that, in case a π-institution is system truth equational, the witnessing equations form a subset of the system core.

Lemma 877 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is system truth equational, with witnessing equations $\tau^\flat \subseteq N^\flat$, then $\tau^\flat \subseteq Z^{\mathcal{I}}$.*

Proof: By system truth equationality, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

Thus, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll} \phi \in T_\Sigma & \text{iff} & (\forall T \leq T' \in \mathrm{ThSys}(\mathcal{I}))(\phi \in T'_\Sigma) \\ & \text{iff} & (\forall T \leq T' \in \mathrm{ThSys}(\mathcal{I}))(\tau^\flat_\Sigma[\phi] \leq \Omega(T')) \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \bigcap\{\Omega(T') : T \leq T' \in \mathrm{ThSys}(\mathcal{I})\} \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \widehat{\Omega}^{\mathcal{I}}(T). \end{array}$$

We conclude, by the definition of $Z^{\mathcal{I}}$, that $\tau^\flat \subseteq Z^{\mathcal{I}}$. ■

Now we prove the necessity of the solubility of the system core for system truth equationality.

Theorem 878 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is system truth equational, then $Z^{\mathcal{I}}$ is soluble.*

Proof: Suppose that $\mathcal{I}$ is system truth equational, with witnessing equations $\tau^\flat$. Then, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll} Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) & \text{implies} & \tau^\flat_\Sigma[\phi] \leq \Omega(T) \quad \text{(Lemma 877)} \\ & \text{iff} & \phi \in T_\Sigma. \quad \text{(system truth equationality)} \end{array}$$

Thus, $Z^{\mathcal{I}}$ is soluble. ■

The reverse implication completes the promised characterization of system truth equationality in terms of the system core.

Theorem 879 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $Z^{\mathcal{I}}$ *is soluble, then* $\mathcal{I}$ *is system truth equational, with witnessing equations* $Z^{\mathcal{I}}$*.*

Proof: It suffices to show that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

The left-to-right implication is given in Proposition 875, whereas the converse is ensured by the postulated solubility of $Z^{\mathcal{I}}$. ■

Theorems 878 and 879 provide the promised characterization of system truth equationality in terms of the solubility of the system core.

$$\mathcal{I} \text{ is System Truth Equational} \longleftrightarrow Z^{\mathcal{I}} \text{ is Soluble.}$$

Theorem 880 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is system truth equational if and only if* $Z^{\mathcal{I}}$ *is soluble.*

Proof: Theorem 878 gives the "only if" and the "if" is by Theorem 879. ■

If $\mathcal{I}$ is system truth equational, then the system core defines theory systems in $\mathcal{I}$ in terms of their Leibniz congruence systems. This proposition may be viewed as a special case of Proposition 871, since $Z^{\mathcal{I}}$ forms a maximal set of witnessing equations of the system truth equationality of $\mathcal{I}$.

Proposition 881 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $Z^{\mathcal{I}}$ *is soluble, then, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,*

$$T = Z^{\mathcal{I}}(\Omega(T)).$$

Proof: If $Z^{\mathcal{I}}$ is soluble, then, by Theorem 879, $Z^{\mathcal{I}}$ forms a set of witnessing equations for the system truth equationality of $\mathcal{I}$. Therefore, by Proposition 871, we get that, for every $T \in \mathrm{ThSys}(\mathcal{I})$, $T = Z^{\mathcal{I}}(\Omega(T))$. ■

This property may be restated as another characterization of system truth equationality. We say that $Z^{\mathcal{I}}$ **defines theory systems** if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $T = Z^{\mathcal{I}}(\Omega(T))$. Then we have:

$$\mathcal{I} \text{ is System Truth Equational} \longleftrightarrow Z^{\mathcal{I}} \text{ Defines Theory Systems.}$$

Theorem 882 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is system truth equational if and only if, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,*

$$T = Z^{\mathcal{I}}(\Omega(T)).$$

Proof: If $\mathcal{I}$ is system truth equational, then, by Theorem 990, $Z^{\mathcal{I}}$ is soluble. Thus, by Proposition 881, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $T = Z^{\mathcal{I}}(\Omega(T))$.

Conversely, if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $T = Z^{\mathcal{I}}(\Omega(T))$, then, $Z^{\mathcal{I}}$ is soluble. Thus, again by Theorem 990, $Z^{\mathcal{I}}$ is a set of witnessing equations and $\mathcal{I}$ is system truth equational. ∎

We finally show that the property that separates system complete reflectivity from system truth equationality is a property of the system core that we call adequacy. In analogy to the adequacy of the Suszko core and to the left adequacy of the left Suszko core, this property ensures that the system core is rich enough to define the congruence system $\widehat{\Omega}^{\mathcal{I}}(T)$ of a theory system T in terms of the Leibniz congruence systems of collections of theory systems that it selects via inclusion.

Recall, once more, that given a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$, based on an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$, and a sentence family $T \in \mathrm{SenFam}(\mathcal{I})$, we denote by $\overrightarrow{T}$ the least sentence system of $\mathcal{I}$ that includes T (see Proposition 2).

Proposition 883 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a π-institution based on* $\mathbf{F}$*. For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \leq \widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $T \in \mathrm{ThSys}(\mathcal{I})$,

$$\begin{array}{llll} \phi \in T_\Sigma & \text{implies} & Z^{\mathcal{I}}_\Sigma[\phi] \leq \widehat{\Omega}^{\mathcal{I}}(T) & \text{(definition of the system core)} \\ & \text{implies} & Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T). & (\widehat{\Omega}^{\mathcal{I}}(T) \leq \Omega(T)) \end{array}$$

Therefore, we have

$$\begin{aligned} &\bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \\ &\quad \leq \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } Z^{\mathcal{I}}_\Sigma[\phi] \leq \widehat{\Omega}^{\mathcal{I}}(T)\} \\ &\quad \leq \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \phi \in T_\Sigma\} \\ &\quad = \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \overrightarrow{\phi} \leq T\} \\ &\quad = \widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})). \end{aligned}$$

Therefore, the displayed inclusion always holds. ∎

It is possible, but not necessary, that the system core of a π-institution satisfies, for every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, the reverse inclusion of that given in Proposition 883:

$$\widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) \leq \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.$$

Intuitively speaking, this means that the system core $Z^{\mathcal{I}}$ is rich enough to allow, for every Σ-sentence ϕ, the determination of those theory systems whose Leibniz congruence systems form a covering of the congruence system $\widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi}))$ associated with $C(\overrightarrow{\phi})$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the system core $Z^{\mathcal{I}}$ of $\mathcal{I}$ is **adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) = \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.$$

Based on our preceding work, it is not difficult to see that, if $Z^{\mathcal{I}}$ is soluble, then it is adequate.

Corollary 884 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $Z^{\mathcal{I}}$ *is soluble, then it is adequate.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rcl}
\widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) & = & \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \overrightarrow{\phi} \leq T\} \\
& & (\text{definition of } \widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi}))) \\
& = & \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \phi \in T\} \\
& & (T \in \mathrm{ThSys}(\mathcal{I})) \\
& = & \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}. \\
& & (\text{solubility of } Z^{\mathcal{I}} \text{ and Proposition 881})
\end{array}$$

We conclude that $Z^{\mathcal{I}}$ is adequate. ■

As a partial converse, in a *system c-reflective* π-institution $\mathcal{I}$, if the system core is adequate, then it is also soluble.

First, we prove the following variant of Lemma 826, giving an alternative characterization of system c-reflectivity in terms of both $\widehat{\Omega}^{\mathcal{I}}$ and the Leibniz operator.

Lemma 885 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is system c-reflective if and only if, for every* $T, T' \in \mathrm{ThSys}(\mathcal{I})$*,*

$$\widehat{\Omega}^{\mathcal{I}}(T) \leq \Omega(T') \quad \textit{implies} \quad T \leq T'.$$

Proof: Assume, first, that $\mathcal{I}$ is system c-reflective and let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\widehat{\Omega}^{\mathcal{I}}(T) \leq \Omega(T')$. By the definition of the hat operator,

$$\bigcap\{\Omega(T'') : T \leq T'' \in \mathrm{ThSys}(\mathcal{I})\} \leq \Omega(T').$$

Using system c-reflectivity, we get that

$$\bigcap\{T'' : T \leq T'' \in \mathrm{ThSys}(\mathcal{I})\} \leq T'.$$

Hence, we conclude $T \leq T'$, as required.

Suppose, conversely, that, for all $T, T' \in \mathrm{ThSys}(\mathcal{I})$, $\widetilde{\Omega}^{\mathcal{I}}(T) \leq \Omega(T')$ implies $T \leq T'$. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. Then we have

$$\begin{array}{lll} \widetilde{\Omega}^{\mathcal{I}}(\bigcap_{T \in \mathcal{T}} T) & \leq & \bigcap_{T \in \mathcal{T}} \widetilde{\Omega}^{\mathcal{I}}(T) \quad \text{(monotonicity of } \widetilde{\Omega}^{\mathcal{I}}) \\ & \leq & \bigcap_{T \in \mathcal{T}} \Omega(T) \quad (\text{since } \widetilde{\Omega}^{\mathcal{I}}(T) \leq \Omega(T)) \\ & \leq & \Omega(T'). \quad \text{(by hypothesis)} \end{array}$$

Using the hypothesis, we conclude that $\bigcap_{T \in \mathcal{T}} T \leq T'$. Therefore, $\mathcal{I}$ is system c-reflective. ■

And now for the promised result showing that in a system c-reflective π-institution $\mathcal{I}$, if the system core is adequate, then it is also soluble.

Proposition 886 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a system c-reflective* π*-institution based on* $\mathbf{F}$*. If the system core* $Z^{\mathcal{I}}$ *is adequate, then it is soluble.*

Proof: Suppose that $\mathcal{I}$ is system c-reflective and that $Z^{\mathcal{I}}$ is adequate. We must show that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

The implication left-to-right is always satisfied by Proposition 875. For the converse, assume that $Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. Then, by the adequacy of $Z^{\mathcal{I}}$, we get that $\widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) \leq \Omega(T)$. Thus, by system c-reflectivity and Lemma 885, we conclude that $C(\overrightarrow{\phi}) \leq T$, which gives $\phi \in T_\Sigma$. ■

We finally show that a π-institution is system truth equational if and only if it is system c-reflective and its system core is adequate.

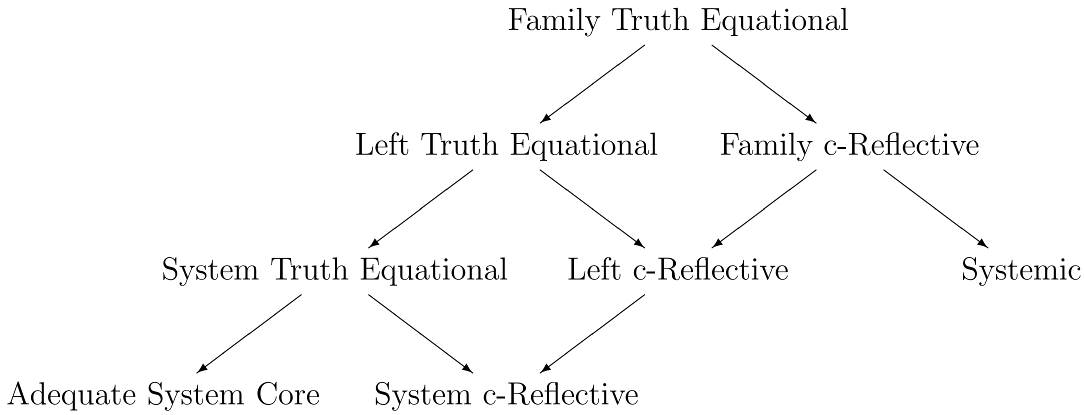
System Truth Equationality	=	$Z^{\mathcal{I}}$ Left Soluble
	=	$Z^{\mathcal{I}}$ Defines Theory Systems
	=	System c-Reflectivity + $Z^{\mathcal{I}}$ Adequate

Theorem 887 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is system truth equational if and only if it is system c-reflective and has an adequate system core.*

Proof: Suppose, first, that $\mathcal{I}$ is system truth equational. Then it is system c-reflective by Theorem 872. Moreover, its system core is soluble by Theorem 990 and, hence, by Corollary 884, its system core is adequate.

Suppose, conversely, that $\mathcal{I}$ is system c-reflective with an adequate system core. Then, by Proposition 886, its system core is soluble and, therefore, by Theorem 990, $\mathcal{I}$ is system truth equational. ■

We have now established the following hierarchy of properties:



Chapter 12

The Syntactic Leibniz Hierarchy: Edifice

12.1 Translations

In this section we discuss translations, interpretations and equivalence that will be used later in the context of algebraizable π-institutions. In the context of algebraizability, the algebraic counterparts of π-institutions may consist of algebraic closure families that lack the property of structurality, i.e., they are not closure systems, as introduced previously. Since these closure families are not structural in general, the corresponding algebraic structures do not constitute π-institutions. To accommodate these, we deal with more general structures that include all π-institutions, but also pairs of algebraic systems and closure families that are non-structural. We call these π-*structures*.

Definition 888 *A* π-**structure** $\mathcal{K} = \langle \mathbf{K}, D \rangle$ *is a pair consisting of:*

- *an algebraic system* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*;*
- *a* $|\mathbf{Sign}|$*-indexed family* $D = \{D_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ *of closure operators* $D_\Sigma : \mathcal{P}\mathrm{SEN}(\Sigma) \to \mathcal{P}\mathrm{SEN}(\Sigma)$*,* $\Sigma \in |\mathbf{Sign}|$*.*

Such a family D *is called a* **closure family** *on* $\mathbf{K}$*.*

Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ and $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ be two algebraic systems. A **translation** $\alpha : \mathbf{K} \to \mathbf{K}'$ is a collection

$$\alpha = \{\alpha_\Sigma\}_{\Sigma \in |\mathbf{Sign}|},$$

where, for all $\Sigma \in |\mathbf{Sign}|$,

$$\alpha_\Sigma : \mathrm{SEN}(\Sigma) \to \mathrm{SenFam}(\mathbf{K}')$$

assigns to each Σ-sentence ϕ of $\mathbf{K}$ a sentence family

$$\alpha_\Sigma[\phi] = \{\alpha_{\Sigma,\Sigma'}[\phi]\}_{\Sigma' \in |\mathbf{Sign}'|}.$$

For $\Sigma \in |\mathbf{Sign}|$, $\Phi \subseteq \mathrm{SEN}(\Sigma)$, we set

$$\alpha_\Sigma[\Phi] = \bigcup \{\alpha_\Sigma[\phi] : \phi \in \Phi\},$$

where the union is, as usual, taken signature-wise and, hence, $\alpha_\Sigma[\Phi] \in \mathrm{SenFam}(\mathbf{K}')$. More generally, for $T \in \mathrm{SenFam}(\mathbf{K})$, we set

$$\alpha[T] = \bigcup \{\alpha_\Sigma[T_\Sigma] : \Sigma \in |\mathbf{Sign}|\}.$$

Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ be algebraic systems and $\mathcal{K} = \langle \mathbf{K}, D \rangle$, $\mathcal{K}' = \langle \mathbf{K}', D' \rangle$ be π-structures based on $\mathbf{K}, \mathbf{K}'$, respectively. An

interpretation $\alpha : \mathcal{K} \to \mathcal{K}'$ is a translation $\alpha : \mathbf{K} \to \mathbf{K}'$, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\phi \in D_\Sigma(\Phi) \quad \text{iff} \quad \alpha_\Sigma[\phi] \leq D'(\alpha_\Sigma[\Phi]).$$

If such an interpretation exists, then $\mathcal{K}$ is said to be **interpretable in** $\mathcal{K}'$.

Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ be algebraic systems and $\mathcal{K} = \langle \mathbf{K}, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}', D'\rangle$ be π-structures based on $\mathbf{K}$, $\mathbf{K}'$, respectively. Let, also,

$$\alpha : \mathcal{K} \to \mathcal{K}' \quad \text{and} \quad \beta : \mathcal{K}' \to \mathcal{K}$$

be interpretations from $\mathcal{K}$ to $\mathcal{K}'$ and from $\mathcal{K}'$ to $\mathcal{K}$, respectively. α and β are said to be **inverses** of each other and the pair $(\alpha, \beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$ is referred to as a **conjugate pair** if:

- for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$D(\phi) = D(\beta[\alpha_\Sigma[\phi]]);$$

- for all $\Sigma' \in |\mathbf{Sign}'|$ and all $\psi \in \mathrm{SEN}'(\Sigma')$,

$$D'(\psi) = D'(\alpha[\beta_{\Sigma'}[\psi]]).$$

The π-structures $\mathcal{K}$ and $\mathcal{K}'$ are called **equivalent** if there exists a conjugate pair $\mathcal{K} \overset{(\alpha,\beta)}{\rightleftarrows} \mathcal{K}'$.

Lemma 889 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *be algebraic systems,* $\mathcal{K} = \langle \mathbf{K}, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}', D'\rangle$ *be π-structures based on* $\mathbf{K}$, $\mathbf{K}'$*, respectively, and* $\alpha : \mathbf{K} \to \mathbf{K}'$, $\beta : \mathbf{K}' \to \mathbf{K}$ *translations. The following are equivalent:*

(i) $\alpha : \mathcal{K} \to \mathcal{K}'$ *is an interpretation and, for all* $\Sigma' \in |\mathbf{Sign}'|$, $\psi \in \mathrm{SEN}'(\Sigma')$, $D'(\psi) = D'(\alpha[\beta_{\Sigma'}[\psi]])$*;*

(ii) $\beta : \mathcal{K}' \to \mathcal{K}$ *is an interpretation and, for all* $\Sigma \in |\mathbf{Sign}|$, $\phi \in \mathrm{SEN}(\Sigma)$, $D(\phi) = D(\beta[\alpha_\Sigma[\phi]])$*.*

Proof: By symmetry, it suffices to show (i)⇒(ii).

Suppose, first, that $\Sigma' \in |\mathbf{Sign}'|$ and $\Psi \cup \{\psi\} \subseteq \mathrm{SEN}'(\Sigma')$. Then, we have

$$\begin{array}{lll} \psi \in D'_{\Sigma'}(\Psi) & \text{iff} & D'(\psi) \leq D'(\Psi) \\ & \text{iff} & D'(\alpha[\beta_{\Sigma'}[\psi]]) \leq D'(\alpha[\beta_{\Sigma'}[\Psi]]) \\ & \text{iff} & \alpha[\beta_{\Sigma'}[\psi]] \leq D'(\alpha[\beta_{\Sigma'}[\Psi]]) \\ & \text{iff} & \beta_{\Sigma'}[\psi] \leq D(\beta_{\Sigma'}[\Psi]). \end{array}$$

So $\beta : \mathcal{K}' \to \mathcal{K}$ is an interpretation.

Assume, next, that $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. Then, by the hypothesis applied to $\alpha_\Sigma[\phi] \in \mathrm{SenFam}(\mathbf{K}')$, we have

$$D'(\alpha[\beta[\alpha_\Sigma[\phi]]]) = D'(\alpha_\Sigma[\phi]).$$

Hence, we get that

$$\alpha_\Sigma[\phi] \le D'(\alpha[\beta[\alpha_\Sigma[\phi]]]) \quad \text{and} \quad \alpha[\beta[\alpha_\Sigma[\phi]]] \le D'(\alpha_\Sigma[\phi]).$$

Therefore, by the fact that α is an interpretation,

$$\phi \in D_\Sigma(\beta[\alpha_\Sigma[\phi]]) \quad \text{and} \quad \beta[\alpha_\Sigma[\phi]] \le D(\phi).$$

So we conclude that $D(\phi) = D(\beta[\alpha_\Sigma[\phi]])$. ∎

Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ be algebraic systems and $\alpha : \mathbf{K} \to \mathbf{K}'$ a translation. Define the **residual α^* of the translation α**,

$$\alpha^* : \mathrm{SenFam}(\mathbf{K}') \to \mathrm{SenFam}(\mathbf{K})$$

by letting, for all $T' \in \mathrm{SenFam}(\mathbf{K}')$,

$$\alpha^*(T') = \{\alpha^*_\Sigma(T')\}_{\Sigma \in |\mathbf{Sign}|}$$

be given, for all $\Sigma \in |\mathbf{Sign}|$, by

$$\alpha^*_\Sigma(T') = \{\phi \in \mathrm{SEN}(\Sigma) : \alpha_\Sigma[\phi] \le T'\}.$$

The following proposition shows that, when applied to interpretations between π-structures the star operator restricts to mappings from theory families to theory families.

Proposition 890 *Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ be algebraic systems, $\mathcal{K} = \langle \mathbf{K}, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}', D'\rangle$ be π-structures based on $\mathbf{K}$, $\mathbf{K}'$, respectively, and $\alpha : \mathcal{K} \to \mathcal{K}'$ an interpretation. Then, for all $T' \in \mathrm{ThFam}(\mathcal{K}')$, $\alpha^*(T') \in \mathrm{ThFam}(\mathcal{K})$.*

Proof: Suppose $T' \in \mathrm{ThFam}(\mathcal{K}')$ and let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in D_\Sigma(\alpha^*_\Sigma(T'))$. Then, since $\alpha : \mathcal{K} \to \mathcal{K}'$ is an interpretation, we have

$$\alpha_\Sigma[\phi] \le D'(\alpha[\alpha^*_\Sigma(T')]) \le D'(T') = T'.$$

Hence $\phi \in \alpha^*_\Sigma(T')$. Since $\Sigma \in |\mathbf{Sign}|$ was arbitrary, we conclude that $\alpha^*_\Sigma(T') \in \mathrm{ThFam}(\mathcal{K})$. ∎

In addition, we show that, when $(\alpha, \beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$ form a conjugate pair, then $\beta^* : \mathrm{ThFam}(\mathcal{K}) \to \mathrm{ThFam}(\mathcal{K}')$ and $\alpha^* : \mathrm{ThFam}(\mathcal{K}') \to \mathrm{ThFam}(\mathcal{K})$ are inverse mappings.

Lemma 891 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *be algebraic systems,* $\mathcal{K} = \langle \mathbf{K}, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}', D'\rangle$ *be* π*-structures based on* $\mathbf{K}$, $\mathbf{K}'$, *respectively, and* $(\alpha,\beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *a conjugate pair. Then, for all* $T \in \mathrm{ThFam}(\mathcal{K})$,

$$\alpha^*(\beta^*(T)) = T.$$

Proof: Suppose $(\alpha,\beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$ is a conjugate pair, $T \in \mathrm{ThFam}(\mathcal{K})$ and let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. Then we have

$$\begin{array}{lll}\phi \in \alpha^*_\Sigma(\beta^*(T)) & \text{iff} & \alpha_\Sigma[\phi] \leq \beta^*(T)\\ & \text{iff} & \beta[\alpha_\Sigma[\phi]] \leq T\\ & \text{iff} & D(\beta[\alpha_\Sigma[\phi]]) \leq T\\ & \text{iff} & D_\Sigma(\phi) \leq T_\Sigma\\ & \text{iff} & \phi \in T_\Sigma.\end{array}$$

Thus, we conclude that $\alpha^*(\beta^*(T)) = T$. ■

Based on Lemma 891, we can show that $\beta^* : \mathrm{ThFam}(\mathcal{K}) \to \mathrm{ThFam}(\mathcal{K}')$ and $\alpha^* : \mathrm{ThFam}(\mathcal{K}') \to \mathrm{ThFam}(\mathcal{K})$ are bijections.

Lemma 892 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *be algebraic systems,* $\mathcal{K} = \langle \mathbf{K}, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}', D'\rangle$ *be* π*-structures based on* $\mathbf{K}$, $\mathbf{K}'$, *respectively, and* $(\alpha,\beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *a conjugate pair. Then* $\alpha^* : \mathrm{ThFam}(\mathcal{K}') \to \mathrm{ThFam}(\mathcal{K})$ *is a bijection.*

Proof: Let $(\alpha,\beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$ be a conjugate pair. First, by Proposition 890, $\alpha^* : \mathrm{ThFam}(\mathcal{K}') \to \mathrm{ThFam}(\mathcal{K})$ is well-defined. To see that it is surjective, let $T \in \mathrm{ThFam}(\mathcal{K})$. Then, by Proposition 890, $\beta^*(T) \in \mathrm{ThFam}(\mathcal{K}')$ and, by Lemma 891, $\alpha^*(\beta^*(T)) = T$. Thus, α^* is indeed surjective. For injectivity, assume $S', T' \in \mathrm{ThFam}(\mathcal{K}')$, such that $\alpha^*(S') = \alpha^*(T')$. Then, by surjectivity, there exist $S, T \in \mathrm{ThFam}(\mathcal{K})$, such that $\beta^*(S) = S'$ and $\beta^*(T) = T'$. Therefore, we get

$$S = \alpha^*(\beta^*(S)) = \alpha^*(S') = \alpha^*(T') = \alpha^*(\beta^*(T)) = T.$$

But then we get $S' = \beta^*(S) = \beta^*(T) = T'$. we conclude that α^* is also injective and, hence, it is a bijection. ■

In the main theorem of this section, it is shown that if $\mathcal{K}$ and $\mathcal{K}'$ are equivalent π-structures via a conjugate pair $(\alpha,\beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$, then $\beta^* : \mathbf{ThFam}(\mathcal{K}) \to \mathbf{ThFam}(\mathcal{K}')$ and $\alpha^* : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ form a pair of mutually inverse order isomorphisms between the complete lattices of the corresponding theory families.

Recall that, given a π-institution $\mathcal{I}$, we denote by

$$\mathbf{ThFam}(\mathcal{I}) = \langle \mathrm{ThFam}(\mathcal{I}), \leq\rangle$$

the complete lattice of theory families of $\mathcal{I}$ ordered by signature-wise inclusion. We extend the notation to the collections of theory families of π-structures. Thus, given a π-structure $\mathcal{K} = \langle \mathbf{K}, D\rangle$, we define

$$\mathbf{ThFam}(\mathcal{K}) = \langle \mathrm{ThFam}(\mathcal{K}), \leq\rangle.$$

Theorem 893 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *be algebraic systems,* $\mathcal{K} = \langle \mathbf{K}, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}', D'\rangle$ *be π-structures based on* $\mathbf{K}$, $\mathbf{K}'$, *respectively, and* $(\alpha, \beta) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *a conjugate pair. Then*

$$\beta^* : \mathbf{ThFam}(\mathcal{K}) \to \mathbf{ThFam}(\mathcal{K}') \quad \textit{and} \quad \alpha^* : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$$

are mutually inverse order isomorphisms.

Proof: We know, by Lemma 892, that β^* and α^* are mutually inverse bijections. Moreover, by definition, they are both order preserving. Thus, each is also order-reflecting, since, e.g., for all $S', T' \in \mathrm{ThFam}(\mathcal{K}')$,

$$\begin{array}{lll}\alpha^*(S') \leq \alpha^*(T') & \text{implies} & \beta^*(\alpha^*(S')) \leq \beta^*(\alpha^*(T')) \\ & \text{implies} & S' \leq T',\end{array}$$

the latter implication following by Lemma 891. ∎

Conversely, it is true that given mutually inverse order isomorphisms between the complete lattices of two π-structures, one may define a conjugate pair between the two that establishes this order-isomorphism via the process that was described above. We provide, next, more details on this inverse process.

Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ be algebraic systems, $\mathcal{K} = \langle \mathbf{K}, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}', D'\rangle$ be π-structures based on $\mathbf{K}$, $\mathbf{K}'$, respectively, and

$$h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$$

an order isomorphism between the corresponding complete lattices of theory families.

Define $\overrightarrow{h} = \{\overrightarrow{h}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ by letting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\overrightarrow{h}_\Sigma : \mathrm{SEN}(\Sigma) \to \mathrm{SenFam}(\mathbf{K}')$$

be given, for all $\phi \in \mathrm{SEN}(\Sigma)$, by

$$\overrightarrow{h}_\Sigma[\phi] = h^{-1}(D(\phi)).$$

Further, define $\overleftarrow{h} = \{\overleftarrow{h}_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}'|}$ by letting, for all $\Sigma' \in |\mathbf{Sign}'|$,

$$\overleftarrow{h}_{\Sigma'} : \mathrm{SEN}'(\Sigma') \to \mathrm{SenFam}(\mathbf{K})$$

be given, for all $\psi \in \mathrm{SEN}'(\Sigma')$, by

$$\overleftarrow{h}_{\Sigma'}[\psi] = h(D'(\psi)).$$

We show that, the two translations $\overrightarrow{h} : \mathbf{K} \to \mathbf{K}'$ and $\overleftarrow{h} : \mathbf{K}' \to \mathbf{K}$, defined above, constitute interpretations between the corresponding π-structures.

Lemma 894 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *be algebraic systems,* $\mathcal{K} = \langle \mathbf{K}, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}', D'\rangle$ *be* π*-structures based on* $\mathbf{K}$, $\mathbf{K}'$*, respectively, and* $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ *an order isomorphism. Then* $\overleftarrow{h} : \mathcal{K}' \to \mathcal{K}$ *is an interpretation.*

Proof: Suppose $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ is an order isomorphism and let $\Sigma' \in |\mathbf{Sign}'|$ and $\Psi \cup \{\psi\} \subseteq \mathrm{SEN}'(\Sigma')$. Then we have

$$\begin{array}{rll} \psi \in D'_{\Sigma'}(\Psi) & \text{iff} & D'(\psi) \leq D'(\Psi) \\ & \text{iff} & h(D'(\psi)) \leq h(D'(\Psi)) \\ & \text{iff} & h(D'(\psi)) \leq h(\bigvee\{D'(\chi) : \chi \in \Psi\}) \\ & \text{iff} & h(D'(\psi)) \leq \bigvee\{h(D'(\chi)) : \chi \in \Psi\} \\ & \text{iff} & \overleftarrow{h}_{\Sigma'}[\psi] \leq \bigvee\{\overleftarrow{h}_{\Sigma'}[\chi] : \chi \in \Psi\} \\ & \text{iff} & \overleftarrow{h}_{\Sigma'}[\psi] \leq D(\overleftarrow{h}_{\Sigma'}[\Psi]). \end{array}$$

Thus, $\overleftarrow{h} : \mathcal{K}' \to \mathcal{K}$ is indeed an interpretation. ■

We now know (by symmetry, based on Lemma 894) that $\overrightarrow{h} : \mathcal{K} \to \mathcal{K}'$ and $\overleftarrow{h} : \mathcal{K}' \to \mathcal{K}$ are interpretations. It is, in fact, the case that $(\overrightarrow{h}, \overleftarrow{h}) : \mathcal{K} \rightleftarrows \mathcal{K}'$ form a conjugate pair, as is shown next.

Lemma 895 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *be algebraic systems,* $\mathcal{K} = \langle \mathbf{K}, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}', D'\rangle$ *be* π*-structures based on* $\mathbf{K}$, $\mathbf{K}'$*, respectively, and* $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ *an order isomorphism. Then* $(\overrightarrow{h}, \overleftarrow{h}) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *is a conjugate pair.*

Proof: By Lemma 889, it suffices to show that $\overleftarrow{h} : \mathcal{K}' \to \mathcal{K}$ is an interpretation and that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $D(\phi) = D(\overleftarrow{h}[\overrightarrow{h}_\Sigma[\phi]])$. The former has been shown in Lemma 894. So it suffices to show the latter. To this end, let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. Then we have

$$\begin{array}{rcl} D(\overleftarrow{h}[\overrightarrow{h}_\Sigma[\phi]]) & = & D(\overleftarrow{h}[h^{-1}(D(\phi))]) \\ & = & D(\bigcup\{\overleftarrow{h}[\chi] : \chi \in h^{-1}(D(\phi))\}) \\ & = & \bigvee\{h(D'(\chi)) : \chi \in h^{-1}(D(\phi))\} \\ & = & h(\bigvee\{D'(\chi) : \chi \in h^{-1}(D(\phi))\}) \\ & = & h(h^{-1}(D(\phi))) \\ & = & D(\phi). \end{array}$$

We conclude that $(\overrightarrow{h},\overleftarrow{h}) : \mathcal{K} \rightleftarrows \mathcal{K}'$ is a conjugate pair. ■

Based on Lemma 895, we can now formulate one of the main theorems of this section to the effect that every order isomorphism between the complete lattices of theory families of two π-structures gives rise to a conjugate pair of interpretations that induce the isomorphism via the star construction.

Theorem 896 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *be algebraic systems,* $\mathcal{K} = \langle \mathbf{K}, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}', D'\rangle$ *be π-structures based on* $\mathbf{K}$, $\mathbf{K}'$, *respectively, and* $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ *an order isomorphism. Then* $(\overrightarrow{h},\overleftarrow{h}) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *is a conjugate pair, such that* $\overrightarrow{h}^* = h$ *and* $\overleftarrow{h}^* = h^{-1}$.

Proof: By Lemma 895, we know that $(\overrightarrow{h},\overleftarrow{h}) : \mathcal{K} \rightleftarrows \mathcal{K}'$ form a conjugate pair. We show that $\overrightarrow{h}^* = h$. The equality $\overleftarrow{h}^* = h^{-1}$ may be proved similarly. To this end, let $T' \in \mathrm{ThFam}(\mathcal{K}')$. Then we have

$$\begin{array}{rcl}\overrightarrow{h}^*_\Sigma(T') & = & \{\phi \in \mathrm{SEN}(\Sigma) : \overrightarrow{h}_\Sigma[\phi] \leq T'\}\\ & = & \{\phi \in \mathrm{SEN}(\Sigma) : h^{-1}(D(\phi)) \leq T'\}\\ & = & D_\Sigma(\{\phi \in \mathrm{SEN}(\Sigma) : h^{-1}(D(\phi)) \leq T'\})\\ & = & D_\Sigma(\{\phi \in \mathrm{SEN}(\Sigma) : D(\phi) \leq h(T')\})\\ & = & D_\Sigma(\{\phi \in \mathrm{SEN}(\Sigma) : \phi \in h_\Sigma(T')\})\\ & = & D_\Sigma(h_\Sigma(T'))\\ & = & h_\Sigma(T').\end{array}$$

Similarly, $\overleftarrow{h}^* = h^{-1}$. ■

12.2 Transformations

Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system and $k \geq 1$ be an integer. Then a **power algebraic system**

$$\mathbf{K}^k = \langle \mathbf{Sign}, \mathrm{SEN}^k, N^k\rangle$$

is the algebraic system whose sentence functor $\mathrm{SEN}^k : \mathbf{Sign} \to \mathbf{Set}$ is the k-th direct power of SEN and whose category N^k of natural transformations consists of k-tuples of natural transformations having the same arity in N.

Let $k, \ell \geq 1$ be integers. A translation $\alpha : \mathbf{K}^k \to \mathbf{K}^\ell$ is called a **transformation** if there exists a set

$$\tau : \mathrm{SEN}^\omega \to \mathrm{SEN}^\ell,$$

in N, with k distinguished arguments, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$,

$$\alpha_\Sigma[\vec{\phi}] = \tau_\Sigma[\vec{\phi}].$$

Moreover a translation $\alpha : \mathbf{K}^k \to \mathbf{K}^\ell$ is called a **natural transformation** if it is a parameter-free transformation, i.e., if there exists $\tau : \mathrm{SEN}^k \to \mathrm{SEN}^\ell$ in N, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$,

$$\alpha_\Sigma[\vec{\phi}] = \tau_\Sigma[\vec{\phi}].$$

Based on the results obtained in Section 12.1, we may formulate some propositions concerning interpretability and equivalence based on transformations.

Proposition 897 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system and* $\mathcal{K} = \langle \mathbf{F}^k, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D'\rangle$ *be two* π*-structures.* $\mathcal{K}$ *is interpretable in* $\mathcal{K}'$ *via a transformation if and only if there exists a set* $\tau : \mathrm{SEN}^\omega \to \mathrm{SEN}^\ell$, *with* k *distinguished arguments, such that, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\Phi \cup \{\vec{\phi}\} \subseteq \mathrm{SEN}(\Sigma)^k$,

$$\vec{\phi} \in D_\Sigma(\Phi) \quad \textit{iff} \quad \tau_\Sigma[\vec{\phi}] \leq D'(\tau_\Sigma[\Phi]).$$

If $\mathcal{K}$ *is interpretable in* $\mathcal{K}'$ *as above, then it is equivalent to* $\mathcal{K}'$ *via a conjugate pair* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *of transformations if and only if, for all* $\Sigma \in |\mathbf{Sign}|$, *all* $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$ *and all* $\Psi \cup \{\vec{\psi}\} \subseteq \mathrm{SEN}(\Sigma)^\ell$,

- $\vec{\psi} \in D'_\Sigma(\Psi)$ *iff* $I_\Sigma[\vec{\psi}] \leq D(I_\Sigma[\Psi])$;
- $D'(\vec{\psi}) = D'(\tau[I_\Sigma[\vec{\psi}]])$;
- $D(\vec{\phi}) = D(I[\tau_\Sigma[\vec{\phi}]])$.

Proof: This is a restatement of the definition of interpretability under the additional hypothesis that the corresponding interpretations are transformations. ■

Proposition 898 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system and* $\mathcal{K} = \langle \mathbf{K}^k, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D'\rangle$ *be two* π*-structures.* $\mathcal{K}$ *is equivalent to* $\mathcal{K}'$ *via a conjugate pair* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *of transformations if and only if one of the following equivalent conditions hold:*

(a) $\tau : \mathcal{K} \to \mathcal{K}'$ *is an interpretation and, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\vec{\psi} \in \mathrm{SEN}(\Sigma)^\ell$, $D'(\vec{\psi}) = D'(\tau[I_\Sigma[\vec{\psi}]])$;

(b) $I : \mathcal{K}' \to \mathcal{K}$ *is an interpretation and, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$, $D(\vec{\phi}) = D(I[\tau_\Sigma[\vec{\phi}]])$.

Proof: Directly by Lemma 889. ■

Taking the point of view of order isomorphisms between lattices of theory families, we would like to have a concept ensuring that such an isomorphism is induced not merely by a conjugate pair of translations, as is asserted by

Theorem 896, but, more emphatically, by a conjugate pair of transformations. We focus on this task next.

Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system and $\mathcal{K} = \langle \mathbf{K}^k, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D'\rangle$ be two π-structures based on $\mathbf{K}^k$, $\mathbf{K}^\ell$, respectively. An order isomorphism $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ is called **transformational** if there exist sets

- $\tau : \mathrm{SEN}^\omega \to \mathrm{SEN}^\ell$ in N, with k distinguished arguments;
- $I : \mathrm{SEN}^\omega \to \mathrm{SEN}^k$ in N, with ℓ distinguished arguments,

such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$ and all $\vec{\psi} \in \mathrm{SEN}(\Sigma)^\ell$,

$$\overrightarrow{h}_\Sigma[\vec{\phi}] = D'(\tau_\Sigma[\vec{\phi}]) \quad \text{and} \quad \overleftarrow{h}_\Sigma[\vec{\psi}] = D(I_\Sigma[\vec{\psi}]).$$

These conditions are, by definition, equivalent, respectively, to the conditions

$$h^{-1}(D(\vec{\phi})) = D'(\tau_\Sigma[\vec{\phi}]) \quad \text{and} \quad h(D'(\vec{\psi})) = D(I_\Sigma[\vec{\psi}]).$$

In this case, we say that h is **induced by** $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$. (Note that, since we will be able to show that (τ, I) is a conjugate pair of transformations, this notation makes sense.)

In fact, the defining conditions yield some crucial relations between theory families, as in shown in the following lemma.

Lemma 899 *Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system, $\mathcal{K} = \langle \mathbf{K}^k, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D'\rangle$ be two π-structures and $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ a transformational order isomorphism induced by $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$. Then, for all $\Sigma \in |\mathbf{Sign}|$, all $\Phi \subseteq \mathrm{SEN}(\Sigma)^k$ and all $\Psi \subseteq \mathrm{SEN}(\Sigma)^\ell$,*

$$h^{-1}(D(\Phi)) = D'(\tau_\Sigma[\Phi]) \quad \textit{and} \quad h(D'(\Psi)) = D(I_\Sigma[\Psi]).$$

Proof: By symmetry, it suffices to show the first equation. We have, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)^k$,

$$\begin{array}{rcll} h^{-1}(D(\Phi)) & = & h^{-1}(\bigvee_{\phi\in\Phi} D(\phi)) & (\text{join in } \mathbf{ThFam}(\mathcal{K})) \\ & = & \bigvee_{\phi\in\Phi} h^{-1}(D(\phi)) & (h^{-1} \text{ order isomorphism}) \\ & = & \bigvee_{\phi\in\Phi} D'(\tau_\Sigma[\phi]) & (h^{-1}(D(\phi)) = \overrightarrow{h}_\Sigma[\phi]) \\ & = & D'(\bigcup_{\phi\in\Phi} \tau_\Sigma[\phi]) & (\text{join in } \mathbf{ThFam}(\mathcal{K}')) \\ & = & D'(\tau_\Sigma[\Phi]). & (\text{by definition}) \end{array}$$

The second equation now follows by symmetry. ■

Now we are in a position to show that a transformational order isomorphism between the lattices of theory families of two π-structures is induced by a conjugate pair of transformations between the two π-structures and, as a consequence, gives rise to an equivalence $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ via a conjugate pair of transformations.

Theorem 900 *Let* $\mathbf{K} = \langle\mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system,* $\mathcal{K} = \langle\mathbf{K}^k, D\rangle$, $\mathcal{K}' = \langle\mathbf{K}^\ell, D'\rangle$ *be two* π*-structures and* $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ *a transformational order isomorphism induced by* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$*. Then* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *is a conjugate pair of transformations.*

Proof: We use Proposition 898. Let $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\vec{\phi}\} \subseteq \mathrm{SEN}(\Sigma)^k$ and $\vec{\psi} \in \mathrm{SEN}(\Sigma)^\ell$. We then have:

$$\begin{array}{lll}\vec{\phi} \in D_\Sigma(\Phi) & \text{iff} & D_\Sigma(\vec{\phi}) \leq D_\Sigma(\Phi)\\ & \text{iff} & h^{-1}(D(\vec{\phi})) \leq h^{-1}(D(\Phi)) \quad (h \text{ order isomorphism})\\ & \text{iff} & D'(\tau_\Sigma[\vec{\phi}]) \leq D'(\tau_\Sigma[\Phi]) \quad (\text{Lemma 899})\\ & \text{iff} & \tau_\Sigma[\vec{\phi}] \leq D'(\tau_\Sigma[\Phi]).\end{array}$$

Thus, $\tau : \mathcal{K} \to \mathcal{K}'$ is an interpretation. Moreover, we have:

$$\begin{array}{lll}D'(\vec{\psi}) & = & h^{-1}(h(D'(\vec{\psi}))) \quad (h \text{ order isomorphism})\\ & = & h^{-1}(D(I_\Sigma[\vec{\psi}])) \quad (h \text{ transformational})\\ & = & D'(\tau[I_\Sigma[\vec{\psi}]]). \quad (\text{Lemma 899})\end{array}$$

We conclude by Proposition 898, that $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ is a conjugate pair of transformations. ∎

As a consequence, we have the following

Theorem 901 *Let* $\mathbf{K} = \langle\mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system,* $\mathcal{K} = \langle\mathbf{K}^k, D\rangle$, $\mathcal{K}' = \langle\mathbf{K}^\ell, D'\rangle$ *be two* π*-structures and* $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ *a transformational order isomorphism induced by* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$*. Then the* π*-structures* $\mathcal{K}$ *and* $\mathcal{K}'$ *are equivalent via the conjugate pair* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *of transformations.*

Proof: This follows directly by Theorem 900. ∎

Similarly, for interpretability and equivalence based on natural transformations, we have the following corresponding propositions.

Proposition 902 *Let* $\mathbf{K} = \langle\mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system and* $\mathcal{K} = \langle\mathbf{K}^k, D\rangle$, $\mathcal{K}' = \langle\mathbf{K}^\ell, D'\rangle$ *be two* π*-structures.* $\mathcal{K}$ *is interpretable in* $\mathcal{K}'$ *via a natural transformation if and only if there exists a set* $\tau : \mathrm{SEN}^k \to \mathrm{SEN}^\ell$ *in* N*, such that, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\Phi \cup \{\vec{\phi}\} \subseteq \mathrm{SEN}(\Sigma)^k$,

$$\vec{\phi} \in D_\Sigma(\Phi) \quad \textit{iff} \quad \tau_\Sigma[\vec{\phi}] \leq D'(\tau_\Sigma[\Phi]).$$

If $\mathcal{K}$ *is interpretable in* $\mathcal{K}'$ *as above, then it is equivalent to* $\mathcal{K}'$ *via a conjugate pair* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *of natural transformations if and only if, for all* $\Sigma \in |\mathbf{Sign}|$*, all* $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$ *and all* $\Psi \cup \{\vec{\psi}\} \subseteq \mathrm{SEN}(\Sigma)^\ell$,

- $\vec{\psi} \in D'_\Sigma(\Psi)$ *iff* $I_\Sigma[\vec{\psi}] \leq D(I_\Sigma[\Psi])$;

- $D'(\vec{\psi}) = D'(\tau[I_\Sigma[\vec{\psi}]])$;
- $D(\vec{\phi}) = D(I[\tau_\Sigma[\vec{\phi}]])$.

Proof: This is a restatement of the definition of interpretability under the additional hypothesis that the corresponding interpretations are natural transformations. ∎

Proposition 903 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system and* $\mathcal{K} = \langle \mathbf{K}^k, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D'\rangle$ *be two* π*-structures.* $\mathcal{K}$ *is equivalent to* $\mathcal{K}'$ *via a conjugate pair* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *of natural transformations if and only if one of the following equivalent conditions hold:*

(a) $\tau : \mathcal{K} \to \mathcal{K}'$ *is an interpretation and, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\vec{\psi} \in \mathrm{SEN}(\Sigma)^\ell$, $D'(\vec{\psi}) = D'(\tau[I_\Sigma[\vec{\psi}]])$;

(b) $I : \mathcal{K}' \to \mathcal{K}$ *is an interpretation and, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$, $D(\vec{\phi}) = D(I[\tau_\Sigma[\vec{\phi}]])$.

Proof: Directly by Lemma 889. ∎

In terms of order isomorphisms between lattices of theory families, we have analogs of preceding results that allow us to work with isomorphisms that are induced by conjugate pairs of natural transformations.

Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system and $\mathcal{K} = \langle \mathbf{K}^k, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D'\rangle$ be two π-structures based on $\mathbf{K}^k$, $\mathbf{K}^\ell$, respectively. An order isomorphism $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ is called **natural** if there exist sets

- $\tau : \mathrm{SEN}^k \to \mathrm{SEN}^\ell$ in N;
- $I : \mathrm{SEN}^\ell \to \mathrm{SEN}^k$ in N,

such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$ and $\vec{\psi} \in \mathrm{SEN}(\Sigma)^\ell$,

$$\overrightarrow{h}_\Sigma[\vec{\phi}] = D'(\tau_\Sigma[\vec{\phi}]) \quad \text{and} \quad \overleftarrow{h}_\Sigma[\vec{\psi}] = D(I_\Sigma[\vec{\psi}]).$$

In this case, we say that h is **induced by** the pair of natural transformations $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$.

Similarly, with the case of a transformational isomorphism, we can show that a natural order isomorphism between the lattices of theory families of two π-structures is induced by a conjugate pair of natural transformations between the two π-structures.

Theorem 904 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system,* $\mathcal{K} = \langle \mathbf{K}^k, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D'\rangle$ *be two* π*-structures and* $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ *a natural order isomorphism induced by* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$*. Then* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *is a conjugate pair of natural transformations.*

Proof: This follows from Theorem 900. ∎

As a consequence, we have the following analog of Theorem 901.

Theorem 905 *Let* $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system,* $\mathcal{K} = \langle \mathbf{K}^k, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D'\rangle$ *be two* π*-structures and* $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ *a natural order isomorphism induced by* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$. *Then the* π*-structures* $\mathcal{K}$ *and* $\mathcal{K}'$ *are equivalent via the conjugate pair* $(\tau, I) : \mathcal{K} \rightleftarrows \mathcal{K}'$ *of natural transformations.*

Proof: This follows directly by Theorem 904. ∎

We now revert to the case of a base algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ and a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ based on $\mathbf{F}$. Our focus, in this standard context, will be on $\mathbf{F}$ itself, on the one hand, and on $\mathbf{F}^2$, on the other. In the context of $\mathbf{F}^2$, given $\Sigma \in |\mathbf{Sign}^\flat|$, we sometimes denote a pair $\langle \phi, \psi\rangle \in \mathrm{SEN}^\flat(\Sigma)^2$ in the equational form

$$\phi \approx \psi.$$

Given a π-structure $\mathcal{Q} = \langle \mathbf{F}^2, D\rangle$, we say that $\mathcal{Q}$ is **equational** if the following five axioms hold:

(R) $\phi \approx \phi \in D_\Sigma(\varnothing)$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$;

(S) $\psi \approx \phi \in D_\Sigma(\phi \approx \psi)$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$;

(T) $\phi \approx \chi \in D_\Sigma(\phi \approx \psi, \psi \approx \chi)$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$;

(C) $\sigma^\flat_\Sigma(\vec{\phi}) \approx \sigma^\flat_\Sigma(\vec{\psi}) \in D_\Sigma(\{\phi_i \approx \psi_i : i < k\})$, for all $\sigma^\flat \in N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi_i, \psi_i \in \mathrm{SEN}^\flat(\Sigma)$, $i < k$;

(I) $\mathrm{SEN}^\flat(f)(\phi) \approx \mathrm{SEN}^\flat(f)(\psi) \in D_{\Sigma'}(\phi \approx \psi)$, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$.

Note that according to the relevant definitions introduced in Chapter 2, the meaning of (I) is that the Σ'-component of the least theory family including $\phi \approx \psi$ in its Σ-component includes $\mathrm{SEN}^\flat(f)(\phi) \approx \mathrm{SEN}^\flat(f)(\psi)$.

These properties are termed **reflexivity**, **symmetry**, **transitivity**, **compatibility** and **invariance**, respectively. The first three ensure that, for all $E \in \mathrm{SenFam}(\mathbf{F}^2)$, $D(E)$ is an equivalence family. The fourth one ensures that $D(E)$ is a congruence family and the last that it is a congruence system, i.e., invariant under the action of signature morphisms. In fact, the following characterization theorem holds, showing that a π-structure is equational if and only if it is structural and all its closure families are congruence systems on $\mathbf{F}$ if and only if it is the equational π-structure relative to a class K of $\mathbf{F}$-algebraic systems according to the definition given in Section 2.17.

Theorem 906 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{Q} = \langle \mathbf{F}^2, D\rangle$ *a* π*-structure. The following statements are equivalent:*

(i) $\mathcal{Q}$ *is equational;*

(ii) *For all* $\theta \in \mathrm{SenFam}(\mathcal{Q})$, $D(\theta) \in \mathrm{ConSys}(\mathbf{F})$*;*

(iii) *For some class* K *of* $\mathbf{F}$*-algebraic systems,* $D = D^{\mathsf{K}}$.

Proof:

(i)⇒(ii) Suppose $\mathcal{Q}$ is equational and let $\theta \in \mathrm{SenFam}(\mathcal{Q})$. We must show that $D(\theta) = \{D_\Sigma(\theta)\}_{\Sigma\in|\mathbf{Sign}^\flat|}$ is a congruence system on $\mathbf{F}$. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$. Since $\mathcal{Q}$ is equational, we have $\phi \approx \phi \in D_\Sigma(\varnothing) \subseteq D_\Sigma(\theta)$. So $D_\Sigma(\theta)$ is reflexive. Suppose, next, that $\phi \approx \psi \in D_\Sigma(\theta)$. Since $\mathcal{Q}$ is equational, we get $\psi \approx \phi \in D_\Sigma(\phi \approx \psi) \subseteq D_\Sigma(\theta)$. Hence, $D_\Sigma(\theta)$ is also symmetric. Further, if $\phi \approx \psi, \psi \approx \chi \in D_\Sigma(\theta)$, then, since $\mathcal{Q}$ is equational, we get $\phi \approx \chi \in D_\Sigma(\phi \approx \psi, \psi \approx \chi) \subseteq D_\Sigma(\theta)$. Thus, $D_\Sigma(\theta)$ is also transitive and, hence, an equivalence relation on $\mathrm{SEN}^\flat(\Sigma)$.

Suppose, now, that $\sigma^\flat \in N^\flat$, $\phi_i, \psi_i \in \mathrm{SEN}^\flat(\Sigma)$, for $i < k$, such that $\phi_i \approx \psi_i \in D_\Sigma(\theta)$, for all $i < k$. Since $\mathcal{Q}$ is equational, we get $\sigma^\flat_\Sigma(\vec{\phi}) \approx \sigma^\flat_\Sigma(\vec{\psi}) \in D_\Sigma(\{\phi_i \approx \psi_i : i < k\}) \subseteq D_\Sigma(\theta)$. Hence, $D_\Sigma(\theta)$ is a congruence family on $\mathbf{F}$. Finally, if $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \approx \psi \in D_\Sigma(\theta)$, then, again based on the fact that $\mathcal{Q}$ is equational, we obtain $\mathrm{SEN}^\flat(f)(\phi) \approx \mathrm{SEN}^\flat(f)(\psi) \in D_{\Sigma'}(\phi \approx \psi) \subseteq D_{\Sigma'}(\theta)$, whence $D(\theta)$ is a congruence system on $\mathbf{F}$, as was to be shown.

(ii)⇒(iii) Suppose D satisfies (ii). We construct a class K of $\mathbf{F}$-algebraic systems as follows. For $\theta \in \mathrm{SenFam}(\mathcal{Q})$, define

$$\mathcal{F}^\theta = \langle \mathbf{F}^\theta, \langle I, \pi^\theta \rangle\rangle := \langle \mathbf{F}/D(\theta), \langle I, \pi^{D(\theta)}\rangle\rangle$$

and set

$$\mathsf{K} = \{\mathcal{F}^\theta : \theta \in \mathrm{SenFam}(\mathcal{Q})\}.$$

Note that the definition of $\mathcal{F}^\theta$ makes sense, since, by hypothesis, $D(\theta) \in \mathrm{ConSys}(\mathbf{F})$, for all $\theta \in \mathrm{SenFam}(\mathcal{Q})$. Our task now is to show that $D = D^{\mathsf{K}}$. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$, $\theta \cup \{\phi \approx \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)^2$.

Suppose, first, that $\phi \approx \psi \in D_\Sigma(\theta)$ and let $\theta' \in \mathrm{SenFam}(\mathcal{Q})$, such that $\pi^{\theta'}_\Sigma(\theta) \subseteq \Delta^{\mathbf{F}/D(\theta')}_\Sigma$. This is equivalent to $\theta_\Sigma \subseteq D_\Sigma(\theta'_\Sigma)$. Hence, we obtain $\phi \approx \psi \in D_\Sigma(\theta) \subseteq D_\Sigma(\theta')$. Thus, $\pi^{\theta'}_\Sigma(\phi) = \pi^{\theta'}_\Sigma(\psi)$. We conclude that $\phi \approx \psi \in D^{\mathsf{K}}_\Sigma(\theta)$. Hence, $D \leq D^{\mathsf{K}}$.

Assume, conversely, that $\phi \approx \psi \notin D_\Sigma(\theta)$. Then, clearly, for $\mathcal{F}^\theta \in \mathsf{K}$, we get $\pi^\theta_\Sigma(D_\Sigma(\theta)) \subseteq \Delta^{\mathbf{F}/D(\theta)}_\Sigma$, but $\pi^\theta_\Sigma(\phi) \neq \pi^\theta_\Sigma(\psi)$. Hence, $\phi \approx \psi \notin D^{\mathsf{K}}_\Sigma(\theta)$. Therefore, $D^{\mathsf{K}} \leq D$ and, hence, $D = D^{\mathsf{K}}$.

(iii)⇒(i) This implication was shown in Proposition 115, which was proven by appealing to the implication (iii)⇒(ii), which was, in turn, the content of Proposition 30.

∎

We have the following useful technical lemma, where, for $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$, we use the abbreviation

$$\vec{\phi} \approx \vec{\psi} = \{\phi_i \approx \psi_i : i < k\}.$$

Lemma 907 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{Q} = \langle \mathbf{F}^2, D \rangle$ *an equational* π*-structure. Then, for all* $\delta^\flat, \epsilon^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\delta^\flat_\Sigma(\vec{\psi}) \approx \epsilon^\flat_\Sigma(\vec{\psi}) \in D_\Sigma(\vec{\phi} \approx \vec{\psi}, \delta^\flat_\Sigma(\vec{\phi}) \approx \epsilon^\flat_\Sigma(\vec{\phi})).$$

Proof: We have, for all $\delta^\flat, \epsilon^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}\delta^\flat_\Sigma(\vec{\psi}) \approx \epsilon^\flat_\Sigma(\vec{\psi}) & \in & D_\Sigma(\delta^\flat_\Sigma(\vec{\psi}) \approx \delta^\flat_\Sigma(\vec{\phi}), \delta^\flat_\Sigma(\vec{\phi}) \approx \epsilon^\flat_\Sigma(\vec{\phi}), \epsilon^\flat_\Sigma(\vec{\phi}) \approx \epsilon^\flat_\Sigma(\vec{\psi})) \\ & & \text{(by transitivity)} \\ & \subseteq & D_\Sigma(\vec{\phi} \approx \vec{\psi}, \delta^\flat_\Sigma(\vec{\phi}) \approx \epsilon^\flat_\Sigma(\vec{\phi})). \\ & & \text{(by symmetry and compatibility)}\end{array}$$

This proves the lemma. ∎

Lemma 907 has the following corollary:

Corollary 908 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{Q} = \langle \mathbf{F}^2, D \rangle$ *an equational* π*-structure. Then, for all* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*, with* k *distinguished arguments, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\tau^\flat_\Sigma[\vec{\psi}] \le D(\vec{\phi} \approx \vec{\psi}, \tau^\flat_\Sigma[\vec{\phi}]).$$

Proof: This follows from Lemma 907, using the reflexivity and the invariance of the closure family D. ∎

We next show that, if a π-institution $\mathcal{I}$, based on an algebraic system $\mathbf{F}$, happens to be equivalent to an equational π-structure $\mathcal{Q}$, based on $\mathbf{F}^2$, via a conjugate pair $(\tau, I) : \mathcal{I} \rightleftarrows \mathcal{Q}$ of transformations, then $\mathcal{I}$ is syntactically protoalgebraic with set of witnessing transformations I.

Theorem 909 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\mathcal{Q} = \langle \mathbf{F}^2, D \rangle$ *an equational* π*-structure. If* $\mathcal{I}$ *is equivalent to* $\mathcal{Q}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}$ *of transformations, then* $\mathcal{I}$ *is syntactically protoalgebraic with witnessing transformations* $I^\flat$*.*

Proof: By definition, it suffices to show that $I^\flat : \mathrm{SEN}^\omega \to \mathrm{SEN}$, with two distinguished arguments, is reflexive, globally family transitive and has the global family compatibility and the global family modus ponens in $\mathcal{I}$. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have, in turn:

- By reflexivity of $\mathcal{Q}$, $\phi \approx \phi \in D_\Sigma(\varnothing)$. Hence, by interpretability, we get $I^\flat_\Sigma[\phi, \phi] \leq C(\varnothing)$. Therefore, $I^\flat$ is reflexive in $\mathcal{I}$;
- By transitivity of $\mathcal{Q}$, $\phi \approx \chi \in D_\Sigma(\phi \approx \psi, \psi \approx \chi)$. Hence, by interpretability, we get $I^\flat_\Sigma[\phi, \chi] \leq C(I^\flat_\Sigma[\phi, \psi], I^\flat_\Sigma[\psi, \chi])$. Therefore, $I^\flat$ is globally family transitive in $\mathcal{I}$;
- By the reflexivity and compatibility of $\mathcal{Q}$, we have, for all $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in N and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, that $\sigma^\flat_\Sigma(\phi, \vec{\chi}) \approx \sigma^\flat_\Sigma(\psi, \vec{\chi}) \in D_\Sigma(\phi \approx \psi)$. Hence, by interpretability,
$$I^\flat_\Sigma[\sigma^\flat_\Sigma(\phi, \vec{\chi}), \sigma^\flat_\Sigma(\psi, \vec{\chi})] \leq C(I^\flat_\Sigma[\phi, \psi]).$$
Therefore, $I^\flat$ has the global family compatibility in $\mathcal{I}$;
- Finally, for global family MP, we have
$$\begin{array}{rcl} C(\psi) & = & C(I^\flat[\tau^\flat_\Sigma[\psi]]) \quad \text{(by equivalence)} \\ & \leq & C(I^\flat_\Sigma[\phi, \psi], I^\flat[\tau^\flat_\Sigma[\phi]]) \\ & & \text{(by Lemma 907 and interpretability)} \\ & = & C(I^\flat_\Sigma[\phi, \psi], \phi). \quad \text{(by equivalence)} \end{array}$$
Thus, for all $T \in \mathrm{ThFam}(\mathcal{I})$, if $\phi \in T_\Sigma$ and $I^\flat_\Sigma[\phi, \psi] \leq T$, then $\psi \in T_\Sigma$, i.e., $I^\flat$ has the global family modus ponens in $\mathcal{I}$.

We conclude that $\mathcal{I}$ is syntactically protoalgebraic with witnessing transformations $I^\flat$. ∎

As a consequence of Theorem 909, we obtain

Corollary 910 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\mathcal{Q} = \langle \mathbf{F}^2, D \rangle$ an equational π-structure. If $\mathcal{I}$ is equivalent to $\mathcal{Q}$ via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}$ of natural transformations, then $\mathcal{I}$ is syntactically equivalential with witnessing transformations $I^\flat$.*

Proof: By Theorem 909, $\mathcal{I}$ is syntactically protoalgebraic with witnessing transformations $I^\flat$. Since $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ is parameter free, we conclude that $\mathcal{I}$ is syntactically equivalential with witnessing transformations $I^\flat$. ∎

Using Theorem 909, we can also show that, if a π-institution $\mathcal{I}$, based on an algebraic system $\mathbf{F}$, happens to be equivalent to an equational π-structure $\mathcal{Q}$, based on $\mathbf{F}^2$, via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}$ of transformations, then $\mathcal{I}$ is family truth equational, with witnessing equations $\tau^\flat$.

Theorem 911 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\mathcal{Q} = \langle \mathbf{F}^2, D \rangle$ *an equational* π*-structure. If* $\mathcal{I}$ *is equivalent to* $\mathcal{Q}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}$ *of transformations, then* $\mathcal{I}$ *is family truth equational, with witnessing equations* $\tau^\flat$*.*

Proof: By definition, it suffices to show that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

We, indeed, have, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll} \phi \in T_\Sigma & \text{iff} & I^\flat[\tau_\Sigma[\phi]] \leq T \quad ((\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q} \text{ an equivalence}) \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(T). \quad (\text{by Theorem 909 and Corollary 791}) \end{array}$$

Therefore, $\mathcal{I}$ is family truth equational, with witnessing equations $\tau^\flat$. ■

We close the section by showing that equivalence between a given π-institution and an equational π-structure established via conjugate pairs of transformations is essentially unique in the sense that both the closure family on $\mathbf{F}^2$ must be unique and the closures of the translations used must be identical. More precisely, we have the following

Theorem 912 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Suppose that* $\mathcal{Q}^1 = \langle \mathbf{F}^2, D^1 \rangle$ *and* $\mathcal{Q}^2 = \langle \mathbf{F}^2, D^2 \rangle$ *are equational* π*-structures that are equivalent to* $\mathcal{I}$ *via the conjugate pairs* $\langle \tau^1, I^1 \rangle : \mathcal{I} \rightleftarrows \mathcal{Q}^1$ *and* $\langle \tau^2, I^2 \rangle : \mathcal{I} \rightleftarrows \mathcal{Q}^2$*, respectively, of transformations. Then, we have:*

(a) $D^1 = D^2$ $(=: D)$ *and, hence,* $\mathcal{Q}^1 = \mathcal{Q}^2$ $(=: \mathcal{Q})$*;*

(b) For all $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,* $C(I^1_\Sigma[\phi, \psi]) = C(I^2_\Sigma[\phi, \psi])$*;*

(c) For all $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,* $D(\tau^1_\Sigma[\phi]) = D(\tau^2_\Sigma[\phi])$*.*

Proof: By Theorem 909, we know that both I^1 and I^2 are witnessing the syntactic protoalgebraicity of $\mathcal{I}$. Thus, by Corollary 791, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^1_\Sigma[\phi, \psi] \leq T \quad \text{iff} \quad \langle \phi, \psi \rangle \in \Omega_\Sigma(T) \quad \text{iff} \quad I^2_\Sigma[\phi, \psi] \leq T.$$

We conclude that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $C(I^1_\Sigma[\phi, \psi]) = C(I^2_\Sigma[\phi, \psi])$, which proves Part (b).

For Part (a), suppose that $\Sigma \in |\mathbf{Sign}^\flat|$ and $E \cup \{\phi \approx \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)^2$. Then, we have

$$\begin{array}{lll} \phi \approx \psi \in D^1_\Sigma(E) & \text{iff} & I^1_\Sigma[\phi, \psi] \leq C(I^1_\Sigma[E]) \quad (\text{interpretability}) \\ & \text{iff} & C(I^1_\Sigma[\phi, \psi]) \leq C(I^1_\Sigma[E]) \\ & \text{iff} & C(I^2_\Sigma[\phi, \psi]) \leq C(I^2_\Sigma[E]) \quad (\text{Part (b)}) \\ & \text{iff} & I^2_\Sigma[\phi, \psi] \leq C(I^2_\Sigma[E]) \\ & \text{iff} & \phi \approx \psi \in D^2_\Sigma(E). \quad (\text{interpretability}) \end{array}$$

Therefore, we get that $D^1 = D^2$. This justifies using $D := D^1 = D^2$ and since the π-structures $\mathcal{Q}^1$ and $\mathcal{Q}^2$, which are both based on $\mathbf{F}^2$, have the same closure families, we obtain $\mathcal{Q} := \mathcal{Q}^1 = \mathcal{Q}^2$.

Finally, for Part (c), suppose that $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, we have

$$\begin{array}{lll} D(\tau^1_\Sigma[\phi]) \leq D(\tau^2_\Sigma[\phi]) & \text{iff} & \tau^1_\Sigma[\phi] \leq D(\tau^2_\Sigma[\phi]) \\ & \text{iff} & I^2[\tau^1_\Sigma[\phi]] \leq C(I^2[\tau^2_\Sigma[\phi]]) \quad \text{(interpretability)} \\ & \text{iff} & I^2[\tau^1_\Sigma[\phi]] \leq C(\phi) \quad \text{(equivalence)} \\ & \text{iff} & I^1[\tau^1_\Sigma[\phi]] \leq C(\phi) \quad \text{(Part (b))} \\ & \text{iff} & \phi \in C_\Sigma(\phi). \quad \text{(equivalence)} \end{array}$$

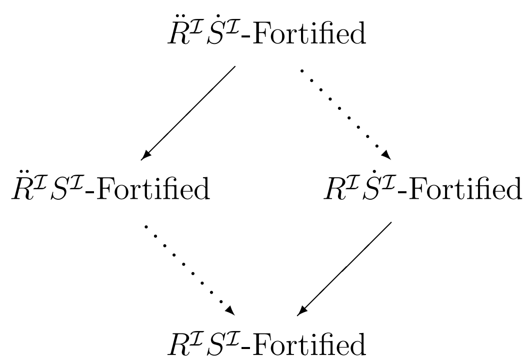
By symmetry, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $D(\tau^1_\Sigma[\phi]) = D(\tau^2_\Sigma[\phi])$. This proves Part (c) and concludes the proof of the theorem. ■

12.3 Syntactic Weak Family Algebraizability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. We say that:

- $\mathcal{I}$ is $R^{\mathcal{I}}S^{\mathcal{I}}$-(**syntactically**) **fortified** if $R^{\mathcal{I}}$ is Leibniz and $S^{\mathcal{I}}$ is adequate;
- $\mathcal{I}$ is $R^{\mathcal{I}}\dot{S}^{\mathcal{I}}$-(**syntactically**) **fortified** if $R^{\mathcal{I}}$ is Leibniz and $\dot{S}^{\mathcal{I}}$ is adequate;
- $\mathcal{I}$ is $\ddot{R}^{\mathcal{I}}S^{\mathcal{I}}$-(**syntactically**) **fortified** if $\ddot{R}^{\mathcal{I}}$ is Leibniz and $S^{\mathcal{I}}$ is adequate;
- $\mathcal{I}$ is $\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$-(**syntactically**) **fortified** if $\ddot{R}^{\mathcal{I}}$ is Leibniz and $\dot{S}^{\mathcal{I}}$ is adequate.

Recall that, by Proposition 997, if $\dot{S}^{\mathcal{I}}$ is adequate, then $S^{\mathcal{I}}$ is adequate. Moreover, since, by Proposition 952, $\ddot{R}^{\mathcal{I}} \subseteq R^{\mathcal{I}}$, it follows that, under the assumption of prealgebraicity, if $\ddot{R}^{\mathcal{I}}$ is Leibniz, then $R^{\mathcal{I}}$ is Leibniz. Thus, we have the following **syntactic fortification hierarchy** (in which the dotted arrows hold under prealgebraicity):



$\mathcal{I}$ is **syntactically weakly family algebraizable** (abbreviated to **syntactically WF algebraizable**) if:

- $\mathcal{I}$ is $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified;
- $\mathcal{I}$ is protoalgebraic;
- $\mathcal{I}$ is family injective.

By Theorem 288, under protoalgebraicity, the properties of family injectivity, family reflectivity and family c-reflectivity coincide. This enables us to formulate the following alternative characterization of syntactic WF algebraizability.

Theorem 913 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically WF algebraizable if and only if it is syntactically protoalgebraic and family truth equational.*

Proof: Assume that $\mathcal{I}$ is syntactically WF algebraizable. Then, on the one hand, it is protoalgebraic and has a Leibniz reflexive core. Thus, by Theorem 805, it is syntactically protoalgebraic. On the other, it is, by Theorem 288, family c-reflective and has an adequate Suszko core. Therefore, by Theorem 847, it is family truth equational.

Assume, conversely, that $\mathcal{I}$ is syntactically protoalgebraic and family truth equational. Then, by Theorem 805, it is protoalgebraic and has a Leibniz reflexive core, and, by Theorem 847, it is family c-reflective and has an adequate Suszko core. Therefore, $\mathcal{I}$ is syntactically WF algebraizable. ∎

Directly from the definitions, we may derive the following relationship between the semantic and syntactic WF algebraizability classes of π-institutions.

Theorem 914 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically WF algebraizable if and only if* $\mathcal{I}$ *is WF algebraizable and* $R^{\mathcal{I}}S^{\mathcal{I}}$*-fortified.*

Proof: $\mathcal{I}$ is syntactically WF algebraizable if and only if, by definition, it is $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified, protoalgebraic and family injective, i.e., iff it is, by definition, $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified and WF algebraizable. ∎

Previous results, put together, also allow us to provide an alternative characterization of syntactic weak family algebraizability in terms of isomorphisms between complete lattices of theory families.

Theorem 915 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically WF algebraizable if and only if it is* $R^{\mathcal{I}}S^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism.

Proof: We have that $\mathcal{I}$ is syntacticaly WF algebraizable if and only if, by Theorem 914, it is $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified and WF algebraizable, if and only if, by Theorem 296, it it $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism. ■

Next, we show that syntactic WF algebraizability may also be characterized by the existence of an equivalence between the π-institution and its algebraic π-structure counterpart via a pair of conjugate transformations.

We embark on the path by defining first the algebraic π-structure $\mathcal{Q}^{\mathcal{I}*}$ associated with a given π-institution $\mathcal{I}$. We recall some concepts that we have already introduced previously which culminate in the definition of $\mathcal{Q}^{\mathcal{I}*}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Recall the definition of the class $\mathrm{AlgSys}^*(\mathcal{I})$ of all reduced $\mathbf{F}$-algebraic systems:

$$\mathrm{AlgSys}^*(\mathcal{I}) = \{\mathcal{A} : (\exists T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}})\}.$$

Given an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, we define the class of $\mathcal{I}^*$**-congruence systems on** $\mathcal{A}$ by

$$\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}) = \{\theta \in \mathrm{ConSys}(\mathbf{A}) : \mathcal{A}/\theta \in \mathrm{AlgSys}^*(\mathcal{I})\}.$$

It turns out that congruence systems in $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ have a straightforward characterization.

Proposition 916 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a π-institution based on* $\mathbf{F}$. *Then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$,

$$\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}) = \{\theta \in \mathrm{ConSys}(\mathbf{A}) : (\exists T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(\Omega^{\mathcal{A}}(T) = \theta)\}.$$

Proof: Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system.

Suppose, first, that $\theta \in \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. By definition, $\mathcal{A}/\theta \in \mathrm{AlgSys}^*(\mathcal{I})$. Thus, there exists $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)$, such that

$$\Omega^{\mathcal{A}/\theta}(T') = \Delta^{\mathcal{A}/\theta}.$$

By applying the inverse of the quotient morphism $\langle I, \pi^\theta\rangle : \mathcal{A} \to \mathcal{A}/\theta$, we get

$$(\pi^\theta)^{-1}(\Omega^{\mathcal{A}/\theta}(T')) = (\pi^\theta)^{-1}(\Delta^{\mathcal{A}/\theta}).$$

Since $\langle I, \pi^\theta\rangle$ is surjective, we get by Proposition 24 and by Corollary 55, that $(\pi^\theta)^{-1}(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and

$$\Omega^{\mathcal{A}}((\pi^\theta)^{-1}(T')) = \theta.$$

Therefore, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \theta$.

Suppose, conversely, that $\theta \in \mathrm{ConSys}(\mathbf{A})$, with $\Omega^{\mathcal{A}}(T) = \theta$, for some $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, we have $\Omega^{\mathcal{A}/\theta}(T/\theta) = \Delta^{\mathcal{A}/\theta}$ and, therefore, by definition, $\mathcal{A}/\theta \in \mathrm{AlgSys}^*(\mathcal{I})$, implying that $\theta \in \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. ■

In general, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ and an $\mathbf{F}$-algebraic system $\mathcal{A}$, the family $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ of $\mathcal{I}^*$-congruence systems on $\mathcal{A}$ need not be closed under signature-wise intersections, i.e., may not form a closure family on $\mathbf{A}^2$. However, we can show that, if $\mathcal{I}$ is protoalgebraic, this is always the case.

Proposition 917 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a protoalgebraic π-institution based on* $\mathbf{F}$*. Then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is closed under arbitrary intersections and, therefore, forms a closure family on* $\mathbf{A}^2$.

Proof: First, note that $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ has a top element $\nabla^{\mathcal{A}}$. To see this, observe that $\mathcal{A}/\nabla^{\mathcal{A}}$ is a trivial algebraic system, which is always a member of $\mathrm{AlgSys}^*(\mathcal{I})$.

It suffices now to show that $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is closed under arbitrary intersections. To this end, suppose $\theta^i \in \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$, for $i \in I$. By Proposition 916, for all $i \in I$, there exists $T^i \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T^i) = \theta^i$. But, by Lemma 23 and protoalgebraicity, we get that

$$\Omega^{\mathcal{A}}(\bigcap_{i\in I} T^i) = \bigcap_{i\in I} \Omega^{\mathcal{A}}(T^i) = \bigcap_{i\in I} \theta^i.$$

Now, again by Proposition 916, we conclude that $\bigcap_{i\in I} \theta^i \in \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. ■

Applying Proposition 917 to the algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$, where $\langle I, \iota\rangle : \mathbf{F} \to \mathbf{F}$ is the identity morphism, we get the following

Corollary 918 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a protoalgebraic π-institution based on* $\mathbf{F}$*. Then,* $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{F})$ *is closed under arbitrary intersections and, therefore, forms a closure family on* $\mathbf{F}^2$.

Proof: This is a special case of Proposition 917. ■

Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a protoalgebraic π-institution. We define, in accordance with Corollary 918, the **algebraic π-structure $\mathcal{Q}^{\mathcal{I}*}$ associated with $\mathcal{I}$** to be the π-structure

$$\mathcal{Q}^{\mathcal{I}*} = \langle \mathbf{F}^2, D^{\mathcal{I}*}\rangle,$$

where $D^{\mathcal{I}*}$ is the closure (operator) family corresponding to the closure family $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{F})$.

Our first result in connecting syntactic WF algebraizability with the associated algebraic π-structure shows that, if a π-institution is syntactically WF algebraizable, then it is equivalent to its associated algebraic π-structure via a conjugate pair of transformations.

Theorem 919 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically WF algebraizable* π*-institution based on* $\mathbf{F}$*. Then* $\mathcal{I}$ *is equivalent to* $\mathcal{Q}^{\mathcal{I}*}$ *via a conjugate pair* $(\tau^\flat, \overleftrightarrow{I}^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$ *of transformations. More precisely:*

- $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, with two distinguished arguments, is a set of witnessing transformations of the syntactic protoalgebraicity of* $\mathcal{I}$*;*
- $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$*, with a single distinguished argument, is a set of witnessing equations for the family truth equationality of* $\mathcal{I}$*.*

Proof: Suppose that $\mathcal{I}$ is syntactically WF algebraizable. Then, by definition, $\mathcal{I}$ is syntactically protoalgebraic and family truth equational. Therefore, there exist a set $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ of natural transformations in $N^\flat$, with two distinguished arguments, witnessing the syntactic protoalgebraicity of $\mathcal{I}$, and a set $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ of natural transformations in $N^\flat$, with a single distinguished argument, witnessing family truth equationality. To verify the conclusion, observe, first, that $\tau^\flat_\Sigma : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SenFam}(\mathbf{F}^2)$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, as the sentence family $\tau^\flat_\Sigma[\phi]$ and $\overleftrightarrow{I}^\flat{}_\Sigma : \mathrm{SEN}^\flat(\Sigma)^2 \to \mathrm{SenFam}(\mathbf{F})$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, as the sentence family $\overleftrightarrow{I}^\flat{}_\Sigma[\phi, \psi]$ are as required. Therefore, by Proposition 898, it suffices to show that:

(a) For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in C_\Sigma(\Phi) \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq D^{\mathcal{I}*}(\tau^\flat_\Sigma[\Phi]);$$

(b) For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$D^{\mathcal{I}*}(\phi \approx \psi) = D^{\mathcal{I}*}(\tau^\flat[\overleftrightarrow{I}^\flat{}_\Sigma[\phi, \psi]]).$$

For (a), let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. Note that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, we have, by family truth equationality,

$$\begin{array}{lll} \Phi \subseteq T_\Sigma & \text{iff} & \tau^\flat_\Sigma[\Phi] \leq \Omega(T); \\ \phi \in T_\Sigma & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(T). \end{array}$$

Therefore, $\phi \in C_\Sigma(\Phi)$ if and only if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Phi \subseteq T_\Sigma$ implies $\phi \in T_\Sigma$, if and only if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\tau^\flat_\Sigma[\Phi] \leq \Omega(T)$ implies $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$, if and only if, by Proposition 916, $\tau^\flat_\Sigma[\phi] \leq D^{\mathcal{I}*}(\tau^\flat_\Sigma[\Phi])$.

For (b), let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\begin{array}{llll} \phi \approx \psi \in \Omega_\Sigma(T) & \text{iff} & \overleftrightarrow{I}^\flat{}_\Sigma[\phi, \psi] \leq T & \text{(Corollary 791)} \\ & \text{iff} & \tau^\flat[\overleftrightarrow{I}^\flat{}_\Sigma[\phi, \psi]] \leq \Omega(T). & \text{(truth equationality)} \end{array}$$

Using again Proposition 916, we conclude that

$$D^{\mathcal{I}*}(\phi \approx \psi) = D^{\mathcal{I}*}(\tau^\flat[\overleftrightarrow{I^\flat}_\Sigma[\phi,\psi]]).$$

Therefore $\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via $(\tau^\flat, \overleftrightarrow{I^\flat}) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$. ∎

Putting together Theorems 909, 911 and 919, we get the following fundamental result to the effect that syntactic WF algebraizability boils down to the equivalence of a π-institution with its associated algebraic π-structure via a conjugate pair of transformations.

Theorem 920 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically WF algebraizable if and only if it is equivalent to* $\mathcal{Q}^{\mathcal{I}*}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$ *of transformations.*

Proof: If $\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair of transformations, then, by Theorem 909, it is syntactically protoalgebraic and, by Theorem 911, it is family truth equational. Therefore, by definition, it is syntactically WF algebraizable.

If, conversely, $\mathcal{I}$ is syntactically WF algebraizable, then, by Theorem 919, it is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair of transformations. ∎

We close the section by slightly generalizing the preceding characterization. Namely, we show that existence of an equivalence with an algebraic π-structure induced by conjugate transformations is sufficient to yield syntactic WF algebraizability.

Theorem 921 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically WF algebraizable if and only if it is equivalent to an algebraic* π*-structure via a conjugate pair of transformations.*

Proof: If $\mathcal{I}$ is syntactically WF algebraizable, then the conclusion follows from Theorem 920. Conversely, if $\mathcal{I}$ is equivalent to an algebraic π-structure via a conjugate pair of transformations, then it is syntactically protoalgebraic by Theorem 909 and family truth equational by Theorem 911, whence it is syntactically WF algebraizable. ∎

Taking into account Theorem 901, we have the following alternative characterization of syntactically WF algebraizable π-institutions:

Theorem 922 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically WF algebraizable if and only if there is a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{I}) \to \mathbf{ThFam}(\mathcal{Q})$*, where* $\mathcal{Q}$ *is an algebraic* π*-structure.*

Proof: The "only if" follows by Theorem 921 and Theorem 893. The "if" is given by Theorem 901 and Theorem 921. ∎

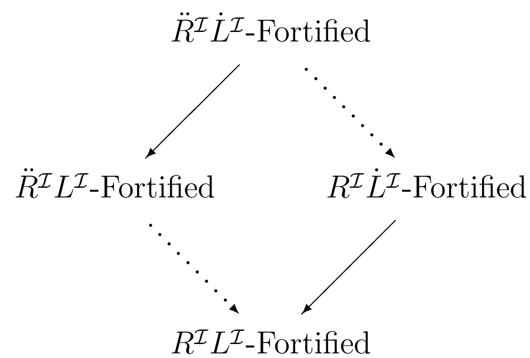
12.4 Syntactic Weak Algebraizability

Syntactic WF algebraizability determines one of the highest levels of the main algebraic hierarchy of π-institutions. Since every syntactically WF algebraizable π-institution is, in particular, family reflective, it follows that every syntactically WF algebraizable π-institution is systemic. To avoid systemicity, one has to weaken the hypothesis of family reflectivity. In this section we follow this line of thought by keeping the assumption of syntactic protoalgebraicity, but insisting only that the π-institution is system truth equational, rather than family truth equational.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that:

- $\mathcal{I}$ is $R^{\mathcal{I}}L^{\mathcal{I}}$-(**syntactically**) **fortified** if $R^{\mathcal{I}}$ is Leibniz and $L^{\mathcal{I}}$ is left adequate;
- $\mathcal{I}$ is $R^{\mathcal{I}}\dot{L}^{\mathcal{I}}$-(**syntactically**) **fortified** if $R^{\mathcal{I}}$ is Leibniz and $\dot{L}^{\mathcal{I}}$ is left adequate;
- $\mathcal{I}$ is $\ddot{R}^{\mathcal{I}}L^{\mathcal{I}}$-(**syntactically**) **fortified** if $\ddot{R}^{\mathcal{I}}$ is Leibniz and $L^{\mathcal{I}}$ is left adequate;
- $\mathcal{I}$ is $\ddot{R}^{\mathcal{I}}\dot{L}^{\mathcal{I}}$-(**syntactically**) **fortified** if $\ddot{R}^{\mathcal{I}}$ is Leibniz and $\dot{L}^{\mathcal{I}}$ is left adequate.

Similarly with the Suszko core, it can be seen that, if $\dot{L}^{\mathcal{I}}$ is left adequate, then $L^{\mathcal{I}}$ is left adequate. Moreover, since, by Proposition 952, $\ddot{R}^{\mathcal{I}} \subseteq R^{\mathcal{I}}$, it follows that, under the assumption of prealgebraicity, if $\ddot{R}^{\mathcal{I}}$ is Leibniz, then $R^{\mathcal{I}}$ is Leibniz. Thus, we have the following **syntactic left fortification hierarchy** (in which the dotted arrows hold under prealgebraicity):



$\mathcal{I}$ is **syntactically weakly algebraizable** (abbreviated to **syntactically W algebraizable**) if:

- $\mathcal{I}$ is $R^{\mathcal{I}}L^{\mathcal{I}}$-fortified;
- $\mathcal{I}$ is protoalgebraic;

- $\mathcal{I}$ is system injective.

By Corollary 300, under protoalgebraicity, the six properties of system injectivity, left injectivity, system reflectivity, left reflectivity, system complete reflectivity and left complete reflectivity coincide. This enables us to formulate the following alternative characterization of syntactic weak algebraizability.

Theorem 923 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically weakly algebraizable if and only if it is syntactically protoalgebraic and system (or, equivalently, left) truth equational.*

Proof: Assume that $\mathcal{I}$ is syntactically weakly algebraizable. Then, on the one hand, it is protoalgebraic and has a Leibniz reflexive core. Thus, by Theorem 805, it is syntactically protoalgebraic. On the other, it is, by Theorem 300, left c-reflective and has a left adequate left Suszko core. Therefore, by Theorem **??**, it is left truth equational.

Assume, conversely, that $\mathcal{I}$ is syntactically protoalgebraic and left truth equational. Then, by Theorem 805, it is protoalgebraic and has a Leibniz reflexive core, and, by Theorem 870, it is left c-reflective and has a left adequate left Suszko core. Therefore, by definition, $\mathcal{I}$ is syntactically weakly algebraizable. ■

Directly from the definitions, we may derive the following relationship between the semantic and syntactic weak algebraizability classes of π-institutions.

Theorem 924 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically weakly algebraizable if and only if* $\mathcal{I}$ *is weakly algebraizable and* $R^{\mathcal{I}} L^{\mathcal{I}}$*-fortified.*

Proof: $\mathcal{I}$ is syntactically weakly algebraizable if and only if, by definition, it is $R^{\mathcal{I}} L^{\mathcal{I}}$-fortified, protoalgebraic and system injective, i.e., iff it is, by definition, $R^{\mathcal{I}} L^{\mathcal{I}}$-fortified and weakly algebraizable. ■

Previous results, put together, also allow us to provide an alternative characterization of syntactic weak algebraizability in terms of isomorphisms between complete lattices of theory systems.

Theorem 925 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically weakly algebraizable if and only if it is* $R^{\mathcal{I}} L^{\mathcal{I}}$*-fortified, stable and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism.

Proof: We have that $\mathcal{I}$ is syntactically weakly algebraizable if and only if, by Theorem 924, it is $R^{\mathcal{I}}L^{\mathcal{I}}$-fortified and weakly algebraizable, if and only if, by Theorem 298, it it $R^{\mathcal{I}}L^{\mathcal{I}}$-fortified, stable and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism. ■

Next, we show that syntactic weak algebraizability may also be characterized by stability in conjunction with the existence of an equivalence between the systemic skeleton of a π-institution and its algebraic π-structure counterpart via a pair of conjugate transformations. To start, we define the *systemic skeleton* of a given π-institution.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Recall that $\mathrm{ThSys}(\mathcal{I})$ forms a complete lattice $\mathbf{ThSys}(\mathcal{I}) = \langle \mathrm{ThSys}(\mathcal{I}), \leq\rangle$ under signature wise inclusion. Therefore, we are justified in defining the π-structure

$$\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}}\rangle$$

of $\mathcal{I}$ by stipulating that $K^{\mathcal{I}} : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}$ is the closure family on $\mathbf{F}$ corresponding to the closed set family $\mathrm{ThSys}(\mathcal{I})$. We call $\mathcal{K}^{\mathcal{I}}$ the **systemic skeleton** of $\mathcal{I}$.

We give an example to show that, in general, $K^{\mathcal{I}}$ is not a π-institution, since $K^{\mathcal{I}} : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}$ may not satisfy structurality.

Example 926 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be defined as follows:*

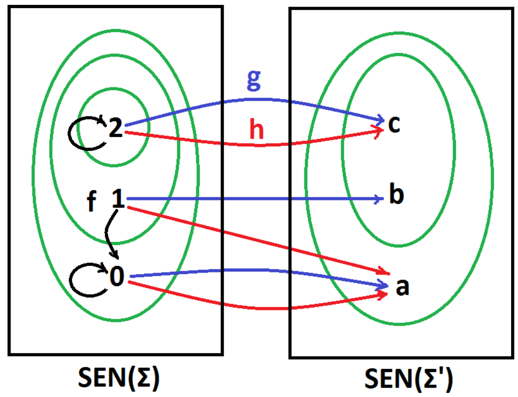
- $\mathbf{Sign}^\flat$ *is the category with objects* Σ, Σ' *and, except the identities, a morphism* $f : \Sigma \to \Sigma$ *and two morphisms* $g, h : \Sigma \to \Sigma'$*, satisfying the following composition rules:*

$$f \circ f = f, \quad gf = h, \quad hf = h.$$

- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by setting* $\mathrm{SEN}^\flat(\Sigma) = \{0,1,2\}$*,* $\mathrm{SEN}^\flat(\Sigma') = \{a,b,c\}$ *and*

$x \in \mathrm{SEN}^\flat(\Sigma)$	$\mathrm{SEN}^\flat(f)(x)$	$\mathrm{SEN}^\flat(g)(x)$	$\mathrm{SEN}^\flat(h)(x)$
0	0	a	a
1	0	b	a
2	2	c	c

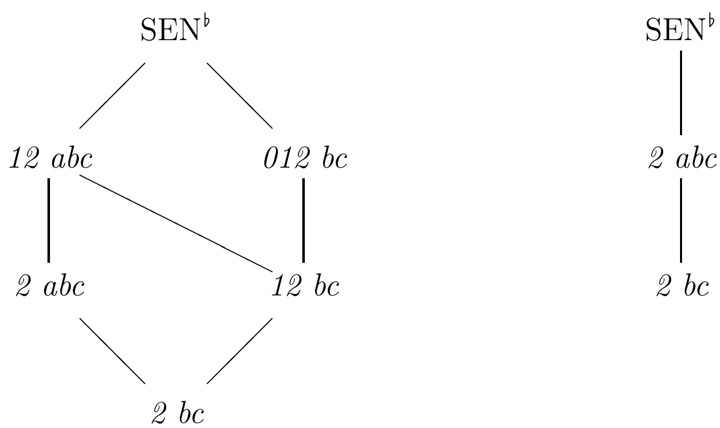
- *Finally,* $N^\flat$ *is the trivial category of natural transformations (consisting of the projections only).*



Next define the π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ by setting

$$\mathcal{C}_\Sigma = \{\{2\}, \{1,2\}, \{0,1,2\}\} \quad \textit{and} \quad \mathcal{C}_{\Sigma'} = \{\{b,c\}, \{a,b,c\}\}.$$

This π-institution has six theory families, having the lattice structure shown on the left below. It has, however, only three theory systems, whose lattice structure is given on the right.



The theory systems of $\mathcal{I}$ are the theory families of the systemic skeleton $\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}} \rangle$. We can see that $\mathcal{K}^{\mathcal{I}}$ is not a π-institution by considering $\Phi = \{1\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. We have

$$\begin{array}{rcl}\mathrm{SEN}^\flat(g)(K^{\mathcal{I}}_\Sigma(\{1\})) & = & \mathrm{SEN}^\flat(g)(\bigcap\{T_\Sigma : \{\{1\}, \varnothing\} \leq T \in \mathrm{ThSys}(\mathcal{I})\}) \\ & = & \mathrm{SEN}^\flat(g)(\{0,1,2\}) \\ & = & \{a,b,c\}.\end{array}$$

On the other hand,

$$\begin{array}{rcl}K^{\mathcal{I}}_{\Sigma'}(\mathrm{SEN}^\flat(g)(\{1\})) & = & K^{\mathcal{I}}_{\Sigma'}(\{b\}) \\ & = & \bigcap\{T_{\Sigma'} : \{\varnothing, \{b\}\} \leq T \in \mathrm{ThSys}(\mathcal{I})\} \\ & = & \{b,c\}.\end{array}$$

Therefore

$$\mathrm{SEN}^\flat(g)(K^{\mathcal{I}}_\Sigma(\{1\})) \notin K^{\mathcal{I}}_{\Sigma'}(\mathrm{SEN}^\flat(g)(\{1\}))$$

showing that $K^{\mathcal{I}}$ *is not structural and, hence,* $\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}}\rangle$ *is a* π*-structure, but not a* π*-institution.*

We now resume our work on the characterization of syntactic weak algebraizability. We will again make use of the algebraic π-structure $\mathcal{Q}^{\mathcal{I}*} = \langle \mathbf{F}^2, D^{\mathcal{I}*}\rangle$ associated with a protoalgebraic π-institution $\mathcal{I}$. Recall that this is the π-structure whose closure family is the one corresponding to the closure set family $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{F})$.

Our first result connecting syntactic weak algebraizability of a π-institution with the associated algebraic π-structure shows that, if a π-institution is syntactically weakly algebraizable, then its systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to its associated algebraic π-structure $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair of transformations.

Theorem 927 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a syntactically weakly algebraizable* π*-institution based on* $\mathbf{F}$*. Then* $\mathcal{K}^{\mathcal{I}}$ *is equivalent to* $\mathcal{Q}^{\mathcal{I}*}$ *via a conjugate pair* $(\tau^\flat, \overleftrightarrow{I}^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$ *of transformations. More precisely:*

- $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, with two distinguished arguments, is a set of witnessing transformations of the syntactic protoalgebraicity of* $\mathcal{I}$*;*
- $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$*, with a single distinguished argument, is a set of witnessing equations for the left truth equationality of* $\mathcal{I}$*.*

Proof: Suppose that $\mathcal{I}$ is syntactically weakly algebraizable. Then, by definition, $\mathcal{I}$ is syntactically protoalgebraic and left truth equational. Therefore, there exist a set $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ of natural transformations in $N^\flat$, with two distinguished arguments, witnessing the syntactic protoalgebraicity of $\mathcal{I}$, and a set $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ of natural transformations in $N^\flat$, with a single distinguished argument, witnessing left truth equationality. To verify the conclusion, observe, first, that $\tau^\flat_\Sigma : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SenFam}(\mathbf{F}^2)$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, as the sentence family $\tau^\flat_\Sigma[\phi]$ and $\overleftrightarrow{I}^\flat_\Sigma : \mathrm{SEN}^\flat(\Sigma)^2 \to \mathrm{SenFam}(\mathbf{F})$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, as the sentence family $\overleftrightarrow{I}^\flat_\Sigma[\phi, \psi]$ are as required. Therefore, by Proposition 898, it suffices to show that:

(a) For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in K^{\mathcal{I}}_\Sigma(\Phi) \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq D^{\mathcal{I}*}(\tau^\flat_\Sigma[\Phi]);$$

(b) For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$D^{\mathcal{I}*}(\phi \approx \psi) = D^{\mathcal{I}*}(\tau^\flat[\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi]]).$$

For (a), let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. Note that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, we have

$$\begin{array}{lll} \Phi \subseteq T_\Sigma & \text{iff} & \Phi \subseteq \overleftarrow{T}_\Sigma \quad (T \in \mathrm{ThSys}(\mathcal{I})) \\ & \text{iff} & \tau^\flat_\Sigma[\Phi] \leq \Omega(T) \quad \text{(left truth equationality)} \end{array}$$

and, similarly,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

Therefore, $\phi \in K^{\mathcal{I}}_\Sigma(\Phi)$ if and only if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\Phi \subseteq T_\Sigma$ implies $\phi \in T_\Sigma$, if and only if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\tau^\flat_\Sigma[\Phi] \leq \Omega(T)$ implies $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$, if and only if, by stability, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\tau^\flat_\Sigma[\Phi] \leq \Omega(T)$ implies $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$, if and only if, by Proposition 916, $\tau^\flat_\Sigma[\phi] \leq D^{\mathcal{I}*}(\tau^\flat_\Sigma[\Phi])$.

For (b), let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have, for all $T \in \mathrm{ThSys}(\mathcal{I})$,

$$\begin{array}{lll} \phi \approx \psi \in \Omega_\Sigma(T) & \text{iff} & \overleftrightarrow{I^\flat}_\Sigma[\phi, \psi] \leq T \quad \text{(Corollary 791)} \\ & \text{iff} & \overleftrightarrow{I^\flat}_\Sigma[\phi, \psi] \leq \overleftarrow{T} \quad (T \in \mathrm{ThSys}(\mathcal{I})) \\ & \text{iff} & \tau^\flat[\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi]] \leq \Omega(T). \quad \text{(left truth equationality)} \end{array}$$

Using again Proposition 916 and stability, we conclude that

$$D^{\mathcal{I}*}(\phi \approx \psi) = D^{\mathcal{I}*}(\tau^\flat[\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi]]).$$

Therefore $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via $(\tau^\flat, \overleftrightarrow{I^\flat}) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$. ■

Towards the converse, we show, first, that, if a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is such that there exists an equivalence $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}$, via a conjugate pair of transformations, between its systemic skeleton and an algebraic π-structure $\mathcal{Q}$, then $I^\flat$ defines Leibniz congruence systems of theory systems of $\mathcal{I}$.

Recall that for a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, based on an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$, and a set $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ of natural transformations in $N^\flat$, with two distinguished arguments, we define, for all $T \in \mathrm{SenFam}(\mathcal{I})$, $I^\flat(T) = \{I^\flat_\Sigma(T)\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$I^\flat_\Sigma(T) = \{\langle \phi, \psi \rangle \in \mathrm{SEN}^\flat(\Sigma)^2 : I^\flat_\Sigma[\phi, \psi] \leq T\}.$$

Proposition 928 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}} \rangle$ is equivalent to an algebraic π-structure $\mathcal{Q}$ via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}$ of transformations, then, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\Omega(T) = I^\flat(T)$.*

Proof: Let $T \in \mathrm{ThSys}(\mathcal{I})$. It suffices to show, by Corollary 98, that $I^\flat(T)$ is a congruence system on $\mathbf{F}$ compatible with T. We know by Lemma 93 that it is a relation system on $\mathbf{F}$.

Suppose $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Since $\mathcal{Q} = \langle \mathbf{F}^2, D\rangle$ is algebraic, we have $\phi \approx \phi \in D_\Sigma(\varnothing)$. Therefore, by interpretability, $I^\flat_\Sigma[\phi,\phi] \le K^{\mathcal{I}}(\varnothing) = C(\varnothing) \le T$. Hence, $\langle \phi,\phi\rangle \in I^\flat_\Sigma(T)$ and $I^\flat(T)$ is reflexive.

Suppose, now, that $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$. Since $\mathcal{Q}$ is algebraic, we have that $\psi \approx \phi \in D_\Sigma(\phi \approx \psi)$. Therefore, by interpretability, $I^\flat_\Sigma[\psi,\phi] \le K^{\mathcal{I}}(I^\flat_\Sigma[\phi,\psi])$. Since $T \in \mathrm{ThSys}(\mathcal{I})$, this implies that, if $I^\flat_\Sigma[\phi,\psi] \le T$, then $I^\flat_\Sigma[\psi,\phi] \le T$. In other words $\langle\phi,\psi\rangle \in I^\flat_\Sigma(T)$ implies $\langle\psi,\phi\rangle \in I^\flat_\Sigma(T)$. Therefore, $I^\flat(T)$ is also symmetric.

Suppose, next, that $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi,\chi \in \mathrm{SEN}^\flat(\Sigma)$. Since $\mathcal{Q}$ is algebraic, we have that $\phi \approx \chi \in D_\Sigma(\phi \approx \psi, \psi \approx \chi)$. Therefore, by interpretability, $I^\flat_\Sigma[\phi,\chi] \le K^{\mathcal{I}}(I^\flat_\Sigma[\phi,\psi], I^\flat_\Sigma[\psi,\chi])$. Since $T \in \mathrm{ThSys}(\mathcal{I})$, this implies that, if $I^\flat_\Sigma[\phi,\psi], I^\flat_\Sigma[\psi,\chi] \le T$, then $I^\flat_\Sigma[\phi,\chi] \le T$. In other words, $\langle\phi,\psi\rangle, \langle\psi,\chi\rangle \in I^\flat_\Sigma(T)$ imply $\langle\phi,\chi\rangle \in I^\flat_\Sigma(T)$. Therefore, $I^\flat(T)$ is transitive.

We have now shown that $I^\flat(T)$ is an equivalence system on $\mathbf{F}$. It remains to show that it satisfies the congruence property and that it is compatible with T.

Suppose that $\sigma^\flat \in N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi},\vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$. Since $\mathcal{Q}$ is algebraic, we have that $\sigma^\flat_\Sigma(\vec{\phi}) \approx \sigma^\flat_\Sigma(\vec{\psi}) \in D_\Sigma(\vec{\phi} \approx \vec{\psi})$ (recall that $\vec{\phi} \approx \vec{\psi}$ means $\{\phi_i \approx \psi_i : i < k\}$). Therefore, by interpretability,

$$I^\flat_\Sigma[\sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi})] \le K^{\mathcal{I}}(\bigcup\{I^\flat_\Sigma[\phi_i,\psi_i] : i < k\}).$$

Since $T \in \mathrm{ThSys}(\mathcal{I})$, this implies that, if, for all $i < k$, $I^\flat_\Sigma[\phi_i,\psi_i] \le T$, then $I^\flat_\Sigma[\sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi})] \le T$. In other words $\langle\phi_i,\psi_i\rangle \in I^\flat_\Sigma(T)$, for all $i < k$, imply $\langle\sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi})\rangle \in I^\flat_\Sigma(T)$. Therefore, $I^\flat(T)$ satisfies the congruence property.

Finally, to see that $I^\flat(T)$ is compatible with T, suppose that $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$. Since $\mathcal{Q}$ is algebraic and $\tau^\flat \in N^\flat$, we have, by Lemma 907,

$$\tau^\flat_\Sigma[\psi] \le D(\tau^\flat_\Sigma[\phi], \phi \approx \psi).$$

By interpretability, this yields

$$I^\flat[\tau^\flat_\Sigma[\psi]] \le K^{\mathcal{I}}(I^\flat[\tau^\flat_\Sigma[\phi]], I^\flat_\Sigma[\phi,\psi]).$$

Since $(\tau^\flat, I^\flat)$ is a conjugate pair, the latter is equivalent to

$$\psi \in K^{\mathcal{I}}_\Sigma(\phi, I^\flat_\Sigma[\phi,\psi]).$$

In other words, for all $T \in \mathrm{ThSys}(\mathcal{I})$,

$$\phi \in T_\Sigma \quad \text{and} \quad \langle\phi,\psi\rangle \in I^\flat_\Sigma(T) \quad \text{imply} \quad \psi \in T_\Sigma.$$

Hence $I^\flat(T)$ is compatible with T. ■

Using Proposition 928, we can show that stability and the existence of an equivalence between the systemic skeleton and an algebraic π-structure ensure syntactic protoalgebraicity.

Theorem 929 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is stable and its systemic skeleton* $\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}} \rangle$ *is equivalent to an algebraic* π*-structure* $\mathcal{Q}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}$ *of transformations, then* $\mathcal{I}$ *is syntactically protoalgebraic, with witnessing transformations* $I^\flat$.

Proof: Suppose that $\mathcal{I}$ is stable and its systemic skeleton $\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}} \rangle$ is equivalent to an algebraic π-structure $\mathcal{Q}$ via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}$ of transformations. Then, we have, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\begin{array}{rcll} \Omega(T) & = & \Omega(\overleftarrow{T}) & \text{(by stability)} \\ & = & I^\flat(\overleftarrow{T}) & \text{(by Proposition 928)} \\ & = & I^\flat(T). & \text{(by Proposition 99)} \end{array}$$

Therefore, $\mathcal{I}$ is syntactically protoalgebraic with witnessing transformations $I^\flat$. ■

Finally, before the main theorem, we show that stability and the existence of a transformational equivalence between the systemic skeleton and an algebraic π-structure ensure left truth equationality.

Theorem 930 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is stable and its systemic skeleton* $\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}} \rangle$ *is equivalent to an algebraic* π*-structure* $\mathcal{Q}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}$ *of transformations, then* $\mathcal{I}$ *is left truth equational, with witnessing equations* $\tau^\flat$.

Proof: We have, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rlll} \phi \in \overleftarrow{T}_\Sigma & \text{iff} & I^\flat[\tau^\flat_\Sigma[\phi]] \le \overleftarrow{T} & ((\tau^\flat, I^\flat) \text{ an equivalence}) \\ & \text{iff} & I^\flat[\tau^\flat_\Sigma[\phi]] \le T & \text{(by Proposition 99)} \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \le \Omega(T). & \text{(by Theorem 929)} \end{array}$$

Therefore, $\mathcal{I}$ is left truth equational, with witnessing equations $\tau^\flat$. ■

Putting together Theorems 929, 930 and 927, we get the following fundamental result to the effect that syntactic weak algebraizability boils down to stability, together with the equivalence of the systemic skeleton of a π-institution with its associated algebraic π-structure via a conjugate pair of transformations.

Theorem 931 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically weakly algebraizable if and only if it is stable and its systemic skeleton* $\mathcal{K}^{\mathcal{I}}$ *is equivalent to* $\mathcal{Q}^{\mathcal{I}*}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$ *of transformations.*

Proof: Suppose, first, that $\mathcal{I}$ is stable and that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair of transformations. Then, by Theorem 929, it is syntactically protoalgebraic and, by Theorem 930, it is left truth equational. Therefore, by definition, it is syntactically weakly algebraizable.

If, conversely, $\mathcal{I}$ is syntactically weakly algebraizable, then, on the one hand, it is protoalgebraic and, therefore, stable, and, on the other, by Theorem 927, it is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair of transformations. ∎

Generalizing again, we show that stability together with the existence of an equivalence of the systemic skeleton with an algebraic π-structure, induced by conjugate transformations, is sufficient to yield syntactic weak algebraizability.

Theorem 932 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically weakly algebraizable if and only if it is stable and its systemic skeleton is equivalent to an algebraic* π*-structure via a conjugate pair of transformations.*

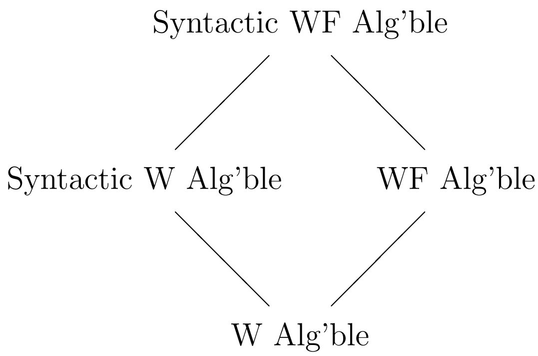
Proof: If $\mathcal{I}$ is syntactically weakly algebraizable, then the conclusion follows from Theorem 931. Conversely, if $\mathcal{K}^{\mathcal{I}}$ is equivalent to an algebraic π-structure via a conjugate pair of transformations, then $\mathcal{I}$ is syntactically protoalgebraic by Theorem 929 and left truth equational by Theorem 930, whence it is syntactically weakly algebraizable. ∎

Finally, in terms of order isomorphisms between theory family lattices, we have the following alternative characterization of syntactically weakly algebraizable π-institutions:

Theorem 933 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically weakly algebraizable if and only if it is stable and there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{K}^{\mathcal{I}}) \to \mathbf{ThFam}(\mathcal{Q})$*, where* $\mathcal{Q}$ *is an algebraic* π*-structure.*

Proof: The "only if" follows by Theorem 932 and Theorem 893. The "if" is given by Theorem 901 and Theorem 932. ∎

Let us give, in closing the section, the picture of the **weak algebraizability hierarchy** that we have established, consisting of both semantic and syntactic classes of π-institutions.



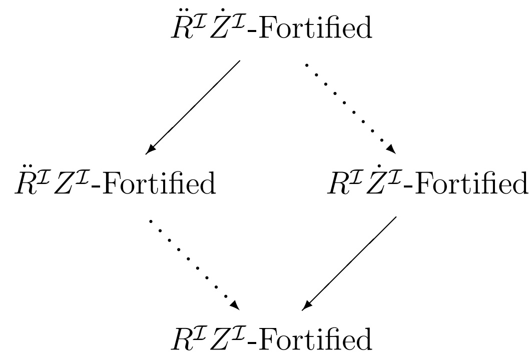
12.5 Syntactic WS PreAlgebraizability

Syntactic WS prealgebraizability, requires, like syntactic WF algebraizability, the monotonicity of the Leibniz operator on theory systems and the injectivity of the Leibniz operator on theory systems but, unlike WF algebraizability, it requires these two properties only on theory systems and not on the entire complete lattice of theory families. As a consequence of this weakened requirement, syntactic WS prealgebraizability implies neither systemicity (as does syntactic WF algebraizability) nor the even weaker condition of stability (as do both kinds of syntactic algebraizability). Thus, as other conditions that were under our scrutiny previously, it allows us to consider for membership π-institutions that are not necessarily stable.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that:

- $\mathcal{I}$ is $R^{\mathcal{I}}Z^{\mathcal{I}}$-(**syntactically**) **fortified** if $R^{\mathcal{I}}$ is Leibniz and $Z^{\mathcal{I}}$ is adequate;
- $\mathcal{I}$ is $R^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-(**syntactically**) **fortified** if $R^{\mathcal{I}}$ is Leibniz and $\dot{Z}^{\mathcal{I}}$ is adequate;
- $\mathcal{I}$ is $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$-(**syntactically**) **fortified** if $\ddot{R}^{\mathcal{I}}$ is Leibniz and $Z^{\mathcal{I}}$ is adequate;
- $\mathcal{I}$ is $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-(**syntactically**) **fortified** if $\ddot{R}^{\mathcal{I}}$ is Leibniz and $\dot{Z}^{\mathcal{I}}$ is adequate.

Similarly with the Suszko core, it can be seen that, if $\dot{Z}^{\mathcal{I}}$ is adequate, then $Z^{\mathcal{I}}$ is adequate. Moreover, since, by Proposition 952, $\ddot{R}^{\mathcal{I}} \subseteq R^{\mathcal{I}}$, it follows that, under the assumption of prealgebraicity, if $\ddot{R}^{\mathcal{I}}$ is Leibniz, then $R^{\mathcal{I}}$ is Leibniz. Thus, we have the following **syntactic system fortification hierarchy** (in which the dotted arrows hold under prealgebraicity):



$\mathcal{I}$ is **syntactically weakly system prealgebraizable** (abbreviated to **syntactically WS prealgebraizable**) if:

- $\mathcal{I}$ is $R^{\mathcal{I}}Z^{\mathcal{I}}$-fortified;
- $\mathcal{I}$ is prealgebraic;
- $\mathcal{I}$ is system injective.

By Theorem 248, under prealgebraicity, the properties of system injectivity, system reflectivity and system complete reflectivity coincide. As a result, we have the following alternative characterization of syntactic weak system prealgebraizability.

Theorem 934 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically weakly system prealgebraizable if and only if it is syntactically prealgebraic and system truth equational.*

Proof: Assume that $\mathcal{I}$ is syntactically weakly system prealgebraizable. Then, on the one hand, it is prealgebraic and has a Leibniz reflexive core. Thus, by Theorem 788, it is syntactically prealgebraic. On the other, it is, by Theorem 248, system c-reflective and has an adequate system core. Therefore, by Theorem 887, it is system truth equational.

Assume, conversely, that $\mathcal{I}$ is syntactically prealgebraic and system truth equational. Then, by Theorem 788, it is prealgebraic and has a Leibniz reflexive core, and, by Theorem 887, it is system c-reflective and has an adequate system core. Therefore, by definition, $\mathcal{I}$ is syntactically weakly system prealgebraizable. ■

Directly from the definitions, we may derive the following relationship between the semantic and syntactic weak system prealgebraizability classes of π-institutions.

Theorem 935 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically weakly system prealgebraizable if and only if* $\mathcal{I}$ *is weakly system prealgebraizable and* $R^{\mathcal{I}} Z^{\mathcal{I}}$*-fortified.*

Proof: $\mathcal{I}$ is syntactically weakly system prealgebraizable if and only if, by definition, it is $R^{\mathcal{I}} Z^{\mathcal{I}}$-fortified, prealgebraic and system injective, i.e., iff it is, by definition, $R^{\mathcal{I}} Z^{\mathcal{I}}$-fortified and weakly system prealgebraizable. ∎

Previous results, put together, also allow us to provide an alternative characterization of syntactic weak system prealgebraizability in terms of morphisms between complete lattices of theory systems.

Theorem 936 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically weakly system prealgebraizable if and only if it is* $R^{\mathcal{I}} Z^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order embedding.

Proof: We have that $\mathcal{I}$ is syntactically weakly system prealgebraizable if and only if, by Theorem 935, it is $R^{\mathcal{I}} Z^{\mathcal{I}}$-fortified and weakly system prealgebraizable, if and only if, by Theorem 256, it it $R^{\mathcal{I}} Z^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order embedding ∎

Next, we show that syntactic weak system prealgebraizability may also be characterized by the existence of an equivalence between the systemic skeleton of a π-institution and an algebraic π-structure associated with the π-institution (different, in general, than $\mathcal{Q}^{\mathcal{I}*}$) via a pair of conjugate transformations.

We embark on the path by defining first the algebraic π-structure $\mathcal{Q}^{\mathcal{I}\bullet}$ associated with a given π-institution $\mathcal{I}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall the definition of the class $\mathrm{AlgSys}^\bullet(\mathcal{I})$ of all $\mathbf{F}$-algebraic systems reduced with respect to $\mathcal{I}$-filter systems:

$$\mathrm{AlgSys}^\bullet(\mathcal{I}) = \{\mathcal{A} : (\exists T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}))(\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}})\}.$$

Given an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, we define the class of $\mathcal{I}^\bullet$-**congruence systems on** $\mathcal{A}$ by

$$\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A}) = \{\theta \in \mathrm{ConSys}(\mathbf{A}) : \mathcal{A}/\theta \in \mathrm{AlgSys}^\bullet(\mathcal{I})\}.$$

It turns out that congruence systems in $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$ have a straightforward characterization.

Proposition 937 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. Then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A}) = \{\theta \in \mathrm{ConSys}(\mathbf{A}) : (\exists T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}))(\Omega^{\mathcal{A}}(T) = \theta)\}.$$

Proof: Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system.

Suppose, first, that $\theta \in \mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$. By definition, $\mathcal{A}/\theta \in \mathrm{AlgSys}^\bullet(\mathcal{I})$. Thus, there exists $T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}/\theta)$, such that

$$\Omega^{\mathcal{A}/\theta}(T') = \Delta^{\mathcal{A}/\theta}.$$

By applying the inverse of the quotient morphism $\langle I, \pi^\theta \rangle : \mathcal{A} \to \mathcal{A}/\theta$, we get

$$(\pi^\theta)^{-1}(\Omega^{\mathcal{A}/\theta}(T')) = (\pi^\theta)^{-1}(\Delta^{\mathcal{A}/\theta}).$$

Since $\langle I, \pi^\theta \rangle$ is surjective, we get by Proposition 24 and Corollary 55, that $(\pi^\theta)^{-1}(T') \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$ and

$$\Omega^{\mathcal{A}}((\pi^\theta)^{-1}(T')) = \theta.$$

Therefore, there exists $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \theta$.

Suppose, conversely, that $\theta \in \mathrm{ConSys}(\mathbf{A})$, with $\Omega^{\mathcal{A}}(T) = \theta$, for some $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$. Then, we have $\Omega^{\mathcal{A}/\theta}(T/\theta) = \Delta^{\mathcal{A}/\theta}$ and, therefore, by definition, $\mathcal{A}/\theta \in \mathrm{AlgSys}^\bullet(\mathcal{I})$, implying that $\theta \in \mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$. ■

In general, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ and an $\mathbf{F}$-algebraic system $\mathcal{A}$, the family $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$ of systemic $\mathcal{I}$-congruence systems on $\mathcal{A}$ need not be closed under signature-wise intersections, i.e., may not form a closure family on $\mathbf{A}^2$. However, we can show that, if $\mathcal{I}$ is prealgebraic, this is always the case.

Proposition 938 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a prealgebraic* π*-institution based on* $\mathbf{F}$*. Then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$ *is closed under arbitrary intersections and, therefore, forms a closure family on* $\mathbf{A}^2$*.*

Proof: First, note that $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$ has a top element $\nabla^{\mathcal{A}}$. To see this, observe that $\mathcal{A}/\nabla^{\mathcal{A}}$ is a trivial algebraic system, which is always a member of $\mathrm{AlgSys}^\bullet(\mathcal{I})$.

It suffices now to show that $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$ is closed under arbitrary intersections. To this end, suppose $\theta^i \in \mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$, for $i \in I$. By Proposition 937, for all $i \in I$, there exists $T^i \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}/\theta^i)$, such that $\Omega^{\mathcal{A}}(T^i) = \theta^i$. But, by Lemma 23 and prealgebraicity, we get that

$$\Omega^{\mathcal{A}}(\bigcap_{i \in I} T^i) = \bigcap_{i \in I} \Omega^{\mathcal{A}}(T^i) = \bigcap_{i \in I} \theta^i.$$

Now, again by Proposition 937, we conclude that $\bigcap_{i\in I}\theta^i \in \mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{A})$. ■

Applying Proposition 938 to the algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$, where $\langle I, \iota\rangle : \mathbf{F} \to \mathbf{F}$ is the identity morphism, we get the following

Corollary 939 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a protoalgebraic π-institution based on $\mathbf{F}$. Then, $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{F})$ is closed under arbitrary intersections and, therefore, forms a closure family on $\mathbf{F}^2$.*

Proof: This is a special case of Proposition 938. ■

Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a prealgebraic π-institution. We define, in accordance with Corollary 939, the **systemic algebraic π-structure $\mathcal{Q}^{\mathcal{I}\bullet}$ associated with $\mathcal{I}$** to be the π-structure $\mathcal{Q}^{\mathcal{I}\bullet} = \langle \mathbf{F}^2, D^{\mathcal{I}\bullet}\rangle$, where $D^{\mathcal{I}\bullet}$ is the closure (operator) family corresponding to the closure family $\mathrm{ConSys}^{\mathcal{I}\bullet}(\mathcal{F})$.

We recall, also, the defining of the systemic skeleton of $\mathcal{I}$, i.e., of the π-structure

$$\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}}\rangle$$

of $\mathcal{I}$, where $K^{\mathcal{I}} : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}$ is the closure family on $\mathbf{F}$ corresponding to the closet set family $\mathrm{ThSys}(\mathcal{I})$.

Now we have the components needed to resume work on the characterization of syntactic weak system prealgebraizability. Our first result connecting syntactic weak system prealgebraizability of a π-institution with the associated systemic algebraic π-structure shows that, if a π-institution is syntactically weakly system prealgebraizable, then its systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to its associated systemic algebraic π-structure $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair of transformations.

Theorem 940 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a syntactically weakly system prealgebraizable π-institution based on $\mathbf{F}$. Then $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair $(\tau^\flat, \overleftrightarrow{I}^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}\bullet}$ of transformations. More precisely:*

- *$I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with two distinguished arguments, is a set of witnessing transformations of the syntactic prealgebraicity of $\mathcal{I}$;*
- *$\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$, with a single distinguished argument, is a set of witnessing equations for the system truth equationality of $\mathcal{I}$.*

Proof: Suppose that $\mathcal{I}$ is syntactically weakly system prealgebraizable. Then, by definition, $\mathcal{I}$ is syntactically prealgebraic and system truth equational. Therefore, there exist a set $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ of natural transformations in $N^\flat$, with two distinguished arguments, witnessing the syntactic prealgebraicity of $\mathcal{I}$, and a set $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ of natural transformations in $N^\flat$, with a single distinguished argument, witnessing system truth equationality. To verify the conclusion, observe, first, that

$\tau^\flat_\Sigma : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SenFam}(\mathbf{F}^2)$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, as the sentence family $\tau^\flat_\Sigma[\phi]$, and $\overleftrightarrow{I}^\flat{}_\Sigma : \mathrm{SEN}^\flat(\Sigma)^2 \to \mathrm{SenFam}(\mathbf{F})$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, as the sentence family $\overleftrightarrow{I}^\flat{}_\Sigma[\phi,\psi]$, are as required. Therefore, by Proposition 898, it suffices to show that:

(a) For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in K^{\mathcal{I}}_\Sigma(\Phi) \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \le D^{\mathcal{I}\bullet}(\tau^\flat_\Sigma[\Phi]);$$

(b) For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$D^{\mathcal{I}\bullet}(\phi \approx \psi) = D^{\mathcal{I}\bullet}(\tau^\flat[\overleftrightarrow{I}^\flat{}_\Sigma[\phi,\psi]]).$$

For (a), let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. Note that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, we have, by system truth equationality,

$$\begin{array}{lll} \Phi \subseteq T_\Sigma & \text{iff} & \tau^\flat_\Sigma[\Phi] \le \Omega(T) \\ \phi \in T_\Sigma & \text{iff} & \tau^\flat_\Sigma[\phi] \le \Omega(T). \end{array}$$

Therefore, $\phi \in K^{\mathcal{I}}_\Sigma(\Phi)$ if and only if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\Phi \subseteq T_\Sigma$ implies $\phi \in T_\Sigma$, if and only if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\tau^\flat_\Sigma[\Phi] \le \Omega(T)$ implies $\tau^\flat_\Sigma[\phi] \le \Omega(T)$, if and only if, by Proposition 937, $\tau^\flat_\Sigma[\phi] \le D^{\mathcal{I}\bullet}(\tau^\flat_\Sigma[\Phi])$.

For (b), let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have, for all $T \in \mathrm{ThSys}(\mathcal{I})$,

$$\begin{array}{llll} \phi \approx \psi \in \Omega_\Sigma(T) & \text{iff} & \overleftrightarrow{I}^\flat{}_\Sigma[\phi,\psi] \le T & \text{(Corollary 770)} \\ & \text{iff} & \tau^\flat[\overleftrightarrow{I}^\flat{}_\Sigma[\phi,\psi]] \le \Omega(T). & \\ & & \text{(system truth equationality)} & \end{array}$$

Using again Proposition 937, we conclude that

$$D^{\mathcal{I}\bullet}(\phi \approx \psi) = D^{\mathcal{I}\bullet}(\tau^\flat[\overleftrightarrow{I}^\flat{}_\Sigma[\phi,\psi]]).$$

Therefore $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via $(\tau^\flat, \overleftrightarrow{I}^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}\bullet}$. ■

We show, next that the existence of an equivalence between the systemic skeleton of a given π-institution and an algebraic π-structure ensures syntactic prealgebraicity.

Theorem 941 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If the systemic skeleton $\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}}\rangle$ of $\mathcal{I}$ is equivalent to an algebraic π-structure $\mathcal{Q}$ via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}$ of transformations, then $\mathcal{I}$ is syntactically prealgebraic, with witnessing transformations $I^\flat$.*

Proof: Suppose that $\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}} \rangle$ is equivalent to an algebraic π-structure $\mathcal{Q}$ via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}$ of transformations. Then, we have, by Proposition 928, that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\Omega(T) = I^\flat(T)$. Therefore, $\mathcal{I}$ is syntactically prealgebraic with witnessing transformations $I^\flat$. ∎

Finally, as a last step before the main theorem, we show that the existence of a transformational equivalence between the systemic skeleton of a given π-institution and an algebraic π-structure ensures system truth equationality.

Theorem 942 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If the systemic skeleton* $\mathcal{K}^{\mathcal{I}} = \langle \mathbf{F}, K^{\mathcal{I}} \rangle$ *of* $\mathcal{I}$ *is equivalent to an algebraic* π*-structure* $\mathcal{Q}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}$ *of transformations, then* $\mathcal{I}$ *is system truth equational, with witnessing equations* $\tau^\flat$.

Proof: We have, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll} \phi \in T_\Sigma & \text{iff} & I^\flat[\tau^\flat_\Sigma[\phi]] \leq T \quad ((\tau^\flat, I^\flat) \text{ an equivalence}) \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(T). \quad (\text{by Theorem 941}) \end{array}$$

Therefore, $\mathcal{I}$ is system truth equational, with witnessing equations $\tau^\flat$. ∎

Putting together Theorems 941, 942 and 940, we get the following fundamental result to the effect that syntactic weak system prealgebraizability boils down to the existence of an equivalence of the systemic skeleton of a π-institution with its associated systemic algebraic π-structure via a conjugate pair of transformations.

Theorem 943 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically weakly system prealgebraizable if and only if its systemic skeleton* $\mathcal{K}^{\mathcal{I}}$ *is equivalent to* $\mathcal{Q}^{\mathcal{I}\bullet}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}\bullet}$ *of transformations.*

Proof: Suppose, first, that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair of transformations. Then, by Theorem 941, it is syntactically prealgebraic and, by Theorem 942, it is system truth equational. Therefore, by definition, it is syntactically weakly system prealgebraizable. If, conversely, $\mathcal{I}$ is syntactically weakly system prealgebraizable, then, by Theorem 940, it is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair of transformations. ∎

It turns out that the existence of an equivalence of the systemic skeleton with an algebraic π-structure, induced by conjugate transformations, is sufficient to yield syntactic weak system prealgebraizability.

Theorem 944 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically weakly system prealgebraizable if and only if its systemic skeleton is equivalent to an algebraic* π*-structure via a conjugate pair of transformations.*

Proof: If $\mathcal{I}$ is syntactically weakly system prealgebraizable, then the conclusion follows from Theorem 943. Conversely, if $\mathcal{K}^{\mathcal{I}}$ is equivalent to an algebraic π-structure via a conjugate pair of transformations, then $\mathcal{I}$ is syntactically prealgebraic by Theorem 941 and system truth equational by Theorem 942, whence it is syntactically weakly system prealgebraizable. ∎

Finally, in terms of order isomorphisms between theory family lattices, we have the following alternative characterization of syntactically weakly system prealgebraizable π-institutions:

Theorem 945 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically weakly system prealgebraizable if and only if there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{K}^{\mathcal{I}}) \to \mathbf{ThFam}(\mathcal{Q})$*, where* $\mathcal{Q}$ *is an algebraic* π*-structure.*

Proof: The "only if" follows by Theorem 944 and Theorem 893. The "if" is given by Theorem 901 and Theorem 944. ∎

12.6 Syntactic WLC PreAlgebraizability

Between syntactic WS prealgebraizability and syntactic weak algebraizability we find the class of syntactic weakly left c-reflective prealgebraizability. This strengthens WS prealgebraizability by replacing system c-reflectivity by the stronger condition of left c-reflectivity. Alternatively, it weakens syntactic weak algebraizability by replacing ptotoalgebraicity by prealgebraicity.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is **syntactically weakly left c-reflectively prealgebraizable** (abbreviated to **syntactically WLC prealgebraizable**) if:

- $\mathcal{I}$ is $R^{\mathcal{I}}L^{\mathcal{I}}$-fortified;
- $\mathcal{I}$ is prealgebraic;
- $\mathcal{I}$ is left c-reflective.

We have the following alternative characterization of syntactic WLC prealgebraizability.

Theorem 946 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically WLC prealgebraizable if and only if it is syntactically prealgebraic and left truth equational.*

Proof: Assume that $\mathcal{I}$ is syntactically WLC prealgebraizable. Then, on the one hand, it is prealgebraic and has a Leibniz reflexive core. Thus, by

Theorem 788, it is syntactically prealgebraic. On the other, it is left c-reflective and has a left adequate left Suszko core. Therefore, by Theorem 870, it is left truth equational.

Assume, conversely, that $\mathcal{I}$ is syntactically prealgebraic and left truth equational. Then, by Theorem 788, it is prealgebraic and has a Leibniz reflexive core, and, by Theorem 870, it is left c-reflective and has a left adequate left Suszko core. Therefore, by definition, $\mathcal{I}$ is syntactically WLC prealgebraizable. ■

Directly from the definitions, we may derive the following relationship between the semantic and syntactic WLC prealgebraizability classes of π-institutions.

Theorem 947 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically WLC prealgebraizable if and only if* $\mathcal{I}$ *is WLC prealgebraizable and* $R^{\mathcal{I}} L^{\mathcal{I}}$*-fortified.*

Proof: $\mathcal{I}$ is syntactically WLC prealgebraizable if and only if, by definition, it is $R^{\mathcal{I}} L^{\mathcal{I}}$-fortified, prealgebraic and left c-reflective, i.e., iff it is, by definition, $R^{\mathcal{I}} L^{\mathcal{I}}$-fortified and WLC prealgebraizable. ■

For an alternative characterization of syntactic WLC prealgebraizability, we take advantage of the corresponding characterization of WLC prealgebraizability in terms of morphisms between complete lattices of theory systems.

Theorem 948 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically WLC prealgebraizable if and only if it is* $R^{\mathcal{I}} L^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left completely order reflecting surjection that restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: We have that $\mathcal{I}$ is syntactically WLC prealgebraizable if and only if, by Theorem 947, it is $R^{\mathcal{I}} L^{\mathcal{I}}$-fortified and WLC prealgebraizable, if and only if, by Theorem 276, it it $R^{\mathcal{I}} L^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is a left completely order reflecting surjection that restricts to an order embedding $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. ■

Recall that syntactic weak system prealgebraizability was characterized by the existence of an equivalence between the systemic skeleton $K^{\mathcal{I}}$ of a

π-institution $\mathcal{I}$ and the systemic algebraic π-structure $\mathcal{Q}^{\mathcal{I}\bullet}$ associated with the π-institution, via a pair of conjugate transformations. To adapt this characterization to capture syntactic WLC prealgebraizability, we need to postulate alongside this equivalence the property of left truth equationality of the π-institution.

Theorem 949 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically WLC prealgebraizable if and only if it is left truth equational and its systemic skeleton* $\mathcal{K}^{\mathcal{I}}$ *is equivalent to* $\mathcal{Q}^{\mathcal{I}\bullet}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}\bullet}$ *of transformations.*

Proof: Suppose, first, that $\mathcal{I}$ is left truth equational and $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair of transformations. Then, $\mathcal{I}$ is left truth equational and, by Theorem 941, it is syntactically prealgebraic. Therefore, by definition, it is syntactically WLC prealgebraizable. If, conversely, $\mathcal{I}$ is syntactically WLC prealgebraizable, then, by Theorem 946, it is left truth equational and it is weakly system prealgebraizable. Thus, by Theorem 940, it is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair of transformations. ∎

Because of Theorem 944, left truth equationality and the existence of an equivalence of the systemic skeleton with an algebraic π-structure, induced by conjugate transformations, is sufficient to yield syntactic WLC prealgebraizability.

Theorem 950 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically WLC prealgebraizable if and only if it is left truth equational and its systemic skeleton is equivalent to an algebraic* π*-structure via a conjugate pair of transformations.*

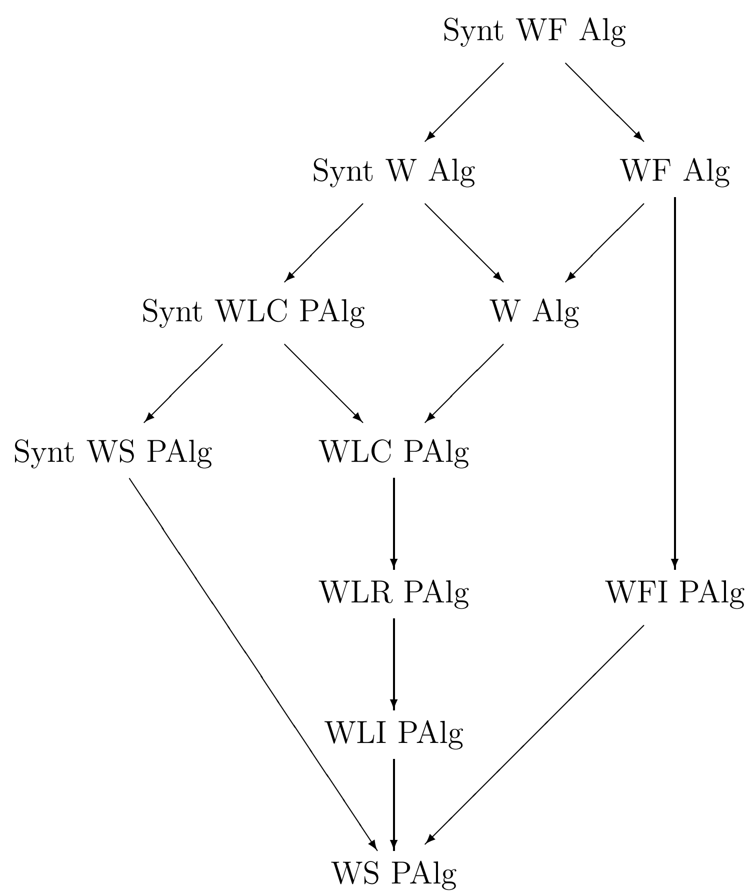
Proof: If $\mathcal{I}$ is syntactically WLC prealgebraizable, then the conclusion follows from Theorem 949. Conversely, if $\mathcal{K}^{\mathcal{I}}$ is equivalent to an algebraic π-structure via a conjugate pair of transformations, then $\mathcal{I}$ is syntactically prealgebraic by Theorem 941. Since, by hypothesis, it is also left truth equational, it is syntactically WLC prealgebraizable. ∎

Finally, in terms of order isomorphisms between theory family lattices, we have the following alternative characterization of syntactically WLC prealgebraizable π-institutions:

Theorem 951 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically WLC prealgebraizable if and only if it is left truth equational and there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{K}^{\mathcal{I}}) \to \mathbf{ThFam}(\mathcal{Q})$*, where* $\mathcal{Q}$ *is an algebraic* π*-structure.*

Proof: The "only if" follows by Theorem 950 and Theorem 893. The "if" is given by Theorem 901 and Theorem 950. ■

Let us give, in closing the section, the picture of the **weak prealgebraizability hierarchy** that we have established, consisting of both semantic and syntactic classes of π-institutions.



Chapter 13

The Syntactic Leibniz Hierarchy: Parameterlessness

13.1 The Binary Reflexive Core

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall that the **reflexive core** of $\mathcal{I}$ is the collection

$$\begin{aligned} R^{\mathcal{I}} &= \{\rho^\flat \in N^\flat : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\rho^\flat_\Sigma[\phi,\phi] \leq \mathrm{Thm}(\mathcal{I}))\} \\ &= \{\rho^\flat \in N^\flat : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \phi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)) \\ &\qquad\qquad (\rho^\flat_\Sigma(\phi,\phi,\vec{\chi}) \subseteq \mathrm{Thm}_\Sigma(\mathcal{I}))\}. \end{aligned}$$

We define the **binary reflexive core** of $\mathcal{I}$ as the collection

$$B^{\mathcal{I}} : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$$

of binary natural transformations in $N^\flat$ given by:

$$\begin{aligned} B^{\mathcal{I}} &= \{\rho^\flat \in N^\flat : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\rho^\flat_\Sigma[\phi,\phi] \leq \mathrm{Thm}(\mathcal{I}))\} \\ &= \{\rho^\flat \in N^\flat : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\rho^\flat_\Sigma(\phi,\phi) \subseteq \mathrm{Thm}_\Sigma(\mathcal{I}))\}. \end{aligned}$$

It turns out that the binary reflexive core of $\mathcal{I}$ coincides with the collection $\ddot{R}^{\mathcal{I}}$.

Proposition 952 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Then $B^{\mathcal{I}} = \ddot{R}^{\mathcal{I}}$.*

Proof: By Theorem 107. ■

In view of Proposition 952, we drop the notation $B^{\mathcal{I}}$ and denote the binary reflexive core of $\mathcal{I}$ by the symbol $\ddot{R}^{\mathcal{I}}$, without fear of ambiguity.

13.2 Syntactic PreEquivalentiality

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. Recall that $\mathcal{I}$ is **preequivalential** if it is prealgebraic and system extensional, i.e., if:

- For all $T, T' \in \mathrm{ThSys}(\mathcal{I})$,

$$T \leq T' \quad \text{implies} \quad \Omega(T) \leq \Omega(T');$$

- For all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(T) \quad \text{iff} \quad \langle \phi, \psi \rangle \in \Omega^{\langle \phi, \psi \rangle}_\Sigma(T \cap \langle \phi, \psi \rangle).$$

We say that $\mathcal{I}$ is **syntactically preequivalential** if there exists $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, without parameters, such that $I^\flat$ has:

- reflexivity;

- global system transitivity;
- global system compatibility; and
- global system modus ponens.

Note that because all these conditions are imposed on theory systems and because $I^\flat$ is parameter-free, they are all equivalent to the corresponding local properties. Therefore, an equivalent definition would require reflexivity, local system transitivity, local system compatibility and local system modus ponens. Because of this, we sometimes omit the global/local specification and simply say "system" in qualifying the corresponding property.

In case $\mathcal{I}$ is syntactically preequivalential, we call $I^\flat$ a **set of witnessing natural transformations**, or, more simply, **witnessing transformations** (of/for the syntactic preequivalentiality of $\mathcal{I}$).

An interesting first observation, that will prove handy later, is that syntactic preequivalentiality is inherited by all π-subinstitutions of a syntactically preequivalential π-institution.

Theorem 953 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\mathcal{I}' = \langle \mathbf{F}', C' \rangle$ *a* π*-subinstitution of* $\mathcal{I}$ *induced by the algebraic subsystem* $\mathbf{F}' = \langle \mathbf{Sign}^\flat, \mathrm{SEN}'^\flat, N'^\flat \rangle \le \mathbf{F}$*. If* $\mathcal{I}$ *is syntactically preequivalential with witnessing transformations* $I^\flat \subseteq N^\flat$*, then* $\mathcal{I}'$ *is also syntactically preequivalential, with witnessing transformations* $I'^\flat \in N'^\flat$*.*

Proof: Suppose that $\mathcal{I}$ is syntactically preequivalential with witnessing transformations $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$. To prove the conclusion, it suffices to show that $I'^\flat$ is reflexive, system transitive and has both the system compatibility and the system modus ponens in $\mathcal{I}'$.

Let, first, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}'^\flat(\Sigma)$. Then, clearly, $I'^\flat_\Sigma[\phi, \phi] \le \mathrm{SEN}'^\flat$ and $I'^\flat_\Sigma[\phi, \phi] = I^\flat_\Sigma[\phi, \phi] \le \mathrm{Thm}(\mathcal{I})$. So we get that

$$I'^\flat_\Sigma[\phi, \phi] \le \mathrm{Thm}(\mathcal{I}) \cap \mathrm{SEN}'^\flat = \mathrm{Thm}(\mathcal{I}').$$

It follows that $I'^\flat$ is reflexive in $\mathcal{I}'$.

Suppose, next, that $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}'^\flat(\Sigma)$, such that

$$I'^\flat_\Sigma[\phi, \psi] \le T \cap \mathrm{SEN}'^\flat \quad \text{and} \quad I'^\flat_\Sigma[\psi, \chi] \le T \cap \mathrm{SEN}'^\flat.$$

Then $I^\flat_\Sigma[\phi, \psi] \le T$ and $I^\flat_\Sigma[\psi, \chi] \le T$, whence, by the global system transitivity of $I^\flat$ in $\mathcal{I}$, we get that $I^\flat_\Sigma[\phi, \chi] \le T$. Since, by hypothesis, $\phi, \chi \in \mathrm{SEN}'^\flat(\Sigma)$, we get that $I'^\flat_\Sigma[\phi, \chi] \le T \cap \mathrm{SEN}'^\flat$. This shows that $I'^\flat$ is globally system transitive in $\mathcal{I}'$.

For system compatibility, let $T \in \mathrm{ThSys}(\mathcal{I})$, $\sigma^\flat \in N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \vec{\chi} \in \mathrm{SEN}'^\flat(\Sigma)$, such that $\overleftrightarrow{I'^\flat}_\Sigma[\phi, \psi] \le T \cap \mathrm{SEN}'^\flat$. Then we get that

$\overset{\leftrightarrow}{I}^\flat{}_\Sigma[\phi,\psi] \le T$, whence, we obtain $I^\flat_\Sigma[\sigma^\flat_\Sigma(\phi,\vec{\chi}),\sigma^\flat_\Sigma(\psi,\vec{\chi})] \le T$. Since $\sigma^\flat \in N^\flat$ and $\phi,\psi,\vec{\chi} \in \mathrm{SEN}'^\flat(\Sigma)$, this yields that

$$I'^\flat_\Sigma[\sigma'^\flat_\Sigma(\phi,\vec{\chi}),\sigma'^\flat_\Sigma(\psi,\vec{\chi})] \le T \cap \mathrm{SEN}'^\flat.$$

Therefore, $I'^\flat$ has the system compatibility in $\mathcal{I}'$.

For the system MP, assume that $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}'^\flat(\Sigma)$, such that $\phi \in T_\Sigma \cap \mathrm{SEN}'^\flat(\Sigma)$ and $I'^\flat_\Sigma[\phi,\psi] \le T \cap \mathrm{SEN}'^\flat$. Then $\phi \in T_\Sigma$ and $I^\flat_\Sigma[\phi,\psi] \le T$, whence we get that $\psi \in T_\Sigma$. Since, by hypothesis, $\psi \in \mathrm{SEN}'^\flat(\Sigma)$, we get that $\psi \in T_\Sigma \cap \mathrm{SEN}'^\flat(\Sigma)$. Therefore, $I'^\flat$ has the system MP in $\mathcal{I}'$.

We conclude that $\mathcal{I}'$ is also syntactically preequivalential with witnessing transformations $I'^\flat$. ∎

Since $I^\flat$ is, a fortiori, a set of witnessing transformations for the syntactic prealgebraicity of $\mathcal{I}$, we get the following result, based on Corollary 770.

Corollary 954 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically preequivalential, with witnessing transformations* $I^\flat$*, if and only if, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,*

$$\overset{\leftrightarrow}{I}^\flat(T) = \Omega(T).$$

Proof: The "only if" is by Corollary 770. The "if" is clear, since the given condition implies that $\overset{\leftrightarrow}{I}^\flat$ satisfies reflexivity, global system transitivity, global system compatibility and global system modus ponens. ∎

Based on Corollary 954, it is easy to see that syntactic preequivalentiality transfers from a π-institution to all its gmatrix families.

Theorem 955 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically preequivalential, with witnessing transformations* $I^\flat$*, if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$*, the* $\mathcal{I}$*-gmatrix family* $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}}\rangle$ *is syntactically preequivalential.*

Proof: The "if" follows by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I,\iota\rangle\rangle$. For the "only if", assume that $\mathcal{I}$ is syntactically preequivalential, with witnessing transformations $I^\flat$, and let $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$. Then, we have

$$\begin{array}{rll}\langle \alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\rangle \in \overset{\leftrightarrow}{I}{}^{\mathcal{A}}_{F(\Sigma)}(T) & \text{iff} & \overset{\leftrightarrow}{I}{}^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)] \le T\\ & \text{iff} & \overset{\leftrightarrow}{I}{}^\flat_\Sigma[\phi,\psi] \le \alpha^{-1}(T)\\ & \text{iff} & \langle \phi,\psi\rangle \in \Omega_\Sigma(\alpha^{-1}(T))\\ & \text{iff} & \langle \phi,\psi\rangle \in \alpha_\Sigma^{-1}(\Omega^{\mathcal{A}}_{F(\Sigma)}(T))\\ & \text{iff} & \langle \alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\rangle \in \Omega^{\mathcal{A}}_{F(\Sigma)}(T).\end{array}$$

Taking into account the surjectivity of $\langle F,\alpha\rangle$, it follows, by Corollary 954, that $\langle\mathcal{A},C^{\mathcal{I},\mathcal{A}}\rangle$ is also syntactically preequivalential, with witnessing transformations $I^{\mathcal{A}}$. ■

It turns out that syntactic preequivalentiality implies preequivalentiality.

Theorem 956 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically preequivalential, then it is preequivalential.*

Proof: Suppose that $\mathcal{I}$ is syntactically preequivalential, with witnessing transformations $I^\flat:(\mathrm{SEN}^\flat)^2\to\mathrm{SEN}^\flat$. Then, it is a fortiori syntactically prealgebraic and, hence, by Theorem 771, prealgebraic. Thus, the Leibniz operator is monotone on theory systems. It suffices, therefore, to show that $\mathcal{I}$ is system extensional. To this end, let $T\in\mathrm{ThSys}(\mathcal{I})$, $\Sigma\in|\mathbf{Sign}^\flat|$ and $\phi,\psi\in\mathrm{SEN}^\flat(\Sigma)$, such that

$$\langle\phi,\psi\rangle\in\Omega_\Sigma^{\langle\phi,\psi\rangle}(T\cap\langle\phi,\psi\rangle).$$

Thus, by Theorem 953 and Corollary 954,

$$\overleftrightarrow{I'^\flat}_\Sigma[\phi,\psi]\leq T\cap\langle\phi,\psi\rangle.$$

Therefore, $\overleftrightarrow{I^\flat}_\Sigma[\phi,\psi]\leq T$, which, again by Corollary 954, implies that $\langle\phi,\psi\rangle\in\Omega_\Sigma(T)$. Since, by Proposition 89, the reverse inclusion always holds, $\mathcal{I}$ is also system extensional and, hence, preequivalential. ■

Apart from the definability of Leibniz congruence systems of theory systems, syntactic preequivalentiality has some additional important consequences. Namely, it implies that the binary reflexive core has the system modus ponens and that it also has the extensionality property. Before we look at those results more closely, we give a key lemma to the effect that in a syntactically preequivalential π-institution, any set of witnessing transformations is included in the binary reflexive core of the π-institution.

Lemma 957 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a syntactically preequivalential* π*-institution, with witnessing transformations* $I^\flat:(\mathrm{SEN}^\flat)^2\to\mathrm{SEN}^\flat$*. Then* $I^\flat\subseteq\ddot{R}^{\mathcal{I}}$*.*

Proof: Since $I^\flat$ is parameter free and reflexive in $\mathcal{I}$, we get, by definition of $B^{\mathcal{I}}$ and Proposition 952, that $I^\flat\subseteq B^{\mathcal{I}}=\ddot{R}^{\mathcal{I}}$. ■

Now we formalize the fact that syntactic preequivalentiality implies the system modus ponens property for the binary reflexive core.

Proposition 958 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically preequivalential, then* $\ddot{R}^{\mathcal{I}}$ *has the system modus ponens in* $\mathcal{I}$*.*

Proof: Suppose that $\mathcal{I}$ is syntactically preequivalential and let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi] \le T$. Then $\phi \in T_\Sigma$ and, by Lemma 957, $I^\flat_\Sigma[\phi, \psi] \le T$. Since $I^\flat$ has the system MP in $\mathcal{I}$, we conclude that $\psi \in T_\Sigma$. Therefore, $\ddot{R}^{\mathcal{I}}$ has the system MP in $\mathcal{I}$. ■

The next property that is implied by syntactic preequivalentiality is the extensionality of the binary reflexive core. Before introducing the concept, we take a look at a technical lemma that will serve to justify its formulation.

Lemma 959 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $X, Y \in \mathrm{SenSys}(\mathcal{I})$. *Then the following conditions are equivalent:*

(a) For all $T \in \mathrm{ThFam}(\mathcal{I})$, $X \le T$ *if and only if* $Y \le T$*;*

(b) For all $T \in \mathrm{ThSys}(\mathcal{I})$, $X \le T$ *if and only if* $Y \le T$.

Proof: That (a)⇒(b) is obvious, since every theory system of $\mathcal{I}$ is also a theory family. For (b)⇒(a), assume that (b) holds and let $T \in \mathrm{ThFam}(\mathcal{I})$, such that $X \le T$. Then, by Lemma 1, $\overleftarrow{X} \le \overleftarrow{T}$. Since $X \in \mathrm{SenSys}(\mathcal{I})$, by Proposition 2, we get that $X \le \overleftarrow{T}$. Therefore, by hypothesis, since $\overleftarrow{T} \in \mathrm{ThSys}(\mathcal{I})$, we obtain $Y \le \overleftarrow{T} \le T$. By symmetry, we conclude that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $X \le T$ if and only if $Y \le T$. ■

Due to Lemma 959 and the fact that both the reflexive core and the binary reflexive core yield sentence systems of $\mathcal{I}$ under substitution, we make the following definition (without the need for distinguishing between a family versus system version):

Definition 960 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$. *The binary reflexive core* $\ddot{R}^{\mathcal{I}}$ *is* **extensional in** $\mathcal{I}$ *if and only if, for all* $T \in \mathrm{ThSys}(\mathcal{I})$ *(or equivalently, by Lemma 959, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*), all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi] \le T \quad \textit{if and only if} \quad R^{\mathcal{I}}_\Sigma[\phi, \psi] \le T.$$

Note that, since, by Lemma 104, $\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi] \le R^{\mathcal{I}}_\Sigma[\phi, \psi]$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, the right-to-left implication in Definition 960 always holds. Therefore one has, equivalently, that $\ddot{R}^{\mathcal{I}}$ is extensional if and only if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi] \le T \quad \text{implies} \quad R^{\mathcal{I}}_\Sigma[\phi, \psi] \le T.$$

This can be taken to justify the name chosen for this property.

As mentioned previously, and shown in the next proposition, syntactic preequivalentiality implies the extensionality of the binary reflexive core:

Proposition 961 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically preequivalential, then* $\ddot{R}^{\mathcal{I}}$ *is extensional.*

Proof: Suppose $\mathcal{I}$ is a syntactically preequivalential π-institution, with witnessing transformations $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$. Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T$. Then, by Lemma 957, we get that $\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi] \leq T$. Thus, by Corollary 954, we get that $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. Since $\mathcal{I}$ is syntactically preequivalential, it is a fortiori syntactically prealgebraic, whence, by Theorems 781 and 782, we get that $R^{\mathcal{I}}$ is a set of witnessing transformations for the prealgebraicity of $\mathcal{I}$ and, hence, by Theorems 782 and 783, $R^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T$. Since the reverse inclusion always holds, we conclude that $\ddot{R}^{\mathcal{I}}$ is indeed extensional in $\mathcal{I}$. ■

As a result of preceding work we obtain the following

Theorem 962 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically preequivalential, then* $\ddot{R}^{\mathcal{I}}$ *has the system modus ponens and is extensional in* $\mathcal{I}$*.*

Proof: By Propositions 958 and 961. ■

We provide, next, a characterization of syntactic preequivalentiality in terms of the preceding two properties of the binary reflexive core of the π-institution, namely system modus ponens and extensionality. Later, we use this characterization to provide an exact description of those preequivalential π-institutions which are syntactically preequivalential.

In proving the reverse implication of that included in Theorem 962, we now show that, if $\ddot{R}^{\mathcal{I}}$ has the system modus ponens and is extensional in $\mathcal{I}$, then $\mathcal{I}$ is syntactically preequivalential.

Theorem 963 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\ddot{R}^{\mathcal{I}}$ *has the system modus ponens and is extensional in* $\mathcal{I}$*, then* $\mathcal{I}$ *is syntactically preequivalential, with witnessing transformations* $\ddot{R}^{\mathcal{I}}$*.*

Proof: If $\ddot{R}^{\mathcal{I}}$ has the system MP, then, by Lemma 104, $R^{\mathcal{I}}$ has a fortiori the global system MP. Thus, by Theorem 781, $\mathcal{I}$ is syntactically prealgebraic with witnessing transformations $R^{\mathcal{I}}$. Thus, $R^{\mathcal{I}}$ is globally system reflexive, globally system transitive, has the global system compatibility property and the global system MP. Moreover, by the extensionality of $\ddot{R}^{\mathcal{I}}$, all these properties transfer from $R^{\mathcal{I}}$ to $\ddot{R}^{\mathcal{I}}$. We conclude that $\mathcal{I}$ is syntactically preequivalential with witnessing transformations $\ddot{R}^{\mathcal{I}}$. ■

Theorems 962 and 963 provide the promised characterization of syntactic preequivalentiality in terms of the system modus ponens and the extensionality of the binary reflexive core.

$$\mathcal{I} \text{ is Syntactically Preequivalential} \longleftrightarrow \begin{array}{ll} & \ddot{R}^{\mathcal{I}} \text{ has System MP} \\ + & \ddot{R}^{\mathcal{I}} \text{ is Extensional.} \end{array}$$

Theorem 964 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically preequivalential if and only if* $\ddot{R}^{\mathcal{I}}$ *has the system modus ponens and is extensional in* $\mathcal{I}$*.*

Proof: Theorem 962 gives the "only if" and the "if" is by Theorem 963. ■

If $\mathcal{I}$ is syntactically preequivalential, then $\ddot{R}^{\mathcal{I}}$ defines Leibniz congruence systems of theory systems in $\mathcal{I}$. This proposition may be viewed as a special case of Corollary 954, since $\ddot{R}^{\mathcal{I}}$ forms a set of witnessing transformations.

Proposition 965 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\ddot{R}^{\mathcal{I}}$ *has system modus ponens and is extensional in* $\mathcal{I}$*, then, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,*

$$\Omega(T) = \ddot{R}^{\mathcal{I}}(T).$$

Proof: Let $T \in \mathrm{ThSys}(\mathcal{I})$. If $\ddot{R}^{\mathcal{I}}$ has the system modus ponens and is extensional, then, by Theorem 963, $\mathcal{I}$ is syntactically preequivalential with witnessing transformations $\ddot{R}^{\mathcal{I}}$. Therefore, by Corollary 954, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\Omega(T) = \ddot{R}^{\mathcal{I}}(T)$. ■

We also get another related characterization of syntactic preequivalentiality.

$$\begin{array}{l} \mathcal{I} \text{ is Syntactically Preequivalential} \\ \qquad \longleftrightarrow \ddot{R}^{\mathcal{I}} \text{ Defines Leibniz Congruence Systems} \\ \qquad\qquad \text{of Theory Systems in } \mathcal{I}. \end{array}$$

Theorem 966 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically preequivalential if and only if, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,*

$$\Omega(T) = \ddot{R}^{\mathcal{I}}(T).$$

Proof: If $\mathcal{I}$ is syntactically preequivalential, then, by Theorem 962, $\ddot{R}^{\mathcal{I}}$ has the system modus ponens and is extensional in $\mathcal{I}$. Thus, by Proposition 965, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\Omega(T) = \ddot{R}^{\mathcal{I}}(T)$.

Conversely, if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\ddot{R}^{\mathcal{I}}(T) = \Omega(T)$, then, $\ddot{R}^{\mathcal{I}}$ is reflexive, system transitive, has the system compatibility and the system modus ponens. Thus, $\mathcal{I}$ is syntactically preequivalential with witnessing transformations $\ddot{R}^{\mathcal{I}}$. ∎

We finally show that the property that separates preequivalentiality from syntactic preequivalentiality is exactly a sort of a local Leibniz compatibility property with respect to the theory system generated by the binary reflexive core.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall that $R^{\mathcal{I}}$ is **Leibniz** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi])).$$

Similarly, we say that $\ddot{R}^{\mathcal{I}}$ is **Leibniz** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega^{\langle \phi, \psi \rangle}_\Sigma(C(\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi]) \cap \langle \phi, \psi \rangle).$$

We show next that, if $\ddot{R}^{\mathcal{I}}$ has the system modus ponens and is extensional in $\mathcal{I}$, then $\ddot{R}^{\mathcal{I}}$ is Leibniz.

Proposition 967 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\ddot{R}^{\mathcal{I}}$ has the system modus ponens and is extensional in $\mathcal{I}$, then $\ddot{R}^{\mathcal{I}}$ is Leibniz.*

Proof: Suppose that $\ddot{R}^{\mathcal{I}}$ has the system MP and is extensional and let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. By Theorem 963, $\mathcal{I}$ is syntactically preequivalential, with witnessing transformations $\ddot{R}^{\mathcal{I}}$. Hence, it is a fortiori syntactically prealgebraic, with witnessing transformations $\ddot{R}^{\mathcal{I}}$. Thus, by Theorem 788, we get

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi])).$$

Then, by Theorem 89, we get

$$\langle \phi, \psi \rangle \in \Omega^{\langle \phi, \psi \rangle}_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi]) \cap \langle \phi, \psi \rangle).$$

By hypothesis (more precisely the extensionality of $\ddot{R}^{\mathcal{I}}$), we get $C(R^{\mathcal{I}}_\Sigma[\phi, \psi]) = C(\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi])$. Therefore, we conclude that $\langle \phi, \psi \rangle \in \Omega^{\langle \phi, \psi \rangle}_\Sigma(C(\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi]) \cap \langle \phi, \psi \rangle)$. So $\ddot{R}^{\mathcal{I}}$ is Leibniz. ∎

In the opposite direction, in a *preequivalential* π-institution $\mathcal{I}$, if the binary reflexive core is Leibniz, then it has the system modus ponens and is extensional in $\mathcal{I}$.

Proposition 968 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a preequivalential π-institution based on $\mathbf{F}$. If $\ddot{R}^{\mathcal{I}}$ is Leibniz, then $\ddot{R}^{\mathcal{I}}$ has the system modus ponens and is extensional in $\mathcal{I}$.*

Proof: Suppose that $\mathcal{I}$ is preequivalential and that $\ddot{R}^{\mathcal{I}}$ is Leibniz.

For the system MP, suppose that $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$, $\ddot{R}^{\mathcal{I}}_\Sigma[\phi,\psi] \leq T$. Then, since $\ddot{R}^{\mathcal{I}}$ is Leibniz,

$$\langle \phi, \psi \rangle \in \Omega^{\langle \phi, \psi \rangle}_\Sigma (C(\ddot{R}^{\mathcal{I}}_\Sigma[\phi,\psi]) \cap \langle \phi, \psi \rangle).$$

Thus, since $\mathcal{I}$ is system extensional,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma (C(\ddot{R}^{\mathcal{I}}_\Sigma[\phi,\psi])).$$

By hypothesis, $C(\ddot{R}^{\mathcal{I}}_\Sigma[\phi,\psi]) \leq T$, whence, by preequivalentiality,

$$\Omega(C(\ddot{R}^{\mathcal{I}}_\Sigma[\phi,\psi])) \leq \Omega(T).$$

We, thus, get that $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. Therefore, by compatibility of $\Omega(T)$ with T, we obtain $\psi \in T_\Sigma$, showing that $\ddot{R}^{\mathcal{I}}$ has the system MP in $\mathcal{I}$.

For extensionality, assume that $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\ddot{R}^{\mathcal{I}}_\Sigma[\phi,\psi] \leq T$. Following the initial argument of the preceding paragraph mutatis mutandis we obtain that $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. But since $\ddot{R}^{\mathcal{I}}$ has the system MP, a fortiori $R^{\mathcal{I}}$ has the global system MP, whence, by Proposition 783, $R^{\mathcal{I}}_\Sigma[\phi,\psi] \leq T$. Since the reverse inclusion always holds, we conclude that $\ddot{R}^{\mathcal{I}}$ is extensional. ∎

We now show that a π-institution is syntactically preequivalential if and only if it is preequivalential and it has a Leibniz binary reflexive core.

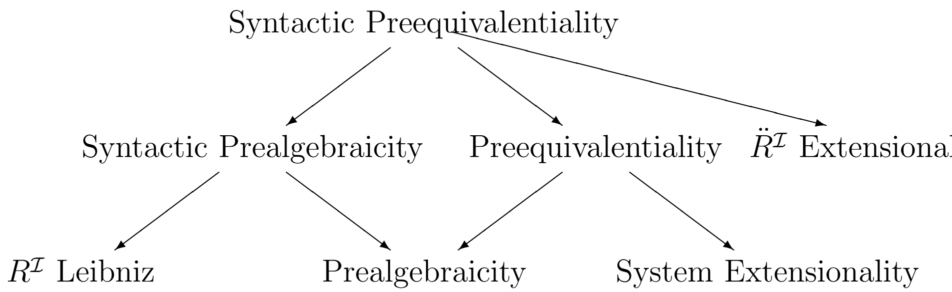
Syntactic Preequivalentiality
= $\ddot{R}^{\mathcal{I}}$ has System MP + $\ddot{R}^{\mathcal{I}}$ is Extensional
= $\ddot{R}^{\mathcal{I}}$ Defines Leibniz Congruence Systems
of Theory Systems in $\mathcal{I}$
= Preequivalentiality + $\ddot{R}^{\mathcal{I}}$ is Leibniz

Theorem 969 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically preequivalential if and only if it is preequivalential and has a Leibniz binary reflexive core.*

Proof: Suppose, first, that $\mathcal{I}$ is syntactically preequivalential. Then it is preequivalential by Theorem 956. Moreover, its binary reflexive core has the system modus ponens and is extensional, by Theorem 964, and, hence, by Proposition 967, its binary reflexive core is Leibniz.

Suppose, conversely, that $\mathcal{I}$ is preequivalential with a Leibniz binary reflexive core. Then, by Proposition 968, its binary reflexive core has the system MP and is extensional. Therefore, by Theorem 964, $\mathcal{I}$ is syntactically preequivalential. ∎

We have the following part of a hierarchy:



13.3 Syntactic Equivalentiality

We now define the class of syntactically equivalential π-institutions. The difference between equivalentiality and preequivalentiality is that the system versions of the properties defining the latter are replaced by their corresponding family versions. Otherwise, the developments are exactly parallel.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$.

Recall that $\mathcal{I}$ is **equivalential** if it is protoalgebraic and family extensional, i.e., if:

- For all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$T \leq T' \quad \text{implies} \quad \Omega(T) \leq \Omega(T');$$

- For all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(T) \quad \text{iff} \quad \langle \phi, \psi \rangle \in \Omega_\Sigma^{\langle \phi, \psi \rangle}(T \cap \langle \phi, \psi \rangle).$$

Recall, also, that protoalgebraicity implies stability. If a π-institution is stable, then it is family extensional if and only if it is system extensional. Thus, under protoalgebraicity, system and family extensionality coincide, and, therefore, $\mathcal{I}$ being equivalential is equivalent to $\mathcal{I}$ being protoalgebraic and system extensional.

We say that $\mathcal{I}$ is **syntactically equivalential** if there exists $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, without parameters, such that $I^\flat$ has:

- reflexivity;
- global family transitivity;
- global family compatibility; and
- global family modus ponens.

We emphasize that, in opposition to the case of the corresponding system properties, in this case, the latter three conditions are not equivalent to the corresponding local properties. So one cannot dispense with the qualification "global" in the defining conditions.

In case $\mathcal{I}$ is syntactically equivalential, we call $I^\flat$ a **set of witnessing natural transformations**, or, more simply, **witnessing transformations** (of the syntactic equivalentiality of $\mathcal{I}$).

As was the case with syntactic preequivalentiality, syntactic equivalentiality is inherited by all π-subinstitutions of a syntactically equivalential π-institution.

Theorem 970 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\mathcal{I}' = \langle \mathbf{F}', C' \rangle$ *a* π*-subinstitution of* $\mathcal{I}$ *induced by the algebraic subsystem* $\mathbf{F}' = \langle \mathbf{Sign}^\flat, \mathrm{SEN}'^\flat, N'^\flat \rangle \le \mathbf{F}$*. If* $\mathcal{I}$ *is syntactically equivalential with witnessing transformations* $I^\flat \subseteq N^\flat$*, then* $\mathcal{I}'$ *is also syntactically equivalential, with witnessing transformations* $I'^\flat \in N'^\flat$*.*

Proof: Suppose that $\mathcal{I}$ is syntactically equivalential with witnessing transformations $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$. To prove the conclusion, it suffices to show that $I'^\flat$ is reflexive, globally family transitive and has both the global family compatibility and the global family modus ponens in $\mathcal{I}'$.

Let, first, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}'^\flat(\Sigma)$. Then, clearly, $I'^\flat_\Sigma[\phi, \phi] \le \mathrm{SEN}'^\flat$ and $I'^\flat_\Sigma[\phi, \phi] = I^\flat_\Sigma[\phi, \phi] \le \mathrm{Thm}(\mathcal{I})$. So we get that

$$I'^\flat_\Sigma[\phi, \phi] \le \mathrm{Thm}(\mathcal{I}) \cap \mathrm{SEN}'^\flat = \mathrm{Thm}(\mathcal{I}').$$

It follows that $I'^\flat$ is reflexive in $\mathcal{I}'$.

Suppose, next, that $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}'^\flat(\Sigma)$, such that

$$I'^\flat_\Sigma[\phi, \psi] \le T \cap \mathrm{SEN}'^\flat \quad \text{and} \quad I'^\flat_\Sigma[\psi, \chi] \le T \cap \mathrm{SEN}'^\flat.$$

Then $I^\flat_\Sigma[\phi, \psi] \le T$ and $I^\flat_\Sigma[\psi, \chi] \le T$, whence by the global family transitivity of $I^\flat$ in $\mathcal{I}$, we get that $I^\flat_\Sigma[\phi, \chi] \le T$. Since, by hypothesis, $\phi, \chi \in \mathrm{SEN}'^\flat(\Sigma)$, we get that $I'^\flat_\Sigma[\phi, \chi] \le T \cap \mathrm{SEN}'^\flat$. This shows that $I'^\flat$ is globally family transitive in $\mathcal{I}'$.

For global family compatibility, let $T \in \mathrm{ThFam}(\mathcal{I})$, $\sigma^\flat \in N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \vec{\chi} \in \mathrm{SEN}'^\flat(\Sigma)$, such that $\overleftrightarrow{I'^\flat}_\Sigma[\phi, \psi] \le T \cap \mathrm{SEN}'^\flat$. Then we get that $\overleftrightarrow{I^\flat}_\Sigma[\phi, \psi] \le T$, whence, we obtain $I^\flat_\Sigma[\sigma^\flat_\Sigma(\phi, \vec{\chi}), \sigma^\flat_\Sigma(\psi, \vec{\chi})] \le T$. Since $\sigma^\flat \in N^\flat$ and $\phi, \psi, \vec{\chi} \in \mathrm{SEN}'^\flat(\Sigma)$, this yields that

$$I'^\flat_\Sigma[\sigma'^\flat_\Sigma(\phi, \vec{\chi}), \sigma'^\flat_\Sigma(\psi, \vec{\chi})] \le T \cap \mathrm{SEN}'^\flat.$$

Therefore, $I'^\flat$ has the global family compatibility in $\mathcal{I}'$.

For the global family MP, assume that $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}'^\flat(\Sigma)$, such that $\phi \in T_\Sigma \cap \mathrm{SEN}'^\flat(\Sigma)$ and $I'^\flat_\Sigma[\phi, \psi] \le T \cap \mathrm{SEN}'^\flat$. Then $\phi \in T_\Sigma$ and $I^\flat_\Sigma[\phi, \psi] \le T$, whence we get that $\psi \in T_\Sigma$. Since, by hypothesis, $\psi \in \mathrm{SEN}'^\flat(\Sigma)$, we get that $\psi \in T_\Sigma \cap \mathrm{SEN}'^\flat(\Sigma)$. Therefore, $I'^\flat$ has the global family MP in $\mathcal{I}'$.

We conclude that $\mathcal{I}'$ is also syntactically equivalential with witnessing transformations $I'^\flat$. ∎

Since $I^\flat$ is, a fortiori, a set of witnessing transformations for the syntactic protoalgebraicity of $\mathcal{I}$, we get the following result, based on Corollary 791.

Corollary 971 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is syntactically equivalential, with witnessing transformations $I^\flat$, if and only if, for all $T \in \mathrm{ThFam}(\mathcal{I})$,*

$$\overleftrightarrow{I^\flat}(T) = \Omega(T).$$

Proof: The "only if" is by Corollary 791. The "if" is clear, since the given condition implies that $\overleftrightarrow{I^\flat}$ satisfies reflexivity, global family transitivity, global family compatibility and global family modus ponens. ∎

Based on Corollary 971, it is easy to see that syntactic equivalentiality also transfers from a π-institution to all its gmatrix families.

Theorem 972 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is syntactically equivalential, with witnessing transformations $I^\flat$, if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, the $\mathcal{I}$-gmatrix family $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ is syntactically equivalential.*

Proof: Analogous to the proof of Theorem 955. ∎

Syntactic equivalentiality implies equivalentiality.

Theorem 973 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is syntactically equivalential, then it is equivalential.*

Proof: Suppose that $\mathcal{I}$ is syntactically equivalential, with witnessing transformations $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$. Then, it is a fortiori syntactically protoalgebraic and, hence, by Theorem 792, protoalgebraic. Thus, the Leibniz operator is monotone on theory families. Since syntactical equivalentiality implies syntactical preequivalentiality, by Theorem 956, we get that $\mathcal{I}$ is also system extensional. ∎

In analogy with syntactic preequivalentiality, syntactic equivalentiality implies that the binary reflexive core has the global family modus ponens and that it also has the extensionality property.

We first formalize the fact that syntactic equivalentiality implies the global family modus ponens property of the binary reflexive core.

Proposition 974 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically equivalential, then* $\ddot{R}^{\mathcal{I}}$ *has the global family modus ponens in* $\mathcal{I}$*.*

Proof: Suppose that $\mathcal{I}$ is syntactically equivalential, with witnessing transformations $I^\flat$, and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T$. Then $\phi \in T_\Sigma$ and, by Lemma 957, $I^\flat_\Sigma[\phi, \psi] \leq T$. Since $I^\flat$ has the global family MP in $\mathcal{I}$, we conclude that $\psi \in T_\Sigma$. Therefore, $\ddot{R}^{\mathcal{I}}$ also has the global family MP in $\mathcal{I}$. ■

We now show that syntactic equivalentiality implies the extensionality of the binary reflexive core.

Corollary 975 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically equivalential, then* $\ddot{R}^{\mathcal{I}}$ *is extensional.*

Proof: Since syntactic equivalentiality implies syntactic preequivalentiality, we get the conclusion by applying Proposition 961. ■

We summarize these two important consequences of syntactic equivalentiality in the following

Theorem 976 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically equivalential, then* $\ddot{R}^{\mathcal{I}}$ *has the global family modus ponens and is extensional in* $\mathcal{I}$*.*

Proof: By Propositions 974 and 975. ■

We provide, next, a characterization of syntactic equivalentiality in terms of the preceding two properties of the binary reflexive core of the π-institution, namely global family modus ponens and extensionality. As with preequivalentiality, we use this characterization to provide an exact description of those equivalential π-institutions which are syntactically equivalential.

In proving the reverse implication of that included in Theorem 976, we show that, if $\ddot{R}^{\mathcal{I}}$ has the global family modus ponens and is extensional in $\mathcal{I}$, then $\mathcal{I}$ is syntactically equivalential.

Theorem 977 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\ddot{R}^{\mathcal{I}}$ *has the global family modus ponens and is extensional in* $\mathcal{I}$*, then* $\mathcal{I}$ *is syntactically equivalential, with witnessing transformations* $\ddot{R}^{\mathcal{I}}$*.*

Proof: If $\ddot{R}^{\mathcal{I}}$ has the global family MP, then, by Lemma 104, $R^{\mathcal{I}}$ has a fortiori the global family MP. Thus, by Theorem 798, $\mathcal{I}$ is syntactically protoalgebraic with witnessing transformations $R^{\mathcal{I}}$. Thus, $R^{\mathcal{I}}$ is reflexive, globally family transitive, has the global family compatibility property and the global family

MP. Moreover, by the extensionality of $\ddot{R}^{\mathcal{I}}$, all these properties transfer from $R^{\mathcal{I}}$ to $\ddot{R}^{\mathcal{I}}$. We conclude that $\mathcal{I}$ is syntactically equivalential with witnessing transformations $\ddot{R}^{\mathcal{I}}$. ∎

Theorems 976 and 977 provide the promised characterization of syntactic equivalentiality in terms of the global family modus ponens and the extensionality of the binary reflexive core.

$$\mathcal{I} \text{ is Syntactically Equivalential } \longleftrightarrow \begin{array}{ll} & \ddot{R}^{\mathcal{I}} \text{ has Global Family MP} \\ + & \ddot{R}^{\mathcal{I}} \text{ is Extensional.} \end{array}$$

Theorem 978 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically equivalential if and only if* $\ddot{R}^{\mathcal{I}}$ *has the global family modus ponens and is extensional in* $\mathcal{I}$.

Proof: Theorem 976 gives the "only if" and the "if" is by Theorem 977. ∎

If $\mathcal{I}$ is syntactically equivalential, then $\ddot{R}^{\mathcal{I}}$ defines Leibniz congruence systems of theory families in $\mathcal{I}$. This proposition may be viewed as a special case of Corollary 971, since $\ddot{R}^{\mathcal{I}}$ forms a set of witnessing transformations.

Proposition 979 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\ddot{R}^{\mathcal{I}}$ *has the global family modus ponens and is extensional in* $\mathcal{I}$*, then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\Omega(T) = \ddot{R}^{\mathcal{I}}(T).$$

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$. If $\ddot{R}^{\mathcal{I}}$ has the global family modus ponens and is extensional, then, by Theorem 977, $\mathcal{I}$ is syntactically equivalential with witnessing transformations $\ddot{R}^{\mathcal{I}}$. Therefore, by Corollary 971, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) = \ddot{R}^{\mathcal{I}}(T)$. ∎

We also get another related characterization of syntactic equivalentiality.

$$\begin{array}{l} \mathcal{I} \text{ is Syntactically Equivalential} \\ \qquad \longleftrightarrow \ddot{R}^{\mathcal{I}} \text{ Defines Leibniz Congruence Systems} \\ \qquad\qquad \text{of Theory Families in } \mathcal{I}. \end{array}$$

Theorem 980 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically equivalential if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\Omega(T) = \ddot{R}^{\mathcal{I}}(T).$$

Proof: If $\mathcal{I}$ is syntactically equivalential, then, by Theorem 976, $\ddot{R}^{\mathcal{I}}$ has the global family modus ponens and is extensional in $\mathcal{I}$. Thus, by Proposition 979, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) = \ddot{R}^{\mathcal{I}}(T)$.

Conversely, if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\ddot{R}^{\mathcal{I}}(T) = \Omega(T)$, then, $\ddot{R}^{\mathcal{I}}$ is reflexive, globally family transitive and has the global family compatibility and the global family modus ponens. Thus, $\mathcal{I}$ is syntactically equivalential with witnessing transformations $\ddot{R}^{\mathcal{I}}$. ■

We finally show that the property that separates equivalentiality from syntactic equivalentiality is the Leibniz property of the binary reflexive core.

We show first that, if $\ddot{R}^{\mathcal{I}}$ has the global family modus ponens and is extensional in $\mathcal{I}$, then $\ddot{R}^{\mathcal{I}}$ is Leibniz.

Corollary 981 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\ddot{R}^{\mathcal{I}}$ *has the global family modus ponens and is extensional in* $\mathcal{I}$*, then* $\ddot{R}^{\mathcal{I}}$ *is Leibniz.*

Proof: Since $\ddot{R}^{\mathcal{I}}$ having the global family MP is stronger than having the system MP, the conclusion follows from Proposition 967. ■

In the opposite direction, in an *equivalential* π-institution $\mathcal{I}$, if the binary reflexive core is Leibniz, then it has the global family modus ponens and is extensional in $\mathcal{I}$.

Proposition 982 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *an equivalential* π*-institution based on* $\mathbf{F}$*. If* $\ddot{R}^{\mathcal{I}}$ *is Leibniz, then* $\ddot{R}^{\mathcal{I}}$ *has the global family modus ponens and is extensional in* $\mathcal{I}$*.*

Proof: Suppose that $\mathcal{I}$ is equivalential and that $\ddot{R}^{\mathcal{I}}$ is Leibniz.

For the global family MP, suppose that $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$, $\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T$. Then, since $\ddot{R}^{\mathcal{I}}$ is Leibniz,

$$\langle \phi, \psi \rangle \in \Omega^{\langle \phi, \psi \rangle}_\Sigma (C(\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi]) \cap \langle \phi, \psi \rangle).$$

Thus, since $\mathcal{I}$ is system extensional,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma (C(\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi])).$$

By hypothesis, $C(\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi]) \leq T$, whence, by equivalentiality,

$$\Omega(C(\ddot{R}^{\mathcal{I}}_\Sigma[\phi, \psi])) \leq \Omega(T).$$

We, thus, get that $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. Therefore, by compatibility of $\Omega(T)$ with T, we obtain $\psi \in T_\Sigma$, showing that $\ddot{R}^{\mathcal{I}}$ has the global family MP in $\mathcal{I}$.

Since equivalentiality implies preequivalentiality, the extensionality of $\ddot{R}^{\mathcal{I}}$ follows from Proposition 968. ■

We now show that a π-institution is syntactically equivalential if and only if it is equivalential and has a Leibniz binary reflexive core.

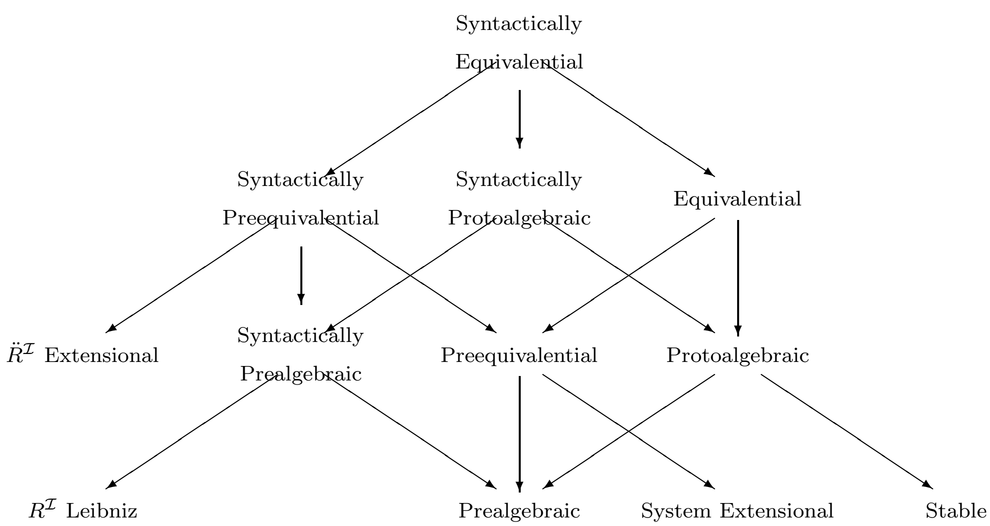
Syntactic Equivalentiality
$= \ddot{R}^{\mathcal{I}}$ has Global Family MP + $\ddot{R}^{\mathcal{I}}$ is Extensional
$= \ddot{R}^{\mathcal{I}}$ Defines Leibniz Congruence Systems of Theory Families in $\mathcal{I}$
= Equivalentiality + $\ddot{R}^{\mathcal{I}}$ is Leibniz

Theorem 983 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is syntactically equivalential if and only if it is equivalential and has a Leibniz binary reflexive core.*

Proof: Suppose, first, that $\mathcal{I}$ is syntactically equivalential. Then it is equivalential by Theorem 973. Moreover, its binary reflexive core has the global family modus ponens and is extensional, by Theorem 978, and, hence, by Corollary 981, its binary reflexive core is Leibniz.

Suppose, conversely, that $\mathcal{I}$ is equivalential with a Leibniz binary reflexive core. Then, by Proposition 982, its binary reflexive core has the global family MP and is extensional. Therefore, by Theorem 978, $\mathcal{I}$ is syntactically equivalential. ■

We have now established the following hierarchy of properties:



13.4 Strong Truth Equationality

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is **strongly (family) truth equational** if there exists a set $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ (with a single distinguished argument), such that, for every $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

In that case, we call $\tau^\flat$ a **set of witnessing equations** (of/for the strong truth equationality of $\mathcal{I}$).

Note that, since $\tau^\flat$ is parameter-free and $\Omega(T)$ is invariant under signature morphisms, strong truth equationality may be defined equivalently by the condition, for every $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma(\phi) \subseteq \Omega_\Sigma(T).$$

Since truth equationality implies systemicity, we get, a fortiori, that strong truth equationality implies systemicity.

We introduce next the unary Suszko core of a π-institution. Analogously with the Suszko core, the unary Suszko core enables one to obtain:

- A characterization of strong truth equationality in terms of the solubility property of the unary Suszko core of the π-institution.
- An exact description of those family c-reflective π-institutions which are strongly truth equational.
- A characterization of those truth equational π-institutions which are strongly truth equational.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. The **unary Suszko core** of $\mathcal{I}$ is the collection

$$\dot{S}^{\mathcal{I}} = \{\sigma^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2 \in N^\flat : (\forall T \in \mathrm{ThFam}(\mathcal{I}))(\sigma^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T)\}.$$

By Lemma 821, this definition is equivalent to setting

$$\begin{aligned}\dot{S}^{\mathcal{I}} \quad = \quad & \{\sigma^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2 \in N^\flat : (\forall \Sigma \in |\mathbf{Sign}^\flat|) \\ & \qquad (\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\sigma^\flat_\Sigma(\phi) \in \widetilde{\Omega}^{\mathcal{I}}_\Sigma(C(\phi))\}.\end{aligned}$$

Note that the unary Suszko core of a π-institution is included in the Suszko core, i.e., we have

Lemma 984 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then* $\dot{S}^{\mathcal{I}} \subseteq S^{\mathcal{I}}$.

Proof: Every pair of unary natural transformations in $N^\flat$ that satisfies the membership criterion for $\dot{S}^{\mathcal{I}}$ also satisfies the condition for membership in $S^{\mathcal{I}}$. ■

Lemma 984 yields immediately the following consequence.

Corollary 985 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $T \in \mathrm{ThFam}(\mathcal{I})$*, we have*

$$S^{\mathcal{I}}(\Omega(T)) \leq \dot{S}^{\mathcal{I}}(\Omega(T)).$$

Proof: We have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rcll} \phi \in S^{\mathcal{I}}_\Sigma(\Omega(T)) & \text{iff} & S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) & \text{(definition)} \\ & \text{implies} & \dot{S}^{\mathcal{I}}[\phi] \leq \Omega(T) & \text{(Lemma 984)} \\ & \text{iff} & \phi \in \dot{S}^{\mathcal{I}}_\Sigma(\Omega(T)). & \text{(definition)} \end{array}$$

Therefore, $S^{\mathcal{I}}(\Omega(T)) \leq \dot{S}^{\mathcal{I}}(\Omega(T))$. ∎

Either directly by the definition or using Proposition 832 together with Corollary 985, we get the following

Proposition 986 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For every* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$T \leq \dot{S}^{\mathcal{I}}(\Omega(T)).$$

Proof: We have $T \leq S^{\mathcal{I}}(\Omega(T)) \leq \dot{S}^{\mathcal{I}}(\Omega(T))$, where the first inclusion is by Lemma 832 and the second by Corollary 985. ∎

Similarly with the Suszko core, the unary Suszko core of a π-institution may or may not satisfy the reverse inclusion of Proposition 986, a property that was called *solubility*.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the unary Suszko core of $\mathcal{I}$ is **soluble** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\dot{S}^{\mathcal{I}}(\Omega(T)) \leq T.$$

Note that $\dot{S}^{\mathcal{I}}$ is soluble if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\dot{S}^{\mathcal{I}}_\Sigma(\phi) \subseteq \Omega_\Sigma(T) \quad \text{implies} \quad \phi \in T_\Sigma.$$

It turns out that possession of the solubility property by the unary Suszko core intrinsically characterizes strong truth equationality. To show the necessity of solubility observe, first, that, in case a π-institution is strongly truth equational, the witnessing equations form a subset of the unary Suszko core.

Lemma 987 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is strongly truth equational, with witnessing equations* $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2 \subseteq N^\flat$*, then* $\tau^\flat \subseteq \dot{S}^{\mathcal{I}}$*.*

Proof: Suppose that $\mathcal{I}$ is strongly truth equational with witnessing equations $\tau^\flat$. Then, $\mathcal{I}$ is, a fortiori, truth equational, with the same witnessing equations. It follows, by Lemma 835, that $\tau^\flat \subseteq S^{\mathcal{I}}$. Since $\tau^\flat$ consists of unary equations and they satisfy the membership criterion for $S^{\mathcal{I}}$, it follows that they also satisfy the condition for membership in $\dot{S}^{\mathcal{I}}$. Therefore, we get that $\tau^\flat \subseteq \dot{S}^{\mathcal{I}}$. ∎

Now we prove the necessity of the solubility of the unary Suszko core for strong truth equationality.

Theorem 988 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is strongly truth equational, then* $\dot{S}^{\mathcal{I}}$ *is soluble.*

Proof: Suppose that $\mathcal{I}$ is strongly truth equational, with witnessing equations $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{llll} \dot{S}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) & \text{implies} & \tau^\flat_\Sigma[\phi] \leq \Omega(T) & \text{(Lemma 987)} \\ & \text{iff} & \phi \in T_\Sigma. & \text{(strong truth equationality)} \end{array}$$

Thus, $\dot{S}^{\mathcal{I}}$ is soluble. ■

The reverse implication also holds and completes the promised characterization of strong truth equationality in terms of the solubility of the unary Suszko core.

Theorem 989 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\dot{S}^{\mathcal{I}}$ *is soluble, then* $\mathcal{I}$ *is strongly truth equational, with witnessing equations* $\dot{S}^{\mathcal{I}}$.

Proof: It suffices to show that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \dot{S}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

The left-to-right implication is given in Proposition 986, whereas the converse is ensured by the postulated solubility of $\dot{S}^{\mathcal{I}}$. ■

Theorems 988 and 989 provide the promised characterization of strong truth equationality in terms of the solubility of the unary Suszko core.

$$\mathcal{I} \text{ is Strongly Truth Equational} \longleftrightarrow \dot{S}^{\mathcal{I}} \text{ is Soluble.}$$

Theorem 990 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is strongly truth equational if and only if* $\dot{S}^{\mathcal{I}}$ *is soluble.*

Proof: Theorem 988 gives the "only if" and the "if" is by Theorem 989. ■

If $\mathcal{I}$ is strongly truth equational, then the unary Suszko core defines theory families in $\mathcal{I}$ in terms of their Leibniz congruence systems. This proposition may be viewed as a special case of Proposition 828, since $\dot{S}^{\mathcal{I}}$ forms a maximal set of witnessing equations of the strong truth equationality of $\mathcal{I}$.

Proposition 991 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\dot{S}^{\mathcal{I}}$ *is soluble, then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$,

$$T = \dot{S}^{\mathcal{I}}(\Omega(T)).$$

Proof: If $\dot{S}^{\mathcal{I}}$ is soluble, then, by Theorem 989, $\dot{S}^{\mathcal{I}}$ forms a set of witnessing equations for the strong truth equationality of $\mathcal{I}$. Therefore, by Proposition 828, we get that, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $T = \dot{S}^{\mathcal{I}}(\Omega(T))$. ∎

This property provides another characterization of strong truth equationality. We say that $\dot{S}^{\mathcal{I}}$ **defines theory families** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T = \dot{S}^{\mathcal{I}}(\Omega(T))$. Then we have:

$$\mathcal{I} \text{ is Strongly Truth Equational} \longleftrightarrow \dot{S}^{\mathcal{I}} \text{ Defines Theory Families.}$$

Theorem 992 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is strongly truth equational if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$T = \dot{S}^{\mathcal{I}}(\Omega(T)).$$

Proof: If $\mathcal{I}$ is truth equational, then, by Theorem 990, $\dot{S}^{\mathcal{I}}$ is soluble. Thus, by Proposition 991, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T = \dot{S}^{\mathcal{I}}(\Omega(T))$.

Conversely, if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T = \dot{S}^{\mathcal{I}}(\Omega(T))$, then, $\dot{S}^{\mathcal{I}}$ is soluble. Thus, again by Theorem 990, $\dot{S}^{\mathcal{I}}$ is a set of witnessing equations and $\mathcal{I}$ is strongly truth equational. ∎

It turns out that the property that separates family complete reflectivity from strong truth equationality is exactly the adequacy property of the unary Suszko core. Roughly speaking, this property ensures that the unary Suszko core is rich enough to define Suszko congruence systems in terms of the Leibniz congruence systems of theory families that it selects via inclusion.

We have the following relationship connecting the unary Suszko core with both Leibniz and Suszko congruence systems.

Proposition 993 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\bigcap\{\Omega(T) : \dot{S}^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)\} \le \widetilde{\Omega}^{\mathcal{I}}(C(\phi)).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, we have, using Lemma 984,

$$S^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T) \quad \text{implies} \quad \dot{S}^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T).$$

Therefore, $\{\Omega(T) : S^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)\} \subseteq \{\Omega(T) : \dot{S}^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)\}$. We conclude that

$$\bigcap\{\Omega(T) : \dot{S}^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)\} \le \bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)\} \le \widetilde{\Omega}^{\mathcal{I}}(C(\phi)),$$

where the last inclusion is based on Proposition 841. ∎

Again it is possible, but not necessary, that the unary Suszko core of a π-institution satisfies, for every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, the reverse inclusion of that given in Proposition 993:

$$\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) \leq \bigcap\{\Omega(T) : \dot{S}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.$$

Intuitively speaking, this means that the unary Suszko core $\dot{S}^{\mathcal{I}}$ is rich enough to allow, for every Σ-sentence ϕ, the determination of those theory families whose Leibniz congruence systems form a covering of the Suszko congruence system of $C(\phi)$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the unary Suszko core $\dot{S}^{\mathcal{I}}$ of $\mathcal{I}$ is **adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) = \bigcap\{\Omega(T) : \dot{S}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.$$

Based on our preceding work, it is not difficult to see that, if $\dot{S}^{\mathcal{I}}$ is soluble, then it is adequate.

Corollary 994 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\dot{S}^{\mathcal{I}}$ is soluble, then it is adequate.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rcll} \widetilde{\Omega}^{\mathcal{I}}(C(\phi)) & = & \bigcap\{\Omega(T) : \phi \in T_\Sigma\} & (\text{definition of } \widetilde{\Omega}^{\mathcal{I}}(C(\phi))) \\ & = & \bigcap\{\Omega(T) : \dot{S}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}. & \\ & & (\text{solubility of } \dot{S}^{\mathcal{I}} \text{ and Proposition 991}) & \end{array}$$

We conclude that $\dot{S}^{\mathcal{I}}$ is adequate. ■

In the opposite direction, in a *family c-reflective* π-institution $\mathcal{I}$, if the unary Suszko core is adequate, then it is also soluble.

Proposition 995 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a family c-reflective π-institution based on $\mathbf{F}$. If $\dot{S}^{\mathcal{I}}$ is adequate, then it is soluble.*

Proof: Suppose that $\mathcal{I}$ is family c-reflective and that $\dot{S}^{\mathcal{I}}$ is adequate. We must show that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$

$$\phi \in T_\Sigma \quad \text{iff} \quad \dot{S}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

The implication left-to-right is always satisfied by Proposition 986. For the converse, assume that $\dot{S}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. Then, by the adequacy of $\dot{S}^{\mathcal{I}}$, we get that $\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) \leq \Omega(T)$. Thus, by family c-reflectivity and Lemma 826, we conclude that $C(\phi) \leq T$, which gives $\phi \in T_\Sigma$. ■

We finally show that a π-institution is strongly truth equational if and only if it is family c-reflective and has an adequate unary Suszko core.

$$\begin{array}{lll} \text{Strong Truth Equationality} & = & \dot{S}^{\mathcal{I}} \text{ Soluble} \\ & = & \dot{S}^{\mathcal{I}} \text{ Defines Theory Families} \\ & = & \text{Family c-Reflectivity} + \dot{S}^{\mathcal{I}} \text{ Adequate} \end{array}$$

Theorem 996 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is strongly truth equational if and only if it is family c-reflective and has an adequate unary Suszko core.*

Proof: Suppose, first, that $\mathcal{I}$ is strongly truth equational. Then it is family c-reflective by Theorem 829. Moreover, its unary Suszko core is soluble by Theorem 990 and, hence, by Corollary 994, its unary Suszko core is adequate.

Suppose, conversely, that $\mathcal{I}$ is family c-reflective with an adequate unary Suszko core. Then, by Proposition 995, its unary Suszko core is soluble and, therefore, by Theorem 990, $\mathcal{I}$ is strongly truth equational. ■

We close the section with a result relating the unary Suszko core with the Suszko core. More precisely, we show that adequacy of the unary Suszko core implies adequacy of the Suszko core.

Proposition 997 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\dot{S}^{\mathcal{I}}$ *is adequate, then* $S^{\mathcal{I}}$ *is adequate.*

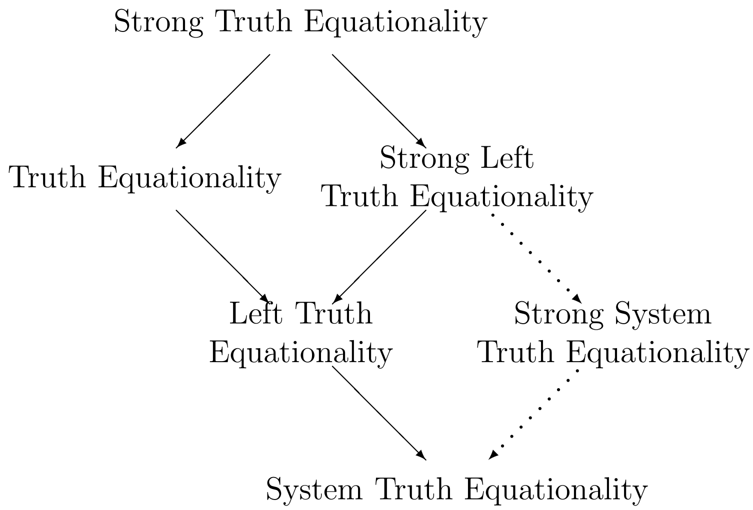
Proof: Suppose that $\dot{S}^{\mathcal{I}}$ is adequate. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rll} \widetilde{\Omega}^{\mathcal{I}}(C(\phi)) & \leq & \bigcap\{\Omega(T) : \dot{S}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \quad (\dot{S}^{\mathcal{I}} \text{ adequate}) \\ & \leq & \bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \quad (\dot{S}^{\mathcal{I}} \subseteq S^{\mathcal{I}}) \\ & \leq & \widetilde{\Omega}^{\mathcal{I}}(C(\phi)). \quad \text{(Proposition 841)} \end{array}$$

Hence, $\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) = \bigcap\{\Omega(T) : S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}$, and $S^{\mathcal{I}}$ is adequate. ■

13.5 Strong Left Truth Equationality

We now undertake the study of strong left truth equationality. This combines, in a certain sense, the study of left truth equationality with that of strong truth equationality. Strong left truth equationality has the same relation to left truth equationality as strong truth equationality has to (family) truth equationality. After this study, we will have the following hierarchy of truth equationality properties, which will be further augmented in the following section by adjoining strong system truth equationality:



Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is **strongly left truth equational** if there exists a set $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ (with a single distinguished argument), such that, for every $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

In that case, we call $\tau^\flat$ a **set of witnessing equations** (of the strong left truth equationality of $\mathcal{I}$).

Note that, similarly to strong truth equationality, since $\tau^\flat$ is parameter-free and $\Omega(T)$ is invariant under signature morphisms, strong left truth equationality may be defined equivalently by the condition, for every $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma(\phi) \subseteq \Omega_\Sigma(T).$$

Recall that strong truth equationality implies systemicity. Therefore, if a π-institution $\mathcal{I}$ is strongly truth equational, we get, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{iff} \quad \phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T),$$

whence $\mathcal{I}$ is also strongly left truth equational.

We introduce next the unary left Suszko core of a π-institution. Analogously with the left Suszko core and the unary Suszko core, the unary left Suszko core enables one to obtain:

- A characterization of strong left truth equationality in terms of the solubility property of the unary left Suszko core of the π-institution.

- An exact description of those left c-reflective π-institutions which are strongly left truth equational.
- A characterization of those left truth equational π-institutions which are strongly left truth equational.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The **unary left Suszko core** of $\mathcal{I}$ is the collection

$$\dot{L}^{\mathcal{I}} = \{\sigma^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2 \in N^\flat : (\forall T \in \mathrm{ThFam}(\mathcal{I}))(\sigma^\flat[\overleftarrow{T}] \leq \widetilde{\Omega}^{\mathcal{I}}(T)\}.$$

There are a couple of different possible characterizations of the unary left Suszko core that can be given. One is in terms of theory systems in place of theory families and another uses theory systems generated by single sentences.

Proposition 998 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$*. Then*

$$\dot{L}^{\mathcal{I}} = \{\sigma^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2 \in N^\flat : (\forall T \in \mathrm{ThSys}(\mathcal{I}))(\sigma^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T)\}.$$

Proof: By Proposition 852, we have that

$$L^{\mathcal{I}} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThSys}(\mathcal{I}))(\sigma^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T)\}.$$

Thus, the conclusion follows by applying the $\dot{}$ operator on both sides, i.e., by intersecting both sides with the set of all pairs of unary natural transformations in $N^\flat$. ∎

With Proposition 998 at hand, the second characterization follows from Lemma 822.

Corollary 999 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$*. Then*

$$\begin{array}{rcl}\dot{L}^{\mathcal{I}} & = & \{\sigma^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2 \in N^\flat : (\forall \Sigma \in |\mathbf{Sign}^\flat|) \\ & & \qquad (\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\sigma^\flat_\Sigma(\phi) \in \widetilde{\Omega}^{\mathcal{I}}_\Sigma(C(\overrightarrow{\phi}))\}.\end{array}$$

Proof: By combining Proposition 998 with Lemma 822. ∎

Note that the unary left Suszko core of a π-institution is included in the left Suszko core, i.e., we have

Lemma 1000 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$*. Then* $\dot{L}^{\mathcal{I}} \subseteq L^{\mathcal{I}}$*.*

Proof: Every pair of unary natural transformations in $N^\flat$ that satisfies the membership criterion for $\dot{L}^{\mathcal{I}}$ also satisfies the condition for membership in $L^{\mathcal{I}}$. ∎

Lemma 1000 yields immediately the following consequence.

Corollary 1001 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. For all $T \in \mathrm{ThFam}(\mathcal{I})$, we have*

$$L^{\mathcal{I}}(\Omega(T)) \leq \dot{L}^{\mathcal{I}}(\Omega(T)).$$

Proof: By Theorem 107 and Corollary 105. ∎

Either directly by the definition or using Proposition 853 together with Corollary 1001, we get the following

Proposition 1002 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. For every $T \in \mathrm{ThFam}(\mathcal{I})$,*

$$\overleftarrow{T} \leq \dot{L}^{\mathcal{I}}(\Omega(T)).$$

Proof: We have $\overleftarrow{T} \leq L^{\mathcal{I}}(\Omega(T)) \leq \dot{L}^{\mathcal{I}}(\Omega(T))$, where the first inclusion is by Lemma 853 and the second by Corollary 1001. ∎

Similarly with the left Suszko core, the unary left Suszko core of a π-institution may or may not satisfy the reverse inclusion of Proposition 1002, a property that was called left solubility.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the unary left Suszko core of $\mathcal{I}$ is **left soluble** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\dot{L}^{\mathcal{I}}(\Omega(T)) \leq \overleftarrow{T}.$$

Note that $\dot{L}^{\mathcal{I}}$ is left soluble if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\dot{L}^{\mathcal{I}}_\Sigma(\phi) \subseteq \Omega_\Sigma(T) \quad \text{implies} \quad \phi \in \overleftarrow{T}_\Sigma.$$

It turns out that possession of left solubility by the unary left Suszko core intrinsically characterizes strong left truth equationality. To show the necessity of left solubility observe, first, that, in case a π-institution is strongly left truth equational, the witnessing equations form a subset of the unary left Suszko core.

Lemma 1003 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is strongly truth equational, with witnessing equations $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2 \subseteq N^\flat$, then $\tau^\flat \subseteq \dot{L}^{\mathcal{I}}$.*

Proof: Suppose that $\mathcal{I}$ is strongly left truth equational with witnessing equations $\tau^\flat$. Then, $\mathcal{I}$ is, a fortiori, left truth equational, with the same witnessing equations. It follows, by Lemma 857, that $\tau^\flat \subseteq L^{\mathcal{I}}$. Since $\tau^\flat$ consists of unary equations and they satisfy the membership criterion for $L^{\mathcal{I}}$, it follows that they also satisfy the condition for membership in $\dot{L}^{\mathcal{I}}$. Therefore, we get that $\tau^\flat \subseteq \dot{L}^{\mathcal{I}}$. ■

Now we prove the necessity of the left solubility of the unary left Suszko core for strong left truth equationality.

Theorem 1004 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is strongly left truth equational, then* $\dot{L}^{\mathcal{I}}$ *is left soluble.*

Proof: Suppose that $\mathcal{I}$ is strongly left truth equational, with witnessing equations $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll} \dot{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) & \text{implies} & \tau^\flat_\Sigma[\phi] \leq \Omega(T) \quad \text{(Lemma 1003)} \\ & \text{iff} & \phi \in \overleftarrow{T}_\Sigma. \quad \text{(strong left truth equationality)} \end{array}$$

Thus, $\dot{L}^{\mathcal{I}}$ is left soluble. ■

The reverse implication also holds and provides the characterization of strong left truth equationality in terms of the left solubility of the unary left Suszko core.

Theorem 1005 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\dot{L}^{\mathcal{I}}$ *is left soluble, then* $\mathcal{I}$ *is strongly left truth equational, with witnessing equations* $\dot{L}^{\mathcal{I}}$*.*

Proof: It suffices to show that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{iff} \quad \dot{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

The left-to-right implication is given in Proposition 1002, whereas the converse is ensured by the postulated left solubility of $\dot{L}^{\mathcal{I}}$. ■

Theorems 1004 and 1005 provide the promised characterization of strong left truth equationality in terms of the left solubility of the unary left Suszko core.

$$\mathcal{I} \text{ is Strongly Left Truth Equational} \longleftrightarrow \dot{L}^{\mathcal{I}} \text{ is Left Soluble.}$$

Theorem 1006 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is strongly left truth equational if and only if* $\dot{L}^{\mathcal{I}}$ *is left soluble.*

Proof: Theorem 1004 gives the "only if" and the "if" is by Theorem 1005. ∎

If $\mathcal{I}$ is strongly left truth equational, then the unary left Suszko core defines theory families in $\mathcal{I}$, up to arrow, in terms of their Leibniz congruence systems. So, analogously to preceding situations, $\dot{L}^{\mathcal{I}}$ forms a maximal set of witnessing equations of the strong left truth equationality of $\mathcal{I}$.

Proposition 1007 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\dot{L}^{\mathcal{I}}$ *is left soluble, then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\overleftarrow{T} = \dot{L}^{\mathcal{I}}(\Omega(T)).$$

Proof: If $\dot{L}^{\mathcal{I}}$ is left soluble, then, by Theorem 1005, $\dot{L}^{\mathcal{I}}$ forms a set of witnessing equations for the strong left truth equationality of $\mathcal{I}$. Therefore, by Proposition 849, we get that, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} = \dot{L}^{\mathcal{I}}(\Omega(T))$. ∎

This property provides another characterization of strong left truth equationality. We say that $\dot{L}^{\mathcal{I}}$ **defines theory families up to arrow** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} = \dot{L}^{\mathcal{I}}(\Omega(T))$. Then we have:

$\mathcal{I}$ is Strongly Left Truth Equational
$\longleftrightarrow$ $\dot{L}^{\mathcal{I}}$ Defines Theory Families Up to Arrow.

Theorem 1008 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is strongly left truth equational if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\overleftarrow{T} = \dot{L}^{\mathcal{I}}(\Omega(T)).$$

Proof: If $\mathcal{I}$ is strongly left truth equational, then, by Theorem 1006, $\dot{L}^{\mathcal{I}}$ is left soluble. Thus, by Proposition 1007, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} = \dot{L}^{\mathcal{I}}(\Omega(T))$.

Conversely, if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{T} = \dot{L}^{\mathcal{I}}(\Omega(T))$, then, $\dot{L}^{\mathcal{I}}$ is left soluble. Thus, again by Theorem 1006, $\dot{L}^{\mathcal{I}}$ is a set of witnessing equations and $\mathcal{I}$ is strongly left truth equational. ∎

In analogy with strong truth equationality and family c-reflectivity, the property that separates left complete reflectivity from strong left truth equationality is exactly the left adequacy of the unary left Suszko core. Roughly speaking, this property ensures that the unary left Suszko core is rich enough to define Suszko congruence systems in terms of the Leibniz congruence systems of theory families that it selects via inclusion.

We have the following relationship connecting the unary left Suszko core with both Leibniz and Suszko congruence systems.

Proposition 1009 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\bigcap\{\Omega(T) : \dot{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \leq \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, we have, using Lemma 1000,

$$L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) \quad \text{implies} \quad \dot{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

Therefore, $\{\Omega(T) : L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \subseteq \{\Omega(T) : \dot{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}$. We conclude that

$$\bigcap\{\Omega(T) : \dot{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \leq \bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \leq \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})),$$

where the last inclusion is based on Proposition 863. ∎

Again it is possible, but not necessary, that the unary left Suszko core of a π-institution satisfies, for every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, the reverse inclusion of that given in Proposition 1009:

$$\widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) \leq \bigcap\{\Omega(T) : \dot{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.$$

Intuitively speaking, this means that the unary left Suszko core $\dot{L}^{\mathcal{I}}$ is rich enough to allow, for every signature Σ and for every Σ-sentence ϕ, the determination of those theory families whose Leibniz congruence systems form a covering of the Suszko congruence system of $C(\overrightarrow{\phi})$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the unary left Suszko core $\dot{L}^{\mathcal{I}}$ of $\mathcal{I}$ is **left adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) = \bigcap\{\Omega(T) : \dot{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.$$

Based on our preceding work, it is not difficult to see that, if $\dot{L}^{\mathcal{I}}$ is left soluble, then it is left adequate.

Corollary 1010 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\dot{L}^{\mathcal{I}}$ *is left soluble, then it is left adequate.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rcll}\widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) & = & \bigcap\{\Omega(T) : \overrightarrow{\phi} \leq T\} & (\text{definition of } \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi}))) \\ & = & \bigcap\{\Omega(T) : \phi \in \overleftarrow{T}_\Sigma\} & (\text{definition of } \overleftarrow{T}) \\ & = & \bigcap\{\Omega(T) : \dot{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}. & \\ & & (\text{left solubility of } \dot{L}^{\mathcal{I}} \text{ and Proposition 1007}) & \end{array}$$

We conclude that $\dot{L}^{\mathcal{I}}$ is left adequate. ∎

In the opposite direction, in a *left c-reflective* π-institution $\mathcal{I}$, if the unary left Suszko core is left adequate, then it is also left soluble.

Proposition 1011 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a left c-reflective* π*-institution based on* $\mathbf{F}$*. If* $\dot{L}^{\mathcal{I}}$ *is left adequate, then it is left soluble.*

Proof: Suppose that $\mathcal{I}$ is left c-reflective and that $\dot{L}^{\mathcal{I}}$ is left adequate. We must show that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{iff} \quad \dot{L}^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T).$$

The implication left-to-right is always satisfied by Proposition 1002. For the converse, assume that $\dot{L}^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)$. Then, by the left adequacy of $\dot{L}^{\mathcal{I}}$, we get that $\widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) \le \Omega(T)$. Thus, by left c-reflectivity and Lemma 868, we conclude that $\overleftarrow{C(\overrightarrow{\phi})} \le \overleftarrow{T}$. This implies $\phi \in \overleftarrow{T}_\Sigma$. ■

We finally show that a π-institution is strongly left truth equational if and only if it is left c-reflective and has a left adequate unary left Suszko core.

Strong Left Truth Equationality
= $\dot{L}^{\mathcal{I}}$ Left Soluble
= $\dot{L}^{\mathcal{I}}$ Defines Theory Families Up to Arrow
= Left c-Reflectivity + $\dot{L}^{\mathcal{I}}$ Left Adequate

Theorem 1012 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is strongly left truth equational if and only if it is left c-reflective and has a left adequate unary left Suszko core.*

Proof: Suppose, first, that $\mathcal{I}$ is strongly left truth equational. Then it is left c-reflective by Theorem 850. Moreover, its unary left Suszko core is left soluble by Theorem 1006 and, hence, by Corollary 1010, its unary left Suszko core is left adequate.

Suppose, conversely, that $\mathcal{I}$ is family c-reflective with a left adequate unary left Suszko core. Then, by Proposition 1011, its unary left Suszko core is left soluble. Hence, by Theorem 1006, $\mathcal{I}$ is strongly left truth equational. ■

We close the section with a result relating the unary left Suszko core with the left Suszko core. More precisely, we show that left adequacy of the unary left Suszko core implies left adequacy of the left Suszko core.

Proposition 1013 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\dot{L}^{\mathcal{I}}$ *is left adequate, then* $L^{\mathcal{I}}$ *is left adequate.*

Proof: Suppose that $\dot{L}^{\mathcal{I}}$ is left adequate. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rcll}\widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) & \leq & \bigcap\{\Omega(T) : \dot{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} & (\dot{L}^{\mathcal{I}} \text{ left adequate})\\ & \leq & \bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} & (\dot{L}^{\mathcal{I}} \subseteq L^{\mathcal{I}})\\ & \leq & \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})). & \text{(Proposition 863)}\end{array}$$

Hence, $\widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) = \bigcap\{\Omega(T) : L^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}$, and $L^{\mathcal{I}}$ is left adequate. ∎

13.6 Strong System Truth Equationality

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is **strongly system truth equational** if there exists a set $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ (with a single distinguished argument), such that, for every $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

In that case, we call $\tau^\flat$ a **set of witnessing equations** (of/for the strong system truth equationality of $\mathcal{I}$).

Again, since $\tau^\flat$ is parameter-free and $\Omega(T)$ is invariant under signature morphisms, strong system truth equationality may be defined equivalently by the condition, for every $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma(\phi) \subseteq \Omega_\Sigma(T).$$

We introduce next the unary system core of a π-institution. Analogously with the system core, the unary system core enables one to obtain:

- A characterization of strong system truth equationality in terms of the solubility property of the unary system core of the π-institution.
- An exact description of those system c-reflective π-institutions which are strongly system truth equational.
- A characterization of those system truth equational π-institutions which are strongly system truth equational.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Recall that, for $T \in \mathrm{ThSys}(\mathcal{I})$, we have introduced the notation

$$\widehat{\Omega}^{\mathcal{I}}(T) = \bigcap\{\Omega(T') : T \leq T' \in \mathrm{ThSys}(\mathcal{I})\}.$$

This is a variant of the Suszko operator, allowing one to zoom in on the theory system structure of the π-institution under consideration, which forms naturally the focus in the present section.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The **unary system core** of $\mathcal{I}$ is the collection

$$\dot{Z}^{\mathcal{I}} = \{\sigma^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2 \in N^\flat : (\forall T \in \mathrm{ThSys}(\mathcal{I}))(\sigma^\flat[T] \leq \widehat{\Omega}^{\mathcal{I}}(T))\}.$$

Note that the unary system core of a π-institution is included in the system core, i.e., we have

Lemma 1014 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Then $\dot{Z}^{\mathcal{I}} \subseteq Z^{\mathcal{I}}$.*

Proof: Every pair of unary natural transformations in $N^\flat$ that satisfies the membership criterion for $\dot{Z}^{\mathcal{I}}$ also satisfies the condition for membership in $Z^{\mathcal{I}}$. ■

Moreover, we have the following relationship between the sentence families defined via the Leibniz congruence systems by the system and the unary system core.

Corollary 1015 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. For all $T \in \mathrm{ThFam}(\mathcal{I})$, we have*

$$Z^{\mathcal{I}}(\Omega(T)) \leq \dot{Z}^{\mathcal{I}}(\Omega(T)).$$

Proof: By Theorem 107 and Corollary 105. ■

The relation between the unary Suszko core, the unary left Suszko core and the unary system core of a π-institution $\mathcal{I}$ is given in the following proposition, forming an analog of Proposition 874, concerning the general (non-unary) analogs of these sets.

Proposition 1016 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\dot{S}^{\mathcal{I}} \subseteq \dot{L}^{\mathcal{I}} \subseteq \dot{Z}^{\mathcal{I}}$;

(b) For every relation family θ on $\mathbf{F}$, $\dot{Z}^{\mathcal{I}}(\theta) \leq \dot{L}^{\mathcal{I}}(\theta) \leq \dot{S}^{\mathcal{I}}(\theta)$.

Proof: From Proposition 874, we have that $S^{\mathcal{I}} \subseteq L^{\mathcal{I}} \subseteq Z^{\mathcal{I}}$. Thus, Part (a) follows by applying the $\dot{}$ operator (which is monotone) to this chain of inclusions. Part (b) follows form Part (a) and the relevant definitions. ■

Either directly by the definition or using Proposition 875 together with Corollary 1015, we get the following

Proposition 1017 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. For every $T \in \mathrm{ThSys}(\mathcal{I})$,*

$$T \leq \dot{Z}^{\mathcal{I}}(\Omega(T)).$$

Proof: We have $T \leq Z^{\mathcal{I}}(\Omega(T)) \leq \dot{Z}^{\mathcal{I}}(\Omega(T))$, where the first inclusion is by Lemma 875 and the second by Corollary 1015. ■

The unary system core of a π-institution may or may not satisfy the reverse inclusion of Proposition 1017, a property that was called previously, in similar contexts, *solubility*.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the unary system core of $\mathcal{I}$ is **soluble** if, for all $T \in \mathrm{ThSys}(\mathcal{I})$,

$$\dot{Z}^{\mathcal{I}}(\Omega(T)) \leq T.$$

Note that $\dot{Z}^{\mathcal{I}}$ is soluble if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\dot{Z}^{\mathcal{I}}_\Sigma(\phi) \subseteq \Omega_\Sigma(T) \quad \text{implies} \quad \phi \in T_\Sigma.$$

It turns out that possession of the solubility property by the unary system core intrinsically characterizes strong system truth equationality. To show the necessity of solubility, we observe, once again, that, in case a π-institution is strongly system truth equational, the witnessing equations form a subset of the unary system core.

Lemma 1018 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is strongly system truth equational, with witnessing equations $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2 \subseteq N^\flat$, then $\tau^\flat \subseteq \dot{Z}^{\mathcal{I}}$.*

Proof: Suppose that $\mathcal{I}$ is strongly system truth equational with witnessing equations $\tau^\flat$. Then, $\mathcal{I}$ is, a fortiori, system truth equational, with the same witnessing equations. It follows, by Lemma 877, that $\tau^\flat \subseteq Z^{\mathcal{I}}$. Since $\tau^\flat$ consists of unary equations and they satisfy the membership criterion for $Z^{\mathcal{I}}$, it follows that they also satisfy the condition for membership in $\dot{Z}^{\mathcal{I}}$. Therefore, we get that $\tau^\flat \subseteq \dot{Z}^{\mathcal{I}}$. ■

Now we prove the necessity of the solubility of the unary system core for strong system truth equationality.

Theorem 1019 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is strongly system truth equational, then $\dot{Z}^{\mathcal{I}}$ is soluble.*

Proof: Suppose that $\mathcal{I}$ is strongly system truth equational, with witnessing equations $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$. Then, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rcll} \dot{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) & \text{implies} & \tau^\flat_\Sigma[\phi] \leq \Omega(T) & \text{(Lemma 1018)} \\ & \text{iff} & \phi \in T_\Sigma, & \end{array}$$

where the last equivalence is based on the postulated strong system truth equationality of $\mathcal{I}$. Thus, $\dot{Z}^{\mathcal{I}}$ is soluble. ■

The reverse implication completes the promised characterization of strong system truth equationality in terms of the solubility of the unary system core.

Theorem 1020 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\dot{Z}^{\mathcal{I}}$ *is soluble, then* $\mathcal{I}$ *is strongly system truth equational, with witnessing equations* $\dot{Z}^{\mathcal{I}}$*.*

Proof: It suffices to show that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \dot{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

The left-to-right implication is given in Proposition 1017, whereas the converse is ensured by the postulated solubility of $\dot{Z}^{\mathcal{I}}$. ■

Theorems 1019 and 1020 provide the promised characterization of strong system truth equationality in terms of the solubility of the unary system core.

$$\mathcal{I} \text{ is Strongly System Truth Equational} \longleftrightarrow \dot{Z}^{\mathcal{I}} \text{ is Soluble.}$$

Theorem 1021 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is strongly system truth equational if and only if* $\dot{Z}^{\mathcal{I}}$ *is soluble.*

Proof: Theorem 1019 gives the "only if" and the "if" is by Theorem 1020. ■

If $\mathcal{I}$ is strongly system truth equational, then the unary system core defines theory systems in $\mathcal{I}$ in terms of their Leibniz congruence systems. This proposition may be viewed as a special case of Proposition 871, since $\dot{Z}^{\mathcal{I}}$ forms a maximal set of witnessing equations of the strong system truth equationality of $\mathcal{I}$.

Proposition 1022 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\dot{Z}^{\mathcal{I}}$ *is soluble, then, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,*

$$T = \dot{Z}^{\mathcal{I}}(\Omega(T)).$$

Proof: If $\dot{Z}^{\mathcal{I}}$ is soluble, then, by Theorem 1020, $\dot{Z}^{\mathcal{I}}$ forms a set of witnessing equations for the strong system truth equationality of $\mathcal{I}$. Therefore, by Proposition 871, we get that, for every $T \in \mathrm{ThSys}(\mathcal{I})$, $T = \dot{Z}^{\mathcal{I}}(\Omega(T))$. ■

This property provides another characterization of strong system truth equationality. We say that $\dot{Z}^{\mathcal{I}}$ **defines theory systems** if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $T = \dot{Z}^{\mathcal{I}}(\Omega(T))$. Then we have:

$$\mathcal{I} \text{ is Strongly System Truth Equational} \\ \longleftrightarrow \dot{Z}^{\mathcal{I}} \text{ Defines Theory Systems.}$$

Theorem 1023 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is strongly system truth equational if and only if, for all* $T \in \mathrm{ThSys}(\mathcal{I})$,

$$T = \dot{Z}^{\mathcal{I}}(\Omega(T)).$$

Proof: If $\mathcal{I}$ is strongly system truth equational, then, by Theorem 1021, $\dot{Z}^{\mathcal{I}}$ is soluble. Thus, by Proposition 1022, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $T = \dot{Z}^{\mathcal{I}}(\Omega(T))$.

Conversely, if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $T = \dot{Z}^{\mathcal{I}}(\Omega(T))$, then, $\dot{Z}^{\mathcal{I}}$ is soluble. Thus, again by Theorem 1021, $\dot{Z}^{\mathcal{I}}$ is a set of witnessing equations and $\mathcal{I}$ is strongly system truth equational. ∎

It turns out that the property that separates system complete reflectivity from strong system truth equationality is exactly the adequacy property of the unary system core. Roughly speaking, this property ensures that the unary system core is rich enough to define the congruence system $\widetilde{\Omega}^{\mathcal{I}}(T)$ of a theory system T in terms of the Leibniz congruence systems of theory systems that it selects via inclusion.

Proposition 1024 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \textit{ and } \dot{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \leq \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $T \in \mathrm{ThSys}(\mathcal{I})$, we have, using Lemma 1014,

$$Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) \quad \text{implies} \quad \dot{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

Therefore,

$$\begin{aligned}\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}\\ \subseteq \{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \dot{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.\end{aligned}$$

We conclude that

$$\begin{aligned}&\bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \dot{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}\\ &\quad \leq \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \leq \widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})),\end{aligned}$$

where the last inclusion is based on Proposition 883. ∎

It is possible, but not necessary, that the unary system core of a π-institution satisfies, for every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, the reverse inclusion of that given in Proposition 1024:

$$\widetilde{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) \leq \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \dot{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.$$

Intuitively speaking, this means that the unary system core $\dot{Z}^{\mathcal{I}}$ is rich enough to allow, for every signature Σ and every Σ-sentence ϕ, the determination of those theory systems whose Leibniz congruence systems form a covering of $\widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi}))$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the unary system core $\dot{Z}^{\mathcal{I}}$ of $\mathcal{I}$ is **adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) = \bigcap \{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \dot{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.$$

Based on our preceding work, it is not difficult to see that, if $\dot{Z}^{\mathcal{I}}$ is soluble, then it is adequate.

Corollary 1025 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\dot{Z}^{\mathcal{I}}$ is soluble, then it is adequate.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rcl}\widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) & = & \bigcap \{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \phi \in T_\Sigma\} \\ & & (\text{definition of } \widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi}))) \\ & = & \bigcap \{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \dot{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}. \\ & & (\text{solubility of } \dot{Z}^{\mathcal{I}} \text{ and Proposition 1022})\end{array}$$

We conclude that $\dot{Z}^{\mathcal{I}}$ is adequate. ∎

In the opposite direction, in a *system c-reflective* π-institution $\mathcal{I}$, if the unary system core is adequate, then it is also soluble.

Proposition 1026 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a system c-reflective π-institution based on $\mathbf{F}$. If $\dot{Z}^{\mathcal{I}}$ is adequate, then it is soluble.*

Proof: Suppose that $\mathcal{I}$ is system c-reflective and that $\dot{Z}^{\mathcal{I}}$ is adequate. We must show that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \dot{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

The implication left-to-right is always satisfied by Proposition 1017. For the converse, assume that $\dot{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. Then, by the adequacy of $\dot{Z}^{\mathcal{I}}$, we get that $\widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) \leq \Omega(T)$. Thus, by system c-reflectivity and Lemma 885, we conclude that $C(\overrightarrow{\phi}) \leq T$, which gives $\phi \in T_\Sigma$. ∎

We finally show that a π-institution is strongly system truth equational if and only if it is system c-reflective and has an adequate unary system core.

Strong System Truth Equationality
= $\dot{Z}^{\mathcal{I}}$ Soluble
= $\dot{Z}^{\mathcal{I}}$ Defines Theory Systems
= System c-Reflectivity + $\dot{Z}^{\mathcal{I}}$ Adequate

Theorem 1027 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is strongly system truth equational if and only if it is system c-reflective and has an adequate unary system core.*

Proof: Suppose, first, that $\mathcal{I}$ is strongly system truth equational. Then it is system c-reflective by Theorem 872. Moreover, its unary system core is soluble by Theorem 1021 and, hence, by Corollary 1025, its unary system core is adequate.

Suppose, conversely, that $\mathcal{I}$ is system c-reflective with an adequate unary system core. Then, by Proposition 1026, its unary system core is soluble and, therefore, by Theorem 1021, $\mathcal{I}$ is strongly system truth equational. ∎

We close the section with a result relating the unary system core with the system core. More precisely, we show that adequacy of the unary system core implies adequacy of the system core.

Proposition 1028 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\dot{Z}^{\mathcal{I}}$ *is adequate, then* $Z^{\mathcal{I}}$ *is adequate.*

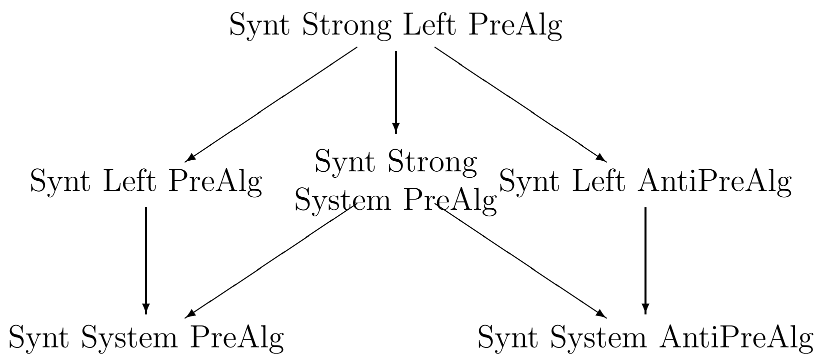
Proof: Suppose that $\dot{Z}^{\mathcal{I}}$ is adequate. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rcl}\widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) & \leq & \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \dot{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \\ & & (\dot{Z}^{\mathcal{I}} \text{ adequate}) \\ & \leq & \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \\ & & (\dot{Z}^{\mathcal{I}} \subseteq Z^{\mathcal{I}}) \\ & \leq & \widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})). \quad \text{(Proposition 883)}\end{array}$$

Hence, $\widehat{\Omega}^{\mathcal{I}}(C(\overrightarrow{\phi})) = \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } Z^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}$, and $Z^{\mathcal{I}}$ is adequate. ∎

13.7 Syntactic Left PreAlgebraizability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution. Recall that $\mathcal{I}$ belongs to one of the classes of the *prealgebraizability hierarchy* when its Leibniz operator is monotone on theory systems and it has a certain kind of extensionality and a certain kind of injectivity or reflectivity or complete reflectivity property. We now turn to corresponding properties defined via "syntactic" means. Keeping a level of consistency, we will call *syntactically prealgebraizable* any π-institution whose Leibniz operator on theory systems is definable via a set of binary natural transformations in $N^\flat$, i.e., a parameter free set of natural transformations in $N^\flat$, and, additionally, has a certain kind of definability property of truth, via a, possibly, parameter free set of equations in $N^\flat$. If the situation is reversed and definability of truth is required to be via a parameter free set of equations, but that is not demanded of the definability of Leibniz congruence systems, then we obtain the classes of *anti-prealgebraizable π-institutions*, a term concocted here to convey a kind of chiral symmetry in applying "parameterlessness". The hierarchy we aim for consists of the six classes depicted in the following diagram.



Membership in the classes of the central column imposes parameter free definability of both the Leibniz operator on theory systems and a kind of parameter free definability of truth. Membership in the classes in the left column insists only on parameter free definability of the Leibniz operator, whereas, symmetrically, membership in the classes of the right column postulates only a kind of parameter free definability of truth.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. In agreement with preestablished nomenclature, we say that $\mathcal{I}$ is $\ddot{R}^{\mathcal{I}}\dot{L}^{\mathcal{I}}$-**fortified** if it has

- a Leibniz binary reflexive core and
- a left adequate unary left Suszko core.

We say that $\mathcal{I}$ is **syntactically strongly left prealgebraizable** if it is

- $\ddot{R}^{\mathcal{I}}\dot{L}^{\mathcal{I}}$-fortified;
- preequivalential (i.e., prealgebraic and system extensional);
- left c-reflective.

Our preceding work in this chapter has paved the way for the following important characterization of syntactic strong left prealgebraizability.

Theorem 1029 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is syntactically strongly left prealgebraizable if and only if it is syntactically preequivalential and strongly left truth equational.*

Proof: We have that $\mathcal{I}$ is syntactically strongly left prealgebraizable if and only if, by definition, it is

- preequivalential and has a Leibniz binary reflexive core;
- left c-reflective and has a left adequate unary left Suszko core;

if and only if, by Theorems 969 and 1012, is it syntactically preequivalential and strongly left truth equational. ∎

An alternative characterization along similar lines relates the syntactic with the corresponding semantic notions introduced in the context of prealgebraizability.

Theorem 1030 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly left prealgebraizable if and only if it is LC prealgebraizable and* $\ddot{R}^{\mathcal{I}}\dot{L}^{\mathcal{I}}$*-fortified.*

Proof: We have that $\mathcal{I}$ is syntactically strongly left prealgebraizable if and only if, by definition,

- it is preequivalential and left c-reflective;
- it has a Leibniz binary reflexive core and a left adequate unary left Suszko core;

if and only if, by definition, it is LC prealgebraizable and $\ddot{R}^{\mathcal{I}}\dot{L}^{\mathcal{I}}$-fortified. ∎

This characterization in terms of semantic properties and preceding work on transference of properties from theory families/systems to filter families/systems on arbitrary algebraic systems yield yet another characterization of syntactic strong left prealgebraizability, which may also be viewed as a kind of transfer property for this class in its own right.

Theorem 1031 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly left prealgebraizable if and only if it is* $\ddot{R}^{\mathcal{I}}\dot{L}^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\Omega^{\mathcal{A}}$ *is monotone on* $\mathcal{I}$*-filter systems, system extensional and left c-reflective.*

Proof: We have that $\mathcal{I}$ is syntactically strongly left prealgebraizable if and only if, by Theorem 1030, it is $\ddot{R}^{\mathcal{I}}\dot{L}^{\mathcal{I}}$-fortified and LC prealgebraizable if and only if, by Theorem 349, it is $\ddot{R}^{\mathcal{I}}\dot{L}^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ is monotone on $\mathcal{I}$-filter systems, system extensional and left c-reflective. ∎

Turning now to characterizations involving property preserving mappings between posets of filter families and congruence systems, we have the following result:

Theorem 1032 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically strongly left prealgebraizable if and only if it is* $\ddot{R}^{\mathcal{I}}\dot{L}^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is a left completely order reflecting surjection that restricts to an order embedding

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

that commutes with inverse logical extensions.

Proof: We have that $\mathcal{I}$ is syntactically strongly left prealgebraizable if and only if, by Theorem 1030, it is $\ddot{R}^{\mathcal{I}}\dot{L}^{\mathcal{I}}$-fortified and LC prealgebraizable if and only if, by Theorem 355 it is $\ddot{R}^{\mathcal{I}}\dot{L}^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is a left completely order reflecting surjection that restricts to an order embedding $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ that commutes with inverse logical extensions. ■

Finally, in terms of conjugate pairs of transformations, we get the following analog of Theorem 949.

Theorem 1033 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically strongly left prealgebraizable if and only if it is strongly left truth equational and its systemic skeleton* $\mathcal{K}^{\mathcal{I}}$ *is equivalent to* $\mathcal{Q}^{\mathcal{I}\bullet}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}\bullet}$ *of natural transformations.*

Proof: Suppose, first, that $\mathcal{I}$ is strongly left truth equational and $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair of natural transformations. Then it is strongly left truth equational and, by Theorem 941, it is syntactically preequivalential. Thus, by Theorem 1029, it is syntactically strongly left prealgebraizable.

Suppose, conversely, that $\mathcal{I}$ is syntactically strongly left prealgebraizable. Then, by Theorem 1029, it is strongly left truth equational and syntactically preequivalential. Hence, by Theorem 934, it is syntactically WS prealgebraizable. Now it follows by Theorem 940 that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a pair $(\tau^\flat, \overleftrightarrow{I}^\flat)$, where $I^\flat$ witnesses the syntactic preequivalentiality and $\tau^\flat$ the syntactic strong left truth equationality of $\mathcal{I}$, and, hence, by definition, they constitute a conjugate pair of natural transformations. ■

Again, the equivalence of the systemic skeleton with some algebraic π-structure via a conjugate pair of natural transformations, coupled with strong left truth equationality, is sufficient to ensure syntactic strong left prealgebraizability.

Theorem 1034 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly left prealgebraizable if and only if it is strongly left truth equational and its systemic skeleton is equivalent to an algebraic* π*-structure via a conjugate pair of natural transformations.*

Proof: If $\mathcal{I}$ is syntactically strongly left prealgebraizable, then, by Theorem 1033, it is strongly left truth equational and its systemic skeleton is equivalent to an algebraic π-structure via a conjugate pair of natural transformations. Suppose, conversely, that $\mathcal{I}$ is strongly left truth equational and its systemic skeleton is equivalent to an algebraic π-structure via a conjugate pair of natural transformations. Then, it is strongly left truth equational and, by Proposition 928, it is syntactically preequivalential. Therefore, by Theorem 1029, it is syntactically strongly left prealgebraizable. ■

Finally, in terms of order isomorphisms between theory family lattices, we have the following alternative characterization of syntactically strongly left prealgebraizable π-institutions:

Theorem 1035 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly left prealgebraizable if and only if it is strongly left truth equational and there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{K}^{\mathcal{I}}) \to \mathbf{ThFam}(\mathcal{Q})$*, induced by a conjugate pair* $(\tau^\flat, I^\flat)$ *of natural transformations, where* $\mathcal{Q}$ *is an algebraic* π*-structure.*

Proof: The "only if" follows by Theorem 1034 and Theorem 893. The "if" is given by Theorem 901 and Theorem 1034. ■

Flanking the class of syntactically strongly left prealgebraizable π-institutions are the classes of *syntactically left prealgebraizable* and *syntactically left antiprealgebraizable* π-institutions. These two classes are defined formally now.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

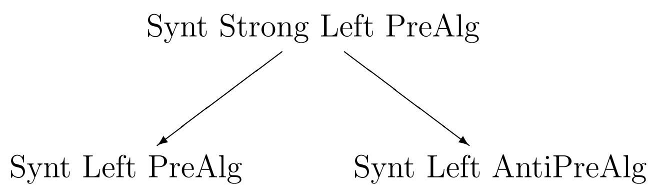
- $\mathcal{I}$ is **syntactically left prealgebraizable** if it is:
  - $\ddot{R}^{\mathcal{I}} L^{\mathcal{I}}$-fortified;
  - preequivalential;
  - left c-reflective;
- $\mathcal{I}$ is **syntactically left antiprealgebraizable** if it is:
  - $R^{\mathcal{I}} \dot{L}^{\mathcal{I}}$-fortified;
  - prealgebraic;

  – left c-reflective.

For both of these classes we have analogs of many of the results proven above for syntactic strong left prealgebraizability. Before formulating them, let us observe that, since:

- preequivalentiality is stronger than prealgebraicity;
- under prealgebraicity, $\ddot{R}^{\mathcal{I}}$ Leibniz implies $R^{\mathcal{I}}$ Leibniz; and
- $\dot{L}^{\mathcal{I}}$ left adequate implies $L^{\mathcal{I}}$ left adequate,

we get, immediately from the definitions, the following hierarchical relations between the upper three classes in the echelon formation of the preceding diagram.



We now provide examples to show that the two inclusions are proper. The first is an example of a π-institution which is syntactically left prealgebraizable but not syntactically strongly left prealgebraizable.

Example 1036 EXAMPLE NOT FOUND YET!!

Next, we give an example of a syntactically left antiprealgebraizable π-institution which fails to be syntactically strongly left prealgebraizable.

Example 1037 EXAMPLE NOT FOUND YET!!

The following analog of Theorem 1029 relates these two chiral types of syntactic left prealgebraizability with various classes introduced previously, providing some important characterizations.

Theorem 1038 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is syntactically left prealgebraizable if and only if it is syntactically preequivalential and left truth equational.*

(b) $\mathcal{I}$ *is syntactically left antiprealgebraizable if and only if it is syntactically prealgebraic and strongly left truth equational.*

Proof:

(a) We have that $\mathcal{I}$ is syntactically left prealgebraizable if and only if, by definition, it is preequivalential, with a Leibniz binary reflexive core, and left c-reflective, with a left adequate left Suszko core, if and only if, by Theorems 969 and 870, is it syntactically preequivalential and left truth equational.

(b) Similarly, $\mathcal{I}$ is syntactically left antiprealgebraizable if and only if, by definition, it is prealgebraic, with a Leibniz reflexive core, and left c-reflective, with a left adequate unary left Suszko core, if and only if, by Theorems 788 and 1012, is it syntactically prealgebraic and strongly left truth equational.

■

An alternative characterization, analogous to that of Theorem 1030, relates the syntactic with the corresponding semantic notions.

Theorem 1039 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is syntactically left prealgebraizable if and only if it is LC prealgebraizable and* $\ddot{R}^{\mathcal{I}} L^{\mathcal{I}}$*-fortified.*

(b) $\mathcal{I}$ *is syntactically left antiprealgebraizable if and only if it is weakly LC prealgebraizable and* $R^{\mathcal{I}} \dot{L}^{\mathcal{I}}$*-fortified.*

Proof:

(a) We have that $\mathcal{I}$ is syntactically left prealgebraizable if and only if, by definition, it is preequivalential and left c-reflective and, moreover, it has a Leibniz binary reflexive core and a left adequate left Suszko core. This happens if and only if, by definition, it is LC prealgebraizable and $\ddot{R}^{\mathcal{I}} L^{\mathcal{I}}$-fortified.

(b) Similarly, $\mathcal{I}$ is syntactically left antiprealgebraizable if and only if, by definition, it is prealgebraic and left c-reflective and, moreover, it has a Leibniz reflexive core and a left adequate unary left Suszko core. This happens if and only if, by definition, it is weakly LC prealgebraizable and $R^{\mathcal{I}} \dot{L}^{\mathcal{I}}$-fortified.

■

This characterization in terms of semantic properties and preceding work on transference of properties from theory families/systems to filter families/systems on arbitrary algebraic systems yield a kind of transfer property for syntactic left (anti)prealgebraizability.

Theorem 1040 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) *$\mathcal{I}$ is syntactically left prealgebraizable if and only if it is $\ddot{R}^{\mathcal{I}}L^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ is monotone on $\mathcal{I}$-filter systems, system extensional and left c-reflective.*

(b) *$\mathcal{I}$ is syntactically left antiprealgebraizable if and only if it is $R^{\mathcal{I}}\dot{L}^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ is monotone on $\mathcal{I}$-filter systems and left c-reflective.*

Proof: We prove only Part (a), since Part (b) is similar. We have that $\mathcal{I}$ is syntactically left prealgebraizable if and only if, by Theorem 1039, it is $\ddot{R}^{\mathcal{I}}L^{\mathcal{I}}$-fortified and LC prealgebraizable if and only if, by Theorem 349, it is $\ddot{R}^{\mathcal{I}}L^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ is monotone on $\mathcal{I}$-filter systems, system extensional and left c-reflective. ■

Turning now to characterizations involving property preserving mappings between posets of filter families and of congruence systems, we have the following result:

Theorem 1041 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) *$\mathcal{I}$ is syntactically left prealgebraizable if and only if it is $\ddot{R}^{\mathcal{I}}L^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is a left completely order reflecting surjection that restricts to an order embedding $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ that commutes with inverse logical extensions.

(b) *$\mathcal{I}$ is syntactically left antiprealgebraizable if and only if it is $R^{\mathcal{I}}\dot{L}^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is a left completely order reflecting surjection that restricts to an order embedding $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$.

Proof: Again we show only Part (a). Part (b) is similar. We have that $\mathcal{I}$ is syntactically left prealgebraizable if and only if, by Theorem 1039, it is $\ddot{R}^{\mathcal{I}}L^{\mathcal{I}}$-fortified and LC prealgebraizable if and only if, by Theorem 355 it is $\ddot{R}^{\mathcal{I}}L^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is a left completely order reflecting surjection that restricts to an order embedding $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ that commutes with inverse logical extensions. ■

Finally, in terms of conjugate pairs of transformations, we get the following analog of Theorem 1033.

Theorem 1042 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is syntactically left prealgebraizable if and only if it is left truth equational and its systemic skeleton* $\mathcal{K}^\mathcal{I}$ *is equivalent to* $\mathcal{Q}^{\mathcal{I}\bullet}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^\mathcal{I} \rightleftarrows \mathcal{Q}^{\mathcal{I}\bullet}$ *of transformations, with* $I^\flat$ *natural.*

(b) $\mathcal{I}$ *is syntactically left antiprealgebraizable if and only if it is strongly left truth equational and its systemic skeleton* $\mathcal{K}^\mathcal{I}$ *is equivalent to* $\mathcal{Q}^{\mathcal{I}\bullet}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^\mathcal{I} \rightleftarrows \mathcal{Q}^{\mathcal{I}\bullet}$ *of transformations, with* $\tau^\flat$ *natural.*

Proof:

(a) Suppose, first, that $\mathcal{I}$ is left truth equational and $\mathcal{K}^\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $I^\flat$ natural. Then it is left truth equational and, by Theorem 941, it is syntactically preequivalential. Thus, by Theorem 1038, it is syntactically left prealgebraizable.

Suppose, conversely, that $\mathcal{I}$ is syntactically left prealgebraizable. Then, by Theorem 1038, it is left truth equational and syntactically preequivalential. Hence, by Theorem 934, it is syntactically WS prealgebraizable. Now it follows by Theorem 940 that $\mathcal{K}^\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a pair $(\tau^\flat, \overleftrightarrow{I^\flat})$, where $I^\flat$ witnesses the syntactic preequivalentiality and $\tau^\flat$ the left truth equationality of $\mathcal{I}$, and, hence, by definition, they constitute a conjugate pair of transformations, with $I^\flat$ natural.

(b) Suppose, first, that $\mathcal{I}$ is strongly left truth equational and $\mathcal{K}^\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $\tau^\flat$ natural. Then it is strongly left truth equational and, by Theorem 941, it is syntactically prealgebraic. Thus, by Theorem 1038, it is syntactically left antiprealgebraizable.

Suppose, conversely, that $\mathcal{I}$ is syntactically left antiprealgebraizable. Then, by Theorem 1038, it is strongly left truth equational and syntactically prealgebraic. Hence, by Theorem 934, it is syntactically WS prealgebraizable. Now it follows by Theorem 940 that $\mathcal{K}^\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a pair $(\tau^\flat, \overleftrightarrow{I^\flat})$, where $I^\flat$ witnesses the syntactic prealgebraicity and $\tau^\flat$ the strong left truth equationality of $\mathcal{I}$, and, hence, by definition, they constitute a conjugate pair of transformations, with $\tau^\flat$ natural.

■

The equivalence of the systemic skeleton with some algebraic π-structure via a conjugate pair of transformations, exhibiting the required one-sided naturality condition, coupled with either left truth equationality or strong left truth equationality, depending on the case considered, is sufficient to ensure syntactic left (anti)prealgebraizability.

Theorem 1043 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is syntactically left prealgebraizable if and only if it is left truth equational and its systemic skeleton is equivalent to an algebraic* π*-structure via a conjugate pair* $(\tau^\flat, I^\flat)$ *of transformations, with* $I^\flat$ *natural.*

(b) $\mathcal{I}$ *is syntactically left antiprealgebraizable if and only if it is strongly left truth equational and its systemic skeleton is equivalent to an algebraic* π*-structure via a conjugate pair* $(\tau^\flat, I^\flat)$ *of transformations, with* $\tau^\flat$ *natural.*

Proof:

(a) If $\mathcal{I}$ is syntactically left prealgebraizable, then, by Theorem 1042, it is left truth equational and its systemic skeleton is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations with $I^\flat$ natural. Suppose, conversely, that $\mathcal{I}$ is left truth equational and its systemic skeleton is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $I^\flat$ natural. Then, it is left truth equational and, by Proposition 928, it is syntactically preequivalential. Therefore, by Theorem 1038, it is syntactically left prealgebraizable.

(b) If $\mathcal{I}$ is syntactically left antiprealgebraizable, then, by Theorem 1042, it is strongly left truth equational and its systemic skeleton is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $\tau^\flat$ natural. Suppose, conversely, that $\mathcal{I}$ is strongly left truth equational and its systemic skeleton is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $\tau^\flat$ natural. Then, it is strongly left truth equational and, by Proposition 928, it is syntactically prealgebraic. Therefore, by Theorem 1038, it is syntactically left antiprealgebraizable.

■

Finally, in terms of order isomorphisms between theory family lattices, we have the following alternative characterization of syntactically left (anti)prealgebraizable π-institutions:

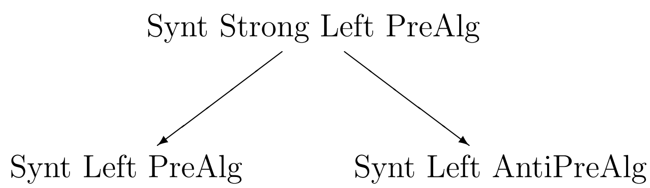
Theorem 1044 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is syntactically left prealgebraizable if and only if it is left truth equational and there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{K}^{\mathcal{I}}) \to \mathbf{ThFam}(\mathcal{Q})$, *induced by a conjugate pair* $(\tau^\flat, I^\flat)$ *of transformations, where* $\mathcal{Q}$ *is an algebraic* π*-structure and* $I^\flat$ *is natural.*

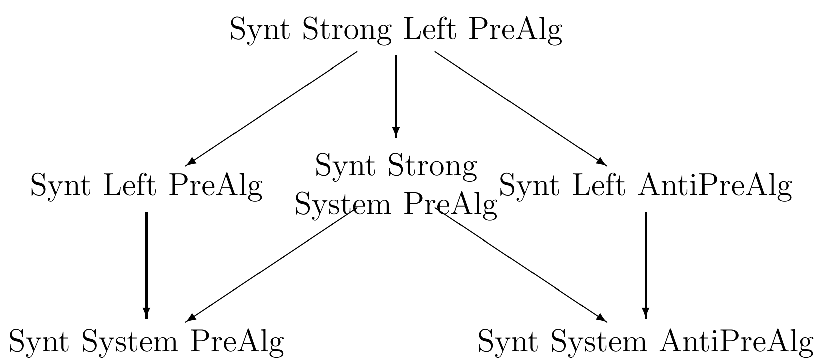
(b) *$\mathcal{I}$ is syntactically left antiprealgebraizable if and only if it is strongly left truth equational and there exists a transformational order isomorphism $h : \mathbf{ThFam}(\mathcal{K}^{\mathcal{I}}) \to \mathbf{ThFam}(\mathcal{Q})$, induced by a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, where $\mathcal{Q}$ is an algebraic π-structure and $\tau^\flat$ is natural.*

Proof: The "only if" follows by Theorem 1043 and Theorem 893. The "if" is given by Theorem 901 and Theorem 1043. ■

In this section we have introduced the three syntactic left prealgebraizability classes



In the next section, we shall introduce, following a similar path, the remaining three syntactic prealgebraizability classes, namely those of the system prealgebraizable π-institutions, in order to complete the syntactic prealgebraizability hierarchy that was described at the beginning of the section:



13.8 Syntactic System PreAlgebraizability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that $\mathcal{I}$ is $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$**-fortified** if it has

- a Leibniz binary reflexive core; and
- an adequate unary system core.

We say that $\mathcal{I}$ is **syntactically strongly system prealgebraizable** if it is

- $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified;
- preequivalential (i.e., prealgebraic and system extensional);

- system c-reflective.

An analog of Theorem 1029 provides an important characterization of syntactic strong system prealgebraizability in terms of lower classes in the syntactic hierarchy.

Theorem 1045 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly system prealgebraizable if and only if it is syntactically preequivalential and strongly system truth equational.*

Proof: We have that $\mathcal{I}$ is syntactically strongly system prealgebraizable if and only if, by definition, it is

- preequivalential and has a Leibniz binary reflexive core;
- system c-reflective and has an adequate unary system core;

if and only if, by Theorems 969 and 1027, is it syntactically preequivalential and strongly system truth equational. ■

An analog of Theorem 1030 gives an alternative characterization of the syntactic notion in terms of the corresponding semantic notions.

Theorem 1046 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly system prealgebraizable if and only if it is system prealgebraizable and* $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$*-fortified.*

Proof: We have that $\mathcal{I}$ is syntactically strongly system prealgebraizable if and only if, by definition,

- it is preequivalential and system c-reflective;
- it has a Leibniz binary reflexive core and an adequate unary system core;

if and only if, by definition, it is system prealgebraizable and, also, $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified. ■

This characterization in terms of semantic properties and preceding work on transference of properties from theory systems to filter systems on arbitrary algebraic systems yield yet another characterization of syntactic strong system prealgebraizability analogous to that of Theorem 1031, which may also be viewed as a kind of transfer property for this class.

Theorem 1047 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly system prealgebraizable if and only if it is* $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\Omega^{\mathcal{A}}$ *is monotone on theory systems, system extensional and system c-reflective.*

Proof: We have that $\mathcal{I}$ is syntactically strongly system prealgebraizable if and only if, by Theorem 1046, it is $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified and system prealgebraizable if and only if, by Theorem 349, it is $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, $\Omega^{\mathcal{A}}$ is monotone on $\mathcal{I}$-filter systems, system extensional and system c-reflective. ∎

Turning now to characterizations involving property preserving mappings between posets of filter families and of congruence systems, we have the following result:

Theorem 1048 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically strongly system prealgebraizable if and only if it is* $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is an order embedding which commutes with inverse logical extensions.

Proof: We have that $\mathcal{I}$ is syntactically strongly system prealgebraizable if and only if, by Theorem 1046, it is $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified and system prealgebraizable if and only if, by Theorem 353, it is $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order embedding which commutes with inverse logical extensions. ∎

Finally, in an analog of Theorem 1033, using conjugate pairs of transformations, we get

Theorem 1049 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically strongly system prealgebraizable if and only if its systemic skeleton* $\mathcal{K}^{\mathcal{I}}$ *is equivalent to* $\mathcal{Q}^{\mathcal{I}\bullet}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}\bullet}$ *of natural transformations.*

Proof: Suppose, first, that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair of natural transformations. Then, by Theorem 942, it is strongly system truth equational and, by Theorem 941, it is syntactically preequivalential. Thus, by Theorem 1045, it is syntactically strongly system prealgebraizable.

Suppose, conversely, that $\mathcal{I}$ is syntactically strongly system prealgebraizable. Then, by Theorem 1045, it is strongly system truth equational and syntactically preequivalential. Hence, by Theorem 934, it is syntactically WS prealgebraizable. Now it follows by Theorem 940 that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a pair $(\tau^\flat, \overleftrightarrow{I}^\flat)$, where $I^\flat$ witnesses the syntactic preequivalentiality and $\tau^\flat$ the syntactic strong system truth equationality of $\mathcal{I}$, and, hence, by definition, they constitute a conjugate pair of natural transformations. ∎

Analogously with Theorem 1034, the equivalence of the systemic skeleton with some algebraic π-structure via a conjugate pair of natural transformations suffices to ensure syntactic strong system prealgebraizability.

Theorem 1050 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly system prealgebraizable if and only if its systemic skeleton is equivalent to an algebraic* π*-structure via a conjugate pair of natural transformations.*

Proof: If $\mathcal{I}$ is syntactically strongly system prealgebraizable, then, by Theorem 1049, its systemic skeleton is equivalent to an algebraic π-structure via a conjugate pair of natural transformations. Suppose, conversely, that the systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to an algebraic π-structure via a conjugate pair of natural transformations. Then, by Proposition 928, it is syntactically preequivalential and, by Theorem 942, it is strongly system truth equational. Therefore, by Theorem 1045, it is syntactically strongly system prealgebraizable. ■

Finally, in terms of order isomorphisms between theory family lattices, we have the following analog of Theorem 1035, providing an alternative characterization of syntactically strongly system prealgebraizable π-institutions.

Theorem 1051 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly system prealgebraizable if and only if there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{K}^{\mathcal{I}}) \to \mathbf{ThFam}(\mathcal{Q})$*, induced by a conjugate pair* $(\tau^\flat, I^\flat)$ *of natural transformations, where* $\mathcal{Q}$ *is an algebraic* π*-structure.*

Proof: The "only if" follows by Theorem 1050 and Theorem 893. The "if" is given by Theorem 901 and Theorem 1050. ■

In the case of syntactic strong left prealgebraizability, studied in the preceding section, below that class sat two wider classes obtained by weakening the naturality requirement either on the side of the witnesses of prealgebraicity or on the side of the witnesses of truth equationality. Similarly here, we get below the class of syntactically strongly system prealgebraizable π-institutions the classes of *syntactically system prealgbebraizable* and *syntactically system antiprealgebraizable* π-institutions. These two classes are defined formally now.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

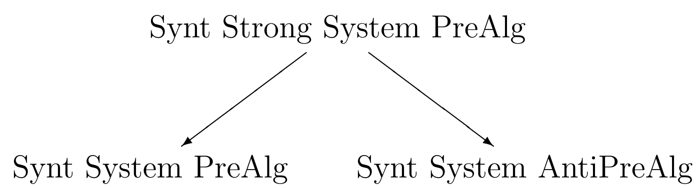
- $\mathcal{I}$ is **syntactically system prealgebraizable** if it is:
  - $\ddot{R}^{\mathcal{I}} Z^{\mathcal{I}}$-fortified;
  - preequivalential;
  - system c-reflective;
- $\mathcal{I}$ is **syntactically system antiprealgebraizable** if it is:

  - $R^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified;
  - prealgebraic;
  - system c-reflective.

For both of these classes we have analogs of many of the results proven above for syntactic strong system prealgebraizability. Again, since:

- preequivalentiality is stronger than prealgebraicity;
- under prealgebraicity, $\ddot{R}^{\mathcal{I}}$ Leibniz implies $R^{\mathcal{I}}$ Leibniz; and
- $\dot{Z}^{\mathcal{I}}$ adequate implies $Z^{\mathcal{I}}$ adequate,

we get, immediately from the definitions the following hierarchical relations between the upper three classes in the echelon formation of the preceding diagram.



We now provide examples to show that the two inclusions are proper. The first is an example of a π-institution which is syntactically system prealgebraizable but not syntactically strongly system prealgebraizable.

Example 1052 EXAMPLE NOT FOUND YET!!

Next, we give an example of a syntactically system antiprealgebraizable π-institution which fails to be syntactically strongly system prealgebraizable.

Example 1053 EXAMPLE NOT FOUND YET!!

The following analog of Theorem 1038 relates these two chiral types of syntactic system prealgebraizability with various classes introduced previously, providing some important characterizations.

Theorem 1054 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is syntactically system prealgebraizable if and only if it is syntactically preequivalential and system truth equational.*

(b) $\mathcal{I}$ *is syntactically system antiprealgebraizable if and only if it is syntactically prealgebraic and strongly system truth equational.*

Proof:

(a) We have that $\mathcal{I}$ is syntactically system prealgebraizable if and only if, by definition, it is preequivalential, with a Leibniz binary reflexive core, and system c-reflective, with an adequate system core, if and only if, by Theorems 969 and 887, is it syntactically preequivalential and system truth equational.

(b) We have that $\mathcal{I}$ is syntactically system antiprealgebraizable if and only if, by definition, it is prealgebraic, with a Leibniz reflexive core, and system c-reflective, with an adequate unary system core, if and only if, by Theorems 788 and 1027, is it syntactically prealgebraic and strongly system truth equational.

■

An alternative characterization, analogous to that of Theorem 1039, relates the syntactic with the corresponding semantic notions.

Theorem 1055 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is syntactically system prealgebraizable if and only if it is system prealgebraizable and* $\ddot{R}^{\mathcal{I}} Z^{\mathcal{I}}$*-fortified.*

(b) $\mathcal{I}$ *is syntactically system antiprealgebraizable if and only if it is weakly system prealgebraizable and* $R^{\mathcal{I}} \dot{Z}^{\mathcal{I}}$*-fortified.*

Proof:

(a) We have that $\mathcal{I}$ is syntactically system prealgebraizable if and only if, by definition, it is preequivalential and system c-reflective and, moreover, it has a Leibniz binary reflexive core and an adequate system core. This happens if and only if, by definition, it is system prealgebraizable and $\ddot{R}^{\mathcal{I}} Z^{\mathcal{I}}$-fortified.

(b) Similarly, $\mathcal{I}$ is syntactically system antiprealgebraizable if and only if, by definition, it is prealgebraic and system c-reflective and, moreover, it has a Leibniz reflexive core and an adequate unary system core. This happens if and only if, by definition, it is weakly system prealgebraizable and $R^{\mathcal{I}} \dot{Z}^{\mathcal{I}}$-fortified.

■

As far as transferring the properties defining syntactic system (anti)prealgebraizability from theory systems to filter systems on arbitrary algebraic systems, we get the following transfer theorem.

Theorem 1056 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is syntactically system prealgebraizable if and only if it is* $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ *is monotone on* $\mathcal{I}$*-filter systems, system extensional and system c-reflective.*

(b) $\mathcal{I}$ *is syntactically system antiprealgebraizable if and only if it is* $R^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ *is monotone on* $\mathcal{I}$*-filter systems and system c-reflective.*

Proof: We prove only Part (a). The proof of Part (b) follows along similar lines. We have that $\mathcal{I}$ is syntactically system prealgebraizable if and only if, by Theorem 1055, it is $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$-fortified and system prealgebraizable if and only if, by Theorem 349, it is $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ is monotone on $\mathcal{I}$-filter systems, system extensional and system c-reflective. ■

As far as characterizations involving property preserving mappings between posets of filter families and of congruence systems, we have the following analog of Theorem 1041.

Theorem 1057 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is syntactically system prealgebraizable if and only if it is* $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is an order embedding which commutes with inverse logical extensions.

(b) $\mathcal{I}$ *is syntactically system antiprealgebraizable if and only if it is* $R^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is an order embedding.

Proof: Again we show only Part (a), since Part (b) follows similar reasoning. We have that $\mathcal{I}$ is syntactically system prealgebraizable if and only if, by Theorem 1055, it is $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$-fortified and system prealgebraizable if and only if, by Theorem 353 it is $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order embedding which commutes with inverse logical extensions. ■

Finally, in terms of conjugate pairs of transformations, we get the following analog of Theorem 1042.

Theorem 1058 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) *$\mathcal{I}$ is syntactically system prealgebraizable if and only if its systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}\bullet}$ of transformations, with $I^\flat$ natural.*

(b) *$\mathcal{I}$ is syntactically system antiprealgebraizable if and only if its systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}\bullet}$ of transformations, with $\tau^\flat$ natural.*

Proof:

(a) Suppose, first, that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $I^\flat$ natural. Then, by Theorem 941, it is syntactically preequivalential and, by Theorem 942, it is system truth equational. Thus, by Theorem 1054, it is syntactically system prealgebraizable.

Suppose, conversely, that $\mathcal{I}$ is syntactically system prealgebraizable. Then, by Theorem 1054, it is syntactically preequivalential and system truth equational. Hence, by Theorem 934, it is syntactically WS prealgebraizable. Now it follows by Theorem 940 that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a pair $(\tau^\flat, \overleftrightarrow{I}^\flat)$, where $I^\flat$ witnesses the syntactic preequivalentiality and $\tau^\flat$ the system truth equationality of $\mathcal{I}$, and, hence, by definition, they constitute a conjugate pair of transformations, with $I^\flat$ natural.

(b) Suppose, first, $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $\tau^\flat$ natural. Then, by Theorem 941, it is syntactically prealgebraic and, by Theorem 942, it is strongly system truth equational. Thus, by Theorem 1054, it is syntactically system antiprealgebraizable.

Suppose, conversely, that $\mathcal{I}$ is syntactically system antiprealgebraizable. Then, by Theorem 1054, it is syntactically prealgebraic and strongly system truth equational. Hence, by Theorem 934, it is syntactically WS prealgebraizable. Now it follows by Theorem 940 that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}\bullet}$ via a pair $(\tau^\flat, \overleftrightarrow{I}^\flat)$, where $I^\flat$ witnesses the syntactic prealgebraicity and $\tau^\flat$ the strong system truth equationality of $\mathcal{I}$, and, hence, by definition, they constitute a conjugate pair of transformations, with $\tau^\flat$ natural.

■

The equivalence of the systemic skeleton with some algebraic π-structure via a conjugate pair of transformations, exhibiting the required one-sided naturality condition, is sufficient to ensure syntactic system (anti)prealgebraizability. This constitutes an analog of Theorem 1043.

Theorem 1059 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) *$\mathcal{I}$ is syntactically system prealgebraizable if and only if its systemic skeleton is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $I^\flat$ natural.*

(b) *$\mathcal{I}$ is syntactically system antiprealgebraizable if and only if its systemic skeleton is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $\tau^\flat$ natural.*

Proof:

(a) If $\mathcal{I}$ is syntactically system prealgebraizable, then, by Theorem 1058, its systemic skeleton is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations with $I^\flat$ natural. Suppose, conversely, that the systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $I^\flat$ natural. Then, by Proposition 928, it is syntactically preequivalential and, by Theorem 942, it is system truth equational. Therefore, by Theorem 1054, it is syntactically system prealgebraizable.

(b) If $\mathcal{I}$ is syntactically system antiprealgebraizable, then, by Theorem 1058, its systemic skeleton is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $\tau^\flat$ natural. Suppose, conversely, that $\mathcal{K}^{\mathcal{I}}$ is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $\tau^\flat$ natural. Then, by Proposition 928, it is syntactically prealgebraic and, by Theorem 942, it is strongly system truth equational. Therefore, by Theorem 1054, it is syntactically system antiprealgebraizable.

■

Finally, in terms of order isomorphisms between theory family lattices, we have the following analog of Theorem 1044, giving an alternative characterization of syntactically system (anti)prealgebraizable π-institutions:

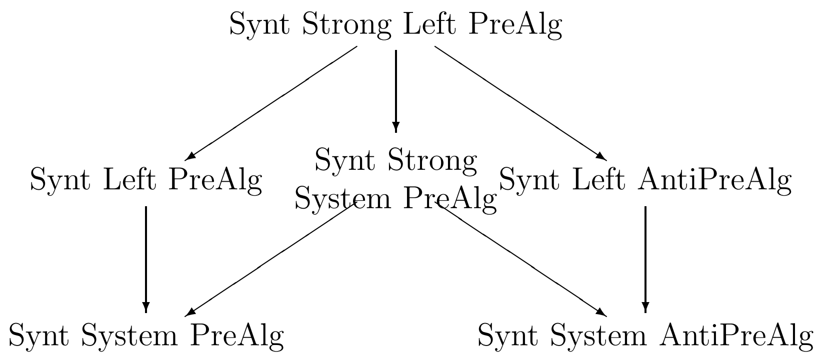
Theorem 1060 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) *$\mathcal{I}$ is syntactically system prealgebraizable if and only if there exists a transformational order isomorphism $h : \mathbf{ThFam}(\mathcal{K}^{\mathcal{I}}) \to \mathbf{ThFam}(\mathcal{Q})$, induced by a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, where $\mathcal{Q}$ is an algebraic π-structure and $I^\flat$ is natural.*

(b) *$\mathcal{I}$ is syntactically system antiprealgebraizable if and only if there exists a transformational order isomorphism $h : \mathbf{ThFam}(\mathcal{K}^{\mathcal{I}}) \to \mathbf{ThFam}(\mathcal{Q})$, induced by a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, where $\mathcal{Q}$ is an algebraic π-structure and $\tau^\flat$ is natural.*

Proof: The "only if" follows by Theorem 1059 and Theorem 893. The "if" is given by Theorem 901 and Theorem 1059. ■

Finally, since we have now described in detail the six classes of the syntactic prealgebraizability hierarchy, it is only appropriate to pause and look for examples that separate the left prealgebraizability from the system prealgebraizability classes, i.e., examples showing that the vertical arrows in the following 6-class diagram

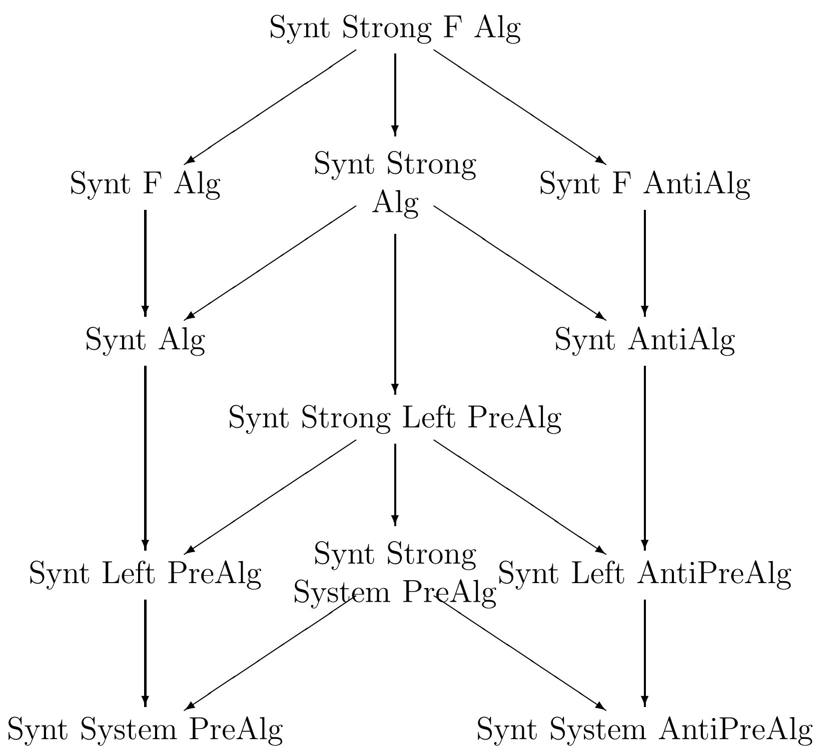


represent, in fact, proper inclusions. We can do this in one swoop by exhibiting an example of a syntactically strong system prealgebraizable π-institution which is neither syntactically left prealgebraizable nor syntactically left antiprealgebraizable.

Example 1061 EXAMPLE NOT FOUND YET!

13.9 Syntactic Family Algebraizability

We now preview the full hierarchy of syntactically prealgebraizable π-institutions that will be established in this section. The bottom six classes are the ones established in the preceding two sections, where prealgebraizability refers to the fact that monotonicity is only applied to theory systems. The top six classes concern syntactic algebraizability, where monotonicity is applied to all theory families. The bottom row of this upper tier consists of those π-institutions, where c-reflectivity is postulated only for theory systems. The very top row above it refers to applying left c-reflectivity to theory families. In the second from top class, system (or, equivalently, left c-reflectivity) is postulated in conjunction with family monotonicity and at the very top row family c-reflectivity is combined with family monotonicity. Finally, as far as columns go, they incorporate meanings similar to the ones described for the cohorts of classes introduced in the preceding sections. The left column applies parameter freeness only to the equivalence natural transformations witnessing syntactic protoalgebraicity. The right column insists on parameter freeness for the defining equations that witness the truth equationality of the π-institution, whereas the middle column is combing those properties and consists of those classes of π-institutions that are syntactically protoalgebraic and syntactic truth equational, with both properties having parameter free witnessing transformations and witnessing equations, respectively. The complete picture that emerges at the end of this and the next section adds to the six-class diagram concluding the previous section six more classes, those positioned at the top two rows.



We start by defining the class at the apex of the diagram. Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that $\mathcal{I}$ is $\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$**-fortified** if it has

- a Leibniz binary reflexive core; and
- an adequate unary Suszko core.

We say that $\mathcal{I}$ is **syntactically strongly family algebraizable** if it is

- $\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$-fortified;
- equivalential (i.e., protoalgebraic and family extensional);
- family c-reflective.

Based on previous work, we can formulate the following important characterization of syntactic strong family algebraizability.

Theorem 1062 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly family algebraizable if and only if it is syntactically equivalential and strongly truth equational.*

Proof: We have that $\mathcal{I}$ is syntactically strongly family algebraizable if and only if, by definition, it is

- equivalential and has a Leibniz binary reflexive core;
- family c-reflective and has an adequate unary Suszko core;

if and only if, by Theorems 983 and 996, is it syntactically preequivalential and strongly left truth equational. ∎

An alternative characterization along similar lines relates the syntactic with the corresponding semantic notions introduced in the context of algebraizability.

Theorem 1063 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly family algebraizable if and only if it is family algebraizable and* $\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$*-fortified.*

Proof: We have that $\mathcal{I}$ is syntactically strongly family algebraizable if and only if, by definition,

- it is equivalential and family c-reflective;
- it has a Leibniz binary reflexive core and an adequate unary Suszko core;

if and only if, by definition, it is family algebraizable (recall that family injectivity and family c-reflectivity coincide under protoalgebraicity) and $\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$-fortified. ∎

This characterization in terms of semantic properties and preceding work on transference of properties from theory families to filter families on arbitrary algebraic systems yields another characterization of syntactic strong family algebraizability, which may also be viewed as a kind of transfer property in its own right.

Theorem 1064 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly family algebraizable if and only if it is* $\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\Omega^{\mathcal{A}}$ *is monotone on* $\mathcal{I}$*-filter families, family extensional and family c-reflective.*

Proof: We have that $\mathcal{I}$ is syntactically strongly family algebraizable if and only if, by Theorem 1063, it is $\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$-fortified and family algebraizable if and only if, by Theorem 364, it is $\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ is monotone and injective (equivalently c-reflective) on $\mathcal{I}$-filter families and family extensional. ∎

Turning now to characterizations involving property preserving mappings between posets of filter families and of congruence systems, we have the following result:

Theorem 1065 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly family algebraizable if and only if it is* $\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is an order isomorphism that commutes with inverse logical extensions.

Proof: We have that $\mathcal{I}$ is syntactically strongly family algebraizable if and only if, by Theorem 1063, it is $\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$-fortified and family algebraizable if and only if, by Theorem 366 it is $\ddot{R}^{\mathcal{I}}\dot{S}^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order isomorphism that commutes with inverse logical extensions. ∎

Finally, in terms of conjugate pairs of transformations, we get the following analog of Theorem 949.

Theorem 1066 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly family algebraizable if and only if it is equivalent to its associated algebraic* π*-structure* $\mathcal{Q}^{\mathcal{I}*}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$ *of natural transformations.*

Proof: Suppose, first, that $\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair of natural transformations. Then it is syntactically equivalential by Corollary 910 and it is family truth equational by Theorem 911. Thus, by Theorem 1062, it is syntactically strongly family algebraizable.

Suppose, conversely, that $\mathcal{I}$ is syntactically strongly family algebraizable. Then, by Theorem 1062, it is strongly family truth equational and syntactically equivalential. Hence, by Theorem 913, it is syntactically WF algebraizable. Now it follows by Theorem 919 that $\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a pair $(\tau^\flat, \overleftrightarrow{I}^\flat)$, where $I^\flat$ witnesses the syntactic equivalentiality and $\tau^\flat$ the syntactic strong truth equationality of $\mathcal{I}$, and, hence, by definition, they constitute a conjugate pair of natural transformations. ∎

It turns out, in this case also, that the equivalence of the π-institution with some algebraic π-structure via a conjugate pair of natural transformations is sufficient to ensure syntactic strong family algebraizability.

Theorem 1067 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly family algebraizable if and only if it is equivalent to an algebraic* π*-structure via a conjugate pair of natural transformations.*

Proof: If $\mathcal{I}$ is syntactically strongly family algebraizable, then, by Theorem 1066, it is equivalent to an algebraic π-structure via a conjugate pair of natural transformations. Suppose, conversely, that $\mathcal{I}$ is equivalent to an algebraic π-structure via a conjugate pair of natural transformations. Then, it is syntactically equivalential by Corollary 910 and it is strongly family truth equational by Theorem 911. Therefore, by Theorem 1062, it is syntactically strongly family algebraizable. ∎

Finally, in terms of order isomorphisms between theory family lattices, we have the following alternative characterization of syntactically strongly family algebraizable π-institutions:

Theorem 1068 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly family algebraizable if and only if there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{I}) \to \mathbf{ThFam}(\mathcal{Q})$*, induced by a conjugate pair* $(\tau^\flat, I^\flat)$ *of natural transformations, where* $\mathcal{Q}$ *is an algebraic* π*-structure.*

Proof: The "only if" follows by Theorem 1067 and Theorem 893. The "if" is given by Theorem 901 and Theorem 1067. ∎

Lying just underneath the class of syntactically strongly family algebraizable π-institutions are the classes of *syntactically family algebraizable* and *syntactically family antialgebraizable* π-institutions. These two classes are defined formally now.

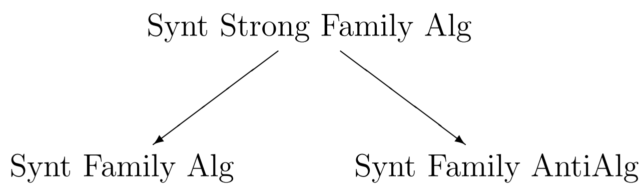
Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

- $\mathcal{I}$ is **syntactically family algebraizable** if it is:
  - $\ddot{R}^{\mathcal{I}} S^{\mathcal{I}}$-fortified;
  - equivalential;
  - family c-reflective;
- $\mathcal{I}$ is **syntactically family antialgebraizable** if it is:
  - $R^{\mathcal{I}} \dot{S}^{\mathcal{I}}$-fortified;
  - protoalgebraic;
  - family c-reflective.

We formulate analogs of many of the results proven previously for the various kinds of syntactic prealgebraizability properties. Observe, first, that, since:

- equivalentiality is stronger than protoalgebraicity;
- under prealgebraicity, $\ddot{R}^{\mathcal{I}}$ Leibniz implies $R^{\mathcal{I}}$ Leibniz; and
- $\dot{S}^{\mathcal{I}}$ adequate implies $S^{\mathcal{I}}$ adequate,

we get, immediately from the definitions the following hierarchical relations between the three topmost classes in the syntactic algebraizability hierarchy.



We now provide examples to show that the two inclusions are proper. The first is an example of a π-institution which is syntactically family algebraizable but not syntactically strongly family algebraizable.

Example 1069 EXAMPLE NOT FOUND YET!!

Next, we give an example of a syntactically family antialgebraizable π-institution which fails to be syntactically strongly family algebraizable.

Example 1070 EXAMPLE NOT FOUND YET!!

The following analog of Theorem 1062 relates these two chiral sorts of syntactic family (anti)algebraizability with various classes introduced previously, providing some important characterizations.

Theorem 1071 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is syntactically family algebraizable if and only if it is syntactically equivalential and family truth equational.*

(b) $\mathcal{I}$ *is syntactically family antialgebraizable if and only if it is syntactically protoalgebraic and strongly family truth equational.*

Proof:

(a) We have that $\mathcal{I}$ is syntactically family algebraizable if and only if, by definition, it is equivalential, with a Leibniz binary reflexive core, and family c-reflective, with an adequate Suszko core, if and only if, by Theorems 983 and 847, is it syntactically equivalential and family truth equational.

(b) We have that $\mathcal{I}$ is syntactically family antialgebraizable if and only if, by definition, it is protoalgebraic, with a Leibniz reflexive core, and family c-reflective, with an adequate unary Suszko core, if and only if, by Theorems 805 and 996, is it syntactically protoalgebraic and strongly family truth equational.

■

An alternative characterization, analogous to that of Theorem 1063, relates the syntactic with the corresponding semantic notions.

Theorem 1072 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is syntactically family algebraizable if and only if it is family algebraizable and* $\ddot{R}^{\mathcal{I}} S^{\mathcal{I}}$*-fortified.*

(b) $\mathcal{I}$ *is syntactically family antialgebraizable if and only if it is family algebraizable and* $R^{\mathcal{I}} \dot{S}^{\mathcal{I}}$*-fortified.*

Proof:

(a) We have that $\mathcal{I}$ is syntactically family algebraizable if and only if, by definition, it is equivalential and family c-reflective and, moreover, it has a Leibniz binary reflexive core and an adequate Suszko core. This happens if and only if, by definition, it is family algebraizable and $\ddot{R}^{\mathcal{I}} S^{\mathcal{I}}$-fortified.

(b) Similarly, $\mathcal{I}$ is syntactically family antialgebraizable if and only if, by definition, it is equivalential and family c-reflective and, moreover, it has a Leibniz reflexive core and an adequate unary Suszko core. This happens if and only if, by definition, it is family algebraizable and $R^{\mathcal{I}} \dot{S}^{\mathcal{I}}$-fortified.

■

The characterization in terms of semantic properties and preceding work on transference of properties from theory families to filter families on arbitrary algebraic systems yield a transfer property for syntactic family (anti)-prealgebraizability.

Theorem 1073 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is syntactically family algebraizable if and only if it is* $\ddot{R}^{\mathcal{I}} S^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\Omega^{\mathcal{A}}$ *is monotone on* $\mathcal{I}$*-filter families, family extensional and family c-reflective.*

(b) $\mathcal{I}$ *is syntactically family antialgebraizable if and only if it is* $R^{\mathcal{I}} \dot{S}^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\Omega^{\mathcal{A}}$ *is monotone on* $\mathcal{I}$*-filter families and family c-reflective.*

Proof: We prove only Part (a), since Part (b) is similar. We have that $\mathcal{I}$ is syntactically family algebraizable if and only if, by Theorem 1072, it is $\ddot{R}^{\mathcal{I}}S^{\mathcal{I}}$-fortified and family algebraizable if and only if, by Theorem 349, it is $\ddot{R}^{\mathcal{I}}S^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle\rangle$, $\Omega^{\mathcal{A}}$ is monotone on $\mathcal{I}$-filter families, family extensional and family c-reflective. ■

Turning now to characterizations involving property preserving mappings between posets of filter families and of congruence systems, we have the following result:

Theorem 1074 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is syntactically family algebraizable if and only if it is* $\ddot{R}^{\mathcal{I}}S^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle\rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is an order isomorphism that commutes with inverse logical extensions.

(b) $\mathcal{I}$ *is syntactically family antialgebraizable if and only if it is* $R^{\mathcal{I}}\dot{S}^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle\rangle$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is an order isomorphism.

Proof: Again we show only Part (a). Part (b) is similar. We have that $\mathcal{I}$ is syntactically family algebraizable if and only if, by Theorem 1072, it is $\ddot{R}^{\mathcal{I}}S^{\mathcal{I}}$-fortified and family algebraizable if and only if, by Theorem 366 it is $\ddot{R}^{\mathcal{I}}S^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle\rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order isomorphism that commutes with inverse logical extensions. ■

In terms of conjugate pairs of transformations, we get the following analog of Theorem 1066.

Theorem 1075 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is syntactically family algebraizable if and only if it is equivalent to* $\mathcal{Q}^{\mathcal{I}*}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$ *of transformations, with* $I^\flat$ *natural.*

(b) $\mathcal{I}$ *is syntactically family antialgebraizable if and only if it is equivalent to* $\mathcal{Q}^{\mathcal{I}*}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$ *of transformations, with* $\tau^\flat$ *natural.*

Proof:

(a) Suppose, first, that $\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $I^\flat$ natural. Then, by Corollary 910, it is syntactically equivalential and, by Theorem 911, it is family truth equational. Thus, by Theorem 1071, it is syntactically family algebraizable.

Suppose, conversely, that $\mathcal{I}$ is syntactically family algebraizable. Then, by Theorem 1071, it is left truth equational and syntactically equivalential. Hence, by Theorem 913, it is syntactically WF prealgebraizable. Now it follows by Theorem 919 that $\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a pair $(\tau^\flat, \overset{\leftrightarrow}{I^\flat})$, where $I^\flat$ witnesses the syntactic equivalentiality and $\tau^\flat$ the syntactic family truth equationality of $\mathcal{I}$, and, hence, by definition, they constitute a conjugate pair of transformations, with $I^\flat$ natural.

(b) Suppose, first, that $\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $\tau^\flat$ natural. Then, by Theorem 909, it is syntactically protoalgebraic and, by Theorem 911, it is strongly family truth equational. Thus, by Theorem 1071, it is syntactically family antialgebraizable.

Suppose, conversely, that $\mathcal{I}$ is syntactically family antialgebraizable. Then, by Theorem 1071, it is strongly left truth equational and syntactically protoalgebraic. Hence, by Theorem 913, it is syntactically WF prealgebraizable. Now it follows by Theorem 919 that $\mathcal{I}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a pair $(\tau^\flat, \overset{\leftrightarrow}{I^\flat})$, where $I^\flat$ witnesses the syntactic protoalgebraicity and $\tau^\flat$ the strong family truth equationality of $\mathcal{I}$, and, hence, by definition, they constitute a conjugate pair of transformations, with $\tau^\flat$ natural.

■

The equivalence of the π-institution with some algebraic π-structure via a conjugate pair of transformations exhibiting the required one-sided naturality condition suffices to ensure syntactic family (anti)algebraizability.

Theorem 1076 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is syntactically family algebraizable if and only if is equivalent to an algebraic* π*-structure via a conjugate pair* $(\tau^\flat, I^\flat)$ *of transformations, with* $I^\flat$ *natural.*

(b) $\mathcal{I}$ *is syntactically family antialgebraizable if and only if it is equivalent to an algebraic* π*-structure via a conjugate pair* $(\tau^\flat, I^\flat)$ *of transformations, with* $\tau^\flat$ *natural.*

Proof:

(a) If $\mathcal{I}$ is syntactically family algebraizable, then, by Theorem 1075, it is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations with $I^\flat$ natural. Suppose, conversely, that $\mathcal{I}$ is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $I^\flat$ natural. Then, by Theorem 909, it is syntactically equivalential and, by Theorem 911, it is family truth equational. Therefore, by Theorem 1071, it is syntactically family algebraizable.

(b) If $\mathcal{I}$ is syntactically family antialgebraizable, then, by Theorem 1075, it is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $\tau^\flat$ natural. Suppose, conversely, that $\mathcal{I}$ is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $\tau^\flat$ natural. Then, by Theorem 909, it is syntactically protoalgebraic and by Theorem 911, it is strongly family truth equational. Therefore, by Theorem 1071, it is syntactically family antialgebraizable.

■

Finally, in terms of order isomorphisms between theory family lattices, we have the following alternative characterization of syntactic family (anti)algebraizability:

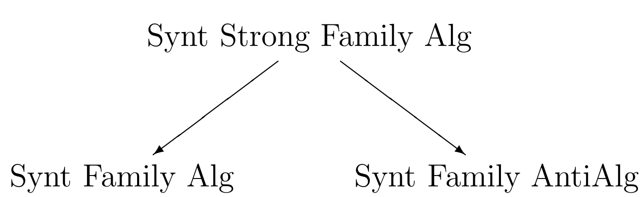
Theorem 1077 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is syntactically family algebraizable if and only if there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{I}) \to \mathbf{ThFam}(\mathcal{Q})$*, induced by a conjugate pair* $(\tau^\flat, I^\flat)$ *of transformations, where* $\mathcal{Q}$ *is an algebraic* π*-structure and* $I^\flat$ *is natural.*

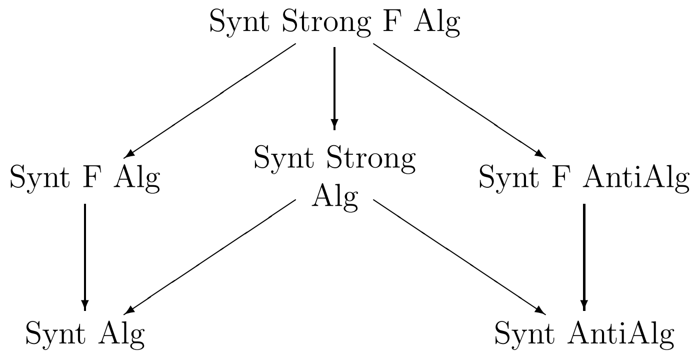
(b) $\mathcal{I}$ *is syntactically family antialgebraizable if and only if there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{I}) \to \mathbf{ThFam}(\mathcal{Q})$*, induced by a conjugate pair* $(\tau^\flat, I^\flat)$ *of transformations, where* $\mathcal{Q}$ *is an algebraic* π*-structure and* $\tau^\flat$ *is natural.*

Proof: The "only if" follows by Theorem 1076 and Theorem 893. The "if" is given by Theorem 901 and Theorem 1076. ■

In this section we have introduced the three syntactic family algebraizability classes



In the next section, we shall introduce the remaining three syntactic algebraizability classes, namely those of the syntactically algebraizable π-institutions, in order to complete the syntactic algebraizability hierarchy that was described at the beginning of the section:



13.10 Syntactic Algebraizability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall that $\mathcal{I}$ is $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$**-fortified** if it has a Leibniz binary reflexive core and an adequate unary system core. We say that $\mathcal{I}$ is **syntactically strongly algebraizable** if it is

- $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified;
- equivalential (i.e., protoalgebraic and family extensional);
- system c-reflective.

An analog of Theorem 1062 provides an important characterization of syntactic strong algebraizability in terms of lower classes in the syntactic hierarchy.

Theorem 1078 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is syntactically strongly algebraizable if and only if it is syntactically equivalential and strongly system truth equational.*

Proof: We have that $\mathcal{I}$ is syntactically strongly algebraizable if and only if, by definition, it is

- equivalential and has a Leibniz binary reflexive core;
- system c-reflective and has an adequate unary system core;

if and only if, by Theorems 983 and 1027, is it syntactically preequivalential and strongly system truth equational. ■

An analog of Theorem 1063 gives an alternative characterization of the syntactic notion in terms of the corresponding semantic notions.

Theorem 1079 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically strongly algebraizable if and only if it is (system) algebraizable and* $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$*-fortified.*

Proof: We have that $\mathcal{I}$ is syntactically strongly algebraizable if and only if, by definition,

- it is equivalential and system c-reflective;
- it has a Leibniz binary reflexive core and an adequate unary system core;

if and only if, by definition, it is algebraizable and $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified. ∎

This characterization in terms of semantic properties and preceding work on transference of properties from theory families/systems to filter families/systems on arbitrary algebraic systems yield another characterization of syntactic strong algebraizability analogous to that of Theorem 1064, which may also be viewed as a kind of transfer property for syntactically strongly algebraizable π-institutions.

Theorem 1080 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically strongly algebraizable if and only if it is* $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ *is monotone on* $\mathcal{I}$*-filter families, family extensional and system c-reflective.*

Proof: We have that $\mathcal{I}$ is syntactically strongly algebraizable if and only if, by Theorem 1079, it is $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified and algebraizable if and only if, by Theorem 349, it is $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ is monotone on $\mathcal{I}$-filter families, family extensional and system c-reflective. ∎

Turning now to characterizations involving property preserving mappings between posets of filter families and of congruence systems, we have the following analog of Theorem 1065.

Theorem 1081 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically strongly algebraizable if and only if it is* $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$*-fortified, stable and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is an order isomorphism that commutes with inverse logical extensions.

Proof: We have that $\mathcal{I}$ is syntactically strongly (system) algebraizable if and only if, by Theorem 1079, it is $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified and algebraizable if and only if, by Theorem 365, it is $\ddot{R}^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified, stable and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order isomorphism that commutes with inverse logical extensions. ■

Finally, in an analog of Theorem 1066, using conjugate pairs of transformations, we get

Theorem 1082 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly algebraizable if and only if it is stable and its systemic skeleton* $\mathcal{K}^{\mathcal{I}}$ *is equivalent to* $\mathcal{Q}^{\mathcal{I}*}$ *via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}*}$ *of natural transformations.*

Proof: Suppose, first, that $\mathcal{I}$ is stable and $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a conjugate pair of natural transformations. Then, by Theorem 929, it is syntactically equivalential and, by Theorem 941, it is strongly system truth equational. Thus, by Theorem 1078, it is syntactically strongly algebraizable.

Suppose, conversely, that $\mathcal{I}$ is syntactically strongly algebraizable. Then, by Theorem 1078, it is syntactically equivalential and strongly system truth equational. Hence, by Theorem 923, it is syntactically weakly algebraizable. Now it follows by Theorem 927 that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}*}$ via a pair $(\tau^\flat, \overleftrightarrow{I}^\flat)$, where $I^\flat$ witnesses the syntactic equivalentiality and $\tau^\flat$ the strong system truth equationality of $\mathcal{I}$, and, hence, by definition, they constitute a conjugate pair of natural transformations. ■

Analogously with Theorem 1067, the equivalence of the systemic skeleton with some algebraic π-structure via a conjugate pair of natural transformations, coupled with stability, suffices to ensure syntactic strong algebraizability.

Theorem 1083 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly algebraizable if and only if it is stable and its systemic skeleton is equivalent to an algebraic* π*-structure via a conjugate pair of natural transformations.*

Proof: If $\mathcal{I}$ is syntactically strongly algebraizable, then, by Theorem 1082, it is stable and its systemic skeleton is equivalent to an algebraic π-structure via a conjugate pair of natural transformations. Suppose, conversely, that $\mathcal{I}$ is stable and that its systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to an algebraic π-structure via a conjugate pair of natural transformations. Then, by Theorem 929, it is syntactically equivalential and, by Theorem 930, it is strongly system truth equational. Therefore, by Theorem 1078, it is syntactically strongly algebraizable. ■

Finally, in terms of order isomorphisms between theory family lattices, we have the following analog of Theorem 1065, providing an alternative characterization of syntactic strong algebraizability.

Theorem 1084 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically strongly algebraizable if and only if it is stable and there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{K}^\mathcal{I}) \to \mathbf{ThFam}(\mathcal{Q})$*, induced by a conjugate pair* $(\tau^\flat, I^\flat)$ *of natural transformations, where* $\mathcal{Q}$ *is an algebraic* π*-structure.*

Proof: The "only if" follows by Theorem 1083 and Theorem 893. The "if" is given by Theorem 901 and Theorem 1083. ∎

As with all other strong (pre)algebraizability classes, studied before, below the class of syntactically strongly algebraizable π-institutions sit two wider classes obtained by weakening the naturality requirement either on the side of the witnesses of prealgebraicity or on the side of the witnesses of truth equationality, namely the classes of *syntactically algebraizable* and *syntactically antialgebraizable* π-institutions.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

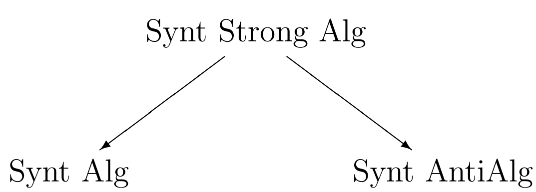
- $\mathcal{I}$ is **syntactically algebraizable** if it is:
  - $\ddot{R}^\mathcal{I} Z^\mathcal{I}$-fortified;
  - equivalential;
  - system c-reflective;
- $\mathcal{I}$ is **syntactically antialgebraizable** if it is:
  - $R^\mathcal{I} \dot{Z}^\mathcal{I}$-fortified;
  - protoalgebraic;
  - system c-reflective.

We now conclude the chapter by formulating analogs of many of the results proven above for syntactic strong algebraizability for these two new classes of π-institutions.

Firs, observe, once more, that, since:

- equivalentiality implies protoalgebraicity;
- under protoalgebraicity, $\ddot{R}^\mathcal{I}$ Leibniz implies $R^\mathcal{I}$ Leibniz; and
- $\dot{Z}^\mathcal{I}$ adequate implies $Z^\mathcal{I}$ adequate,

we get the following hierarchical relations between the three classes in the second-from-top tier of the syntactic algebraizability hierarchy.



Examples are in order to show that the two inclusions are proper. The first is an example of a π-institution which is syntactically algebraizable but not syntactically strongly algebraizable.

Example 1085 EXAMPLE NOT FOUND YET!!

Next, we give an example of a syntactically antialgebraizable π-institution which fails to be syntactically strongly algebraizable.

Example 1086 EXAMPLE NOT FOUND YET!!

The following analog of Theorem 1078 relates the two chiral types of syntactic algebraizability with various classes introduced previously, providing some important characterizations.

Theorem 1087 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is syntactically algebraizable if and only if it is syntactically equivalential and system truth equational.*

(b) $\mathcal{I}$ *is syntactically antialgebraizable if and only if it is syntactically protoalgebraic and strongly system truth equational.*

Proof:

(a) We have that $\mathcal{I}$ is syntactically algebraizable if and only if, by definition, it is equivalential, with a Leibniz binary reflexive core, and system c-reflective, with an adequate system core, if and only if, by Theorems 983 and 887, is it syntactically equivalential and system truth equational.

(b) We have that $\mathcal{I}$ is syntactically antialgebraizable if and only if, by definition, it is protoalgebraic, with a Leibniz reflexive core, and system c-reflective, with an adequate unary system core, if and only if, by Theorems 805 and 1027, is it syntactically protoalgebraic and strongly system truth equational.

■

An alternative characterization, analogous to that of Theorem 1079, relates the syntactic with the corresponding semantic notions.

Theorem 1088 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is syntactically algebraizable if and only if it is algebraizable and* $\ddot{R}^{\mathcal{I}} Z^{\mathcal{I}}$*-fortified.*

(b) $\mathcal{I}$ *is syntactically antialgebraizable if and only if it is weakly algebraizable and* $R^{\mathcal{I}} \dot{Z}^{\mathcal{I}}$*-fortified.*

Proof:

(a) We have that $\mathcal{I}$ is syntactically algebraizable if and only if, by definition, it is equivalential and system c-reflective and, moreover, it has a Leibniz binary reflexive core and an adequate system core. This happens if and only if, by definition, it is algebraizable and $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$-fortified.

(b) Similarly, $\mathcal{I}$ is syntactically antialgebraizable if and only if, by definition, it is protoalgebraic and system c-reflective and, moreover, it has a Leibniz reflexive core and an adequate unary system core. This happens if and only if, by definition, it is weakly algebraizable and $R^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified.

■

The properties defining syntactic (anti)algebraizability transfer from theory families/systems to filter families/systems on arbitrary algebraic systems. More precisely, we obtain the following analog of Theorem 1080.

Theorem 1089 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is syntactically algebraizable if and only if it is* $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ *is monotone on* $\mathcal{I}$*-filter families, family extensional and system c-reflective.*

(b) $\mathcal{I}$ *is syntactically antialgebraizable if and only if it is* $R^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$*-fortified and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ *is monotone on* $\mathcal{I}$*-filter families and system c-reflective.*

Proof: We prove only Part (a). Part (b) is similar. We have that $\mathcal{I}$ is syntactically algebraizable if and only if, by Theorem 1088, it is $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$-fortified and algebraizable if and only if, by Theorem 363, it is $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$-fortified and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ is monotone on $\mathcal{I}$-filter families, family extensional and system c-reflective. ■

Turning now to characterizations involving property preserving mappings between posets of filter families and of congruence systems, we have the following analog of Theorem 1081.

Theorem 1090 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is syntactically algebraizable if and only if it is* $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$*-fortified, stable and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is an order isomorphism that commutes with inverse logical extensions.

(b) *$\mathcal{I}$ is syntactically antialgebraizable if and only if it is $R^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified, stable and, for every* **F***-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,*

$$\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is an order isomorphism.

Proof:

(a) $\mathcal{I}$ is syntactically algebraizable if and only if, by Theorem 1088, it is $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$-fortified and algebraizable if and only if, by Theorem 365 it is $\ddot{R}^{\mathcal{I}}Z^{\mathcal{I}}$-fortified, stable and, for every **F**-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order isomorphism that commutes with inverse logical extensions.

(b) $\mathcal{I}$ is syntactically antialgebraizable if and only if, by Theorem 1088, it is $R^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified and weakly algebraizable if and only if, by Theorem 298 it is $R^{\mathcal{I}}\dot{Z}^{\mathcal{I}}$-fortified, stable and, for every **F**-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order isomorphism. ■

Finally, in terms of conjugate pairs of transformations, we get the following analog of Theorem 1082.

Theorem 1091 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on* **F***.*

(a) *$\mathcal{I}$ is syntactically algebraizable if and only if it is stable and its systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}}$ via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}}$ of transformations, with $I^\flat$ natural.*

(b) *$\mathcal{I}$ is syntactically antialgebraizable if and only if it is stable and its systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}}$ via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{K}^{\mathcal{I}} \rightleftarrows \mathcal{Q}^{\mathcal{I}}$ of transformations, with $\tau^\flat$ natural.*

Proof:

(a) Suppose, first, that $\mathcal{I}$ is stable and that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}}$ via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $I^\flat$ natural. Then, by Theorem 929, it is syntactically equivalential and, by Theorem 930, it is system truth equational. Thus, by Theorem 1087, it is syntactically algebraizable.

Suppose, conversely, that $\mathcal{I}$ is syntactically algebraizable. Then, by Theorem 1087, it is syntactically equivalential and system truth equational. Hence, by Theorem 923, it is syntactically weakly algebraizable. Now it follows by Theorem 927 that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}}$ via a pair $(\tau^\flat, \overleftrightarrow{I}^\flat)$, where $I^\flat$ witnesses the syntactic equivalentiality and $\tau^\flat$ the system truth equationality of $\mathcal{I}$, and, hence, by definition, they constitute a conjugate pair of transformations, with $I^\flat$ natural.

(b) Suppose, first, that $\mathcal{I}$ is stable and that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}}$ via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $\tau^\flat$ natural. Then, by Theorem 929, it is syntactically protoalgebraic and, by Theorem 930, it is strongly system truth equational. Thus, by Theorem 1087, it is syntactically antialgebraizable.

Suppose, conversely, that $\mathcal{I}$ is syntactically antialgebraizable. Then, by Theorem 1087, it is syntactically protoalgebraic and strongly system truth equational. Hence, by Theorem 923, it is syntactically weakly algebraizable. Now it follows by Theorem 927 that $\mathcal{K}^{\mathcal{I}}$ is equivalent to $\mathcal{Q}^{\mathcal{I}}$ via a pair $(\tau^\flat, \overleftrightarrow{I}^\flat)$, where $I^\flat$ witnesses the syntactic protoalgebraicity and $\tau^\flat$ the strong system truth equationality of $\mathcal{I}$, and, hence, by definition, they constitute a conjugate pair of transformations, with $\tau^\flat$ natural.

■

The equivalence of the systemic skeleton with some algebraic π-structure via a conjugate pair of transformations, exhibiting the required one-sided naturality condition, is sufficient to ensure syntactic system (anti)algebraizability. This constitutes an analog of Theorem 1083.

Theorem 1092 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is syntactically algebraizable if and only if it is stable and its systemic skeleton is equivalent to an algebraic* π*-structure via a conjugate pair* $(\tau^\flat, I^\flat)$ *of transformations, with* $I^\flat$ *natural.*

(b) $\mathcal{I}$ *is syntactically antialgebraizable if and only if it is stable and its systemic skeleton is equivalent to an algebraic* π*-structure via a conjugate pair* $(\tau^\flat, I^\flat)$ *of transformations, with* $\tau^\flat$ *natural.*

Proof:

(a) If $\mathcal{I}$ is syntactically algebraizable, then, by Theorem 1091, it is stable and its systemic skeleton is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations with $I^\flat$ natural. Suppose, conversely, that $\mathcal{I}$ is stable and that its systemic skeleton $\mathcal{K}^{\mathcal{I}}$ is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $I^\flat$ natural. Then, by Proposition 929, it is syntactically equivalential and, by Theorem 930, it is system truth equational. Therefore, by Theorem 1087, it is syntactically algebraizable.

(b) If $\mathcal{I}$ is syntactically antialgebraizable, then, by Theorem 1091, it is stable and its systemic skeleton is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with $\tau^\flat$ natural. Suppose, conversely, that $\mathcal{I}$ is stable and that $\mathcal{K}^{\mathcal{I}}$ is equivalent to an algebraic π-structure via a conjugate pair $(\tau^\flat, I^\flat)$ of transformations, with

$\tau^\flat$ natural. Then, by Proposition 929, it is syntactically protoalgebraic and, by Theorem 930, it is strongly system truth equational. Therefore, by Theorem 1087, it is syntactically antialgebraizable. ■

Finally, in terms of order isomorphisms between theory family lattices, we have the following analog of Theorem 1084, giving an alternative characterization of syntactically (anti)algebraizable π-institutions:

Theorem 1093 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) *$\mathcal{I}$ is syntactically algebraizable if and only if it is stable and there exists a transformational order isomorphism* $h : \mathbf{ThFam}(\mathcal{K}^{\mathcal{I}}) \to \mathbf{ThFam}(\mathcal{Q})$, *induced by a conjugate pair* $(\tau^\flat, I^\flat)$ *of transformations, where* $\mathcal{Q}$ *is an algebraic* π*-structure and* $I^\flat$ *is natural.*

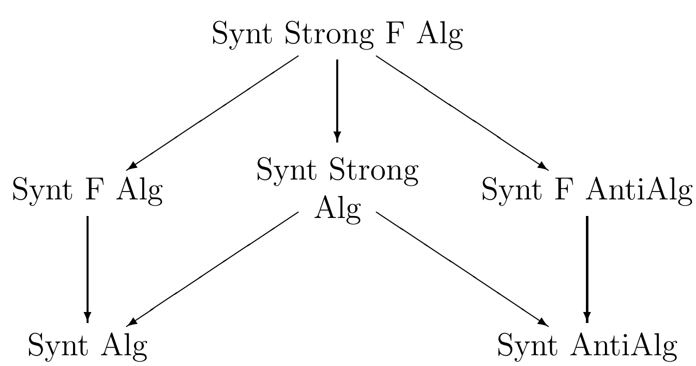
(b) *$\mathcal{I}$ is syntactically antialgebraizable if and only if it is stable and there exists a transformational order isomorphism*

$$h : \mathbf{ThFam}(\mathcal{K}^{\mathcal{I}}) \to \mathbf{ThFam}(\mathcal{Q}),$$

induced by a conjugate pair $(\tau^\flat, I^\flat)$ *of transformations, where* $\mathcal{Q}$ *is an algebraic* π*-structure and* $\tau^\flat$ *is natural.*

Proof: The "only if" follows by Theorem 1092 and Theorem 893. The "if" is given by Theorem 901 and Theorem 1092. ■

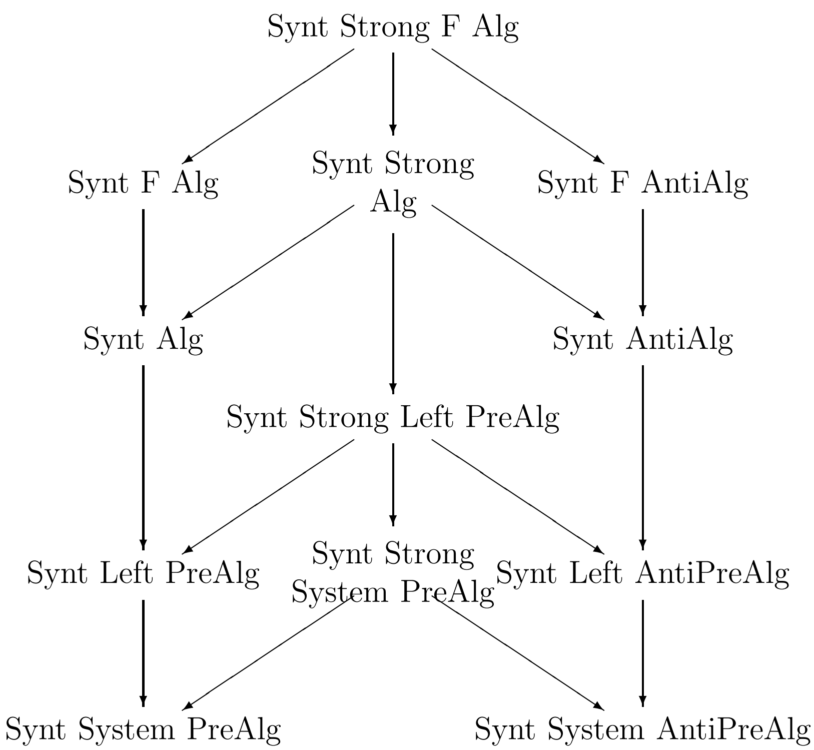
To close the chapter, we have some class separating work to do. First of all, since we have now described in detail the six classes of the syntactic algebraizability hierarchy, it is only appropriate to pause and look for examples that separate the family algebraizability classes, i.e., those in the top-most tier, from the algebraizability classes, that is those immediately below them. In other words, we are looking for examples that show that the vertical arrows in the accompanying diagram



represent, in fact, proper inclusions. We can do this all at once by exhibiting an example of a syntactically strongly algebraizable π-institution which is neither syntactically family algebraizable nor syntactically family antialgebraizable.

Example 1094 EXAMPLE NOT FOUND YET!

Last, since the syntactic algebraizability classes, shown in the bottom row of the preceding diagram, dominate the syntactic left prealgebraizability classes in the 12-class hierarchy, we also need examples to separate syntactically algebraizable from syntactically left prealgebraizable π-institutions, i.e., examples showing that the longish vertical arrows in the diagram



represent proper inclusions. Again, in a single strike, this can be accomplished by providing an example of a syntactically strongly left prealgebraizable π-institution which is neither syntactically algebraizable nor syntactically antialgebraizable.

Example 1095 EXAMPLE NOT FOUND YET!

Chapter 14

The Syntactic Leibniz Hierarchy: Basement I

14.1 Rough/Narrow Truth Equationality

In this section, we study *rough/narrow truth equationality*, the syntactic analog of rough c-reflectivity, which, recalling Corollary 482, coincides with narrow c-reflectivty. It has the same relation to truth equationality as rough c-reflectivity has to c-reflectivity. In other words, it mimics truth equationality, but it is applied only to theory families of a π-institution all of whose components are nonempty.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that $\mathcal{I}$ is **roughly** or **narrowly (family) truth equational** if there exists $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with a single distinguished argument, such that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi/\Omega_\Sigma(T) \in \widetilde{T}_\Sigma/\Omega_\Sigma(T) \quad \text{iff} \quad \tau_\Sigma^{\mathcal{F}/\Omega(T)}[\phi/\Omega_\Sigma(T)] \leq \Delta^{\mathcal{F}/\Omega(T)}.$$

Recall that, by Proposition 369, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(\widetilde{T}) = \Omega(T)$. Thus, $\Omega(T)$ is compatible with $\widetilde{T}$ and, hence, the preceding definition makes sense. The collection $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ is referred to as a set of **witnessing equations** (of/for the rough/narrow truth equationality of $\mathcal{I}$).

Paralleling Proposition 816, we get the following alternative characterization.

Proposition 1096 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ a collection of natural transformations in $N^\flat$, with a single distinguished argument. $\mathcal{I}$ is roughly truth equational, with witnessing equations $\tau^\flat$, if and only if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,*

$$\phi \in \widetilde{T}_\Sigma \quad \textit{iff} \quad \tau_\Sigma^\flat[\phi] \leq \Omega(T).$$

Proof: Suppose $\mathcal{I}$ is roughly truth equational and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{lll}
\phi \in \widetilde{T}_\Sigma & \text{iff} & \phi/\Omega_\Sigma(T) \in \widetilde{T}_\Sigma/\Omega_\Sigma(T) \quad \text{(Proposition 369 and compatibility)} \\
& \text{iff} & \tau_\Sigma^{\mathcal{F}/\Omega(T)}[\phi/\Omega_\Sigma(T)] \leq \Delta^{\mathcal{F}/\Omega(T)} \quad \text{(rough truth equationality)} \\
& \text{iff} & \tau_\Sigma^\flat[\phi]/\Omega(T) \leq \Delta^{\mathcal{F}/\Omega(T)} \quad \text{(by definition)} \\
& \text{iff} & \tau_\Sigma^\flat[\phi] \leq \Omega(T).
\end{array}$$

Suppose, conversely, that the given condition holds. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{lll}
\phi/\Omega_\Sigma(T) \in \widetilde{T}_\Sigma/\Omega_\Sigma(T) & \text{iff} & \phi \in \widetilde{T}_\Sigma \quad \text{(Proposition 369 and compatibility)} \\
& \text{iff} & \tau_\Sigma^\flat[\phi] \leq \Omega(T) \quad \text{(by hypothesis)} \\
& \text{iff} & \tau_\Sigma^\flat[\phi]/\Omega(T) \leq \Delta^{\mathcal{F}/\Omega(T)} \\
& \text{iff} & \tau_\Sigma^{\mathcal{F}/\Omega(T)}[\phi/\Omega_\Sigma(T)] \leq \Delta^{\mathcal{F}/\Omega(T)}. \quad \text{(definition)}
\end{array}$$

Therefore, $\mathcal{I}$ is roughly truth equational. ■

It is not difficult to see that an alternative way to express rough truth equationality is to assert the same condition that defines truth equationality, excluding, however, those theory families with at least one empty component.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall from Chapter 6 that we denote by $\mathrm{ThFam}^{\not\varnothing}(\mathcal{I})$ the collection of all theory families T of $\mathcal{I}$, such that $T_\Sigma \neq \varnothing$, for all $\Sigma \in |\mathbf{Sign}^\flat|$:

$$\mathrm{ThFam}^{\not\varnothing}(\mathcal{I}) = \{T \in \mathrm{ThFam}(\mathcal{I}) : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(T_\Sigma \neq \varnothing)\}.$$

Recall, also, that, if $\mathcal{I}$ has theorems, then $\mathrm{ThFam}^{\not\varnothing}(\mathcal{I}) = \mathrm{ThFam}(\mathcal{I})$. In particular, this is the case if $\mathcal{I}$ happens to be truth equational.

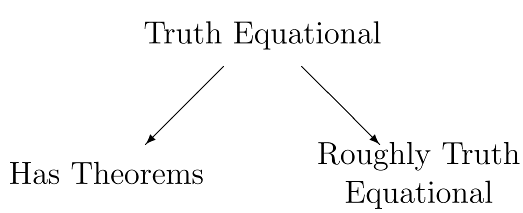
Proposition 1097 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ a collection of natural transformations in $N^\flat$, with a single distinguished argument. $\mathcal{I}$ is roughly truth equational, with witnessing equations $\tau^\flat$, if and only if, for all $T \in \mathrm{ThFam}^{\not\varnothing}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,*

$$\phi \in T_\Sigma \quad \textit{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

Proof: Suppose $\mathcal{I}$ is roughly truth equational, with witnessing equations $\tau^\flat$. Let $T \in \mathrm{ThFam}^{\not\varnothing}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then $\widetilde{T} = T$, whence, by Proposition 1096, $\phi \in T_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$.

Suppose, conversely, that the displayed condition holds. Consider $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, since, by definition of $\widetilde{T}$, we have $\widetilde{T} \in \mathrm{ThFam}^{\not\varnothing}(\mathcal{I})$, we get, by hypothesis, $\phi \in \widetilde{T}_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(\widetilde{T})$, whence, using Proposition 369, we conclude that $\phi \in \widetilde{T}_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Therefore, $\mathcal{I}$ is roughly truth equational. ■

As a corollary, we obtain the following key relationship between rough truth equationality and truth equationality.



Corollary 1098 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is truth equational if and only if it is roughly truth equational and has theorems.*

Proof: Suppose, first, that $\mathcal{I}$ is roughly truth equational, with witnessing equations $\tau^\flat$, and that it has theorems. Availability of theorems implies that $\mathrm{ThFam}^{\nrightarrow}(\mathcal{I}) = \mathrm{ThFam}(\mathcal{I})$. Thus, by Proposition 1097, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in T_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Thus, $\mathcal{I}$ is truth equational, with the same witnessing equations $\tau^\flat$.

Assume, conversely, that $\mathcal{I}$ is truth equational, with witnessing equations $\tau^\flat$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, and, hence, a fortiori, for all $T \in \mathrm{ThFam}^{\nrightarrow}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in T_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Hence, again by Proposition 1097, $\mathcal{I}$ is roughly truth equational. Finally, by Theorem 829, $\mathcal{I}$ is family c-reflective and, by Proposition 243, it is family reflective and, hence, family injective. Thus, it must have theorems. ∎

Our next goal is to prove an analog of the characterization theorem, Theorem 838, of truth equationality in terms of the solubility of the Suszko core for rough truth equationality.

Rough truth equationality allows the following expression for all theory families with nonempty components, forming an analog of Proposition 828.

Proposition 1099 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following conditions are equivalent:*

(i) $\mathcal{I}$ *is roughly truth equational, with witnessing equations* $\tau^\flat$*;*

(ii) For all $T \in \mathrm{ThFam}(\mathcal{I})$*,* $\tau^\flat(\Omega(T)) = \widetilde{T}$*;*

(iii) For all $T \in \mathrm{ThFam}^{\nrightarrow}(\mathcal{I})$*,* $\tau^\flat(\Omega(T)) = T$*.*

Proof:

(i)⇒(ii) Suppose $\mathcal{I}$ is roughly truth equational, with witnessing equations $\tau^\flat$, and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{llll} \phi \in \tau^\flat_\Sigma(\Omega(T)) & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(T) & \text{(definition)} \\ & \text{iff} & \phi \in \widetilde{T}_\Sigma. & \text{(rough truth equationality)} \end{array}$$

(ii)⇒(iii) Suppose Condition (ii) holds. Let $T \in \mathrm{ThFam}^{\nrightarrow}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Since $T \in \mathrm{ThFam}^{\nrightarrow}(\mathcal{I})$, $T = \widetilde{T}$, whence, by hypothesis, $T = \tau^\flat(\Omega(T))$.

(iii)⇒(i) If, conversely, for all $T \in \mathrm{ThFam}^{\nrightarrow}(\mathcal{I})$, $T = \tau^\flat(\Omega(T))$, then, by Proposition 1097, $\mathcal{I}$ is roughly truth equational, with witnessing equations $\tau^\flat$.

∎

Recall from Chapter 6 that, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, $\mathcal{I}$ is called *roughly family c-reflective* if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,

$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \quad \text{implies} \quad \bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T'}.$$

We are now able to show that rough truth equationality implies rough family c-reflectivity. This is an analog of Theorem 829.

Theorem 1100 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly truth equational, then it is roughly family c-reflective.*

Proof: Suppose $\mathcal{I}$ is roughly truth equational, with witnessing equations $\tau^\flat$. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. Then we have

$$\begin{array}{rcll} \bigcap_{T\in\mathcal{T}} \widetilde{T} & = & \bigcap_{T\in\mathcal{T}} \tau^\flat(\Omega(T)) & \text{(Proposition 1099)} \\ & = & \tau^\flat(\bigcap_{T\in\mathcal{T}} \Omega(T)) & \text{(set theory)} \\ & \leq & \tau^\flat(\Omega(T')) & \text{(hypothesis)} \\ & = & \widetilde{T}'. & \text{(Proposition 1099)} \end{array}$$

Thus, $\mathcal{I}$ is roughly family c-reflective. ∎

In the context of rough truth equationality, the notion paralleling the Suszko core is the *rough Suszko core*, a modification of the original which is defined, naturally enough and as, perhaps, was to be expected, by circumventing theory families with empty components.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The **rough Suszko core** $S^{\mathcal{I}\not\,}$ of $\mathcal{I}$ is the collection

$$S^{\mathcal{I}\not\,} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}(\mathcal{I}))(\sigma^\flat[\widetilde{T}] \leq \widetilde{\Omega}^{\mathcal{I}}(\widetilde{T}))\}.$$

As before, an alternative characterization avoids $\widetilde{\ }$ at the expense of restricting quantification over $\mathrm{ThFam}^{\not\,}(\mathcal{I})$.

Proposition 1101 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then*

$$S^{\mathcal{I}\not\,} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}^{\not\,}(\mathcal{I}))(\sigma^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T))\}.$$

Proof: Inside this proof we set

$$M^{\mathcal{I}} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}^{\not\,}(\mathcal{I}))(\sigma^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T))\}.$$

Our goal is to show that $S^{\mathcal{I}\not\,} = M^{\mathcal{I}}$. Suppose, first, that $\sigma^\flat \in S^{\mathcal{I}\not\,}$ and let $T \in \mathrm{ThFam}^{\not\,}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$. Since $T \in \mathrm{ThFam}^{\not\,}(\mathcal{I})$, we get $\widetilde{T} = T$. Hence, by hypothesis, $\phi \in \widetilde{T}_\Sigma$. Thus, since $\sigma^\flat \in S^{\mathcal{I}\not\,}$, we get

$$\sigma^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(\widetilde{T}) = \widetilde{\Omega}^{\mathcal{I}}(T).$$

This proves that $\sigma^\flat \in M^{\mathcal{I}}$. Assume, conversely, that $\sigma^\flat \in M^{\mathcal{I}}$ and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \widetilde{T}_\Sigma$. Since $\widetilde{T} \in \mathrm{ThFam}^{\not\,}(\mathcal{I})$ and $\sigma^\flat \in M^{\mathcal{I}}$, we get $\sigma^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(\widetilde{T})$, whence, $\sigma^\flat \in S^{\mathcal{I}\not\,}$. This proves that $S^{\mathcal{I}\not\,} = M^{\mathcal{I}}$. ∎

From the definition, it is not difficult to see that any theory family T with all its components nonempty is always included in $S^{\mathcal{I}\not\,}(\Omega(T))$. This forms an analog in the rough context of Proposition 832.

Proposition 1102 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$*,*

$$T \leq S^{\mathcal{I}\lightning}(\Omega(T)).$$

Proof: Suppose $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$, and $\sigma^\flat \in S^{\mathcal{I}\lightning}$. Then, by Proposition 1101, $\sigma^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T) \leq \Omega(T)$. Hence, $S^{\mathcal{I}\lightning}_\Sigma[\phi] \leq \Omega(T)$. By definition, then, $\phi \in S^{\mathcal{I}\lightning}_\Sigma(\Omega(T))$. Since Σ and $\phi \in T_\Sigma$ were arbitrary, we conclude that $T \leq S^{\mathcal{I}\lightning}(\Omega(T))$. ■

The reverse inclusion may or may not hold. If it does, for all $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, we say that the rough Suszko core of $\mathcal{I}$ is *soluble*.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The rough Suszko core $S^{\mathcal{I}\lightning}$ of $\mathcal{I}$ is said to be **soluble** if, for all $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$S^{\mathcal{I}\lightning}_\Sigma[\phi] \leq \Omega(T) \quad \text{implies} \quad \phi \in T_\Sigma.$$

An alternative way to express solubility is to again expand the view to all theory families at the balancing expense of adding rough equivalence representatives.

Lemma 1103 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $S^{\mathcal{I}\lightning}$ *is soluble if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\widetilde{T} = S^{\mathcal{I}\lightning}(\Omega(T)).$$

Proof: $S^{\mathcal{I}\lightning}$ is soluble if and only if, by definition and Proposition 1102, for all $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, $T = S^{\mathcal{I}\lightning}(\Omega(T))$, if and only if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{T} = S^{\mathcal{I}\lightning}(\Omega(\widetilde{T}))$, if and only if, by Proposition 369, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{T} = S^{\mathcal{I}\lightning}(\Omega(T))$. ■

As was the case with truth equationality (see Lemma 835), it turns out that, if a given π-institution is roughly truth equational, then any collection of witnessing equations must be included in the rough Suszko core of $\mathcal{I}$. Differently put, in case of rough truth equationality, the rough Suszko core is a candidate for the largest set of witnessing equations.

Lemma 1104 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly truth equational, with witnessing equations* $\tau^\flat$*, then* $\tau^\flat \subseteq S^{\mathcal{I}\lightning}$*.*

Proof: Suppose $\mathcal{I}$ is roughly truth equational, with witnessing equations $\tau^\flat$. Let $T \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$. Then, for all $T \leq T' \in \mathrm{ThFam}(\mathcal{I})$, we have $T' \in \mathrm{ThFam}^{\lightning}(\mathcal{I})$ and $\phi \in T'_\Sigma$. Thus, by rough truth equationality, and Proposition 1097, $\tau^\flat_\Sigma[\phi] \leq \Omega(T')$. Since T', with the postulated properties was arbitrary,

$$\tau^\flat_\Sigma[\phi] \leq \bigcap\{\Omega(T') : T \leq T'\} = \widetilde{\Omega}^{\mathcal{I}}(T).$$

We conclude, using Proposition 1101, that $\tau^\flat \subseteq S^{\mathcal{I}\nleq}$. ∎

We are now ready to prove the equivalence between rough truth equationality and the solubility of the rough Suszko core. In the next theorem, we show that truth equationality implies the solubility of the rough Suszko core. This forms a rough analog of Theorem 836.

Theorem 1105 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly truth equational, then* $S^{\mathcal{I}\nleq}$ *is soluble.*

Proof: Suppose $\mathcal{I}$ is roughly truth equational, with witnessing equations $\tau^\flat$. Let $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $S^{\mathcal{I}\nleq}_\Sigma[\phi] \leq \Omega(T)$. Then, by rough truth equationality and Lemma 1104, $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Again, using rough truth equationality and Proposition 1097, we conclude that $\phi \in T_\Sigma$. This shows that $S^{\mathcal{I}\nleq}$ is soluble. ∎

Conversely, in a rough analog of Theorem 837, we show that the solubility of the rough Suszko core of a π-institution implies rough truth equationality.

Theorem 1106 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $S^{\mathcal{I}\nleq}$ *is soluble, then* $\mathcal{I}$ *is roughly truth equational, with witnessing equations* $S^{\mathcal{I}\nleq}$*.*

Proof: Assume $S^{\mathcal{I}\nleq}$ is soluble and let $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. By Proposition 1097, it suffices to show that

$$\phi \in T_\Sigma \quad \text{iff} \quad S^{\mathcal{I}\nleq}_\Sigma[\phi] \leq \Omega(T).$$

If $\phi \in T_\Sigma$, then, by Proposition 1102, $\phi \in S^{\mathcal{I}\nleq}_\Sigma(\Omega(T))$, i.e., $S^{\mathcal{I}\nleq}_\Sigma[\phi] \leq \Omega(T)$. On the other hand, the reverse inclusion is guaranteed by the solubility of $S^{\mathcal{I}\nleq}$. Thus, $\mathcal{I}$ is roughly truth equational, with witnessing equations $S^{\mathcal{I}\nleq}$. ∎

Theorems 1105 and 1106 provide the first characterization of rough truth equationality in terms of the solubility of the rough Suszko core. This parallels Theorem 838, which asserted a similar characterization for truth equationality in terms of the solubility of the Suszko core of a π-institution.

$$\mathcal{I} \text{ Roughly Truth Equational} \longleftrightarrow S^{\mathcal{I}\nleq} \text{ Soluble}$$

Theorem 1107 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is roughly truth equational if and only if* $S^{\mathcal{I}\nleq}$ *is soluble.*

Proof: The "if" is by Theorem 1106. The "only if" by Theorem 1105. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the rough Suszko core $S^{\mathcal{I}\nmid}$ of $\mathcal{I}$ **roughly defines theory families** if, for al $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$,

$$T = S^{\mathcal{I}\nmid}(\Omega(T)).$$

Another characterization of rough truth equationality, along the lines of Theorem 840, asserts that it is equivalent to the rough definability of the theory families by the rough Suszko core.

$$\begin{array}{l}\mathcal{I} \text{ Roughly Truth Equational} \\ \qquad\qquad \longleftrightarrow\ S^{\mathcal{I}\nmid} \text{ Roughly Defines Theory Families}\end{array}$$

Theorem 1108 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is roughly truth equational if and only if $S^{\mathcal{I}\nmid}$ roughly defines theory families in $\mathcal{I}$.*

Proof: Suppose $\mathcal{I}$ is roughly truth equational. By Theorem 1107, $S^{\mathcal{I}\nmid}$ is soluble. Hence, by definition, for all $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $S^{\mathcal{I}\nmid}(\Omega(T)) \le T$. Since, by Proposition 1102, the reverse always holds, we get, for all $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $T = S^{\mathcal{I}\nmid}(\Omega(T))$. Thus, $S^{\mathcal{I}\nmid}$ roughly defines theory families in $\mathcal{I}$. Conversely, if, for all $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $T = S^{\mathcal{I}\nmid}(\Omega(T))$, then $S^{\mathcal{I}\nmid}$ is soluble and, therefore, by Theorem 1107, $\mathcal{I}$ is roughly truth equational. ■

We embark, next, in the process of establishing a connection between rough truth equationality and rough family c-reflectivity by means of the Suszko operator. We start by showing that, in every π-institution $\mathcal{I}$, $T \le S^{\mathcal{I}\nmid}(\Omega(T))$ actually holds for every theory family of $\mathcal{I}$ and not only for those theory families in $\mathrm{ThFam}^{\nmid}(\mathcal{I})$.

Lemma 1109 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. For all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,*

$$\phi \in T_\Sigma \quad \textit{implies} \quad S^{\mathcal{I}\nmid}_\Sigma[\phi] \le \Omega(T).$$

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{llll}\phi \in T_\Sigma & \text{implies} & \phi \in \widetilde{T}_\Sigma & (T \le \widetilde{T}) \\ & \text{implies} & S^{\mathcal{I}\nmid}_\Sigma[\phi] \le \widetilde{\Omega}^{\mathcal{I}}(\widetilde{T}) & (\text{definition of } S^{\mathcal{I}\nmid}) \\ & \text{implies} & S^{\mathcal{I}\nmid}_\Sigma[\phi] \le \Omega(\widetilde{T}) & (\widetilde{\Omega}^{\mathcal{I}} \le \Omega) \\ & \text{iff} & S^{\mathcal{I}\nmid}_\Sigma[\phi] \le \Omega(T). & (\text{Proposition 369})\end{array}$$

This establishes the displayed implication. ■

In the sequel, in dealing with intersections of Leibniz congruence systems, as, e.g., when computing a Suszko congruence system, we shall have the need to switch between arbitrary collections of theory families and collections of theory families having all components nonempty. In all those situations, the following straightforward technical lemma is quite useful.

Lemma 1110 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $X \in \mathrm{SenFam}(\mathbf{F})$ *and* $\theta \in \mathrm{SenFam}(\mathbf{F}^2)$*.*

(a) $\{\Omega(T) : X \leq T \in \mathrm{ThFam}(\mathcal{I})\} = \{\Omega(T) : X \leq T \in \mathrm{ThFam}^{\nparallel}(\mathcal{I})\}$*;*

(b) $\{\Omega(T) : X \leq T \in \mathrm{ThFam}(\mathcal{I})$ *and* $\theta \leq \Omega(T)\} = \{\Omega(T) : X \leq T \in \mathrm{ThFam}^{\nparallel}(\mathcal{I})$ *and* $\theta \leq \Omega(T)\}$*.*

Proof:

(a) Since $\mathrm{ThFam}^{\nparallel}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I})$, it is clear that

$$\{\Omega(T) : X \leq T \in \mathrm{ThFam}^{\nparallel}(\mathcal{I})\} \subseteq \{\Omega(T) : X \leq T \in \mathrm{ThFam}(\mathcal{I})\}.$$

To prove the reverse inclusion, let $T \in \mathrm{ThFam}(\mathcal{I})$, such that $X \leq T$. Consider $\widetilde{T} \in \mathrm{ThFam}^{\nparallel}(\mathcal{I})$. We get $X \leq T \leq \widetilde{T}$ and, moreover, by Proposition 369, $\Omega(\widetilde{T}) = \Omega(T)$. This proves that $\{\Omega(T) : X \leq T \in \mathrm{ThFam}(\mathcal{I})\} \subseteq \{\Omega(T) : X \leq T \in \mathrm{ThFam}^{\nparallel}(\mathcal{I})\}$.

(b) As in Part (a), the right-to-left inclusion is obvious. For the reverse, consider $T \in \mathrm{ThFam}(\mathcal{I})$, such that $X \leq T$ and $\theta \leq \Omega(T)$. Then, again, $\widetilde{T} \in \mathrm{ThFam}^{\nparallel}(\mathcal{I})$, such that both $X \leq T \leq \widetilde{T}$ and $\theta \leq \Omega(T) = \Omega(\widetilde{T})$. This shows that the left-to-right inclusion also holds.

■

As a corollary, we obtain, for instance, an alternative expression for the Suszko congruence system associated with a given theory family of a π-institution $\mathcal{I}$.

Corollary 1111 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\widetilde{\Omega}^{\mathcal{I}}(T) = \bigcap \{\Omega(T') : T \leq T' \in \mathrm{ThFam}^{\nparallel}(\mathcal{I})\}.$$

Proof: Immediate by the definition of $\widetilde{\Omega}^{\mathcal{I}}$ and Lemma 1110. ■

Based on Lemma 1109, we may show that, for every theory family T, $T \leq S^{\mathcal{I}\nparallel}(\widetilde{\Omega}^{\mathcal{I}}(T))$.

Proposition 1112 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $T \in \mathrm{ThFam}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\phi \in T_\Sigma \quad \text{implies} \quad S^{\mathcal{I}\nparallel}_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T).$$

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{rll}\phi \in T_\Sigma & \text{implies} & \phi \in T'_\Sigma, \text{ for all } T \le T' \in \mathrm{ThFam}(\mathcal{I})\\ & \text{implies} & S^{\mathcal{I}\natural}_\Sigma[\phi] \le \Omega(T'), \text{ for all } T \le T' \in \mathrm{ThFam}(\mathcal{I})\\ & & \quad\text{(by Lemma 1109)}\\ & \text{iff} & S^{\mathcal{I}\natural}_\Sigma[\phi] \le \widetilde{\Omega}^{\mathcal{I}}(T). \quad \text{(definition of } \widetilde{\Omega}^{\mathcal{I}})\end{array}$$

■

In analogy with the case of rough truth equationality, we may introduce the notion of *adequacy* of the rough Suszko core, which will help in characterizing the relationship between rough truth equationality and rough c-reflectivity. The following proposition, a rough analog of Proposition 841, partially justifies the notion of adequacy that will follow.

Proposition 1113 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\bigcap\{\Omega(T) : S^{\mathcal{I}\natural}_\Sigma[\phi] \le \Omega(T)\} \le \widetilde{\Omega}^{\mathcal{I}}(C(\phi)).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\begin{array}{rll}\phi \in T_\Sigma & \text{implies} & S^{\mathcal{I}\natural}_\Sigma[\phi] \le \widetilde{\Omega}^{\mathcal{I}}(T) \quad \text{(Proposition 1112)}\\ & \text{implies} & S^{\mathcal{I}\natural}_\Sigma[\phi] \le \Omega(T). \quad (\widetilde{\Omega}^{\mathcal{I}} \le \Omega)\end{array}$$

Hence,

$$\begin{array}{rcl}\bigcap\{\Omega(T) : S^{\mathcal{I}\natural}_\Sigma[\phi] \le \Omega(T)\} & \le & \bigcap\{\Omega(T) : S^{\mathcal{I}\natural}_\Sigma[\phi] \le \widetilde{\Omega}^{\mathcal{I}}(T)\}\\ & \le & \bigcap\{\Omega(T) : \phi \in T_\Sigma\}\\ & = & \widetilde{\Omega}^{\mathcal{I}}(C(\phi)).\end{array}$$

This is the displayed formula in the statement. ■

If the reverse inclusion of that proven in Proposition 1113 holds, then we say that the rough Suszko core of $\mathcal{I}$ is *adequate*.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. We say that the rough Suszko core $S^{\mathcal{I}\natural}$ of $\mathcal{I}$ is **adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) \le \bigcap\{\Omega(T) : S^{\mathcal{I}\natural}_\Sigma[\phi] \le \Omega(T)\}.$$

We can show right away that solubility of the rough Suszko core implies adequacy.

Corollary 1114 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $S^{\mathcal{I}\natural}$ *is soluble, then it is adequate.*

Proof: Suppose $S^{\mathcal{I}\natural}$ is soluble and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) & = & \bigcap\{\Omega(T) : T \in \mathrm{ThFam}(\mathcal{I}) \text{ and } \phi \in T_\Sigma\} \\ & & \text{(definition of } \widetilde{\Omega}^{\mathcal{I}}) \\ & = & \bigcap\{\Omega(T) : T \in \mathrm{ThFam}^{\natural}(\mathcal{I}) \text{ and } \phi \in T_\Sigma\} \\ & & \text{(Lemma 1110)} \\ & = & \bigcap\{\Omega(T) : T \in \mathrm{ThFam}^{\natural}(\mathcal{I}) \text{ and } S^{\mathcal{I}\natural}_\Sigma[\phi] \le \Omega(T)\} \\ & & \text{(solubility of } S^{\mathcal{I}\natural}) \\ & = & \bigcap\{\Omega(T) : T \in \mathrm{ThFam}(\mathcal{I}) \text{ and } S^{\mathcal{I}\natural}_\Sigma[\phi] \le \Omega(T)\}. \\ & & \text{(Lemma 1110)}\end{array}$$

Thus, $S^{\mathcal{I}\natural}$ is adequate. ■

We prove, next, the converse of Corollary 1114, under the additional assumption that the π-institution $\mathcal{I}$ under consideration is roughly family c-reflective. This constitutes an analog of Proposition 846.

Proposition 1115 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a roughly family c-reflective π-institution based on $\mathbf{F}$. If $S^{\mathcal{I}\natural}$ is adequate, then it is soluble.*

Proof: Suppose $\mathcal{I}$ is roughly family c-reflective and $S^{\mathcal{I}\natural}$ is adequate. Let $T \in \mathrm{ThFam}^{\natural}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $S^{\mathcal{I}\natural}_\Sigma[\phi] \le \Omega(T)$. By the adequacy of $S^{\mathcal{I}\natural}$, we get that $\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) \le \Omega(T)$. By Lemma 1110,

$$\bigcap\{\Omega(T) : T \in \mathrm{ThFam}^{\natural}(\mathcal{I}) \text{ and } \phi \in T_\Sigma\} \le \Omega(T).$$

By rough family c-reflectivity, $\bigcap\{T \in \mathrm{ThFam}^{\natural}(\mathcal{I}) \text{ and } \phi \in T_\Sigma\} \le T$. Hence, $\phi \in T_\Sigma$. We conclude that $S^{\mathcal{I}\natural}$ is soluble. ■

We are now in a position to prove the main characterization theorem relating rough truth equationality with rough family c-reflectivity, an analog of Theorem 847, which characterized truth equationality in terms of family c-reflectivity and the adequacy of the Suszko core.

Rough Truth Equationality	=	$S^{\mathcal{I}\natural}$ Soluble
	=	$S^{\mathcal{I}\natural}$ Roughly Defines Theory Families
	=	Rough Family c-Reflectivity
		+ $S^{\mathcal{I}\natural}$ Adequate

Theorem 1116 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is roughly truth equational if and only if it is roughly family c-reflective and has an adequate rough Suszko core.*

Proof: Suppose, first, that $\mathcal{I}$ is roughly truth equational. By Theorem 1100, it is roughly family c-reflective. By Theorem 1105, its rough Suszko core is soluble. Thus, by Corollary 1114, its rough Suszko core is also adequate.

Assume, conversely, that $\mathcal{I}$ is roughly family c-reflective and has an adequate rough Suszko core. Then, by Proposition 1115, its rough Suszko core is also soluble. Hence, by Theorem 1107, $\mathcal{I}$ is roughly truth equational. ■

Even though Theorem 847 formed the inspiration for the formulation of Theorem 1116, we show that it can be obtained as a corollary of the latter. This also exhibits the close connection between the two results which should have been anticipated, given the fact that the work done here is intended to mimic the former, while circumventing potential obstacles due to the absence of theorems.

Corollary 1117 (Theorem 847) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is truth equational if and only if it is family c-reflective and has an adequate Suszko core.*

Proof: $\mathcal{I}$ is truth equational if and only if, by Corollary 1098, it is roughly truth equational and has theorems if and only if, by Theorem 1116, it is roughly family c-reflective, has theorems and has an adequate rough Suszko core if and only if, by Theorem 468 and the definitions of the Suszko core, the rough Suszko core and their adequacy properties, $\mathcal{I}$ is family c-reflective and its Suszko core is adequate. ■

We close the section by looking at a couple of results that may be perceived either as alternative characterizations of rough truth equationality, involving arbitrary $\mathbf{F}$-algebraic systems, or as transfer theorems.

Theorem 1118 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is roughly truth equational, with witnessing equations* $\tau^\flat$*, if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*, all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi \in \mathrm{SEN}(\Sigma)$*,*

$$\phi \in \widetilde{T}_\Sigma \quad \textit{iff} \quad \tau^{\mathcal{A}}_\Sigma[\phi] \leq \Omega^{\mathcal{A}}(T).$$

Proof: If the postulated condition holds, then it holds, in particular, for the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$. This yields immediately that $\mathcal{I}$ is roughly truth equational.

Suppose, conversely, that $\mathcal{I}$ is roughly truth equational and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rll}
\alpha_\Sigma(\phi) \in \widetilde{T}_{F(\Sigma)} & \text{iff} & \phi \in \alpha^{-1}_\Sigma(\widetilde{T}_{F(\Sigma)}) \\
& \text{iff} & \phi \in \widetilde{\alpha^{-1}_\Sigma(T_{F(\Sigma)})} \quad \text{(Theorem 377)} \\
& \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(\alpha^{-1}(T)) \quad \text{(hypothesis)} \\
& \text{iff} & \tau^\flat_\Sigma[\phi] \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T)) \quad \text{(Proposition 24)} \\
& \text{iff} & \tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi)] \leq \Omega^{\mathcal{A}}(T). \quad \text{(Lemma 95)}
\end{array}$$

Hence, taking into account the surjectivity of $\langle F,\alpha\rangle$, we conclude that the displayed condition holds. ∎

In analogy with the notation $\mathrm{ThFam}^{\nmid}(\mathcal{I})$, we introduce the following for filter families over arbitrary $\mathbf{F}$-algebraic systems all of whose components are nonempty.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, an $\mathbf{F}$-algebraic system. Define $\mathrm{FiFam}^{\mathcal{I}\nmid}(\mathcal{A})$ to be the collection of all $\mathcal{I}$-filter families T on $\mathcal{A}$, such that $T_\Sigma \neq \varnothing$, for all $\Sigma \in |\mathbf{Sign}|$:

$$\mathrm{FiFam}^{\mathcal{I}\nmid}(\mathcal{A}) = \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : (\forall \Sigma \in |\mathbf{Sign}|)(T_\Sigma \neq \varnothing)\}.$$

We now get immediately the following corollary.

Corollary 1119 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is roughly truth equational, with witnessing equations $\tau^\flat$, if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, all $T \in \mathrm{FiFam}^{\mathcal{I}\nmid}(\mathcal{A})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,*

$$\phi \in T_\Sigma \quad \textit{iff} \quad \tau^{\mathcal{A}}_\Sigma[\phi] \leq \Omega^{\mathcal{A}}(T).$$

Proof: Suppose that $\mathcal{I}$ is roughly truth equational. Then, if $\mathcal{A}$ is an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}\nmid}(\mathcal{A})$, we get

$$\begin{array}{rcll} T & = & \widetilde{T} & (T \in \mathrm{FiFam}^{\mathcal{I}\nmid}(\mathcal{A})) \\ & = & \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T)). & \text{(Theorem 1118)} \end{array}$$

Suppose, conversely, that the displayed condition holds. Then, if $\mathcal{A}$ is an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get, taking into account that $\widetilde{T} \in \mathrm{FiFam}^{\mathcal{I}\nmid}(\mathcal{A})$,

$$\begin{array}{rcll} \widetilde{T} & = & \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(\widetilde{T})) & \text{(hypothesis)} \\ & = & \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T)). & \text{(Proposition 369)} \end{array}$$

This establishes the claimed equivalence. ∎

14.2 Rough Left Truth Equationality

We now turn to *rough left truth equationality*. As the terminology suggests:

- It is in the same relation to rough left c-reflectivity as rough truth equationality is to rough c-reflectivity;
- It is in the same relation to rough truth equationality as left truth equationality is to truth equationality.

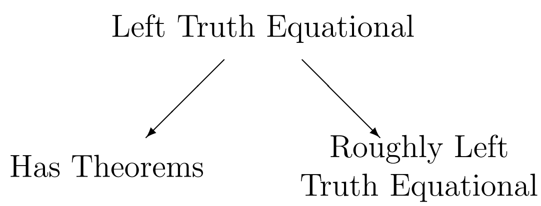
Roughly speaking (in both senses), rough left truth equationality is defined analogously to left truth equationality, but it is applied to rough representatives of theory families so as to avoid theory families with empty components.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that $\mathcal{I}$ is **roughly left truth equational** if there exists $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with a single distinguished argument, such that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in \widetilde{\overleftarrow{T}}_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

The collection $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ is referred to as a set of **witnessing equations** (of/for the rough left truth equationality of $\mathcal{I}$).

The following relationship between rough left truth equationality and left truth equationality, an analog of the relationship between rough truth equationality and truth equationality, presented in Corollary 1098, holds.



Proposition 1120 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is left truth equational if and only if it is roughly left truth equational and has theorems.*

Proof: Suppose, first, that $\mathcal{I}$ is roughly left truth equational, with witnessing equations $\tau^\flat$, and that it has theorems. Availability of theorems implies that $\mathrm{ThFam}^\lightning(\mathcal{I}) = \mathrm{ThFam}(\mathcal{I})$. Thus, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in \overleftarrow{T}_\Sigma$ if and only if $\phi \in \widetilde{\overleftarrow{T}}_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Thus, $\mathcal{I}$ is left truth equational, with the same witnessing equations $\tau^\flat$.

Assume, conversely, that $\mathcal{I}$ is left truth equational, with witnessing equations $\tau^\flat$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in \overleftarrow{T}_\Sigma$ iff $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. This clearly implies that $\mathcal{I}$ has theorems, since, otherwise, given that $\Omega(\overline{\varnothing}) = \nabla^{\mathbf{F}} = \Omega(\mathrm{SEN}^\flat)$, we would get $\mathrm{SEN}^\flat = \overleftarrow{\mathrm{SEN}^\flat} = \overleftarrow{\overline{\varnothing}} = \overline{\varnothing}$, a contradiction. Moreover, due to the availability of theorems, we get, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in \widetilde{\overleftarrow{T}}_\Sigma$ if and only if $\phi \in \overleftarrow{T}_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Thus, $\mathcal{I}$ is roughly left truth equational. ∎

Our next goal is to prove an analog of the characterization theorem, Theorem 860, of left truth equationality in terms of the left solubility of the left Suszko core for rough left truth equationality.

Rough left truth equationality allows an expression for $\widetilde{\overleftarrow{T}}$, for all theory families T, in terms of the Leibniz congruence system of T. The following proposition forms an analog of Proposition 1099.

Proposition 1121 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is roughly left truth equational, with witnessing equations* $\tau^\flat$*, if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,* $\widetilde{\overleftarrow{T}} = \tau^\flat(\Omega(T))$.

Proof: $\mathcal{I}$ is roughly left truth equational, with witnessing equations $\tau^\flat$, if and only if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in \widetilde{\overleftarrow{T}}_\Sigma$ iff $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$, if and only if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\overleftarrow{T}} = \tau^\flat(\Omega(T))$. ■

Recall from Chapter 6 that, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, $\mathcal{I}$ is called *roughly left c-reflective* if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,

$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \quad \text{implies} \quad \bigcap_{T \in \mathcal{T}} \widetilde{\overleftarrow{T}} \leq \widetilde{\overleftarrow{T'}}.$$

We are now able to show that rough left truth equationality implies rough left c-reflectivity. This is an analog of Theorem 1100.

Theorem 1122 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly left truth equational, then it is roughly left c-reflective.*

Proof: Suppose $\mathcal{I}$ is roughly left truth equational, with witnessing equations $\tau^\flat$. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. Then we have

$$\begin{array}{rcll}
\bigcap_{T \in \mathcal{T}} \widetilde{\overleftarrow{T}} & = & \bigcap_{T \in \mathcal{T}} \tau^\flat(\Omega(T)) & \text{(Proposition 1121)} \\
 & = & \tau^\flat(\bigcap_{T \in \mathcal{T}} \Omega(T)) & \text{(set theory)} \\
 & \leq & \tau^\flat(\Omega(T')) & \text{(hypothesis)} \\
 & = & \widetilde{\overleftarrow{T'}}. & \text{(Proposition 1121)}
\end{array}$$

Thus, $\mathcal{I}$ is roughly left c-reflective. ■

In the context of rough left truth equationality, the notion paralleling the left Suszko core is the *rough left Suszko core*, a modification of the original, which is defined below.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The **rough left Suszko core** $\widetilde{L}^{\mathcal{I}}$ of $\mathcal{I}$ is the collection

$$\begin{array}{rcl}
\widetilde{L}^{\mathcal{I}} & = & \{\sigma^\flat \in N^\flat : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \phi \in \mathrm{SEN}^\flat(\Sigma)) \\
 & & \quad (\sigma^\flat_\Sigma[\phi] \leq \bigcap\{\Omega(T) : \phi \in \widetilde{\overleftarrow{T}}_\Sigma\})\}.
\end{array}$$

From the definition, it is not difficult to see that, for any theory family T, $\widetilde{\overleftarrow{T}}$ is always included in $\widetilde{L}^{\mathcal{I}}(\Omega(T))$. This forms an analog in the rough left context of Proposition 1102.

Proposition 1123 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *For all* $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\widetilde{\overleftarrow{T}} \leq \widetilde{L}^{\mathcal{I}}(\Omega(T)).$$

Proof: Suppose $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \widetilde{\overleftarrow{T}}_\Sigma$, and $\sigma^\flat \in \widetilde{L}^{\mathcal{I}}$. Then, by the definition of $\widetilde{L}^{\mathcal{I}}$, $\sigma^\flat_\Sigma[\phi] \leq \Omega(T)$. Hence, $\widetilde{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. Thus, by definition of $\widetilde{L}^{\mathcal{I}}(\Omega(T))$, $\phi \in \widetilde{L}^{\mathcal{I}}_\Sigma(\Omega(T))$. Since Σ and $\phi \in \widetilde{\overleftarrow{T}}_\Sigma$ were arbitrary, we conclude that $\widetilde{\overleftarrow{T}} \leq \widetilde{L}^{\mathcal{I}}(\Omega(T))$. ∎

The reverse inclusion may or may not hold. If it does, for all $T \in \mathrm{ThFam}(\mathcal{I})$, we say that the rough left Suszko core of $\mathcal{I}$ is *left soluble*.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The rough left Suszko core $\widetilde{L}^{\mathcal{I}}$ of $\mathcal{I}$ is said to be **left soluble** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widetilde{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) \quad \text{implies} \quad \phi \in \widetilde{\overleftarrow{T}}_\Sigma.$$

As was the case with rough truth equationality (see Lemma 1104), it turns out that, if a given π-institution is roughly left truth equational, then any collection of witnessing equations must be included in the rough left Suszko core of $\mathcal{I}$. In other words, in case of rough left truth equationality, the rough left Suszko core forms a candidate for the largest collection of witnessing equations.

Lemma 1124 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is roughly left truth equational, with witnessing equations* $\tau^\flat$, *then* $\tau^\flat \subseteq \widetilde{L}^{\mathcal{I}}$.

Proof: Suppose $\mathcal{I}$ is roughly left truth equational, with witnessing equations $\tau^\flat$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \widetilde{\overleftarrow{T}}_\Sigma$. Then, by rough left truth equationality, $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Since T was arbitrary,

$$\tau^\flat_\Sigma[\phi] \leq \bigcap \{\Omega(T) : \phi \in \widetilde{\overleftarrow{T}}_\Sigma\}.$$

Hence, since $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$ were arbitrary, we conclude that $\tau^\flat \subseteq \widetilde{L}^{\mathcal{I}}$. ∎

We are now ready to prove the equivalence between rough left truth equationality and the left solubility of the rough left Suszko core. In the next theorem, we show that rough left truth equationality implies the left solubility of the rough left Suszko core. This forms an analog of Theorem 1105.

Theorem 1125 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly left truth equational, then* $\widetilde{L}^{\mathcal{I}}$ *is left soluble.*

Proof: Suppose $\mathcal{I}$ is roughly left truth equational, with witnessing equations $\tau^\flat$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\widetilde{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. Then, by rough left truth equationality and Lemma 1124, $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Again, using rough left truth equationality, we conclude that $\phi \in \overleftarrow{\widetilde{T}}_\Sigma$. This shows that $\widetilde{L}^{\mathcal{I}}$ is left soluble. ■

Conversely, in an analog of Theorem 1106, we show that the left solubility of the rough left Suszko core of a π-institution implies rough left truth equationality.

Theorem 1126 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\widetilde{L}^{\mathcal{I}}$ *is left soluble, then* $\mathcal{I}$ *is roughly left truth equational, with witnessing equations* $\widetilde{L}^{\mathcal{I}}$*.*

Proof: Assume $\widetilde{L}^{\mathcal{I}}$ is left soluble and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. We must show that

$$\phi \in \overleftarrow{\widetilde{T}}_\Sigma \quad \text{iff} \quad \widetilde{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

If $\phi \in \overleftarrow{\widetilde{T}}_\Sigma$, then, by Proposition **??**, $\phi \in \widetilde{L}^{\mathcal{I}}_\Sigma(\Omega(T))$, i.e., $\widetilde{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. On the other hand, the reverse inclusion is guaranteed by the postulated left solubility of $\widetilde{L}^{\mathcal{I}}$. Thus, $\mathcal{I}$ is indeed roughly left truth equational, with witnessing equations $\widetilde{L}^{\mathcal{I}}$. ■

Theorems 1125 and 1126 provide the first characterization of rough left truth equationality in terms of the left solubility of the rough left Suszko core. This parallels Theorem 1107, which asserted a similar characterization for rough truth equationality in terms of the solubility of the rough Suszko core of a π-institution.

$$\mathcal{I} \text{ Roughly Left Truth Equational } \longleftrightarrow \widetilde{L}^{\mathcal{I}} \text{ Left Soluble}$$

Theorem 1127 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is roughly left truth equational if and only if* $\widetilde{L}^{\mathcal{I}}$ *is left soluble.*

Proof: The "if" is by Theorem 1126. The "only if" by Theorem 1125. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the rough left Suszko core $\widetilde{L}^{\mathcal{I}}$ of $\mathcal{I}$ **roughly defines theory families up to arrow** if, for al $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\overleftarrow{\widetilde{T}} = \widetilde{L}^{\mathcal{I}}(\Omega(T)).$$

Another characterization of rough left truth equationality, along the lines of Theorem 1108, asserts that it is equivalent to the rough definability up to arrow of the theory families by the rough left Suszko core.

$$\mathcal{I} \text{ Roughly Left Truth Equational} \\ \longleftrightarrow \widetilde{L}^{\mathcal{I}} \text{ Roughly Defines Theory Families Up to Arrow}$$

Theorem 1128 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is roughly left truth equational if and only if* $\widetilde{L}^{\mathcal{I}}$ *roughly defines theory families in* $\mathcal{I}$ *up to arrow.*

Proof: Suppose $\mathcal{I}$ is roughly left truth equational. By Theorem 1125, $\widetilde{L}^{\mathcal{I}}$ is left soluble. Hence, by definition, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{L}^{\mathcal{I}}(\Omega(T)) \leq \overleftarrow{\overleftarrow{T}}$. Since, by Proposition 1123, the reverse always holds, we get, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{\overleftarrow{T}} = \widetilde{L}^{\mathcal{I}}(\Omega(T))$. Thus, $\widetilde{L}^{\mathcal{I}}$ roughly defines theory families in $\mathcal{I}$ up to arrow. Conversely, if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{\overleftarrow{T}} = \widetilde{L}^{\mathcal{I}}(\Omega(T))$, then $\widetilde{L}^{\mathcal{I}}$ is left soluble and, therefore, by Theorem 1126, $\mathcal{I}$ is roughly left truth equational. ■

We establish, next, a connection between rough left truth equationality and rough left c-reflectivity by means of the rough left Suszko core. To help us is this task, in analogy with the case of rough truth equationality, we introduce the notion of *left adequacy* of the rough left Suszko core. The following proposition, a "left" analog of Proposition 1113, motivates and, in a sense, justifies, the notion of left adequacy that will follow. Its role parallels that of Proposition 1113 in motivating the definition of adequacy of the rough Suszko core.

Proposition 1129 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\bigcap\{\Omega(T) : \widetilde{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \leq \bigcap\{\Omega(T) : \phi \in \overleftarrow{\overleftarrow{T}}_\Sigma\}.$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\phi \in \overleftarrow{\overleftarrow{T}}_\Sigma \quad \text{implies} \quad \widetilde{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T). \quad (\text{Definition of } \widetilde{L}^{\mathcal{I}})$$

Hence,

$$\bigcap\{\Omega(T) : \widetilde{L}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\} \quad \leq \quad \bigcap\{\Omega(T) : \phi \in \overleftarrow{\overleftarrow{T}}_\Sigma\}.$$

This is the displayed formula in the statement. ■

If the reverse inclusion of that proven in Proposition 1129 holds, then we say that the rough left Suszko core of $\mathcal{I}$ is *left adequate*.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the rough left Suszko core $\widetilde{L}^{\mathcal{I}}$ of $\mathcal{I}$ is **left adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\bigcap\{\Omega(T) : \phi \in \overleftarrow{\overleftarrow{T}}_\Sigma\} \le \bigcap\{\Omega(T) : \widetilde{L}^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)\}.$$

We can show, in analogy with Corollary 1114, that the left solubility of the rough left Suszko core implies left adequacy.

Corollary 1130 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\widetilde{L}^{\mathcal{I}}$ is left soluble, then it is left adequate.*

Proof: Suppose $\widetilde{L}^{\mathcal{I}}$ is left soluble and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, by left solubility and Proposition 1123, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\phi \in \overleftarrow{\overleftarrow{T}}_\Sigma$ if and only if $\widetilde{L}^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)$. Therefore,

$$\bigcap\{\Omega(T) : \phi \in \overleftarrow{\overleftarrow{T}}_\Sigma\} \quad = \quad \bigcap\{\Omega(T) : \widetilde{L}^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)\}.$$

Thus, $\widetilde{L}^{\mathcal{I}}$ is left adequate. ■

We prove, next, the converse of Corollary 1130, under the additional assumption that the π-institution $\mathcal{I}$ under consideration is roughly left c-reflective. This constitutes an analog of Proposition 1115.

Proposition 1131 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a roughly left c-reflective π-institution based on $\mathbf{F}$. If $\widetilde{L}^{\mathcal{I}}$ is left adequate, then it is left soluble.*

Proof: Suppose $\mathcal{I}$ is roughly left c-reflective and $\widetilde{L}^{\mathcal{I}}$ is left adequate. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\widetilde{L}^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)$. By the postulated left adequacy of $\widetilde{L}^{\mathcal{I}}$, we get that $\bigcap\{\Omega(T) : \phi \in \overleftarrow{\overleftarrow{T}}_\Sigma\} \le \Omega(T)$. By rough left truth equationality, $\bigcap\{\overleftarrow{\overleftarrow{T}} : \phi \in \overleftarrow{\overleftarrow{T}}_\Sigma\} \le \overleftarrow{\overleftarrow{T}}$. Therefore, $\phi \in \overleftarrow{\overleftarrow{T}}_\Sigma$. We conclude that $\widetilde{L}^{\mathcal{I}}$ is left soluble. ■

We are now in a position to prove the main characterization theorem relating rough left truth equationality with rough left c-reflectivity, an analog of Theorem 1116, which characterized rough truth equationality in terms of rough family c-reflectivity and the adequacy of the rough Suszko core.

Rough Left Truth Equationality	=	$\widetilde{L}^{\mathcal{I}}$ Left Soluble
	=	$\widetilde{L}^{\mathcal{I}}$ Roughly Defines Theory Families Up to Arrow
	=	Rough Left c-Reflectivity + $\widetilde{L}^{\mathcal{I}}$ Left Adequate

Theorem 1132 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is roughly left truth equational if and only if it is roughly left c-reflective and has a left adequate rough left Suszko core.*

Proof: Suppose, first, that $\mathcal{I}$ is roughly left truth equational. By Theorem 1122, it is roughly left c-reflective. By Theorem 1125, its rough left Suszko core is left soluble. Thus, by Corollary 1130, its rough left Suszko core is also left adequate.

Assume, conversely, that $\mathcal{I}$ is roughly left c-reflective and has a left adequate rough left Suszko core. Then, by Proposition 1131, its rough left Suszko core is also left soluble. Hence, by Theorem 1126, $\mathcal{I}$ is roughly left truth equational. ∎

Based on Proposition 1120 and Theorem 468, it is not difficult to show, in an analog of Corollary 1117, that the characterization theorem, Theorem 870, of left truth equationality in terms of left c-reflectivity and the left adequacy of the left Suszko core, can be inferred from Theorem 1132.

Corollary 1133 (Theorem 870) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is left truth equational if and only if it is left c-reflective and has a left adequate left Suszko core.*

Proof: $\mathcal{I}$ is left truth equational if and only if, by Proposition 1120, it is roughly left truth equational and has theorems, if and only if, by Theorem 1132, it is roughly left c-reflective, with a left adequate rough left Suszko core and has theorems, if and only if, by Theorem 468 and the definitions of left Suszko core and rough left Suszko core, it is left c-reflective and has a left adequate left Suszko core. ∎

We close the section by looking at a result, an analog of Theorem 1118, which may be perceived either as an alternative characterization of rough left truth equationality, involving arbitrary $\mathbf{F}$-algebraic systems, or as a transfer theorem.

Theorem 1134 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is roughly left truth equational, with witnessing equations* $\tau^\flat$*, if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi \in \mathrm{SEN}(\Sigma)$*,*

$$\phi \in \overset{\approx}{\overleftarrow{T}}_\Sigma \quad \text{iff} \quad \tau^{\mathcal{A}}_\Sigma[\phi] \leq \Omega^{\mathcal{A}}(T).$$

Proof: If the postulated condition holds, then it holds, in particular, for the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. This yields immediately that $\mathcal{I}$ is roughly left truth equational.

Suppose, conversely, that $\mathcal{I}$ is roughly left truth equational and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rll}
\alpha_\Sigma(\phi) \in \widetilde{\widetilde{\overleftarrow{T}}}_{F(\Sigma)} & \text{iff} & \phi \in \alpha_\Sigma^{-1}(\underset{\sim}{\widetilde{\overleftarrow{T}}_{F(\Sigma)}}) \\
& \text{iff} & \phi \in \alpha_\Sigma^{-1}(\underset{\sim}{\overleftarrow{T}_{F(\Sigma)}}) \quad \text{(Theorem 377)} \\
& \text{iff} & \phi \in \overleftarrow{\alpha_\Sigma^{-1}(T_{F(\Sigma)})} \quad \text{(Lemma 6)} \\
& \text{iff} & \tau_\Sigma^\flat[\phi] \leq \Omega(\alpha^{-1}(T)) \quad \text{(hypothesis)} \\
& \text{iff} & \tau_\Sigma^\flat[\phi] \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T)) \quad \text{(Proposition 24)} \\
& \text{iff} & \tau_{F(\Sigma)}^{\mathcal{A}}[\alpha_\Sigma(\phi)] \leq \Omega^{\mathcal{A}}(T). \quad \text{(Lemma 95)}
\end{array}$$

Hence, taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that the displayed condition holds. ■

14.3 Narrow Left Truth Equationality

We now turn to *narrow left truth equationality*. As the terminology suggests:

- It is in the same relation to narrow left c-reflectivity as rough left truth equationality is to rough left c-reflectivity;
- It is in the same relation to rough/narrow truth equationality as left truth equationality is to truth equationality.

In a nutshell, narrow left truth equationality is defined analogously to left truth equationality, but care is taken to bypass theory families with empty components.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that $\mathcal{I}$ is **narrowly left truth equational** if there exists $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with a single distinguished argument, such that, for all $T \in \mathrm{ThFam}^{\nsubseteq}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{iff} \quad \tau_\Sigma^\flat[\phi] \leq \Omega(T).$$

The collection $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ is referred to as a set of **witnessing equations** (of/for the narrow left truth equationality of $\mathcal{I}$).

An alternative characterization quantifies the relevant condition over all theory families, but it does so at the expense of using the rough operator on one side (and implicitly also on the other).

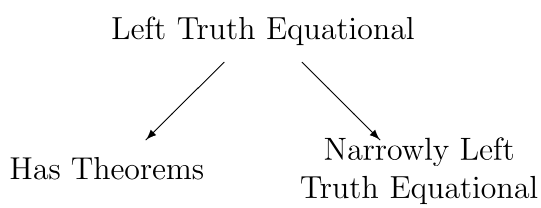
Lemma 1135 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is narrowly left truth equational if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\phi \in \overleftarrow{\widetilde{T}}_\Sigma \quad \textit{iff} \quad \tau_\Sigma^\flat[\phi] \leq \Omega(T).$$

Proof: Suppose, first, that $\mathcal{I}$ is narrowly left truth equational, with witnessing equations $\tau^\flat$, and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, since $\widetilde{T} \in \mathrm{ThFam}^{\not\emptyset}(\mathcal{I})$, we get, by hypothesis, $\phi \in \overleftarrow{\widetilde{T}}_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \le \Omega(\widetilde{T})$. Therefore, by Proposition 369, $\phi \in \overleftarrow{\widetilde{T}}_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \le \Omega(T)$.

Suppose, conversely, that the displayed equivalence holds and let $T \in \mathrm{ThFam}^{\not\emptyset}(\mathcal{I})$. Then $\widetilde{T} = T$. Thus, by hypothesis, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in \overleftarrow{T}_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \le \Omega(T)$. Therefore, $\mathcal{I}$ is narrowly left truth equational. ■

The following relationship between rough left truth equationality and left truth equationality, an analog of the relationship between rough truth equationality and truth equationality, presented in Corollary 1098, holds. Note that narrow left truth equationality is in the same relationship to left truth equationality as rough left truth equationality is to left truth equationality, as detailed in Proposition 1120.



Proposition 1136 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is left truth equational if and only if it is narrowly left truth equational and has theorems.*

Proof: Suppose, first, that $\mathcal{I}$ is narrowly left truth equational, with witnessing equations $\tau^\flat$, and that it has theorems. Availability of theorems implies that $\mathrm{ThFam}^{\not\emptyset}(\mathcal{I}) = \mathrm{ThFam}(\mathcal{I})$. Thus, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in \overleftarrow{T}_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \le \Omega(T)$. Thus, $\mathcal{I}$ is left truth equational, with the same witnessing equations $\tau^\flat$.

Assume, conversely, that $\mathcal{I}$ is left truth equational, with witnessing equations $\tau^\flat$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in \overleftarrow{T}_\Sigma$ iff $\tau^\flat_\Sigma[\phi] \le \Omega(T)$. This clearly implies that $\mathcal{I}$ has theorems, since, otherwise, given that $\Omega(\overline{\varnothing}) = \nabla^{\mathbf{F}} = \Omega(\mathrm{SEN}^\flat)$, we would get $\mathrm{SEN}^\flat = \overleftarrow{\mathrm{SEN}^\flat} = \overleftarrow{\overline{\varnothing}} = \overline{\varnothing}$, a contradiction. Moreover, since $\mathrm{ThFam}^{\not\emptyset}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I})$, left truth equationality implies trivially narrow left truth equationality. ■

Our next goal is to prove an analog of the characterizations, Theorem 860 and Proposition 1121, of left truth equationality and rough left truth equationality, respectively, for narrow left truth equationality.

Narrow left truth equationality allows an expression for $\overleftarrow{T}$, for all theory families T without empty components, or alternatively, for $\overleftarrow{\widetilde{T}}$, for all theory families T, in terms of the Leibniz congruence system of T. The following proposition forms an analog of Proposition 1121.

Proposition 1137 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then the following statements are equivalent:*

(i) $\mathcal{I}$ *is narrowly left truth equational, with witnessing equations* $\tau^\flat$*;*

(ii) For all $T \in \mathrm{ThFam}^{\not\emptyset}(\mathcal{I})$, $\overleftarrow{T} = \tau^\flat(\Omega(T))$*;*

(iii) For all $T \in \mathrm{ThFam}(\mathcal{I})$, $\overleftarrow{\widetilde{T}} = \tau^\flat(\Omega(T))$*.*

Proof: Suppose, first, that $\mathcal{I}$ is narrowly left truth equational, with witnessing equations $\tau^\flat$, and let $T \in \mathrm{ThFam}^{\not\emptyset}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{lll} \phi \in \tau^\flat_\Sigma(\Omega(T)) & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(T) \quad \text{(definition)} \\ & \text{iff} & \phi \in \overleftarrow{T}_\Sigma. \quad \text{(hypothesis)} \end{array}$$

Suppose, next, that Condition (ii) holds and let $T \in \mathrm{ThFam}(\mathcal{I})$. Then $\widetilde{T} \in \mathrm{ThFam}^{\not\emptyset}(\mathcal{I})$, whence, by hypothesis, $\overleftarrow{\widetilde{T}} = \tau^\flat(\Omega(\widetilde{T})) = \tau^\flat(\Omega(T))$, where the last equality holds by Proposition 369. Finally, suppose that Condition (iii) holds and let $T \in \mathrm{ThFam}^{\not\emptyset}(\mathcal{I})$. Then $\widetilde{T} = T$, whence, we get, by hypothesis, $\overleftarrow{T} = \tau^\flat(\Omega(T))$, showing that $\mathcal{I}$ is narrowly left truth equational. ■

Recall from Chapter 6 that, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, $\mathcal{I}$ is called *narrowly left c-reflective* if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\not\emptyset}(\mathcal{I})$,

$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \quad \text{implies} \quad \bigcap_{T \in \mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}.$$

We are now able to show that narrow left truth equationality implies narrow left c-reflectivity. This is an analog of Theorem 1122.

Theorem 1138 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is narrowly left truth equational, then it is narrowly left c-reflective.*

Proof: Suppose $\mathcal{I}$ is narrowly left truth equational, with witnessing equations $\tau^\flat$. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\not\emptyset}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. Then we have

$$\begin{array}{lll} \bigcap_{T \in \mathcal{T}} \overleftarrow{T} & = & \bigcap_{T \in \mathcal{T}} \tau^\flat(\Omega(T)) \quad \text{(Proposition 1137)} \\ & = & \tau^\flat(\bigcap_{T \in \mathcal{T}} \Omega(T)) \quad \text{(set theory)} \\ & \leq & \tau^\flat(\Omega(T')) \quad \text{(hypothesis)} \\ & = & \overleftarrow{T'}. \quad \text{(Proposition 1137)} \end{array}$$

Thus, $\mathcal{I}$ is narrowly left c-reflective. ■

In the context of narrow left truth equationality, the notion paralleling the left Suszko core is the *narrow left Suszko core*, a modification of the original, which is defined below.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The **narrow left Suszko core** $L^{\mathcal{I}\nmid}$ of $\mathcal{I}$ is the collection

$$L^{\mathcal{I}\nmid} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}^{\nmid}(\mathcal{I}))(\sigma^\flat[\overleftarrow{T}] \leq \widetilde{\Omega}^{\mathcal{I}}(T))\}.$$

From the definition, it is not difficult to see that, for any theory family T, with all components nonempty, $\overleftarrow{T}$ is always included in $L^{\mathcal{I}\nmid}(\Omega(T))$. This forms an analog in the narrow left context of Propositions 1102 and 1123.

Proposition 1139 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$*,*

$$\overleftarrow{T} \leq L^{\mathcal{I}\nmid}(\Omega(T)).$$

Proof: Suppose $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \overleftarrow{T}_\Sigma$, and $\sigma^\flat \in L^{\mathcal{I}\nmid}$. Then, by the definition of $L^{\mathcal{I}\nmid}$, $\sigma^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T) \leq \Omega(T)$. Hence, $L^{\mathcal{I}\nmid}_\Sigma[\phi] \leq \Omega(T)$. Thus, by definition of $L^{\mathcal{I}\nmid}(\Omega(T))$, $\phi \in L^{\mathcal{I}\nmid}_\Sigma(\Omega(T))$. Since Σ and $\phi \in \overleftarrow{T}_\Sigma$ were arbitrary, we conclude that $\overleftarrow{T} \leq L^{\mathcal{I}\nmid}(\Omega(T))$. ■

The reverse inclusion may or may not hold. If it does, for all $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, we say that the narrow left Suszko core of $\mathcal{I}$ is *left soluble*.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The narrow left Suszko core $L^{\mathcal{I}\nmid}$ of $\mathcal{I}$ is said to be **left soluble** if, for all $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$L^{\mathcal{I}\nmid}_\Sigma[\phi] \leq \Omega(T) \quad \text{implies} \quad \phi \in \overleftarrow{T}_\Sigma.$$

As was the case with rough left truth equationality (see Lemma 1124), it turns out that, if a given π-institution is narrowly left truth equational, then any collection of witnessing equations must be included in the narrow left Suszko core of $\mathcal{I}$; differently put, in case of narrow left truth equationality, the narrow left Suszko core forms a candidate for the largest collection of witnessing equations.

Lemma 1140 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is narrowly left truth equational, with witnessing equations* $\tau^\flat$*, then* $\tau^\flat \subseteq L^{\mathcal{I}\nmid}$*.*

Proof: Suppose $\mathcal{I}$ is narrowly left truth equational, with witnessing equations $\tau^\flat$. Let $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \overleftarrow{T}_\Sigma$. Then, for all $T \leq T' \in \mathrm{ThFam}(\mathcal{I})$, $\phi \in \overleftarrow{T'}_\Sigma$, whence, by narrow left truth equationality, $\tau^\flat_\Sigma[\phi] \leq \Omega(T')$. Since T', with the postulated properties was arbitrary,

$$\tau^\flat_\Sigma[\phi] \leq \bigcap\{\Omega(T') : T \leq T' \in \mathrm{ThFam}(\mathcal{I})\} = \widetilde{\Omega}^{\mathcal{I}}(T).$$

Hence, $\tau^\flat[\overleftarrow{T}] \leq \widetilde{\Omega}^{\mathcal{I}}(T)$. Since $T \in \mathrm{ThFam}^\natural(\mathcal{I})$ was arbitrary, we conclude that $\tau^\flat \subseteq L^{\mathcal{I}\natural}$. ∎

We are now ready to prove the equivalence between narrow left truth equationality and the left solubility of the narrow left Suszko core. In the next theorem, we show that narrow left truth equationality implies the left solubility of the narrow left Suszko core. This forms an analog of Theorem 1125.

Theorem 1141 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is narrowly left truth equational, then* $L^{\mathcal{I}\natural}$ *is left soluble.*

Proof: Suppose $\mathcal{I}$ is narrowly left truth equational, with witnessing equations $\tau^\flat$. Let $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $L^{\mathcal{I}\natural}_\Sigma[\phi] \leq \Omega(T)$. Then, by narrow left truth equationality and Lemma 1140, $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Again, using narow left truth equationality, we conclude that $\phi \in \overleftarrow{T}_\Sigma$. This shows that $L^{\mathcal{I}\natural}$ is left soluble. ∎

Conversely, in an analog of Theorem 1126, we show that the left solubility of the narrow left Suszko core of a π-institution implies narrow left truth equationality.

Theorem 1142 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $L^{\mathcal{I}\natural}$ *is left soluble, then* $\mathcal{I}$ *is narrowly left truth equational, with witnessing equations* $L^{\mathcal{I}\natural}$.

Proof: Assume $L^{\mathcal{I}\natural}$ is left soluble and let $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. We must show that

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{iff} \quad L^{\mathcal{I}\natural}_\Sigma[\phi] \leq \Omega(T).$$

If $\phi \in \overleftarrow{T}_\Sigma$, then, by Proposition 1139, $\phi \in L^{\mathcal{I}\natural}_\Sigma(\Omega(T))$, i.e., $L^{\mathcal{I}\natural}_\Sigma[\phi] \leq \Omega(T)$. On the other hand, the reverse inclusion is guaranteed by the postulated left solubility of $L^{\mathcal{I}\natural}$. Thus, $\mathcal{I}$ is indeed narrowly left truth equational, with witnessing equations $L^{\mathcal{I}\natural}$. ∎

Theorems 1141 and 1142 provide the first characterization of narrow left truth equationality in terms of the left solubility of the narrow left Suszko core. This parallels Theorem 1127, which asserted a similar characterization for rough left truth equationality in terms of the left solubility of the rough left Suszko core of a π-institution.

$$\mathcal{I} \text{ Narrowly Left Truth Equational } \longleftrightarrow L^{\mathcal{I}\nmid} \text{ Left Soluble}$$

Theorem 1143 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is narrowly left truth equational if and only if* $L^{\mathcal{I}\nmid}$ *is left soluble.*

Proof: The "if" is by Theorem 1142. The "only if" by Theorem 1141. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the narrow left Suszko core $L^{\mathcal{I}\nmid}$ of $\mathcal{I}$ **narrowly defines theory families up to arrow** if, for al $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$,

$$\overleftarrow{T} = L^{\mathcal{I}\nmid}(\Omega(T)).$$

Another characterization of narrow left truth equationality, along the lines of Theorem 1128, asserts that it is equivalent to the narrow definability up to arrow of the theory families by the narrow left Suszko core.

$$\begin{array}{l} \mathcal{I} \text{ Narrowly Left Truth Equational} \\ \qquad \longleftrightarrow L^{\mathcal{I}\nmid} \text{ Narrowly Defines Theory} \\ \qquad\qquad \text{Families Up to Arrow} \end{array}$$

Theorem 1144 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is narrowly left truth equational if and only if* $L^{\mathcal{I}\nmid}$ *narrowly defines theory families in* $\mathcal{I}$ *up to arrow.*

Proof: Suppose $\mathcal{I}$ is narrowly left truth equational. By Theorem 1141, $L^{\mathcal{I}\nmid}$ is left soluble. Hence, by definition, for all $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $L^{\mathcal{I}\nmid}(\Omega(T)) \leq \overleftarrow{T}$. Since, by Proposition 1139, the reverse always holds, we get, for all $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $\overleftarrow{T} = L^{\mathcal{I}\nmid}(\Omega(T))$. Thus, $L^{\mathcal{I}\nmid}$ narrowly defines theory families in $\mathcal{I}$ up to arrow. Conversely, if, for all $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $\overleftarrow{T} = L^{\mathcal{I}\nmid}(\Omega(T))$, then $L^{\mathcal{I}\nmid}$ is left soluble and, therefore, by Theorem 1142, $\mathcal{I}$ is narrowly left truth equational. ∎

We would like, next to establish a connection between narrow left truth equationality and narrow left c-reflectivity by means of the narrow left Suszko core. To accomplish this, we introduce an apparently modified version of the Suszko operator, which, however, is identical to the Suszko operator itself. This modified version is convenient for the purpose of handling proofs in a more straightforward and efficient way.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We define the **narrow Suszko operator** $\widetilde{\Omega}^{\mathcal{I}\nleq}$ by setting, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\widetilde{\Omega}^{\mathcal{I}\nleq}(T) = \bigcap\{\Omega(T') : T \leq T' \in \mathrm{ThFam}^{\nleq}(\mathcal{I})\}.$$

By Corollary 1111, we have, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\Omega}^{\mathcal{I}\nleq}(T) = \widetilde{\Omega}^{\mathcal{I}}(T)$. So this is indeed an apparent and not a substantial change and one can think, without any loss, of $\widetilde{\Omega}^{\mathcal{I}\nleq}$ as the Suszko operator.

In analogy with the case of rough truth equationality and rough left truth equationality, we may introduce the notion of *left adequacy* of the narrow left Suszko core, which will help in characterizing the relationship between narrow left truth equationality and narrow left c-reflectivity. The following proposition, a "left" analog of Proposition 1113 and an analog of Proposition 1129, justifies the notion of left adequacy that will follow.

Proposition 1145 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,*

$$\bigcap\{\Omega(T) : L^{\mathcal{I}\nleq}_\Sigma[\phi] \leq \Omega(T)\} \leq \widetilde{\Omega}^{\mathcal{I}\nleq}(C(\overrightarrow{\phi})).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$,

$$\begin{array}{llll}\phi \in \overleftarrow{T}_\Sigma & \text{implies} & L^{\mathcal{I}\nleq}_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T) & (\text{Definition of } L^{\mathcal{I}\nleq}) \\ & \text{implies} & L^{\mathcal{I}\nleq}_\Sigma[\phi] \leq \Omega(T). & (\widetilde{\Omega}^{\mathcal{I}} \leq \Omega)\end{array}$$

Hence,

$$\begin{array}{l}\bigcap\{\Omega(T) : L^{\mathcal{I}\nleq}_\Sigma[\phi] \leq \Omega(T)\} \\ \quad = \bigcap\{\Omega(T) : T \in \mathrm{ThFam}^{\nleq}(\mathcal{I}) \text{ and } L^{\mathcal{I}\nleq}_\Sigma[\phi] \leq \Omega(T)\} \\ \quad \leq \bigcap\{\Omega(T) : T \in \mathrm{ThFam}^{\nleq}(\mathcal{I}) \text{ and } L^{\mathcal{I}\nleq}_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T)\} \\ \quad \leq \bigcap\{\Omega(T) : T \in \mathrm{ThFam}^{\nleq}(\mathcal{I}) \text{ and } \phi \in \overleftarrow{T}_\Sigma\} \\ \quad = \bigcap\{\Omega(T) : T \in \mathrm{ThFam}^{\nleq}(\mathcal{I}) \text{ and } \overrightarrow{\phi} \leq T\} \\ \quad = \widetilde{\Omega}^{\mathcal{I}\nleq}(C(\overrightarrow{\phi})).\end{array}$$

This is the displayed formula in the statement. ■

If the reverse inclusion of that proven in Proposition 1145 holds, then we say that the narrow left Suszko core of $\mathcal{I}$ is *left adequate*.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the narrow left Suszko core $L^{\mathcal{I}\nleq}$ of $\mathcal{I}$ is **left adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widetilde{\Omega}^{\mathcal{I}\nleq}(C(\overrightarrow{\phi})) \leq \bigcap\{\Omega(T) : L^{\mathcal{I}\nleq}_\Sigma[\phi] \leq \Omega(T)\}.$$

We can show, in analogy with Corollary 1130, that the left solubility of the narrow left Suszko core implies left adequacy.

Corollary 1146 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $L^{\mathcal{I}\nmid}$ *is left soluble, then it is left adequate.*

Proof: Suppose $L^{\mathcal{I}\nmid}$ is left soluble and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{aligned}
\widetilde{\Omega}^{\mathcal{I}\nmid}(C(\overrightarrow{\phi})) &= \bigcap\{\Omega(T) : T \in \mathrm{ThFam}^{\nmid}(\mathcal{I}) \text{ and } \overrightarrow{\phi} \leq T\} \\
&\quad \text{(definition of } \widetilde{\Omega}^{\mathcal{I}\nmid}) \\
&= \bigcap\{\Omega(T) : T \in \mathrm{ThFam}^{\nmid}(\mathcal{I}) \text{ and } \phi \in \overleftarrow{T}_\Sigma\} \\
&\quad \text{(Definition of } \overrightarrow{\phi} \text{ and } \overleftarrow{T}) \\
&= \bigcap\{\Omega(T) : T \in \mathrm{ThFam}^{\nmid}(\mathcal{I}) \text{ and } L^{\mathcal{I}\nmid}_\Sigma[\phi] \leq \Omega(T)\} \\
&\quad \text{(Left solubility of } L^{\mathcal{I}\nmid}) \\
&= \bigcap\{\Omega(T) : L^{\mathcal{I}\nmid}_\Sigma[\phi] \leq \Omega(T)\}. \quad \text{(Lemma 1110)}
\end{aligned}$$

Thus, $L^{\mathcal{I}\nmid}$ is left adequate. ∎

In order to prove a partial converse of Corollary 1146, we will employ the following characterization of narrow left c-reflectivity.

Lemma 1147 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is narrowly left c-reflective if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$ *and all* $T' \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$*,*

$$\widetilde{\Omega}^{\mathcal{I}\nmid}(T) \leq \Omega(T') \quad \textit{implies} \quad \overleftarrow{T} \leq \overleftarrow{T'}.$$

Proof: Suppose, first, that $\mathcal{I}$ is narrowly left c-reflective and let $T \in \mathrm{ThFam}(\mathcal{I})$ and $T' \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, such that $\widetilde{\Omega}^{\mathcal{I}\nmid}(T) \leq \Omega(T')$. Then, by definition,

$$\bigcap\{\Omega(T'') : T \leq T'' \in \mathrm{ThFam}^{\nmid}(\mathcal{I})\} \leq \Omega(T').$$

Hence, by narrow left c-reflectivity, $\bigcap\{\overleftarrow{T''} : T \leq T'' \in \mathrm{ThFam}^{\nmid}(\mathcal{I})\} \leq \overleftarrow{T'}$. However, $T \leq T''$ implies that $\overleftarrow{T} \leq \overleftarrow{T''}$. Hence, we obtain $\overleftarrow{T} \leq \overleftarrow{T'}$.

Suppose, conversely, that the displayed condition in the statement holds and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}^{\nmid}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. Then, since $\mathcal{T} \subseteq \mathrm{ThFam}^{\nmid}(\mathcal{I})$, we get that

$$\bigcap\{\Omega(T'') : \bigcap\mathcal{T} \leq T'' \in \mathrm{ThFam}^{\nmid}(\mathcal{I})\} \leq \Omega(T').$$

By definition, then, $\widetilde{\Omega}^{\mathcal{I}\nmid}(\bigcap\mathcal{T}) \leq \Omega(T')$, whence, by hypothesis, $\overleftarrow{\bigcap\mathcal{T}} \leq \overleftarrow{T'}$. Therefore, $\bigcap_{T\in\mathcal{T}} \overleftarrow{T} \leq \overleftarrow{T'}$. This shows that $\mathcal{I}$ is narrowly left c-reflective. ∎

We prove, next, the converse of Corollary 1146, under the additional assumption that the π-institution $\mathcal{I}$ under consideration is narrowly left c-reflective. This constitutes an analog of Proposition 1131.

Proposition 1148 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a narrowly left c-reflective* π*-institution based on* $\mathbf{F}$. *If* $L^{\mathcal{I}\natural}$ *is left adequate, then it is left soluble.*

Proof: Suppose $\mathcal{I}$ is narrowly left c-reflective and $L^{\mathcal{I}\natural}$ is left adequate. Let $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $L^{\mathcal{I}\natural}_\Sigma[\phi] \leq \Omega(T)$. By the postulated left adequacy of $L^{\mathcal{I}\natural}$, we get that $\widetilde{\Omega}^{\mathcal{I}\natural}(C(\overrightarrow{\phi})) \leq \Omega(T)$. By narrow left truth equationality and Lemma 1147, $\overleftarrow{C(\overrightarrow{\phi})} \leq \overleftarrow{T}$. Therefore, $\phi \in \overleftarrow{T}_\Sigma$. We conclude that $L^{\mathcal{I}\natural}$ is left soluble. ■

We are now in a position to prove the main characterization theorem relating narrow left truth equationality with narrow left c-reflectivity, an analog of Theorem 1132, which characterized rough left truth equationality in terms of rough left c-reflectivity and the adequacy of the rough left Suszko core.

$$\begin{array}{rcl}\text{Narrow Left Truth Equationality} & = & L^{\mathcal{I}\natural} \text{ Left Soluble}\\ & = & L^{\mathcal{I}\natural} \text{ Narrowly Defines Theory}\\ & & \quad\text{Families Up to Arrow}\\ & = & \text{Narrow Left c-Reflectivity}\\ & & + \; L^{\mathcal{I}\natural} \text{ Left Adequate}\end{array}$$

Theorem 1149 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is narrowly left truth equational if and only if it is narrowly left c-reflective and has a left adequate narrow left Suszko core.*

Proof: Suppose, first, that $\mathcal{I}$ is narrowly left truth equational. By Theorem 1138, it is narrowly left c-reflective. By Theorem 1141, its narrow left Suszko core is left soluble. Thus, by Corollary 1146, its narrow left Suszko core is also left adequate.

Assume, conversely, that $\mathcal{I}$ is narrowly left c-reflective and has a left adequate narrow left Suszko core. Then, by Proposition 1148, its narrow left Suszko core is also left soluble. Hence, by Theorem 1142, $\mathcal{I}$ is narrowly left truth equational. ■

Based on Proposition 1136 and Theorem 468, it is not difficult to show, in an analog of Corollary 1133, that the characterization theorem, Theorem 870, of left truth equationality in terms of left c-reflectivity and the left adequacy of the left Suszko core, can be inferred from Theorem 1149.

Corollary 1150 (Theorem 870) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is left truth equational if and only if it is left c-reflective and has a left adequate left Suszko core.*

Proof: $\mathcal{I}$ is left truth equational if and only if, by Proposition 1136, it is narrowly left truth equational and has theorems, if and only if, by Theorem 1149, it is narrowly left c-reflective, with a left adequate narrow left Suszko core and has theorems, if and only if, by Theorem 468 and the definitions of left Suszko core and narrow left Suszko core, it is left c-reflective and has a left adequate left Suszko core. ■

We close the section by looking at a result, an analog of Theorem 1134, which may be perceived either as an alternative characterization of narrow left truth equationality, involving arbitrary **F**-algebraic systems, or as a transfer theorem.

Theorem 1151 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is narrowly left truth equational, with witnessing equations* $\tau^\flat$*, if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, all* $T \in \mathrm{FiFam}^{\mathcal{I}\natural}(\mathcal{A})$*, all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi \in \mathrm{SEN}(\Sigma)$*,*

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{iff} \quad \tau_\Sigma^{\mathcal{A}}[\phi] \leq \Omega^{\mathcal{A}}(T).$$

Proof: If the postulated condition holds, then it holds, in particular, for the **F**-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. This yields immediately that $\mathcal{I}$ is narrowly left truth equational.

Suppose, conversely, that $\mathcal{I}$ is narrowly left truth equational and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an **F**-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}\natural}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{lll}
\alpha_\Sigma(\phi) \in \overleftarrow{T}_{F(\Sigma)} & \text{iff} & \phi \in \alpha_\Sigma^{-1}(\overleftarrow{T}_{F(\Sigma)}) \\
& \text{iff} & \phi \in \overleftarrow{\alpha_\Sigma^{-1}(T_{F(\Sigma)})} \quad \text{(Lemma 6)} \\
& \text{iff} & \tau_\Sigma^\flat[\phi] \leq \Omega(\alpha^{-1}(T)) \quad \text{(hypothesis)} \\
& \text{iff} & \tau_\Sigma^\flat[\phi] \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T)) \quad \text{(Proposition 24)} \\
& \text{iff} & \tau_{F(\Sigma)}^{\mathcal{A}}[\alpha_\Sigma(\phi)] \leq \Omega^{\mathcal{A}}(T). \quad \text{(Lemma 95)}
\end{array}$$

Hence, taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that the displayed condition holds. ■

14.4 Rough System Truth Equationality

We now turn to *rough system truth equationality*. As the terminology suggests:

- It is in the same relation to rough system c-reflectivity as rough left truth equationality is to rough left c-reflectivity;
- It is in the same relation to rough left truth equationality as system truth equationality is to left truth equationality.

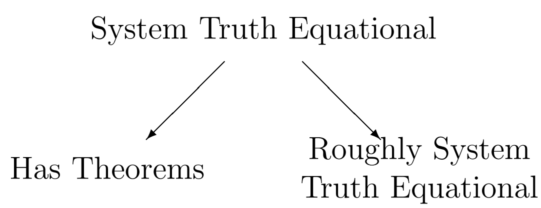
Roughly speaking, rough system truth equationality is defined analogously to system truth equationality, but it is applied to rough representatives of theory systems so as to avoid theory systems with empty components.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that $\mathcal{I}$ is **roughly system truth equational** if there exists $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with a single distinguished argument, such that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in \widetilde{T}_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

The collection $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ is referred to as a set of **witnessing equations** (of/for the rough system truth equationality of $\mathcal{I}$).

The following relationship between rough system truth equationality and system truth equationality, an analog of the relationship between rough truth equationality and truth equationality, presented in Corollary 1098, holds.



Proposition 1152 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is system truth equational if and only if it is roughly system truth equational and has theorems.*

Proof: Suppose, first, that $\mathcal{I}$ is roughly system truth equational, with witnessing equations $\tau^\flat$, and that it has theorems. Availability of theorems implies that $\mathrm{ThSys}^{\lightning}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$. Thus, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in T_\Sigma$ if and only if $\phi \in \widetilde{T}_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Thus, $\mathcal{I}$ is system truth equational, with the same witnessing equations $\tau^\flat$.

Assume, conversely, that $\mathcal{I}$ is system truth equational, with witnessing equations $\tau^\flat$. Then, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in T_\Sigma$ iff $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. This clearly implies that $\mathcal{I}$ has theorems. Moreover, due to the availability of theorems, we get, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in \widetilde{T}_\Sigma$ if and only if $\phi \in T_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Thus, $\mathcal{I}$ is roughly system truth equational. ■

Our next goal is to prove an analog of the characterization theorem, Theorem 1127, of rough left truth equationality in terms of the left solubility of the rough left Suszko core for rough system truth equationality.

Rough system truth equationality allows an expression for $\widetilde{T}$, for all theory systems T, in terms of the Leibniz congruence system of T. The following proposition forms an analog of Proposition 1121.

Proposition 1153 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is roughly system truth equational, with witnessing equations* $\tau^\flat$*, if and only if, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,* $\widetilde{T} = \tau^\flat(\Omega(T))$*.*

Proof: $\mathcal{I}$ is roughly system truth equational, with witnessing equations $\tau^\flat$, if and only if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in \widetilde{T}_\Sigma$ iff $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$, if and only if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\widetilde{T} = \tau^\flat(\Omega(T))$. ■

Recall from Chapter 6 that, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, $\mathcal{I}$ is called *roughly system c-reflective* if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$,

$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \quad \text{implies} \quad \bigcap_{T \in \mathcal{T}} \widetilde{T} \leq \widetilde{T'}.$$

We are now able to show that rough system truth equationality implies rough system c-reflectivity. This is an analog of Theorem 1122.

Theorem 1154 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is roughly system truth equational, then it is roughly system c-reflective.*

Proof: Suppose $\mathcal{I}$ is roughly system truth equational, with witnessing equations $\tau^\flat$. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. Then we have

$$\begin{array}{rcll} \bigcap_{T \in \mathcal{T}} \widetilde{T} & = & \bigcap_{T \in \mathcal{T}} \tau^\flat(\Omega(T)) & \text{(Proposition 1153)} \\ & = & \tau^\flat(\bigcap_{T \in \mathcal{T}} \Omega(T)) & \text{(set theory)} \\ & \leq & \tau^\flat(\Omega(T')) & \text{(hypothesis)} \\ & = & \widetilde{T'}. & \text{(Proposition 1153)} \end{array}$$

Thus, $\mathcal{I}$ is roughly system c-reflective. ■

In the context of rough system truth equationality, the notion paralleling the rough left Suszko core is the *rough system core*, a modification of the system core, which is defined below.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The **rough system core** $\widetilde{Z}^{\mathcal{I}}$ of $\mathcal{I}$ is the collection

$$\begin{array}{rcl} \widetilde{Z}^{\mathcal{I}} & = & \{\sigma^\flat \in N^\flat : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \phi \in \mathrm{SEN}^\flat(\Sigma)) \\ & & \quad (\sigma^\flat_\Sigma[\phi] \leq \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \phi \in \widetilde{T}_\Sigma\})\}. \end{array}$$

From the definition, it is not difficult to see that, for any theory system T, $\widetilde{T}$ is always included in $\widetilde{Z}^{\mathcal{I}}(\Omega(T))$. This forms an analog in the rough left context of Proposition 1123.

Proposition 1155 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $T \in \mathrm{ThSys}(\mathcal{I})$*,*

$$\widetilde{T} \leq \widetilde{Z}^{\mathcal{I}}(\Omega(T)).$$

Proof: Suppose $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \widetilde{T}_\Sigma$, and $\sigma^\flat \in \widetilde{Z}^{\mathcal{I}}$. Then, by the definition of $\widetilde{Z}^{\mathcal{I}}$, $\sigma^\flat_\Sigma[\phi] \leq \Omega(T)$. Hence, $\widetilde{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. Thus, by definition of $\widetilde{Z}^{\mathcal{I}}(\Omega(T))$, $\phi \in \widetilde{Z}^{\mathcal{I}}_\Sigma(\Omega(T))$. Since Σ and $\phi \in \widetilde{T}_\Sigma$ were arbitrary, we conclude that $\widetilde{T} \leq \widetilde{Z}^{\mathcal{I}}(\Omega(T))$. ∎

The reverse inclusion may or may not hold. If it does, for all $T \in \mathrm{ThSys}(\mathcal{I})$, we say that the rough system core of $\mathcal{I}$ is *soluble*.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The rough system core $\widetilde{Z}^{\mathcal{I}}$ of $\mathcal{I}$ is said to be **soluble** if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widetilde{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) \quad \text{implies} \quad \phi \in \widetilde{T}_\Sigma.$$

As was the case with rough left truth equationality (see Lemma 1124), if a given π-institution is roughly system truth equational, then any collection of witnessing equations must be included in the rough system core of $\mathcal{I}$. In other words, in case of rough system truth equationality, the rough system core forms a candidate for the largest collection of witnessing equations.

Lemma 1156 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is roughly system truth equational, with witnessing equations $\tau^\flat$, then $\tau^\flat \subseteq \widetilde{Z}^{\mathcal{I}}$.*

Proof: Suppose $\mathcal{I}$ is roughly system truth equational, with witnessing equations $\tau^\flat$. Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \widetilde{T}_\Sigma$. Then, by rough system truth equationality, $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Since T was arbitrary,

$$\tau^\flat_\Sigma[\phi] \leq \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \phi \in \widetilde{T}_\Sigma\}.$$

Hence, since $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$ were arbitrary, we conclude that $\tau^\flat \subseteq \widetilde{Z}^{\mathcal{I}}$. ∎

We are now ready to prove the equivalence between rough system truth equationality and the solubility of the rough system core. In the next theorem, we show that rough system truth equationality implies the solubility of the rough system core. This forms an analog of Theorem 1125.

Theorem 1157 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is roughly system truth equational, then $\widetilde{Z}^{\mathcal{I}}$ is soluble.*

Proof: Suppose $\mathcal{I}$ is roughly system truth equational, with witnessing equations $\tau^\flat$. Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\widetilde{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. Then, by rough system truth equationality and Lemma 1156, $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Again, using rough system truth equationality, we conclude that $\phi \in \widetilde{T}_\Sigma$. This shows that $\widetilde{Z}^{\mathcal{I}}$ is soluble. ∎

Conversely, in an analog of Theorem 1126, we show that the solubility of the rough system core of a π-institution implies rough system truth equationality.

Theorem 1158 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\widetilde{Z}^{\mathcal{I}}$ *is soluble, then* $\mathcal{I}$ *is roughly system truth equational, with witnessing equations* $\widetilde{Z}^{\mathcal{I}}$*.*

Proof: Assume $\widetilde{Z}^{\mathcal{I}}$ is soluble and let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. We must show that

$$\phi \in \widetilde{T}_\Sigma \quad \text{iff} \quad \widetilde{Z}^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T).$$

If $\phi \in \widetilde{T}_\Sigma$, then, by Proposition 1155, $\phi \in \widetilde{Z}^{\mathcal{I}}_\Sigma(\Omega(T))$, i.e., $\widetilde{Z}^{\mathcal{I}}_\Sigma[\phi] \le \Omega(T)$. On the other hand, the reverse inclusion is guaranteed by the postulated solubility of $\widetilde{Z}^{\mathcal{I}}$. Thus, $\mathcal{I}$ is indeed roughly system truth equational, with witnessing equations $\widetilde{Z}^{\mathcal{I}}$. ∎

Theorems 1157 and 1158 provide the first characterization of rough system truth equationality in terms of the solubility of the rough system core. This parallels Theorem 1127, which asserted a similar characterization for rough left truth equationality in terms of the left solubility of the rough left Suszko core of a π-institution.

$$\mathcal{I} \text{ Roughly System Truth Equational } \longleftrightarrow \widetilde{Z}^{\mathcal{I}} \text{ Soluble}$$

Theorem 1159 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is roughly system truth equational if and only if* $\widetilde{Z}^{\mathcal{I}}$ *is left soluble.*

Proof: The "if" is by Theorem 1158. The "only if" by Theorem 1157. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the rough system core $\widetilde{Z}^{\mathcal{I}}$ of $\mathcal{I}$ **roughly defines theory systems** if, for al $T \in \mathrm{ThSys}(\mathcal{I})$,

$$\widetilde{T} = \widetilde{Z}^{\mathcal{I}}(\Omega(T)).$$

Another characterization of rough system truth equationality, along the lines of Theorem 1128, asserts that it is equivalent to the rough definability of the theory systems by the rough system core.

$$\begin{array}{l}\mathcal{I} \text{ Roughly System Truth Equational} \\ \qquad \longleftrightarrow \widetilde{Z}^{\mathcal{I}} \text{ Roughly Defines Theory Systems}\end{array}$$

Theorem 1160 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is roughly system truth equational if and only if* $\widetilde{Z}^{\mathcal{I}}$ *roughly defines theory systems in* $\mathcal{I}$*.*

Proof: Suppose $\mathcal{I}$ is roughly system truth equational. By Theorem 1157, $\widetilde{Z}^{\mathcal{I}}$ is soluble. Hence, by definition, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\widetilde{Z}^{\mathcal{I}}(\Omega(T)) \leq \widetilde{T}$. Since, by Proposition 1155, the reverse inclusion always holds, we get, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\widetilde{T} = \widetilde{Z}^{\mathcal{I}}(\Omega(T))$. Thus, $\widetilde{Z}^{\mathcal{I}}$ roughly defines theory systems in $\mathcal{I}$. Conversely, if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\widetilde{T} = \widetilde{Z}^{\mathcal{I}}(\Omega(T))$, then $\widetilde{Z}^{\mathcal{I}}$ is soluble and, therefore, by Theorem 1158, $\mathcal{I}$ is roughly system truth equational. ∎

We establish, next, a connection between rough system truth equationality and rough system c-reflectivity by means of the rough system core. In analogy with the case of rough left truth equationality, we introduce, first, the notion of *adequacy* of the rough system core. The following proposition, a system analog of Proposition 1129, motivates the notion of adequacy that will follow.

Proposition 1161 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{aligned}\bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \widetilde{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}\\ \leq \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \phi \in \widetilde{T}_\Sigma\}.\end{aligned}$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $T \in \mathrm{ThSys}(\mathcal{I})$,

$$\phi \in \widetilde{T}_\Sigma \quad \text{implies} \quad \widetilde{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T). \quad (\text{Definition of } \widetilde{Z}^{\mathcal{I}})$$

Hence,

$$\begin{aligned}\bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \widetilde{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}\\ \leq \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \phi \in \widetilde{T}_\Sigma\}.\end{aligned}$$

This is the displayed formula in the statement. ∎

If the reverse inclusion of that proven in Proposition 1161 holds, then we say that the rough system core of $\mathcal{I}$ is *adequate*.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the rough system core $\widetilde{Z}^{\mathcal{I}}$ of $\mathcal{I}$ is **adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{aligned}\bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \phi \in \widetilde{T}_\Sigma\}\\ \leq \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \widetilde{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.\end{aligned}$$

We can show, in analogy with Corollary 1130, that the solubility of the rough system core implies its adequacy.

Corollary 1162 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\widetilde{Z}^{\mathcal{I}}$ *is soluble, then it is adequate.*

Proof: Suppose $\widetilde{Z}^{\mathcal{I}}$ is soluble and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, by solubility and Proposition 1155, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\phi \in \widetilde{T}_\Sigma$ if and only if $\widetilde{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. Therefore,

$$\begin{aligned}&\bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \phi \in \widetilde{T}_\Sigma\}\\ &\qquad = \bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \widetilde{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.\end{aligned}$$

Thus, $\widetilde{Z}^{\mathcal{I}}$ is adequate. ■

We prove, next, the converse of Corollary 1162, under the additional assumption that the π-institution $\mathcal{I}$ under consideration is roughly system c-reflective. This constitutes an analog of Proposition 1131.

Proposition 1163 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a roughly system c-reflective* π*-institution based on* $\mathbf{F}$*. If* $\widetilde{Z}^{\mathcal{I}}$ *is adequate, then it is soluble.*

Proof: Suppose $\mathcal{I}$ is roughly system c-reflective and $\widetilde{Z}^{\mathcal{I}}$ is adequate. Let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\widetilde{Z}^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. By the postulated adequacy of $\widetilde{Z}^{\mathcal{I}}$, we get that $\bigcap\{\Omega(T) : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \phi \in \widetilde{T}_\Sigma\} \leq \Omega(T)$. By rough system truth equationality, $\bigcap\{\widetilde{T} : T \in \mathrm{ThSys}(\mathcal{I}) \text{ and } \phi \in \widetilde{T}_\Sigma\} \leq \widetilde{T}$. Therefore, $\phi \in \widetilde{T}_\Sigma$. We conclude that $\widetilde{Z}^{\mathcal{I}}$ is soluble. ■

We are now in a position to prove the main characterization theorem relating rough system truth equationality with rough system c-reflectivity, an analog of Theorem 1132, which characterized rough left truth equationality in terms of rough left c-reflectivity and the left adequacy of the rough left Suszko core.

Rough System Truth Equationality
= $\widetilde{Z}^{\mathcal{I}}$ Soluble
= $\widetilde{Z}^{\mathcal{I}}$ Roughly Defines Theory Systems
= Rough System c-Reflectivity + $\widetilde{Z}^{\mathcal{I}}$ Adequate

Theorem 1164 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is roughly system truth equational if and only if it is roughly system c-reflective and has an adequate rough system core.*

Proof: Suppose, first, that $\mathcal{I}$ is roughly system truth equational. By Theorem 1154, it is roughly system c-reflective. By Theorem 1157, its rough system core is soluble. Thus, by Corollary 1162, its rough system core is also adequate.

Assume, conversely, that $\mathcal{I}$ is roughly system c-reflective and has an adequate rough system core. Then, by Proposition 1163, its rough system core is also soluble. Hence, by Theorem 1158, $\mathcal{I}$ is roughly system truth equational. ■

Based on Proposition 1152 and Theorem 468, it is not difficult to show, in an analog of Corollary 1133, that the characterization theorem, Theorem 887, of system truth equationality in terms of system c-reflectivity and the adequacy of the system core, can be inferred from Theorem 1164.

Corollary 1165 (Theorem 887) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is system truth equational if and only if it is system c-reflective and has an adequate system core.*

Proof: $\mathcal{I}$ is system truth equational if and only if, by Proposition 1152, it is roughly system truth equational and has theorems, if and only if, by Theorem 1164, it is roughly system c-reflective, with an adequate rough system core and has theorems, if and only if, by Theorem 468 and the definitions of system core and rough system core, it is system c-reflective and has an adequate system core. ■

We close the section by looking at a result, an analog of Theorem 1134, which may be perceived either as an alternative characterization of rough system truth equationality, involving arbitrary $\mathbf{F}$-algebraic systems, or as a transfer theorem.

Theorem 1166 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is roughly system truth equational, with witnessing equations* $\tau^\flat$*, if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, all* $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$*, all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi \in \mathrm{SEN}(\Sigma)$*,*

$$\phi \in \widetilde{T}_\Sigma \quad \textit{iff} \quad \tau^{\mathcal{A}}_\Sigma[\phi] \leq \Omega^{\mathcal{A}}(T).$$

Proof: If the postulated condition holds, then it holds, in particular, for the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. This yields immediately that $\mathcal{I}$ is roughly system truth equational.

Suppose, conversely, that $\mathcal{I}$ is roughly system truth equational and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rll}
\alpha_\Sigma(\phi) \in \widetilde{T}_{F(\Sigma)} & \text{iff} & \phi \in \alpha^{-1}_\Sigma(\widetilde{T}_{F(\Sigma)}) \\
& \text{iff} & \phi \in \widetilde{\alpha^{-1}_\Sigma(T_{F(\Sigma)})} \quad \text{(Theorem 377)} \\
& \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(\alpha^{-1}(T)) \quad \text{(hypothesis)} \\
& \text{iff} & \tau^\flat_\Sigma[\phi] \leq \alpha^{-1}(\Omega^{\mathcal{A}}(T)) \quad \text{(Proposition 24)} \\
& \text{iff} & \tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi)] \leq \Omega^{\mathcal{A}}(T). \quad \text{(Lemma 95)}
\end{array}$$

Hence, taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that the displayed condition holds. ■

14.5 Narrow System Truth Equationality

Finally, we discuss *narrow system truth equationality*, the weakest of all rough/narrow truth equationality conditions. As the terminology suggests:

- It is in the same relation to narrow system c-reflectivity as rough system truth equationality is to rough system c-reflectivity;
- It is in the same relation to narrow truth equationality and narrow left truth equationality as rough system truth equationality is to rough truth equationality and rough left truth equationality, respectively.

In a nutshell, narrow system truth equationality is defined analogously to system truth equationality, but care is taken to bypass theory systems with empty components.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that $\mathcal{I}$ is **narrowly system truth equational** if there exists $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with a single distinguished argument, such that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi/\Omega_\Sigma(T) \in \widetilde{T}_\Sigma/\Omega_\Sigma(T) \quad \text{iff} \quad \tau_\Sigma^{\mathcal{F}/\Omega(T)}[\phi/\Omega_\Sigma(T)] \le \Delta^{\mathcal{F}/\Omega(T)}.$$

Once more, since, by Proposition 369, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(\widetilde{T}) = \Omega(T)$, $\Omega(T)$ is compatible with $\widetilde{T}$ and, hence, the preceding definition makes sense. The collection $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ is referred to as a set of **witnessing equations** (of/for the narrow system truth equationality of $\mathcal{I}$).

As in Proposition 1096, we get the following alternative characterization.

Proposition 1167 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ a collection of natural transformations in $N^\flat$, with a single distinguished argument. $\mathcal{I}$ is narrowly system truth equational, with witnessing equations $\tau^\flat$, if and only if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,*

$$\phi \in \widetilde{T}_\Sigma \quad \textit{iff} \quad \tau^\flat_\Sigma[\phi] \le \Omega(T).$$

Proof: Suppose that $\mathcal{I}$ is narrowly truth equational and let $T \in \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{rll}
\phi \in \widetilde{T}_\Sigma & \text{iff} & \phi/\Omega_\Sigma(T) \in \widetilde{T}_\Sigma/\Omega_\Sigma(T) \quad \text{(Proposition 369 and compatibility)}\\
& \text{iff} & \tau_\Sigma^{\mathcal{F}/\Omega(T)}[\phi/\Omega_\Sigma(T)] \le \Delta^{\mathcal{F}/\Omega(T)} \quad \text{(by hypothesis)}\\
& \text{iff} & \tau^\flat_\Sigma[\phi]/\Omega(T) \le \Delta^{\mathcal{F}/\Omega(T)} \quad \text{(by definition)}\\
& \text{iff} & \tau^\flat_\Sigma[\phi] \le \Omega(T).
\end{array}$$

Suppose, conversely, that the displayed condition holds. Let $T \in \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{lll}\phi/\Omega_\Sigma(T) \in \widetilde{T}_\Sigma/\Omega_\Sigma(T) & \text{iff} & \phi \in \widetilde{T}_\Sigma \quad \text{(Proposition 369 and compatibility)}\\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(T) \quad \text{(by hypothesis)}\\ & \text{iff} & \tau^\flat_\Sigma[\phi]/\Omega(T) \leq \Delta^{\mathcal{F}/\Omega(T)}\\ & \text{iff} & \tau^{\mathcal{F}/\Omega(T)}_\Sigma[\phi/\Omega_\Sigma(T)] \leq \Delta^{\mathcal{F}/\Omega(T)}. \quad \text{(definition)}\end{array}$$

Therefore, $\mathcal{I}$ is narrowly system truth equational. ∎

It is not difficult to see that an alternative way to express narrow system truth equationality is to assert the same condition that defines system truth equationality, excluding, however, those theory systems with at least one empty component.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Recall from Chapter 6 that we denote by $\mathrm{ThSys}^{\not\varnothing}(\mathcal{I})$ the collection of all theory systems T of $\mathcal{I}$, such that $T_\Sigma \neq \varnothing$, for all $\Sigma \in |\mathbf{Sign}^\flat|$:

$$\mathrm{ThSys}^{\not\varnothing}(\mathcal{I}) = \{T \in \mathrm{ThSys}(\mathcal{I}) : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(T_\Sigma \neq \varnothing)\}.$$

Recall, also, that, if $\mathcal{I}$ has theorems, then $\mathrm{ThSys}^{\not\varnothing}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$. In particular, this is case if $\mathcal{I}$ happens to be system truth equational.

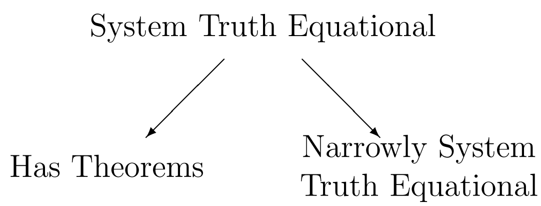
Proposition 1168 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$ and $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ a collection of natural transformations in $N^\flat$, with a single distinguished argument. $\mathcal{I}$ is narrowly system truth equational, with witnessing equations $\tau^\flat$, if and only if, for all $T \in \mathrm{ThSys}^{\not\varnothing}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,*

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega(T).$$

Proof: Suppose $\mathcal{I}$ is narrowly system truth equational, with witnessing equations $\tau^\flat$. Let $T \in \mathrm{ThSys}^{\not\varnothing}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then $\widetilde{T} = T \in \mathrm{ThSys}(\mathcal{I})$, whence, taking into account Proposition 1167, $\phi \in T_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$.

Suppose, conversely, that the displayed condition holds. Consider $T \in \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, since, by definition of $\widetilde{T}$, we have $\widetilde{T} \in \mathrm{ThSys}^{\not\varnothing}(\mathcal{I})$, we get, by hypothesis, $\phi \in \widetilde{T}_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(\widetilde{T})$, whence, using Proposition 369, we conclude that $\phi \in \widetilde{T}_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Therefore, $\mathcal{I}$ is narrowly system truth equational. ∎

As a corollary, we obtain the following key relationship between narrow system truth equationality and system truth equationality, paralleling the one established between system truth equationality and rough system truth equationality in Corollary 1152.



Corollary 1169 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is system truth equational if and only if it is narrowly system truth equational with theorems.*

Proof: Suppose, first, that $\mathcal{I}$ is narrowly system truth equational, with witnessing equations $\tau^\flat$, and that it has theorems. Availability of theorems implies that $\mathrm{ThSys}^{\nmid}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$. Thus, by Proposition 1168, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in T_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Thus, $\mathcal{I}$ is system truth equational, with the same witnessing equations $\tau^\flat$.

Assume, conversely, that $\mathcal{I}$ is system truth equational, with witnessing equations $\tau^\flat$. Then, for all $T \in \mathrm{ThSys}(\mathcal{I})$, and, hence, a fortiori, for all $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \in T_\Sigma$ if and only if $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Hence, again by Proposition 1168, $\mathcal{I}$ is narrowly system truth equational. Finally, by Theorem 872, $\mathcal{I}$ is system c-reflective and, by Proposition 243, it is system reflective and, therefore, system injective. Thus, it must have theorems. ■

Our next goal is to prove an analog of the characterization theorem, Theorem 1159, of rough system truth equationality in terms of the solubility of the rough system core for the case of narrow system truth equationality.

Narrow system truth equationality allows the following expression for all theory systems with nonempty components, forming an analog of Proposition 1153.

Proposition 1170 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following conditions are equivalent:*

(i) $\mathcal{I}$ *is narrowly system truth equational, with witnessing equations* $\tau^\flat$*;*

(ii) For all $T \in \mathrm{ThSys}(\mathcal{I})$*, such that* $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$*,* $\widetilde{T} = \tau^\flat(\Omega(T))$*;*

(iii) For all $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$*,* $T = \tau^\flat(\Omega(T))$*.*

Proof: Suppose $\mathcal{I}$ is narrowly system truth equational, with witnessing equations $\tau^\flat$, and let $T \in \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{rll} \phi \in \tau^\flat_\Sigma(\Omega(T)) & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(T) \quad \text{(definition)} \\ & \text{iff} & \phi \in \widetilde{T}_\Sigma. \quad \text{(hypothesis and Proposition 1167)} \end{array}$$

This proves Condition (ii). If Condition (ii) holds and $T \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$, then $\widetilde{T} = T \in \mathrm{ThSys}(\mathcal{I})$, whence, by hypothesis, $T = \widetilde{T} = \tau^\flat(\Omega(T))$. Thus, Condition (iii) holds. Finally, assume Condition (iii) holds. Then, by Proposition **??**, $\mathcal{I}$ is narrowly system truth equational, with witnessing equations $\tau^\flat$. ■

Recall from Chapter 6 that, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$, $\mathcal{I}$ is called *narrowly system c-reflective* if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^{\nleq}(\mathcal{I})$,

$$\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T') \quad \text{implies} \quad \bigcap_{T\in\mathcal{T}} T \leq T'.$$

We are now able to show that narrow system truth equationality implies narrow system c-reflectivity. This is an analog of Theorem 1154.

Theorem 1171 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is narrowly system truth equational, then it is narrowly system c-reflective.*

Proof: Suppose $\mathcal{I}$ is narrowly system truth equational, with witnessing equations $\tau^\flat$. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^{\nleq}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Omega(T) \leq \Omega(T')$. Then we have

$$\begin{array}{rcll}
\bigcap_{T\in\mathcal{T}} T & = & \bigcap_{T\in\mathcal{T}} \tau^\flat(\Omega(T)) & \text{(Proposition 1170)}\\
& = & \tau^\flat(\bigcap_{T\in\mathcal{T}} \Omega(T)) & \text{(set theory)}\\
& \leq & \tau^\flat(\Omega(T')) & \text{(hypothesis)}\\
& = & T'. & \text{(Proposition 1170)}
\end{array}$$

Thus, $\mathcal{I}$ is narrowly system c-reflective. ■

In the context of narrow system truth equationality, the notion paralleling the rough system core, introduced in Section 14.3, is the *narrow system core*, a modification of the original definition of the system core from Chapter 11, which is defined by circumventing theory systems with empty components.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. The **narrow system core** $Z^{\mathcal{I}\nleq}$ of $\mathcal{I}$ is the collection

$$Z^{\mathcal{I}\nleq} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThSys}(\mathcal{I}))(\widetilde{T} \in \mathrm{ThSys}(\mathcal{I}) \Rightarrow \sigma^\flat[\widetilde{T}] \leq \widehat{\Omega}^{\mathcal{I}}(\widetilde{T}))\}.$$

As before, an alternative characterization avoids $\widetilde{\ }$ at the expense of restricting quantification over $\mathrm{ThSys}^{\nleq}(\mathcal{I})$.

Proposition 1172 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Then*

$$Z^{\mathcal{I}\nleq} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThSys}^{\nleq}(\mathcal{I}))(\sigma^\flat[T] \leq \widehat{\Omega}^{\mathcal{I}}(T))\}.$$

Proof: Inside this proof we set

$$M^{\mathcal{I}\nleq} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThSys}^{\nleq}(\mathcal{I}))(\sigma^\flat[T] \leq \widehat{\Omega}^{\mathcal{I}}(T))\}.$$

Our goal is to show that $Z^{\mathcal{I}\nmid} = M^{\mathcal{I}\nmid}$. Suppose, first, that $\sigma^\flat \in Z^{\mathcal{I}\nmid}$ and let $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$. Since $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, we get $\widetilde{T} = T$. Thus, on the one hand, $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$ and, on the other, by hypothesis, $\phi \in \widetilde{T}_\Sigma$. Thus, since $\sigma^\flat \in Z^{\mathcal{I}\nmid}$, we get

$$\sigma^\flat_\Sigma[\phi] \le \widehat{\Omega}^{\mathcal{I}}(\widetilde{T}) = \widehat{\Omega}^{\mathcal{I}}(T).$$

This proves that $\sigma^\flat \in M^{\mathcal{I}\nmid}$. Assume, conversely, that $\sigma^\flat \in M^{\mathcal{I}\nmid}$ and let $T \in \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \widetilde{T}_\Sigma$. Since $\widetilde{T} \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$ and $\sigma^\flat \in M^{\mathcal{I}\nmid}$, we get $\sigma^\flat_\Sigma[\phi] \le \widehat{\Omega}^{\mathcal{I}}(\widetilde{T})$, whence, $\sigma^\flat \in Z^{\mathcal{I}\nmid}$. This proves that $Z^{\mathcal{I}\nmid} = M^{\mathcal{I}\nmid}$. ■

From the definition, it is not difficult to see that any theory system T with all its components nonempty is always included in $Z^{\mathcal{I}\nmid}(\Omega(T))$. This forms an analog in the rough system context of Proposition 1155.

Proposition 1173 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$*,*

$$T \le Z^{\mathcal{I}\nmid}(\Omega(T)).$$

Proof: Suppose $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$, and $\sigma^\flat \in Z^{\mathcal{I}\nmid}$. Then, by Proposition **??**, $\sigma^\flat_\Sigma[\phi] \le \widehat{\Omega}^{\mathcal{I}}(T) \le \Omega(T)$. Hence, $Z^{\mathcal{I}\nmid}_\Sigma[\phi] \le \Omega(T)$. By definition, then, $\phi \in Z^{\mathcal{I}\nmid}_\Sigma(\Omega(T))$. Since Σ and $\phi \in T_\Sigma$ were arbitrary, we conclude that $T \le Z^{\mathcal{I}\nmid}(\Omega(T))$. ■

The reverse inclusion may or may not hold. If it does, for all $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, we say that the narrow system core $Z^{\mathcal{I}\nmid}$ of $\mathcal{I}$ is *soluble*.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The narrow system core $Z^{\mathcal{I}\nmid}$ of $\mathcal{I}$ is said to be **soluble** if, for all $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$Z^{\mathcal{I}\nmid}_\Sigma[\phi] \le \Omega(T) \quad \text{implies} \quad \phi \in T_\Sigma.$$

An alternative way to express solubility is to again expand the view to all theory systems, with nonempty components, at the balancing expense of adding rough equivalence representatives. We obtain, thus, an analog of Lemma 1103.

Lemma 1174 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $Z^{\mathcal{I}\nmid}$ *is soluble if and only if, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*, such that* $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$*,*

$$\widetilde{T} = Z^{\mathcal{I}\nmid}(\Omega(T)).$$

Proof: $Z^{\mathcal{I}\nleq}$ is soluble if and only if, by definition and Proposition 1173, for all $T \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$, $T = Z^{\mathcal{I}\nleq}(\Omega(T))$. It is easy to see that this holds if and only if, for all $T \in \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$, $\widetilde{T} = Z^{\mathcal{I}\nleq}(\Omega(\widetilde{T}))$. And this is equivalent, by Proposition 369, to the statement that, for all $T \in \mathrm{ThSys}(\mathcal{I})$, such that $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$, $\widetilde{T} = Z^{\mathcal{I}\nleq}(\Omega(T))$. ■

As was the case with rough system truth equationality (see Lemma 1156), it turns out that, if a given π-institution is narrowly system truth equational, then any collection of witnessing equations must be included in the narrow system core of $\mathcal{I}$. That is, in case of narrow system truth equationality, the narrow system core is a candidate for the largest set of witnessing equations.

Lemma 1175 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$*. If $\mathcal{I}$ is narrowly system truth equational, with witnessing equations $\tau^\flat$, then $\tau^\flat \subseteq Z^{\mathcal{I}\nleq}$.*

Proof: Suppose $\mathcal{I}$ is narrowly system truth equational, with witnessing equations $\tau^\flat$. Let $T \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$. Then, for all $T \leq T' \in \mathrm{ThSys}(\mathcal{I})$, we have $T' \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$ and $\phi \in T'_\Sigma$. Thus, by narrow system truth equationality and Proposition 1168, $\tau^\flat_\Sigma[\phi] \leq \Omega(T')$. Since T', with the postulated properties was arbitrary,

$$\tau^\flat_\Sigma[\phi] \leq \bigcap\{\Omega(T') : T \leq T' \in \mathrm{ThSys}(\mathcal{I})\} = \widehat{\Omega}^{\mathcal{I}}(T).$$

We conclude, using Proposition 1172, that $\tau^\flat \subseteq Z^{\mathcal{I}\nleq}$. ■

We are now ready to prove the equivalence between narrow system truth equationality and the solubility of the narrow system core, an analog of Theorem 1159. First, we show that narrow system truth equationality implies the solubility of the narrow system core. This forms an analog of Theorem 1157.

Theorem 1176 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$*. If $\mathcal{I}$ is narrowly system truth equational, then $Z^{\mathcal{I}\nleq}$ is soluble.*

Proof: Suppose $\mathcal{I}$ is narrowly system truth equational, with witnessing equations $\tau^\flat$. Let $T \in \mathrm{ThSys}^{\nleq}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $Z^{\mathcal{I}\nleq}_\Sigma[\phi] \leq \Omega(T)$. Then, by narrow system truth equationality and Lemma 1175, $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Again, using narrow system truth equationality and Proposition 1168, we conclude that $\phi \in T_\Sigma$. This shows that $Z^{\mathcal{I}\nleq}$ is soluble. ■

Conversely, the solubility of the narrow system core of a π-institution implies narrow system truth equationality.

Theorem 1177 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $Z^{\mathcal{I}\nmid}$ *is soluble, then* $\mathcal{I}$ *is narrowly system truth equational, with witnessing equations* $Z^{\mathcal{I}\nmid}$*.*

Proof: Assume $Z^{\mathcal{I}\nmid}$ is soluble and let $T \in \mathrm{ThSys}^\nmid(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. By Proposition 1168, it suffices to show that

$$\phi \in T_\Sigma \quad \text{iff} \quad Z^{\mathcal{I}\nmid}_\Sigma[\phi] \leq \Omega(T).$$

If $\phi \in T_\Sigma$, then, by Proposition 1173, $\phi \in Z^{\mathcal{I}\nmid}_\Sigma(\Omega(T))$, i.e., $Z^{\mathcal{I}\nmid}_\Sigma[\phi] \leq \Omega(T)$. On the other hand, the reverse inclusion is guaranteed by the postulated solubility of $Z^{\mathcal{I}\nmid}$. Thus, $\mathcal{I}$ is indeed narrowly system truth equational, with witnessing equations $Z^{\mathcal{I}\nmid}$. ■

Theorems 1176 and 1177 provide the first characterization of narrow system truth equationality in terms of the solubility of the narrow system core. This parallels Theorem 1159, which asserted a similar characterization for rough system truth equationality in terms of the solubility of the rough system core of a π-institution.

$$\mathcal{I} \text{ Narrowly System Truth Equational } \longleftrightarrow\ Z^{\mathcal{I}\nmid} \text{ Soluble}$$

Theorem 1178 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is narrowly system truth equational if and only if* $Z^{\mathcal{I}\nmid}$ *is soluble.*

Proof: The "if" is by Theorem 1177. The "only if" by Theorem 1176. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the narrow system core $Z^{\mathcal{I}\nmid}$ of $\mathcal{I}$ **narrowly defines theory systems** if, for al $T \in \mathrm{ThSys}^\nmid(\mathcal{I})$,

$$T = Z^{\mathcal{I}\nmid}(\Omega(T)).$$

Another characterization of narrow system truth equationality, along the lines of Theorem 1160, asserts that it is equivalent to the narrow definability of the theory systems by the narrow system core.

$$\begin{array}{l}\mathcal{I} \text{ Narrowly System Truth Equational} \\ \qquad\qquad \longleftrightarrow\ Z^{\mathcal{I}\nmid} \text{ Narrowly Defines Theory Systems}\end{array}$$

Theorem 1179 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is narrowly system truth equational if and only if* $Z^{\mathcal{I}\nmid}$ *narrowly defines theory systems in* $\mathcal{I}$*.*

Proof: Suppose $\mathcal{I}$ is narrowly system truth equational. By Theorem 1176, $Z^{\mathcal{I}\lightning}$ is soluble. Hence, by definition, for all $T \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$, $Z^{\mathcal{I}\lightning}(\Omega(T)) \leq T$. Since, by Proposition 1173, the reverse inclusion always holds, we get, for all $T \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$, $T = Z^{\mathcal{I}\lightning}(\Omega(T))$. Thus, $Z^{\mathcal{I}\lightning}$ narrowly defines theory systems in $\mathcal{I}$. Conversely, if, for all $T \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$, $T = Z^{\mathcal{I}\lightning}(\Omega(T))$, then $Z^{\mathcal{I}\lightning}$ is soluble and, therefore, by Theorem 1178, $\mathcal{I}$ is narrowly system truth equational. ■

We establish, next, a connection between narrow system truth equationality and narrow system c-reflectivity by means of a variant of the systemic Suszko operator. This variant of the systemic Suszko operator, denoted $\widehat{\Omega}^{\mathcal{I}\lightning}$, is not necessarily identical to the systemic Suszko operator $\widehat{\Omega}^{\mathcal{I}}$ itself, unlike the version of the Suszko operator $\widetilde{\Omega}^{\mathcal{I}\lightning}$, defined in the preceding section, which was introduced only for convenience, but was actually shown to be equivalent to the original version $\widetilde{\Omega}^{\mathcal{I}}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We define the **narrow systemic Suszko operator** $\widetilde{\Omega}^{\mathcal{I}\lightning}$ by setting, for all $T \in \mathrm{ThSys}(\mathcal{I})$,

$$\widehat{\Omega}^{\mathcal{I}\lightning}(T) = \bigcap \{\Omega(T') : T \leq T' \in \mathrm{ThSys}^{\lightning}(\mathcal{I})\}.$$

Note that Lemma 1110 and, hence, a hypothetical analog of Corollary 1111, are not applicable in the case of theory systems, since, given $T \in \mathrm{ThSys}(\mathcal{I})$, it may not be the case that $\widetilde{T} \in \mathrm{ThSys}(\mathcal{I})$. On the other hand, as the following lemma shows, for all $T \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$, $\widehat{\Omega}^{\mathcal{I}\lightning}(T) = \widehat{\Omega}^{\mathcal{I}}(T)$. So, for the case of theory systems, all of whose components are nonempty, the two operators do coincide.

Lemma 1180 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $T \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$*,*

$$\widehat{\Omega}^{\mathcal{I}}(T) = \bigcap \{\Omega(T') : T \leq T' \in \mathrm{ThSys}^{\lightning}(\mathcal{I})\}.$$

Proof: Since $\{T' \in \mathrm{ThSys}^{\lightning}(\mathcal{I}) : T \leq T'\} \subseteq \{T' \in \mathrm{ThSys}(\mathcal{I}) : T \leq T'\}$, we get

$$\widehat{\Omega}^{\mathcal{I}}(T) \leq \bigcap \{\Omega(T') : T \leq T' \in \mathrm{ThSys}^{\lightning}(\mathcal{I})\}.$$

But, if $T \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$, then, for all $T' \in \mathrm{ThSys}(\mathcal{I})$, such that $T \leq T'$, we have $T' \in \mathrm{ThSys}^{\lightning}(\mathcal{I})$. Thus, in this particular case, the two collections above are identical and, therefore, equality holds between the two sides in the displayed formula, which proves the lemma. ■

In analogy with the case of rough system truth equationality, we may introduce the notion of *adequacy* of the narrow system core, which will help in characterizing the relationship between narrow system truth equationality and narrow system c-reflectivity. The following proposition, a "narrow" analog of Proposition 1161, justifies the notion of adequacy that will follow.

Proposition 1181 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\bigcap\{\Omega(T) : T \in \mathrm{ThSys}^{\sharp}(\mathcal{I}) \text{ and } Z_\Sigma^{\mathcal{I}\sharp}[\phi] \leq \Omega(T)\} \leq \widehat{\Omega}^{\mathcal{I}\sharp}(C(\overrightarrow{\phi})).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $T \in \mathrm{ThSys}^{\sharp}(\mathcal{I})$,

$$\begin{array}{lll} \phi \in T_\Sigma & \text{implies} & Z_\Sigma^{\mathcal{I}\sharp}[\phi] \leq \widehat{\Omega}^{\mathcal{I}}(T) \quad (\text{Definition of } Z^{\mathcal{I}\sharp}) \\ & \text{implies} & Z_\Sigma^{\mathcal{I}\sharp}[\phi] \leq \Omega(T). \quad (T \in \mathrm{ThSys}^{\sharp}(\mathcal{I})) \end{array}$$

Hence,

$$\begin{array}{l} \bigcap\{\Omega(T) : T \in \mathrm{ThSys}^{\sharp}(\mathcal{I}) \text{ and } Z_\Sigma^{\mathcal{I}\sharp}[\phi] \leq \Omega(T)\} \\ \quad \leq \bigcap\{\Omega(T) : T \in \mathrm{ThSys}^{\sharp}(\mathcal{I}) \text{ and } Z_\Sigma^{\mathcal{I}\sharp}[\phi] \leq \widehat{\Omega}^{\mathcal{I}}(T)\} \\ \quad \leq \bigcap\{\Omega(T) : T \in \mathrm{ThSys}^{\sharp}(\mathcal{I}) \text{ and } \phi \in T_\Sigma\} \\ \quad = \bigcap\{\Omega(T) : T \in \mathrm{ThSys}^{\sharp}(\mathcal{I}) \text{ and } \overrightarrow{\phi} \leq T\} \\ \quad = \widetilde{\Omega}^{\mathcal{I}\sharp}(C(\overrightarrow{\phi})). \end{array}$$

This is the displayed formula in the statement. ∎

If the reverse inclusion of that proven in Proposition 1181 holds, then we say that the narrow system core of $\mathcal{I}$ is *adequate*.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the narrow system core $Z^{\mathcal{I}\sharp}$ of $\mathcal{I}$ is **adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widehat{\Omega}^{\mathcal{I}\sharp}(C(\overrightarrow{\phi})) \leq \bigcap\{\Omega(T) : T \in \mathrm{ThSys}^{\sharp}(\mathcal{I}) \text{ and } Z_\Sigma^{\mathcal{I}\sharp}[\phi] \leq \Omega(T)\}.$$

We can show, in analogy with Corollary 1162, that the solubility of the narrow system core implies its adequacy.

Corollary 1182 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $Z^{\mathcal{I}\sharp}$ *is soluble, then it is adequate.*

Proof: Suppose $Z^{\mathcal{I}\sharp}$ is soluble and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{lll} \widehat{\Omega}^{\mathcal{I}\sharp}(C(\overrightarrow{\phi})) & = & \bigcap\{\Omega(T) : T \in \mathrm{ThSys}^{\sharp}(\mathcal{I}) \text{ and } \overrightarrow{\phi} \leq T\} \\ & & (\text{definition of } \widehat{\Omega}^{\mathcal{I}\sharp}) \\ & = & \bigcap\{\Omega(T) : T \in \mathrm{ThSys}^{\sharp}(\mathcal{I}) \text{ and } \phi \in T_\Sigma\} \\ & & (T \in \mathrm{ThSys}(\mathcal{I})) \\ & = & \bigcap\{\Omega(T) : T \in \mathrm{ThSys}^{\sharp}(\mathcal{I}) \text{ and } Z_\Sigma^{\mathcal{I}\sharp}[\phi] \leq \Omega(T)\}. \\ & & (\text{Solubility of } Z^{\mathcal{I}\sharp}) \end{array}$$

Thus, $Z^{\mathcal{I}\sharp}$ is adequate. ∎

In order to prove a partial converse of Corollary 1182, we will employ the following characterization of narrow system c-reflectivity.

Lemma 1183 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is narrowly system c-reflective if and only if, for all* $T \in \mathrm{ThSys}(\mathcal{I})$ *and all* $T' \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$*,*

$$\widehat{\Omega}^{\mathcal{I}\nmid}(T) \leq \Omega(T') \quad \textit{implies} \quad T \leq T'.$$

Proof: Suppose, first, that $\mathcal{I}$ is narrowly system c-reflective and let $T \in \mathrm{ThSys}(\mathcal{I})$ and $T' \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, such that $\widehat{\Omega}^{\mathcal{I}\nmid}(T) \leq \Omega(T')$. Then, by definition,

$$\bigcap\{\Omega(T'') : T \leq T'' \in \mathrm{ThSys}^{\nmid}(\mathcal{I})\} \leq \Omega(T').$$

By narrow system c-reflectivity, $\bigcap\{T'' : T \leq T'' \in \mathrm{ThSys}^{\nmid}(\mathcal{I})\} \leq T'$. Thus, $T \leq T'$.

Suppose, conversely, that the displayed condition in the statement holds and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThSys}^{\nmid}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T')$. Then, since $\mathcal{T} \subseteq \mathrm{ThSys}^{\nmid}(\mathcal{I})$, we get that

$$\bigcap\{\Omega(T) : \bigcap \mathcal{T} \leq T'' \in \mathrm{ThSys}^{\nmid}(\mathcal{I})\} \leq \Omega(T').$$

By definition, then, $\widehat{\Omega}^{\mathcal{I}\nmid}(\bigcap \mathcal{T}) \leq \Omega(T')$, whence, by hypothesis, $\bigcap \mathcal{T} \leq T'$. This shows that $\mathcal{I}$ is narrowly system c-reflective. ■

We prove, next, a partial converse of Corollary 1182, under the additional assumption that the π-institution $\mathcal{I}$ under consideration is narrowly system c-reflective. This constitutes an analog of Proposition 1163.

Proposition 1184 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a narrowly system c-reflective* π*-institution based on* $\mathbf{F}$*. If* $Z^{\mathcal{I}\nmid}$ *is adequate, then it is soluble.*

Proof: Suppose $\mathcal{I}$ is narrowly system c-reflective and $Z^{\mathcal{I}\nmid}$ is adequate. Let $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $Z^{\mathcal{I}\nmid}_\Sigma[\phi] \leq \Omega(T)$. By the postulated adequacy of $Z^{\mathcal{I}\nmid}$, we get that $\widehat{\Omega}^{\mathcal{I}\nmid}(C(\overrightarrow{\phi})) \leq \Omega(T)$. By narrow system truth equationality and Lemma 1183, $C(\overrightarrow{\phi}) \leq T$. Therefore, $\phi \in T_\Sigma$. We conclude that $Z^{\mathcal{I}\nmid}$ is soluble. ■

We are now in a position to prove the main characterization theorem relating narrow system truth equationality with narrow system c-reflectivity, an analog of Theorem 1164, which characterized rough system truth equationality in terms of rough system c-reflectivity and the adequacy of the rough system core.

Narrow System Truth Equationality
= $Z^{\mathcal{I}\nmid}$ Soluble
= $Z^{\mathcal{I}\nmid}$ Narrowly Defines Theory Systems
= Narrow System c-Reflectivity
+ $Z^{\mathcal{I}\nmid}$ Adequate

Theorem 1185 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is narrowly system truth equational if and only if it is narrowly system c-reflective and has an adequate narrow system core.*

Proof: Suppose, first, that $\mathcal{I}$ is narrowly system truth equational. By Theorem 1171, it is narrowly system c-reflective. By Theorem 1176, its narrow system core is soluble. Thus, by Corollary 1182, its narrow system core is also adequate.

Assume, conversely, that $\mathcal{I}$ is narrowly system c-reflective and has an adequate narrow system core. Then, by Proposition 1184, its narrow system core is soluble. Hence, by Theorem 1177, $\mathcal{I}$ is narrowly system truth equational. ■

Based on Proposition 1152 and Theorem 468, it is not difficult to show, in an analog of Corollary 1165, that the characterization theorem, Theorem 887, of system truth equationality in terms of system c-reflectivity and the adequacy of the system core, can be inferred from Theorem 1185.

Corollary 1186 (Theorem 887) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is system truth equational if and only if it is system c-reflective and has an adequate system core.*

Proof: $\mathcal{I}$ is system truth equational if and only if, by Proposition 1152, it is narrowly system truth equational and has theorems, if and only if, by Theorem 1185, it is narrowly system c-reflective, with an adequate narrow system core and has theorems, if and only if, by Theorem 468 and the definitions of system core and narrow system core, it is system c-reflective and has an adequate system core. ■

Finally, we prove an analog of Theorem 1166, which may be perceived either as an alternative characterization of narrow system truth equationality, involving arbitrary $\mathbf{F}$-algebraic systems, or as a transfer theorem.

Theorem 1187 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is narrowly system truth equational, with witnessing equations* $\tau^\flat$*, if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, all* $T \in \mathrm{FiSys}^{\mathcal{I}\sharp}(\mathcal{A})$*, all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi \in \mathrm{SEN}(\Sigma)$*,*

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau_\Sigma^{\mathcal{A}}[\phi] \leq \Omega^{\mathcal{A}}(T).$$

Proof: If the postulated condition holds, then it holds, in particular, for the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. This yields immediately that $\mathcal{I}$ is narrowly system truth equational.

Suppose, conversely, that $\mathcal{I}$ is narrowly system truth equational and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiSys}^{\mathcal{I}\natural}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{lll}\alpha_\Sigma(\phi) \in T_{F(\Sigma)} & \text{iff} & \phi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)}) \\ & \text{iff} & \tau_\Sigma^\flat[\phi] \le \Omega(\alpha^{-1}(T)) \\ & & \text{(Lemma 6 and hypothesis)} \\ & \text{iff} & \tau_\Sigma^\flat[\phi] \le \alpha^{-1}(\Omega^{\mathcal{A}}(T)) \quad \text{(Proposition 24)} \\ & \text{iff} & \tau_{F(\Sigma)}^{\mathcal{A}}[\alpha_\Sigma(\phi)] \le \Omega^{\mathcal{A}}(T). \quad \text{(Lemma 95)}\end{array}$$

Hence, taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that the displayed condition holds. ■

14.6 Availability of Natural Theorems

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall that, by convention, if $\mathcal{I}$ has theorems, then, for every $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathcal{I}$ has a Σ-theorem, i.e., there exists $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\varnothing)$.

On the other hand, recall from Section 2.6 that we say that a π-institution $\mathcal{I}$ has *natural theorems* if there exists a $\vartheta^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)^k$,

$$\vartheta_\Sigma^\flat(\vec{\phi}) \in C_\Sigma(\varnothing).$$

Furthermore, recall that we denote by $\mathrm{NThm}(\mathcal{I})$ the collection of natural theorems of $\mathcal{I}$.

It is straightforward that having natural theorems is a stronger property than having theorems.

Lemma 1188 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ has natural theorems, then it has theorems.*

Proof: Suppose $\vartheta^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$ is a natural theorem. Let $\Sigma \in |\mathbf{Sign}^\flat|$. By convention $\mathrm{SEN}^\flat(\Sigma) \neq \varnothing$. Let $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$. Then, we get $\vartheta_\Sigma^\flat(\vec{\phi}) \in \mathrm{Thm}_\Sigma(\mathcal{I})$. This shows that $\mathcal{I}$ has theorems. ■

On the other hand, it is easy to find examples of π-institutions with theorems that do not possess natural theorems. For example, every π-institution with at least one non-trivial set of sentences $\mathrm{SEN}^\flat(\Sigma)$, containing both a Σ-theorem and a Σ-non theorem, and with a trivial category of natural transformations, cannot have natural theorems. This follows from the fact that, under these circumstances, no projection natural transformation can be a natural theorem and projection natural transformations are the only ones available because of the triviality of $N^\flat$.

Another useful observation is that every π-institution with natural theorems has at least one at-most-unary natural theorem.

Lemma 1189 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *has natural theorems, then it has at least one at-most-unary natural theorem.*

Proof: Suppose $\mathcal{I}$ has natural theorems and let $\vartheta^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ be a natural theorem in $N^\flat$. If $k = 0$ or 1, then there is nothing to prove. If $k > 1$, then we define $\vartheta'^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\vartheta'^\flat_\Sigma(\phi) = \vartheta^\flat_\Sigma(\underbrace{\phi, \phi, \dots, \phi}_{k}).$$

Since $\vartheta'^\flat = \vartheta \circ \langle p^{1,0}, p^{1,0}, \dots, p^{1,0}\rangle$ and $\vartheta^\flat$ is in $N^\flat$, we get that $\vartheta'^\flat$ is in $N^\flat$ also. Moreover, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\vartheta'^\flat_\Sigma(\phi) = \vartheta^\flat_\Sigma(\phi, \dots, \phi) \in \mathrm{Thm}_\Sigma(\mathcal{I}).$$

Hence $\vartheta'^\flat$ is a unary natural theorem. ∎

We have the following characterization of natural theorems involving the local Frege operator.

Theorem 1190 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*, and* $\vartheta^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ *a natural transformation in* $N^\flat$*. Then the following conditions are equivalent:*

(i) $\vartheta^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ *is a natural theorem;*

(ii) For every $T \in \mathrm{ThFam}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\phi \in T_\Sigma \quad \textit{iff} \quad \langle \phi, \vartheta^\flat_\Sigma(\vec{\chi})\rangle \in \lambda_\Sigma(T);$$

(iii) For every $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\phi \in \mathrm{Thm}_\Sigma(\mathcal{I}) \quad \textit{iff} \quad \langle \phi, \vartheta^\flat_\Sigma(\vec{\chi})\rangle \in \lambda_\Sigma(\mathrm{Thm}(\mathcal{I})).$$

Proof:

(i)⇒(ii) Assume that $\vartheta^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ is a natural theorem. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$.

- Suppose $\phi \in T_\Sigma$. Then, since $\vartheta^\flat_\Sigma(\vec{\chi}) \in \mathrm{Thm}_\Sigma(\mathcal{I}) \subseteq T_\Sigma$, we get, by definition of $\lambda(T)$, $\langle \phi, \vartheta^\flat_\Sigma(\vec{\chi})\rangle \in \lambda_\Sigma(T)$.

- On the other hand, assume $\langle\phi,\vartheta^\flat_\Sigma(\vec{\chi})\rangle\in\lambda_\Sigma(T)$. Since $\vartheta^\flat_\Sigma(\vec{\chi})\in \mathrm{Thm}_\Sigma(\mathcal{I})\subseteq T_\Sigma$, we get, by the definition of $\lambda(T)$, $\phi\in T_\Sigma$.

(ii)⇒(iii) Trivial.

(iii)⇒(i) Suppose that (iii) holds. Let $\Sigma\in|\mathbf{Sign}^\flat|$, $\phi\in\mathrm{SEN}^\flat(\Sigma)$ and $\vec{\chi}\in\mathrm{SEN}^\flat(\Sigma)$.

- If $\phi\in\mathrm{Thm}_\Sigma(\mathcal{I})$, then, by hypothesis, $\langle\phi,\vartheta^\flat_\Sigma(\vec{\chi})\rangle\in\lambda_\Sigma(\mathrm{Thm}(\mathcal{I}))$, whence, $\vartheta^\flat_\Sigma(\vec{\chi})\in\mathrm{Thm}_\Sigma(\mathcal{I})$.
- If $\phi\notin\mathrm{Thm}_\Sigma(\mathcal{I})$, then, by hypothesis, $\langle\phi,\vartheta^\flat_\Sigma(\vec{\chi})\rangle\notin\lambda_\Sigma(\mathrm{Thm}(\mathcal{I}))$. Thus, $\theta^\flat_\Sigma(\vec{\chi})\in\mathrm{Thm}_\Sigma(\mathcal{I})$.

We conclude that, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\vec{\chi}\in\mathrm{SEN}^\flat(\Sigma)$, $\vartheta^\flat_\Sigma(\vec{\chi})\in\mathrm{Thm}_\Sigma(\mathcal{I})$. Therefore, $\vartheta^\flat$ is a natural theorem.

■

We provide two additional equivalent conditions in the theorem following the next lemma.

Lemma 1191 *Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, $\mathcal{I}=\langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$, and $\vartheta^\flat:(\mathrm{SEN}^\flat)^k\to\mathrm{SEN}^\flat$ a natural transformation in $N^\flat$. If $\vartheta^\flat$ is a natural theorem, then, for every $\mathbf{F}$-algebraic system $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$, all $\Sigma\in|\mathbf{Sign}|$ and all $\vec{\chi}\in\mathrm{SEN}(\Sigma)$, $\vartheta_\Sigma(\vec{\chi})\in C^{\mathcal{I},\mathcal{A}}_\Sigma(\varnothing)$, i.e., ϑ is a natural theorem of $\langle\mathcal{A},C^{\mathcal{I},\mathcal{A}}\rangle$.*

Proof: Since $\vartheta^\flat$ is a natural theorem, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\vec{\chi}\in\mathrm{SEN}^\flat(\Sigma)$,

$$\vartheta_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi}))=\alpha_\Sigma(\vartheta^\flat_\Sigma(\vec{\chi}))\in\alpha_\Sigma(\mathrm{Thm}_\Sigma(\mathcal{I}))\subseteq C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\varnothing).$$

By the surjectivity of $\langle F,\alpha\rangle$ the conclusion follows. ■

Theorem 1192 *Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, $\mathcal{I}=\langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$, and $\vartheta^\flat:(\mathrm{SEN}^\flat)^k\to\mathrm{SEN}^\flat$ a natural transformation in $N^\flat$. Then the following conditions are equivalent:*

(i) $\vartheta^\flat:(\mathrm{SEN}^\flat)^k\to\mathrm{SEN}^\flat$ *is a natural theorem;*

(ii) For every $\mathbf{F}$-algebraic system $\mathcal{A}$, all $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, all $\Sigma\in|\mathbf{Sign}|$ and all $\phi\in\mathrm{SEN}(\Sigma)$,

$$\phi\in T_\Sigma\quad\textit{iff}\quad(\forall\vec{\chi}\in\mathrm{SEN}(\Sigma))(\langle\phi,\vartheta_\Sigma(\vec{\chi})\rangle\in\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T));$$

(iii) For every $\mathbf{F}$-algebraic system $\mathcal{A}$, all $\Sigma\in|\mathbf{Sign}|$ and all $\phi\in\mathrm{SEN}(\Sigma)$,

$$\phi\in C^{\mathcal{I},\mathcal{A}}_\Sigma(\varnothing)\quad\textit{iff}\quad(\forall\vec{\chi}\in\mathrm{SEN}(\Sigma))(\langle\phi,\vartheta_\Sigma(\vec{\chi})\rangle\in\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(C^{\mathcal{I},\mathcal{A}}(\varnothing))).$$

Proof:

(i)⇒(ii) Assume that $\vartheta^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ is a natural theorem and let $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. By Lemma 1191, for all $\vec{\chi} \in \mathrm{SEN}(\Sigma)$, $\vartheta^\flat_\Sigma(\vec{\chi}) \in C^{\mathcal{I},\mathcal{A}}_\Sigma(\varnothing)$.

- If $\phi \in T_\Sigma$, then, clearly, for all $\vec{\chi} \in \mathrm{SEN}(\Sigma)$, and all $T \le T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we have $\phi \in T'_\Sigma$ and $\vartheta_\Sigma(\vec{\chi}) \in T'_\Sigma$. Hence, $\langle \phi, \vartheta_\Sigma(\vec{\chi})\rangle \in \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T)$.
- If, for all $\vec{\chi} \in \mathrm{SEN}(\Sigma)$, $\langle \phi, \vartheta_\Sigma(\vec{\chi})\rangle \in \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T)$, then, in particular, for all $\vec{\chi} \in \mathrm{SEN}(\Sigma)$, $\langle \phi, \vartheta_\Sigma(\vec{\chi})\rangle \in \lambda^{\mathcal{A}}_\Sigma(T)$. Since $\vartheta_\Sigma(\vec{\chi}) \in T_\Sigma$, we conclude that $\phi \in T_\Sigma$.

(ii)⇒(iii) Trivial.

(iii)⇒(iv) Suppose that (iii) holds. Consider, first, the trivial algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, with the single signature object $*$ and such that $\mathrm{SEN}(*) = \{0\}$. Then, we have $\langle 0, \vartheta_\Sigma(\vec{0})\rangle = \langle 0,0\rangle \in \{\langle 0,0\rangle\} = \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_*(\varnothing)$. If $\varnothing \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, then this would imply, by hypothesis, that $0 \in \varnothing$, a contradiction. Thus, $\varnothing \notin \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A})$. This shows that $\mathcal{I}$ has theorems.

Let, now, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\chi} \in \mathrm{SEN}(\Sigma)$. Take a theorem $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$. Then, by hypothesis, $\langle t, \vartheta^\flat_\Sigma(\vec{\chi})\rangle \in \widetilde{\lambda}^{\mathcal{I}}_\Sigma(\mathrm{Thm}(\mathcal{I})) \subseteq \lambda_\Sigma(\mathrm{Thm}(\mathcal{I}))$. Thus, since $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$, we must have $\vartheta^\flat_\Sigma(\vec{\chi}) \in \mathrm{Thm}_\Sigma(\mathcal{I})$. But $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$ were arbitrary, whence, we conclude that $\vartheta^\flat$ is a natural theorem.

■

We saw that availability of natural theorems is a strictly stronger condition that availability of theorems. We have the following theorem, which follows from preceding results.

Theorem 1193 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an* $\mathbf{F}$*-algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *has natural theorems, then there exists* $\tau : (\mathrm{SEN}^\flat)^k \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*, such that, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A}$*, all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi \in \mathrm{SEN}(\Sigma)$*,*

$$\phi \in T_\Sigma \quad \textit{iff}, \quad \text{for all } \vec{\chi} \in \mathrm{SEN}(\Sigma), \tau_\Sigma(\phi, \vec{\chi}) \subseteq \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T);$$

(b) If the condition in the conclusion of (a) holds, then $\mathcal{I}$ *has theorems.*

Proof:

(a) Suppose $\mathcal{I}$ has natural theorems and let $\vartheta^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ be a natural theorem. Then, we define $\tau^\flat : (\mathrm{SEN}^\flat)^{k+1} \to (\mathrm{SEN}^\flat)^2$, by setting

$$\tau^\flat := \{p^{k+1,0} \approx \vartheta^\flat \circ \langle p^{k+1,1}, \dots, p^{k+1,k}\rangle\}.$$

Then the conclusion follows from Theorem 1192.

(b) Suppose that the conclusion of Part (a) holds. Consider the trivial algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, with the single signature object $*$ and such that $\mathrm{SEN}(*) = \{0\}$. If $\mathcal{I}$ does not have theorems, then $\varnothing \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Since $0 \notin \varnothing$, we must have, by hypothesis, $\langle 0, 0\rangle = \langle 0, \vartheta_\Sigma(\vec{0})\rangle \notin \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_*(\varnothing) = \{\langle 0, 0\rangle\}$, a contradiction. Therefore, $\mathcal{I}$ has theorems.

■

We may think of a π-institution that has theorems, but not natural theorems, as having a syntactic deficiency, i.e., not having enough natural transformations in its category of natural transformations to express theoremhood. So in an analogous way with the one used to formulate similar properties through the Leibniz property of the reflexive core and the adequacy of the Suszko core, we make the following definition, taking cue from Theorem 1190.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. The **Frege core** $F^{\mathcal{I}}$ of $\mathcal{I}$ is defined by

$$\begin{array}{rcl} F^{\mathcal{I}} & = & \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}(\mathcal{I}))(\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)) \\ & & (T_\Sigma \approx \sigma^\flat_\Sigma(\vec{\chi}) \subseteq \widetilde{\lambda}^{\mathcal{I}}_\Sigma(T))\}. \end{array}$$

It is not difficult to show that, in case $\mathcal{I}$ has theorems, $F^{\mathcal{I}} = \mathrm{NThm}(\mathcal{I})$.

Proposition 1194 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ has theorems, then $F^{\mathcal{I}} = \mathrm{NThm}(\mathcal{I})$.*

Proof: Suppose that $\mathcal{I}$ has theorems.

Assume $\sigma^\flat \in F^{\mathcal{I}}$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$. Since $\mathcal{I}$ has theorems, there exists $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$. Then, by hypothesis and the definition of $F^{\mathcal{I}}$,

$$\langle t, \sigma^\flat_\Sigma(\vec{\chi})\rangle \in \widetilde{\lambda}^{\mathcal{I}}_\Sigma(\mathrm{Thm}(\mathcal{I})) \subseteq \lambda_\Sigma(\mathrm{Thm}(\mathcal{I})).$$

Thus, since $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$, $\sigma^\flat_\Sigma(\vec{\chi}) \in \mathrm{Thm}_\Sigma(\mathcal{I})$. Since $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$ were arbitrary, $\sigma^\flat \in \mathrm{NThm}(\mathcal{I})$. Therefore, we get that $F^{\mathcal{I}} \subseteq \mathrm{NThm}(\mathcal{I})$.

Suppose, conversely, that $\sigma^\flat \in \mathrm{NThm}(\mathcal{I})$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $T \leq T' \in \mathrm{ThFam}(\mathcal{I})$. Then, we get $\phi \in T'_\Sigma$ and $\sigma^\flat_\Sigma(\vec{\chi}) \in T'_\Sigma$, whence

$$\phi \in T'_\Sigma \quad \text{iff} \quad \sigma^\flat_\Sigma(\vec{\chi}) \in T'_\Sigma.$$

That is, for all $T \leq T' \in \mathrm{ThFam}(\mathcal{I})$, $\langle \phi, \sigma^\flat_\Sigma(\vec{\chi})\rangle \in \lambda_\Sigma(T)$. Hence, $\langle \phi, \sigma^\flat_\Sigma(\vec{\chi})\rangle \in \widetilde{\lambda}^{\mathcal{I}}_\Sigma(T)$. This shows that $\sigma^\flat \in F^{\mathcal{I}}$, whence $\mathrm{NThm}(\mathcal{I}) \subseteq F^{\mathcal{I}}$. ■

In the remainder of the section, we show that a property analogous to adequacy, coupled with possession of theorems, guarantees the existence of natural theorems. The following lemma partly justifies the definition of adequacy.

Proposition 1195 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\bigcap\{\lambda(T) : (\forall \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma))(\phi \approx F^{\mathcal{I}}_\Sigma(\vec{\chi}) \subseteq \lambda_\Sigma(T))\} \leq \widetilde{\lambda}^{\mathcal{I}}(C(\phi)).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. By the definition of the Frege core, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{implies} \quad \phi \approx F^{\mathcal{I}}_\Sigma(\vec{\chi}) \subseteq \widetilde{\lambda}^{\mathcal{I}}_\Sigma(T) \subseteq \lambda_\Sigma(T).$$

Therefore, we get

$$\begin{array}{l}\{T \in \mathrm{ThFam}(\mathcal{I}) : \phi \in T_\Sigma\} \\ \quad \subseteq \{T \in \mathrm{ThFam}(\mathcal{I}) : (\forall \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma))(\phi \approx F^{\mathcal{I}}_\Sigma(\vec{\chi}) \subseteq \lambda_\Sigma(T))\}.\end{array}$$

This, now, yields

$$\bigcap\{\lambda(T) : (\forall \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma))(\phi \approx F^{\mathcal{I}}_\Sigma(\vec{\chi}) \subseteq \lambda_\Sigma(T))\} \leq \widetilde{\lambda}^{\mathcal{I}}(C(\phi)),$$

i.e., the displayed formula in the statement. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that the Frege core $F^{\mathcal{I}}$ is **adequate** if the reverse inclusion of the one proved in Proposition 1195 holds, i.e., if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widetilde{\lambda}(C(\phi)) \leq \bigcap\{\lambda(T) : (\forall \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma))(\phi \approx F^{\mathcal{I}}_\Sigma(\vec{\chi}) \subseteq \lambda_\Sigma(T))\}.$$

Theorem 1196 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *has natural theorems if and only if it has theorems and its Frege core is adequate.*

Proof: If $\mathcal{I}$ has natural theorems, then, by Lemma 1188, it has theorems. Moreover, by Proposition 1194, $F^{\mathcal{I}} = \mathrm{NThm}(\mathcal{I})$. Now consider $\tau^\flat \in \mathrm{NThm}(\mathcal{I})$ and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \approx F^{\mathcal{I}}_\Sigma(\vec{\chi}) \subseteq \lambda_\Sigma(T)$. Since $F^{\mathcal{I}} = \mathrm{NThm}(\mathcal{I})$, we get $F^{\mathcal{I}}_\Sigma(\vec{\chi}) \subseteq \mathrm{Thm}_\Sigma(\mathcal{I}) \subseteq T_\Sigma$. Thus, $\phi \in T_\Sigma$. This shows that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{r}\{T \in \mathrm{ThFam}(\mathcal{I}) : (\forall \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma))(\phi \approx F^{\mathcal{I}}_\Sigma(\vec{\chi}) \subseteq \lambda_\Sigma(T))\} \\ \leq \{T \in \mathrm{ThFam}(\mathcal{I}) : \phi \in T_\Sigma\}.\end{array}$$

This proves that $F^{\mathcal{I}}$ is adequate.

Assume, conversely, that $\mathcal{I}$ has theorems and $F^{\mathcal{I}}$ is adequate.

Note that, if $\mathrm{Thm}(\mathcal{I}) = \mathrm{SEN}^\flat$, then $p^{1,0} : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ is a natural theorem. So we may assume that $\overline{\varnothing} \lneq \mathrm{Thm}(\mathcal{I}) \lneq \mathrm{SEN}^\flat$. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$ and $\phi \in \mathrm{SEN}^\flat(\Sigma) \backslash \mathrm{Thm}_\Sigma(\mathcal{I})$. Then, we get

$$\langle \phi, t \rangle \in \lambda_\Sigma(C(\phi)) \quad \text{but} \quad \langle \phi, t \rangle \notin \lambda_\Sigma(\mathrm{Thm}(\mathcal{I})).$$

Thus, if $F^{\mathcal{I}} = \varnothing$, then

$$\mathrm{Thm}(\mathcal{I}) \in \{T \in \mathrm{ThFam}(\mathcal{I}) : (\forall \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma))(\phi \approx F^{\mathcal{I}}_\Sigma(\vec{\chi}) \subseteq \lambda_\Sigma(T))\}.$$

So $\langle \phi, t\rangle \notin \bigcap\{\lambda_\Sigma(T) : (\forall \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma))(\phi \approx F^{\mathcal{I}}_\Sigma(\vec{\chi}) \subseteq \lambda_\Sigma(T))\}$. Since $\langle \phi, t\rangle \in \bigcap\{\lambda_\Sigma(T) : \phi \in T_\Sigma\}$, we get that

$$\widetilde{\lambda}(C(\phi)) \nleq \bigcap\{\lambda(T) : (\forall \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma))(\phi \approx F^{\mathcal{I}}_\Sigma(\vec{\chi}) \subseteq \lambda_\Sigma(T))\},$$

contrary to the postulated adequacy of $F^{\mathcal{I}}$. ■

We close the section by showing that having natural theorems is a property that transfers from a π-institution to all $\mathcal{I}$-gmatrix families.

Theorem 1197 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *has natural theorems if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*, the* $\mathcal{I}$*-gmatrix* $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}}\rangle$ *has natural theorems.*

Proof: The "if" is clear by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$ and taking into account the fact that $C^{\mathcal{I},\mathcal{F}} = C$. The "only if" was proven in Lemma 1191. ■

Chapter 15

The Syntactic Leibniz Hierarchy: Basement II

15.1 Syntactic Narrow Family Monotonicity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

Recall that $\mathcal{I}$ is **roughly/narrowly family monotone** if, for all $T, T' \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$,

$$T \leq T' \quad \text{implies} \quad \Omega(T) \leq \Omega(T').$$

In this section we introduce and study a syntactic analog of this concept.

First, we relativize family reflexivity, family symmetry, family transitivity, family compatibility and family modus ponens to $\mathrm{ThFam}^{\nmid}(\mathcal{I})$.

Let, as above, $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Moreover, suppose that $I^\flat \subseteq N^\flat$ is a collection of natural transformations in $N^\flat$, with two distinguished arguments.

- $I^\flat$ is **roughly family reflexive** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,
$$I^\flat_\Sigma[\phi, \phi] \leq \widetilde{T};$$
- $I^\flat$ is **narrowly family reflexive** if, for all $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,
$$I^\flat_\Sigma[\phi, \phi] \leq T.$$

As the following lemma establishes rough and narrow family reflexivity are identical properties.

Lemma 1198 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat \subseteq N^\flat$ a family of natural transformations in $N^\flat$, with two distinguished arguments. $I^\flat$ is roughly family reflexive if and only if it is narrowly family reflexive.*

Proof: Suppose, first, that $I^\flat$ is roughly family reflexive and consider $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Since $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, we have $\widetilde{T} = T$, whence, by rough family reflexivity, $I^\flat_\Sigma[\phi, \phi] \leq \widetilde{T} = T$. Thus, $I^\flat$ is narrowly family reflexive.

Suppose, conversely, that $I^\flat$ is narrowly family reflexive and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Since $\widetilde{T} \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, we get, by narrow family reflexivity, $I^\flat_\Sigma[\phi, \phi] \leq \widetilde{T}$. Thus, $I^\flat$ is roughly family reflexive. ■

Let, again, $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat \subseteq N^\flat$ a collection of natural transformations in $N^\flat$, with two distinguished arguments.

- $I^\flat$ is **roughly family symmetric** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$I^\flat_\Sigma[\phi,\psi] \le \widetilde{T} \quad \text{implies} \quad I^\flat_\Sigma[\psi,\phi] \le \widetilde{T};$$
- $I^\flat$ is **narrowly family symmetric** if, for all $T \in \mathrm{ThFam}^{\not\sim}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$I^\flat_\Sigma[\phi,\psi] \le T \quad \text{implies} \quad I^\flat_\Sigma[\psi,\phi] \le T.$$

Similarly to rough and narrow family reflexivity, rough and narrow family symmetry coincide.

Lemma 1199 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat \subseteq N^\flat$ a family of natural transformations in $N^\flat$, with two distinguished arguments. $I^\flat$ is roughly family symmetric if and only if it is narrowly family symmetric.*

Proof: Suppose, first, that $I^\flat$ is roughly family symmetric and consider $T \in \mathrm{ThFam}^{\not\sim}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $I^\flat_\Sigma[\phi,\psi] \le T$. Since $T \in \mathrm{ThFam}^{\not\sim}(\mathcal{I})$, we have $\widetilde{T} = T$, whence, by hypothesis, $I^\flat_\Sigma[\phi,\psi] \le \widetilde{T}$. Applying rough family symmetry, we get $I^\flat_\Sigma[\psi,\phi] \le \widetilde{T} = T$. Thus, $I^\flat$ is narrowly family symmetric.

Suppose, conversely, that $I^\flat$ is narrowly family symmetric and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $I^\flat_\Sigma[\phi,\psi] \le \widetilde{T}$. Since $\widetilde{T} \in \mathrm{ThFam}^{\not\sim}(\mathcal{I})$, we get, by narrow family symmetry, $I^\flat_\Sigma[\psi,\phi] \le \widetilde{T}$. Thus, $I^\flat$ is roughly family symmetric. ■

Let, once more, $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat \subseteq N^\flat$ a collection of natural transformations in $N^\flat$, with two distinguished arguments.

- $I^\flat$ is **roughly family transitive** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$,
$$I^\flat_\Sigma[\phi,\psi] \cup I^\flat_\Sigma[\psi,\chi] \le \widetilde{T} \quad \text{implies} \quad I^\flat_\Sigma[\phi,\chi] \le \widetilde{T};$$
- $I^\flat$ is **narrowly family transitive** if, for all $T \in \mathrm{ThFam}^{\not\sim}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$,
$$I^\flat_\Sigma[\phi,\psi] \cup I^\flat_\Sigma[\psi,\chi] \le T \quad \text{implies} \quad I^\flat_\Sigma[\phi,\chi] \le T.$$

Rough and narrow family transitivity also coincide.

Lemma 1200 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat \subseteq N^\flat$ a family of natural transformations in $N^\flat$, with two distinguished arguments. $I^\flat$ is roughly family transitive if and only if it is narrowly family transitive.*

Proof: Similar to the proof of Lemma 1199. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat \subseteq N^\flat$ a collection of natural transformations in $N^\flat$, with two distinguished arguments.

- $I^\flat$ is **roughly family compatible** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\sigma^\flat \in N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,
$$\bigcup_{i<k} \overleftrightarrow{I^\flat}_\Sigma[\phi_i, \psi_i] \leq \widetilde{T} \quad \text{implies} \quad I^\flat_\Sigma[\sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi})] \leq \widetilde{T};$$
- $I^\flat$ is **narrowly family compatible** if, for all $T \in \mathrm{ThFam}^{\nsim}(\mathcal{I})$, all $\sigma^\flat \in N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,
$$\bigcup_{i<k} \overleftrightarrow{I^\flat}_\Sigma[\phi_i, \psi_i] \leq T \quad \text{implies} \quad I^\flat_\Sigma[\sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi})] \leq T.$$

Rough and narrow family transitivity also coincide.

Lemma 1201 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat \subseteq N^\flat$ a family of natural transformations in $N^\flat$, with two distinguished arguments. $I^\flat$ is roughly family compatible if and only if it is narrowly family compatible.*

Proof: Similar to the proof of Lemma 1199. ∎

Finally, we define the property of possessing the rough and the narrow family modus ponens. Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat \subseteq N^\flat$ a collection of natural transformations in $N^\flat$, with two distinguished arguments.

- $I^\flat$ has the **rough family MP** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi \in \widetilde{T}_\Sigma \quad \text{and} \quad I^\flat_\Sigma[\phi, \psi] \leq \widetilde{T} \quad \text{imply} \quad \psi \in \widetilde{T}_\Sigma;$$
- $I^\flat$ has the **narrow family MP** if, for all $T \in \mathrm{ThFam}^{\nsim}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi \in T_\Sigma \quad \text{and} \quad I^\flat_\Sigma[\phi, \psi] \leq T \quad \text{imply} \quad \psi \in T_\Sigma.$$

As with all preceding properties, the rough and narrow family MP turn out to be identical properties.

Lemma 1202 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat \subseteq N^\flat$ a family of natural transformations in $N^\flat$, with two distinguished arguments. $I^\flat$ has the rough family MP if and only if it has the narrow family MP.*

Proof: The proof again follows the lines of the proof of Lemma 1199, but we describe it also in detail.

Suppose, first, that $I^\flat$ has the rough family MP and let $T \in \mathrm{ThFam}^{\nsim}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $I^\flat_\Sigma[\phi,\psi] \leq T$. Again, by hypothesis, $\widetilde{T} = T$, whence, we get $\phi \in \widetilde{T}_\Sigma$ and $I^\flat_\Sigma[\phi,\psi] \leq \widetilde{T}$. Thus, by rough family MP, we get that $\psi \in \widetilde{T}_\Sigma$, i.e., $\psi \in T_\Sigma$. Thus, $I^\flat$ has the narrow family MP.

Assume, conversely, that $I^\flat$ has the narrow family MP and consider $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \widetilde{T}_\Sigma$ and $I^\flat_\Sigma[\phi,\psi] \leq \widetilde{T}$. Since $\widetilde{T} \in \mathrm{ThFam}^{\nsim}(\mathcal{I})$, we may apply narrow family MP to conclude that $\psi \in \widetilde{T}_\Sigma$. This proves that $I^\flat$ has the rough family MP. ■

We say that $\mathcal{I}$ is **syntactically roughly/narrowly family monotone** if there exists $I^\flat \subseteq N^\flat$, with two distinguished arguments, such that $I^\flat$ satisfies:

- narrow family reflexivity;
- narrow family transitivity;
- narrow family compatibility; and
- narrow family MP.

In that case, we call $I^\flat$ a **set of witnessing natural transformations**, or, more simply, **witnessing transformations** (of the syntactic rough/narrow family monotonicity of $\mathcal{I}$).

It turns out that, if $\mathcal{I}$ is a syntactically narrowly family monotone π-institution, with witnessing transformations $I^\flat$, then $\overleftrightarrow{I^\flat}(T)$ is a congruence system on $\mathbf{F}$ compatible with T, for all $T \in \mathrm{ThFam}^{\nsim}(\mathcal{I})$. This forms a "narrow" analog of Proposition 790.

Proposition 1203 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is syntactically narrowly family monotone, with witnessing transformations* $I^\flat$, *then, for all* $T \in \mathrm{ThFam}^{\nsim}(\mathcal{I})$, $\overleftrightarrow{I^\flat}(T)$ *is a congruence system on* $\mathbf{F}$ *compatible with* T.

Proof: The proof follows along the lines of the proof of Proposition 790. So we give an outline. Let $T \in \mathrm{ThFam}^{\nsim}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$. The narrow family reflexivity of $I^\flat$ ensures that $\langle \phi, \phi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$. The fact that $\overleftrightarrow{I^\flat}$ is the symmetrization of $I^\flat$ ensures that $\langle \phi, \psi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$ implies that $\langle \psi, \phi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$. The narrow family transitivity of $I^\flat$ guarantees that $\langle \phi, \psi \rangle, \langle \psi, \chi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$ imply $\langle \phi, \chi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$.

Suppose, next, that $\sigma^\flat \in N^\flat$, $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$. Then, the narrow family compatibility of $I^\flat$ ensures that, if, for all $i < k$, $\langle \phi_i, \psi_i \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$, then

$\langle\sigma^\flat_\Sigma(\vec{\phi}),\sigma^\flat_\Sigma(\vec{\psi})\rangle\in I^\flat_\Sigma(T)$. Thus, $\overleftrightarrow{I}^\flat(T)$ is a congruence family on $\mathbf{F}$. However, by Lemma 93, $\overleftrightarrow{I}^\flat(T)$ is a relation system on $\mathbf{F}$. Hence, $\overleftrightarrow{I}^\flat(T)$ is a congruence system on $\mathbf{F}$.

It only remains to show that $\overleftrightarrow{I}^\flat(T)$ is compatible with T. Assume that $\phi\in T_\Sigma$ and $\langle\phi,\psi\rangle\in\overleftrightarrow{I}^\flat{}_\Sigma(T)$. Since $I^\flat\subseteq\overleftrightarrow{I}^\flat$, we get, by the narrow family MP of $I^\flat$, that $\psi\in T_\Sigma$. Thus, $\overleftrightarrow{I}^\flat(T)$ is also compatible with T. ■

Proposition 1203 shows that $\overleftrightarrow{I}^\flat$ defines Leibniz congruence systems of theory families in $\mathrm{ThFam}^{\nmid}(\mathcal{I})$. Following similar terminology adopted in Chapter 14, we say that $I^\flat$ **roughly** or **narrowly defines Leibniz congruence systems** of theory families in $\mathcal{I}$ if, for all $T\in\mathrm{ThFam}^{\nmid}(\mathcal{I})$,

$$\overleftrightarrow{I}^\flat(T)=\Omega(T).$$

Then, in what is an analog of Corollary 791, we obtain

Corollary 1204 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is syntactically narrowly family monotone, with witnessing transformations* $I^\flat$*, then* $I^\flat$ *narrowly defines Leibniz congruence systems of theory families in* $\mathcal{I}$.

Proof: By Proposition 1203 and Corollary 98. ■

Corollary 1204 allows establishing the fact that syntactic narrow family monotonicity implies (semantic) narrow family monotonicity. This forms an analog of Theorem 792.

Theorem 1205 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is syntactically narrowly family monotone, then it is narrowly family monotone.*

Proof: Suppose that $\mathcal{I}$ is syntactically narrowly family monotone with witnessing transformations $I^\flat$. Let $T,T'\in\mathrm{ThFam}^{\nmid}(\mathcal{I})$, such that $T\le T'$. Then

$$\begin{array}{rcll}\Omega(T)&=&\overleftrightarrow{I}^\flat(T)&\text{(by Corollary 1204)}\\&\le&\overleftrightarrow{I}^\flat(T')&\text{(by Lemma 94)}\\&=&\Omega(T').&\text{(by Corollary 1204)}\end{array}$$

Thus, $\mathcal{I}$ is narrowly family monotone. ■

We now introduce the notion of the rough/narrow reflexive core of a π-institution $\mathcal{I}$ in a way analogous to the reflexive core, which was introduced in Chapter 11. Its introduction will enable us to provide a characterization of the syntactical narrow family monotonicity property and to establish a relationship between this property and its semantic counterpart.

Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and $\mathcal{I}=\langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$.

- The **rough reflexive core** of $\mathcal{I}$ is the collection

$$\begin{array}{rcl}\widetilde{R}^{\mathcal{I}} & = & \{\rho^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}(\mathcal{I}))(\forall \Sigma \in |\mathbf{Sign}^\flat|) \\ & & (\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\rho^\flat_\Sigma[\phi,\phi] \le \widetilde{T})\};\end{array}$$

- The **narrow reflexive core** of $\mathcal{I}$ is the collection

$$\begin{array}{rcl}R^{\mathcal{I}\natural} & = & \{\rho^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}^\natural(\mathcal{I}))(\forall \Sigma \in |\mathbf{Sign}^\flat|) \\ & & (\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\rho^\flat_\Sigma[\phi,\phi] \le T)\}.\end{array}$$

These two notions are identical, as shown in the following proposition, and this justifies the usage of the terms rough and narrow reflexive core interchangeably in this context.

Proposition 1206 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then* $\widetilde{R}^{\mathcal{I}} = R^{\mathcal{I}\natural}$*.*

Proof: On the one hand, if $\rho^\flat \in \widetilde{R}^{\mathcal{I}}$, $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, then, by the definition of the rough reflexive core, $\rho^\flat_\Sigma[\phi,\phi] \le \widetilde{T} = T$, where the equality follows from the assumption that $T \in \mathrm{ThFam}^\natural(\mathcal{I})$. This shows that $\rho^\flat \in R^{\mathcal{I}\natural}$. On the other hand, if $\rho^\flat \in R^{\mathcal{I}\natural}$, $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, then, since $\widetilde{T} \in \mathrm{ThFam}^\natural(\mathcal{I})$, we get by the definition of $R^{\mathcal{I}\natural}$, $\rho^\flat_\Sigma[\phi,\phi] \le \widetilde{T}$. This shows that $\rho^\flat \in \widetilde{R}^{\mathcal{I}}$. ■

Given any theory family in $\mathrm{ThFam}^\natural(\mathcal{I})$, the relation system $R^{\mathcal{I}\natural}(T)$ is a reflexive relation system on $\mathbf{F}$. This forms an analog of Lemma 773.

Lemma 1207 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then, for all* $T \in \mathrm{ThFam}^\natural(\mathcal{I})$*,* $R^{\mathcal{I}\natural}(T)$ *is a reflexive relation system on* $\mathbf{F}$*.*

Proof: Let $T \in \mathrm{ThFam}^\natural(\mathcal{I})$. By Lemma 93, $R^{\mathcal{I}\natural}(T)$ is a relation system on $\mathbf{F}$. For reflexivity, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. By the definition of the narrow reflexive core, $R^{\mathcal{I}\natural}_\Sigma[\phi,\phi] \le T$. Thus, $\langle\phi,\phi\rangle \in R^{\mathcal{I}\natural}_\Sigma(T)$ and, therefore, $R^{\mathcal{I}\natural}(T)$ is reflexive. ■

As in Lemma 775, it may also be established that $R^{\mathcal{I}\natural}(T)$ is a symmetric relation system on $\mathbf{F}$, for all $T \in \mathrm{ThFam}^\natural(\mathcal{I})$.

Lemma 1208 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then, for all* $T \in \mathrm{ThFam}^\natural(\mathcal{I})$*,* $R^{\mathcal{I}\natural}(T)$ *is a symmetric relation system on* $\mathbf{F}$*.*

Proof: Let $T \in \mathrm{ThFam}^\natural(\mathcal{I})$. Again, Lemma 93 shows that $R^{\mathcal{I}\natural}(T)$ is a relation system. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle\phi,\psi\rangle \in R^{\mathcal{I}\natural}_\Sigma(T)$. Equivalently, $R^{\mathcal{I}\natural}_\Sigma[\phi,\psi] \le T$. Consider any $\rho^\flat \in R^{\mathcal{I}\natural}$. By the definition of

$R^{\mathcal{I}\natural}$, we get that $\overline{\rho^\flat} \in R^{\mathcal{I}\natural}$. Therefore, by the hypothesis, $\overline{\rho^\flat}_\Sigma[\phi,\psi] \le T$. But this gives $\rho^\flat_\Sigma[\psi,\phi] \le T$. Since this holds for all $\rho^\flat \in R^{\mathcal{I}\natural}$, we conclude that $R^{\mathcal{I}\natural}_\Sigma[\psi,\phi] \le T$. Hence, $\langle\psi,\phi\rangle \in R^{\mathcal{I}\natural}_\Sigma(T)$. Therefore, $R^{\mathcal{I}\natural}(T)$ is a symmetric relation system on $\mathbf{F}$. ■

Continuing the study of sequence of properties of $R^{\mathcal{I}\natural}(T)$, we show that, for all theory families $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, $R^{\mathcal{I}\natural}(T)$ has the compatibility property in $\mathbf{F}$.

Lemma 1209 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Then, for all $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, $R^{\mathcal{I}\natural}(T)$ has the compatibility property in $\mathbf{F}$.*

Proof: Let $T \in \mathrm{ThFam}^\natural(\mathcal{I})$. We rely on Corollary 12. Let $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ is in $N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle\phi,\psi\rangle \in R^{\mathcal{I}\natural}_\Sigma(T)$ or, equivalently, $R^{\mathcal{I}\natural}_\Sigma[\phi,\psi] \le T$. Let $\rho^\flat : (\mathrm{SEN}^\flat)^n \to \mathrm{SEN}^\flat$ be arbitrary in $R^{\mathcal{I}\natural}$. We consider the natural transformation $\rho'^\flat : (\mathrm{SEN}^\flat)^{n+k} \to \mathrm{SEN}^\flat$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\zeta,\eta,\vec{\chi},\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$, by

$$\rho'^\flat_\Sigma(\zeta,\eta,\vec{\chi},\vec{\xi}) = \rho^\flat_\Sigma(\sigma^\flat_\Sigma(\zeta,\vec{\chi}),\sigma^\flat_\Sigma(\eta,\vec{\chi}),\vec{\xi}).$$

Note that, since $\sigma^\flat \in N^\flat$, $\rho^\flat \in N^\flat$ and

$$\begin{aligned}\rho'^\flat \;=\; \rho^\flat \circ \langle &\sigma^\flat \circ \langle p^{n+k,0}, p^{n+k,2},\dots,p^{n+k,k}\rangle, \sigma^\flat \circ \langle p^{n+k,1},p^{n+k,2},\dots,p^{n+k,k}\rangle,\\ &p^{n+k,k+1},\dots,p^{n+k,n+k-1}\rangle,\end{aligned}$$

we get that $\rho'^\flat \in N^\flat$. Moreover, for all $T' \in \mathrm{ThFam}^\natural(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\zeta,\vec{\chi},\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{aligned}\rho'^\flat_\Sigma(\zeta,\zeta,\vec{\chi},\vec{\xi}) &= \rho^\flat_\Sigma(\sigma^\flat_\Sigma(\zeta,\vec{\chi}),\sigma^\flat_\Sigma(\zeta,\vec{\chi}),\vec{\xi}) \quad \text{(by definition of } \rho'^\flat)\\ &\in T'_\Sigma. \quad (\text{since } \rho^\flat \in R^{\mathcal{I}\natural}).\end{aligned}$$

Thus, by the definition of the narrow reflexive core, we get that $\rho'^\flat \in R^{\mathcal{I}\natural}$.

Now since $\rho'^\flat \in R^{\mathcal{I}\natural}$ and, by hypothesis, $R^{\mathcal{I}\natural}_\Sigma[\phi,\psi] \le T$, we get, in particular, that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\chi},\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}),\vec{\xi}) \in T_{\Sigma'}.$$

Hence, a fortiori, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\sigma^\flat_\Sigma(\phi,\vec{\chi})),\mathrm{SEN}^\flat(f)(\sigma^\flat_\Sigma(\psi,\vec{\chi})),\vec{\xi}) \in T_{\Sigma'}.$$

This proves that

$$\rho^\flat_\Sigma[\sigma^\flat_\Sigma(\phi,\vec{\chi}),\sigma^\flat_\Sigma(\psi,\vec{\chi})] \le T.$$

Since this holds for all $\rho^\flat \in R^{\mathcal{I}\natural}$, we get that $R^{\mathcal{I}\natural}_\Sigma[\sigma^\flat_\Sigma(\phi,\vec{\chi}),\sigma^\flat_\Sigma(\psi,\vec{\chi})] \le T$ or, equivalently, $\langle\sigma^\flat_\Sigma(\phi,\vec{\chi}),\sigma^\flat_\Sigma(\psi,\vec{\chi})\rangle \in R^{\mathcal{I}\natural}_\Sigma(T)$. Therefore, $R^{\mathcal{I}\natural}(T)$ has the congruence compatibility property in $\mathbf{F}$. ■

We now show, in an analog of Theorem 799, that possession of the narrow family modus ponens by the narrow reflexive core intrinsically characterizes syntactic narrow family monotonicity. We start by showing that possession of the narrow family MP by the narrow reflexive core is necessary for syntactic narrow family monotonicity. This forms an analog of Theorem 796.

Theorem 1210 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically narrowly family monotone, then* $R^{\mathcal{I}\nmid}$ *has the narrow family MP.*

Proof: Suppose that $\mathcal{I}$ is syntactically narrowly family monotone with witnessing transformations $I^\flat$. Since, by definition, $I^\flat$ is narrowly family reflexive, we get, by definition of $R^{\mathcal{I}\nmid}$, $I^\flat \subseteq R^{\mathcal{I}\nmid}$. Thus, since $I^\flat$ has narrow family MP in $\mathcal{I}$, we get that, a fortiori, $R^{\mathcal{I}\nmid}$ also satisfies the narrow family MP. ∎

Possession of narrow family MP by $R^{\mathcal{I}\nmid}$ implies that $R^{\mathcal{I}\nmid}$ has the narrow family transitivity in $\mathcal{I}$. This proposition forms an analog of Proposition 797.

Proposition 1211 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}\nmid}$ *has the narrow family MP, then it also has the narrow family transitivity in* $\mathcal{I}$*.*

Proof: Suppose that $R^{\mathcal{I}\nmid}$ has the narrow family MP and let $T \in \mathrm{ThFam}^\nmid(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle, \langle \psi, \chi \rangle \in R^{\mathcal{I}\nmid}_\Sigma(T)$. This means that $R^{\mathcal{I}\nmid}_\Sigma[\phi, \psi] \leq T$ and $R^{\mathcal{I}\nmid}_\Sigma[\psi, \chi] \leq T$. Then, by Lemma 1209, we get that, for all $\rho^\flat \in R^{\mathcal{I}\nmid}$, and all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$R^{\mathcal{I}\nmid}_{\Sigma'}[\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), \vec{\xi}), \\ \rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi})] \leq T.$$

But, by hypothesis, $R^{\mathcal{I}\nmid}_\Sigma[\phi, \psi] \leq T$ and $R^{\mathcal{I}\nmid}$ has the narrow family MP. Therefore, for all $\rho^\flat \in R^{\mathcal{I}\nmid}$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi}) \subseteq T_{\Sigma'},$$

i.e., $R^{\mathcal{I}\nmid}_\Sigma[\phi, \chi] \leq T$. This shows $\langle \phi, \chi \rangle \in R^{\mathcal{I}\nmid}_\Sigma(T)$ and, hence, $R^{\mathcal{I}\nmid}$ is narrowly family transitive in $\mathcal{I}$. ∎

We are now ready to show a converse of Theorem 1210, i.e., that possession of the narrow family MP by $R^{\mathcal{I}\nmid}$ suffices to establish the syntactic narrow family monotonicity of $\mathcal{I}$, since, in that case, $R^{\mathcal{I}\nmid}$ serves as a family of witnessing transformations. The following constitutes an analog of Theorem 798.

Theorem 1212 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}\nleq}$ *has the narrow family MP, then* $\mathcal{I}$ *is syntactically narrowly family monotone, with witnessing transformations* $R^{\mathcal{I}\nleq}$*.*

Proof: By Lemma 1207, $R^{\mathcal{I}\nleq}$ is narrowly family reflexive in $\mathcal{I}$. By Lemma 1208, $R^{\mathcal{I}\nleq}$ is narrowly family symmetric in $\mathcal{I}$. By hypothesis and Proposition 1211, it is narrowly family transitive in $\mathcal{I}$. By Lemma 1209 it has the narrow family compatibility property in $\mathcal{I}$. Finally, by hypothesis, it has the narrow family MP in $\mathcal{I}$. We conclude that $\mathcal{I}$ is syntactically narrowly family monotone, with witnessing transformations $R^{\mathcal{I}\nleq}$. ∎

Theorems 1210 and 1212 provide the promised characterization of syntactic narrow family monotonicity in terms of the narrow family MP of the narrow reflexive core.

$$\begin{array}{c} \mathcal{I} \text{ is Syntactically Narrow} \\ \text{Family Monotone} \end{array} \longleftrightarrow \begin{array}{c} R^{\mathcal{I}\nleq} \text{ has Narrow Family} \\ \text{Modus Ponens} \end{array}.$$

Theorem 1213 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically narrowly family monotone if and only if* $R^{\mathcal{I}\nleq}$ *has the narrow family MP in* $\mathcal{I}$*.*

Proof: Theorem 1210 gives the "only if" and the "if" is by Theorem 1212. ∎

A related alternative characterization asserts that syntactic narrow family monotonicity amounts to the narrow definability of Leibniz congruence systems of theory families by the narrow reflexive core. This result forms an analog of Theorem 801.

$$\begin{array}{c} \mathcal{I} \text{ is Syntactically Narrow} \\ \text{Family Monotone} \end{array} \longleftrightarrow \begin{array}{c} R^{\mathcal{I}\nleq} \text{ Defines Leibniz Congruence} \\ \text{Systems of Theory Families} \end{array}.$$

Theorem 1214 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically narrowly family monotone if and only if, for all* $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$*,*

$$\Omega(T) = R^{\mathcal{I}\nleq}(T).$$

Proof: If $\mathcal{I}$ is syntactically narrowly family monotone, then, by Theorem 1210, $R^{\mathcal{I}\nleq}$ has the narrow family MP in $\mathcal{I}$. Thus, by Theorem 1212, $R^{\mathcal{I}\nleq}$ is a family of witnessing transformations for the syntactic narrow family monotonicity of $\mathcal{I}$. Thus, by Corollary 1204, for all $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, $\Omega(T) = R^{\mathcal{I}\nleq}(T)$.

Suppose, conversely, that the displayed condition holds. Then $R^{\mathcal{I}\natural}$ is narrowly family reflexive, narrowly family transitive and has the narrow family compatibility property and the narrow family MP. Hence, it constitutes a collection of witnessing transformations and, therefore, $\mathcal{I}$ is syntactically narrowly family monotone. ∎

In the case of syntactic protoalgebraicity, in Chapter 11, it was shown that the property that separates syntactic protoalgebraicity from protoalgebraicity is the Leibniz compatibility property with respect to the theory family generated by the reflexive core, i.e., the property that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle\phi,\psi\rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi,\psi])).$$

The task of characterizing those π-institutions that are syntactically narrowly family monotone among those that are narrowly family monotone is more involved. The additional complications arise from the fact that the class of theory families $\mathrm{ThFam}^{\natural}(\mathcal{I})$ may not be, in general, closed under (signature-wise) intersections and, hence, may not possess a least element. Therefore, to pinpoint syntactic narrow family monotonicity inside the class of narrow family monotone π-institutions, we need to devise a suitable analog of the Leibniz compatibility property with respect to the theory family generated by the narrow reflexive core.

To introduce this analog and to understand how it comes about and how it extends the Leibniz property, we interject a small discussion. Recall that a π-institution $\mathcal{I}$ is *protoalgebraic* if its Leibniz operator is monotone on theory families. Recall, also, that its reflexive core $R^{\mathcal{I}}$ is said to be *Leibniz* if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle\phi,\psi\rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi,\psi])).$$

If a π-institution is protoalgebraic and has a Leibniz reflexive core, then it satisfies the global family modus ponens. This was shown in Chapter 11 using the following method. Considering $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $R^{\mathcal{I}}_\Sigma[\phi,\psi] \leq T$, we get

- $\langle\phi,\psi\rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi,\psi]))$ first, by applying the Leibniz property;
- $\Omega(C(R^{\mathcal{I}}_\Sigma[\phi,\psi])) \leq \Omega(T)$, by applying the hypothesis that $R^{\mathcal{I}}_\Sigma[\phi,\psi] \leq T$ and the postulated protoalgebraicity of $\mathcal{I}$.

However, in case of narrow family monotonicity, the plausibility of $R^{\mathcal{I}\natural}_\Sigma[\phi,\psi]$ having some empty components makes it likely that, in the second stage, narrow family monotonicity may not be applicable to ensure the inclusion $\Omega(C(R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])) \leq \Omega(T)$.

An obvious remedy is to restrict attention to those π-institutions in which $C(R^{\mathcal{I}\natural}_\Sigma[\phi,\psi]) \in \mathrm{ThFam}^{\natural}(\mathcal{I})$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, and leave the Leibniz property unaltered. A more relaxed approach is to assume that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, the poset

$$[R^{\mathcal{I}\natural}_\Sigma[\phi,\psi]) \coloneqq \{T \in \mathrm{ThFam}^\natural(\mathcal{I}) : R^{\mathcal{I}\natural}_\Sigma[\phi,\psi] \le T\}$$

satisfies the descending chain condition and to postulate that every minimal element $T \in [R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$ satisfies $\langle\phi,\psi\rangle \in \Omega_\Sigma(T)$.

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.

- For $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, define

$$[R^{\mathcal{I}\natural}_\Sigma[\phi,\psi]) \coloneqq \{T \in \mathrm{ThFam}^\natural(\mathcal{I}) : R^{\mathcal{I}\natural}_\Sigma[\phi,\psi] \le T\};$$

- For $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $\mathcal{I}$ is called $\langle\Sigma,\phi,\psi\rangle$-**reflexively covered** if, for every theory family $T \in [R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$, there exists minimal $T' \in [R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$, such that $T' \le T$;
- $\mathcal{I}$ is called **reflexively covered** if it is $\langle\Sigma,\phi,\psi\rangle$-reflexively covered, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$.

Given $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, we write

$$\min[R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$$

for the collection of minimal elements in $[R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$.

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. We say that the narrow reflexive core $R^{\mathcal{I}\natural}$ of $\mathcal{I}$ is **Leibniz** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$ and all $T \in \min[R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$,

$$\langle\phi,\psi\rangle \in \Omega_\Sigma(T).$$

We show, in an analog of Proposition 785, that, if $R^{\mathcal{I}\natural}$ has the narrow family MP, then it is Leibniz. In fact, the proof demonstrates that, under the narrow family MP, a stronger property than that of being Leibniz holds; more concretely, that for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$ and all $T \in [R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$,

$$\langle\phi,\psi\rangle \in \Omega_\Sigma(T).$$

Proposition 1215 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $R^{\mathcal{I}\natural}$ has the narrow family MP, then for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$ and all $T \in [R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$, $\langle\phi,\psi\rangle \in \Omega_\Sigma(T)$.*

Proof: Suppose $R^{\mathcal{I}\natural}$ has the narrow family MP and let $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $R^{\mathcal{I}\natural}_\Sigma[\phi,\psi] \le T$. To verify that $\langle\phi,\psi\rangle \in \Omega_\Sigma(T)$, we use Theorem 19. Let $\sigma^\flat \in N^\flat$, $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and

$\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$, such that $\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \in T_{\Sigma'}$. Since $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, by Lemma 1209,

$$R^{\mathcal{I}\nleq}_{\Sigma'}[\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}), \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi})] \leq T.$$

Thus, since, by hypothesis, $R^{\mathcal{I}\nleq}$ has the narrow family MP, we obtain

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}) \in T_{\Sigma'}.$$

By symmetry, we conclude that, for all $\sigma^\flat \in N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \in T_{\Sigma'} \quad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}) \in T_{\Sigma'}.$$

Hence, by Theorem 19, $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$ and, therefore, $R^{\mathcal{I}\nleq}$ is Leibniz. ∎

Corollary 1216 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $R^{\mathcal{I}\nleq}$ has the narrow family MP, then it is Leibniz.*

Proof: Directly by Proposition 1215. ∎

In the opposite direction, when dealing with reflexively covered π-institutions, we may show that narrow family monotonicity combined with the Leibniz property of the narrow reflexive core imply that the narrow reflexive core has the narrow family modus ponens in $\mathcal{I}$.

Proposition 1217 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a reflexively covered, narrowly family monotone π-institution based on $\mathbf{F}$. If $R^{\mathcal{I}\nleq}$ is Leibniz, then it has the narrow family MP in $\mathcal{I}$.*

Proof: Let $\mathcal{I}$ be a reflexively covered π-institution. Suppose that $\mathcal{I}$ is narrowly family monotone and that $R^{\mathcal{I}\nleq}$ is Leibniz. Let $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $R^{\mathcal{I}\nleq}_\Sigma[\phi, \psi] \leq T$. Since $\mathcal{I}$ is reflexively covered, there exists $T' \in \min[R^{\mathcal{I}\nleq}_\Sigma[\phi, \psi]]$, such that $T' \leq T$. Now we have

$$\begin{array}{rcll} \langle \phi, \psi \rangle & \in & \Omega_\Sigma(T') & (\text{since } R^{\mathcal{I}\nleq} \text{ is Leibniz and } T' \in \min[R^{\mathcal{I}\nleq}_\Sigma[\phi, \psi]]) \\ & \subseteq & \Omega_\Sigma(T). & (\text{since } T' \leq T \text{ and } \mathcal{I} \text{ is narrowly family monotone}) \end{array}$$

Therefore, since $\phi \in T_\Sigma$, we get, by the compatibility of $\Omega(T)$ with T, that $\psi \in T_\Sigma$. We conclude that $R^{\mathcal{I}\nleq}$ has the narrow family MP in $\mathcal{I}$. ∎

Thus, at least for reflexively covered π-institutions, it is possible to show that the class of syntactically narrowly monotone ones inside the class of the narrowly monotone ones can be characterized exactly by the Leibniz property of the narrow reflexive core. This forms a partial analog of Theorem 805 in the narrow context.

Theorem 1218 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a reflexively covered* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically narrowly family monotone if and only if it is narrowly family monotone and has a Leibniz narrow reflexive core.*

Proof: Let $\mathcal{I}$ be a reflexively covered π-institution.

Suppose, first, that $\mathcal{I}$ is syntactically narrowly family monotone. Then it is narrowly family monotone by Theorem 1205. Moreover, its narrow reflexive core has the narrow family MP by Theorem 1210 and, hence, by Corollary 1216, its narrow reflexive core is Leibniz.

Suppose, conversely, that $\mathcal{I}$ is narrowly family monotone with a Leibniz narrow reflexive core. Then, by Proposition 1217, its narrow reflexive core has the narrow family MP and, therefore, by Theorem 1212, $\mathcal{I}$ is syntactically narrowly family monotone, with witnessing transformations $R^{\mathcal{I}\nmid}$. ∎

15.2 Syntactic Narrow System Monotonicity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

Recall that $\mathcal{I}$ is **narrowly system monotone** if, for all $T, T' \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$,

$$T \le T' \quad \text{implies} \quad \Omega(T) \le \Omega(T').$$

In this section, in analogy with Section 15.1, we introduce and study a syntactic analog of this concept.

First, the concepts of narrow family reflexivity, narrow family symmetry, narrow family transitivity, narrow family compatibility and narrow family modus ponens can all be relativized to $\mathrm{ThSys}^{\nmid}(\mathcal{I})$.

Let, as above, $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Moreover, suppose that $I^\flat \subseteq N^\flat$ is a collection of natural transformations in $N^\flat$, with two distinguished arguments.

- $I^\flat$ is **narrowly system reflexive** if, for all $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma[\phi, \phi] \le T;$$

- $I^\flat$ is **narrowly system symmetric** if, for all $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma[\phi, \psi] \le T \quad \text{implies} \quad I^\flat_\Sigma[\psi, \phi] \le T;$$

- $I^\flat$ is **narrowly system transitive** if, for all $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma[\phi, \psi] \cup I^\flat_\Sigma[\psi, \chi] \le T \quad \text{implies} \quad I^\flat_\Sigma[\phi, \chi] \le T;$$

- $I^\flat$ is **narrowly system compatible** if, for all $T \in \mathrm{ThSys}^\nmid(\mathcal{I})$, all $\sigma^\flat \in N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\bigcup_{i<k} \overleftrightarrow{I^\flat}_\Sigma[\phi_i, \psi_i] \leq T \quad \text{implies} \quad I^\flat_\Sigma[\sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi})] \leq T;$$

- $I^\flat$ has the **narrow system MP** if, for all $T \in \mathrm{ThSys}^\nmid(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{and} \quad I^\flat_\Sigma[\phi, \psi] \leq T \quad \text{imply} \quad \psi \in T_\Sigma.$$

We say that $\mathcal{I}$ is **syntactically narrowly system monotone** if there exists $I^\flat \subseteq N^\flat$, with two distinguished arguments, such that $I^\flat$ satisfies:

- narrow system reflexivity;
- narrow system transitivity;
- narrow system compatibility; and
- narrow system MP.

In that case, we call $I^\flat$ a **set of witnessing natural transformations**, or, more simply, **witnessing transformations** (of the syntactic narrow system monotonicity of $\mathcal{I}$).

It turns out that, if $\mathcal{I}$ is a syntactically narrowly system monotone π-institution, with witnessing transformations $I^\flat$, then $\overleftrightarrow{I^\flat}(T)$ is a congruence system on $\mathbf{F}$ compatible with T, for all $T \in \mathrm{ThSys}^\nmid(\mathcal{I})$. This forms a system analog of Proposition 1203.

Proposition 1219 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically narrowly system monotone, with witnessing transformations* $I^\flat$*, then, for all* $T \in \mathrm{ThSys}^\nmid(\mathcal{I})$*,* $\overleftrightarrow{I^\flat}(T)$ *is a congruence system on* $\mathbf{F}$ *compatible with* T*.*

Proof: The proof is similar to that of Proposition 1203. Let $T \in \mathrm{ThSys}^\nmid(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$. The narrow system reflexivity of $I^\flat$ ensures that $\langle \phi, \phi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$. The fact that $\overleftrightarrow{I^\flat}$ is the symmetrization of $I^\flat$ ensures that $\langle \phi, \psi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$ implies that $\langle \psi, \phi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$. The narrow system transitivity of $I^\flat$ guarantees that $\langle \phi, \psi \rangle, \langle \psi, \chi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$ imply $\langle \phi, \chi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$.

Suppose, next, that $\sigma^\flat \in N^\flat$, $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$. Then, the narrow system compatibility of $I^\flat$ ensures that, if, for all $i < k$, $\langle \phi_i, \psi_i \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$, then $\langle \sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi}) \rangle \in I^\flat_\Sigma(T)$. Thus, $\overleftrightarrow{I^\flat}(T)$ is a congruence family on $\mathbf{F}$. However, by Lemma 93, $\overleftrightarrow{I^\flat}(T)$ is a relation system on $\mathbf{F}$. Hence, $\overleftrightarrow{I^\flat}(T)$ is a congruence system on $\mathbf{F}$.

It only remains to show that $\overleftrightarrow{I^\flat}(T)$ is compatible with T. Assume that $\phi \in T_\Sigma$ and $\langle \phi, \psi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$. Since $I^\flat \subseteq \overleftrightarrow{I^\flat}$, we get, by the narrow system MP of $I^\flat$, that $\psi \in T_\Sigma$. Thus, $\overleftrightarrow{I^\flat}(T)$ is also compatible with T. ∎

Proposition 1219 shows that $\overleftrightarrow{I^\flat}$ defines Leibniz congruence systems of theory systems in $\mathrm{ThSys}^{\nmid}(\mathcal{I})$. Again, following terminology adopted in Section 15.1, we say that $I^\flat$ **narrowly defines Leibniz congruence systems** of theory systems in $\mathcal{I}$ if, for all $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$,

$$\overleftrightarrow{I^\flat}(T) = \Omega(T).$$

Then, in what is an analog of Corollary 1204, we obtain

Corollary 1220 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically narrowly system monotone, with witnessing transformations* $I^\flat$*, then* $I^\flat$ *narrowly defines Leibniz congruence systems of theory systems in* $\mathcal{I}$*.*

Proof: By Proposition 1219 and Corollary 98. ∎

Corollary 1220 shows that syntactic narrow system monotonicity implies (semantic) narrow system monotonicity. This forms an analog of Theorem 1205.

Theorem 1221 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically narrowly system monotone, then it is narrowly system monotone.*

Proof: Suppose that $\mathcal{I}$ is syntactically narrowly system monotone with witnessing transformations $I^\flat$. Let $T, T' \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, such that $T \leq T'$. Then

$$\begin{array}{rcll} \Omega(T) & = & \overleftrightarrow{I^\flat}(T) & \text{(by Corollary 1220)} \\ & \leq & \overleftrightarrow{I^\flat}(T') & \text{(by Lemma 94)} \\ & = & \Omega(T'). & \text{(by Corollary 1220)} \end{array}$$

Thus, $\mathcal{I}$ is narrowly system monotone. ∎

We now introduce the notion of the narrow reflexive system core of a π-institution $\mathcal{I}$ in a way analogous to the narrow reflexive core, which was introduced in Section 15.1. Its introduction will enable us to provide a characterization of the syntactical narrow system monotonicity property and to establish a relationship between this property and its semantic counterpart.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The **narrow reflexive system core** of $\mathcal{I}$ is the collection

$$\begin{aligned} R^{\mathcal{I}s} = \{\rho^\flat \in N^\flat : (\forall T \in \mathrm{ThSys}^{\nmid}(\mathcal{I}))(\forall \Sigma \in |\mathbf{Sign}^\flat|) \\ (\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\rho^\flat_\Sigma[\phi,\phi] \leq T)\}. \end{aligned}$$

Given any theory system in $\mathrm{ThSys}^{\nmid}(\mathcal{I})$, the relation system $R^{\mathcal{I}s}(T)$ is a reflexive relation system on $\mathbf{F}$. This forms an analog of Lemma 1207.

Lemma 1222 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then, for all* $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$*,* $R^{\mathcal{I}s}(T)$ *is a reflexive relation system on* $\mathbf{F}$*.*

Proof: Let $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$. By Lemma 93, $R^{\mathcal{I}s}(T)$ is a relation system on $\mathbf{F}$. For reflexivity, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. By the definition of the narrow reflexive system core, $R^{\mathcal{I}s}_\Sigma[\phi,\phi] \leq T$. Thus, $\langle \phi,\phi \rangle \in R^{\mathcal{I}s}_\Sigma(T)$ and, therefore, $R^{\mathcal{I}s}(T)$ is reflexive. ■

As in Lemma 1208, we establish that $R^{\mathcal{I}s}(T)$ is a symmetric relation system on $\mathbf{F}$, for all $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$.

Lemma 1223 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then, for all* $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$*,* $R^{\mathcal{I}s}(T)$ *is a symmetric relation system on* $\mathbf{F}$*.*

Proof: Let $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$. Again, Lemma 93 shows that $R^{\mathcal{I}s}(T)$ is a relation system. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi,\psi \rangle \in R^{\mathcal{I}s}_\Sigma(T)$. Equivalently, $R^{\mathcal{I}s}_\Sigma[\phi,\psi] \leq T$. Consider any $\rho^\flat \in R^{\mathcal{I}s}$. By the definition of $R^{\mathcal{I}s}$, we get that $\overline{\rho^\flat} \in R^{\mathcal{I}s}$. Therefore, by the hypothesis, $\overline{\rho^\flat}_\Sigma[\phi,\psi] \leq T$. But this gives $\rho^\flat_\Sigma[\psi,\phi] \leq T$. Since this holds for all $\rho^\flat \in R^{\mathcal{I}s}$, we conclude that $R^{\mathcal{I}s}_\Sigma[\psi,\phi] \leq T$. Hence, $\langle \psi,\phi \rangle \in R^{\mathcal{I}s}_\Sigma(T)$. Therefore, $R^{\mathcal{I}s}(T)$ is a symmetric relation system on $\mathbf{F}$. ■

We now show that, for all theory systems $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, $R^{\mathcal{I}s}(T)$ has the compatibility property in $\mathbf{F}$. This forms an analog of Lemma 1209.

Lemma 1224 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then, for all* $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$*,* $R^{\mathcal{I}s}(T)$ *has the compatibility property in* $\mathbf{F}$*.*

Proof: Let $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$. We rely on Corollary 12. Let $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ is in $N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi,\psi \rangle \in R^{\mathcal{I}s}_\Sigma(T)$ or, equivalently, $R^{\mathcal{I}s}_\Sigma[\phi,\psi] \leq T$. Let $\rho^\flat : (\mathrm{SEN}^\flat)^n \to \mathrm{SEN}^\flat$ be arbitrary in $R^{\mathcal{I}s}$. We consider the natural transformation $\rho'^\flat : (\mathrm{SEN}^\flat)^{n+k} \to \mathrm{SEN}^\flat$, defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\zeta,\eta,\vec{\chi},\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$, by

$$\rho'^\flat_\Sigma(\zeta,\eta,\vec{\chi},\vec{\xi}) = \rho^\flat_\Sigma(\sigma^\flat_\Sigma(\zeta,\vec{\chi}),\sigma^\flat_\Sigma(\eta,\vec{\chi}),\vec{\xi}).$$

Note that, since $\sigma^\flat \in N^\flat$, $\rho^\flat \in N^\flat$ and

$$\rho'^\flat = \rho^\flat \circ \langle \sigma^\flat \circ \langle p^{n+k,0}, p^{n+k,2}, \dots, p^{n+k,k} \rangle, \sigma^\flat \circ \langle p^{n+k,1}, p^{n+k,2}, \dots, p^{n+k,k} \rangle, p^{n+k,k+1}, \dots, p^{n+k,n+k-1} \rangle,$$

we get that $\rho'^\flat \in N^\flat$. Moreover, for all $T' \in \mathrm{ThSys}^\nmid(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\zeta, \vec{\chi}, \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rcll} \rho'^\flat_\Sigma(\zeta, \zeta, \vec{\chi}, \vec{\xi}) & = & \rho^\flat_\Sigma(\sigma^\flat_\Sigma(\zeta, \vec{\chi}), \sigma^\flat_\Sigma(\zeta, \vec{\chi}), \vec{\xi}) & \text{(by definition of } \rho'^\flat) \\ & \in & T'_\Sigma. \quad (\text{since } \rho^\flat \in R^{\mathcal{I}s}). & \end{array}$$

Thus, by the definition of the narrow reflexive system core, we get that $\rho'^\flat \in R^{\mathcal{I}s}$.

Now since $\rho'^\flat \in R^{\mathcal{I}s}$ and, by hypothesis, $R^{\mathcal{I}s}_\Sigma[\phi, \psi] \leq T$, we get, in particular, that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi}, \vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}), \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}), \vec{\xi}) \in T_{\Sigma'}.$$

Hence, a fortiori, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\sigma^\flat_\Sigma(\phi, \vec{\chi})), \mathrm{SEN}^\flat(f)(\sigma^\flat_\Sigma(\psi, \vec{\chi})), \vec{\xi}) \in T_{\Sigma'}.$$

This proves that

$$\rho^\flat_\Sigma[\sigma^\flat_\Sigma(\phi, \vec{\chi}), \sigma^\flat_\Sigma(\psi, \vec{\chi})] \leq T.$$

Since this holds for all $\rho^\flat \in R^{\mathcal{I}s}$, we get that $R^{\mathcal{I}s}_\Sigma[\sigma^\flat_\Sigma(\phi, \vec{\chi}), \sigma^\flat_\Sigma(\psi, \vec{\chi})] \leq T$ or, equivalently, $\langle \sigma^\flat_\Sigma(\phi, \vec{\chi}), \sigma^\flat_\Sigma(\psi, \vec{\chi}) \rangle \in R^{\mathcal{I}s}_\Sigma(T)$. Therefore, $R^{\mathcal{I}s}(T)$ has the congruence compatibility property in $\mathbf{F}$. ∎

We now show, in an analog of Theorem 1213, that possession of the narrow system modus ponens by the narrow reflexive system core intrinsically characterizes syntactic narrow system monotonicity. We start by showing that possession of the narrow system MP by the narrow reflexive core is necessary for syntactic narrow system monotonicity. This forms an analog of Theorem 1210.

Theorem 1225 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is syntactically narrowly system monotone, then $R^{\mathcal{I}s}$ has the narrow system MP.*

Proof: Suppose that $\mathcal{I}$ is syntactically narrowly system monotone with witnessing transformations $I^\flat$. Since, by definition, $I^\flat$ is narrowly system reflexive, we get, by definition of $R^{\mathcal{I}s}$, $I^\flat \subseteq R^{\mathcal{I}s}$. Thus, since $I^\flat$ has the narrow system MP in $\mathcal{I}$, we get that, a fortiori, $R^{\mathcal{I}s}$ also satisfies the narrow system MP. ∎

If $R^{\mathcal{I}s}$ has the narrow system MP, then it has the narrow system transitivity in $\mathcal{I}$. This proposition forms an analog of Proposition 1211.

Proposition 1226 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $R^{\mathcal{I}s}$ *has the narrow system MP, then it also has the narrow system transitivity in* $\mathcal{I}$.

Proof: Suppose that $R^{\mathcal{I}s}$ has the narrow system MP and let $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle, \langle \psi, \chi \rangle \in R^{\mathcal{I}s}_\Sigma(T)$. This means that $R^{\mathcal{I}s}_\Sigma[\phi, \psi] \leq T$ and $R^{\mathcal{I}s}_\Sigma[\psi, \chi] \leq T$. Then, by Lemma 1224, we get that, for all $\rho^\flat \in R^{\mathcal{I}s}$, and all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$R^{\mathcal{I}s}_{\Sigma'}[\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), \vec{\xi}), \\ \rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi})] \leq T.$$

But, by hypothesis, $R^{\mathcal{I}s}_\Sigma[\phi, \psi] \leq T$ and $R^{\mathcal{I}s}$ has the narrow system MP. Therefore, for all $\rho^\flat \in R^{\mathcal{I}s}$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi}) \subseteq T_{\Sigma'},$$

i.e., $R^{\mathcal{I}s}_\Sigma[\phi, \chi] \leq T$. This shows $\langle \phi, \chi \rangle \in R^{\mathcal{I}s}_\Sigma(T)$ and, hence, $R^{\mathcal{I}s}$ is narrowly system transitive in $\mathcal{I}$. ■

We are now ready to show a converse of Theorem 1225, i.e., that possession of the narrow system MP by $R^{\mathcal{I}s}$ suffices to establish the syntactic narrow system monotonicity of $\mathcal{I}$, since, in that case, $R^{\mathcal{I}s}$ serves as a family of witnessing transformations. The following constitutes an analog of Theorem 1212.

Theorem 1227 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $R^{\mathcal{I}s}$ *has the narrow system MP, then* $\mathcal{I}$ *is syntactically narrowly system monotone, with witnessing transformations* $R^{\mathcal{I}s}$.

Proof: By Lemma 1222, $R^{\mathcal{I}s}$ is narrowly system reflexive in $\mathcal{I}$. By Lemma 1223, $R^{\mathcal{I}s}$ is narrowly system symmetric in $\mathcal{I}$. By hypothesis and Proposition 1226, it is narrowly system transitive in $\mathcal{I}$. By Lemma 1224 it has the narrow system compatibility property in $\mathcal{I}$. Finally, by hypothesis, it has the narrow system MP in $\mathcal{I}$. We conclude that $\mathcal{I}$ is syntactically narrowly system monotone, with witnessing transformations $R^{\mathcal{I}s}$. ■

Theorems 1225 and 1227 provide the promised characterization of syntactic narrow system monotonicity in terms of the narrow system MP of the narrow reflexive system core.

$$\begin{array}{c}\mathcal{I}\text{ is Syntactically Narrow}\\ \text{System Monotone}\end{array} \longleftrightarrow \begin{array}{c}R^{\mathcal{I}s}\text{ has Narrow System}\\ \text{Modus Ponens}\end{array}.$$

Theorem 1228 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically narrowly system monotone if and only if* $R^{\mathcal{I}s}$ *has the narrow system MP in* $\mathcal{I}$*.*

Proof: Theorem 1225 gives the "only if" and the "if" is by Theorem 1227. ■

A related alternative characterization asserts that syntactic narrow system monotonicity amounts to the narrow definability of Leibniz congruence systems of theory systems by the narrow reflexive system core. This result forms an analog of Theorem 1214.

$$\begin{array}{c}\mathcal{I} \text{ is Syntactically Narrow}\\ \text{System Monotone}\end{array} \longleftrightarrow \begin{array}{c}R^{\mathcal{I}s} \text{ Defines Leibniz Congruence}\\ \text{Systems of Theory Systems}\end{array}.$$

Theorem 1229 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically narrowly system monotone if and only if, for all* $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$*,*

$$\Omega(T) = R^{\mathcal{I}s}(T).$$

Proof: If $\mathcal{I}$ is syntactically narrowly system monotone, then, by Theorem 1225, $R^{\mathcal{I}s}$ has the narrow system MP in $\mathcal{I}$. Thus, by Theorem 1227, $R^{\mathcal{I}s}$ is a family of witnessing transformations for the syntactic narrow system monotonicity of $\mathcal{I}$. Thus, by Corollary 1220, for all $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, $\Omega(T) = R^{\mathcal{I}s}(T)$.

Suppose, conversely, that the displayed condition holds. Then $R^{\mathcal{I}s}$ is narrowly system reflexive, narrowly system transitive and has the narrow system compatibility property and the narrow system MP. Hence, it constitutes a collection of witnessing transformations and, therefore, $\mathcal{I}$ is syntactically narrowly system monotone. ■

To prove an analog of Theorem 1218, which, in a certain restricted sense characterizes syntactic narrow family monotonicity inside the class of narrow family monotone π-institutions, we create a suitable analog of the Leibniz compatibility property with respect to the theory family generated by the narrow reflexive system core. Once more, the difficulty in this case, similarly with that described in some detail in Section 15.1, arises from the fact that $\mathrm{ThSys}^{\nmid}(\mathcal{I})$ may not be, in general, closed under signature-wise intersections.

To introduce this analog and to understand how it comes about and how it extends the Leibniz property, we elaborate further on the relevant discussion initiated in Section 15.1. Recall that a π-institution $\mathcal{I}$ is *prealgebraic* if its Leibniz operator is monotone on theory systems. Recall, also, once more, that its reflexive core $R^{\mathcal{I}}$ is said to be *Leibniz* if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi])).$$

If a π-institution is prealgebraic and has a Leibniz reflexive core, then it satisfies the global system modus ponens. This was shown in Chapter 11 using the following method. Considering $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $R^{\mathcal{I}}_\Sigma[\phi,\psi] \leq T$, we get

- $\langle\phi,\psi\rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi,\psi]))$ first, by applying the Leibniz property;
- $\Omega(C(R^{\mathcal{I}}_\Sigma[\phi,\psi])) \leq \Omega(T)$, by applying the hypothesis that $R^{\mathcal{I}}_\Sigma[\phi,\psi] \leq T$ and the postulated prealgebraicity of $\mathcal{I}$ and observing at the same time that $C(R^{\mathcal{I}}_\Sigma[\phi,\psi]) \in \mathrm{ThSys}(\mathcal{I})$, since $R^{\mathcal{I}}_\Sigma[\phi,\psi]$ is a sentence system.

However, in case of narrow system monotonicity, the plausibility of $R^{\mathcal{I}s}_\Sigma[\phi,\psi]$ having some empty components makes it likely that, in the second stage, narrow system monotonicity may not be applicable to ensure the inclusion $\Omega(C(R^{\mathcal{I}s}_\Sigma[\phi,\psi])) \leq \Omega(T)$. To deal with this plausibility, we assume, in a similar way as before, that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, the poset

$$[R^{\mathcal{I}s}_\Sigma[\phi,\psi]) \coloneqq \{T \in \mathrm{ThSys}^{\not\emptyset}(\mathcal{I}) : R^{\mathcal{I}s}_\Sigma[\phi,\psi] \leq T\}$$

satisfies the descending chain condition and to postulate that every minimal element $T \in [R^{\mathcal{I}s}_\Sigma[\phi,\psi])$ satisfies $\langle\phi,\psi\rangle \in \Omega_\Sigma(T)$.

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.

- For $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, define
$$[R^{\mathcal{I}s}_\Sigma[\phi,\psi]) \coloneqq \{T \in \mathrm{ThSys}^{\not\emptyset}(\mathcal{I}) : R^{\mathcal{I}s}_\Sigma[\phi,\psi] \leq T\};$$
- For $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, $\mathcal{I}$ is called **$\langle\Sigma,\phi,\psi\rangle$-reflexively system covered** if, for every theory system $T \in [R^{\mathcal{I}s}_\Sigma[\phi,\psi])$, there exists minimal $T' \in [R^{\mathcal{I}s}_\Sigma[\phi,\psi])$, such that $T' \leq T$;
- $\mathcal{I}$ is called **reflexively system covered** if it is $\langle\Sigma,\phi,\psi\rangle$-reflexively system covered, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$.

Given $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, we write

$$\min[R^{\mathcal{I}s}_\Sigma[\phi,\psi])$$

for the collection of minimal elements in $[R^{\mathcal{I}s}_\Sigma[\phi,\psi])$.

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. We say that the narrow reflexive system core $R^{\mathcal{I}s}$ of $\mathcal{I}$ is **Leibniz** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$ and all $T \in \min[R^{\mathcal{I}s}_\Sigma[\phi,\psi])$,

$$\langle\phi,\psi\rangle \in \Omega_\Sigma(T).$$

We show, in an analog of Proposition 1215, that, if $R^{\mathcal{I}s}$ has the narrow system MP, then it is Leibniz. In fact, the proof demonstrates that, under the narrow system MP, a stronger property than that of being Leibniz holds; more concretely, that for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$ and all $T \in [R^{\mathcal{I}s}_\Sigma[\phi, \psi])$ (and not only for $T \in \min[R^{\mathcal{I}s}_\Sigma[\phi, \psi]))$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(T).$$

Proposition 1230 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}s}$ *has the narrow system MP, then for all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$ *and all* $T \in [R^{\mathcal{I}s}_\Sigma[\phi, \psi])$*,* $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$*.*

Proof: Suppose $R^{\mathcal{I}s}$ has the narrow system MP and let $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $R^{\mathcal{I}s}_\Sigma[\phi, \psi] \leq T$. To verify that $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$, we use Theorem 19. Let $\sigma^\flat \in N^\flat$, $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$, such that $\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \in T_{\Sigma'}$. Since $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, by Lemma 1224,

$$R^{\mathcal{I}s}_{\Sigma'}[\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}), \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi})] \leq T.$$

Thus, since, by hypothesis, $R^{\mathcal{I}s}$ has the narrow system MP, we obtain

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}) \in T_{\Sigma'}.$$

By symmetry, we conclude that, for all $\sigma^\flat \in N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \in T_{\Sigma'} \quad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}) \in T_{\Sigma'}.$$

Hence, by Theorem 19, $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. ■

Corollary 1231 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}s}$ *has the narrow system MP, then it is Leibniz.*

Proof: Directly by Proposition 1230. ■

In the opposite direction, when dealing with reflexively system covered π-institutions, we may show that narrow system monotonicity combined with the Leibniz property of the narrow reflexive system core imply that the narrow reflexive system core has the narrow system modus ponens in $\mathcal{I}$. The following proposition forms an analog of Proposition 1217 in the system context.

Proposition 1232 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a reflexively system covered, narrowly system monotone* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}s}$ *is Leibniz, then it has the narrow system MP in* $\mathcal{I}$*.*

Proof: Let $\mathcal{I}$ be a reflexively system covered π-institution. Suppose that $\mathcal{I}$ is narrowly system monotone and that $R^{\mathcal{I}s}$ is Leibniz. Let $T \in \mathrm{ThSys}^{\nmid}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $R^{\mathcal{I}s}_\Sigma[\phi, \psi] \leq T$. Since $\mathcal{I}$ is reflexively system covered, there exists $T' \in \min[R^{\mathcal{I}s}_\Sigma[\phi, \psi])$, such that $T' \leq T$. Now we have

$$\begin{array}{lll}\langle \phi, \psi \rangle & \in & \Omega_\Sigma(T') \quad (\text{since } R^{\mathcal{I}s} \text{ is Leibniz and } T' \in \min[R^{\mathcal{I}s}_\Sigma[\phi, \psi])) \\ & \subseteq & \Omega_\Sigma(T). \quad (\text{since } T' \leq T \text{ and } \mathcal{I} \text{ is narrowly system monotone})\end{array}$$

Therefore, since $\phi \in T_\Sigma$, we get, by the compatibility of $\Omega(T)$ with T, that $\psi \in T_\Sigma$. We conclude that $R^{\mathcal{I}s}$ has the narrow system MP in $\mathcal{I}$. ∎

Thus, at least for reflexively system covered π-institutions, it is possible to show that the class of syntactically narrowly system monotone ones inside the class of the narrowly system monotone ones can be characterized exactly by the Leibniz property of the narrow reflexive system core. This forms a partial analog of Theorem 1218.

Theorem 1233 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a reflexively system covered π-institution based on $\mathbf{F}$. $\mathcal{I}$ is syntactically narrowly system monotone if and only if it is narrowly system monotone and has a Leibniz narrow reflexive system core.*

Proof: Let $\mathcal{I}$ be a reflexively system covered π-institution.

Suppose, first, that $\mathcal{I}$ is syntactically narrowly system monotone. Then it is narrowly system monotone by Theorem 1221. Moreover, its narrow reflexive system core has the narrow system MP by Theorem 1225 and, hence, by Corollary 1231, its narrow reflexive system core is Leibniz.

Suppose, conversely, that $\mathcal{I}$ is narrowly system monotone with a Leibniz narrow reflexive system core. Then, by Proposition 1232, its narrow reflexive system core has the narrow system MP and, therefore, by Theorem 1227, $\mathcal{I}$ is syntactically narrowly system monotone, with witnessing transformations $R^{\mathcal{I}s}$. ∎

15.3 Syntactic Narrow Right Monotonicity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $I^\flat \subseteq N^\flat$ a collection of natural transformations in $N^\flat$, with two distinguished arguments. Recall from Proposition 99, that, for all $T \in \mathrm{SenFam}(\mathbf{F})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma[\phi, \psi] \leq T \quad \text{iff} \quad I^\flat_\Sigma[\phi, \psi] \leq \overleftarrow{T}. \tag{15.1}$$

Let, now, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. We may attempt to define "syntactic narrow left monotonicity" as the existence of a collection

$I^\flat \subseteq N^\flat$, with two distinguished arguments, such that, for all $T \in \mathrm{ThFam}^{\not\,}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma[\phi,\psi] \leq \overleftarrow{T} \quad \text{iff} \quad \langle \phi,\psi\rangle \in \Omega_\Sigma(T).$$

Because of the the preceding remark, however, this condition would amount exactly to defining syntactic narrow family monotonicity. On the other hand, syntactic narrow system monotonicity is equivalent, again based on the remark above, to asserting the existence of $I^\flat \subseteq N^\flat$, with two distinguished arguments, such that, for all $T \in \mathrm{ThFam}^{\not\,}(\mathcal{I})$, with $\overleftarrow{T} \in \mathrm{ThSys}^{\not\,}(\mathcal{I})$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma[\phi,\psi] \leq T \quad \text{iff} \quad \langle \phi,\psi\rangle \in \Omega_\Sigma(\overleftarrow{T}).$$

If we drop the restriction that $\overleftarrow{T}$ be in $\mathrm{ThSys}^{\not\,}(\mathcal{I})$, thus allowing the condition above to be imposed on the wider class of all $T \in \mathrm{ThFam}^{\not\,}(\mathcal{I})$, we obtain a concept slightly more general that syntactic narrow system monotonicity, which we term *syntactic narrow right monotonicity*. We study this notion in more detail in this section, following the study of syntactic narrow family (and system) monotonicity, carried out in the preceding sections of the chapter.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Recall that $\mathcal{I}$ is **narrowly right monotone** if, for all $T, T' \in \mathrm{ThFam}^{\not\,}(\mathcal{I})$,

$$T \leq T' \quad \text{implies} \quad \Omega(\overleftarrow{T}) \leq \Omega(\overleftarrow{T'}).$$

In this section, following the work on syntactic narrow family monotonicity of Section 15.1, we introduce and study a syntactic analog of narrow right monotonicity.

First, the concepts of narrow system reflexivity, narrow system symmetry, narrow system transitivity, narrow system compatibility and narrow system modus ponens are recast to accommodate theory systems that arise by applying the arrow operator $\leftarrow$ on theory families in $\mathrm{ThFam}^{\not\,}(\mathcal{I})$. Note that such theory systems include, of course, all theory systems in $\mathrm{ThSys}^{\not\,}(\mathcal{I})$, since these arise by applying the arrow operator on themselves.

Let, as above, $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Moreover, suppose that $I^\flat \subseteq N^\flat$ is a collection of natural transformations in $N^\flat$, with two distinguished arguments.

- $I^\flat$ is **narrowly right reflexive** if, for all $T \in \mathrm{ThFam}^{\not\,}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma[\phi,\phi] \leq \overleftarrow{T};$$

- $I^\flat$ is **narrowly right symmetric** if, for all $T \in \mathrm{ThFam}^{\nsucc}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma[\phi, \psi] \leq \overleftarrow{T} \quad \text{implies} \quad I^\flat_\Sigma[\psi, \phi] \leq \overleftarrow{T};$$

- $I^\flat$ is **narrowly right transitive** if, for all $T \in \mathrm{ThFam}^{\nsucc}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat_\Sigma[\phi, \psi] \cup I^\flat_\Sigma[\psi, \chi] \leq \overleftarrow{T} \quad \text{implies} \quad I^\flat_\Sigma[\phi, \chi] \leq \overleftarrow{T};$$

- $I^\flat$ is **narrowly right compatible** if, for all $T \in \mathrm{ThFam}^{\nsucc}(\mathcal{I})$, all $\sigma^\flat \in N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\bigcup_{i<k} \overleftrightarrow{I^\flat}_\Sigma[\phi_i, \psi_i] \leq \overleftarrow{T} \quad \text{implies} \quad I^\flat_\Sigma[\sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi})] \leq \overleftarrow{T};$$

- $I^\flat$ has the **narrow right MP** if, for all $T \in \mathrm{ThFam}^{\nsucc}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in \overleftarrow{T}_\Sigma \quad \text{and} \quad I^\flat_\Sigma[\phi, \psi] \leq \overleftarrow{T} \quad \text{imply} \quad \psi \in \overleftarrow{T}_\Sigma.$$

Note that, because of Equivalence (15.1), narrow right reflexivity, narrow right symmetry, narrow right transitivity and narrow right compatibility are equivalent, respectively, to narrow family reflexivity, narrow family symmetry, narrow family transitivity and narrow family compatibility. They are simply recast involving the arrow operator, but the change is inessential. On the other hand, narrow right modus ponens is an essentially different property than narrow family modus ponens and it is the critical property that differentiates syntactic narrow right monotonicity from syntactic narrow family monotonicity.

Note, also, that, based on Equivalence (15.1), for all $T \in \mathrm{ThFam}^{\nsucc}(\mathcal{I})$,

$$I^\flat(T) = I^\flat(\overleftarrow{T}).$$

We say that $\mathcal{I}$ is **syntactically narrowly right monotone** if there exists $I^\flat \subseteq N^\flat$, with two distinguished arguments, such that $I^\flat$ satisfies:

- narrow right reflexivity;
- narrow right transitivity;
- narrow right compatibility; and
- narrow right MP.

In that case, we call $I^\flat$ a **set of witnessing natural transformations**, or, more simply, **witnessing transformations** (of the syntactic narrow right monotonicity of $\mathcal{I}$).

It turns out that, if $\mathcal{I}$ is a syntactically narrowly right monotone π-institution, with witnessing transformations $I^\flat$, then $\overleftrightarrow{I^\flat}(T)$ $(:= \overleftrightarrow{I^\flat}(\overleftarrow{T}))$ is a congruence system on $\mathbf{F}$ compatible with $\overleftarrow{T}$, for all $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$. This forms a system analog of Proposition 1203.

Proposition 1234 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically narrowly right monotone, with witnessing transformations* $I^\flat$*, then, for all* $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$*,* $\overleftrightarrow{I^\flat}(T)$ *is a congruence system on* $\mathbf{F}$ *compatible with* $\overleftarrow{T}$*.*

Proof: The proof is similar to that of Proposition 1203. Let $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$. The narrow right reflexivity of $I^\flat$ ensures that $\langle \phi, \phi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$. The fact that $\overleftrightarrow{I^\flat}$ is the symmetrization of $I^\flat$ ensures that $\langle \phi, \psi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$ implies that $\langle \psi, \phi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$. The narrow right transitivity of $I^\flat$ guarantees that $\langle \phi, \psi \rangle, \langle \psi, \chi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$ imply $\langle \phi, \chi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$.

Suppose, next, that $\sigma^\flat \in N^\flat$, $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$. Then, the narrow right compatibility of $I^\flat$ ensures that, if, for all $i < k$, $\langle \phi_i, \psi_i \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$, then $\langle \sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi}) \rangle \in I^\flat_\Sigma(T)$. Thus, $\overleftrightarrow{I^\flat}(T)$ is a congruence family on $\mathbf{F}$. However, by Lemma 93, $\overleftrightarrow{I^\flat}(T)$ is a relation system on $\mathbf{F}$. Hence, $\overleftrightarrow{I^\flat}(T)$ is a congruence system on $\mathbf{F}$.

It only remains to show that $\overleftrightarrow{I^\flat}(T)$ is compatible with $\overleftarrow{T}$. Assume that $\phi \in \overleftarrow{T}_\Sigma$ and $\langle \phi, \psi \rangle \in \overleftrightarrow{I^\flat}_\Sigma(T)$. Since $I^\flat \subseteq \overleftrightarrow{I^\flat}$, we get, by the narrow right MP of $I^\flat$, that $\psi \in \overleftarrow{T}_\Sigma$. Thus, $\overleftrightarrow{I^\flat}(T)$ is also compatible with $\overleftarrow{T}$. ■

Proposition 1234 shows that $\overleftrightarrow{I^\flat}$ defines Leibniz congruence systems of those theory systems of the form $\overleftarrow{T}$, for $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$. We say that $I^\flat$ **narrowly defines Leibniz congruence systems** of theory families in $\mathcal{I}$ **up to arrow** if, for all $T \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$,

$$\overleftrightarrow{I^\flat}(T) = \Omega(\overleftarrow{T}).$$

Then, in what is an analog of Corollary 1204, we obtain

Corollary 1235 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically narrowly right monotone, with witnessing transformations* $I^\flat$*, then* $I^\flat$ *narrowly defines Leibniz congruence systems of theory families in* $\mathcal{I}$ *up to arrow.*

Proof: By Proposition 1219 and Corollary 98. ∎

This corollary has as immediate consequence the fact that syntactic narrow right monotonicity implies (semantic) narrow right monotonicity. This forms an analog of Theorem 1205.

Theorem 1236 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically narrowly right monotone, then it is narrowly right monotone.*

Proof: Suppose that $\mathcal{I}$ is syntactically narrowly right monotone with witnessing transformations $I^\flat$. Let $T, T' \in \mathrm{ThFam}^{\nmid}(\mathcal{I})$, such that $T \leq T'$. Then

$$\begin{array}{rcll} \Omega(\overleftarrow{T}) & = & \overleftrightarrow{I}^\flat(T) & \text{(by Corollary 1235)} \\ & \leq & \overleftrightarrow{I}^\flat(T') & \text{(by Lemma 94)} \\ & = & \Omega(\overleftarrow{T'}). & \text{(by Corollary 1235)} \end{array}$$

Thus, $\mathcal{I}$ is narrowly right monotone. ∎

We now introduce the notion of the narrow reflexive system core of a π-institution $\mathcal{I}$ in a way analogous to the narrow reflexive core, which was introduced in Section 15.1. Its introduction will enable us to provide a characterization of the syntactical narrow system monotonicity property and to establish a relationship between this property and its semantic counterpart.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall from Section 15.1 that the **narrow reflexive core** of $\mathcal{I}$ is the collection

$$\begin{array}{rcl} R^{\mathcal{I}\nmid} & = & \{\rho^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}^{\nmid}(\mathcal{I}))(\forall \Sigma \in |\mathbf{Sign}^\flat|) \\ & & \qquad (\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\rho^\flat_\Sigma[\phi, \phi] \leq T)\}. \end{array}$$

Recall, also, from Lemmas 1207, 1208 and 1209, that, given any theory family in $\mathrm{ThFam}^{\nmid}(\mathcal{I})$, the relation system $R^{\mathcal{I}\nmid}(T)$ is a reflexive and symmetric relation system on $\mathbf{F}$ that has the congruence compatibility property in $\mathbf{F}$.

We now show, in an analog of Theorem 1213, that possession of the narrow right modus ponens by the narrow reflexive core intrinsically characterizes syntactic narrow right monotonicity. We start by showing that possession of the narrow right MP by the narrow reflexive core is necessary for syntactic narrow right monotonicity. This forms an analog of Theorem 1210.

Theorem 1237 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically narrowly right monotone, then* $R^{\mathcal{I}\nmid}$ *has the narrow right MP.*

Proof: Suppose that $\mathcal{I}$ is syntactically narrowly right monotone with witnessing transformations $I^\flat$. Since, by definition, $I^\flat$ is narrowly right reflexive, which is equivalent to being narrowly family reflexive, we get, by definition of $R^{\mathcal{I}\nleq}$, $I^\flat \subseteq R^{\mathcal{I}\nleq}$. Thus, since $I^\flat$ has the narrow right MP in $\mathcal{I}$, we get that, a fortiori, $R^{\mathcal{I}\nleq}$ also satisfies the narrow right MP. ■

If $R^{\mathcal{I}\nleq}$ has the narrow right MP, then it has the narrow right transitivity in $\mathcal{I}$. This proposition forms an analog of Proposition 1211.

Proposition 1238 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $R^{\mathcal{I}\nleq}$ *has the narrow right MP, then it also has the narrow right transitivity in* $\mathcal{I}$.

Proof: Suppose that $R^{\mathcal{I}\nleq}$ has the narrow right MP and let $T \in \mathrm{ThFam}^{\nleq}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle, \langle \psi, \chi \rangle \in R^{\mathcal{I}\nleq}_\Sigma(T)$. This means that $R^{\mathcal{I}\nleq}_\Sigma[\phi, \psi] \leq T$ and $R^{\mathcal{I}\nleq}_\Sigma[\psi, \chi] \leq T$. Then, by Lemma 1224, we get that, for all $\rho^\flat \in R^{\mathcal{I}\nleq}$, and all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$R^{\mathcal{I}\nleq}_{\Sigma'}[\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), \vec{\xi}), \\ \rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi})] \leq T.$$

But, by hypothesis, $R^{\mathcal{I}\nleq}_\Sigma[\phi, \psi] \leq T$ and $R^{\mathcal{I}\nleq}$ has the narrow right MP. Therefore, for all $\rho^\flat \in R^{\mathcal{I}\nleq}$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\rho^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\xi}) \subseteq T_{\Sigma'},$$

i.e., $R^{\mathcal{I}\nleq}_\Sigma[\phi, \chi] \leq T$. This shows $\langle \phi, \chi \rangle \in R^{\mathcal{I}\nleq}_\Sigma(T)$ and, hence, $R^{\mathcal{I}\nleq}$ is narrowly right transitive in $\mathcal{I}$. ■

We are now ready to show a converse of Theorem 1237, i.e., that possession of the narrow right MP by $R^{\mathcal{I}\nleq}$ suffices to establish the syntactic narrow right monotonicity of $\mathcal{I}$, since, in that case, $R^{\mathcal{I}\nleq}$ serves as a family of witnessing transformations. The following constitutes an analog of Theorem 1212.

Theorem 1239 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $R^{\mathcal{I}\nleq}$ *has the narrow right MP, then* $\mathcal{I}$ *is syntactically narrowly right monotone, with witnessing transformations* $R^{\mathcal{I}\nleq}$.

Proof: By Lemma 1207, $R^{\mathcal{I}\nleq}$ is narrowly right reflexive in $\mathcal{I}$. By Lemma 1208, $R^{\mathcal{I}\nleq}$ is narrowly right symmetric in $\mathcal{I}$. By hypothesis and Proposition 1238, it is narrowly right transitive in $\mathcal{I}$. By Lemma 1209 it has the narrow right compatibility property in $\mathcal{I}$. Finally, by hypothesis, it has the narrow right MP in $\mathcal{I}$. We conclude that $\mathcal{I}$ is syntactically narrowly right monotone, with witnessing transformations $R^{\mathcal{I}\nleq}$. ■

Theorems 1237 and 1239 provide the promised characterization of syntactic narrow right monotonicity in terms of the narrow right MP of the narrow reflexive core.

$$\begin{array}{c}\mathcal{I} \text{ is Syntactically Narrow}\\ \text{Right Monotone}\end{array} \longleftrightarrow \begin{array}{c}R^{\mathcal{I}\natural} \text{ has Narrow Right}\\ \text{Modus Ponens}\end{array}.$$

Theorem 1240 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically narrowly right monotone if and only if* $R^{\mathcal{I}\natural}$ *has the narrow right MP in* $\mathcal{I}$*.*

Proof: Theorem 1237 gives the "only if" and the "if" is by Theorem 1239. ■

A related alternative characterization asserts that syntactic narrow right monotonicity amounts to the narrow definability of Leibniz congruence systems of theory families up to arrow by the narrow reflexive core. This result forms an analog of Theorem 1214.

$$\begin{array}{c}\mathcal{I} \text{ is Syntactically Narrow}\\ \text{Right Monotone}\end{array} \longleftrightarrow \begin{array}{c}R^{\mathcal{I}\natural} \text{ Defines Leibniz Congruence Systems}\\ \text{of Theory Families up to Arrow}\end{array}$$

Theorem 1241 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically narrowly right monotone if and only if, for all* $T \in \mathrm{ThFam}^{\natural}(\mathcal{I})$*,*

$$\Omega(\overleftarrow{T}) = R^{\mathcal{I}\natural}(T).$$

Proof: If $\mathcal{I}$ is syntactically narrowly right monotone, then, by Theorem 1237, $R^{\mathcal{I}\natural}$ has the narrow right MP in $\mathcal{I}$. Thus, by Theorem 1239, $R^{\mathcal{I}\natural}$ is a family of witnessing transformations for the syntactic narrow right monotonicity of $\mathcal{I}$. Thus, by Corollary 1235, for all $T \in \mathrm{ThFam}^{\natural}(\mathcal{I})$, $\Omega(\overleftarrow{T}) = R^{\mathcal{I}\natural}(T)$.

Suppose, conversely, that the displayed condition holds. Then $R^{\mathcal{I}\natural}$ is narrowly right reflexive, narrowly right transitive and has the narrow right compatibility property and the narrow right MP. Hence, it constitutes a collection of witnessing transformations and, therefore, $\mathcal{I}$ is syntactically narrowly right monotone. ■

To prove an analog of Theorem 1218, which, in a sense analogous to that seen for syntactic narrow family monotonicity, characterizes syntactic narrow right monotonicity inside the class of narrow right monotone π-institutions, we create a suitable analog of the Leibniz compatibility property with respect to the theory family generated by the narrow reflexive core. Once more, the difficulty in this case, similarly with that described in some detail in Section 15.1, arises from the fact that $\mathrm{ThFam}^{\natural}(\mathcal{I})$ may not be, in general, closed under signature-wise intersections.

To introduce this analog and to understand how it comes about and how it extends the Leibniz property, we reembark, once more, on a discussion initiated in Section 15.1 and revisit some of the points with relevance in treating the "right" case.

Recall, again, the definition of prealgebraicity and the Leibniz property of the reflexive core of a π-institution. Also recall the method employed to show that, if a π-institution is prealgebraic and has a Leibniz reflexive core, then it satisfies the global system modus ponens, which is done by first applying the Leibniz property and then prealgebraicity. However, in case of narrow right monotonicity, the plausibility of $R^{\mathcal{I}\natural}_\Sigma[\phi,\psi]$ having some empty components makes it likely that, when one attempts to apply narrow right monotonicity in place of prealgebraicity in the second stage of the argument outlined above, its application in order to derive the inclusion $\Omega(C(R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])) \leq \Omega(\overleftarrow{T})$ may not be possible. To deal with this plausibility, we assume, in a similar way as before, that the π-institution under consideration is reflexively covered and postulate that every minimal element $T \in [R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$ satisfies $\langle\phi,\psi\rangle \in \Omega_\Sigma(T)$, for every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. For $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, recall the notation

$$[R^{\mathcal{I}\natural}_\Sigma[\phi,\psi]) \coloneqq \{T \in \mathrm{ThFam}^\natural(\mathcal{I}) : R^{\mathcal{I}\natural}_\Sigma[\phi,\psi] \leq T\}.$$

Recall, also that $\mathcal{I}$ is said to be *reflexively covered* if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, it is $\langle\Sigma,\phi,\psi\rangle$-reflexively covered, i.e., for every theory family $T \in [R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$, there exists minimal $T' \in [R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$, such that $T' \leq T$. Recall, furthermore, that, given $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, we write $\min[R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$ for the collection of minimal elements in $[R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. We say that the narrow reflexive core $R^{\mathcal{I}\natural}$ of $\mathcal{I}$ is **right Leibniz** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$ and all $T \in \min[R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$,

$$\langle\phi,\psi\rangle \in \Omega_\Sigma(\overleftarrow{T}).$$

We show, in an analog of Proposition 1215, that, if $R^{\mathcal{I}\natural}$ has the narrow right MP, then it is right Leibniz. In fact, the proof demonstrates that, under the narrow right MP, a stronger property than that of being right Leibniz holds; more concretely, that for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$ and all $T \in [R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$ (not only for $T \in \min[R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$), $\langle\phi,\psi\rangle \in \Omega_\Sigma(\overleftarrow{T})$.

Proposition 1242 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $R^{\mathcal{I}\natural}$ has the narrow right MP, then for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$ and all $T \in [R^{\mathcal{I}\natural}_\Sigma[\phi,\psi])$, $\langle\phi,\psi\rangle \in \Omega_\Sigma(\overleftarrow{T})$.*

Proof: Suppose $R^{\mathcal{I}\natural}$ has the narrow right MP and let $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $R^{\mathcal{I}\natural}_\Sigma[\phi,\psi] \leq T$. To verify that $\langle\phi,\psi\rangle \in \Omega_\Sigma(\overleftarrow{T})$, we use Theorem 19. Let $\sigma^\flat \in N^\flat$, $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$, such that $\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}) \in \overleftarrow{T}_{\Sigma'}$. Since $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, by Lemma 1209,

$$R^{\mathcal{I}\natural}_{\Sigma'}[\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi})] \leq T.$$

Thus, since, by hypothesis, $R^{\mathcal{I}\natural}$ has the narrow right MP, we obtain

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}) \in \overleftarrow{T}_{\Sigma'}.$$

By symmetry, we conclude that, for all $\sigma^\flat \in N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}) \in \overleftarrow{T}_{\Sigma'} \quad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}) \in \overleftarrow{T}_{\Sigma'}.$$

Hence, by Theorem 19, $\langle\phi,\psi\rangle \in \Omega_\Sigma(\overleftarrow{T})$. ■

Corollary 1243 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$. If $R^{\mathcal{I}\natural}$ has the narrow right MP, then it is right Leibniz.*

Proof: Directly by Proposition 1242. ■

To prove a converse, we restrict attention to reflexively covered π-institutions. Inside this class, we may show that narrow right monotonicity combined with the right Leibniz property of the narrow reflexive core imply that the narrow reflexive core has the narrow right modus ponens in $\mathcal{I}$. The following proposition forms an analog of Propositions 1217 and 1232 in the "right" context.

Proposition 1244 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F},C\rangle$ a reflexively covered, narrowly right monotone π-institution based on $\mathbf{F}$. If $R^{\mathcal{I}\natural}$ is right Leibniz, then it has the narrow right MP in $\mathcal{I}$.*

Proof: Let $\mathcal{I}$ be a reflexively covered π-institution. Suppose that $\mathcal{I}$ is narrowly right monotone and that $R^{\mathcal{I}\natural}$ is right Leibniz. Let $T \in \mathrm{ThFam}^\natural(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \overleftarrow{T}_\Sigma$ and $R^{\mathcal{I}\natural}_\Sigma[\phi,\psi] \leq T$. Since $\mathcal{I}$ is reflexively covered, there exists $T' \in \min[R^{\mathcal{I}\natural}_\Sigma[\phi,\psi]]$, such that $T' \leq T$. Now we have

$$\begin{array}{lll}\langle\phi,\psi\rangle & \in & \Omega_\Sigma(\overleftarrow{T'}) \quad (\text{since } R^{\mathcal{I}\natural} \text{ is right Leibniz and } T' \in \min[R^{\mathcal{I}\natural}_\Sigma[\phi,\psi]])\\ & \subseteq & \Omega_\Sigma(\overleftarrow{T}). \quad (\text{since } T' \leq T \text{ and } \mathcal{I} \text{ is narrowly right monotone})\end{array}$$

Therefore, since $\phi \in \overleftarrow{T}_\Sigma$, we get, by the compatibility of $\Omega(\overleftarrow{T})$ with $\overleftarrow{T}$, that $\psi \in \overleftarrow{T}_\Sigma$. We conclude that $R^{\mathcal{I}\natural}$ has the narrow right MP in $\mathcal{I}$. ■

Thus, at least for reflexively covered π-institutions, it is possible to show that the class of syntactically narrowly right monotone ones inside the class of the narrowly right monotone ones can be characterized exactly by the right Leibniz property of the narrow reflexive core. This forms a partial analog of Theorems 1218 and 1233.

Theorem 1245 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a reflexively covered* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically narrowly right monotone if and only if it is narrowly right monotone and has a right Leibniz narrow reflexive core.*

Proof: Let $\mathcal{I}$ be a reflexively covered π-institution.

Suppose, first, that $\mathcal{I}$ is syntactically narrowly right monotone. Then it is narrowly right monotone by Theorem 1236. Moreover, its narrow reflexive core has the narrow right MP by Theorem 1237 and, hence, by Corollary 1243, its narrow reflexive core is right Leibniz.

Suppose, conversely, that $\mathcal{I}$ is narrowly right monotone with a right Leibniz narrow reflexive core. Then, by Proposition 1244, its narrow reflexive core has the narrow right MP and, therefore, by Theorem 1239, $\mathcal{I}$ is syntactically narrowly right monotone, with witnessing transformations $R^{\mathcal{I}\natural}$. ■

Chapter 16

The Syntactic Leibniz Hierarchy: Attic I

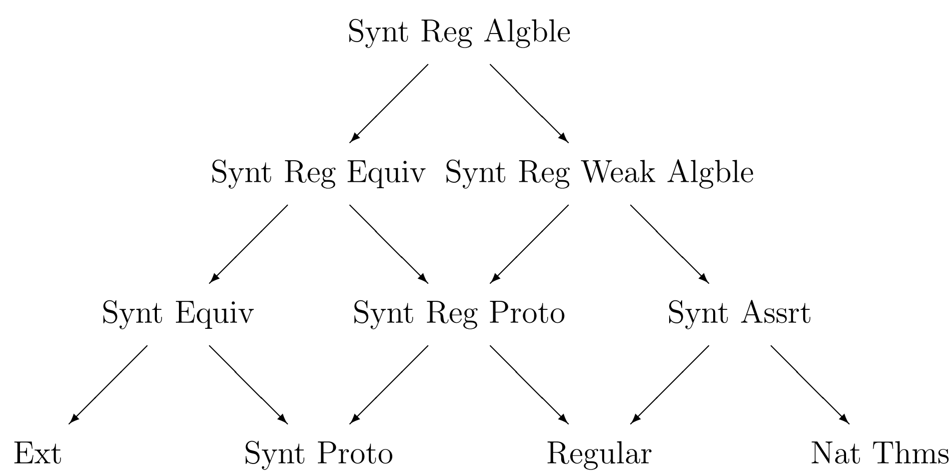
16.1 Introduction

In this chapter our goal is to develop a hierarchy analogous to the one developed in Chapter 8, but on the syntactic side. The key on the semantic side, developed in Chapter 8, was the property of regularity of a π-institution. The family version of the property asserts that, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, based on an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$, $\mathcal{I}$ is *family regular* if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi, \psi \in T_\Sigma \quad \text{implies} \quad \langle \phi, \psi \rangle \in \Omega_\Sigma(T).$$

By combining this property with pre- or proto-algebraicity, on the one hand, and with the existence of theorems, on the other, which subsumes complete reflectivity, one obtains various classes in the regular (weak) (pre)algebraizability hierarchy, which were studied in some detail in Chapter 8.

In this chapter, as our interest shifts to the syntactic side, the role played by of pre- and proto-algebraicity is assumed by syntactic pre- and proto-algebraicity, respectively, and the existence of theorems is replaced by the existence of natural theorems. By adding these features to regularity, one obtains the classes of the syntactically regularly (weakly) (pre)algebraizable π-institutions, which dominate, in general, the corresponding semantic classes. Roughly speaking, the hierarchy that we are aiming for here has the general shape depicted in the accompanying diagram. Of course various classes are present at each level, since the properties shown have various flavors, or versions, that may be used at each of the combinations depicted.



16.2 Regularity of Transformations

To prepare us for the main developments, we start by looking closely at the various versions of the regularity property of a family of natural transformations in a given π-institution.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Moreover, let $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ be a collection of natural transformations in $N^\flat$, having two distinguished arguments. We define the following properties:

- $I^\flat$ has the **family regularity in** $\mathcal{I}$, or is **family regular in** $\mathcal{I}$, if, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi, \psi \in T_\Sigma \quad \text{implies} \quad I^\flat_\Sigma[\phi, \psi] \leq T;$$
- $I^\flat$ has the **left regularity in** $\mathcal{I}$, or is **left regular in** $\mathcal{I}$, if, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi, \psi \in \overleftarrow{T}_\Sigma \quad \text{implies} \quad I^\flat_\Sigma[\phi, \psi] \leq T;$$
- $I^\flat$ has the **right regularity in** $\mathcal{I}$, or is **right regular in** $\mathcal{I}$, if, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi, \psi \in T_\Sigma \quad \text{implies} \quad I^\flat_\Sigma[\phi, \psi] \leq \overleftarrow{T};$$
- $I^\flat$ has the **system regularity in** $\mathcal{I}$, or is **system regular in** $\mathcal{I}$, if, for all $T \in \mathrm{ThSys}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\phi, \psi \in T_\Sigma \quad \text{implies} \quad I^\flat_\Sigma[\phi, \psi] \leq T.$$

Recalling that, by Proposition 99, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, we have
$$I^\flat_\Sigma[\phi, \psi] \leq T \quad \text{iff} \quad I^\flat_\Sigma[\phi, \psi] \leq \overleftarrow{T},$$
it is easy to see that the four properties defined above collapse in pairs and, therefore, there are only two distinct ones. This is detailed in the following:

Proposition 1246 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a collection of natural transformations in $N^\flat$, with two distinguished arguments.*

(a) $I^\flat$ is family regular in $\mathcal{I}$ if and only if it is right regular in $\mathcal{I}$;

(b) $I^\flat$ is system regular in $\mathcal{I}$ if and only if it is left regular in $\mathcal{I}$.

Proof: By Proposition 99, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, we have
$$I^\flat_\Sigma[\phi, \psi] \leq T \quad \text{iff} \quad I^\flat_\Sigma[\phi, \psi] \leq \overleftarrow{T}.$$

Thus, taking into account the definitions of family and right regularity, the equivalence of Part (a) becomes clear. We turn now to Part (b).

Assume, first, that $I^\flat$ is left regular in $\mathcal{I}$ and let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Since $T \in \mathrm{ThSys}(\mathcal{I})$, $\overleftarrow{T} = T$, whence, by hypothesis, $\phi, \psi \in \overleftarrow{T}_\Sigma$. Thus, by left regularity, $I^\flat_\Sigma[\phi, \psi] \le T$. This shows that $I^\flat$ has the system regularity in $\mathcal{I}$.

Suppose, conversely, that $I^\flat$ is system regular in $\mathcal{I}$ and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in \overleftarrow{T}_\Sigma$. Since $\overleftarrow{T} \in \mathrm{ThSys}(\mathcal{I})$, we get, by system regularity, $I^\flat_\Sigma[\phi, \psi] \le \overleftarrow{T}$. Therefore, by Proposition 99, $I^\flat_\Sigma[\phi, \psi] \le T$, showing that $I^\flat$ has the left regularity in $\mathcal{I}$. ∎

Based on Proposition 1246, we use the term **family regular** to refer to family/right regularity and the term **system regular** for system/left regularity. As far as the relation between these two distinct properties, it is straightforward to see that, as is typical with almost all properties studied in the monograph, system regularity is weaker than family regularity.

Proposition 1247 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a collection of natural transformations in $N^\flat$, having two distinguished arguments. If $I^\flat$ is family regular in $\mathcal{I}$, then it is system regular in $\mathcal{I}$.*

Proof: This is clear from the definitions, since the condition defining system regularity is a specialization of that defining family regularity, where T is allowed to range over theory systems only. ∎

Thus, the following hierarchy of regularity properties emerges.

$I^\flat$ Family/Right Regular

↓

$I^\flat$ System/Left Regular

It is also easy to see that, in case $\mathcal{I}$ is systemic, the two properties of being family and system regular are identified.

Proposition 1248 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a collection of natural transformations in $N^\flat$. If $\mathcal{I}$ is systemic, then $I^\flat$ is system regular if and only if it is family regular in $\mathcal{I}$.*

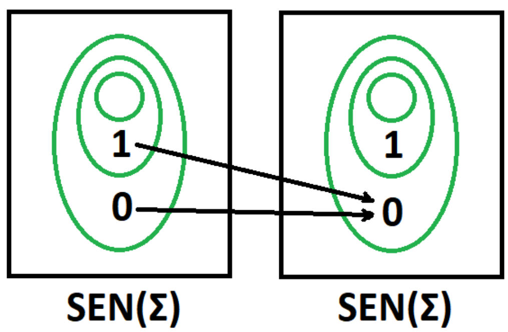
Proof: If $\mathcal{I}$ is systemic, then $\mathrm{ThFam}(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$, whence the two conditions defining family and system regularity are identical. ∎

And it is not difficult to show that this hierarchy does not collapse, in general.

Example 1249 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the category with the single object* Σ *and a single (non-identity) morphism* $f : \Sigma \to \Sigma$*, such that* $f \circ f = f$*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1\}$ *and* $\mathrm{SEN}^\flat(f)(0) = 0$, $\mathrm{SEN}^\flat(f)(1) = 0$*;*
- $N^\flat$ *is the clone of natural transformations generated by the binary natural transformation* $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$*, specified by*

$$\sigma^\flat_\Sigma(x, y) = 0, \quad \textit{for all } x, y \in \mathrm{SEN}^\flat(\Sigma).$$



Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\varnothing, \{1\}, \{0, 1\}\}.$$

$\mathcal{I}$ *has three theory families* $\overline{\varnothing}$, $\{\{1\}\}$ *and* $\mathrm{SEN}^\flat$*, but only two theory systems,* $\overline{\varnothing}$ *and* $\mathrm{SEN}^\flat$*. Consider* $I^\flat = \{\sigma^\flat\}$*. Since, the only theory systems are* $\overline{\varnothing}$ *and* $\mathrm{SEN}^\flat$*,* $I^\flat$ *is trivially system regular. On the other hand, for* $T = \{\{1\}\}$*, we get,* $1 \in T_\Sigma$*, but* $I^\flat_\Sigma[1, 1] = \{\{0\}\} \not\leq \{\{1\}\}$*, whence* $I^\flat$ *is not family regular in* $\mathcal{I}$*.*

We close the section by showing that the two versions of regularity transfer from the family $I^\flat$ to $I^\mathcal{A}$, for all $\mathbf{F}$-algebraic systems $\mathcal{A}$.

Proposition 1250 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a collection of natural transformations in* $N^\flat$*, with two distinguished arguments.*

(a) $I^\flat$ *is family regular in* $\mathcal{I}$ *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $I^\mathcal{A}$ *is family regular in* $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$*;*

(b) $I^\flat$ *is system regular in* $\mathcal{I}$ *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $I^\mathcal{A}$ *is system regular in* $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$*.*

Proof:

(a) The "if" follows easily by considering the $\mathbf{F}$-algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and recalling from Lemma 51 that $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}) = \mathrm{ThFam}(\mathcal{I})$.

Assume, conversely, that $I^\flat$ is family regular in $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \in T_{F(\Sigma)}$. Then, $\phi, \psi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})$. By Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, whence, by the postulated family regularity of $I^\flat$ in $\mathcal{I}$, we get that $I^\flat_\Sigma[\phi, \psi] \leq \alpha^{-1}(T)$. Thus, by Lemma 95, we get $I^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi), \alpha_\Sigma(\psi)] \leq T$. Taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, if $\phi, \psi \in T_\Sigma$, then $I^{\mathcal{A}}_\Sigma[\phi, \psi] \leq T$. Therefore, $I^{\mathcal{A}}$ is family regular in $\langle \mathcal{A}, C^{\mathcal{I}, \mathcal{A}} \rangle$.

(b) This follows along very similar lines.

■

16.3 Syntactic Regular PreAlgebraicity

In the next result, we connect the property of regularity of a collection of natural transformations with the property of regularity of a π-institution $\mathcal{I}$, studied in Chapter 8. More specifically, we show that, in case the π-institution under consideration is syntactically pre- (proto-)algebraic with $I^\flat$ a collection of witnessing transformations, then family (system) regularity of $I^\flat$ is equivalent to $\mathcal{I}$ being family (system) regular. Since the combination of syntactic pre- and proto-algebraicity with regularity turns out to be an important property in its own right, we give it a name, partly inspired by the results that follow. Recall that there are two kinds of syntactic monotonicity, namely syntactic prealgebraicity and syntactic protoalgebraicity, and two kinds of regularity properties of collections of natural transformations, namely family regularity and system regularity. Thus, by combining syntactic monotonicity properties with regularity properties, we obtain, a priori, four versions.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

- $\mathcal{I}$ is said to be **syntactically family regularly protoalgebraic** if it is syntactically protoalgebraic, with a witnessing collection $I^\flat$ of transformations, which is family regular in $\mathcal{I}$;

- $\mathcal{I}$ is said to be **syntactically system regularly protoalgebraic** if it is syntactically protoalgebraic, with a witnessing collection $I^\flat$ of transformations, which is system regular in $\mathcal{I}$;

- $\mathcal{I}$ is said to be **syntactically family regularly prealgebraic** if it is syntactically prealgebraic, with a witnessing collection $I^\flat$ of transformations, which is family regular in $\mathcal{I}$;
- $\mathcal{I}$ is said to be **syntactically system regularly prealgebraic** if it is syntactically prealgebraic, with a witnessing collection $I^\flat$ of transformations, which is system regular in $\mathcal{I}$.

The definitions are partially justified by the following propositions that relate them to the semantical notions of family, right, left and system regularity.

Proposition 1251 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically protoalgebraic* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$*.*

(a) $\mathcal{I}$ *is family regular if and only if* $I^\flat$ *is family regular in* $\mathcal{I}$*;*

(b) $\mathcal{I}$ *is left regular if and only if* $I^\flat$ *is system regular in* $\mathcal{I}$*.*

Proof: Let $\mathcal{I}$ be a syntactically protoalgebraic π-institution, with witnessing transformations $I^\flat$.

(a) This part is easy to see, since, by syntactic protoalgebraicity, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(T) \quad \text{iff} \quad I^\flat_\Sigma[\phi, \psi] \leq T.$$

(b) Suppose, first, that $\mathcal{I}$ is left regular and let $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Since $T \in \mathrm{ThSys}(\mathcal{I})$, $\phi, \psi \in \overleftarrow{T}_\Sigma$. By the left regularity of $\mathcal{I}$, we get $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$, whence, by syntactic protoalgebraicity, $I^\flat_\Sigma[\phi, \psi] \leq T$. This shows that $I^\flat$ is system regular in $\mathcal{I}$.

Assume, conversely, that $I^\flat$ is system regular in $\mathcal{I}$ and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in \overleftarrow{T}_\Sigma$. Then, since $\overleftarrow{T} \in \mathrm{ThSys}(\mathcal{I})$, by the system regularity of $I^\flat$, $I^\flat_\Sigma[\phi, \psi] \leq \overleftarrow{T} \leq T$, whence, by syntactic protoalgebraicity, $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$. Therefore, $\mathcal{I}$ is left regular.

■

Proposition 1252 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically prealgebraic* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$*.*

(a) $\mathcal{I}$ *is right regular if and only if* $I^\flat$ *is family regular in* $\mathcal{I}$*;*

(b) $\mathcal{I}$ is system regular if and only if $I^\flat$ is system regular in $\mathcal{I}$.

Proof: Let $\mathcal{I}$ be a syntactically prealgebraic π-institution, with witnessing transformations $I^\flat$.

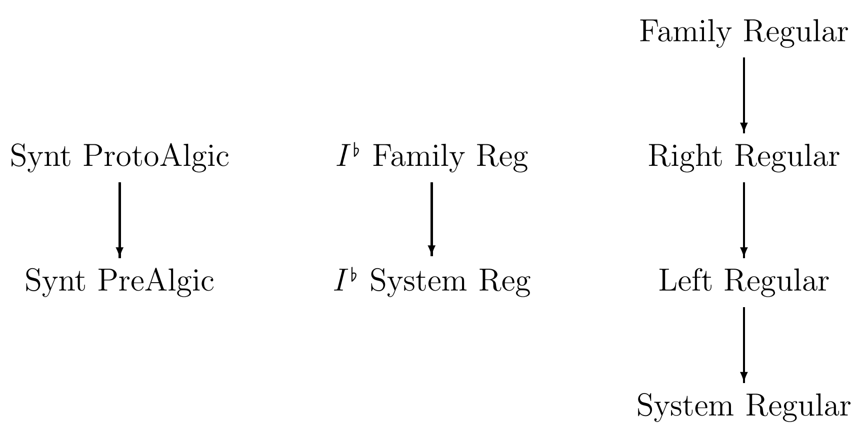
(a) Suppose, first, that $\mathcal{I}$ is right regular and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. By the right regularity of $\mathcal{I}$, we get $\langle \phi, \psi \rangle \in \Omega_\Sigma(\overleftarrow{T})$, whence, since $\overleftarrow{T} \in \mathrm{ThSys}(\mathcal{I})$, we get, by syntactic prealgebraicity, $I^\flat_\Sigma[\phi, \psi] \leq \overleftarrow{T} \leq T$. This shows that $I^\flat$ is family regular in $\mathcal{I}$.

Assume, conversely, that $I^\flat$ is family regular in $\mathcal{I}$ and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Then, by the family regularity of $I^\flat$, $I^\flat_\Sigma[\phi, \psi] \leq T$. Hence, by Proposition 99, we get $I^\flat_\Sigma[\phi, \psi] \leq \overleftarrow{T}$. Since $\overleftarrow{T} \in \mathrm{ThSys}(\mathcal{I})$, by syntactic prealgebraicity, $\langle \phi, \psi \rangle \in \Omega_\Sigma(\overleftarrow{T})$. Therefore, $\mathcal{I}$ is right regular.

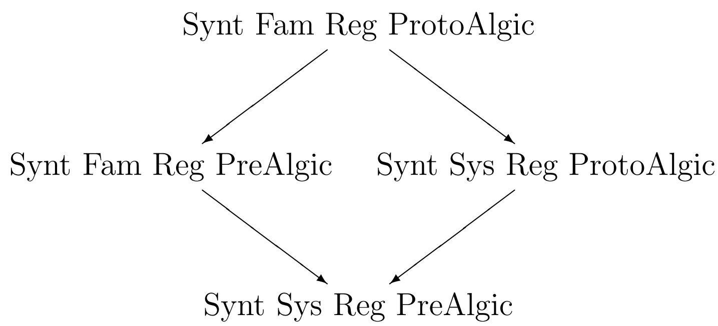
(b) This part is straightforward, since, by syntactic prealgebraicity, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $\langle \phi, \psi \rangle \in \Omega_\Sigma(T)$ if and only if $I^\flat_\Sigma[\phi, \psi] \leq T$.

■

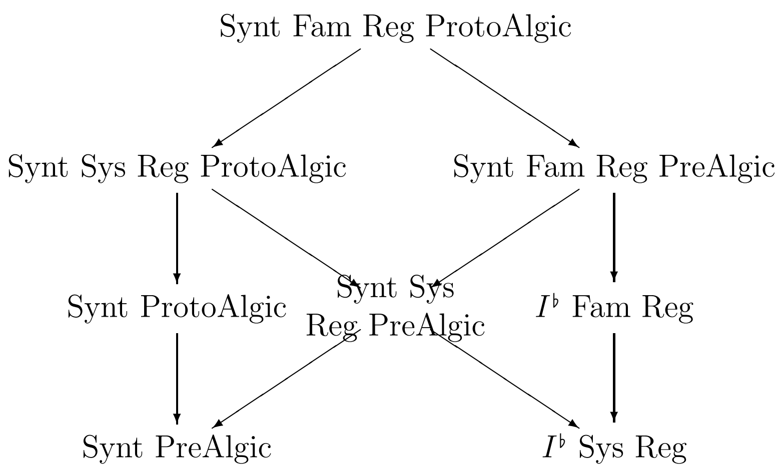
Propositions 1251 and 1252 may be viewed as partial justifications for the definitions of syntactic regular pre- and proto-algebraicity Moreover, recalling the following hierarchies of syntactic pre- and protoalgebraicity, of the regularity properties of $I^\flat$ and of semantic regularity,



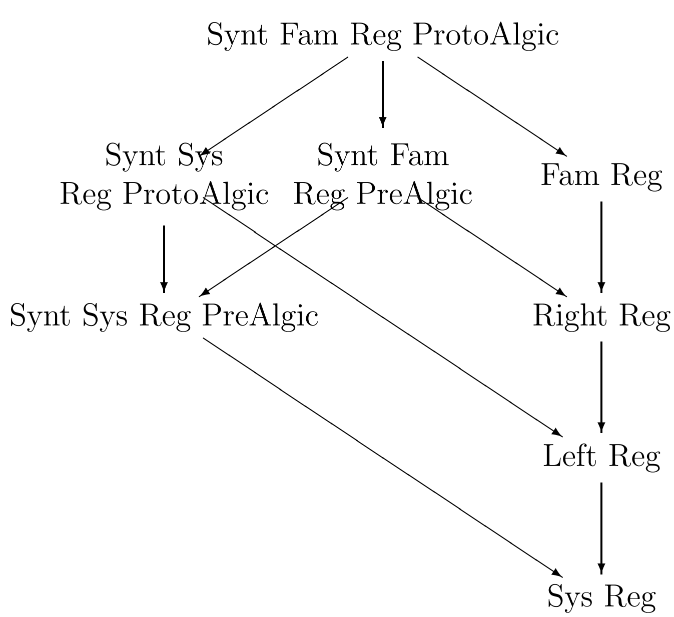
the following hierarchy of syntactic classes of regularly pre- and protoalgebraic π-institutions emerges.



Furthermore, these four classes relate with their immediate subordinate properties on the syntactic side, as shown in the following diagram



and with the four semantic regularity classes, as revealed by Propositions 1251 and 1252, as shown in the following diagram.



Theorem 584, which provided a characterization of both family and of system regularity in terms of the Suszko operator and of a system version of the Suszko operator, respectively, gives rise to the following characterizations of family and system regularity of witnessing collections of natural transformations for the proto- and pre-algebraicity, respectively, of a π-institution.

Corollary 1253 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *a collection of natural transformations in* $N^\flat$*, with two distinguished arguments.*

(a) If $\mathcal{I}$ *is syntactically protoalgebraic, with witnessing transformations* $I^\flat$*, then* $I^\flat$ *is family regular in* $\mathcal{I}$ *if and only if, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\langle \phi, \psi \rangle \in \widetilde{\Omega}^{\mathcal{I}}_\Sigma(C(\phi, \psi));$$

(ba) If $\mathcal{I}$ *is syntactically prealgebraic, with witnessing transformations* $I^\flat$*, then* $I^\flat$ *is system regular in* $\mathcal{I}$ *if and only if, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\langle \phi, \psi \rangle \in \widetilde{\Omega}^{\mathcal{I}}_\Sigma(\overrightarrow{C}(\phi, \psi)).$$

Proof: Part (a) follows by combining Part (a) of Proposition 1251 with the characterization of family regularity given in Theorem 584. Similarly, Part (b) follows by combing Part (b) of Proposition 1252 with the characterization of system regularity given in Theorem 584. ■

The next results form transfer theorems, asserting that all four types of syntactic regularity, studied here, transfer from a π-institution to all its generalized matrix families/systems. We start with the two types obtained by strengthening syntactic protoalgebraicity.

Theorem 1254 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically family (system, respectively) regularly protoalgebraic, with witnessing transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$*, if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *(and all* $T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$*, respectively), all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,*

- $\langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_\Sigma(T)$ *iff* $I^{\mathcal{A}}_\Sigma[\phi, \psi] \leq T$*;*
- $\phi, \psi \in T_\Sigma$ *implies* $I^{\mathcal{A}}_\Sigma[\phi, \psi] \leq T$ *(*$\phi, \psi \in T'_\Sigma$ *implies* $I^{\mathcal{A}}_\Sigma[\phi, \psi] \leq T'$*, respectively).*

Proof: $\mathcal{I}$ is syntactically regularly protoalgebraic if and only if, by definition, it is syntactically protoalgebraic, with witnessing transformations $I^\flat$, which are family regular, if and only if, by Theorem 810 and Proposition 1250, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ is syntactically protoalgebraic, with witnessing transformations $I^{\mathcal{A}}$, which are family regular in $\langle\mathcal{A}, C^{\mathcal{I},\mathcal{A}}\rangle$, if and only if, for every **F**-algebraic system $\mathcal{A}$, the two conditions asserted in the statement hold.

The case of system regularity may be treated similarly. ■

We close with the two types that only require syntactic prealgebraicity.

Theorem 1255 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically family (system, respectively) regularly prealgebraic with witnessing transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$*, if and only if, for every* **F***-algebraic system* $\mathcal{A} = \langle\mathbf{A}, \langle F, \alpha\rangle\rangle$*, with* $\mathbf{A} = \langle\mathbf{Sign}, \mathrm{SEN}, N\rangle$*, all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and* $T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$*, all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,*

- $\langle\phi, \psi\rangle \in \Omega^{\mathcal{A}}_\Sigma(T')$ *iff* $I^{\mathcal{A}}_\Sigma[\phi, \psi] \leq T'$*;*
- $\phi, \psi \in T_\Sigma$ *implies* $I^{\mathcal{A}}_\Sigma[\phi, \psi] \leq T$ *(*$\phi, \psi \in T'_\Sigma$ *implies* $I^{\mathcal{A}}_\Sigma[\phi, \psi] \leq T'$*, respectively).*

Proof: Similar to the proof of Theorem 1254. ■

16.4 Syntactic Regular (Pre-)Equivalentiality

Syntactic regular pre- and proto-algebraicity were defined by combining syntactic pre- and proto-algebraicity, respectively, with versions of regularity. If we upgrade syntactic pre- and proto-algebraicity to syntactic preequivalentiality and equivalentiality, respectively, then we obtain, analogously, versions of syntactic regular preequivalentiality and syntactic regular equivalentiality, respectively.

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F}, C\rangle$ a π-institution based on **F**.

- $\mathcal{I}$ is said to be **syntactically family regularly equivalential** if it is syntactically equivalential, with a witnessing collection $I^\flat$ of transformations, which is family regular in $\mathcal{I}$;
- $\mathcal{I}$ is said to be **syntactically system regularly equivalential** if it is syntactically equivalential, with a witnessing collection $I^\flat$ of transformations, which is system regular in $\mathcal{I}$;
- $\mathcal{I}$ is said to be **syntactically family regularly preequivalential** if it is syntactically preequivalential, with a witnessing collection $I^\flat$ of transformations, which is family regular in $\mathcal{I}$;
- $\mathcal{I}$ is said to be **syntactically system regularly preequivalential** if it is syntactically preequivalential, with a witnessing collection $I^\flat$ of transformations, which is system regular in $\mathcal{I}$.

Analogs of Propositions 1251 and 1252 may be proven. They follow the same lines of proof, the only difference being that the witnessing collections of transformations we are dealing with in this case, as opposed to the cases of syntactic pre- and proto-algebraicity, are parameter free.

Corollary 1256 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically equivalential* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$.

(a) *$\mathcal{I}$ is family regular if and only if $I^\flat$ is family regular in $\mathcal{I}$;*

(b) *$\mathcal{I}$ is left regular if and only if $I^\flat$ is system regular in $\mathcal{I}$.*

Proof: By Proposition 1251, taking into account the fact that syntactic equivalentiality is equivalent to syntactic protoalgebraicity via a parameter free collection of transformations. ■

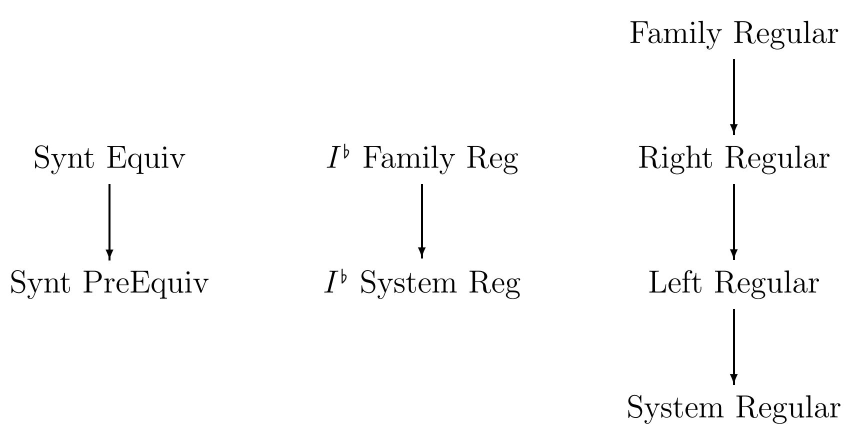
Corollary 1257 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically preequivalential* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$.

(a) *$\mathcal{I}$ is right regular if and only if $I^\flat$ is family regular in $\mathcal{I}$;*

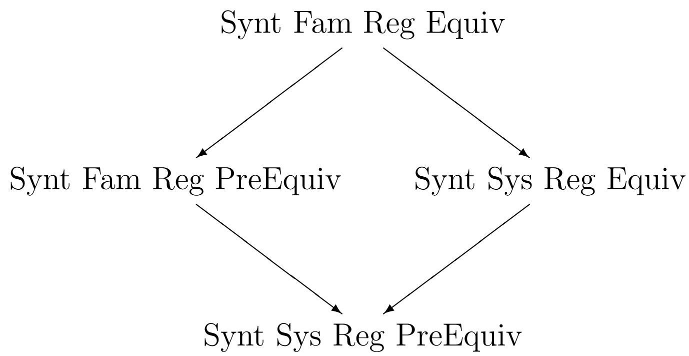
(b) *$\mathcal{I}$ is system regular if and only if $I^\flat$ is system regular in $\mathcal{I}$.*

Proof: By Proposition 1252, taking into account the fact that syntactic preequivalentiality is equivalent to syntactic prealgebraicity via a parameter free collection of transformations. ■

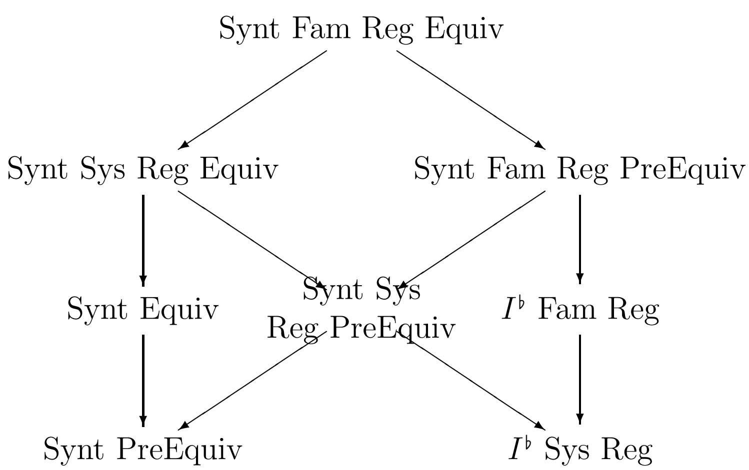
Recalling the following hierarchies of syntactic (pre)equivalentiality, of the regularity properties of $I^\flat$ and of semantic regularity,



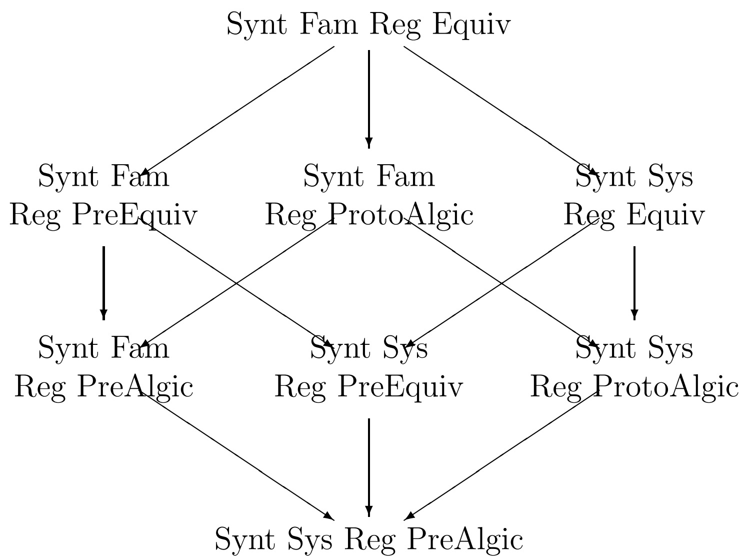
the following hierarchy of syntactic classes of regularly (pre)equivalential π-institutions arises.



Furthermore, these four classes relate to their immediate subordinate properties on the syntactic side, as shown in the following diagram



Moreover, from the fact that syntactic equivalentiality is equivalent to syntactic protoalgebraicity, with a parameter free witnessing collection of transformations, and, similarly for preequivalentiality and prealgebraicity, we get, immediately from the definitions. the following hierarchy of classes of π-institutions involving syntactic regular pre- and proto-algebraicity and syntactic regular (pre)equivalentiality.



An analog to Corollary 1253 adjusts its contents to address the special case in which the collection $I^\flat$ witnessing syntactic pre- or proto-algebraicity is parameter free, thus giving rise, instead, to syntactic preequivalentiality or equivalentiality, respectively.

Corollary 1258 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *a collection of natural transformations in* $N^\flat$ *(with both arguments distinguished).*

(a) If $\mathcal{I}$ *is syntactically equivalential, with witnessing transformations* $I^\flat$*, then* $I^\flat$ *is family regular in* $\mathcal{I}$ *if and only if, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*
$$\langle \phi, \psi \rangle \in \widetilde{\Omega}^{\mathcal{I}}_\Sigma(C(\phi, \psi));$$

(b) If $\mathcal{I}$ *is syntactically preequivalential, with witnessing transformations* $I^\flat$*, then* $I^\flat$ *is system regular in* $\mathcal{I}$ *if and only if, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*
$$\langle \phi, \psi \rangle \in \widehat{\Omega}^{\mathcal{I}}_\Sigma(\overrightarrow{C}(\phi, \psi)).$$

Proof: Each part is a consequence of the corresponding part of Corollary 1253 and the fact that $I^\flat$ is assumed to be parameter free. ■

Finally, the transfer theorems for syntactic regular pre- and proto-algebraicity, Theorems 1254 and 1255, may also be easily adapted to provide analogous transfer theorems for syntactic regular equivalentiality and preequivalentiality, respectively.

Corollary 1259 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically family (system, respectively) regularly equivalential, with witnessing transformations* $I^\flat : (\mathrm{SEN}^\flat)^2 \to$

$\mathrm{SEN}^\flat$, *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, *with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, *all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *(and all* $T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, *respectively), all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

- $\langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_\Sigma(T)$ *iff* $I^{\mathcal{A}}_\Sigma[\phi, \psi] \leq T$;
- $\phi, \psi \in T_\Sigma$ *implies* $I^{\mathcal{A}}_\Sigma[\phi, \psi] \leq T$ *(*$\phi, \psi \in T'_\Sigma$ *implies* $I^{\mathcal{A}}_\Sigma[\phi, \psi] \leq T'$, *respectively).*

Proof: Directly from Theorem 1254. ■

We close with the two types that only require syntactic preequivalentiality.

Corollary 1260 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically family (system, respectively) regularly pre-equivalential with witnessing transformations* $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$, *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, *with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, *all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and* $T' \in \mathrm{FiSys}^{\mathcal{I}}(\mathcal{A})$, *all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

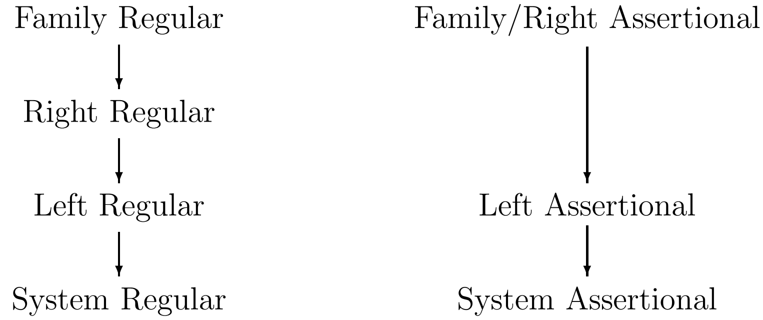
- $\langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_\Sigma(T')$ *iff* $I^{\mathcal{A}}_\Sigma[\phi, \psi] \leq T'$;
- $\phi, \psi \in T_\Sigma$ *implies* $I^{\mathcal{A}}_\Sigma[\phi, \psi] \leq T$ *(*$\phi, \psi \in T'_\Sigma$ *implies* $I^{\mathcal{A}}_\Sigma[\phi, \psi] \leq T'$, *respectively).*

Proof: Follows from Theorem 1255. ■

16.5 Syntactic Assertionality

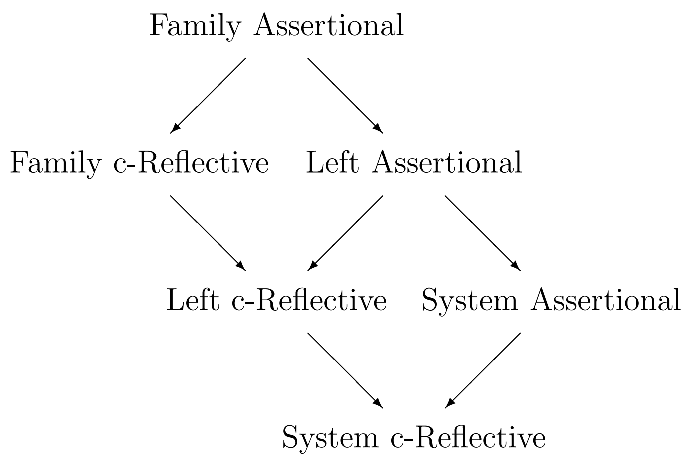
In this section, we study some of the consequences of adding to the various versions of semantic regularity, studied in detail in Section 8.2, the property of having natural theorems.

Recall, first, from Section 8.2, that there are four distinct types of semantic regularity, namely, family, right, left and system, which form the hierarchy depicted in the left diagram (obtained in Section 8.2).



If to the various regularity conditions, one adds the existence of theorems, then one obtains the semantic assertionality classes, which were studied in detail in Section 8.3, where it was shown that they form the hierarchy depicted in the diagram on the right.

Additionally, it was shown in Section 8.3 that these three assertionality classes dominate, respectively, the three corresponding complete reflectivity classes. This is shown in the third diagram, reproduced here from Section 8.3.



In this section, we study the classes arising by adding to the various flavors of semantic regularity the property of possessing natural theorems. Since the property of possessing natural theorems is strictly stronger that having theorems, there are, in accordance with the results recalled from Section 8.3 above, only three potentially different classes of π-institutions arising. These, of course, dominate the corresponding assertionality classes. The π-institution members of these classes are termed *syntactically assertional*. A strong motivation for introducing these three classes lies in the fact that lifting the possession of theorems to that of the existence of natural theorems, in tandem with semantic regularity, is enough to allow passing from the semantic classes of completely reflective π-institutions to the corresponding syntactic classes of truth equational π-institutions.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

- $\mathcal{I}$ is **syntactically family assertional** if it is family regular and has natural theorems;
- $\mathcal{I}$ is **syntactically left assertional** if it is left regular and has natural theorems;
- $\mathcal{I}$ is **syntactically right assertional** if it is right regular and has natural theorems;

- $\mathcal{I}$ is **syntactically system assertional** if it is system regular and has natural theorems.

First, it is easy to see that syntactic family and syntactic right assertionality coincide.

Proposition 1261 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically family assertional if and only if it is syntactically right assertional.*

Proof: $\mathcal{I}$ is syntactically family assertional iff, by definition, it is family assertional and has natural theorems iff, by Proposition 591, it is right assertional and has natural theorems iff, by definition, it is syntactically right assertional. ■

Given Proposition 1261, asserting that syntactic family and syntactic right assertionality coincide, we may establish the hierarchy of syntactic assertionality classes.

Proposition 1262 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is syntactically family/right assertional, then it is syntactically left assertional;*

(b) If $\mathcal{I}$ *is syntactically left assertional, then it is syntactically system assertional.*

Proof: If $\mathcal{I}$ is syntactically family assertional, then it is, by definition, family assertional and has natural theorems, whence, by Proposition 592, it is left assertional and has natural theorems, i.e., it is syntactically left assertional. Similarly, if $\mathcal{I}$ is syntactically left assertional, then it is, by definition, left assertional and has natural theorems, whence, by Proposition 592, it is system assertional and has natural theorems, i.e., it is syntactically system assertional. ■

Proposition 1262 establishes the following hierarchy of **syntactic assertionality** classes, paralleling the corresponding semantic hierarchy established in Section 8.3.

Syntactically Family/Right Assertional

↓

Syntactically Left Assertional

↓

Syntactically System Assertional

It is not difficult to see that the bottom classes of the hierarchy collapse, if restricted to stable π-institutions, and that the entire hierarchy collapses when considering only systemic π-institutions.

Proposition 1263 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\mathcal{I}$ *is stable and syntactically system assertional, then it is syntactically left assertional;*

(b) If $\mathcal{I}$ *is systemic and syntactically system assertional, then it is syntactically family assertional.*

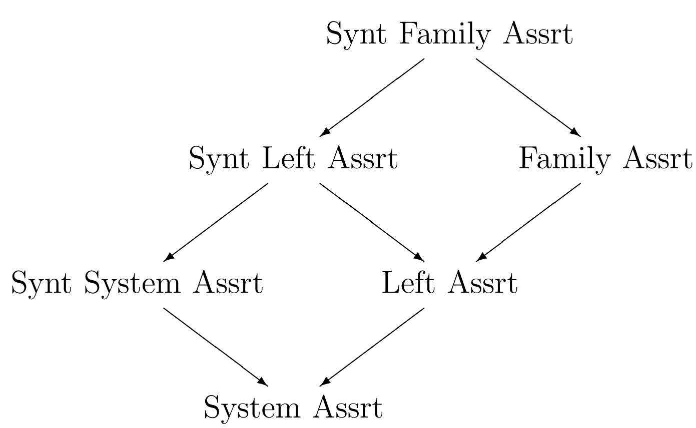
Proof: The first statement follows directly from Proposition 579, whereas the second implication is a consequence of Proposition 580. ■

We formalize, next, a result, which is straightforward, establishing the close interrelationships between the syntactic and semantic assertionality classes.

Proposition 1264 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically family (respectively left, system) assertional if and only if it is family (respectively left, system) assertional and has natural theorems.*

Proof: These equivalences follow by the definitions involved, since existence of natural theorems implies having theorems, as was shown in Lemma 1188. ■

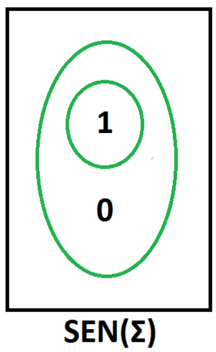
Thus, Proposition 1264 establishes the following relationships between the semantic assertionality and the corresponding syntactic assertionality classes.



It is not difficult to show, by providing an example, that the syntactic classes are properly included in the semantic ones. More precisely, we provide an example of a π-institution which is family assertional but fails to be syntactically system assertional. Thus, it belongs to all three semantic assertionality classes but in none of the three syntactic assertionality steps of the hierarchy.

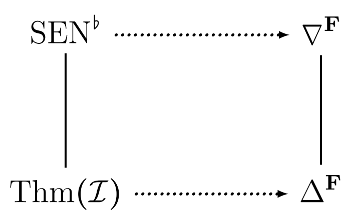
Example 1265 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system determined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with a single object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is specified by* $\mathrm{SEN}^\flat(\Sigma) = \{0,1\}$*;*
- $N^\flat$ *is the trivial category of natural transformations.*



Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be the* π*-institution determined by* $\mathcal{C}_\Sigma = \{\{1\},\{0,1\}\}$*.*

$\mathcal{I}$ *has two theory families,* $\mathrm{Thm}(\mathcal{I})$ *and* $\mathrm{SEN}^\flat$*, which are also theory systems. Moreover,* $\Omega(\mathrm{Thm}(\mathcal{I})) = \Delta^{\mathbf{F}}$ *and* $\Omega(\mathrm{SEN}^\flat) = \nabla^{\mathbf{F}}$*.*



Clearly, $\mathcal{I}$ *is family regular, i.e., for all* $T \in \mathrm{ThFam}(\mathcal{I})$*, and all* $x, y \in \{0,1\}$*, if* $x, y \in T_\Sigma$*, then* $\langle x, y \rangle \in \Omega_\Sigma(T)$*. Further, obviously,* $\mathcal{I}$ *has theorems. Finally, since there are no nontrivial natural transformations in* $N^\flat$*,* $\mathcal{I}$ *does not have natural theorems. Therefore,* $\mathcal{I}$ *is family assertional but it fails to be syntactically system assertional.*

A corollary of the connections established in Proposition 1264 and the characterizations of semantic assertionality classes, given in Proposition 588, provides similar characterizations of the three syntactic assertionality classes.

Corollary 1266 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution having natural theorems and* $\tau : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ *a natural theorem.*

(a) *$\mathcal{I}$ is syntactically family assertional if and only if, for all* $T \in \text{ThFam}(\mathcal{I})$, $T = \tau/\Omega(T)$;

(b) *$\mathcal{I}$ is syntactically left assertional if and only if, for all* $T \in \text{ThFam}(\mathcal{I})$, $\overleftarrow{T} = \tau/\Omega(T)$;

(c) *$\mathcal{I}$ is syntactically system assertional if and only if, for all* $T \in \text{ThSys}(\mathcal{I})$, $T = \tau/\Omega(T)$.

Proof: By combining Propositions 1264 and 588. ∎

We conclude the section by establishing the relationships between the three syntactic assertionality classes and the three truth equationality classes, introduced and studied in detail in Chapter 11.

Theorem 1267 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$.

(a) *If $\mathcal{I}$ is syntactically family assertional, with a natural theorem* $\tau : \text{SEN}^\flat \to \text{SEN}^\flat$, *then it is family truth equational, with witnessing equation* $\iota \approx \tau$;

(b) *If $\mathcal{I}$ is syntactically left assertional, with a natural theorem* $\tau : \text{SEN}^\flat \to \text{SEN}^\flat$, *then it is left truth equational, with witnessing equation* $\iota \approx \tau$;

(c) *If $\mathcal{I}$ is syntactically system assertional, with a natural theorem* $\tau : \text{SEN}^\flat \to \text{SEN}^\flat$, *then it is system truth equational, with witnessing equation* $\iota \approx \tau$.

Proof: We prove Part (a). The other parts can be proven similarly. Suppose that $\mathcal{I}$ is syntactically family assertional, with $\tau : \text{SEN}^\flat \to \text{SEN}^\flat$ a natural theorem. Let $T \in \text{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \text{SEN}^\flat(\Sigma)$. To show that $\mathcal{I}$ is family truth equational, with witnessing equation $\iota \approx \tau$, we must establish the equivalence

$$\phi \in T_\Sigma \quad \text{iff} \quad (\iota \approx \tau)_\Sigma[\phi] \leq \Omega(T).$$

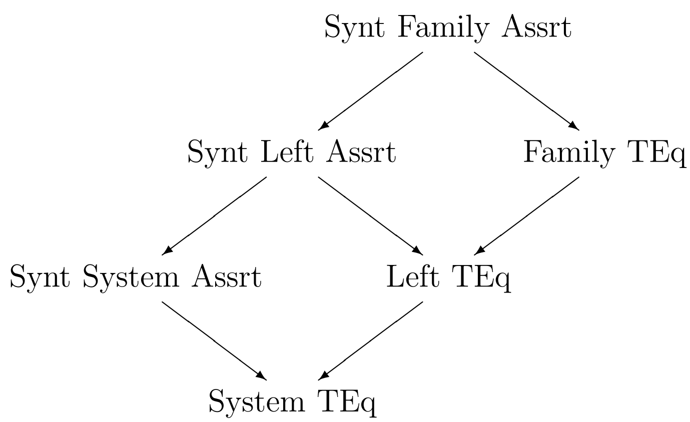
Suppose, first, that $\phi \in T_\Sigma$. Since τ is a natural theorem, we also have $\tau_\Sigma(\phi) \in T_\Sigma$. Thus, by family regularity (part of syntactic family assertionality), $\langle \phi, \tau_\Sigma(\phi) \rangle \in \Omega_\Sigma(T)$. But $\Omega(T)$ is a congruence system on $\mathbf{F}$, whence, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\langle \text{SEN}^\flat(f)(\phi), \tau_{\Sigma'}(\text{SEN}^\flat(f)(\phi)) \rangle \in \Omega_{\Sigma'}(T),$$

i.e., $(\iota \approx \tau)_\Sigma[\phi] \leq \Omega(T)$.

Assume, conversely, that $(\iota \approx \tau)_\Sigma[\phi] \leq \Omega(T)$. In particular, $\langle \phi, \tau_\Sigma(\phi) \rangle \in \Omega_\Sigma(T)$. However, since τ is a natural theorem, $\tau_\Sigma(\phi) \in T_\Sigma$. Therefore, by the compatibility of $\Omega(T)$ with T, we get that $\phi \in T_\Sigma$. ∎

Theorem 1267 establishes the following mixed hierarchy of syntactic assertionality and truth equationality classes.



It is not difficult to see that the syntactic assertionality classes are properly included in the corresponding truth equationality classes. This is accomplished by exhibiting a π-institution which is family truth equational but fails to be syntactically system assertional.

Example 1268 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be the algebraic system defined as follows:*

- $\mathbf{Sign}^\flat$ *is the trivial category with object* Σ*;*
- $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$ *is defined by* $\mathrm{SEN}^\flat(\Sigma) = \{0, 1, 2\}$*;*
- $N^\flat$ *is the clone generated by the unary natural transformations* $\sigma^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, specified by*

$$\sigma^\flat_\Sigma(0) = 0, \quad \sigma^\flat_\Sigma(1) = 1, \quad \sigma^\flat_\Sigma(2) = 0,$$

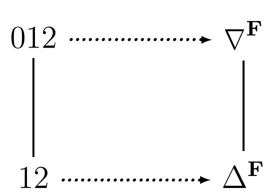
  and $\tau^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, given by*

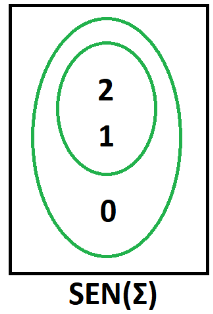
$$\tau^\flat_\Sigma(0) = 2, \quad \tau^\flat_\Sigma(1) = 1, \quad \tau^\flat_\Sigma(2) = 2.$$

Define the π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *by stipulating that*

$$\mathcal{C}_\Sigma = \{\{1, 2\}, \{0, 1, 2\}\}.$$

$\mathcal{I}$ *is systemic and its lattice of theory families and corresponding Leibniz congruence systems are shown in the diagram.*



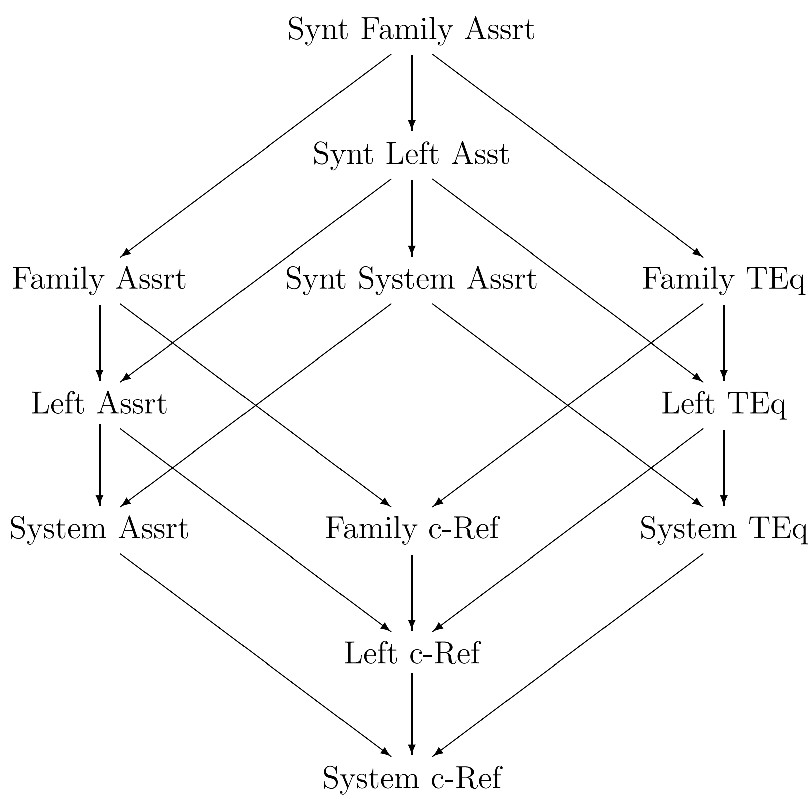


It is not difficult to check that $\mathcal{I}$ *is family truth equational, with witnessing equation* $\iota \approx \tau^\flat$.

On the other hand, $\mathcal{I}$ *is not system regular, since, for* $T = \{\{1,2\}\}$, *we have* $1,2 \in T_\Sigma$, *but* $\langle 1,2\rangle \notin \Delta^{\mathbf{F}}_\Sigma = \Omega_\Sigma(T)$.

Thus, $\mathcal{I}$ *belongs to all three truth equationality classes, but does not satisfy any of the three regularity conditions and, hence, belongs to none of the three syntactic assertionality classes.*

Finally, if we add the corresponding semantic classes of those depicted in the preceding diagram, we get a bigger view of the hierarchy consisting of assertionality (semantic and syntactic) and of complete reflectivity (semantic) and truth equationality (syntactic) classes.



Finally, based on previously established results, we can easily show that the three types of syntactic assertionality transfer from a π-institution to all its generalized matrix families. This constitutes an analog of Theorem 599 in the syntactic context.

Theorem 1269 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically family (respectively, left, system) assertional if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\langle \mathbf{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ *is syntactically family (respectively, left, system) assertional.*

Proof: This follows by putting together Theorem 585, asserting that regularity transfers, and Theorem 1197, asserting that the existence of natural theorems transfers. ■

16.6 Syntactic RW Prealgebraizability

In this section, we deal with three versions of syntactic regular weak prealgebraizability. These arise by combining syntactic prealgebraicity with each of the three versions of syntactic assertionality.

Definition 1270 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is* **syntactically regularly weakly family prealgebraizable***, or* **syntactically RWF prealgebraizable** *for short, if it is syntactically prealgebraic and syntactically family assertional;*
- $\mathcal{I}$ *is* **syntactically regularly weakly left prealgebraizable***, or* **syntactically RWL prealgebraizable** *for short, if it is syntactically prealgebraic and syntactically left assertional;*
- $\mathcal{I}$ *is* **syntactically regularly weakly system prealgebraizable***, or* **syntactically RWS prealgebraizable** *for short, if it is syntactically prealgebraic and syntactically system assertional.*

Based on the syntactic assertionality hierarchy established in Proposition 1262, we have the following

Proposition 1271 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is syntactically regularly weakly family prealgebraizable, then it is syntactically regularly weakly left prealgebraizable;*

(b) *If $\mathcal{I}$ is syntactically regularly weakly left prealgebraizable, then it is syntactically regularly weakly system prealgebraizable.*

Proof: Straightforward by combining Definition 1270 and Proposition 1262. ■

Proposition 1271 establishes the syntactic regular weak prealgebraizability hierarchy depicted in the following diagram.

Syntactic Regular Weak Family Prealgebraizable

↓

Syntactic Regular Weak Left Prealgebraizable

↓

Syntactic Regular Weak System Prealgebraizable

Being very close to the apex of the Leibniz hierarchy, just below the other classes that are studied in detail in the remaining sections of the present chapter, it compares favorably (meaning is stronger) to many of the other classes, semantic and syntactic introduced so far.

First, we look at the extant relationships between syntactic regular weak prealgebraizability classes and the four syntactic regular prealgebraicity classes of Section 16.3. It turns out that syntactic regular weak family prealgebraizability implies syntactic family regular protoalgebraicity and that syntactically regular weak system prealgebraizability implies syntactic system regular prealgebraicity. The only implication one can draw from the middle class of syntactically regular weak left prealgebraizability is the trivial one of being syntactically prealgebraic and left regular, which, strictly speaking, lies outside the syntactic hierarchy of Section 16.3.

Proposition 1272 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) *If $\mathcal{I}$ is syntactically RWF prealgebraizable, then it is syntactically family regularly protoalgebraic;*

(b) *If $\mathcal{I}$ is syntactically RWS prealgebraizable, then it is syntactically system regularly prealgebraic.*

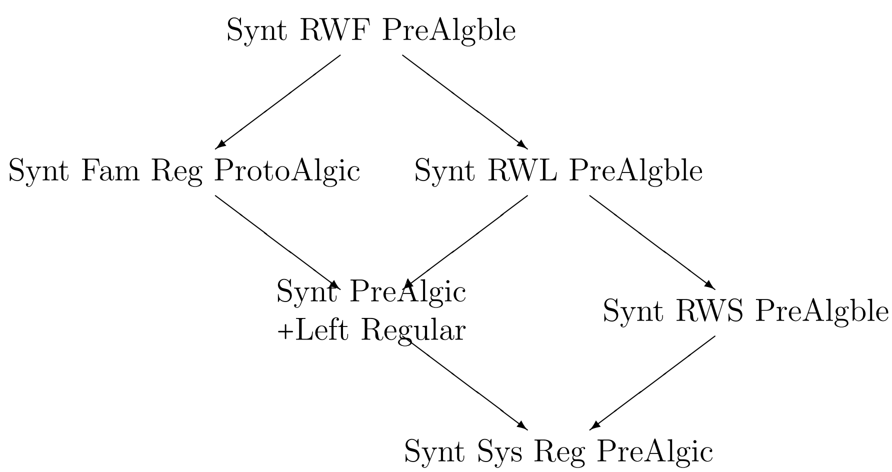
Proof:

(a) Suppose that $\mathcal{I}$ is syntactically RWF prealgebraizable. Note that, by definition, $\mathcal{I}$ is syntactically family assertional, i.e., it is family regular and has natural theorems. Thus, by Theorem 1267, it is family truth equational. Thus, by Theorem 829, it is family c-reflective, whence, by Proposition 237, it is systemic. Thus, since, by definition, it is syntactically prealgebraic, it must be syntactically protoalgebraic. This proves that it is syntactically family regularly protoalgebraic.

(b) By definition $\mathcal{I}$ is syntactically system assertional, whence it is system regular. And it is syntactically prealgebraic, also by definition. Thus, it is syntactically system regularly prealgebraic.

■

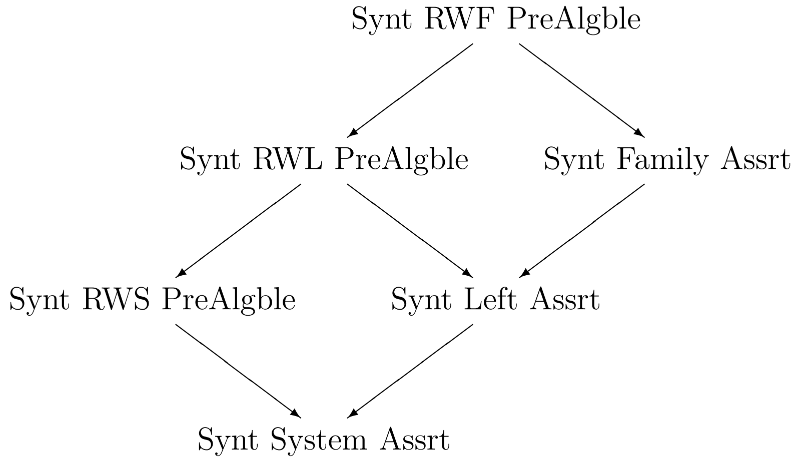
Thus, according to Proposition 1272, we get the mixed hierarchy depicted in the diagram.



As far as relationships between the syntactic regular weak prealgebraizability hierarchy and the syntactic assertionality hierarchy are concerned, we have, directly by definition, the following inclusions.

Proposition 1273 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is syntactically regularly weakly family (left, system, respectively) prealgebraizable, then it is syntactically family (left, system, respectively) assertional.*

Proof: Directly from Definition 1270. ■



Finally we look at closer relationships with other classes that are placed relatively high in the Leibniz hierarchy. Still staying with syntactically defined classes, we have the following relationships between the classes in the syntactic regular weak prealgebraizability hierarchy and the classes in the syntactic weak prealgebraizability hierarchy, which were defined in Chapter 12.

Proposition 1274 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

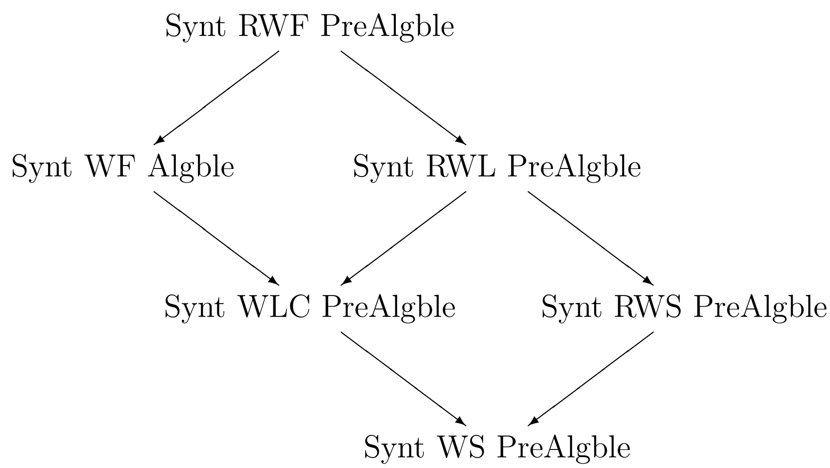
(a) If $\mathcal{I}$ *is syntactically regularly weakly family prealgebraizable, then it is syntactically weakly family algebraizable;*

(b) If $\mathcal{I}$ *is syntactically regularly weakly left prealgebraizable, then it is syntactically weakly left c-reflective prealgebraizable;*

(c) If $\mathcal{I}$ *is syntactically regularly weakly system prealgebraizable, then it is syntactically weakly system prealgebraizable.*

Proof: We only prove Part (a). Parts (b) and (c) can be proven similarly and are easier. Suppose $\mathcal{I}$ is syntactically regularly weakly family prealgebraizable. Then, it is, by definition syntactically family assertional. Thus, by Theorem 1267, it is family truth equational and, therefore, systemic. Thus, on the one hand, $\mathcal{I}$ is syntactically prealgebraic, and, hence, by systemicity, syntactically protoalgebraic, and, on the other, it is family truth equational. Therefore, it is syntactically weakly family algebraizable. ■

Proposition 1274, establishes the following hierarchy of syntactically regularly weakly prealgebraizable and syntactically weakly prealgebraizable π-institutions.



Finally, we reach across to bridge the gap between syntactically and semantically defined prealgebraizability classes. We establish relationaships that govern the syntactic regular weak prealgebraizability classes and the regular weak prealgebraizability classes that were defined in Chapter 8.

Proposition 1275 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

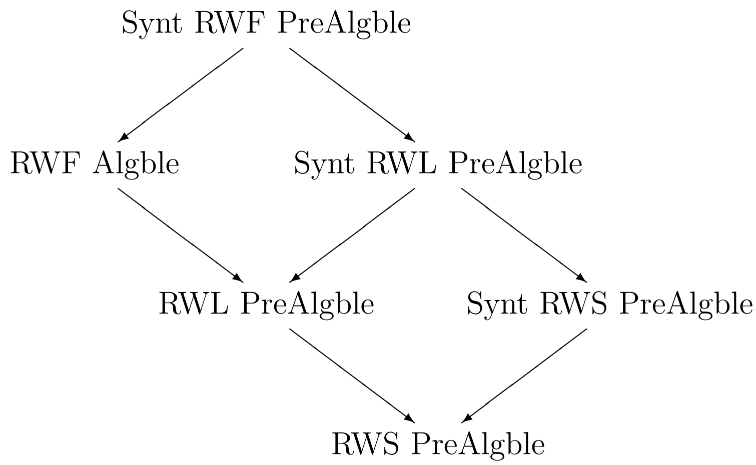
(a) If $\mathcal{I}$ *is syntactically regularly weakly family prealgebraizable, then it is regularly weakly family algebraizable;*

(b) If $\mathcal{I}$ *is syntactically regularly weakly left prealgebraizable, then it is regularly weakly left prealgebraizable;*

(c) If $\mathcal{I}$ *is syntactically regularly weakly system prealgebraizable, then it is regularly weakly system prealgebraizable.*

Proof: This follows from the facts that, on the one hand, syntactic prealgebraicity implies prealgebraicity and, on the other hand, syntactic family (left, system, respectively) assertionality implies family (left, system, respectively) assertionality. ∎

Proposition 1275 gives rise to the following mixed, semantic and syntactic, hierarchy of regularly weakly prealgebraizable π-institutions.



Based on existing results, we can show that all three kinds of syntactic regular weak prealgebraizability transfer from theory families/systems to filter families/systems over arbitrary **F**-algebraic systems. This is the syntactic analog of Theorem 609, which asserted that regular weak prealgebraizability properties transfer from a π-institution to all its generalized matrix families.

Theorem 1276 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically regularly weakly family (left, system, respectively) prealgebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, the* $\mathcal{I}$*-gmatrix family* $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ *is syntactically regularly weakly family (left, system, respectively) prealgebraizable.*

Proof: By Theorem 789, syntactic prealgebraicity transfers. By Theorem 585, the three regularity properties transfer. Finally, by Theorem 1197, the property of possessing natural theorems also transfers. Thus, the properties of being syntactically regularly weakly family, left and system prealgebraizable all transfer from $\mathcal{I}$ to $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$, for all $\mathbf{F}$-algebraic systems $\mathcal{A}$. ■

Finally, we adapt previously obtained results characterizing regular weak prealgebraizability to obtain similar characterizations of syntactic regular weak prealgebraizability in terms of mappings between posets of filter families/systems (including theory families/systems) and congruence systems. Essentially, to the characterizations obtained in Theorems 610, 611 and 612, we add the conditions of having enough natural transformations so that syntactic prealgebraicity is ensured and also the existence of natural theorems so that truth equationality is obtained, rather than having only complete reflectivity.

Theorem 1277 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(i) $\mathcal{I}$ *is syntactically regularly weakly family prealgebraizable;*

(ii) $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is an order isomorphism,* $\mathcal{I}$ *has a Leibniz reflexive core and a natural theorem* $\tau : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, such that, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,* $T = \tau/\Omega(T)$*;*

(iii) For every $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, the clauses of Part (ii) hold for the* π*-institution* $\langle \mathcal{A}.C^{\mathcal{I},\mathcal{A}}\rangle$*.*

Proof: By Theorem 1299, $\mathcal{I}$ is syntactically regularly weakly family prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, the π-institution $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}}\rangle$ is also syntactically regularly weakly family prealgebraizable. Thus, to prove the statement, it suffices to consider the equivalence (i)$\Leftrightarrow$(ii).

Suppose, first, that $\mathcal{I}$ is syntactically regularly weakly family prealgebraizable. Then it is, by definition, syntactically prealgebraic. Moreover, it is, by definition, syntactically family assertional. Thus, it has a natural theorem τ and it is, by Theorem 1267, family truth equational. Thus, by Theorem 829, it is family c-reflective and, hence, by Proposition 237, systemic. This implies that it is syntactically protoalgebraic and family truth equational. Using Theorem 610, we conclude that Ω is an order isomorphism. By Theorem 788, it has a Leibniz reflexive core and, by Corollary 1266, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T = \tau/\Omega(T)$.

Assume, conversely, that the postulated conditions hold. By Proposition 1275, $\mathcal{I}$ is regularly weakly family prealgebraizable. Hence it is protoalgebraic, which, together with the postulated Leibniz property of the reflexive core, gives, by Corollary 809, that it is syntactically protoalgebraic. Further, by hypothesis and Corollary 1266, it is syntactically family assertional. Thus, by definition, it is syntactically regularly weakly family prealgebraizable. ■

Analogous characterization theorems may be provided for syntactical regular weak left and system prealgebraizability. The proofs are analogous and are omitted.

Theorem 1278 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(i) $\mathcal{I}$ *is syntactically regularly weakly left prealgebraizable;*

(ii) $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is an order embedding,* $\mathcal{I}$ *has a Leibniz reflexive core and a natural theorem* $\tau : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, such that, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,* $\overleftarrow{T} = \tau/\Omega(T)$*;*

(iii) For every $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, the clauses of Part (ii) hold for the* π*-institution* $\langle \mathcal{A}.C^{\mathcal{I},\mathcal{A}}\rangle$*.*

Theorem 1279 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(i) $\mathcal{I}$ *is syntactically regularly weakly system prealgebraizable;*

(ii) $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is an order embedding,* $\mathcal{I}$ *has a Leibniz reflexive core and a natural theorem* $\tau : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, such that, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,* $T = \tau/\Omega(T)$*;*

(iii) *For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, the clauses of Part (ii) hold for the* π*-institution* $\langle \mathcal{A}.C^{\mathcal{I},\mathcal{A}} \rangle$*.*

16.7 Syntactic RW Algebraizability

In this section, we deal with three versions of syntactic regular weak algebraizability. These arise by combining syntactic protoalgebraicity with each of the three versions of syntactic assertionality.

Definition 1280 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is* **syntactically regularly weakly family algebraizable***, or* **syntactically RWF algebraizable** *for short, if it is syntactically protoalgebraic and syntactically family assertional;*
- $\mathcal{I}$ *is* **syntactically regularly weakly left algebraizable***, or* **syntactically RWL algebraizable** *for short, if it is syntactically protoalgebraic and syntactically left assertional;*
- $\mathcal{I}$ *is* **syntactically regularly weakly system algebraizable***, or* **syntactically RWS algebraizable** *for short, if it is syntactically protoalgebraic and syntactically system assertional.*

One of the immediate consequences of family assertionality is that the π-institution under consideration must be systemic and, therefore, that pre- and protoalgebraicity coincide. This reasoning has been applied a few times already in the preceding section. It shows that syntactic regular weak family algebraizability coincides with syntactic weak family prealgebraizability.

Proposition 1281 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically regularly weakly family algebraizable if and only if it is syntactically regularly weakly family prealgebraizable.*

Proof: Suppose $\mathcal{I}$ is syntactically regularly weakly family prealgebraizable. Then, by definition, it is family assertional. Thus, by Theorem 1267, it is family truth equational. Hence, by Theorem 829, it is family completely reflective and, hence, by Proposition 237, it is systemic. Since, by definition, it is syntactically prealgebraic, it is, by systemicity, syntactically protoalgebraic. Therefore, being syntactically protoalgebraic and syntactically family assertional, it is syntactically regularly weakly family algebraizable. The reverse implication is trivial. So, equivalence of the two conditions is established. ■

The second important observation that one can make is that syntactic regular weak left and system algebraizability coincide. This is due to the fact that protoalgebraicity implies stability.

Proposition 1282 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically regularly weakly left algebraizable if and only if it is syntactically regularly weakly system algebraizable.*

Proof: It is easy to see that the left version implies the system version. This follows directly from the fact that syntactic left assertionality implies syntactic system assertionality, established in Proposition 1262. For the converse, assume that $\mathcal{I}$ is syntactically regularly weakly system algebraizable. Then it is, by definition, syntactically protoalgebraic. This implies, by Theorem 805, that it is protoalgebraic. Hence, by Theorem 175, it is stable. Now, also by definition, $\mathcal{I}$ is syntactically system assertional. Thus, by Proposition 1263, it is syntactically left assertional. Being syntactically protoalgebraic and syntactically left assertional, $\mathcal{I}$ is, by definition, syntactically regularly weakly left algebraizable. ■

Based on the syntactic assertionality hierarchy established in Proposition 1262, we have the following

Corollary 1283 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically regularly weakly family algebraizable, then it is syntactically regularly weakly system algebraizable.*

Proof: Straightforward by combining Definition 1280 and Proposition 1262. ■

Proposition 1283 establishes the syntactic regular weak algebraizability hierarchy depicted in the following diagram.

Syntactic Regular Weak Family Algebraizable

↓

Syntactic Regular Weak System Algebraizable

It is easy to see how the two classes introduced in this section fit within a mixed syntactic regular weak (pre)algebraizability hierarchy. Given Proposition 1281, which showed that the top classes in each of the two hierarchies coincide, the picture is completed by the following easy consequence of the definitions involved.

Proposition 1284 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically regularly weakly system algebraizable, then it is syntactically regularly weakly left prealgebraizable.*

Proof: If $\mathcal{I}$ is syntactically regularly weakly system algebraizable, then, by Proposition 1282, it is syntactically regularly weakly left algebraizable, whence, since syntactic protoalgebraicity implies syntactic prealgebraicity, we conclude that $\mathcal{I}$ is syntactically regularly weakly left algebraizable. ∎

Thus, the following diagram presents the complete picture consisting of the four syntactic regular weak (pre)algebraizability classes of π-institutions. Compare this with the identical hierarchy revealed on the semantic side in Section 8.5.

Syntactic RWF (Pre)Algebraizable

↓

Syntactic RWS Algebraizable

↓

Syntactic RWL Prealgebraizable

↓

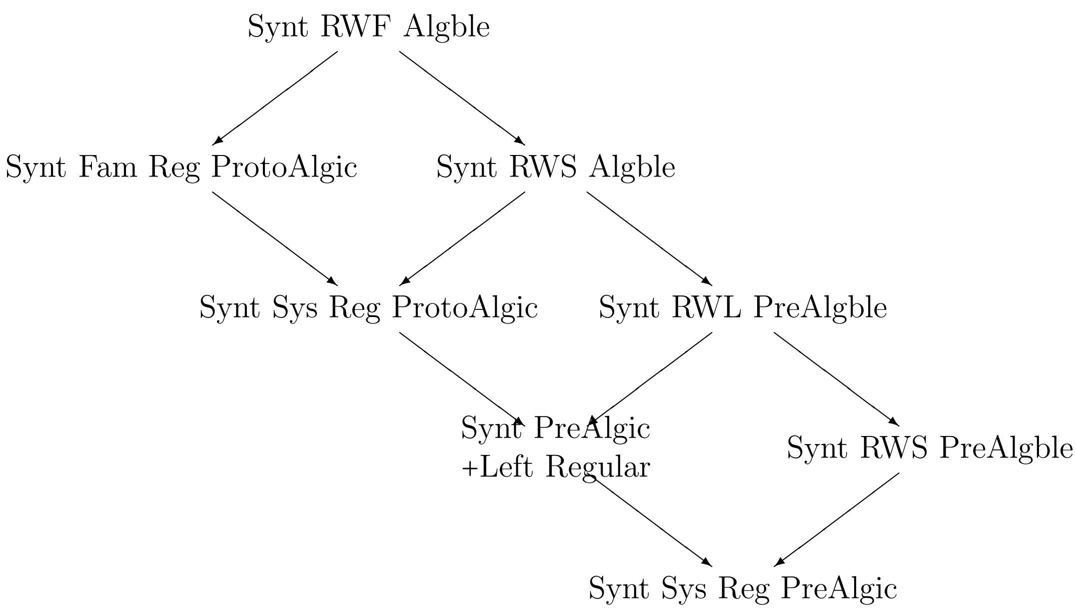
Syntactic RWS Prealgebraizable

To complete the puzzle of the relationships between syntactic regular weak prealgebraizability and syntactic regular pre- and protoalgebraicity classes, it suffices to observe that syntactic regular weak system algebraizability implies, rather trivially, syntactic system regular protoalgebraicity.

Proposition 1285 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically RWS algebraizable, then it is syntactically system regularly protoalgebraic.*

Proof: Suppose that $\mathcal{I}$ is syntactically RWS algebraizable. Note that, by definition, $\mathcal{I}$ is syntactically system assertional, and syntactically protoalgebraic. Hence, it is syntactically system regularly protoalgebraic. ∎

Thus, according to both Proposition 1272 and Proposition 1285, we get the following hierarchy, which completes the diagram given in the preceding section after Proposition 1272.

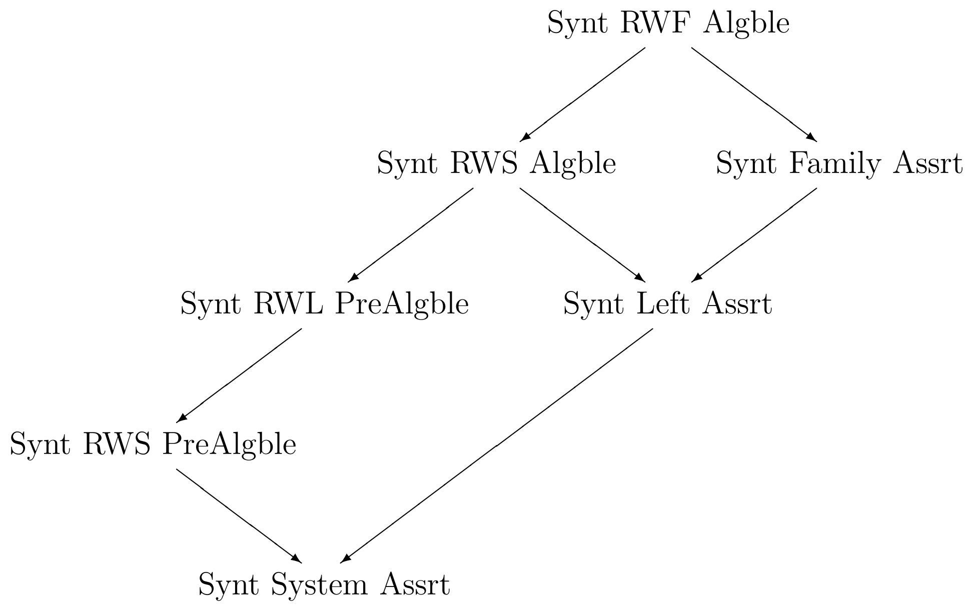


As far as relationships between the syntactic regular weak prealgebraizability hierarchy and the syntactic assertionality hierarchy are concerned, the picture is completed by realizing that syntactic regular weak system algebraizability implies syntactic left assertionality.

Corollary 1286 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically RWS algebraizable, then it is syntactically left assertional.*

Proof: The conclusion follows directly by Proposition 1282. ∎

Thus, according to Corollary 1286, and the hierarchy obtained in the preceding section, the interactions with syntactic assertionality properties are as shown in the diagram.

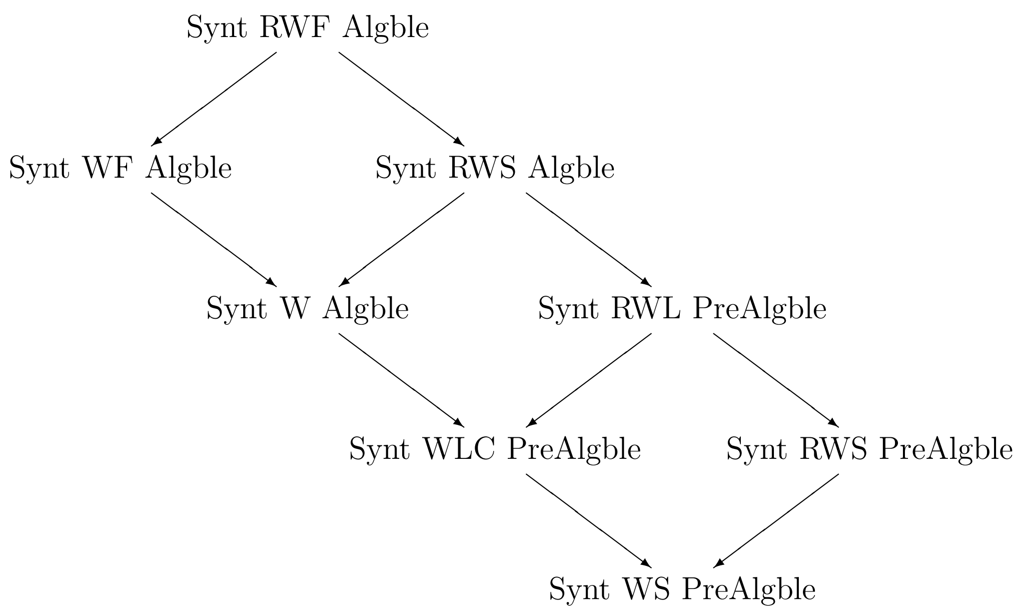


Finally we look at completing the hierarchy diagrams examining the relationships with other classes that are placed relatively high in the Leibniz hierarchy. Staying with syntactically defined classes, we have the following extra relationship between syntactically regularly weakly system algebraizable π-institutions and syntactically weakly (system) algebraizable ones.

Proposition 1287 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically regularly weakly system algebraizable, then it is syntactically weakly (system) algebraizable.*

Proof: Suppose $\mathcal{I}$ is syntactically regularly weakly system algebraizable. Then, it is, by definition syntactically protoalgebraic and system assertional. Thus, by Theorem 1267, it is syntactically protoalgebraic and system truth equational. Therefore, it is, by definition, syntactically weakly (system) algebraizable. ■

Proposition 1287, in conjunction with Proposition 1274, completes the hierarchy of syntactically regularly weakly (pre)algebraizable and syntactically weakly (pre)algebraizable π-institutions, part of which was shown following Proposition 1274.

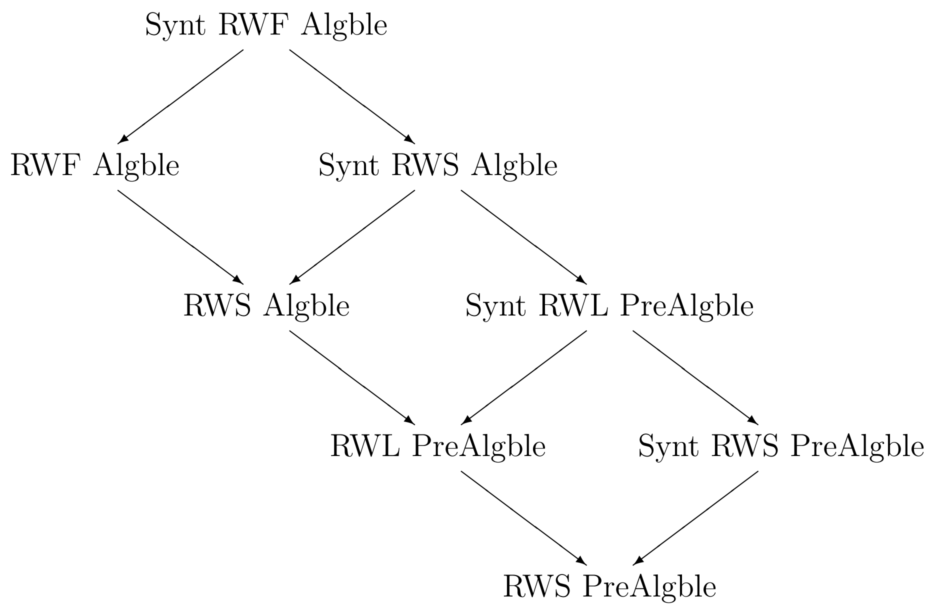


Finally, we revisit the relationships between syntactically and semantically defined (pre)algebraizability classes. We show that syntactic regular weak system algebraizability implies regular weak system algebraizability. This completes the picture established in Proposition 1275.

Proposition 1288 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically regularly weakly system algebraizable, then it is regularly weakly system algebraizable.*

Proof: This follows from the facts that, on the one hand, by Theorem 805, syntactic protoalgebraicity implies protoalgebraicity and, on the other hand, by Proposition 1264, syntactic system assertionality implies system assertionality. ■

Propositions 1275 and 1288 give rise to the following mixed, semantic and syntactic, hierarchy of regularly weakly (pre)algebraizable π-institutions, which completes the hierarchy shown after Proposition 1275.



As was the case with the three syntactic regular weak prealgebraizability classes, we may show that syntactic regular weak system algebraizability also transfers from theory families/systems to filter families/systems over arbitrary **F**-algebraic systems. This completes Transfer Theorem 1276.

Theorem 1289 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is syntactically regularly weakly system algebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, the* $\mathcal{I}$*-gmatrix family* $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ *is syntactically regularly weakly system algebraizable.*

Proof: By Theorem 810, syntactic protoalgebraicity transfers. By Theorem 585, system regularity transfers. Finally, by Theorem 1197, the property of possessing natural theorems also transfers. Thus, syntactic regular weak system algebraizability transfers from $\mathcal{I}$ to $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$, for all **F**-algebraic systems $\mathcal{A}$. This establishes the theorem. ■

Finally, we adapt previously obtained results characterizing syntactic regular weak prealgebraizability to obtain a similar characterization of syntactic regular weak system algebraizability in terms of mappings between posets of filter families/ systems (including theory families/systems) and congruence systems. This completes Theorem 1277.

Theorem 1290 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(i) $\mathcal{I}$ *is syntactically regularly weakly system algebraizable;*

(ii) $\mathcal{I}$ *is stable,* $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is an order isomorphism,* $\mathcal{I}$ *has a Leibniz reflexive core and a natural theorem* $\tau : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, such that, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,* $T = \tau/\Omega(T)$*;*

(iii) For every $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, the clauses of Part (ii) hold for the* π*-institution* $\langle \mathcal{A}.C^{\mathcal{I},\mathcal{A}} \rangle$*.*

Proof: By Theorem 1289, $\mathcal{I}$ is syntactically regularly weakly system algebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, the π-institution $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ is also syntactically regularly weakly system algebraizable. Thus, to prove the statement, it suffices to consider the equivalence (i)⇔(ii).

Suppose, first, that $\mathcal{I}$ is syntactically regularly weakly system algebraizable. Then, it is, in particular, by Proposition 1288, regularly weakly system algebraizable, and, by definition, syntactically protoalgebraic and syntactically system assertional. By Theorem 624, $\mathcal{I}$ is stable, $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is an order isomorphism and, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $T = \tau/\Omega(T)$, where τ is a natural theorem, whose existence is guaranteed by syntactic assertionality. Finally, syntactic protoalgebraicity implies, by Theorem 805, that $\mathcal{I}$ has a Leibniz reflexive core.

Assume, conversely, that the postulated conditions hold. By Theorem 624, $\mathcal{I}$ is regularly weakly system algebraizable. Hence it is protoalgebraic, which, together with the postulated Leibniz property of the reflexive core, gives, by Theorem 805, that it is syntactically protoalgebraic. Further, since it is regularly weakly system algebraizable, it is, in particular, system regular and, by hypothesis, has natural theorems. Thus, it is syntactically system assertional. Hence, being syntactically protoalgebraic and syntactically system assertional, it is, by definition, syntactically regularly weakly system algebraizable. ∎

16.8 Syntactic Regular (Pre)Algebraizability

In this section, we deal with the four versions of syntactic regular (pre)algebraizability, corresponding to the four versions of syntactic regular weak (pre)algebraizability that were studied in the preceding two sections. These arise by combining syntactic (pre)equivalentiality with each of the three versions of syntactic assertionality. They give rise to a four-element linear hierarchy that parallels that of syntactically regularly weakly (pre)algebraizable π-institutions and lies directly above it. The four classes, introduced and studied in the present section, lie at the very apex of the Leibniz hierarchies that were studied in detail in the monograph, and which form the backbone of the field of categorical abstract algebraic logic.

A priori, one may define six different classes of syntactically regularly (pre)algebraizable π-institutions. Three of these classes use syntactic preequivalentiality.

Definition 1291 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

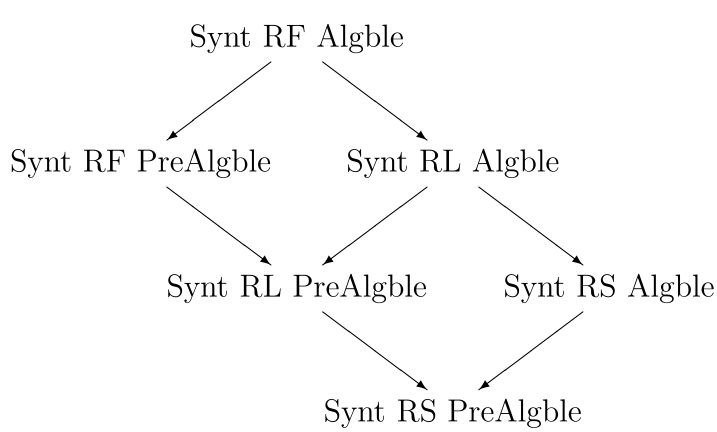
- $\mathcal{I}$ *is* **syntactically regularly family prealgebraizable***, or* **syntactically RF prealgebraizable** *for short, if it is syntactically preequivalential and syntactically family assertional;*
- $\mathcal{I}$ *is* **syntactically regularly left prealgebraizable***, or* **syntactically RL prealgebraizable** *for short, if it is syntactically preequivalential and syntactically left assertional;*
- $\mathcal{I}$ *is* **syntactically regularly system prealgebraizable***, or* **syntactically RS prealgebraizable** *for short, if it is syntactically preequivalential and syntactically system assertional.*

Three more classes use syntactic equivalentiality.

Definition 1292 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- $\mathcal{I}$ *is* **syntactically regularly family algebraizable***, or* **syntactically RF algebraizable** *for short, if it is syntactically equivalential and syntactically family assertional;*
- $\mathcal{I}$ *is* **syntactically regularly left algebraizable***, or* **syntactically RL algebraizable** *for short, if it is syntactically equivalential and syntactically left assertional;*
- $\mathcal{I}$ *is* **syntactically regularly system algebraizable***, or* **syntactically RS algebraizable** *for short, if it is syntactically equivalential and syntactically system assertional.*

We can show that similar relationships to those holding between the syntactic regular weak (pre)algebraizability classes are valid in this case also, leading to the collapsing of the six-class hierarchy (which, a priori, would look as in the accompanying figure)



to only four classes forming a linear hierarchy.

The top classes of syntactically regularly family prealgebraizable and algebraizable π-institutions coincide. Moreover, in the algebraizability case, syntactic regular left algebraizability turns out to be identical with syntactic regular system algebraizability. These relationships are presented formally in the following proposition.

Proposition 1293 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is syntactically regularly family prealgebraizable if and only if it is syntactically regularly family algebraizable;*

(b) $\mathcal{I}$ *is syntactically regularly left algebraizable if and only if it is syntactically regularly system algebraizable.*

Proof:

(a) Of course the right-to-left implication is trivial, since, by definition (see Section 13.2 and 13.3), syntactic equivalentiality implies syntactic preequivalentiality. On the other hand, by Theorem 1267, syntactic family assertionality implies family truth equationality, which, in turn, implies, by Theorem 829, family c-reflectivity and, hence, by Lemma 233, systemicity. Thus, under the given hypothesis, syntactic preequivalentiality coincides with syntactic equivalentiality.

(b) Again, since it is obvious that syntactical regular left algebraizability implies syntactical system algebraizability, in view of the fact (Proposition 1262) that syntactical left assertionality implies syntactical system assertionality, one must focus on the reverse implication. However, syntactic system algebraizability entails syntactic protoalgebraicity, which implies, by Theorem 792, protoalgebraicity, which, in turn, by Lemma 170, implies stability. And under stability, by Proposition 1263, syntactic left assertionality and syntactic system assertionality coincide. ■

Now the following implications are straightforward and establish the hierarchy obtained from the preceding diagram, if one takes into account the pairwise identification of classes proven in Proposition 1293.

Proposition 1294 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is syntactically regularly family (pre)algebraizable, then it is syntactically regularly system (left) algebraizable;*

(b) If $\mathcal{I}$ *is syntactically regularly system (left) algebraizable, then it is syntactically regularly left prealgebraizable;*

(c) *If $\mathcal{I}$ is syntactically regularly left prealgebraizable, then it is syntactically regularly system prealgebraizable.*

Proof: Part (a) relies on the fact that, by Proposition 1262, syntactic family assertionality is stronger than syntactic system assertionality. Part (b) relies on the fact that syntactic equivalentiality implies syntactic preequivalentiality. Finally, Part (c) is a direct consequence of syntactic system assertionality being dominated by syntactic left assertionality (Proposition 1262). ■

Proposition 1294, which takes into account the identifications of Proposition 1293, establishes the syntactic regular (pre)algebraizability hierarchy depicted in the following diagram.

Syntactic Regular Family Algebraizable

↓

Syntactic Regular System Algebraizable

↓

Syntactic Regular Left Prealgebraizable

↓

Syntactic Regular System Prealgebraizable

We look, next, at the relationships between syntactic regular (pre)algebraizability classes and the four syntactic regular (pre)equivalentiality classes of Section 16.4. Syntactic regular family algebraizability implies syntactic family regular equivalentiality, syntactic regular system algebraizability implies syntactic system regular equivalentiality and syntactic regular system prealgebraizability implies syntactic system regular preequivalentiality. However, from syntactic regular left prealgebraizability we can only make the trivial deduction of syntactic preequivalentiality and left regularity. Strictly speaking, the combination of these two properties does not form a class in the syntactic hierarchy of Section 16.4.

Proposition 1295 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

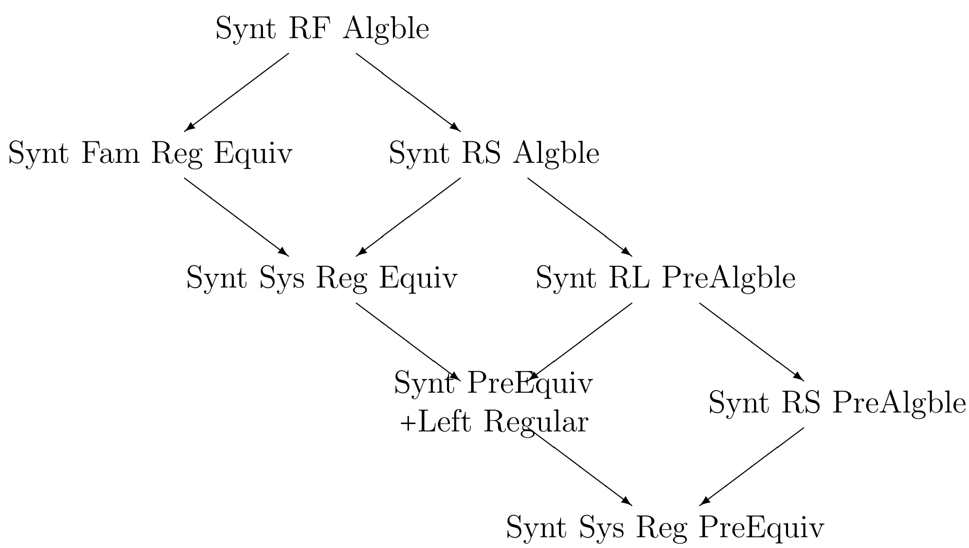
(a) *If $\mathcal{I}$ is syntactically RF algebraizable, then it is syntactically family regularly equivalential;*

(b) *If $\mathcal{I}$ is syntactically RS algebraizable, then it is syntactically system regularly equivalential;*

(c) *If $\mathcal{I}$ is syntactically RS prealgebraizable, then it is syntactically system regularly preequivalential.*

Proof: For Part (a) observe that, by definition, $\mathcal{I}$ is syntactically equivalential and syntactically family assertional, which implies that it is family regular. Thus, it is syntactically family regularly equivalential. Similarly, for Part (b), $\mathcal{I}$ is, by definition, syntactically equivalential and syntactically system assertional, which implies system regularity. Thus, it is syntactically system regularly equivalential. Finally, in Part (c), $\mathcal{I}$ is, by definition, syntactically preequivalential and syntactically system assertional, whence, once more, it is also syntactically system regular. Hence, it is syntactically system regularly preequivalential. ∎

Thus, according to Proposition 1295, we get the mixed hierarchy depicted in the diagram.



We do not dwell on relationships between the syntactic regular (pre)algebraizability classes and the syntactic assertionlity classes, since those are direct consequences of the relationships, already established in the preceding section, between syntactic regular weak (pre)algebraizability classes and the syntactic assertionality classes, once the following, also easily obtainable, relations between syntactic regular (pre)algebraizability classes and syntactic regular weak (pre)algebraizability classes are established.

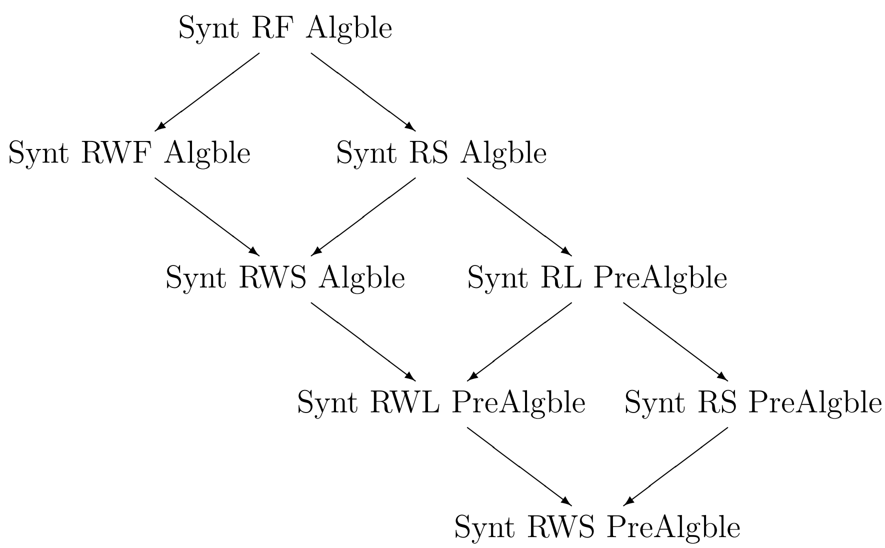
Proposition 1296 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) *If $\mathcal{I}$ is syntactically regularly family (system, respectively) algebraizable, then it is syntactically regularly weakly family (system, respectively) algebraizable;*

(b) *If $\mathcal{I}$ is syntactically regularly left (system, respectively) prealgebraizable, then it is syntactically regularly weakly left (system, respectively) prealgebraizable.*

Proof: Directly from the definitions involved. ∎

Thus, we get a comprehensive picture of the syntactic regular prealgebraizability hierarchy, including both weak and "strong" (meaning non-weak) classes.



Finally we look at the relationships with other classes that are placed just below syntactically regularly (pre)algebraizable π-institutions, namely, the classes in the syntactic (pre)algebraizablity hierarchy and those in the (semantic) regular (pre)algebraizability hierarchy. The former hierarchy was studied in detail in Chapter 12, whereas the latter was studied in Chapter 8. Starting with the relationships between the syntactic regular (pre)algebraizability and the syntactic (pre)algebraizability classes, we get the following

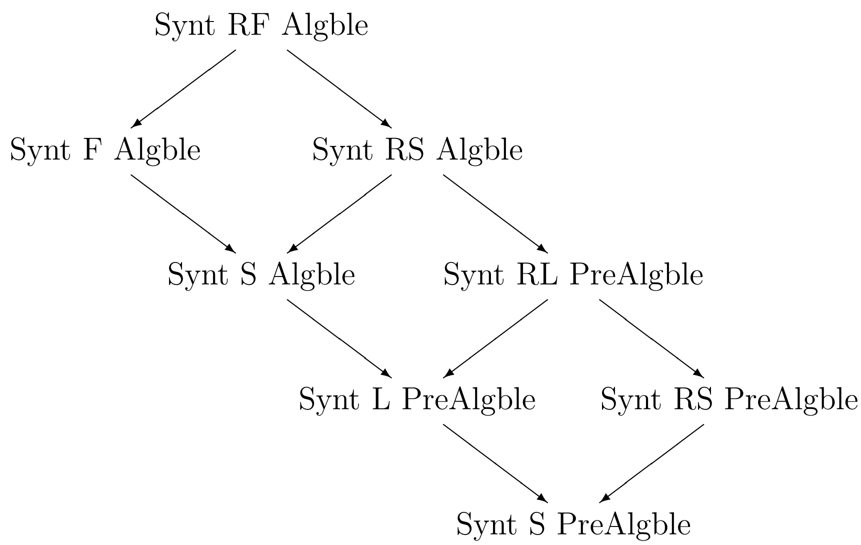
Proposition 1297 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$.

(a) *If $\mathcal{I}$ is syntactically regularly family (system, respectively) algebraizable, then it is syntactically family (system, respectively) algebraizable;*

(b) *If $\mathcal{I}$ is syntactically regularly left (system, respectively) prealgebraizable, then it is syntactically left (system, respectively) prealgebraizable.*

Proof: Part (a) follows from the fact that syntactic family and system assertionality imply, respectively, family and system truth equationality. Part (b), similarly, follows from the fact that syntactic left and system assertionality imply, respectively, left and system truth equationality. All the aforementioned implications, forming the key to the inclusions in the statement, are the subject of Theorem 1267. ■

Proposition 1297, establishes the following mixed hierarchy of syntactically regularly (pre)algebraizable and syntactically (pre)algebraizable π-institutions.



We close with the relationships between syntactically and semantically defined regular (pre)algebraizability classes.

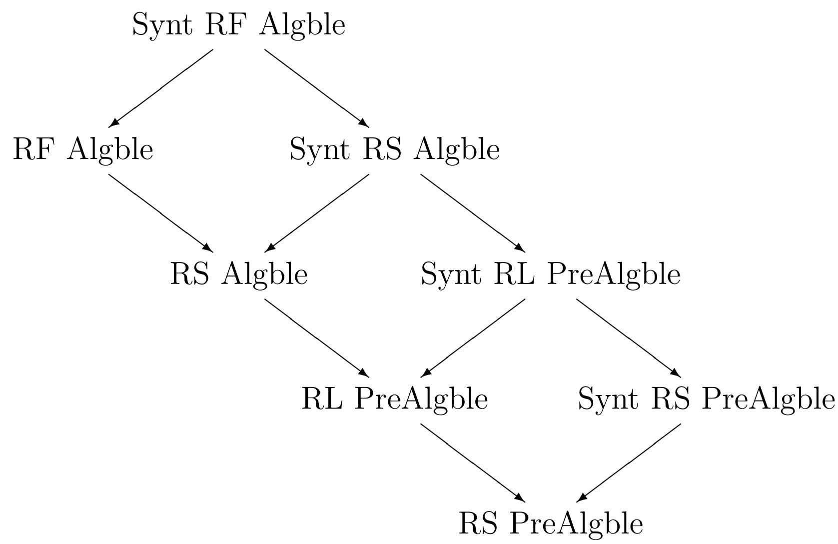
Proposition 1298 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\mathcal{I}$ *is syntactically regularly family (system, respectively) algebraizable, then it is regularly family (system, respectively) algebraizable;*

(b) If $\mathcal{I}$ *is syntactically regularly left (system, respectively) prealgebraizable, then it is regularly left (system, respectively) prealgebraizable.*

Proof: This follows from the facts that, on the one hand, syntactic pre- and protoalgebraicity imply respectively pre- and protoalgebraicity, and, on the other hand, syntactic family (left, system, respectively) assertionality implies family (left, system, respectively) assertionality. The former implications are established in Theorems 771 and 792. The latter are by Proposition 1264. ■

Proposition 1298 gives rise to the following mixed, semantic and syntactic, hierarchy of regularly (pre)algebraizable π-institutions.



As was the case with syntactic regular weak (pre)algebraizability, all four flavors of syntactic regular (pre)algebraizability transfer from theory families/systems to filter families/systems over arbitrary $\mathbf{F}$-algebraic systems. This is a "strong" analog of Theorems 1276 and 1289, which asserted that syntactic regular weak (pre)algebraizability properties transfer from a π-institution to all its generalized matrix families.

Theorem 1299 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is syntactically regularly family (system, respectively) algebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, the $\mathcal{I}$-gmatrix family $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ is syntactically regularly family (system, respectively) algebraizable;

(b) $\mathcal{I}$ is syntactically regularly left (system, respectively) prealgebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, the $\mathcal{I}$-gmatrix family $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ is syntactically regularly left (system, respectively) prealgebraizable.

Proof: By Theorems 955 and 972, syntactic preequivalentiality and syntactic equivalentiality transfer. By Theorem 585, the three regularity properties transfer. Finally, by Theorem 1197, the property of possessing natural theorems also transfers. Thus, all four syntactic regular (pre)algebraizability properties transfer from $\mathcal{I}$ to $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$, for all $\mathbf{F}$-algebraic systems $\mathcal{A}$. ■

Finally, we obtain characterizations of syntactically regular (pre)algebraizability in terms of mappings between posets of filter families/ systems (including theory families/systems) and congruence systems.

Theorem 1300 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(i) $\mathcal{I}$ *is syntactically regularly family algebraizable;*

(ii) $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is an order isomorphism commuting with inverse logical extensions,* $\mathcal{I}$ *has a Leibniz binary reflexive core and a natural theorem* $\tau : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, such that, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,* $T = \tau/\Omega(T)$*;*

(iii) For every $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, the clauses of Part (ii) hold for the* π*-institution* $\langle \mathcal{A}.C^{\mathcal{I},\mathcal{A}} \rangle$*.*

Proof: By Theorem 1299, $\mathcal{I}$ is syntactically regularly family algebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, the π-institution $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ is also syntactically regularly family algebraizable. Thus, to prove the statement, it suffices to consider the equivalence (i)$\Leftrightarrow$(ii).

Suppose, first, that $\mathcal{I}$ is syntactically regularly family algebraizable. Then it is, by definition, syntactically equivalential and syntactically family assertional. Thus, it has a natural theorem τ, it is family regular and it is, by Theorem 1267, family truth equational. Using Corollary 649, we conclude that Ω is an order isomorphism commuting with inverse logical extensions, by Theorem 983, that it has a Leibniz binary reflexive core and, by Corollary 1266, that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T = \tau/\Omega(T)$.

Assume, conversely, that the postulated conditions hold. By Corollary 649, $\mathcal{I}$ is regularly family algebraizable. Hence it is equivalential, whence, together with the postulated Leibniz property of the binary reflexive core, we obtain, by Corollary 983, that it is syntactically equivalential. Further, by hypothesis and Corollary 1266, it is syntactically family assertional. Thus, by definition, it is syntactically regularly family algebraizable. ∎

Theorem 1301 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(i) $\mathcal{I}$ *is syntactically regularly system algebraizable;*

(ii) $\mathcal{I}$ *is stable,* $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is an order isomorphism commuting with inverse logical extensions,* $\mathcal{I}$ *has a Leibniz binary reflexive core and a natural theorem* $\tau : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, such that, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,* $T = \tau/\Omega(T)$*;*

(iii) For every $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, the clauses of Part (ii) hold for the* π*-institution* $\langle \mathcal{A}.C^{\mathcal{I},\mathcal{A}} \rangle$*.*

Proof: Similar to that of Theorem 1300. ■

Analogous characterization theorems may be provided for the syntactic regular prealgebraizability properties. The proofs are also similar and are, therefore, omitted.

Theorem 1302 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(i) $\mathcal{I}$ *is syntactically regularly left prealgebraizable;*

(ii) $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is an order embedding commuting with inverse logical extensions,* $\mathcal{I}$ *has a Leibniz binary reflexive core and a natural theorem* $\tau : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, such that, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,* $\overleftarrow{T} = \tau/\Omega(T)$*;*

(iii) For every $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, the clauses of Part (ii) hold for the* π*-institution* $\langle \mathcal{A}.C^{\mathcal{I},\mathcal{A}} \rangle$*.*

Theorem 1303 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(i) $\mathcal{I}$ *is syntactically regularly system prealgebraizable;*

(ii) $\Omega : \mathrm{ThSys}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ *is an order embedding commuting with inverse logical extensions,* $\mathcal{I}$ *has a Leibniz binary reflexive core and a natural theorem* $\tau : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$*, such that, for all* $T \in \mathrm{ThSys}(\mathcal{I})$*,* $T = \tau/\Omega(T)$*;*

(iii) For every $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, the clauses of Part (ii) hold for the* π*-institution* $\langle \mathcal{A}.C^{\mathcal{I},\mathcal{A}} \rangle$*.*

Chapter 17

The Syntactic Leibniz Hierarchy: Attic II

17.1 Finitary Companions Revisited

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall from Chapter 9 the construction of the *finitary companion* $\mathcal{I}^f = \langle \mathbf{F}, C^f \rangle$ of $\mathcal{I}$. It is defined, by setting, for all $\Sigma in|\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$C^f_\Sigma(\Phi) = \bigcup\{C_\Sigma(\Phi') : \Phi' \subseteq_f \Phi\},$$

where $\subseteq_f$ denotes the finite subset relation. It was shown in Corollary 653 that $\mathcal{I}^f$ is the largest finitary π-institution based on $\mathbf{F}$ that lies below $\mathcal{I}$ is the $\leq$ ordering. Furthermore, even though it is obvious, based on $\mathcal{I}^f \leq \mathcal{I}$, that $\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$, Proposition 655 provided a characterization of those sentence families of $\mathbf{F}$ that are theory families of $\mathcal{I}^f$. More concretely, it asserted that $T \in \mathrm{ThFam}(\mathcal{I}^f)$ if and only if it is the union of a directed locally finitely generated collection of theory families of $\mathcal{I}$.

Turning now to the Leibniz hierarchy, some of the semantic aspects of which, in relation to finitarity, were studied in some detail in Chapter 9, it was proven in Lemma 656 that protoalgebraicity is inherited by $\mathcal{I}$ from $\mathcal{I}^f$, i.e., if $\mathcal{I}^f$ is protoalgebraic, then so is $\mathcal{I}$ itself. This is a rather simple consequence of the fact that $\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$.

Recall from Chapter 11 the definition of the *reflexive core* $R^{\mathcal{I}}$ of a π-institution $\mathcal{I}$. It consists of all natural transformations $\rho^\flat$ in $N^\flat$, with two distinguished arguments, having the property that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\rho^\flat_\Sigma[\phi, \phi] \leq \mathrm{Thm}(\mathcal{I}).$$

It is not very difficult to show that the reflexive core of the finitary companion $\mathcal{I}^f$ of a π-institution $\mathcal{I}$ is included in that of $\mathcal{I}$.

Lemma 1304 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then*

$$R^{\mathcal{I}^f} \subseteq R^{\mathcal{I}}.$$

Proof: Suppose $\rho^\flat \in R^{\mathcal{I}^f}$ and consider $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. We have

$$\begin{array}{rcll}\rho^\flat_\Sigma[\phi, \phi] & \leq & \mathrm{Thm}(\mathcal{I}^f) & (\rho^\flat \in R^{\mathcal{I}^f}) \\ & \leq & \mathrm{Thm}(\mathcal{I}). & (\mathrm{Thm}(\mathcal{I}) \in \mathrm{ThFam}(\mathcal{I}^f))\end{array}$$

Thus, by definition, $\rho^\flat \in R^{\mathcal{I}}$. It follows that $R^{\mathcal{I}^f} \subseteq R^{\mathcal{I}}$. ■

Recall that the reflexive core $R^{\mathcal{I}}$ is said to be *Leibniz* if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi])).$$

From the fact that $R^{\mathcal{I}^f} \subseteq R^{\mathcal{I}}$ it follows at once that, if $\mathcal{I}^f$ is protoalgebraic and $R^{\mathcal{I}^f}$ is Leibniz in $\mathcal{I}^f$, then $R^{\mathcal{I}}$ is Leibniz in $\mathcal{I}$.

Proposition 1305 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}^f$ *is protoalgebraic and* $R^{\mathcal{I}^f}$ *is Leibniz in* $\mathcal{I}^f$*, then so is* $R^{\mathcal{I}}$ *in* $\mathcal{I}$*.*

Proof: Suppose that $\mathcal{I}^f$ is protoalgebraic and $R^{\mathcal{I}^f}$ is Leibniz in $\mathcal{I}^f$. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. We then have

$$\begin{array}{lll}\langle \phi, \psi \rangle & \in & \Omega_\Sigma(C^f(R^{\mathcal{I}^f}_\Sigma[\phi, \psi])) \quad (R^{\mathcal{I}^f} \text{ Leibniz in } \mathcal{I}^f) \\ & \subseteq & \Omega_\Sigma(C^f(R^{\mathcal{I}}_\Sigma[\phi, \psi])) \quad (\text{Lemma 1304 and hypothesis}) \\ & \subseteq & \Omega_\Sigma(C(R^{\mathcal{I}}_\Sigma[\phi, \psi])). \quad (\text{Corollary 653 and hypothesis})\end{array}$$

Therefore, $R^{\mathcal{I}}$ is Leibniz in $\mathcal{I}$. ∎

We can now show that syntactic protoalgebraicity is inherited by a π-institution $\mathcal{I}$ from its finitary companion $\mathcal{I}^f$. This forms an analog in the syntactic context of Lemma 656.

Theorem 1306 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}^f$ *is syntactically protoalgebraic, then so is* $\mathcal{I}$*.*

Proof: Suppose $\mathcal{I}^f$ is syntactically protoalgebraic. By Theorem 805, it is protoalgebraic and its reflexive core $R^{\mathcal{I}^f}$ is Leibniz in $\mathcal{I}^f$. Therefore, by Lemma 656, $\mathcal{I}$ is protoalgebraic and, by Proposition 1305, $R^{\mathcal{I}}$ is Leibniz in $\mathcal{I}$. Therefore, again by Theorem 805, $\mathcal{I}$ is syntactically protoalgebraic. ∎

Recalling Theorem 799, which characterizes syntactic protoalgebraicity in terms of the global family modus ponens property of the reflexive core, we derive the following

Corollary 1307 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}^f}$ *has the global family MP in* $\mathcal{I}^f$*, then* $R^{\mathcal{I}}$ *has the global family MP in* $\mathcal{I}$*.*

Proof: If $R^{\mathcal{I}^f}$ has the global family MP in $\mathcal{I}^f$, then, by Theorem 799, $\mathcal{I}^f$ is syntactically protoalgebraic. Thus, by Theorem 1306, $\mathcal{I}$ is syntactically protoalgebraic, whence, again by Theorem 799, applied in the opposite direction, $R^{\mathcal{I}}$ has the global family MP in $\mathcal{I}$. ∎

Alternatively, instead of deriving the implication in Corollary 1307 by applying Theorem 1306, we may prove it first and then use Theorem 799 to establish that syntactic protoalgebraicity of $\mathcal{I}^f$ implies the syntactic protoalgebraicity of $\mathcal{I}$. We outline this reasoning also, at the expense of having to repeat Corollary 1307 and Theorem 1306.

Lemma 1308 (Corollary 1307) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $R^{\mathcal{I}^f}$ *has the global family MP in* $\mathcal{I}^f$*, then* $R^{\mathcal{I}}$ *has the global family MP in* $\mathcal{I}$*.*

Proof: Suppose $R^{\mathcal{I}^f}$ has the global family MP in $\mathcal{I}^f$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$\phi \in T_\Sigma \quad \text{and} \quad R^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T.$$

By Lemma 1304, we get

$$\phi \in T_\Sigma \quad \text{and} \quad R^{\mathcal{I}^f}_\Sigma[\phi, \psi] \leq T.$$

But $\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$ and $R^{\mathcal{I}^f}$ is assumed to have the global family MP in $\mathcal{I}^f$. Thus, $\psi \in T_\Sigma$. This proves that $R^{\mathcal{I}}$ has the global family MP in $\mathcal{I}$. ■

Corollary 1309 (Theorem 1306) *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}^f$ is syntactically protoalgebraic, then so is $\mathcal{I}$.*

Proof: Suppose $\mathcal{I}^f$ is syntactically protoalgebraic. Then, by Theorem 799, $R^{\mathcal{I}^f}$ has the global family MP in $\mathcal{I}^f$. Thus, by Lemma 1308, $R^{\mathcal{I}}$ has the global family MP in $\mathcal{I}$. Hence, again by applying Theorem 799, only now in the reverse direction, $\mathcal{I}$ is syntactically protoalgebraic. ■

A similar work can be undertaken concerning truth equationality, based on an analog of Lemma 657, but referring to family c-reflectivity, which can be proved in a similar fashion as Lemma 657. We now provide the details.

It is straightforward to see, first of all, that family complete reflectivity is also inherited by $\mathcal{I}$ itself by its finitary companion.

Lemma 1310 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}^f$ is family c-reflective, then so is $\mathcal{I}$.*

Proof: If $\mathcal{I}^f$ is family c-reflective, then, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I}^f)$,

$$\bigcap_{T \in \mathcal{T}} \Omega(T) \leq \Omega(T') \quad \text{implies} \quad \bigcap_{T \in \mathcal{T}} T \leq T'.$$

In particular, the condition holds if quantification is restricted over the collection $\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$. Therefore, $\mathcal{I}$ is family c-reflective. ■

It is not very hard either to see that the the Suszko core of the finitary companion $\mathcal{I}^f$ of a π-institution $\mathcal{I}$ is contained in the Suszko core of $\mathcal{I}$ itself, just as was the case with the reflexive core. Recall that the Suszko core $S^{\mathcal{I}}$ of a π-institution $\mathcal{I}$ consists of those natural transformations $\sigma^\flat$ in $N^\flat$, with a single distinguished argument, such that, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\sigma^\flat[T] \leq \widetilde{\Omega}^{\mathcal{I}}(T).$$

This means, of course, that, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{implies} \quad \sigma^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T).$$

Lemma 1311 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then*

$$S^{\mathcal{I}^f} \subseteq S^{\mathcal{I}}.$$

Proof: Suppose that $\sigma^\flat \in S^{\mathcal{I}^f}$ and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$. Then, since $\sigma^\flat \in S^{\mathcal{I}^f}$ and $T \in \mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$, we get $\sigma^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}^f}(T) \leq \widetilde{\Omega}^{\mathcal{I}}(T)$, where the second inclusion follows from the fact that $\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$. Therefore, we conclude that $\sigma^\flat \in S^{\mathcal{I}}$. Hence, $S^{\mathcal{I}^f} \subseteq S^{\mathcal{I}}$. ■

With this result available, we can see that, if $\mathcal{I}^f$ is family c-reflective and its Suszko core is adequate, then the Suszko core of $\mathcal{I}$ is also adequate. Recall that adequacy of the Suszko core $S^{\mathcal{I}}$ means that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) = \bigcap\{\Omega(T) : T \in \mathrm{ThFam}(\mathcal{I}) \text{ and } S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}.$$

Recall also, that the right-to-left inclusion always holds. So the definition is tantamount to the assertion that the left-to-right inclusion also holds.

Proposition 1312 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}^f$ *is family c-reflective and* $S^{\mathcal{I}^f}$ *is adequate in* $\mathcal{I}^f$*, then so is* $S^{\mathcal{I}}$ *in* $\mathcal{I}$*.*

Proof: Suppose $\mathcal{I}^f$ is family c-reflective and that $S^{\mathcal{I}^f}$ is adequate. Then, by Theorem 847, $\mathcal{I}^f$ is truth equational, whence, by Theorem 840, for all $T \in \mathrm{ThFam}(\mathcal{I}^f)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad S^{\mathcal{I}^f}_\Sigma[\phi] \leq \Omega(T).$$

Consider $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. We have

$$\begin{array}{lll}
\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) & = & \bigcap\{\Omega(T) : T \in \mathrm{ThFam}(\mathcal{I}) \text{ and } \phi \in T_\Sigma\} \\
& & (\text{Definition of } \widetilde{\Omega}^{\mathcal{I}}) \\
& = & \bigcap\{\Omega(T) : T \in \mathrm{ThFam}(\mathcal{I}) \text{ and } S^{\mathcal{I}^f}_\Sigma[\phi] \leq \Omega(T)\} \\
& & (\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f) \text{ and displayed equivalence}) \\
& \leq & \bigcap\{\Omega(T) : T \in \mathrm{ThFam}(\mathcal{I}) \text{ and } S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)\}. \\
& & (\text{Lemma 1311})
\end{array}$$

Thus, by definition, $S^{\mathcal{I}}$ is also adequate in $\mathcal{I}$. ■

We can now show that truth equationality is inherited by a π-institution $\mathcal{I}$ from its finitary companion $\mathcal{I}^f$. This forms an analog of Lemma 1306.

Theorem 1313 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}^f$ *is truth equational, then so is* $\mathcal{I}$*.*

Proof: Suppose $\mathcal{I}^f$ is truth equational. By Theorem 847, it is family c-reflective and its Suszko core $S^{\mathcal{I}^f}$ is adequate in $\mathcal{I}^f$. Therefore, by Lemma 1310, $\mathcal{I}$ is family c-reflective and, by Proposition 1312, $S^{\mathcal{I}}$ is adequate in $\mathcal{I}$. Therefore, again by Theorem 847, $\mathcal{I}$ is truth equational. ∎

Theorem 840 characterized truth equationality in terms of the solubility property of the Suszko core. In fact, the solubility of the Suszko core is the condition asserting that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T) \quad \text{implies} \quad \phi \in T_\Sigma.$$

Since the reverse implication always holds, the condition is equivalent to the assertion that, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\phi \in T_\Sigma \quad \text{iff} \quad S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T).$$

Corollary 1314 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $S^{\mathcal{I}^f}$ *is soluble in* $\mathcal{I}^f$*, then* $S^{\mathcal{I}}$ *is soluble in* $\mathcal{I}$*.*

Proof: If $S^{\mathcal{I}^f}$ is soluble in $\mathcal{I}^f$, then, by Theorem 838, $\mathcal{I}^f$ is truth equational. Thus, by Theorem 1313, $\mathcal{I}$ is also truth equational, whence, again by Theorem 838, applied in the opposite direction, $S^{\mathcal{I}}$ is soluble in $\mathcal{I}$. ∎

Once more, as was the case with syntactic protoalgebraicity, instead of deriving the implication in Corollary 1314 by applying Theorem 1313, we may prove it first and then use Theorem 838 to establish that truth equationality of $\mathcal{I}^f$ implies truth equationality of $\mathcal{I}$.

Lemma 1315 (Corollary 1314) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $S^{\mathcal{I}^f}$ *is soluble in* $\mathcal{I}^f$*, then* $S^{\mathcal{I}}$ *is soluble in* $\mathcal{I}$*.*

Proof: Suppose $S^{\mathcal{I}^f}$ is soluble in $\mathcal{I}^f$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $S^{\mathcal{I}}_\Sigma[\phi] \leq \Omega(T)$. hence, by Lemma 1311, we get $S^{\mathcal{I}^f}_\Sigma[\phi] \leq \Omega(T)$. But $\mathrm{ThFam}(\mathcal{I}) \subseteq \mathrm{ThFam}(\mathcal{I}^f)$ and $S^{\mathcal{I}^f}$ is assumed to be soluble in $\mathcal{I}^f$. Thus, $\phi \in T_\Sigma$. This proves that $S^{\mathcal{I}}$ is soluble in $\mathcal{I}$. ∎

Corollary 1316 (Theorem 1313) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}^f$ *is truth equational, then so is* $\mathcal{I}$*.*

Proof: Suppose $\mathcal{I}^f$ is truth equational. Then, by Theorem 838, $S^{\mathcal{I}^f}$ is soluble in $\mathcal{I}^f$. Thus, by Lemma 1315, $S^{\mathcal{I}}$ is soluble in $\mathcal{I}$. Hence, again by applying Theorem 838, only now in the reverse direction, $\mathcal{I}$ is truth equational. ∎

We conclude the section by synthesizing Theorems 1306 and 1313. Recall that a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is syntactically weakly family algebraizable if it is

- protoalgebraic;
- family c-reflective;
- $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified, i.e., has a Leibniz reflexive core and an adequate Suszko core.

By Theorem 913, $\mathcal{I}$ is syntactically weakly family algebraizable if and only if it is syntactically protoalgebraic and family truth equational. Thus, we get

Theorem 1317 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}^f$ is syntactically weakly family algebraizable, then so is $\mathcal{I}$.*

Proof: If $\mathcal{I}^f$ is syntactically weakly family algebraizable, then, by Theorem 913, it is syntactically protoalgebraic and family truth equational. Hence, by Theorems 1306 and 1313, $\mathcal{I}$ possesses the same properties. Therefore, applying again Theorem 913 in the reverse direction, we conclude that $\mathcal{I}$ is also syntactically weakly family algebraizable. ∎

In Section 9.4, we saw that the continuity of the Leibniz operator is one of the key properties when studying finitarity conditions. Lemma 660 showed that, if $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is continuous, then $\mathcal{I}$ is protoalgebraic. That is asserting the continuity of the Leibniz operator strengthens protoalgebraicity. Additionally, it was proven in Lemma 661 that, if $\mathbf{Sign}^\flat$ is finite, then continuity of Ω also ensures that the finitary companion $\mathcal{I}^f$ of $\mathcal{I}$ is also protoalgebraic.

We begin, here, our parallel treatment on the syntactic side by showing that, maintaining the finiteness of $\mathbf{Sign}^\flat$, the condition that $\mathcal{I}$ be syntactically protoalgebraic, with a finite collection of parameter-free witnessing transformations $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$, constitutes an additional strengthening on protoalgebraicity, on top of the continuity of Ω.

Proposition 1318 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\mathbf{Sign}^\flat$ finite, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is syntactically protoalgebraic, with a finite parameter-free collection $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ of witnessing transformations, then $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ is continuous.*

Proof: Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\mathbf{Sign}^\flat$ finite, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a syntactically protoalgebraic π-institution based on $\mathbf{F}$, with a finite parameter-free collection $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ of witnessing transformations. Suppose $\{T^i : i \in I\}$ is a directed collection of theory families of $\mathcal{I}$, such that $\bigcup_{i\in I} T^i \in \mathrm{ThFam}(\mathcal{I})$. Then, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rll}\langle \phi, \psi \rangle \in \Omega_\Sigma(\bigcup_{i\in I} T^i) & \text{iff} & I^\flat_\Sigma[\phi,\psi] \leq \bigcup_{i\in I} T^i \\ & \text{iff} & I^\flat_\Sigma[\phi,\psi] \leq T^i, \text{ some } i \in I, \\ & \text{iff} & \langle \phi, \psi \rangle \in \Omega_\Sigma(T^i), \text{ some } i \in I, \\ & \text{iff} & \langle \phi, \psi \rangle \in \bigcup_{i\in I} \Omega_\Sigma(T^i).\end{array}$$

Note that the second equivalence employs both the fact that $\mathbf{Sign}^\flat$ is finite and the fact that $I^\flat$ is finite and parameter-free. Thus, $\Omega(\bigcup_{i\in I} T^i) = \bigcup_{i\in I} \Omega(T^i)$ and, hence, Ω is indeed continuous. ■

We next see that this stronger condition than the continuity of the Leibniz operator suffices to ensure that $\mathcal{I}^f$ is also syntactically protoalgebraic, with the same collection of witnessing transformations. Thus, the following proposition may be viewed as a syntactic analog of Lemma 661.

Proposition 1319 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is syntactically protoalgebraic, with a finite and parameter-free collection* $I^\flat$ *of witnessing transformations, then* $\mathcal{I}^f$ *is also syntactically protoalgebraic, with the same collection of witnessing transformations.*

Proof: Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\mathbf{Sign}^\flat$ finite, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a syntactically protoalgebraic π-institution, with a finite and parameter-free collection $I^\flat$ of witnessing transformations. Let $T \in \mathrm{ThFam}(\mathcal{I}^f)$. Then, by Proposition 655, there exists a directed locally finitely generated collection $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $T = \bigcup_{i\in I} T^i$. Now we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rll}\langle \phi, \psi \rangle \in \Omega_\Sigma(T) & \text{iff} & \langle \phi, \psi \rangle \in \Omega_\Sigma(\bigcup_{i\in I} T^i) \\ & \text{iff} & \langle \phi, \psi \rangle \in \bigcup_{i\in I} \Omega_\Sigma(T^i) \quad \text{(Proposition 1318)} \\ & \text{iff} & \langle \phi, \psi \rangle \in \Omega_\Sigma(T^i), \text{ some } i \in I, \\ & \text{iff} & \overleftrightarrow{I}^\flat_\Sigma[\phi, \psi] \leq T^i, \text{ some } i \in I, \\ & \text{iff} & \overleftrightarrow{I}^\flat_\Sigma[\phi, \psi] \leq \bigcup_{i\in I} T^i \\ & \text{iff} & \overleftrightarrow{I}^\flat_\Sigma[\phi, \psi] \leq T.\end{array}$$

Again, note that the one-before-the-last equivalence employs both the fact that $\mathbf{Sign}^\flat$ is finite and the fact that $I^\flat$ is finite and parameter-free. Therefore, by Corollary 791, $\mathcal{I}^f$ is also syntactically protoalgebraic, with the same collection $I^\flat$ of witnessing transformations. ■

Suppose, now, that $\mathbf{Sign}^\flat$ is finite and $\mathcal{I}$ is weakly family algebraizable, so that $\Omega^{-1} : \mathrm{ConSys}^*(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$ be defined. An analog of Proposition 1318 asserts that, if $\mathcal{I}$ is truth equational, with a finite and parameter-free witnessing family $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ of equations, then the inverse Leibniz operator Ω^{-1} is continuous. Thus, under these hypotheses, the truth equationality of $\mathcal{I}$ via a finite, parameter-free collection of witnessing equations is stronger than the continuity of Ω^{-1}.

Proposition 1320 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a weakly family algebraizable* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is truth equational, with a finite parameter-free collection* $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *of witnessing equations, then* $\Omega^{-1} : \mathrm{ConSys}^*(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$ *is continuous.*

Proof: Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\mathbf{Sign}^\flat$ finite, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a weakly family algebraizable π-institution, which is, in addition, truth equational, with a finite parameter-free collection $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ of witnessing equations. Let $\{\theta^i : i \in I\}$ be a directed collection of $\mathcal{I}^*$-congruence systems, such that $\bigcup_{i \in I} \theta^i \in \mathrm{ConSys}^*(\mathcal{I})$. Now we get, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll} \phi \in \Omega_\Sigma^{-1}(\bigcup_{i \in I} \theta^i) & \text{iff} & \tau_\Sigma^\flat[\phi] \leq \bigcup_{i \in I} \theta^i \\ & \text{iff} & \tau_\Sigma^\flat[\phi] \leq \theta^i, \text{ some } i \in I, \\ & \text{iff} & \phi \in \Omega_\Sigma^{-1}(\theta^i), \text{ some } i \in I, \\ & \text{iff} & \phi \in \bigcup_{i \in I} \Omega_\Sigma^{-1}(\theta^i). \end{array}$$

Thus, Ω^{-1} is indeed continuous. ■

Recall from Theorem 663 that given a weakly family algebraizable π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, based on an algebraic system $\mathbf{F}$ over a finite category of signatures, the continuity of both $\Omega : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^*(\mathcal{I})$ and $\Omega^{-1} : \mathrm{ConSys}^*(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I})$ are sufficient to ensure that $\mathcal{I}^f$ is also weakly family algebraizable. In Propositions 1318 and 1320, by comparison, it was shown that the continuities of Ω and Ω^{-1} are strengthened by assuming, respectively, that $\mathcal{I}$ is syntactically protoalgebraic, with a finite, parameter-free witnessing family of transformations, and that $\mathcal{I}$ is family truth equational, with a finite, parameter-free witnessing family of equations. We show, next, in an analog of Theorem 663, that imposing these two stronger conditions on $\mathcal{I}$ suffices to ensure that syntactic strong algebraizability transfers from $\mathcal{I}$ to its finitary companion $\mathcal{I}^f$.

Proposition 1321 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\mathbf{Sign}^\flat$ finite, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a syntactically protoalgebraic π-institution, with a finite parameter-free collection $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ of witnessing transformations. If $\mathcal{I}$ is family truth equational, with a finite and parameter-free collection $\tau^\flat$ of witnessing equations, then $\mathcal{I}^f$ is also family truth equational, with the same collection of witnessing equations.*

Proof: Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\mathbf{Sign}^\flat$ finite, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a weakly family algebraizable π-institution, which is family truth equational, with a finite and parameter-free collection $I^\flat$ of witnessing equations. Let $T \in \mathrm{ThFam}(\mathcal{I}^f)$. Then, by Proposition 655, there exists a directed locally finitely generated collection $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $T = \bigcup_{i \in I} T^i$. Now we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll} \phi \in T_\Sigma & \text{iff} & \phi \in \bigcup_{i \in I} T_\Sigma^i \\ & \text{iff} & \phi \in T_\Sigma^i, \text{ some } i \in I, \\ & \text{iff} & \tau_\Sigma^\flat[\phi] \leq \Omega(T^i), \text{ some } i \in I, \\ & \text{iff} & \tau_\Sigma^\flat[\phi] \leq \bigcup_{i \in I} \Omega(T^i) \\ & \text{iff} & \tau_\Sigma^\flat[\phi] \leq \Omega(\bigcup_{i \in I} T^i) \quad \text{(Proposition 1318)} \\ & \text{iff} & \tau_\Sigma^\flat[\phi] \leq \Omega(T). \end{array}$$

Again, note that the fourth equivalence employs both the fact that $\mathbf{Sign}^\flat$ is finite and the fact that $\tau^\flat$ is finite and parameter-free. We conclude that $\mathcal{I}^f$ is also family truth equational, with the same collection $\tau^\flat$ of witnessing equations. ■

Putting together Propositions we finally obtain the promised analog of Theorem 663.

Theorem 1322 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically strongly family algebraizable* π*-institution, via a conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathsf{K}}$ *consisting of finite and parameter-free collections of transformations. Then* $\mathcal{I}^f$ *is also syntactically strongly family algebraizable, via the same conjugate pair of transformations.*

Proof: We simply put together Propositions 1319 and 1321. ■

17.2 Natural Finitarity

This section deals with concepts analogous to those studied in Section 9.4, but in the syntactic, rather than in the semantic, context. In the semantic context, the four key ingredients of our study were the finitarity of the π-institutions involved as well as the continuity of the Leibniz operator and its inverse. Recall that for the inverse to be defined in the context under consideration, the general underlying hypothesis that the π-institution $\mathcal{I}$ be weakly family algebraizable was adhered to. In the present, syntactic, context, we assume that $\mathcal{I}$ is syntactically strongly family algebraizable, that is, syntactically family algebraizable via a conjugate pair $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathsf{K}}$, where both $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ and $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ are parameter-free witnessing collections of equations and of transformations, respectively. The four notions involved are the properties of $\mathcal{I}$ and $\mathcal{Q}^{\mathsf{K}}$ being naturally finitary, a strengthening of finitarity, and those of $\tau^\flat$ and $I^\flat$ being finite, also strengthening the continuity of the Leibniz operator and its inverse operator. But let us embark on the developments so as to clarify these introductory remarks and to make the concepts and the details involved precise.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall that $\mathcal{I}$ is *finitary* if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$, there exists $\Phi' \subseteq_f \Phi$, such that $\phi \in C_\Sigma(\Phi')$. Equivalently, $\mathcal{I}$ is finitary if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$C_\Sigma(\Phi) = \bigcup\{C_\Sigma(\Phi') : \Phi' \subseteq_f \Phi\}.$$

We say that $\mathcal{I}$ is **naturally finitary** if it is finitary and, in addition, the following condition holds:

(NATFIN) If, for some collections $\mu, \nu : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ of natural transformations in $N^\flat$, such that $|\mu| < \infty$, it holds that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\mu_\Sigma[\vec{\phi}] \leq C(\nu_\Sigma[\vec{\phi}]),$$

then, there exists a finite subset $\nu' \subseteq \nu$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\mu_\Sigma[\vec{\phi}] \leq C(\nu'_\Sigma[\vec{\phi}]).$$

It is not difficult to see that, if $\mathcal{I}$ is naturally finitary, the implication resulting from (NATFIN) by replacing the two inclusions by equalities of closure families also holds.

Lemma 1323 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is naturally finitary, then, for all $\mu, \nu : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with $|\mu| < \infty$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$, $C(\mu_\Sigma[\vec{\phi}]) = C(\nu_\Sigma[\vec{\phi}])$, there exists a finite $\nu' \subseteq \nu$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$, $C(\mu_\Sigma[\vec{\phi}]) = C(\nu'_\Sigma[\vec{\phi}])$.*

Proof: Suppose $\mathcal{I}$ is naturally finitary and let $\mu, \nu : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with $|\mu| < \infty$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$, $C(\mu_\Sigma[\vec{\phi}]) = C(\nu_\Sigma[\vec{\phi}])$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$, $\mu_\Sigma[\vec{\phi}] \leq C(\nu_\Sigma[\vec{\phi}])$. Thus, by natural finitarity, there exists a finite subset $\nu' \subseteq \nu$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$, $\mu_\Sigma[\vec{\phi}] \leq C(\nu'_\Sigma[\vec{\phi}])$. But, then, we obtain, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$C(\nu_\Sigma[\vec{\phi}]) = C(\mu_\Sigma[\vec{\phi}]) \leq C(\nu'_\Sigma[\vec{\phi}]) \leq C(\nu_\Sigma[\vec{\phi}]).$$

We conclude that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$, $C(\mu_\Sigma[\vec{\phi}]) = C(\nu'_\Sigma[\vec{\phi}])$. ■

Starting to take advantage of natural finitarity, we show that it allows to draw the conclusion that, in case of syntactic family algebraizability, the existence of a finite witnessing family of transformations ensures that every witnessing family possesses a finite witnessing subfamily. More precisely, we have

Lemma 1324 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a naturally finitary π-institution based on $\mathbf{F}$. Suppose $\mathcal{I}$ is syntactically family algebraizable, with equivalent guasivariety K. If $\mathcal{I}$ has a finite witnessing family $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ of transformations, then every witnessing family $J^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ possesses a finite witnessing subfamily $J'^\flat$.*

Proof: Suppose that $\mathcal{I}$ is naturally finitary and syntactically family algebraizable, with equivalent guasivariety K. Let $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ be a finite set of witnessing transformations and $J^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ a family of witnessing transformations. By Theorem 912, we get that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C(I^\flat_\Sigma[\phi,\psi]) = C(J^\flat_\Sigma[\phi,\psi]).$$

Since $\mathcal{I}$ is naturally finitary and, by hypothesis, $|I^\flat| < \infty$, we get, by Lemma 1323, that there exists finite $J'^\flat \subseteq J^\flat$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C(J'^\flat_\Sigma[\phi,\psi]) = C(I^\flat_\Sigma[\phi,\psi]).$$

Thus, applying Proposition 903, we conclude that $J'^\flat$ is also a witnessing family of transformations. ■

Dually, we may also prove a corresponding result concerning the witnessing equations for the truth equationality of $\mathcal{I}$.

Lemma 1325 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically strongly family algebraizable* π*-institution based on* $\mathbf{F}$*, with equivalent guasivariety* K*. If* $\mathcal{Q}^\mathsf{K}$ *is naturally finitary and* $\mathcal{I}$ *has a finite witnessing family* $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *of equations, then every witnessing family* $\rho^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *of equations possesses a finite witnessing subfamily* $\rho'^\flat$*.*

Proof: Follows along the lines of the proof of Lemma 1324. Suppose that $\mathcal{I}$ is syntactically strongly family algebraizable, with equivalent guasivariety K, such that $\mathcal{Q}^\mathsf{K}$ is naturally finitary. Let $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ be a finite set of witnessing equations and $\rho^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ a family of witnessing equations. By Theorem 912, we get that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$D^\mathsf{K}(\tau^\flat_\Sigma[\phi]) = D^\mathsf{K}(\rho^\flat_\Sigma[\phi]).$$

Since $\mathcal{Q}^\mathsf{K}$ is naturally finitary and, by hypothesis, $|\tau^\flat| < \infty$, we get, by Lemma 1323, that there exists finite $\rho'^\flat \subseteq \rho^\flat$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$D^\mathsf{K}(\rho'^\flat_\Sigma[\phi]) = D^\mathsf{K}(\tau^\flat_\Sigma[\phi]).$$

Thus, applying Proposition 903, we conclude that $\rho'^\flat$ is also a witnessing family of equations. ■

We now establish a theorem to the effect that, under natural finitarity and syntactic strong family algebraizability, every witnessing family of equations contains a finite witnessing subfamily. This is the first main result in a series of finitarity results that we aim to prove in the present section, with the ultimate goal of obtaining a hierarchy on the syntactic side, analogous to that obtained on the semantic side at the end of Section 9.4.

Theorem 1326 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a naturally finitary, syntactically strongly family algebraizable* π*-institution, with equivalent guasivariety* K*. Then every witnessing collection* $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *of equations contains a finite subcollection* $\tau'^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$*, which is also a witnessing collection.*

Proof: Suppose $\mathcal{I}$ is naturally finitary and syntactically strongly family algebraizable, with equivalent guasivariety K. Let $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ be a collection of witnessing equations. Then, by definition, there exists a collection $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C(\iota_\Sigma[\phi]) = C(\phi) = C(I^\flat[\tau^\flat_\Sigma[\phi]]).$$

Since $\mathcal{I}$ is naturally finitary, there exist finite $I'^\flat \subseteq I^\flat$ and $\tau'^\flat \subseteq \tau^\flat$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C(\phi) = C(I'^\flat[\tau'^\flat_\Sigma[\phi]]).$$

Thus, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C(\phi) = C(I^\flat[\tau'^\flat_\Sigma[\phi]]).$$

Thus, since, by the properties of $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathsf{K}}$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$,

$$\phi \approx \psi \in D^{\mathsf{K}}_\Sigma(E) \quad \text{iff} \quad I^\flat_\Sigma[\phi, \psi] \leq C(I^\flat_\Sigma[E]),$$

we get, by Proposition 903, that $\tau'^\flat$ is a witnessing family of equations. ■

Dually, we may prove that, under syntactic strong family algebraizability and natural finitarity of the algebraic counterpart $\mathcal{Q}^{\mathsf{K}}$, every witnessing family of transformations contains a finite witnessing subfamily.

Theorem 1327 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically strongly family algebraizable* π*-institution, with equivalent guasivariety* K*, such that* $\mathcal{Q}^{\mathsf{K}}$ *is naturally finitary. Then every witnessing collection* $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *of transformations contains a finite subcollection* $I'^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$*, which is also a witnessing collection.*

Proof: Suppose $\mathcal{I}$ is syntactically strongly family algebraizable, with equivalent guasivariety K, such that $\mathcal{Q}^{\mathsf{K}}$ is naturally finitary. Let $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ be a collection of witnessing transformations. Then, by definition, there exists a collection $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$D^{\mathsf{K}}(\langle p^{2,0}, p^{2,1}\rangle_\Sigma[\phi, \psi]) = D^{\mathsf{K}}(\phi \approx \psi) = D^{\mathsf{K}}(\tau^\flat[I^\flat_\Sigma[\phi, \psi]]).$$

Since $\mathcal{I}$ is naturally finitary, there exist finite $I'^\flat \subseteq I^\flat$ and $\tau'^\flat \subseteq \tau^\flat$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$D^{\mathsf{K}}(\phi \approx \psi) = D^{\mathsf{K}}(\tau'^\flat[I'^\flat_\Sigma[\phi, \psi]]).$$

Thus, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$D^{\mathsf{K}}(\phi \approx \psi) = D^{\mathsf{K}}(\tau^\flat[I'^\flat_\Sigma[\phi, \psi]]).$$

Thus, since, by the properties of $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathsf{K}}$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in C_\Sigma(\Phi) \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq D^{\mathsf{K}}(\tau^\flat_\Sigma[\Phi]),$$

we get, by Proposition 903, that $I'^\flat$ is also a witnessing family of transformations. ■

The following proposition asserts that, under similar hypotheses, but adding finiteness of the signature category, the finitarity of $\mathcal{I}$ and of the witnessing collection $I^\flat$ imply the finitarity of the algebraic counterpart $\mathcal{Q}^{\mathsf{K}}$.

Proposition 1328 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a finitary, syntactically family algebraizable* π*-institution, with equivalent guasivariety* K*. If* $\mathcal{I}$ *has a finite witnessing set* $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *of transformations, then its algebraic counterpart* $\mathcal{Q}^{\mathsf{K}}$ *is also finitary.*

Proof: Suppose $\mathcal{I}$ is a finitary π-institution based on an algebraic system $\mathbf{F}$ over a finite category of signatures. Assume that $\mathcal{I}$ is syntactically family algebraizable, with equivalent guasivariety K and that it has a finite witnessing collection $I^\flat$ of transformations. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$, such that

$$\phi \approx \psi \in D^{\mathsf{K}}_\Sigma(E).$$

Since $I^\flat$ is a witnessing collection of transformations,

$$I^\flat_\Sigma[\phi, \psi] \leq C(I^\flat_\Sigma[E]).$$

Since $\mathbf{Sign}^\flat$ is finite and $I^\flat$ is finite, we get, by the finitarity of $\mathcal{I}$ that there exists finite $E' \subseteq E$, such that $I^\flat_\Sigma[\phi, \psi] \leq C(I^\flat_\Sigma[E'])$. Thus, again by the fact that $I^\flat$ is a set of witnessing transformations, we obtain $\phi \approx \psi \in D^{\mathsf{K}}_\Sigma(E')$. Thus, $\mathcal{Q}^{\mathsf{K}}$ is indeed finitary. ■

A similar result can also be established when focus is shifted from finitarity to natural finitarity.

Proposition 1329 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a naturally finitary, syntactically family algebraizable* π*-institution, with equivalent guasivariety* K*. If* $\mathcal{I}$ *has a finite witnessing set* $I^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *of transformations, then its algebraic counterpart* $\mathcal{Q}^{\mathsf{K}}$ *is also naturally finitary.*

Proof: Suppose $\mathcal{I}$ is a naturally finitary π-institution based on an algebraic system $\mathbf{F}$ over a finite category of signatures. Assume that $\mathcal{I}$ is syntactically family algebraizable, with equivalent guasivariety K and that it has a finite witnessing collection $I^\flat$ of transformations. By Proposition 1328, we know that $\mathcal{Q}^\mathsf{K}$ is finitary. Let $\mu,\nu : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ be collections of natural transformations in $N^\flat$, with $|\mu| < \infty$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\mu_\Sigma[\vec{\phi}] \leq D^\mathsf{K}(\nu_\Sigma[\vec{\phi}]).$$

Since $I^\flat$ is a witnessing collection of transformations, we get, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat[\mu_\Sigma[\vec{\phi}]] \leq C(I^\flat[\nu_\Sigma[\vec{\phi}]]).$$

But both μ and $I^\flat$ are finite and, also, $\mathbf{Sign}^\flat$ is assumed to be finite. Hence, since $\mathcal{I}$ is naturally finitary, there exists finite $\nu' \subseteq \nu$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^\flat[\mu_\Sigma[\vec{\phi}]] \leq C(I^\flat[\nu'_\Sigma[\vec{\phi}]]).$$

Therefore, again by the fact that $I^\flat$ is a set of witnessing transformations, we obtain, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\mu_\Sigma[\vec{\phi}] \leq D^\mathsf{K}(\nu'_\Sigma[\vec{\phi}]).$$

Thus, $\mathcal{Q}^\mathsf{K}$ is indeed naturally finitary. ∎

We turn, next, to results dual to those established in Propositions 1328 and 1329. We start with a dual to Proposition 1328 to the effect that, if $\mathcal{Q}^\mathsf{K}$ is finitarty and $\mathcal{I}$ has a finite witnessing collection of equations, then $\mathcal{I}$ is itself finitary.

Proposition 1330 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically strongly family algebraizable* π*-institution, with equivalent guasivariety* K*. If* $\mathcal{Q}^\mathsf{K}$ *is finitary and* $\mathcal{I}$ *has a finite witnessing set* $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *of equations, then* $\mathcal{I}$ *is also finitary.*

Proof: Suppose $\mathcal{I}$ is a π-institution based on an algebraic system $\mathbf{F}$ over a finite category of signatures. Assume that $\mathcal{I}$ is syntactically strongly family algebraizable, with equivalent guasivariety K, such that $\mathcal{Q}^\mathsf{K}$ is finitary, and that it has a finite witnessing collection $\tau^\flat$ of equations. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that

$$\phi \in C_\Sigma(\Phi).$$

Since $\tau^\flat$ is a witnessing collection of equations,

$$\tau^\flat_\Sigma[\phi] \leq D^\mathsf{K}(\tau^\flat_\Sigma[\Phi]).$$

Since $\mathbf{Sign}^\flat$ is finite and $\tau^\flat$ is finite, we get, by the finitarity of $\mathcal{Q}^\mathsf{K}$ that there exists finite $\Phi' \subseteq \Phi$, such that $\tau^\flat_\Sigma[\phi] \leq D^\mathsf{K}(\tau^\flat_\Sigma[\Phi'])$. Thus, again by the fact that $\tau^\flat$ is a set of witnessing equations, we obtain $\phi \in C_\Sigma(\Phi')$. Thus, $\mathcal{I}$ is indeed finitary. ■

A dual of Proposition 1329 addresses the case of natural finitarity.

Proposition 1331 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically strongly family algebraizable* π*-institution, with equivalent guasivariety* K*. If* $\mathcal{Q}^\mathsf{K}$ *is naturally finitary and* $\mathcal{I}$ *has a finite witnessing set* $\tau^\flat : \mathrm{SEN}^\flat \to (\mathrm{SEN}^\flat)^2$ *of equations, then* $\mathcal{I}$ *is also naturally finitary.*

Proof: Suppose $\mathcal{I}$ is a π-institution based on an algebraic system $\mathbf{F}$ over a finite category of signatures. Assume that $\mathcal{I}$ is syntactically strongly family algebraizable, with equivalent guasivariety K, such that $\mathcal{Q}^\mathsf{K}$ is naturally finitary, and that it has a finite witnessing collection $\tau^\flat$ of equations. Let $\mu, \nu : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ be collections of natural transformations in $N^\flat$, with $|\mu| < \infty$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\mu_\Sigma[\vec{\phi}] \leq C(\nu_\Sigma[\vec{\phi}]).$$

Since $\tau^\flat$ is a witnessing collection of equations, we get, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\tau^\flat[\mu_\Sigma[\vec{\phi}]] \leq D^\mathsf{K}(\tau^\flat[\nu_\Sigma[\vec{\phi}]]).$$

But both μ and $\tau^\flat$ are finite and, also, $\mathbf{Sign}^\flat$ is assumed to be finite. Hence, since $\mathcal{Q}^\mathsf{K}$ is naturally finitary, there exists finite $\nu' \subseteq \nu$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\tau^\flat[\mu_\Sigma[\vec{\phi}]] \leq D^\mathsf{K}(\tau^\flat[\nu'_\Sigma[\vec{\phi}]]).$$

Therefore, again by the fact that $\tau^\flat$ is a set of witnessing equations, we obtain, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\mu_\Sigma[\vec{\phi}] \leq C(\nu'_\Sigma[\vec{\phi}]).$$

Thus, $\mathcal{I}$ is naturally finitary. ■

Finally, we present a syntactic analog of Corollary 668, which summarizes the conclusions drawn from the study of the various finitarity properties, at the center of the investigations carried out in the present chapter.

Corollary 1332 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\mathbf{Sign}^\flat$ *finite, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically strongly family algebraizable* π*-institution, via the conjugate pair* $(\tau^\flat, I^\flat) : \mathcal{I} \rightleftarrows \mathcal{Q}^\mathsf{K}$*.*

(a) If both $\tau^\flat$ *and* $I^\flat$ *are finite, then* $\mathcal{I}$ *is naturally finitary if and only if* $\mathcal{Q}^\mathsf{K}$ *is naturally finitary;*

(b) *If $\mathcal{I}$ is naturally finitary, then $\mathcal{Q}^{\mathsf{K}}$ is naturally finitary if and only if $I^\flat$ can be taken to be finite;*

(c) *If $\mathcal{Q}^{\mathsf{K}}$ is naturally finitary, then $\mathcal{I}$ is naturally finitary if and only if $\tau^\flat$ can be taken to be finite.*

In each case, if the equivalent alternatives hold, then all four "finitarity" conditions hold.

Proof:

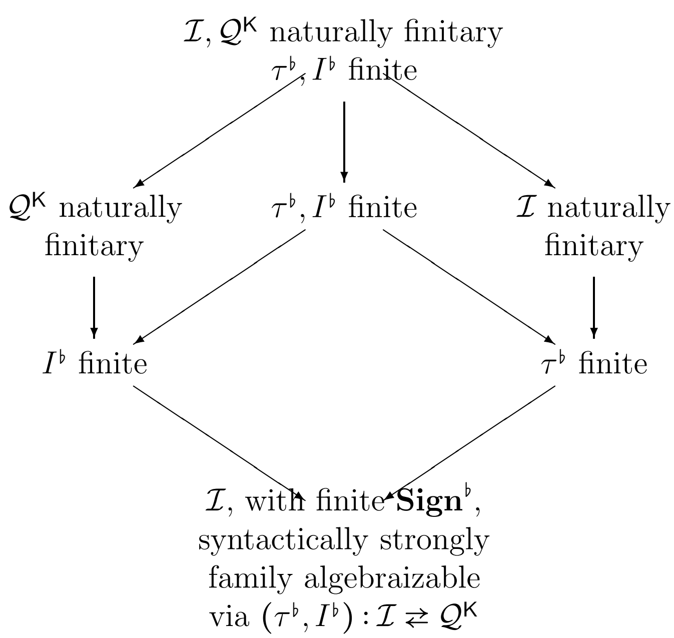
(a) Suppose both $\tau^\flat$ and $I^\flat$ are finite. If $\mathcal{I}$ is naturally finitary, then, by Proposition 1329, $\mathcal{Q}^{\mathsf{K}}$ is also naturally finitary. If, on the other hand, $\mathcal{Q}^{\mathsf{K}}$ is naturally finitary, then, by Proposition 1331, $\mathcal{I}$ is naturally finitary.

(b) Assume that $\mathcal{I}$ is naturally finitary. If $\mathcal{Q}^{\mathsf{K}}$ is naturally finitary, then, by Theorem 1327, $I^\flat$ may be taken to be finite. If, on trhe other hand, $I^\flat$ can be taken to be finite, then, by Proposition 1329, $\mathcal{Q}^{\mathsf{K}}$ is naturally finitary.

(c) Assume $\mathcal{Q}^{\mathsf{K}}$ is naturally finitary. If $\mathcal{I}$ is naturally finitary, then, by Theorem 1326, $\tau^\flat$ may be taken to be finite. If, on the other hand, $\tau^\flat$ may be taken to be finite, then, by Proposition 1331, $\mathcal{I}$ is naturally finitary.

■

In summary, Corollary 1332 establishes the hierarchy depicted below, which parallels in the syntactic context the hierarchy pictured at the end of Chapter 9, concerning the semantic side.



Chapter 18

Properties of Selected Classes

18.1 Protoalgebraic π-Institutions

18.1.1 The Correspondence Theorem

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

Recall that $\mathcal{I}$ is **protoalgebraic** if the Leibniz operator is monotone on theory families, i.e., if, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$T \leq T' \quad \text{implies} \quad \Omega(T) \leq \Omega(T').$$

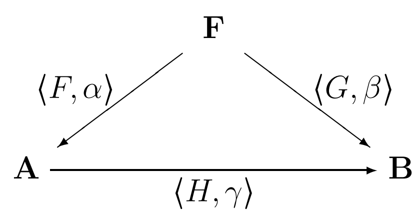
Recall, also, that, by Theorem 175, every protoalgebraic π-institution is stable and that, moreover, by Theorem 179, $\mathcal{I}$ is protoalgebraic if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$T \leq T' \quad \text{implies} \quad \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T').$$

The π-institution $\mathcal{I}$ has the **compatibility property** if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, with $T \leq T'$, and all $\theta \in \mathrm{ConSys}(\mathbf{A})$,

$$\theta \text{ compatible with } T \quad \text{implies} \quad \theta \text{ compatible with } T'.$$

The π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ has the **filter correspondence property** if, for all $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$, and surjective morphism $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$, with $H : \mathbf{Sign} \to \mathbf{Sign}'$ an isomorphism,



and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$,

$$\gamma^{-1}(\widehat{\gamma}(T) \vee T') = T \vee \gamma^{-1}(T'),$$

where $\widehat{\gamma}(T) = C^{\mathcal{I},\mathcal{B}}(\gamma(T))$ is the least $\mathcal{I}$-filter family on $\mathcal{B}$ that includes $\gamma(T)$.

Our goal is to show that both the compatibility property and the filter correspondence property characterize protoalgebraic π-institutions. We start with a lemma to the effect that, for every $\mathcal{I}$-filter family T of $\mathcal{A}$, if the kernel of $\langle H, \gamma \rangle$ happens to be compatible with T, then $\gamma(T)$ is already an $\mathcal{I}$-filter family of $\mathcal{B}$ and, therefore, $\widehat{\gamma}(T) = \gamma(T)$.

Lemma 1333 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ $\mathbf{F}$-algebraic systems, and $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism, with $H : \mathbf{Sign} \to \mathbf{Sign}'$ an isomorphism. If $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\mathrm{Ker}(\langle H, \gamma \rangle)$ is compatible with T, then $\widehat{\gamma}(T) = \gamma(T)$.*

Proof: By definition, $\gamma(T) \leq \widehat{\gamma}(T)$ always holds. To show the reverse inequality, it suffices to show that $\gamma(T)$, under the hypothesis of the compatibility of $\mathrm{Ker}(\langle H,\gamma\rangle)$ with T, is an $\mathcal{I}$-filter family of $\mathcal{B}$. So assume $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$, and let $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$, such that

$$\beta_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq \gamma_{F(\Sigma')}(T_{F(\Sigma')}).$$

This gives

$$\gamma_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))) \subseteq \gamma_{F(\Sigma')}(T_{F(\Sigma')}).$$

By the postulated compatibility of $\mathrm{Ker}(\langle H,\gamma\rangle)$ with T, we obtain

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq T_{F(\Sigma')}.$$

Since $\phi \in C_\Sigma(\Phi)$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get that

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \in T_{F(\Sigma')}.$$

Thus, $\gamma_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi))) \in \gamma_{F(\Sigma')}(T_{F(\Sigma')})$, and, therefore,

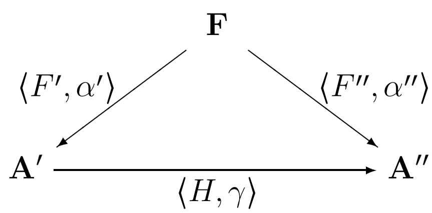
$$\beta_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \in \gamma_{F(\Sigma')}(T_{F(\Sigma')}).$$

This shows that $\gamma(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$ and, hence, $\widehat{\gamma}(T) = \gamma(T)$. ∎

Next, we give an equivalent formulation of the Filter Correspondence Property.

Proposition 1334 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *has the filter correspondence property iff*

> *for all* $\mathcal{I}$*-matrix families* $\mathfrak{A} = \langle \mathcal{A}', T'\rangle$, $\mathfrak{A}'' = \langle \mathcal{A}'', T''\rangle$ *and strict surjective matrix morphism* $\langle H,\gamma\rangle : \mathfrak{A}' \to \mathfrak{A}''$*, with* $H : \mathbf{Sign}' \to \mathbf{Sign}''$ *an isomorphism,*



$$T = \gamma^{-1}(\gamma(T)), \quad \textit{for all } T' \leq T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}').$$

Proof: Suppose, first, that $\mathcal{I}$ has the Filter Correspondence Property and consider $\mathcal{I}$-matrix families $\mathfrak{A} = \langle \mathcal{A}', T'\rangle$, $\mathfrak{A}'' = \langle \mathcal{A}'', T''\rangle$, a strict surjective matrix morphism $\langle H,\gamma\rangle:\mathfrak{A}'\to\mathfrak{A}''$, with $H:\mathbf{Sign}'\to\mathbf{Sign}''$ an isomorphism, and $T'\leq T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$. Then we have

$$\begin{array}{rcll}\gamma^{-1}(\gamma(T)) & \leq & \gamma^{-1}(\widehat{\gamma}(T)\vee T'') & (\gamma(T)\leq\widehat{\gamma}(T))\\ & = & T\vee\gamma^{-1}(T'') & \text{(Filter Correspondence)}\\ & = & T\vee T' & (\langle H,\gamma\rangle\text{ strict})\\ & = & T. & (T'\leq T\text{ by hypothesis})\end{array}$$

Thus, the displayed property holds. Assume, conversely, that the displayed property holds. We must show that $\mathcal{I}$ has the Filter Correspondence Property. So suppose that $\mathcal{A}=\langle\mathbf{A}',\langle F',\alpha'\rangle\rangle$, $\mathcal{A}''=\langle\mathbf{A}'',\langle F'',\alpha''\rangle\rangle$ are $\mathbf{F}$-algebraic systems, $\langle H,\gamma\rangle:\mathcal{A}'\to\mathcal{A}''$ a surjective morphism, with $H:\mathbf{Sign}'\to\mathbf{Sign}''$ an isomorphism, and let $T'\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$ and $T''\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}'')$. Our goal is to show that

$$\gamma^{-1}(\widehat{\gamma}(T')\vee T'')=T'\vee\gamma^{-1}(T'').$$

Notice that $\langle H,\gamma\rangle:\langle\mathcal{A}',\gamma^{-1}(T'')\rangle\to\langle\mathcal{A}'',T''\rangle$ is a strict surjective morphism, with H an isomorphism, and $\gamma^{-1}(T'')\leq T'\vee\gamma^{-1}(T'')$. Thus, we fit the setup of the hypothesis, which allows us to conclude that

$$\gamma^{-1}(\gamma(T'\vee\gamma^{-1}(T'')))=T'\vee\gamma^{-1}(T'').$$

So, it suffices, in turn, to show that $\widehat{\gamma}(T')\vee T''=\gamma(T'\vee\gamma^{-1}(T''))$ and, since, $\mathrm{Ker}(\langle H,\gamma\rangle)$ is compatible with T' (having $\gamma^{-1}(\gamma(T'))=T'$, by hypothesis), it suffices, by Lemma 1333, to show that

$$\gamma(T')\vee T''=\gamma(T'\vee\gamma^{-1}(T'')).$$

The left to right inclusion is obvious, since $\gamma(T'),T''\leq\gamma(T'\vee\gamma^{-1}(T''))$. Conversely, note that, taking into account the hypothesis, $T',\gamma^{-1}(T'')\leq\gamma^{-1}(\gamma(T')\vee T'')$. Therefore, $T'\vee\gamma^{-1}(T'')\leq\gamma^{-1}(\gamma(T')\vee T'')$ and, therefore, $\gamma(T'\vee\gamma^{-1}(T''))\leq\gamma(T')\vee T''$ and, hence, the right to left inclusion also holds. Thus, the Filter Correspondence Property holds. ■

Now we proceed with the formulation and proof of the main theorem.

Theorem 1335 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(i) $\mathcal{I}$ *is protoalgebraic;*

(ii) $\mathcal{I}$ *has the compatibility property;*

(iii) $\mathcal{I}$ *has the filter correspondence property.*

Proof:

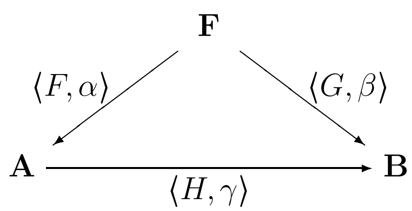
(i)⇒(ii) Suppose $\mathcal{I}$ is protoalgebraic and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system, $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, with $T \le T'$, and $\theta \in \mathrm{ConSys}(\mathbf{A})$, such that θ is compatible with T. Then we have

$$\begin{array}{rcll} \theta & \le & \Omega(T) & \text{(by the compatibility of } \theta \text{ with } T) \\ & \le & \Omega(T'). & \text{(by protoalgebraicity)} \end{array}$$

We conclude that θ is also compatible with T' and, hence, $\mathcal{I}$ has the compatibility property.

(ii)⇒(i) Suppose that $\mathcal{I}$ has the compatibility property and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \le T'$. Then $\Omega(T) \in \mathrm{ConSys}(\mathbf{A})$ and, by the definition of a Leibniz congruence system, it is compatible with T. Now it follows by the compatibility property, that $\Omega(T)$ is also compatible with T'. Hence $\Omega(T) \le \Omega(T')$. We conclude that $\mathcal{I}$ is protoalgebraic.

(ii)⇒(iii) Suppose that $\mathcal{I}$ has the compatibility property and consider $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta\rangle\rangle$, a commutative triangle



with $H : \mathbf{Sign} \to \mathbf{Sign}'$ an isomorphism, and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. Note that it is always the case that

$$T \vee \gamma^{-1}(T') \le \gamma^{-1}(\widehat{\gamma}(T) \vee T').$$

Thus, it suffices to show that, under the hypothesis of compatibility, the reverse inclusion also holds.

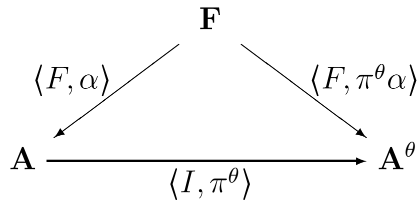
Consider, temporarily, $X \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\gamma^{-1}(T') \le X$. Since $\mathrm{Ker}(\langle H, \gamma\rangle)$ is compatible with $\gamma^{-1}(T')$, by the postulated compatibility property, it is also compatible with X. Thus, by Lemma 1333, $\widehat{\gamma}(X) = \gamma(X)$. Moreover, we have $T' \le \gamma(X) = \widehat{\gamma}(X)$.

Now set $X = T \vee \gamma^{-1}(T')$ and reason as follows:

$$\begin{array}{rcll} \gamma^{-1}(\widehat{\gamma}(T) \vee T') & \le & \gamma^{-1}(\widehat{\gamma}(X) \vee T') & (T \le X) \\ & = & \gamma^{-1}(\widehat{\gamma}(X)) & (T' \le \widehat{\gamma}(X)) \\ & = & \gamma^{-1}(\gamma(X)) & (\widehat{\gamma}(X) = \gamma(X)) \\ & = & X. & (\mathrm{Ker}(\langle H, \gamma\rangle) \text{ compatible with } X) \end{array}$$

So we get $\gamma^{-1}(\widehat{\gamma}(T) \vee T') = T \vee \gamma^{-1}(T')$ and $\mathcal{I}$ has the correspondence property.

(iii)⇒(ii) Suppose that $\mathcal{I}$ has the correspondence property and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system, $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, with $T \leq T'$, and $\theta \in \mathrm{ConSys}(\mathbf{A})$, such that θ is compatible with T. We look at the commutative diagram



and calculate

$$\begin{array}{rcl}(\pi^\theta)^{-1}(\widehat{\pi^\theta}(T')) & = & (\pi^\theta)^{-1}(\widehat{\pi^\theta}(T') \vee \widehat{\pi^\theta}(T)) \quad (T \leq T') \\ & = & T' \vee (\pi^\theta)^{-1}(\widehat{\pi^\theta}(T)) \quad \text{(correspondence property)} \\ & \leq & T' \vee T \quad (\theta \text{ compatible with } T) \\ & = & T'. \quad (T \leq T')\end{array}$$

Thus, θ is also compatible with T' and $\mathcal{I}$ has the compatibility property. ■

As a consequence we obtain the *Correspondence Theorem*, which asserts that, under the same hypothesis, $\langle H, \gamma \rangle$ induces an order isomorphism between the principal filter of the lattice $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})$ generated by $\gamma^{-1}(T')$ and the principal filter of the lattice $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{B})$ generated by T'.

Theorem 1336 (Correspondence Theorem) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a protoalgebraic* π*-institution based on* $\mathbf{F}$*. Let, also,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ *be* $\mathbf{F}$*-algebraic systems and* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism, with* $H : \mathbf{Sign} \to \mathbf{Sign}'$ *an isomorphism, and* $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$*. Then,* $Y \mapsto \gamma^{-1}(Y)$*,* $T' \leq Y \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$*, defines an order isomorphism* $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'} \cong \mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})^{\gamma^{-1}(T')}$*.*

Proof: $\gamma^{-1} : \mathbf{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'} \to \mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})^{\gamma^{-1}(T')}$ is well defined by Corollary 55 and it is clearly monotone. Furthermore, $\widehat{\gamma} : \mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})^{\gamma^{-1}(T')} \to \mathbf{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$ is also well-defined and monotone. So it suffices to show that, for all $T' \leq Y \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, $\widehat{\gamma}(\gamma^{-1}(Y)) = Y$ and that, for all $\gamma^{-1}(T') \leq X \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\gamma^{-1}(\widehat{\gamma}(X)) = X$.

First, for $T' \leq Y \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, since $\langle H, \gamma \rangle$ is surjective, $\gamma(\gamma^{-1}(Y)) = Y$ and, therefore, $\widehat{\gamma}(\gamma^{-1}(Y)) = Y$, since $Y \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. For the other equation, if $\gamma^{-1}(T') \leq X \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we have,

$$\begin{array}{rcl}\gamma^{-1}(\widehat{\gamma}(X)) & = & \gamma^{-1}(\widehat{\gamma}(X) \vee T') \quad (\gamma^{-1}(T') \leq X \Rightarrow T' \leq \widehat{\gamma}(X)) \\ & = & X \vee \gamma^{-1}(T') \quad \text{(correspondence property)} \\ & = & X. \quad (\gamma^{-1}(T') \leq X)\end{array}$$

So $\gamma^{-1} : \mathbf{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'} \cong \mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})^{\gamma^{-1}(T')}$. ■

18.1.2 The Homomorphism Theorem

We show that, in the case of protoalgebraic π-institutions $\mathcal{I}$, every surjective morphism of $\mathcal{I}$-matrix families gives rise to a corresponding surjective morphism between their reductions. This establishes a "reduction" functor and, moreover, gives rise to a version of the Homomorphism Theorem of Universal Algebra.

Recall that, given a base algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ and an $\mathbf{F}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, we denote by $\mathfrak{A}^*$ the reduction of $\mathfrak{A}$, i.e.,

$$\mathfrak{A}^* = \langle \mathcal{A}/\Omega^{\mathcal{A}}(T), T/\Omega^{\mathcal{A}}(T) \rangle.$$

Moreover, extending this notation, given $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, we set

$$\phi^* = \phi/\Omega^{\mathcal{A}}_\Sigma(T) \in \mathrm{SEN}^*(\Sigma).$$

Theorem 1337 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a protoalgebraic π-institution based on $\mathbf{F}$. Further, let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$, be $\mathbf{F}$-algebraic systems, $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ and $\mathfrak{A}' = \langle \mathcal{A}', T' \rangle$ be $\mathcal{I}$-matrix families and $\langle H, \gamma \rangle : \mathfrak{A} \to \mathfrak{A}'$ a surjective morphism. Then, there exists a surjective morphism $\langle H, \gamma^* \rangle : \mathfrak{A}^* \to \mathfrak{A}'^*$, given, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, by*

$$\gamma^*_\Sigma(\phi^*) = \gamma_\Sigma(\phi)^*.$$

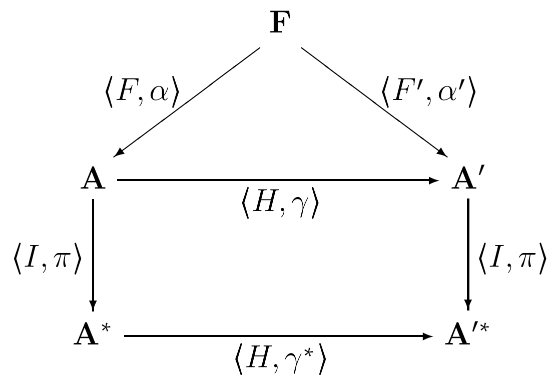
Proof: First, we show that, for all $\Sigma \in |\mathbf{Sign}|$, $\gamma^*_\Sigma : \mathrm{SEN}^*(\Sigma) \to \mathrm{SEN}'^*(H(\Sigma))$ is well-defined. Indeed, suppose $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\phi^* = \psi^*$, i.e., $\langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_\Sigma(T)$. Then, since $T \leq \gamma^{-1}(T')$, we get, by protoalgebraicity, $\langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_\Sigma(\gamma^{-1}(T'))$, whence, by Proposition 24, $\langle \phi, \psi \rangle \in \gamma^{-1}_\Sigma(\Omega^{\mathcal{A}'}_{H(\Sigma)}(T'))$, and, hence, $\langle \gamma_\Sigma(\phi), \gamma_\Sigma(\psi) \rangle \in \Omega^{\mathcal{A}'}_{H(\Sigma)}(T')$, or, equivalently, $\gamma_\Sigma(\phi)^* = \gamma_\Sigma(\psi)^*$.

Next we see that $\gamma^* : \mathrm{SEN}^* \to \mathrm{SEN}'^* \circ H$ is a natural transformation. To this end, let $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\phi \in \mathrm{SEN}(\Sigma)$. Then we have

$$\begin{array}{ccc}
\mathrm{SEN}^*(\Sigma) & \xrightarrow{\gamma^*_\Sigma} & \mathrm{SEN}'^*(H(\Sigma)) \\
\mathrm{SEN}^*(f) \downarrow & & \downarrow \mathrm{SEN}'^*(H(f)) \\
\mathrm{SEN}^*(\Sigma') & \xrightarrow[\gamma^*_{\Sigma'}]{} & \mathrm{SEN}'^*(H(\Sigma'))
\end{array}$$

$$\begin{array}{rcl}
\mathrm{SEN}'^*(H(f))(\gamma^*_\Sigma(\phi^*)) & = & \mathrm{SEN}'^*(H(f))(\gamma_\Sigma(\phi)^*) \\
& = & \mathrm{SEN}'(H(f))(\gamma_\Sigma(\phi))^* \\
& = & \gamma_{\Sigma'}(\mathrm{SEN}(f)(\phi))^* \\
& = & \gamma^*_{\Sigma'}(\mathrm{SEN}(f)(\phi)^*) \\
& = & \gamma^*_{\Sigma'}(\mathrm{SEN}^*(f)(\phi^*)).
\end{array}$$

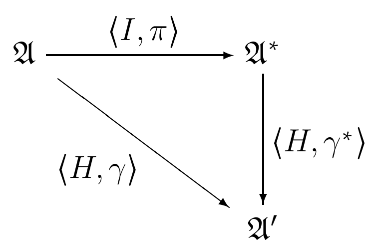
Surjectivity of $\langle H,\gamma^*\rangle : \mathcal{A}^* \to \mathcal{A}'^*$ follows from the fact that $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{A}'$ is surjective. So it suffices to show that $\langle F',\pi\alpha'\rangle = \langle H,\gamma^*\rangle \circ \langle F,\pi,\alpha\rangle$ and that $\langle H,\gamma^*\rangle : \mathfrak{A}^* \to \mathfrak{A}'^*$ is a matrix family morphism. The first equation follows from the fact that the upper triangle of the diagram commutes by hypothesis and the rectangle commutes by the definition of $\langle H,\gamma^*\rangle$.



To finish the proof, we calculate, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $\phi^* \in T_\Sigma/\Omega^{\mathcal{A}}_\Sigma(T)$ if and only if, by compatibility, $\phi \in T_\Sigma$ implies, by hypothesis, $\phi \in \gamma_\Sigma^{-1}(T'_{H(\Sigma)})$ if and only if $\gamma_\Sigma(\phi) \in T'_{H(\Sigma)}$ if and only if, by compatibility, $\gamma_\Sigma(\phi)^* \in T'_{H(\Sigma)}/\Omega^{\mathcal{A}'}_{H(\Sigma)}(T')$ if and only if, by the definition of γ^*, $\gamma^*_\Sigma(\phi^*) \in T'_{H(\Sigma)}/\Omega^{\mathcal{A}'}_{H(\Sigma)}(T')$ if and only if $\phi^* \in (\gamma^*_\Sigma)^{-1}(T'_{H(\Sigma)}/\Omega^{\mathcal{A}'}_{H(\Sigma)}(T'))$. ■

We also have the following construction.

Corollary 1338 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a protoalgebraic* π*-institution based on* $\mathbf{F}$*. Let, also* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, $\mathcal{A}' = \langle \mathbf{A}', \langle F',\alpha'\rangle\rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$*, be* $\mathbf{F}$*-algebraic systems,* $\mathfrak{A} = \langle \mathcal{A}, T\rangle$ *an* $\mathcal{I}$*-matrix family and* $\mathfrak{A}' = \langle \mathcal{A}', T'\rangle$ *a reduced* $\mathcal{I}$*-matrix family and* $\langle H,\gamma\rangle : \mathfrak{A} \to \mathfrak{A}'$ *a surjective morphism.*



There exists a unique surjective morphism $\langle H,\gamma^*\rangle : \mathfrak{A}^* \to \mathfrak{A}'$ *that makes the triangle commute.*

Proof: By Theorem 1337, there exists a surjective matrix morphism $\langle H,\gamma^*\rangle :$

$\mathfrak{A}^* \to \mathfrak{A}'^*$, such that the following rectangle commutes:

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\langle I,\pi\rangle} & \mathfrak{A}^* \\ {\scriptstyle\langle H,\gamma\rangle}\downarrow & & \downarrow{\scriptstyle\langle H,\gamma^*\rangle} \\ \mathfrak{A}' & \xrightarrow[\langle I,\pi\rangle]{} & \mathfrak{A}'^* \end{array}$$

But, by hypothesis, $\mathfrak{A}'$ is reduced, whence $\mathfrak{A}'^* = \mathfrak{A}'$ and $\langle I,\pi\rangle = \langle I,\iota\rangle : \mathfrak{A}' \to \mathfrak{A}'^*$ is the identity morphism. We now obtain the triangle depicted in the diagram of the statement. ■

Let us denote by $\mathbf{MatFam}(\mathcal{I})$ the category of $\mathcal{I}$-matrix families with surjective matrix morphisms between them and, similarly, $\mathbf{MatFam}^*(\mathcal{I})$ the category of reduced $\mathcal{I}$-matrix families with surjective matrix morphisms between them. Then, based on Theorem 1337, we obtain the following functor.

Theorem 1339 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then*

$$^* : \mathbf{MatFam}(\mathcal{I}) \to \mathbf{MatFam}^*(\mathcal{I})$$

is a functor. The subcategory $\mathbf{MatFam}^*(\mathcal{I})$ *is a reflective subcategory of* $\mathbf{MatFam}(\mathcal{I})$ *with* * *a reflector from* $\mathbf{MatFam}(\mathcal{I})$ *to* $\mathbf{MatFam}^*(\mathcal{I})$*.*

Proof: Given $\mathcal{A} \in \mathrm{MatFam}(\mathcal{I})$, it is easy to see that $\langle I,\iota^*\rangle : \mathfrak{A}^* \to \mathfrak{A}^*$ is the identity matrix morphism. For the composition property, assume $\mathfrak{A}$, $\mathfrak{A}'$, $\mathfrak{A}'' \in \mathrm{MathFam}(\mathcal{I})$, and $\langle G,\beta\rangle : \mathfrak{A} \to \mathfrak{A}'$, $\langle H,\gamma\rangle : \mathfrak{A}' \to \mathfrak{A}''$ be matrix morphisms. Then, we have, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{rcl} (\gamma_{G(\Sigma)} \circ \beta_\Sigma)^*(\phi^*) & = & \gamma_{G(\Sigma)}(\beta_\Sigma(\phi))^* \\ & = & \gamma^*_{G(\Sigma)}(\beta_\Sigma(\phi)\star) \\ & = & \gamma^*_{G(\Sigma)}(\beta^*_\Sigma(\phi^*)). \end{array}$$

Thus, $(\langle H,\gamma\rangle \circ \langle G,\beta\rangle)^* = \langle H,\gamma\rangle^* \circ \langle G,\beta\rangle^*$. Therefore, $^* : \mathbf{MatFam}(\mathcal{I}) \to \mathbf{MatFam}^*(\mathcal{I})$ is a functor.

As far a s reflectivity is concerned, for every $\mathfrak{A} \in \mathrm{MatFam}^*(\mathcal{I})$, we consider the natural quotient morphism $\langle I,\pi\rangle : \mathfrak{A} \to \mathfrak{A}^*$. Given reduced $\mathfrak{B} \in \mathrm{MatFam}^*(\mathcal{I})$ and a surjective morphism $\langle H,\gamma\rangle : \mathfrak{A} \to \mathfrak{B}$, the surjective morphism $\langle H,\gamma^*\rangle : \mathfrak{A}^* \to \mathfrak{B}$ of Corollary 1338 is the unique surjective morphism such that the following diagram commutes.

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\langle I,\pi\rangle} & \mathfrak{A}^* \\ & {\scriptstyle\langle H,\gamma\rangle}\searrow & \downarrow{\scriptstyle\langle H,\gamma^*\rangle} \\ & & \mathfrak{B} \end{array}$$

Thus, $\mathbf{MatFam}^*(\mathcal{I})$ is a reflective subcategory of $\mathbf{MatFam}(\mathcal{I})$ with * a reflector from $\mathbf{MatFam}(\mathcal{I})$ to $\mathbf{MatFam}^*(\mathcal{I})$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. Given an $\mathcal{I}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, we denote by $\mathbf{FiFam}^{\mathcal{I}}(\mathfrak{A})$ the principal filter of the complete lattice $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})$ generated by the $\mathcal{I}$-filter family T:

$$\mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}) = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : T \leq T'\}.$$

Recall that this set is also denoted by $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$, without explicit reference to the matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$.

The Correspondence Theorem allows us to prove the following.

Theorem 1340 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a protoalgebraic π-institution based on $\mathbf{F}$. For every $\mathcal{I}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,*

$$\mathbf{FiFam}^{\mathcal{I}}(\mathfrak{A}) \cong \mathbf{FiFam}^{\mathcal{I}}(\mathfrak{A}^*).$$

Proof: By The Correspondence Theorem 1336, with $\mathcal{B} = \mathcal{A}/\Omega^{\mathcal{A}}(T)$, $\langle H, \gamma \rangle = \langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T)$ and $T' = T/\Omega^{\mathcal{A}}(T)$, we get

$$\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T))^{T/\Omega^{\mathcal{A}}(T)} \cong \mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})^{\pi^{-1}(T/\Omega^{\mathcal{A}}(T))}.$$

But this amounts to $\mathbf{FiFam}^{\mathcal{I}}(\mathfrak{A}^*) \cong \mathbf{FiFam}(\mathfrak{A})$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Consider $\mathbf{F}$-matrix families $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ and $\mathfrak{B} = \langle \mathcal{B}, T' \rangle$ and a surjective matrix morphism $\langle H, \gamma \rangle : \mathfrak{A} \to \mathfrak{B}$. By definition, we have $T \leq \gamma^{-1}(T')$. We call $\gamma^{-1}(T')$ the **filter kernel** of $\langle H, \gamma \rangle$. By the inclusion relation above, we can see that, if $\mathfrak{B} \in \mathrm{MatFam}(\mathcal{I})$, then $\gamma^{-1}(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$.

Given $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ and $X \in \mathrm{SenFam}(\mathcal{A})$, such that $T \leq X$, we define

$$\mathfrak{A}/X := \langle \mathcal{A}, X \rangle^* = \langle \mathcal{A}/\Omega^{\mathcal{A}}(X), X/\Omega^{\mathcal{A}}(X) \rangle.$$

We call $\mathfrak{A}/X$ the **quotient of $\mathfrak{A}$ by X**. We note that, if $\mathfrak{A} \in \mathrm{MatFam}(\mathcal{I})$, then

$$X \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}) \quad \text{iff} \quad \mathfrak{A}/X \in \mathrm{MatFam}^*(\mathcal{I}).$$

The following is an analog in the context of $\mathcal{I}$-matrix families of the Homomorphism Theorem of Universal Algebra.

Theorem 1341 (Homomorphism Theorem) *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a protoalgebraic π-institution based on $\mathbf{F}$. Let also $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, $\mathfrak{A}' = \langle \mathcal{A}', T' \rangle \in \mathrm{MatFam}(\mathcal{I})$ and $\langle H, \gamma \rangle : \mathfrak{A} \to \mathfrak{A}'$ a surjective morphism.*

(i) *There exists a strict surjective morphism* $\langle H, \gamma' \rangle : \mathfrak{A}/\gamma^{-1}(T') \to \mathfrak{A}'^*$ *with isomorphic components;*

(ii) *If* $X \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$ *and* $X \le \gamma^{-1}(T')$, *then, there exists a surjective morphism* $\langle H, \gamma^X \rangle : \mathfrak{A}/X \to \mathfrak{A}'^*$, *such that*

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\langle H,\gamma\rangle} & \mathfrak{A}' \\ {\scriptstyle \langle I,\pi^X\rangle}\downarrow & & \downarrow{\scriptstyle \langle I,\pi\rangle} \\ \mathfrak{A}/X & \xrightarrow[\langle H,\gamma^X\rangle]{} & \mathfrak{A}'^* \end{array}$$

$\langle H, \gamma^X \rangle \circ \langle I, \pi^X \rangle = \langle I, \pi \rangle \circ \langle H, \gamma \rangle$.

Proof:

(i) First, note that $\gamma^{-1}(T') \le \gamma^{-1}(T')$, whence, $\langle H, \gamma \rangle : \langle \mathcal{A}, \gamma^{-1}(T') \rangle \to \mathfrak{A}'$ is also a surjective matrix morphism. Thus, taking into account that $T \le \gamma^{-1}(T')$, we get, by Theorem 1337, a surjective matrix morphism $\langle H, \gamma^* \rangle : \mathfrak{A}/\gamma^{-1}(T') \to \mathfrak{A}'^*$, such that the following diagram commutes.

$$\begin{array}{ccc} \langle \mathcal{A}, \gamma^{-1}(T') \rangle & \xrightarrow{\langle H,\gamma\rangle} & \mathfrak{A}' \\ {\scriptstyle \langle I,\pi\rangle}\downarrow & & \downarrow{\scriptstyle \langle I,\pi\rangle} \\ \mathfrak{A}/\gamma^{-1}(T') & \xrightarrow[\langle H,\gamma^*\rangle]{} & \mathfrak{A}'^* \end{array}$$

It remains to show that, for every $\Sigma \in |\mathbf{Sign}|$,

$$\gamma^*_\Sigma : \mathrm{SEN}^{\gamma^{-1}(T')}(\Sigma) \to \mathrm{SEN}'^*(H(\Sigma))$$

is a bijection and that $\langle H, \gamma^* \rangle$ is strict. To see that γ^*_Σ is a bijection, let $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that

$$\gamma^*_\Sigma(\phi/\Omega^{\mathcal{A}}_\Sigma(\gamma^{-1}(T'))) = \gamma^*_\Sigma(\psi/\Omega^{\mathcal{A}}_\Sigma(\gamma^{-1}(T'))).$$

Then, by the commutativity of the rectangle,

$$\gamma_\Sigma(\phi)/\Omega^{\mathcal{A}'}_{H(\Sigma)}(T') = \gamma_\Sigma(\psi)/\Omega^{\mathcal{A}'}_{H(\Sigma)}(T').$$

This gives that $\langle \phi, \psi \rangle \in \gamma^{-1}_\Sigma(\Omega^{\mathcal{A}'}_{H(\Sigma)}(T'))$. Thus, $\langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_\Sigma(\gamma^{-1}(T'))$ and, hence,

$$\phi/\Omega^{\mathcal{A}}_\Sigma(\gamma^{-1}(T')) = \phi/\Omega^{\mathcal{A}}_\Sigma(\gamma^{-1}(T')).$$

Therefore, $\gamma^*_\Sigma : \mathrm{SEN}^{\gamma^{-1}(T')}(\Sigma) \to \mathrm{SEN}'^*(H(\Sigma))$ is a bijection, for all $\Sigma \in |\mathbf{Sign}|$.

To prove strictness, assume $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi/\Omega^{\mathcal{A}}_\Sigma(\gamma^{-1}(T')) \in \gamma^{*-1}_\Sigma(T'_{H(\Sigma)}/\Omega^{\mathcal{A}'}_{H(\Sigma)}(T'))$. Then $\gamma^*_\Sigma(\phi/\Omega^{\mathcal{A}}_\Sigma(\gamma^{-1}(T'))) \in T'_{H(\Sigma)}/\Omega^{\mathcal{A}'}_{H(\Sigma)}(T')$. Hence, by the definition of γ^*, we get $\gamma_\Sigma(\phi)^* \in T'_{H(\Sigma)}/\Omega^{\mathcal{A}'}_{H(\Sigma)}(T')$. By compatibility, we obtain $\gamma_\Sigma(\phi) \in T'_{H(\Sigma)}$, whence $\phi \in \gamma^{-1}_\Sigma(T'_{H(\Sigma)})$. This, finally, yields

$$\phi/\Omega^{\mathcal{A}}_\Sigma(\gamma^{-1}(T')) \in \gamma^{-1}_\Sigma(T'_{H(\Sigma)})/\Omega^{\mathcal{A}}_\Sigma(\gamma^{-1}(T')),$$

proving strictness.

(ii) This part is proven by the following diagram chase:

$$\begin{array}{ccccc}
\mathfrak{A} & & \xrightarrow{\langle H,\gamma\rangle} & & \mathfrak{A}' \\
\langle I,\pi^X\rangle \downarrow & & & & \downarrow \langle I,\pi\rangle \\
\mathfrak{A}/X & \xrightarrow[\langle I,\pi\rangle]{} & \mathfrak{A}/\gamma^{-1}(T') & \xrightarrow[\langle H,\gamma'\rangle]{} & \mathfrak{A}'^*
\end{array}$$

where $\langle I,\pi\rangle : \mathfrak{A}/X \to \mathfrak{A}/\gamma^{-1}(T')$ is the canonical projection morphism, defined because of the hypothesis $X \le \gamma^{-1}(T')$ and protoalgebraicity, and

$$\langle H,\gamma'\rangle : \mathfrak{A}/\gamma^{-1}(T') \to \mathfrak{A}'^*$$

is the morphism obtained in Part (i).

■

18.2 Pointed Classes of Algebraic Systems

Proposition 1342 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a protoalgebraic π-institution based on $\mathbf{F}$, having theorems. Then the following conditions are equivalent:*

(i) $\mathcal{I}$ is family regular;

(ii) For all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in T_\Sigma$, $T_\Sigma = \phi/\Omega_\Sigma(T)$;

(iii) For all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$, $T_\Sigma = t/\Omega_\Sigma(T)$;

(iv) For all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and some $\phi \in T_\Sigma$, $T_\Sigma = \phi/\Omega_\Sigma(T)$;

(v) For all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and some $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$, $T_\Sigma = t/\Omega_\Sigma(T)$.

Proof:

(i)⇒(ii) Suppose $\mathcal{I}$ is family regular and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in T_\Sigma$. Then, we have, for all $\psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rll} \psi \in T_\Sigma \quad \text{iff} & \phi, \psi \in T_\Sigma & (\phi \in T_\Sigma) \\ \text{iff} & C(\phi,\psi) \le T & (\text{definition of } C(\phi,\psi)) \\ \text{implies} & \Omega_\Sigma(C(\phi,\psi)) \le \Omega_\Sigma(T) & (\mathcal{I} \text{ protoalgebraic}) \\ \text{implies} & \langle \phi,\psi\rangle \in \Omega_\Sigma(T) & (\mathcal{I} \text{ family regular}) \\ \text{iff} & \psi \in \phi/\Omega_\Sigma(T). & (\text{definition of } \phi/\Omega_\Sigma(T)) \end{array}$$

On the other hand, if $\psi \in \phi/\Omega_\Sigma(T)$, then, since $\phi \in T_\Sigma$, by the compatibility of $\Omega(T)$ with T, $\psi \in T_\Sigma$. Thus, we conclude that $T_\Sigma = \phi/\Omega_\Sigma(T)$.

(ii)⇒(iii) Suppose (ii) holds and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$. Then, since $\mathrm{Thm}(\mathcal{I}) \le T$, we get that $t \in T_\Sigma$ and, hence, by hypothesis, $T_\Sigma = t/\Omega_\Sigma(T)$.

(iii)⇒(iv) Assume (iii) holds and let $T \in \mathrm{ThFam}(\mathcal{I})$ and $\Sigma \in |\mathbf{Sign}^\flat|$. Since $\mathcal{I}$ has theorems, there exists $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$. Then, $t \in T_\Sigma$ and, by hypothesis, $T_\Sigma = t/\Omega_\Sigma(T)$.

(iv)⇒(v) Assume (iv) holds and let $T \in \mathrm{ThFam}(\mathcal{I})$ and $\Sigma \in |\mathbf{Sign}^\flat|$. Then, by hypothesis, there exists $\phi \in T_\Sigma$, such that $T_\Sigma = \phi/\Omega_\Sigma(T)$. Moreover, $\mathcal{I}$ has theorems, whence, there exists $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$. Then, we have $t \in T_\Sigma = \phi/\Omega_\Sigma(T)$, whence $\langle \phi, t\rangle \in \Omega_\Sigma(T)$ and, therefore, $T_\Sigma = \phi/\Omega_\Sigma(T) = t/\Omega_\Sigma(T)$.

(v)⇒(i) Assume that (v) holds and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in T_\Sigma$. By hypothesis, for some $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$, $T_\Sigma = t/\Omega_\Sigma(T)$. Hence, $\phi,\psi \in t/\Omega_\Sigma(T)$, i.e., $\langle \phi,t\rangle \in \Omega_\Sigma(T)$ and $\langle t,\psi\rangle \in \Omega_\Sigma(T)$. By transitivity, $\langle \phi,\psi\rangle \in \Omega_\Sigma(T)$. Therefore, $\mathcal{I}$ is family regular.

■

We show now that a protoalgebraic family assertional π-institution $\mathcal{I}$ is weakly family algebraizable.

Theorem 1343 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a protoalgebraic π-institution based on* $\mathbf{F}$. *If $\mathcal{I}$ is family assertional, then it is weakly family algebraizable.*

Proof: By Definition 613, protoalgebraicity and family assertionality are equivalent to regular weak family algebraizability. By Proposition 620, this entails weak family algebraizability.

More directly, assume $\mathcal{I}$ is family assertional. Since it is protoalgebraic, by hypothesis, it suffices to show that $\mathcal{I}$ is family injective. To this end, let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\Omega(T) = \Omega(T')$. Then, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$,

$$T_\Sigma = t/\Omega_\Sigma(T) = t/\Omega_\Sigma(T') = T'_\Sigma.$$

Therefore $T = T'$. Hence $\mathcal{I}$ is family injective and, therefore, it is weakly family algebraizable. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a class of $\mathbf{F}$-algebraic systems. We say that K is $\top^\flat$-**pointed** if, for all $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle \in \mathsf{K}$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)$,

$$\top^{\mathcal{A}}_\Sigma(\vec{\phi}) = \top^{\mathcal{A}}_\Sigma(\vec{\psi}).$$

K is called **pointed** if it is $\top^\flat$-pointed with respect to some $\top^\flat$ in $N^\flat$.

If a class K is pointed, then, for every $\mathcal{A} \in \mathsf{K}$, we write $\top^{\mathcal{A}} = \{\top^{\mathcal{A}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, where $\top^{\mathcal{A}}_\Sigma \coloneqq \top^{\mathcal{A}}_\Sigma(\vec{\phi})$, for some $\vec{\phi} \in \mathrm{SEN}(\Sigma)$, this value being independent of the choice of $\vec{\phi} \in \mathrm{SEN}(\Sigma)$.

We focus now on protoalgebraic, family regular π-institutions that have natural theorems. Recall that this means that there exists a natural transformation $\top^\flat$ in $N^\flat$, such that $\top^\flat$ is evaluated to a theorem in every signature and at all tuples of sentences. Of course, by definition, all π-institutions that fit this description are regularly weakly family algebraizable. We show that for such π-institutions, the class $\mathrm{AlgSys}^*(\mathcal{I})$ of their reduced algebraic systems is a pointed class of $\mathbf{F}$-algebraic systems, where any natural theorem may serve as the "point".

Proposition 1344 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a protoalgebraic family regular π-institution based on $\mathbf{F}$, having natural theorems. Then, the class $\mathrm{AlgSys}^*(\mathcal{I})$ is a pointed class of $\mathbf{F}$-algebraic systems.*

Proof: Suppose $\mathcal{I}$ is protoalgebraic and family regular, with a natural theorem $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$, i.e., such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$, $\top^\flat_\Sigma(\vec{\phi}) \in \mathrm{Thm}_\Sigma(\mathcal{I})$. By family regularity, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$, $\langle \top^\flat_\Sigma(\vec{\phi}), \top^\flat_\Sigma(\vec{\psi}) \rangle \in \Omega_\Sigma(\mathrm{Thm}(\mathcal{I}))$. Now, let $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I})$. Thus, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. Therefore, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$, we get, by what was shown above and protoalgebraicity, and taking into account Lemma 51, $\langle \top^\flat_\Sigma(\vec{\phi}), \top^\flat_\Sigma(\vec{\psi}) \rangle \in \Omega_\Sigma(\alpha^{-1}(T))$, whence, by Proposition 24, $\langle \top^\flat_\Sigma(\vec{\phi}), \top^\flat_\Sigma(\vec{\psi}) \rangle \in \alpha^{-1}_\Sigma(\Omega^{\mathcal{A}}_{F(\Sigma)}(T))$. Thus,

$$\langle \alpha_\Sigma(\top^\flat_\Sigma(\vec{\phi})), \alpha_\Sigma(\top^\flat_\Sigma(\vec{\psi})) \rangle \in \Omega^{\mathcal{A}}_{F(\Sigma)}(T) = \Delta^{\mathcal{A}}_{F(\Sigma)},$$

i.e., since $\top^\flat$ is a natural transformation, $\top^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})) = \top^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\psi}))$. Taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that $\mathrm{AlgSys}^*(\mathcal{I})$ is a pointed class of $\mathbf{F}$-algebraic systems, with any natural theorem serving as the "point" natural transformation. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$ and K a $\top^\flat$-pointed class of $\mathbf{F}$-algebraic systems. We say that K

is **relatively point regular** if, for every $\theta,\theta' \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$,

$$\top^\flat/\theta = \top^\flat/\theta' \quad \text{implies} \quad \theta = \theta'.$$

It is not difficult to show that the defining property transfers from K-congruence systems on $\mathcal{F}$ to K-congruence systems on every $\mathbf{F}$-algebraic system, under the proviso that K be an abstract class.

Lemma 1345 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ *a natural transformation in* $N^\flat$*, and* K *a* $\top^\flat$*-pointed abstract class of* $\mathbf{F}$*-algebraic systems. If* K *is relatively point regular, then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $\theta,\theta' \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$*,*

$$\top^{\mathcal{A}}/\theta = \top^{\mathcal{A}}/\theta' \quad \textit{implies} \quad \theta = \theta'.$$

Proof: Suppose $\mathcal{A}$ is an $\mathbf{F}$-algebraic system, $\theta,\theta' \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$, such that $\top^{\mathcal{A}}/\theta = \top^{\mathcal{A}}/\theta'$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rll}\langle \phi, \top^\flat_\Sigma\rangle \in \alpha_\Sigma^{-1}(\theta_{F(\Sigma)}) & \text{iff} & \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\top^\flat_\Sigma)\rangle \in \theta_{F(\Sigma)}\\ & \text{iff} & \langle \alpha_\Sigma(\phi), \top^{\mathcal{A}}_{F(\Sigma)}\rangle \in \theta_{F(\Sigma)}\\ & \text{iff} & \langle \alpha_\Sigma(\phi), \top^{\mathcal{A}}_{F(\Sigma)}\rangle \in \theta'_{F(\Sigma)}\\ & \text{iff} & \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\top^\flat_\Sigma)\rangle \in \theta'_{F(\Sigma)}\\ & \text{iff} & \langle \phi, \top^\flat_\Sigma\rangle \in \alpha_\Sigma^{-1}(\theta'_{F(\Sigma)}).\end{array}$$

Thus, $\top^\flat/\alpha^{-1}(\theta) = \top^\flat/\alpha^{-1}(\theta')$. Since K is abstract and $\mathcal{A}/\theta, \mathcal{A}/\theta' \in \mathsf{K}$, we get that $\mathcal{F}/\alpha^{-1}(\theta), \mathcal{F}/\alpha^{-1}(\theta') \in \mathsf{K}$. It follows that $\alpha^{-1}(\theta), \alpha^{-1}(\theta') \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$. Since K is relatively point regular, by definition, $\alpha^{-1}(\theta) = \alpha^{-1}(\theta')$. Therefore, by surjectivity of $\langle F, \alpha\rangle$, $\theta = \theta'$. ∎

Moreover, we can show that, for a protoalgebraic family regular π-institution $\mathcal{I}$, having natural theorems, the associated class $\mathrm{AlgSys}^*(\mathcal{I})$ of its reduced algebraic systems is a relatively point regular class.

Proposition 1346 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a protoalgebraic family regular* π*-institution based on* $\mathbf{F}$*, having natural theorems. Then, the class* $\mathrm{AlgSys}^*(\mathcal{I})$ *is a relatively point regular class of* $\mathbf{F}$*-algebraic systems.*

Proof: We know, by Proposition 1344, that $\mathrm{AlgSys}^*(\mathcal{I})$ is pointed, with any natural theorem $\top^\flat$ serving as a "point". Consider $\theta,\theta' \in \mathrm{ConSys}^*(\mathcal{I})$, such that $\top^\flat/\theta = \top^\flat/\theta'$. Since $\theta,\theta' \in \mathrm{ConSys}^*(\mathcal{I})$, there exist $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\theta = \Omega(T)$ and $\theta' = \Omega(T')$. But then, since $\mathcal{I}$ is protoalgebraic and family regular, with theorems, we get, by Proposition 1342,

$$\begin{array}{rcll}\theta & = & \Omega(T) & \\ & = & \Omega(\top^\flat/\Omega(T)) & \text{(Proposition 1342)}\\ & = & \Omega(\top^\flat/\theta) & \\ & = & \Omega(\top^\flat/\theta') & \text{(hypothesis)}\\ & = & \Omega(\top^\flat/\Omega(T')) & \\ & = & \Omega(T') & \text{(Proposition 1342)}\\ & = & \theta'. & \end{array}$$

Hence $\mathrm{AlgSys}^*(\mathcal{I})$ is indeed relatively point regular. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a $\top^\flat$-pointed class of $\mathbf{F}$-algebraic systems. Define on $\mathbf{F}$ the family $C^{\mathsf{K},\top} = \{C^{\mathsf{K},\top}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$C^{\mathsf{K},\top}_\Sigma : \mathcal{P}(\mathrm{SEN}^\flat(\Sigma)) \to \mathcal{P}(\mathrm{SEN}^\flat(\Sigma)),$$

be given, for all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, by

$$\phi \in C^{\mathsf{K},\top}_\Sigma(\Phi) \quad \text{iff} \quad \phi \approx \top^\flat_\Sigma \in C^{\mathsf{K}}_\Sigma(\Phi \approx \top^\flat_\Sigma),$$

i.e., $\phi \in C^{\mathsf{K},\top}_\Sigma(\Phi)$ if and only if, for all $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle \in \mathsf{K}$, all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq \{\top^{\mathcal{A}}_{F(\Sigma')}\} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) = \top^{\mathcal{A}}_{F(\Sigma')}.$$

In the next proposition, it is shown that $C^{\mathsf{K},\top}$ is a closure system on $\mathbf{F}$. In this way the pointed class K of $\mathbf{F}$-algebraic systems defines a bona fide π-institution based on $\mathbf{F}$.

Proposition 1347 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a $\top^\flat$-pointed class of $\mathbf{F}$-algebraic systems. $C^{\mathsf{K},\top}$ is a closure system on $\mathbf{F}$.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$. It is obvious from the definition that

$$C^{\mathsf{K},\top}_\Sigma : \mathcal{P}(\mathrm{SEN}^\flat(\Sigma)) \to \mathcal{P}(\mathrm{SEN}^\flat(\Sigma))$$

is inflationary and monotone. To show that it is also idempotent, let $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C^{\mathsf{K},\top}_\Sigma(C^{\mathsf{K},\top}_\Sigma(\Phi))$. Thus, we have, by definition, for all $\mathcal{A} \in \mathsf{K}$, all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(C^{\mathsf{K},\top}_\Sigma(\Phi))) \subseteq \{\top^{\mathcal{A}}_{F(\Sigma')}\} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) = \top^{\mathcal{A}}_{F(\Sigma')}.$$

But, also by definition, we have, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq \{\top^{\mathcal{A}}_{F(\Sigma')}\} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(C^{\mathsf{K},\top}_\Sigma(\Phi))) \subseteq \{\top^{\mathcal{A}}_{F(\Sigma')}\}.$$

Therefore, we get that, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq \{\top^{\mathcal{A}}_{F(\Sigma')}\} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) = \top^{\mathcal{A}}_{F(\Sigma')},$$

showing that $\phi \in C^{\mathsf{K},\top}_\Sigma(\Phi)$.

It remains, finally, to show that $C^{\mathsf{K},\top}$ is structural. Let $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C^{\mathsf{K},\top}_\Sigma(\Phi)$. Consider $\mathcal{A} \in \mathsf{K}$, such that, for all $\Sigma'' \in |\mathbf{Sign}^\flat|$ and all $g \in \mathbf{Sign}^\flat(\Sigma', \Sigma'')$,

$$\Sigma \xrightarrow{\ f\ } \Sigma' \xrightarrow{\ g\ } \Sigma''$$

$\alpha_{\Sigma''}(\mathrm{SEN}^\flat(g)(\mathrm{SEN}^\flat(f)(\Phi))) \subseteq \{\top^{\mathcal{A}}_{F(\Sigma'')}\}$. This gives $\alpha_{\Sigma''}(\mathrm{SEN}^\flat(gf)(\Phi)) \subseteq \{\top^{\mathcal{A}}_{F(\Sigma'')}\}$, whence, by hypothesis, $\alpha_{\Sigma''}(\mathrm{SEN}^\flat(gf)(\phi)) = \top^{\mathcal{A}}_{F(\Sigma'')}$. Thus, for all $\Sigma'' \in |\mathbf{Sign}^\flat|$ and all $g \in \mathbf{Sign}^\flat(\Sigma', \Sigma'')$, $\alpha_{\Sigma''}(\mathrm{SEN}^\flat(g)(\mathrm{SEN}^\flat(f)(\phi))) = \top^{\mathcal{A}}_{F(\Sigma'')}$. We conclude that $\mathrm{SEN}^\flat(f)(\phi) \in C^{\mathsf{K},\top}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))$ and, therefore, $C^{\mathsf{K},\top}$ is also structural. ■

Based on Proposition 1347, it makes sense, given an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$, with $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a $\top^\flat$-pointed class of $\mathbf{F}$-algebraic systems, to define the **assertional π-institution of K** as the pair

$$\mathcal{I}^{\mathsf{K},\top} = \langle \mathbf{F}, C^{\mathsf{K},\top} \rangle.$$

We have seen in Proposition 1346 that, if $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is a protoalgebraic and family regular π-institution, having natural theorems, then its class $\mathrm{AlgSys}^*(\mathcal{I})$ of reduced $\mathbf{F}$-algebraic systems is a relatively point regular class. We show next, in a form of converse, that if K is a relatively point regular guasivariety of $\mathbf{F}$-algebraic systems, then the assertional π-institution $\mathcal{I}^{\mathsf{K},\top}$, associated with K, is a protoalgebraic family regular π-institution that has natural theorems.

First, we establish possession of natural theorems, under the assumption that K is pointed.

Proposition 1348 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a pointed class of $\mathbf{F}$-algebraic systems. Then $\mathcal{I}^{\mathsf{K},\top}$ has natural theorems.*

Proof: Let K be a pointed class of $\mathbf{F}$-algebraic systems. Since K is pointed, there exists $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$, all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\top^{\mathcal{A}}_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}))) = \top^{\mathcal{A}}_{F(\Sigma')}.$$

This implies that $\alpha_{\Sigma'}(\top^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\phi}))) = \top^{\mathcal{A}}_{F(\Sigma')}$ and, hence, we obtain $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\top^\flat_\Sigma(\vec{\phi}))) = \top^{\mathcal{A}}_{F(\Sigma')}$. Thus, by definition, $\top^\flat_\Sigma(\vec{\phi}) \in C^{\mathsf{K},\top}_\Sigma(\varnothing)$ and, therefore, $\top^\flat$ is a natural theorem. ■

Next, we turn to proving family regularity, again under the assumption of pointedness.

Proposition 1349 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a pointed class of $\mathbf{F}$-algebraic systems. Then $\mathcal{I}^{\mathsf{K},\top}$ is a family regular π-institution.*

Proof: Let K be a pointed class of $\mathbf{F}$-algebraic systems. We know, by Proposition 1348, that $\mathcal{I}^{\mathsf{K},\top}$ has a natural theorem $\top^\flat$, where $\top^\flat$ is a point in K. We show that $\mathcal{I}^{\mathsf{K},\top}$ is family regular. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, we have

$$\mathrm{SEN}^\flat(f)(\phi) \approx \top^\flat_{\Sigma'}, \mathrm{SEN}^\flat(f)(\psi) \approx \top^\flat_{\Sigma'} \in C^{\mathsf{K}}_{\Sigma'}(\phi \approx \top^\flat_\Sigma, \psi \approx \top^\flat_\Sigma).$$

This implies that, for all $\sigma^\flat$ in $N^\flat$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \approx \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}) \in C^{\mathsf{K}}_{\Sigma'}(\phi \approx \top^\flat_\Sigma, \psi \approx \top^\flat_\Sigma).$$

Now we get

$$\begin{array}{l}\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \approx \top^\flat_{\Sigma'} \in C^{\mathsf{K}}_{\Sigma'}(\phi \approx \top^\flat_\Sigma, \psi \approx \top^\flat_\Sigma) \\ \qquad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}) \approx \top^\flat_{\Sigma'} \in C^{\mathsf{K}}_{\Sigma'}(\phi \approx \top^\flat_\Sigma, \psi \approx \top^\flat_\Sigma).\end{array}$$

Hence, by definition,

$$\begin{array}{l}\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \in C^{\mathsf{K},\top}_{\Sigma'}(\phi, \psi) \\ \qquad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}) \in C^{\mathsf{K},\top}_{\Sigma'}(\phi, \psi).\end{array}$$

Therefore, by Theorem 19, $\langle \phi, \psi \rangle \in \Omega_\Sigma(C(\phi, \psi))$. ■

Before establishing protoalgebraicity, we need a couple of lemmas. We show, first, that, if K is a pointed class, then all theory families of $\mathcal{I}^{\mathsf{K},\top}$ are fully determined by the corresponding Leibniz class of the point.

Lemma 1350 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with a natural transformation $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a pointed class of $\mathbf{F}$-algebraic systems. Then, for all $T \in \mathrm{ThFam}(\mathcal{I}^{\mathsf{K},\top})$,*

$$T = \top^\flat / \Omega(T).$$

Proof: Let K be a pointed class of $\mathbf{F}$-algebraic systems, $T \in \mathrm{ThFam}(\mathcal{I}^{\mathsf{K},\top})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$.

- Suppose $\phi \in \top^\flat_\Sigma / \Omega_\Sigma(T)$. This means that $\langle \phi, \top^\flat_\Sigma \rangle \in \Omega_\Sigma(T)$. But, by definition, $\top^\flat_\Sigma \in \mathrm{Thm}_\Sigma(\mathcal{I}^{\mathsf{K},\top}) \subseteq T_\Sigma$, whence, by the compatibility property of $\Omega(T)$ with T, we get that $\phi \in T_\Sigma$.

- Suppose $\phi \in T_\Sigma$. Then $\phi \approx \top^\flat_\Sigma \in C^{\mathsf{K}}_\Sigma(T \approx \top^\flat)$. This implies that, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\begin{array}{l}\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \approx \top^\flat_{\Sigma'} \in C^{\mathsf{K}}_{\Sigma'}(T \approx \top^\flat) \\ \qquad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\top^\flat_\Sigma), \vec{\chi}) \approx \top^\flat_{\Sigma'} \in C^{\mathsf{K}}_{\Sigma'}(T \approx \top^\flat).\end{array}$$

  This is, by definition, equivalent to the statement that, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \in C^{\mathsf{K},\top}_{\Sigma'}(T) \quad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\top^\flat_\Sigma), \vec{\chi}) \in C^{\mathsf{K},\top}_{\Sigma'}(T).$$

  We conclude that $\langle \phi, \top^\flat_\Sigma \rangle \in \Omega_\Sigma(T)$, i.e., that $\phi \in \top^\flat_\Sigma / \Omega_\Sigma(T)$.

Thus, we get that $T = \top^\flat/\Omega(T)$. ■

Next, we show that, if K is a relatively point regular guasivariety, then, for every theory family of $\mathcal{I}^{\mathsf{K},\top}$, the quotient of $\mathcal{F}$ by the Leibniz congruence system of T, belongs to K and, therefore, for every theory family T of $\mathcal{I}^{\mathsf{K},\top}$, the Leibniz congruence system $\Omega(T)$ is a K-congruence system on $\mathcal{F}$.

Lemma 1351 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with a natural transformation* $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* K *a relatively point regular guasivariety of* $\mathbf{F}$*-algebraic systems. Then, for all* $T \in \mathrm{ThFam}(\mathcal{I}^{\mathsf{K},\top})$*,* $\mathcal{F}/\Omega(T) \in \mathsf{K}$*.*

Proof: Suppose that K is a relatively point regular guasivariety of $\mathbf{F}$-algebraic systems and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi_i, \psi_i \in \mathrm{SEN}^\flat(\Sigma)$, $i \in I$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$\langle\{\phi_i \approx \psi_i : i \in I\}, \phi \approx \psi\rangle \in \mathrm{GEq}_\Sigma(\mathsf{K}).$$

This is equivalent to the statement $\langle\phi, \psi\rangle \in \Theta^{\mathsf{K},\mathcal{F}}_\Sigma(\{\langle\phi_i, \psi_i\rangle : i \in I\})$. Since $\Theta^{\mathsf{K},\mathcal{F}}(\{\langle\phi_i, \psi_i\rangle : i \in I\}) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$ and K is relatively point regular, $\Theta^{\mathsf{K},\mathcal{F}}(\{\langle\phi_i, \psi_i\rangle : i \in I\})$ is completely determined by its $\top^\flat$-equivalence class. So it suffices to consider guasiequations of the form

$$\langle\{\phi_i \approx \top^\flat_\Sigma : i \in I\}, \phi \approx \top^\flat_\Sigma\rangle \in \mathrm{GEq}_\Sigma(\mathsf{K}).$$

Now, let $T \in \mathrm{ThFam}(\mathcal{I}^{\mathsf{K},\top})$, such that $\langle\phi_i, \top^\flat_\Sigma\rangle \in \Omega_\Sigma(T)$, for all $i \in I$. Then, taking into account Lemma 1350, $\phi_i \in \top^\flat_\Sigma/\Omega_\Sigma(T) = T_\Sigma$, for all $i \in I$. Therefore, by definition, $\phi_i \approx \top^\flat_\Sigma \in C^{\mathsf{K}}_\Sigma(T \approx \top^\flat)$, for all $i \in I$. Since, by hypothesis, $\langle\{\phi_i \approx \top^\flat_\Sigma : i \in I\}, \phi \approx \top^\flat_\Sigma\rangle \in \mathrm{GEq}_\Sigma(\mathsf{K})$, we get $\phi \approx \top^\flat_\Sigma \in C^{\mathsf{K}}_\Sigma(T \approx \top^\flat)$, i.e., $\phi \in C^{\mathsf{K},\top}_\Sigma(T)$. Since $T \in \mathrm{ThFam}(\mathcal{I}^{\mathsf{K},\top})$, $\phi \in T_\Sigma = \top^\flat_\Sigma/\Omega_\Sigma(T)$. Therefore, $\langle\phi, \top^\flat_\Sigma\rangle \in \Omega_\Sigma(T)$. We conclude that $\mathcal{F}/\Omega(T)$ satisfies all guasiequations of K and, hence, since K is a guasivariety, $\mathcal{F}/\Omega(T) \in \mathsf{K}$. ■

Finally, we establish protoalgebraicity of $\mathcal{I}^{\mathsf{K},\top}$, under the hypotheses that K is a relatively point regular guasivariety of $\mathbf{F}$-algebraic systems.

Proposition 1352 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* K *a relatively point regular guasivariety of* $\mathbf{F}$*-algebraic systems. Then* $\mathcal{I}^{\mathsf{K},\top}$ *is a protoalgebraic* π*-institution.*

Proof: Let K be a relatively point regular guasivariety of $\mathbf{F}$-algebraic systems. We know, by Proposition 1348, that $\mathcal{I}^{\mathsf{K},\top}$ has a natural theorem $\top^\flat$, where $\top^\flat$ is a point in K, and, by Proposition 1349, that $\mathcal{I}^{\mathsf{K},\top}$ is a family regular π-institution.

Now we show that $\mathcal{I}^{\mathsf{K},\top}$ is protoalgebraic. Suppose that $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \le T'$. Then, by Lemma 1350, we get $\top^\flat/\Omega(T) \le \top^\flat/\Omega(T')$. Since, by Lemma 1351, $\Omega(T)$ and $\Omega(T')$ are K-congruence systems on $\mathcal{F}$ and K is relatively point regular, they are completely determined (generated) by their $\top^\flat$-classes and, hence, we get $\Omega(T) \le \Omega(T')$. Thus, $\mathcal{I}^{\mathsf{K},\top}$ is protoalgebraic. ■

We show, next, that, for a protoalgebraic family regular π-institution $\mathcal{I}$, having natural theorems, the assertional π-institution of its class $\mathrm{AlgSys}^*(\mathcal{I})$ of reduced $\mathbf{F}$-algebraic systems coincides with $\mathcal{I}$.

Theorem 1353 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a family regular protoalgebraic* π*-institution based on* $\mathbf{F}$*, having a natural theorem* $\top$*. Then*

$$\mathcal{I}^{\mathrm{AlgSys}^*(\mathcal{I}),\top} = \mathcal{I}.$$

Proof: Set, for brevity in the course of this proof, $\mathsf{K} := \mathrm{AlgSys}^*(\mathcal{I})$. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{lll}\phi \in C^{\mathsf{K},\top}_\Sigma(\Phi) & \text{iff} & \phi \approx \top^\flat_\Sigma \in C^{\mathsf{K}}_\Sigma(\Phi \approx \top^\flat_\Sigma)\\ & \text{iff} & \text{for all } T \in \mathrm{ThFam}(\mathcal{I}), \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma,\Sigma'),\\ & & \mathrm{SEN}^\flat(f)(\Phi) \approx \top^\flat_{\Sigma'} \in \Omega_{\Sigma'}(T)\\ & & \qquad\qquad \text{implies } \mathrm{SEN}^\flat(f)(\phi) \approx \top^\flat_{\Sigma'} \in \Omega_{\Sigma'}(T)\\ & \text{iff} & \text{for all } T \in \mathrm{ThFam}(\mathcal{I}), \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma,\Sigma'),\\ & & \mathrm{SEN}^\flat(f)(\Phi) \in T_{\Sigma'} \text{ implies } \mathrm{SEN}^\flat(f)(\phi) \in T_{\Sigma'}\\ & \text{iff} & \phi \in C_\Sigma(\Phi).\end{array}$$

We conclude that $C^{\mathsf{K},\top} = C$ and, therefore, $\mathcal{I}^{\mathrm{AlgSys}^*(\mathcal{I}),\top} = \mathcal{I}$. ■

Moreover, starting with a relatively point regular guasivariety of $\mathbf{F}$-algebraic systems, the class of all reduced $\mathbf{F}$-algebraic systems of its assertional π-institution coincides with the original class.

Theorem 1354 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* K *a relatively point regular guasivariety of* $\mathbf{F}$*-algebraic systems. Then*

$$\mathrm{AlgSys}^*(\mathcal{I}^{\mathsf{K},\top}) = \mathsf{K}.$$

Proof: Let K be a relatively point regular guasivariety of $\mathbf{F}$-algebraic systems. Assume that $\mathcal{A} \in \mathsf{K}$ and consider $\{\top^\mathcal{A}\} := \{\top^\mathcal{A}_\Sigma\}_{\Sigma\in|\mathbf{Sign}|} \in \mathrm{SenFam}(\mathcal{A})$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C^{\mathsf{K},\top}_\Sigma(\Phi)$ and $\alpha_\Sigma(\Phi) \subseteq \{\top^\mathcal{A}_{F(\Sigma)}\}$, we get, by the definition of $C^{\mathsf{K},\top}$, $\alpha_\Sigma(\phi) = \top^\mathcal{A}_{F(\Sigma)}$. Therefore, $\{\top^\mathcal{A}\} \in \mathrm{ThFam}(\mathcal{I}^{\mathsf{K},\top})$. Moreover, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll}\langle \phi, \top^\mathcal{A}_\Sigma\rangle \in \Omega^\mathcal{A}_\Sigma(\{\top^\mathcal{A}\}) & \text{iff} & \phi = \top^\mathcal{A}_\Sigma \quad \text{(Lemma 1350)}\\ & \text{iff} & \langle \phi, \top^\mathcal{A}_\Sigma\rangle \in \Delta^\mathcal{A}_\Sigma.\end{array}$$

Thus, $\top^\mathcal{A}/\Omega^\mathcal{A}(\{\top^\mathcal{A}\}) = \top^\mathcal{A}/\Delta^\mathcal{A}$. Therefore, by relative point regularity, we obtain $\Omega^\mathcal{A}(\{\top^\mathcal{A}\}) = \Delta^\mathcal{A}$. This yields $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I}^{\mathsf{K},\top})$.

Assume, conversely, that $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I}^{\mathsf{K},\top})$. Then, by definition, there exists $T \in \mathrm{FiFam}^{\mathcal{I}^{\mathsf{K},\top}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. Suppose that $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that

$$\langle \Phi \approx \top^\flat_\Sigma, \phi \approx \top^\flat_\Sigma \rangle \in \mathrm{GEq}_\Sigma(\mathsf{K})$$

and $\alpha_\Sigma(\Phi) \subseteq \{\top^{\mathcal{A}}_{F(\Sigma)}\}$. Then, since $T \in \mathrm{FiFam}^{\mathcal{I}^{\mathsf{K},\top}}(\mathcal{A})$, $\alpha_\Sigma(\Phi) \subseteq T_{F(\Sigma)}$. Hence, $\Phi \subseteq \alpha_\Sigma^{-1}(T_{F(\Sigma)})$. Since $T \in \mathrm{ThFam}^{\mathcal{I}^{\mathsf{K},\top}}(\mathcal{A})$, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I}^{\mathsf{K},\top})$, whence, by Lemma 1350, $\alpha^{-1}(T) = \top^\flat/\Omega(\alpha^{-1}(T))$. Thus, we get $\Phi \subseteq \top^\flat_\Sigma/\Omega_\Sigma(\alpha^{-1}(T))$. Hence, $\Phi \approx \top^\flat_\Sigma \in \Omega_\Sigma(\alpha^{-1}(T))$. By Lemma 1351, $\Omega(\alpha^{-1}(T)) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{F})$, whence, since $\langle \Phi \approx \top^\flat_\Sigma, \phi \approx \top^\flat_\Sigma \rangle \in \mathrm{GEq}_\Sigma(\mathsf{K})$, $\phi \approx \top^\flat_\Sigma \in \Omega_\Sigma(\alpha^{-1}(T))$. By Proposition 24, $\phi \approx \top^\flat_\Sigma \in \alpha_\Sigma^{-1}(\Omega^{\mathcal{A}}_{F(\Sigma)}(T))$, i.e., $\alpha_\Sigma(\phi) \approx \top^{\mathcal{A}}_{F(\Sigma)} \in \Omega^{\mathcal{A}}_{F(\Sigma)}(T) = \Delta^{\mathcal{A}}_{F(\Sigma)}$. Thus, $\alpha_\Sigma(\phi) = \top^{\mathcal{A}}_{F(\Sigma)}$. We conclude that $\langle \Phi \approx \top^\flat_\Sigma, \phi \approx \top^\flat_\Sigma \rangle \in \mathrm{GEq}_\Sigma(\mathcal{A})$. Since $\mathcal{A}$ satisfies all guasiequations in $\mathrm{GEq}(\mathsf{K})$ and K is, by hypothesis, a guasivariety, we get that $\mathcal{A} \in \mathsf{K}$. Therefore, $\mathrm{AlgSys}^*(\mathcal{I}^{\mathsf{K},\top}) = \mathsf{K}$. ■

Now we can formulate the main theorems of the section.

Theorem 1355 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is protoalgebraic family regular, with natural theorems, if and only if it is the assertional* π*-institution of a relatively point regular guasivariety of* $\mathbf{F}$*-algebraic systems.*

More precisely, $\mathcal{I}$ *is protoalgebraic family regular, with natural theorems, if and only if* $\mathrm{AlgSys}^*(\mathcal{I})$ *is a relatively point regular guasivariety and* $\mathcal{I} = \mathcal{I}^{\mathrm{AlgSys}^*(\mathcal{I}),\top}$*, where* $\top^\flat$ *is any natural theorem.*

Proof: Suppose $\mathcal{I}$ is protoalgebraic family regular, with natural theorems. Then, by Proposition 1346, $\mathrm{AlgSys}^*(\mathcal{I})$ is a relatively point regular class of $\mathbf{F}$-algebraic systems and, by protoalgebraicity, Proposition 68 and Theorem **??**, it is a guasivariety. Moreover, by Theorem 1353, $\mathcal{I} = \mathcal{I}^{\mathrm{AlgSys}^*(\mathcal{I}),\top}$.

Assume, conversely, that $\mathcal{I}^{\mathsf{K},\top}$ is the assertional π-institution of a relatively point regular guasivariety K of $\mathbf{F}$-algebraic systems. Then, by Proposition 1349, it is family regular, by Proposition 1352, it is protoalgebraic and, by Proposition 1348, it has natural theorems. ■

Theorem 1356 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ *in* $N^\flat$*. Then, there exists a one to one correspondence between relatively point regular guasivarieties, with point* $\top^\flat$*, and family regular protoalgebraic* π*-institutions, with a natural theorem* $\top^\flat$*.*

Every relatively point regular guasivariety with point $\top^\flat$ *determines a unique family regular protoalgebraic* π*-institution with natural theorems, its assertional* π*-institution.*

Every family regular protoalgebraic π-institution with natural theorems is the assertional π-institution of a unique relatively point regular guasivariety, the guasivariety $\mathrm{AlgSys}^*(\mathcal{I})$ *of all its reduced* $\mathbf{F}$*-algebraic systems.*

For each family regular protoalgebraic π-institution, with a natural theorem $\top^\flat$, we have $\mathcal{I} = \mathcal{I}^{\mathrm{AlgSys}^*(\mathcal{I}),\top}$ and, conversely, for every relatively point regular guasivariety K, with point $\top^\flat$, we have $\mathsf{K} = \mathrm{AlgSys}^*(\mathcal{I}^{\mathsf{K},\top})$.

Proof: This is a recap of Theorems 1353 and 1354. ■

Chapter 19

Full Models of π-Institutions

19.1 π-Structures Revisited

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system. An $N^\flat$-**structure** is a pair $\mathbb{L} = \langle \mathbf{A}, D \rangle$, where $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ is an $N^\flat$-algebraic system and $D : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}$ is a closure (operator) family (not necessarily a system, i.e., not necessarily structural) on $\mathbf{A}$. An $\mathbf{F}$-**structure** is a pair $\mathbb{L} = \langle \mathcal{A}, D \rangle$, where $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ is an $\mathbf{F}$-algebraic system and $D : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}$ is a closure family on $\mathcal{A}$.

We give a condition pinpointing exactly when a closure family is a closure system and, as a consequence, when a π-structure becomes a π-institution.

Proposition 1357 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *be an algebraic system and* $D : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}$ *a closure family on* SEN. *Then* D *is a closure system, if and only if, for all* $\Sigma, \Sigma' \in |\mathbf{Sign}|$ *and all* $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\mathrm{SEN}(f)^{-1}(\mathcal{D}_{\Sigma'}) \subseteq \mathcal{D}_\Sigma$.

Proof: Suppose, first, that D is structural and let $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $X' \subseteq \mathrm{SEN}(\Sigma')$, such that $D_{\Sigma'}(X') = X'$. Then we have

$$\begin{array}{rcl}\mathrm{SEN}(f)(D_\Sigma(\mathrm{SEN}(f)^{-1}(X'))) & \subseteq & D_{\Sigma'}(\mathrm{SEN}(f)(\mathrm{SEN}(f)^{-1}(X'))) \\ & \subseteq & D_{\Sigma'}(X') \\ & = & X'.\end{array}$$

So $D_\Sigma(\mathrm{SEN}(f)^{-1}(X')) \subseteq \mathrm{SEN}(f)^{-1}(X')$ and $\mathrm{SEN}(f)^{-1}(X') \in \mathcal{D}_\Sigma$.

Suppose, conversely, that, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\mathrm{SEN}(f)^{-1}(\mathcal{D}_{\Sigma'}) \subseteq \mathcal{D}_\Sigma$. Let $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathrm{Sign}(\Sigma, \Sigma')$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\phi \in D_\Sigma(\Phi)$. Let $T' \in \mathcal{D}_{\Sigma'}$, such that $\mathrm{SEN}(f)(\Phi) \subseteq T'$. Thus, $\Phi \subseteq \mathrm{SEN}(f)^{-1}(T')$. By hypothesis, $\mathrm{SEN}(f)^{-1}(T') \in \mathcal{D}_\Sigma$, whence, since $\phi \in D_\Sigma(\Phi)$ and $\Phi \subseteq \mathrm{SEN}(f)^{-1}(T')$, we get that $\phi \in \mathrm{SEN}(f)^{-1}(T')$ and, therefore, $\mathrm{SEN}(f)(\phi) \in T'$. We conclude that $\mathrm{SEN}(f)(\phi) \in D_{\Sigma'}(\mathrm{SEN}(f)(\Phi))$. Thus, D is a structural closure family on SEN. ■

Let $\mathbb{L} = \langle \mathbf{A}, D \rangle$ and $\mathbb{L}' = \langle \mathbf{A}', D' \rangle$ be two $N^\flat$-structures, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ and $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$, and consider an $N^\flat$-algebraic system morphism $\langle F, \alpha \rangle : \mathbf{A} \to \mathbf{A}'$. We say that

> $\langle F, \alpha \rangle$ is a **logical morphism from $\mathbb{L}$ to $\mathbb{L}'$** , denoted $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$, if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,
>
> $$\phi \in D_\Sigma(\Phi) \quad \text{implies} \quad \alpha_\Sigma(\phi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)),$$
>
> or, equivalently, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)$,
>
> $$\alpha_\Sigma(D_\Sigma(\Phi)) \subseteq D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

Proposition 1358 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system,* $\mathbb{L} = \langle \mathbf{A}, D \rangle$ *and* $\mathbb{L}' = \langle \mathbf{A}', D' \rangle$ *be two* $N^\flat$*-structures, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *and* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$, *and consider an* $N^\flat$*-algebraic system morphism* $\langle F, \alpha \rangle : \mathbf{A} \to \mathbf{A}'$. $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ *is a logical morphism if and only if, for every* $T' \in \mathrm{ThFam}(\mathbb{L}')$, $\alpha^{-1}(T') \in \mathrm{ThFam}(\mathbb{L})$.

Proof: Suppose, first, that $\langle F,\alpha\rangle : \mathbb{L}\rangle\!\!-\mathbb{L}'$ is a logical morphism and let $T' \in \mathrm{ThFam}(\mathbb{L}')$, $\Sigma \in |\mathbf{Sign}|$, $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in D_\Sigma(\alpha_\Sigma^{-1}(T'_{F(\Sigma)}))$. Then, we have

$$\begin{array}{rcl}\alpha_\Sigma(\phi) & \in & \alpha_\Sigma(D_\Sigma(\alpha_\Sigma^{-1}(T'_{F(\Sigma)})))\\ & \subseteq & D'_{F(\Sigma)}(\alpha_\Sigma(\alpha_\Sigma^{-1}(T'_{F(\Sigma)})))\\ & \subseteq & D'_{F(\Sigma)}(T'_{F(\Sigma)})\\ & = & T'_{F(\Sigma)}.\end{array}$$

Therefore, $\phi \in \alpha_\Sigma^{-1}(T'_{F(\Sigma)})$ and we conclude that $\alpha^{-1}(T') \in \mathrm{ThFam}(\mathbb{L})$.

Suppose, conversely, that, for every $T' \in \mathrm{ThFam}(\mathbb{L}')$, we have $\alpha^{-1}(T') \in \mathrm{ThFam}(\mathbb{L})$ and let $\Sigma \in |\mathbf{Sign}|$, $\Phi\cup\{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\phi \in D_\Sigma(\Phi)$. Then, for all $T' \in \mathrm{ThFam}(\mathbb{L}')$, such that $\alpha_\Sigma(\Phi) \subseteq T'_{F(\Sigma)}$, we get $\Phi \subseteq \alpha_\Sigma^{-1}(T'_{F(\Sigma)})$. Since $\phi \in D_\Sigma(\Phi)$ and, by hypothesis, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathbb{L})$, we get $\phi \in \alpha_\Sigma^{-1}(T'_{F(\Sigma)})$. Hence, $\alpha_\Sigma(\phi) \in T'_{F(\Sigma)}$. Since $T' \in \mathrm{ThFam}(\mathbb{L}')$ was arbitrary, we conclude that $\alpha_\Sigma(\phi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$. Thus, $\langle F,\alpha\rangle : \mathbb{L}\rangle\!\!-\mathbb{L}'$ is a logical morphism. ∎

Let $\mathbb{L} = \langle \mathbf{A}, D\rangle$ and $\mathbb{L}' = \langle \mathbf{A}', D'\rangle$ be two $N^\flat$-structures, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ and $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$, and consider an $N^\flat$-algebraic system morphism $\langle F,\alpha\rangle : \mathbf{A} \to \mathbf{A}'$. We say that

> $\langle F,\alpha\rangle$ is a **bilogical morphism from $\mathbb{L}$ to $\mathbb{L}'$** , denoted $\langle F,\alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$, if $\langle F,\alpha\rangle : \mathbf{A} \to \mathbf{A}'$ is surjective and, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,
>
> $$\phi \in D_\Sigma(\Phi) \quad \text{iff} \quad \alpha_\Sigma(\phi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)),$$
>
> or, equivalently, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)$,
>
> $$\alpha_\Sigma(D_\Sigma(\Phi)) = D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

Proposition 1359 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *and* $\mathbb{L}' = \langle \mathbf{A}', D'\rangle$ *be two* $N^\flat$*-structures, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *and* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$*, and consider a surjective* $N^\flat$*-algebraic system morphism* $\langle F,\alpha\rangle : \mathbf{A} \to \mathbf{A}'$. $\langle F,\alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$ *is a bilogical morphism if and only if* $\mathrm{ThFam}(\mathbb{L}) = \alpha^{-1}(\mathrm{ThFam}(\mathbb{L}'))$.

Proof: Suppose, first, that $\langle F,\alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$ is a bilogical morphism. Then, by Proposition 1358, if $T' \in \mathrm{ThFam}(\mathbb{L}')$, then $\alpha^{-1}(T') \in \mathrm{ThFam}(\mathbb{L})$, whence $\alpha^{-1}(\mathrm{ThFam}(\mathbb{L}')) \subseteq \mathrm{ThFam}(\mathbb{L})$. To show the converse, suppose that $T \in \mathrm{ThFam}(\mathbb{L})$. For every $\Sigma' \in |\mathbf{Sign}'|$, choose $\Sigma \in |\mathbf{Sign}|$, such that $F(\Sigma) = \Sigma'$ and let $T'_{\Sigma'} = D'_{\Sigma'}(\alpha_\Sigma(T_\Sigma))$. Then set

$$T' = \{T'_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}'|}.$$

Clearly, $T' \in \mathrm{ThFam}(\mathbb{L}')$ and, since $\langle F,\alpha\rangle$ is a bilogical morphism, $\alpha^{-1}(T') \in \mathrm{ThFam}(\mathbb{L})$. But we also have. for all $\Sigma \in |\mathbf{Sign}|$,

$$\alpha_\Sigma^{-1}(T'_{F(\Sigma)}) = \alpha_\Sigma^{-1}(D'_{F(\Sigma)}(\alpha_\Sigma(T_\Sigma))) = D_\Sigma(T_\Sigma) = T_\Sigma.$$

Therefore, we conclude that $\mathrm{ThFam}(\mathbb{L}) \subseteq \alpha^{-1}(\mathrm{ThFam}(\mathbb{L}'))$.

Suppose, conversely, that $\mathrm{ThFam}(\mathbb{L}) = \alpha^{-1}(\mathrm{ThFam}(\mathbb{L}'))$ and let $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$. We have $\phi \in D_\Sigma(\Phi)$ iff, for all $T \in \mathrm{ThFam}(\mathbb{L})$,

$$\Phi \subseteq T_\Sigma \quad \text{implies} \quad \phi \in T_\Sigma,$$

iff, for all $T' \in \mathrm{ThFam}(\mathbb{L}')$,

$$\Phi \subseteq \alpha_\Sigma^{-1}(T'_{F(\Sigma)}) \quad \text{implies} \quad \phi \in \alpha_\Sigma^{-1}(T'_{F(\Sigma)}),$$

iff, for all $T' \in \mathrm{ThFam}(\mathbb{L}')$,

$$\alpha_\Sigma(\Phi) \subseteq T'_{F(\Sigma)} \quad \text{implies} \quad \alpha_\Sigma(\phi) \in T'_{F(\Sigma)},$$

iff $\alpha_\Sigma(\phi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$. Therefore, $\langle F,\alpha\rangle$ is a bilogical morphism. ∎

Let $\mathbb{L} = \langle \mathbf{A}, D\rangle$ and $\mathbb{L}' = \langle \mathbf{A}', D'\rangle$ be two $N^\flat$-structures, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ and $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$, and consider an $N^\flat$-algebraic system morphism $\langle F,\alpha\rangle : \mathbf{A} \to \mathbf{A}'$. We say that

> $\langle F,\alpha\rangle$ is an **α-isomorphism from $\mathbb{L}$ to $\mathbb{L}'$** , denoted $\langle F,\alpha\rangle : \mathbb{L} \vdash^\alpha \mathbb{L}'$, if it is a bilogical morphism $\langle F,\alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$, such that, for all $\Sigma \in |\mathbf{Sign}|$, $\alpha_\Sigma : \mathrm{SEN}(\Sigma) \to \mathrm{SEN}'(F(\Sigma))$ is a bijection.

Finally, $\langle F,\alpha\rangle : \mathbb{L} \to \mathbb{L}'$ is an **isomorphism**, denoted $\langle F,\alpha\rangle : \mathbb{L} \cong \mathbb{L}'$, if it is an α-isomorphism and $F : \mathbf{Sign} \to \mathbf{Sign}'$ is also an isomorphism.

In most instances, when a result holds for $F : \mathbf{Sign} \to \mathbf{Sign}'$ an isomorphism, we will formulate it, for simplicity, for the identity functor $I_{\mathbf{Sign}} : \mathbf{Sign} \to \mathbf{Sign}$, which will be sufficient for most of our purposes.

The following is an important characterization result for bilogical morphisms containing many equivalent formulations.

Given an $N^\flat$-structure $\mathbb{L} = \langle \mathbf{A}, D\rangle$, a congruence system $\theta \in \mathrm{ConSys}(\mathbf{A})$ is called a **logical congruence system of** $\mathbb{L}$ if it is compatible with every theory family of $\mathbb{L}$, i.e., if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi,\psi \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi,\psi\rangle \in \theta_\Sigma \quad \text{implies} \quad D_\Sigma(\phi) = D_\Sigma(\psi).$$

If this is the case, we write $\theta \in \mathrm{ConSys}(\mathbb{L})$.

Proposition 1360 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *and* $\mathbb{L}' = \langle \mathbf{A}', D'\rangle$ *be two* $N^\flat$*-structures, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *and* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$*, and consider a surjective* $N^\flat$*-algebraic system morphism* $\langle F,\alpha\rangle : \mathbf{A} \to \mathbf{A}'$*. Then the following are equivalent:*

(i) $\langle F,\alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$ *is a bilogical morphism;*

(ii) *For all* $\Sigma \in |\mathbf{Sign}|$, $\Phi \subseteq \mathrm{SEN}(\Sigma)$, $D_\Sigma(\Phi) = \alpha_\Sigma^{-1}(D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)))$*;*

(iii) *For all* $\Sigma \in |\mathbf{Sign}|$, $\Phi \subseteq \mathrm{SEN}(\Sigma)$, $\alpha_\Sigma(D_\Sigma(\Phi)) = D'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$ *and* $\mathrm{Ker}(\langle F,\alpha\rangle) \in \mathrm{ConSys}(\mathbb{L})$*;*

(iv) *For all* $\Sigma \in |\mathbf{Sign}|$, $\Psi \subseteq \mathrm{SEN}'(F(\Sigma))$, $D'_{F(\Sigma)}(\Psi) = \alpha_\Sigma(D_\Sigma(\alpha_\Sigma^{-1}(\Psi)))$ *and* $\mathrm{Ker}(\langle F,\alpha\rangle) \in \mathrm{ConSys}(\mathbb{L})$*;*

(v) *For all* $\Sigma \in |\mathbf{Sign}|$, $\mathrm{Th}_{F(\Sigma)}(\mathbb{L}') = \alpha_\Sigma(\mathrm{Th}_\Sigma(\mathbb{L}))$ *and, also,* $\mathrm{Ker}(\langle F,\alpha\rangle) \in \mathrm{ConSys}(\mathbb{L})$*;*

(vi) $\mathrm{ThFam}(\mathbb{L}) = \alpha^{-1}(\mathrm{ThFam}(\mathbb{L}'))$.

Proof:

(i)⇒(ii) Suppose $\langle F,\alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$ and let $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$. Then, we have

$$\begin{array}{lll}\phi \in D_\Sigma(\Phi) & \text{iff} & \alpha_\Sigma(\phi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \\ & \text{iff} & \phi \in \alpha_\Sigma^{-1}(D'_{F(\Sigma)}(\alpha_\Sigma(\Phi))).\end{array}$$

We conclude that $D_\Sigma(\Phi) = \alpha_\Sigma^{-1}(D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)))$.

(ii)⇒(iii) Let $\Sigma \in |\mathbf{Sign}|$ and $\Phi \subseteq \mathrm{SEN}(\Sigma)$. Then, by the hypothesis (ii), $D_\Sigma(\Phi) = \alpha_\Sigma^{-1}(D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)))$, whence, by surjectivity of $\langle F,\alpha\rangle$, $\alpha_\Sigma(D_\Sigma(\Phi)) = D'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$. For the second claim, suppose $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. Then

$$\alpha_\Sigma^{-1}(D'_{F(\Sigma)}(\alpha_\Sigma(\phi))) = \alpha_\Sigma^{-1}(D'_{F(\Sigma)}(\alpha_\Sigma(\psi))).$$

Thus, by hypothesis, $D_\Sigma(\phi) = D_\Sigma(\psi)$. It follows that $\mathrm{Ker}(\langle F,\alpha\rangle)$ is a logical congruence system of $\mathbb{L}$.

(iii)⇒(iv) Let $\Sigma \in |\mathbf{Sign}|$ and $\Psi \in \mathrm{SEN}'(F(\Sigma))$. Then we have

$$\begin{array}{rcl}D'_{F(\Sigma)}(\Psi) & = & D'_{F(\Sigma)}(\alpha_\Sigma(\alpha_\Sigma^{-1}(\Psi))) \\ & = & \alpha_\Sigma(D_\Sigma(\alpha_\Sigma^{-1}(\Psi))).\end{array}$$

(iv)⇒(v) Let $\Sigma \in |\mathbf{Sign}|$ and assume, first, that $T' \in \mathrm{Th}_{F(\Sigma)}(\mathbb{L}')$. Then, we have

$$T' = D'_{F(\Sigma)}(T') = \alpha_\Sigma(D_\Sigma(\alpha_\Sigma^{-1}(T'))) \in \alpha_\Sigma(\mathrm{Th}_\Sigma(\mathbb{L})).$$

Suppose, conversely, that $T \in \mathrm{Th}_\Sigma(\mathbb{L})$. Then, we have

$$\begin{array}{rcl}D'_{F(\Sigma)}(\alpha_\Sigma(T)) & = & \alpha_\Sigma(D_\Sigma(\alpha_\Sigma^{-1}(\alpha_\Sigma(T)))) \\ & = & \alpha_\Sigma(D_\Sigma(T)) \\ & = & \alpha_\Sigma(T).\end{array}$$

Therefore, $\alpha_\Sigma(T) \in \mathrm{Th}_{F(\Sigma)}(\mathbb{L}')$.

(v)⇒(vi) It suffices to show that, for all $\Sigma \in |\mathbf{Sign}|$, $\mathrm{Th}_\Sigma(\mathbb{L}) = \alpha_\Sigma^{-1}(\mathrm{Th}_{F(\Sigma)}(\mathbb{L}'))$.

Suppose, first, $T \in \mathrm{Th}_\Sigma(\mathbb{L})$. Then, by hypothesis, $\alpha_\Sigma(T) \in \mathrm{Th}_{F(\Sigma)}(\mathbb{L}')$. But, since $\mathrm{Ker}(\langle F,\alpha\rangle) \in \mathrm{ConSys}(\mathbb{L})$, we now get

$$T = \alpha_\Sigma^{-1}(\alpha_\Sigma(T)) \in \alpha_\Sigma^{-1}(\mathrm{Th}_{F(\Sigma)}(\mathbb{L}')).$$

Therefore, $\mathrm{Th}_\Sigma(\mathbb{L}) \subseteq \alpha_\Sigma^{-1}(\mathrm{Th}_{F(\Sigma)}(\mathbb{L}'))$.

Suppose, conversely, $T' \in \mathrm{Th}_{F(\Sigma)}(\mathbb{L}')$. Then, by hypothesis, there exists $T \in \mathrm{Th}_\Sigma(\mathbb{L})$, such that $T' = \alpha_\Sigma(T)$. Thus, since $\mathrm{Ker}(\langle F,\alpha\rangle) \in \mathrm{ConSys}(\mathbb{L})$, we now get

$$\alpha_\Sigma^{-1}(T') = \alpha_\Sigma^{-1}(\alpha_\Sigma(T)) = T \in \mathrm{Th}_\Sigma(\mathbb{L}).$$

We conclude that $\alpha_\Sigma^{-1}(\mathrm{Th}_{F(\Sigma)}(\mathbb{L}')) \subseteq \mathrm{Th}_\Sigma(\mathbb{L})$ and, hence, $\mathrm{ThFam}(\mathbb{L}) = \alpha^{-1}(\mathrm{ThFam}(\mathbb{L}'))$.

(vi)⇒(i) This is one part of Proposition 1359. ∎

A consequence of the preceding characterization is that, when the categories of signatures of the $N^\flat$-structures that are connected via a bilogical morphism are isomorphic, then the complete lattices of their theory families are order isomorphic.

Proposition 1361 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *and* $\mathbb{L}' = \langle \mathbf{A}', D'\rangle$ *be two* $N^\flat$*-structures, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *and* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *and consider a surjective* $N^\flat$*-algebraic system morphism* $\langle F,\alpha\rangle : \mathbf{A} \to \mathbf{A}'$*. Then* $\langle F,\alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$ *is a bilogical morphism if and only if, for all* $\Sigma \in |\mathbf{Sign}|$*,* $\alpha_\Sigma : \mathbf{Th}_\Sigma(\mathbb{L}) \to \mathbf{Th}_{F(\Sigma)}(\mathbb{L}')$ *is an order isomorphism.*

Proof: First, by Part (v) of Proposition 1360, α is a well defined surjection from $\mathrm{Th}_\Sigma(\mathbb{L})$ onto $\mathrm{Th}_{F(\Sigma)}(\mathbb{L}')$. Second, by Part (ii) of Proposition 1360, it is an injection. Therefore, it is a bijection, whose inverse, also by Part (ii) of Proposition 1360, is α_Σ^{-1}. That both α_Σ and α_Σ^{-1} are order preserving is straightforward.

Conversely, note that Part (vi) of Proposition 1360 is automatically satisfied in case $\alpha_\Sigma : \mathbf{Th}_\Sigma(\mathbb{L}) \to \mathbf{Th}_{F(\Sigma)}(\mathbb{L}')$ is an order isomorphism, for all $\Sigma \in |\mathbf{Sign}|$. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system and consider an $N^\flat$-algebraic system $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$. Extending the concept and notation from the framework of closure systems and π-institutions, given two closure families D and D' on $\mathbf{A}$ and corresponding $N^\flat$-structures $\mathbb{L} = \langle \mathbf{A}, D\rangle$ and $\mathbb{L}' = \langle \mathbf{A}, D'\rangle$, we write $D \le D'$ and $\mathbb{L} \le \mathbb{L}'$ to signify that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)$,

$$D_\Sigma(\Phi) \subseteq D'_\Sigma(\Phi).$$

Under this ordering, the collection of all closure families on the algebraic system $\mathbf{A}$ forms a complete lattice, which will be denoted by

$$\mathbf{ClFam}(\mathbf{A}) = \langle \mathrm{ClFam}(\mathbf{A}), \leq \rangle.$$

Given $D \in \mathrm{ClFam}(\mathbf{A})$ and corresponding $N^\flat$-structure $\mathbb{L} = \langle \mathbf{A}, D\rangle$, we write $\mathrm{ClFam}(\mathbb{L}) = \mathrm{ClFam}^D(\mathbf{A})$ to denote the principal filter of $\mathbf{ClFam}(\mathbf{A})$ generated by D, i.e., we set

$$\mathrm{ClFam}(\mathbb{L}) = \{D' \in \mathrm{ClFam}(\mathbf{A}) : D \leq D'\}.$$

Then, we have the following corollary, expressed partially in terms of the closed set families corresponding in the standard way with closure operator families.

Corollary 1362 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system, $\mathbb{L} = \langle \mathbf{A}, D\rangle$ and $\mathbb{L}' = \langle \mathbf{A}', D'\rangle$ be two $N^\flat$-structures, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ and $\mathbf{A}' = \langle \mathbf{Sign}, \mathrm{SEN}', N'\rangle$. If $\langle I, \alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$, where $I : \mathbf{Sign} \to \mathbf{Sign}$ is the identity functor, is a bilogical morphism, then*

$$\mathcal{T} \mapsto \alpha(\mathcal{T}) := \{\alpha(T) : T \in \mathcal{T}\}$$

is an isomorphism between $\mathbf{ClFam}(\mathbb{L})$ and $\mathbf{ClFam}(\mathbb{L}')$.

Proof: Directly from Proposition 1361. ∎

Recall that given an algebraic system $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ and a closure family $\mathcal{D}$ on $\mathbf{A}$, with corresponding $N^\flat$-structure $\mathbb{L} = \langle \mathbf{A}, \mathcal{D}\rangle$, and $T \in \mathrm{ThFam}(\mathbb{L})$, we denote by $\mathbb{L}^T = \langle \mathbf{A}, \mathcal{D}^T\rangle$ the $N^\flat$-structure whose theory families are those closure families of $\mathbb{L}$ that contain T. Moreover, we denote by $\widetilde{\Omega}(\mathbb{L}^T)$ or $\widetilde{\Omega}^{\mathbf{A}}(\mathcal{D}^T)$ the Tarski congruence system of $\mathbb{L}^T$, i.e., the large congruence system on $\mathbf{A}$ compatible with all theory families in $\mathcal{D}^T$.

Connecting Tarski congruence systems and bilogical morphisms, we obtain the following:

Proposition 1363 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system, $\mathbb{L} = \langle \mathbf{A}, D\rangle$ and $\mathbb{L}' = \langle \mathbf{A}', D'\rangle$ be two $N^\flat$-structures, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ and $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$. If $\langle F, \alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$ is a bilogical morphism, then, for all $T' \in \mathrm{ThFam}(\mathbb{L}')$,*

$$\alpha^{-1}(\widetilde{\Omega}(\mathbb{L}'^{T'})) = \widetilde{\Omega}(\mathbb{L}^{\alpha^{-1}(T')}).$$

Proof: We have

$$\begin{array}{rcl}
\alpha^{-1}(\widetilde{\Omega}(\mathbb{L}'^{T'})) & = & \alpha^{-1}(\bigcap\{\Omega^{\mathbf{A}'}(T'') : T' \leq T'' \in \mathrm{ThFam}(\mathbb{L}')\}) \\
& = & \bigcap\{\alpha^{-1}(\Omega^{\mathbf{A}'}(T'')) : T' \leq T'' \in \mathrm{ThFam}(\mathbb{L}')\}) \\
& = & \bigcap\{\Omega^{\mathbf{A}}(\alpha^{-1}(T'')) : T' \leq T'' \in \mathrm{ThFam}(\mathbb{L}')\}) \\
& = & \bigcap\{\Omega^{\mathbf{A}}(T) : \alpha^{-1}(T') \leq T \in \mathrm{ThFam}(\mathbb{L})\}) \\
& = & \widetilde{\Omega}(\mathbb{L}^{\alpha^{-1}(T')}).
\end{array}$$

∎

In particular, we obtain

Corollary 1364 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system,* $\mathbb{L} = \langle \mathbf{A}, D \rangle$ *and* $\mathbb{L}' = \langle \mathbf{A}', D' \rangle$ *be two* $N^\flat$*-structures, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *and* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$*. If* $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ *is a bilogical morphism, then*

$$\alpha^{-1}(\widetilde{\Omega}(\mathbb{L}')) = \widetilde{\Omega}(\mathbb{L}).$$

Proof: By Proposition 1361, we have that $\alpha^{-1}(\mathrm{Thm}(\mathbb{L}')) = \mathrm{Thm}(\mathbb{L})$. So the result follows by applying Proposition 1363 with $T' = \mathrm{Thm}(\mathbb{L}')$. ∎

We close the section by proving that two important properties of $N^\flat$-structures are preserved under bilogical morphisms.

First, we show that finitarity is preserved across bilogical morphisms. Given a base algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$, an $N^\flat$-algebraic system $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ and an $N^\flat$-structure $\mathbb{L} = \langle \mathbf{A}, D \rangle$, we say that $\mathbb{L}$ is **finitary** if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)$,

$$D_\Sigma(\Phi) = \bigcup \{D_\Sigma(\Psi) : \Psi \subseteq_f \Phi\},$$

where $\subseteq_f$ denoted the finite subset relation.

Proposition 1365 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ $N^\flat$*-algebraic systems,* $\mathbb{L} = \langle \mathbf{A}, D \rangle$*,* $\mathbb{L}' = \langle \mathbf{A}', D' \rangle$ $N^\flat$*-structures, based on* $\mathbf{A}$*,* $\mathbf{A}'$*, respectively, and* $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ *a bilogical morphism. Then* $\mathbb{L}$ *is finitary if and only if* $\mathbb{L}'$ *is finitary.*

Proof: Since $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ is a bilogical morphism, it is surjective and, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\phi \in D_\Sigma(\Phi) \quad \text{iff} \quad \alpha_\Sigma(\phi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

Now we have $\mathbb{L}$ finitary iff, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\phi \in D_\Sigma(\Phi) \quad \text{implies} \quad \phi \in D_\Sigma(\Psi), \text{ some } \Psi \subseteq_f \Phi,$$

iff, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\alpha_\Sigma(\phi) \subseteq D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \quad \text{implies} \quad \alpha_\Sigma(\phi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Psi)), \text{ some } \Psi \subseteq_f \Phi,$$

which, taking into account the surjectivity of $\langle F, \alpha \rangle$, is equivalent to $\mathbb{L}'$ being finitary. ∎

Finally, we show that structurality is also preserved by bilogical morphisms.

Proposition 1366 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ $N^\flat$*-algebraic systems,* $\mathbb{L} = \langle \mathbf{A}, D \rangle$*,* $\mathbb{L}' = \langle \mathbf{A}', D' \rangle$ $N^\flat$*-structures, based on* $\mathbf{A}$*,* $\mathbf{A}'$*, respectively, and* $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ *a bilogical morphism. Then* D *is structural if and only if* D' *is structural.*

Proof: Suppose, first, that D is structural. We will use Proposition 1357. Consider $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $T' \in \mathcal{D}'_{F(\Sigma')}$. Then we have

$$\begin{array}{rcl}
D'_{F(\Sigma)}(\mathrm{SEN}'(F(f))^{-1}(T')) & = & \alpha_\Sigma(D_\Sigma(\alpha_\Sigma^{-1}(\mathrm{SEN}'(F(f))^{-1}(T')))) \\
& & \text{(Proposition 1360)}
\end{array}$$

$$\begin{array}{ccc}
\mathrm{SEN}(\Sigma) & \xrightarrow{\alpha_\Sigma} & \mathrm{SEN}'(F(\Sigma)) \\
{\scriptstyle \mathrm{SEN}(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{SEN}'(F(f))} \\
\mathrm{SEN}(\Sigma') & \xrightarrow[\alpha_{\Sigma'}]{} & \mathrm{SEN}'(F(\Sigma'))
\end{array}$$

$$\begin{array}{rl}
= & \alpha_\Sigma(D_\Sigma(\mathrm{SEN}(f)^{-1}(\alpha_{\Sigma'}^{-1}(T')))) \\
& \text{(Commutativity of Rectangle)} \\
= & \alpha_\Sigma(\mathrm{SEN}(f)^{-1}(\alpha_{\Sigma'}^{-1}(T'))) \\
& \text{(Propositions 1360 and 1357)} \\
= & \alpha_\Sigma(\alpha_{\Sigma'}^{-1}(\mathrm{SEN}'(F(f))^{-1}(T'))) \\
& \text{(Commutativity of Rectangle)} \\
= & \mathrm{SEN}'(F(f))^{-1}(T'). \\
& \text{(Surjectivity of } \langle F, \alpha\rangle)
\end{array}$$

By Proposition 1357 and taking into account the surjectivity of $\langle F, \alpha\rangle$, we conclude that D' is structural.

Assume, conversely, that D' is structural. Let $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $T \in \mathcal{D}_{\Sigma'}$. Then, there exists, by Proposition 1360, $T' \in \mathcal{D}'_{F(\Sigma')}$, such that $T = \alpha_{\Sigma'}^{-1}(T')$. So we have

$$\begin{array}{rcl}
D_\Sigma(\mathrm{SEN}(f)^{-1}(T)) & = & D_\Sigma(\mathrm{SEN}(f)^{-1}(\alpha_{\Sigma'}^{-1}(T'))) \\
& = & D_\Sigma(\alpha_\Sigma^{-1}(\mathrm{SEN}'(F(f))^{-1}(T'))) \\
& & \text{(Commutativity of Rectangle)} \\
& = & \alpha_\Sigma^{-1}(\mathrm{SEN}'(F(f))^{-1}(T')) \\
& & \text{(Propositions 1357 and 1360)} \\
& = & \mathrm{SEN}(f)^{-1}(\alpha_{\Sigma'}^{-1}(T')) \\
& & \text{(Commutativity of Rectangle)} \\
& = & \mathrm{SEN}(f)^{-1}(T).
\end{array}$$

We conclude, using Proposition 1357, that D is structural. ∎

19.2 Quotients and Morphisms

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system and $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an $N^\flat$-algebraic system. Given a congruence system $\theta \in \mathrm{ConSys}(\mathbf{A})$, we may define the quotient $\mathbf{A}^\theta := \mathbf{A}/\theta$ and the quotient morphism $\langle I, \pi^\theta\rangle : \mathbf{A} \to \mathbf{A}^\theta$. Moreover, given an $N^\flat$-structure $\mathbb{L} = \langle \mathbf{A}, D\rangle$, we define on the quotient $\mathbf{A}^\theta$ the closure family $D^\theta : \mathcal{P}\mathrm{SEN}^\theta \to \mathcal{P}\mathrm{SEN}^\theta$ by stipulating that its corresponding closure family $\mathcal{D}^\theta \subseteq \mathcal{P}\mathrm{SEN}^\theta$ is given by

$$\mathcal{D}^\theta := \{T \in \mathrm{SenFam}(\mathbf{A}^\theta) : (\pi^\theta)^{-1}(T) \in \mathcal{D}\}.$$

It is not difficult to see that $\mathcal{D}^\theta$ is indeed a closure family on $\mathbf{A}^\theta$. Indeed, for all $T^i \in \mathcal{D}^\theta$, $i \in I$, we have

$$(\pi^\theta)^{-1}(\bigcap_{i\in I} T^i) = \bigcap_{i\in I} (\pi^\theta)^{-1}(T^i) \in \mathcal{D},$$

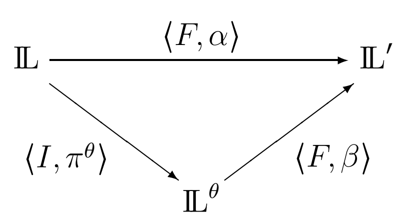
since $\mathcal{D}$ is, by hypothesis, a closure family on $\mathbf{A}$. The $N^\flat$-structure $\mathbb{L}^\theta = \langle \mathbf{A}^\theta, D^\theta\rangle$ is called the **quotient of** $\mathbb{L}$ **by** θ.

Consider, again, the quotient morphism $\langle I, \pi^\theta\rangle : \mathbf{A} \to \mathbf{A}^\theta$. It is not difficult to see either that $\langle I, \pi^\theta\rangle : \mathbb{L} \succ \mathbb{L}^\theta$ is a logical morphism. This simply follows from the definition of $\mathbb{L}^\theta$ and the characterization in Proposition 1358. This logical morphism is also termed the **quotient morphism** from $\mathbb{L}$ onto $\mathbb{L}^\theta$.

Suppose, now, that, in addition to being a congruence system on $\mathbf{A}$, θ is a logical congruence system of $\mathbb{L}$, $\theta \in \mathrm{ConSys}(\mathbb{L})$. An equivalent formalization is to say that $\theta \leq \widetilde{\Omega}(\mathbb{L})$. This hypothesis ensures that $\mathcal{D}^\theta = \pi^\theta(\mathcal{D})$ and that, moreover, $\mathrm{Ker}(\langle I, \pi^\theta\rangle) = \theta \in \mathrm{ConSys}(\mathbb{L})$. Therefore, by Part (v) of Proposition 1360, the quotient morphism $\langle I, \pi^\theta\rangle : \mathbb{L} \to \mathbb{L}^\theta$ becomes a bilogical morphism.

Having behind us this short introduction, we proceed to formulate and prove the Morphism Theorems, which correspond for $N^\flat$-structures to the Homomorphism, Second Isomorphism and Correspondence Theorems of Universal Algebra in forms reminiscent of the versions applicable in the context of abstract logics of abstract algebraic logic.

Theorem 1367 (Morphism Theorem) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *two* $N^\flat$*-algebraic systems, and* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *and* $\mathbb{L}' = \langle \mathbf{A}', D'\rangle$ *two* $N^\flat$*-structures. If* $\langle F, \alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$ *is a bilogical morphism, with* $\theta = \mathrm{Ker}(\langle F, \alpha\rangle)$*, then there exists an* α*-isomorphism* $\langle F, \beta\rangle : \mathbb{L}^\theta \vdash^\alpha \mathbb{L}'$*, that makes the following diagram commute*



Proof: Since $\langle F, \alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$ is a bilogical morphism, we get that $\theta = \mathrm{Ker}(\langle F, \alpha\rangle)$ is a congruence system of $\mathbb{L}$. Thus, $\langle I, \pi^\theta\rangle : \mathbb{L} \vdash \mathbb{L}^\theta$ is also a bilogical morphism. Define $\langle F,\beta\rangle : \mathbf{A}^\theta \to \mathbf{A}'$ by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\beta_\Sigma(\phi/\theta_\Sigma) = \alpha_\Sigma(\phi).$$

First, $\langle F,\beta\rangle$ is well-defined: This is straightforward, since, if $\langle\phi,\psi\rangle \in \theta_\Sigma = \mathrm{Ker}_\Sigma(\langle F,\alpha\rangle)$, then, by definition, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$.

Second, $\beta : \mathrm{SEN}^\theta \to \mathrm{SEN}' \circ F$ is natural: We have, for all $\Sigma,\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma,\Sigma')$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{ccc}
\mathrm{SEN}^\theta(\Sigma) & \xrightarrow{\beta_\Sigma} & \mathrm{SEN}'(F(\Sigma)) \\
{\scriptstyle \mathrm{SEN}^\theta(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{SEN}'(F(f))} \\
\mathrm{SEN}^\theta(\Sigma') & \xrightarrow[\beta_{\Sigma'}]{} & \mathrm{SEN}'(F(\Sigma'))
\end{array}$$

$$\begin{array}{lll}
\mathrm{SEN}'(F(f))(\beta_\Sigma(\phi/\theta_\Sigma)) & = & \mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi)) \\
& = & \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \\
& = & \beta_{\Sigma'}(\mathrm{SEN}(f)(\phi)/\theta_{\Sigma'}) \\
& = & \beta_{\Sigma'}(\mathrm{SEN}^\theta(f)(\phi/\theta_\Sigma)).
\end{array}$$

Third, $\langle F,\beta\rangle : \mathbf{A}^\theta \to \mathbf{A}'$ is surjective: This is also clear, based on the fact that $\langle F,\alpha\rangle : \mathbf{A} \to \mathbf{A}'$ is surjective.

Fourth, $\langle F,\beta\rangle : \mathbb{L}^\theta \vdash \mathbb{L}'$ is bilogical: Since surjectivity was pointed out above, we only have to show Part (iii) of Proposition 1360. First, note that $\mathrm{Ker}(\langle F,\beta\rangle) \in \mathrm{ConSys}(\mathbb{L}^\theta)$, since, for all $\Sigma \in |\mathbf{Sign}|$, $\phi,\psi \in \mathrm{SEN}(\Sigma)$, if $\langle\phi/\theta_\Sigma,\psi/\theta_\Sigma\rangle \in \mathrm{Ker}_\Sigma(\langle F,\beta\rangle)$, then $\beta_\Sigma(\phi/\theta_\Sigma) = \beta_\Sigma(\psi/\theta_\Sigma)$, whence $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. Thus, since $\mathrm{Ker}(\langle F,\alpha\rangle) \in \mathrm{ConSys}(\mathbb{L})$, $D_\Sigma(\phi) = D_\Sigma(\psi)$ and, hence, $D^\theta_\Sigma(\phi/\theta_\Sigma) = D^\theta_\Sigma(\psi/\theta_\Sigma)$. This proves that $\mathrm{Ker}(\langle F,\beta\rangle) \in \mathrm{ConSys}(\mathbb{L}^\theta)$. Finally, we have, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll}
\beta_\Sigma(D^\theta_\Sigma(\Phi/\theta_\Sigma)) & = & \beta_\Sigma(D_\Sigma(\Phi)/\theta_\Sigma) \\
& = & \alpha_\Sigma(D_\Sigma(\Phi)) \\
& = & D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \\
& = & D'_{F(\Sigma)}(\beta_\Sigma(\Phi/\theta_\Sigma)).
\end{array}$$

This shows that both conditions in Part (iii) of Proposition 1360 are satisfied and, hence, $\langle F,\beta\rangle : \mathbb{L}^\theta \vdash \mathbb{L}'$ is a bilogical morphism.

Fifth, for all $\Sigma \in |\mathbf{Sign}|$ $\beta_\Sigma : \mathrm{SEN}^\theta(\Sigma) \to \mathrm{SEN}'(F(\Sigma))$ is injective. If $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\beta_\Sigma(\phi/\theta_\Sigma) = \beta_\Sigma(\psi/\theta_\Sigma)$, then $\alpha_\Sigma(\phi) = \beta_\Sigma(\phi)$, whence $\phi/\theta_\Sigma = \psi/\theta_\Sigma$. We now conclude that $\langle F,\beta\rangle : \mathbb{L}^\theta \vdash^\alpha \mathbb{L}'$ is an α-isomorphism.

Finally, the triangle commutes: This is clear, since, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $\beta_\Sigma(\pi^\theta_\Sigma(\phi)) = \beta_\Sigma(\phi/\theta_\Sigma) = \alpha_\Sigma(\phi)$. ■

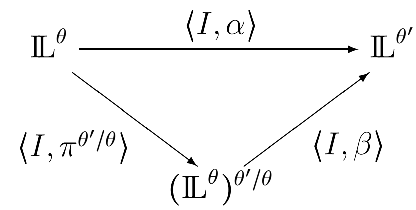
Theorem 1368 (Isomorphism Theorem) *Suppose* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *is a base algebraic system and* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *an* $N^\flat$*-algebraic system. If* $\mathbb{L} = \langle \mathbf{A}, D \rangle$ *is an* $N^\flat$*-structure and* $\theta, \theta' \in \mathrm{ConSys}(\mathbb{L})$*, such that* $\theta \leq \theta'$*, then* $\theta'/\theta \in \mathrm{ConSys}(\mathbb{L}^\theta)$ *and* $(\mathbb{L}^\theta)^{\theta'/\theta} \cong \mathbb{L}^{\theta'}$*.*

Proof: First, we show that $\theta'/\theta \in \mathrm{ConSys}(\mathbb{L}^\theta)$. To this end, let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi/\theta_\Sigma, \psi/\theta_\Sigma \rangle \in \theta'_\Sigma/\theta_\Sigma$. Then $\langle \phi, \psi \rangle \in \theta'_\Sigma$, whence, since $\theta' \in \mathrm{ConSys}(\mathbb{L})$, $D_\Sigma(\phi) = D_\Sigma(\psi)$. Hence, $D^\theta_\Sigma(\phi/\theta_\Sigma) = D^\theta_\Sigma(\psi/\theta_\Sigma)$. So $\theta'/\theta \in \mathrm{ConSys}(\mathbb{L}^\theta)$.

To finish the proof, we define $\langle I, \alpha \rangle : \mathbb{L}^\theta \vdash \mathbb{L}^{\theta'}$, by setting, for all $\Sigma \in |\mathbf{Sign}|$, $\phi \in \mathrm{SEN}(\Sigma)$,

$$\alpha_\Sigma(\phi/\theta_\Sigma) = \phi/\theta'_\Sigma.$$

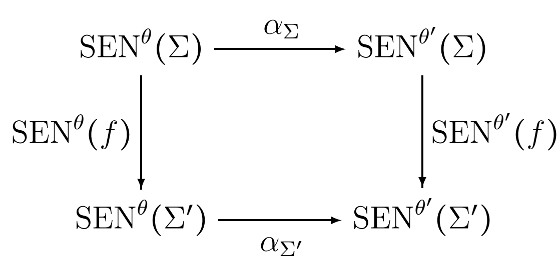
If we show that $\langle I, \alpha \rangle : \mathbb{L}^\theta \vdash \mathbb{L}^{\theta'}$ is a bilogical morphism, then, by noting that $\mathrm{Ker}(\langle I, \alpha \rangle) = \theta'/\theta$ and applying Theorem 1367,



we will have the sought after isomorphism $\langle I, \beta \rangle : (\mathbb{L}^\theta)^{\theta'/\theta} \cong \mathbb{L}^{\theta'}$.

First, $\langle I, \alpha \rangle : \mathrm{SEN}^\theta \to \mathrm{SEN}^{\theta'}$ is well-defined, since, for all $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, if $\langle \phi, \psi \rangle \in \theta_\Sigma$, then, by hypothesis, $\langle \phi, \psi \rangle \in \theta'_\Sigma$, showing that $\alpha_\Sigma(\phi/\theta_\Sigma) = \alpha_\Sigma(\psi/\theta_\Sigma)$.

Second, $\alpha : \mathrm{SEN}^\theta \to \mathrm{SEN}^{\theta'}$ is natural, since, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\phi \in \mathrm{SEN}(\Sigma)$,



$$\begin{array}{rcl}
\mathrm{SEN}^{\theta'}(f)(\alpha_\Sigma(\phi/\theta_\Sigma)) & = & \mathrm{SEN}^{\theta'}(f)(\phi/\theta'_\Sigma) \\
 & = & \mathrm{SEN}(f)(\phi)/\theta'_{\Sigma'} \\
 & = & \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)/\theta_{\Sigma'}) \\
 & = & \alpha_{\Sigma'}(\mathrm{SEN}^\theta(\phi/\theta_\Sigma)).
\end{array}$$

Third, it is clear that $\langle I, \alpha \rangle : \mathbf{A}^\theta \to \mathbf{A}^{\theta'}$ is surjective. So it suffices now to show that the conditions in Part (iii) of Proposition 1360 are satisfied.

First, $\text{Ker}(\langle I,\alpha\rangle) = \theta'/\theta \in \text{ConSys}(\mathbb{L}^\theta)$, as was shown above. Finally, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \subseteq \text{SEN}(\Sigma)$, we have

$$\begin{array}{rcl}\alpha_\Sigma(D^\theta_\Sigma(\Phi/\theta_\Sigma)) & = & \alpha_\Sigma(D_\Sigma(\Phi)/\theta_\Sigma)\\ & = & D_\Sigma(\Phi)/\theta'_\Sigma\\ & = & D^{\theta'}_\Sigma(\Phi/\theta'_\Sigma)\\ & = & D^{\theta'}_\Sigma(\alpha_\Sigma(\Phi/\theta_\Sigma)).\end{array}$$

Therefore, $\langle I,\alpha\rangle : \mathbb{L}^\theta \vdash \mathbb{L}^{\theta'}$ is indeed a bilogical morphism. ■

Theorem 1369 (Correspondence Theorem) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \text{SEN}, N\rangle$ *an* $N^\flat$*-algebraic system,* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *an* $N^\flat$*-structure and* $\theta \in \text{ConSys}(\mathbb{L})$. *Then* $\theta' \mapsto \theta'/\theta$ *defines an order isomorphism between the principal filter* $[\theta, \widetilde{\Omega}(\mathbb{L})]$ *in* $\mathbf{ConSys}(\mathbb{L})$ *and the complete lattice* $\mathbf{ConSys}(\mathbb{L}^\theta)$.

Proof: By Theorem 1368, the mapping $\theta' \mapsto \theta'/\theta$ is a well defined mapping from $[\theta, \widetilde{\Omega}(\mathbb{L})]$ into $\text{ConSys}(\mathbb{L}^\theta)$. The mapping is also one-to-one. To see this, assume $\theta'/\theta = \theta''/\theta$ and let $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi \in \text{SEN}(\Sigma)$, such that $\langle \phi,\psi\rangle \in \theta'_\Sigma$. Then $\langle \phi/\theta_\Sigma, \psi/\theta_\Sigma\rangle \in \theta'_\Sigma/\theta_\Sigma = \theta''_\Sigma/\theta_\Sigma$ and, therefore, $\langle\phi,\psi\rangle \in \theta''_\Sigma$. Thus, $\theta' \leq \theta''$ and, hence, by symmetry, $\theta' = \theta''$. The mapping is also surjective. To prove surjectivity, Let $\eta \in \text{ConSys}(\mathbb{L}^\theta)$. Define $\theta' = \{\theta'_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\theta'_\Sigma = \{\langle\phi,\psi\rangle \in \text{SEN}(\Sigma)^2 : \langle \phi/\theta_\Sigma, \psi/\theta_\Sigma\rangle \in \eta_\Sigma\}.$$

It is easy to see that θ' is a congruence system on $\mathbf{A}$. It is also easy to see that $\theta \leq \theta'$. Furthermore, θ' is a congruence system of $\mathbb{L}$, since, if $\Sigma \in |\mathbf{Sign}|$, $\phi,\psi \in \text{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle \in \theta'_\Sigma$, we get $\langle \phi/\theta_\Sigma, \psi/\theta_\Sigma\rangle \in \eta_\Sigma \in \text{ConSys}(\mathbb{L}^\theta)$, whence $D^\theta_\Sigma(\phi/\theta_\Sigma) = D^\theta_\Sigma(\psi/\theta_\Sigma)$, i.e., $D_\Sigma(\phi)/\theta_\Sigma = D_\Sigma(\psi)/\theta_\Sigma$ and, since $\theta \in \text{ConSys}(\mathbb{L})$, $D_\Sigma(\phi) = D_\Sigma(\psi)$. Since $\theta' \mapsto \theta'/\theta = \eta$, it follows that the mapping is also surjective. Finally, it is obvious that both it and its inverse are monotone, which establishes that it is an order isomorphism. ■

The Correspondence Theorem implies immediately a relation between the quotient of a Tarski congruence system and the Tarski system of the corresponding quotient structure.

Corollary 1370 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \text{SEN}, N\rangle$ *an* $N^\flat$*-algebraic system,* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *an* $N^\flat$*-structure and* $\theta \in \text{ConSys}(\mathbb{L})$. *Then*

$$\widetilde{\Omega}(\mathbb{L}^\theta) = \widetilde{\Omega}(\mathbb{L})/\theta.$$

Proof: We take $\theta' = \widetilde{\Omega}(\mathbb{L})$ and apply the Correspondence Theorem 1369. ■

Corollary 1370 allows us also to conclude that the quotient of any $N^\flat$-structure by its Tarski congruence system has an identity Tarski congruence system. More precisely,

$$\widetilde{\Omega}(\mathbb{L}^{\widetilde{\Omega}(\mathbb{L})}) = \widetilde{\Omega}(\mathbb{L})/\widetilde{\Omega}(\mathbb{L}) = \Delta^{\mathbf{A}/\widetilde{\Omega}(\mathbb{L})}.$$

This leads to the definition of a reduced $N^\flat$-structure and of the reduction of an $N^\flat$-structure.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system, $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ an $N^\flat$-algebraic system and $\mathbb{L} = \langle \mathbf{A}, D\rangle$ an $N^\flat$-structure. We call $\mathbb{L}$ **reduced** if $\widetilde{\Omega}(\mathbb{L}) = \Delta^{\mathbf{A}}$. More generally, we set $\mathbb{L}^* = \mathbb{L}^{\widetilde{\Omega}(\mathbb{L})}$ and call $\mathbb{L}^*$ the **reduction** of $\mathbb{L}$. Moreover, for a class L of $N^\flat$-structures, we set

$$\mathsf{L}^* = \{\mathbb{L}^* : \mathbb{L} \in \mathsf{L}\}.$$

By the comments following Corollary 1370, $\mathbb{L}^*$ is reduced for any $N^\flat$-structure $\mathbb{L}$. In case $\mathbb{L}$ is reduced to start with, then $\mathbb{L}^* \cong \mathbb{L}$ and, in this case, $\mathbb{L}^*$ will be identified with $\mathbb{L}$.

Another important consequence of the Correspondence Theorem is that reducing a quotient of a structure results in a reduced structure that is isomorphic (and, thus, can be identified) with the reduction of the originally given structure.

Proposition 1371 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system, $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ an $N^\flat$-algebraic system, $\mathbb{L} = \langle \mathbf{A}, D\rangle$ an $N^\flat$-structure and $\theta \in \mathrm{ConSys}(\mathbb{L})$. Then*

$$(\mathbb{L}^\theta)^* \cong \mathbb{L}^*.$$

Proof: We have

$$\begin{array}{rcl}(\mathbb{L}^\theta)^* & = & (\mathbb{L}^\theta)^{\widetilde{\Omega}(\mathbb{L}^\theta)} \quad \text{(Definition of Reduction)}\\ & = & (\mathbb{L}^\theta)^{\widetilde{\Omega}(\mathbb{L})/\theta} \quad \text{(Corollary 1370)}\\ & \cong & \mathbb{L}^{\widetilde{\Omega}(\mathbb{L})} \quad \text{(Theorem 1368)}\\ & = & \mathbb{L}^*. \quad \text{(Definition of Reduction)}\end{array}$$

■

Generalizing Proposition 1371, we can show that a similar relation holds between the reductions of two $N^\flat$-structures that are related via a bilogical morphism.

Proposition 1372 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system, $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ and $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ $N^\flat$-algebraic systems and $\mathbb{L} = \langle \mathbf{A}, D\rangle$ and $\mathbb{L}' = \langle \mathbf{A}, D'\rangle$ $N^\flat$-structures, based on $\mathbf{A}$ and $\mathbf{A}'$, respectively. If there exists a bilogical morphism $\langle F, \alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$, then there exists an α-isomorphism*

$$\mathbb{L}^* \vdash^\alpha \mathbb{L}'^*.$$

Proof: We define $\langle F, \beta\rangle : \mathbf{A}^* \to \mathbf{A}'^*$ by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\beta_\Sigma(\phi/\widetilde{\Omega}_\Sigma(\mathbb{L})) = \alpha_\Sigma(\phi)/\widetilde{\Omega}_{F(\Sigma)}(\mathbb{L}'),$$

i.e., $\langle F, \beta\rangle$ is the morphism that makes the following rectangle commute

$$\begin{array}{ccc}
\mathbf{A} & \xrightarrow{\langle F,\alpha\rangle} & \mathbf{A}' \\
{\scriptstyle \langle I,\pi^{\widetilde{\Omega}(\mathbb{L})}\rangle}\downarrow & & \downarrow{\scriptstyle \langle I,\pi^{\widetilde{\Omega}(\mathbb{L}')}\rangle} \\
\mathbf{A}^* & \xrightarrow[\langle F,\beta\rangle]{} & \mathbf{A}'^*
\end{array}$$

First, $\langle I, \beta\rangle$ is well-defined: In fact, if $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi\rangle \in \widetilde{\Omega}_\Sigma(\mathbb{L})$, then, since, by Corollary 1364, $\Omega(\mathbb{L}) = \alpha^{-1}(\widetilde{\Omega}(\mathbb{L}'))$, we get that $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in \widetilde{\Omega}_{F(\Sigma)}(\mathbb{L}')$.

Second $\beta : \mathrm{SEN}^{\widetilde{\Omega}(\mathbb{L})} \to \mathrm{SEN}'^{\widetilde{\Omega}(\mathbb{L}')} \circ F$ is a natural transformation: Let $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\phi \in \mathrm{SEN}(\Sigma)$. We have

$$\begin{array}{ccc}
\mathrm{SEN}^{\widetilde{\Omega}(\mathbb{L})}(\Sigma) & \xrightarrow{\beta_\Sigma} & \mathrm{SEN}'^{\widetilde{\Omega}(\mathbb{L}')}(F(\Sigma)) \\
{\scriptstyle \mathrm{SEN}^{\widetilde{\Omega}(\mathbb{L})}(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{SEN}'^{\widetilde{\Omega}(\mathbb{L}')}(F(f))} \\
\mathrm{SEN}^{\widetilde{\Omega}(\mathbb{L})}(\Sigma') & \xrightarrow[\beta_{\Sigma'}]{} & \mathrm{SEN}'^{\widetilde{\Omega}(\mathbb{L}')}(F(\Sigma'))
\end{array}$$

$$\begin{array}{l}
\mathrm{SEN}'^{\widetilde{\Omega}(\mathbb{L}')}(F(f))(\beta_\Sigma(\phi/\widetilde{\Omega}_\Sigma(\mathbb{L}))) \\
\quad = \mathrm{SEN}'^{\widetilde{\Omega}(\mathbb{L}')}(F(f))(\alpha_\Sigma(\phi)/\widetilde{\Omega}_{F(\Sigma)}(\mathbb{L}')) \\
\quad = \mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi))/\widetilde{\Omega}_{F(\Sigma')}(\mathbb{L}') \\
\quad = \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi))/\widetilde{\Omega}_{F(\Sigma')}(\mathbb{L}') \\
\quad = \beta_{\Sigma'}(\mathrm{SEN}(f)(\phi)/\widetilde{\Omega}_{\Sigma'}(\mathbb{L})) \\
\quad = \beta_{\Sigma'}(\mathrm{SEN}^{\widetilde{\Omega}(\mathbb{L})}(f)(\phi/\widetilde{\Omega}_\Sigma(\mathbb{L})).
\end{array}$$

Third, for every $\Sigma \in |\mathbf{Sign}|$, $\beta_\Sigma : \mathrm{SEN}^{\widetilde{\Omega}(\mathbb{L})}(\Sigma) \to \mathrm{SEN}'^{\widetilde{\Omega}(\mathbb{L}')}(F(\Sigma))$ is a bijection. Surjectivity is immediate and follows from the fat that both $\langle F, \alpha\rangle$ and $\langle I, \pi^{\widetilde{\Omega}(\mathbb{L}')}\rangle$ are surjective. For injectivity, if $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\beta_\Sigma(\phi/\widetilde{\Omega}_\Sigma(\mathbb{L})) = \beta_\Sigma(\psi/\widetilde{\Omega}_\Sigma(\mathbb{L}))$, then $\alpha_\Sigma(\phi)/\widetilde{\Omega}_{F(\Sigma)}(\mathbb{L}') = \alpha_\Sigma(\psi)/\widetilde{\Omega}_{F(\Sigma)}(\mathbb{L}')$, whence

$$\langle \phi, \psi\rangle \in \alpha_\Sigma^{-1}(\widetilde{\Omega}_{F(\Sigma)}(\mathbb{L}')) = \widetilde{\Omega}_\Sigma(\mathbb{L}).$$

This proves that β_Σ is indeed a bijection.

Finally, we use Part (iii) of Proposition 1360 to show that it is a bilogical morphism. Of course, since $\langle F, \beta\rangle$ has injective components, we get

$\mathrm{Ker}(\langle F,\beta\rangle) = \Delta^{\mathbf{A}^*}$ and, hence it is a congruence system of $\mathbb{L}^*$. Finally, if $\Sigma \in |\mathbf{Sign}|$ and $\Phi \subseteq \mathrm{SEN}(\Sigma)$, we have

$$\begin{array}{rcl}\beta_\Sigma(D^*_\Sigma(\Phi/\widetilde{\Omega}_\Sigma(\mathbb{L}))) & = & \beta_\Sigma(D_\Sigma(\Phi)/\widetilde{\Omega}_\Sigma(\mathbb{L}))\\ & = & \alpha_\Sigma(D_\Sigma(\Phi))/\widetilde{\Omega}_{F(\Sigma)}(\mathbb{L}')\\ & = & D'_{F(\Sigma)}(\alpha_\Sigma(\Phi))/\widetilde{\Omega}_{F(\Sigma)}(\mathbb{L}')\\ & = & D'^*_{F(\Sigma)}(\alpha_\Sigma(\Phi)/\widetilde{\Omega}_{F(\Sigma)}(\mathbb{L}'))\\ & = & D'^*_{F(\Sigma)}(\beta_\Sigma(\Phi/\widetilde{\Omega}_\Sigma(\mathbb{L})).\end{array}$$

Thus, $\langle F,\beta\rangle : \mathbb{L}^* \vdash^\alpha \mathbb{L}'^*$ is an α-isomorphism, as claimed. ∎

In case $\mathbf{Sign}' = \mathbf{Sign}$ and $\langle I,\alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$ is a bilogical morphism, with $I : \mathbf{Sign} \to \mathbf{Sign}$ the identity functor, then we obtain

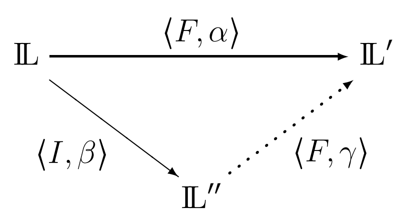
Corollary 1373 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *and* $\mathbf{A}' = \langle \mathbf{Sign}, \mathrm{SEN}', N'\rangle$ $N^\flat$*-algebraic systems and* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *and* $\mathbb{L}' = \langle \mathbf{A}, D'\rangle$ $N^\flat$*-structures, based on* $\mathbf{A}$ *and* $\mathbf{A}'$*, respectively. If there exists a bilogical morphism* $\langle I,\alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$*, then*

$$\mathbb{L}^* \cong \mathbb{L}'^*.$$

Proof: Immediate by Proposition 1372. ∎

The next result is a "fill-in" lemma that provides sufficient conditions under which one can find a morphism that "fills-in" the third side of a commutative triangle, given arrows emanating from one of its vertices.

Proposition 1374 (Fill-In Lemma) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *and* $\mathbf{A}'' = \langle \mathbf{Sign}, \mathrm{SEN}'', N''\rangle$ $N^\flat$*-algebraic systems and* $\mathbb{L} = \langle \mathbf{A}, D\rangle$*,* $\mathbb{L}' = \langle \mathbf{A}', D'\rangle$ *and* $\mathbb{L}'' = \langle \mathbf{A}'', D''\rangle$ $N^\flat$*-structures, based on* $\mathbf{A}$*,* $\mathbf{A}'$ *and* $\mathbf{A}''$*, respectively. Given a logical morphism* $\langle F,\alpha\rangle : \mathbb{L} \succ \mathbb{L}'$ *and a bilogical morphism* $\langle I,\beta\rangle : \mathbb{L} \vdash \mathbb{L}''$*, such that* $\ker(\langle I,\beta\rangle) \le \mathrm{Ker}(\langle F,\alpha\rangle)$*,*



there exists a unique logical morphism $\langle F,\gamma\rangle : \mathbb{L}'' \succ \mathbb{L}'$*, such that the triangle commutes. Moreover,* $\langle F,\gamma\rangle$ *is bilogical if and only if* $\langle F,\alpha\rangle$ *is bilogical.*

Proof: Define, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}''(\Sigma)$,

$$\gamma_\Sigma(\phi) = \alpha_\Sigma(\psi),$$

where $\psi \in \mathrm{SEN}(\Sigma)$, such that $\beta_\Sigma(\psi) = \phi$.

First, since, for all $\Sigma \in |\mathbf{Sign}|$, $\psi, \psi' \in \mathrm{SEN}(\Sigma)$, such that $\beta_\Sigma(\psi) = \beta_\Sigma(\psi')$, we have, by hypothesis, $\alpha_\Sigma(\psi) = \alpha_\Sigma(\psi')$, this definition is sound.

Second, $\gamma : \mathrm{SEN}'' \to \mathrm{SEN}' \circ F$ is a natural transformation: For all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\phi \in \mathrm{SEN}''(\Sigma)$, such that $\phi = \beta_\Sigma(\psi)$, for some $\psi \in \mathrm{SEN}(\Sigma)$, we have

$$\begin{array}{ccc}
\mathrm{SEN}''(\Sigma) & \xrightarrow{\gamma_\Sigma} & \mathrm{SEN}'(F(\Sigma)) \\
{\scriptstyle \mathrm{SEN}''(f)} \downarrow & & \downarrow {\scriptstyle \mathrm{SEN}'(F(f))} \\
\mathrm{SEN}''(\Sigma') & \xrightarrow[\gamma_{\Sigma'}]{} & \mathrm{SEN}'(F(\Sigma'))
\end{array}$$

$$\begin{array}{rcl}
\mathrm{SEN}'(F(f))(\gamma_\Sigma(\phi)) & = & \mathrm{SEN}'(F(f))(\alpha_\Sigma(\psi)) \\
& = & \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi)) \\
& = & \gamma_{\Sigma'}(\mathrm{SEN}''(f)(\phi)),
\end{array}$$

where the last equality follows from

$$\beta_{\Sigma'}(\mathrm{SEN}(f)(\psi)) = \mathrm{SEN}''(f)(\beta_\Sigma(\psi)) = \mathrm{SEN}''(f)(\phi)$$

and the definition of $\gamma_{\Sigma'}$.

Now it is clear that the triangle of the diagram commutes. Moreover, for all $T' \in \mathrm{ThFam}(\mathbb{L}')$, since $\langle F, \alpha \rangle$ is a logical morphism, $\alpha^{-1}(T') \in \mathrm{ThFam}(\mathbb{L})$ and, hence, by commutativity, $\beta^{-1}(\gamma^{-1}(T')) \in \mathrm{ThFam}(\mathbb{L})$. Hence, since $\langle I, \beta \rangle$ is a bilogical morphism, $\gamma^{-1}(T') \in \mathrm{ThFam}(\mathbb{L}'')$. This proves that $\langle F, \gamma \rangle$ is also a logical morphism.

Finally, for the last statement, note that $\langle F, \gamma \rangle$ is surjective if and only if $\langle F, \alpha \rangle$ is surjective, and, furthermore, $\mathrm{ThFam}(\mathbb{L}'') = \gamma^{-1}(\mathrm{ThFam}(\mathbb{L}'))$ if and only if $\beta^{-1}(\mathrm{ThFam}(\mathbb{L}'')) = \alpha^{-1}(\mathrm{ThFam}(\mathbb{L}'))$ if and only if $\mathrm{ThFam}(\mathbb{L}) = \alpha^{-1}(\mathrm{ThFam}(\mathbb{L}'))$. We conclude, taking into account Part (vi) of Proposition 1360, that $\langle F, \gamma \rangle$ is bilogical if and only if $\langle F, \alpha \rangle$ is. ∎

19.3 Filter Families and π-Structures

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ an $\mathbf{F}$-algebraic system. We have seen that the collection $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ of $\mathcal{I}$-filter families on $\mathcal{A}$ forms a complete lattice $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A}) = \langle \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}), \leq \rangle$ under signature-wise inclusion. Therefore, the pair $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ constitutes an $\mathbf{F}$-structure. This $\mathbf{F}$-structure will also be denoted interchangeably by $\langle \mathcal{A}, C^{\mathcal{I},\mathcal{A}} \rangle$ or $\langle \mathcal{A}, \mathcal{C}^{\mathcal{I},\mathcal{A}} \rangle$, with reference to the closure (operator) family or the closed set family corresponding to $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Such pairs will play an important role in this chapter, since they will be used as models of $\mathcal{I}$ that provide a semantics for the logical system formalized by $\mathcal{I}$.

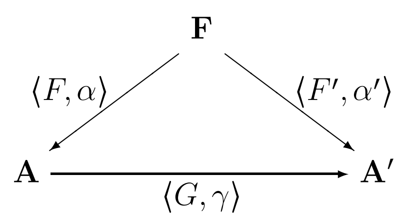
It is clear that the closure families of $\mathbf{F}$-structures of this form are structural and, hence, $\mathbf{F}$-structures of this form are actually π-institutions.

Proposition 1375 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, an* $\mathbf{F}$*-algebraic system. Then* $C^{\mathcal{I},\mathcal{A}} : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}$ *is a structural closure operator on* SEN*.*

Proof: We use Proposition 1357. Let $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, $T \in \mathcal{C}^{\mathcal{I},\mathcal{A}}_{F(\Sigma')}$. Then, by Lemma 51, $\alpha^{-1}_{\Sigma'}(T) \in \mathcal{C}_{\Sigma'}$. Since C is structural, by Proposition 1357, $\mathrm{SEN}(f)^{-1}(\alpha^{-1}_{\Sigma'}(T)) \in \mathcal{C}_\Sigma$. By the naturality of $\alpha : \mathrm{SEN}^\flat \to \mathrm{SEN} \circ F$, we get $\alpha^{-1}_\Sigma(\mathrm{SEN}(F(f))^{-1}(T)) \in \mathcal{C}_\Sigma$, whence, again by Lemma 51, $\mathrm{SEN}(F(f))^{-1}(T) \in \mathcal{C}^{\mathcal{I},\mathcal{A}}_\Sigma$. Using the surjectivity of $\langle F, \alpha \rangle$ and Proposition 1357, we conclude that $C^{\mathcal{I},\mathcal{A}}$ is structural. ∎

Our next result characterizes bilogical morphisms between $\mathbf{F}$-structures of the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$.

Proposition 1376 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$ *with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ $\mathbf{F}$*-algebraic systems, such that, there exists a surjective* $\langle G, \gamma \rangle : \mathbf{A} \to \mathbf{A}'$*, such that*



$\langle G, \gamma \rangle \circ \langle F, \alpha \rangle = \langle F', \alpha' \rangle$*. Then the following statements are equivalent:*

(i) $\langle G, \gamma \rangle : \langle \mathcal{A}, \mathcal{C}^{\mathcal{I},\mathcal{A}} \rangle \vdash \langle \mathcal{A}', \mathcal{C}^{\mathcal{I},\mathcal{A}'} \rangle$ *is a bilogical morphism;*

(ii) For every $\Sigma \in |\mathbf{Sign}|$*,* $\gamma_\Sigma : \mathcal{C}^{\mathcal{I},\mathcal{A}}_\Sigma \to \mathcal{C}^{\mathcal{I},\mathcal{A}'}_{G(\Sigma)}$ *is an order isomorphism;*

(iii) For every $\Sigma \in |\mathbf{Sign}|$*, and all* $T \in \mathcal{C}^{\mathcal{I},\mathcal{A}}_\Sigma$*,* $\gamma_\Sigma(T) \in \mathcal{C}^{\mathcal{I},\mathcal{A}'}_{G(\Sigma)}$ *and, in addition, we have* $\mathrm{Ker}(\langle G, \gamma \rangle) \in \mathrm{ConSys}(\langle \mathcal{A}, \mathcal{C}^{\mathcal{I},\mathcal{A}} \rangle)$*.*

Proof:

(i)⇒(ii) This is a special case of Proposition 1361.

(ii)⇒(iii) The first assertion is obvious. For the second, if $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\gamma_\Sigma(\phi) = \gamma_\Sigma(\psi)$, then, for every $T' \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A}')$,

$$\gamma_\Sigma(\phi) \in T'_{G(\Sigma)} \quad \text{iff} \quad \gamma_\Sigma(\psi) \in T'_{G(\Sigma)}.$$

Hence, for every $T' \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A}')$,

$$\phi \in \gamma_\Sigma^{-1}(T'_{G(\Sigma)}) \quad \text{iff} \quad \phi \in \gamma_\Sigma^{-1}(T'_{G(\Sigma)}).$$

Therefore, by hypothesis, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

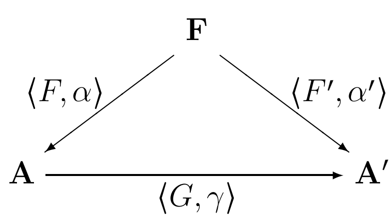
$$\phi \in T_\Sigma \quad \text{iff} \quad \psi \in T_\Sigma.$$

It now follows that $\mathrm{Ker}(\langle I, \gamma\rangle) \in \mathrm{ConSys}(\langle \mathcal{A}, \mathcal{C}^{\mathcal{I},\mathcal{A}}\rangle)$.

(iii)⇒(i) By hypothesis, we get $\gamma(\mathcal{C}_\Sigma^{\mathcal{I},\mathcal{A}}) \subseteq \mathcal{C}_{G(\Sigma)}^{\mathcal{I},\mathcal{A}'}$ and also that $\mathrm{Ker}(\langle I, \gamma\rangle) \in \mathrm{ConSys}(\langle \mathcal{A}, \mathcal{C}^{\mathcal{I},\mathcal{A}}\rangle)$. Therefore, by Part (v) of Proposition 1360, it suffices to show that $\mathcal{C}_{G(\Sigma)}^{\mathcal{I},\mathcal{A}'} \subseteq \gamma(\mathcal{C}_\Sigma^{\mathcal{I},\mathcal{A}})$. But this follows from the fact that, if $T' \in \mathcal{C}_{G(\Sigma)}^{\mathcal{I},\mathcal{A}'}$, then by Corollary 55, $\gamma_\Sigma^{-1}(T') \in \mathcal{C}_\Sigma^{\mathcal{I},\mathcal{A}}$ and, then, by surjectivity, $T' = \gamma_\Sigma(\gamma_\Sigma^{-1}(T'))$. ■

We also have the following related result that, roughly speaking, forces the closure family of a structure that is the bilogical morphism image of a structure whose closure family consists of all filter families to also consist of the entirely of filter families.

Proposition 1377 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*,* $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha'\rangle\rangle$ *with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ $\mathbf{F}$*-algebraic systems, such that, there exists a surjective* $\langle G, \gamma\rangle : \mathbf{A} \to \mathbf{A}'$*, such that*



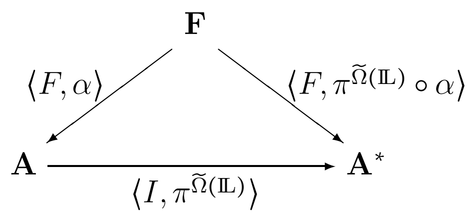
$\langle G, \gamma\rangle \circ \langle F, \alpha\rangle = \langle F', \alpha'\rangle$*. If* $\langle G, \gamma\rangle : \langle \mathcal{A}, \mathcal{C}^{\mathcal{I},\mathcal{A}}\rangle \vdash \langle \mathcal{A}', \mathcal{C}'\rangle$ *is a bilogical morphism, then* $\mathcal{C}' = \mathcal{C}^{\mathcal{I},\mathcal{A}'}$*.*

Proof: First, since $\langle G, \gamma\rangle$ is a bilogical morphism, for all $T' \in \mathcal{C}'$, we have $\gamma^{-1}(T') \in \mathcal{C}^{\mathcal{I},\mathcal{A}}$. Thus, by Corollary 55, $T' \in \mathcal{C}^{\mathcal{I},\mathcal{A}'}$. This proves that $\mathcal{C}' \subseteq \mathcal{C}^{\mathcal{I},\mathcal{A}'}$. Suppose, conversely, that $T' \in \mathcal{C}^{\mathcal{I},\mathcal{A}'}$. Then, again by Corollary 55, $\gamma^{-1}(T') \in \mathcal{C}^{\mathcal{I},\mathcal{A}}$. Therefore, since $\langle G, \gamma\rangle$ is a bilogical morphism, $T' = \gamma(\gamma^{-1}(T')) \in \mathcal{C}'$. We conclude that $\mathcal{C}' = \mathcal{C}^{\mathcal{I},\mathcal{A}'}$. ■

This result has the following immediate corollaries, one addressing reductions and the other isomorphisms.

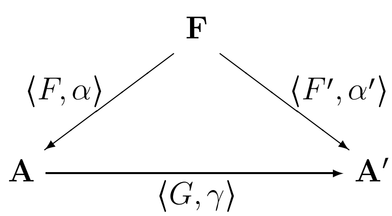
Corollary 1378 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, an* $\mathbf{F}$*-algebraic system. Then* $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$*.*

Proof: Let $\mathbb{L} = \langle \mathcal{A}, \mathcal{C}^{\mathcal{I},\mathcal{A}} \rangle$ and apply Proposition 1377 to the special configuration of morphisms depicted in the diagram:



■

Corollary 1379 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$ *with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ $\mathbf{F}$*-algebraic systems, such that, there exists an isomorphism* $\langle G, \gamma \rangle : \mathbf{A} \cong \mathbf{A}'$*, such that*

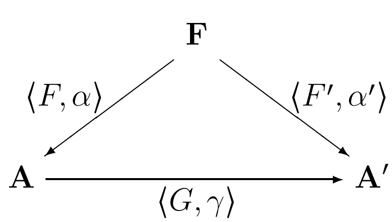


$\langle G, \gamma \rangle \circ \langle F, \alpha \rangle = \langle F', \alpha' \rangle$*. If* $\langle G, \gamma \rangle : \langle \mathcal{A}, \mathcal{D} \rangle \vdash \langle \mathcal{A}', \mathcal{D}' \rangle$ *is a bilogical morphism, then* $\mathcal{D} = \mathcal{C}^{\mathcal{I},\mathcal{A}}$ *if and only if* $\mathcal{D}' = \mathcal{C}^{\mathcal{I},\mathcal{A}'}$*.*

Proof: We apply Proposition 1377 twice; once using $\langle G, \gamma \rangle : \mathcal{A} \to \mathcal{A}'$ and once using $\langle G, \gamma \rangle^{-1} : \mathcal{A}' \to \mathcal{A}$. ■

Corollary 1379 can be strengthened slightly but, to accomplish this, we need the following proposition, which is a sort of symmetric to Proposition 1377.

Proposition 1380 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$ *two* $\mathbf{F}$*-algebraic systems.*



If $\langle G, \gamma \rangle : \langle \mathcal{A}, \mathcal{D} \rangle \vdash^\alpha \langle \mathcal{A}', C^{\mathcal{I},\mathcal{A}'} \rangle$ *is an* α*-isomorphism, then* $\mathcal{D} = \mathcal{C}^{\mathcal{I},\mathcal{A}}$*.*

Proof: We show, first, that $\mathcal{D} \subseteq \mathcal{C}^{\mathcal{I},\mathcal{A}}$. Suppose $T \in \mathcal{D}$. Then, since $\langle G, \gamma\rangle$ is an α-isomorphism, there exists, by Proposition 1360, $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$, such that $T = \gamma^{-1}(T')$. Now, by Corollary 55, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Hence, $\mathcal{D} \subseteq \mathcal{C}^{\mathcal{I},\mathcal{A}}$.

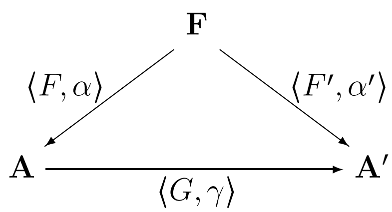
Suppose, conversely, that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Since $\langle G, \gamma\rangle$ is an α-isomorphism, there exists a unique $T' \in \mathrm{SenFam}(\mathcal{A}')$, such that $T = \gamma^{-1}(T')$. Thus, we have

$$\alpha'^{-1}(T') = \alpha^{-1}(\gamma^{-1}(T')) = \alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I}).$$

Hence, $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$ and, since $\langle G, \gamma\rangle$ is a bilogical morphism, $T = \gamma^{-1}(T') \in \mathcal{D}$. We conclude that $\mathcal{C}^{\mathcal{I},\mathcal{A}} \subseteq \mathcal{D}$ and equality follows. ■

A generalization of Corollary 1379 relaxes the requirement that there exists an isomorphism between **F**-algebraic systems to the requirement that there exists an α-isomorphism.

Corollary 1381 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*,* $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha'\rangle\rangle$ *with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ **F***-algebraic systems, such that, there exists a surjective morphism* $\langle G, \gamma\rangle : \mathbf{A} \cong \mathbf{A}'$*, such that*



$\langle G, \gamma\rangle \circ \langle F, \alpha\rangle = \langle F', \alpha'\rangle$. *If* $\langle G, \gamma\rangle : \langle \mathcal{A}, \mathcal{D}\rangle \vdash^\alpha \langle \mathcal{A}', \mathcal{D}'\rangle$ *is an* α*-isomorphism, then* $\mathcal{D} = \mathcal{C}^{\mathcal{I},\mathcal{A}}$ *if and only if* $\mathcal{D}' = \mathcal{C}^{\mathcal{I},\mathcal{A}'}$.

Proof: We put together Proposition 1377 and Proposition 1380. ■

19.4 $\mathcal{I}$-Structures

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, an **F**-algebraic system and $\mathbb{L} = \langle \mathcal{A}, D\rangle$ an **F**-structure. Define $C^{\mathbb{L}} = \{C^{\mathbb{L}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$C^{\mathbb{L}}_\Sigma : \mathcal{P}\mathrm{SEN}^\flat \to \mathcal{P}\mathrm{SEN}^\flat$$

be defined, for all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}\phi \in C^{\mathbb{L}}_\Sigma(\Phi) & \text{iff} & \text{for all } \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma, \Sigma'),\\ & & \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \subseteq D_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))).\end{array}$$

More generally, given a class L of **F**-structures, we set

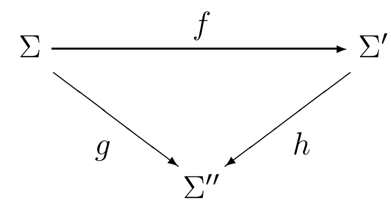
$$C^{\mathsf{L}} = \bigcap\{C^{\mathbb{L}} : \mathbb{L} \in \mathsf{L}\}.$$

We show that C^{L} is a closure system on **F** and, as a result, $\mathcal{I}^{\mathsf{L}} = \langle \mathbf{F}, C^{\mathsf{L}}\rangle$ qualifies as a π-institution.

Proposition 1382 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* L *a class of* $\mathbf{F}$*-structures. The collection* $C^{\mathsf{L}} : \mathcal{P}\mathrm{SEN}^\flat \to \mathcal{P}\mathrm{SEN}^\flat$ *is a closure system on* $\mathbf{F}$*.*

Proof: Inflationarity, monotonicity and idempotency of C^{L} follow immediately from the corresponding properties of each of the operators of the $\mathbf{F}$-structures in L. We show structurality in more detail. Suppose $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C^{\mathsf{L}}_\Sigma(\Phi)$. Then, for all $\langle \mathcal{A}, D \rangle \in \mathsf{L}$, all $\Sigma'' \in |\mathbf{Sign}^\flat|$ and all $g \in \mathbf{Sign}^\flat(\Sigma, \Sigma'')$,

$$\alpha_{\Sigma''}(\mathrm{SEN}^\flat(g)(\phi)) \in D_{F(\Sigma'')}(\alpha_{\Sigma''}(\mathrm{SEN}^\flat(g)(\Phi))).$$



Thus, for all $\langle \mathcal{A}, D \rangle \in \mathsf{L}$, all $\Sigma'' \in |\mathbf{Sign}^\flat|$ and all $h \in \mathbf{Sign}^\flat(\Sigma', \Sigma'')$,

$$\alpha_{\Sigma''}(\mathrm{SEN}^\flat(h)(\mathrm{SEN}^\flat(f)(\phi))) \in D_{F(\Sigma'')}(\alpha_{\Sigma''}(\mathrm{SEN}^\flat(h)(\mathrm{SEN}^\flat(f)(\Phi)))).$$

This proves that $\mathrm{SEN}^\flat(f)(\phi) \in C^{\mathsf{L}}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))$, and, hence, that C^{L} is structural. ■

C^{L} is termed the **closure system on F generated by** L and we denote by $\mathcal{I}^{\mathsf{L}} = \langle \mathbf{F}, C^{\mathsf{L}} \rangle$ the π-institution corresponding to C^{L}.

Next, it is shown that $\mathbf{F}$-structures related by bilogical morphism generate identical closure systems.

Proposition 1383 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$ *two* $\mathbf{F}$*-algebraic systems,* $\mathbb{L} = \langle \mathcal{A}, D \rangle$*,* $\mathbb{L}' = \langle \mathcal{A}', D' \rangle$ *two* $\mathbf{F}$*-structures and* $\langle G, \gamma \rangle : \mathbb{L} \vdash \mathbb{L}'$ *a bilogical morphism, such that* $\langle F', \alpha' \rangle = \langle G, \gamma \rangle \circ \langle F, \alpha \rangle$*. Then* $C^{\mathbb{L}} = C^{\mathbb{L}'}$*.*

Proof: We have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

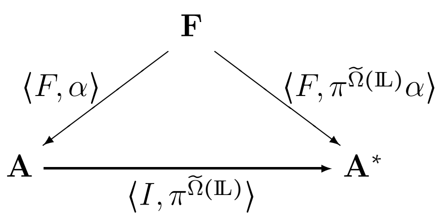
$$\begin{array}{lll}
\phi \in C^{\mathbb{L}}_\Sigma(\Phi) & \text{iff} & \text{for all } \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma, \Sigma') \\
& & \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \subseteq D_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))) \\
& \text{iff} & \text{for all } \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma, \Sigma') \\
& & \gamma_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi))) \\
& & \qquad \subseteq D'_{G(F(\Sigma'))}(\gamma_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)))) \\
& \text{iff} & \text{for all } \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma, \Sigma') \\
& & \alpha'_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \subseteq D'_{F'(\Sigma')}(\alpha'_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))) \\
& \text{iff} & \phi \in C^{\mathbb{L}'}_\Sigma(\Phi).
\end{array}$$

We conclude that $C^{\mathbb{L}} = C^{\mathbb{L}'}$. ■

As a special case of Proposition 1383, we get that both an $\mathbf{F}$-structure and its reduction generate the same closure system on the base algebraic system $\mathbf{F}$.

Corollary 1384 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $\mathbb{L} = \langle \mathcal{A}, D \rangle$ *an* $\mathbf{F}$*-structure. Then* $C^{\mathbb{L}} = C^{\mathbb{L}^*}$*.*

Proof: This is obtained directly by Proposition 1383 once we recall that, since $\widetilde{\Omega}(\mathbb{L})$ is a congruence system of $\mathbb{L}$, $\langle I, \pi^{\widetilde{\Omega}(\mathbb{L})} \rangle : \mathbb{L} \to \mathbb{L}^*$ is a bilogical morphism.



And this gives the configuration of the diagram that matches the setup in the hypothesis of Proposition 1383. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, an $\mathbf{F}$-algebraic system and $\mathbb{L} = \langle \mathcal{A}, D \rangle$ an $\mathbf{F}$-structure. We say that $\mathbb{L}$ is an $\mathcal{I}$-**structure** or a **model of** $\mathcal{I}$ if $C \leq C^{\mathbb{L}}$, i.e., if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in C_\Sigma(\Phi) \quad \text{implies} \quad \phi \in C^{\mathbb{L}}_\Sigma(\Phi).$$

Of course, $C \leq C^{\mathbb{L}}$ requires that, for all $T \in \mathrm{ThFam}(\mathbb{L})$, all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq T_{F(\Sigma')} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \in T_{F(\Sigma')},$$

i.e., that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Therefore, we obtain the following characterization:

Proposition 1385 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, an* $\mathbf{F}$*-algebraic system and* $\mathbb{L} = \langle \mathcal{A}, D \rangle$ *an* $\mathbf{F}$*-structure.* $\mathbb{L}$ *is an* $\mathcal{I}$*-structure if and only if, for all* $T \in \mathrm{ThFam}(\mathbb{L})$*,* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, i.e., if and only if* $\mathrm{ThFam}(\mathbb{L}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

Again, the defining condition of an $\mathcal{I}$-structure may be simplified due to the structurality of $\mathcal{I}$. More precisely, based on Lemma 50, we have:

Lemma 1386 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, an* $\mathbf{F}$*-algebraic system and* $\mathbb{L} = \langle \mathcal{A}, D \rangle$ *an* $\mathbf{F}$*-structure. Then, the following conditions are equivalent:*

(a) $\mathbb{L}$ *is an* $\mathcal{I}$*-structure;*

(b) For all $T \in \mathrm{ThFam}(\mathbb{L})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$*, such that* $\phi \in C_\Sigma(\Phi)$*,*

$$\alpha_\Sigma(\Phi) \subseteq T_{F(\Sigma)} \quad \textit{implies} \quad \alpha_\Sigma(\phi) \in T_{F(\Sigma)};$$

(c) For all $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$*,* $\alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi))$*.*

Proof: Condition (a) clearly implies (b) and (b) and (c) are equivalent. So it remains to show that (b) implies (a). But, if Condition (*b*) holds, then, by Lemma 50, $\mathrm{ThFam}(\mathbb{L}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, whence, by Proposition 1385, $\mathbb{L}$ is an $\mathcal{I}$-structure. ■

We denote by $\mathrm{Str}(\mathcal{I})$ the class of all $\mathcal{I}$-structures and let

$$\mathrm{Str}^*(\mathcal{I}) = (\mathrm{Str}(\mathcal{I}))^*$$

be the class of all reduced $\mathcal{I}$-structures.

Since we know that $\mathbb{L} \in \mathrm{Str}(\mathcal{I})$ is and only if $\mathrm{ThFam}(\mathbb{L}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, it follows that, given an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ is the weakest $\mathcal{I}$-structure of $\mathcal{A}$, i.e., the one with the finest closure family.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We say that $\mathcal{I}$ is **complete with respect to a given class** L of $\mathbf{F}$-structures if $C = C^{\mathsf{L}}$, i.e., if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in C_\Sigma(\Phi) \quad \text{iff} \quad \phi \in C^{\mathsf{L}}_\Sigma(\Phi).$$

As consequences of Proposition 1383 and of its Corollary 1384, we have the following results about models of π-institutions and about classes of structures with respect to which a π-institution is complete.

Proposition 1387 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) If $\langle \mathcal{A}, D \rangle$*,* $\mathbb{L}' = \langle \mathcal{A}', D' \rangle$ *are* $\mathbf{F}$*-structures and* $\langle G, \gamma \rangle : \mathbb{L} \vdash \mathbb{L}'$ *a bilogical morphism, the* $\mathbb{L}$ *is an* $\mathcal{I}$*-structure if and only if* $\mathbb{L}'$ *is an* $\mathcal{I}$*-structure.*

(b) If $\mathbb{L} = \langle \mathcal{A}, D \rangle$ *is an* $\mathbf{F}$*-structure, then* $\mathbb{L}$ *is an* $\mathcal{I}$*-structure if and only if* $\mathbb{L}^*$ *is an* $\mathcal{I}$*-structure.*

(c) If $\mathcal{I}$ *is complete with respect to a class* L *of* $\mathbf{F}$*-structures, then it is also complete with respect to* L^**.*

Proof: the first part is a consequence of Proposition 1383, whereas Parts (b) and (c) follow directly from Corollary 1384. ■

Proposition 1388 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* L *a class of* $\mathcal{I}$*-structures. If* L *includes* $\langle \mathcal{F}, C \rangle$ *or* $\langle \mathcal{F}, C \rangle^*$*, then* $\mathcal{I}$ *is complete with respect to both* L *and* L^**. In particular* $\mathcal{I}$ *is complete with respect to both* $\mathrm{Str}(\mathcal{I})$ *and* $\mathrm{Str}^*(\mathcal{I})$*.*

Proof: The key here is to notice that $C = C^{\langle \mathcal{F}, C \rangle} = C^{\langle \mathcal{F}, C \rangle^*}$. Then, the rest is easy because we have

$$C \le C^{\mathsf{L}} = C^{\mathsf{L}^*} \le C^{\langle \mathcal{F}, C \rangle} = C^{\langle \mathcal{F}, C \rangle^*} = C.$$

Therefore, we conclude $C = C^{\mathsf{L}} = C^{\mathsf{L}^*}$ and, hence, $\mathcal{I}$ is complete with respect to both L and L^*. ■

19.5 Full $\mathcal{I}$-Structures

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and $\mathbb{L} = \langle \mathcal{A}, D \rangle$ an $\mathbf{F}$-structure. $\mathbb{L}$ is a **full $\mathcal{I}$-structure** or a **full model of $\mathcal{I}$** if

$$\mathbb{L}^* = \langle \mathcal{A}^*, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*) \rangle,$$

i.e., if the closure family of the reduction of $\mathbb{L}$ consists of all $\mathcal{I}$-filter families on the $\mathbf{F}$-algebraic system $\mathcal{A}/\widetilde{\Omega}(\mathbb{L})$.

We denote the class of all full $\mathcal{I}$-structures by $\mathrm{FStr}(\mathcal{I})$ and the class of all reduced full $\mathcal{I}$-structures by $\mathrm{FStr}^*(\mathcal{I})$. We also write $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$ for the collection of all full $\mathcal{I}$-structures on the $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$.

We show that full $\mathcal{I}$-structures are fully deserving of the name $\mathcal{I}$-structures.

Proposition 1389 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $\mathbb{L} = \langle \mathcal{A}, D \rangle$ *a full* $\mathcal{I}$*-structure.*

(a) D *is structural;*

(b) $\mathbb{L}$ *is an* $\mathcal{I}$*-structure;*

(c) $\mathbb{L}$ *has theorems if and only if* $\mathcal{I}$ *has theorems.*

Proof:

(a) By Proposition 1375, D^* is structural. Therefore, by Proposition 1366, D is also structural.

(b) Suppose $\mathbb{L} \in \mathrm{FStr}(\mathcal{I})$. Then, by definition, $\mathrm{ThFam}(\mathbb{L}^*) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$. Thus, by Proposition 1385, $\mathbb{L}^* \in \mathrm{Str}(\mathcal{I})$. Therefore, by Proposition 1387, $\mathbb{L} \in \mathrm{Str}(\mathcal{I})$.

(c) If $\mathcal{I}$ does not have theorems, then $\varnothing \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$. Therefore, by the definition of a full $\mathcal{I}$-structure, $\varnothing \in \mathrm{ThFam}(\mathbb{L}^*)$ and, hence $\varnothing \in \mathrm{ThFam}(\mathbb{L})$. Conversely, if $\varnothing \notin \mathrm{ThFam}(\mathcal{I})$, then $\varnothing \notin \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and, hence, $\varnothing \notin \mathrm{ThFam}(\mathbb{L})$.

■

We now show that, for every **F**-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, the pair $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ is always a full $\mathcal{I}$-structure and, thus, the weakest such structure on $\mathcal{A}$.

Proposition 1390 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$. *For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ *is the weakest full* $\mathcal{I}$*-structure on* $\mathcal{A}$.

Proof: Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an **F**-algebraic system. Then, by Corollary 1378,

$$(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*).$$

So $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ is a full $\mathcal{I}$-structure. Moreover, since, by Proposition 1389, every full $\mathcal{I}$-structure is an $\mathcal{I}$-structure, by Proposition 1385, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is the largest possible set of theory families of a full $\mathcal{I}$-structure. So $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ is the weakest full $\mathcal{I}$-structure. ■

Specializing to the algebraic system $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$, where $\langle I, \iota \rangle : \mathbf{F} \to \mathbf{F}$ is the identity morphism, we get

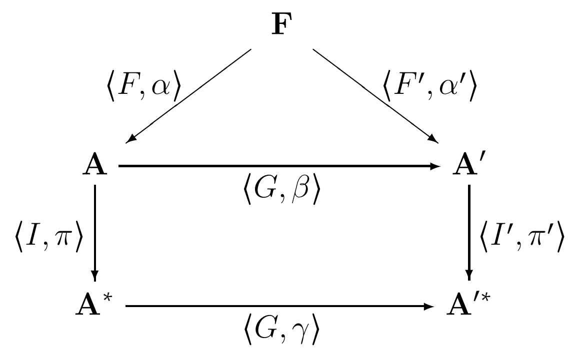
Corollary 1391 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$. *Then* $\langle \mathcal{F}, C \rangle$ *is the weakest full* $\mathcal{I}$*-structure on* $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$.

Proof: By taking $\mathcal{A} = \mathcal{F}$ in Proposition 1390. ■

Next, we see that bilogical morphisms between **F**-structures preserve the property of being a full model in both directions.

Proposition 1392 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$ *two* $\mathbf{F}$*-algebraic systems and* $\mathbb{L} = \langle \mathcal{A}, D \rangle$, $\mathbb{L}' = \langle \mathcal{A}', D' \rangle$ *two* $\mathbf{F}$*-structures. If there exists a bilogical morphism* $\langle G, \beta \rangle : \mathbb{L} \vdash \mathbb{L}'$, *then* $\mathbb{L}$ *is a full* $\mathcal{I}$*-structure if and only if* $\mathbb{L}'$ *is a full* $\mathcal{I}$*-structure.*

Proof: Suppose $\mathbb{L} = \langle \mathcal{A}, D\rangle$ and $\mathbb{L}' = \langle \mathcal{A}', D'\rangle$ are two $\mathbf{F}$-structures and let $\langle G, \beta\rangle : \mathbb{L} \vdash \mathbb{L}'$ be a bilogical morphism. Then, by Proposition 1372, there exists an α-isomorphism $\langle G, \gamma\rangle : \mathbb{L}^* \vdash^\alpha \mathbb{L}'^*$, such that the following diagram commutes.



where $\langle I, \pi\rangle : \mathbf{A} \to \mathbf{A}/\widetilde{\Omega}(\mathbb{L})$ and $\langle I', \pi'\rangle : \mathbf{A}' \to \mathbf{A}'/\widetilde{\Omega}(\mathbb{L}')$ denote the quotient morphisms. If $\mathbb{L}$ is a full $\mathcal{I}$-structure, then, by definition, $\mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$. Thus, by Proposition 1377, $\mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}'^*)$. This shows that $\mathbb{L}'$ is a full $\mathcal{I}$-structure. If, conversely, $\mathbb{L}'$ is a full $\mathcal{I}$-structure, then, by definition $\mathcal{D}'^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}'^*)$. Thus, by Proposition 1380, $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$ and, therefore, $\mathbb{L}$ is a full $\mathcal{I}$-structure, by definition. ■

Proposition 1392 allows the formulation of a characterizing property of full $\mathcal{I}$-structures in terms of bilogical morphisms and weakest full $\mathcal{I}$-structures.

Corollary 1393 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ an $\mathbf{F}$-algebraic system and $\mathbb{L} = \langle \mathcal{A}, D\rangle$ an $\mathbf{F}$-structure. $\mathbb{L}$ is a full $\mathcal{I}$-structure if and only if there exists a bilogical morphism from $\mathbb{L}$ onto an $\mathbf{F}$-structure of the form $\langle \mathcal{A}', \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')\rangle$, for some $\mathbf{F}$-algebraic system $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha'\rangle\rangle$.*

Proof: The "only if" is clear, since, if $\mathbb{L} = \langle \mathcal{A}, D\rangle$ is a full $\mathcal{I}$-structure, then $\langle I, \pi\rangle : \mathbb{L} \vdash \mathbb{L}^*$ is a bilogical morphism and, moreover, by the definition of fullness, $\mathbb{L}^* = \langle \mathcal{A}^*, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)\rangle$.

Assume, conversely, that $\langle H, \gamma\rangle : \mathbb{L} \vdash \langle \mathcal{A}', \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')\rangle$ is a bilogical morphism. By Proposition 1390, $\langle \mathcal{A}', \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')\rangle \in \mathrm{FStr}(\mathcal{I})$. Therefore, by Proposition 1392, $\mathbb{L} \in \mathrm{FStr}(\mathcal{I})$, as well. ■

We now formulate a result that can be used to show that a property of $\mathbf{F}$-structures for every full $\mathcal{I}$-structure of a π-institution $\mathcal{I}$ based on $\mathbf{F}$. It characterizes $\mathrm{FStr}(\mathcal{I})$ as the smallest class of $\mathbf{F}$-structures containing all $\mathbf{F}$-structures of the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$, with $\mathcal{A}$ ranging over all $\mathbf{F}$-algebraic systems, and closed under bilogical morphisms. It follows that to prove that a property holds for all members of $\mathrm{FStr}(\mathcal{I})$ it suffices to show that it holds for all $\mathbf{F}$-structures of the specific form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ and that it is preserved under bilogical morphisms.

Corollary 1394 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$. $\mathrm{FStr}(\mathcal{I})$ *is the smallest class containing* $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$*, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and closed under both images and preimages under bilogical morphisms.*

Proof: By Proposition 1390, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, the pair $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle \in \mathrm{FStr}(\mathcal{I})$. Moreover, by Proposition 1392, $\mathrm{FStr}(\mathcal{I})$ is closed under both images and preimages under bilogical morphisms. On the other hand, let L be a class satisfying these properties and let $\langle \mathcal{A}, D \rangle \in \mathrm{FStr}(\mathcal{I})$. By Corollary 1393, there exists an $\mathbf{F}$-algebraic system $\mathcal{A}'$ and a bilogical morphism

$$\langle H, \gamma \rangle : \langle \mathcal{A}, D \rangle \vdash \langle \mathcal{A}', \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}') \rangle.$$

By hypothesis, $\langle \mathcal{A}', \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}') \rangle \in \mathsf{L}$ and, again by hypothesis, $\langle \mathcal{A}, D \rangle \in \mathsf{L}$. Thus, we conclude that $\mathrm{FStr}(\mathcal{I}) \subseteq \mathsf{L}$. This proves that $\mathrm{FStr}(\mathcal{I})$ is indeed the smallest class satisfying the given properties. ∎

An alternative characterization of full $\mathcal{I}$-structures uses both the Leibniz and the Tarski operators.

Theorem 1395 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $\mathbb{L} = \langle \mathcal{A}, D \rangle$ *an* $\mathbf{F}$*-structure. Then* $\mathbb{L}$ *is a full* $\mathcal{I}$*-structure if and only if*

$$\mathcal{D} = \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}(\mathbb{L}) \leq \Omega^{\mathcal{A}}(T)\}.$$

Proof: Let $\mathbb{L}$ be an $\mathbf{F}$-structure and set

$$\mathcal{T} = \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}(\mathbb{L}) \leq \Omega^{\mathcal{A}}(T)\}.$$

Suppose, first, that $\mathbb{L} = \langle \mathcal{A}, D \rangle \in \mathrm{FStr}(\mathcal{I})$. We must show $\mathcal{D} = \mathcal{T}$. To this end, let $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, by Proposition 1385, $T \in \mathrm{FIFam}^{\mathcal{I}}(\mathcal{A})$ and, by the definition of the Tarski congruence system, $\widetilde{\Omega}(\mathbb{L}) \leq \Omega^{\mathcal{A}}(T)$. Thus, $\mathcal{D} \subseteq \mathcal{T}$. Conversely, if $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}(\mathbb{L}) \leq \Omega^{\mathcal{A}}(T)$, then $\widetilde{\Omega}(\mathbb{L})$ is compatible with T. Setting $T' = T/\widetilde{\Omega}(\mathbb{L})$, we have, by Corollary 56, that $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}(\mathbb{L}))$ and $T = \pi^{-1}(T')$, where $\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\widetilde{\Omega}(\mathbb{L})$ is the quotient morphism, which is also a bilogical morphism $\langle I, \pi \rangle : \mathbb{L} \vdash \mathbb{L}^*$. Since, by hypothesis $\mathbb{L}$ is full, we get that $\mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}(\mathbb{L}))$, whence, $T = \pi^{-1}(T') \in \pi^{-1}(\mathcal{D}^*) = \mathcal{D}$. We conclude that $\mathcal{T} \subseteq \mathcal{D}$.

Assume, conversely, that $\mathcal{D} = \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}(\mathbb{L}) \leq \Omega^{\mathcal{A}}(T)\}$. Then, by Proposition 1360,

$$\langle I, \pi \rangle : \mathbb{L} \vdash \langle \mathcal{A}/\widetilde{\Omega}(\mathbb{L}), \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}(\mathbb{L})) \rangle$$

is a bilogical morphism. Therefore, $\mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}(\mathbb{L}))$, showing that $\mathbb{L} \in \mathrm{FStr}(\mathcal{I})$. ∎

19.6 $\mathcal{I}$-Algebraic Systems

Since $\langle\mathcal{A},\mathcal{D}\rangle$ is a full $\mathcal{I}$-structure if and only if $\mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$, we conclude that the reduced full $\mathcal{I}$-structures are exactly those structures of the form $\langle\mathcal{A},\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$, which are reduced.

Definition 1396 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F},C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*. An* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle$ *if an* $\mathcal{I}$-**algebraic system** *if and only if the* $\mathbf{F}$*-structure* $\langle\mathcal{A},\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ *is reduced, i.e., if* $\mathcal{A}$ *is the underlying* $\mathbf{F}$*-algebraic system of a reduced full* $\mathcal{I}$*-structure.*

We denote by $\mathrm{AlgSys}(\mathcal{I})$ *the class of all* $\mathcal{I}$*-algebraic systems.*

Since $\mathcal{I}$-algebraic systems are determined based on reduced full $\mathcal{I}$-structures, the following characterization is useful in this context.

Proposition 1397 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $\mathbb{L} = \langle\mathcal{A},D\rangle$ *an* $\mathbf{F}$*-structure. Then the following are equivalent:*

(i) $\mathbb{L}$ *is a reduced full* $\mathcal{I}$*-structure;*

(ii) $\mathbb{L}$ *is reduced and* $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*;*

(iii) $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$ *and* $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

Proof:

(i)⇒(ii) Suppose that $\mathbb{L} = \langle\mathcal{A},D\rangle$ is a reduced full $\mathcal{I}$-structure. Since $\mathbb{L}$ is full, $\mathbb{L}^* = \langle\mathcal{A}^*,\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)\rangle$. Since $\mathbb{L}$ is reduced, $\mathbb{L}^* = \mathbb{L}$. Thus, $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

(ii)⇒(ii) Assume $\mathbb{L} = \langle\mathcal{A},D\rangle$ is reduced and $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Since $\mathbb{L}$ is reduced, $\mathbb{L}^* = \mathbb{L} = \langle\mathcal{A},\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$. Therefore, $\mathbb{L}$ is also full and, consequently, $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$.

(iii)⇒(i) Let $\mathbb{L} = \langle\mathcal{A},D\rangle$, with $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$ and $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Since $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, there exists a closure family D' on $\mathcal{A}$, such that $\langle\mathcal{A},D'\rangle$ is a reduced full $\mathcal{I}$-structure. Since $\langle\mathcal{A},D'\rangle$ is full and reduced, we have $\mathcal{D}' = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Since, by hypothesis, $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get that $\mathcal{D}' = \mathcal{D}$. Hence, $\mathbb{L} = \langle\mathcal{A},\mathcal{D}\rangle = \langle\mathcal{A},\mathcal{D}'\rangle$ is a reduced full $\mathcal{I}$-structure. ■

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$. Let, also, $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle$ an $\mathbf{F}$-algebraic system. We denote by $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ the collection of all congruence systems θ on $\mathcal{A}$, such that the quotient algebraic system $\mathcal{A}^\theta$ is in $\mathrm{AlgSys}(\mathcal{I})$:

$$\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}) = \{\theta \in \mathrm{ConSys}(\mathcal{A}) : \mathcal{A}^\theta \in \mathrm{AlgSys}(\mathcal{I})\}.$$

A congruence system $\theta \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is called an $\mathcal{I}$-**congruence system on** $\mathcal{A}$.

It turns out that the Tarski congruence systems of full $\mathcal{I}$-structures are all $\mathcal{I}$-congruence systems.

Proposition 1398 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $\mathbb{L} = \langle \mathcal{A}, D \rangle$ *an* $\mathbf{F}$*-structure. If* $\mathbb{L}$ *is full, then* $\mathcal{A}^* \in \mathrm{AlgSys}(\mathcal{I})$ *and, therefore,* $\widetilde{\Omega}(\mathbb{L}) \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$*.*

Proof: Suppose that $\mathbb{L} = \langle \mathcal{A}, D \rangle$ is a full $\mathcal{I}$-structure. Then, by definition, $\mathbb{L}^* = \langle \mathcal{A}^*, \mathcal{D}^* \rangle = \langle \mathcal{A}^*, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*) \rangle$ is a reduced full $\mathcal{I}$-structure. Hence $\mathcal{A}^* \in \mathrm{AlgSys}(\mathcal{I})$ and, therefore, by definition, $\widetilde{\Omega}(\mathbb{L}) \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$. ■

Even though, according to the definition, $\mathcal{I}$-algebraic systems are determined as the $\mathbf{F}$-algebraic system reducts of reduced full $\mathcal{I}$-structures, they can also be characterized without reference to fullness.

Proposition 1399 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathrm{AlgSys}(\mathcal{I})$ *is the class of all underlying* $\mathbf{F}$*-algebraic systems of all reduced* $\mathcal{I}$*-structures:*

$$\mathrm{AlgSys}(\mathcal{I}) = \{\mathcal{A} : (\exists \langle \mathcal{A}, D \rangle \in \mathrm{Str}^{\mathcal{I}}(\mathcal{A}))(\widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) = \Delta^{\mathcal{A}})\}.$$

Proof: If $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, then, by Proposition 1397, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ is a reduced full $\mathcal{I}$-structure. Conversely, if $\mathbb{L} = \langle \mathcal{A}, D \rangle$ is a reduced $\mathcal{I}$-structure, then $\mathcal{D} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and, therefore,

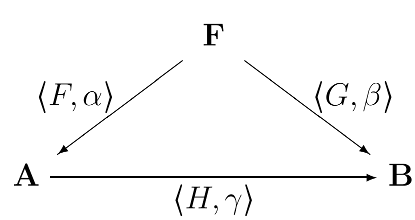
$$\widetilde{\Omega}(\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle) \leq \widetilde{\Omega}(\mathbb{L}) = \Delta^{\mathcal{A}}.$$

Thus, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ is a reduced full $\mathcal{I}$-structure and $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$. ■

The class of all $\mathcal{I}$-algebraic systems is closed under isomorphisms.

Proposition 1400 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathrm{AlgSys}(\mathcal{I})$ *is closed under isomorphisms.*

Proof: Assume $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle \in \mathrm{AlgSys}(\mathcal{I})$ and let $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ be an isomorphism.



Then, we have that $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \gamma^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}))$, which shows that

$$\langle H,\gamma\rangle : \langle\mathcal{A},\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle \cong \langle\mathcal{B},\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})\rangle.$$

Now we can use Proposition 1363 to see that $\langle\mathcal{A},\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ is reduced if and only if $\langle\mathcal{B},\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})\rangle$ is reduced and, therefore, by Proposition 1399, $\mathcal{A}\in\mathrm{AlgSys}(\mathcal{I})$ if and only if $\mathcal{B}\in\mathrm{AlgSys}(\mathcal{I})$. ■

Proposition 1397 gave characterizing conditions for reduced full $\mathcal{I}$-structures. An analog for full $\mathcal{I}$-structures is the following:

Proposition 1401 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $\mathbb{L} = \langle\mathcal{A},D\rangle$ *an* $\mathbf{F}$*-structure. Then the following are equivalent:*

(i) $\mathbb{L}$ *is a full* $\mathcal{I}$*-structure;*

(ii) $\mathcal{A}^*\in\mathrm{AlgSys}(\mathcal{I})$ *and* $\mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$*;*

(iii) There exists a bilogical morphism $\langle H,\gamma\rangle : \mathbb{L}\vdash\langle\mathcal{A}',D'\rangle$*, such that* $\mathcal{A}'\in\mathrm{AlgSys}(\mathcal{I})$ *and* $\mathcal{D}' = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$*.*

Proof:

(i)⇒(ii) Suppose $\mathbb{L} = \langle\mathcal{A},D\rangle$ is a full $\mathcal{I}$-structure. Then, by definition, $\mathbb{L}^* = \langle\mathcal{A}^*,\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)\rangle$. Thus, $\mathcal{A}^*\in\mathrm{AlgSys}(\mathcal{I})$ and $\mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$.

(ii)⇒(iii) Obvious, since $\langle H,\gamma\rangle : \mathbb{L}\vdash\mathbb{L}^*$ is a bilogical morphism.

(iii)⇒(i) Assume that $\langle H,\gamma\rangle : \mathbb{L}\to\langle\mathcal{A}',\mathcal{D}'\rangle$ is a bilogical morphism, such that $\mathcal{A}'\in\mathrm{AlgSys}(\mathcal{I})$ and $\mathcal{D}' = \mathrm{Fifam}^{\mathcal{I}}(\mathcal{A}')$. Then, by Proposition 1372, there exists an α-isomorphism $\mathbb{L}^*\vdash^\alpha\langle\mathcal{A}'^*,\mathcal{D}'^*\rangle$. Since $\mathcal{A}'\in\mathrm{AlgSys}(\mathcal{I})$ and $\mathcal{D}' = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$, it follows, by Proposition 1397, that $\langle\mathcal{A}',\mathcal{D}'\rangle$ is a reduced full $\mathcal{I}$-structure. So we have

$$\langle\mathcal{A}'^*,\mathcal{D}'^*\rangle = \langle\mathcal{A}',\mathcal{D}'\rangle = \langle\mathcal{A}',\mathrm{FiFam}(\mathcal{A}')\rangle.$$

Hence, by Proposition 1380, $\mathbb{L}^* = \langle\mathcal{A}^*,\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)\rangle$ and, therefore, $\mathbb{L}$ is a full $\mathcal{I}$-structure.

■

It turns out that, given a π-institution $\mathcal{I}$, the class of all full $\mathcal{I}$-structures, the class of all $\mathbf{F}$-structures of the form $\langle\mathcal{A},\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$, where $\mathcal{A}$ ranges over all $\mathbf{F}$-algebraic systems, as well as the class of all reduced full $\mathcal{I}$-structures are complete $\mathbf{F}$-structure semantics for $\mathcal{I}$.

Theorem 1402 (Completeness Theorem) *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is complete with respect to the following classes of* $\mathbf{F}$*-structures:*

(i) *The class of all full* $\mathcal{I}$*-structures;*

(ii) *The class of all* $\mathbf{F}$*-structures of the form* $\langle\mathcal{A},\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$*;*

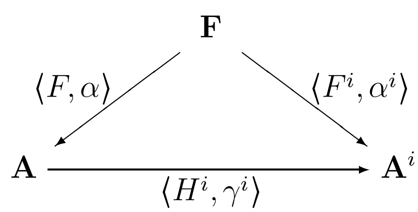
(iii) *The class of all reduced full* $\mathcal{I}$*-structures, i.e., structures of the form* $\langle\mathcal{A},\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$*, with* $\mathcal{A}\in\mathrm{AlgSys}(\mathcal{I})$*.*

Proof: Note that all three classes of $\mathbf{F}$-structures consist of $\mathcal{I}$-structures and include $\langle\mathcal{F},C\rangle^*$. Therefore, by Proposition 1388, $\mathcal{I}$ is complete with respect to each one of them. ∎

Let $\mathcal{I}=\langle\mathbf{F},C\rangle$ be a π-institution. To $\mathcal{I}$ we have associated (among others) two classes of $\mathbf{F}$-algebraic systems. One is the class $\mathrm{AlgSys}^*(\mathcal{I})$ of underlying $\mathbf{F}$-algebraic systems of reduced $\mathcal{I}$-matrix families. The other is the class $\mathrm{AlgSys}(\mathcal{I})$ of underlying $\mathbf{F}$-algebraic systems of reduced $\mathcal{I}$-structures (according to Proposition 1399). To explore an important relationship between these two classes, we introduce an operator on $\mathbf{F}$-algebraic systems, which is related to an operator on $\mathbf{F}$-matrix families, given the same name.

Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and consider $\mathbf{F}$-algebraic systems $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$ and $\mathcal{A}^i=\langle\mathbf{A}^i,\langle F^i,\alpha^i\rangle\rangle$, $i\in I$, and a system of surjective morphisms

$$\langle H^i,\gamma^i\rangle:\mathcal{A}\to\mathcal{A}^i,\quad i\in I.$$



We say $\{\langle H^i,\gamma^i\rangle:i\in I\}$ is a **subdirect intersection (system)** and call the $\langle H^i,\gamma^i\rangle$ **subdirect intersection morphisms** if

$$\bigcap_{i\in I}\mathrm{Ker}(\langle H^i,\gamma^i\rangle)=\Delta^{\mathcal{A}}.$$

If such a system exists, we say that $\mathcal{A}$ is a **subdirect intersection** of the $\mathbf{F}$-algebraic systems $\{\mathcal{A}^i:i\in I\}$. Given a class K of $\mathbf{F}$-algebraic systems and an $\mathbf{F}$-algebraic system $\mathcal{A}$, we write

$$\mathcal{A}\in\overset{\triangleleft}{\mathbb{\Pi}}(\mathsf{K})$$

to signify that $\mathcal{A}$ is a subdirect intersection of a collection $\{\mathcal{A}^i:i\in I\}$, with $\mathcal{A}^i\in\mathsf{K}$, for all $i\in I$.

We can show that the operator $\overset{\triangleleft}{\mathbb{\Pi}}$ is a closure operator on classes of $\mathbf{F}$-algebraic systems.

Proposition 1403 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system. Then*

$$\overset{\triangleleft}{\mathbb{I}\mathbb{P}} : \mathcal{P}(\mathrm{AlgSys}(\mathbf{F})) \to \mathcal{P}(\mathrm{AlgSys}(\mathbf{F}))$$

is a closure operator.

Proof: Suppose, first, that $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$ and $\mathcal{A} \in \mathsf{K}$. Since the identity morphism $\langle I, \iota \rangle : \mathcal{A} \to \mathcal{A}$ is a subdirect intersection, we get that $\mathcal{A} \in \overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\mathsf{K})$. Thus, $\overset{\triangleleft}{\mathbb{I}\mathbb{P}}$ is inflationary.

Suppose, next, that $\mathsf{K} \subseteq \mathsf{K}' \subseteq \mathrm{AlgSys}(\mathbf{F})$ and $\mathcal{A} \in \overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\mathsf{K})$. Thus, $\mathcal{A}$ is a subdirect intersection of a collection $\{\mathcal{A}^i : i \in I\} \subseteq \mathsf{K}$. Then $\mathcal{A}$ is a subdirect intersection of $\{\mathcal{A}^i : i \in I\} \subseteq \mathsf{K}'$. Hence, $\overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\mathsf{K}) \subseteq \overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\mathsf{K}')$ and, therefore, $\overset{\triangleleft}{\mathbb{I}\mathbb{P}}$ is also monotone.

Assume, finally, that $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$ and let $\mathcal{A} \in \overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\mathsf{K}))$. Thus, there exists a subdirect intersection system

$$\langle H^i, \gamma^i \rangle : \mathcal{A} \to \mathcal{A}^i, \quad i \in I,$$

where $\mathcal{A}^i \in \overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\mathsf{K})$, for all $i \in I$. Consequently, for all $i \in I$, there exists a subdirect intersection system

$$\langle H^{ij}, \gamma^{ij} \rangle : \mathcal{A}^i \to \mathcal{A}^{ij}, \quad j \in J_i,$$

where $\mathcal{A}^{ij} \in \mathsf{K}$, for all $i \in I, j \in J_i$. We consider the collection

$$\langle H^{ij}, \gamma^{ij} \rangle \circ \langle H^i, \gamma^i \rangle : \mathcal{A} \to \mathcal{A}^{ij}, \quad i \in I, j \in J_i.$$

We have

$$\begin{array}{rcl}
\bigcap_{i\in I} \bigcap_{j\in J_i} \mathrm{Ker}(\langle H^{ij}, \gamma^{ij} \rangle \circ \langle H^i, \gamma^i \rangle) & = & \bigcap_{i\in I} \bigcap_{j\in J_i} (\gamma^i)^{-1}(\mathrm{Ker}(\langle H^{ij}, \gamma^{ij} \rangle)) \\
& = & \bigcap_{i\in I} (\gamma^i)^{-1}(\bigcap_{j\in J_i} (\mathrm{Ker}(\langle H^{ij}, \gamma^{ij} \rangle))) \\
& = & \bigcap_{i\in I} (\gamma^i)^{-1}(\Delta^{\mathcal{A}^i}) \\
& = & \bigcap_{i\in I} \mathrm{Ker}(\langle H^i, \gamma^i \rangle) \\
& = & \Delta^{\mathcal{A}}.
\end{array}$$

Thus, the system $\{\langle H^{ij}, \gamma^{ij} \rangle \circ \langle H^i, \gamma^o \rangle : i \in I, j \in J_i\}$ is a subdirect intersection system, showing that $\mathcal{A} \in \overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\mathsf{K})$. We conclude that $\overset{\triangleleft}{\mathbb{I}\mathbb{P}}$ is also idempotent. ∎

Using subdirect intersections, we can give the exact relationship between the classes $\mathrm{AlgSys}(\mathcal{I})$ and $\mathrm{AlgSys}^*(\mathcal{I})$. Namely, we show that the former is the class of all subdirect intersections of collections of algebraic systems in the latter class. In particular $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$.

Theorem 1404 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *Then*

$$\mathrm{AlgSys}(\mathcal{I}) = \overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\mathrm{AlgSys}^*(\mathcal{I})).$$

Proof: Assume, first, that $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$. Then, we have

$$\bigcap\{\Omega^{\mathcal{A}}(T) : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} = \widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}.$$

Now, consider the collection

$$\langle I, \pi^{\Omega^{\mathcal{A}}(T)}\rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T), \quad T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}).$$

By the preceding equation, this collection constitutes a subdirect intersection. Moreover, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we have $\langle \mathcal{A}/\Omega^{\mathcal{A}}(T), T/\Omega^{\mathcal{A}}(T)\rangle \in \mathrm{MatFam}^*(\mathcal{I})$ and, hence, $\mathcal{A}/\Omega^{\mathcal{A}}(T) \in \mathrm{AlgSys}^*(\mathcal{I})$. Therefore, we get that $\mathcal{A} \in \overset{\triangleleft}{\mathbb{I}\mathrm{I}}(\mathrm{AlgSys}^*(\mathcal{I}))$.

Suppose, conversely, that $\mathcal{A} \in \overset{\triangleleft}{\mathbb{I}\mathrm{I}}(\mathrm{AlgSys}^*(\mathcal{I}))$. Thus, there exists a subdirect intersection

$$\langle H^i \gamma^i\rangle : \mathcal{A} \to \mathcal{A}^i, \quad i \in I,$$

with $\mathcal{A}^i \in \mathrm{AlgSys}^*(\mathcal{I})$, for all $i \in I$. Thus, for all $i \in I$, there exists $T^i \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^i)$, such that $\Omega^{\mathcal{A}^i}(T^i) = \Delta^{\mathcal{A}^i}$. Now, we calculate:

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) & = & \bigcap\{\Omega^{\mathcal{A}}(T) : T \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A})\}\\ & \subseteq & \bigcap_{i\in I}\Omega^{\mathcal{A}}((\gamma^i)^{-1}(T^i))\\ & = & \bigcap_{i\in I}(\gamma^i)^{-1}(\Omega^{\mathcal{A}^i}(T^i))\\ & = & \bigcap_{i\in I}(\gamma^i)^{-1}(\Delta^{\mathcal{A}^i})\\ & = & \bigcap_{i\in I}\mathrm{Ker}(\langle H^i, \gamma^i\rangle)\\ & = & \Delta^{\mathcal{A}}.\end{array}$$

We conclude that $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$. Thus, $\mathrm{AlgSys}(\mathcal{I}) \subseteq \overset{\triangleleft}{\mathbb{I}\mathrm{I}}(\mathrm{AlgSys}^*(\mathcal{I}))$. ■

Corollary 1405 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *Then* $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$ *and, moreover,* $\mathrm{ALgSys}^*(\mathcal{I}) = \mathrm{AlgSys}(\mathcal{I})$ *if and only if* $\mathrm{AlgSys}^*(\mathcal{I})$ *is closed under subdirect intersections.*

Proof: We have

$$\begin{array}{rcll}\mathrm{AlgSys}^*(\mathcal{I}) & \subseteq & \overset{\triangleleft}{\mathbb{I}\mathrm{I}}(\mathrm{AlgSys}^*(\mathcal{I})) & \text{(by Proposition 1403)}\\ & = & \mathrm{AlgSys}(\mathcal{I}). & \text{(by Theorem 1404)}\end{array}$$

If $\mathrm{AlgSys}^*(\mathcal{I})$ is closed under subdirect intersections,

$$\mathrm{AlgSys}(\mathcal{I}) = \overset{\triangleleft}{\mathbb{I}\mathrm{I}}(\mathrm{AlgSys}^*(\mathcal{I})) \subseteq \mathrm{AlgSys}^*(\mathcal{I}).$$

Conversely, if $\mathrm{AlgSys}^*(\mathcal{I}) = \mathrm{AlgSys}(\mathcal{I})$, then $\overset{\triangleleft}{\mathbb{I}\mathrm{I}}(\mathrm{AlgSys}^*(\mathcal{I})) = \mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$. ■

Finally, we give a relation between the classes of algebraic systems associated in this way with π-institutions based on the same algebraic system that are related by $\leq$. Recall that, given $\mathcal{I} = \langle \mathbf{F}, C\rangle$ and $\mathcal{I}' = \langle \mathbf{F}, C'\rangle$, we write $C \leq C'$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$, $C_\Sigma(\Phi) \subseteq C'_\Sigma(\Phi)$. If this is the case, we also write $\mathcal{I} \leq \mathcal{I}'$ and say that $\mathcal{I}'$ is stronger than $\mathcal{I}$ and that $\mathcal{I}$ is weaker than $\mathcal{I}'$. Recall, also, that, $\mathcal{I} \leq \mathcal{I}'$ if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

Proposition 1406 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$, $\mathcal{I}' = \langle \mathbf{F}, C'\rangle$ *be two π-institutions based on* $\mathbf{F}$. *If* $\mathcal{I} \leq \mathcal{I}'$, *then*

$$\mathrm{AlgSys}(\mathcal{I}') \subseteq \mathrm{AlgSys}(\mathcal{I}) \quad \textit{and} \quad \mathrm{AlgSys}^*(\mathcal{I}') \subseteq \mathrm{AlgSys}^*(\mathcal{I}).$$

Proof: If $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I}')$, then, there exists $T' \in \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T') = \Delta^{\mathcal{A}}$. But, since $\mathcal{I} \leq \mathcal{I}'$, we have $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Therefore $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I})$. We now conclude that $\mathrm{AlgSys}^*(\mathcal{I}') \subseteq \mathrm{AlgSys}^*(\mathcal{I})$.

For the second inclusion, we get

$$\begin{array}{rcll}
\mathrm{AlgSys}(\mathcal{I}') & = & \overset{\triangleleft}{\mathrm{I}\mathrm{P}}(\mathrm{AlgSys}^*(\mathcal{I}')) & \text{(Theorem 1404)}\\
 & \subseteq & \overset{\triangleleft}{\mathrm{I}\mathrm{P}}(\mathrm{AlgSys}^*(\mathcal{I})) & \text{(Proposition 1403)}\\
 & = & \mathrm{AlgSys}(\mathcal{I}). & \text{(Theorem 1404)}
\end{array}$$

■

19.7 Lattice of Full $\mathcal{I}$-Structures

In this section we show that, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ and an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{F}, \langle F, \alpha\rangle\rangle$, the poset $\langle \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}), \leq\rangle$ of full $\mathcal{I}$-structures on $\mathcal{A}$ and the poset $\langle \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}), \leq\rangle$ of $\mathcal{I}$-congruence systems on $\mathcal{A}$ are isomorphic through the Tarski operator

$$\langle \mathcal{A}, D\rangle \mapsto \widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}).$$

We start by defining an operator which will turn out to be the inverse of the Tarski operator.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ an $\mathbf{F}$-algebraic system. Given $\theta \in \mathrm{ConSys}(\mathcal{A})$, define

$$\widetilde{H}^{\mathcal{A}}(\theta) = \langle \mathcal{A}, \mathcal{D}^\theta\rangle,$$

by setting

$$\mathcal{D}^\theta = (\pi^\theta)^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta)),$$

where $\langle I, \pi^\theta\rangle : \mathcal{A} \to \mathcal{A}^\theta$ is the quotient morphism.

Note that, by definition of $\widetilde{H}^{\mathcal{A}}(\theta)$, the morphism

$$\langle I, \pi^\theta\rangle : \widetilde{H}^{\mathcal{A}}(\theta) \to \langle \mathcal{A}^\theta, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta)\rangle$$

is a bilogical morphism.

We have the following properties concerning the operator $\widetilde{H}$.

Lemma 1407 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system.*

(a) For every $\theta \in \mathrm{ConSys}(\mathcal{A})$*,*

- *(i)* $\theta \in \mathrm{ConSys}(\widetilde{H}^{\mathcal{A}}(\theta))$*;*
- *(ii)* $\widetilde{H}^{\mathcal{A}}(\theta)/\theta = \langle \mathcal{A}^\theta, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta) \rangle$*;*
- *(iii)* $\widetilde{H}^{\mathcal{A}}(\theta) \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$*;*

(b) $\theta \mapsto \widetilde{H}^{\mathcal{A}}(\theta)$ *is order preserving, i.e., for all* $\theta, \theta' \in \mathrm{ConSys}(\mathcal{A})$*,* $\theta \le \theta'$ *implies* $\widetilde{H}^{\mathcal{A}}(\theta) \le \widetilde{H}^{\mathcal{A}}(\theta')$*.*

Proof:

(a) For Part (i) we must show that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $\langle \phi, \psi \rangle \in \theta_\Sigma$ implies that $D^\theta_\Sigma(\phi) = D^\theta_\Sigma(\psi)$. Suppose, to this end, that $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \theta_\Sigma$. Then, we have

$$C^{\mathcal{I},\mathcal{A}^\theta}_\Sigma(\phi/\theta_\Sigma) = C^{\mathcal{I},\mathcal{A}^\theta}_\Sigma(\psi/\theta_\Sigma),$$

i.e., $C^{\mathcal{I},\mathcal{A}^\theta}_\Sigma(\pi^\theta_\Sigma(\phi)) = C^{\mathcal{I},\mathcal{A}^\theta}_\Sigma(\pi^\theta_\Sigma(\psi))$. This gives that

$$(\pi^\theta_\Sigma)^{-1}(C^{\mathcal{I},\mathcal{A}^\theta}_\Sigma(\pi^\theta_\Sigma(\phi))) = (\pi^\theta_\Sigma)^{-1}(C^{\mathcal{I},\mathcal{A}^\theta}_\Sigma(\pi^\theta_\Sigma(\psi))).$$

Since $\langle I, \pi^\theta \rangle : \widetilde{H}^{\mathcal{A}}(\theta) \to \langle \mathcal{A}^\theta, C^{\mathcal{I},\mathcal{A}^\theta} \rangle$ is a bilogical morphism, we get by Proposition 1360, $D^\theta_\Sigma(\phi) = D^\theta_\Sigma(\psi)$. We conclude that $\theta \in \mathrm{ConSys}(\widetilde{H}^{\mathcal{A}}(\theta))$.

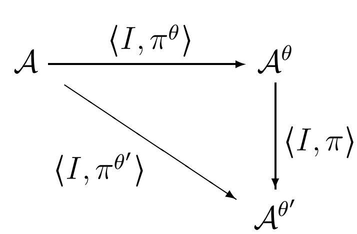
For Part (ii), we have

$$\pi^\theta(\mathcal{D}^\theta) = \pi^\theta((\pi^\theta)^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta))) = \mathrm{FiFam}^{\mathcal{A}}(\mathcal{A}^\theta),$$

where the last equality follows from the fact that $\langle I, \pi^\theta \rangle$ is a bilogical morphism, by applying Proposition 1360.

Part (iii) follows from the fact that the morphism $\langle I, \pi^\theta \rangle : \widetilde{H}^{\mathcal{A}}(\theta) \to \langle \mathcal{A}^\theta, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta) \rangle$ is a bilogical morphism and Corollary 1393.

(b) Suppose $\theta, \theta' \in \mathrm{ConSys}(\mathcal{A})$, such that $\theta \le \theta'$. Then we have the following commutative diagram of $\mathbf{F}$-algebraic systems.



where, $\langle I, \pi \rangle : \mathcal{A}^\theta \to \mathcal{A}^{\theta'}$ is defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, by

$$\pi_\Sigma(\phi/\theta_\Sigma) = \phi/\theta'_\Sigma.$$

Taking this diagram into account, we have

$$\begin{array}{rcll} \mathcal{D}^{\theta'} & = & (\pi^{\theta'})^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^{\theta'})) & (\text{Definition of } \mathcal{D}^{\theta'}) \\ & = & (\pi^{\theta})^{-1}(\pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^{\theta'}))) & (\pi \circ \pi^{\theta} = \pi^{\theta'}) \\ & \subseteq & (\pi^{\theta})^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^{\theta})) & (\text{Corollary 55}) \\ & = & \mathcal{D}^{\theta}. & (\text{Definition of } \mathcal{D}^{\theta}) \end{array}$$

Thus, we get $\widetilde{H}^{\mathcal{A}}(\theta) \leq \widetilde{H}^{\mathcal{A}}(\theta')$. ■

We are ready now for the main isomorphism theorem that was promised at the beginning of the section.

Theorem 1408 (Isomorphism Theorem) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, an* $\mathbf{F}$*-algebraic system. Then*

$$\widetilde{\Omega}^{\mathcal{A}} : \langle \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}), \leq \rangle \to \langle \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}), \leq \rangle$$

is an order isomorphism, with inverse

$$\widetilde{H}^{\mathcal{A}} : \langle \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}), \leq \rangle \to \langle \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}), \leq \rangle.$$

Proof: By Proposition 1398, if $\mathbb{L} \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$, then $\widetilde{\Omega}^{\mathcal{A}}(\mathbb{L}) \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$. Moreover, by Lemma 1407, if $\theta \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$, then $\widetilde{H}^{\mathcal{A}}(\theta) \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$. So, both $\widetilde{\Omega}^{\mathcal{A}}$ and $\widetilde{H}^{\mathcal{A}}$ are well-defined, with domains and codomains as indicated.

We show, next, that they are mutually inverse mappings.

Suppose, first, that $\mathbb{L} = \langle \mathcal{A}, D \rangle \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$. Then, by Proposition 1398, $\mathcal{A}^* \in \mathrm{AlgSys}^{\mathcal{I}}(\mathcal{A})$ and $\widetilde{\Omega}^{\mathcal{A}}(\mathbb{L}) \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$. By fullness, $\mathcal{D} = \pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*))$, where $\langle I, \pi \rangle : \mathbb{L} \vdash \mathbb{L}^*$ is the quotient bilogical morphism. Then, by definition of $\widetilde{H}^{\mathcal{A}}$, we get that $\widetilde{H}^{\mathcal{A}}(\widetilde{\Omega}^{\mathcal{A}}(\mathbb{L})) = \mathbb{L}$.

Suppose, on the other hand, that $\theta \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$. By definition, $\mathcal{A}^\theta \in \mathrm{AlgSys}(\mathcal{I})$. Thus, by definition,

$$\langle \mathcal{A}^\theta, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta) \rangle \in \mathrm{FStr}(\mathcal{I}) \quad \text{and} \quad \widetilde{\Omega}^{\mathcal{A}^\theta}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta)) = \Delta^{\mathcal{A}^\theta}.$$

Now, we get

$$\begin{array}{rcll} \widetilde{\Omega}^{\mathcal{A}}(\widetilde{H}^{\mathcal{A}}(\theta)) & = & \widetilde{\Omega}^{\mathcal{A}}((\pi^\theta)^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta))) & (\text{Definition of } \widetilde{H}^{\mathcal{A}}(\theta)) \\ & = & (\pi^\theta)^{-1}(\widetilde{\Omega}^{\mathcal{A}^\theta}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta))) & (\text{Corollary 1364}) \\ & = & (\pi^\theta)^{-1}(\Delta^{\mathcal{A}^\theta}) & (\text{Hypothesis}) \\ & = & \theta. & (\text{Set Theory}) \end{array}$$

Since, by definition $\widetilde{\Omega}^{\mathcal{A}}$ is order preserving and, by Lemma 1407, $\widetilde{H}^{\mathcal{A}}$ is also order preserving, we conclude that $\widetilde{\Omega}^{\mathcal{A}}$ is an order isomorphism with inverse $\widetilde{H}^{\mathcal{A}}$. ■

We show next that the poset of $\mathcal{I}$-congruence systems on an $\mathbf{F}$-algebraic system $\mathcal{A}$ is a complete lattice with infimum given by signature-wise intersection. In conjunction with the Isomorphism Theorem, this will yield that the poset of full $\mathcal{I}$-structures on $\mathcal{A}$ is also a complete lattice.

Theorem 1409 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, the poset* $\langle \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}), \leq \rangle$ *is a complete lattice with infimum given by signature-wise intersection.*

Proof: First, note that $\nabla^{\mathcal{A}} \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$, since $\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\nabla^{\mathcal{A}})) = \Delta^{\mathcal{A}/\nabla^{\mathcal{A}}}$ and, hence, $\mathcal{A}/\nabla^{\mathcal{A}} \in \mathrm{AlgSys}(\mathcal{I})$. So $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ has a largest element. Assume, next, that, for all $i \in I$, $\theta^i \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$. We must show that $\bigcap_{i\in I} \theta^i \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$. To this end, set $\theta := \bigcap_{i\in I} \theta^i$ and consider the projection morphisms

$$\langle I, \pi^i \rangle : \mathcal{A}^\theta \to \mathcal{A}^{\theta^i}, \quad i \in I,$$

which are bilogical morphisms

$$\langle I, \pi^i \rangle : \langle \mathcal{A}^\theta, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta) \rangle \vdash \langle \mathcal{A}^{\theta^i}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^{\theta^i}) \rangle, \quad i \in I.$$

By hypothesis, $\langle \mathcal{A}^{\theta^i}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^{\theta^i}) \rangle$ is reduced, i.e.,

$$\widetilde{\Omega}^{\mathcal{A}^{\theta^i}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^{\theta^i})) = \Delta^{\mathcal{A}^{\theta^i}}, \quad i \in I.$$

Now we have, for all $i \in I$,

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathcal{A}^\theta}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta)) & \leq & \widetilde{\Omega}^{\mathcal{A}^\theta}((\pi^i)^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^{\theta^i}))) \\ & = & (\pi^i)^{-1}(\widetilde{\Omega}^{\mathcal{A}^{\theta^i}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^{\theta^i}))) \\ & = & (\pi^i)^{-1}(\Delta^{\mathcal{A}^{\theta^i}}) \\ & = & \theta^i/\theta.\end{array}$$

Thus, we get

$$\widetilde{\Omega}^{\mathcal{A}^\theta}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta)) \leq \bigcap_{i\in I}(\theta^i/\theta) = (\bigcap_{i\in I}\theta^i)/\theta = \theta/\theta = \Delta^{\mathcal{A}^\theta}.$$

We conclude that $\langle \mathcal{A}^\theta, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta) \rangle$ is reduced and, hence, $\mathcal{A}^\theta \in \mathrm{AlgSys}(\mathcal{I})$, giving that $\theta \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$.

The conclusion of the theorem now follows. ∎

As a consequence of the Isomorphism Theorem 1408 and Theorem 1409, we get

Corollary 1410 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* $\langle \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}), \leq \rangle$ *is a complete lattice and*

$$\widetilde{\Omega}^{\mathcal{A}} : \langle \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}), \leq \rangle \to \langle \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}), \leq \rangle$$

is a lattice isomorphism.

Proof: By Theorems 1408 and 1409. ∎

It follows from the preceding results that, given a collection $\{\mathbb{L}^i : i \in I\} \subseteq \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$, with $\mathbb{L}^i = \langle \mathcal{A}, D^i \rangle$, $i \in I$, its infimum in $\langle \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}), \leq \rangle$ is the $\mathcal{I}$-structure

$$\langle \mathcal{A}, (\pi^\theta)^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^\theta)\rangle,$$

where $\theta = \bigcap_{i \in I} \widetilde{\Omega}^{\mathcal{A}}(\mathbb{L}^i)$, and $\langle I, \pi^\theta \rangle : \mathcal{A} \to \mathcal{A}^\theta$ is the quotient morphism. It is not necessarily the case, however, that this system be the signature-wise intersection of the $\mathbb{L}^i$'s. In other words, $\langle \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}), \leq \rangle$ is not, in general, a sublattice of the complete lattice of all $\mathcal{I}$-structures on $\mathcal{A}$.

It turns out that bilogical morphisms with isomorphic functor components induce isomorphisms between principal ideals of the corresponding full structure lattices and, similarly isomorphisms between principal ideals of the corresponding lattices of congruence systems.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. Let also $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, be an $\mathbf{F}$-algebraic system. Consider the complete lattice $\langle \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}), \leq \rangle$ and let $\mathbb{L} = \langle \mathcal{A}, D \rangle \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$. Recall that the ordering $\leq$ reflects the ordering of the closure operators on $\mathcal{A}$, which is dual to the inclusion ordering of the corresponding closure set systems. So, when we refer to an ideal in $\langle \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}), \leq \rangle$ we mean in the form of closure (operator) families and this translates to a filter, when one views structures in the form of their theory families. Keeping this in mind, we introduce the notation $\mathrm{FStr}^{\mathcal{I}}(\mathbb{L})$ to refer to the principal ideal of all full $\mathcal{I}$-structures on $\mathcal{A}$ generated by $\mathbb{L}$. These are full $\mathcal{I}$-structures whose collection of theory families include $\mathcal{D}$.

$$\begin{array}{rcl}\mathrm{FStr}^{\mathcal{I}}(\langle \mathcal{A}, D \rangle) & = & \{\langle \mathcal{A}, D' \rangle \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) : D' \leq D\} \\ & = & \{\langle \mathcal{A}, \mathcal{D}' \rangle \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) : \mathcal{D} \leq \mathcal{D}'\}.\end{array}$$

Then we have the following.

Proposition 1411 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$. *Let also* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{A}' = \langle \mathbf{Sign}, \mathrm{SEN}', N' \rangle$ *be* $N^\flat$*-algebraic systems, over the same category of signatures,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$ $\mathbf{F}$*-algebraic systems,* $\mathbb{L} = \langle \mathcal{A}, D \rangle$, $\mathbb{L}' = \langle \mathcal{A}', D' \rangle$ *full* $\mathcal{I}$*-structures and* $\langle I, \gamma \rangle : \mathbb{L} \vdash \mathbb{L}'$ *a bilogical morphism. Then*

$$\langle \mathcal{A}, \mathcal{X} \rangle \mapsto \langle \mathcal{A}', \gamma(\mathcal{X}) \rangle$$

is an isomorphism from $\mathrm{FStr}^{\mathcal{I}}(\mathbb{L})$ *to* $\mathrm{FStr}^{\mathcal{I}}(\mathbb{L}')$.

Moreover, the principal ideals of $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ *and of* $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}')$, *generated by* $\widetilde{\Omega}^{\mathcal{A}}(\mathbb{L})$ *and* $\widetilde{\Omega}^{\mathcal{A}'}(\mathbb{L}')$, *respectively, are isomorphic.*

Proof: By Corollary 1362, the displayed mapping is an isomorphism between $\mathbf{ClFam}(\mathbb{L})$ and $\mathbf{ClFam}(\mathbb{L}')$. Proposition 1392 gives the statement, since

$\langle I, \gamma \rangle$ induces bilogical morphisms between the corresponding elements in $\text{ClFam}(\mathbb{L})$ and $\text{ClFam}(\mathbb{L}')$. The second statement now follows by applying the Isomorphism Theorem 1408. ∎

Corollary 1412 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. Let also* $\mathbf{A} = \langle \mathbf{Sign}, \text{SEN}, N \rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}, \text{SEN}', N' \rangle$ *be* $N^\flat$*-algebraic systems, over the same category of signatures,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$ $\mathbf{F}$*-algebraic systems and* $\langle I, \gamma \rangle : \mathcal{A} \to \mathcal{A}'$ *a surjective morphism, such that*

$$\langle I, \gamma \rangle : \langle \mathcal{A}, \text{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle \vdash \langle \mathcal{A}', \text{FiFam}^{\mathcal{I}}(\mathcal{A}') \rangle$$

is a bilogical morphism. Then $\langle \mathcal{A}, \mathcal{X} \rangle \mapsto \langle \mathcal{A}', \gamma(\mathcal{X}) \rangle$ *is an isomorphism from* $\mathbf{FStr}^{\mathcal{I}}(\mathcal{A})$ *to* $\mathbf{FStr}^{\mathcal{I}}(\mathcal{A}')$*. Moreover,* $\mathbf{ConSys}^{\mathcal{I}}(\mathcal{A}) \cong \mathbf{ConSys}^{\mathcal{I}}(\mathcal{A}')$*.*

Proof: This follows by Proposition 1411, since, by Proposition 1390, the $\mathcal{I}$-structures $\langle \mathcal{A}, \text{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ and $\langle \mathcal{A}', \text{FiFam}^{\mathcal{I}}(\mathcal{A}') \rangle$ are the weakest full $\mathcal{I}$-structures on $\mathcal{A}$ and $\mathcal{A}'$, respectively. The last isomorphism follows by the second statement of Proposition 1411. ∎

We close the section by looking at some functors that relate the categories having as objects the structures that we have focused upon and with surjective homomorphism running between them.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We describe three categories related to $\mathcal{I}$.

- The category $\mathbf{FStr}(\mathcal{I})$:
  - The objects are full $\mathcal{I}$-structures $\mathbb{L} = \langle \mathcal{A}, D \rangle$;
  - Given objects $\mathbb{L} = \langle \mathcal{A}, D \rangle$ and $\mathbb{L}' = \langle \mathcal{A}', D' \rangle$, a morphism in $\mathbf{FStr}(\mathcal{I})$

    $$\langle H, \gamma \rangle : \mathbb{L} \to \mathbb{L}'$$

    is a surjective morphism $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{A}'$, which is also a logical morphism $\langle H, \gamma \rangle : \mathbb{L} \vdash \mathbb{L}'$.

  It is not difficult to verify that these two clauses specify indeed a category, with composition being ordinary composition of morphisms.

- The category $\mathbf{FStr}^*(\mathcal{I})$:

  This is the full subcategory of $\mathbf{FStr}(\mathcal{I})$, with objects all full $\mathcal{I}$-structures.

- The category $\mathbf{AlgSys}(\mathcal{I})$:
  - The objects are $\mathcal{I}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$;

 – Given objects $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ and $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha'\rangle\rangle$, a morphism in $\mathbf{AlgSys}(\mathcal{I})$

$$\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{A}'$$

 is a surjective $\mathbf{F}$-algebraic system morphism.

It is not difficult in this case either to verify that these two clauses specify indeed a category, with composition being ordinary composition of morphisms.

The following picture gives an overview of the relationships that hold between these categories and will be established shortly. The categories $\mathbf{AlgSys}(\mathcal{I})$ and $\mathbf{FStr}^*(\mathcal{I})$ are isomorphic through an isomorphism

$$\Phi : \mathbf{AlgSys}(\mathcal{I}) \cong \mathbf{FStr}^*(\mathcal{I}),$$

which will be defined in the upcoming Theorem 1413. Moreover, the category $\mathbf{FStr}^*(\mathcal{I})$ is a reflective subcategory of the category $\mathbf{FStr}(\mathcal{I})$, with reflector the reduction functor $^* : \mathbf{FStr}(\mathcal{I}) \to \mathbf{FStr}^*(\mathcal{I})$ that will be visited in detail in the last Theorem 1414 of the section.

$$\mathbf{AlgSys}(\mathcal{I}) \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} \mathbf{FStr}^*(\mathcal{I}) \underset{*}{\overset{J}{\hookrightarrow\!\!\!\leftarrow}} \mathbf{FStr}(\mathcal{I})$$

Theorem 1413 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then the categories* $\mathbf{AlgSys}(\mathcal{I})$ *and* $\mathbf{FStr}^*(\mathcal{I})$ *are isomorphic.*

Proof: We define the functor

$$\Phi : \mathbf{AlgSys}(\mathcal{I}) \to \mathbf{FStr}^*(\mathcal{I})$$

by setting:

- For all $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle \in \mathrm{AlgSys}(\mathcal{I})$,

$$\Phi(\mathcal{A}) = \langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle;$$

- For all $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{A}'$ in $\mathbf{AlgSys}(\mathcal{I})$,

$$\Phi(\langle H, \gamma\rangle) = \langle H, \gamma\rangle : \langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle \vdash \langle \mathcal{A}', \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')\rangle.$$

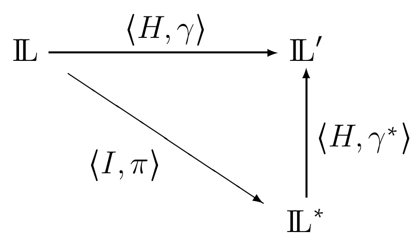
First, observe that, by Proposition 1397, if $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, then $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ is a reduced full $\mathcal{I}$-structure. So Φ is correctly defined. Moreover, if $\langle \mathcal{A}, D\rangle \in \mathrm{FStr}^*(\mathcal{I})$, then, again by Proposition 1397, $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$. Thus, Φ is a bijection on objects.

Finally, by Corollary 55, if $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{A}'$ is a surjective morphism, then, for all $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}')$, $\gamma^{-1}(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Therefore, $\Phi(\langle H,\gamma\rangle)$ is a well-defined logical morphism, by Proposition 1358. Since it is clear that Φ is bijective on morphisms as well, we get that $\Phi : \mathbf{AlgSys}(\mathcal{I}) \to \mathbf{FStr}^*(\mathcal{I})$ is indeed an isomorphism of categories. ■

Finally, for the reflection:

Theorem 1414 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then* $\mathbf{FStr}^*(\mathcal{I})$ *is a full reflective subcategory of* $\mathbf{FStr}(\mathcal{I})$ *with reflector the reduction functor* $^* : \mathbf{FStr}(\mathcal{I}) \to \mathbf{FStr}^*(\mathcal{I})$.

Proof: It is obvious that $\mathbf{FStr}^*(\mathcal{I})$ is a full subcategory of $\mathbf{FStr}(\mathcal{I})$. We must show that, given $\mathbb{L} = \langle \mathcal{A}, D\rangle \in \mathrm{FStr}(\mathcal{I})$, the pair $\langle \mathbb{L}^*, \langle I,\pi\rangle : \mathbb{L} \to \mathbb{L}^*\rangle$ is a reflector, i.e., that, given $\mathbb{L}' = \langle \mathcal{A}', D'\rangle \in \mathrm{FStr}^*(\mathcal{I})$ and $\langle H,\gamma\rangle : \mathbb{L} \to \mathbb{L}'$ in $\mathbf{FStr}(\mathcal{I})$, there exists a unique $\langle H,\gamma^*\rangle : \mathbb{L}^* \to \mathbb{L}'$ in $\mathbf{FStr}^*(\mathcal{I})$, such that the following diagram commutes.



Consider $\mathbb{L}^\gamma := \langle \mathcal{A}, \gamma^{-1}(\mathcal{D}')\rangle$. Clearly, since, by hypothesis and Proposition 1358, $\gamma^{-1}(\mathcal{D}') \subseteq \mathcal{D}$, we have that $\mathbb{L} \le \mathbb{L}^\gamma$. Now we have

$$\begin{array}{rcll}
\mathrm{Ker}(\langle I,\pi\rangle) & = & \widetilde{\Omega}^{\mathcal{A}}(\mathbb{L}) & \text{(Set Theory)}\\
& \le & \widetilde{\Omega}^{\mathcal{A}}(\mathbb{L}^\gamma) & (\mathbb{L} \le \mathbb{L}^\gamma)\\
& = & \gamma^{-1}(\widetilde{\Omega}^{\mathcal{A}'}(\mathbb{L}')) & \text{(Corollary 1364)}\\
& = & \gamma^{-1}(\Delta^{\mathcal{A}'}) & (\mathbb{L}' \in \mathrm{FStr}^*(\mathcal{I}))\\
& = & \mathrm{Ker}(\langle H,\gamma\rangle). & \text{(Set Theory)}
\end{array}$$

By the Fill-in Lemma (Proposition 1374), there exists a unique logical morphism $\langle H,\gamma^*\rangle : \mathbb{L}^* \to \mathbb{L}'$, such that the displayed diagram commutes, which, in addition, is surjective by the commutativity of the triangle. ■

19.8 Frege Relations Revisited

We revisit here in more detail the types of Frege relations and Frege operators one may consider in conjunction with π-institutions or π-structures, more generally.

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system and $T \in \mathrm{SenFam}(\mathbf{A})$.

- The **local Frege relation family** $\lambda^{\mathbf{A}}(T) = \{\lambda^{\mathbf{A}}_\Sigma(T)\}_{\Sigma\in|\mathbf{Sign}|}$ **of** T **on** $\mathbf{A}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\lambda^{\mathbf{A}}_\Sigma(T) = \{\langle\phi,\psi\rangle \in \mathrm{SEN}(\Sigma)^2 : \phi \in T_\Sigma \text{ iff } \psi \in T_\Sigma\}.$$

- The **global Frege relation family** $\Lambda^{\mathbf{A}}(T) = \{\Lambda^{\mathbf{A}}_\Sigma(T)\}_{\Sigma\in|\mathbf{Sign}|}$ **of** T **on** $\mathbf{A}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\begin{aligned}\Lambda^{\mathbf{A}}_\Sigma(T) &= \{\langle\phi,\psi\rangle \in \mathrm{SEN}(\Sigma)^2 : \text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma'),\\ &\qquad \mathrm{SEN}(f)(\phi) \in T_{\Sigma'} \text{ iff } \mathrm{SEN}(f)(\psi) \in T_{\Sigma'}\}.\end{aligned}$$

The operators $\lambda^{\mathbf{A}}, \Lambda^{\mathbf{A}} : \mathrm{SenFam}(\mathbf{A}) \to \mathrm{RelFam}(\mathbf{A})$ are called the **local** and **global Frege operators on A**, respectively.

Let now $\mathbf{A} = \langle\mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system and $\mathbb{L} = \langle\mathbf{A}, D\rangle$ be a π-structure.

- The **local Frege relation family** $\widetilde{\lambda}^{\mathbf{A}}(\mathbb{L}) = \widetilde{\lambda}^{\mathbf{A}}(D) = \{\widetilde{\lambda}^{\mathbf{A}}_\Sigma(D)\}_{\Sigma\in|\mathbf{Sign}|}$ **of** $\mathbb{L}$, or **of** D **on A**, is defined by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\widetilde{\lambda}^{\mathbf{A}}_\Sigma(D) = \{\langle\phi,\psi\rangle \in \mathrm{SEN}(\Sigma)^2 : D_\Sigma(\phi) = D_\Sigma(\psi)\}.$$

- The **global Frege relation family** $\widetilde{\Lambda}^{\mathbf{A}}(\mathbb{L}) = \widetilde{\Lambda}^{\mathbf{A}}(D) = \{\widetilde{\Lambda}^{\mathbf{A}}_\Sigma(D)\}_{\Sigma\in|\mathbf{Sign}|}$ **of** $\mathbb{L}$, or **of** D **on A**, is defined by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\begin{aligned}\widetilde{\Lambda}^{\mathbf{A}}_\Sigma(D) &= \{\langle\phi,\psi\rangle \in \mathrm{SEN}(\Sigma)^2 : \text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma'),\\ &\qquad D_{\Sigma'}(\mathrm{SEN}(f)(\phi)) = D_{\Sigma'}(\mathrm{SEN}(f)(\psi))\}.\end{aligned}$$

Consider again an algebraic system $\mathbf{A} = \langle\mathbf{Sign}, \mathrm{SEN}, N\rangle$, a π-structure $\mathbb{L} = \langle\mathbf{A}, D\rangle$ and $X \in \mathrm{SenFam}(\mathbf{A})$. Recall the notation $D^X : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}$ denoting the closure family on $\mathbf{A}$ that is defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)$, by

$$D^X_\Sigma(\Phi) = D_\Sigma(X_\Sigma \cup \Phi).$$

- The **local Frege relation family**

$$\widetilde{\lambda}^{\mathbb{L}}(X) = \widetilde{\lambda}^{\mathbf{A},D}(X) = \{\widetilde{\lambda}^{\mathbf{A},D}_\Sigma(X)\}_{\Sigma\in|\mathbf{Sign}|}$$

of X **in** $\mathbb{L}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\widetilde{\lambda}^{\mathbf{A},D}_\Sigma(X) = \{\langle\phi,\psi\rangle \in \mathrm{SEN}(\Sigma)^2 : D^X_\Sigma(\phi) = D^X_\Sigma(\psi)\}.$$

- The **global Frege relation family**

$$\widetilde{\Lambda}^{\mathbb{L}}(X) = \widetilde{\Lambda}^{\mathbf{A},D}(X) = \{\widetilde{\Lambda}^{\mathbf{A},D}_\Sigma(X)\}_{\Sigma\in|\mathbf{Sign}|}$$

of X **in** $\mathbb{L}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\begin{aligned}\widetilde{\Lambda}^{\mathbf{A},D}_\Sigma(X) &= \{\langle\phi,\psi\rangle \in \mathrm{SEN}(\Sigma)^2 : \text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma'),\\ &\qquad D^X_{\Sigma'}(\mathrm{SEN}(f)(\phi)) = D^X_{\Sigma'}(\mathrm{SEN}(f)(\psi))\}.\end{aligned}$$

The operators $\widetilde{\lambda}^{\mathbf{A},D}, \widetilde{\Lambda}^{\mathbf{A},D} : \mathrm{SenFam}(\mathbf{A}) \to \mathrm{RelFam}(\mathbf{A})$ are called the **local** and **global Frege operators on** $\mathbb{L}$, respectively.

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system, $\mathbb{L} = \langle \mathbf{A}, D\rangle$ be a π-structure and $X \in \mathrm{SenFam}(\mathbf{A})$. Some obvious relationships hold between several of the notions defined above. We denote by $\mathrm{Thm}(\mathbb{L}) = \{\mathrm{Thm}_\Sigma(\mathbb{L})\}_{\Sigma\in|\mathbf{Sign}|}$, where $\mathrm{Thm}_\Sigma(\mathbb{L}) = D_\Sigma(\varnothing)$, an obvious generalization of the corresponding notion from π-institutions. Note, however, that, since D is not necessarily structural, in this case $\mathrm{Thm}(\mathbb{L})$ is a theory family, but not necessarily a theory system. Then, we have the following:

$$\begin{array}{rcl}\widetilde{\lambda}^{\mathbf{A}}(D) & = & \widetilde{\lambda}^{\mathbf{A},D}(\mathrm{Thm}(\mathbb{L}));\\ \widetilde{\Lambda}^{\mathbf{A}}(D) & = & \widetilde{\Lambda}^{\mathbf{A},D}(\mathrm{Thm}(\mathbb{L}));\\ & & \\ \widetilde{\lambda}^{\mathbf{A},D}(X) & = & \bigcap\{\lambda^{\mathbf{A}}(T) : X \le T \in \mathrm{ThFam}(\mathbb{L})\};\\ \widetilde{\Lambda}^{\mathbf{A},D}(X) & = & \bigcap\{\Lambda^{\mathbf{A}}(T) : X \le T \in \mathrm{ThFam}(\mathbb{L})\}.\end{array}$$

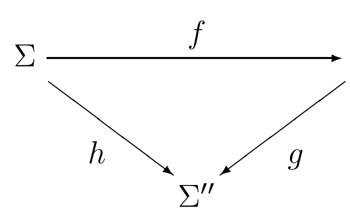
We show that all three local Frege operators give rise to equivalence families, whereas all three global operators give rise to equivalence systems on the underlying algebraic system $\mathbf{A}$.

Lemma 1415 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system,* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *a* π*-structure and* $X \in \mathrm{SenFam}(\mathbf{A})$.

(a) $\lambda^{\mathbf{A}}(X)$, $\widetilde{\lambda}^{\mathbf{A}}(D)$ *and* $\widetilde{\lambda}^{\mathbf{A},D}(X)$ *are equivalence families on* $\mathbf{A}$*;*

(b) $\Lambda^{\mathbf{A}}(X)$, $\widetilde{\Lambda}^{\mathbf{A}}(D)$ *and* $\widetilde{\Lambda}^{\mathbf{A},D}(X)$ *are equivalence systems on* $\mathbf{A}$*.*

Proof: Because of the interdependencies between these concepts, pointed out before the lemma, it suffices to prove the statements only for $\lambda^{\mathbf{A}}(X)$ and $\Lambda^{\mathbf{A}}(X)$. That both $\lambda^{\mathbf{A}}(X)$ and $\Lambda^{\mathbf{A}}(X)$ are equivalence families is obvious because of the properties of the equivalence connective used in their definitions. So it suffices to show only that $\Lambda^{\mathbf{A}}(X)$ is a system, i.e., that it is invariant under signature morphisms. So suppose $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi\rangle \in \Lambda^{\mathbf{A}}_\Sigma(X)$ and let $\Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma, \Sigma')$.



By the definition of $\Lambda^{\mathbf{A}}(X)$, we have that, for all $\Sigma'' \in |\mathbf{Sign}|$ and all $h \in \mathbf{Sign}(\Sigma, \Sigma'')$,

$$D_{\Sigma''}(\mathrm{SEN}(h)(\phi)) = D_{\Sigma''}(\mathrm{SEN}(h)(\psi)).$$

A fortiori, for all $\Sigma'' \in |\mathbf{Sign}|$ and all $g \in \mathbf{Sign}(\Sigma', \Sigma'')$, we have

$$D_{\Sigma''}(\mathrm{SEN}(g)(\mathrm{SEN}(f)(\phi))) = D_{\Sigma''}(\mathrm{SEN}(g)(\mathrm{SEN}(f)(\psi))).$$

By the definition of $\Lambda^{\mathbf{A}}(X)$, this proves that $\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)\rangle \in \Lambda^{\mathbf{A}}_{\Sigma'}(X)$. Thus $\Lambda^{\mathbf{A}}(X)$ is indeed an equivalence system on $\mathbf{A}$. ■

The next lemma shows that all four "tilde" Frege operators are monotone.

Lemma 1416 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system and* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *a* π*-structure on* $\mathbf{A}$*.*

(a) $\widetilde{\lambda}^{\mathbf{A}}, \widetilde{\Lambda}^{\mathbf{A}} : \mathbf{ClFam}(\mathbf{A}) \to \mathbf{EqvFam}(\mathbf{A})$ *are monotone;*

(b) $\widetilde{\lambda}^{\mathbf{A},D}, \widetilde{\Lambda}^{\mathbf{A},D} : \mathbf{SenFam}(\mathbf{A}) \to \mathbf{EqvFam}(\mathbf{A})$ *are monotone.*

Proof: Suppose $D, D' \in \mathrm{ClFam}(\mathbf{A})$, such that $D \le D'$, and let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi\rangle \in \widetilde{\lambda}^{\mathbf{A}}_{\Sigma}(D)$. Then $D_\Sigma(\phi) = D_\Sigma(\psi)$, whence $D'_\Sigma(\phi) = D'_\Sigma(\psi)$. Thus, $\langle \phi, \psi\rangle \in \widetilde{\lambda}^{\mathbf{A}}_{\Sigma}(D')$. We conclude that $\widetilde{\lambda}^{\mathbf{A}}(D) \le \widetilde{\lambda}^{\mathbf{A}}(D')$. The proof for $\widetilde{\Lambda}^{\mathbf{A}} : \mathbf{ClFam}(\mathbf{A}) \to \mathbf{EqvFam}(\mathbf{A})$ is similar.

Suppose, next, that $X, X' \in \mathrm{SenFam}(\mathbf{A})$, such that $X \le X'$. Note that, in this situation, we have

$$\{T \in \mathrm{ThFam}(\mathbb{L}) : X' \le T\} \subseteq \{T \in \mathrm{ThFam}(\mathbb{L}) : X \le T\}.$$

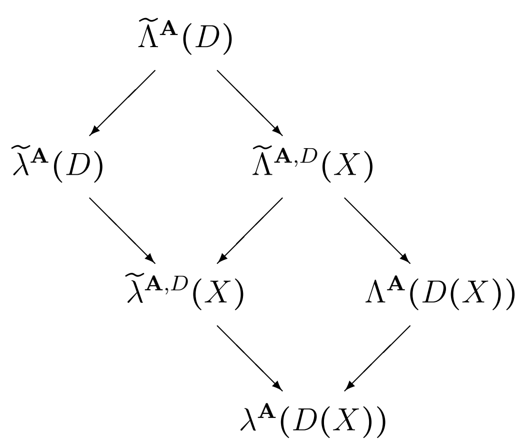
Therefore, we have

$$\begin{array}{rcl}\widetilde{\lambda}^{\mathbf{A},D}(X) & = & \bigcap\{\lambda^{\mathbf{A}}(T) : X \le T \in \mathrm{ThFam}(\mathbb{L})\}\\ & \le & \bigcap\{\lambda^{\mathbf{A}}(T) : X' \le T \in \mathrm{ThFam}(\mathbb{L})\}\\ & = & \widetilde{\lambda}^{\mathbf{A},D}(X').\end{array}$$

The proof for $\widetilde{\Lambda}^{\mathbf{A},D} : \mathbf{SenFam}(\mathbf{A}) \to \mathbf{EqvFam}(\mathbf{A})$ is similar. ■

The equivalence families produced by applying the six Frege operators form a hierarchy under inclusion that we now make explicit.

Proposition 1417 *Let* $\mathbf{A} = \langle \mathbf{A}, \mathrm{SEN}, N\rangle$ *be an algebraic system,* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *a* π*-structure and* $X \in \mathrm{SenFam}(\mathbf{A})$*. Then, we have the following inclusions between equivalence families on* $\mathbf{A}$*:*



Proof: First, note that the three southwest inclusions are obvious, since the conditions defining $\widetilde{\lambda}^{\mathbf{A}}$, $\widetilde{\lambda}^{\mathbf{A},D}$ and $\lambda^{\mathbf{A}}$ are special cases of the ones defining $\widetilde{\Lambda}^{\mathbf{A}}$, $\widetilde{\Lambda}^{\mathbf{A},D}$ and $\Lambda^{\mathbf{A}}$, respectively.

We show, next, the southeast inclusions between the λ's, since the ones between the Λ's may shown similarly. We have

$$\widetilde{\lambda}^{\mathbf{A}}(D) = \widetilde{\lambda}^{\mathbf{A},D}(\mathrm{Thm}(\mathbb{L})) \leq \widetilde{\lambda}^{\mathbf{A},D}(D(X)) = \widetilde{\lambda}^{\mathbf{A},D}(X).$$

Moreover,

$$\begin{array}{rcl}\widetilde{\lambda}^{\mathbf{A},D}(X) & = & \bigcap\{\lambda^{\mathbf{A}}(T) : X \leq T \in \mathrm{ThFam}(\mathbb{L})\}\\ & \leq & \lambda^{\mathbf{A}}(D(X)).\end{array}$$

Therefore, we have $\widetilde{\lambda}^{\mathbf{A}}(D) \leq \widetilde{\lambda}^{\mathbf{A},D}(X) \leq \lambda^{\mathbf{A}}(D(X))$. ■

In the case of structural π-structures, i.e., π-institutions, the hierarchy collapses to a smaller one, the top pair collapses and in the case of a sentence system, the middle pair does also. More precisely, we have

Proposition 1418 *Let* $\mathbf{A} = \langle\mathbf{A}, \mathrm{SEN}, N\rangle$ *be an algebraic system,* $\mathbb{L} = \langle\mathbf{A}, D\rangle$ *a* π*-structure and* $X \in \mathrm{SenSys}(\mathbf{A})$*. If* D *is structural, then*

$$\widetilde{\Lambda}^{\mathbf{A}}(D) = \widetilde{\lambda}^{\mathbf{A}}(D) \quad \textit{and} \quad \widetilde{\Lambda}^{\mathbf{A},D}(X) = \widetilde{\lambda}^{\mathbf{A},D}(X).$$

Proof: By the remarks preceding Lemma 1415, it suffices to show that the second equation holds. Since it is always the case that $\widetilde{\Lambda}^{\mathbf{A},D}(X) \leq \widetilde{\lambda}^{\mathbf{A},D}(X)$, we must prove the opposite inclusion. Let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi, \psi\rangle \in \widetilde{\lambda}^{\mathbf{A},D}_{\Sigma}(X)$. Then, we have, by definition, $D_\Sigma(X_\Sigma, \phi) = D_\Sigma(X_\Sigma, \psi)$. By the structurality of D, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

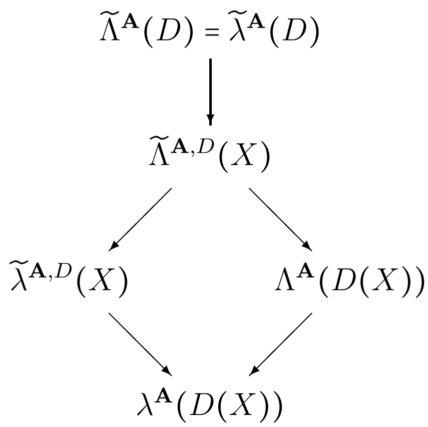
$$D_{\Sigma'}(\mathrm{SEN}(f)(X_\Sigma), \mathrm{SEN}(f)(\phi)) = D_{\Sigma'}(\mathrm{SEN}(f)(X_\Sigma), \mathrm{SEN}(f)(\psi)).$$

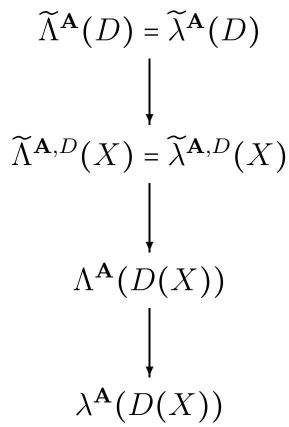
Therefore, is X is a sentence system, we get

$$D_{\Sigma'}(X_{\Sigma'}, \mathrm{SEN}(f)(\phi)) = D_{\Sigma'}(X_{\Sigma'}, \mathrm{SEN}(f)(\psi)).$$

We conclude that $\langle\phi, \psi\rangle \in \widetilde{\Lambda}^{\mathbf{A},D}_{\Sigma}(X)$. ■

Thus, if $\mathbb{L} = \langle\mathbf{A}, D\rangle$ is a π-institution, and $X \in \mathrm{SenFam}(\mathbf{A})$, we obtain the simplified hierarchy of Frege relation families shown on the left and, if, in addition, $X \in \mathrm{SenSys}(\mathbf{A})$, we get the linear hierarchy shown on the right.





We look next at how finitarity of a closure family relates to continuity of Frege operators.

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an algebraic system. Recall that:

- An $X \in \mathrm{SenFam}(\mathbf{A})$ is called **locally finite** if, for all $\Sigma \in |\mathbf{Sign}|$, X_Σ is finite. We write $Y \leq_{lf} X$ to suggest that Y is a locally finite sentence subfamily of X.

- A collection $\mathcal{X} \subseteq \mathrm{SenFam}(\mathbf{A})$ is said to be **locally directed** if, for every $\Sigma \in |\mathbf{Sign}|$ and finite $\mathcal{Y} \subseteq \mathcal{X}$, there exists $X \in \mathcal{X}$, such that $Y_\Sigma \leq X_\Sigma$, for all $Y \in \mathcal{Y}$.

Let $\mathbb{L} = \langle \mathbf{A}, D \rangle$ be a π-structure based on $\mathbf{A}$.

- $\mathbb{L}$ is **finitary** if, for all $X \in \mathrm{SenFam}(\mathbf{A})$,
$$D(X) = \bigcup \{D(Y) : Y \leq_{lf} X\}.$$

- The operator $\widetilde{\lambda}^{\mathbf{A},D} : \mathrm{SenFam}(\mathbf{A}) \to \mathrm{EqvFam}(\mathbf{A})$ is **locally continuous** if, for every locally directed $\{X^i : i \in I\} \subseteq \mathrm{SenFam}(\mathbf{A})$,
$$\widetilde{\lambda}^{\mathbf{A},D}(\bigcup_{i \in I} X^i) = \bigcup_{i \in I} \widetilde{\lambda}^{\mathbf{A},D}(X^i).$$

Proposition 1419 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *be an algebraic system and* $\mathbb{L} = \langle \mathbf{A}, D \rangle$ *a* π*-structure based on* $\mathbf{A}$. $\mathbb{L}$ *is finitary if and only if*
$$\widetilde{\lambda}^{\mathbf{A},D} : \mathrm{SenFam}(\mathbf{A}) \to \mathrm{EqvFam}(\mathbf{A})$$
is locally continuous.

Proof: Suppose, first, that $\mathbb{L}$ is finitary and let $\{X^i : i \in I\} \subseteq \mathrm{SenFam}(\mathbf{A})$ be locally directed. Since, by Lemma 1416, $\widetilde{\lambda}^{\mathbf{A},D}$ is monotone, we have

$$\bigcup_{i\in I}\widetilde{\lambda}^{\mathbf{A},D}(X^i) \le \widetilde{\lambda}^{\mathbf{A},D}(\bigcup_{i\in I}X^i).$$

To show the reverse inclusion, let $\Sigma \in |\mathbf{Sign}|$, $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle \in \widetilde{\lambda}^{\mathbf{A},D}_\Sigma(\bigcup_{i\in I}X^i)$. Then, by definition, $D_\Sigma(\bigcup_{i\in I}X^i_\Sigma,\phi) = D_\Sigma(\bigcup_{i\in I}X^i_\Sigma,\psi)$. Since $\mathbb{L}$ is finitary, there exists finite $\Phi \le_f \bigcup_{i\in I}X^i_\Sigma$, such that $D_\Sigma(\Phi,\phi) = D_\Sigma(\Phi,\psi)$. Hence, since $\{X^i : i \in I\}$ is locally directed, there exists $i \in I$, such that $\Phi \subseteq X^i_\Sigma$. Hence, $D_\Sigma(X^i_\Sigma,\phi) = D_\Sigma(X^i_\Sigma,\psi)$, i.e., $\langle\phi,\psi\rangle \in \widetilde{\lambda}^{\mathbf{A},D}_\Sigma(X^i)$. We conclude that $\widetilde{\lambda}^{\mathbf{A},D}(\bigcup_{i\in I}X^i) \le \bigcup_{i\in I}\widetilde{\lambda}^{\mathbf{A},D}(X^i)$.

Assume, conversely, that $\widetilde{\lambda}^{\mathbf{A},D}$ is locally continuous and consider $X \in \mathrm{SenFam}(\mathbf{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in D_\Sigma(X_\Sigma)$. Let $\mathcal{Z} = \{Z \in \mathrm{SenFam}(\mathbf{A}) : Z \le_{lf} X\}$. $\mathcal{Z}$ is a locally directed family, such that $\bigcup\mathcal{Z} = X$. For all $\psi \in X_\Sigma$, we have $D_\Sigma(X_\Sigma,\phi) = D_\Sigma(X_\Sigma,\psi) = D_\Sigma(X_\Sigma)$. So we get

$$\langle\phi,\psi\rangle \in \widetilde{\lambda}^{\mathbf{A},D}_\Sigma(X) = \widetilde{\lambda}^{\mathbf{A},D}_\Sigma(\bigcup\mathcal{Z}).$$

By the local directedness of $\mathcal{Z}$ and local continuity of $\widetilde{\lambda}^{\mathbf{A},D}$, we get $\langle\phi,\psi\rangle \in \bigcup_{Z\in\mathcal{Z}}\widetilde{\lambda}^{\mathbf{A},D}_\Sigma(Z)$. Therefore, we get, for some $Z \le_{lf} X$, $\phi \in D_\Sigma(Z_\Sigma,\psi) = D_\Sigma(Z_\Sigma)$. This shows that $\mathbb{L}$ is finitary. ∎

Among the key properties of Frege relations, which partly explains their usefulness in the algebraic study of logical systems, is that, loosely speaking, they are approximated from below by the Leibniz, the Tarski and the Suszko congruence systems.

Proposition 1420 *Let* $\mathbf{A} = \langle\mathbf{Sign},\mathrm{SEN},N\rangle$ *be an algebraic system and* $\mathbb{L} = \langle\mathbf{A},D\rangle$ *a* π*-structure based on* $\mathbf{A}$ *and* $T \in \mathrm{ThFam}(\mathbb{L})$.

(a) The Leibniz congruence system $\Omega^{\mathbf{A}}(T)$ *is the largest congruence system on* $\mathbf{A}$ *included in* $\Lambda^{\mathbf{A}}(T)$ *and in* $\lambda^{\mathbf{A}}(T)$*;*

(b) The Tarski congruence system $\widetilde{\Omega}^{\mathbf{A}}(D)$ *is the largest congruence system on* $\mathbf{A}$ *included in* $\widetilde{\Lambda}^{\mathbf{A}}(D)$ *and in* $\widetilde{\lambda}^{\mathbf{A}}(D)$*;*

(c) The Suszko congruence system $\widetilde{\Omega}^{\mathbf{A},D}(T)$ *is the largest congruence system on* $\mathbf{A}$ *included in* $\widetilde{\Lambda}^{\mathbf{A},D}(T)$ *and in* $\widetilde{\lambda}^{\mathbf{A},D}(T)$*.*

Proof: Let $\Sigma \in |\mathbf{Sign}|$, $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle \in \Omega^{\mathbf{A}}_\Sigma(T)$. Since $\Omega^{\mathbf{A}}(T)$ is a congruence system, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma,\Sigma')$, $\langle\mathrm{SEN}(f)(\phi),\mathrm{SEN}(f)(\psi)\rangle \in \Omega^{\mathbf{A}}_{\Sigma'}(T)$. Thus, by the compatibility property of $\Omega^{\mathbf{A}}(T)$ with T, we get

$$\mathrm{SEN}(f)(\phi) \in T_{\Sigma'} \quad \text{iff} \quad \mathrm{SEN}(f)(\psi) \in T_{\Sigma'},$$

i.e., $\langle\phi,\psi\rangle \in \Lambda^{\mathbf{A}}_\Sigma(T)$. We conclude that $\Omega^{\mathbf{A}}(T) \le \Lambda^{\mathbf{A}}(T) \le \lambda^{\mathbf{A}}(T)$.

Suppose, next, that $\theta \in \mathrm{ConSys}(\mathbf{A})$, such that $\theta \leq \lambda^{\mathbf{A}}(T)$. If $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \theta_\Sigma$ and $\phi \in T_\Sigma$, then $\langle \phi, \psi \rangle \in \lambda^{\mathbf{A}}(T)$ and $\phi \in T_\Sigma$, whence by the definition of $\lambda^{\mathbf{A}}(T)$, $\psi \in T_\Sigma$. Thus, θ is a congruence system on $\mathbf{A}$ compatible with T and, therefore, $\theta \leq \Omega^{\mathbf{A}}(T)$, by the maximality property of $\Omega^{\mathbf{A}}(T)$.

Parts (b) and (c) can be proved similarly. ∎

We show next that Frege relations are preserved under inverse surjective morphisms.

Lemma 1421 *Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ be algebraic systems, $\mathbb{L} = \langle \mathbf{A}, D \rangle$, $\mathbb{L}' = \langle \mathbf{A}', D' \rangle$ be π-structures based on $\mathbf{A}$, $\mathbf{A}'$, respectively, and $\langle F, \alpha \rangle : \mathbf{A} \to \mathbf{A}'$ a surjective morphism.*

(a) For every $X \in \mathrm{SenFam}(\mathbf{A}')$, $\Lambda^{\mathbf{A}}(\alpha^{-1}(X)) = \alpha^{-1}(\Lambda^{\mathbf{A}'}(X))$ and, also, $\lambda^{\mathbf{A}}(\alpha^{-1}(X)) = \alpha^{-1}(\lambda^{\mathbf{A}'}(X))$;

(b) If $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ is a bilogical morphism, then $\widetilde{\Lambda}^{\mathbf{A}}(D) = \alpha^{-1}(\widetilde{\Lambda}^{\mathbf{A}'}(D'))$ and, also, $\widetilde{\lambda}^{\mathbf{A}}(D) = \alpha^{-1}(\widetilde{\lambda}^{\mathbf{A}'}(D'))$.

Proof:

(a) Let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$. We have $\langle \phi, \psi \rangle \in \alpha_\Sigma^{-1}(\Lambda^{\mathbf{A}'}(X))$ iff $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \Lambda^{\mathbf{A}'}_{F(\Sigma)}(X)$ iff, by surjectivity, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi)) \in X_{F(\Sigma')} \quad \text{iff} \quad \mathrm{SEN}'(F(f))(\alpha_\Sigma(\psi)) \in X_{F(\Sigma')},$$

iff, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \in X_{F(\Sigma')} \quad \text{iff} \quad \alpha_{\Sigma'}(\mathrm{SEN}'(f)(\psi)) \in X_{F(\Sigma')},$$

iff, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}(f)(\phi) \in \alpha_{\Sigma'}^{-1}(X_{F(\Sigma')}) \quad \text{iff} \quad \mathrm{SEN}(f)(\psi) \in \alpha_{\Sigma'}^{-1}(X_{F(\Sigma')}),$$

iff, $\langle \phi, \psi \rangle \in \Lambda^{\mathbf{A}}_\Sigma(\alpha^{-1}(X))$.

The proof of $\lambda^{\mathbf{A}}(\alpha^{-1}(X)) = \alpha^{-1}(\lambda^{\mathbf{A}'}(X))$ is similar.

(b) We have

$$\begin{array}{rcl}
\widetilde{\Lambda}(D) & = & \bigcap\{\Lambda^{\mathbf{A}}(T) : T \in \mathrm{ThFam}(\mathbb{L})\} \\
& = & \bigcap\{\Lambda(\alpha^{-1}(T')) : T' \in \mathrm{ThFam}(\mathbb{L}')\} \\
& = & \bigcap\{\alpha^{-1}(\Lambda^{\mathbf{A}'}(T')) : T' \in \mathrm{ThFam}(\mathbb{L}')\} \\
& = & \alpha^{-1}(\bigcap\{\Lambda^{\mathbf{A}'}(T') : T' \in \mathrm{ThFam}(\mathbb{L}')\}) \\
& = & \alpha^{-1}(\widetilde{\Lambda}^{\mathbf{A}'}(D')).
\end{array}$$

The proof of $\widetilde{\lambda}^{\mathbf{A}}(D) = \alpha^{-1}(\widetilde{\lambda}^{\mathbf{A}'}(D'))$ is similar.

∎

19.9 Fullness and Metalogical Properties

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system.

An **F-rule** is a pair $\langle P, \rho \rangle$, with $P \cup \{\rho\} : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a finite set of natural transformations in $N^\flat$.

A **generalized** or **Gentzen F-rule**, or **F-grule** for short, is a pair

$$\langle \{\langle P^i, \rho^i \rangle : i \in I\}, \langle P, \rho \rangle \rangle,$$

where $\{\langle P^i, \rho^i \rangle : i \in I\} \cup \{\langle P, \rho \rangle\}$ is a finite set of **F**-rules. We sometimes write an **F**-grule in the "two-line" format

$$\frac{\langle P^i, \rho^i \rangle : i \in I}{\langle P, \rho \rangle} \quad \text{or} \quad \frac{P^i \vdash \rho^i : i \in I}{P \vdash \rho}.$$

Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on **F**. $\mathcal{I}$ **satisfies** $\dfrac{\langle P^i, \rho^i \rangle : i \in I}{\langle P, \rho \rangle}$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\rho^i_\Sigma(\vec{\chi}) \in C_\Sigma(P^i_\Sigma(\vec{\chi})),\ i \in I, \quad \text{impies} \quad \rho_\Sigma(\vec{\chi}) \in C_\Sigma(P_\Sigma(\vec{\chi})).$$

Similarly, an **F**-structure $\mathbb{L} = \langle \mathcal{A}, D \rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, **satisfies** $\dfrac{\langle P^i, \rho^i \rangle : i \in I}{\langle P, \rho \rangle}$ if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma)$,

$$\rho^i_\Sigma(\vec{\chi}) \in D_\Sigma(P^i_\Sigma(\vec{\chi})),\ i \in I, \quad \text{impies} \quad \rho_\Sigma(\vec{\chi}) \in D_\Sigma(P_\Sigma(\vec{\chi})).$$

Since an **F**-rule can be perceived as a a special case of an **F**-grule (with empty set of premises), this notion of satisfaction applies in particular to **F**-rules.

It turns out that satisfaction of **F**-rules is transferred from a π-institution to all its structure models.

Proposition 1422 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* **F**. *If* $\mathcal{I}$ *satisfies an* **F***-rule* $\langle P, \rho \rangle$*, then every* $\mathcal{I}$*-structure satisfies the same* **F***-rule.*

Proof: Suppose $\mathcal{I}$ satisfies $\langle P, \rho \rangle$ and let $\mathbb{L} = \langle \mathcal{A}, D \rangle$ be an $\mathcal{I}$-structure, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$. Then, for all $T \in \mathrm{ThFam}(\mathbb{L})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, we have $P_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi})) \subseteq T_{F(\Sigma)}$ if and only if $\alpha_\Sigma(P_\Sigma(\vec{\chi})) \subseteq T_{F(\Sigma)}$ if and only if $P_\Sigma(\vec{\chi}) \subseteq \alpha^{-1}_\Sigma(T_{F(\Sigma)})$. Thus, since, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, we get, by hypothesis, $\rho_\Sigma(\vec{\chi}) \in \alpha^{-1}_\Sigma(T_{F(\Sigma)})$. This is equivalent to $\alpha_\Sigma(\rho_\Sigma(\vec{\chi})) \in T_{F(\Sigma)}$ and, in turn, to $\rho_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi})) \in T_{F(\Sigma)}$. We conclude, by the surjectivity of $\langle F, \alpha \rangle$, that $\mathbb{L}$ satisfies $\langle P, \rho \rangle$ as well. ■

Moreover, it turns out that satisfaction of an **F**-grule, in general, is preserved by bilogical morphisms.

Proposition 1423 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ $N^\flat$*-algebraic systems,* $\mathbb{L} = \langle \mathbf{A}, D \rangle$*,* $\mathbb{L}' = \langle \mathbf{A}', D' \rangle$ π*-structures based on* $\mathbf{A}$*,* $\mathbf{A}'$*, respectively, and* $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ *a bilogical morphism. Then* $\mathbb{L}$ *satisfies an* $\mathbf{F}$*-grule* $\langle \{\langle P^i, \rho^i \rangle : i \in I\}, \langle P, \rho \rangle \rangle$ *if and only if* $\mathbb{L}'$ *satisfies the same* $\mathbf{F}$*-grule.*

Proof: Suppose, first, that $\mathbb{L}$ satisfies the $\mathbf{F}$-grule and let $\Sigma \in |\mathbf{Sign}|$, $\vec{\chi} \in \mathrm{SEN}(\Sigma)$, such that, for all $i \in I$,

$$\rho'^i_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi})) \in D'_{F(\Sigma)}(P'^i_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi}))).$$

This is equivalent to

$$\alpha_\Sigma(\rho^i_\Sigma(\vec{\chi})) \in D'_{F(\Sigma)}(\alpha_\Sigma(P^i_{F(\Sigma)}(\vec{\chi}))).$$

Since $\langle F, \alpha \rangle$ is a bilogical morphism, we get $\rho^i_\Sigma(\vec{\chi}) \in D_\Sigma(P^i_\Sigma(\vec{\chi}))$. Since, by hypothesis, $\mathbb{L}$ satisfies the given $\mathbf{F}$-grule, we get that $\rho_\Sigma(\vec{\chi}) \in D_\Sigma(P_\Sigma(\vec{\chi}))$. Reversing the steps above, we conclude that

$$\rho'_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi})) \in D'_{F(\Sigma)}(P'_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi}))).$$

Since $\langle F, \alpha \rangle$ is surjective, this shows that $\mathbb{L}'$ satisfies the $\mathbf{F}$-grule as well.

Suppose, conversely, that $\mathbb{L}'$ satisfies the $\mathbf{F}$-grule $\langle \{\langle P^i, \rho^i \rangle : i \in I\}, \langle P, \rho \rangle \rangle$. Let $\Sigma \in |\mathbf{Sign}|$ and $\vec{\chi} \in \mathrm{SEN}(\Sigma)$, such that, for all $i \in I$,

$$\rho^i_\Sigma(\vec{\chi}) \in D_\Sigma(P^i_\Sigma(\vec{\chi})).$$

Since, $\langle F, \alpha \rangle$ is a bilogical morphism, we get

$$\alpha_\Sigma(\rho^i_\Sigma(\vec{\chi})) \in D'_{F(\Sigma)}(\alpha_\Sigma(P^i_\Sigma(\vec{\chi}))),$$

which gives $\rho'^i_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi})) \in D'_{F(\Sigma)}(P'^i_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi})))$. Since, by hypothesis, $\mathbb{L}'$ satisfies the given $\mathbf{F}$-grule, we now get

$$\rho'_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi})) \in D'_{F(\Sigma)}(P'_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi}))).$$

Reversing again the preceding steps, we finally obtain that

$$\rho_\Sigma(\vec{\chi}) \in D_\Sigma(P_\Sigma(\vec{\chi})).$$

Thus, $\mathbb{L}$ satisfies the same $\mathbf{F}$-grule as well. ■

19.9.1 The Congruence Property

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an algebraic system and $\mathbb{L} = \langle \mathbf{A}, D \rangle$ a π-structure.

- $\mathbb{L}$ has the **Congruence Property** if $\widetilde{\Lambda}^{\mathbf{A}}(D)$ is a congruence system on $\mathbf{A}$, i.e., by Proposition 1420, if and only if
$$\widetilde{\Lambda}^{\mathbf{A}}(D) = \widetilde{\Omega}^{\mathbf{A}}(D);$$
- $\mathbb{L}$ has the **strong Congruence Property** if $\widetilde{\lambda}^{\mathbf{A}}(D)$ is a congruence system on $\mathbf{A}$, i.e., by Proposition 1420, if and only if
$$\widetilde{\lambda}^{\mathbf{A}}(D) = \widetilde{\Omega}^{\mathbf{A}}(D).$$

Of course, the strong Congruence Property implies the Congruence Property.

Proposition 1424 *Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an algebraic system and $\mathbb{L} = \langle \mathbf{A}, D \rangle$ a π-structure. If $\mathcal{I}$ has the strong Congruence Property, then it has the Congruence Property.*

Proof: We know that $\widetilde{\Omega}^{\mathbf{A}}(D) \le \widetilde{\Lambda}^{\mathbf{A}}(T) \le \widetilde{\lambda}^{\mathbf{A}}(T)$. If $\mathbb{L}$ has the strong Congruence Property, $\widetilde{\Omega}^{\mathbf{A}}(D) = \widetilde{\lambda}^{\mathbf{A}}(T)$, whence, also, $\widetilde{\Omega}^{\mathbf{A}}(D) = \widetilde{\Lambda}^{\mathbf{A}}(T)$. Thus, $\mathbb{L}$ has the Congruence Property. ■

Corollary 1425 *Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an algebraic system and $\mathbb{L} = \langle \mathbf{A}, D \rangle$ a reduced π-structure based on $\mathbf{A}$.*

(a) $\mathbb{L}$ has the Congruence Property if and only if $\widetilde{\Lambda}^{\mathbf{A}}(D) = \Delta^{\mathbf{A}}$.

(b) $\mathbb{L}$ has the strong Congruence Property if and only if $\widetilde{\lambda}^{\mathbf{A}}(D) = \Delta^{\mathbf{A}}$.

Proof: $\mathbb{L}$ has the Congruence Property if and only if $\widetilde{\Lambda}^{\mathbf{A}}(D) = \widetilde{\Omega}^{\mathbf{A}}(D)$ if and only if, since $\mathbb{L}$ is reduced, $\widetilde{\Lambda}^{\mathbf{A}}(D) = \Delta^{\mathbf{A}}$. Part (b) is similar. ■

It turns out that the Congruence Property is preserved in both directions under bilogical morphisms.

Proposition 1426 *Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ be algebraic systems, $\mathbb{L} = \langle \mathbf{A}, D \rangle$, $\mathbb{L}' = \langle \mathbf{A}', D' \rangle$ π-structures based on $\mathbf{A}$, $\mathbf{A}'$, respectively, and $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ a bilogical morphism.*

(a) $\mathbb{L}$ has the Congruence Property if and only if $\mathbb{L}'$ has the Congruence Property;

(b) $\mathbb{L}$ has the strong Congruence Property if and only if $\mathbb{L}'$ has the strong Congruence Property.

Proof:

(a) We have

$$\begin{aligned}\widetilde{\Lambda}^{\mathbf{A}}(D) = \widetilde{\Omega}^{\mathbf{A}}(D) \quad &\text{iff} \quad \alpha^{-1}(\widetilde{\Lambda}^{\mathbf{A}'}(D')) = \alpha^{-1}(\widetilde{\Omega}^{\mathbf{A}'}(D'))\\ &\text{iff} \quad \widetilde{\Lambda}^{\mathbf{A}'}(D') = \widetilde{\Omega}^{\mathbf{A}'}(D'),\end{aligned}$$

the first equivalence by Corollary 1364 and Lemma 1421, and the second equivalence by the surjectivity of $\langle F, \alpha\rangle$. Therefore, $\mathbb{L}$ has the Congruence Property if and only if $\mathbb{L}'$ has the Congruence Property.

(b) Similarly,

$$\begin{aligned}\widetilde{\lambda}^{\mathbf{A}}(D) = \widetilde{\Omega}^{\mathbf{A}}(D) \quad &\text{iff} \quad \alpha^{-1}(\widetilde{\lambda}^{\mathbf{A}'}(D')) = \alpha^{-1}(\widetilde{\Omega}^{\mathbf{A}'}(D'))\\ &\text{iff} \quad \widetilde{\lambda}^{\mathbf{A}'}(D') = \widetilde{\Omega}^{\mathbf{A}'}(D'),\end{aligned}$$

the first equivalence by Corollary 1364 and Lemma 1421, and the second equivalence by the surjectivity of $\langle F, \alpha\rangle$. Therefore, $\mathbb{L}$ has the strong Congruence Property if and only if $\mathbb{L}'$ has the strong Congruence Property. ■

Using the Congruence Property, we are now able to introduce the first two classes of the Frege hierarchy of π-institutions, which will be looked more closely at in a subsequent chapter.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, \mathbb{N}^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.

- $\mathcal{I}$ is **selfextensional** if it has the Congruence Property, i.e., if

$$\widetilde{\Lambda}(\mathcal{I}) = \widetilde{\Omega}(\mathcal{I}).$$

  Recall from Proposition 1418, that, since $\mathcal{I}$, is structural, this is equivalent to having the strong Congruence Property.

- $\mathcal{I}$ is **fully selfextensional** if every full $\mathcal{I}$-structure $\mathbb{L} = \langle \mathcal{A}, D\rangle$ has the Congruence Property, i.e., if, for all $\mathbb{L} = \langle \mathcal{A}, D\rangle \in \mathrm{FStr}(\mathcal{I})$,

$$\widetilde{\Lambda}^{\mathcal{A}}(D) = \widetilde{\Omega}^{\mathcal{A}}(D).$$

  Recall, also, from Proposition 1389 and Proposition 1418, that, since every full $\mathcal{I}$-structure is structural, this amounts to every full $\mathcal{I}$-structure having the strong Congruence Property.

We give a characterization of selfextensional π-institutions next.

Proposition 1427 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is selfextensional if and only if, for all $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi_i, \psi_i \in \mathrm{SEN}^\flat(\Sigma)$, $i < k$,*

$$\begin{aligned}&C_\Sigma(\phi_i) = C_\Sigma(\psi_i), \; i < k,\\ &\qquad \textit{imply} \quad C_\Sigma(\sigma^\flat_\Sigma(\phi_0, \dots, \phi_{k-1})) = C_\Sigma(\sigma^\flat_\Sigma(\psi_0, \dots, \psi_{k-1})).\end{aligned}$$

Proof: Suppose that $\mathcal{I}$ is selfextensional and let $\sigma^\flat \in N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi_i, \psi_i \in \mathrm{SEN}^\flat(\Sigma)$, $i < k$, such that $C_\Sigma(\phi_i) = C_\Sigma(\psi_i)$, $i < k$. Thus, for all $i < k$, $\langle \phi_i, \psi_i \rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}) = \widetilde{\Omega}_\Sigma(\mathcal{I})$, by selfextensionality. Since $\widetilde{\Omega}(\mathcal{I})$ is a congruence system,

$$\langle \sigma^\flat_\Sigma(\phi_0, \dots, \phi_{k-1}), \sigma^\flat_\Sigma(\psi_0, \dots, \psi_{k-1}) \rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I}) = \widetilde{\lambda}_\Sigma(\mathcal{I}).$$

We conclude that $C_\Sigma(\sigma^\flat_\Sigma(\phi_0, \dots, \phi_{k-1})) = C_\Sigma(\sigma^\flat_\Sigma(\psi_0, \dots, \psi_{k-1}))$.

Suppose, conversely, that the displayed condition holds. Then $\widetilde{\lambda}(\mathcal{I})$ is a congruence system on $\mathbf{F}$. Therefore, by Proposition 1420, we have that $\widetilde{\Omega}(\mathcal{I}) = \widetilde{\lambda}(\mathcal{I})$. We conclude that $\mathcal{I}$ is selfextensional. ∎

And also a characterization of fully selfextensional π-institutions.

Proposition 1428 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is fully selfextensional if and only if every $\mathbf{F}$-structure of the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ has the Congruence Property.*

Proof: Assume, first, that $\mathcal{I}$ is fully selfextensional. By definition, every full $\mathcal{I}$-structure has the Congruence Property. By Proposition 1390, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ is full, for every $\mathbf{F}$-algebraic system $\mathcal{A}$. Therefore, every $\mathbf{F}$-structure of the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ has the Congruence Property.

Assume, conversely, that every $\mathbf{F}$-structure of the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ has the Congruence Property. Let $\mathbb{L} = \langle \mathcal{A}, D \rangle$ be a full $\mathcal{I}$-structure. Then, by definition, the reduction morphism is a bilogical morphism

$$\langle I, \pi \rangle : \langle \mathcal{A}, D \rangle \vdash \langle \mathcal{A}^*, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*) \rangle.$$

By hypothesis, $\langle \mathcal{A}^*, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*) \rangle$ has the Congruence Property. Thus, by Proposition 1426, $\mathbb{L}$ also has the Congruence Property. Therefore, every full $\mathcal{I}$-structure has the Congruence Property and we conclude that $\mathcal{I}$ is fully selfextensional. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. Recall that the **Lindenbaum-Tarski algebraic system of** $\mathcal{I}$ is the algebraic system $\mathcal{F}/\widetilde{\Omega}(\mathcal{I})$, where $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. Recall, also, that, given a class of $\mathbf{F}$-algebraic systems, $Q(\mathsf{K})$ denotes the syntactic variety generated by K, i.e., those $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, such that $\bigcap \{\mathrm{Ker}(\mathcal{K}) : \mathcal{K} \in \mathsf{K}\} \leq \mathrm{Ker}(\mathcal{A})$. We denoted $\mathsf{Q}(\mathcal{I}) = Q(\mathcal{F}/\widetilde{\Omega}(\mathcal{I}))$, the syntactic variety generated by the Lindenbaum-Tarski $\mathbf{F}$-algebraic system of $\mathcal{I}$. Moreover, given $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, we write $\mathsf{K} \vDash_\Sigma \phi \approx \psi$ for $\langle \phi, \psi \rangle \in \bigcap_{\mathcal{K} \in \mathsf{K}} \mathrm{Ker}_\Sigma(\mathcal{K})$.

Using these conventions, we can formulate a proposition to the effect that, for a selfextensional π-institution $\mathcal{I}$, an equation is satisfied in $\mathsf{Q}(\mathcal{I})$ if and only if it is in the Frege equivalence family $\widetilde{\lambda}(\mathcal{I})$.

Proposition 1429 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is selfextensional, then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\mathsf{Q}(\mathcal{I}) \vDash_\Sigma \phi \approx \psi \quad \textit{if and only if} \quad \langle \phi, \psi \rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}).$$

Proof: By the definition of $\mathsf{Q}(\mathcal{I})$, it is easy to see that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\mathsf{Q}(\mathcal{I}) \vDash_\Sigma \phi \approx \psi \quad \text{iff} \quad \langle \phi, \psi \rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I}).$$

Since $\mathcal{I}$ is assumed selfextensional, this happens if and only if $\langle \phi, \psi \rangle \in \widetilde{\Lambda}_\Sigma(\mathcal{I})$, i.e., due to the structurality of C, if and only if $\langle \phi, \psi \rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I})$. ■

19.9.2 The Property of Conjunction

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an algebraic system, such that, in N, there exists a binary natural transformation

$$\wedge : \mathrm{SEN}^2 \to \mathrm{SEN},$$

and $\mathbb{L} = \langle \mathbf{A}, D \rangle$ a π-structure. We say that $\mathbb{L}$ has the **Conjunction Property with respect to** $\wedge$ if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$D_\Sigma(\phi \wedge_\Sigma \psi) = D_\Sigma(\phi, \psi),$$

where $\phi \wedge_\Sigma \psi := \wedge_\Sigma(\phi, \psi)$. In this case, we also say $\wedge$ is a **conjunction** for $\mathbb{L}$ and that $\mathbb{L}$ is **conjunctive**.

Lemma 1430 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *be an algebraic system, with* $\wedge : \mathrm{SEN}^2 \to \mathrm{SEN}$ *in* N*, and* $\mathbb{L} = \langle \mathbf{A}, D \rangle$ *a* π*-structure based on* $\mathbf{A}$*.* $\mathbb{L}$ *has the Conjunction Property with respect to* $\wedge$ *if and only if, for every* $T \in \mathrm{ThFam}(\mathbb{L})$*, all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,*

$$\phi \wedge_\Sigma \psi \in T_\Sigma \quad \textit{iff} \quad \phi \in T_\Sigma \textit{ and } \psi \in T_\Sigma.$$

Proof: Suppose that $\mathbb{L}$ has the Conjunction Property with respect to $\wedge$ and let $T \in \mathrm{ThFam}(\mathbb{L})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$.

If $\phi \wedge_\Sigma \psi \in T_\Sigma$, then

$$\phi \in D_\Sigma(\phi, \psi) = D_\Sigma(\phi \wedge_\Sigma \psi) \subseteq D_\Sigma(T_\Sigma) = T_\Sigma.$$

Similarly, $\psi \in T_\Sigma$.

Conversely, if $\phi, \psi \in T_\Sigma$, then

$$\phi \wedge_\Sigma \psi \in D_\Sigma(\phi \wedge_\Sigma \psi) = D_\Sigma(\phi, \psi) \subseteq D_\Sigma(T_\Sigma) = T_\Sigma.$$

Thus, $\phi \wedge_\Sigma \psi \in T_\Sigma$ if and only if $\phi, \psi \in T_\Sigma$.

Suppose, conversely, that the displayed condition in the statement is satisfied. Then, for all $\Sigma \in |\mathbf{Sign}|$, and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{rcl} D_\Sigma(\phi \wedge_\Sigma \psi) & = & \bigcap\{T_\Sigma : T \in \mathrm{ThFam}(\mathbb{L}) \text{ and } \phi \wedge_\Sigma \psi \in T_\Sigma\} \\ & = & \bigcap\{T_\Sigma : T \in \mathrm{Thfam}(\mathbb{L}) \text{ and } \phi, \psi \in T_\Sigma\} \\ & = & D_\Sigma(\phi, \psi). \end{array}$$

So $\wedge : \mathrm{SEN}^2 \to \mathrm{SEN}$ is a conjunction for $\mathbb{L}$. ■

In terms of **F**-rules one can characterize the Conjunction Property as follows.

Proposition 1431 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$ *and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *based on* $\mathbf{F}$. $\mathcal{I}$ *has the Conjunction Property if and only if it satisfies*

$$p^{2,0}, p^{2,1} \vdash \wedge^\flat \circ \langle p^{2,0}, p^{2,1}\rangle, \quad \wedge^\flat \circ \langle p^{2,0}, p^{2,1}\rangle \vdash p^{2,0}, \quad \wedge^\flat \circ \langle p^{2,0}, p^{2,1}\rangle \vdash p^{2,1}.$$

Note that, in practice, we write these **F**-grules in the more familiar form

$$x, y \vdash x \wedge^\flat y, \quad x \wedge^\flat y \vdash x, \quad x \wedge^\flat y \vdash y,$$

where $x, y, z, \ldots$ stand for the corresponding projection natural transformations in $N^\flat$.

Proof: We have that $\mathcal{I}$ satisfies

$$x, y \vdash x \wedge^\flat y, \quad x \wedge^\flat y \vdash x, \quad x \wedge^\flat y \vdash y,$$

iff, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \wedge^\flat_\Sigma \psi \in C_\Sigma(\phi, \psi), \quad \phi \in C_\Sigma(\phi \wedge^\flat_\Sigma \psi), \quad \psi \in C_\Sigma(\phi \wedge^\flat_\Sigma \psi),$$

iff, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C_\Sigma(\phi \wedge^\flat_\Sigma \psi) = C_\Sigma(\phi, \psi)$$

iff $\mathcal{I}$ has the Conjunction Property with respect to $\wedge^\flat$. ■

Having the Conjunction Property is preserved under bilogical morphisms in both directions.

Proposition 1432 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system with a binary* $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$. *Suppose* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ *are* $N^\flat$*-algebraic systems* $\mathbb{L} = \langle \mathbf{A}, D\rangle$, $\mathbb{L}' = \langle \mathbf{A}', D'\rangle$ π*-structures based on* $\mathbf{A}$, $\mathbf{A}'$*, respectively, and* $\langle F, \alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$ *a bilogical morphism.* $\mathbb{L}$ *has the Conjunction Property with respect to* $\wedge$ *if and only if* $\mathbb{L}'$ *has the Conjunction Property with respect to* $\wedge'$.

Proof: This follows by Proposition 1431 and Proposition 1423. ∎

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system and $\wedge : \mathrm{SEN}^2 \to \mathrm{SEN}$ a binary natural transformation in N. We denote by $N^\wedge$ the category of natural transformations on SEN generated by $\wedge$. Clearly, since $\wedge$ is in N, we have that $N^\wedge$ is a wide subcategory of N. Moreover, we denote

$$\mathbf{A}^\wedge = \langle \mathbf{Sign}, \mathrm{SEN}, N^\wedge\rangle$$

the algebraic system that results by taking $N^\wedge$ instead of N as its category of natural transformations. This corresponds to the well-known operation of reducing the type of an algebra.

Proposition 1433 *Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system, with $\wedge : \mathrm{SEN}^2 \to \mathrm{SEN}$ in N, and $\mathbb{L} = \langle \mathbf{A}, D\rangle$ be a π-structure based on $\mathbf{A}$. If $\mathbb{L}$ has the Conjunction Property with respect to $\wedge$, then $\widetilde{\Lambda}^{\mathbf{A}}(D)$ is a congruence system on $\mathbf{A}^\wedge$. Moreover, for all $X \in \mathrm{SenFam}(\mathbf{A})$, $\widetilde{\Lambda}^{\mathbf{A},D}(X)$ is also a congruence system on $\mathbf{A}^\wedge$.*

Proof: It suffices to show that, for all $T \in \mathrm{ThFam}(\mathbb{L})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \phi', \psi, \psi' \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi, \phi'\rangle, \langle \psi, \psi'\rangle \in \widetilde{\Lambda}^{\mathbf{A}}_\Sigma(T) \quad \text{implies} \quad \langle \phi \wedge_\Sigma \psi, \phi' \wedge_\Sigma \psi'\rangle \in \widetilde{\Lambda}^{\mathbf{A}}_\Sigma(T).$$

To this end, suppose $\Sigma \in |\mathbf{Sign}|$, $\phi, \phi', \psi, \psi' \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \phi'\rangle$, $\langle \psi, \psi'\rangle \in \widetilde{\Lambda}^{\mathbf{A}}_\Sigma(T)$. Then, by definition, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\begin{array}{rcl} \mathrm{SEN}(f)(\phi) \in T_{\Sigma'} & \text{iff} & \mathrm{SEN}(f)(\phi') \in T_{\Sigma'}, \\ \mathrm{SEN}(f)(\psi) \in T_{\Sigma'} & \text{if} & \mathrm{SEN}(f)(\psi') \in T_{\Sigma'}. \end{array}$$

Thus, we have, by the Conjunction Property and Lemma 1430,

$$\begin{array}{rcl} \mathrm{SEN}(f)(\phi \wedge_\Sigma \psi) \in T_{\Sigma'} & \text{iff} & \mathrm{SEN}(f)(\phi) \wedge_{\Sigma'} \mathrm{SEN}(f)(\psi) \in T_{\Sigma'} \\ & \text{iff} & \mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi) \in T_{\Sigma'} \\ & \text{iff} & \mathrm{SEN}(f)(\phi'), \mathrm{SEN}(f)(\psi') \in T_{\Sigma'} \\ & \text{iff} & \mathrm{SEN}(f)(\phi') \wedge_{\Sigma'} \mathrm{SEN}(f)(\psi') \in T_{\Sigma'} \\ & \text{iff} & \mathrm{SEN}(f)(\phi' \wedge_\Sigma \psi') \in T_{\Sigma'}. \end{array}$$

Thus, $\langle \phi \wedge_\Sigma \psi, \phi' \wedge_\Sigma \psi'\rangle \in \widetilde{\Lambda}^{\mathbf{A}}_\Sigma(T)$ and $\widetilde{\Lambda}^{\mathbf{A}}(T)$ is a congruence system on $\mathbf{A}^\wedge$.

The fact that $\widetilde{\Lambda}^{\mathbf{A}}(D)$ and $\widetilde{\Lambda}^{\mathbf{A},D}(X)$ are congruence systems on $\mathbf{A}^\wedge$ now follow from the relationships outlined before Lemma 1415. ∎

The Conjunction Property also satisfies a transfer property to the effect that a given π-institution has the Conjunction Property if and only if all its π-structure models have the Conjunction Property.

Proposition 1434 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *has the Conjunction Property with respect to* $\wedge^\flat$ *if and only if, for every* $\mathcal{I}$*-structure* $\mathbb{L} = \langle \mathcal{A}, D \rangle$*,* $\mathbb{L}$ *has the Conjunction Property with respect to* $\wedge$*.*

Proof: Suppose that $\mathcal{I}$ has the Conjunction Property with respect to $\wedge^\flat$. Then, by Propositions 1431 and 1422, $\mathbb{L}$ has the Conjunction Property with respect to $\wedge$.

The converse is trivial, since $\langle \mathcal{F}, C \rangle \in \mathrm{Str}(\mathcal{I})$, where $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$. ∎

Finally, we show that if a π-institution $\mathcal{I}$ has the Conjunction Property, then any finitary $\mathcal{I}$-structure with the Congruence Property is full.

Proposition 1435 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, and* $\mathbb{L} = \langle \mathcal{A}, D \rangle$ *a finitary* $\mathcal{I}$*-structure, which has no theorems if* $\mathcal{I}$ *has no theorems. If* $\mathcal{I}$ *has the Conjunction Property with respect to* $\wedge^\flat$ *and* $\mathbb{L}$ *has the strong Congruence Property, then* $\mathbb{L}$ *is a full* $\mathcal{I}$*-structure.*

Proof: Suppose $\mathbb{L} = \langle \mathcal{A}, D \rangle$ is a finitary $\mathcal{I}$-structure, without theorems, if $\mathcal{I}$ has no theorems, satisfying the strong Congruence Property. Our goal is to show that $\mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$. We denote by $\langle I, \pi \rangle : \mathbb{L} \vdash \mathbb{L}^*$ the bilogical quotient morphism.

Assume, first, that $T \in \mathcal{D}^*$. Then $\pi^{-1}(T) \in \mathcal{D} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{D})$, by Proposition 1385. Hence, by Corollary 55, $T \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A}^*)$. We conclude that $\mathcal{D}^* \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$.

Conversely, assume that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$. If $T = \varnothing$, then $\mathcal{I}$ does not have theorems. Thus, by hypothesis, $\varnothing \in \mathcal{D}$ and, therefore, $\varnothing \in \mathcal{D}^*$. Suppose, next, that $T \neq \varnothing$. Let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in D^*_\Sigma(T_\Sigma)$. By hypothesis and Proposition 1365, there exist $\phi_0, \ldots, \phi_{n-1} \in T_\Sigma$, such that $\phi \in D^*_\Sigma(\phi_0, \ldots, \phi_{n-1})$. By the Conjunction Property and Proposition 1434, $\phi \in D^*_\Sigma(\phi_0 \wedge_\Sigma (\cdots \wedge_\Sigma \phi_{n-1}))$. Therefore,

$$D^*_\Sigma(\phi \wedge_\Sigma (\phi_0 \wedge_\Sigma (\cdots \wedge_\Sigma \phi_{n-1}))) = D^*_\Sigma(\phi_0 \wedge_\Sigma (\cdots \wedge_\Sigma \phi_{n-1})).$$

By hypothesis, Proposition 1426 and Corollary 1425, we get that

$$\phi \wedge_\Sigma (\phi_0 \wedge_\Sigma (\cdots \wedge_\Sigma \phi_{n-1})) = \phi_0 \wedge_\Sigma (\cdots \wedge_\Sigma \phi_{n-1}).$$

Since $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$ and $\mathcal{I}$ has the Conjunction Property with respect to $\wedge^\flat$, $\phi_0, \ldots, \phi_{n-1} \in T_\Sigma$ imply that $\phi_0 \wedge_\Sigma (\cdots \wedge \phi_{n-1}) \in T_\Sigma$. Thus, by the displayed equation above, $\phi \wedge_\Sigma (\phi_0 \wedge_\Sigma (\cdots \wedge_\Sigma \phi_{n-1})) \in T_\Sigma$. By Proposition 1434, $\phi \in T_\Sigma$. So we have $D^*_\Sigma(T_\Sigma) = T_\Sigma$ and, hence, $T \in \mathcal{D}^*$. Thus, $\mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$ and $\mathbb{L}$ is a full $\mathcal{I}$-structure. ∎

19.9.3 The Deduction-Detachment Theorem

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system, with $\rightarrow: \mathrm{SEN}^2 \to \mathrm{SEN}$ a binary natural transformation in N, and $\mathbb{L} = \langle \mathbf{A}, D\rangle$ a π-structure.

- $\mathbb{L}$ has the **Modus Ponens** or **Detachment with respect to** $\rightarrow$ if, for all $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\phi, \psi\} \subseteq \mathrm{SEN}(\Sigma)$,
$$\phi \rightarrow_\Sigma \psi \in D_\Sigma(\Phi) \quad \text{implies} \quad \psi \in D_\Sigma(\Phi, \phi).$$
- $\mathbb{L}$ has the **Deduction Theorem with respect to** $\rightarrow$ if, for all $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\phi, \psi\} \subseteq \mathrm{SEN}(\Sigma)$,
$$\psi \in D_\Sigma(\Phi, \phi) \quad \text{implies} \quad \phi \rightarrow_\Sigma \psi \in D_\Sigma(\Phi).$$
- $\mathbb{L}$ has the **Deduction Detachment Theorem with respect to** $\rightarrow$ if it has both the Modus Ponens and the Deduction Theorem with respect to $\rightarrow$.

Structures that have the Deduction Detachment Theorem always have theorems. The following proposition gives a few of those theorems that are inspired by classical propositional calculus.

Proposition 1436 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system, with* $\rightarrow: \mathrm{SEN}^2 \to \mathrm{SEN}$ *a binary natural transformation in* N*, and* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *a* π*-structure that has the Deduction Detachment Theorem with respect to* $\rightarrow$*. Then, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi, \chi \in \mathrm{SEN}(\Sigma)$*,*

(a) $\phi \rightarrow_\Sigma \phi \in \mathrm{Thm}_\Sigma(\mathbb{L})$*;*

(b) $\phi \rightarrow_\Sigma (\psi \rightarrow_\Sigma \phi) \in \mathrm{Thm}_\Sigma(\mathbb{L})$*;*

(c) $(\phi \rightarrow_\Sigma (\psi \rightarrow_\Sigma \chi)) \rightarrow_\Sigma ((\phi \rightarrow_\Sigma \psi) \rightarrow_\Sigma (\phi \rightarrow_\Sigma \chi)) \in \mathrm{Thm}_\Sigma(\mathbb{L})$*.*

Proof: Let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi, \chi \in \mathrm{SEN}(\Sigma)$.

(a) Since $\phi \in D_\Sigma(\phi)$, we get by the Deduction Theorem, $\phi \rightarrow_\Sigma \phi \in D_\Sigma(\varnothing)$. So $\phi \rightarrow_\Sigma \phi \in \mathrm{Thm}_\Sigma(\mathbb{L})$.

(b) Since $\phi \in D_\Sigma(\phi, \psi)$, we get, by the Deduction Theorem, $\psi \rightarrow_\Sigma \phi \in D_\Sigma(\phi)$. By yet another application of the Deduction Theorem, we conclude that $\phi \rightarrow_\Sigma (\psi \rightarrow_\Sigma \phi) \in D_\Sigma(\varnothing)$. Therefore, $\phi \rightarrow_\Sigma (\psi \rightarrow_\Sigma \phi) \in \mathrm{Thm}_\Sigma(\mathbb{L})$.

(c) Since $\phi \rightarrow_\Sigma \psi \in D_\Sigma(\phi \rightarrow_\Sigma \psi)$ and $\phi \rightarrow_\Sigma (\psi \rightarrow_\Sigma \chi) \in D_\Sigma(\phi \rightarrow_\Sigma (\psi \rightarrow_\Sigma \chi))$, we get, by Modus Ponens, $\psi \in D_\Sigma(\phi \rightarrow_\Sigma \psi, \phi)$ and $\psi \rightarrow_\Sigma \chi \in$

$D_\Sigma(\phi \to_\Sigma (\psi \to_\Sigma \chi), \phi)$. Moreover, since $\psi \to_\Sigma \chi \in D_\Sigma(\psi \to_\Sigma \chi)$, we get, by Modus Ponens, $\chi \in D_\Sigma(\psi \to_\Sigma \chi, \psi)$. Thus, we obtain

$$\chi \in D_\Sigma(\psi \to_\Sigma \chi, \psi) \subseteq D_\Sigma(\phi \to_\Sigma (\psi \to_\Sigma \chi), \phi \to_\Sigma \psi, \phi).$$

By the Deduction Theorem, $\phi \to_\Sigma \chi \in D_\Sigma(\phi \to_\Sigma (\psi \to_\Sigma \chi), \phi \to_\Sigma \psi)$. By another application of the Deduction Theorem, $(\phi \to_\Sigma \psi) \to_\Sigma (\phi \to_\Sigma \chi) \in D_\Sigma(\phi \to_\Sigma (\psi \to_\Sigma \chi))$. A final application of the Deduction Theorem yields

$$(\phi \to_\Sigma (\psi \to_\Sigma \chi)) \to_\Sigma ((\phi \to_\Sigma \psi) \to_\Sigma (\phi \to_\Sigma \chi)) \in D_\Sigma(\varnothing).$$

Therefore, $(\phi \to_\Sigma (\psi \to_\Sigma \chi)) \to_\Sigma ((\phi \to_\Sigma \psi) \to_\Sigma (\phi \to_\Sigma \chi)) \in \mathrm{Thm}_\Sigma(\mathbb{L})$. ■

Corollary 1437 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ a binary natural transformation in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, which has the Deduction Detachment Theorem with respect to $\to^\flat$. Then, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, and every $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T \neq \varnothing$. Consequently, for every $\mathbb{L} = \langle \mathcal{A}, D \rangle \in \mathrm{Str}(\mathcal{I})$, $\mathrm{Thm}(\mathbb{L}) \neq \varnothing$.*

Proof: Clear by Proposition 1436. ■

We now give a characterization of the Podus Ponens property.

Proposition 1438 *Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an algebraic system, with $\to : \mathrm{SEN}^2 \to \mathrm{SEN}$ in N, and $\mathbb{L} = \langle \mathcal{A}, D \rangle$ a π-structure. $\mathbb{L}$ has the Modus Ponens with respect to $\to$ if and only if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,*

$$\psi \in D_\Sigma(\phi, \phi \to_\Sigma \psi)$$

if and only if, for all $T \in \mathrm{ThFam}(\mathbb{L})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{and} \quad \phi \to_\Sigma \psi \in T_\Sigma \quad \text{imply} \quad \psi \in T_\Sigma.$$

Proof: Suppose, first, that $\mathbb{L}$ has the Modus Ponens with respect to $\to$ and let $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$. Then $\phi \to_\Sigma \psi \in D_\Sigma(\phi \to_\Sigma \psi)$, whence, by the Modus Ponens, $\psi \in D_\Sigma(\phi \to_\Sigma \psi, \phi)$. Conversely, suppose, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $\psi \in D_\Sigma(\phi, \phi \to_\Sigma \psi)$ and let $\Phi \subseteq \mathrm{SEN}(\Sigma)$, such that $\phi \to_\Sigma \psi \in D_\Sigma(\Phi)$. Then, we have

$$\psi \in D_\Sigma(\phi, \phi \to_\Sigma \psi) \subseteq D_\Sigma(\Phi, \phi).$$

So $\mathbb{L}$ has the Modus Ponens with respect to $\to$.

The second equivalence is straightforward. ■

The Modus Ponens in a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ may also be characterized in terms of $\mathbf{F}$-rules.

Proposition 1439 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\rightarrow^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *has the Modus Ponens with respect to* $\rightarrow^\flat$ *if and only if it satisfies the* $\mathbf{F}$*-rule*

$$x, x \rightarrow^\flat y \vdash y.$$

Proof: $\mathcal{I}$ satisfies $x, x \rightarrow^\flat y \vdash y$ if and only if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\psi \in C_\Sigma(\phi, \phi \rightarrow^\flat_\Sigma \psi)$$

if and only if, by Proposition 1438, $\mathcal{I}$ has the Modus Ponens with respect to $\rightarrow^\flat$. ■

Since the Modus Ponens in a π-institution is expressible in terms of $\mathbf{F}$-rules, we may use Propositions 1422 and 1423 to draw the conclusions that the Modus Ponens transfers to all models and, moreover, that it is preserved by all bilogical morphisms.

Corollary 1440 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\rightarrow^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *has the Modus Ponens with respect to* $\rightarrow^\flat$*, then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and every* $\mathbb{L} = \langle \mathcal{A}, D \rangle \in \mathrm{Str}(\mathcal{I})$*,* $\mathbb{L}$ *has the Modus Ponens with respect to* $\rightarrow$*.*

Proof: This follows by combing Proposition 1439 with Proposition 1422. ■

Corollary 1441 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\rightarrow^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ *be* $N^\flat$*-algebraic systems,* $\mathbb{L} = \langle \mathbf{A}, D \rangle$*,* $\mathbb{L}' = \langle \mathbf{A}', D' \rangle$ *be* $N^\flat$*-structures based on* $\mathbf{A}$*,* $\mathbf{A}'$*, respectively, and* $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ *a bilogical morphism.* $\mathbb{L}$ *has the Modus Ponens with respect to* $\rightarrow$ *if and only if* $\mathbb{L}'$ *has the Modus Ponens with respect to* $\rightarrow'$*.*

Proof: The conclusion follows by combining Proposition 1439 with Proposition 1423. ■

It turns out that the Deduction Theorem also transfers under bilogical morphisms.

Proposition 1442 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat$ *be an algebraic system, with* $\rightarrow^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ *be* $N^\flat$*-algebraic systems,* $\mathbb{L} = \langle \mathbf{A}, D \rangle$*,* $\mathbb{L}' = \langle \mathbf{A}', D' \rangle$ *be* $N^\flat$*-structures based on* $\mathbf{A}$*,* $\mathbf{A}'$*, respectively, and* $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ *a bilogical morphism.* $\mathbb{L}$ *has the Deduction Theorem with respect to* $\rightarrow$ *if and only if* $\mathbb{L}'$ *has the Deduction Theorem with respect to* $\rightarrow'$*.*

Proof: Suppose, first, that $\mathbb{L}$ has the Deduction Theorem with respect to $\to$ and let $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\phi, \psi\} \subseteq \mathrm{SEN}(\Sigma)$, such that

$$\alpha_\Sigma(\psi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi)).$$

Then, since $\langle F, \alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$ is a bilogical morphism, $\psi \in D_\Sigma(\Phi, \phi)$. Thus, since $\mathbb{L}$ has the Deduction Theorem, $\phi \to_\Sigma \psi \in D_\Sigma(\Phi)$. Again, by the fact that $\langle F, \alpha\rangle$ is a bilogical morphism, we obtain $\alpha_\Sigma(\phi \to_\Sigma \psi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$ or, equivalently, $\alpha_\Sigma(\phi) \to'_{F(\Sigma)} \alpha_\Sigma(\psi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$. Since $\langle F, \alpha\rangle$ is surjective, we conclude that $\mathbb{L}'$ also has the Deduction Theorem with respect to $\to'$.

Suppose, conversely, that $\mathbb{L}'$ has the Deduction Theorem with respect to $\to'$ and let $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\phi, \psi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\psi \in D_\Sigma(\Phi, \phi)$. Since $\langle F, \alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$ is a bilogical morphism, we get that

$$\alpha_\Sigma(\phi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi)).$$

Since $\mathbb{L}'$ has the Deduction Theorem with respect to $\to'$, $\alpha_\Sigma(\phi) \to'_{F(\Sigma)} \alpha_\Sigma(\psi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$ or, equivalently, $\alpha_\Sigma(\phi \to_\Sigma \psi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$. Again by the fact that $\langle F, \alpha\rangle$ is a bilogical morphism, we get $\phi \to_\Sigma \psi \in D_\Sigma(\Phi)$. Therefore, $\mathbb{L}$ also has the Deduction Theorem with respect to $\to$. ■

In an analog of Theorem 1433, we prove that, in case a π-structure has the Deduction Detachment Theorem, with respect to a binary natural transformation, then, Frege relation systems defined on the π-structure are congruence systems if one restricts to the category of natural transformations generated by the binary natural transformation.

Proposition 1443 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system, with* $\to: \mathrm{SEN}^2 \to \mathrm{SEN}$ *in* N*, and* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *be a* π*-structure based on* $\mathbf{A}$*. If* $\mathbb{L}$ *has the Deduction Detachment Property with respect to* $\to$*, then* $\widetilde{\Lambda}^{\mathbf{A}}(D)$ *is a congruence system on* $\mathbf{A}^\to$*. Moreover, for all* $X \in \mathrm{SenFam}(\mathbf{A})$*,* $\widetilde{\Lambda}^{\mathbf{A},D}(X)$ *is also a congruence system on* $\mathbf{A}^\to$*.*

Proof: It suffices to show that, for all $X \in \mathrm{SenFam}(\mathbf{A})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \phi', \psi, \psi' \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi, \phi'\rangle, \langle \psi, \psi'\rangle \in \widetilde{\Lambda}^{\mathbf{A},D}_\Sigma(X) \quad \text{implies} \quad \langle \phi \to_\Sigma \psi, \phi' \to_\Sigma \psi'\rangle \in \widetilde{\Lambda}^{\mathbf{A},D}_\Sigma(X).$$

So, suppose $X \in \mathrm{SenFam}(\mathbf{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi, \phi', \psi, \psi' \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \phi'\rangle, \langle \psi, \psi'\rangle \in \widetilde{\Lambda}^{\mathbf{A},D}_\Sigma(X)$. Thus, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\begin{array}{rcl} D_{\Sigma'}(X_{\Sigma'}, \mathrm{SEN}(f)(\phi)) & = & D_{\Sigma'}(X_{\Sigma'}, \mathrm{SEN}(f)(\phi')),\\ D_{\Sigma'}(X_{\Sigma'}, \mathrm{SEN}(f)(\psi)) & = & D_{\Sigma'}(X_{\Sigma'}, \mathrm{SEN}(f)(\psi')).\end{array}$$

Now, using the Modus Ponens with respect to $\to$ and the displayed equations, we get

$$\begin{array}{c}\mathrm{SEN}(f)(\psi) \in D_{\Sigma'}(X_{\Sigma'}, \mathrm{SEN}(f)(\phi) \to_{\Sigma'} \mathrm{SEN}(f)(\psi), \mathrm{SEN}(f)(\phi)),\\ \mathrm{SEN}(f)(\phi) \in D_{\Sigma'}(X_{\Sigma'}, \mathrm{SEN}(f)(\phi')).\end{array}$$

Therefore,

$$\begin{array}{rcl}\mathrm{SEN}(f)(\psi') & \in & D_{\Sigma'}(X_{\Sigma'},\mathrm{SEN}(f)(\psi))\\ & \subseteq & D_{\Sigma'}(X_{\Sigma'},\mathrm{SEN}(f)(\phi)\to_{\Sigma'}\mathrm{SEN}(f)(\psi),\mathrm{SEN}(f)(\phi))\\ & \subseteq & D_{\Sigma'}(X_{\Sigma'},\mathrm{SEN}(f)(\phi)\to_{\Sigma'}\mathrm{SEN}(f)(\psi),\mathrm{SEN}(f)(\phi')).\end{array}$$

By the Deduction Theorem, we now get

$$\mathrm{SEN}(f)(\phi')\to_{\Sigma'}\mathrm{SEN}(f)(\psi')\in D_{\Sigma'}(X_{\Sigma'},\mathrm{SEN}(f)(\phi)\to_{\Sigma'}\mathrm{SEN}(f)(\psi)).$$

This is equivalent to $\mathrm{SEN}(f)(\phi'\to_\Sigma\psi')\in D_{\Sigma'}(X_{\Sigma'},\mathrm{SEN}(f)(\phi\to_\Sigma\psi))$. Thus, by symmetry, we get that, for all $\Sigma'\in|\mathbf{Sign}|$ and all $f\in\mathbf{Sign}(\Sigma,\Sigma')$, $D_{\Sigma'}(X_{\Sigma'},\mathrm{SEN}(f)(\phi\to_\Sigma\psi))=D_{\Sigma'}(X_{\Sigma'},\mathrm{SEN}(f)(\phi'\to_\Sigma\psi'))$ and, we conclude that $\langle\phi\to_\Sigma\psi,\phi'\to_\Sigma\psi'\rangle\in\widetilde{\Lambda}^{\mathbf{A},D}_\Sigma(X)$. ■

Our next goal is to show that, if a finitary π-institution $\mathcal{I}$ has the Deduction Detachment Theorem, then every full $\mathcal{I}$-structure also has the Deduction Detachment Theorem.

Theorem 1444 *Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, with $\to^\flat$: $(\mathrm{SEN}^\flat)^2\to\mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I}=\langle\mathbf{F},C\rangle$ a finitary π-institution based on $\mathbf{F}$. If $\mathcal{I}$ has the Deduction Detachment Theorem with respect to $\to^\flat$, then every full $\mathcal{I}$-structure $\mathbb{L}=\langle\mathcal{A},D\rangle$ has the Deduction Detachment Theorem with respect to $\to$.*

Proof: Suppose $\mathcal{I}$ has the Deduction Detachment Theorem with respect to $\to^\flat$. By Corollary 1393, Corollary 1441 and Proposition 1442, it is enough to show that every $\mathcal{I}$-structure of the form $\langle\mathcal{A},\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ has the Deduction Detachment Theorem with respect to $\to$. By Corollary 1440, every $\mathcal{I}$-structure has the Modus Ponens with respect to $\to$. So it suffices to show that $\langle\mathcal{A},\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ has the Deduction Theorem with respect to $\to$, i.e., that, for all $\Sigma'\in|\mathbf{Sign}|$ and all $\Phi'\cup\{\phi',\psi'\}\subseteq\mathrm{SEN}(\Sigma')$,

$$\psi'\in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi',\phi')\quad\text{implies}\quad\phi'\to_{\Sigma'}\psi'\in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi').$$

We do this, using Proposition 114, by applying induction on $n<\omega$ to show that, for all $n<\omega$,

$$\psi'\in\Xi^n_{\Sigma'}(\Phi',\phi')\quad\text{implies}\quad\phi'\to_{\Sigma'}\psi'\in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi').$$

For $n=0$, we get $\psi'\in\Xi^0_{\Sigma'}(\Phi',\phi')=\Phi'\cup\{\phi'\}$.

- If $\psi'=\phi'$, then $\phi'\to_{\Sigma'}\psi'=\phi'\to_{\Sigma'}\phi'\in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\varnothing)\subseteq C^{\mathcal{A},\mathcal{I}}_{\Sigma'}(\Phi')$, because of Proposition 1436.
- If $\psi'\in\Phi'$, then $\psi'\to_{\Sigma'}(\phi'\to_{\Sigma'}\psi')\in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\varnothing)\subseteq C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi')$, again by Proposition 1436. Since $\psi'\in\Phi'\subseteq C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi')$, we get by the Modus Ponens, $\phi'\to_{\Sigma'}\psi'\in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi')$.

Assume, next that, if, for some $i < n$, $\psi' \in \Xi^i_\Sigma(\Phi', \phi')$, then $\phi' \to_{\Sigma'} \psi' \in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi')$. Consider $\psi' \in \Xi^n_{\Sigma'}(\Phi', \phi')$. By definition, there exists $\Sigma \in |\mathbf{Sign}^\flat|$, such that $F(\Sigma) = \Sigma'$, and $\Phi \cup \{\phi, \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that

$$\psi \in C_\Sigma(\Phi, \phi), \quad \alpha_\Sigma(\Phi) \subseteq \Xi^{n-1}_{\Sigma'}(\Phi', \phi'), \quad \alpha_\Sigma(\phi) = \phi', \quad \alpha_\Sigma(\psi) = \psi'.$$

Now, we have, on the one hand, by the Induction Hypothesis, $\phi' \to_{\Sigma'} \alpha_\Sigma(\chi) \in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi')$, for all $\chi \in \Phi$. On the other hand, since $\psi \in C_\Sigma(\Phi, \phi)$, we get, using Modus Ponens,

$$\psi \in C_\Sigma(\Phi, \phi) \subseteq C_\Sigma(\{\phi \to^\flat_\Sigma \chi : \chi \in \Phi\}, \phi)$$

and, therefore, by the Deduction Theorem, $\phi \to^\flat_\Sigma \psi \in C_\Sigma(\{\phi \to^\flat_\Sigma \chi : \chi \in \Phi\})$. Therefore, we obtain

$$\phi' \to_{\Sigma'} \psi' \in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\{\phi' \to_{\Sigma'} \alpha_\Sigma(\chi) : \chi \in \Phi\}).$$

Finally, we obtain

$$\begin{array}{rcl} \phi' \to_{\Sigma'} \psi' & \in & C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\{\phi' \to_{\Sigma'} \alpha_\Sigma(\chi) : \chi \in \Phi\}) \\ & \subseteq & C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi')) \\ & = & C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi'). \end{array}$$

We conclude that $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ has the Deduction Detachment Theorem and therefore, every full $\mathcal{I}$-structure does also. ∎

Finally, we show that if a π-institution has the Deduction Detachment Theorem, then every finitary $\mathcal{I}$-structure, with the Deduction Theorem and the Congruence Property is a full $\mathcal{I}$-structure.

Proposition 1445 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution that has the Deduction Detachment Theorem with respect to $\to^\flat$. If $\mathbb{L} = \langle \mathcal{A}, D\rangle$ is a finitary $\mathcal{I}$-structure with the Deduction Theorem and the strong Congruence Property, then $\mathbb{L}$ is a full $\mathcal{I}$-structure.*

Proof: Suppose $\mathbb{L} = \langle \mathcal{A}, D\rangle$ is a finitary $\mathcal{I}$-structure with the Deduction Theorem and the Congruence Property. Then, by Proposition 1385, $\mathcal{D} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and our goal is to show that $\mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$.

Suppose, first, that $T' \in \mathcal{D}^*$. Consider the bilgical quotient morphism $\langle I, \pi\rangle : \mathbb{L} \vdash \mathbb{L}^*$. Then we have $T = \pi^{-1}(T') \in \mathcal{D} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Thus, by Corollary 55, $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$. We conclude that $\mathcal{D}^* \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$.

Suppose, conversely, that $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$. Since $\mathcal{I}$ has the Deduction Detachment Theorem, by Corollary 1437, $T' \neq \varnothing$. Let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi^* \in D^*_\Sigma(T'_\Sigma)$. By the finitarity of $\mathbb{L}$ and Proposition 1365, there exist $\phi_0, \ldots, \phi_{n-1} \in \mathrm{SEN}(\Sigma)$, such that $\phi^*_0, \ldots, \phi^*_{n-1} \in T'_\Sigma$ and

$\phi^* \in D^*_\Sigma(\phi^*_0,\dots,\phi^*_{n-1})$. Since $\mathbb{L}$ has the Deduction Theorem, by Proposition 1442, so does $\mathbb{L}^*$, whence

$$\phi^*_0 \to^*_\Sigma (\cdots(\phi^*_{n-1} \to^*_\Sigma \phi^*)\cdots) \in D^*_\Sigma(\varnothing) = D^*_\Sigma(\phi^* \to^*_\Sigma \phi^*),$$

the last equality, by Proposition 1436. Now we get

$$D^*_\Sigma(\phi^*_0 \to^*_\Sigma (\cdots(\phi^*_{n-1} \to^*_\Sigma \phi^*)\cdots)) = D^*_\Sigma(\phi^* \to^*_\Sigma \phi^*).$$

Since $\mathbb{L}$ has the strong Congruence Property, by Proposition 1426, so does $\mathbb{L}^*$. Hence, by Corollary 1425,

$$\phi^*_0 \to^*_\Sigma (\cdots(\phi^*_{n-1} \to^*_\Sigma \phi^*)\cdots) = \phi^* \to^*_\Sigma \phi^* \in T'_\Sigma.$$

Since $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$ and $\phi^*_0,\dots,\phi^*_{n-1} \in T'_\Sigma$, we get by the Modus Ponens, $\phi^* \in T'_\Sigma$. We conclude that $D^*_\Sigma(T'_\Sigma) \subseteq T'_\Sigma$ and, therefore, $T' \in \mathcal{D}^*$. So $\mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$. Hence, $\mathbb{L} = \langle \mathcal{A}, D\rangle$ is a full $\mathcal{I}$-structure. ■

19.9.4 The Property of Disjunction

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system, with $\vee : \mathrm{SEN}^2 \to \mathrm{SEN}$ a binary natural transformation in N, and $\mathbb{L} = \langle \mathcal{A}, D\rangle$ a π-structure.

$\mathbb{L}$ has the **Disjunction Property with respect to** $\vee$ if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi,\psi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$D_\Sigma(\Phi, \phi \vee_\Sigma \psi) = D_\Sigma(\Phi,\phi) \cap D_\Sigma(\Phi,\psi).$$

In the next proposition, we discuss some of the $\mathbf{F}$(g)rules that a π-institution having the Disjunction Property satisfies.

Proposition 1446 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\vee^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ has the Disjunction Property with respect to $\vee^\flat$, then $\mathcal{I}$ satisfies the following $\mathbf{F}$-grules:*

(a) $x \vdash x \vee^\flat y$ *and* $y \vdash x \vee^\flat y$*;*

(b) $\dfrac{X, x \vdash z, \quad X, y \vdash z}{X, x \vee^\flat y \vdash z}$*, where X consists of any set of projections;*

(c) $x \vee^\flat y \vdash y \vee^\flat x$*;*

(d) $x \vdash x \vee^\flat x$ *and* $x \vee^\flat x \vdash x$*.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi,\psi,\chi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$.

(a) By the Disjunction Property, $\phi \vee^\flat_\Sigma \psi \in C_\Sigma(\phi) \cap C_\Sigma(\psi)$.

(b) Suppose $\chi \in C_\Sigma(\Phi,\phi)$ and $\chi \in C_\Sigma(\Phi,\psi)$. Then, by the Disjunction Property,
$$\chi \in C_\Sigma(\Phi,\phi) \cap C_\Sigma(\Phi,\psi) = C_\Sigma(\Phi,\phi \vee^\flat_\Sigma \psi).$$

(c) We have, using the Disjunction Property,
$$\begin{array}{rcl}\psi \vee^\flat_\Sigma \phi & \in & C_\Sigma(\psi \vee^\flat_\Sigma \phi) = C_\Sigma(\psi) \cap C_\Sigma(\phi)\\ & = & C_\Sigma(\phi) \cap C_\Sigma(\psi) = C_\Sigma(\phi \vee^\flat_\Sigma \psi).\end{array}$$

(d) We have
$$\phi \vee^\flat_\Sigma \phi \in C_\Sigma(\phi \vee^\flat_\Sigma \phi) = C_\Sigma(\phi) \cap C_\Sigma(\phi) = C_\Sigma(\phi)$$
and, also,
$$\phi \in C_\Sigma(\phi) = C_\Sigma(\phi) \cap C_\Sigma(\phi) = C_\Sigma(\phi \vee^\flat_\Sigma \phi).$$
■

It is not difficult to see that the Disjunction Property is also preserved under bilogical morphisms.

Proposition 1447 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\vee^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ *be* $N^\flat$*-algebraic systems,* $\mathbb{L} = \langle \mathbf{A}, D \rangle$*,* $\mathbb{L}' = \langle \mathbf{A}', D' \rangle$ *be* $N^\flat$*-structures based on* $\mathbf{A}$*,* $\mathbf{A}'$*, respectively, and* $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ *a bilogical morphism.* $\mathbb{L}$ *has the Disjunction Property with respect to* $\vee$ *if and only if* $\mathbb{L}'$ *has the Disjunction Property with respect to* $\vee'$*.*

Proof: Let $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\phi,\psi\} \subseteq \mathrm{SEN}(\Sigma)$. Then, since $\langle F,\alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ is a bilogical morphism, we have, using Proposition 1360,
$$\begin{array}{rcl}D_\Sigma(\Phi,\phi \vee_\Sigma \psi) & = & \alpha_\Sigma^{-1}(D'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi) \vee'_{F(\Sigma)} \alpha_\Sigma(\psi));\\ D_\Sigma(\Phi,\phi) \cap D_\Sigma(\Phi,\psi) & = & \alpha_\Sigma^{-1}(D'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi))\\ & & \quad \cap\, \alpha_\Sigma^{-1}(D'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\psi))\\ & = & \alpha_\Sigma^{-1}(D'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi))\\ & & \quad \cap\, D'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\psi))).\end{array}$$
Now, using the surjectivity of $\langle F,\alpha \rangle$, we get that
$$D_\Sigma(\Phi,\phi \vee_\Sigma \psi) = D_\Sigma(\Phi,\phi) \cap D_\Sigma(\Phi,\psi)$$
if and only if
$$\begin{array}{l}D'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi) \vee'_{F(\Sigma)} \alpha_\Sigma(\psi))\\ \quad = D'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi)) \cap D'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\psi))\end{array}$$
Once more, taking into account the surjectivity of $\langle F,\alpha \rangle$, we conclude that $\mathbb{L}$ has the Disjunction Property with respect to $\vee$ if and only if $\mathbb{L}'$ has the Disjunction Property with respect to $\vee'$. ■

Using induction, we can extend the defining equation of the Disjunction Property so that we can accommodate a finite number of conjunctions instead of only a single conjunction.

Proposition 1448 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system, with* $\vee : \mathrm{SEN}^2 \to \mathrm{SEN}$ *in* N*, and* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *a* π*-structure. If* $\mathbb{L}$ *has the Disjunction Property with respect to* $\vee$*, then, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\Phi \cup \{\phi_0, \dots, \phi_{n-1}\} \subseteq \mathrm{SEN}(\Sigma)$*,*

$$D_\Sigma(\Phi, \phi_0 \vee_\Sigma \psi, \dots, \phi_{n-1} \vee_\Sigma \psi) = D_\Sigma(\Phi, \phi_0, \dots, \phi_{n-1}) \cap D_\Sigma(\Phi, \psi).$$

Proof: First, note that, for all $\Sigma \in |\mathbf{Sign}|$, all $\Phi \cup \{\phi_0, \dots, \phi_{n-1}, \psi\} \subseteq \mathrm{SEN}(\Sigma)$ and all $i < n$,

$$\begin{array}{rcl}\phi_i \vee_\Sigma \psi & \in & D_\Sigma(\Phi, \phi_i) \cap D_\Sigma(\Phi, \psi)\\ & \subseteq & D_\Sigma(\Phi, \phi_0, \dots, \phi_{n-1}) \cap D_\Sigma(\Phi, \psi).\end{array}$$

Thus, we get

$$D_\Sigma(\Phi, \phi_0 \vee_\Sigma \psi, \dots, \phi_{n-1} \vee_\Sigma \psi) \subseteq D_\Sigma(\Phi, \phi_0, \dots, \phi_{n-1}) \cap D_\Sigma(\Phi, \psi).$$

For the reverse inclusion, we use induction on n.

For $n = 1$, by the Disjunction Property, we get $D_\Sigma(\Phi, \phi_0) \cap D_\Sigma(\Phi, \psi) = D_\Sigma(\Phi, \phi_0 \vee_\Sigma \psi)$.

Assume that the inclusion holds for n.

Let $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\phi_0, \dots, \phi_n, \psi\} \subseteq \mathrm{SEN}(\Sigma)$. Then we have

$$\begin{array}{l}D_\Sigma(\Phi, \phi_0, \phi_1, \dots, \phi_n) \cap D_\Sigma(\Phi, \psi)\\ = D_\Sigma(\Phi, \phi_0, \phi_1, \dots, \phi_n) \cap D_\Sigma(\Phi, \psi) \cap D_\Sigma(\Phi, \psi)\\ \subseteq D_\Sigma(\Phi, \phi_0, \phi_1, \dots, \phi_n) \cap D_\Sigma(\Phi, \psi, \phi_1, \dots, \phi_n) \cap D_\Sigma(\Phi, \psi)\\ \subseteq D_\Sigma(\Phi, \phi_0 \vee_\Sigma \psi, \phi_1, \dots, \phi_n) \cap D_\Sigma(\Phi, \phi_0 \vee_\Sigma \psi, \psi)\\ \subseteq D_\Sigma(\Phi, \phi_0, \vee_\Sigma \psi, \phi_1 \vee_\Sigma \psi, \dots, \phi_n \vee_\Sigma \psi).\end{array}$$

Hence the inclusion - and, therefore, the equation - holds for all $n < \omega$. ∎

Another property of structures having disjunction is that, if a specific entailment with a finite number of premises holds and the hypotheses are disjuncted with the same sentence, then the disjunction of the conclusion with the same sentence follows from the disjuncted hypotheses.

Lemma 1449 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system, with* $\vee : \mathrm{SEN}^2 \to \mathrm{SEN}$ *in* N*, and* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *a* π*-structure. If* $\mathbb{L}$ *has the Disjunction Property with respect to* $\vee$*, then, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi_0, \phi_{n-1}, \phi, \psi \in \mathrm{SEN}(\Sigma)$*,*

$$\phi \in D_\Sigma(\phi_0, \dots, \phi_{n-1}) \quad \textit{implies} \quad \phi \vee_\Sigma \psi \in D_\Sigma(\phi_0 \vee_\Sigma \psi, \dots, \phi_{n-1} \vee_\Sigma \psi).$$

Proof: Let $\Sigma \in |\mathbf{Sign}|$ and $\phi_0, \phi_{n-1}, \phi, \psi \in \mathrm{SEN}(\Sigma)$. By Proposition 1448,

$$D_\Sigma(\phi_0 \vee_\Sigma \psi, \dots, \phi_{n-1} \vee_\Sigma \psi) = D_\Sigma(\phi_0, \dots, \phi_{n-1}) \cap D_\Sigma(\psi).$$

But, by hypothesis, $\phi \in D_\Sigma(\phi_0,\dots,\phi_{n-1})$. So we get

$$\begin{array}{rcl}\phi \vee_\Sigma \psi & \in & D_\Sigma(\phi) \cap D_\Sigma(\psi)\\ & \subseteq & D_\Sigma(\phi_0,\dots,\phi_{n-1}) \cap D_\Sigma(\psi)\\ & = & D_\Sigma(\phi_0 \vee_\Sigma \psi,\dots,\phi_{n-1} \vee_\Sigma \psi).\end{array}$$

■

Our final goal is to show that, if a finitary π-institution has the Disjunction Property, then every full $\mathcal{I}$-structure also has the Disjunction Property. To accomplish this, we prove, first, a lemma to the effect that, if an entailment holds in a model then the disjunct of the conclusion with an arbitrary sentence is entailed by the same premises except for one, which is replaced by the disjunct of the original with the same sentence.

Lemma 1450 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\vee^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a finitary* π*-institution based on* $\mathbf{F}$*, which has the Disjunction Property with respect to* $\vee^\flat$*. Then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$*, all* $\Sigma' \in |\mathbf{Sign}|$ *and all* $\Phi' \cup \{\phi',\psi',\chi'\} \subseteq \mathrm{SEN}(\Sigma')$*,*

$$\chi' \in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi',\phi') \quad \textit{implies} \quad \chi' \vee_{\Sigma'} \psi' \in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi',\phi' \vee_{\Sigma'} \psi').$$

Proof: Let $\Sigma' \in |\mathbf{Sign}|$ and $\Phi' \cup \{\phi',\psi',\chi'\} \subseteq \mathrm{SEN}(\Sigma')$. By Proposition 114, $C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi',\phi') = \Xi_{\Sigma'}(\Phi',\phi') = \bigcup_{n<\omega} \Xi^n_{\Sigma'}(\Phi',\phi')$. We show by induction on $n < \omega$ that, for all $n < \omega$

$$\chi' \in \Xi^n_{\Sigma'}(\Phi',\phi') \quad \text{implies} \quad \chi' \vee_{\Sigma'} \psi' \in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi',\phi' \vee_{\Sigma'} \psi').$$

If $n = 0$, the hypothesis is $\chi' \in \Xi^0_{\Sigma'}(\Phi',\phi') = \Phi' \cup \{\phi'\}$.

- If $\chi' = \phi'$, then the conclusion follows trivially.
- Suppose that $\chi' \in \Phi'$. Then, we have $\chi' \vee_{\Sigma'} \psi' \in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\chi') \subseteq C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi') \subseteq C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi',\phi' \vee_{\Sigma'} \psi')$, where the first inclusion follows by Propositions 1446 and 1422.

Assume that the displayed implication holds for all $i < n$. Let $\chi' \in \Xi^n_{\Sigma'}(\Phi',\phi')$. By definition, there exists $\Sigma \in |\mathbf{Sign}^\flat|$, such that $F(\Sigma) = \Sigma'$, and $\phi_0,\dots,\phi_{k-1},\chi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$\chi \in C_\Sigma(\phi_0,\dots,\phi_{k-1}), \quad \alpha_\Sigma(\chi) = \chi', \quad \alpha_\Sigma(\phi_i) \in \Xi^{n-1}_{\Sigma'}(\Phi',\phi'),\ i < k.$$

By the induction hypothesis, $\alpha_\Sigma(\phi_i) \vee_{\Sigma'} \psi' \in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi',\phi' \vee_{\Sigma'} \psi')$, for all $i < k$. Note that, by the surjectivity of $\langle F,\alpha\rangle$, there exists $\psi \in \mathrm{SEN}(\Sigma)$, such that $\alpha_\Sigma(\psi) = \psi'$. Since $\chi \in C_\Sigma(\phi_0,\dots,\phi_{k-1})$, we get, by Lemma 1449, $\chi \vee^\flat_\Sigma \psi \in C_\Sigma(\phi_0 \vee^\flat_\Sigma \psi,\dots,\phi_{k-1} \vee^\flat_\Sigma \psi)$. Therefore, applying $\langle F,\alpha\rangle$,

$$\begin{array}{rcl}\chi' \vee_{\Sigma'} \psi' & \in & C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\alpha_\Sigma(\phi_0) \vee_{\Sigma'} \psi',\dots,\alpha_\Sigma(\phi_{k-1}) \vee_{\Sigma'} \psi')\\ & \subseteq & C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\Phi',\phi' \vee_{\Sigma'} \psi').\end{array}$$

Thus, the displayed formula holds for all $n < \omega$, yielding the conclusion. ∎

Finally, we show that all full models of a given finitary π-institution with the Disjunction Property also have the Disjunction Property.

Theorem 1451 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\vee^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a finitary π-institution based on $\mathbf{F}$. If $\mathcal{I}$ has the Disjunction Property with respect to $\vee^\flat$, then every full $\mathcal{I}$-structure $\mathbb{L} = \langle \mathcal{A}, D \rangle$ has the Disjunction Property with respect to $\vee$.*

Proof: By Corollary 1393 and Proposition 1447, it suffices to show that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi, \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{aligned}&C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi) \vee_{F(\Sigma)} \alpha_\Sigma(\psi)) \\ &\qquad = C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi)) \cap C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\psi)).\end{aligned}$$

By Propositions Propositions 1446 and 1422, we have

$$\begin{aligned}&C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi) \vee_{F(\Sigma)} \alpha_\Sigma(\psi)) \\ &\qquad C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi)) \cap C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\psi)).\end{aligned}$$

Conversely, suppose that, for some $\chi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\alpha_\Sigma(\chi) \in C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi)) \cap C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\psi)).$$

Then, by Lemma 1450,

$$\begin{aligned}\alpha_\Sigma(\chi) \vee_{F(\Sigma)} \alpha_\Sigma(\psi) &\in C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi) \vee_{F(\Sigma)} \alpha_\Sigma(\psi)), \\ \alpha_\Sigma(\chi) \vee_{F(\Sigma)} \alpha_\Sigma(\chi) &\in C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\psi) \vee_{F(\Sigma)} \alpha_\Sigma(\chi)).\end{aligned}$$

Now we get

$$\begin{array}{rcl}\alpha_\Sigma(\chi) & \in & C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\chi) \vee_{F(\Sigma)} \alpha_\Sigma(\chi)) \\ & \subseteq & C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\psi) \vee_{F(\Sigma)} \alpha_\Sigma(\chi)) \\ & \subseteq & C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\chi) \vee_{F(\Sigma)} \alpha_\Sigma(\psi)) \\ & \subseteq & C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi) \vee_{F(\Sigma)} \alpha_\Sigma(\psi)).\end{array}$$

Taking into account the surjectivity of $\langle F, \alpha \rangle$, we conclude that $\mathbb{L}$ has the Disjunction Property with respect to $\vee$. ∎

19.9.5 Reductio ad Absurdum

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system, with $\neg : \mathrm{SEN} \to \mathrm{SEN}$ a unary natural transformation in N, and $\mathbb{L} = \langle \mathbf{A}, D\rangle$ a π-structure based on $\mathbf{A}$.

$\mathbb{L}$ has the **Intuitionistic Reductio ad Absurdum with respect to** $\neg$ if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\neg_\Sigma \phi \in D_\Sigma(\Phi) \quad \text{if and only if} \quad D_\Sigma(\Phi, \phi) = \mathrm{SEN}(\Sigma).$$

$\mathbb{L}$ has the **Reductio ad Absurdum with respect to** $\neg$ if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\phi \in D_\Sigma(\Phi) \quad \text{if and only if} \quad D_\Sigma(\Phi, \neg_\Sigma \phi) = \mathrm{SEN}(\Sigma).$$

The two properties of the Reductio ad Absurdum are related.

Proposition 1452 *Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system, with $\neg : \mathrm{SEN} \to \mathrm{SEN}$ in N, and $\mathbb{L} = \langle \mathbf{A}, D\rangle$ a π-structure based on $\mathbf{A}$. $\mathbb{L}$ has the Reductio as Absurdum with respect to $\neg$ if and only if it has the Intuitionistic Reductio ad Absurdum with respect to $\neg$ and, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $\phi \in D_\Sigma(\neg_\Sigma \neg_\Sigma \phi)$.*

Proof: Suppose, first, that $\mathbb{L}$ has the Reduction ad Absurdum with respect to $\neg$ and let $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$.

Since $\neg_\Sigma \phi \in D_\Sigma(\neg_\Sigma \phi)$, we get $D_\Sigma(\neg_\Sigma \phi, \neg_\Sigma \neg_\Sigma \phi) = \mathrm{SEN}(\Sigma)$. Therefore, $\phi \in D_\Sigma(\neg_\Sigma \neg_\Sigma \phi)$.

Suppose, next, that $\neg_\Sigma \phi \in D_\Sigma(\Phi)$. Note that, since $\phi \in D_\Sigma(\phi)$, we also have $D_\Sigma(\phi, \neg_\Sigma \phi) = \mathrm{SEN}(\Sigma)$. So, finally, $\mathrm{SEN}(\Sigma) = D_\Sigma(\phi, \neg_\Sigma \phi) \subseteq D_\Sigma(\Phi, \phi)$ and equality follows.

On the other hand, if $D_\Sigma(\Phi, \phi) = \mathrm{SEN}(\Sigma)$, then $D_\Sigma(\Phi, \neg_\Sigma \neg_\Sigma \phi) = \mathrm{SEN}(\Sigma)$, whence $\neg_\Sigma \phi \in D_\Sigma(\phi)$.

Suppose, conversely, that $\mathbb{L}$ has the Intuitionistic Reductio ad Absurdum and that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $\phi \in D_\Sigma(\neg_\Sigma \neg_\Sigma \phi)$, and let $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$.

Suppose, first, $\phi \in D_\Sigma(\Phi)$. Since $\neg_\Sigma \phi \in D_\Sigma(\neg_\Sigma \phi)$, we get $D_\Sigma(\phi, \neg_\Sigma \phi) = \mathrm{SEN}(\Sigma)$. Therefore, $\mathrm{SEN}(\Sigma) = D_\Sigma(\phi, \neg_\Sigma \phi) \subseteq D_\Sigma(\Phi, \neg_\Sigma \phi)$.

On the other hand, if $D_\Sigma(\Phi, \neg_\Sigma \phi) = \mathrm{SEN}(\Sigma)$, then $\neg_\Sigma \neg_\Sigma \phi \in D_\Sigma(\Phi)$, whence $\phi \in D_\Sigma(\neg_\Sigma \neg_\Sigma \phi) \subseteq D_\Sigma(\phi)$. ■

We also show for future reference that the Intuitionistic Reductio ad Absurdum is preserved under bilogical morphisms.

Proposition 1453 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\neg^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ in $N^\flat$, $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ be $N^\flat$-algebraic systems, $\mathbb{L} = \langle \mathbf{A}, D\rangle$, $\mathbb{L}' = \langle \mathbf{A}', D'\rangle$ be $N^\flat$-structures based on $\mathbf{A}$, $\mathbf{A}'$, respectively, and $\langle F, \alpha\rangle : \mathbb{L} \vdash \mathbb{L}'$ a bilogical morphism. $\mathbb{L}$ has the Intuitionistic Reductio ad Absurdum with respect to $\neg$ if and only if $\mathbb{L}'$ has the Intuitionistic Reduction ad Absurdum with respect to $\neg'$.*

Proof: Suppose $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$. Then, we have

$$\begin{array}{rll} \neg_\Sigma\phi \in D_\Sigma(\Phi) & \text{iff} & \alpha_\Sigma(\neg_\Sigma\phi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \\ & \text{iff} & \neg'_{F(\Sigma)}(\alpha_\Sigma(\phi)) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)). \end{array}$$

Moreover, using the surjectivity of $\langle F, \alpha\rangle$,

$$D_\Sigma(\Phi,\phi) = \mathrm{SEN}(\Sigma) \quad \text{iff} \quad D'_{F(\Sigma)}(\alpha_\Sigma(\Phi),\alpha_\Sigma(\phi)) = \mathrm{SEN}'(F(\Sigma)).$$

Thus, the equivalence

$$\neg_\Sigma\phi \in D_\Sigma(\Phi) \quad \text{iff} \quad D_\Sigma(\Phi,\phi) = \mathrm{SEN}(\Sigma)$$

holds if and only if the equivalence

$$\neg'_{F(\Sigma)}\alpha_\Sigma(\phi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \quad \text{iff} \quad D'_{F(\Sigma)}(\alpha_\Sigma(\Phi),\alpha_\Sigma(\phi)) = \mathrm{SEN}'(F(\Sigma))$$

holds. In other words, $\mathbb{L}$ has the Intuitionistic Reduction ad Absurdum with respect to $\neg$ if and only if $\mathbb{L}'$ has the Intuitionistic Reductio ad Absurdum with respect to $\neg'$. ■

The Intuitionistic Reductio ad Absurdum is also closely related with are called inconsistent elements or inconsistencies.

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system, with $\bot : \mathrm{SEN} \to \mathrm{SEN}$ a unary natural transformation in N, and $\mathbb{L} = \langle \mathbf{A}, D\rangle$ a π-structure based on $\mathbf{A}$. $\bot$ is an **inconsistency** in $\mathbb{L}$ if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$D_\Sigma(\bot_\Sigma\phi) = \mathrm{SEN}(\Sigma).$$

Having an inconsistency is clearly expressible by an $\mathbf{F}$-rule and, therefore, if a π-institution has an inconsistency then all its models do also.

Lemma 1454 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\bot^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. $\bot^\flat$ is an inconsistency in $\mathcal{I}$ if and only if $\mathcal{I}$ satisfies the $\mathbf{F}$-rule $\bot^\flat x \vdash y$.*

Proof: We have that $\mathcal{I}$ satisfies $\bot^\flat x \vdash y$ if and only if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, $\psi \in C_\Sigma(\bot^\flat_\Sigma\phi)$, if and only if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $C_\Sigma(\bot^\flat_\Sigma\phi) = \mathrm{SEN}(\Sigma)$, if and only if $\bot^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ is an inconsistency in $\mathcal{I}$. ■

Corollary 1455 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\bot^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $\bot^\flat$ is an inconsistency in $\mathbb{L}$, then, $\bot$ is an inconsistency in every $\mathcal{I}$-structure $\mathbb{L} = \langle \mathcal{A}, D\rangle$, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$.*

Proof: By Lemma 1454 and Proposition 1422. ■

The following proposition exhibits the relation between the Intuitionistic Reductio ad Absurdum and inconsistencies.

Proposition 1456 *Let* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ *be an algebraic system, with* $\to : \mathrm{SEN}^2 \to \mathrm{SEN}$, *and* $\mathbb{L} = \langle \mathbf{A}, D\rangle$ *a* π*-structure that has the Deduction Detachment Theorem with respect to* $\to$. $\mathbb{L}$ *has the Intuitionistic Reductio ad Absurdum with respect to some* $\neg : \mathrm{SEN} \to \mathrm{SEN}$ *in* N *if and only if it has an inconsistency* $\bot : \mathrm{SEN} \to \mathrm{SEN}$ *in* N. *Moreover, in that case, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi \in \mathrm{SEN}(\Sigma)$,

$$D_\Sigma(\neg_\Sigma \phi) = D_\Sigma(\phi \to_\Sigma \bot_\Sigma \phi).$$

Proof: Suppose that $\mathbb{L}$ has the Deduction Detachment Theorem with respect to $\to$.

Assume, first, that $\mathbb{L}$ also has the Intuitionistic Reductio ad Absurdum with respect to $\neg$. Let $\bot : \mathrm{SEN} \to \mathrm{SEN}$ be defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, by

$$\bot_\Sigma \phi = \neg_\Sigma(\phi \to_\Sigma \phi).$$

First, note, that, since $\bot = \neg \circ \to \circ \langle p^{1,0}, p^{1,0}\rangle$ and both $\to$ and $\neg$ are in N, we get that $\bot$ is also in N. Moreover, we have, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{ll} & \neg_\Sigma(\phi \to_\Sigma \phi) \in D_\Sigma(\neg_\Sigma(\phi \to_\Sigma \phi)) \\ \text{iff} & D_\Sigma(\phi \to_\Sigma \phi, \neg_\Sigma(\phi \to_\Sigma \phi)) = \mathrm{SEN}(\Sigma) \\ & \text{(by the Intuitionistic Reductio ad Absurdum)} \\ \text{iff} & D_\Sigma(\neg_\Sigma(\phi \to_\Sigma \phi)) = \mathrm{SEN}(\Sigma). \\ & \text{(by Proposition 1436)} \end{array}$$

Thus, $\bot$ is an inconsistency in $\mathbb{L}$.

Assume, conversely, that $\bot : \mathrm{SEN} \to \mathrm{SEN}$ is an inconsistency in $\mathbb{L}$. Define $\neg : \mathrm{SEN} \to \mathrm{SEN}$, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, by

$$\neg_\Sigma \phi = \phi \to_\Sigma \bot_\Sigma \phi.$$

Since $\neg = \to \circ \langle p^{1,0}, \bot\rangle$ and, both $\to$ and $\bot$ are in N, it follows that $\neg$ is also in N. Moreover, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, we have

$$\begin{array}{lll} \neg_\Sigma \phi \in D_\Sigma(\Phi) & \text{iff} & \phi \to_\Sigma \bot_\Sigma \phi \in D_\Sigma(\Phi) \\ & \text{iff} & \bot_\Sigma \phi \in D_\Sigma(\Phi, \phi) \\ & \text{iff} & D_\Sigma(\Phi, \phi) = \mathrm{SEN}(\Sigma). \end{array}$$

Thus, $\mathbb{L}$ has the Intuitionistic Reductio ad Absurdum with respect to $\neg$.

Finally, it remains to prove the last equality. Let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. On the one hand, we have, by the Modus Ponens, $\bot_\Sigma \phi \in D_\Sigma(\phi, \phi \to_\Sigma$

$\bot_\Sigma\phi$), whence, since $\bot$ is an inconsistency, $D_\Sigma(\phi, \phi \to_\Sigma \bot_\Sigma\phi) = \mathrm{SEN}(\Sigma)$ and, hence, by the Intuitionistic Reductio ad Absurdum, $\neg_\Sigma\phi \in D_\Sigma(\phi \to_\Sigma \bot_\Sigma\phi)$. On the other hand, since $\neg_\Sigma\phi \in D_\Sigma(\neg_\Sigma\phi)$, by the Intuitionistic Reductio ad Absurdum, $D_\Sigma(\phi, \neg_\Sigma\phi) = \mathrm{SEN}(\Sigma)$ and, hence, $\neg_\Sigma\phi \in D_\Sigma(\phi, \neg_\Sigma\phi)$. Therefore, by the Deduction Theorem, $\phi \to_\Sigma \bot_\Sigma\phi \in D_\Sigma(\neg_\Sigma\phi)$. These two parts allow us to conclude that $D_\Sigma(\neg_\Sigma\phi) = D_\Sigma(\phi \to_\Sigma \bot_\Sigma\phi)$. ■

Since both the Deduction Detachment Theorem and inconsistencies are inherited by the full models of a finitary π-institution, we obtain the following concerning transference of the Intuitionistic Reductio ad Absurdum by full models.

Corollary 1457 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ and $\neg^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a finitary π-institution based on $\mathbf{F}$. If $\mathcal{I}$ has the Deduction Detachment Theorem with respect to $\to^\flat$ and the Intuitionistic Reductio ad Absurdum with respect to $\neg^\flat$, then, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, every full $\mathcal{I}$-structure $\mathbb{L} = \langle \mathcal{A}, D \rangle$ also has the Deduction Detachment Theorem with respect to $\to$ and the Intuitionistic Reductio ad Absurdum with respect to $\neg$.*

Proof: Assume the hypothesis and let $\mathbb{L} = \langle \mathcal{A}, D \rangle \in \mathrm{FStr}(\mathcal{I})$. By Theorem 1444, $\mathbb{L}$ has the Deduction Detachment Theorem with respect to $\to$. By Proposition 1456, $\bot^\flat = \neg^\flat \circ \to^\flat \circ \langle p^{1,0}, p^{1,0} \rangle$ is an inconsistency in $\mathcal{I}$. Therefore, by Corollary 1455, $\bot$ is an inconsistency in $\mathbb{L}$. Finally, we use again Proposition 1456 to conclude that $\mathbb{L}$ has both the Deduction Detachment Theorem with respect to $\to$ and the Intuitionistic Reductio ad Absurdum with respect to $\neg$. ■

19.9.6 Modality Introduction

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an algebraic system, with $\# : \mathrm{SEN} \to \mathrm{SEN}$ a unary natural transformation in N, and $\mathbb{L} = \langle \mathbf{A}, D \rangle$ a π-structure based on $\mathbf{A}$. $\mathbb{L}$ has **Modality Introduction with respect to** $\#$ if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\phi \in D_\Sigma(\Phi) \quad \text{implies} \quad \#_\Sigma\phi \in D_\Sigma(\#_\Sigma\Phi),$$

where $\#_\Sigma\Phi = \{\#_\Sigma\chi : \chi \in \Phi\}$.

It turns out that Modality Introduction is also preserved under bilogical morphisms.

Proposition 1458 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\#^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ in $N^\flat$, $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\mathbf{A}' = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ be $N^\flat$-algebraic systems, $\mathbb{L} = \langle \mathbf{A}, D \rangle$, $\mathbb{L}' = \langle \mathbf{A}', D' \rangle$ be $N^\flat$-structures based on $\mathbf{A}$, $\mathbf{A}'$, respectively, and $\langle F, \alpha \rangle : \mathbb{L} \vdash \mathbb{L}'$ a bilogical morphism. $\mathbb{L}$ has the Modality Introduction with respect to $\#$ if and only if $\mathbb{L}'$ has the Modality Introduction with respect to $\#'$.*

Proof: Let $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$. Then, since $\langle F, \alpha\rangle$ is a bilogical morphism, we have

$$\begin{array}{rcl}\phi \in D_\Sigma(\Phi) & \text{iff} & \alpha_\Sigma(\phi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi));\\ \#_\Sigma\phi \in D_\Sigma(\#_\Sigma\Phi) & \text{iff} & \#'_{F(\Sigma)}\alpha_\Sigma(\phi) \in D'_{F(\Sigma)}(\#'_{F(\Sigma)}\alpha_\Sigma(\Phi)).\end{array}$$

Therefore, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\phi \in D_\Sigma(\Phi) \quad \text{implies} \quad \#_\Sigma\phi \in D_\Sigma(\#_\Sigma\Phi)$$

is equivalent to

$$\alpha_\Sigma(\phi) \in D'_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \quad \text{implies} \quad \#'_{F(\Sigma)}\alpha_\Sigma(\phi) \in D'_{F(\Sigma)}(\#'_{F(\Sigma)}\alpha_\Sigma(\Phi)).$$

Taking into account the surjectivity of $\langle F, \alpha\rangle$, we conclude that $\mathbb{L}$ has Modality Introduction with respect to $\#$ if and only if $\mathbb{L}'$ has Modality Introduction with respect to $\#'$. ∎

We conclude this section by showing that modality introduction is inherited by all full $\mathcal{I}$-structures if $\mathcal{I}$ is a finitary π-institution possessing the property.

Proposition 1459 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\#^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a finitary π-institution based on $\mathbf{F}$. If $\mathcal{I}$ has Modality Introduction with respect to $\#^\flat$, then every full $\mathcal{I}$-structure $\mathbb{L} = \langle \mathcal{A}, D\rangle$ has the Modality Introduction with respect to $\#$.*

Proof: By Corollary 1393 and Proposition 1458, it suffices to show that every $\mathcal{I}$-structure of the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ has the Modality Introduction with respect to $\#$. Let $\Sigma' \in |\mathbf{Sign}|$ and $\Phi' \cup \{\phi'\} \subseteq \mathrm{SEN}(\Sigma')$. By Proposition 114, it suffices to show that

$$\phi' \in \Xi_{\Sigma'}(\Phi') \quad \text{implies} \quad \#_{\Sigma'}\phi' \in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\#_{\Sigma'}\Phi').$$

We do this by applying induction on $n < \omega$ to show that, for all $n < \omega$,

$$\phi' \in \Xi^n_{\Sigma'}(\Phi') \quad \text{implies} \quad \#_{\Sigma'}\phi' \in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\#_{\Sigma'}\Phi').$$

If $n = 0$, then the hypothesis is $\phi' \in \Xi^0_{\Sigma'}(\Phi') = \Phi'$ and the conclusion is trivial. Assume that the displayed formula holds, for all $i < n$ and assume $\Sigma' \in |\mathbf{Sign}|$ and $\Phi' \cup \{\phi'\} \subseteq \mathrm{SEN}(\Sigma')$, such that $\phi' \in \Xi^n_{\Sigma'}(\Phi')$. Then, by definition, there exists $\Sigma \in |\mathbf{Sign}^\flat|$, such that $F(\Sigma) = \Sigma'$, and $\phi_0, \ldots, \phi_{k-1}, \phi \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$\phi \in C_\Sigma(\phi_0, \ldots, \phi_{k-1}), \quad \alpha_\Sigma(\phi) = \phi', \quad \alpha_\Sigma(\phi_i) \in \Xi^{n-1}_{\Sigma'}(\Phi'),\ i < k.$$

By the induction hypothesis, for all $i < k$, $\#_{\Sigma'}\alpha_\Sigma(\phi_i) \in C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\#_{\Sigma'}\Phi')$. Moreover, since $\mathcal{I}$ has Modality Introduction with respect to $\#^\flat$, we get $\#^\flat_\Sigma\phi \in C_\Sigma(\#^\flat_\Sigma\phi_0,\dots,\#^\flat_\Sigma\phi_{k-1})$. Therefore, applying $\langle F,\alpha\rangle$,

$$\begin{array}{lcl}\#_{F(\Sigma)}\phi' & \in & C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\#_{F(\Sigma)}\alpha_\Sigma(\phi_0),\dots,\#_{F(\Sigma)}\alpha_\Sigma(\phi_{k-1}))\\ & \subseteq & C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(\#_\Sigma\Phi').\end{array}$$

This proves the induction step and shows that $\mathbb{L}$ has the Modality Introduction with respect to $\#$. ■

19.10 $\mathcal{I}$-Structures and Protoalgebraicity

We now work with an arbitrary π-institution $\mathcal{I}$ and look at its $\mathcal{I}$-structures and their properties. We start with a characterization of protoalgebraicity involving $\mathcal{I}$-structures.

Proposition 1460 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Then the following conditions are equivalent.*

(i) $\mathcal{I}$ is protoalgebraic;

(ii) For every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ and every $\mathcal{I}$-structure $\mathbb{L} = \langle \mathcal{A}, D\rangle$, $\widetilde{\Omega}^{\mathcal{A}}(D) = \Omega^{\mathcal{A}}(\mathrm{Thm}(\mathbb{L}))$;

(iii) For every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ and every $\mathcal{I}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, T\rangle$, $\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})) = \Omega^{\mathcal{A}}(T)$;

(iv) For every $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\Omega}(\mathcal{I}^T) = \Omega(T)$.

Proof:

(i)⇒(ii) Assume $\mathcal{I}$ is protoalgebraic and let $\mathbb{L} = \langle \mathcal{A}, D\rangle$ be an $\mathcal{I}$-structure. Then, by Proposition 1385, $\mathcal{D} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Moreover, by Theorem 179, $\Omega^{\mathcal{A}}$ is monotone on $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Hence, we get

$$\widetilde{\Omega}^{\mathcal{A}}(D) = \bigcap\{\Omega^{\mathcal{A}}(T) : T \in \mathcal{D}\} = \Omega^{\mathcal{A}}(\bigcap\mathcal{D}) = \Omega^{\mathcal{A}}(\mathrm{Thm}(\mathbb{L})).$$

(ii)⇒(iii) Follows by applying (ii) to $\mathbb{L} = \langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$.

(iii)⇒(iv) Follows by applying (iii) to $\mathcal{A} = \langle \mathcal{F}, T\rangle$, where, as usual, $\mathcal{F} = \langle \mathbf{F}, \langle I,\iota\rangle\rangle$, $\langle I,\iota\rangle : \mathbf{F} \to \mathbf{F}$ the identity morphism.

(iv)⇒(i) Suppose that, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\Omega}(\mathcal{I}^T) = \Omega(T)$ and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \le T'$. Then $T' \in \mathrm{ThFam}(\mathcal{I}^T)$, whence $\widetilde{\Omega}(\mathcal{I}^T) \le \Omega(T')$. But, by hypothesis, $\widetilde{\Omega}(\mathcal{I}^T) = \Omega(T)$. Thus, we get $\Omega(T) \le \Omega(T')$. We conclude that Ω is monotone on theory families and, therefore, $\mathcal{I}$ is protoalgebraic.

■

Recall that to a π-institution $\mathcal{I}$, we have associated two different classes of algebraic systems. On the one hand, the class $\mathrm{AlgSys}^*(\mathcal{I})$ consists of the $\mathbf{F}$-algebraic system reducts of the reduced $\mathcal{I}$-matrix families. On the other, the class $\mathrm{AlgSys}(\mathcal{I})$ consists of the $\mathbf{F}$-algebraic system reducts of the reduced full $\mathcal{I}$-structures, or, equivalently, as was shown in Proposition 1399, by the $\mathbf{F}$-algebraic system reducts of the reduced $\mathcal{I}$-structures. Under the hypothesis of protoalgebraicity, it turns out that the two classes $\mathrm{AlgSys}(\mathcal{I})$ and $\mathrm{AlgSys}^*(\mathcal{I})$ coincide.

Proposition 1461 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is protoalgebraic, then* $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$.

Proof: By Theorem 1404, we know that $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$. Assume, conversely, that $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle \in \mathrm{AlgSys}(\mathcal{I})$. Then, there exists, by Proposition 1399, $\mathcal{D} \in \mathrm{ClFam}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) = \Delta^{\mathcal{A}}$. Thus, by Proposition 1460, $\widetilde{\Omega}^{\mathcal{A}}(\bigcap \mathcal{D}) = \widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) = \Delta^{\mathcal{A}}$, whence, since $\bigcap \mathcal{D} \in \mathcal{D} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I})$. ■

Protoalgebraicity is strong enough to allow full $\mathcal{I}$-structures on an algebraic system to be determined by their theorem systems.

Lemma 1462 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system. If* $\mathcal{I}$ *is protoalgebraic and* $\mathbb{L} = \langle \mathcal{A}, D \rangle$*,* $\mathbb{L}' = \langle \mathcal{A}, D' \rangle$ *are full* $\mathcal{I}$ *structures based on* $\mathcal{A}$*, such that* $\mathrm{Thm}(\mathbb{L}) = \mathrm{Thm}(\mathbb{L}')$*, then* $\mathbb{L} = \mathbb{L}'$.

Proof: Since $\mathcal{I}$ is protoalgebraic, we have, by Proposition 1460,

$$\widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) = \Omega^{\mathcal{A}}(\mathrm{Thm}(\mathbb{L})) = \Omega^{\mathcal{A}}(\mathrm{Thm}(\mathbb{L}')) = \widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}').$$

By the Isomorphism Theorem 1408, $\widetilde{\Omega}^{\mathcal{A}} : \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order isomorphism, in particular one-to-one. So we get that $\mathbb{L} = \mathbb{L}'$. ■

For protoalgebraic π-institutions, it follows that all full $\mathcal{I}$-models have the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle = \langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{A}}(\langle \mathcal{A}, T \rangle) \rangle$, i.e., their closure systems are principal filters in the lattice of $\mathcal{I}$-filter families of the underlying algebraic system.

Theorem 1463 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is protoalgebraic if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, all* $\mathcal{I}$*-structures in* $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$ *have the form* $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}) \rangle$*, for some* $\mathfrak{A} = \langle \mathcal{A}, T \rangle \in \mathrm{MatFam}^{\mathcal{I}}(\mathcal{A})$.

Proof: Assume, first, that $\mathcal{I}$ is protoalgebraic and let $\mathbb{L} = \langle \mathcal{A}, D \rangle \in \mathrm{FStr}(\mathcal{I})$. Let $T = \mathrm{Thm}(\mathbb{L})$ and set $\mathfrak{A} = \langle \mathcal{A}, T \rangle$. Clearly, $\mathcal{D} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$. By protoalgebracity and Proposition 1460, $\widetilde{\Omega}^{\mathcal{A}}(\mathbb{L}) = \Omega^{\mathcal{A}}(T)$. Therefore, if $\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T)$ denotes the quotient morphism, $\langle I, \pi \rangle : \mathbb{L} \vdash \mathbb{L}^*$ is a bilogical morphism. Since $\mathbb{L} \in \mathrm{FStr}(\mathcal{I})$, $\mathcal{D}^* = \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A}^*)$. But then, if $T' \in \mathrm{ThFam}^{\mathcal{I}}(\mathfrak{A})$, $T \leq T'$, whence $\Omega^{\mathcal{A}}(T)$ is compatible with T' and, hence, by Corollary 56, $T'/\Omega^{\mathcal{A}}(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*) = \mathcal{D}^*$. Therefore, $T' = \pi^{-1}(T'/\Omega^{\mathcal{A}}(T)) \in \mathcal{D}$. So $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \mathcal{D}$.

Suppose, conversely, that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, all $\mathcal{I}$-structures in $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$ have the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}) \rangle$, for some $\mathfrak{A} = \langle \mathcal{A}, T \rangle \in \mathrm{MatFam}^{\mathcal{I}}(\mathcal{A})$. Let $T, T' \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \leq T'$. Since, by Theorem 1404, $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$, $\Omega^{\mathcal{A}}(T) \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$. Therefore, by the Isomorphism Theorem 1408, there exists $\mathbb{L} = \langle \mathcal{A}, \mathcal{D} \rangle \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) = \Omega^{\mathcal{A}}(T)$. Let $\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T)$ be the quotient morphism. Then $\langle I, \pi \rangle : \mathbb{L} \vdash \mathbb{L}^*$ is a bilogical morphism and, since $\mathbb{L}$ is full, $\mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*)$. Thus, $T = \pi^{-1}(T/\Omega^{\mathcal{A}}(T)) \in \mathcal{D}$. By hypothesis, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$, for some $\mathfrak{A} = \langle \mathcal{A}, T'' \rangle \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Thus, $T'' \leq T'$, whence $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A}) = \mathcal{D}$. We now get $\Omega^{\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) \leq \Omega^{\mathcal{A}}(T')$. So $\Omega^{\mathcal{A}}$ is monotone on $\mathcal{A}$. Since $\mathcal{A}$ was arbitrary, we conclude that $\mathcal{I}$ is protoalgebraic. ∎

We proved that, for a protoalgebraic π-institution $\mathcal{I}$, all full $\mathcal{I}$-structures have the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle$, for some $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. We seek now to characterize those $\mathcal{I}$-filter families T for which the pair $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle$ is a full $\mathcal{I}$-model, i.e., those $\mathcal{I}$-filter families T that give rise, through the principal filters they determine in $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})$ to full $\mathcal{I}$-structures.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system. We define

$$\mathrm{FiFam}^{\mathcal{I},f}(\mathcal{A}) = \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T) \rangle \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})\}.$$

Using the Isomorphism Theorem 1408, it is not difficult to see that, under protoalgebraicity, there exists an order isomorphism between the poset determined by $\mathrm{FiFam}^{\mathcal{I},f}(\mathcal{A})$ and the lattice of all $\mathcal{I}$-congruence systems on $\mathcal{A}$.

Proposition 1464 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is protoalgebraic, then, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$,*

$$\Omega^{\mathcal{A}} : \mathbf{FiFam}^{\mathcal{I},f}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism.

Proof: Consider the mapping $T \mapsto \langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^T) \rangle$. This is a mapping from $\mathrm{FiFam}^{\mathcal{I},f}(\mathcal{A})$ into $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$, by the definition of $\mathrm{FiFam}^{\mathcal{I},f}(\mathcal{A})$. Clearly, it is one-to-one and both order preserving and order reflecting. If $\mathcal{I}$ is protoalgebraic, by Theorem 1463, it is also surjective. Hence, it is an order isomorphism. By the Isomorphism Theorem 1408, $\widetilde{\Omega}^{\mathcal{A}} : \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is also an order isomorphism. Thus, $T \mapsto \widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T)$ is an order isomorphism from $\mathrm{FiFam}^{\mathcal{I},f}(\mathcal{A})$ onto $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$. By Protoalgebraicity and Proposition 1460, we have $\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T) = \Omega^{\mathcal{A}}(T)$ and. moreover, by Proposition 1461, $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}) = \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. Hence, we conclude that $\Omega^{\mathcal{A}} : \mathbf{FiFam}^{\mathcal{I},f}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order-isomorphism. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system. Define

$$\sim^{\mathcal{I},\mathcal{A}} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^2$$

by setting, for all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$T \sim^{\mathcal{I},\mathcal{A}} T' \quad \text{iff} \quad \Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T'),$$

i.e., $\sim^{\mathcal{I},\mathcal{A}}$ is the kernel of the Leibniz operator on $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

It is clear from the definition that $\sim^{\mathcal{I},\mathcal{A}}$ is an equivalence relation on $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. In case $\mathcal{I}$ is protoalgebraic, we have another important property.

Lemma 1465 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, SEN^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system. If $\mathcal{I}$ is protoalgebraic, then each equivalence class of $\sim^{\mathcal{I},\mathcal{A}}$ has a minimum element.*

Proof: Let $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and consider the equivalence class $[T]$ of T under $\sim^{\mathcal{I},\mathcal{A}}$. Then we have $\bigcap[T] \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and, moreover,

$$\begin{array}{rcl}\Omega^{\mathcal{A}}(\bigcap[T]) & = & \bigcap\{\Omega^{\mathcal{A}}(T'); T' \in [T]\} \\ & = & \bigcap\{\Omega^{\mathcal{A}}(T) : T' \in [T]\} \\ & = & \Omega^{\mathcal{A}}(T).\end{array}$$

So $\bigcap[T] \in [T]$ and, therefore, $\bigcap[T]$ is the minimum element of $[T]$. ∎

The next proposition provides the promised characterization of those $\mathcal{I}$-filter families that determine full $\mathcal{I}$-structures through their principal filters in $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})$, in the case of a protoalgebraic $\mathcal{I}$.

Proposition 1466 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a protoalgebraic π-institution based on $\mathbf{F}$. For every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and every $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, the following conditions are equivalent:*

(i) $T \in \mathrm{FiFam}^{\mathcal{I},f}(\mathcal{A})$, *i.e.,* $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle \in \mathrm{FStr}(\mathcal{I})$;

(ii) $T = \min[T]$, *where* $[T]$ *is the equivalence class of* T *under* $\sim^{\mathcal{I},\mathcal{A}}$;

(iii) $T/\Omega^{\mathcal{A}}(T) = \min \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T))$.

Proof:

(ii)⇒(iii) Assume that $T = \min [T]$ and let $Y \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T))$. Our goal is to show that $T/\Omega^{\mathcal{A}}(T) \le Y$. Consider the quotient morphism $\langle I,\pi\rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T)$ and let $X = \pi^{-1}(Y) \cap T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then

$$X = \pi^{-1}(Y) \cap \pi^{-1}(\pi(T)) = \pi^{-1}(Y \cap \pi(T)).$$

It follows that $\Omega^{\mathcal{A}}(T)$ is compatible with X and, hence, $\Omega^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(X)$. But, by definition, $X \le T$ and, thus, by protoalgebraicity, $\Omega^{\mathcal{A}}(X) \le \Omega^{\mathcal{A}}(T)$. We conclude that $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(X)$ and, hence, $T \sim^{\mathcal{I},\mathcal{A}} X$. By hypothesis, we now get $T \le \pi^{-1}(Y)$, i.e., $T/\Omega^{\mathcal{A}}(T) \le X$. Since $Y \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T))$ was arbitrary, we conclude that $T/\Omega^{\mathcal{A}}(T) = \min \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T))$.

(iii)⇒(i) Suppose that $T/\Omega^{\mathcal{A}}(T) = \min \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T))$. By protoalgebraicity (see the Correspondence Theorem 1336),

$$\pi : \mathbf{FiFam}^{\mathcal{I}}(\langle \mathcal{A}, T\rangle) \cong \mathbf{FiFam}^{\mathcal{I}}(\langle \mathcal{A}/\Omega^{\mathcal{A}}(T), T/\Omega^{\mathcal{A}}(T)\rangle).$$

By hypothesis,

$$\mathrm{FiFam}^{\mathcal{I}}(\langle \mathcal{A}/\Omega^{\mathcal{A}}(T), T/\Omega^{\mathcal{A}}(T)\rangle) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T)).$$

Since, by Proposition 1460, $\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\langle \mathcal{A}, T\rangle)) = \Omega^{\mathcal{A}}(T)$, we obtain

$$\mathrm{FiFam}^{\mathcal{I}}(\langle \mathcal{A}, T\rangle)^* = \mathrm{FiFam}^{\mathcal{I}}(\langle \mathcal{A}, T\rangle^*).$$

This proves that $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\langle \mathcal{A}, T\rangle)\rangle$ is a full $\mathcal{I}$-structure. We conclude that $T \in \mathrm{FiFam}^{\mathcal{I},f}(\mathcal{A})$.

(i)⇒(ii) Suppose $T \in \mathrm{FiFam}^{\mathcal{I},f}(\mathcal{A})$. Since $\mathcal{I}$ is protoalgebraic, by Lemma 1465, there exists $T' = \min [T]$. By the proofs of the two preceding implications (ii)⇒(iii)⇒(i), $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T'}\rangle$ is a full $\mathcal{I}$-structure. But, by hypothesis, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T}\rangle$ is also a full $\mathcal{I}$-structure. Now observe that, by Proposition 1460,

$$\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T'}) = \Omega^{\mathcal{A}}(T') = \Omega^{\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T}).$$

By the Isomorphism Theorem 1408, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T'} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T}$ and, therefore, $T' = T$. So we conclude that $T = \min [T]$.

■

Since, for a protoalgebraic π-institution $\mathcal{I}$, the filter families determining full $\mathcal{I}$-structures are the ones that are minimal in their equivalence classes under $\sim^{\mathcal{I},\mathcal{A}}$, we can easily conclude that the class of those filter families consists of all filter families just in case all equivalence classes are singletons.

Proposition 1467 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a protoalgebraic* π*-institution based on* $\mathbf{F}$. *Then* $\mathrm{FiFam}^{\mathcal{I},f}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *(i.e., for all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle$ *is a full* $\mathcal{I}$*-structure) if and only if* $\Omega^{\mathcal{A}}$ *is injective on* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

Proof: By Proposition 1466, $\mathrm{FiFam}^{\mathcal{I},f}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ if and only if, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T = \min[T]$, if and only if, for all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$ implies $T = T'$, if and only if $\Omega^{\mathcal{A}}$ is injective on $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. ■

Recall that a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is called **weakly family algebraizable**, or **WF algebraizable** for short, if the Leibniz operator Ω is monotone and injective on the theory families of $\mathcal{I}$. Equivalently, by Theorem 295, $\mathcal{I}$ is WF algebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, the Leibniz operator on $\mathcal{A}$ is monotone and injective on $\mathcal{I}$-filter families.

The following theorem provides additional characterizations in terms of $\mathcal{I}$-structures and $\mathcal{I}$-congruence systems.

Theorem 1468 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *Then the following conditions are equivalent.*

(i) $\mathcal{I}$ *is protoalgebraic and, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T/\Omega^{\mathcal{A}}(T) = \min \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T))$;

(ii) *For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ *is monotone and injective on* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$;

(iii) *For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $T \mapsto \langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle$ *is a bijection between* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and* $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$ *and, hence, an order isomorphism from* $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})$ *to* $\mathbf{FStr}^{\mathcal{I}}(\mathcal{A})$;

(iv) *For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathbf{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}}(\mathcal{A})$ *is an order isomorphism;*

(v) *For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}} : \mathbf{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is an order isomorphism.*

Proof:

(i)⇔(ii) By Propositions 1466 and 1467.

(i)⇒(iii) It is clear that $T \to \langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle$ is injective. By Proposition 1466, it is into $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$ and, by Theorem 1463, it is onto $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$. Hence it is a bijection, as claimed.

(iii)⇒(iv) Since, by hypothesis, every full $\mathcal{I}$-structure is of the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle$, by Theorem 1463, $\mathcal{I}$ is protoalgebraic. The composition of the given isomorphism

$$\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A}) \cong \mathbf{FStr}^{\mathcal{I}}(\mathcal{A})$$

with the isomorphism established in the Isomorphism Theorem 1408,

$$\widetilde{\Omega}^{\mathcal{A}} : \mathbf{FStr}^{\mathcal{I}}(\mathcal{A}) \to \mathbf{ConSys}^{\mathcal{I}}(\mathcal{A})$$

gives an isomorphism

$$\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A}) \cong \mathbf{ConSys}^{\mathcal{I}}(\mathcal{A}),$$

which by protoalgebraicity and Proposition 1460 is identical to the Leibniz operator.

(iv)⇒(v) By Corollary 1405, $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$. Thus, $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}) \subseteq \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$. By the hypothesis, every $\theta \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is of the form $\Omega^{\mathcal{A}}(T)$, for some $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Therefore, $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}) \subseteq \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$.

(v)⇒(ii) is trivial.

■

In the context of weakly family algebraizable π-institutions, we look, also at the local continuity of the Leibniz and the Tarski operators.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system.

- $\Omega^{\mathcal{A}}$ is **continuous** if, for every directed collection $\{T^i : i \in I\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\bigcup_{i \in I} T^i \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we have
$$\Omega^{\mathcal{A}}(\bigcup_{i \in I} T^i) = \bigcup_{i \in I} \Omega^{\mathcal{A}}(T^i).$$
- $\widetilde{\Omega}^{\mathcal{A}}$ is **continuous** if, for every directed family $\{\mathbb{L}^i : i \in I\} \subseteq \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$,
$$\widetilde{\Omega}^{\mathcal{A}}(\sup\{\mathbb{L}^i : i \in I\}) = \bigcup_{i \in I} \widetilde{\Omega}^{\mathcal{A}}(\mathbb{L}^i).$$

Proposition 1469 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a finitary π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is weakly family algebraizable, then, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\Omega^{\mathcal{A}}$ is continuous if and only if $\widetilde{\Omega}^{\mathcal{A}}$ is continuous.*

Proof: Let $\Phi : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$ be the bijection of Theorem 1468. Then, by Proposition 1460,

$$\Omega^{\mathcal{A}} = \widetilde{\Omega}^{\mathcal{A}} \circ \Phi \quad \text{and} \quad \widetilde{\Omega}^{\mathcal{A}} = \Omega^{\mathcal{A}} \circ \Phi^{-1}.$$

Suppose, first, that $\Omega^{\mathcal{A}}$ is continuous and let $\{\mathbb{L}^i : i \in I\} \subseteq \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$ be directed. If $T^i = \Phi^{-1}(\mathbb{L}^i)$, $i \in I$, then $\{T^i : i \in I\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is also directed. Directedness implies local directedness and, therefore, by Proposition 112, $\bigcup_{i\in I} T^i \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Now we get

$$\Phi(\bigcup_{i\in I} T^i) = \langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{\bigcup_{i\in I} T^i}\rangle = \langle \mathcal{A}, \bigcap_{i\in I} \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T^i}\rangle,$$

whence $\Phi(\bigcup_{i\in I} T^i) = \sup_{i\in I} \mathbb{L}^i$. Therefore, we get

$$\widetilde{\Omega}^{\mathcal{A}}(\sup_{i\in I} \mathbb{L}^i) = (\Omega^{\mathcal{A}} \circ \Phi^{-1})(\Phi(\bigcup_{i\in I} T^i)) = \Omega^{\mathcal{A}}(\bigcup_{i\in I} T^i) = \bigcup_{i\in I} \Omega^{\mathcal{A}}(T^i) = \bigcup_{i\in I} \widetilde{\Omega}^{\mathcal{A}}(\mathbb{L}^i).$$

So $\widetilde{\Omega}^{\mathcal{A}}$ is also continuous.

Assume, conversely, $\widetilde{\Omega}^{\mathcal{A}}$ is continuous. Let $\{T^i : i \in I\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ be directed. Then $\{\Phi(T^i) : i \in I\} \subseteq \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$ is also directed and we have

$$\Omega^{\mathcal{A}}(\bigcup_{i\in I} T^i) = \widetilde{\Omega}^{\mathcal{A}}(\Phi(\bigcup_{i\in I} T^i)) = \widetilde{\Omega}^{\mathcal{A}}(\sup_{i\in I} \mathbb{L}^i) = \bigcup_{i\in I} \widetilde{\Omega}^{\mathcal{A}}(\mathbb{L}^i) = \bigcup_{i\in I} \Omega^{\mathcal{A}}(T^i).$$

Therefore $\Omega^{\mathcal{A}}$ is also continuous. ■

19.11 $\mathcal{I}$-Structures and Fregeanity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is called **Fregean** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, the π-structure $\mathcal{I}^T$ has the Congruence Property, i.e., for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\widetilde{\Lambda}^{\mathcal{I}}(T) = \widetilde{\Omega}^{\mathcal{I}}(T).$$

Clearly, $\mathcal{I}$ is Fregean if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, if, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$C_{\Sigma'}(T_{\Sigma'}, \mathrm{SEN}^\flat(f)(\phi)) = C_{\Sigma'}(T_{\Sigma'}, \mathrm{SEN}^\flat(f)(\psi)),$$

then, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$C_{\Sigma'}(T_{\Sigma'}, \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi})) = C_{\Sigma'}(T_{\Sigma'}, \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi})).$$

$\mathcal{I}$ is called **strongly Fregean** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, the π-structure $\mathcal{I}^T$ has the strong Congruence Property, i.e., for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\widetilde{\lambda}^{\mathcal{I}}(T) = \widetilde{\Omega}^{\mathcal{I}}(T).$$

In this case, we get that $\mathcal{I}$ is strongly Fregean if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C_\Sigma(T_\Sigma, \phi) = C_\Sigma(T_\Sigma, \psi)$$

implies, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$C_{\Sigma'}(T_{\Sigma'},\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi})) = C_{\Sigma'}(T_{\Sigma'},\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi})).$$

A consequence of strong Fregeanity is that every reduced matrix family model has either an empty filter family or a filter family all of whose components are singletons.

Proposition 1470 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is strongly Fregean, then, for every reduced $\mathcal{I}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, T\rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ and $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, we have, for all $\Sigma \in |\mathbf{Sign}|$, $|T_\Sigma| = 0$, or, for all $\Sigma \in |\mathbf{Sign}|$, $|T_\Sigma| = 1$.*

We abbreviate the first disjunct of the conclusion, as usual, by $T = \varnothing$ and the second by writing $|T| = 1$.

Proof: Assume that $T \neq \varnothing$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \in T_{F(\Sigma)}$. Then $\phi,\psi \in \alpha^{-1}_\Sigma(T_{F(\Sigma)})$. Hence,

$$C_\Sigma(\alpha^{-1}_\Sigma(T_{F(\Sigma)}),\phi) = C_\Sigma(\alpha^{-1}_\Sigma(T_{F(\Sigma)}),\psi),$$

i.e., $\langle \phi,\psi\rangle \in \widetilde{\lambda}^{\mathcal{I}}_\Sigma(\alpha^{-1}(T))$. By strong Fregeanity, $\langle\phi,\psi\rangle \in \widetilde{\Omega}^{\mathcal{I}}_\Sigma(\alpha^{-1}(T))$. Thus, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\begin{array}{l} C_{\Sigma'}(\alpha^{-1}_{\Sigma'}(T_{F(\Sigma')}),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi})) \\ \qquad = C_{\Sigma'}(\alpha^{-1}_{\Sigma'}(T_{F(\Sigma')}),\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi})).\end{array}$$

Now we get

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}) \in \alpha^{-1}_{\Sigma'}(T_{F(\Sigma')}) \quad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}) \in \alpha^{-1}_{\Sigma'}(T_{F(\Sigma')}).$$

Equivalently,

$$\begin{array}{l} \sigma_{F(\Sigma')}(\mathrm{SEN}(F(f))(\alpha_\Sigma(\phi)),\alpha_{\Sigma'}(\vec{\chi})) \in T_{F(\Sigma')} \\ \qquad \text{iff} \quad \sigma_{F(\Sigma')}(\mathrm{SEN}(F(f))(\alpha_\Sigma(\psi)),\alpha_{\Sigma'}(\vec{\chi})) \in T_{F(\Sigma')}.\end{array}$$

Taking into account the surjectivity of $\langle F,\alpha\rangle$, we get that $\langle \alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\rangle \in \Omega^{\mathcal{A}}_{F(\Sigma)}(T) = \Delta^{\mathcal{A}}_{F(\Sigma)}$, the last equation holding since $\mathfrak{A}$ is reduced. Hence, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$ and, therefore, for all $\Sigma \in |\mathbf{Sign}|$, $|T_\Sigma| = 1$. ∎

Of course, in the case of Fregeanity and protoalgebraicity, the role of the Tarski operator may be substituted by the Leibniz operator.

Proposition 1471 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I}$ *is protoalgebraic and Fregean if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) = \widetilde{\Lambda}^{\mathcal{I}}(T)$;

(b) $\mathcal{I}$ *is protoalgebraic and strongly Fregean if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) = \widetilde{\lambda}^{\mathcal{I}}(T)$.

Proof: We only prove Part (a), since Part (b) can be proven by following a similar reasoning.

If $\mathcal{I}$ is protoalgebraic, then, by Proposition 1460, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) = \widetilde{\Omega}^{\mathcal{I}}(T)$. If $\mathcal{I}$ is Fregean, then ,by definition, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\Omega}^{\mathcal{I}}(T) = \widetilde{\Lambda}^{\mathcal{I}}(T)$. Therefore, if $\mathcal{I}$ is protoalgebraic and Fregean, then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) = \widetilde{\Lambda}^{\mathcal{I}}(T)$.

Assume, conversely, that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, we have $\Omega(T) = \widetilde{\Lambda}^{\mathcal{I}}(T)$. Then, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$, we have

$$\Omega(T) = \widetilde{\Lambda}^{\mathcal{I}}(T) \overset{\text{Lemma 1416}}{\leq} \widetilde{\Lambda}^{\mathcal{I}}(T') = \Omega(T').$$

Thus, Ω is monotone on $\mathrm{ThFam}(\mathcal{I})$ and $\mathcal{I}$ is protoalgebraic. Moreover, since, by Proposition 1460, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\Omega}^{\mathcal{I}}(T) = \Omega(T)$, we get $\widetilde{\Omega}^{\mathcal{I}}(T) = \widetilde{\Lambda}^{\mathcal{I}}(T)$ and, therefore, $\mathcal{I}$ is also Fregean. ■

Recall that a π-institution $\mathcal{I}$ is self extensional if

$$\widetilde{\Omega}(\mathcal{I}) = \widetilde{\Lambda}(\mathcal{I}) \ (= \widetilde{\lambda}(\mathcal{I})).$$

It turns out that Fregeanity (and, therefore, strong Fregeanity) implies self extensionality.

Corollary 1472 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is Fregean, then it is self extensional.*

Proof: We have

$$\begin{aligned} \widetilde{\Omega}(\mathcal{I}) &= \widetilde{\Omega}^{\mathcal{I}}(\mathrm{Thm}(\mathcal{I})) \quad \text{(by definition)} \\ &= \widetilde{\Lambda}^{\mathcal{I}}(\mathrm{Thm}(\mathcal{I})) \quad \text{(by Fregeanity)} \\ &= \widetilde{\Lambda}(\mathcal{I}). \quad \text{(by definition)} \end{aligned}$$

So $\mathcal{I}$ is self extensional. ■

If a π-institution $\mathcal{I}$ is strongly Fregean and has theorems, then the mapping $T \mapsto \mathcal{I}^T$ establishes an order embedding from the lattice of the theory families of $\mathcal{I}$ into the lattice of full $\mathcal{I}$-structures on the algebraic system $\mathcal{F}$.

Proposition 1473 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is strongly Fregean with theorems, then* $T \mapsto \mathcal{I}^T$ *is an order embedding of* $\mathbf{ThFam}(\mathcal{I})$ *into* $\mathbf{FStr}^{\mathcal{I}}(\mathcal{F})$.

Proof: We start by showing that the proposed mapping is indeed well-defined into $\mathrm{FStr}^{\mathcal{I}}(\mathcal{F})$, i.e., that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\mathcal{I}^T = \langle \mathcal{F}, \mathrm{ThFam}(\mathcal{I})^T \rangle$ is a full $\mathcal{I}$-structure. To this end, let $T \in \mathrm{ThFam}(\mathcal{I})$ and set $\theta = \widetilde{\Omega}^{\mathcal{I}}(T) = \widetilde{\lambda}^{\mathcal{I}}(T)$. To verify that $\mathcal{I}^T$ is a full $\mathcal{I}$-structure, it suffices to show that $\mathrm{ThFam}(\mathcal{I}^T)/\theta = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}/\theta)$.

If $T' \in \mathrm{ThFam}(\mathcal{I}^T)$, then, by definition of θ, θ is compatible with T'. Therefore, by Corollary 56, $T'/\theta \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}/\theta)$. Thus, $\mathrm{ThFam}(\mathcal{I}^T)/\theta \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}/\theta)$.

If, on the other hand, $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}/\theta)$, then, setting $\langle I, \pi \rangle : \mathcal{F} \to \mathcal{F}/\theta$ the quotient morphism, we have, by Corollary 55, $\pi^{-1}(T') \in \mathrm{Fifam}^{\mathcal{I}}(\mathcal{F})$, i.e., $\pi^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$. We also have, taking into account that $\mathcal{I}$ has theorems, that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi \in T_\Sigma$ and $\psi \in T_\Sigma \cap \pi_\Sigma^{-1}(T'_\Sigma)$,

$$\langle \phi, \psi \rangle \in \widetilde{\lambda}^{\mathcal{I}}_\Sigma(T) = \theta_\Sigma.$$

So $\pi_\Sigma(\phi) = \pi_\Sigma(\psi)$, whence $\phi \in \pi_\Sigma^{-1}(\pi_\Sigma(\psi)) \in \pi_\Sigma^{-1}(T'_\Sigma)$. Since $\phi \in T_\Sigma$ was arbitrary, $T \le \pi^{-1}(T')$ and, hence, $\pi^{-1}(T') \in \mathrm{ThFam}(\mathcal{I}^T)$. This shows that $T' \in \mathrm{ThFam}(\mathcal{I}^T)/\theta$ and allows us to conclude that $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}/\theta) \subseteq \mathrm{ThFam}(\mathcal{I}^T)/\theta$.

As for the rest, everything follows, since $T \mapsto \mathcal{I}^T$ is clearly one-to-one and both order preserving and order reflecting. ∎

If one adds protoalgebraicity into the mix, then the order embedding of Proposition 1473 becomes an order isomorphism.

Proposition 1474 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a strongly Fregean protoalgebraic* π*-institution with theorems, based on* $\mathbf{F}$*. Then* $T \mapsto \mathcal{I}^T$ *is an isomorphism between* $\mathbf{ThFam}(\mathcal{I})$ *and* $\mathbf{FStr}^{\mathcal{I}}(\mathcal{F})$*.*

Proof: By Proposition 1473, it suffices to show that the mapping $T \mapsto \mathcal{I}^T$ is also onto $\mathrm{FStr}^{\mathcal{I}}(\mathcal{F})$. The latter follows from Theorem 1463. ∎

Strong Fregeanity, protoalgebraicity and the existence of theorems have very strong consequences for a π-institution. They ensure that the π-institution is weakly family algebraizable, that the Leibniz operator is continuous (in case of finitarity) and that all reduced matrix families have singleton filter families.

Proposition 1475 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a strongly Fregean protoalgebraic* π*-institution with theorems.*

(a) $\mathcal{I}$ *is family injective and hence weakly family algebraizable;*

(b) If $\mathcal{I}$ *is finitary, then* Ω *is locally continuous;*

(c) For every $\mathfrak{A} = \langle \mathcal{A}, T \rangle \in \mathrm{MatFam}^*(\mathcal{I})$*,* $|T| = 1$*.*

Proof: By Proposition 1460, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Omega(T) = \widetilde{\Omega}^{\mathcal{I}}(T)$. Thus, composing the mapping $T \mapsto \mathcal{I}^T$ of Proposition 1474, with the isomorphism of Theorem 1408, we obtain an isomorphism $\Omega : \mathbf{ThFam}(\mathcal{I}) \to \mathbf{ConSys}^{\mathcal{I}}(\mathcal{F})$. By Proposition 1471, $\Omega = \widetilde{\lambda}^{\mathcal{I}}$ and, hence, by Proposition 1419, Ω is locally continuous. Finally, if $\mathfrak{A} = \langle \mathcal{A}, T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$, then, since $\mathcal{I}$ has theorems, $T \neq \varnothing$ and, therefore, by Proposition 1470, $|T| = 1$. ■

We saw in Corollary 1472 that Fregeanity implies self extensionality. On the other hand, even though we cannot prove that strong Fregeanity, coupled with protoalgebraicity, are strong enough to guarantee full self extensionality, we can show that they imply a weaker property, namely a version of full self extensionality applying only to full $\mathcal{I}$-structures with isomorphic functor components. We start with a technical lemma.

Lemma 1476 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a strongly Fregean protoalgebraic π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ an $\mathbf{F}$-algebraic system, with $F : \mathbf{Sign}^\flat \to \mathbf{Sign}$ an isomorphism. Then, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,*

$$\Omega^{\mathcal{A}}(T) = \widetilde{\lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T).$$

Proof: Note that, by protoalgebraicity, $\Omega^{\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T)$. Therefore, by compatibility, $\Omega^{\mathcal{A}}(T) \leq \widetilde{\lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T)$. It therefore suffices to show the reverse inclusion. To this end and taking into account the surjectivity of $\langle F, \alpha\rangle$, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in \widetilde{\lambda}^{\mathcal{A}}_{F(\Sigma)}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T)$. Then, by definition, for all $T \leq T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we have

$$\alpha_\Sigma(\phi) \in T''_{F(\Sigma)} \quad \text{iff} \quad \alpha_\Sigma(\psi) \in T''_{F(\Sigma)}.$$

However, since $\mathcal{I}$ is protoalgebraic, we get, by the Correspondence Theorem 1336, that, for all $\alpha^{-1}(T) \leq T' \in \mathrm{ThFam}(\mathcal{I})$,

$$\phi \in T'_\Sigma \quad \text{iff} \quad \psi \in T'_\Sigma.$$

Hence, by definition,

$$\begin{array}{rcll}\langle \phi, \psi\rangle & \in & \widetilde{\lambda}^{\mathcal{I}}_\Sigma(\alpha^{-1}(T)) & \\ & = & \Omega_\Sigma(\alpha^{-1}(T)) & \text{(strong Fregeanity)} \\ & = & \alpha_\Sigma^{-1}(\Omega^{\mathcal{A}}_{F(\Sigma)}(T)). & \text{(Proposition 24)}\end{array}$$

Hence $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in \Omega^{\mathcal{A}}_{F(\Sigma)}(T)$. Taking into account the surjectivity of $\langle F, \alpha\rangle$, we now conclude that $\widetilde{\lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T) \leq \Omega^{\mathcal{A}}(T)$. ■

Proposition 1477 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is strongly Fregean and protoalgebraic, then, every full $\mathcal{I}$-structure, with an isomorphic functor component, has the Congruence Property.*

Proof: We deal first with the case $\mathrm{Thm}(\mathcal{I}) = \varnothing$. Then, since $\mathcal{I}$ is protoalgebraic, the only option is

$$\mathrm{ThFam}(\mathcal{I}) = \{T : T_\Sigma = \varnothing \text{ or } \mathrm{SEN}^\flat(\Sigma), \text{ for all } \Sigma \in |\mathbf{Sign}^\flat|\}.$$

In this case, given an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, the only full $\mathcal{I}$-structures on $\mathcal{A}$ are of the form $\langle \mathcal{A}, \mathcal{D}\rangle$, with

$$\mathcal{D} = \{T : T_\Sigma = \varnothing \text{ or } \mathrm{SEN}(\Sigma), \text{ for all } \Sigma \in |\mathbf{Sign}|\}.$$

All those have the Congruence Property.

Assume, next, that $\mathcal{I}$ has theorems. By Lemma 1476, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, with $F : \mathbf{Sign}^\flat \to \mathbf{Sign}$ an isomorphism, and every $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$\Omega^{\mathcal{A}}(T) = \widetilde{\lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T).$$

Thus, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T\rangle$ has the strong Congruence Property. Since, by Theorem 1463, every full $\mathcal{I}$-structure, with an isomorphic functor component has this form, we conclude that every full $\mathcal{I}$-structure, with an isomorphic functor component, has the Congruence Property. ∎

For finitary fully self extensional π-institutions, we obtain the following characterizations of weak family algebraizability.

Proposition 1478 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a finitary, fully self extensional π-institution based on $\mathbf{F}$. Then the following conditions are equivalent.*

(i) $\mathcal{I}$ is strongly Fregean, protoalgebraic and has theorems;

(ii) $\mathcal{I}$ is weakly family algebraizable and Ω is locally continuous;

(iii) $\mathcal{I}$ is weakly family algebraizable.

Proof: (i)⇒(ii) follows from Proposition 1475. (ii)⇒(iii) is trivial. For (iii)⇒(i) note, first, that, by hypothesis $\mathcal{I}$ is family monotone and family injective. Thus, $\mathcal{I}$ is protoalgebraic. By Proposition 1468, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\mathcal{I}^T \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{F})$. Hence, by full self extensionality, $\mathcal{I}^T$ has the strong Congruence Property and, hence, $\mathcal{I}$ is strongly Fregean. Finally, since $\Omega(\varnothing) = \Omega(\mathrm{SEN}^\flat) = \nabla^{\mathcal{F}}$, we get, by injectivity, $\varnothing \notin \mathrm{ThFam}(\mathcal{I})$ and $\mathcal{I}$ has theorems. ∎

Chapter 20

Full Adequacy

20.1 Gentzen π-Institutions

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\Sigma \in |\mathbf{Sign}^\flat|$. A Σ-**sequent** is a pair

$$\langle \Phi, \phi \rangle,$$

where $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$ (with Φ possibly empty). Sometimes we write

$$\Phi \rhd_\Sigma \phi \quad \text{or} \quad \Phi \vdash_\Sigma \phi$$

to denote the Σ-sequent $\langle \Phi, \phi \rangle$. The set Φ is called **set of antecedents of** $\langle \Phi, \phi \rangle$ and ϕ is called the **consequent of** $\langle \Phi, \phi \rangle$.

The collection of Σ-sequents is denoted by $\mathrm{Seq}_\Sigma(\mathbf{F})$ and the set of all Σ-sequents with nonempty set of antecedents is denoted by $\mathrm{Seq}^0_\Sigma(\mathbf{F})$. We then set

$$\mathrm{Seq}(\mathbf{F}) = \{\mathrm{Seq}_\Sigma(\mathbf{F})\}_{\Sigma \in |\mathbf{Sign}^\flat|} \quad \text{and} \quad \mathrm{Seq}^0(\mathbf{F}) = \{\mathrm{Seq}^0_\Sigma(\mathbf{F})\}_{\Sigma \in |\mathbf{Sign}^\flat|}.$$

We sometimes use boldface Greek letters such as $\boldsymbol{\gamma}, \boldsymbol{\delta}, \ldots$ to denote Σ-sequents and boldface capital Greek letters such as $\boldsymbol{\Gamma}, \boldsymbol{\Delta}, \ldots$ for sets of Σ-sequents. Moreover, we write $\Gamma \vdash_\Sigma \Phi$ to stand for the set $\{\Gamma \vdash_\Sigma \phi : \phi \in \Phi\}$ of Σ-sequents.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. A **Gentzen π-institution based on F of type 1** (**of type 0**, respectively) is a pair

$$\mathfrak{G} = \langle \mathbf{F}, G \rangle,$$

where $G : \mathcal{P}(\mathrm{Seq}(\mathbf{F})) \to \mathcal{P}(\mathrm{Seq}(\mathbf{F}))$ ($G : \mathcal{P}(\mathrm{Seq}^0(\mathbf{F})) \to \mathcal{P}(\mathrm{Seq}^0(\mathbf{F}))$, respectively) is a closure system on $\mathrm{Seq}(\mathbf{F})$ ($\mathrm{Seq}^0(\mathbf{F})$, respectively) that, in addition, satisfies the following **structural rules**, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \Psi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$:

$$\begin{array}{rl} \text{(Axiom)} & \phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing); \\ \text{(Weakening)} & \Phi, \Psi \vdash_\Sigma \phi \in G_\Sigma(\Phi \vdash_\Sigma \phi); \\ \text{(Cut)} & \Phi \vdash_\Sigma \phi \in G_\Sigma(\Phi \vdash_\Sigma \Psi,\ \Phi, \Psi \vdash_\Sigma \phi). \end{array}$$

If $\Phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$ we call $\Phi \vdash_\Sigma \phi$ a Σ-**theorem** or a **derivable Σ-sequent** of $\mathfrak{G}$.

Each Gentzen π-institution based on an algberaic system $\mathbf{F}$ defines in a natural way a π-institution based on $\mathbf{F}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$. The **π-institution $\mathcal{I}^{\mathfrak{G}} = \langle F, C^{\mathfrak{G}} \rangle$ defined** or **determined by $\mathfrak{G}$** is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in C^{\mathfrak{G}}_\Sigma(\Phi) \quad \text{iff} \quad \Phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing).$$

Proposition 1479 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution based on* $\mathbf{F}$. $C^{\mathfrak{G}} : \mathcal{P}\mathrm{SEN}^\flat \to \mathcal{P}\mathrm{SEN}^\flat$ *is a closure system on* $\mathrm{SEN}^\flat$ *and, hence,* $\mathcal{I}^{\mathfrak{G}} = \langle \mathbf{F}, C^{\mathfrak{G}} \rangle$ *is a* π*-institution.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \Psi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$.

If $\phi \in \Phi$, then, by (Axiom) $\phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$ and by (Weakening) $\Phi \vdash_\Sigma \phi \in G_\Sigma(\phi \vdash_\Sigma \phi)$, whence $\Phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$. Therefore $\phi \in C^{\mathfrak{G}}_\Sigma(\Phi)$.

If $\Phi \subseteq \Psi$ and $\phi \in C^{\mathfrak{G}}_\Sigma(\Phi)$, then $\Phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$ and, by (Weakening), $\Psi \vdash_\Sigma \phi \in G_\Sigma(\Phi \vdash_\Sigma \phi)$, whence $\Psi \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$, giving $\phi \in C^{\mathfrak{G}}_\Sigma(\Psi)$.

If $\phi \in C^{\mathfrak{G}}_\Sigma(C^{\mathfrak{G}}_\Sigma(\Phi))$, then $C^{\mathfrak{G}}_\Sigma(\Phi) \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$ and, by (Weakening),

$$\Phi, C^{\mathfrak{G}}_\Sigma(\Phi) \vdash_\Sigma \phi \in G_\Sigma(C^{\mathfrak{G}}_\Sigma(\Phi) \vdash_\Sigma \phi) \subseteq G_\Sigma(\varnothing).$$

Moreover, by definition $\Phi \vdash_\Sigma C^{\mathfrak{G}}_\Sigma(\Phi) \subseteq G_\Sigma(\varnothing)$, whence, by (Cut),

$$\Phi \vdash_\Sigma \phi \in G_\Sigma(\Phi \vdash_\Sigma C^{\mathfrak{G}}_\Sigma(\Phi),\ \Phi, C^{\mathfrak{G}}_\Sigma(\Phi) \vdash_\Sigma \phi) \subseteq G_\Sigma(\varnothing).$$

Therefore, $\phi \in C^{\mathfrak{G}}_\Sigma(\Phi)$.

Finally, suppose $\phi \in C^{\mathfrak{G}}_\Sigma(\Phi)$, $\Sigma' \in |\mathbf{Sign}^\flat|$ and $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$. Then, by definition, $\Phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$ and, by structurality,

$$\mathrm{SEN}^\flat(f)(\Phi) \vdash_{\Sigma'} \mathrm{SEN}^\flat(f)(\phi) \in G_{\Sigma'}(\varnothing).$$

This shows that $\mathrm{SEN}^\flat(f)(\phi) \in C^{\mathfrak{G}}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))$ and, therefore, $C^{\mathfrak{G}}$ is a closure system on $\mathrm{SEN}^\flat$, as was to be shown. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution, also based on $\mathbf{F}$. We say that $\mathfrak{G}$ is **adequate for** $\mathcal{I}$ if $C = C^{\mathfrak{G}}$ and, moreover,

- $\mathfrak{G}$ is of type 1 if $\mathcal{I}$ has theorems and
- $\mathfrak{G}$ is of type 0 if $\mathcal{I}$ does not have theorems.

The following proposition clarifies the distinction imposed on the type, since it reveals the fact that, if $\mathcal{I}$ has no theorems, then it is sufficient to assume that a Gentzen π-institution adequate for $\mathcal{I}$ has type 0.

Proposition 1480 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution based on* $\mathbf{F}$.

(a) If $\mathfrak{G}$ *is of type 0, then* $\mathcal{I}^{\mathfrak{G}}$ *does not have theorems.*

(b) If $\mathfrak{G}$ *is of type 1, then its restriction* $\mathfrak{G}^0 = \langle \mathbf{F}, G^0 \rangle$ *to* $\mathrm{Seq}^0(\mathbf{F})$ *is a Gentzen* π*-institution of type 0.*

(c) If $\mathfrak{G}$ *is of type 1 and* $\mathcal{I}^{\mathfrak{G}}$ *has no theorems, then* $\mathcal{I}^{\mathfrak{G}} = \mathcal{I}^{\mathfrak{G}^0}$.

Proof:

(a) Suppose $\mathcal{I}^{\mathfrak{G}}$ has theorems. Thus, for all $\Sigma \in |\mathbf{Sign}^\flat|$, there exists $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C^{\mathfrak{G}}_\Sigma(\varnothing)$. Thus, by definition, $\varnothing \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$. Therefore, $\mathfrak{G}$ cannot be of type 0 (since it admits a sequent with an empty set of antecedents).

(b) Suppose $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ is of type 1. Consider $\mathfrak{G}^0 = \langle \mathbf{F}, G^0\rangle$. We must show that $G^0 : \mathcal{P}(\mathrm{Seq}^0(\mathbf{F})) \to \mathcal{P}(\mathrm{Seq}^0(\mathbf{F}))$ is a closure system on $\mathrm{Seq}^0(\mathbf{F})$ that satisfies the structural rules.

- Suppose $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathbf{\Gamma} \cup \{\boldsymbol{\gamma}\} \subseteq \mathrm{Seq}^0_\Sigma(\mathbf{F})$, such that $\boldsymbol{\gamma} \in \mathbf{\Gamma}$. Then $\boldsymbol{\gamma} \in G_\Sigma(\mathbf{\Gamma})$ and, hence, $\boldsymbol{\gamma} \in G^0_\Sigma(\mathbf{\Gamma})$.
- Suppose $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathbf{\Gamma} \cup \mathbf{\Delta} \cup \{\boldsymbol{\gamma}\} \subseteq \mathrm{Seq}^0_\Sigma(\mathbf{F})$, such that $\boldsymbol{\gamma} \in G^0_\Sigma(\mathbf{\Gamma})$ and $\mathbf{\Gamma} \subseteq \mathbf{\Delta}$. Then, by definition, $\boldsymbol{\gamma} \in G_\Sigma(\mathbf{\Gamma})$ and $\mathbf{\Gamma} \subseteq \mathbf{\Delta}$, whence $\boldsymbol{\gamma} \in G_\Sigma(\mathbf{\Delta})$. So $\boldsymbol{\gamma} \in G^0_\Sigma(\mathbf{\Delta})$.
- Suppose $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathbf{\Gamma} \cup \{\boldsymbol{\gamma}\} \subseteq \mathrm{Seq}^0_\Sigma(\mathbf{F})$, such that $\boldsymbol{\gamma} \in G^0_\Sigma(G^0_\Sigma(\mathbf{\Gamma}))$. Then $\boldsymbol{\gamma} \in G_\Sigma(G_\Sigma(\mathbf{\Gamma})) = G_\Sigma(\mathbf{\Gamma})$. As $\mathbf{\Gamma} \cup \{\boldsymbol{\gamma}\} \subseteq \mathrm{Seq}^0_\Sigma(\mathbf{F})$, it follows that $\boldsymbol{\gamma} \in G^0_\Sigma(\mathbf{\Gamma})$.
- Suppose $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathbf{\Gamma} \cup \{\boldsymbol{\gamma}\} \subseteq \mathrm{Seq}^0_\Sigma(\mathbf{F})$, such that $\boldsymbol{\gamma} \in G^0_\Sigma(\mathbf{\Gamma})$, $\Sigma' \in |\mathbf{Sign}^\flat|$ and $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$. Then $\boldsymbol{\gamma} \in G_\Sigma(\mathbf{\Gamma})$, whence $\mathrm{SEN}^\flat(f)(\boldsymbol{\gamma}) \in G_{\Sigma'}(\mathrm{SEN}^\flat(f)(\mathbf{\Gamma}))$. Observing that, if $\mathbf{\Gamma} \cup \{\boldsymbol{\gamma}\} \subseteq \mathrm{Seq}^0_\Sigma(\mathbf{F})$, then $\mathrm{SEN}^\flat(f)(\mathbf{\Gamma}) \cup \{\mathrm{SEN}^\flat(f)(\boldsymbol{\gamma})\} \subseteq \mathrm{Seq}^0_{\Sigma'}(\mathbf{F})$, we conclude that
$$\mathrm{SEN}^\flat(f)(\boldsymbol{\gamma}) \in G^0_{\Sigma'}(\mathrm{SEN}^\flat(f)(\mathbf{\Gamma})).$$

Next, for the structural rules:

(Axiom) For $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \vdash_\Sigma \phi \in \mathrm{Seq}^0_\Sigma(\mathbf{F})$, whence, since, by (Axiom), $\phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$, $\phi \vdash_\Sigma \phi \in G^0_\Sigma(\varnothing)$.

(Weakening) Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \Psi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\Phi \neq \varnothing$. Then, since $\Phi \cup \Psi \neq \varnothing$ and since, by (Weakening), $\Phi, \Psi \vdash_\Sigma \phi \in G_\Sigma(\Phi \vdash_\Sigma \phi)$, we conclude that $\Phi, \Psi \vdash_\Sigma \phi \in G^0_\Sigma(\Phi \vdash_\Sigma \phi)$.

(Cut) Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \Psi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, with $\Phi \neq \varnothing$. Then $\Phi \cup \Psi \neq \varnothing$ and, since, by (Cut), $\Phi \vdash_\Sigma \phi \in G_\Sigma(\Phi \vdash_\Sigma \Psi,\ \Phi, \Psi \vdash_\Sigma \phi)$, we get $\Phi \vdash_\Sigma \phi \in G^0_\Sigma(\Phi \vdash_\Sigma \Psi,\ \Phi, \Psi \vdash_\Sigma \phi)$.

(c) Suppose $\mathfrak{G}$ is of type 1 and $\mathcal{I}^{\mathfrak{G}}$ has no theorems. Clearly, $G^0 \leq G$ so that $C^{\mathfrak{G}^0} \leq C^{\mathfrak{G}}$. On the other hand, let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C^{\mathfrak{G}}_\Sigma(\Phi)$. Since $\mathcal{I}^{\mathfrak{G}}$ has no theorems, $\Gamma \neq \varnothing$. Moreover, by definition, $\Phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$. Thus, $\Phi \vdash_\Sigma \phi \in G^0_\Sigma(\varnothing)$. We conclude that $\phi \in C^{\mathfrak{G}^0}_\Sigma(\Phi)$. So $C^{\mathfrak{G}} \leq C^{\mathfrak{G}^0}$ and equality follows.

■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\Sigma \in |\mathbf{Sign}^\flat|$.

- A **Gentzen Σ-axiom** is a Σ-sequent $\gamma \in \mathrm{Seq}_\Sigma(\mathbf{F})$;
- A **Gentzen Σ-rule** is a pair $\langle \mathbf{\Gamma}, \gamma \rangle$, where $\mathbf{\Gamma} \cup \{\gamma\} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$.

A **Gentzen axiom system** is a collection $\mathrm{Ax} = \{\mathrm{Ax}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, where Ax_Σ is a set of Gentzen Σ-axioms, which is $\mathbf{Sign}^\flat$-invariant.

A **Gentzen rule system** is a collection $\mathrm{Ir} = \{\mathrm{Ir}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, where Ir_Σ is a set of Gentzen Σ-rules, which is also $\mathbf{Sign}^\flat$-invariant. Set

$$R = \mathrm{Ax} \cup \mathrm{Ir}.$$

The **Gentzen closure system** $G^R \subseteq \mathcal{P}(\mathrm{Seq}(\mathbf{F})) \to \mathcal{P}(\mathrm{Seq}(\mathbf{F}))$ **generated by** R is the least closure system on $\mathrm{Seq}(\mathbf{F})$, satisfying the structural rules, that contains R, i.e., such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

- $\gamma \in G^R_\Sigma(\varnothing)$, for all $\gamma \in \mathrm{Ax}_\Sigma$, and
- $\gamma \in G^R_\Sigma(\mathbf{\Gamma})$, for all $\langle \mathbf{\Gamma}, \gamma \rangle \in \mathrm{Ir}_\Sigma$.

We denote by $\mathfrak{G}^R = \langle \mathbf{F}, G^R \rangle$ the corresponding Gentzen π-institution, called the **Gentzen π-institution generated by** R.

A Finitary Parenthesis

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $R = \mathrm{Ax} \cup \mathrm{Ir}$ a set of finitary Gentzen axioms and rules of inference (i.e., such that the set of antecedents of all sequents involved is finite and the set of hypotheses of each rule of inference is also finite), and $\mathbf{\Gamma} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$ a set of Σ-sequents. We define a family

$$\Xi^R_\Sigma(\mathbf{\Gamma}) = \bigcup \{\Xi^{R,n}_\Sigma(\mathbf{\Gamma}) : n < \omega\},$$

where $\Xi^{R,n}_\Sigma(\mathbf{\Gamma})$ is defined by induction on $n < \omega$ as follows:

- $\Xi^{R,0}_\Sigma(\mathbf{\Gamma}) = \{\phi \vdash_\Sigma \phi : \phi \in \mathrm{SEN}^\flat(\Sigma)\} \cup \mathrm{Ax}_\Sigma \cup \mathbf{\Gamma}$;
- For all $n \geq 0$, $\Phi \cup \Psi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rcl}\Xi^{R,n+1}_\Sigma(\mathbf{\Gamma}) & = & \{\Phi, \Psi \vdash_\Sigma \phi : \Phi \vdash_\Sigma \phi \in \Xi^{R,n}_\Sigma(\mathbf{\Gamma})\} \\ & & \cup \{\Phi \vdash_\Sigma \phi : \Phi \vdash_\Sigma \Psi,\ \Phi, \Psi \vdash_\Sigma \phi \in \Xi^{R,n}_\Sigma(\mathbf{\Gamma})\} \\ & & \cup \{\Phi \vdash_\Sigma \phi : \langle \mathbf{\Delta}, \Phi \vdash_\Sigma \phi \rangle \in \mathrm{Ir}_\Sigma, \mathbf{\Delta} \subseteq \Xi^{R,n}_\Sigma(\mathbf{\Gamma})\}.\end{array}$$

We define $\Xi^R : \mathcal{P}(\mathrm{Seq}(\mathbf{F})) \to \mathcal{P}(\mathrm{Seq}(\mathbf{F}))$, by letting $\Xi^R \coloneqq \{\Xi^R_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, where $\Xi^R_\Sigma : \mathcal{P}(\mathrm{Seq}_\Sigma(\mathbf{F})) \to \mathcal{P}(\mathrm{Seq}_\Sigma(\mathbf{F}))$ as defined above.

We show, next, that this is closure system on $\mathrm{Seq}(\mathbf{F})$, which satisfies the structural rules, includes Ax and is closed under Ir.

Lemma 1481 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $R = \mathrm{Ax} \cup \mathrm{Ir}$ *a collection of finitary axioms and rules of inference.* $\Xi^R : \mathcal{P}(\mathrm{Seq}(\mathbf{F})) \to \mathcal{P}(\mathrm{Seq}(\mathbf{F}))$ *is a closure system on* $\mathrm{Seq}(\mathbf{F})$*, satisfying the structural rules and including* R*.*

Proof: We show, first, that Ξ^R is a closure system on $\mathrm{Seq}(\mathbf{F})$.

- Suppose $\Sigma \in |\mathbf{Sign}^\flat|$ and $\mathbf{\Gamma} \cup \{\boldsymbol{\gamma}\} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$, such that $\boldsymbol{\gamma} \in \mathbf{\Gamma}$. Then $\boldsymbol{\gamma} \in \Xi^{R,0}_\Sigma(\mathbf{\Gamma})$ and, hence, $\boldsymbol{\gamma} \in \Xi^R_\Sigma(\mathbf{\Gamma})$.
- Suppose $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathbf{\Gamma} \cup \mathbf{\Delta} \cup \{\boldsymbol{\gamma}\} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$, such that $\boldsymbol{\gamma} \in \Xi^R_\Sigma(\mathbf{\Gamma})$ and $\mathbf{\Gamma} \subseteq \mathbf{\Delta}$. Then, for some $n < \omega$, $\boldsymbol{\gamma} \in \Xi^{R,n}_\Sigma(\mathbf{\Gamma})$ and $\mathbf{\Gamma} \subseteq \mathbf{\Delta}$. We show by induction on n, that then $\boldsymbol{\gamma} \in \Xi^{R,n}_\Sigma(\mathbf{\Delta})$.
  - If $n = 0$, then the conclusion follows directly from the inclusion $\mathbf{\Gamma} \subseteq \mathbf{\Delta}$.
  - Now suppose that $n > 0$ and that the conclusion holds for $n - 1$.
    * If $\boldsymbol{\gamma} = \Phi, \Psi \vdash_\Sigma \phi$, with $\Phi \vdash_\Sigma \phi \in \Xi^{R,n-1}_\Sigma(\mathbf{\Gamma})$, then, by the induction hypothesis, $\Phi \vdash_\Sigma \phi \in \Xi^{R,n-1}_\Sigma(\mathbf{\Delta})$, whence, it follows that $\Phi, \Psi \vdash_\Sigma \phi \in \Xi^{R,n}_\Sigma(\mathbf{\Delta})$.
    * If $\boldsymbol{\gamma} = \Phi \vdash_\Sigma \phi$, with $\Phi \vdash_\Sigma \Psi$, $\Phi, \Psi \vdash_\Sigma \phi \in \Xi^{R,n-1}_\Sigma(\mathbf{\Gamma})$, we get, by the induction hypothesis, $\Phi \vdash_\Sigma \Psi$, $\Phi, \Psi \vdash_\Sigma \phi \in \Xi^{R,n-1}_\Sigma(\mathbf{\Delta})$, whence $\Phi \vdash_\Sigma \phi \in \Xi^{R,n}_\Sigma(\mathbf{\Delta})$.
    * If $\boldsymbol{\gamma} = \Phi \vdash_\Sigma \phi$, with $\langle \mathbf{\Upsilon}, \Phi \vdash_\Sigma \phi \rangle \in \mathrm{Ir}_\Sigma$ and $\mathbf{\Upsilon} \subseteq \Xi^{R,n-1}_\Sigma(\mathbf{\Gamma})$, then, by the induction hypothesis, $\mathbf{\Upsilon} \subseteq \Xi^{R,n-1}_\Sigma(\mathbf{\Delta})$, whence, again, $\Phi \vdash_\Sigma \phi \in \Xi^{R,n}_\Sigma(\mathbf{\Delta})$.
- Suppose $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathbf{\Gamma} \cup \{\boldsymbol{\gamma}\} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$, such that $\boldsymbol{\gamma} \in \Xi^R_\Sigma(\Xi^R_\Sigma(\mathbf{\Gamma}))$. Then, for some $n < \omega$, $\boldsymbol{\gamma} \in \Xi^{R,n}_\Sigma(\Xi^R_\Sigma(\mathbf{\Gamma}))$. We show by induction on n, that then $\boldsymbol{\gamma} \in \Xi^R_\Sigma(\mathbf{\Gamma})$.
  - If $n = 0$, then $\boldsymbol{\gamma}$ is of the form $\phi \vdash_\Sigma \phi$ or is in Ax_Σ or in $\Xi^R_\Sigma(\mathbf{\Gamma})$. In the first two cases, it is in $\Xi^{R,0}_\Sigma(\mathbf{\Gamma}) \subseteq \Xi^R_\Sigma(\mathbf{\Gamma})$ and in the last in $\Xi^R_\Sigma(\mathbf{\Gamma})$.
  - Suppose $n > 0$ and the conclusion holds for $n - 1$.
    * If $\boldsymbol{\gamma} = \Phi, \Psi \vdash_\Sigma \phi$, with $\Phi \vdash_\Sigma \phi \in \Xi^{R,n-1}_\Sigma(\Xi^R_\Sigma(\mathbf{\Gamma}))$, then, by the induction hypothesis, $\Phi \vdash_\Sigma \phi \in \Xi^R_\Sigma(\mathbf{\Gamma})$, i.e., $\Phi \vdash_\Sigma \phi \in \Xi^{R,m}_\Sigma(\mathbf{\Gamma})$, for some $m < \omega$. Thus $\Phi, \Psi \vdash_\Sigma \phi \in \Xi^{R,m+1}_\Sigma(\mathbf{\Gamma}) \subseteq \Xi^R_\Sigma(\mathbf{\Gamma})$.
    * If $\boldsymbol{\gamma} = \Phi \vdash_\Sigma \phi$, with $\Phi \vdash_\Sigma \Psi$, $\Phi, \Psi \vdash_\Sigma \phi \in \Xi^{R,n-1}_\Sigma(\Xi^R_\Sigma(\mathbf{\Gamma}))$, we get, by the induction hypothesis, $\Phi \vdash_\Sigma \Psi$, $\Phi, \Psi \vdash_\Sigma \phi \in \Xi^R_\Sigma(\mathbf{\Gamma})$. Since Ψ is finite, there exists $m > 0$, such that $\Phi \vdash_\Sigma \Psi$, $\Phi, \Psi \vdash_\Sigma \phi \in \Xi^{R,m}_\Sigma(\mathbf{\Gamma})$. Thus, $\Phi \vdash_\Sigma \phi \in \Xi^{R,m+1}_\Sigma(\mathbf{\Gamma}) \subseteq \Xi^R_\Sigma(\mathbf{\Gamma})$.

- * If $\gamma = \Phi \vdash_\Sigma \phi$, with $\langle \mathbf{\Delta}, \Phi \vdash_\Sigma \phi \rangle \in \mathrm{Ir}_\Sigma$ and $\mathbf{\Delta} \subseteq \Xi_\Sigma^{R,n-1}(\Xi_\Sigma^R(\mathbf{\Gamma}))$, then, by the induction hypothesis, $\mathbf{\Delta} \subseteq \Xi_\Sigma^R(\mathbf{\Gamma})$, whence, again, since $\mathbf{\Delta}$ is finite, there exists $m > 0$, such that $\mathbf{\Delta} \subseteq \Xi_\Sigma^{R,m}(\mathbf{\Gamma})$. Therefore, $\Phi \vdash_\Sigma \phi \in \Xi_\Sigma^{R,m+1}(\mathbf{\Gamma}) \subseteq \Xi_\Sigma^R(\mathbf{\Gamma})$.

- Suppose $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathbf{\Gamma} \cup \{\gamma\} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$, such that $\gamma \in \Xi_\Sigma^R(\mathbf{\Gamma})$, and let $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$. Then, for some $n < \omega$, $\gamma \in \Xi_\Sigma^{R,n}(\mathbf{\Gamma})$. We show by induction on n, that then $\mathrm{SEN}^\flat(f)(\gamma) \in \Xi_{\Sigma'}^{R,n}(\mathrm{SEN}^\flat(f)(\mathbf{\Gamma}))$.

  - If $n = 0$, then γ is of the form $\phi \vdash_\Sigma \phi$ or in Ax_Σ or in $\mathbf{\Gamma}$. In the first case, $\mathrm{SEN}^\flat(f)(\gamma) = \mathrm{SEN}^\flat(f)(\phi) \vdash_{\Sigma'} \mathrm{SEN}^\flat(f)(\phi) \in \Xi_{\Sigma'}^{R,0}(\mathrm{SEN}^\flat(f)(\mathbf{\Gamma}))$, by definition. In the second case, the conclusion holds by the postulated invariance of Ax under $\mathbf{Sign}^\flat$. In the last case, it holds because, by definition, $\mathrm{SEN}^\flat(f)(\gamma) \in \Xi_{\Sigma'}^{R,0}(\mathrm{SEN}^\flat(f)(\mathbf{\Gamma}))$.
  - Suppose $n > 0$ and the conclusion holds for $n - 1$.
    * If $\gamma = \Phi, \Psi \vdash_\Sigma \phi$, with $\Phi \vdash_\Sigma \phi \in \Xi_\Sigma^{R,n-1}(\mathbf{\Gamma})$, then, by the induction hypothesis,

      $$\mathrm{SEN}^\flat(f)(\Phi) \vdash_{\Sigma'} \mathrm{SEN}^\flat(f)(\phi) \in \Xi_{\Sigma'}^{R,n-1}(\mathrm{SEN}^\flat(f)(\mathbf{\Gamma})),$$

      whence, by definition. $\mathrm{SEN}^\flat(f)(\Phi \cup \Psi) \vdash_{\Sigma'} \mathrm{SEN}^\flat(f)(\phi) \in \Xi_{\Sigma'}^{R,n}(\mathrm{SEN}^\flat(f)(\mathbf{\Gamma}))$.
    * If $\gamma = \Phi \vdash_\Sigma \phi$, with $\Phi \vdash_\Sigma \Psi$, $\Phi, \Psi \vdash_\Sigma \phi \in \Xi_\Sigma^{R,n-1}(\mathbf{\Gamma})$, we get, by the induction hypothesis, $\mathrm{SEN}^\flat(f)(\Phi) \vdash_{\Sigma'} \mathrm{SEN}^\flat(f)(\Psi)$, $\mathrm{SEN}^\flat(f)(\Phi \cup \Psi) \vdash_{\Sigma'} \mathrm{SEN}^\flat(f)(\phi) \in \Xi_{\Sigma'}^{R,n-1}(\mathrm{SEN}^\flat(f)(\mathbf{\Gamma}))$. So, again by definition,

      $$\mathrm{SEN}^\flat(f)(\Phi) \vdash_{\Sigma'} \mathrm{SEN}^\flat(f)(\phi) \in \Xi_{\Sigma'}^{R,n}(\mathrm{SEN}^\flat(f)(\mathbf{\Gamma})).$$

    * If $\gamma = \Phi \vdash_\Sigma \phi$, with $\langle \mathbf{\Delta}, \Phi \vdash_\Sigma \phi \rangle \in \mathrm{Ir}_\Sigma$ and $\mathbf{\Delta} \subseteq \Xi_\Sigma^{R,n-1}(\mathbf{\Gamma})$, then, by the induction hypothesis,

      $$\mathrm{SEN}^\flat(f)(\mathbf{\Delta}) \subseteq \Xi_{\Sigma'}^{R,n}(\mathrm{SEN}^\flat(f)(\mathbf{\Gamma})),$$

      whence, since Ir is invariant under $\mathbf{Sign}^\flat$, we get, by definition, $\mathrm{SEN}^\flat(f)(\Phi) \vdash_{\Sigma'} \mathrm{SEN}^\flat(f)(\phi) \in \Xi_{\Sigma'}^{R,n}(\mathrm{SEN}^\flat(f)(\mathbf{\Gamma}))$.

We have concluded the proof that Ξ^R is a closure system on $\mathrm{Seq}(\mathbf{F})$.

Next, we show that it satisfies the structural rules.

- For (Axiom), if $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, then, by definition, $\phi \vdash_\Sigma \phi \in \Xi_\Sigma^{R,0}(\varnothing) \subseteq \Xi_\Sigma^R(\varnothing)$.

- For (Weakening), if $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \Psi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$, such that $\Phi \vdash_\Sigma \phi \in \Xi_\Sigma^R(\mathbf{\Gamma})$, then, there exists $n < \omega$, such that $\Phi \vdash_\Sigma \phi \in \Xi_\Sigma^{R,n}(\mathbf{\Gamma})$. Therefore, by definition, $\Phi, \Psi \vdash_\Sigma \phi \in \Xi_\Sigma^{R,n+1}(\mathbf{\Gamma}) \subseteq \Xi_\Sigma^R(\mathbf{\Gamma})$.

- For (Cut), if $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \Psi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$, such that $\Phi \vdash_\Sigma \Psi$, $\Phi, \Psi \vdash_\Sigma \phi \in \Xi^R_\Sigma(\mathbf{\Gamma})$, then, since $\Psi \subseteq_f \mathrm{SEN}^\flat(\Sigma)$, there exists $n < \omega$, such that $\Phi \vdash_\Sigma \Psi$, $\Phi, \Psi \vdash_\Sigma \phi \in \Xi^{R,n}_\Sigma(\mathbf{\Gamma})$. Thus, by definition, $\Phi \vdash_\Sigma \phi \in \Xi^{R,n+1}_\Sigma(\mathbf{\Gamma}) \subseteq \Xi^R_\Sigma(\mathbf{\Gamma})$.

So Ξ^R does satisfy all three structural rules.

Finally, it does include all rules in R:

- For $\Sigma \in |\mathbf{Sign}^\flat|$, $\boldsymbol{\gamma} \in \mathrm{Ax}_\Sigma$, we have $\boldsymbol{\gamma} \in \mathrm{Ax}_\Sigma \subseteq \Xi^{R,0}_\Sigma(\varnothing) \subseteq \Xi^R_\Sigma(\varnothing)$.
- For $\Sigma \in |\mathbf{Sign}^\flat|$, $\langle \mathbf{\Gamma}, \boldsymbol{\gamma}\rangle \in \mathrm{Ir}_\Sigma$, such that $\mathbf{\Gamma} \subseteq \Xi^R_\Sigma(\mathbf{\Delta})$, since $\mathbf{\Gamma}$ is finite, there exists $n < \omega$, such that $\mathbf{\Gamma} \subseteq \Xi^{R,n}_\Sigma(\mathbf{\Delta})$. Thus, by definition, $\boldsymbol{\gamma} \in \Xi^{R,n+1}_\Sigma(\mathbf{\Delta}) \subseteq \Xi^R_\Sigma(\mathbf{\Delta})$.

This concludes the proof of the statement. ■

We show that, given a system R of finitary Gentzen axioms and rules of inference, the closure $G^R_\Sigma(\mathbf{\Gamma})$ of a set $\mathbf{\Gamma}$ of Σ-sequents in the least Gentzen π-institution generated by R is exactly $\Xi^R_\Sigma(\mathbf{\Gamma})$.

Proposition 1482 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $R = \mathrm{Ax} \cup \mathrm{Ir}$ *a set of finitary Gentzen axioms and rules of inference. Then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\mathbf{\Gamma} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$,

$$G^R_\Sigma(\mathbf{\Gamma}) = \Xi^R_\Sigma(\mathbf{\Gamma}).$$

Proof: Suppose, first, that $\boldsymbol{\gamma} \in \mathrm{Seq}_\Sigma(\mathbf{F})$, such that $\boldsymbol{\gamma} \in \Xi^R_\Sigma(\mathbf{\Gamma})$. Then, $\boldsymbol{\gamma} \in \Xi^{R,n}_\Sigma(\mathbf{\Gamma})$, for some $n < \omega$. We show by induction on $n < \omega$ that, if $\boldsymbol{\gamma} \in \Xi^{R,n}_\Sigma(\mathbf{\Gamma})$, then $\boldsymbol{\gamma} \in G^R_\Sigma(\mathbf{\Gamma})$.

- The conclusion is obvious for $n = 0$, since, by definition, $G^R_\Sigma(\mathbf{\Gamma})$ satisfies the structural rules, contains Ax_Σ and, clearly, includes $\mathbf{\Gamma}$;
- A similar clause applies for the induction step:
  - If $\boldsymbol{\gamma} = \Phi, \Psi \vdash_\Sigma \phi$, with $\Phi \vdash_\Sigma \phi \in \Xi^{R,n-1}_\Sigma(\mathbf{\Gamma})$, then, by the induction hypothesis, $\Phi \vdash_\Sigma \phi \in G^R_\Sigma(\mathbf{\Gamma})$ and, since G^R satisfies the structural rules, $\Phi, \Psi \vdash_\Sigma \phi \in G^R_\Sigma(\mathbf{\Gamma})$ also.
  - If $\boldsymbol{\gamma} = \Phi \vdash_\Sigma \phi$, with $\Phi \vdash_\Sigma \Psi \subseteq \Xi^{R,n-1}_\Sigma(\mathbf{\Gamma})$ and $\Psi \vdash_\Sigma \phi \in \Xi^{R,n-1}_\Sigma(\mathbf{\Gamma})$, then, again by the induction hypothesis, $\Phi \vdash_\Sigma \Psi \subseteq G^R_\Sigma(\mathbf{\Gamma})$ and $\Psi \vdash_\Sigma \phi \in G^R_\Sigma(\mathbf{\Gamma})$, whence, since G^R satisfies the structural rules, $\Phi \vdash_\Sigma \phi \in G^R_\Sigma(\mathbf{\Gamma})$.
  - If $\boldsymbol{\gamma} = \Phi \vdash_\Sigma \phi$, with $\langle \mathbf{\Delta}, \boldsymbol{\gamma}\rangle \in \mathrm{Ir}_\Sigma$ and $\mathbf{\Delta} \subseteq \Xi^{R,n-1}_\Sigma(\mathbf{\Gamma})$, then, by the induction hypothesis, $\mathbf{\Delta} \subseteq G^R_\Sigma(\mathbf{\Gamma})$ and, since G^R is closed under the rules of inference, we get that $\Phi \vdash_\Sigma \phi \in G^R_\Sigma(\mathbf{\Gamma})$.

We conclude that $\Xi^{R,n}_\Sigma(\mathbf{\Gamma}) \subseteq G^R_\Sigma(\mathbf{\Gamma})$, for all $n < \omega$, and, therefore, $\Xi^R_\Sigma(\mathbf{\Gamma}) \subseteq G^R_\Sigma(\mathbf{\Gamma})$.

Conversely, since, by Lemma 1481, $\Xi^R : \mathcal{P}(\mathrm{Seq}(\mathbf{F})) \to \mathcal{P}(\mathrm{Seq}(\mathbf{F}))$ is a closure system on $\mathrm{Seq}(\mathbf{F})$, which satisfies the structural rules, contains Ax and is closed under Ir, we conclude by the minimality of G^R, that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\mathbf{\Gamma} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$, $G^R_\Sigma(\mathbf{\Gamma}) \subseteq \Xi^R_\Sigma(\mathbf{\Gamma})$. From this, the conclusion follows. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. Define a Gentzen π-institution $\mathfrak{G}^{\mathcal{I}} = \langle \mathbf{F}, G^{\mathcal{I}} \rangle$, as follows:

1. If $\mathcal{I}$ has theorems, $\mathfrak{G}^{\mathcal{I}}$ is of type 1 and if $\mathcal{I}$ does not have theorems, then $\mathfrak{G}^{\mathcal{I}}$ is of type 0;
2. Set $\mathrm{Ax}^{\mathcal{I}} = \{\mathrm{Ax}^{\mathcal{I}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, where, for all $\Sigma \in |\mathbf{Sign}^\flat|$,
$$\mathrm{Ax}^{\mathcal{I}}_\Sigma = \{\Phi \vdash_\Sigma \phi : \phi \in C_\Sigma(\Phi)\}.$$
Let $R^{\mathcal{I}} := \mathrm{Ax}^{\mathcal{I}}$. Then set $\mathfrak{G}^{\mathcal{I}} := \mathfrak{G}^{R^{\mathcal{I}}}$.

Of course, $\mathfrak{G}^{\mathcal{I}}$ is a Gentzen π-institution. Moreover, it turns out that, if $\mathcal{I}$ is finitary, then $\mathfrak{G}^{\mathcal{I}}$ is adequate for the π-institution $\mathcal{I}$.

Lemma 1483 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a finitary π-institution based on $\mathbf{F}$. Then $\mathfrak{G}^{\mathcal{I}} = \langle \mathbf{F}, G^{\mathcal{I}} \rangle$ is a Gentzen π-institution adequate for $\mathcal{I}$.*

Proof: Note that, by hypothesis and Proposition 1482, $\mathfrak{G}^{\mathcal{I}} = \langle \mathbf{F}, \Xi^{R^{\mathcal{I}}} \rangle$.

Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$. We must show that
$$\phi \in C_\Sigma(\Phi) \quad \text{iff} \quad \Phi \vdash_\Sigma \phi \in \Xi^{R^{\mathcal{I}}}_\Sigma(\varnothing).$$

First, if $\phi \in C_\Sigma(\Phi)$, then, by definition $\Phi \vdash_\Sigma \phi \in \mathrm{Ax}^{\mathcal{I}}_\Sigma$. Therefore, since $\mathrm{Ax}^{\mathcal{I}}_\Sigma \subseteq \Xi^{R^{\mathcal{I}}}_\Sigma(\varnothing)$, we get $\Phi \vdash_\Sigma \phi \in \Xi^{R^{\mathcal{I}}}_\Sigma(\varnothing)$.

Conversely, we must show that, if $\Phi \vdash_\Sigma \phi \in \Xi^{R^{\mathcal{I}}}_\Sigma(\varnothing)$, then $\phi \in C_\Sigma(\Phi)$. We do this by showing, using induction on $n < \omega$, that
$$\Phi \vdash_\Sigma \phi \in \Xi^{R^{\mathcal{I}},n}_\Sigma(\varnothing) \quad \text{implies} \quad \phi \in C_\Sigma(\Phi).$$

- If $\Phi \vdash_\Sigma \phi \in \Xi^{R^{\mathcal{I}},0}_\Sigma(\varnothing)$, then it is either of the form $\phi \vdash_\Sigma \phi$ or in $\mathrm{Ax}^{\mathcal{I}}_\Sigma$. In the first case, the conclusion follows by the inflationarity of C and, in the second, by the definition of $\mathrm{Ax}^{\mathcal{I}}$.
- Suppose $n > 0$ and that the conclusion holds for $n - 1$. Then, since $\mathrm{Ir}^{\mathcal{I}} = \varnothing$, there are only two cases to consider.

- If $\Phi \vdash_\Sigma \phi$ is of the form $\Phi_1, \Phi_2 \vdash_\Sigma \phi \in \Xi_\Sigma^{R^{\mathcal{I}},n}(\varnothing)$, with $\Phi_1 \vdash_\Sigma \phi \in \Xi_\Sigma^{R^{\mathcal{I}},n-1}(\varnothing)$, then, by the induction hypothesis, $\phi \in C_\Sigma(\Phi_1)$ and, hence, by the monotonicity of C, $\phi \in C_\Sigma(\Phi_1, \Phi_2)$.
- If $\Phi \vdash_\Sigma \phi \in \Xi_\Sigma^{R^{\mathcal{I}},n}(\varnothing)$, with $\Phi \vdash_\Sigma \Psi$, $\Phi, \Psi \vdash_\Sigma \phi \in \Xi_\Sigma^{R^{\mathcal{I}},n-1}(\varnothing)$, then, by the induction hypothesis, $\Psi \subseteq C_\Sigma(\Phi)$ and $\phi \in C_\Sigma(\Phi, \Psi)$, whence

$$\begin{array}{rcll} \phi & \in & C_\Sigma(\Phi, \Psi) & \text{(hypothesis)} \\ & \subseteq & C_\Sigma(\Phi, C_\Sigma(\Phi)) & \text{(monotonicity)} \\ & \subseteq & C_\Sigma(C_\Sigma(\Phi)) & \text{(monotonicity)} \\ & = & C_\Sigma(\Phi). & \text{(idempotency)} \end{array}$$

This finishes the induction and concludes the proof. ■

End of the Finitary Parenthesis

20.2 $\mathfrak{G}$-Structures and $\mathfrak{G}$-Algebraic Systems

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and $\mathbb{L} = \langle \mathcal{A}, D \rangle$ an $\mathbf{F}$-structure. $\mathbb{L}$ is a **$\mathfrak{G}$-structure** or a **model of $\mathfrak{G}$** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\{\Phi_i \vdash_\Sigma \phi_i : i \in I\} \cup \{\Phi \vdash_\Sigma \phi\} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$,

$$\begin{array}{c} \Phi \vdash_\Sigma \phi \in G_\Sigma(\{\Phi_i \vdash_\Sigma \phi_i : i \in I\}) \text{ and } \alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi_i)),\ i \in I, \\ \text{imply} \quad \alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi)). \end{array}$$

In relation to $\mathfrak{G}$-structures, we use the following notation:

- $D \in \mathrm{ClFam}^{\mathfrak{G}}(\mathcal{A})$ if $\langle \mathcal{A}, D \rangle$ is a $\mathfrak{G}$-structure;
- $\mathrm{Str}(\mathfrak{G})$ is the collection of all $\mathfrak{G}$-structures;
- $\mathrm{Str}^{\mathfrak{G}}(\mathcal{A})$ is the collection of all $\mathfrak{G}$-structures on the $\mathbf{F}$-algebraic system $\mathcal{A}$.

Let, again, $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$ and $\mathbf{\Gamma} = \{\mathbf{\Gamma}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|} \in \mathrm{ThFam}(\mathfrak{G})$. Define $D^{\mathbf{\Gamma}} : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}$, by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$D_\Sigma^{\mathbf{\Gamma}}(\Phi) = \{\phi \in \mathrm{SEN}^\flat(\Sigma) : \Phi \vdash_\Sigma \phi \in \mathbf{\Gamma}_\Sigma\}.$$

We show that $D^{\mathbf{\Gamma}}$, thus defined, is a closure family on $\mathrm{SEN}^\flat$ and, therefore, $\langle \mathcal{F}, D^{\mathbf{\Gamma}} \rangle$ is an $\mathbf{F}$-structure. In fact, $\langle \mathcal{F}, D^{\mathbf{\Gamma}} \rangle$ is a $\mathfrak{G}$-structure.

Lemma 1484 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen π-institution based on* $\mathbf{F}$ *and* $\mathbf{\Gamma} \in \mathrm{ThFam}(\mathfrak{G})$*. Then* $\mathbb{L}^{\mathbf{\Gamma}} = \langle \mathcal{F}, C^{\mathbf{\Gamma}} \rangle$ *is a $\mathfrak{G}$-structure.*

Proof: We show, first, that $D^{\mathbf{\Gamma}}$ is a closure family on $\mathcal{F}$.

- Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \Phi$. Then, by (Axiom) $\phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$. By (Weakening), $\Phi \vdash_\Sigma \phi \in G_\Sigma(\phi \vdash_\Sigma \phi)$. Therefore, by (Cut), $\Phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$. Therefore, $\Phi \vdash_\Sigma \phi \in \mathbf{\Gamma}_\Sigma$ and, hence, $\phi \in D^{\mathbf{\Gamma}}_\Sigma(\Phi)$.

- Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \Psi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in D^{\mathbf{\Gamma}}_\Sigma(\Phi)$ and $\Phi \subseteq \Psi$. By definition, $\Phi \vdash_\Sigma \phi \in \mathbf{\Gamma}_\Sigma$. Hence, by (Weakening) $\Psi \vdash_\Sigma \phi \in G_\Sigma(\Phi \vdash_\Sigma \phi) \subseteq G_\Sigma(\mathbf{\Gamma}_\Sigma) = \mathbf{\Gamma}_\Sigma$. We conclude that $\phi \in D^{\mathbf{\Gamma}}_\Sigma(\Psi)$.

- Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in D^{\mathbf{\Gamma}}_\Sigma(D^{\mathbf{\Gamma}}_\Sigma(\Phi))$. Then, by definition, $D^{\mathbf{\Gamma}}_\Sigma(D^{\mathbf{\Gamma}}_\Sigma(\Phi)) \vdash_\Sigma \phi \in \mathbf{\Gamma}_\Sigma$ and $D^{\mathbf{\Gamma}}_\Sigma(\Phi) \vdash_\Sigma D^{\mathbf{\Gamma}}_\Sigma(D^{\mathbf{\Gamma}}_\Sigma(\Phi)) \subseteq \mathbf{\Gamma}_\Sigma$. Now we get

$$\begin{array}{rcl} D^{\mathbf{\Gamma}}_\Sigma(\Phi) \vdash_\Sigma \phi & \in & G_\Sigma(D^{\mathbf{\Gamma}}_\Sigma(\Phi), D^{\mathbf{\Gamma}}_\Sigma(D^{\mathbf{\Gamma}}_\Sigma(\Phi)) \vdash_\Sigma \phi, \\ & & \qquad D^{\mathbf{\Gamma}}_\Sigma(\Phi) \vdash_\Sigma D^{\mathbf{\Gamma}}_\Sigma(D^{\mathbf{\Gamma}}_\Sigma(\Phi))) \\ & \subseteq & G_\Sigma(D^{\mathbf{\Gamma}}_\Sigma(D^{\mathbf{\Gamma}}_\Sigma(\Phi)) \vdash_\Sigma \phi, \\ & & \qquad D^{\mathbf{\Gamma}}_\Sigma(\Phi) \vdash_\Sigma D^{\mathbf{\Gamma}}_\Sigma(D^{\mathbf{\Gamma}}_\Sigma(\Phi))) \\ & \subseteq & G_\Sigma(\mathbf{\Gamma}_\Sigma) \\ & = & \mathbf{\Gamma}_\Sigma. \end{array}$$

  Therefore, by definition, $\phi \in D^{\mathbf{\Gamma}}_\Sigma(\Phi)$.

We conclude that $\mathbb{L}^{\mathbf{\Gamma}} = \langle \mathcal{F}, D^{\mathbf{\Gamma}} \rangle$ is an $\mathbf{F}$-structure. We show, next, that $\mathbb{L}^{\mathbf{\Gamma}}$ is a $\mathfrak{G}$-structure. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\{\Phi_i \vdash_\Sigma \phi_i : i \in I\} \cup \{\Phi \vdash_\Sigma \phi\} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$, such that

- $\Phi \vdash_\Sigma \phi \in G_\Sigma(\{\Phi_i \vdash_\Sigma \phi_i : i \in I\})$ and
- $\phi_i \in D^{\mathbf{\Gamma}}_\Sigma(\Phi_i)$, for all $i \in I$.

Then, by definition, $\Phi_i \vdash_\Sigma \phi_i \in \mathbf{\Gamma}_\Sigma$, for all $i \in I$. Since $\mathbf{\Gamma} \in \mathrm{ThFam}(\mathfrak{G})$, we get $\Phi \vdash_\Sigma \phi \in \mathbf{\Gamma}_\Sigma$. Thus, by definition, $\phi \in D^{\mathbf{\Gamma}}_\Sigma(\Phi)$. So $\mathbb{L}^{\mathbf{\Gamma}} = \langle \mathcal{F}, D^{\mathbf{\Gamma}} \rangle$ is a $\mathfrak{G}$-structure. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$, and $\mathbb{L} = \langle \mathcal{F}, D \rangle$ a $\mathfrak{G}$-structure. We define $\mathbf{\Gamma}^{\mathbb{L}} = \{\mathbf{\Gamma}^{\mathbb{L}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathbf{\Gamma}^{\mathbb{L}}_\Sigma = \{\Phi \vdash_\Sigma \phi \in \mathrm{Seq}_\Sigma(\mathbf{F}) : \phi \in D_\Sigma(\Phi)\}.$$

We show that $\mathbf{\Gamma}^{\mathbb{L}}$, thus defined, is a theory family of the Gentzen π-institution $\mathfrak{G}$.

Lemma 1485 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen π-institution based on* $\mathbf{F}$*, and* $\mathbb{L} = \langle \mathcal{F}, D \rangle$ *a $\mathfrak{G}$-structure. Then* $\mathbf{\Gamma}^{\mathbb{L}} \in \mathrm{ThFam}(\mathfrak{G})$.

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\{\Phi_i \vdash_\Sigma \phi_i : i \in I\} \cup \{\Phi \vdash_\Sigma \phi\} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$, such that

- $\Phi \vdash_\Sigma \phi \in G_\Sigma(\{\Phi_i \vdash_\Sigma \phi_i : i \in I\})$ and
- $\Phi_i \vdash_\Sigma \phi_i \in \mathbf{\Gamma}^{\mathbb{L}}_\Sigma$, for all $i \in I$.

Then, by definition, $\phi_i \in D_\Sigma(\Phi_i)$, for all $i \in I$. Thus, since $\mathbb{L}$ is a $\mathfrak{G}$-structure, $\phi \in D_\Sigma(\Phi)$. Therefore, $\Phi \vdash_\Sigma \phi \in \mathbf{\Gamma}^{\mathbb{L}}_\Sigma$. We conclude $\mathbf{\Gamma}^{\mathbb{L}} \in \mathrm{ThFam}(\mathfrak{G})$. ■

We show next that the two preceding constructions, of a $\mathfrak{G}$-structure $\mathbb{L}^{\mathbf{\Gamma}}$ out of a given theory family $\mathbf{\Gamma}$ of $\mathfrak{G}$ and of a theory family $\mathbf{\Gamma}^{\mathbb{L}}$ out of a given $\mathfrak{G}$-structure $\mathbb{L} = \langle \mathcal{F}, D\rangle$ are inverses of one another.

Proposition 1486 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ a Gentzen π-institution based on $\mathbf{F}$ and $\mathbb{L} = \langle \mathcal{F}, D\rangle$ an $\mathbf{F}$-structure.*

(a) $\mathbb{L} \in \mathrm{Str}(\mathfrak{G})$ if and only if $\mathbf{\Gamma}^{\mathbb{L}} \in \mathrm{ThFam}(\mathfrak{G})$ and $\mathbb{L} = \mathbb{L}^{\mathbf{\Gamma}^{\mathbb{L}}}$;

(b) $\mathbf{\Gamma} \in \mathrm{ThFam}(\mathfrak{G})$ if and only if $\mathbb{L}^{\mathbf{\Gamma}} \in \mathrm{Str}(\mathfrak{G})$ and $\mathbf{\Gamma} = \mathbf{\Gamma}^{\mathbb{L}^{\mathbf{\Gamma}}}$;

(c) $\mathcal{I}^{\mathfrak{G}} = \langle \mathbf{F}, C^{\mathfrak{G}}\rangle$ is the smallest $\mathfrak{G}$-structure on $\mathcal{F}$ and $C^{\mathfrak{G}} = D^{\mathrm{Thm}(\mathfrak{G})}$.

Proof:

(a) Suppose, first, that $\mathbb{L} = \langle \mathcal{F}, D\rangle \in \mathrm{Str}(\mathfrak{G})$. Then, by Lemma 1485, $\mathbf{\Gamma}^{\mathbb{L}} \in \mathrm{ThFam}(\mathfrak{G})$. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. We have

$$\begin{array}{rll} \phi \in D^{\mathbf{\Gamma}^{\mathbb{L}}}_\Sigma(\Phi) & \text{iff} & \Phi \vdash_\Sigma \phi \in \mathbf{\Gamma}^{\mathbb{L}}_\Sigma \\ & \text{iff} & \phi \in D_\Sigma(\Phi). \end{array}$$

So $D = D^{\mathbf{\Gamma}^{\mathbb{L}}}$.

Assume, conversely, that $\mathbf{\Gamma}^{\mathbb{L}} \in \mathrm{ThFam}(\mathfrak{G})$ and $\mathbb{L} = \mathbb{L}^{\mathbf{\Gamma}^{\mathbb{L}}}$. By Lemma 1484, $\mathbb{L}^{\mathbf{\Gamma}^{\mathbb{L}}} \in \mathrm{Str}(\mathfrak{G})$. Thus, $\mathbb{L} = \mathbb{L}^{\mathbf{\Gamma}^{\mathbb{L}}} \in \mathrm{Str}(\mathfrak{G})$.

(b) Suppose, first, that $\mathbf{\Gamma} \in \mathrm{ThFam}(\mathfrak{G})$. Then, by Lemma 1484, $\mathbb{L}^{\mathbf{\Gamma}} \in \mathrm{Str}(\mathfrak{G})$. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rll} \Phi \vdash_\Sigma \phi \in \mathbf{\Gamma}^{\mathbb{L}^{\mathbf{\Gamma}}}_\Sigma & \text{iff} & \phi \in D^{\mathbf{\Gamma}}_\Sigma(\Phi) \\ & \text{iff} & \Phi \vdash_\Sigma \phi \in \mathbf{\Gamma}_\Sigma. \end{array}$$

So $\mathbf{\Gamma} = \mathbf{\Gamma}^{\mathbb{L}^{\mathbf{\Gamma}}}$.

Suppose, conversely, that $\mathbb{L}^{\mathbf{\Gamma}} \in \mathrm{Str}(\mathfrak{G})$ and $\mathbf{\Gamma} = \mathbf{\Gamma}^{\mathbb{L}^{\mathbf{\Gamma}}}$. Then, by Lemma 1485, $\mathbf{\Gamma}^{\mathbb{L}^{\mathbf{\Gamma}}} \in \mathrm{ThFam}(\mathfrak{G})$ and, hence, $\mathbf{\Gamma} \in \mathrm{ThFam}(\mathfrak{G})$.

(c) By Parts (a) and (b),

$$\mathbb{L} \longmapsto \mathbf{\Gamma}^{\mathbb{L}}$$
$$\mathbb{L}^{\mathbf{\Gamma}} \longleftarrow\!\!\dashv \mathbf{\Gamma}$$

are mutually inverse mappings between $\mathrm{ThFam}(\mathfrak{G})$ and $\mathrm{Str}^{\mathfrak{G}}(\mathcal{F})$ and both are clearly order-preserving. Thus $\mathbb{L}^{\mathrm{Thm}(\mathfrak{G})} = \mathcal{I}^{\mathfrak{G}}$ is the least $\mathfrak{G}$-structure on $\mathcal{F}$.

■

The next result shows that a Gentzen π-institution is complete with respect to class of all $\mathfrak{G}$-structures.

Proposition 1487 (Completeness Theorem) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution based on* $\mathbf{F}$*. For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\{\Phi_i \vdash_\Sigma \phi_i : i \in I\} \cup \{\Phi \vdash_\Sigma \phi\} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$*,* $\Phi \vdash_\Sigma \phi \in G_\Sigma(\{\Phi_i \vdash_\Sigma \phi_i : i \in I\})$ *if and only if, for every* $\mathfrak{G}$*-structure* $\mathbb{L} = \langle \mathcal{A}, D \rangle$*, with* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,*

$$\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi_i)),\ i \in I, \quad \textit{imply} \quad \alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\{\Phi_i \vdash_\Sigma \phi_i : i \in I\} \cup \{\Phi \vdash_\Sigma \phi\} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$.

Suppose, first, that $\Phi \vdash_\Sigma \phi \in G_\Sigma(\{\Phi_i \vdash_\Sigma \phi_i : i \in I\})$ and let $\mathbb{L} = \langle \mathcal{A}, D \rangle \in \mathrm{Str}(\mathfrak{G})$, such that $\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi_i))$, for all $i \in I$. Then, by the definition of a $\mathfrak{G}$-structure, $\alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi))$.

Assume, conversely, that the displayed condition in the statement holds. Let $\mathbf{\Gamma} \in \mathrm{ThFam}(\mathfrak{G})$, such that $\{\Phi_i \vdash_\Sigma \phi_i : i \in I\} \subseteq \mathbf{\Gamma}_\Sigma$. Then, by definition, $\phi_i \in D^{\mathbf{\Gamma}}_\Sigma(\Phi_i)$, for all $\in I$. Since, by Lemma 1484, $\mathbb{L}^{\mathbf{\Gamma}}$ is a $\mathfrak{G}$-structure, we get, by hypothesis, $\phi \in D^{\mathbf{\Gamma}}_\Sigma(\Phi)$. Therefore, $\Phi \vdash_\Sigma \phi \in \mathbf{\Gamma}_\Sigma$. We conclude that $\Phi \vdash_\Sigma \phi \in G_\Sigma(\{\Phi_i \vdash_\Sigma \phi_i : i \in I\})$. ■

Next, we show that the property of being a model of a Gentzen π-institution is preserved under bilogical morphisms.

Proposition 1488 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$ *two* $\mathbf{F}$*-algebraic systems,* $\mathbb{L} = \langle \mathcal{A}, D \rangle$*,* $\mathbb{L}' = \langle \mathcal{A}', D' \rangle$ *two* $\mathbf{F}$*-structures and* $\langle H, \gamma \rangle : \mathbb{L} \vdash \mathbb{L}'$ *a bilogical morphism.* $\mathbb{L}$ *is a* $\mathfrak{G}$*-structure if and only if* $\mathbb{L}'$ *is a* $\mathfrak{G}$*-structure.*

Proof: For the proof, it suffices to notice that, since $\langle H, \gamma \rangle$ is a bilogical morphism, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rl} \alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi)) & \text{iff} \quad \gamma_{F(\Sigma)}(\alpha_\Sigma(\phi)) \in D'_{H(F(\Sigma))}(\gamma_{F(\Sigma)}(\alpha_\Sigma(\Phi))) \\ & \text{iff} \quad \alpha'_\Sigma(\phi) \in D'_{F'(\Sigma)}(\alpha'_\Sigma(\Phi)). \end{array}$$

The rest of the argument is straightforward: If $\mathbb{L}$ is a $\mathfrak{G}$-structure, then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\{\Phi_i \vdash_\Sigma \phi_i : i \in I\} \cup \{\Phi \vdash_\Sigma \phi\} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$, such that $\Phi \vdash_\Sigma \phi \in G_\Sigma(\{\Phi_i \vdash \phi_i : i \in I\})$ and $\alpha'_\Sigma(\phi_i) \in D'_{F'(\Sigma)}(\alpha'_\Sigma(\Phi_i))$, for all $i \in I$, we get $\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi_i))$, for all $i \in I$, whence, by hypothesis, $\alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi))$, which gives $\alpha'_\Sigma(\phi) \in D'_{F'(\Sigma)}(\alpha'_\Sigma(\Phi))$. We conclude that $\mathbb{L}'$ is also a $\mathfrak{G}$-structure. If, conversely, $\mathbb{L}'$ is a $\mathfrak{G}$-structure, then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\{\Phi_i \vdash_\Sigma \phi_i : i \in I\} \cup \{\Phi \vdash_\Sigma \phi\} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$, such that $\Phi \vdash_\Sigma \phi \in G_\Sigma(\{\Phi_i \vdash \phi_i : i \in I\})$ and $\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi_i))$, for all $i \in I$, we get $\alpha'_\Sigma(\phi_i) \in D'_{F'(\Sigma)}(\alpha'_\Sigma(\Phi_i))$, for all $i \in I$, whence, by hypothesis, $\alpha'_\Sigma(\phi) \in D'_{F'(\Sigma)}(\alpha'_\Sigma(\Phi))$, which gives $\alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi))$. We conclude that $\mathbb{L}$ is also a $\mathfrak{G}$-structure. ■

In particular, we obtain

Corollary 1489 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$. An $\mathbf{F}$-structure $\mathbb{L} = \langle \mathcal{A}, D \rangle$ is a $\mathfrak{G}$-structure if and only if its reduction $\mathbb{L}^*$ is a $\mathfrak{G}$-structure.*

Proof: This follows directly from Proposition 1488, since the quotient morphism $\langle I, \pi \rangle : \mathbb{L} \vdash \mathbb{L}^*$ is a bilogical morphism. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system. $\mathcal{A}$ is a **$\mathfrak{G}$-algebraic system** if it is the underlying algebraic system of a reduced $\mathfrak{G}$-structure. We denote the class of all $\mathfrak{G}$-algebraic systems by $\mathrm{AlgSys}(\mathfrak{G})$, i.e., we have

$$\mathrm{AlgSys}(\mathfrak{G}) = \{\mathcal{A} : (\exists D \in \mathrm{ClFam}^{\mathfrak{G}}(\mathcal{A}))(\widetilde{\Omega}^{\mathcal{A}}(D) = \Delta^{\mathcal{A}})\}.$$

We show that, if a Gentzen π-institution $\mathfrak{G}$ happens to be adequate for a π-institution $\mathcal{I}$, then every $\mathfrak{G}$-algebraic system is also an $\mathcal{I}$-algebraic system.

Lemma 1490 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$ that is adequate for $\mathcal{I}$. Then:*

(a) Every $\mathfrak{G}$-structure is an $\mathcal{I}$-structure;

(b) $\mathrm{AlgSys}(\mathfrak{G}) \subseteq \mathrm{AlgSys}(\mathcal{I})$.

Proof:

(a) Let $\mathbb{L} = \langle \mathcal{A}, D \rangle \in \mathrm{Str}(\mathfrak{G})$. Suppose $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$. By the adequacy of $\mathfrak{G}$ for $\mathcal{I}$, $\Phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$. Since $\mathbb{L} \in \mathrm{Str}(\mathfrak{G})$, $\alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi))$. Thus, by Lemma 50, $\mathbb{L}$ is an $\mathcal{I}$-structure.

(b) Assume that $\mathfrak{G}$ is adequate for $\mathcal{I}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle \in \mathrm{AlgSys}(\mathfrak{G})$. Then, there exists a $\mathfrak{G}$-structure $\mathbb{L} = \langle \mathcal{A}, D \rangle$, such that $\widetilde{\Omega}^{\mathcal{A}}(D) = \Delta^{\mathcal{A}}$. To conclude that $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, it suffices, by Proposition 1399, to show that $\mathbb{L} \in \mathrm{Str}^{\mathcal{I}}(\mathcal{A})$. But this was done in Part (a).

■

20.3 Fully Adequate Gentzen π-Institutions

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$, such that $\mathfrak{G}$ is adequate for $\mathcal{I}$. Then, with $\mathcal{I}$ may be associated two classes of $\mathbf{F}$-algebraic systems and two classes of $\mathcal{I}$-structures:

- $\mathcal{I}$-algebraic systems and full $\mathcal{I}$-structures;
- $\mathfrak{G}$-algebraic systems and $\mathfrak{G}$-structures.

We devise certain conditions that, when possible to enforce, would guarantee that a Gentzen π-institution $\mathfrak{G}$ adequate for $\mathcal{I}$ can be picked in such a way as to have $\mathrm{AlgSys}(\mathfrak{G}) = \mathrm{AlgSys}(\mathcal{I})$ and $\mathrm{Str}(\mathfrak{G}) = \mathrm{FStr}(\mathcal{I})$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$. $\mathfrak{G}$ is said to be **fully adequate for** $\mathcal{I}$ if one of the following two conditions holds:

- $\mathcal{I}$ has theorems, $\mathfrak{G}$ is of type 1 and, for every $\mathbf{F}$-structure $\mathbb{L} = \langle \mathcal{A}, D \rangle$, $\mathbb{L} \in \mathrm{FStr}(\mathcal{I})$ if and only if $\mathbb{L} \in \mathrm{Str}(\mathfrak{G})$;
- $\mathcal{I}$ does not have theorems, $\mathfrak{G}$ is of type 0 and, for every $\mathbf{F}$-structure $\mathbb{L} = \langle \mathcal{A}, D \rangle$, $\mathbb{L} \in \mathrm{FStr}(\mathcal{I})$ if and only if $\mathbb{L} \in \mathrm{Str}(\mathfrak{G})$ and $\mathbb{L}$ does not have theorems.

We show that, if $\mathfrak{G}$ is fully adequate for $\mathcal{I}$, then it is also adequate for $\mathcal{I}$.

Proposition 1491 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$. If $\mathfrak{G}$ is fully adequate for $\mathcal{I}$, then $\mathfrak{G}$ is adequate for $\mathcal{I}$.*

Proof: Assume that $\mathfrak{G}$ is fully adequate for $\mathcal{I}$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$.

If $\phi \in C^{\mathfrak{G}}_\Sigma(\Phi)$, then, by definition, $\Phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing)$. Since, by Corollary 1391, $\langle \mathcal{F}, C \rangle \in \mathrm{FStr}(\mathcal{I})$, we get, by hypothesis, $\langle \mathcal{F}, C \rangle \in \mathrm{Str}(\mathfrak{G})$. Therefore, $\phi \in C_\Sigma(\Phi)$.

Assume, conversely, that $\phi \in C_\Sigma(\Phi)$. Since, by Proposition 1486, $\langle \mathcal{F}, C^{\mathfrak{G}} \rangle \in \mathrm{Str}(\mathfrak{G})$, which, additionally, does not have theorems, if $\mathcal{I}$ has no theorems, we get, by hypothesis, $\langle \mathcal{F}, C^{\mathfrak{G}} \rangle \in \mathrm{FStr}(\mathcal{I})$. But, by Corollary 1391, $\langle \mathcal{F}, C \rangle$ is the weakest full $\mathcal{I}$-structure on $\mathcal{F}$. Therefore, since $\phi \in C_\Sigma(\Phi)$, we get that $\phi \in C^{\mathfrak{G}}_\Sigma(\Phi)$. ■

We provide next a characterization of full adequacy, which also showcases its features and hints at why it is a useful notion in trying to connect π-institutions with Gentzen π-institutions.

Proposition 1492 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen* π*-institution based on* $\mathbf{F}$*.* $\mathfrak{G}$ *is fully adequate for* $\mathcal{I}$ *if and only if*

1. $\mathrm{AlgSys}(\mathfrak{G}) = \mathrm{AlgSys}(\mathcal{I})$;
2. *For all* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ *is the only reduced* $\mathfrak{G}$*-structure on* $\mathcal{A}$ *(without theorems if* $\mathcal{I}$ *does not have any);*
3. $\mathcal{I}$ *has theorems and* $\mathfrak{G}$ *is of type 1 or* $\mathcal{I}$ *does not have theorems and* $\mathfrak{G}$ *is of type 0.*

Proof: Assume $\mathfrak{G}$ is fully adequate for $\mathcal{I}$. Note that Condition 3 holds by definition. By Proposition 1491, $\mathcal{I}$ is adequate for $\mathcal{I}$. By Lemma 1490, $\mathrm{AlgSys}(\mathfrak{G}) \subseteq \mathrm{AlgSys}(\mathcal{I})$. If, on the other hand, $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, then $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ is a reduced full $\mathcal{I}$-structure. Thus, by hypothesis, it is a reduced $\mathfrak{G}$-structure. It follows that $\mathcal{A} \in \mathrm{AlgSys}(\mathfrak{G})$. This shows that Condition 1 also holds. It remains now to prove Condition 2. To this end, suppose $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$. Then $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ is a reduced full $\mathcal{I}$-structure. By hypothesis, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ is a reduced $\mathfrak{G}$-structure. By the Isomorphism Theorem 1408, it is the only full $\mathcal{I}$-structure on $\mathcal{A}$ that is reduced. Hence, by hypothesis, it is the only reduced $\mathfrak{G}$-structure on $\mathcal{A}$. This proves Condition 2 and concludes the "only if".

Assume, conversely, that Conditions 1-3 hold. Then, for all $\mathbf{F}$-structures $\mathbb{L} = \langle \mathcal{A}, D\rangle$,

$$\begin{array}{lll} \mathbb{L} \in \mathrm{FStr}(\mathcal{I}) & \text{iff} & \mathcal{A}^* \in \mathrm{AlgSys}(\mathcal{I}) \text{ and } \mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*) \\ & \text{iff} & \mathcal{A}^* \in \mathrm{AlgSys}(\mathfrak{G}) \text{ and } \langle \mathcal{A}^*, \mathcal{D}^*\rangle \in \mathrm{Str}(\mathfrak{G}) \\ & & \text{(w/o theorems if } \mathcal{I} \text{ does not have any)} \\ & \text{iff} & \langle \mathcal{A}, \mathcal{D}\rangle \in \mathrm{Str}(\mathfrak{G}) \\ & & \text{(w/o theorems if } \mathcal{I} \text{ does not have any).} \end{array}$$

This, combined with Condition 3, gives that $\mathfrak{G}$ is fully adequate for $\mathcal{I}$. ■

If a π-institution $\mathcal{I}$ has a fully adequate Gentzen π-institution, then that Gentzen π-institution is unique.

Proposition 1493 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$*,* $\mathfrak{G}' = \langle \mathbf{F}, G'\rangle$ *two Gentzen* π*-institutions based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *and* $\mathfrak{G}'$ *are fully adequate for* $\mathcal{I}$*, then* $\mathfrak{G} = \mathfrak{G}'$*.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\{\Phi_i \vdash_\Sigma \phi_i : i \in I\} \cup \{\Phi \vdash_\Sigma \phi\} \subseteq \mathrm{Seq}_\Sigma(\mathbf{F})$. Then, we get $\Phi \vdash_\Sigma \phi \in G_\Sigma(\{\Phi_i \vdash_\Sigma \phi_i : i \in I\})$ if and only if, by Proposition 1487, for every $\langle \mathcal{A}, D\rangle \in \mathrm{Str}(\mathfrak{G})$,

$$\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi_i)),\ i \in I, \quad \text{imply} \quad \alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi))$$

if and only if, by full adequacy, for all $\langle \mathcal{A}, D\rangle \in \mathrm{FStr}(\mathcal{I})$,

$$\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi_i)),\ i \in I, \quad \text{imply} \quad \alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi))$$

if and only if, by full adequacy, for every $\langle \mathcal{A}, D\rangle \in \mathrm{Str}(\mathfrak{G}')$,

$$\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi_i)),\ i \in I, \quad \text{imply} \quad \alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi))$$

if and only if, by Proposition 1487, $\Phi \vdash_\Sigma \phi \in G'_\Sigma(\{\Phi_i \vdash_\Sigma \phi_i : i \in I\})$. Therefore, $G = G'$ and, hence, $\mathfrak{G} = \mathfrak{G}'$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system. Recall the notation for the family of $\mathbf{F}$-equations $\mathrm{Eq}(\mathbf{F}) = \{\mathrm{Eq}_\Sigma(\mathbf{F})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, where $\mathrm{Eq}_\Sigma(\mathbf{F}) = \mathrm{SEN}^\flat(\Sigma)^2$. Let K be a class of $\mathbf{F}$-algebraic systems and recall the relative equational consequence of K

$$C^{\mathsf{K}} = \{C^{\mathsf{K}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|} : \mathcal{P}(\mathrm{Eq}(\mathbf{F})) \to \mathcal{P}(\mathrm{Eq}(\mathbf{F}))$$

given, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$, by

$$\phi \approx \psi \in C^{\mathsf{K}}_\Sigma(E) \quad \text{iff} \quad \begin{array}{l}\text{for all } \mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle \in \mathsf{K},\\ E \subseteq \mathrm{Ker}_\Sigma(\mathcal{A}) \text{ implies } \langle \phi, \psi\rangle \in \mathrm{Ker}_\Sigma(\mathcal{A}).\end{array}$$

We show that the structure $\mathcal{Q}^{\mathsf{K}} = \langle \mathbf{F}^2, C^{\mathsf{K}}\rangle$ is a π-structure.

Lemma 1494 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and let K be a class of $\mathbf{F}$-algebraic systems. Then $\mathcal{Q}^{\mathsf{K}} = \langle \mathbf{F}^2, C^{\mathsf{K}}\rangle$ is a π-structure.*

Proof: By Lemma **??**. ■

Recall from Proposition 115, that $\mathcal{Q}^{\mathsf{K}}$ satisfies the properties of reflexivity, symmetry, transitivity, congruence and invariance. So we have

Corollary 1495 *et $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and let K be a class of $\mathbf{F}$-algebraic systems. Then $\mathcal{Q}^{\mathsf{K}} = \langle \mathbf{F}^2, C^{\mathsf{K}}\rangle$ is an equational π-structure.*

Proof: By Lemma 1494 and Proposition 115. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ a Gentzen π-institution based on $\mathbf{F}$ and K a class of $\mathbf{F}$-algebraic systems. According to the framework developed in Chapter 12, we say that $\mathfrak{G}$ is **equivalent to** $\mathcal{Q}^{\mathsf{K}}$ if there exists a conjugate pair of translations $(t,s):\mathfrak{G}\rightleftarrows\mathcal{Q}^{\mathsf{K}}$, where

$$t:\quad \mathfrak{G}\longrightarrow\mathcal{Q}^{\mathsf{K}}$$
$$\mathfrak{G}\longleftarrow\mathcal{Q}^{\mathsf{K}}\quad :s$$

We will focus specifically on the case in which the translation $sq:\mathrm{Eq}(\mathbf{F})\to\mathrm{SenFam}(\mathfrak{G})$ is natural and given by the natural transformation $\kappa:\mathrm{Eq}(\mathbf{F})\to\mathcal{P}(\mathrm{Seq}(\mathbf{F}))$, determined, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\phi,\psi\in\mathrm{SEN}^\flat(\Sigma)$, by

$$\kappa_\Sigma(\phi,\psi)=\{\phi\vdash_\Sigma\psi,\psi\vdash_\Sigma\phi\}.$$

Recall that, in this case, since κ does not have any parameters, we have that, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\phi,\psi\in\mathrm{SEN}^\flat(\Sigma)$,

$$sq_\Sigma[\phi\approx\psi]=\{sq_{\Sigma,\Sigma'}[\phi\approx\psi]\}_{\Sigma'\in|\mathbf{Sign}^\flat|},$$

where

$$sq_{\Sigma,\Sigma'}[\phi\approx\psi]=\{\mathrm{SEN}^\flat(f)(\phi\vdash_\Sigma\psi),\mathrm{SEN}^\flat(\psi\vdash_\Sigma\phi):f\in\mathbf{Sign}^\flat(\Sigma,\Sigma')\}.$$

Finally, we say that the Gentzen π-institution $\mathfrak{G}$ **has** or **satisfies Congruence** if, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\phi_i,\psi_i\in\mathrm{SEN}^\flat(\Sigma)$, $i<k$,

$$\sigma^\flat_\Sigma(\vec{\phi})\vdash_\Sigma\sigma^\flat_\Sigma(\vec{\psi})\in G_\Sigma(\bigcup_{i<k}sq_\Sigma[\phi_i\approx\psi_i]).$$

We show that the equivalence of a Gentzen π-institution $\mathfrak{G}$ with an equational π-institution $\mathcal{Q}^{\mathsf{K}}$ implies that $\mathfrak{G}$ satisfies Congruence and, moreover, that it has interesting consequences for any π-institution for which $\mathfrak{G}$ happens to be adequate. More precisely, such a π-institution must be self extensional and the variety generated by its Lindenbaum-Tarski $\mathbf{F}$-algebraic system must coincide with the variety generated by the class K.

Proposition 1496 *Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, $\mathfrak{G}=\langle\mathbf{F},G\rangle$ a Gentzen π-institution based on $\mathbf{F}$ and K a class of $\mathbf{F}$-algebraic systems. If $\mathfrak{G}$ is equivalent to $\mathcal{Q}^{\mathsf{K}}$ via a conjugate pair $(t,sq):\mathfrak{G}\rightleftarrows\mathcal{Q}^{\mathsf{K}}$, then $\mathfrak{G}$ satisfies Congruence. If, in addition, $\mathfrak{G}$ is adequate for a π-institution $\mathcal{I}=\langle\mathbf{F},C\rangle$, then $\mathcal{I}$ is self extensional and $Q(\mathsf{K})=\mathsf{K}^{\mathcal{I}}$.*

Proof: Suppose $\mathfrak{G}$ is equivalent to $\mathcal{Q}^{\mathsf{K}}$ via a conjugate pair $(t,sq):\mathfrak{G}\rightleftarrows\mathcal{Q}^{\mathsf{K}}$ and let $\sigma^\flat:(\mathrm{SEN}^\flat)^k\to\mathrm{SEN}^\flat$ in $N^\flat$, $\Sigma\in|\mathbf{Sign}^\flat|$ and $\phi_i,\psi_i\in\mathrm{SEN}^\flat(\Sigma)$, $i<k$. By Proposition 115,

$$\sigma^\flat_\Sigma(\vec{\phi})\approx\sigma^\flat_\Sigma(\vec{\psi})\in C^{\mathsf{K}}_\Sigma(\{\phi_i\approx\psi_i:i<k\}).$$

Thus, by the hypothesis,

$$\sigma^\flat_\Sigma(\vec{\phi})\vdash_\Sigma\sigma^\flat_\Sigma(\vec{\psi})\in G_\Sigma(\bigcup_{i<k}sq_\Sigma[\phi_i\approx\psi_i]).$$

Thus, $\mathfrak{G}$ satisfies Congruence.

Suppose, next, that $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is a π-institution, for which $\mathfrak{G}$ is adequate, and let $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi_i, \psi_i \in \mathrm{SEN}^\flat(\Sigma)$, $i < k$, such that $C_\Sigma(\phi_i) = C_\Sigma(\psi_i)$. By structurality, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, $C_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) = C_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi))$. Then, by adequacy,

$$\mathrm{SEN}^\flat(f)(\phi_i \vdash_\Sigma \psi_i), \mathrm{SEN}^\flat(f)(\psi_i \vdash_\Sigma \phi_i) \in G_{\Sigma'}(\varnothing).$$

Since $\mathfrak{G}$ has Congruence, we get

$$\begin{array}{rcl}\sigma^\flat_\Sigma(\vec{\phi}) \vdash_\Sigma \sigma^\flat_\Sigma(\vec{\psi}), \sigma^\flat_\Sigma(\vec{\psi}) \vdash_\Sigma \sigma^\flat_\Sigma(\vec{\phi}) & \in & G_\Sigma(\bigcup_{i<k} sq_\Sigma[\phi_i \approx \psi_i]) \\ & \subseteq & G_\Sigma(G_\Sigma(\varnothing)) \\ & = & G_\Sigma(\varnothing)\end{array}.$$

Again using adequacy, $C_\Sigma(\sigma^\flat_\Sigma(\vec{\phi})) = C_\Sigma(\sigma^\flat_\Sigma(\vec{\psi}))$. Therefore, $\widetilde{\lambda}(\mathcal{I})$ is a congruence system on $\mathbf{F}$ and, by Proposition 1427, $\mathcal{I}$ is self extensional.

For the last claim, recall that

$$\begin{array}{rcl} Q(\mathsf{K}) & = & \{\mathcal{A} : \mathrm{Ker}(\mathsf{K}) \leq \mathrm{Ker}(\mathcal{A})\}; \\ \mathsf{K}^\mathcal{I} = Q(\mathcal{F}/\widetilde{\Omega}(\mathcal{I})) & = & \{\mathcal{A} : \widetilde{\Omega}(\mathcal{I}) \leq \mathrm{Ker}(\mathcal{A})\}. \end{array}$$

Moreover, note that $\mathrm{Ker}(\mathsf{K}) = \mathrm{Thm}(\mathcal{Q}^\mathsf{K})$. Therefore, to see that the claim holds, it suffices to show that $\mathrm{Thm}(\mathcal{Q}^\mathsf{K}) = \widetilde{\Omega}(\mathcal{I})$. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{lll} \phi \approx \psi \in D^\mathsf{K}_\Sigma(\varnothing) & \text{iff} & sq_\Sigma[\phi \approx \psi] \leq G(\varnothing) \quad \text{(by hypothesis)} \\ & \text{iff} & C_\Sigma(\phi) = C_\Sigma(\psi) \quad \text{(by adequacy)} \\ & \text{iff} & \langle \phi, \psi \rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}) \quad \text{(by definition)} \\ & \text{iff} & \langle \phi, \psi \rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I}). \quad \text{(by self extensionality)} \end{array}$$

Thus, we have $Q(\mathsf{K}) = \mathsf{K}^\mathcal{I}$, as claimed. ■

In closing the section, we show that, given a π-institution $\mathcal{I}$ that has an adequate finitary Gentzen π-institution $\mathfrak{G}$, satisfying Congruence, the equational consequence based on the variety $\mathsf{K}^\mathcal{I}$ is translated into the consequence of the Gentzen π-institution via sq.

Proposition 1497 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a finitary Gentzen π-institution, having Congruence, that is adequate for $\mathcal{I}$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $E \cup \{\phi, \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,*

$$\phi \approx \psi \in D^{\mathsf{K}^\mathcal{I}}_\Sigma(E) \quad \textit{implies} \quad sq_\Sigma[\phi \approx \psi] \leq G(sq_\Sigma[E]).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$, $E \cup \{\phi, \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \approx \psi \in D^{\mathsf{K}^\mathcal{I}}_\Sigma(E)$. By Theorem 119, we have $D^{\mathsf{K}^\mathcal{I}} = \Xi^{\mathrm{Ker}(\mathsf{K}^\mathcal{I})} = \Xi^{\widetilde{\Omega}(\mathcal{I})}$. So, we get $\phi \approx \psi \in \Xi^{\widetilde{\Omega}(\mathcal{I})}_\Sigma(E)$. We show by induction on $n < \omega$, that, for all $n < \omega$,

$$\phi \approx \psi \in \Xi^{\widetilde{\Omega}(\mathcal{I}),n}_\Sigma(E) \quad \text{implies} \quad sq_\Sigma[\phi \approx \psi] \leq G(sq_\Sigma[E]).$$

- For $n = 0$, we must have $\phi = \psi$ or $\langle\phi,\psi\rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I})$ or $\phi \approx \psi \in E$.

  In the first case the conclusion follows by (Axiom).

  In the second case, we have that $C_\Sigma(\phi) = C_\Sigma(\psi)$, whence, by adequacy, $sq_\Sigma[\phi \approx \psi] \leq G(\varnothing) \leq G(sq_\Sigma[E])$.

  In the last case, the conclusion follows by the inflationarity of G.

- Suppose, now, that the implication holds for $n > 0$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$, such that $\phi \approx \psi \in \Xi_\Sigma^{\widetilde{\Omega}(\mathcal{I}),n+1}(E)$.

  If $\psi \approx \phi \in \Xi_\Sigma^{\widetilde{\Omega}(\mathcal{I}),n}(E)$, then, by the induction hypothesis, $sq_\Sigma[\psi \approx \phi] \leq G(sq_\Sigma[E])$. Since $sq_\Sigma[\phi \approx \psi] = sq_\Sigma[\psi \approx \phi]$, we conclude that $sq_\Sigma[\phi \approx \psi] \leq G(sq_\Sigma[E])$.

  If $\phi \approx \chi, \chi \approx \psi \in \Xi_\Sigma^{\widetilde{\Omega}(\mathcal{I}),n}(E)$, then, by the induction hypothesis,

  $$sq_\Sigma[\phi \approx \chi], sq_\Sigma[\chi \approx \psi] \leq G(sq_\Sigma[E]).$$

  Using (Cut) and monotonicity, we get

  $$\begin{array}{rcl} sq_\Sigma[\phi \approx \psi] & \leq & G(sq_\Sigma[\phi \approx \chi], sq_\Sigma[\chi \approx \psi]) \\ & \leq & G(sq_\Sigma[E]). \end{array}$$

  If $\phi \approx \psi$ is of the form $\sigma_\Sigma^\flat(\vec{\phi}) \approx \sigma_\Sigma^\flat(\vec{\psi})$, with $\phi_i \approx \psi_i \in \Xi_\Sigma^{\widetilde{\Omega}(\mathcal{I}),n}(E)$, $i < k$, then, by the induction hypothesis, $sq_\Sigma[\phi_i \approx \psi_i] \leq G(sq_\Sigma[E])$ $i < k$. Then, since $\mathfrak{G}$ has Congruence, we conclude

  $$\begin{array}{rcl} sq_\Sigma[\sigma_\Sigma^\flat(\vec{\phi}) \approx \sigma_\Sigma^\flat(\vec{\psi})] & \leq & G(\bigcup_{i<k} sq_\Sigma[\phi_i \approx \psi_i]) \\ & \leq & G(sq_\Sigma[E]). \end{array}$$

  Last, assume that $\phi \approx \psi$ has the form $\mathrm{SEN}^\flat(f)(\phi' \approx \psi')$, for some $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma',\Sigma)$, such that $\phi' \approx \psi' \in \Xi_{\Sigma'}^{\widetilde{\Omega}(\mathcal{I}),n}(E)$. Then, by the induction hypothesis, $sq_{\Sigma'}[\phi' \approx \psi'] \leq G(sq_\Sigma[E])$. But, note that $sq_\Sigma[\phi \approx \psi] = sq_\Sigma[\mathrm{SEN}^\flat(f)(\phi' \approx \psi')] \leq sq_{\Sigma'}[\phi' \approx \psi']$. Thus, we get $sq_\Sigma[\phi \approx \psi] \leq G(sq_\Sigma[E])$.

We conclude that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$, $\phi \approx \psi \in D_\Sigma^{\mathsf{K}^\mathcal{I}}(E)$ implies that $sq_\Sigma[\phi \approx \psi] \leq G(sq_\Sigma[E])$. ■

20.4 Smoothness and Finitary Adaptations

In this section we define smooth Gentzen π-institutions and we also adapt some of the preceding results to the case of finitary π-institutions. This work is meant to pave the way for upcoming results on self extensionality and conjunction, presented in the next section, and on self extensionality and the deduction detachment theorem, which follow in the section after that.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$. We say that $\mathfrak{G}$ is **smooth** if G operates on finite sequents and it is systemic, i.e., by Proposition 149, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\mathrm{SEN}^\flat(f)(\Phi \vdash_\Sigma \phi) \in G_{\Sigma'}(\Phi \vdash_\Sigma \phi).$$

In the case of smooth Gentzen systems, the equivalence of the Gentzen system with an algebraic π-structure may be simplified as follows.

Proposition 1498 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a smooth Gentzen π-institution and K a class of $\mathbf{F}$-algebraic systems. Then $\mathfrak{G}$ is equivalent to $\mathcal{Q}^{\mathsf{K}}$ via the conjugate pair $(t, sq) : \mathfrak{G} \rightleftarrows \mathcal{Q}^{\mathsf{K}}$ if and only if its is equivalent to $\mathcal{Q}^{\mathsf{K}}$ via the conjugate pair $(t, \kappa) : \mathfrak{G} \rightleftarrows \mathcal{Q}^{\mathsf{K}}$.*

Proof: By Lemma 889, it is enough to show that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$G(sq_\Sigma[\phi \approx \psi]) = G(\kappa_\Sigma(\phi \approx \psi)).$$

This is, however, a consequence of smoothness. ■

Moreover, for a smooth Gentzen π-institution $\mathfrak{G}$, satisfying Congruence is equivalent to an apparently simpler condition.

Proposition 1499 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a smooth Gentzen π-institution. $\mathfrak{G}$ satisfies Congruence if and only if, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi_i, \psi_i \in \mathrm{SEN}^\flat(\Sigma)$, $i < k$,*

$$\sigma^\flat_\Sigma(\vec{\phi}) \vdash_\Sigma \sigma^\flat_\Sigma(\vec{\psi}) \in G_\Sigma(\{\phi_i \vdash_\Sigma \psi_i, \psi_i \vdash_\Sigma \phi_i : i < k\}).$$

Proof: Assume, first, that $\mathfrak{G}$ satisfies Congruence. Then, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi_i, \psi_i \in \mathrm{SEN}^\flat(\Sigma)$, $i < k$,

$$\begin{array}{rcl}\sigma^\flat_\Sigma(\vec{\phi}) \vdash_\Sigma \sigma^\flat_\Sigma(\vec{\psi}) & \in & G_\Sigma(\bigcup_{i<k} sq_\Sigma[\phi_i \approx \psi_i]) \\ & & \text{(by Congruence)} \\ & \subseteq & G_\Sigma(\{\phi_i \vdash_\Sigma \psi_i, \psi_i \vdash_\Sigma \phi_i : i < k\}). \\ & & \text{(by Smoothness)}\end{array}$$

Assume, conversely, that the given condition holds. Then, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi_i, \psi_i \in \mathrm{SEN}^\flat(\Sigma)$, $i < k$,

$$\begin{array}{rcl}\sigma^\flat_\Sigma(\vec{\phi}) \vdash_\Sigma \sigma^\flat_\Sigma(\vec{\psi}) & \in & G_\Sigma(\{\phi_i \vdash_\Sigma \psi_i, \psi_i \vdash_\Sigma \phi_i : i < k\}) \\ & & \text{(Hypothesis)} \\ & \subseteq & G_\Sigma(\bigcup_{i<k} sq_\Sigma[\phi_i \approx \psi_i]). \\ & & \text{(by Monotonicity)}\end{array}$$

We conclude that $\mathfrak{G}$ has Congruence. ∎

If a π-institution $\mathcal{I}$ is finitary, any $\mathcal{I}$-structure must also be finitary. Therefore, for any Gentzen π-institution $\mathfrak{G}$, no infinitary $\mathfrak{G}$-structure can be a full $\mathcal{I}$-structure. It is this observation that leads to the following modification of the definition of full adequacy for finitary π-institutions.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a finitary π-institution based on $\mathbf{F}$ and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$. $\mathfrak{G}$ is said to be **fully adequate for** $\mathcal{I}$ if one of the following two conditions holds:

- $\mathcal{I}$ has theorems, $\mathfrak{G}$ is of type 1 and, for every $\mathbf{F}$-structure $\mathbb{L} = \langle \mathcal{A}, D \rangle$, $\mathbb{L} \in \mathrm{FStr}(\mathcal{I})$ if and only if $\mathbb{L}$ is finitary and $\mathbb{L} \in \mathrm{Str}(\mathfrak{G})$;
- $\mathcal{I}$ does not have theorems, $\mathfrak{G}$ is of type 0 and, for every $\mathbf{F}$-structure $\mathbb{L} = \langle \mathcal{A}, D \rangle$, $\mathbb{L} \in \mathrm{FStr}(\mathcal{I})$ if and only if $\mathbb{L} \in \mathrm{Str}(\mathfrak{G})$ and $\mathbb{L}$ is finitary without theorems.

For the sequel we need a finitary adaptation of Proposition 1492. This is a characterization of full adequacy of a Gentzen system for a finitary π-institution $\mathcal{I}$.

Proposition 1500 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a finitary π-institution based on* $\mathbf{F}$ *and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen π-institution based on* $\mathbf{F}$*.* $\mathfrak{G}$ *is fully adequate for* $\mathcal{I}$ *if and only if*

1. $\mathrm{AlgSys}(\mathfrak{G}) = \mathrm{AlgSys}(\mathcal{I})$*;*
2. *For all* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*,* $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ *is the only finitary and reduced* $\mathfrak{G}$*-structure on* $\mathcal{A}$ *(without theorems if* $\mathcal{I}$ *does not have any);*
3. $\mathcal{I}$ *has theorems and* $\mathfrak{G}$ *is of type 1 or* $\mathcal{I}$ *does not have theorems and* $\mathfrak{G}$ *is of type 0.*

Proof: Assume $\mathfrak{G}$ is fully adequate for $\mathcal{I}$. Then, by Proposition 1492, Conditions 1-3 hold, where in Condition 2 $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ is finitary by Proposition 114. Thus, the "only if" holds.

Assume, conversely, that Conditions 1-3 hold. Then, for all $\mathbf{F}$-structures $\mathbb{L} = \langle \mathcal{A}, D \rangle$,

$$\begin{array}{lll} \mathbb{L} \in \mathrm{FStr}(\mathcal{I}) & \text{iff} & \mathcal{A}^* \in \mathrm{AlgSys}(\mathcal{I}), \mathcal{D}^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^*) \\ & \text{iff} & \mathcal{A}^* \in \mathrm{AlgSys}(\mathfrak{G}) \text{ and } \langle \mathcal{A}^*, \mathcal{D}^* \rangle \in \mathrm{Str}(\mathfrak{G}) \\ & & \text{finitary (w/o theorems if } \mathcal{I} \text{ does not have any)} \\ & \text{iff} & \langle \mathcal{A}, \mathcal{D} \rangle \in \mathrm{Str}(\mathfrak{G}) \text{ finitary} \\ & & \text{(w/o theorems if } \mathcal{I} \text{ does not have any).} \end{array}$$

This, combined with Condition 3, gives that $\mathfrak{G}$ is fully adequate for $\mathcal{I}$. ∎

20.5 IsoFull Adequacy and the DD Theorem

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and, for all $n < \omega$, $\Delta^n : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a collection of natural transformations in $N^\flat$, with $n+1$ distinguished arguments. Set

$$\Delta = \{\Delta^n : n < \omega\}.$$

Given a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, based on $\mathbf{F}$, and $T \in \mathrm{ThFam}(\mathcal{I})$, Δ is a **Parameterized Graded Deduction Detachment** (**PGDD**) **system for** $\mathcal{I}$ **over** T if, for all $n < \omega$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi_0, \dots, \phi_{n-1}, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\psi \in C_\Sigma(T_\Sigma, \phi_0, \dots, \phi_{n-1}) \quad \text{iff} \quad \Delta^n_\Sigma[\phi_0, \dots, \phi_{n-1}, \psi] \le T.$$

The left-to-right implication is the **Graded Deduction Property over** T and the right-to-left implication is the **Graded Detachment Property over** T.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\Delta = \{\Delta^n : n < \omega\}$ in $N^\flat$. Define a family $r^{\Delta^n} = \{r^{\Delta^n}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ of Gentzen $\mathbf{F}$-rules by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$r^{\Delta^n}_\Sigma = \{\langle \{\phi_0, \dots, \phi_{n-1} \vdash_\Sigma \psi\}, \vdash_\Sigma \Delta^n_\Sigma(\phi_0, \dots, \phi_{n-1}, \psi, \vec{\chi}) \rangle : \vec{\phi}, \psi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)\}.$$

Existence of a PGDD system Δ over a theory family T guarantees that the $\mathcal{I}$-structure $\langle \mathcal{F}, C^T \rangle$ satisfies all Gentzen $\mathbf{F}$-rules in r^{Δ^n}, $n < \omega$.

Lemma 1501 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $T \in \mathrm{ThFam}(\mathcal{I})$ and $\Delta = \{\Delta^n : n < \omega\}$ a PGDD system for $\mathcal{I}$ over T. Then, for all $n < \omega$,*

$$\langle \mathcal{F}, \mathrm{ThFam}(\mathcal{I})^T \rangle \vDash r^{\Delta^n}.$$

Proof: Suppose $\Sigma \in |\mathbf{Sign}^\flat|$, $\vec{\phi}, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\psi \in C^T_\Sigma(\phi_0, \dots, \phi_{n-1})$. Equivalently, we get $\psi \in C_\Sigma(T_\Sigma, \phi_0, \dots, \phi_{n-1})$. By hypothesis, since Δ is a PGDD system for $\mathcal{I}$ over T, we get $\Delta^n_\Sigma[\phi_0, \dots, \phi_{n-1}, \psi] \le T$. In particular, we get, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $\Delta^n_\Sigma(\phi_0, \dots, \phi_{n-1}, \psi, \vec{\chi}) \subseteq T_\Sigma$. Equivalently, $\vdash_\Sigma \Delta^n_\Sigma(\phi_0, \dots, \phi_{n-1}, \psi, \vec{\chi}) \in C^T_\Sigma(\varnothing)$. Thus, $\langle \mathcal{F}, \mathrm{ThFam}(\mathcal{I})^T \rangle \vDash r^{\Delta^n}$. ∎

We show, next, that all Gentzen $\mathbf{F}$-rules are preserved by bilogical morphisms between $\mathbf{F}$-structures.

Lemma 1502 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$ $\mathbf{F}$-algebraic systems, $\mathbb{L} = \langle \mathcal{A}, D \rangle$, $\mathbb{L}' = \langle \mathcal{A}', D' \rangle$ two $\mathbf{F}$-structures and $\langle H, \gamma \rangle : \mathbb{L} \vdash \mathbb{L}'$ a bilogical morphism. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$, every $\mathbf{F}$-sequent $\Psi \vdash_\Sigma \psi$ and every Gentzen $\mathbf{F}$-rule $r := \langle \{\Phi_i \vdash_\Sigma \phi_i : i \in I\}, \Phi \vdash_\Sigma \phi \rangle$,*

(a) $\mathbb{L} \vDash_\Sigma \Psi \vdash_\Sigma \psi$ *if and only if* $\mathbb{L}' \vDash_\Sigma \Psi \vdash_\Sigma \psi$;

(b) $\mathbb{L} \vDash_\Sigma r$ *if and only if* $\mathbb{L}' \vDash_\Sigma r$.

Proof:

(a) We have

$$\begin{array}{lll}\mathbb{L} \vDash_\Sigma \Psi \vdash_\Sigma \psi & \text{iff} & \alpha_\Sigma(\psi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Psi)) \\ & \text{iff} & \gamma_{F(\Sigma)}(\alpha_\Sigma(\psi)) \in D'_{H(F(\Sigma))}(\gamma_{H(F(\Sigma))}(\alpha_\Sigma(\Psi))) \\ & \text{iff} & \alpha'_\Sigma(\psi) \in D'_{F'(\Sigma)}(\alpha'_\Sigma(\Psi)) \\ & \text{iff} & \mathbb{L}' \vDash_\Sigma \Psi \vdash_\Sigma \psi.\end{array}$$

(b) This part follows easily from Part (a).

($\Rightarrow$) If $\alpha'_\Sigma(\phi_i) \in D'_{F'(\Sigma)}(\alpha'_\Sigma(\Phi_i))$, $i \in I$, then $\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi))$, $i \in I$, whence $\alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi))$. So $\alpha'_\Sigma(\phi) \in D'_{F'(\Sigma)}(\alpha'_\Sigma(\Phi))$.

($\Leftarrow$) If $\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi_i))$, $i \in I$, then $\alpha'_\Sigma(\phi_i) \in D'_{F'(\Sigma)}(\alpha'_\Sigma(\Phi_i))$, $i \in I$, whence $\alpha'_\Sigma(\phi) \in D'_{F'(\Sigma)}(\alpha'_\Sigma(\Phi_i))$, $i \in I$, and, therefore, $\alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi))$.

■

Some of the elements of the discussion that follows will be revisited in Chapter 21 on $\mathcal{I}$-operators in a more general context. We give a preview of a few results here, as needed, restricting the discussion mostly to protoalgebraic π-institutions. This restriction will be lifted in Chapter 21, where the concepts will be revisited in full generality.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Given an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and an $\mathcal{I}$-filter family $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we let

$$[T] = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \Omega^{\mathcal{A}}(T') = \Omega^{\mathcal{A}}(T)\},$$

the **equi-Leibniz class of** T. If $[T]$ has a smallest member, it is denoted by T^*. T is called a **Leibniz filter** if $T = T^*$, i.e., if it is the smallest filter in its equi-Leibniz class. We denote by $\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$ the collection of all Leibniz $\mathcal{I}$-filter families of $\mathcal{A}$.

We show that Leibniz filter families are preserved under inverse surjective morphisms with isomorphic functor components. For a more general result, see Corollary 1575 in Chapter 21.

Lemma 1503 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ $\mathbf{F}$*-algebraic systems and* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism, with* H *an isomorphism. Then*

$$\gamma^{-1}(\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{B})) \subseteq \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: Let $T' \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{B})$ and $T = \gamma^{-1}(T')$. Then $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Moreover, $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(\gamma^{-1}(T')) = \gamma^{-1}(\Omega^{\mathcal{B}}(T'))$. Consider $X \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $X \in [T]$, i.e., such that $\Omega^{\mathcal{A}}(X) = \Omega^{\mathcal{A}}(T)$. Then, since $\Omega^{\mathcal{A}}(T) = \gamma^{-1}(\Omega^{\mathcal{B}}(T'))$, $\mathrm{Ker}(\langle H, \gamma\rangle)$ is compatible with X. Hence $\gamma(X) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. Furthermore,

$$\gamma^{-1}(\Omega^{\mathcal{B}}(\gamma(X))) = \Omega^{\mathcal{A}}(\gamma^{-1}(\gamma(X))) = \Omega^{\mathcal{A}}(X) = \Omega^{\mathcal{A}}(T) = \gamma^{-1}(\Omega^{\mathcal{B}}(T')).$$

So $\Omega^{\mathcal{B}}(\gamma(X)) = \Omega^{\mathcal{B}}(T')$. Thus, since $T' \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{B})$, $T' \le \gamma(X)$. Now we get, taking again into account the compatibility of $\mathrm{Ker}(\langle H, \gamma\rangle)$ with X, $T = \gamma^{-1}(T') \le \gamma^{-1}(\gamma(X)) = X$. This proves that $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. ■

As a corollary, we obtain

Corollary 1504 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta\rangle\rangle$ $\mathbf{F}$-algebraic systems and $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism, with H an isomorphism. Then, for all $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, such that T'^* exists,*

$$\gamma^{-1}(T'^*) = \gamma^{-1}(T')^*.$$

Proof: Suppose $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, such that T'^* exists. Then $\gamma^{-1}(T'^*) \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$, by Lemma 1503. Hence, we have

$$\Omega^{\mathcal{A}}(\gamma^{-1}(T'^*)) = \gamma^{-1}(\Omega^{\mathcal{B}}(T'^*)) = \gamma^{-1}(\Omega^{\mathcal{B}}(T')) = \Omega^{\mathcal{A}}(\gamma^{-1}(T')).$$

Thus, since $\gamma^{-1}(T'^*)$ is Leibniz, we get $\gamma^{-1}(T'^*) = \gamma^{-1}(T')^*$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. An $\mathcal{I}$-structure $\mathbb{L} = \langle \mathcal{A}, D\rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, is called **isofull** if it is full and F is an isomorphism.

We show, next, that, if $\Delta = \{\Delta^n : n < \omega\}$ is a PGDD system for $\mathcal{I}$ over every $\mathcal{I}$-theory family, then every isofull $\mathcal{I}$-structure satisfies the Gentzen $\mathbf{F}$-rules r^{Δ^n}.

Lemma 1505 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a protoalgebraic π-institution based on $\mathbf{F}$ and $\Delta = \{\Delta^n : n < \omega\}$ a PGDD system for $\mathcal{I}$ over every Leibniz $\mathcal{I}$-theory family. Then every isofull $\mathcal{I}$-structure satisfies r^{Δ^n}, for all $n < \omega$.*

Proof: By Lemma 1502, it suffices to show that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, with F an isomorphism, and every $n < \omega$,

$$\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle \vDash r^{\Delta^n}.$$

Let $T = C^{\mathcal{I},\mathcal{A}}(\varnothing)$ be the smallest $\mathcal{I}$-filter family on $\mathcal{A}$. By the Correspondence Theorem for protoalgebraic π-institutions, we have $\alpha^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) =$

$\mathrm{ThFam}(\mathcal{I})^{\alpha^{-1}(T)}$. Since T is least among all $\mathcal{I}$-filter families of $\mathcal{A}$, we have $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. Therefore, by Lemma 1503, $\alpha^{-1}(T) \in \mathrm{ThFam}^*(\mathcal{I})$. Thus, by the hypothesis and Lemma 1501, we get that $\langle\mathcal{F}, \mathrm{ThFam}(\mathcal{I})^{\alpha^{-1}(T)}\rangle \vDash r^{\Delta^n}$. However, $\langle F,\alpha\rangle : \langle\mathcal{F}, \mathrm{ThFam}(\mathcal{I})^{\alpha^{-1}(T)}\rangle \vdash \langle\mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ is a bilogical morphism, whence, by Lemma 1502, we get $\langle\mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle \vDash r^{\Delta^n}$. ∎

In the next lemma, it is shown that, in case the π-institution $\mathcal{I}$ is syntactically protoalgebraic, the witnessing transformations may be used to generate Leibniz filter families.

Lemma 1506 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *a syntactically protoalgebraic* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*,* $\mathcal{A} = \langle\mathbf{A}, \langle F,\alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. Then*

$$T^* = C^{\mathcal{I},\mathcal{A}}(\bigcup\{\overset{\leftrightarrow\mathcal{A}}{I}_\Sigma[\phi,\psi] : \Sigma \in |\mathbf{Sign}|, \langle\phi,\psi\rangle \in \Omega^{\mathcal{A}}_\Sigma(T)\}).$$

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. We set

$$\widetilde{T} = C^{\mathcal{I},\mathcal{A}}(\bigcup\{\overset{\leftrightarrow\mathcal{A}}{I}_\Sigma[\phi,\psi] : \Sigma \in |\mathbf{Sign}|, \langle\phi,\psi\rangle \in \Omega^{\mathcal{A}}_\Sigma(T)\}).$$

Our goal is to show that $T^* = \widetilde{T}$. First, let $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T' \in [T]$, i.e., $\Omega^{\mathcal{A}}(T') = \Omega^{\mathcal{A}}(T)$. Then, we have, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi,\psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{rll}\langle\phi,\psi\rangle \in \Omega^{\mathcal{A}}_\Sigma(T) & \text{iff} & \langle\phi,\psi\rangle\Omega^{\mathcal{A}}_\Sigma(T')\\ & \text{iff} & \overset{\leftrightarrow\mathcal{A}}{I}_\Sigma[\phi,\psi] \leq T'.\end{array}$$

We conclude that $\widetilde{T} \leq T^*$ and, by protoalgebraicity, $\Omega^{\mathcal{A}}(\widetilde{T}) \leq \Omega^{\mathcal{A}}(T^*)$. On the other hand, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle \in \Omega^{\mathcal{A}}_\Sigma(T)$, $\overset{\leftrightarrow\mathcal{A}}{I}_\Sigma[\phi,\psi] \leq \widetilde{T}$. Thus, $\Omega^{\mathcal{A}}(T^*) = \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(\widetilde{T})$. Therefore, $\Omega^{\mathcal{A}}(\widetilde{T}) = \Omega^{\mathcal{A}}(T^*)$ and, since we showed that $\widetilde{T} \leq T^*$, we get by the minimality property of T^* in $[T]$, $T^* = \widetilde{T}$. ∎

Corollary 1507 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *a syntactically protoalgebraic* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*,* $\mathcal{A} = \langle\mathbf{A}, \langle F,\alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. Then* $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$ *if and only if, there exists* $X \leq \mathrm{SEN}^2$*, such that*

$$T = C^{\mathcal{I},\mathcal{A}}(\bigcup\{\overset{\leftrightarrow\mathcal{A}}{I}_\Sigma[\phi,\psi] : \Sigma \in |\mathbf{Sign}|, \langle\phi,\psi\rangle \in X_\Sigma\}).$$

Proof: For the left-to-right implication, assume $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. Take $X = \Omega^{\mathcal{A}}(T)$. Then we have, using the hypothesis and Lemma 1506, $T = T^* = C^{\mathcal{I},\mathcal{A}}(\bigcup\{\overset{\leftrightarrow\mathcal{A}}{I}_\Sigma[\phi,\psi] : \Sigma \in |\mathbf{Sign}|, \langle\phi,\psi\rangle \in X_\Sigma\})$.

Suppose, conversely, that $T = C^{\mathcal{I},\mathcal{A}}(\bigcup\{\overset{\leftrightarrow\mathcal{A}}{I}_\Sigma[\phi,\psi] : \Sigma \in |\mathbf{Sign}|, \langle\phi,\psi\rangle \in X_\Sigma\})$, for some $X \leq \mathrm{SEN}$. Then, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi,\psi \in \mathrm{SEN}(\Sigma)$, $\langle\phi,\psi\rangle \in X_\Sigma$ implies $\overset{\leftrightarrow\mathcal{A}}{I}_\Sigma[\phi,\psi] \leq T$. Thus, $\langle\phi,\psi\rangle \in \Omega^{\mathcal{A}}_\Sigma(T)$. Therefore, by Lemma 1506, $T \leq C^{\mathcal{I},\mathcal{A}}(\bigcup\{\overset{\leftrightarrow\mathcal{A}}{I}_\Sigma[\phi,\psi] : \Sigma \in |\mathbf{Sign}|, \langle\phi,\psi\rangle \in X_\Sigma\}) = T^*$. Since, it is always the case that $T^* \leq T$, we get that $T = T^*$ and $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. ∎

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. A collection $\Delta = \{\Delta^n : n < \omega\}$, where $\Delta^n : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}$ in $N^\flat$, with $n+1$ distinguished arguments, is called **Leibniz generating over** $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi},\psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C(\Delta^n_\Sigma[\vec{\phi},\psi]) \in \mathrm{ThFam}^*(\mathcal{I}),$$

for all $n < \omega$.

We show that, for a syntactically protoalgebraic π-institution $\mathcal{I}$, the property of being Leibniz generating over $\mathcal{I}$, transfers, in certain sense, to the filter families over arbitrary $\mathbf{F}$-algebraic systems.

Lemma 1508 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle\mathbf{F}, C\rangle$ a syntactically protoalgebraic π-institution based on $\mathbf{F}$, with witnessing transformations $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\Delta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ a Leibniz generating collection in $N^\flat$, with $n+1$ distinguished arguments. Then for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle\mathbf{A}, \langle F,\alpha\rangle\rangle$, $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi},\psi \in \mathrm{SEN}(\Sigma)$,*

$$C^{\mathcal{I},\mathcal{A}}(\Delta^{\mathcal{A}}_\Sigma[\vec{\phi},\psi]) \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: By hypothesis, Δ is Leibniz generating. Hence, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\vec{\phi},\psi \in \mathrm{SEN}^\flat(\Sigma)$, $C(\Delta_\Sigma[\vec{\phi},\psi]) \in \mathrm{ThFam}^*(\mathcal{I})$. Thus, by Corollary 1507, there exists $X \leq (\mathrm{SEN}^\flat)^2$, such that

$$C(\Delta_\Sigma[\vec{\phi},\psi]) = C(\bigcup\{\overset{\leftrightarrow\flat}{I}_{\Sigma'}[\phi',\psi'] : \Sigma' \in |\mathbf{Sign}^\flat|, \langle\phi',\psi'\rangle \in X_{\Sigma'}\}).$$

Now we get

$$\begin{aligned}
&C^{\mathcal{I},\mathcal{A}}(\Delta^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\vec{\phi}),\alpha_\Sigma(\psi)])\\
&= C^{\mathcal{I},\mathcal{A}}(\alpha(\Delta_\Sigma[\phi,\psi]))\\
&= C^{\mathcal{I},\mathcal{A}}(\alpha(C(\bigcup\{\overset{\leftrightarrow\flat}{I}_{\Sigma'}[\phi',\psi'] : \Sigma' \in |\mathbf{Sign}^\flat|, \langle\phi',\psi'\rangle \in X_{\Sigma'}\})))\\
&= C^{\mathcal{I},\mathcal{A}}(\alpha(\bigcup\{\overset{\leftrightarrow\flat}{I}_{\Sigma'}[\phi',\psi'] : \Sigma' \in |\mathbf{Sign}^\flat|, \langle\phi',\psi'\rangle \in X_{\Sigma'}\}))\\
&= C^{\mathcal{I},\mathcal{A}}(\bigcup\{\overset{\leftrightarrow\mathcal{A}}{I}_{F(\Sigma')}[\alpha_{\Sigma'}(\phi'),\alpha_{\Sigma'}(\psi')] : \Sigma' \in |\mathbf{Sign}^\flat|,\\
&\qquad\qquad \langle\alpha_{\Sigma'}(\phi'),\alpha_{\Sigma'}(\psi')\rangle \in \alpha_{\Sigma'}(X_{\Sigma'})\}).
\end{aligned}$$

Thus, taking into account the surjectivity of $\langle F,\alpha\rangle$, we obtain, using again Corollary 1507, that for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi},\psi \in \mathrm{SEN}(\Sigma)$, $C^{\mathcal{I},\mathcal{A}}(\Delta^{\mathcal{A}}_\Sigma[\vec{\phi},\psi]) \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. A PGDD system $\Delta = \{\Delta^n : n < \omega\}$ for $\mathcal{I}$ is called **Leibniz generating** if Δ^n is Leibniz generating, for every $n < \omega$.

It is not difficult to see that Leibniz generating PGDD systems have a graded Modus Ponens property, in the sense detailed in the following

Lemma 1509 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\Delta = \{\Delta^n : n < \omega\}$ is a PGDD system for $\mathcal{I}$ over every Leibniz theory family, then, for all $n < \omega$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \psi \in \mathrm{SEN}^\flat(\Sigma)$,*

$$\psi \in C_\Sigma(\Delta^n_\Sigma[\vec{\phi}, \psi], \vec{\phi}).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi}, \psi \in \mathrm{SEN}^\flat(\Sigma)$ and set $T = C(\Delta^n_\Sigma[\vec{\phi}, \psi])$. By hypothesis, $T \in \mathrm{ThFam}^*(\mathcal{I})$. Since Δ is a PGDD system for $\mathcal{I}$ over every Leibniz theory family, we get

$$\psi \in C_\Sigma(T_\Sigma, \vec{\phi}) \quad \text{iff} \quad \Delta^n_\Sigma[\vec{\phi}, \psi] \leq C(T) = T.$$

Thus, since the right hand side of the equivalence holds, we obtain $\psi \in C_\Sigma(T_\Sigma, \vec{\phi}) = C_\Sigma(C_\Sigma(\Delta^n_\Sigma[\vec{\phi}, \psi]), \vec{\phi}) = C_\Sigma(\Delta^n_\Sigma[\vec{\phi}, \psi], \vec{\phi})$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\Delta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with a single distinguished argument. We say that Δ **isodefines Leibniz filter families over** $\mathcal{I}$ if, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with F an isomorphism, all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and all $\Sigma \in |\mathbf{Sign}|$,

$$T^*_\Sigma = \{\phi \in \mathrm{SEN}(\Sigma) : \Delta^{\mathcal{A}}_\Sigma[\phi] \leq T\}.$$

We show, next, that, in a syntactically protoalgebraic π-institution $\mathcal{I}$, which has a Leibniz generating PGDD system $\Delta = \{\Delta^n : n < \omega\}$ over every Leibniz theory family, the 0-th component Δ^0 does isodefine Leibniz filter families over $\mathcal{I}$.

A couple of lemmas are needed first.

Lemma 1510 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a protoalgebraic π-institution based on $\mathbf{F}$. For every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T \leq T'$ implies $T^* \leq T'^*$.*

Proof: Suppose $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \leq T'$. By protoalgebraicity of $\mathcal{I}$, we get

$$\Omega^{\mathcal{A}}(T \cap T'^*) = \Omega^{\mathcal{A}}(T) \cap \Omega^{\mathcal{A}}(T'^*) = \Omega^{\mathcal{A}}(T) \cap \Omega^{\mathcal{A}}(T') = \Omega^{\mathcal{A}}(T).$$

Thus $T^* \leq T \cap T'^* \leq T'^*$. ■

Lemma 1511 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a protoalgebraic* π*-institution based on* $\mathbf{F}$*. Then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle \in \mathrm{FStr}(\mathcal{I}) \quad \textit{iff} \quad T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: We have, using Theorem 1395 and protoalgebraicity, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle \in \mathrm{FStr}(\mathcal{I})$ if and only if

$$\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\}.$$

Since, under protoalgebraicity, it always holds that

$$\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \subseteq \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\},$$

it suffices to show that

$$\begin{array}{lll} T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) & \text{iff} & \text{for all } T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}), \\ & & \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T') \text{ implies } T \leq T'. \end{array}$$

The right to left implication is trivial, since the condition on the right implies that T is smallest among all filter families sharing the same Leibniz congruence system with T. For the converse, suppose T is a Leibniz filter family of $\mathcal{A}$ and that $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. Then, using protoalgebraicity, we get $\Omega(T \cap T') = \Omega^{\mathcal{A}}(T) \cap \Omega^{\mathcal{A}}(T') = \Omega^{\mathcal{A}}(T)$. Thus, we conclude that $T = T^* \leq T \cap T' \leq T'$. ∎

Theorem 1512 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a syntactically protoalgebraic* π*-institution based on* $\mathbf{F}$*, with* $\Delta = \{\Delta^n : n < \omega\}$ *a Leibniz generating PGDD system for* $\mathcal{I}$ *over every Leibniz theory family. Then* Δ^0 *isodefines Leibniz filter families over* $\mathcal{I}$*.*

Proof: Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system, with F an isomorphism, and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$.

Suppose, first, that $\Delta^0_\Sigma[\phi] \leq T$. Let $T' = C^{\mathcal{I},\mathcal{A}}(\Delta^0_\Sigma[\phi])$. By hypothesis and Lemma 1508, $T' \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. Since $T' \leq T$, by Lemma 1510, $T' \leq T^*$. Hence, $\Delta^0_\Sigma[\phi] \leq T^*$. Therefore, by Lemma 1509, $\phi \in T^*_\Sigma$.

Suppose, conversely, that $\phi \in T^*_\Sigma$. Then, by definition, the $\mathcal{I}$-structure $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T^*} \rangle$ satisfies $\vdash_\Sigma \phi$. By Lemma 1511, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T^*} \rangle \in \mathrm{FStr}(\mathcal{I})$. Thus, by Lemma 1505, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{A}}(\mathcal{A})^{T^*} \rangle$ satisfies r^{Δ^0}. Hence, for all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')$, $\Delta^0_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi}) \in T^*_{\Sigma'} \subseteq T_{\Sigma'}$. We conclude that $\Delta^0_\Sigma[\phi] \leq T$. ∎

We are ready now to prove one half of the main result of this section. We would like to show that, for a syntactically protoalgebraic finitary π-institution, the existence of a Leibniz generating PGDD system over all

Leibniz theory families implies the existenc of an isofully adequate Gentzen π-institution.

We define that institution, first, preceding the statement of the theorem that involves it in its proof.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\Delta = \{\Delta^n : n < \omega\}$ in $N^\flat$, where Δ^n has $n+1$ distinguished arguments, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Define:

- $\mathrm{Ax}^{\mathcal{I}} = \{\mathrm{Ax}^{\mathcal{I}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, where, for all $\Sigma \in |\mathbf{Sign}^\flat|$,
$$\mathrm{Ax}^{\mathcal{I}}_\Sigma = \{\Phi \vdash_\Sigma \phi : \phi \in C_\Sigma(\Phi)\};$$
- $\mathrm{Ir}^{\mathcal{I}} = \{\mathrm{Ir}^{\mathcal{I}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, where, for all $\Sigma \in |\mathbf{Sign}^\flat|$,
$$\mathrm{Ir}^{\mathcal{I}}_\Sigma = \{r^{\Delta^n}_\Sigma : n < \omega\}.$$

Set $R^{\mathcal{I}} = \mathrm{Ax}^{\mathcal{I}} \cup \mathrm{Ir}^{\mathcal{I}}$ and let
$$\mathfrak{G}^{\mathcal{I}} = \langle \mathbf{F}, G^{\mathcal{I}} \rangle$$
be the Gentzen π-institution generated by $R^{\mathcal{I}}$ (recall that $G^{\mathcal{I}}$ is required to be a structural closed system on $\mathrm{Seq}(\mathbf{F})$ and, therefore, it is assumed to satisfy, by default, (Axiom), (Weakening) and (Cut)).

Theorem 1513 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a syntactically protoalgebraic finitary π-institution based on $\mathbf{F}$. If $\mathcal{I}$ has a Leibniz generating PGDD system $\Delta = \{\Delta^n : n < \omega\}$ over all Leibniz theory families, then it has an isofully adequate Gentzen π-institution, namely the Gentzen π-institution $\mathfrak{G}^{\mathcal{I}} = \langle \mathbf{F}, G^{\mathcal{I}} \rangle$.*

Proof: We must show that, for every $\mathbf{F}$-structure $\mathbb{L} = \langle \mathcal{A}, D \rangle$, where $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with F an isomorphism, we have
$$\mathbb{L} \in \mathrm{Str}(\mathfrak{G}^{\mathcal{I}}) \quad \text{iff} \quad \mathbb{L} \in \mathrm{FStr}(\mathcal{I}).$$
Suppose, first, that $\mathbb{L} \in \mathrm{FStr}(\mathcal{I})$. Then $\mathbb{L}$ is, in particular, an $\mathcal{I}$-structure. Therefore, it satisfies $\mathrm{Ax}^{\mathcal{I}}$. Moreover, by Lemma 1505, $\mathbb{L}$ satisfies r^{Δ^n}, for all $n < \omega$. Hence, it also satisfies $\mathrm{Ir}^{\mathcal{I}}$. We conclude that $\mathbb{L} \in \mathrm{Str}(\mathfrak{G}^{\mathcal{I}})$.

Suppose, conversely, that $\mathbb{L} \in \mathrm{Str}(\mathfrak{G}^{\mathcal{I}})$. Clearly, $\mathbb{L} \in \mathrm{Str}(\mathcal{I})$, since it satisfies $\mathrm{Ax}^{\mathcal{I}}$. So it suffices to show that it is also full. Assume, to the contrary, that $\mathbb{L}$ is not full and let $T = D(\varnothing)$. Since $\mathcal{I}$ is protoalgebraic and $\mathbb{L}$ is not full, we have, using Lemma 1511, $\mathcal{D} \subsetneq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T^*}$. Consider $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T^*} - \mathcal{D}$. Then we get $D(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T^*}$ and $T^* \leq T' \lneq D(T')$. Let $\Sigma \in |\mathbf{Sign}|$, $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in D_\Sigma(T') - T'_\Sigma$. Then, there exists $\Phi \subseteq_f T'_\Sigma$, such that $\phi \in D_\Sigma(\Phi)$. Since $\mathbb{L}$ satisfies $\mathrm{Ir}^{\mathcal{I}}$, we get $\Delta^n_\Sigma[\Phi, \phi] \leq T$. But Δ is also Leibniz generating, whence, by Lemma 1508,

$D(\Delta^n_\Sigma[\Phi,\phi]) \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. Therefore, by Lemma 1510, $\Delta^n_\Sigma[\Phi,\phi] \le T^*$. Now we get $\phi \in D_\Sigma(T'_\Sigma,\Phi) \subseteq D_\Sigma(T'_\Sigma) = T'_\Sigma$, which contradicts our assumption. Therefore, $\mathbb{L}$ is also full, as was to be shown. ■

Suppose, now, that $\mathcal{I}$ is a syntactically protoalgebraic, finitary π-institution with an isofully adequate Gentzen π-institution $\mathfrak{G} = \langle \mathbf{F}, G\rangle$. Then, for all $\langle \mathcal{F}, D\rangle \in \mathrm{Str}(\mathfrak{G})$, we must have$\langle \mathcal{F}, D\rangle \in \mathrm{FStr}(\mathcal{I})$ and, therefore, taking into account Lemma 1511, we get that $\mathcal{D} = \mathrm{ThFam}(\mathcal{I})^T$, where $T \in \mathrm{ThFam}^*(\mathcal{I})$. We denote by

$$h^{\mathfrak{G}} : \mathrm{Str}^{\mathfrak{G}}(\mathcal{F}) \to \mathrm{ThFam}^*(\mathcal{I})$$

the bijection that is established by this association, which is, in addiction an order isomorphism.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{G} = \langle \mathbf{F}, G\rangle$ a Gentzen π-institution based on $\mathbf{F}$. Given a theory family $\mathbf{\Gamma} \in \mathrm{ThFam}(\mathfrak{G})$, recall the $\mathbf{F}$-structure $\mathbb{L}^{\mathbf{\Gamma}} = \langle \mathcal{F}, D^{\mathbf{\Gamma}}\rangle$, which was shown in Lemma 1484 to be a $\mathfrak{G}$-structure. For notational purposes, given $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi_0,\dots,\phi_{n-1},\psi \in \mathrm{SEN}^\flat(\Sigma)$, let us write

$$\mathfrak{G}[\phi_0,\dots,\phi_{n-1} \vdash_\Sigma \psi] \coloneqq \mathbb{L}^{G(\phi_0,\dots,\phi_{n-1}\vdash_\Sigma\psi)},$$

where, as usual, $G(\phi_0,\dots,\phi_{n-1} \vdash_\Sigma \psi)$ denotes the least theory family of $\mathfrak{G}$ including the $\mathbf{F}$-sequent $\phi_0,\dots,\phi_{n-1} \vdash_\Sigma \psi$.

We call the Gentzen π-institution $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ **transformational** if, for all all $n < \omega$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi_0,\dots,\phi_{n-1},\psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$h^{\mathfrak{G}}(\mathfrak{G}[\phi_0,\dots,\phi_{n-1} \vdash_\Sigma \psi]) = C(\Delta^n_\Sigma[\phi_0,\dots,\phi_{n-1},\psi]),$$

for some $\Delta^n : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with $n+1$ distinguished arguments.

We can show that the isofully adequate Gentzen π-institution $\mathfrak{G}^{\mathcal{I}}$ associated with a syntactically protoalgebraic finitary π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ that has a Leibniz generating PGDD system $\Delta = \{\Delta^n : n < \omega\}$ over all Leibniz theory families, as in Theorem 1513, is, in fact, a transformational Gentzen π-institution.

Theorem 1514 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a syntactically protoalgebraic finitary π-institution based on $\mathbf{F}$. If $\mathcal{I}$ has a Leibniz generating PGDD system $\Delta = \{\Delta^n : n < \omega\}$ over all Leibniz theory families, then the isofully adequate Gentzen π-institution $\mathfrak{G}^{\mathcal{I}} = \langle \mathbf{F}, G^{\mathcal{I}}\rangle$ for $\mathcal{I}$ is transformational.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi_0,\dots,\phi_{n-1},\psi \in \mathrm{SEN}^\flat(\Sigma)$ and denote

$$T \coloneqq C(\Delta^n_\Sigma[\phi_0,\dots,\phi_{n-1},\psi]).$$

By hypothesis, we have $T \in \mathrm{ThFam}^*(\mathcal{I})$. Since T is a Leibniz $\mathcal{I}$-theory family and Δ is a PGDD system over all Leibniz theory families, T is closed under all axioms and rules of $\mathfrak{G}^{\mathcal{I}}$. Moreover, if $\phi_0,\dots,\phi_{n-1} \in T_\Sigma$, then, since $\Delta^n_\Sigma[\phi_0,\dots,\phi_{n-1},\psi] \le T$, we get, by Lemma 1509, $\psi \in T_\Sigma$. Hence, T is a theory family of the $\mathbf{F}$-structure $\mathfrak{G}^{\mathcal{I}}[\phi_0,\dots,\phi_{n-1},\psi]$. Since, by definition, $h^{\mathfrak{G}}(\mathfrak{G}[\phi_0,\dots,\phi_{n-1} \vdash_\Sigma \psi])$ is its least theory family, we get that $h^{\mathfrak{G}}(\mathfrak{G}[\phi_0,\dots,\phi_{n-1} \vdash_\Sigma \psi]) \le C(\Delta^n_\Sigma[\phi_0,\dots,\phi_{n-1},\psi])$.

On the other hand, by definition,

$$\phi_0,\dots,\phi_{n-1} \vdash_\Sigma \psi \in G^{\mathcal{I}}_\Sigma(\phi_0,\dots,\phi_{n-1} \vdash_\Sigma \psi),$$

whence, writing $\mathfrak{G}^{\mathcal{I}}[\phi_0,\dots,\phi_{n-1} \vdash_\Sigma \psi] \coloneqq \langle \mathcal{F}, D\rangle$, we get $\psi \in D_\Sigma(\phi_0,\dots,\phi_{n-1})$. Recalling that every full model is structural, we get, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$,

$$\mathrm{SEN}^\flat(f)(\psi) \in D_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi_0),\dots,\mathrm{SEN}^\flat(f)(\phi_{n-1})).$$

Thus, since $\mathfrak{G}^{\mathcal{I}}$ satisfies r^{Δ^n}, we get, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$,

$$\Delta_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi_0),\dots,\mathrm{SEN}^\flat(f)(\phi_{n-1}),\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}) \subseteq D_\Sigma(\varnothing),$$

i.e., that $\Delta^n_\Sigma[\phi_0,\dots,\phi_{n-1},\psi] \le h^{\mathfrak{G}}(\mathfrak{G}[\phi_0,\dots,\phi_{n-1} \vdash_\Sigma \psi])$. We now conclude that

$$C(\Delta^n_\Sigma[\phi_0,\dots,\phi_{n-1},\psi]) \le h^{\mathfrak{G}}(\mathfrak{G}[\phi_0,\dots,\phi_{n-1} \vdash_\Sigma \psi]).$$

Therefore, for all all $n < \omega$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi_0,\dots,\phi_{n-1},\psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$h^{\mathfrak{G}}(\mathfrak{G}[\phi_0,\dots,\phi_{n-1} \vdash_\Sigma \psi]) = C(\Delta^n_\Sigma[\phi_0,\dots,\phi_{n-1},\psi]),$$

showing that $\mathfrak{G}^{\mathcal{I}}$ is transformational. ∎

Finally, we show that, for a syntactically protoalgebraic, finitary π-institution $\mathcal{I}$, the existence of an isofully adequate, transformational Gentzen π-institution $\mathfrak{G}$ for $\mathcal{I}$ implies that $\mathcal{I}$ has a Leibniz generating PGDD system over every Leibniz theory family.

Theorem 1515 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a syntactically protoalgebraic, finitary π-institution based on $\mathbf{F}$. If $\mathcal{I}$ has an isofully adequate transformational Gentzen π-institution, then $\mathcal{I}$ has a Leibniz generating PGDD system over every Leibniz theory family.*

Proof: Suppose that $\mathcal{I}$ has an isofully adequate transformational Gentzen π-institution $\mathfrak{G} = \langle \mathbf{F}, G\rangle$. Thus, by definition, for all $n < \omega$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi_0,\dots,\phi_{n-1},\psi \in \mathrm{SEN}^\flat(\Sigma)$, there exists a collection $\Delta^n : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with $n+1$ distinguished arguments, such that

$$h^{\mathfrak{G}}(\mathfrak{G}[\phi_0,\dots,\phi_{n-1},\psi]) = C(\Delta^n_\Sigma[\phi_0,\dots,\phi_{n-1},\psi]).$$

By the fact that $h^{\mathfrak{G}}$ maps, by hypothesis and Lemma 1511, into $\mathrm{ThFam}^*(\mathcal{I})$, ensures that $\Delta = \{\Delta^n : n < \omega\}$ is Leibniz generating. So it suffices to show that Δ is a PGDD system for $\mathcal{I}$ over every Leibniz theory family. To this end, assume that $T \in \mathrm{ThFam}^*(\mathcal{I})$, $n < \omega$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi_0, \ldots, \phi_{n-1}, \psi \in \mathrm{SEN}^\flat(\Sigma)$. We must show that

$$\psi \in C_\Sigma(T_\Sigma, \phi_0, \ldots, \phi_{n-1}) \quad \text{iff} \quad \Delta^n_\Sigma[\phi_0, \ldots, \phi_{n-1}, \psi] \leq T.$$

We have

$$\begin{array}{lll}\psi \in C_\Sigma(T_\Sigma, \phi_0, \ldots, \phi_{n-1}) & \text{iff} & \psi \in C^T_\Sigma(\phi_0, \ldots, \phi_{n-1}) \\ & \text{iff} & \mathfrak{G}[\phi_0, \ldots, \phi_{n-1} \vdash_\Sigma \psi] \leq \mathcal{C}^T \\ & \text{iff} & h^{\mathfrak{G}}(\mathfrak{G}(\phi_0, \ldots, \phi_{n-1} \vdash_\Sigma \psi]) \leq T \\ & \text{iff} & C(\Delta^n_\Sigma[\phi_0, \ldots, \phi_{n-1}, \psi]) \leq T \\ & \text{iff} & \Delta^n_\Sigma[\phi_0, \ldots, \phi_{n-1}, \psi] \leq T.\end{array}$$

We conclude that Δ is indeed Leibniz generating PGDD system for $\mathcal{I}$ over every Leibniz theory family. ■

In conclusion, we have

Theorem 1516 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system. A syntactically protoalgebraic, finitary* π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *based on* $\mathbf{F}$ *has an isofully adequate transformational Gentzen* π*-institution if and only if it has a Leibniz generating PGDD system over every Leibniz theory family.*

Proof: The "if" was proven in Theorems 1513 and 1514. The "only if" is by Theorem 1515. ■

Chapter 21

Operators on π-Institutions

21.1 $\mathcal{I}$-Operators

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system.

An **$\mathcal{I}$-operator on $\mathcal{A}$** is a map

$$O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{I}) \to \mathrm{EqvFam}(\mathcal{A}),$$

where $\mathrm{EqvFam}(\mathcal{A})$ is the collection of equivalence families on $\mathcal{A}$.

Given an $\mathcal{I}$-operator $O^{\mathcal{A}}$ on $\mathcal{A}$, we define three derived operators (functions) as follows:

- The **lifting of** $O^{\mathcal{A}}$, $\widetilde{O}^{\mathcal{A}} : \mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \to \mathrm{EqvFam}(\mathcal{A})$, is given, for all $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, by
$$\widetilde{O}^{\mathcal{A}}(\mathcal{T}) = \bigcap \{O^{\mathcal{A}}(T) : T \in \mathcal{T}\};$$
- The **relativization of** $O^{\mathcal{A}}$ **to** $\mathcal{I}$, $\widetilde{O}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$, is given, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, by
$$\widetilde{O}^{\mathcal{I},\mathcal{A}}(T) = \bigcap \{O^{\mathcal{A}}(T') : T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} = \widetilde{O}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T);$$
- $O^{\mathcal{A}^{-1}} : \mathrm{EqvFam}(\mathcal{A}) \to \mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$ is given, for all $\theta \in \mathrm{EqvFam}(\mathcal{A})$, by
$$O^{\mathcal{A}^{-1}}(\theta) = \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \theta \leq O^{\mathcal{A}}(T)\}.$$

Note that the lifting of the Leibniz operator $\Omega^{\mathcal{A}}$ on $\mathcal{A}$ is the Tarski operator $\widetilde{\Omega}^{\mathcal{A}}$ on $\mathcal{A}$, whereas the relativization of the Leibniz operator on $\mathcal{A}$ is the Suszko operator $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ on $\mathcal{A}$.

Immediately from the definitions, we obtain the following:

Lemma 1517 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ an $\mathcal{I}$-operator on $\mathcal{A}$.*

(a) $\widetilde{O}^{\mathcal{I},\mathcal{A}}(T) \leq O^{\mathcal{A}}(T)$, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$;

(b) $\widetilde{O}^{\mathcal{A}}(\mathcal{T}) \leq O^{\mathcal{A}}(T)$, for all $T \in \mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

Proof: Obvious from the definitions. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, and, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, let there be given an $\mathcal{I}$-operator $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$. We write

$$O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$$

and refer to this family as a **family of $\mathcal{I}$-operators**.

Since $\mathcal{I}$-operators are meant to abstract the operators of abstract algebraic logic, those properties that were studied in preceding chapters concerning the Leibniz operator play also a significant role when it comes to $\mathcal{I}$-operators.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and

$$O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$$

an $\mathcal{I}$-operator on $\mathcal{A}$.

- $O^{\mathcal{A}}$ is **order-preserving** or **monotone** if, for all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,
$$T \leq T' \quad \text{implies} \quad O^{\mathcal{A}}(T) \leq O^{\mathcal{A}}(T');$$
- $O^{\mathcal{A}}$ is **order-reflecting** or **reflective** if, for all $T, T' \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A})$,
$$O^{\mathcal{A}}(T) \leq O^{\mathcal{A}}(T') \quad \text{implies} \quad T \leq T';$$
- $O^{\mathcal{A}}$ is **completely order reflecting** or **c-reflective** if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,
$$\bigcap_{T \in \mathcal{T}} O^{\mathcal{A}}(T) \leq O^{\mathcal{A}}(T') \quad \text{implies} \quad \bigcap \mathcal{T} \leq T'.$$

Some important characterizations are related to these properties.

Lemma 1518 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ an $\mathcal{I}$-operator on $\mathcal{A}$.*

$$O^{\mathcal{A}} \textit{ is monotone if and only if } O^{\mathcal{A}} = \widetilde{O}^{\mathcal{I},\mathcal{A}}.$$

Proof: Suppose, first, that $O^{\mathcal{A}}$ is monotone. Then, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$\begin{array}{rcl} O^{\mathcal{A}}(T) & = & \bigcap\{O^{\mathcal{A}}(T') : T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \\ & = & \widetilde{O}^{\mathcal{I},\mathcal{A}}(T). \end{array}$$

If, conversely, $O^{\mathcal{A}} = \widetilde{O}^{\mathcal{I},\mathcal{A}}$, then, for all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \leq T'$, we get

$$\begin{array}{rcl} O^{\mathcal{A}}(T) & = & \widetilde{O}^{\mathcal{I},\mathcal{A}}(T) \\ & = & \bigcap\{O^{\mathcal{A}}(T'') : T \leq T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \\ & \leq & \bigcap\{O^{\mathcal{A}}(T'') : T' \leq T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \\ & = & \widetilde{O}^{\mathcal{I},\mathcal{A}}(T') \\ & = & O^{\mathcal{A}}(T'). \end{array}$$

Therefore, $O^{\mathcal{A}}$ is monotone. ■

Lemma 1519 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ *an* $\mathcal{I}$*-operator on* $\mathcal{A}$*.* $O^{\mathcal{A}}$ *is c-reflective if and only if, for all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$\widetilde{O}^{\mathcal{I},\mathcal{A}}(T) \le O^{\mathcal{A}}(T') \quad \textit{implies} \quad T \le T'.$$

Proof: Assume, first, that $O^{\mathcal{A}}$ is c-reflective and let $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{O}^{\mathcal{I},\mathcal{A}}(T) \le O^{\mathcal{A}}(T')$. Then, by definition, $\bigcap\{O^{\mathcal{A}}(T'') : T \le T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \le O^{\mathcal{A}}(T')$. By c-reflectivity, $\bigcap\{T'' : T \le T'' \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A})\} \le T'$, i.e., $T \le T'$.

Suppose, conversely, that the displayed condition holds and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\bigcap_{T \in \mathcal{T}} O^{\mathcal{A}}(T) \le O^{\mathcal{A}}(T')$. Then we get

$$\widetilde{O}^{\mathcal{I},\mathcal{A}}(\bigcap \mathcal{T}) \le \bigcap_{T \in \mathcal{T}} O^{\mathcal{A}}(T) \le O^{\mathcal{A}}(T').$$

Hence, by the hypothesis, $\bigcap \mathcal{T} \le T'$ and $O^{\mathcal{A}}$ is c-reflective. ∎

We now show that the operators $\widetilde{O}^{\mathcal{A}}$ and $O^{\mathcal{A}^{-1}}$, associated with a given $\mathcal{I}$-operator $O^{\mathcal{A}}$, establish a Galois connection between the class $\mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$ of bundles of $\mathcal{I}$-filters on $\mathcal{A}$ and the class $\mathrm{EqvFam}(\mathcal{A})$ of equivalence families on $\mathcal{A}$. This will yield several important consequences.

Proposition 1520 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ *an* $\mathcal{I}$*-operator on* $\mathcal{A}$*. The maps*

$$\begin{array}{rcl} \widetilde{O}^{\mathcal{A}} : \ \mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) & \longrightarrow & \mathrm{EqvFam}(\mathcal{A}) \\ \mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) & \longleftarrow & \mathrm{EqvFam}(\mathcal{A}) \quad : O^{\mathcal{A}^{-1}} \end{array}$$

establish a Galois connection, where $\mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$ *is ordered under the subclass relation and* $\mathrm{EqvFam}(\mathcal{A})$ *under signature-wise inclusion.*

Proof: We must show that, for all $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $\theta \in \mathrm{EqvFam}(\mathcal{A})$,

$$\mathcal{T} \subseteq O^{\mathcal{A}^{-1}}(\theta) \quad \text{iff} \quad \widetilde{O}^{\mathcal{A}}(\mathcal{T}) \ge \theta.$$

In fact, we have

$$\begin{array}{rll} \mathcal{T} \subseteq O^{\mathcal{A}^{-1}}(\theta) & \text{iff} & \theta \le O^{\mathcal{A}}(T), \text{ for all } T \in \mathcal{T}, \\ & \text{iff} & \theta \le \bigcap\{O^{\mathcal{A}}(T) : T \in \mathcal{T}\} \\ & \text{iff} & \theta \le \widetilde{O}^{\mathcal{A}}(\mathcal{T}). \end{array}$$

Thus $(\widetilde{O}^{\mathcal{A}}, O^{\mathcal{A}^{-1}}) : \mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \rightleftarrows \mathrm{EqvFam}(\mathcal{A})$ is, in fact, a Galois connection. ∎

Corollary 1521 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ *an* $\mathcal{I}$*-operator on* $\mathcal{A}$*.*

(a) The operators

$$\widetilde{O}^{\mathcal{A}} : \mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \to \mathrm{EqvFam}(\mathcal{A})$$
$$O^{\mathcal{A}^{-1}} : \mathrm{EqvFam}(\mathcal{A}) \to \mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$$

are order reversing;

(b) The operators

$$O^{\mathcal{A}^{-1}} \circ \widetilde{O}^{\mathcal{A}} : \mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \to \mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$$
$$\widetilde{O}^{\mathcal{A}} \circ O^{\mathcal{A}^{-1}} : \mathrm{EqvFam}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$$

are closure operators;

(c) The collection of fixed-points of $O^{\mathcal{A}^{-1}} \circ \widetilde{O}^{\mathcal{A}}$ *is the range of* $O^{\mathcal{A}^{-1}}$ *and the collection of fixed-points of* $\widetilde{O}^{\mathcal{A}} \circ O^{\mathcal{A}^{-1}}$ *is the range of* $\widetilde{O}^{\mathcal{A}}$*;*

(d) $\widetilde{O}^{\mathcal{A}}$ *and* $O^{\mathcal{A}^{-1}}$ *restrict to mutually inverse order isomorphisms between the collections of fixed-points of* $O^{\mathcal{A}^{-1}} \circ \widetilde{O}^{\mathcal{A}}$ *and of fixed-points of* $\widetilde{O}^{\mathcal{A}} \circ O^{\mathcal{A}^{-1}}$*.*

Proof: Known facts about Galois connections. ■

We capture the elements described in Part (c) of Corollary 1521, by making the following definitions.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and

$$O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$$

an $\mathcal{I}$-operator on $\mathcal{A}$.

- A family $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is called $O^{\mathcal{A}}$-**full** if $\mathcal{T} = O^{\mathcal{A}^{-1}}(\widetilde{O}^{\mathcal{A}}(\mathcal{T}))$ if and only if $\mathcal{T} \in \mathrm{Ran}(O^{\mathcal{A}^{-1}})$;
- An equivalence family $\theta \in \mathrm{EqvFam}(\mathcal{A})$ is $O^{\mathcal{A}}$-**full** if $\theta = \widetilde{O}^{\mathcal{A}}(O^{\mathcal{A}^{-1}}(\theta))$ if and only if $\theta \in \mathrm{Ran}(\widetilde{O}^{\mathcal{A}})$.

The following statements provide a justification of the terminology used.

Proposition 1522 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ *an* $\mathcal{I}$*-operator on* $\mathcal{A}$*.*

(a) *A collection* $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *is* $O^{\mathcal{A}}$*-full if and only if it is the largest* $\mathcal{D} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, such that* $\widetilde{O}^{\mathcal{A}}(\mathcal{D}) = \widetilde{O}^{\mathcal{A}}(\mathcal{T})$*;*

(b) *An equivalence family* $\theta \in \mathrm{EqvFam}(\mathcal{A})$ *is* $O^{\mathcal{A}}$*-full if and only if it is the largest* $\eta \in \mathrm{EqvFam}(\mathcal{A})$*, such that* $O^{\mathcal{A}^{-1}}(\eta) = O^{\mathcal{A}^{-1}}(\theta)$*.*

Proof: We do Part (a). Part (b) can be proved analogously. Suppose, first, that $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is $O^{\mathcal{A}}$-full and let $\mathcal{D} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{O}^{\mathcal{A}}(\mathcal{D}) = \widetilde{O}^{\mathcal{A}}(\mathcal{T})$. Then, we have

$$\mathcal{D} \subseteq O^{\mathcal{A}^{-1}}(\widetilde{O}^{\mathcal{A}}(\mathcal{D})) = O^{\mathcal{A}^{-1}}(\widetilde{O}^{\mathcal{A}}(\mathcal{T})) = \mathcal{T}.$$

Suppose, conversely, that $\mathcal{T}$ is the largest among $\mathcal{D} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{O}^{\mathcal{A}}(\mathcal{D}) = \widetilde{O}^{\mathcal{A}}(\mathcal{T})$ and let $T \in O^{\mathcal{A}^{-1}}(\widetilde{O}^{\mathcal{A}}(\mathcal{T}))$. Then, by definition, $\widetilde{O}^{\mathcal{A}}(\mathcal{T}) \le O^{\mathcal{A}}(T)$. Hence, $\widetilde{O}^{\mathcal{A}}(\mathcal{T} \cup \{T\}) = \widetilde{O}^{\mathcal{A}}(\mathcal{T})$. By the maximality of $\mathcal{T}$, we conclude that $T \in \mathcal{T}$. This shows that $O^{\mathcal{A}^{-1}}(\widetilde{O}^{\mathcal{A}}(\mathcal{T})) \subseteq \mathcal{T}$. Since the opposite inclusion always holds, we conclude that $\mathcal{T}$ is a fixed point of $O^{\mathcal{A}^{-1}} \circ \widetilde{O}^{\mathcal{A}}$ and, hence, it is $O^{\mathcal{A}}$-full. ■

We have the following consequences:

Corollary 1523 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ *an* $\mathcal{I}$*-operator on* $\mathcal{A}$*.*

(a) $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *is* $O^{\mathcal{A}}$*-full;*

(b) $\nabla^{\mathcal{A}}$ *is* $O^{\mathcal{A}}$*-full;*

(c) *If* $O^{\mathcal{A}}$ *is monotone and* $\mathcal{T}$ *is* $O^{\mathcal{A}}$*-full, then* $\mathcal{T}$ *is an upset in the poset* $\mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$*.*

Proof: All three statements are direct consequences of Proposition 1522. ■

21.2 Congruential $\mathcal{I}$-Operators

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system. An $\mathcal{I}$-operator $O^{\mathcal{A}}$: $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ is called **congruential** if, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $O^{\mathcal{A}}(T) \in \mathrm{ConSys}(\mathcal{A})$. Thus a congruential $\mathcal{I}$-operator is an operator $O^{\mathcal{A}}$: $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}(\mathcal{A})$.

Proposition 1524 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system,* $\theta \in \mathrm{ConSys}(\mathcal{A})$ *and* $\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\theta$ *the quotient natural transformation.*

(a) $\Omega^{\mathcal{A}^{-1}}(\theta) = \pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta))$ *and* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta) = \pi(\Omega^{\mathcal{A}^{-1}}(\theta))$*;*

(b) The mappings

$$\begin{array}{l}\pi : \mathrm{SenFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{SenFam}(\mathcal{A}/\theta)\\ \pi^{-1} : \mathrm{SenFam}(\mathcal{A}/\theta) \to \mathrm{SenFam}(\mathcal{A})\end{array}$$

restrict to mutually inverse order isomorphisms between $\Omega^{\mathcal{A}^{-1}}(\theta)$ *and* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)$*.*

Proof:

(a) Suppose $T \in \Omega^{\mathcal{A}^{-1}}(\theta)$. Then $\theta \le \Omega^{\mathcal{A}}(T)$. Hence θ is compatible with T, which implies, by Corollary 57, $\pi(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)$. Suppose, conversely, that $T \in \pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta))$. Then $\pi(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)$ and, hence, by Corollary 57, $\pi^{-1}(\pi(T)) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Therefore, T is compatible with θ, showing that $\theta \le \Omega^{\mathcal{A}}(T)$. This gives $T \in \Omega^{\mathcal{A}^{-1}}(\theta)$.

The second equality of Part (a) is obtained from the first, using the surjectivity of $\langle I, \pi\rangle$.

(b) By Part (a), the mappings

$$\begin{array}{l}\pi \restriction_{\Omega^{\mathcal{A}^{-1}}(\theta)}: \Omega^{\mathcal{A}^{-1}}(\theta) \to \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\\ \pi^{-1} \restriction_{\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})}: \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \Omega^{\mathcal{A}^{-1}}(\theta)\end{array}$$

are well-defined. Moreover, they are clearly inverses of one another and order preserving. Thus, they establish an order isomorphism between $\Omega^{\mathcal{A}^{-1}}(\theta)$ and $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)$.

■

21.3 O-Classes and O-Filter Families

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ an $\mathbf{F}$-algebraic system and

$$O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$$

an $\mathcal{I}$-operator on $\mathcal{A}$. For all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, the O-**class of** T, denoted $[\![T]\!]^O$, is the collection

$$[\![T]\!]^O = \Omega^{\mathcal{A}^{-1}}(O^{\mathcal{A}}(T)) = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : O^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')\}.$$

It turns out that this class forms a closure family on $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

Proposition 1525 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ *an* $\mathcal{I}$*-operator on* $\mathcal{A}$*. For all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $[\![T]\!]^O$ *is a closure family on* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

Proof: First, observe that $\Omega^{\mathcal{A}}(\mathrm{SEN}) = \nabla^{\mathcal{A}}$, whence $\mathrm{SEN} \in [\![T]\!]^O$. Next, let $\{T^i : i \in I\} \subseteq [\![T]\!]^O$. Then, we have

$$O^{\mathcal{A}}(T) \leq \bigcap_{i\in I} \Omega^{\mathcal{A}}(T^i) \leq \Omega^{\mathcal{A}}(\bigcap_{i\in I} T^i).$$

So $\bigcap_{i\in I} T^i \in [\![T]\!]^O$ and $[\![T]\!]^O$ is a closure family on $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. ■

Something even stronger is true in case $O^{\mathcal{A}}$ happens to be a congruential $\mathcal{I}$-operator. In that case, the pair $\langle \mathcal{A}, [\![T]\!]^O\rangle$ turns out to be a full $\mathcal{I}$-structure.

Proposition 1526 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}(\mathcal{A})$ *a congruential* $\mathcal{I}$*-operator on* $\mathcal{A}$*. For all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $\langle \mathcal{A}, [\![T]\!]^O\rangle$ *is a full* $\mathcal{I}$*-structure.*

Proof: Let $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then $[\![T]\!]^O = \Omega^{\mathcal{A}^{-1}}(O^{\mathcal{A}}(T))$. By hypothesis, $O^{\mathcal{A}}(T) \in \mathrm{ConSys}(\mathcal{A})$. Thus, by Proposition 1524,

$$[\![T]\!]^O = \Omega^{\mathcal{A}^{-1}}(O^{\mathcal{A}}(T)) = \pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T))),$$

where $\langle I, \pi\rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T)$ is the quotient natural transformation. Thus, by definition, $\langle \mathcal{A}, [\![T]\!]^O\rangle$ is a full $\mathcal{I}$-structure. ■

As a corollary, we obtain the fact that $[\![T]\!]^O$ is a closure system on $\mathcal{A}$ and, therefore, $\langle \mathcal{A}, [\![T]\!]^O\rangle$ is a π-institution and not merely a π-structure.

Corollary 1527 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}(\mathcal{A})$ *a congruential* $\mathcal{I}$*-operator on* $\mathcal{A}$*. For all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $[\![T]\!]^O$ *is a closure system on* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

Proof: By Propositions 1526 and 1389. ■

Corollary 1527 justifies the following definition.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ an $\mathbf{F}$-algebraic system and

$$O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}(\mathcal{A})$$

a congruential $\mathcal{I}$-operator on $\mathcal{A}$. For all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, the least element of the O-class of T is denoted by T^O:

$$T^O = \bigcap [\![T]\!]^O.$$

A $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is called an O-**filter family** if $T = T^O$. Note that, by Corollary 1527, an O-filter family must be an $\mathcal{I}$-filter system.

The collection of all O-filter systems of $\mathcal{A}$ is denoted by $\mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A})$.

Proposition 1528 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ *an* $\mathcal{I}$*-operator on* $\mathcal{A}$*.* $O^{\mathcal{A}}$ *is reflective (and, hence, injective) on* $\mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A})$*.*

Proof: Let $T, T' \in \mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A})$, such that $O^{\mathcal{A}}(T) \leq O^{\mathcal{A}}(T')$. Then, $[\![T']\!]^O \subseteq [\![T]\!]^O$. Therefore,

$$\begin{array}{rcl} T & = & T^O \quad (T \in \mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A})) \\ & = & \bigcap [\![T]\!]^O \quad \text{(definition)} \\ & \leq & \bigcap [\![T']\!]^O \quad ([\![T']\!]^O \subseteq [\![T]\!]^O) \\ & = & T'^O \quad \text{(definition)} \\ & = & T'. \quad (T' \in \mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A})) \end{array}$$

We conclude that $O^{\mathcal{A}}$ is reflective. ■

Proposition 1529 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ *a monotone* $\mathcal{I}$*-operator on* $\mathcal{A}$*. Then the mapping* $T \mapsto T^O$ *is monotone, i.e., for all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$T \leq T' \quad \textit{implies} \quad T^O \leq T'^O.$$

Proof: We have, for all $T, T' \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A})$,

$$\begin{array}{rll} T \leq T' & \text{implies} & O^{\mathcal{A}}(T) \leq O^{\mathcal{A}}(T') \quad \text{(hypothesis)} \\ & \text{implies} & [\![T']\!]^O \subseteq [\![T]\!]^O \quad (\text{definitions of } [\![T]\!]^O, [\![T']\!]^O) \\ & \text{implies} & T^O \leq T'^O. \quad (\text{definitions of } T^O, T'^O) \end{array}$$

So $T \mapsto T^O$ is a monotone mapping. ■

Proposition 1530 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system,* $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}(\mathcal{A})$ *a congruential* $\mathcal{I}$*-operator on* $\mathcal{A}$ *and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.* $T \in \mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A})$ *if and only if* $T/O^{\mathcal{A}}(T)$ *is the least* $\mathcal{I}$*-filter family of* $\mathcal{A}/O^{\mathcal{A}}(T)$*.*

Proof: By hypothesis, $O^{\mathcal{A}}(T) \in \mathrm{ConSys}(\mathcal{A})$. Consider the quotient natural transformation

$$\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/O^{\mathcal{A}}(T).$$

Since $\Omega^{\mathcal{A}^{-1}}(O^{\mathcal{A}}(T)) = [\![T]\!]^O$, we get, by Proposition 1524, that

$$\pi : [\![T]\!]^O \to \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/O^{\mathcal{A}}(T))$$

is an order isomorphism. Thus, $T^O/O^{\mathcal{A}}(T)$ is the least $\mathcal{I}$-filter family on $\mathcal{A}/O^{\mathcal{A}}(T)$. ■

21.4 Compatibility $\mathcal{I}$-Operators

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ an $\mathcal{I}$-operator on $\mathcal{A}$. $O^{\mathcal{A}}$ is called a **compatibility $\mathcal{I}$-operator** if, for all $T \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A})$,

$$O^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T).$$

Clearly, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}(\mathcal{A})$ is the largest compatibility $\mathcal{I}$-operator on $\mathcal{A}$. If one assumes monotonicity, then this role is played by the Suszko operator instead:

Lemma 1531 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. For every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, the Suszko operator $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}(\mathcal{A})$ is the largest monotone compatibility $\mathcal{I}$-operator on $\mathcal{A}$.*

Proof: Suppose that $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ is a monotone compatibility $\mathcal{I}$-operator on $\mathcal{A}$. Then, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$\begin{array}{rcl} O^{\mathcal{A}}(T) & = & \widetilde{O}^{\mathcal{I},\mathcal{A}}(T) \quad \text{(by Lemma 1518)} \\ & = & \bigcap\{O^{\mathcal{A}}(T') : T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \quad \text{(by Definition)} \\ & \leq & \bigcap\{\Omega^{\mathcal{A}}(T') : T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \quad \text{(by Compatibility)} \\ & = & \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T). \quad \text{(by Definition)} \end{array}$$

So $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ is the largest monotone compatibility $\mathcal{I}$-operator on $\mathcal{A}$. ■

For compatibility $\mathcal{I}$-operators, we have the following properties pertaining to O-classes and O-filter systems.

Lemma 1532 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ a compatibility $\mathcal{I}$-operator on $\mathcal{A}$. Then, for every $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,*

$$T \in [\![T]\!]^O \quad \textit{and} \quad T^O \leq T.$$

Proof: Let $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Since $O^{\mathcal{A}}$ is a compatibility $\mathcal{I}$-operator, $O^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T)$. Thus, by definition of $[\![T]\!]^O$, we get $T \in [\![T]\!]^O$. Moreover, since $T \in [\![T]\!]^O$, we now get $T^O = \bigcap [\![T]\!]^O \leq T$. ■

For monotone compatibility $\mathcal{I}$-operators, we have the following properties.

Lemma 1533 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ a monotone compatibility $\mathcal{I}$-operator on $\mathcal{A}$. Then, for every $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$:*

(a) $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \subseteq [\![T]\!]^O$*;*

(b) $[\![T]\!]^O = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$ *iff* $T = T^O$ *iff* $T \in \mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A})$.

Proof:

(a) Suppose $T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then

$$\begin{array}{rcll} O^{\mathcal{A}}(T) & \leq & O^{\mathcal{A}}(T') & \text{(by Monotonicity)} \\ & \leq & \Omega^{\mathcal{A}}(T'). & \text{(by Compatibility)} \end{array}$$

So, by definition of $[\![T]\!]^O$, $T' \in [\![T]\!]^O$.

(b) The second equivalence if simply the definition of $\mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A})$. So it suffices to prove the first equivalence. Assume, first, that $[\![T]\!]^O = \mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A})$. Then, we have $T^O = \bigcap [\![T]\!]^O = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T = T$.

Assume, conversely, that $T = T^O$. Then, if $T' \in [\![T]\!]^O$, we get $T = T^O = \bigcap [\![T]\!]^O \leq T'$. Thus, $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$. Since, by Part (a), the converse always holds, we get $[\![T]\!]^O = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$.

■

In the case of compatibility $\mathcal{I}$-operators, there are also close relationships between their classes and their filter families and those associated to the Leibniz operator. More precisely, we get:

Lemma 1534 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ *a compatibility* $\mathcal{I}$*-operator on* $\mathcal{A}$*. Then, for every* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*:*

(a) $[\![T]\!]^\Omega \subseteq [\![T]\!]^O$*;*

(b) $T^O \leq T^\Omega$*;*

(c) $\mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I},\Omega}(\mathcal{A})$*.*

Proof:

(a) Let $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then we have

$$\begin{array}{rlll} T' \in [\![T]\!]^\Omega & \text{implies} & \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T') & \text{(by Definition of } [\![T]\!]^\Omega) \\ & \text{implies} & O^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T') & \text{(by Compatibility)} \\ & \text{implies} & T' \in [\![T]\!]^O. & \text{(by Definition of } [\![T]\!]^O) \end{array}$$

Thus, $[\![T]\!]^\Omega \subseteq [\![T]\!]^O$.

(b) Using Part (a), we get $T^O = \bigcap [\![T]\!]^O \leq \bigcap [\![T]\!]^\Omega = T^\Omega$.

(c) Assume $T' \in \mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A})$. Then, by definition, $T'^{O} = T'$. Thus, by Part (b), $T' \le T'^{\Omega}$. Since, by Lemma 1532, $T'^{\Omega} \le T'$, we get $T'^{\Omega} = T'$ and, therefore, $T' \in \mathrm{FiFam}^{\mathcal{I},\Omega}(\mathcal{A})$. We conclude that $\mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I},\Omega}(\mathcal{A})$.

■

For monotone compatibility $\mathcal{I}$-operators, we have similar relationships between their classes and their filter families and those associated to the Suszko operator.

Lemma 1535 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ a monotone compatibility $\mathcal{I}$-operator on $\mathcal{A}$. Then, for every $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$:*

(a) $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{I}}} \le [\![T]\!]^{O}$;

(b) $T^{O} \le T^{\widetilde{\Omega}^{\mathcal{I}}}$;

(c) $\mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I},\widetilde{\Omega}^{\mathcal{I}}}(\mathcal{A})$.

Proof:

(a) Let $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then we have

$$\begin{array}{lll} T' \in [\![T]\!]^{\widetilde{\Omega}^{\mathcal{I}}} & \text{implies} & \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T') \quad (\text{by Definition of } [\![T]\!]^{\widetilde{\Omega}^{\mathcal{I}}}) \\ & \text{implies} & O^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T') \quad (\text{by Lemma 1531}) \\ & \text{implies} & T' \in [\![T]\!]^{O}. \quad (\text{by Definition of } [\![T]\!]^{O}) \end{array}$$

Thus, $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{I}}} \subseteq [\![T]\!]^{O}$.

(b) Using Part (a), we get $T^{O} = \bigcap [\![T]\!]^{O} \le \bigcap [\![T]\!]^{\widetilde{\Omega}^{\mathcal{I}}} = T^{\widetilde{\Omega}^{\mathcal{I}}}$.

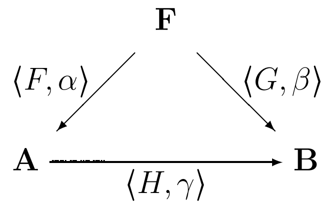
(c) Assume $T' \in \mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A})$. Then, by definition, $T'^{O} = T'$. Thus, by Part (b), $T' \le T'^{\widetilde{\Omega}^{\mathcal{I}}}$. Since, by Lemma 1532, $T'^{\widetilde{\Omega}^{\mathcal{I}}} \le T'$, we get $T'^{\widetilde{\Omega}^{\mathcal{I}}} = T'$ and, therefore, $T' \in \mathrm{FiFam}^{\mathcal{I},\widetilde{\Omega}^{\mathcal{I}}}(\mathcal{A})$. We conclude that $\mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I},\widetilde{\Omega}^{\mathcal{I}}}(\mathcal{A})$.

■

21.5 Commuting $\mathcal{I}$-Operators

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ $\mathbf{F}$-algebraic systems and $\langle H, \gamma \rangle$:

$\mathcal{A} \to \mathcal{B}$ a surjective morphism.



Let, also $O^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ and $O^{\mathcal{B}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}) \to \mathrm{EqvFam}(\mathcal{B})$ be $\mathcal{I}$-operators on $\mathcal{A}$ and on $\mathcal{B}$, respectively. We say that the pair $(O^{\mathcal{A}}, O^{\mathcal{B}})$ is **commuting** if, for all $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$,

$$O^{\mathcal{A}}(\gamma^{-1}(T')) = \gamma^{-1}(O^{\mathcal{B}}(T')).$$

More generally, let $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ be a family of $\mathcal{I}$-operators. We say that O is a **commuting family** if, for every pair $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ of $\mathbf{F}$-algebraic systems, and every surjective morphism $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$, the pair $(O^{\mathcal{A}}, O^{\mathcal{B}})$ is commuting.

A slightly more relaxed version, which will be of use to us later, is that of semi-commutation. We say that a family of $\mathcal{I}$-operators $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ is a **semi-commuting family** if, for every pair $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ of $\mathbf{F}$-algebraic systems, and every surjective morphism $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$, with H an isomorphism, the pair $(O^{\mathcal{A}}, O^{\mathcal{B}})$ is commuting.

It turns out that semi-commutation is too restrictive when applied to compatibility $\mathcal{I}$-operators, since there is only one semi-commuting family of compatibility $\mathcal{I}$-operators, namely, the Leibniz operator.

Theorem 1536 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, and* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-commuting family of compatibility* $\mathcal{I}$*-operators. Then* $O = \Omega$*.*

Proof: Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Consider the quotient morphism

$$\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T).$$

We get, using Compatibility,

$$O^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(T/\Omega^{\mathcal{A}}(T)) \le \Omega^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(T/\Omega^{\mathcal{A}}(T)) = \Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}.$$

So, we get $O^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(T/\Omega^{\mathcal{A}}(T)) = \Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}$. Since, by hypothesis, O is a semi-commuting family, we now get

$$\begin{array}{rcl} O^{\mathcal{A}}(T) & = & O^{\mathcal{A}}(\pi^{-1}(T/\Omega^{\mathcal{A}}(T))) \\ & = & \pi^{-1}(O^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(T/\Omega^{\mathcal{A}}(T))) \\ & = & \pi^{-1}(\Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}) \\ & = & \Omega^{\mathcal{A}}(T). \end{array}$$

We conclude that $O = \Omega$. ■

In particular, we have

Corollary 1537 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The Suszko operator* $\widetilde{\Omega}^{\mathcal{I}}$ *is semi-commuting if and only if* $\widetilde{\Omega}^{\mathcal{I}} = \Omega$*.*

Proof: If $\widetilde{\Omega}^{\mathcal{I}} = \Omega$, then, by Proposition 24, $\widetilde{\Omega}^{\mathcal{I}}$ is commuting and, hence,semi-commuting. If conversely, $\widetilde{\Omega}^{\mathcal{I}}$ is semi-commuting, then, by Theorem 1536, $\widetilde{\Omega}^{\mathcal{I}} = \Omega$. ∎

21.6 Coherent $\mathcal{I}$-Operators

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ a family of $\mathcal{I}$-operators. Moreover, let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ be $\mathbf{F}$-algebraic systems, $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

- The morphism $\langle H, \gamma \rangle$ is said to be O-**compatible with** T if
$$\mathrm{Ker}(\langle H, \gamma \rangle) \leq O^{\mathcal{A}}(T);$$
- The morphism $\langle H, \gamma \rangle$ is said to be O-**compatible with** $\mathcal{T}$ if
$$\mathrm{Ker}(\langle H, \gamma \rangle) \leq O^{\mathcal{A}}(T), \text{ for all } T \in \mathcal{T},$$
i.e., if and only if
$$\mathrm{Ket}(\langle H, \gamma \rangle) \leq \widetilde{O}^{\mathcal{A}}(\mathcal{T}).$$

For the Leibniz operator, we have

Corollary 1538 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A}, \mathcal{B}$ $\mathbf{F}$*-algebraic systems and* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism.* $\langle H, \gamma \rangle$ *is* Ω*-compatible with* T *if and only if* $\mathrm{Ker}(\langle H, \gamma \rangle)$ *is compatible with* T*.*

Proof: We have $\langle H, \gamma \rangle$ is Ω-compatible with T if and only if, by definition, $\mathrm{Ker}(\langle H, \gamma \rangle) \leq \Omega^{\mathcal{A}}(T)$ if and only if, by the compatibility of $\Omega^{\mathcal{A}}(T)$ with T, $\mathrm{Ker}(\langle H, \gamma \rangle)$ is compatible with T. ∎

Moreover, for a family O of compatibility $\mathcal{I}$-operators, we get

Corollary 1539 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a family of compatibility* $\mathcal{I}$*-operators,* $\mathcal{A}, \mathcal{B}$ $\mathbf{F}$*-algebraic systems,* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. If* $\langle H, \gamma \rangle$ *is* O*-compatible with* T*, then:*

(a) $\langle H, \gamma \rangle$ *is* Ω*-compatible with* T*;*

(b) *If H is an isomorphism, $T = \gamma^{-1}(\gamma(T))$;*

(c) *If H is an isomorphism, $O^{\mathcal{A}}(T) = \gamma^{-1}(\gamma(O^{\mathcal{A}}(T))$.*

Proof:

(a) We have

$$\begin{array}{rcll}\mathrm{Ker}(\langle H,\gamma\rangle) & \leq & O^{\mathcal{A}}(T) & \text{(hypothesis)}\\ & \leq & \Omega^{\mathcal{A}}(T). & \text{(by Compatibility)}\end{array}$$

Thus, $\langle H,\gamma\rangle$ is Ω-compatible with T.

(b) Suppose H is an isomorphism. By Part (a) and Corollary 1539, we get $\mathrm{Ker}(\langle H,\gamma\rangle)$ is compatible with T. Therefore, $\gamma^{-1}(\gamma(T)) \leq T$. Since the reverse inclusion is always satisfied, we get $T = \gamma^{-1}(\gamma(T))$.

(c) Again the inclusion $O^{\mathcal{A}}(T) \leq \gamma^{-1}(\gamma(O^{\mathcal{A}}(T)))$ is always satisfied. So it suffices to show the reverse inclusion. So assume $\Sigma \in |\mathbf{Sign}|$, $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle \in \gamma_\Sigma^{-1}(\gamma_\Sigma(O_\Sigma^{\mathcal{A}}(T)))$. Thus, by definition, $\langle\gamma_\Sigma(\phi),\gamma_\Sigma(\psi)\rangle \in \gamma_\Sigma(O_\Sigma^{\mathcal{A}}(T))$. Therefore, there exist $\phi',\psi' \in \mathrm{SEN}(\Sigma)$, with $\langle\phi',\psi'\rangle \in O_\Sigma^{\mathcal{A}}(T)$, such that $\langle\gamma_\Sigma(\phi),\gamma_\Sigma(\psi)\rangle = \langle\gamma_\Sigma(\phi'),\gamma_\Sigma(\psi')\rangle$. This shows that

$$\langle\phi,\phi'\rangle,\langle\psi,\psi'\rangle \in \mathrm{Ker}_\Sigma(\langle H,\gamma\rangle) \leq O_\Sigma^{\mathcal{A}}(T).$$

Since $\langle\phi',\psi'\rangle \in O_\Sigma^{\mathcal{A}}(T)$ and $O^{\mathcal{A}}(T)$ is an equivalence family, we get, using symmetry and transitivity, that $\langle\phi,\psi\rangle \in O_\Sigma^{\mathcal{A}}(T)$. We conclude that $\gamma^{-1}(\gamma(O^{\mathcal{A}}(T))) \leq O^{\mathcal{A}}(T)$.

■

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$ and $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ a family of $\mathcal{I}$-operators.

- O is called **coherent** if, for all $\mathbf{F}$-algebraic systems $\mathcal{A}$, $\mathcal{B}$, every surjective morphism $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$ and all $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$,

$$\langle H,\gamma\rangle\ O\text{-compatible with } \gamma^{-1}(T') \text{ implies } O^{\mathcal{A}}(\gamma^{-1}(T')) = \gamma^{-1}(O^{\mathcal{B}}(T')).$$

- O is called **semi-coherent** if, for all $\mathbf{F}$-algebraic systems $\mathcal{A}$, $\mathcal{B}$, every surjective morphism $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$, with H an isomorphism, and all $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$,

$$\langle H,\gamma\rangle\ O\text{-compatible with } \gamma^{-1}(T') \text{ implies } O^{\mathcal{A}}(\gamma^{-1}(T')) = \gamma^{-1}(O^{\mathcal{B}}(T')).$$

Clearly, if O is coherent, then it is also semi-coherent.

We define the **identity** $\mathcal{I}$-operator

$$I = \{I^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\},$$

by letting, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, $I^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$, be given, for all $T \in \mathrm{Fifam}^{\mathcal{I}}(\mathcal{A})$, by

$$I^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}.$$

Lemma 1540 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The identity* I *is a coherent family of compatibility* $\mathcal{I}$*-operators.*

Proof: It is clear that $I^{\mathcal{A}}$ is a compatibility $\mathcal{I}$-operator, for every $\mathbf{F}$-algebraic system $\mathcal{A}$. So it suffices to prove coherence. To this end, let $\mathcal{A}$, $\mathcal{B}$ be $\mathbf{F}$-algebraic systems, $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism and $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, such that $\mathrm{Ker}(\langle H, \gamma \rangle) \leq I^{\mathcal{A}}(\gamma^{-1}(T')) = \Delta^{\mathcal{A}}$. Thus, we have $\mathrm{Ker}(\langle H, \gamma \rangle) = \Delta^{\mathcal{A}}$. Now we get

$$\gamma^{-1}(I^{\mathcal{B}}(T')) = \gamma^{-1}(\Delta^{\mathcal{B}}) = \mathrm{Ker}(\langle H, \gamma \rangle) = \Delta^{\mathcal{A}} = I^{\mathcal{A}}(\gamma^{-1}(T)).$$

Thus, I is a coherent family of compatibility $\mathcal{I}$-operators. ∎

Another example of a coherent $\mathcal{I}$-operator is the Leibniz operator.

Lemma 1541 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The Leibniz operator* Ω *is a coherent family of compatibility operators.*

Proof: By definition Ω is a family of compatibility $\mathcal{I}$-operators. For coherence, assume that $\mathcal{A}$, $\mathcal{B}$ are $\mathbf{F}$-algebraic systems, $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism and $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. Since, by Proposition 24, $\Omega^{\mathcal{A}}(\gamma^{-1}(T')) = \gamma^{-1}(\Omega^{\mathcal{B}}(T'))$, we get that the coherence implication is trivially satisfied and, hence Ω is a coherent family of compatibility $\mathcal{I}$-operators. ∎

For semi-coherence of compatibility $\mathcal{I}$-operators, we get the following characterization.

Lemma 1542 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a family of compatibility* $\mathcal{I}$*-operators.* O *is semi-coherent if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A}$*,* $\mathcal{B}$*, all surjective morphisms* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$*, with* H *an isomorphism, and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, if* $\langle H, \gamma \rangle$ *is* O*-compatible with* T*, then* $\gamma(O^{\mathcal{A}}(T)) = O^{\mathcal{B}}(\gamma(T))$*.*

Proof: Suppose, first, that O is semi-coherent and let $\mathcal{A}$, $\mathcal{B}$ be $\mathbf{F}$-algebraic systems, $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism, with H an isomorphism and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that

$$\mathrm{Ker}(\langle H,\gamma\rangle) \le O^{\mathcal{A}}(T).$$

Then, by Corollary 1539, $\mathrm{Ker}(\langle H,\gamma\rangle) \le O^{\mathcal{A}}(\gamma^{-1}(\gamma(T)))$. Applying semi-coherence gives

$$\begin{array}{rcl}\gamma^{-1}(O^{\mathcal{B}}(\gamma(T))) & = & O^{\mathcal{A}}(\gamma^{-1}(\gamma(T))) \quad \text{(by semi-coherence)}\\ & = & O^{\mathcal{A}}(T). \quad \text{(by Corollary 1539)}\end{array}$$

By the surjectivity of $\langle H,\gamma\rangle$, $O^{\mathcal{B}}(\gamma(T)) = \gamma(O^{\mathcal{A}}(T))$.

Assume, conversely, that the condition in the statement holds. Let $\mathcal{A}$, $\mathcal{B}$ be $\mathbf{F}$-algebraic systems, $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism, with H an isomorphism, and $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, such that

$$\mathrm{Ker}(\langle H,\gamma\rangle) \le O^{\mathcal{A}}(\gamma^{-1}(T')).$$

Then, we get

$$\begin{array}{rcl}O^{\mathcal{A}}(\gamma^{-1}(T')) & = & \gamma^{-1}(\gamma(O^{\mathcal{A}}(\gamma^{-1}(T')))) \quad \text{(by Corollary 1539)}\\ & = & \gamma^{-1}(O^{\mathcal{B}}(\gamma(\gamma^{-1}(T'))) \quad \text{(by hypothesis)}\\ & = & \gamma^{-1}(O^{\mathcal{B}}(T')). \quad \text{(by surjectivity of } \langle H,\gamma\rangle)\end{array}$$

So O is a semi-coherent family of compatibility $\mathcal{I}$-operators. ∎

We also have the following alternative characterization for semi-coherence of compatibility $\mathcal{I}$-operators.

Lemma 1543 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$ and $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ a family of compatibility $\mathcal{I}$-operators. O is semi-coherent if and only if, for all $\mathbf{F}$-algebraic systems $\mathcal{A}$, $\mathcal{B}$ and all surjective morphisms $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$, with H an isomorphism,*

$$O^{\mathcal{A}^{-1}}(\mathrm{Ker}(\langle H,\gamma\rangle)) = \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \gamma^{-1}(O^{\mathcal{B}}(\gamma(T))) = O^{\mathcal{A}}(T)\}.$$

Proof: Suppose O is semi-coherent and let $\mathcal{A}$, $\mathcal{B}$ be $\mathbf{F}$-algebraic systems, $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism, with H an isomorphism, and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

- If $T \in O^{\mathcal{A}^{-1}}(\mathrm{Ker}(\langle H,\gamma\rangle))$, then, by definition, $\mathrm{Ker}(\langle H,\gamma\rangle) \le O^{\mathcal{A}}(T)$. Thus, by Lemma 1542, $\gamma(O^{\mathcal{A}}(T)) = O^{\mathcal{B}}(\gamma(T))$. Hence,

$$\gamma^{-1}(\gamma(O^{\mathcal{A}}(T))) = \gamma^{-1}(O^{\mathcal{B}}(\gamma(T))).$$

  Thus, by Corollary 1539, $O^{\mathcal{A}}(T) = \gamma^{-1}(O^{\mathcal{B}}(\gamma(T)))$.

- If $\gamma^{-1}(O^{\mathcal{B}}(\gamma(T))) = O^{\mathcal{A}}(T)$, then we get $\mathrm{Ker}(\langle H,\gamma\rangle) = \gamma^{-1}(\Delta^{\mathcal{B}}) \le \gamma^{-1}(O^{\mathcal{B}}(\gamma(T))) = O^{\mathcal{A}}(T)$. Thus, $T \in O^{\mathcal{A}^{-1}}(\mathrm{Ker}(\langle H,\gamma\rangle))$.

We conclude that $O^{\mathcal{A}^{-1}}(\mathrm{Ker}(\langle H,\gamma\rangle)) = \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \gamma^{-1}(O^{\mathcal{B}}(\gamma(T))) = O^{\mathcal{A}}(T)\}$.

Assume, conversely, that the condition of the statement holds. Let $\mathcal{A}$, $\mathcal{B}$ be $\mathbf{F}$-algebraic systems, $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism, with H an isomorphism and $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, such that $\mathrm{Ker}(\langle H,\gamma\rangle) \le O^{\mathcal{A}}(\gamma^{-1}(T'))$. Then $\gamma^{-1}(T') \in O^{\mathcal{A}^{-1}}(\mathrm{Ker}(\langle H,\gamma\rangle))$, whence, by hypothesis,

$$\gamma^{-1}(O^{\mathcal{B}}(\gamma(\gamma^{-1}(T')))) = O^{\mathcal{A}}(\gamma^{-1}(T')).$$

By surjectivity of $\langle H,\gamma\rangle$, $\gamma^{-1}(O^{\mathcal{B}}(T')) = O^{\mathcal{A}}(\gamma^{-1}(T'))$ and, hence, O is a semi-coherent family of compatibility $\mathcal{I}$-operators. ■

Next we show that semi-coherence of compatibility $\mathcal{I}$-operators is preserved under relativization.

Proposition 1544 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$ and $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ a semi-coherent family of compatibility $\mathcal{I}$-operators. Then*

$$\widetilde{O}^{\mathcal{I}} = \{\widetilde{O}^{\mathcal{I},\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$$

is also a semi-coherent family of compatibility $\mathcal{I}$-operators.

Proof: It is easy to see that $\widetilde{O}^{\mathcal{I}}$ is also a compatibility operator. We have, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$\begin{array}{rcll} \widetilde{O}^{\mathcal{I},\mathcal{A}}(T) & = & \bigcap\{O^{\mathcal{A}}(T') : T \le T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} & \text{(definition)}\\ & \le & \bigcap\{\Omega^{\mathcal{A}}(T') : T \le T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} & \text{(compatibility)}\\ & \le & \Omega^{\mathcal{A}}(T). \quad \text{(set theory)} & \end{array}$$

Thus, $\widetilde{O}^{\mathcal{I}}$ is indeed a compatibility $\mathcal{I}$-operator.

For semi-coherence, assume $\mathcal{A}$, $\mathcal{B}$ are $\mathbf{F}$-algebraic systems, $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism, with H an isomorphism, and $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, such that $\mathrm{Ker}(\langle H,\gamma\rangle) \le \widetilde{O}^{\mathcal{I},\mathcal{A}}(\gamma^{-1}(T'))$. We must show that

$$\widetilde{O}^{\mathcal{I},\mathcal{A}}(\gamma^{-1}(T')) = \gamma^{-1}(\widetilde{O}^{\mathcal{I},\mathcal{B}}(T')).$$

Claim: We have

$$\{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \gamma^{-1}(T') \le T\} = \{\gamma^{-1}(T'') : T' \le T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})\}.$$

- Suppose $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\gamma^{-1}(T') \le T$. Then, by Corollary 1539, $T = \gamma^{-1}(\gamma(T))$, where, by Corollary 56, $\gamma(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. Moreover, by hypothesis and the surjectivity of $\langle H,\gamma\rangle$, $T' = \gamma(\gamma^{-1}(T')) \le \gamma(T)$. This proves the left-to-right inclusion.

- Let $T' \leq T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. Then, by Corollary 55, we obtain $\gamma^{-1}(T'') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and, by hypothesis, $\gamma^{-1}(T') \leq \gamma^{-1}(T'')$. This shows that the right-to-left inclusion also holds.

This proves the Claim. Now, based on the Claim, we reason as follows:

$$\begin{array}{rcl}\gamma^{-1}(\widetilde{O}^{\mathcal{I},\mathcal{B}}(T')) & = & \gamma^{-1}(\bigcap\{O^{\mathcal{B}}(T'') : T' \leq T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})\}) \\ & = & \bigcap\{\gamma^{-1}(O^{\mathcal{B}}(T'')) : T' \leq T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})\} \\ & = & \bigcap\{O^{\mathcal{A}}(\gamma^{-1}(T'')) : T' \leq T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})\} \\ & & \text{(by Semi-Coherence of } O) \\ & = & \bigcap\{O^{\mathcal{A}}(T) : \gamma^{-1}(T') \leq T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \\ & & \text{(by the Claim)} \\ & = & \widetilde{O}^{\mathcal{I},\mathcal{A}}(\gamma^{-1}(T')).\end{array}$$

Thus, $\widetilde{O}^{\mathcal{I}}$ is indeed semi-coherent. ■

Proposition 1545 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a coherent family of compatibility* $\mathcal{I}$*-operators. Let, also,* $\mathcal{A}$*,* $\mathcal{B}$ *be* $\mathbf{F}$*-algebraic systems and* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism. For all* $\mathcal{T}' \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}')$*, such that* $\langle H, \gamma \rangle$ *is* O*-compatible with* $\gamma^{-1}(\mathcal{T}')$*,*

$$\widetilde{O}^{\mathcal{A}}(\gamma^{-1}(\mathcal{T}')) = \gamma^{-1}(\widetilde{O}^{\mathcal{B}}(\mathcal{T}')).$$

Proof: We have

$$\begin{array}{rcl}\gamma^{-1}(\widetilde{O}^{\mathcal{B}}(\mathcal{T}')) & = & \gamma^{-1}(\bigcap\{O^{\mathcal{B}}(T') : T' \in \mathcal{T}'\}) \\ & = & \bigcap\{\gamma^{-1}(O^{\mathcal{B}}(T')) : T' \in \mathcal{T}'\} \\ & = & \bigcap\{O^{\mathcal{A}}(\gamma^{-1}(T')) : T' \in \mathcal{T}'\} \\ & & \text{(hypothesis and coherence)} \\ & = & \bigcap\{O^{\mathcal{A}}(T) : T \in \gamma^{-1}(\mathcal{T}')\} \\ & = & \widetilde{O}^{\mathcal{A}}(\gamma^{-1}(\mathcal{T}')).\end{array}$$

■

Proposition 1546 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$ *and* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of compatibility* $\mathcal{I}$*-operators. Let, also,* $\mathcal{A}$*,* $\mathcal{B}$ *be* $\mathbf{F}$*-algebraic systems and* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism, with* H *an isomorphism. For all* $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, such that* $\langle H, \gamma \rangle$ *is* O*-compatible with* $\mathcal{T}$*,* $\gamma(\widetilde{O}^{\mathcal{A}}(\mathcal{T})) = \widetilde{O}^{\mathcal{B}}(\gamma(\mathcal{T}))$*.*

Proof: By the hypothesis and Corollary 1539, we get that $\mathcal{T} = \gamma^{-1}(\gamma(\mathcal{T}))$. So exploiting Proposition 1545, we get

$$\begin{array}{rcl}\gamma(\widetilde{O}^{\mathcal{A}}(\mathcal{T})) & = & \gamma(\widetilde{O}^{\mathcal{A}}(\gamma^{-1}(\gamma(\mathcal{T})))) \\ & = & \gamma(\gamma^{-1}(\widetilde{O}^{\mathcal{B}}(\gamma(\mathcal{T})))) \\ & = & \widetilde{O}^{\mathcal{B}}(\gamma(\mathcal{T})).\end{array}$$

■

21.7 Semi-Coherence and Full Objects

We start by providing a characterization of the inverse operator associated with a semi-coherent family of compatibility $\mathcal{I}$-operators.

Proposition 1547 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, and* $O = \{O^\mathcal{A} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of compatibility* $\mathcal{I}$*-operators. Then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $\theta \in \mathrm{ConSys}(\mathcal{A})$*,*

$$\begin{array}{rcl} O^{\mathcal{A}^{-1}}(\theta) & = & \{T \in \mathrm{FiFam}^\mathcal{I}(\mathcal{A}) : \pi^{-1}(O^{\mathcal{A}/\theta}(T/\theta)) = O^\mathcal{A}(T)\} \\ & = & \pi^{-1}(\{T' \in \mathrm{FiFam}^\mathcal{I}(\mathcal{A}/\theta) : \pi^{-1}(O^{\mathcal{A}/\theta}(T')) = O^\mathcal{A}(\pi^{-1}(T'))\}), \end{array}$$

where $\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\theta$ *denotes the quotient morphism.*

Proof: We have by hypothesis and Lemma 1543,

$$O^{\mathcal{A}^{-1}}(\theta) = \{T \in \mathrm{FiFam}^\mathcal{I}(\mathcal{A}) : \pi^{-1}(O^{\mathcal{A}/\theta}(T/\theta)) = O^\mathcal{A}(T)\}.$$

For the second equality, if $T \in \mathrm{FiFam}^\mathcal{I}(\mathcal{A})$, such that $\pi^{-1}(O^{\mathcal{A}/\theta}(T/\theta)) = O^\mathcal{A}(T)$, then $T = \pi^{-1}(T/\theta)$ and, also,

$$\pi^{-1}(O^{\mathcal{A}/\theta}(T/\theta)) = O^\mathcal{A}(T) = O^\mathcal{A}(\pi^{-1}(T/\theta)).$$

This proves the left-to-right inclusion, since, by Corollary 57, we have $T/\theta \in \mathrm{FiFam}^\mathcal{I}(\mathcal{A}/\theta)$.

Assume, conversely, that $T' \in \mathrm{FiFam}^\mathcal{I}(\mathcal{A}/\theta)$, such that $\pi^{-1}(O^{\mathcal{A}/\theta}(T')) = O^\mathcal{A}(\pi^{-1}(T'))$. Then, by Corollary, 57, $\pi^{-1}(T') \in \mathrm{FiFam}^\mathcal{I}(\mathcal{A})$ and, moreover,

$$\pi^{-1}(O^{\mathcal{A}/\theta}(\pi^{-1}(T')/\theta)) = \pi^{-1}(O^{\mathcal{A}/\theta}(T')) = O^\mathcal{A}(\pi^{-1}(T')).$$

This proves the right-to-left-inclusion. ■

We now give a characterization of O-full $\mathcal{I}$-classes for semi-coherent families of congruential compatibility $\mathcal{I}$-operators.

Corollary 1548 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $O = \{O^\mathcal{A} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of congruential compatibility* $\mathcal{I}$*-operators,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $\mathcal{T} \subseteq \mathrm{FiFam}^\mathcal{I}(\mathcal{A})$*.* $\mathcal{T}$ *is* $O^\mathcal{A}$*-full if and only if, for some surjective* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$*, with* H *an isomorphism, which may be taken to be the quotient morphism* $\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\widetilde{O}(\mathcal{T})$*,*

$$\mathcal{T} = \{T \in \mathrm{FiFam}^\mathcal{I}(\mathcal{A}) : \gamma^{-1}(O^\mathcal{B}(\gamma(T))) = O^\mathcal{A}(T)\}.$$

Proof: Suppose $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is $O^{\mathcal{A}}$-full. By definition, $\mathcal{T} = O^{\mathcal{A}^{-1}}(\widetilde{O}^{\mathcal{A}}(\mathcal{T}))$. Let $\langle I,\pi\rangle : \mathcal{A} \to \mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T})$ be the quotient morphism. Then we have $\mathcal{T} = O^{\mathcal{A}^{-1}}(\mathrm{Ker}(\langle I,\pi\rangle))$ whence, by Proposition 1547,

$$\mathcal{T} = \{T \in \mathrm{Fifam}^{\mathcal{I}}(\mathcal{A}) : \pi^{-1}(O^{\mathcal{A}/\widetilde{O}^{\mathcal{A}}(T)}(\pi(T))) = O^{\mathcal{A}}(T)\}.$$

Assume, conversely, that $\mathcal{T} = \{T \in \mathrm{Fifam}^{\mathcal{I}}(\mathcal{A}) : \gamma^{-1}(O^{\mathcal{B}}(\gamma(T))) = O^{\mathcal{A}}(T)\}$, for some surjective $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$, with H an isomorphism. By Proposition 1547, $\mathcal{T} = O^{\mathcal{A}^{-1}}(\mathrm{Ker}(\langle H,\gamma\rangle))$, whence $\mathcal{T} \in \mathrm{Ran}(O^{\mathcal{A}^{-1}})$, showing that $\mathcal{T}$ is $O^{\mathcal{A}}$-full. ∎

Corollary 1549 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of congruential compatibility* $\mathcal{I}$*-operators,* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.* $\mathcal{T}$ *is* $O^{\mathcal{A}}$*-full if and only if, for some* $\theta \in \mathrm{ConSys}(\mathcal{I})$*, which can be taken to be* $\widetilde{O}^{\mathcal{A}}(\mathcal{T})$*, and with* $\langle I,\pi\rangle : \mathcal{A} \to \mathcal{A}/\theta$ *the corresponding quotient morphism,*

$$\mathcal{T} = \pi^{-1}(\{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta) : \pi^{-1}(O^{\mathcal{A}/\theta}(T')) = O^{\mathcal{A}}(\pi^{-1}(T'))\}).$$

Proof: Assume, first, that $\mathcal{T}$ is $O^{\mathcal{A}}$-full. Then, by definition, we have $\mathcal{T} = O^{\mathcal{A}^{-1}}(\widetilde{O}^{\mathcal{A}}(\mathcal{T}))$. Take $\theta = \widetilde{O}^{\mathcal{A}}(\mathcal{T})$. Then $\mathcal{T} = O^{\mathcal{A}^{-1}}(\theta)$, whence, by Proposition 1547,

$$\mathcal{T} = \pi^{-1}(\{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta) : \pi^{-1}(O^{\mathcal{A}/\theta}(T')) = O^{\mathcal{A}}(\pi^{-1}(T'))\}).$$

Assume, conversely, that $\mathcal{T}$ is given by the displayed expression above, for some $\theta \in \mathrm{ConSys}(\mathcal{A})$ and $\langle I,\pi\rangle : \mathcal{A} \to \mathcal{A}/\theta$ the quotient morphism. Then, by Proposition 1547, $\mathcal{T} = O^{\mathcal{A}^{-1}}(\theta) \in \mathrm{Ran}(O^{\mathcal{A}^{-1}})$ and, therefore, $\mathcal{T}$ is $O^{\mathcal{A}}$-full, by definition. ∎

Turning, next, to the full congruence systems, we obtain the following characterization.

Proposition 1550 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of compatibility* $\mathcal{I}$*-operators,* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $\theta \in \mathrm{ConSys}(\mathcal{A})$*.* θ *is* $O^{\mathcal{A}}$*-full if and only if*

$$\widetilde{O}^{\mathcal{A}/\theta}(\{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta) : \pi^{-1}(O^{\mathcal{A}/\theta}(T')) = O^{\mathcal{A}}(\pi^{-1}(T'))\}) = \Delta^{\mathcal{A}/\theta},$$

where $\langle I,\pi\rangle : \mathcal{A} \to \mathcal{A}/\theta$ *is the quotient morphism.*

Proof: Let $\mathcal{T}' = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta) : \pi^{-1}(O^{\mathcal{A}/\theta}(T')) = O^{\mathcal{A}}(\pi^{-1}(T'))\}$. Then $\langle I,\pi\rangle$ is compatible with $\pi^{-1}(\mathcal{T}')$, since, for all $T' \in \mathcal{T}'$,

$$\mathrm{Ker}(\langle I,\pi\rangle) = \pi^{-1}(\Delta^{\mathcal{A}/\theta}) \leq \pi^{-1}(O^{\mathcal{A}/\theta}(T')) = O^{\mathcal{A}}(\pi^{-1}(T')).$$

Thus, by Propositions 1547 and 1545, θ is $O^{\mathcal{A}}$-full if and only if

$$\theta = \widetilde{O}^{\mathcal{A}}(O^{\mathcal{A}^{-1}}(\theta)) = \widetilde{O}^{\mathcal{A}}(\pi^{-1}(\mathcal{T}')) = \pi^{-1}(\widetilde{O}^{\mathcal{A}/\theta}(\mathcal{T}')).$$

Now, if θ is $O^{\mathcal{A}}$-full, then we get, using the surjectivity of the quotient morphism,

$$\begin{array}{rcl}\widetilde{O}^{\mathcal{A}/\theta}(\mathcal{T}') & = & \pi(\pi^{-1}(\widetilde{O}^{\mathcal{A}/\theta}(\mathcal{T}'))) \\ & = & \pi(\theta) = \Delta^{\mathcal{A}/\theta}.\end{array}$$

If, conversely, $\widetilde{O}^{\mathcal{A}/\theta}(\mathcal{T}') = \Delta^{\mathcal{A}/\theta}$, then $\theta = \pi^{-1}(\Delta^{\mathcal{A}/\theta}) = \pi^{-1}(\widetilde{O}^{\mathcal{A}/\theta}(\mathcal{T}'))$. Hence, by the equivalence detailed above, θ is $O^{\mathcal{A}}$-full. ■

Since Ω is a semi-coherent family of compatibility $\mathcal{I}$-operators, we now get

Corollary 1551 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $\theta \in \mathrm{ConSys}(\mathcal{A})$*. Then*

$$\Omega^{\mathcal{A}^{-1}}(\theta) = \pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)),$$

where $\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\theta$ *is the quotient morphism.*

Proof: By Proposition 1547,

$$\Omega^{\mathcal{A}^{-1}}(\theta) = \pi^{-1}(\{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta') : \pi^{-1}(\Omega^{A/\theta}(T')) = \Omega^{\mathcal{A}}(\pi^{-1}(T'))\}).$$

But, by Proposition 24, Ω is commuting and, hence,

$$\{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta') : \pi^{-1}(\Omega^{A/\theta}(T')) = \Omega^{\mathcal{A}}(\pi^{-1}(T'))\} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta).$$

Therefore, $\Omega^{\mathcal{A}^{-1}}(\theta) = \pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta))$. ■

21.8 The General Correspondence Theorem

Theorem 1552 (General Correspondence Theorem) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of compatibility* $\mathcal{I}$*-operators. Then, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$*, all surjective morphisms* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$*, with* H *an isomorphism, and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, if* $\langle H, \gamma \rangle$ *is* O*-compatible with* T*, then* γ *induces an order isomorphism from* $[\![T]\!]^{O^{\mathcal{A}}}$ *onto* $[\![\gamma(T)]\!]^{O^{\mathcal{B}}}$*, with inverse* γ^{-1}*.*

Proof: Assume that $\langle H, \gamma \rangle$ is O-compatible with T. By Corollary 1539, $\langle H, \gamma \rangle$ is Ω-compatible with T. By the same Corollary and by Corollary 56, $T = \gamma^{-1}(\gamma(T))$ and $\gamma(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$.

Suppose, next, that $T' \in [\![T]\!]^{O^{\mathcal{A}}}$. Then, $\mathrm{Ker}(\langle H,\gamma\rangle) \le O^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')$. Again, based on Corollaries 1539 and 56, we get $\gamma^{-1}(\gamma(T')) = T'$ and $\gamma(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. Moreover, we get

$$\begin{array}{rcll} O^{\mathcal{B}}(\gamma(T)) & = & \gamma(O^{\mathcal{A}}(T)) & \text{(by Lemma 1542)} \\ & \le & \gamma(\Omega^{\mathcal{A}}(T')) & \\ & = & \Omega^{\mathcal{B}}(\gamma(T')). & \text{(by Lemma 1542)} \end{array}$$

Thus, $\gamma(T') \in [\![\gamma(T)]\!]^{O^{\mathcal{B}}}$.

Suppose, next, that $T'' \in [\![\gamma(T)]\!]^{O^{\mathcal{B}}}$. Then, we get $O^{\mathcal{B}}(\gamma(T)) \le \Omega^{\mathcal{B}}(T')$. By Corollary 55, $\gamma^{-1}(T'') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and by surjectivity, $\gamma(\gamma^{-1}(T'')) = T''$. Since $\langle H,\gamma\rangle$ is O-compatible with $T = \gamma^{-1}(\gamma(T))$, we get, using semi-coherence,

$$\begin{array}{rcl} O^{\mathcal{A}}(T) & = & O^{\mathcal{A}}(\gamma^{-1}(\gamma(T)) \\ & = & \gamma^{-1}(O^{\mathcal{B}}(\gamma(T))) \\ & \le & \gamma^{-1}(\Omega^{\mathcal{B}}(T')) \\ & = & \Omega^{\mathcal{A}}(\gamma^{-1}(T')). \end{array}$$

Hence, $\gamma^{-1}(T'') \in [\![T]\!]^{O^{\mathcal{A}}}$. We conclude that γ is a bijection from $[\![T]\!]^{O^{\mathcal{A}}}$ onto $[\![\gamma(T)]\!]^{O^{\mathcal{B}}}$, with inverse γ^{-1}. But, clearly, both γ and γ^{-1} are order preserving functions, whence they establish an order isomorphism between these two ordered sets. ■

The General Correspondence Theorem has the following consequence concerning O-filter systems on different $\mathbf{F}$-algebraic systems.

Corollary 1553 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of compatibility* $\mathcal{I}$*-operators. Then, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G,\beta\rangle\rangle$, *all surjective morphisms* $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$, *with* H *an isomorphism, and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, *if* $\langle H,\gamma\rangle$ *is* O*-compatible with* T*, then*

$$T \in \mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A}) \quad \textit{iff} \quad \gamma(T) \in \mathrm{FiFam}^{\mathcal{I},O}(\mathcal{B}).$$

Proof: We have the following chain of equivalences:

$$\begin{array}{rll} T \in \mathrm{FiFam}^{\mathcal{I},O}(\mathcal{A}) & \text{iff} & T = T^O \\ & \text{iff} & T = \bigcap [\![T]\!]^{O^{\mathcal{A}}} \\ & \text{iff} & \gamma(T) = \bigcap [\![\gamma(T)]\!]^{O^{\mathcal{B}}} \quad \text{(by Theorem 1552)} \\ & \text{iff} & \gamma(T) = \gamma(T)^O \\ & \text{iff} & \gamma(T) \in \mathrm{FiFam}^{\mathcal{I},O}(\mathcal{B}). \end{array}$$

Thus, the claim is established. ■

For semi-coherent congruential compatibility $\mathcal{I}$-operators, we obtain a relation between the O-filter systems on an algebraic system and those on the quotient algebraic system.

Corollary 1554 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of congruential compatibility* $\mathcal{I}$*-operators. Then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$T^O / O^{\mathcal{A}}(T) = (T / O^{\mathcal{A}}(T))^O$$

and it is the least $\mathcal{I}$*-filter family on* $\mathcal{A}/O^{\mathcal{A}}(T)$*.*

Proof: Consider the quotient morphism $\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/O^{\mathcal{A}}(T)$. $\langle I, \pi \rangle$ is surjective, with I an isomorphism, and it is O-compatible with T. By Theorem 1552, $\pi : [\![T]\!]^{O^{\mathcal{A}}} \to [\![T/O^{\mathcal{A}}(T)]\!]^{O^{\mathcal{A}/O^{\mathcal{A}}(T)}}$ is an order isomorphism with inverse π^{-1}. Since T^O is the least $\mathcal{I}$-filter family of $[\![T]\!]^{O^{\mathcal{A}}}$, it follows that $T^O/O^{\mathcal{A}}(T)$ must be the least $\mathcal{I}$-filter family of $[\![T/O^{\mathcal{A}}(T)]\!]^{O^{\mathcal{A}/O^{\mathcal{A}}(T)}}$, which is, by definition, $(T/O^{\mathcal{A}}(T))^O$. Finally, since $O^{\mathcal{A}/O^{\mathcal{A}}(T)}(T/O^{\mathcal{A}}(T)) = \Delta^{\mathcal{A}/O^{\mathcal{A}}(T)}$, it follows that $[\![T/O^{\mathcal{A}}(T)]\!]^{O^{\mathcal{A}/O^{\mathcal{A}}(T)}} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/O^{\mathcal{A}}(T))$. Thus, $(T/O^{\mathcal{A}}(T))^O$ is the least $\mathcal{I}$-filter family on $\mathcal{A}/O^{\mathcal{A}}(T)$. ■

Finally, applying the General Correspondence Theorem to the relativization of an operator, we obtain the following:

Theorem 1555 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of compatibility* $\mathcal{I}$*-operators. Then, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$*, all surjective morphisms* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$*, with* H *an isomorphism, and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, if* $\langle H, \gamma \rangle$ *is* $\widetilde{O}^{\mathcal{I}}$*-compatible with* T*, then* γ *induces an order isomorphism from* $[\![T]\!]^{\widetilde{O}^{\mathcal{I},\mathcal{A}}}$ *onto* $[\![\gamma(T)]\!]^{\widetilde{O}^{\mathcal{I},\mathcal{B}}}$*, with inverse* γ^{-1}*.*

Proof: It is clear that if O is a compatibility $\mathcal{I}$-operator, the same holds for $\widetilde{O}^{\mathcal{I}}$. Moreover, by Proposition 1544, if O is a semi-coherent family, then $\widetilde{O}^{\mathcal{I}}$ is also semi-coherent. Therefore, under the given hypotheses, we can apply Theorem 1552 with $\widetilde{O}^{\mathcal{I}}$ in place of O and the result immediately follows. ■

21.9 Algebraic Systems of $\mathcal{I}$-Operators

With a given family of congruential operators, there are associated several classes of algebraic systems, which it is the purpose of this section to study closely, in analogy to the various classes ensued from applications of the Leibniz operator, and to explore their interrelationships.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$ and $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ a family of congruential $\mathcal{I}$-operators. We define the following classes of $\mathbf{F}$-algebraic systems associated with O (assuming closure under isomorphisms):

- $\mathrm{AlgSys}^O(\mathcal{I}) = \{\mathcal{A}/O^{\mathcal{A}}(T) : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}), T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}$;
- $\mathrm{AlgSys}_O(\mathcal{I}) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : (\exists T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(O^{\mathcal{A}}(T) = \Delta^{\mathcal{A}})\}$;
- $\mathrm{AlgSys}^{\widetilde{O}^{\mathcal{I}}}(\mathcal{I}) = \{\mathcal{A}/\widetilde{O}^{\mathcal{I},\mathcal{A}}(T) : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}), T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}$;
- $\mathrm{AlgSys}_{\widetilde{O}^{\mathcal{I}}}(\mathcal{I}) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : (\exists T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(\widetilde{O}^{\mathcal{I},\mathcal{A}}(T) = \Delta^{\mathcal{A}})\}$;
- $\mathrm{AlgSys}^{\widetilde{O}}(\mathcal{I}) = \{\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T}) : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}), \mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}$;
- $\mathrm{AlgSys}_{\widetilde{O}}(\mathcal{I}) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : (\exists \mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(\widetilde{O}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathcal{A}})\}$.

Names corresponding to these classes go as follows:

- $\mathrm{AlgSys}^O(\mathcal{I})$ is the class of O-**reduced F-algebraic systems**;
- $\mathrm{AlgSys}_O(\mathcal{I})$ is the class of O-**reductions of F-algebraic systems**;
- $\mathrm{AlgSys}^{\widetilde{O}^{\mathcal{I}}}(\mathcal{I})$ is the class of $\widetilde{O}^{\mathcal{I}}$-**reduced F-algebraic systems**;
- $\mathrm{AlgSys}_{\widetilde{O}^{\mathcal{I}}}(\mathcal{I})$ is the class of $\widetilde{O}^{\mathcal{I}}$-**reductions of F-algebraic systems**;
- $\mathrm{AlgSys}^{\widetilde{O}}(\mathcal{I})$ is the class of $\widetilde{O}$-**reduced F-algebraic systems**;
- $\mathrm{AlgSys}_{\widetilde{O}}(\mathcal{I})$ is the class of $\widetilde{O}$-**reductions of F-algebraic systems**.

We provide some alternative characterizations for the classes associated with the lifting $\widetilde{O}$ of the operator O.

Lemma 1556 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$ and $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ a family of congruential $\mathcal{I}$-operators.*

(a) $\mathrm{AlgSys}^{\widetilde{O}}(\mathcal{I}) = \{\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T}) : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}), \mathcal{T}\ O^{\mathcal{A}}\text{-full}\}$*;*

(b) $\mathrm{AlgSys}_{\widetilde{O}}(\mathcal{I}) = \{\mathcal{A} : \widetilde{O}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}\}$*;*

(c) $\mathrm{AlgSys}_{\widetilde{O}}(\mathcal{I}) = \{\mathcal{A} : (\exists \mathcal{T}\ O^{\mathcal{A}}\text{-full})(\widetilde{O}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathcal{A}})\}$.

Proof: Note that, for all three equalities claimed, the right-to-left inclusions are trivial, given the definitions of the corresponding classes on the left. Therefore, in working out the various parts, it suffices to show the left-to-right inclusions.

(a) Suppose that $\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T}) \in \mathrm{AlgSys}^{\widetilde{O}}(\mathcal{I})$, for some $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{A}}(\mathcal{A})$. Since $\widetilde{O}^{\mathcal{A}}(\mathcal{T})$ is by definition, an O-full congruence system on $\mathcal{A}$, there exists, by Corollary 1521, an O-full $\mathcal{T}' \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{O}^{\mathcal{A}}(\mathcal{T}') = \widetilde{O}^{\mathcal{A}}(\mathcal{T})$. Thus, we get $\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T}) = \mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T}') \in \{\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T}) : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}), T\ O^{\mathcal{A}}\text{-full}\}$.

(b) Assume $\mathcal{A} \in \mathrm{AlgSys}_{\widetilde{O}}(\mathcal{I})$. Then, by definition, there exists a collection $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{O}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathcal{A}}$. Therefore,

$$\widetilde{O}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{A}}(\mathcal{A})) \leq \widetilde{O}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathcal{A}}.$$

Thus, $\widetilde{O}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}$. We get that $\mathcal{A} \in \{\mathcal{A} : \widetilde{O}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}\}$.

(c) This follows directly from Part (b) and Corollary 1523. ∎

We now show that the three pairs of classes of reduced - classes of reductions, associated with the same operator, consist of identical classes of **F**-algebraic systems. This is due to the fact that the reduction of an **F**-algebraic system results in a reduced **F**-algebraic system, taken with respect to the same operator.

Lemma 1557 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of congruential compatibility* $\mathcal{I}$*-operators and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system. For all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and all* $\theta \in \mathrm{ConSys}(\mathcal{A})$*,*

$$\theta \leq O^{\mathcal{A}}(T) \quad \textit{implies} \quad O^{\mathcal{A}/\theta}(T/\theta) = O^{\mathcal{A}}(T)/\theta.$$

In particular, $O^{\mathcal{A}/O^{\mathcal{A}}(T)}(T/O^{\mathcal{A}}(T)) = \Delta^{\mathcal{A}/O^{\mathcal{A}}(T)}$*.*

Proof: Let $\mathcal{A}$ be an **F**-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $\theta \in \mathrm{ConSys}(\mathcal{A})$, such that $\theta \leq O^{\mathcal{A}}(T)$. Consider the quotient morphism $\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\theta$. It is surjective and, by hypothesis, O-compatible with T. By the assumption of semi-coherence and Lemma 1542, we get

$$O^{\mathcal{A}/\theta}(T/\theta) = O^{\mathcal{A}/\theta}(\pi(T)) = \pi(O^{\mathcal{A}}(T)) = O^{\mathcal{A}}(T)/\theta.$$

The last assertion in the statement is the specialization of what was just proven for $\theta = O^{\mathcal{A}}(T)$. ∎

Proposition 1558 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, and* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of congruential compatibility* $\mathcal{I}$*-operators. Then*

$$\mathrm{AlgSys}^{O}(\mathcal{I}) = \mathrm{AlgSys}_{O}(\mathcal{I}).$$

Proof: Suppose $\mathcal{A} \in \mathrm{AlgSys}_{O}(\mathcal{I})$. Then, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $O^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. But then $\mathcal{A} \cong \mathcal{A}/\Delta^{\mathcal{A}} = \mathcal{A}/O^{\mathcal{A}}(T) \in \mathrm{AlgSys}^{O}(\mathcal{I})$.

On the other hand, if $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ so that $\mathcal{A}/O^{\mathcal{A}}(T) \in \mathrm{AlgSys}^{O}(\mathcal{I})$, then, for $T/O^{\mathcal{A}}(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/O^{\mathcal{A}}(T))$, we get, by Lemma 1557,

$$O^{\mathcal{A}/O^{\mathcal{A}}(T)}(T/O^{\mathcal{A}}(T)) = \Delta^{\mathcal{A}/O^{\mathcal{A}}(T)},$$

whence, by definition $\mathcal{A}/O^{\mathcal{A}}(T) \in \mathrm{AlgSys}_{O}(\mathcal{I})$. ∎

Corollary 1559 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, and* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of congruential compatibility* $\mathcal{I}$*-operators. Then*

$$\mathrm{AlgSys}^{\widetilde{O}^{\mathcal{I}}}(\mathcal{I}) = \mathrm{AlgSys}_{\widetilde{O}^{\mathcal{I}}}(\mathcal{I}).$$

Proof: By Proposition 1544 together with Proposition 1558. ∎

Lemma 1560 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of congruential compatibility* $\mathcal{I}$*-operators and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system. For all* $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$

$$\widetilde{O}^{\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T})}(\mathcal{T}/\widetilde{O}^{\mathcal{A}}(\mathcal{T})) = \Delta^{\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T})}.$$

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Consider the quotient morphism $\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T})$. It is surjective and O-compatible with $\mathcal{T}$. By the assumption of semi-coherence and Proposition 1546, we get

$$\begin{array}{rcl}\widetilde{O}^{\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T})}(\mathcal{T}/\widetilde{O}^{\mathcal{A}}(\mathcal{T})) & = & \widetilde{O}^{\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T})}(\pi(\mathcal{T})) \\ & = & \pi(\widetilde{O}^{\mathcal{A}}(\mathcal{T})) \\ & = & \widetilde{O}^{\mathcal{A}}(\mathcal{T})/\widetilde{O}^{\mathcal{A}}(\mathcal{T}) \\ & = & \Delta^{\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T})}.\end{array}$$

This concludes the proof of the statement. ∎

Proposition 1561 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, and* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of congruential compatibility* $\mathcal{I}$*-operators. Then*

$$\mathrm{AlgSys}^{\widetilde{O}}(\mathcal{I}) = \mathrm{AlgSys}_{\widetilde{O}}(\mathcal{I}).$$

Proof: Suppose $\mathcal{A} \in \mathrm{AlgSys}_{\widetilde{O}}(\mathcal{I})$. Then, there exists $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{O}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathcal{A}}$. But then $\mathcal{A} \cong \mathcal{A}/\Delta^{\mathcal{A}} = \mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T}) \in \mathrm{AlgSys}^{\widetilde{O}}(\mathcal{I})$.

On the other hand, if $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ so that $\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T}) \in \mathrm{AlgSys}^{\widetilde{O}}(\mathcal{I})$, then, for $\mathcal{T}/\widetilde{O}^{\mathcal{A}}(\mathcal{T}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T}))$, we get, by Lemma 1560,

$$\widetilde{O}^{\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T})}(\mathcal{T}/\widetilde{O}^{\mathcal{A}}(\mathcal{T})) = \Delta^{\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T})},$$

whence, by definition $\mathcal{A}/\widetilde{O}^{\mathcal{A}}(\mathcal{T}) \in \mathrm{AlgSys}_{\widetilde{O}}(\mathcal{I})$. ∎

Proposition 1562 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, and* $O = \{O^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ *a semi-coherent family of congruential compatibility* $\mathcal{I}$*-operators. Then*

$$\mathrm{AlgSys}^{\widetilde{O}}(\mathcal{I}) = \mathrm{AlgSys}_{\widetilde{O}}(\mathcal{I}) = \mathrm{AlgSys}^{\widetilde{O}^{\mathcal{I}}}(\mathcal{I}) = \mathrm{AlgSys}_{\widetilde{O}^{\mathcal{I}}}(\mathcal{I}).$$

Proof: For every **F**-algebraic system and every $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we have $\widetilde{O}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{O}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T)$. This equality gives that

$$\mathrm{AlgSys}_{\widetilde{O}^{\mathcal{I}}}(\mathcal{I}) \subseteq \mathrm{AlgSys}_{\widetilde{O}}(\mathcal{I}) \quad \text{and} \quad \mathrm{AlgSys}^{\widetilde{O}^{\mathcal{I}}}(\mathcal{I}) \subseteq \mathrm{AlgSys}^{\widetilde{O}}(\mathcal{I}).$$

Assume, conversely, in the first case, that $\mathrm{AlgSys}_{\widetilde{O}}(\mathcal{I})$. By Lemma 1556, $\widetilde{O}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}$. Let $T = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then we get

$$\widetilde{O}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{O}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T) = \widetilde{O}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}.$$

This shows that $\mathcal{A} \in \mathrm{AlgSys}_{\widetilde{O}^{\mathcal{I}}}(\mathcal{I})$. Due to Corollary 1559 and Proposition 1561 the equality just proven suffices to guarantee the conclusion. ∎

21.10 Leibniz Operator as an $\mathcal{I}$-Operator

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on **F**. We consider in this section the Leibniz operator

$$\Omega = \{\Omega^{\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\},$$

which is a coherent, congruential, compatibility $\mathcal{I}$-operator. We saw that its lifting is the Tarski operator $\widetilde{\Omega}$ and its relativization is the Suszko operator $\widetilde{\Omega}^{\mathcal{I}}$. Using the definition, for every **F**-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $\theta \in \mathrm{ConSys}(\mathcal{A})$, we have

$$\begin{aligned} \Omega^{\mathcal{A}^{-1}}(\theta) &= \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \theta \le \Omega^{\mathcal{A}}(T)\} \\ &= \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \theta \text{ is compatible with } T\}. \end{aligned}$$

We have the following characterizations of Ω-full objects:

Proposition 1563 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on **F**, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an **F**-algebraic system and $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. $\mathcal{T}$ is Ω-full if and only if $\langle \mathcal{A}, \mathcal{T} \rangle$ is a full $\mathcal{I}$-structure.*

Proof: We have

$$\begin{aligned} \mathcal{T} \text{ is } \Omega\text{-full} \quad &\text{iff} \quad \mathcal{T} = \Omega^{\mathcal{A}^{-1}}(\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})) \\ &\qquad \text{(by definition of } \Omega\text{-full)} \\ &\text{iff} \quad \mathcal{T} = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')\} \\ &\qquad \text{(by definition of } \Omega^{\mathcal{A}^{-1}}) \\ &\text{iff} \quad \langle \mathcal{A}, \mathcal{T} \rangle \text{ is a full } \mathcal{I}\text{-structure.} \\ &\qquad \text{(by Theorem 1395)} \end{aligned}$$

∎

Recall that $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ denotes the collection of all $\mathrm{AlgSys}(\mathcal{I})$-congruence systems on an **F**-algebraic system $\mathcal{A}$, i.e., those congruence systems θ on $\mathcal{A}$, such that $\mathcal{A}/\theta \in \mathrm{ConSys}(\mathcal{I})$. For Ω-full congruence systems, we get

Proposition 1564 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $\theta \in \mathrm{ConSys}(\mathcal{A})$*.* θ *is* Ω*-full if and only if* $\theta \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$*.*

Proof: We have

$$\begin{array}{lll} \theta \text{ is } \Omega\text{-full} & \text{iff} & \widetilde{\Omega}^{\mathcal{A}/\theta}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}/\theta} \\ & & \text{(by Proposition 1550)} \\ & \text{iff} & \mathcal{A}/\theta \in \mathrm{AlgSys}(\mathcal{I}) \\ & & \text{(by Proposition 1399)} \\ & \text{iff} & \theta \in \mathrm{ConSys}(\mathcal{A}). \\ & & \text{(by definition).} \end{array}$$

■

As a corollary of these two characterizations, we can derive from our work on Galois connections (more precisely Corollary 1521) the Isomorphism Theorem 1408 between full $\mathcal{I}$-structures and $\mathcal{I}$-congruence systems.

Corollary 1565 (Isomorphism Theorem 1408) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system. The operators* $\widetilde{\Omega}^{\mathcal{A}}$ *and* $\Omega^{\mathcal{A}^{-1}}$ *establish a Galois connection between* $\mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$ *and* $\mathrm{EqvFam}(\mathcal{A})$*, which restricts to mutually inverse isomorphisms between* $\langle \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}), \leq \rangle$ *and* $\langle \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}), \leq \rangle$*.*

Proof: By Corollary 1521 and Propositions 1563 and 1564, noting that the order on $\langle \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}), \leq \rangle$ is the converse from that inherited by $\langle \mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})), \subseteq \rangle$. ■

By applying Proposition 1522 to the Leibniz operator, we get a characterization of full $\mathcal{I}$-structures.

Proposition 1566 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.* $\langle \mathcal{A}, \mathcal{T} \rangle$ *is a full* $\mathcal{I}$*-structure if and only if* $\mathcal{T}$ *is the largest collection* $\mathcal{D} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, such that* $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) = \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})$*.*

Proof: By instantiating Proposition 1522 to the Leibniz operator. ■

Moreover, directly from Lemma 1518, we get:

Proposition 1567 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is protoalgebraic if and only if* $\widetilde{\Omega}^{\mathcal{I}} = \Omega$*.*

Proof: By instantiating Lemma 1518 to the Leibniz operator. ■

We turn now to Ω-classes and Ω-filter families. Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

The Ω-**class of** T or **Leibniz class of** T is

$$[\![T]\!]^* := \Omega^{\mathcal{A}^{-1}}(\Omega^{\mathcal{A}}(T)) = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\}.$$

The **Leibniz filter family of** T is the $\mathcal{I}$-filter family

$$T^* = \bigcap [\![T]\!]^*.$$

We say that T is a **Leibniz filter family** if $T^* = T$. The collection of all Leibniz filter families of $\mathcal{A}$ is denoted by $\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$.

We further denote by $[T]$ the **equi-Leibniz class of** T, i.e., the collection of all $\mathcal{I}$-filter families of $\mathcal{A}$ that share the same Leibniz congruence system with T:

$$[T] = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \Omega^{\mathcal{A}}(T') = \Omega^{\mathcal{A}}(T)\} \subseteq [\![T]\!]^*.$$

Some basic properties involving these concepts follow.

Lemma 1568 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

(a) $T^* \leq \bigcap [T] \leq T$*;*

(b) If $T^* = T$*, then* $T = \bigcap [T]$*;*

(c) If $\mathcal{I}$ *is protoalgebraic, then* $T = T^*$ *if and only if* $T = \bigcap [T]$*.*

Proof:

(a) We have $T^* = \bigcap [\![T]\!]^* \leq \bigcap [T] \leq T$.

(b) If $T^* = T$, then, by Part (a), $T = \bigcap [T]$.

(c) Suppose that $\mathcal{I}$ is protoalgebraic. The necessity is given by Part (b). For the sufficiency, assume that $T = \bigcap [T]$. Since, by Part (a), $T^* \leq T$, by protoalgebraicity, $\Omega^{\mathcal{A}}(T^*) \leq \Omega^{\mathcal{A}}(T)$. Since $T^* \in [\![T]\!]^*$, we get, by definition, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T^*)$. Hence, $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T^*)$ and, therefore, $T^* \in [T]$. Now we conclude that $T = \bigcap [T] \leq T^*$. By Part (a), the reverse inclusion always holds.

■

Proposition 1569 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

(a) $\langle \mathcal{A}, [\![T]\!]^* \rangle \in \mathrm{FStr}(\mathcal{I})$*;*

(b) $\widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^*) = \Omega^{\mathcal{A}}(T)$*.*

Proof: By Proposition 1563, $\langle\mathcal{A},[\![T]\!]^*\rangle$ is a full $\mathcal{I}$-structure. Since $T\in[\![T]\!]^*$, it follows that $\widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^*)\le\Omega^{\mathcal{A}}(T)$. On the other hand, for all $T'\in[\![T]\!]^*$, $\Omega^{\mathcal{A}}(T)\le\Omega^{\mathcal{A}}(T')$. Thus, $\Omega^{\mathcal{A}}(T)\le\bigcap_{T'\in[\![T]\!]^*}\Omega^{\mathcal{A}}(T')=\widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^*)$. ■

It turns out that, for every theory family, its Leibniz counterpart is in fact a Leibniz theory family.

Proposition 1570 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. Then* $T^*\in\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$*.*

Proof: By Lemma 1568, we have $(T^*)^*\le T^*$. On the other hand, $T^*\in[\![T]\!]^*$. So, by definition $\Omega^{\mathcal{A}}(T)\le\Omega^{\mathcal{A}}(T^*)$. This shows that $[\![T^*]\!]^*\subseteq[\![T]\!]^*$. This, in turn, yields $T^*=\bigcap[\![T]\!]^*\le\bigcap[\![T^*]\!]^*=(T^*)^*$. We conclude that $(T^*)^*=T^*$ and, hence, $T^*\in\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. ■

We also have a characterization of Leibniz filter families in terms of full structures.

Proposition 1571 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.* $T\in\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$ *if and only if, there exists* $\langle\mathcal{A},\mathcal{T}\rangle\in\mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$*, such that* $T=\bigcap\mathcal{T}$*.*

Proof: Suppose, first, that $T\in\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. Then $T=T^*=\bigcap[\![T]\!]^*$ and, by Proposition 1569, $\langle\mathcal{A},[\![T]\!]^*\rangle$ is a full $\mathcal{I}$-structure.

Assume, conversely, that $T=\bigcap\mathcal{T}$, with $\langle\mathcal{A},\mathcal{T}\rangle\in\mathrm{FStr}(\mathcal{I})$. Since $T=\bigcap\mathcal{T}\in\mathcal{T}$, we get $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})\le\Omega^{\mathcal{A}}(T)$. Thus, we get

$$[\![T]\!]^*=\Omega^{\mathcal{A}^{-1}}(\Omega^{\mathcal{A}}(T))\subseteq\Omega^{\mathcal{A}^{-1}}(\widetilde{\Omega}^{\mathcal{A}}(T))=\mathcal{T}.$$

So $T=\bigcap\mathcal{T}\le\bigcap[\![T]\!]^*=T^*$. Since, by Lemma 1568, $T^*\le T$, we conclude that $T^*=T$ and, hence, $T\in\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. ■

Corollary 1554, applied to the Leibniz operator, gives another characterization of Leibniz filter families.

Proposition 1572 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.* $T\in\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$ *if and only if* $T/\Omega^{\mathcal{A}}(T)$ *is the least filter family in* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T))$*.*

Proof: By specializing Corollary 1554 to the Leibniz operator. ■

Leibniz filter families may also be used in characterizing the reflectivity of the Leibniz operator, which characterizes family reflective π-institutions.

Proposition 1573 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. Ω *is reflective if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

Proof: We have that Ω is reflective if and only if, by definition, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$\Omega^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T') \quad \text{implies} \quad T \le T',$$

if and only if, by definition of $[\![T]\!]^*$, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T' \in [\![T]\!]^*$ implies $T \le T'$, if and only if, since $T^* = \min [\![T]\!]^*$, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T = T^*$, if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. ■

Surjective morphisms between algebraic systems, with isomorphic signature components, that satisfy a compatibility condition, induce order isomorphisms between Leibniz classes, which restrict to order isomorphisms between equi-Leibniz classes.

Theorem 1574 (Correspondence Theorem for Leibniz Classes) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ *two* $\mathbf{F}$*-algebraic systems,* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism, with* H *an isomorphism, and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. *If* $\langle H, \gamma \rangle$ *is* Ω*-compatible with* T*, then* γ *induces an order isomorphism from* $[\![T]\!]^*$ *onto* $[\![\gamma(T)]\!]^*$*, with inverse* γ^{-1}*. In addition, for all* $T' \in [\![T]\!]^*$, γ *induces an order isomorphism from* $[T']$ *onto* $[\gamma(T')]$.

Proof: The first statement follows from the General Correspondence Theorem 1552 by instantiation to the Leibniz operator. So we undertake the proof of the additional statement. Suppose that $T', T'' \in [\![T]\!]^*$. Since $T' \in [\![T]\!]^*$, we get $[T'] \subseteq [\![T']\!]^* \subseteq [\![T]\!]^*$. Thus, by the first statement, $\gamma^{-1}(\gamma(T')) = T'$ and $\gamma^{-1}(\gamma(T'')) = T''$. Thus, we get

$$\begin{array}{rll} \Omega^{\mathcal{A}}(T'') = \Omega^{\mathcal{A}}(T') & \text{iff} & \Omega^{\mathcal{A}}(\gamma^{-1}(\gamma(T''))) = \Omega^{\mathcal{A}}(\gamma^{-1}(\gamma(T'))) \\ & \text{iff} & \gamma^{-1}(\Omega^{\mathcal{B}}(\gamma(T''))) = \gamma^{-1}(\Omega^{\mathcal{B}}(\gamma(T'))) \\ & \text{iff} & \Omega^{\mathcal{B}}(\gamma(T'')) = \Omega^{\mathcal{B}}(\gamma(T')). \end{array}$$

So $T'' \in [T']$ if and only if $\gamma(T'') \in [\gamma(T')]$. Thus, the order isomorphism $\gamma : [\![T]\!]^* \to [\![\gamma(T)]\!]^*$ restricts to an order isomorphism $\gamma : [T] \to [\gamma(T)]$. ■

As a consequence of Correspondence Theorem, we get a correspondence between Leibniz filter families.

Corollary 1575 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ *two* $\mathbf{F}$*-algebraic systems,* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism, with* H *an isomorphism, and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. *If* $\langle H, \gamma \rangle$ *is* Ω*-compatible with* T*, then*

$$T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \quad \textit{iff} \quad \gamma(T) \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{B}).$$

Proof: By Theorem 1574, under the isomorphism $\gamma : [\![T]\!]^* \to [\![\gamma(T)]\!]^*$, the least theory family T^* of $[\![T]\!]^*$ corresponds to the least theory family $\gamma(T)^*$ of $[\![\gamma(T)]\!]^*$. Therefore, $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$ if and only if $T = T^*$ if and only if $\gamma(T) = \gamma(T)^*$ if and only if $\gamma(T) \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{B})$. ■

Rephrased in terms of strict surjective morphisms Corollary 1575 yields

Corollary 1576 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ *two* $\mathbf{F}$*-algebraic systems,* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $T' \in \mathrm{FiFam}^{\mathcal{A}}(\mathcal{B})$ *and* $\langle H, \gamma \rangle : \langle \mathcal{A}, T \rangle \to \langle \mathcal{B}, T' \rangle$ *a strict surjective morphism, with* H *an isomorphism. Then*

$$T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \quad \textit{iff} \quad T' \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{B}).$$

Proof: It suffices to show that $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ is Ω-compatible with T. If that is the case, then, since $T = \gamma^{-1}(T')$, we get, $T' = \gamma(\gamma^{-1}(T')) = \gamma(T)$, and the statement follows by applying Corollary 1575. We have, indeed

$$\begin{array}{rcl}\mathrm{Ker}(\langle H, \gamma \rangle) & = & \gamma^{-1}(\Delta^{\mathcal{B}}) \\ & \leq & \gamma^{-1}(\Omega^{\mathcal{B}}(T')) \\ & = & \Omega^{\mathcal{A}}(\gamma^{-1}(T')) \\ & = & \Omega^{\mathcal{A}}(T).\end{array}$$

Therefore, $\langle H, \gamma \rangle$ is indeed compatible with T. ■

The Correspondence Theorem 1574 allows us to formulate a Correspondence Theorem for the special case of protoalgebraic π-institutions that, as it turns out, provides an additional characterization of protoalgebraicity.

Theorem 1577 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is protoalgebraic if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$*, all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$ *and every strict surjective morphism* $\langle H, \gamma \rangle : \langle \mathcal{A}, T \rangle \to \langle \mathcal{B}, T' \rangle$*, with* H *an isomorphism,* γ *induces an order isomorphism from* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$ *onto* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$*, with inverse* γ^{-1}*.*

Proof: Suppose, first, that $\mathcal{I}$ is protoalgebraic and let $\langle H, \gamma \rangle : \langle \mathcal{A}, T \rangle \to \langle \mathcal{B}, T' \rangle$ be a strict surjective morphism, with H an isomorphism. Then, we get $T = \gamma^{-1}(T')$ and $T' = \gamma(\gamma^{-1}(T')) = \gamma(T)$. So $T = \gamma^{-1}(\gamma(T))$. This implies that $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ is compatible with T. By the Correspondence Theorem for Leibniz Classes 1574, γ induces an order isomorphism $\gamma : [\![T]\!]^* \to [\![T']\!]^*$, with inverse γ^{-1}. But, by protoalgebraicity, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$ and $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$ are upsets of $[\![T]\!]^*$ and $[\![T']\!]^*$, respectively and T corresponds to T' under γ. Therefore, γ restricts to an order isomorphism $\gamma : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \to \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$, with inverse γ^{-1}.

Suppose, conversely, that the given condition holds. Let $T,T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$. Consider the quotient morphism $\langle I,\pi\rangle : \mathcal{F} \to \mathcal{F}/\Omega(T)$. It gives a strict surjective morphism

$$\langle I,\pi\rangle : \langle \mathcal{F},T\rangle \to \langle \mathcal{F}/\Omega(T), T/\Omega(T)\rangle.$$

Since, by hypothesis, $\pi : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F})^T \to \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}/\Omega(T))^{T/\Omega(T)}$ is an order isomorphism. with inverse π^{-1} and, clearly, $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F})^T$, we get that $\pi(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}/\Omega(T))^{T/\Omega(T)}$ and $T' = \pi^{-1}(\pi(T'))$. Now we get

$$\begin{array}{rcl}\Omega(T) & = & \mathrm{Ker}(\langle I,\pi\rangle)\\ & = & \pi^{-1}(\Delta^{\mathcal{F}/\Omega(T)})\\ & \leq & \pi^{-1}(\Omega^{\mathcal{F}/\Omega(T)}(\pi(T')))\\ & = & \Omega(\pi^{-1}(\pi(T')))\\ & = & \Omega(T').\end{array}$$

Since Ω is monotone, we conclude that $\mathcal{I}$ is a protoalgebraic π-institution. ■

Now we get a characterization of those full $\mathcal{I}$-structures whose closure families are Leibniz classes.

Proposition 1578 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, such that* $\langle \mathcal{A},\mathcal{T}\rangle$ *is a full* $\mathcal{I}$*-structure. Then* $\mathcal{T} = [\![T]\!]^*$*, for some* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, if and only if* $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \in \mathrm{AlgSys}^*(\mathcal{I})$*.*

Proof: Suppose, first, that $\mathcal{T} = [\![T]\!]^*$, for some $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, using Proposition 1569, we get

$$\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^*) = \Omega^{\mathcal{A}}(T).$$

Therefore, $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \mathcal{A}/\Omega^{\mathcal{A}}(T) \in \mathrm{AlgSys}^*(\mathcal{I})$.

Assume, conversely, that $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \in \mathrm{AlgSys}^*(\mathcal{I})$. By definition, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}))$, such that $\Omega^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})}(T) = \Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})}$. This equality implies that $[\![T]\!]^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}))$. Now consider the quotient morphism $\langle I,\pi\rangle : \mathcal{A} \to \mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})$. Since, by hypothesis $\langle \mathcal{A},\mathcal{T}\rangle$ is a full $\mathcal{I}$-structure, we get

$$\mathcal{T} = \pi^{-1}(\mathcal{T}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})) = \pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}))) = \pi^{-1}([\![T]\!]^*).$$

Moreover,

$$\mathrm{Ker}(\langle I,\pi\rangle) = \pi^{-1}(\Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})}) = \pi^{-1}(\Omega^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})}(T)) = \Omega^{\mathcal{A}}(\pi^{-1}(T)).$$

So $\langle I,\pi\rangle$ is Ω-compatible with $\pi^{-1}(T)$. By the Correspondence Theorem for Leibniz Classes 1574, we get an order isomorphism $\pi : [\![\pi^{-1}(T)]\!]^* \to [\![T]\!]^*$. This gives $\mathcal{T} = \pi^{-1}([\![T]\!]^*) = [\![\pi^{-1}(T)]\!]^*$. ■

We get, as a consequence, a characterization of those π-institutions for which all full $\mathcal{I}$-structures are determined by Leibniz classes of $\mathcal{I}$-filter families.

Proposition 1579 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) = \{\langle \mathcal{A}, [\![T]\!]^* \rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}$ *if and only if* $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$*.*

Proof: Suppose, first, that, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) = \{\langle \mathcal{A}, [\![T]\!]^* \rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}$. Since $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$ holds in general, it suffices to show the reverse inclusion. To this end, let $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$. Thus, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. Since $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{I}}} \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$, we get, by hypothesis, $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{I}}} = [\![T']\!]^*$. Now notice the following:

- $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \subseteq [\![T]\!]^{\widetilde{\Omega}^{\mathcal{I}}}$, whence $\Omega^{\mathcal{A}}(T') \leq \bigcap_{T \leq T''} \Omega^{\mathcal{A}}(T'') = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$;
- $T' \in [\![T']\!]^*$ implies $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$.

We conclude that $\Omega^{\mathcal{A}}(T') = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. Hence, we have $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I})$.

Assume, conversely, that $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$. Since, by Proposition 1569, we have, in general, $\{\langle \mathcal{A}, [\![T]\!]^* \rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \subseteq \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$, it suffices to show the reverse inclusion. To this end, let $\langle \mathcal{A}, \mathcal{T} \rangle \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$. Then $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \in \mathrm{AlgSys}(\mathcal{I})$. By hypothesis, $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \in \mathrm{AlgSys}^*(\mathcal{I})$. Therefore, by Proposition 1578, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\mathcal{T} = [\![T]\!]^*$. ■

So, one way to characterize π-institutions $\mathcal{I}$ for which $\mathcal{I}$-algebraic systems and $\mathcal{I}^*$-algebraic systems coincide is to look at the form of full $\mathcal{I}$-structures. An alternative characterization uses the Leibniz and Suszko operators.

Proposition 1580 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$ *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, there exists* $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, such that* $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$*.*

Proof: Assume $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$. Let $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, so that $\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \mathrm{AlgSys}(\mathcal{I})$. By Proposition 1579, there exists $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{I}}} = [\![T']\!]^*$. We now get

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^{\widetilde{\Omega}^{\mathcal{I}}}) = \widetilde{\Omega}^{\mathcal{A}}([\![T']\!]^*) = \Omega^{\mathcal{A}}(T').$$

Assume, conversely, that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, there exists $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$. Let $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$. Then, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. By hypothesis, there exists $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T') = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. We conclude that $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I})$. The reverse inclusion always holds. Therefore, $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$. ■

Proposition 1580, gives the following feature of protoalgebraic π-institutions.

Corollary 1581 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is protoalgebraic, then every full* $\mathcal{I}$*-structure is of the form* $\langle \mathcal{A}, [\![T]\!]^* \rangle$*, for some* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and some* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

Proof: We know that, if $\mathcal{I}$ is protoalgebraic and $\mathcal{A}$ is an $\mathbf{F}$-algebraic system, then, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we have $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T)$. Therefore, by Proposition 1580 and Proposition 1579, every full $\mathcal{I}$-structure has the form claimed in the statement. ∎

This property of the full $\mathcal{I}$-structures in a more precise form, yields a characterization of protoalgebraicity.

Theorem 1582 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following conditions are equivalent:*

(i) $\mathcal{I}$ *is protoalgebraic;*

(ii) Every full $\mathcal{I}$*-structure is of the form* $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle$*, for some* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and some* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*;*

(iii) Every full $\mathcal{I}$*-structure is of the form* $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle$*, for some* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and some* $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$*;*

(iv) $[\![T]\!]^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T^*}$*, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

Proof:

(i)⇒(ii) Suppose that $\mathcal{I}$ is protoalgebraic and let $\langle \mathcal{A}, \mathcal{T} \rangle \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$. By Proposition 1563, $\mathcal{T} = \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \leq \Omega^{\mathcal{A}}(T)\}$. By protoalgebraicity, $\mathcal{T}$ is an upset in $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Moreover, $\mathcal{T}$ has a least element, $T = \bigcap \mathcal{T}$. Thus, we have $\mathcal{T} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{\cap \mathcal{T}}$.

(ii)⇒(iii) Assume (ii) holds and let $\langle \mathcal{A}, \mathcal{T} \rangle \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$. Then, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\mathcal{T} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$. By Proposition 1571, $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$.

(iii)⇒(iv) Assume (iii) holds and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. By Proposition 1569, $\langle \mathcal{A}, [\![T]\!]^* \rangle \in \mathrm{FStr}(\mathcal{I})$. By Proposition 1570, $T^* \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$ and, by definition $T^* = \bigcap [\![T]\!]^*$. Thus, by (iii), we get $[\![T]\!]^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T^*}$.

(iv)⇒(i) Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \leq T'$. Then, by Lemma 1568, $T^* \leq T \leq T'$. By hypothesis, $T' \in [\![T]\!]^*$. So we get, by definition of $[\![T]\!]^*$, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. Since Ω is monotone on every $\mathbf{F}$-algebraic system, we get that $\mathcal{I}$ is protoalgebraic. ∎

21.11 Suszko Operator as an $\mathcal{I}$-Operator

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

The $\widetilde{\Omega}^{\mathcal{I}}$**-class of** T or **Suszko class of** T is

$$[\![T]\!]^{\mathsf{Su}} = \Omega^{\mathcal{A}^{-1}}(\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)) = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\}.$$

The **Suszko filter family of** T is

$$T^{\mathsf{Su}} = \bigcap [\![T]\!]^{\mathsf{Su}}.$$

T is a **Suszko filter family** if $T^{\mathsf{Su}} = T$. The collection of all Suszko filter families of $\mathcal{A}$ is denoted by $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$.

The following lemma gives some of the basic properties of Suszko classes and Suszko theory families.

Lemma 1583 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

(a) $T^{\mathsf{Su}} \leq T^* \leq T$*;*

(b) $T^{\mathsf{Su}} = T$ *implies* $T^* = T$*;*

(c) If $T \leq T'$*, then* $[\![T']\!]^{\mathsf{Su}} \subseteq [\![T]\!]^{\mathsf{Su}}$ *and* $T^{\mathsf{Su}} \leq T'^{\mathsf{Su}}$*;*

(d) $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \subseteq [\![T]\!]^{\mathsf{Su}} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T^{\mathsf{Su}}}$*;*

(e) $[\![T]\!]^{\mathsf{Su}} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$ *if and only if* $T^{\mathsf{Su}} = T$*.*

Proof:

(a) We have $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T)$. Hence $[\![T]\!]^* \subseteq [\![T]\!]^{\mathsf{Su}}$. This gives

$$T^{\mathsf{Su}} = \bigcap [\![T]\!]^{\mathsf{Su}} \leq [\![T]\!]^* = T^*.$$

The last inequality is by Lemma 1568.

(b) If $T = T^{\mathsf{Su}}$, then, by Part (a), $T = T^*$.

(c) If $T \leq T'$, by the monotonicity of the Suszko operator, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T')$. Thus, we get $[\![T']\!]^{\mathsf{Su}} \subseteq [\![T]\!]^{\mathsf{Su}}$. Finally, $T^{\mathsf{Su}} = \bigcap [\![T]\!]^{\mathsf{Su}} \leq \bigcap [\![T']\!]^{\mathsf{Su}} = T'^{\mathsf{Su}}$.

(d) Suppose $T \leq T'$. Then $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T') \leq \Omega^{\mathcal{A}}(T')$. So $T' \in [\![T]\!]^{\mathsf{Su}}$. Moreover, if $T' \in [\![T]\!]^{\mathsf{Su}}$, then $T^{\mathsf{Su}} = \bigcap [\![T]\!]^{\mathsf{Su}} \leq T'$.

(e) By specializing Lemma 1533. ∎

For the Suszko classes, we get an analogous result to Proposition 1569, to the effect that they are closure families of full $\mathcal{I}$-structures and their Tarski congruence systems equal the Suszko congruence system of their generating theory family.

Proposition 1584 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

(a) $\langle \mathcal{A}, [\![T]\!]^{\mathsf{Su}} \rangle \in \mathrm{FStr}(\mathcal{I})$*;*

(b) $\widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^{\mathsf{Su}}) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$*.*

Proof: Part (a) is a specialization of Proposition 1526.

Since, by Lemma 1583, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \subseteq [\![T]\!]^{\mathsf{Su}}$, we get $\widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^{\mathsf{Su}}) \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. On the other hand, if $T' \in [\![T]\!]^{\mathsf{Su}}$, then $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. Hence $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \bigcap_{T' \in [\![T]\!]^{\mathsf{Su}}} \Omega^{\mathcal{A}}(T') = \widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^{\mathsf{Su}})$. Equality now follows. ∎

The mapping $T \mapsto T^{\mathsf{Su}}$ is monotone.

Lemma 1585 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system. For all* $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$T \leq T' \quad \textit{implies} \quad T^{\mathsf{Su}} \leq T'^{\mathsf{Su}}.$$

Proof: By Proposition 1529. ∎

Moreover, even though T^{Su} is not necessarily a Suszko theory family, in case it happens to be, it is the largest such below T.

Lemma 1586 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. For all* $T' \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$*, such that* $T' \leq T$*, we have* $T' \leq T^{\mathsf{Su}}$*.*

Proof: Suppose $T' \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$, such that $T' \leq T$. Then, by the hypothesis and Lemma 1585, $T' = T'^{\mathsf{Su}} \leq T^{\mathsf{Su}}$. ∎

As far as characterizing Suszko theory families, we have the following

Proposition 1587 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.* $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$ *if and only if* $T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$ *is the least* $\mathcal{I}$*-filter family of* $\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$*.*

Proof: By Proposition 1530. ∎

It turns out that the collection of Suszko theory families of a π-institution forms a join complete subsemilattice of the lattice of all theory families.

Lemma 1588 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$ *is a join complete subsemilattice of* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

Proof: Suppose $\{T^i : i \in I\} \subseteq \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. By Lemma 1585, we get, for all $i \in I$,

$$T^i = (T^i)^{\mathsf{Su}} \leq (\bigvee_{i \in I} T)^{\mathsf{Su}}.$$

This gives $\bigvee_{i \in I} T^i \leq (\bigvee_{i \in I} T^i)^{\mathsf{Su}}$. But, by Lemma 1583, $(\bigvee_{i \in I} T^i)^{\mathsf{Su}} \leq \bigvee_{i \in I} T^i$. Hence, we conclude that $\bigvee_{i \in I} T^i \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. ∎

For an $\mathbf{F}$-algebraic system $\mathcal{A}$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, it turns out that T is a Suszko $\mathcal{I}$-filter family exactly when it is the least filter family of a full $\mathcal{I}$-structure, whose closure family consists of the upset in the lattice of $\mathcal{I}$-theory families generated by T.

Theorem 1589 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. *The following conditions are equivalent:*

(i) $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$*;*

(ii) $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle \in \mathrm{FStr}(\mathcal{I})$*;*

(iii) $T = \bigcap \mathcal{T}$*, where* $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *is an upset and* $\langle \mathcal{A}, \mathcal{T} \rangle \in \mathrm{FStr}(\mathcal{I})$.

Proof:

(i)⇒(ii) Assume that $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. Then, by Lemma 1583, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T = [\![T]\!]^{\mathsf{Su}}$ and, moreover, by Proposition 1584, $\langle \mathcal{A}, [\![T]\!]^{\mathsf{Su}} \rangle \in \mathrm{FStr}(\mathcal{I})$.

(ii)⇒(iii) Assume (ii) holds and set $\mathcal{T} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$. Then, $T = \bigcap \mathcal{T}$, $\mathcal{T}$ is an upset in $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and, by hypothesis, $\langle \mathcal{A}, \mathcal{T} \rangle \in \mathrm{FStr}(\mathcal{I})$.

(iii)⇒(i) Suppose, finally, that $T = \bigcap \mathcal{T}$, where $\mathcal{T}$ is an upset in $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{T})$ and $\langle \mathcal{A}, \mathcal{T} \rangle$ is a full $\mathcal{I}$-structure. We then have $T = \bigcap \mathcal{T} \in \mathcal{T}$, since $\mathcal{T}$ is a closure family. Hence, since $\mathcal{T}$ is an upset, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \subseteq \mathcal{T}$. But, by hypothesis $T = \bigcap \mathcal{T}$, whence $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$. Thus, we get that $\mathcal{T} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$. Since $\langle \mathcal{A}, \mathcal{T} \rangle$ is a full $\mathcal{I}$-structure, we have, by Theorem 1395,

$$\mathcal{T} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T) \leq \Omega^{\mathcal{A}}(T')\}.$$

But $\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$, whence $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T = [\![T]\!]^{\mathsf{Su}}$. Now we get $T = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T = \bigcap [\![T]\!]^{\mathsf{Su}} = T^{\mathsf{Su}}$ and $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$.

■

It turns out that requiring that all $\mathcal{I}$-filter families on all $\mathbf{F}$-algebraic systems be Suszko filter families is tantamount to $\mathcal{I}$ being family completely reflective.

Theorem 1590 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following conditions are equivalent:*

(i) $\mathcal{I}$ *is family c-reflective;*

(ii) For every $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*;*

(iii) For every $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*,* $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

Proof:

(i)⇒(iii) Assume that $\mathcal{I}$ is family c-reflective and let $\mathcal{A}$ be an $\mathcal{I}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, for all $T' \in [\![T]\!]^{\mathsf{Su}}$, we have $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. By family c-reflectivity and Lemma 826, we get $T \leq T'$, Thus, $T \leq \bigcap [\![T]\!]^{\mathsf{Su}} = T^{\mathsf{Su}}$. Since, by Lemma 1583, $T^{\mathsf{Su}} \leq T$, we get $T = T^{\mathsf{Su}}$, i.e., $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. We conclude that $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

(iii)⇒(ii) Suppose that (iii) holds and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Consider the quotient morphism

$$\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T).$$

Set $T' \in \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))$. Since $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$ is compatible with T, by Corollary 57, $T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))$. Thus, by definition, $T' \leq T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. Thus, we get

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(T') \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)) = \Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}.$$

By hypothesis, since $\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \mathrm{AlgSys}(\mathcal{I})$, we get that

$$T', T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)).$$

By Proposition 1528, $T' = T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. Thus, $T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$ is the least $\mathcal{I}$-theory family on $\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. Therefore, by Proposition 1587, $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$.

(ii)⇒(i) Assume (ii) and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system, $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. By definition, $T' \in [\![T]\!]^{\mathsf{Su}}$. Since $T = T^{\mathsf{Su}}$, we get that $T = \bigcap [\![T]\!]^{\mathsf{Su}} \leq T'$. By Lemma 826, $\mathcal{I}$ is family c-reflective. ■

Using Theorem 1590, we get additional characterizations of family c-reflectivity.

Corollary 1591 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following conditions are equivalent:*

(i) $\mathcal{I}$ *is family c-reflective;*

(ii) $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle \in \mathrm{FStr}(\mathcal{I})$*, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*;*

(iii) $[\![T]\!]^{\mathsf{Su}} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$*, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

Proof:

(i)⇒(ii) Assume $\mathcal{I}$ is family c-reflective and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. By Theorem 1590, $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. Thus, by Theorem 1589, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle \in \mathrm{FStr}(\mathcal{I})$.

(ii)⇒(iii) Assume (ii). Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. By hypothesis, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle \in \mathrm{FStr}(\mathcal{I})$. Thus, by Theorem 1589, $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. Therefore, by Lemma 1583, $[\![T]\!]^{\mathsf{Su}} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$.

(iii)⇒(i) Assume (iii). Then, by Lemma 1583, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. Therefore, by Theorem 1590, $\mathcal{I}$ is family c-reflective.

■

The condition that all full $\mathcal{I}$-structures are of the form given in Part (ii) of Theorem 1591 is tantamount to weak family algebraizability.

Corollary 1592 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is weakly family algebraizable if and only if* $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) = \{\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}$*, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

Proof: By definition $\mathcal{I}$ is WF algebraizable if and only if it is protoalgebraic and family c-reflective, if and only if, by Theorem 1582 and by Corollary 1591, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) \subseteq \{\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}$ and $\{\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \subseteq \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$, if and only if $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) = \{\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}$, for every $\mathbf{F}$-algebraic system $\mathcal{A}$. ■

Moreover, as far as characterizations of WF algberaizability we obtain one that involves both Suszko classes and Suszko filter families.

Proposition 1593 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is weakly family algebraizable if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$ *and, for every* $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, such that* $\langle \mathcal{A}, \mathcal{T} \rangle \in \mathrm{FStr}(\mathcal{I})$*, there exists* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, such that* $\mathcal{T} = [\![T]\!]^{\mathsf{Su}}$*.*

Proof: Suppose that $\mathcal{I}$ is weakly family algebraizable. Since it is protoalgebraic, by Theorem 1582, for every **F**-algebraic system $\mathcal{A}$, if $\langle\mathcal{A},\mathcal{T}\rangle \in \mathrm{FStr}(\mathcal{I})$, then $\mathcal{T} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$, for some $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Since $\mathcal{I}$ is family c-reflective, by Corollary 1591, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T = [\![T]\!]^{\mathsf{Su}}$. Hence, if $\langle\mathcal{A},\mathcal{T}\rangle \in \mathrm{FStr}(\mathcal{I})$, then $\mathcal{T} = [\![T]\!]^{\mathsf{Su}}$, for some $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Finally, by Theorem 1590, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$.

Suppose, conversely, that the given property holds. Since, for every **F**-algebraic system $\mathcal{A}$, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$, by Theorem 1590, $\mathcal{I}$ is family c-reflective. By, Corollary 1591, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $[\![T]\!]^{\mathsf{Su}} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$. By hypothesis, for all $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\langle\mathcal{A},\mathcal{T}\rangle \in \mathrm{FStr}(\mathcal{I})$, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\mathcal{T} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$. Hence, by Theorem 1582, $\mathcal{I}$ is also protoalgebraic. We conclude that $\mathcal{I}$ is WF algebraizable. ■

As far as Suszko classes go, we have a special correspondence theorem that follows from the General Correspondence Theorem 1552 for O-classes.

Theorem 1594 (Correspondence Theorem for Suszko Classes) *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle$*,* $\mathcal{B} = \langle\mathbf{B},\langle G,\beta\rangle\rangle$ **F***-algebraic systems,* $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism, with* H *an isomorphism, and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. If* $\langle H,\gamma\rangle$ *is* $\widetilde{\Omega}^{\mathcal{I}}$*-compatible with* T*, then* γ *induces an order isomorphism from* $[\![T]\!]^{\mathsf{Su}}$ *to* $[\![\gamma(T)]\!]^{\mathsf{Su}}$*, with inverse* γ^{-1}*.*

Proof: By Proposition 1544, $\widetilde{\Omega}^{\mathcal{I}}$ is a semi-coherent family of compatibility $\mathcal{I}$-operators, whence, by Theorem 1552, we get the conclusion. ■

Under the hypotheses of Theorem 1594, we also obtain a correspondence between Suszko filter families:

Corollary 1595 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle$*,* $\mathcal{B} = \langle\mathbf{B},\langle G,\beta\rangle\rangle$ **F***-algebraic systems,* $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism, with* H *an isomorphism, and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, such that* $\langle H,\gamma\rangle$ *is* $\widetilde{\Omega}^{\mathcal{I}}$*-compatible with* T*. Then*

$$T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \quad \textit{iff} \quad \gamma(T) \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{B}).$$

Proof: We have

$$\begin{array}{lll} T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) & \text{iff} & T = T^{\mathsf{Su}} \\ & \text{iff} & T = \bigcap [\![T]\!]^{\mathsf{Su}} \\ & \text{iff} & \gamma(T) = \bigcap [\![\gamma(T)]\!]^{\mathsf{Su}} \quad \text{(by Theorem 1594)} \\ & \text{iff} & \gamma(T) = \gamma(T)^{\mathsf{Su}} \\ & \text{iff} & \gamma(T) \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{B}). \end{array}$$

■

Analogously to Theorem 1577, characterizing protoalgebraicity via a correspondence between posets of filter families of $\mathcal{F}$-algebraic systems related via surjective strict morphisms, we get a correspondence theorem characterizing family c-reflectivity.

Theorem 1596 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is family c-reflective if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$, *all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and* $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, *and all strict surjective* $\langle H, \gamma \rangle : \langle \mathcal{A}, T \rangle \to \langle \mathcal{B}, T' \rangle$, *with* H *an isomorphism, such that* $\langle H, \gamma \rangle$ *is* $\widetilde{\Omega}^{\mathcal{I}}$*-compatible with* T, γ *induces an order isomorphism from* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$ *onto* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'^{\mathsf{Su}}}$, *with inverse* γ^{-1}.

Proof: Suppose, first, that $\mathcal{I}$ is family c-reflective. Let $\langle H, \gamma \rangle : \langle \mathcal{A}, T \rangle \to \langle \mathcal{B}, T' \rangle$ be a strict surjective morphism, with H an isomorphism, such that $\langle H, \gamma \rangle$ is $\widetilde{\Omega}^{\mathcal{I}}$-compatible with T. By Theorem 1594, $\gamma : [\![T]\!]^{\mathsf{Su}} \to [\![T']\!]^{\mathsf{Su}}$ is an order isomorphism with inverse γ^{-1}. By Corollary 1591, $[\![T]\!]^{\mathsf{Su}} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^T$ and $[\![T']\!]^{\mathsf{Su}} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$. Moreover, by Theorem 1590, $T' = T'^{\mathsf{Su}}$. Thus, we get the conclusion.

Assume, conversely, that the given condition holds. It suffices, by Theorem 1590, to show that every $\mathcal{I}$-filter family on every $\mathbf{F}$-algebraic system is a Suszko $\mathcal{I}$-filter family. So let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Consider the quotient morphism

$$\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T).$$

Then $\langle I, \pi \rangle : \langle \mathcal{A}, T \rangle \to \langle \mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T), T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \rangle$ is a strict surjective morphism, with I an isomorphism and it is $\widetilde{\Omega}^{\mathcal{I}}$-compatible with T. By hypothesis,

$$\pi : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \to \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))^{(T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))^{\mathsf{Su}}}$$

is an order isomorphism with inverse π^{-1}.

- By Lemma 1557, we get $\widetilde{\Omega}^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}$. Thus, by the definition of a Suszko class,

$$[\![T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)]\!]^{\mathsf{Su}} = \mathrm{FiFam}^{\mathcal{I}}(T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)),$$

  whence $(T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))^{\mathsf{Su}} = \bigcap \mathrm{FiFam}^{\mathcal{I}}(T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))$ and, therefore,

$$\mathrm{FiFam}^{\mathcal{I}}(T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))^{(T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))^{\mathsf{Su}}} = \mathrm{FiFam}^{\mathcal{I}}(T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)).$$

- By Theorem 1594, $\pi : [\![T]\!]^{\mathsf{Su}} \to [\![T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)]\!]^{\mathsf{Su}}$ is an order isomorphism with inverse π^{-1}.

We conclude that $[\![T]\!]^{\mathsf{Su}} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$. By Lemma 1583, $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. Thus, every $\mathcal{I}$-filter family on every $\mathbf{F}$-algebraic system is a Suszko $\mathcal{I}$-filter family and, by Theorem 1590, $\mathcal{I}$ is family c-reflective. ∎

Along similar lines, for weakly family algebraizable π-institutions, we get the following characterization, which consists of strengthening the condition in Theorem 1596 by requiring that it holds for all strict surjective morphisms with isomorphic signature components, without additional compatibility requirements.

Theorem 1597 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is weakly family algebraizable if and only if, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$, *all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and* $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, *and all strict surjective morphisms* $\langle H, \gamma \rangle : \langle \mathcal{A}, T \rangle \to \langle \mathcal{B}, T' \rangle$, *with* H *an isomorphism,* γ *induces an order isomorphism from* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$ *onto* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'^{\mathsf{Su}}}$, *with inverse* γ^{-1}.

Proof: Suppose, first, that $\mathcal{I}$ is weakly family algebraizable. On the one hand, it is protoalgberaic, whence, by Theorem 1577, $\gamma : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \to \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$ is an order isomorphism with inverse γ^{-1}. On the other hand, it is family c-reflective, whence by Theorem 1590, $T' = T'^{\mathsf{Su}}$. This establishes the conclusion.

Assume, conversely, that the property in the statement holds. Then, by Theorem 1596, $\mathcal{I}$ is family c-reflective. Thus, by Theorem 1590, $T'^{\mathsf{Su}} = T'$. So $\gamma : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \to \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$ is an order isomorphism with inverse γ^{-1}. Hence, by Theorem 1577, $\mathcal{I}$ is also protoalgebraic. Therefore, $\mathcal{I}$, being both protoalgebraic and family c-reflective, is weakly family algebraizable. ∎

Next, in analogy with Proposition 1579, we give a characterization of those π-institutions $\mathcal{I}$ all of whose $\mathcal{I}$-structures correspond to closure families consisting of Suszko classes.

Proposition 1598 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *The following conditions are equivalent:*

(i) $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) = \{\langle \mathcal{A}, [\![T]\!]^{\mathsf{Su}} \rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}$, *for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*;*

(ii) $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ *is surjective, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$.

Proof:

(i)⇒(ii) Suppose (i) holds. Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $\theta \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$. Then, by Corollary 1565, there exists $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\langle \mathcal{A}, \mathcal{T} \rangle \in$

$\mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$ and $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \theta$. By hypothesis, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\mathcal{T} = [\![T]\!]^{\mathsf{Su}}$. Now we get, using Proposition 1584,

$$\theta = \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^{\mathsf{Su}}) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T).$$

Thus, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ is indeed surjective.

(ii)⇒(i) Assume that (ii) holds. Since, by Proposition 1584, the right-to-left inclusion in (i) always holds, it suffices to show the reverse inclusion. To this end, let $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\langle\mathcal{A},\mathcal{T}\rangle \in \mathrm{FStr}(\mathcal{I})$. Then $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \in \mathrm{AlgSys}(\mathcal{I})$, which gives that $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \in \mathrm{ConSys}^{\mathcal{A}}(\mathcal{A})$. Therefore, by hypothesis, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. Since, by Proposition 1584, $\widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^{\mathsf{Su}}) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$ and, by Proposition 1584, $\langle\mathcal{A}.[\![T]\!]^{\mathsf{Su}}\rangle \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$, we get, by the isomorphism established in Corollary 1565, that $\mathcal{T} = [\![T]\!]^{\mathsf{Su}}$.

■

The next proposition provides a characterization of weakly family algebraizable π-institution inside the class of family c-reflective ones, based on the form of their full $\mathcal{I}$-structures.

Proposition 1599 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F},C\rangle$ a family c-reflective π-institution based on $\mathbf{F}$. $\mathcal{I}$ is weakly family algebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$,*

$$\mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) = \{\langle\mathcal{A},[\![T]\!]^{\mathsf{Su}}\rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}.$$

Proof: Suppose, first, that $\mathcal{I}$ is weakly family algebraizable. Since this implies that $\mathcal{I}$ is protoalgebraic, we get that $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$. Thus, by Proposition 1579, $\mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) = \{\langle\mathcal{A},[\![T]\!]^{\mathsf{Su}}\rangle : T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}$, for every $\mathbf{F}$-algebraic system $\mathcal{A}$.

Suppose, conversely, that the condition given in the statement holds and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Since $\Omega^{\mathcal{A}}(T) \in \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A}) \subseteq \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$, by hypothesis and Proposition 1598, there exists $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T')$. Hence, $[\![T]\!]^* = [\![T']\!]^{\mathsf{Su}}$. Since $\mathcal{I}$ is family c-reflective, by Theorem 1590 and Lemma 1583, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. Thus, $T = T^* = T'^{\mathsf{Su}} = T'$. We conclude that $\Omega^{\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. By Proposition 1567, we conclude that $\mathcal{I}$ is protoalgebraic. Since, by hypothesis, it is family c-reflective, we conclude that $\mathcal{I}$ is weakly family algebraizable. ■

We see, next, that family c-reflectivity is characterized by the property that all principal filters in the lattice of filter families are Suszko full classes and, also, by the reflectivity of the Suszko operator on every $\mathbf{F}$-algebraic system.

Proposition 1600 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following conditions are equivalent:*

(i) $\mathcal{I}$ *is family c-reflective;*

(ii) $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$ *is Suszko full for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*;*

(iii) $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}(\mathcal{A})$ *is order reflecting, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

Proof:

(i)⇒(iii) Suppose that $\mathcal{I}$ is family c-reflective and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T')$. Then we get

$$\bigcap\{\Omega^{\mathcal{A}}(T'') : T \leq T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T') \leq \Omega^{\mathcal{A}}(T').$$

By hypothesis and Lemma 826, $\bigcap\{T'' : T \leq T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \leq T'$, i.e., $T \leq T'$. Thus, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ is order reflecting.

(iii)⇒(i) If $\widetilde{\Omega}^{\mathcal{I}}$ is order reflecting, then it is a fortiori injective. Thus, by Theorem 827, $\mathcal{I}$ is family c-reflective.

(ii)⇒(iii) Assume (ii) holds. Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T')$. Then, by hypothesis,

$$\begin{array}{rcl}\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T & = & \widetilde{\Omega}^{\mathcal{I},\mathcal{A}^{-1}}(\widetilde{\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T)) \\ & = & \widetilde{\Omega}^{\mathcal{I},\mathcal{A}^{-1}}(\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)) \\ & = & \{T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T'')\}.\end{array}$$

Similarly, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T'} = \{T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T') \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T'')\}$. Therefore, $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$, i.e., $T \leq T'$ and $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ is order reflecting.

(iii)⇒(ii) Assume $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ is order reflecting for every $\mathbf{F}$-algebraic system $\mathcal{A}$. Then

$$\begin{array}{l}\widetilde{\Omega}^{\mathcal{I},\mathcal{A}^{-1}}(\widetilde{\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T)) \\ \quad = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}^{-1}}(\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)) \\ \quad = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T')\} \\ \quad = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T.\end{array}$$

Hence $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$ is Suszko full. ■

Finally, we conclude the section with a characterization of protoalgebraicity in terms of the form of full $\mathcal{I}$-structures and, also, by the coincidence of Leibniz and Suszko classes.

Proposition 1601 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following conditions are equivalent:*

(i) $\mathcal{I}$ *is protoalgebraic;*

(ii) $\mathrm{FStr}(\mathcal{I}) = \{\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I},\widetilde{\Omega}^{\mathcal{I}}}(\mathcal{A})\rangle : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$*;*

(iii) $[\![T]\!]^* = [\![T]\!]^{\mathsf{Su}}$*, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

Proof:

(i)⇒(ii) Suppose $\mathcal{I}$ is protoalgebraic. Then, by Theorem 1582,

$$\mathrm{FStr}(\mathcal{I}) = \{\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I},\Omega}(\mathcal{A})\rangle : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}.$$

But, by Lemma 1518, $\widetilde{\Omega}^{\mathcal{I}} = \Omega$, whence, the conclusion follows.

(i)⇒(iii) If $\mathcal{I}$ is protoalgebraic, then, by Lemma 1518, $\widetilde{\Omega}^{\mathcal{I}} = \Omega$. Therefore, $[\![T]\!]^* = [\![T]\!]^{\mathsf{Su}}$, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

(ii)⇒(i) Suppose (ii) holds. Then, for every $\mathbf{F}$-algebraic system $\mathcal{A}$,

$$\mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) = \{\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T\rangle : T \in \mathrm{FiFam}^{\mathcal{I},\widetilde{\Omega}^{\mathcal{I}}}(\mathcal{A})\}.$$

So by Theorem 1582, $\mathcal{I}$ is protoalgebraic.

(iii)⇒(i) Assume (iii). Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \le T'$. By Lemma 1583,

$$T' \in [\![T']\!]^{\mathsf{Su}} \subseteq [\![T]\!]^{\mathsf{Su}} = [\![T]\!]^*.$$

So $\Omega^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')$. Thus, Ω is monotone and, therefore, $\mathcal{I}$ is protoalgebraic.

■

21.12 Frege Operator as an $\mathcal{I}$-Operator

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system.

Recall that $\lambda^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$ is given, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, by setting $\lambda^{\mathcal{A}}(T) = \{\lambda^{\mathcal{A}}_\Sigma(T)\}_{\Sigma \in |\mathbf{Sign}|}$, where, for all $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\lambda^{\mathcal{A}}_\Sigma(T) = \{\langle \phi, \psi \rangle \in \mathrm{SEN}(\Sigma)^2 : \phi \in T_\Sigma \text{ iff } \psi \in T_\Sigma\}.$$

Its lifting is the operator $\widetilde{\lambda}^{\mathcal{A}} : \mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \to \mathrm{EqvFam}(\mathcal{A})$, given, for all $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$\widetilde{\lambda}^{\mathcal{A}}(\mathcal{T}) = \bigcap\{\lambda^{\mathcal{A}}(T') : T' \in \mathcal{T}\}.$$

Its relativization is the operator $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{EqvFam}(\mathcal{A})$, given, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, by

$$\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) = \bigcap\{\lambda^{\mathcal{A}}(T') : T \le T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}.$$

Given $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, the $\widetilde{\lambda}^{\mathcal{I}}$**-class of** T or **Frege class of** T is

$$[\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}} = \Omega^{\mathcal{A}^{-1}}(\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T)) = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')\}.$$

Since λ is not a compatibility $\mathcal{I}$-operator, $[\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$ may not be the closure family of a full $\mathcal{I}$-structure. But, nevertheless, it is still a closure family on $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

Proposition 1602 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. Then* $[\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$ *is a closure family on* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

Proof: This is specialization of Proposition 1525. ∎

Given $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, based on Proposition 1602, we denote by $T^{\widetilde{\lambda}^{\mathcal{I}}}$ the least $\mathcal{I}$-filter family of $[\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$, i.e.,

$$T^{\widetilde{\lambda}^{\mathcal{I}}} = \bigcap [\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}.$$

Moreover, we say that T is a **Frege filter family** if $T = T^{\widetilde{\lambda}^{\mathcal{I}}}$. The collection of all Frege $\mathcal{I}$-filter families of $\mathcal{A}$ is denoted by $\mathrm{FiFam}^{\mathcal{I},\widetilde{\lambda}^{\mathcal{I}}}(\mathcal{A})$.

We give, now, a characterization of Frege filter families for π-institutions with theorems.

Lemma 1603 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*, having theorems,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

$$T \in \mathrm{FiFam}^{\mathcal{I},\widetilde{\lambda}^{\mathcal{I}}}(\mathcal{A}) \quad \textit{iff} \quad \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T) \quad \textit{iff} \quad T \in [\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}.$$

Proof: The last equivalence is by the definition of $[\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$. So it suffices to show the first equivalence.

Suppose, first, that $T \in \mathrm{FiFam}^{\mathcal{I},\widetilde{\lambda}^{\mathcal{I}}}(\mathcal{A})$. Then, we have

$$\begin{array}{lll} T & = & T^{\widetilde{\lambda}^{\mathcal{I}}} \\ & = & \bigcap [\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}} \\ & = & \bigcap\{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')\}. \end{array}$$

Thus, taking into account Proposition 1602, $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T)$.

Suppose, conversely, that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T)$. Let $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T' \in [\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$, i.e., $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')$. Let $\Sigma \in |\mathbf{Sign}|$ and $t \in C_\Sigma^{\mathcal{I},\mathcal{A}}(\varnothing)$, which exists, since $\mathcal{I}$ is assumed to have theorems. Then, if $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in T_\Sigma$, we get $C_\Sigma^{\mathcal{I},\mathcal{A}}(T_\Sigma,\phi) = T_\Sigma = C_\Sigma^{\mathcal{I},\mathcal{A}}(T_\Sigma,t)$. Thus, $\langle\phi,t\rangle \in \widetilde{\lambda}_\Sigma^{\mathcal{I},\mathcal{A}}(T) \subseteq \Omega_\Sigma^{\mathcal{A}}(T')$. Since $t \in T'_\Sigma$, by compatibility, $\phi \in T'_\Sigma$. Therefore, $T \le T'$. Now we have

$$\begin{array}{rcll} T & \le & \bigcap [\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}} & (T \le T', \text{ for all } T' \in [\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}) \\ & = & T^{\widetilde{\lambda}^{\mathcal{I}}} & (\text{by definition}) \\ & \le & T. & (\text{since } T \in [\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}) \end{array}$$

Hence, we conclude that $T = T^{\widetilde{\lambda}^{\mathcal{I}}}$ and $T \in \mathrm{FiFam}^{\mathcal{I},\widetilde{\lambda}^{\mathcal{I}}}(\mathcal{A})$. ∎

Assuming that the π-institution $\mathcal{I}$ is protoalgebraic, gives the following characterization of Frege filter families.

Corollary 1604 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle\mathbf{F},C\rangle$ be a protoalgebraic π-institution based on $\mathbf{F}$, having theorems, $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle$ an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.*

$$T \in \mathrm{FiFam}^{\mathcal{I},\widetilde{\lambda}^{\mathcal{I}}}(\mathcal{A}) \quad \textit{iff} \quad \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T).$$

Proof: If $T \in \mathrm{FiFam}^{\mathcal{I},\widetilde{\lambda}^{\mathcal{I}}}(\mathcal{A})$, then

$$\begin{array}{rcll} \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) & \le & \Omega^{\mathcal{A}}(T) & (\text{by Lemma \textbf{??}}) \\ & = & \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) & (\text{by protoalgebraicity}) \\ & \le & \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T). & (\text{by compatibility}) \end{array}$$

Thus, $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T)$. The converse is by Lemma 1603. ∎

Each component of any $\mathcal{I}$-filter family is determined by any of its elements modulo the Frege operator.

Proposition 1605 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle\mathbf{F},C\rangle$ be a π-institution based on $\mathbf{F}$, having theorems, $\mathcal{A} = \langle\mathbf{A},\langle F,\alpha\rangle\rangle$ an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in T_\Sigma$, $T_\Sigma = \phi/\widetilde{\lambda}_\Sigma^{\mathcal{I},\mathcal{A}}(T)$.*

Proof: Suppose that $T \in \mathrm{FiFam}^{\mathcal{I},\widetilde{\lambda}^{\mathcal{I}}}(\mathcal{A})$ and let $\Sigma \in |\mathbf{Sign}|$, $\phi \in T_\Sigma$.

- Let $\psi \in T_\Sigma$. Then, we have $C_\Sigma^{\mathcal{I},\mathcal{A}}(T_\Sigma,\phi) = T_\Sigma = C_\Sigma^{\mathcal{I},\mathcal{A}}(T_\Sigma,\psi)$. Thus, $\langle\phi,\psi\rangle \in \widetilde{\lambda}_\Sigma^{\mathcal{I},\mathcal{A}}(T)$, i.e., $\psi \in \phi/\widetilde{\lambda}_\Sigma^{\mathcal{I},\mathcal{A}}(T)$.
- Conversely, if $\psi \in \phi/\widetilde{\lambda}_\Sigma^{\mathcal{I},\mathcal{A}}(T)$, then $\langle\phi,\psi\rangle \in \widetilde{\lambda}_\Sigma^{\mathcal{I},\mathcal{A}}(T)$, which gives $C_\Sigma^{\mathcal{I},\mathcal{A}}(T_\Sigma,\phi) = C_\Sigma^{\mathcal{I},\mathcal{A}}(T_\Sigma,\psi)$. Since $\phi \in T_\Sigma$, we get $\psi \in T_\Sigma$.

We conclude that $T_\Sigma = \phi/\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T)$. ■

Every Frege filter family is also a Leibniz filter family.

Lemma 1606 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*, having theorems. Then*

$$\mathrm{FiFam}^{\mathcal{I},\widetilde{\lambda}^{\mathcal{I}}}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}^*}(\mathcal{A}).$$

Proof: Suppose $T \in \mathrm{FiFam}^{\mathcal{I},\widetilde{\lambda}^{\mathcal{I}}}(\mathcal{A})$. Then, by Lemma 1603, $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T)$. Since $T^* \in [\![T]\!]^*$, we also have $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T^*)$. Therefore, $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T^*)$. Thus, by definition, $T^* \in [\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$. Now we have

$$T = T^{\lambda^{\mathcal{I}}} = \bigcap [\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}} \leq T^*$$

and, since, by Lemma 1568, $T^* \leq T$ always holds, we get $T = T^*$, i.e., $T \in \mathrm{FiFam}^{\mathcal{I}^*}(\mathcal{A})$. ■

We saw that, in general, the Leibniz and Suszko filter families of a given filter family T are included in T, i.e., $T^*, T^{\mathsf{Su}} \leq T$. On the other hand, for Frege filter families, we have the reverse inclusion.

Lemma 1607 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*, having theorems,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system. For all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $T \leq T^{\widetilde{\lambda}^{\mathcal{I}}}$*.*

Proof: By Proposition 1602, $T^{\lambda^{\mathcal{I}}} \in [\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$. Thus, $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T^{\lambda^{\mathcal{I}}})$. Let $\Sigma \in |\mathbf{Sign}|$ and $t \in C^{\mathcal{I},\mathcal{A}}_\Sigma(\varnothing)$ and assume $\phi \in T_\Sigma$. Then, we have $C^{\mathcal{I},\mathcal{A}}_\Sigma(T_\Sigma, t) = T_\Sigma = C^{\mathcal{I},\mathcal{A}}_\Sigma(T_\Sigma, \phi)$, i.e., $\langle t, \phi\rangle \in \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T)$. By the preceding inequality, $\langle t, \phi\rangle \in \Omega^{\mathcal{A}}_\Sigma(T^{\widetilde{\lambda}^{\mathcal{I}}})$. But $t \in T^{\widetilde{\lambda}^{\mathcal{I}}}_\Sigma$, whence, by compatibility, $\phi \in T^{\lambda^{\mathcal{I}}}_\Sigma$. We conclude that $T \leq T^{\widetilde{\lambda}^{\mathcal{I}}}$. ■

Strong Fregeanity is characterized by compatibility of the Frege operator on theory families and, similarly, full strong Fregeanity by the compatibility of the Frege operator on filter families of arbitrary algebraic systems.

Proposition 1608 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *is strongly Fregean if and only if* $\widetilde{\lambda}^{\mathcal{I},\mathcal{F}}$ *is a compatibility* $\mathcal{I}$*-operator on* $\mathcal{F}$*;*

(b) $\mathcal{I}$ *is fully strongly Fregean if and only if* $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}$ *is a compatibility* $\mathcal{I}$*-operator on every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

Proof: We only prove Part (a) in detail. Part (b) may be proved similarly, by working on an arbitrary $\mathbf{F}$-algebraic system $\mathcal{A}$ instead of on $\mathcal{F}$.

Suppose $\mathcal{I}$ is strongly Fregean. Then, by definition, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\lambda}^{\mathcal{I},\mathcal{F}}(T) \le \widetilde{\Omega}^{\mathcal{I},\mathcal{F}}(T) \le \Omega^{\mathcal{F}}(T)$. So $\lambda^{\mathcal{I},\mathcal{F}}$ is a compatibility $\mathcal{I}$-operator on $\mathcal{F}$.

Suppose, conversely, that $\widetilde{\lambda}^{\mathcal{I},\mathcal{F}}$ is a compatibility $\mathcal{I}$-operator on $\mathcal{F}$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\lambda}^{\mathcal{I},\mathcal{F}}(T) \le \Omega^{\mathcal{F}}(T)$. Therefore,

$$\begin{array}{lcll}\widetilde{\lambda}^{\mathcal{I},\mathcal{F}}(T) & = & \bigcap\{\widetilde{\lambda}^{\mathcal{I},\mathcal{F}}(T') : T \le T' \in \mathrm{ThFam}(\mathcal{I})\} & (\text{monotonicity of } \widetilde{\lambda}^{\mathcal{I},\mathcal{F}}) \\ & \le & \bigcap\{\Omega^{\mathcal{F}}(T') : T \le T' \in \mathrm{ThFam}(\mathcal{I})\} & (\text{by the hypothesis}) \\ & = & \widetilde{\Omega}^{\mathcal{I},\mathcal{F}}(T). & (\text{by definition})\end{array}$$

Since, by compatibility, $\widetilde{\Omega}^{\mathcal{I},\mathcal{F}}(T) \le \widetilde{\lambda}^{\mathcal{I},\mathcal{F}}(T)$ always holds, we conclude that $\mathcal{I}$ is strongly Fregean. ■

The characterizations of Proposition 1608 imply that a π-institution if strongly Fregean if and only if every theory family is Frege and that it is fully strongly Fregean if and only if every filter family of any algebraic system is a Frege filter family.

Corollary 1609 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a π-institution based on* $\mathbf{F}$*, having theorems.* $\mathcal{I}$ *is strongly Fregean if and only if* $\mathrm{ThFam}(\mathcal{I}) = \mathrm{ThFam}^{\widetilde{\lambda}^{\mathcal{I}}}(\mathcal{I})$.

Proof: Suppose $\mathcal{I}$ is strongly Fregean and let $T \in \mathrm{ThFam}(\mathcal{I})$. By Proposition 1608, $\widetilde{\lambda}^{\mathcal{I},\mathcal{F}}(T) \le \Omega^{\mathcal{F}}(T)$. Thus, by Lemma 1603, $T \in \mathrm{ThFam}^{\widetilde{\lambda}^{\mathcal{I}}}(\mathcal{I})$.

Assume, conversely, that every theory family of $\mathcal{I}$ is Frege. Then, by Lemma 1603, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\lambda}^{\mathcal{I},\mathcal{F}}(T) \le \Omega^{\mathcal{F}}(T)$. Thus, by Proposition 1608, $\mathcal{I}$ is strongly Fregean. ■

Corollary 1610 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a π-institution based on* $\mathbf{F}$*, having theorems.* $\mathcal{I}$ *is fully strongly Fregean if and only if* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I},\widetilde{\lambda}^{\mathcal{I}}}(\mathcal{A})$*, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$.

Proof: The proof follows along the same lines as that of Corollary 1609, using Proposition 1608 and Lemma 1603, but applied over an arbitrary $\mathbf{F}$-algebraic system $\mathcal{A}$ instead of over $\mathcal{F}$. ■

Our next goal is to show that the Frege operator $\widetilde{\lambda}^{\mathcal{I}}$ is a semi-coherent family of $\mathcal{I}$-operators. But, first, we need to have available an isomorphism theorem involving this operator. So we embark on some preparatory work.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ $\mathbf{F}$-algebraic systems, and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. A surjective morphism $\langle H, \gamma \rangle : \langle \mathcal{A}, T \rangle \to \langle \mathcal{B}, T' \rangle$ is called **deductive** if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\gamma_\Sigma(\phi) = \gamma_\Sigma(\psi) \quad \text{implies} \quad C_\Sigma^{\mathcal{I},\mathcal{A}}(T_\Sigma, \phi) = C_\Sigma^{\mathcal{I},\mathcal{A}}(T_\Sigma, \psi).$$

Equivalently, $\langle H,\gamma\rangle$ is deductive if and only if

$$\mathrm{Ker}(\langle H,\gamma\rangle) \le \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T),$$

i.e., if and only if $\langle H,\gamma\rangle$ is $\widetilde{\lambda}^{\mathcal{I}}$-compatible with T.

We now show that for a surjective morphism, with an isomorphic signature component, compatibility properties and deductive morphisms are very closely interrelated.

Lemma 1611 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G,\beta\rangle\rangle$ $\mathbf{F}$-algebraic systems, and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. For a surjective morphism $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$, with H an isomorphism, the following statements are equivalent:*

(i) $\langle H,\gamma\rangle$ *is* $\widetilde{\Omega}^{\mathcal{I}}$*-compatible with* T*;*

(ii) $\langle H,\gamma\rangle$ *is* $\widetilde{\lambda}^{\mathcal{I}}$*-compatible with* T*;*

(iii) $\langle H,\gamma\rangle : \langle \mathcal{A}, T\rangle \to \langle \mathcal{B}, \gamma(T)\rangle$ *is deductive.*

Proof:

(i)⇒(ii) Suppose $\mathrm{Ker}(\langle H,\gamma\rangle) \le \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. Since, by compatibility, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \le \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T)$, we get that $\mathrm{Ker}(\langle H,\gamma\rangle) \le \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T)$. Thus $\langle H,\gamma\rangle$ is $\widetilde{\lambda}^{\mathcal{I}}$-compatible with T.

(ii)⇒(iii) Suppose $\mathrm{Ker}(\langle H,\gamma\rangle) \le \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T)$. This implies that $\mathrm{Ker}(\langle H,\gamma\rangle)$ is compatible with T. Indeed, if $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\gamma_\Sigma(\phi) = \gamma_\Sigma(\psi)$ and $\phi \in T_\Sigma$, then, by the hypothesis, $\langle \phi,\psi\rangle \in \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(T)$, i.e., $C^{\mathcal{I},\mathcal{A}}_\Sigma(T_\Sigma,\phi) = C^{\mathcal{I},\mathcal{A}}_\Sigma(T_\Sigma,\psi)$. Since $\phi \in T_\Sigma$, we get that $\psi \in T_\Sigma$. Thus, by Corollary 56, $\gamma(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. Moreover, by hypothesis and the comments preceding the lemma, $\langle H,\gamma\rangle : \langle \mathcal{A}, T\rangle \to \langle \mathcal{B}, \gamma(T)\rangle$ is a deductive morphism.

(iii)⇒(i) Suppose that $\mathrm{Ker}(\langle H,\gamma\rangle) \le \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T)$. Then, since, by Corollary 17, $\mathrm{Ker}(\langle H,\gamma\rangle)$ is a congruence system on $\mathcal{A}$ and, by Proposition 1420, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$ is the largest congruence system on $\mathcal{A}$ included in $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T)$, we conclude that $\mathrm{Ker}(\langle H,\gamma\rangle) \le \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. Thus, $\langle H,\gamma\rangle$ is $\widetilde{\Omega}^{\mathcal{I}}$-compatible with T.

■

We now show that each deductive morphism, with an isomorphic signature component, induces an order isomorphism between the principal filter of the lattice of filter families generated by the domain and the principal filter of the lattice of theory families generate by its codomain.

Theorem 1612 (Correspondence Theorem for Deductive Morphisms) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ $\mathbf{F}$*-algebraic systems,* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$ *and* $\langle H, \gamma \rangle : \langle \mathcal{A}, T \rangle \to \langle \mathcal{B}, T' \rangle$ *a deductive morphism, with* H *an isomorphism. Then* γ *induces an order isomorphism from* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{\gamma^{-1}(T')}$ *onto* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$*, with inverse* γ^{-1}*.*

Proof: By Lemma 1611, $\langle H, \gamma \rangle$ is $\widetilde{\Omega}^{\mathcal{I}}$-compatible with T. This implies that, for every $T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{\gamma^{-1}(T')}$, $\mathrm{Ker}(\langle H, \gamma \rangle)$ is compatible with T''. It follows by Corollary 56 that, for all $T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{\gamma^{-1}(T')}$, $\gamma(T'') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$. Moreover, the same compatibility property implies that $\gamma^{-1}(\gamma(T'')) = T''$, for all $T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{\gamma^{-1}(T')}$. Finally, by surjectivity of $\langle H, \gamma \rangle$, we get, for all $T''' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$, $\gamma(\gamma^{-1}(T''')) = T'''$. Therefore, $\gamma : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{\gamma^{-1}(T')} \to \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$ is a bijection and, since, clearly, both γ and γ^{-1} are order preserving, they are mutually inverse order isomorphisms, as claimed. ∎

Now we are ready to return to the main line of work and establish that $\widetilde{\lambda}^{\mathcal{I}}$ constitutes a semi-coherent family of $\mathcal{I}$-operators.

Theorem 1613 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The Frege operator* $\widetilde{\lambda}^{\mathcal{I}}$ *is a semi-coherent family of* $\mathcal{I}$*-operators.*

Proof: Let $\mathcal{A}$, $\mathcal{B}$ be $\mathbf{F}$-algebraic systems, $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism, with H an isomorphism, and $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, such that $\langle H, \gamma \rangle$ is $\widetilde{\lambda}^{\mathcal{I}}$-compatible with $\gamma^{-1}(T')$. Then, by Lemma 1611 and Theorem 1612, $\gamma : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{\gamma^{-1}(T')} \to \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$ is an order isomorphism with inverse γ^{-1}. Thus, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{\gamma^{-1}(T')}$, $\gamma(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})^{T'}$ and $\gamma^{-1}(\gamma(T)) = T$. Now we get, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{rcl}
C^{\mathcal{I},\mathcal{B}}_{H(\Sigma)}(T'_{H(\Sigma)}, \gamma_\Sigma(\phi)) & = & C^{\mathcal{I},\mathcal{B}}_{H(\Sigma)}(\gamma_\Sigma(\gamma^{-1}_\Sigma(T'_{H(\Sigma)})), \gamma_\Sigma(\phi)) \\
& = & C^{\mathcal{I},\mathcal{B}}_{H(\Sigma)}(\gamma_\Sigma(C^{\mathcal{I},\mathcal{A}}_\Sigma(\gamma^{-1}_\Sigma(T'_{H(\Sigma)}), \phi))) \\
& = & \gamma_\Sigma(C^{\mathcal{I},\mathcal{A}}_\Sigma(\gamma^{-1}_\Sigma(T'_{H(\Sigma)}), \phi)).
\end{array}$$

Therefore, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{ll}
\multicolumn{2}{l}{\langle \phi, \psi \rangle \in \gamma^{-1}_\Sigma(\widetilde{\lambda}^{\mathcal{I},\mathcal{B}}_{H(\Sigma)}(T'))} \\
\text{iff} & \langle \gamma_\Sigma(\phi), \gamma_\Sigma(\psi) \rangle \in \widetilde{\lambda}^{\mathcal{I},\mathcal{B}}_{H(\Sigma)}(T') \\
\text{iff} & C^{\mathcal{I},\mathcal{B}}_{H(\Sigma)}(T'_{H(\Sigma)}, \gamma_\Sigma(\phi)) = C^{\mathcal{I},\mathcal{B}}_{H(\Sigma)}(T'_{H(\Sigma)}, \gamma_\Sigma(\psi)) \\
\text{iff} & \gamma_\Sigma(C^{\mathcal{I},\mathcal{A}}_\Sigma(\gamma^{-1}_\Sigma(T'_{H(\Sigma)}), \phi)) = \gamma_\Sigma(C^{\mathcal{I},\mathcal{A}}_\Sigma(\gamma^{-1}_\Sigma(T'_{H(\Sigma)}), \psi)) \\
\text{iff} & C^{\mathcal{I},\mathcal{A}}_\Sigma(\gamma^{-1}_\Sigma(T'_{H(\Sigma)}), \phi) = C^{\mathcal{I},\mathcal{A}}_\Sigma(\gamma^{-1}_\Sigma(T'_{H(\Sigma)}), \psi) \\
\text{iff} & \langle \phi, \psi \rangle \in \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(\gamma^{-1}(T')).
\end{array}$$

We conclude that $\widetilde{\lambda}^{\mathcal{I}}$ is semi-coherent. ∎

It turns out that if we strengthen the semi-coherence condition by requiring that $\widetilde{\lambda}^{\mathcal{I}}$ be commuting over all morphisms, with isomorphic signature components (regardless of compatibility), then we get a characterization of protoalgebraicity.

Theorem 1614 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is protoalgebraic if and only if* $\widetilde{\lambda}^{\mathcal{I}}$ *is a semi-commuting family of* $\mathcal{I}$*-operators.*

Proof: Suppose, first, that $\mathcal{I}$ is protoalgebraic and let $\mathcal{A}$, $\mathcal{B}$ be $\mathbf{F}$-algebraic systems, $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism, with H an isomorphism, and $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. Then, by Corollary 55, $\gamma^{-1}(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$. Then

$$\begin{aligned}
\langle \phi, \psi \rangle \in \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}_\Sigma(\gamma^{-1}(T')) & \\
\text{iff} \quad & C^{\mathcal{I},\mathcal{A}}_\Sigma(\gamma^{-1}_\Sigma(T'_{H(\Sigma)}), \phi) = C^{\mathcal{I},\mathcal{A}}_\Sigma(\gamma^{-1}_\Sigma(T'_{H(\Sigma)}), \psi) \\
\text{iff} \quad & C^{\mathcal{I},\mathcal{B}}_{H(\Sigma)}(T'_{H(\Sigma)}, \gamma_\Sigma(\phi)) = C^{\mathcal{I},\mathcal{B}}_{H(\Sigma)}(T'_{H(\Sigma)}, \gamma_\Sigma(\psi)) \\
\text{iff} \quad & \langle \gamma_\Sigma(\phi), \gamma_\Sigma(\psi) \rangle \in \widetilde{\lambda}^{\mathcal{I},\mathcal{B}}_{H(\Sigma)}(T') \\
\text{iff} \quad & \langle \phi, \psi \rangle \in \gamma^{-1}_\Sigma(\widetilde{\lambda}^{\mathcal{I},\mathcal{B}}_{H(\Sigma)}(T')).
\end{aligned}$$

Therefore, $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\gamma^{-1}(T')) = \gamma^{-1}(\widetilde{\lambda}^{\mathcal{I},\mathcal{B}}(T'))$ and $\widetilde{\lambda}^{\mathcal{I}}$ is a semi-commuting family of $\mathcal{I}$-operators.

Assume, conversely, that $\widetilde{\lambda}^{\mathcal{I}}$ is semi-commuting and let $\mathcal{A}$, $\mathcal{B}$ be $\mathbf{F}$-algebraic systems, $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism, with H an isomorphism, and $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. Since $\widetilde{\Omega}^{\mathcal{I},\mathcal{B}}(T') \leq \widetilde{\lambda}^{\mathcal{I},\mathcal{B}}(T')$, we get

$$\gamma^{-1}(\widetilde{\Omega}^{\mathcal{I},\mathcal{B}}(T')) \leq \gamma^{-1}(\widetilde{\lambda}^{\mathcal{I},\mathcal{B}}(T')) = \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\gamma^{-1}(T')).$$

By Corollary 17, $\gamma^{-1}(\widetilde{\Omega}^{\mathcal{I},\mathcal{B}}(T'))$ is a congruence system on $\mathcal{A}$. By Proposition 1420, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(\gamma^{-1}(T'))$ is the largest congruence system below $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\gamma^{-1}(T'))$. Therefore, we get $\gamma^{-1}(\widetilde{\Omega}^{\mathcal{I},\mathcal{B}}(T')) \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(\gamma^{-1}(T'))$. Since the converse inclusion always holds, $\widetilde{\Omega}^{\mathcal{I}}$ is semi-commuting. Thus, by Corollary 1537, we get that $\widetilde{\Omega}^{\mathcal{I}} = \Omega$ and, therefore, by Lemma 1518, $\mathcal{I}$ is protoalgebraic. ■

We also get a commutativity property with direct images.

Lemma 1615 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ $\mathbf{F}$*-algebraic systems,* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism, with* H *an isomorphism and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. *If* $\langle H, \gamma \rangle$ *is* $\widetilde{\lambda}^{\mathcal{I}}$*-compatible with* T*, then*

$$\gamma(\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T)) = \widetilde{\lambda}^{\mathcal{I},\mathcal{B}}(\gamma(T)).$$

Proof: By hypothesis and Lemma 1611, $\langle H, \gamma \rangle$ is $\widetilde{\Omega}^{\mathcal{I}}$-compatible with T. Therefore, by Corollary 56, $\gamma(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$ and, also, $T = \gamma^{-1}(\gamma(T))$. Since, by Theorem 1613, $\widetilde{\lambda}^{\mathcal{I}}$ is semi-coherent, we get

$$\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\gamma^{-1}(\gamma(T))) = \gamma^{-1}(\widetilde{\lambda}^{\mathcal{I},\mathcal{B}}(\gamma(T))).$$

Hence, by the surjectivity of $\langle H,\gamma\rangle$, we get $\gamma(\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T)) = \widetilde{\lambda}^{\mathcal{I},\mathcal{B}}(\gamma(T))$. ■

In analogy with previous correspondence theorems we have the following one regarding correspondence between Frege classes.

Theorem 1616 (Correspondence Theorem for Frege Classes) *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$*,* $\mathcal{B} = \langle \mathbf{B}, \langle G,\beta\rangle\rangle$ $\mathbf{F}$*-algebraic systems,* $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism, with* H *an isomorphism and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. If* $\langle H,\gamma\rangle$ *is* $\widetilde{\lambda}^{\mathcal{I}}$*-compatible with* T*, then* γ *induces an order isomorphism from* $[\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$ *onto* $[\![\gamma(T)]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$*, with inverse* γ^{-1}*.*

Proof: Since $\langle H,\gamma\rangle$ is $\widetilde{\lambda}^{\mathcal{I}}$-compatible with T, we get, by Lemma 1611, that $\langle H,\gamma\rangle$ is $\widetilde{\Omega}^{\mathcal{I}}$-compatible with T. Therefore, by Corollary 56, $\gamma(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$ and, also, $T = \gamma^{-1}(\gamma(T))$.

Now, let $T' \in [\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$. Then $\mathrm{Ker}(\langle H,\gamma\rangle) \le \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')$. As a consequence, we get $\gamma(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$ and $\gamma^{-1}(\gamma(T')) = T'$. Now we get

$$\begin{array}{rcll}\widetilde{\lambda}^{\mathcal{I},\mathcal{B}}(\gamma(T)) & = & \gamma(\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T)) & \text{(by Lemma 1615)}\\ & \le & \gamma(\Omega^{\mathcal{A}}(T')) & (\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T'))\\ & = & \Omega^{\mathcal{B}}(\gamma(T')). & \text{(by Lemma 1542).}\end{array}$$

We conclude that $\gamma(T') \in [\![\gamma(T)]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$.

Suppose, conversely, that $T' \in [\![\gamma(T)]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$. Then $\widetilde{\lambda}^{\mathcal{I},\mathcal{B}}(\gamma(T)) \le \Omega^{\mathcal{B}}(T')$. By Corollary 55, $\gamma^{-1}(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and, by surjectivity, $\gamma(\gamma^{-1}(T')) = T'$. Moreover, $\langle H,\gamma\rangle$ is $\widetilde{\lambda}^{\mathcal{I}}$-compatible with $\gamma^{-1}(\gamma(T)) = T$. Hence, we have

$$\begin{array}{rcll}\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) & = & \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\gamma^{-1}(\gamma(T))) & \\ & = & \gamma^{-1}(\widetilde{\lambda}^{\mathcal{I},\mathcal{B}}(T)) & \text{(by Theorem 1613)}\\ & \le & \gamma^{-1}(\Omega^{\mathcal{B}}(T')) & (\widetilde{\lambda}^{\mathcal{I},\mathcal{B}}(\gamma(T)) \le \Omega^{\mathcal{B}}(T'))\\ & = & \Omega^{\mathcal{A}}(\gamma^{-1}(T')). & \text{(by Proposition 24)}\end{array}$$

Hence, $\gamma^{-1}(T') \in [\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$.

Thus, $\gamma : [\![T]\!]^{\widetilde{\lambda}^{\mathcal{I}}} \to [\![\gamma(T)]\!]^{\widetilde{\lambda}^{\mathcal{I}}}$ is a bijection, with inverse γ^{-1}. Since both mappings are order-preserving, we conclude that they form a pair of mutually inverse order isomorphisms. ■

This correspondence theorem allows us to provide characterizations of full self extensionality and full strong Fregeanity in the following two corollaries.

Corollary 1617 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is fully self extensional if and only if, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*,* $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}$*.*

Proof: Suppose, first, that $\mathcal{I}$ is fully self extensional and let $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$. Then we have

$$\begin{array}{lll}\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) & = & \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \\ & & \text{(by full self extensionality)} \\ & = & \Delta^{\mathcal{A}}. \quad (\text{since } \mathcal{A} \in \mathrm{AlgSys}(\mathcal{I}))\end{array}$$

Suppose, conversely, that, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}$, and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system. Set, for notational convenience and brevity, $\mathcal{B} = \mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$, and consider the quotient morphism

$$\langle I,\pi\rangle : \mathcal{A} \to \mathcal{B}.$$

Then $\mathrm{Ker}(\langle I,\pi\rangle) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$, whence $\mathrm{Ker}(\langle I,\pi\rangle)$ is compatible with $\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Hence, we get, by Corollary 56, $\pi(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$ and $\pi^{-1}(\pi(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))) = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Since, $\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is the least $\mathcal{I}$-family of $\mathcal{A}$, it must be a Suszko $\mathcal{I}$-filter family. Hence, by Corollary 1554, $\pi(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$ is the least $\mathcal{I}$-filter family on $\mathcal{B}$, i.e., we have

$$\pi(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}).$$

Since $\mathrm{Ker}(\langle I,\pi\rangle) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \leq \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$, $\langle I,\pi\rangle$ is $\widetilde{\lambda}^{\mathcal{I}}$-compatible with $\pi^{-1}(\pi(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))) = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Moreover, since $\mathcal{B} \in \mathrm{AlgSys}(\mathcal{I})$, we get, by hypothesis, $\widetilde{\lambda}^{\mathcal{I},\mathcal{B}}(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})) = \Delta^{\mathcal{B}}$. Now we get

$$\begin{array}{lll}\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) & = & \mathrm{Ker}(\langle I,\pi\rangle) \quad (\text{definition of } \langle I,\pi\rangle) \\ & = & \pi^{-1}(\Delta^{\mathcal{B}}) \quad (\text{definition of kernel}) \\ & = & \pi^{-1}(\widetilde{\lambda}^{\mathcal{I},\mathcal{B}}(\pi(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})))) \\ & & (\text{shown above}) \\ & = & \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\pi^{-1}(\pi(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})))) \\ & & (\text{Theorem 1614}) \\ & = & \widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})).\end{array}$$

We conclude that $\mathcal{I}$ is fully self extensional. ■

Corollary 1618 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is fully strongly Fregean if and only if, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$.

Proof: The left-to-right implication follows directly by the definition of full strong Fregeanity. Assume, conversely, that, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. Let $\mathcal{A}$ be an arbitrary $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and consider the quotient morphism

$$\langle I,\pi\rangle : \mathcal{A} \to \mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T).$$

$\mathrm{Ker}(\langle I,\pi\rangle)=\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$ is compatible with T. Hence $\pi(T)\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))$ and $\pi^{-1}(\pi(T))=T$. Moreover, $\mathrm{Ker}(\langle I,\pi\rangle)=\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)\le\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T)$. Thus, $\langle I,\pi\rangle$ is $\widetilde{\lambda}^{\mathcal{I}}$-compatible with T. Since $\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)\in\mathrm{AlgSys}(\mathcal{I})$, we get, by hypothesis,

$$\widetilde{\lambda}^{\mathcal{I},\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(\pi(T))=\widetilde{\Omega}^{\mathcal{I},\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(\pi(T))=\Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}.$$

Now we have

$$\begin{array}{rcll}\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)&=&\mathrm{Ker}(\langle I,\pi\rangle) & \text{(definition of } \langle I,\pi\rangle)\\ &=&\pi^{-1}(\Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}) & \text{(definition of kernel)}\\ &=&\pi^{-1}(\widetilde{\lambda}^{\mathcal{I},\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(\pi(T))) & \text{(shown above)}\\ &=&\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(\pi^{-1}(\pi(T))) & \text{(Theorem 1614)}\\ &=&\widetilde{\lambda}^{\mathcal{I},\mathcal{A}}(T).\end{array}$$

We conclude that $\mathcal{I}$ is fully strongly Fregean. ■

On the other hand, strong Fregeanity, combined with the existence of natural theorems, implies assertionality.

Corollary 1619 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a π-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is strongly Fregean and has natural theorems, then it is syntactically family assertional.*

Proof: Assume $\mathcal{I}$ is strongly Fregean and has natural theorems. Let $\vartheta^\flat:(\mathrm{SEN}^\flat)^k\to\mathrm{SEN}^\flat$ be a natural theorem. Then define $\tau^\flat:(\mathrm{SEN}^\flat)^{k+1}\to\mathrm{SEN}^\flat$ by

$$\tau^\flat:=\{p^{k+1,0}\approx\vartheta^\flat\circ\langle p^{k+1,1},\dots,p^{k+1,k}\rangle\}.$$

We show, first, that, for all $T\in\mathrm{ThFam}(\mathcal{I})$, all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\vec{\chi},\vec{\chi}'\in\mathrm{SEN}^\flat(\Sigma)$, $\langle\vartheta^\flat_\Sigma(\vec{\chi}),\vartheta^\flat_\Sigma(\vec{\chi}'\rangle\in\Omega_\Sigma(T)$. Since, $\vartheta^\flat$ is a natural theorem, we have, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\vec{\chi},\vec{\chi}'\in\mathrm{SEN}^\flat(\Sigma)$, $\vartheta^\flat_\Sigma(\vec{\chi}),\vartheta^\flat_\Sigma(\vec{\chi}')\in\mathrm{Thm}_\Sigma(\mathcal{I})$. Therefore, we get $\langle\vartheta^\flat_\Sigma(\vec{\chi}),\vartheta^\flat_\Sigma(\vec{\chi}')\rangle\in\widetilde{\lambda}^{\mathcal{I}}_\Sigma(T)$. Thus, by strong Fregeanity, $\langle\vartheta^\flat_\Sigma(\vec{\chi}),\vartheta^\flat_\Sigma(\vec{\chi}')\rangle\in\Omega^{\mathcal{I}}_\Sigma(T)\le\Omega_\Sigma(T)$.

We show, next, that $\mathcal{I}$ is systemic. To this end, let $T\in\mathrm{ThFam}(\mathcal{I})$, $\Sigma,\Sigma'\in|\mathbf{Sign}^\flat|$, $f\in\mathbf{Sign}^\flat(\Sigma,\Sigma')$ and $\phi\in\mathrm{SEN}^\flat(\Sigma)$, such that $\phi\in T_\Sigma$. Let $t\in\mathrm{Thm}_\Sigma(\mathcal{I})$. Then, we have $\langle\phi,t\rangle\in\widetilde{\lambda}^{\mathcal{I}}_\Sigma(T)=\widetilde{\Omega}^{\mathcal{I}}_\Sigma(T)$. Hence, since $\widetilde{\Omega}^{\mathcal{I}}(T)$ is a congruence system, we get $\langle\mathrm{SEN}^\flat(f)(\phi),\mathrm{SEN}^\flat(f)(t)\rangle\in\widetilde{\Omega}^{\mathcal{I}}_{\Sigma'}(T)=\widetilde{\lambda}^{\mathcal{I}}_{\Sigma'}(T)$. But $\mathrm{SEN}^\flat(f)(t)\in\mathrm{Thm}_{\Sigma'}(\mathcal{I})\subseteq T_{\Sigma'}$ and, therefore, by compatibility, $\mathrm{SEN}^\flat(f)(\phi)\in T_{\Sigma'}$. Hence, $T\in\mathrm{ThSys}(\mathcal{I})$ and $\mathcal{I}$ is systemic.

Finally, we show that, for all $T\in\mathrm{ThFam}(\mathcal{I})$, $\Sigma\in|\mathbf{Sign}^\flat|$ and $\phi\in\mathrm{SEN}^\flat(\Sigma)$,

$$\phi\in T_\Sigma\quad\text{iff}\quad\tau^\flat_\Sigma[\phi]\le\Omega_\Sigma(T).$$

Assume, first $\phi\in T_\Sigma$. By systemicity, for every $\Sigma'\in|\mathbf{Sign}^\flat|$ and all $f\in\mathbf{Sign}^\flat(\Sigma,\Sigma')$, $\mathrm{SEN}^\flat(f)(\phi)\in T_{\Sigma'}$. Therefore, for all $\vec{\chi}'\in\mathrm{SEN}^\flat(\Sigma')$, $\langle\mathrm{SEN}^\flat(f)(\phi),\vartheta^\flat_{\Sigma'}(\vec{\chi}')\rangle\in\widetilde{\lambda}^{\mathcal{I}}_{\Sigma'}(T)=\widetilde{\Omega}^{\mathcal{I}}_{\Sigma'}(T)\le\Omega_{\Sigma'}(T)$. If, conversely, for all $\Sigma'\in|\mathbf{Sign}^\flat|$, all $f\in$

$\mathbf{Sign}(\Sigma,\Sigma')$ and all $\vec{\chi}' \in \mathrm{SEN}^\flat(\Sigma')$, we have $\langle \mathrm{SEN}^\flat(\phi), \vartheta^\flat_{\Sigma'}(\vec{\chi}')\rangle \in \Omega_{\Sigma'}(T)$, then, in particular for $f = i_\Sigma$, we get, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $\langle \phi, \vartheta^\flat_\Sigma(\vec{\chi})\rangle \in \Omega_\Sigma(T)$. Since $\vartheta^\flat_\Sigma(\vec{\chi}) \in \mathrm{Thm}_\Sigma(\mathcal{I}) \subseteq T_\Sigma$, we get, by compatibility, that $\phi \in T_\Sigma$. ■

Finally, combining this work with previously obtained results, we get the following corollary comparing the injectivity of the various $\mathcal{I}$-operators we have studied.

Corollary 1620 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) If $\widetilde{\Omega}^{\mathcal{I}}$ *is injective, then* $\widetilde{\Omega}$ *is injective.*

(b) If Ω *is injective, then* $\widetilde{\lambda}^{\mathcal{I}}$ *is injective.*

Proof:

(a) If $\widetilde{\Omega}^{\mathcal{I}}$ is injective, then, by Theorem 827, Ω is c-reflective. Thus, it is, a fortiori, injective.

(b) If Ω is injective, then, necessarily, $\mathcal{I}$ has theorems. Therefore, by Theorem 495, we get that $\widetilde{\lambda}^{\mathcal{I}}$ is injective.

■

21.13 Leibniz Hierarchy Revisited

Using the Isomorphism Theorem 1408 between full $\mathcal{I}$-structures and $\mathcal{I}$-congruence systems, we obtain, in the case of protoalgebraic π-institutions, the following special isomorphism theorem between Leibniz $\mathcal{I}$-filter families and $\mathcal{I}^*$-congruence systems.

Proposition 1621 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a protoalgebraic* π*-institution based on* $\mathbf{F}$. *Then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, the Leibniz operator* $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is an order isomorphism.*

Proof: By Theorem 1408, for every $\mathbf{F}$-algebraic system $\mathcal{A}$,

$$\widetilde{\Omega}^{\mathcal{A}} : \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is an order isomorphism. By protoalgebraicity and Theorem 1582,

$$\mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) = \{\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T\rangle : T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})\}.$$

Moreover, by protoalgebraicity, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T) = \Omega^{\mathcal{A}}(T)$ and, also, $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}) = \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. Therefore, we get that $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism. ■

We now show that $\Omega^{\mathcal{A}}$, as a mapping from Leibniz $\mathcal{I}$-filter families to $\mathcal{I}^*$-congruence systems on $\mathcal{I}$-algebraic systems, being an order isomorphism is sufficient to establish that the same mapping is an order isomorphism for all $\mathbf{F}$-algebraic systems.

Proposition 1622 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then, the following conditions are equivalent:*

(i) For every $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is an order isomorphism;*

(ii) For every $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*,* $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is an order isomorphism.*

Proof: Since (i)⇒(ii) is trivial, assume (ii) holds and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an arbitrary $\mathbf{F}$-algebraic system.

By Proposition 1528, $\Omega^{\mathcal{A}}$ is injective on $\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$.

To show surjectivity, assume $\theta \in \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. By definition, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\theta = \Omega^{\mathcal{A}}(T)$. Consider the quotient morphism $\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T)$. Since $\mathrm{Ker}(\langle I, \pi \rangle) = \Omega^{\mathcal{A}}(T)$ is compatible with T, by Corollary 56, $\pi(T) \in \mathrm{Fifam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T))$. Since $\mathcal{A}/\Omega^{\mathcal{A}}(T) \in \mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$, we get, by hypothesis, that there exists $T' \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}/\Omega^{\mathcal{A}}(T))$, such that $\Omega^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\pi(T)) = \Omega^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(T')$. Now we have

$$\begin{array}{rcll}\Omega^{\mathcal{A}}(T) & = & \Omega^{\mathcal{A}}(\pi^{-1}(\pi(T))) & (\mathrm{Ker}(\langle I, \pi \rangle) \text{ compatible with } T)\\ & = & \pi^{-1}(\Omega^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\pi(T))) & \\ & = & \pi^{-1}(\Omega^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(T')) & \\ & = & \Omega^{\mathcal{A}}(\pi^{-1}(T')). & \end{array}$$

Since $\mathrm{Ker}(\langle I, \pi \rangle) = \Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(\pi^{-1}(T'))$, we get that $\mathrm{Ker}(\langle I, \pi \rangle)$ is compatible with $\pi^{-1}(T')$, and, hence, $\pi(\pi^{-1}(T')) = T' \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}/\Omega^{\mathcal{A}}(T))$. by Corollary 1575, $\pi^{-1}(T') \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. We showed that $\theta = \Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(\pi^{-1}(T'))$, with $\pi^{-1}(T') \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. Therefore, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is surjective.

Next, we turn to monotonicity. To this end, let $T, T' \in \mathrm{Fifam}^{\mathcal{I}*}(\mathcal{A})$, such that $T \leq T'$. Consider the quotient morphism

$$\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T).$$

We have $\mathrm{Ker}(\langle I, \pi \rangle) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T)$ and, also, $\mathrm{Ker}(\langle I, \pi \rangle) = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T') \leq \Omega^{\mathcal{A}}(T')$. Thus, by Corollary 56,

$$\pi(T), \pi(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)).$$

Since $\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \text{AlgSys}(\mathcal{I})$, we get, by hypothesis, $\Omega^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(\pi(T)) \le \Omega^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(\pi(T'))$. Therefore,

$$\begin{array}{rcll} \Omega^{\mathcal{A}}(T) & = & \Omega^{\mathcal{A}}(\pi^{-1}(\pi(T))) & \text{(compatibility)} \\ & = & \pi^{-1}(\Omega^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(\pi(T))) & \\ & \le & \pi^{-1}(\Omega^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(\pi(T'))) & \\ & = & \Omega^{\mathcal{A}}(\pi^{-1}(\pi(T'))) & \\ & = & \Omega^{\mathcal{A}}(T'). & \end{array}$$

Hence $\Omega^{\mathcal{A}} : \text{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \text{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is monotone. Finally, by Proposition 1528, $\Omega^{\mathcal{A}}$ is reflective on $\text{FiFam}^{\mathcal{I}*}(\mathcal{A})$. Thus, we conclude that $\Omega^{\mathcal{A}} : \text{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \text{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism. ■

Next, we show that, if $\Omega^{\mathcal{A}}$ from the Leibniz filter families onto the $\mathcal{I}^*$-congruence systems happens to be an order isomorphism on every $\mathcal{I}$-algebraic system, then the class of $\mathcal{I}$-algebraic systems coincides with the class of $\mathcal{I}^*$-algebraic systems.

Lemma 1623 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If, for all* $\mathcal{A} \in \text{AlgSys}(\mathcal{I})$*,*

$$\Omega^{\mathcal{A}} : \text{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \text{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism, then $\text{AlgSys}(\mathcal{I}) = \text{AlgSys}^*(\mathcal{I})$*.*

Proof: By Corollary 1405, we know that $\text{AlgSys}^*(\mathcal{I}) \subseteq \text{AlgSys}(\mathcal{I})$ always holds. So it suffices to show the reverse inclusion. To this end, let $\mathcal{A} \in \text{AlgSys}(\mathcal{I})$ and let $T^m = \bigcap \text{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, for all $T \in \text{FiFam}^{\mathcal{I}*}(\mathcal{A})$, we have, by the hypothesis, $\Omega^{\mathcal{A}}(T^m) \le \Omega^{\mathcal{A}}(T)$, which yields $[\![T]\!]^* \subseteq [\![T^m]\!]^*$.

Now let $T' \in \text{ThFam}^{\mathcal{I}}(\mathcal{A})$. By hypothesis, there exists $T \in \text{ThFam}^{\mathcal{I}*}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T') = \Omega^{\mathcal{A}}(T)$. Thus, we get $T' \in [\![T']\!]^* = [\![T]\!]^* \subseteq [\![T^m]\!]^*$. Since this holds for every $T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, we conclude that $[\![T^m]\!]^* = \text{FiFam}^{\mathcal{I}}(\mathcal{A})$. By Proposition 1578, we get $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\text{FiFam}^{\mathcal{I}}(\mathcal{A})) \in \text{AlgSys}^*(\mathcal{I})$ and, as, by hypothesis, $\mathcal{A} \in \text{AlgSys}(\mathcal{I})$ and, hence, $\widetilde{\Omega}^{\mathcal{A}}(\text{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}$, we get $\mathcal{A} = \mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\text{FiFam}^{\mathcal{I}}(\mathcal{A})) \in \text{AlgSys}^*(\mathcal{I})$. We conclude that $\text{AlgSys}(\mathcal{I}) \subseteq \text{AlgSys}^*(\mathcal{I})$ and, therefore, the two classes of algebraic systems coincide. ■

The same conclusion may be drawn if we assume that $\Omega^{\mathcal{A}}$ is an order isomorphism from the collection of Suszko $\mathcal{I}$-filter families to the collection of all $\mathcal{I}^*$-congruence systems and, in fact, the proof follows along very similar lines.

Lemma 1624 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If, for all* $\mathcal{A} \in \text{AlgSys}(\mathcal{I})$*,*

$$\Omega^{\mathcal{A}} : \text{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \to \text{ConSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism, then $\text{AlgSys}(\mathcal{I}) = \text{AlgSys}^*(\mathcal{I})$*.*

Proof: By Corollary 1405, we know that $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$ always holds. So it suffices to show the reverse inclusion. To this end, let $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$ and let $T^m = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, for all $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$, we have, by the hypothesis, $\Omega^{\mathcal{A}}(T^m) \leq \Omega^{\mathcal{A}}(T)$, which yields $[\![T]\!]^* \subseteq [\![T^m]\!]^*$.

Now let $T' \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A})$. By hypothesis, there exists $T \in \mathrm{ThFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T') = \Omega^{\mathcal{A}}(T)$. Thus, we get $T' \in [\![T']\!]^* = [\![T]\!]^* \subseteq [\![T^m]\!]^*$. Since this holds for every $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we conclude that $[\![T^m]\!]^* = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. By Proposition 1578, we get $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \in \mathrm{AlgSys}^*(\mathcal{I})$ and, as, by hypothesis, $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$ and, hence, $\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}$, we get $\mathcal{A} = \mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \in \mathrm{AlgSys}^*(\mathcal{I})$. We conclude that $\mathrm{AlgSys}(\mathcal{I}) \subseteq \mathrm{AlgSys}^*(\mathcal{I})$ and, therefore, the two classes of algebraic systems coincide. ■

We know that, under protoalgebraicity, $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$. The converse is true when family c-reflectivity is also assumed.

Proposition 1625 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a family completely reflective* π*-institution based on* $\mathbf{F}$. *If* $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$, *then* $\mathcal{I}$ *is protoalgebraic.*

Proof: Assume that $\mathcal{I}$ is family c-reflective and that $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$. By Proposition 1579, every full $\mathcal{I}$-structure on an $\mathbf{F}$-algebraic system $\mathcal{A}$ has the form $\langle \mathcal{A}, [\![T]\!]^* \rangle$, for some $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. By Proposition 1584, then, for every $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, there exists $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $[\![T]\!]^{\mathsf{Su}} = [\![T']\!]^*$. Hence, $T^{\mathsf{Su}} = T'^*$. Now we have

$$\begin{array}{rcll} T & = & T^{\mathsf{Su}} & \text{(by Theorem 1590)} \\ & = & T'^* & \text{(shown above)} \\ & = & T'. & \text{(by Lemma 1583)} \end{array}$$

We conclude that $[\![T]\!]^{\mathsf{Su}} = [\![T]\!]^*$. Since this holds, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $T \in \mathrm{Fifam}^{\mathcal{I}}(\mathcal{A})$, we conclude, by Proposition 1601, that $\mathcal{I}$ is protoalgebraic. ■

By Lemma 1623, we may replace equality of the two classes of algebraic system in Proposition 1625 by the condition that the Leibniz operator be an order isomorphism.

Proposition 1626 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a family completely reflective* π*-institution based on* $\mathbf{F}$. *If, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is an order isomorphism, then* $\mathcal{I}$ *is protoalgebraic.*

Proof: By Proposition 1625 and Lemma 1623. ■

We also get a characterization of weak family algebraizability.

Corollary 1627 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is weakly family algebraizable if and only if it is family c-reflective and* $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$*.*

Proof: If $\mathcal{I}$ is weakly family algebraizable, then it is, by definition, family c-reflective and protoalgebraic. By protoalgebraicity, $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$. On the other hand, if $\mathcal{I}$ is family c-reflective and $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$, then it is family c-reflective and, by Proposition 1625, it is also protoalgebraic. Hence, $\mathcal{I}$ is weakly family algebraizable. ■

If, in Proposition 1626, we drop the hypothesis of $\mathcal{I}$ being family c-reflective, but compensate by assuming that $\Omega^{\mathcal{A}}$ is an order isomorphism between the collection of Suszko filter families and $\mathcal{I}^*$-congruence systems on all $\mathbf{F}$-algebraic systems, then we can still infer the protoalgebraicity of $\mathcal{I}$.

Theorem 1628 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is protoalgebraic if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$ *is an order isomorphism.*

Proof: By Proposition 1621, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$ is an order isomorphism. By protoalgebraicity and Theorem 1601, $\mathrm{FiFam}^{\mathcal{I}^*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. Therefore, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{I})$ is an order isomorphism.

Conversely, assume that, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$ is an order isomorphism. By Lemma 1518, it suffices to show that the Leibniz and Suszko operators on an arbitrary $\mathbf{F}$-algebraic system coincide. To this end, let $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

- Note that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$. By Lemma 1624 and the hypothesis, there exists $T' \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$. Hence, $[\![T]\!]^{\mathsf{Su}} = [\![T']\!]^*$ and, therefore, by Lemma 1583, $T^{\mathsf{Su}} = T'^* = T'$. We conclude that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T^{\mathsf{Su}})$.

- Note that $\Omega^{\mathcal{A}}(T) \in \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$. Thus, there exists, by hypothesis, $T'' \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T'')$. So $[\![T]\!]^* = [\![T'']\!]^*$. Hence, by Lemma 1583, $T^* = T''^* = T''$. This gives $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T^*)$. Since $T^* = T'' \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$, $(T^*)^{\mathsf{Su}} = T^*$. But, by Lemma 1568, $T^* \le T$. Hence, $[\![T]\!]^{\mathsf{Su}} \subseteq [\![T^*]\!]^{\mathsf{Su}}$ and, thus, $T^* = (T^*)^{\mathsf{Su}} \le T^{\mathsf{Su}}$. By Lemma 1583, the reverse inclusion always holds, whence $T^* = T^{\mathsf{Su}}$. Now we get $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T^{\mathsf{Su}})$.

Since, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T^{\mathsf{Su}}) = \Omega^{\mathcal{A}}(T)$, we get that $\mathcal{I}$ is a protoalgebraic π-institution. ■

Theorem 1628 allows us to give a related characterization of equivalential π-institutions.

Corollary 1629 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *$\mathcal{I}$ is family equivalential if and only if it is family commuting and, for every $\mathbf{F}$-algebraic system $\mathcal{A}$,* $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is an order isomorphism.*

Proof: Suppose, first, that $\mathcal{I}$ is family equivalential. Then, by definition, $\mathcal{I}$ is family extensional and protoalgebraic. Thus, by Theorem 327, it is family inverse commuting and, by Theorem 325, it is family commuting. Moreover, by Theorem 1628, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism.

Assume, conversely, that $\mathcal{I}$ is family commuting and that, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism. Then, by Theorem 1628, it is protoalgebraic. Therefore, by Theorem 325, it is family inverse commuting and, by Theorem 327, it is family extensional. Being protoalgebraic and family extensional, it is, by definition, family equivalential. ∎

We turn now to establishing some characterizations of semantic classes in the Leibniz hierarchy via the use of the Suszko operator. First, we show that the family c-reflectivity of the Leibniz operator is equivalent with the universal injectivity of the Suszko operator and, in turn, a sufficient (and, trivially, necessary) condition fr it is the injectivity of the Suszko operator on an $\mathcal{I}$-algebraic systems.

Theorem 1630 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$. *The following conditions are equivalent:*

(i) *$\mathcal{I}$ is family c-reflective;*

(ii) $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ *is injective, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$*;*

(iii) $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ *is injective, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$.

Proof:

(i)⇒(ii) By Proposition 1528, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ is injective on $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. By hypothesis and Theorem 1590, $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Thus, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ is injective, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$.

(ii)⇒(iii) Trivial.

(iii)⇒(i) We use again Theorem 1590, showing that for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. To this end. let $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Consider $T^m = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))$. We have $\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \mathrm{AlgSys}(\mathcal{I})$ and, by Corollary 56, $T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))$.

Hence, by assumption, $T^m \le T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. By monotonicity of the Suszko operator, Proposition 1544 and Lemma 1557,

$$\widetilde{\Omega}^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(T^m) \le \widetilde{\Omega}^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I}<\mathcal{A}}(T)}(T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)) = \Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}.$$

Hence, by hypothesis, $T/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = T^m$. Therefore, by Proposition 1587, $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. This proves that $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and, by Theorem 1590, yields that $\mathcal{I}$ is family c-reflective. ∎

As regards protoalgebraicity, we have the following characterization.

Theorem 1631 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is protoalgebraic if and only if* $\widetilde{\Omega}^{\mathcal{I}}$ *is commuting.*

Proof: If $\mathcal{I}$ is protoalgebraic, then, by Lemma 1518, $\widetilde{\Omega}^{\mathcal{I}} = \Omega$ and, by Proposition 24, $\widetilde{\Omega}^{\mathcal{I}}$ is commuting. If, conversely, $\widetilde{\Omega}^{\mathcal{I}}$ is commuting, then, by Corollary 1537, $\widetilde{\Omega}^{\mathcal{I}} = \Omega$. Therefore, by Lemma 1518, $\mathcal{I}$ is protoalgebraic. ∎

We also get characterizations for equivalential, weakly algebraizable and algebraizable π-institutions.

Theorem 1632 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathcal{I}$ *is family equivalential if and only if* $\widetilde{\Omega}^{\mathcal{I}}$ *is commuting and family extensional.*

(b) $\mathcal{I}$ *is weakly family algebraizable if and only if* $\widetilde{\Omega}^{\mathcal{I}}$ *is injective and commuting.*

(c) $\mathcal{I}$ *is family algebraizable if and only if* $\widetilde{\Omega}^{\mathcal{I}}$ *is injective and commuting and family extensional.*

Proof:

(a) Suppose $\mathcal{I}$ is equivalential. Then, by Theorem 334, Ω is monotone and family extensional. By Lemma 1518, $\Omega = \widetilde{\Omega}^{\mathcal{I}}$. Thus, by Proposition 24, $\widetilde{\Omega}^{\mathcal{I}}$ is commuting and family extensional. Conversely, if $\widetilde{\Omega}^{\mathcal{I}}$ is commuting and family extensional, then, by Corollary 1537, $\widetilde{\Omega}^{\mathcal{I}} = \Omega$. Thus, Ω is monotone and family extensional. By Theorem 334, $\mathcal{I}$ is family equivalential.

(b) By Theorems 1630 and 1631.

(c) By Part (a) and Theorem 1630.

■

In general, it is not hard to show that the Suszko operator on an **F**-algebraic system is an order embedding from the collection of Suszko $\mathcal{I}$-filter families into the family of $\mathcal{I}$-congruence systems.

Proposition 1633 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,*

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$$

is an order embedding.

Proof: The Suszko operator $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ is always into $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$. It is monotone by definition, and it is order-reflecting on $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$ by Proposition 1528. Therefore, it is an order embedding, as claimed. ■

Requiring the preceding embedding to be an order isomorphism turns out to be equivalent to the protoalgebraicity of $\mathcal{I}$.

Theorem 1634 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following conditions are equivalent:*

(i) $\mathcal{I}$ *is protoalgebraic;*

(ii) $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ *is an order isomorphism, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*;*

(iii) $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ *is surjective, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*.*

Proof:

(i)⇒(ii) By hypothesis and Lemma 1518, $\Omega^{\mathcal{A}} = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$. Thus, by Proposition 1580, $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}^*(\mathcal{I})$. It follows that $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}) = \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$. Now, by Theorem 1628, we get that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order isomorphism.

(ii)⇒(iii) Trivial.

(iii)⇒(i) Assume (iii). We show that the Leibniz operator $\Omega^{\mathcal{A}}$ is monotone on the $\mathcal{I}$-filter families of every $\mathbf{F}$-algebraic system $\mathcal{A}$. To this end, let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \leq T'$. Since $\Omega^{\mathcal{A}}(T) \in \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A}) \subseteq \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$, there exists, by hypothesis, $T'' \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T'') = \Omega^{\mathcal{A}}(T)$. Thus, we have

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathcal{A}}([\![T'']\!]^{\mathsf{Su}}) & = & \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T'') \\ & = & \Omega^{\mathcal{A}}(T) \\ & = & \widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^*).\end{array}$$

Since $\langle\mathcal{A},[\![T'']\!]^{\mathsf{Su}}\rangle$, $\langle\mathcal{A},[\![T]\!]^*\rangle\in\mathrm{FStr}(\mathcal{I}|)$, by Theorem 1408, $[\![T'']\!]^{\mathsf{Su}}=[\![T]\!]^*$. Moreover, since $T''\in\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$, by Lemma 1583, we obtain $[\![T'']\!]^{\mathsf{Su}}=\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T''}$. Since $T\in[\![T]\!]^*=[\![T'']\!]^{\mathsf{Su}}=\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T''}$, we get $T''\le T\le T'$. Thus, $T'\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^{T''}=[\![T'']\!]^{\mathsf{Su}}=[\![T]\!]^*$. In other words, $\Omega^{\mathcal{A}}(T)\le\Omega^{\mathcal{A}}(T')$. We conclude that $\Omega^{\mathcal{A}}$ is monotone on every $\mathcal{A}$, whence $\mathcal{I}$ is protoalgebraic.

■

In closing the section, we exploit Theorem 1634 to provide characterizations of some of the classes of the semantic Leibniz hierarchy.

Theorem 1635 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) *$\mathcal{I}$ is protoalgebraic if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}:\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})\to\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order isomorphism;*

(b) *$\mathcal{I}$ is family c-reflective if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}:\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\to\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order embedding;*

(c) *$\mathcal{I}$ is weakly family algebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}:\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\to\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order isomorphism;*

(d) *$\mathcal{I}$ is family algebraizable if and only if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}:\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\to\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order isomorphism and $\mathcal{I}$ is family extensional.*

Proof:

(a) By Theorem 1634.

(b) By Proposition 1633, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}:\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})\to\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is always an order embedding. By Theorem 1590, family c-reflectivity implies $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})=\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. We conclude that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}:\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\to\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order embedding. If, conversely, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}:\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\to\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order embedding, then it is injective on $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, whence, by Theorem 1630, $\mathcal{I}$ is family c-reflective.

(c) Assume, first, that $\mathcal{I}$ is weakly family algebraizable. By Theorem 1628, $\Omega^{\mathcal{A}}:\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})\to\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism. By Theorem 1590, $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})=\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Therefore, $\Omega^{\mathcal{A}}:\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\to\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism. Finally, by protoalgebraicity and Lemma 1518, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}=\Omega^{\mathcal{A}}$, and by protoalgebraicity and Proposition 1580, $\mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})=\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$. Thus, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}:\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\to\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order isomorphism.

If, conversely, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}:\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\to\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order isomorphism, then, by Theorem 1630, $\mathcal{I}$ is family c-reflective, whence, by

Theorem 1590, $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and, hence, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ is onto $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. Thus, by Theorem 1634, $\mathcal{I}$ is protoalgebraic. We conclude that $\mathcal{I}$ is weakly family algebraizable.

(d) By Part (c) and the definition of family algebraizability.

■

21.14 Suszko Operator and Truth Equationality

Recall that by Proposition 68 and Proposition 28, it makes sense, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, to consider the relative congruence system $\Theta^{\mathcal{I},\mathcal{A}}(R) := \Theta^{\mathrm{AlgSys}(\mathcal{I}),\mathcal{A}}(R)$ on $\mathcal{A}$ generated by a relation family $R \in \mathrm{RelFam}(\mathcal{A})$.

Lemma 1636 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a family truth equational* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $X \in \mathrm{SenFam}(\mathcal{A})$*, if* $\theta = \Theta^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}[X])$ *and* $\langle I, \pi\rangle : \mathcal{A} \to \mathcal{A}/\theta$ *is the quotient morphism, then*

$$\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta) = \pi(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^X) \text{ and } \pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^X.$$

Proof: Let us set

$$\mathcal{T} = \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \Theta^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}[X]) \le \Omega^{\mathcal{A}}(T)\}.$$

By Proposition 1524, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta) = \pi(\mathcal{T})$ and $\pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)) = \mathcal{T}$. But we also have

$$\begin{array}{rcl}
\mathcal{T} & = & \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \theta \le \Omega^{\mathcal{A}}(T)\} \\
 & & \text{(definition of } \mathcal{T}) \\
 & = & \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \tau^{\mathcal{A}}[X] \le \Omega^{\mathcal{A}}(T)\} \\
 & & \text{(since } \theta = \Theta^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}[X]) \text{ and } \Omega^{\mathcal{A}}(T) \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})) \\
 & = & \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : X \le T\} \\
 & & \text{(by family truth equationality)} \\
 & = & \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^X. \quad \text{(definition of } \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^X)
\end{array}$$

The conclusion follows. ■

We show, next that, under the same hypotheses, the Suszko congruence system of an $\mathcal{I}$-filter family generated by a sentence family X equals the least $\mathcal{I}$-congruence system on $\mathcal{A}$ generated by the relation family $\tau^{\mathcal{A}}[X]$.

Proposition 1637 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a family truth equational* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $X \in \mathrm{SenFam}(\mathcal{A})$*,*

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(C^{\mathcal{I},\mathcal{A}}(X)) = \Theta^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}[X]).$$

In particular, if $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Theta^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}[T])$*.*

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system, $X \in \mathrm{SenFam}(\mathcal{A})$ and set $\theta = \Theta^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}[X])$. Since $\theta \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$, we have $\mathcal{A}/\theta \in \mathrm{AlgSys}(\mathcal{I})$. Therefore,

$$\widetilde{\Omega}^{\mathcal{A}/\theta}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)) = \Delta^{\mathcal{A}/\theta}.$$

Consider the quotient morphism

$$\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\theta.$$

We have

$$\begin{array}{rcll}
\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(C^{\mathcal{I},\mathcal{A}}(X)) & = & \widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^X) & \\
& = & \widetilde{\Omega}^{\mathcal{A}}(\pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta))) & \text{(Lemma 1636)} \\
& = & \pi^{-1}(\widetilde{\Omega}^{\mathcal{A}/\theta}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta))) & \\
& = & \pi^{-1}(\Delta^{\mathcal{A}/\theta}) & \\
& = & \theta. &
\end{array}$$

Therefore, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(C^{\mathcal{I},\mathcal{A}}(X)) = \Theta^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}[X])$, as was to be shown. ∎

Proposition 1638 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a family truth equational* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*. For every* $\Sigma \in |\mathbf{Sign}^\flat|$ *and* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) = \Theta^{\mathcal{I}}(\tau^\flat_\Sigma[\phi]).$$

Proof: Directly from Proposition 1637, letting $X = \{X_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, where $X_\Sigma = \{\phi\}$ and $X_{\Sigma'} = \varnothing$, for all $\Sigma' \neq \Sigma$. ∎

Another property is that the Suszko congruence family of the $\mathcal{I}$-filter family generated by a sentence family X can be obtained as the join of the Suszko congruence families of the $\mathcal{I}$-filter families generated by each singleton in X.

Proposition 1639 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a family truth equational* π*-institution based on* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $X \in \mathrm{SenFam}(\mathcal{A})$*,*

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(C^{\mathcal{I},\mathcal{A}}(X)) = \bigvee \{\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(C^{\mathcal{I},\mathcal{A}}(\phi)) : \phi \in X_\Sigma, \Sigma \in |\mathbf{Sign}|\}.$$

Proof: Suppose $\mathcal{I}$ is family truth equational, with witnessing transformations $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$. Then, we have, for every **F**-algebraic system $\mathcal{A}$ and all $X \in \mathrm{SenFam}(\mathcal{A})$,

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(C^{\mathcal{I},\mathcal{A}}(X)) & = & \Theta^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}[X]) \\ & & \text{(by Proposition 1637)} \\ & = & \bigvee\{\Theta^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}_\Sigma[\phi]) : \phi \in X_\Sigma, \Sigma \in |\mathbf{Sign}|\} \\ & & \text{(by Proposition 35)} \\ & = & \bigvee\{\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(C^{\mathcal{I},\mathcal{A}}(\phi)) : \phi \in X_\Sigma, \Sigma \in |\mathbf{Sign}|\}. \\ & & \text{(by Proposition 1637)}\end{array}$$

This proves the statement. ∎

More generally, we have

Proposition 1640 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a family truth equational* π*-institution based on* **F**. *For every* **F***-algebraic system* $\mathcal{A}$ *and all* $\{T^i : i \in I\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(\bigvee_{i\in I}^{\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})} T^i) = \bigvee_{i\in I}^{\mathbf{ConSys}^{\mathcal{I}}(\mathcal{A})} \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T^i).$$

Proof: Suppose $\mathcal{I}$ is family truth equational, with witnessing transformations $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, and let $\mathcal{A}$ be an **F**-algebraic system and $\{T^i : i \in I\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, we have

$$\begin{array}{rcll}\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(\bigvee_{i\in I} T^i) & = & \widetilde{\Omega}(C^{\mathcal{I},\mathcal{A}}(\bigcup_{i\in I} T^i)) & \text{(joins in } \mathbf{FiFam}^{\mathcal{I}}(\mathcal{A})) \\ & = & \Theta^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}[\bigcup_{i\in I} T^i]) & \text{(Proposition 1637)} \\ & = & \Theta^{\mathcal{I},\mathcal{A}}(\bigcup_{i\in I} \tau^{\mathcal{A}}[T^i]) & \\ & = & \bigvee_{i\in I} \Theta^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}[T^i]) & \text{(joins in } \mathbf{ConSys}^{\mathcal{I}}(\mathcal{A})) \\ & = & \bigvee_{i\in I} \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T^i). & \text{(Proposition 1637)}\end{array}$$

This proves the statement. ∎

Another property is the commutativity of the Suszko operator with surjective morphisms with isomorphic functor components.

Proposition 1641 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a family truth equational* π*-institution based on* **F**. *For all* **F***-algebraic systems* $\mathcal{A}$, $\mathcal{B}$*, all surjective morphisms* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$*, with* H *an isomorphism, and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{B}}(C^{\mathcal{I},\mathcal{B}}(\gamma(T))) = \Theta^{\mathcal{I},\mathcal{B}}(\gamma(\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))).$$

Proof: Suppose $\mathcal{I}$ is family truth equational, with witnessing transformations $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, and let $\mathcal{A}$, $\mathcal{B}$ be **F**-algebraic systems, $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism, with H an isomorphism, and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. We now get

$$\begin{array}{rcll}\widetilde{\Omega}^{\mathcal{I},\mathcal{B}}(C^{\mathcal{I},\mathcal{B}}(\gamma(T))) & = & \Theta^{\mathcal{I},\mathcal{B}}(\tau^{\mathcal{B}}[\gamma(T)]) & \text{(by Proposition 1637)}\\ & = & \Theta^{\mathcal{I},\mathcal{B}}(\gamma(\Theta^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}[T]))) & \text{(by Proposition 34)}\\ & = & \Theta^{\mathcal{I},\mathcal{B}}(\gamma(\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))). & \text{(by Proposition 1637)}\end{array}$$

This proves the equality in the statement. ∎

We now build a little further on our work of Section 12.2 in order to give another characterization of family truth equationality.

Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ and $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ be algebraic systems and $\mathcal{K} = \langle \mathbf{K}, D\rangle$ and $\mathcal{K}' = \langle \mathbf{K}', D'\rangle$ be two π-structures based on $\mathbf{K}$ and $\mathbf{K}'$, respectively. Consider an order embedding

$$h : \mathbf{ThFam}(\mathcal{K}) \to \mathbf{ThFam}(\mathcal{K}').$$

Recall that $\overleftarrow{h} = \{\overleftarrow{h}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ is defined, for all $\Sigma \in |\mathbf{Sign}|$, by letting

$$\overleftarrow{h}_\Sigma : \mathrm{SEN}(\Sigma) \to \mathrm{SenFam}(\mathbf{K}')$$

be given, for all $\phi \in \mathrm{SEN}(\Sigma)$, by

$$\overleftarrow{h}_\Sigma[\phi] = h(D(\phi)).$$

Then we have the following analog of Lemma 894.

Lemma 1642 *Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{K}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ be algebraic systems, $\mathcal{K} = \langle \mathbf{K}, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}', D'\rangle$ be π-structures based on $\mathbf{K}$, $\mathbf{K}'$, respectively, and $h : \mathbf{ThFam}(\mathcal{K}) \to \mathbf{ThFam}(\mathcal{K}')$ an order embedding, which preserves suprema. Then $\overleftarrow{h} : \mathcal{K} \to \mathcal{K}'$ is an interpretation.*

Proof: Suppose $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ is an order embedding and let $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$. Then we have

$$\begin{array}{rll}\phi \in D_\Sigma(\Phi) & \text{iff} & D(\phi) \leq D(\Phi)\\ & \text{iff} & h(D(\phi)) \leq h(D(\Phi))\\ & \text{iff} & h(D(\phi)) \leq h(\bigvee\{D(\chi) : \chi \in \Phi\})\\ & \text{iff} & h(D(\phi)) \leq \bigvee\{h(D(\chi)) : \chi \in \Phi\}\\ & \text{iff} & \overleftarrow{h}_\Sigma[\phi] \leq \bigvee\{\overleftarrow{h}_\Sigma[\chi] : \chi \in \Phi\}\\ & \text{iff} & \overleftarrow{h}_\Sigma[\phi] \leq D'(\overleftarrow{h}_\Sigma[\Phi]).\end{array}$$

Thus, $\overleftarrow{h} : \mathcal{K}' \to \mathcal{K}$ is indeed an interpretation. ∎

Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system and $\mathcal{K} = \langle \mathbf{K}^k, D\rangle$ and $\mathcal{K}' = \langle \mathbf{K}^\ell, D'\rangle$ be two π-structures based on $\mathbf{K}^k$ and $\mathbf{K}^\ell$, respectively. Consider a suprema preserving order embedding

$$h : \mathbf{ThFam}(\mathcal{K}) \to \mathbf{ThFam}(\mathcal{K}').$$

We say that the order embedding $h : \mathbf{ThFam}(\mathcal{K}) \to \mathbf{ThFam}(\mathcal{K}')$ is **transformational** if there exists $\tau : \mathrm{SEN}^\omega \to \mathrm{SEN}^\ell$, with k distinguished arguments, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$,

$$\overleftarrow{h}_\Sigma[\vec{\phi}] = D'(\tau_\Sigma[\vec{\phi}]).$$

We have the following analog of Lemma 899.

Lemma 1643 *Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system, $\mathcal{K} = \langle \mathbf{K}^k, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D'\rangle$ be two π-structures and $h : \mathbf{ThFam}(\mathcal{K}) \to \mathbf{ThFam}(\mathcal{K}')$ a transformational suprema preserving order embedding induced by $\tau : \mathbf{K}^k \to \mathbf{K}^\ell$. Then, for all $\Sigma \in |\mathbf{Sign}|$, all $\Phi \subseteq \mathrm{SEN}(\Sigma)^k$,*

$$h(D(\Phi)) = D'(\tau_\Sigma[\Phi]).$$

Proof: We have, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)^k$,

$$\begin{array}{rcll} h(D(\Phi)) & = & h(\bigvee_{\phi\in\Phi} D(\phi)) & (\text{join in } \mathbf{ThFam}(\mathcal{K})) \\ & = & \bigvee_{\phi\in\Phi} h(D(\phi)) & (h \text{ suprema preserving}) \\ & = & \bigvee_{\phi\in\Phi} D'(\tau_\Sigma[\phi]) & (\overleftarrow{h}_\Sigma[\phi] = D'(\tau_\Sigma[\phi])) \\ & = & D'(\bigcup_{\phi\in\Phi} \tau_\Sigma[\phi]) & (\text{join in } \mathbf{ThFam}(\mathcal{K}')) \\ & = & D'(\tau_\Sigma[\Phi]). & (\text{by definition}) \end{array}$$

This proves the equality of the statement. ∎

Furthermore, we have an analog of Theorem 900:

Theorem 1644 *Let $\mathbf{K} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system, $\mathcal{K} = \langle \mathbf{K}^k, D\rangle$, $\mathcal{K}' = \langle \mathbf{K}^\ell, D'\rangle$ be two π-structures and $h : \mathbf{ThFam}(\mathcal{K}') \to \mathbf{ThFam}(\mathcal{K})$ a transformational suprema preserving order embedding induced by $\tau : \mathbf{K}^k \to \mathbf{K}^\ell$. Then $\tau : \mathcal{K} \to \mathcal{K}'$ is an interpretation.*

Proof: Let $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)^k$. We then have:

$$\begin{array}{rll} \phi \in D_\Sigma(\Phi) & \text{iff} & D_\Sigma(\phi) \leq D_\Sigma(\Phi) \\ & \text{iff} & h(D(\phi)) \leq h(D(\Phi)) \quad (h \text{ order embedding}) \\ & \text{iff} & D'(\tau_\Sigma[\phi]) \leq D'(\tau_\Sigma[\Phi]) \quad (\text{Lemma 1643}) \\ & \text{iff} & \tau_\Sigma[\phi] \leq D'(\tau_\Sigma[\Phi]). \end{array}$$

Thus, $\tau : \mathcal{K} \to \mathcal{K}'$ is an interpretation. ∎

Now, we obtain the following theorem characterizing family truth equationality in terms of transformational suprema preserving order embeddings.

Theorem 1645 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is family truth equational if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ *is a transformational suprema preserving order embedding.*

Proof: Suppose, first, that $\mathcal{I}$ is family truth equational, with witnessing transformations $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$. Then, it is, a fortiori, family c-reflective, whence, by Theorem 1635, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is an order embedding, for every $\mathbf{F}$-algebraic system $\mathcal{A}$. By Proposition 1637, $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}$ is transformational and, by Proposition 1640, it is suprema preserving.

Assume, conversely, that $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$ is a transformational suprema preserving order embedding. Then, on the one hand, by Theorem 1635, $\mathcal{I}$ is family c-reflective, and, on the other, by definition, there exists $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$, such that $\widetilde{\Omega}^{\mathcal{I}} : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}^{\mathcal{I}}(\mathcal{F})$ is induced by $\tau^\flat$. Thus, by Theorem 1644, $\tau^\flat : \mathcal{I} \to \mathcal{Q}^{\mathrm{AlgSys}(\mathcal{I})}$ is an interpretation. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. we have

$$\begin{array}{lll} \phi \in T_\Sigma & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Theta^{\mathcal{I},\mathcal{F}}(\tau^\flat_\Sigma[T_\Sigma]) \quad (\tau^\flat \text{ an interpretation}) \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \widetilde{\Omega}^{\mathcal{I}}(T) \quad (\text{by Lemma 1643}) \\ & \text{implies} & \tau^\flat_\Sigma[\phi] \leq \Omega(T). \quad (\widetilde{\Omega}^{\mathcal{I}}(T) \leq \Omega(T)) \end{array}$$

If, conversely, $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$, then $\Theta^{\mathcal{I},\mathcal{F}}(\tau^\flat_\Sigma[\phi]) \leq \Omega(T)$, whence, by Lemma 1643, $\widetilde{\Omega}^{\mathcal{I}}(C(\phi)) \leq \Omega(T)$. Thus, by family c-reflectivity and Lemma 1519, $\phi \in T_\Sigma$. Therefore, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \Omega_\Sigma(T).$$

We conclude that $\mathcal{I}$ is family truth-equational, with witnessing transformations $\tau^\flat$. ∎

21.15 Relations With Algebraic Semantics

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

Given a class K of $\mathbf{F}$-algebraic systems, recall the definition oof the closure system $C^{\mathsf{K},\tau} = \{C^{\mathsf{K},\tau}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, where, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $C^{\mathsf{K},\tau}_\Sigma : \mathcal{P}(\mathrm{SEN}^\flat(\Sigma)) \to \mathcal{P}(\mathrm{SEN}^\flat(\Sigma))$ is given, for all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, by

$$\phi \in C^{\mathsf{K},\tau}_\Sigma(\Phi) \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq C^{\mathsf{K}}(\tau^\flat_\Sigma[\Phi]).$$

Define the class $\mathsf{K}(\mathcal{I}, \tau)$ of $\mathbf{F}$-algebraic systems by

$$\mathsf{K}(\mathcal{I}, \tau) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : C \leq C^{\mathcal{A},\tau}\}.$$

The following proposition gives a characterization of this class.

Proposition 1646 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then*

$$\mathsf{K}(\mathcal{I},\tau) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : \tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}.$$

Proof: Suppose, first, that $\mathcal{A} \in \mathsf{K}(\mathcal{I},\tau)$. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$ and $\alpha_\Sigma(\Phi) \subseteq \tau^{\mathcal{A}}_{F(\Sigma)}(\Delta^{\mathcal{A}})$. Then, by definition, $\tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\Phi)] \leq \Delta^{\mathcal{A}}$. This implies $\alpha(\tau^\flat_\Sigma[\Phi]) \leq \Delta^{\mathcal{A}}$. Since, by hypothesis, $\phi \in C_\Sigma(\Phi)$ and $C \leq C^{\mathcal{A},\tau}$, we get $\alpha(\tau^\flat_\Sigma[\phi]) \leq \Delta^{\mathcal{A}}$. Equivalently, $\tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi)] \leq \Delta^{\mathcal{A}}$, i.e., $\alpha_\Sigma(\phi) \in \tau^{\mathcal{A}}_{F(\Sigma)}(\Delta^{\mathcal{A}})$. We conclude that $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. This proves the left-to-right inclusion.

Assume, conversely, that $\mathcal{A}$ is an $\mathbf{F}$-algebraic system, such that $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$ and $\alpha(\tau^\flat_\Sigma[\Phi]) \leq \Delta^{\mathcal{A}}$. Then $\tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\Phi)] \leq \Delta^{\mathcal{A}}$, i.e., $\alpha_\Sigma(\Phi) \subseteq \tau^{\mathcal{A}}_{F(\Sigma)}(\Delta^{\mathcal{A}})$. Since, by hypothesis, $\phi \in C_\Sigma(\Phi)$ and $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get $\alpha_\Sigma(\phi) \in \tau^{\mathcal{A}}_{F(\Sigma)}(\Delta^{\mathcal{A}})$, whence $\tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi)] \leq \Delta^{\mathcal{A}}$ or, equivalently, $\alpha(\tau^\flat_\Sigma[\phi]) \leq \Delta^{\mathcal{A}}$. We conclude that $\phi \in C^{\mathcal{A},\tau}_\Sigma(\Phi)$ and, hence, $\mathcal{A} \in \mathsf{K}(\mathcal{I},\tau)$. ∎

It is readily inferred from the definition that, provided $\mathcal{I}$ has a $\tau^\flat$-algebraic semantics, then the class $\mathsf{K}(\mathcal{I},\tau)$ is the largest such.

Corollary 1647 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *has a* $\tau^\flat$*-algebraic semantics, then* $\mathsf{K}(\mathcal{I},\tau)$ *is its largest* $\tau^\flat$*-algebraic semantics.*

Proof: Suppose K is a $\tau^\flat$-algebraic semantics for $\mathcal{I}$ and let $\mathcal{A} \in \mathsf{K}$. Then, by the definition of $\tau^\flat$-algebraic semantics and taking into account the membership $\mathcal{A} \in \mathsf{K}$, we get $C = C^{\mathsf{K},\tau} \leq C^{\mathcal{A},\tau}$. Therefore, by definition of $\mathsf{K}(\mathcal{I},\tau)$, $\mathcal{A} \in \mathsf{K}(\mathcal{I},\tau)$. We conclude that $\mathsf{K} \subseteq \mathsf{K}(\mathcal{I},\tau)$. ∎

we can also show that, if $\mathcal{I}$ is family truth equational, with witnessing transformations $\tau^\flat$, then $\mathrm{AlgSys}(\mathcal{I})$ is a $\tau^\flat$-algebraic semantics for $\mathcal{I}$.

Proposition 1648 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is family truth equational, with witnessing transformations* $\tau^\flat$*, then* $\mathrm{AlgSys}(\mathcal{I})$ *is a* $\tau^\flat$*-algebraic semantics for* $\mathcal{I}$*.*

Proof: We must show that $C = C^{\mathrm{AlgSys}(\mathcal{I}),\tau}$.

Let $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$ and $\alpha(\tau^\flat_\Sigma[\Phi]) \leq \Delta^{\mathcal{A}}$. Since $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, there exists $\mathcal{T} \subseteq$

$\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \Delta^{\mathcal{A}}$. Hence, we get

$$\begin{array}{rcl}\alpha(\tau^\flat_\Sigma[\Phi]) \leq \Delta^{\mathcal{A}} & \text{iff} & \alpha(\tau^\flat_\Sigma[\Phi]) \leq \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \\ & \text{iff} & \alpha(\tau^\flat_\Sigma[\Phi]) \leq \Omega^{\mathcal{A}}(T), \text{ for all } T \in \mathcal{T}, \\ & \text{iff} & \tau^\flat_\Sigma[\Phi] \leq \Omega(\alpha^{-1}(T)), \text{ for all } T \in \mathcal{T}, \\ & \text{iff} & \Phi \subseteq \alpha^{-1}_\Sigma(T), \text{ for all } T \in \mathcal{T}, \\ & \text{implies} & \phi \subseteq \alpha^{-1}_\Sigma(T), \text{ for all } T \in \mathcal{T}, \\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \Omega(\alpha^{-1}(T)), \text{ for all } T \in \mathcal{T}, \\ & \text{iff} & \alpha(\tau^\flat_\Sigma[\phi]) \leq \Omega^{\mathcal{A}}(T), \text{ for all } T \in \mathcal{T}, \\ & \text{iff} & \alpha(\tau^\flat_\Sigma[\phi]) \leq \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \\ & \text{iff} & \alpha(\tau^\flat_\Sigma[\phi]) \leq \Delta^{\mathcal{A}}.\end{array}$$

We conclude that $\phi \in C^{\mathrm{AlgSys}(\mathcal{I}),\tau}_\Sigma(\Phi)$. Therefore, $C \leq C^{\mathrm{AlgSys}(\mathcal{I}),\tau}$.

Suppose, conversely, that $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C^{\mathrm{AlgSys}(\mathcal{I}),\tau}_\Sigma(\Phi)$ and $T \in \mathrm{ThFam}(\mathcal{I})$, such that $\Phi \subseteq T_\Sigma$. Then, we have $\tau^\flat_\Sigma[\Phi] \leq \Omega(T)$, i.e., $\tau^{\mathcal{F}/\Omega(T)}_\Sigma[\Phi/\Omega_\Sigma(T)] \leq \Delta^{\mathcal{F}/\Omega(T)}$. But $\mathcal{F}/\Omega(T) \in \mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$. Therefore, since $\phi \in C^{\mathrm{AlgSys}(\mathcal{I}),\tau}_\Sigma(\Phi)$, we get that $\tau^{\mathcal{F}/\Omega(T)}_\Sigma[\phi/\Omega_\Sigma(T)] \leq \Delta^{\mathcal{F}/\Omega(T)}$, whence $\tau^\flat_\Sigma[\phi] \leq \Omega(T)$. Therefore, $\phi \in T_\Sigma$ and we conclude that $\phi \in C_\Sigma(\Phi)$. This proves that $C^{\mathrm{AlgSys}(\mathcal{I}),\tau} \leq C$ and, as a result, equality follows.

We have now shown that $\mathrm{AlgSys}(\mathcal{I})$ is a $\tau^\flat$-algebraic semantics for $\mathcal{I}$. ∎

Corollary 1649 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$, *and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is family truth equational, with witnessing transformations* $\tau^\flat$, *then* $\mathrm{AlgSys}(\mathcal{I}) \subseteq \mathsf{K}(\mathcal{I},\tau)$.

Proof: By Proposition 1648 and Corollary 1647. ∎

For family truth equational π-institutions we have the following characterization of the least $\mathcal{I}$-filter families on arbitrary algebraic systems and on $\mathcal{I}$-algebraic systems.

Lemma 1650 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a family truth equational* π*-institution based on* $\mathbf{F}$, *with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$.

(a) For every $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, $C^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}(\Delta^{\mathcal{A}})) = C^{\mathcal{I},\mathcal{A}}(\varnothing)$;

(b) For every $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) = C^{\mathcal{I},\mathcal{A}}(\varnothing)$.

Proof:

(a) Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system. We have $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \leq \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(C^{\mathcal{I},\mathcal{A}}(\varnothing)))$. By family truth equationality, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \leq C^{\mathcal{I},\mathcal{A}}(\varnothing)$. It follows that $C^{\mathcal{I},\mathcal{A}}(\tau^{\mathcal{A}}(\Delta^{\mathcal{A}})) = C^{\mathcal{I},\mathcal{A}}(\varnothing)$.

(b) Let $\mathcal{A} \in \text{AlgSys}(\mathcal{I})$. Then, by Part (a), $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \leq C^{\mathcal{I},\mathcal{A}}(\varnothing)$. Assume, conversely, that $\Sigma \in |\mathbf{Sign}|$, $\phi \in \text{SEN}(\Sigma)$, such that $\phi \in C^{\mathcal{I},\mathcal{A}}_{\Sigma}(\varnothing)$. Then, for all $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\phi \in T_\Sigma = \tau^{\mathcal{A}}_{\Sigma}(\Omega^{\mathcal{A}}(T))$, i.e., for all $T \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\tau^{\mathcal{A}}_{\Sigma}[\phi] \leq \Omega^{\mathcal{A}}(T)$. We conclude that $\tau^{\mathcal{A}}_{\Sigma}[\phi] \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(C^{\mathcal{I},\mathcal{A}}(\varnothing)) = \Delta^{\mathcal{A}}$. This shows that $\phi \in \tau^{\mathcal{A}}_{\Sigma}(\Delta^{\mathcal{A}})$. Thus, $C^{\mathcal{I},\mathcal{A}}(\varnothing) \leq \tau^{\mathcal{A}}(\Delta^{\mathcal{A}})$. Equality now follows.

■

So in the case of family truth equational π-institutions, we may strengthen the characterization of the class $\mathsf{K}(\mathcal{I},\tau)$ given in Proposition 1646.

Proposition 1651 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a family truth equational* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $\tau^\flat : (\text{SEN}^\flat)^\omega \to (\text{SEN}^\flat)^2$ *in* $N^\flat$*. Then*

$$\mathsf{K}(\mathcal{I},\tau) = \{\mathcal{A} \in \text{AlgSys}(\mathbf{F}) : \tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) = C^{\mathcal{I},\mathcal{A}}(\varnothing)\}.$$

Proof: Note that, taking into account Proposition 1646,

$$\begin{aligned}\{\mathcal{A} \in \text{AlgSys}(\mathbf{F}) : \tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) = C^{\mathcal{I},\mathcal{A}}(\varnothing)\}&\\ \subseteq \{\mathcal{A} \in \text{AlgSys}(\mathbf{F}) : \tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})\}&\\ = \mathsf{K}(\mathcal{I},\tau).&\end{aligned}$$

Assume, conversely, that $\mathcal{A} \in \mathsf{K}(\mathcal{I},\tau)$. Then, by Proposition 1646, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$ and, by definition of $\mathsf{K}(\mathcal{I},\tau)$, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \leq C^{\mathcal{I},\mathcal{A}}(\varnothing)$. Hence $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) = C^{\mathcal{I},\mathcal{A}}(\varnothing)$. ■

Proposition 1651 has some interesting consequences. First, any two sets of witnessing transformations for truth equationality are, roughly speaking, deductively equivalent over any $\mathcal{I}$-algebraic system.

Corollary 1652 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\tau^\flat \tau'^\flat : (\text{SEN}^\flat)^\omega \to (\text{SEN}^\flat)^2$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is family truth equational, with witnessing transformations* $\tau^\flat$ *and* $\tau'^\flat$*, then, for every* $\mathcal{A} \in \text{AlgSys}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \text{SEN}^\flat(\Sigma)$*,*

$$C^{\mathcal{A}}(\tau^\flat_\Sigma[\phi]) = C^{\mathcal{A}}(\tau'^\flat_\Sigma[\phi]).$$

Proof: Suppose $\mathcal{I}$ is family truth equational, with witnessing transformations $\tau^\flat$ and $\tau'^\flat$ and let $\mathcal{A} \in \text{AlgSys}(\mathcal{I})$ and $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \text{SEN}^\flat(\Sigma)$, such that $\alpha(\tau'^\flat_\Sigma[\phi]) \leq \Delta^{\mathcal{A}}$. This is equivalent to $\tau'^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi)] \leq \Delta^{\mathcal{A}}$, i.e., $\alpha_\Sigma(\phi) \in \tau'^{\mathcal{A}}_{F(\Sigma)}(\Delta^{\mathcal{A}})$. By Proposition 1651, $\alpha_\Sigma(\phi) \in C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\varnothing)$. Again by Proposition 1651, $\alpha_\Sigma(\phi) \in \tau^{\mathcal{A}}_{F(\Sigma)}(\Delta^{\mathcal{A}})$. Thus, $\tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi)] \leq \Delta^{\mathcal{A}}$. Hence, $\alpha_\Sigma(\tau^\flat_\Sigma[\phi]) \leq \Delta^{\mathcal{A}}$. This shows that $\tau^\flat_\Sigma[\phi] \leq C^{\mathcal{A}}(\tau'^\flat_\Sigma[\phi])$. By symmetry, we conclude that $C^{\mathcal{A}}(\tau^\flat_\Sigma[\phi]) = C^{\mathcal{A}}(\tau'^\flat_\Sigma[\phi])$. ■

Finally, the Suszko core $S^\flat$ has a special position among all witnessing transformations. Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} =$

$\langle\mathbf{F},C\rangle$ be a π-institution based on $\mathbf{F}$. Recall that the Suszko core of $\mathcal{I}$ is the collection

$$S^{\mathcal{I}}=\{\sigma^\flat\in N^\flat:(\forall T\in\mathrm{ThFam}(\mathcal{I}))(\sigma^\flat[T]\leq\widetilde{\Omega}(T)\}.$$

Recall, also, that, by Lemma 835, if $\mathcal{I}$ is truth equational, with witnessing equations $\tau^\flat\subseteq N^\flat$, then $\tau^\flat\subseteq S^{\mathcal{I}}$.

Corollary 1653 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a family truth equational* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $\tau^\flat:(\mathrm{SEN}^\flat)^\omega\to(\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*. Then*

$$\mathsf{K}(\mathcal{I},S^{\mathcal{I}})\subseteq\mathsf{K}(\mathcal{I},\tau).$$

Proof: Suppose $\mathcal{A}\in\mathsf{K}(\mathcal{I},S^{\mathcal{I}})$. By hypothesis and Lemma 835, $\tau^\flat\subseteq S^{\mathcal{I}}$. Hence $S^{\mathcal{I}}(\Delta^{\mathcal{A}})\leq\tau^\flat(\Delta^{\mathcal{A}})$. But, by hypothesis, Theorem 840 and Proposition 1651, $S^{\mathcal{I}}(\Delta^{\mathcal{A}})=C^{\mathcal{I},\mathcal{A}}(\varnothing)$ and, by hypothesis and Lemma 1650, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}})\leq C^{\mathcal{I},\mathcal{A}}(\varnothing)$. Hence, we have

$$C^{\mathcal{I},\mathcal{A}}(\varnothing)=S^{\mathcal{I}}(\Delta^{\mathcal{A}})\leq\tau^{\mathcal{A}}(\Delta^{\mathcal{A}})\leq C^{\mathcal{I},\mathcal{A}}(\varnothing).$$

Therefore, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}})=C^{\mathcal{I},\mathcal{A}}(\varnothing)$ and, thus, by Proposition 1651, $\mathcal{A}\in\mathsf{K}(\mathcal{I},\tau)$. We conclude that $\mathsf{K}(\mathcal{I},S^{\mathcal{I}})\subseteq\mathsf{K}(\mathcal{I},\tau)$. ■

Corollary 1654 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a family truth equational* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $\tau^\flat:(\mathrm{SEN}^\flat)^\omega\to(\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*. Then* $\mathsf{K}(\mathcal{I},S^{\mathcal{I}})$ *is a* $\tau^\flat$*-algebraic semantics for* $\mathcal{I}$*.*

Proof: Observe that we have

$$\begin{array}{rcll}\mathrm{AlgSys}(\mathcal{I})&\subseteq&\mathsf{K}(\mathcal{I},S^{\mathcal{I}})&\text{(by Theorem 840 and Corollary 1649)}\\&\subseteq&\mathsf{K}(\mathcal{I},\tau).&\text{(by Corollary 1653)}\end{array}$$

Since, by Proposition 1648, $\mathrm{AlgSys}(\mathcal{I})$ is a $\tau^\flat$-algebraic semantics for $\mathcal{I}$ and, by Corollary 1647, $\mathsf{K}(\mathcal{I},\tau)$ is also a $\tau^\flat$-algebraic semantics for $\mathcal{I}$, we conclude that $\mathsf{K}(\mathcal{I},S^{\mathcal{I}})$ is one also. ■

21.16 The $\mathcal{I}$-Operator $\Psi^{\mathsf{K},\tau}$

Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, with $\tau^\flat:(\mathrm{SEN}^\flat)^\omega\to(\mathrm{SEN}^\flat)^2$ in $N^\flat$, and $\mathcal{I}=\langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$, with a $\tau^\flat$-algebraic semantics K, such that $\mathrm{AlgSys}^*(\mathcal{I})\subseteq\mathsf{K}$. For every $\mathbf{F}$-algebraic system $\mathcal{A}$, we define the operator

$$\Psi^{\mathsf{K},\tau,\mathcal{A}}:\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\to\mathrm{EqvFam}(\mathcal{A})$$

by setting, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$\Psi^{\mathsf{K},\tau,\mathcal{A}}(T) = \Theta^{\overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K}),\mathcal{A}}(\tau^{\mathcal{A}}[T]).$$

Note that, by the hypotheses and Proposition 28, $\Psi^{\mathsf{K},\tau,\mathcal{A}}$ is well-defined, since $\overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K})$-congruence systems on $\mathcal{A}$ form a closure system on $\mathcal{A}^2$.

It is the case that if a class K of $\mathbf{F}$-algebraic systems is a $\tau^\flat$-algebraic semantics for a π-institution $\mathcal{I}$, then so is the larger class $\overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K})$.

Proposition 1655 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If a class K of $\mathbf{F}$-algebraic systems is a $\tau^\flat$-algebraic semantics for $\mathcal{I}$, then so is $\overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K})$.*

Proof: First, observe that $\mathsf{K} \subseteq \overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K})$, whence $C^{\overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K}),\tau} \leq C^{\mathsf{K},\tau} = C$. To show the converse, let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$. Let

$$\langle H^i, \gamma^i\rangle : \mathcal{A} \to \mathcal{A}^i, \quad i \in I,$$

be a subdirect intersection, with $\mathcal{A}^i \in \mathsf{K}$, for all $i \in I$, and assume that $\alpha(\tau^\flat_\Sigma[\Phi]) \leq \Delta^{\mathcal{A}}$. Since, by definition $\Delta^{\mathcal{A}} = \bigcap_{i\in I}\mathrm{Ker}(\langle H^i, \gamma^i\rangle)$, we get that $\alpha(\tau^\flat_\Sigma[\Phi]) \leq \mathrm{Ker}(\langle H^i, \gamma^i\rangle)$, for all $i \in I$, i.e., $\gamma^i(\alpha(\tau^\flat_\Sigma[\Phi])) \leq \Delta^{\mathcal{A}^i}$, $i \in I$, or, equivalently, $\alpha^i(\tau^\flat_\Sigma[\Phi]) \leq \Delta^{\mathcal{A}^i}$, $i \in I$. Since $\phi \in C_\Sigma(\Phi)$, $\mathcal{A}^i \in \mathsf{K}$, for all $i \in I$ and K is a $\tau^\flat$-algebraic semantics for $\mathcal{I}$, we get $\alpha^i(\tau^\flat_\Sigma[\phi]) \leq \Delta^{\mathcal{A}^i}$, for all $i \in I$. We now reverse the steps above. We get $\gamma^i(\alpha(\tau^\flat_\Sigma[\phi])) \subseteq \Delta^{\mathcal{A}^i}$, $i \in I$, then $\alpha(\tau^\flat_\Sigma[\phi]) \leq \mathrm{Ker}(\langle H^i, \gamma^i\rangle)$, $i \in I$, and, finally, $\alpha(\tau^\flat_\Sigma[\phi]) \leq \Delta^{\mathcal{A}}$. Thus, $\phi \in C^{\mathcal{A},\tau}_\Sigma(\Phi)$. Since, for all $\mathcal{A} \in \overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K})$, $C \leq C^{\mathcal{A},\tau}$, we conclude that $C \leq C^{\overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K}),\tau}$. Therefore, $\overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K})$ is also a $\tau^\flat$-algebraic semantics for $\mathcal{I}$. ■

Tying the operator $\Psi^{\mathsf{K},\tau,\mathcal{A}}$ with our preceding work in this Chapter, we show that it is a congruential monotone compatibility $\mathcal{I}$-operator on $\mathcal{A}$.

Proposition 1656 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$, with a $\tau^\flat$-algebraic semantics K, such that $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathsf{K}$. For every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Psi^{\mathsf{K},\tau,\mathcal{A}}$ is a congruential, monotone, compatibility $\mathcal{I}$-operator on $\mathcal{A}$.*

Proof: $\Psi^{\mathsf{K},\tau,mcA}$ is, by definition, an $\mathcal{I}$-operator on $\mathcal{A}$. It is congruential, since, again by definition, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Theta^{\overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K}),\mathcal{A}}(\tau^{\mathcal{A}}[T]) \in \mathrm{ConSys}(\mathcal{A})$. It is monotone, since, for all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, with $T \leq T'$, we get $\tau^{\mathcal{A}}[T] \leq \tau^{\mathcal{A}}[T']$ and, therefore, $\Theta^{\overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K}),\mathcal{A}}(\tau^{\mathcal{A}}[T]) \leq \Theta^{\overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K}),\mathcal{A}}(\tau^{\mathcal{A}}[T'])$.

To see that it is also a compatibility $\mathcal{I}$-operator, consider $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Note, first, that

$$\begin{array}{rcll}\mathrm{AlgSys}(\mathcal{I}) & = & \overset{\triangleleft}{\mathbb{P}}(\mathrm{AlgSys}^*(\mathcal{I}) & \text{(by Theorem 1404)}\\ & \subseteq & \overset{\triangleleft}{\mathbb{P}}(\mathsf{K}). & \text{(since } \mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathsf{K})\end{array}$$

Thus, we get $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}) \subseteq \mathrm{ConSys}^{\overset{\triangleleft}{\mathbb{P}}(\mathsf{K})}(\mathcal{A})$. Since, by Corollary 824, $\tau^{\mathcal{A}}[T] \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$, we get

$$\Theta^{\overset{\triangleleft}{\mathbb{P}}(\mathsf{K}),\mathcal{A}}(\tau^{\mathcal{A}}[T]) \leq \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T).$$

Therefore, $\Psi^{\mathsf{K},\tau,\mathcal{A}}$ is also a compatibility $\mathcal{I}$-operator on $\mathcal{A}$. ■

It turns out that $\Psi^{\mathsf{K},\tau} = \{\Psi^{\mathsf{K},\tau,\mathcal{A}} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$ is also semi-coherent. To show this, we formulate two technical lemmas on the way.

Lemma 1657 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$. *For all* $\mathbf{F}$*-algebraic systems* $\mathcal{A}$, $\mathcal{B}$, *every surjective morphism* $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$, *with* H *an isomorphism, and all* $T' \in \mathrm{SenFam}(\mathcal{B})$,

(a) $\tau^{\mathcal{B}}[T'] = \gamma(\tau^{\mathcal{A}}[\gamma^{-1}(T')])$;

(b) $\tau^{\mathcal{A}}[\gamma^{-1}(T')] \leq \gamma^{-1}(\tau^{\mathcal{B}}[T'])$.

Proof: First, note that, for all $T \in \mathrm{SenFam}(\mathcal{A})$, we have, taking into account the fact that $\langle H, \gamma\rangle$ is a surjective morphism, $\gamma(\tau^{\mathcal{A}}[T]) = \tau^{\mathcal{B}}[\gamma(T)]$. Now, we set $T = \gamma^{-1}(T')$. This gives

$$\gamma(\tau^{\mathcal{A}}[\gamma^{-1}(T')]) = \tau^{\mathcal{B}}[\gamma(\gamma^{-1}(T'))] = \tau^{\mathcal{B}}[T'],$$

which conclude the proof of Part (a). For Part (b), we have, using Part (a),

$$\tau^{\mathcal{A}}[\gamma^{-1}(T')] \leq \gamma^{-1}(\gamma(\tau^{\mathcal{A}}[\gamma^{-1}(T')])) = \gamma^{-1}(\tau^{\mathcal{B}}[T']).$$

■

Lemma 1658 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$, *and* K *a class of* $\mathbf{F}$*-algebraic systems, such that* $\overset{\triangleleft}{\mathbb{P}}(\mathsf{K}) \subseteq \mathsf{K}$. *For all* $\mathbf{F}$*-algebraic systems* $\mathcal{A}$, $\mathcal{B}$, *every surjective morphism* $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$, *with* H *an isomorphism, and all* $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$,

$$\begin{array}{l}\{\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A}) : \mathrm{Ker}(\langle H, \gamma\rangle) \leq \theta \text{ and } \tau^{\mathcal{A}}[\gamma^{-1}(T')] \leq \theta\}\\ \qquad = \{\gamma^{-1}(\theta') : \theta' \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{B}) \text{ and } \tau^{\mathcal{B}}[T'] \leq \theta'\}.\end{array}$$

Proof: Suppose K is closed under subdirect intersections and let $\mathcal{A}$, $\mathcal{B}$ be $\mathbf{F}$-algebraic systems, $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism, with H an isomorphism, and $T' \in \mathrm{SenFam}(\mathcal{B})$.

(⊆) Let $\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A})$, such that $\mathrm{Ker}(\langle H,\gamma\rangle) \leq \theta$ and $\tau^{\mathcal{A}}[\gamma^{-1}(T')] \leq \theta$. By Lemma 1657, $\tau^{\mathcal{B}}[T'] = \gamma(\tau^{\mathcal{A}}[\gamma^{-1}(T')]) \leq \gamma(\theta)$. By Proposition 33, $\gamma(\theta) \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{B})$. Finally, by Lemma 25, $\theta = \gamma^{-1}(\gamma(\theta))$. Hence, we get

$$\theta = \gamma^{-1}(\gamma(\theta)) \in \{\gamma^{-1}(\theta') : \theta' \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{B}) \text{ and } \tau^{\mathcal{B}}[T'] \leq \theta'\}.$$

(⊇) Suppose, now, $\theta' \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{B})$, such that $\tau^{\mathcal{B}}[T'] \leq \theta'$. By Lemma 1657, $\tau^{\mathcal{A}}[\gamma^{-1}(T')] \leq \gamma^{-1}(\tau^{\mathcal{B}}[T']) \leq \gamma^{-1}(\theta')$. Finally, $\mathrm{Ker}(\langle H,\gamma\rangle) = \gamma^{-1}(\Delta^{\mathcal{B}}) \leq \gamma^{-1}(\theta')$. So we get

$$\gamma^{-1}(\theta') \in \{\theta \in \mathrm{ConSys}^{\mathsf{K}}(\mathcal{A}) : \mathrm{Ker}(\langle H,\gamma\rangle) \leq \theta \text{ and } \tau^{\mathcal{A}}[\gamma^{-1}(T')] \leq \theta\}.$$ ■

Now, for the main theorem to the effect that $\Psi^{\mathsf{K},\tau}$ is a semi-coherent family of congruential monotone compatibility $\mathcal{I}$-operators.

Theorem 1659 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$, with a $\tau^\flat$-algebraic semantics K, such that $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathsf{K}$. $\Psi^{\mathsf{K},\tau}$ is a semi-coherent family of congruential monotone compatibility $\mathcal{I}$-operators.*

Proof: Let $\mathcal{A}$, $\mathcal{B}$ be $\mathbf{F}$-algebraic systems, $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$ a surjective morphism, with H an isomorphism, and $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, such that $\langle H,\gamma\rangle$ is $\Psi^{\mathsf{K},\tau}$-compatible with $\gamma^{-1}(T')$. Then, by definition,

$$\mathrm{Ker}(\langle H,\gamma\rangle) \leq \Psi^{\mathsf{K},\tau,\mathcal{A}}(\gamma^{-1}(T')) = \Theta^{\overset{\triangleleft}{\mathrm{II}}(\mathsf{K}),\mathcal{A}}(\tau^{\mathcal{A}}[\gamma^{-1}(T')]).$$

So we have

$$\begin{array}{rcl}
\Psi^{\mathsf{K},\tau,\mathcal{A}}(\gamma^{-1}(T')) & = & \Theta^{\overset{\triangleleft}{\mathrm{II}}(\mathsf{K}),\mathcal{A}}(\tau^{\mathcal{A}}[\gamma^{-1}(T')]) \\
& = & \bigcap\{\theta \in \mathrm{ConSys}^{\overset{\triangleleft}{\mathrm{II}}(\mathsf{K})}(\mathcal{A}) : \\
& & \qquad \mathrm{Ker}(\langle H,\gamma\rangle) \leq \theta \text{ and } \tau^{\mathcal{A}}[\gamma^{-1}(T')] \leq \theta\} \\
& = & \bigcap\{\gamma^{-1}(\theta') : \theta' \in \mathrm{ConSys}^{\overset{\triangleleft}{\mathrm{II}}(\mathsf{K})}(\mathcal{B}) \text{ and } \tau^{\mathcal{B}}[T'] \leq \theta'\} \\
& = & \gamma^{-1}(\bigcap\{\theta' \in \mathrm{ConSys}^{\overset{\triangleleft}{\mathrm{II}}(\mathsf{K})}(\mathcal{B}) : \tau^{\mathcal{B}}[T'] \leq \theta'\}) \\
& = & \gamma^{-1}(\Theta^{\overset{\triangleleft}{\mathrm{II}}(\mathsf{K}),\mathcal{B}}(\tau^{\mathcal{B}}[T'])) \\
& = & \gamma^{-1}(\Psi^{\mathsf{K},\tau,\mathcal{B}}(T')).
\end{array}$$

This proves that $\Psi^{\mathsf{K},\tau}$ is also semi-coherent (the remaining properties having been demonstrated in Proposition 1656). ■

Since, by Proposition 1648, every family truth equational π-institution $\mathcal{I}$ has $\mathrm{AlgSys}(\mathcal{I})$ as a $\tau^\flat$-algebraic semantics and $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$, setting $\mathsf{K} := \mathrm{AlgSys}(\mathcal{I})$, we get that $\Psi^{\mathsf{K},\tau}$ is a semi-coherent family of monotone congruential compatibility $\mathcal{I}$-operators.

Our last result shows that the classes of **F**-algebraic systems associated with $\Psi^{\mathsf{K},\tau}$ (which are equal by Proposition 1558) coincide with $\overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K})$.

First, however, we show that, for any π-institution $\mathcal{I}$, with $\tau^\flat$ in $N^\flat$, the class $\mathsf{K}(\mathcal{I},\tau)$ is closed under subdirect intersections.

Lemma 1660 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then*

$$\overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K}(\mathcal{I},\tau)) \subseteq \mathsf{K}(\mathcal{I},\tau).$$

Proof: Let $\mathcal{A}^i \in \mathsf{K}(\mathcal{I},\tau)$, for all $i \in I$, and

$$\langle H^i, \gamma^i\rangle : \mathcal{A} \to \mathcal{A}^i, \quad i \in I,$$

be a subdirect intersection. Then, we have, by definition of subdirect intersection, $\bigcap_{i\in I} \mathrm{Ker}(\langle H^i,\gamma^i\rangle) = \Delta^{\mathcal{A}}$ and, by Proposition 1646, $\tau^{\mathcal{A}^i}(\Delta^{\mathcal{A}^i}) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^i)$, for all $i \in I$. These give

$$\begin{array}{rcl}
\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) & = & \tau^{\mathcal{A}}(\bigcap_{i\in I}\mathrm{Ker}(\langle H^i,\gamma^i\rangle))\\
& = & \bigcap_{i\in I}\tau^{\mathcal{A}}(\mathrm{Ker}(\langle H^i,\gamma^i\rangle))\\
& = & \bigcap_{i\in I}\tau^{\mathcal{A}}((\gamma^i)^{-1}(\Delta^{\mathcal{A}^i}))\\
& = & \bigcap_{i\in I}(\gamma^i)^{-1}(\tau^{\mathcal{A}^i}(\Delta^{\mathcal{A}^i}))\\
& \in & \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}),
\end{array}$$

where membership follows from the fact that $\tau^{\mathcal{A}^i}(\Delta^{\mathcal{A}^i}) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}^i)$, for all $i \in I$, by Corollary 55 and by closure of $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ under intersections. We conclude, using again Proposition 1646, that $\mathcal{A} \in \mathsf{K}(\mathcal{I},\tau)$. ■

Recall the classes of **F**-algebraic systems

$$\begin{array}{rcl}
\mathrm{AlgSys}_{\Psi^{\mathsf{K},\tau}}(\mathcal{I}) & = & \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : (\exists T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(\Psi^{\mathsf{K},\tau,\mathcal{A}}(T) = \Delta^{\mathcal{A}})\};\\
\mathrm{AlgSys}^{\Psi^{\mathsf{K},\tau}}(\mathcal{I}) & = & \{\mathcal{A}/\Psi^{\mathsf{K},\tau,\mathcal{A}}(T) : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}), T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}.
\end{array}$$

Proposition 1661 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*, with a* $\tau^\flat$*-algebraic semantics* K*, such that* $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathsf{K}$*. Then*

$$\mathrm{AlgSys}_{\Psi^{\mathsf{K},\tau}}(\mathcal{I}) = \mathrm{AlgSys}^{\Psi^{\mathsf{K},\tau}}(\mathcal{I}) = \overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K}).$$

Proof: First, by Proposition 1558, $\mathrm{AlgSys}_{\Psi^{\mathsf{K},\tau}}(\mathcal{I}) = \mathrm{AlgSys}^{\Psi^{\mathsf{K},\tau}}(\mathcal{I})$. So it suffices to show that $\mathrm{AlgSys}_{\Psi^{\mathsf{K},\tau}}(\mathcal{I}) = \overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K})$.

Suppose, first, that $\mathcal{A} \in \overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K})$. By Corollary 1647, $\mathsf{K} \subseteq \mathsf{K}(\mathcal{I},\tau)$. By Lemma 1660, $\overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K}) \subseteq \overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K}(\mathcal{I},\tau)) \subseteq \mathsf{K}(\mathcal{I},\tau)$, whence $\mathcal{A} \in \mathsf{K}(\mathcal{I},\tau)$. Thus, by

Proposition 1646, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Moreover, $\Delta^{\mathcal{A}} \in \mathrm{ConSys}^{\overset{\triangleleft}{\amalg}(\mathsf{K})}(\mathcal{A})$. We now get

$$\Psi^{\mathsf{K},\tau,\mathcal{A}}(\tau^{\mathcal{A}}(\Delta^{\mathcal{A}})) = \Theta^{\overset{\triangleleft}{\amalg}(\mathsf{K}),\mathcal{A}}(\tau^{\mathcal{A}}[\tau^{\mathcal{A}}(\Delta^{\mathcal{A}})]) \leq \Theta^{\overset{\triangleleft}{\amalg}(\mathsf{K}),\mathcal{A}}(\Delta^{\mathcal{A}}) = \Delta^{\mathcal{A}}.$$

We conclude that $\mathcal{A} \in \mathrm{AlgSys}_{\Psi^{\mathsf{K},\tau}}(\mathcal{I})$.

Suppose, conversely, that $\mathcal{A} \in \mathrm{AlgSys}_{\Psi^{\mathsf{K},\tau}}(\mathcal{I})$. Then, there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Psi^{\mathsf{K},\tau,\mathcal{A}}(T) = \Delta^{\mathcal{A}}$, that is, $\Theta^{\overset{\triangleleft}{\amalg}(\mathsf{K}),\mathcal{A}}(\tau^{\mathcal{A}}[T]) = \Delta^{\mathcal{A}}$. This shows that $\Delta^{\mathcal{A}}$ is an $\overset{\triangleleft}{\amalg}(\mathsf{K})$-congruence system on $\mathcal{A}$. Hence $\mathcal{A} \in \overset{\triangleleft}{\amalg}(\mathsf{K})$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$, with a $\tau^\flat$-algebraic semantics K, such that $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathsf{K}$. Then we have

$$\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathsf{K} \subseteq \mathsf{K}(\mathcal{I},\tau).$$

Assume, now, that $\mathcal{I}$ is family truth equational, with witnessing transformations $\tau^\flat$. By Proposition 1648, $\mathrm{AlgSys}(\mathcal{I})$ is a $\tau^\flat$-algebraic semantics for $\mathcal{I}$ and, by Proposition 65, $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$. Thus, in the case of truth equationality $\mathcal{I}$ has a $\tau^\flat$-algebraic semantics K, such that $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathsf{K} \subseteq \mathsf{K}(\mathcal{I},\tau)$.

- If $\mathsf{K} = \mathrm{AlgSys}^*(\mathcal{I})$, then, by Proposition 1637 and Theorem 1404, we would have $\Psi^{\mathsf{K},\tau} = \widetilde{\Omega}^{\mathcal{I}}$;
- At the other extreme, if $\mathsf{K} = \mathsf{K}(\mathcal{I},\tau)$, then, we get, by Proposition 1661 and Lemma 1660, a semi-coherent family of congruential monotone compatibility $\mathcal{I}$-operators $\Psi^{\mathsf{K},\tau}$, such that, similarly, $\mathrm{AlgSys}_{\Psi^{\mathsf{K},\tau}}(\mathcal{I}) = \mathsf{K}(\mathcal{I},\tau)$.

Chapter 22

The Strong Version of a π-Institution

22.1 The Strong Version of a π-Institution

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We define the following classes of $\mathcal{I}$-matrix families.

$$\begin{array}{rcl} \mathsf{M}^{\mathcal{I}*} & = & \{\langle \mathcal{A}, T\rangle : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}), T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})\}; \\ \mathsf{M}^{\mathcal{I},\mathsf{Su}} & = & \{\langle \mathcal{A}, T\rangle : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}), T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})\}; \\ \mathsf{M}^{\mathcal{I},m} & = & \{\langle \mathcal{A}, T\rangle : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}), T = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}. \end{array}$$

We show that all three classes of $\mathcal{I}$-matrix families generate the same closure system on $\mathbf{F}$.

Proposition 1662 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Then $\mathcal{I}^{\mathsf{M}^{\mathcal{I}*}} = \mathcal{I}^{\mathsf{M}^{\mathcal{I},m}}$.*

Proof: By Lemma 1568, we have that, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, $\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. Thus, $\mathsf{M}^{\mathcal{I},m} \subseteq \mathsf{M}^{\mathcal{I}*}$. This implies that $\mathcal{I}^{\mathsf{M}^{\mathcal{I}*}} \leq \mathcal{I}^{\mathsf{M}^{\mathcal{I},m}}$. To show the converse, assume that $\langle \mathcal{A}, T\rangle \in \mathsf{M}^{\mathcal{I}*}$ and consider the quotient morphism $\langle I, \pi\rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T)$. By Corollary 1554, $\pi(T^*)$ is the least $\mathcal{I}$-filter family of $\mathcal{A}/\Omega^{\mathcal{A}}(T)$. By hypothesis $T = T^*$, whence $\pi(T) = \pi(T^*)$ and, hence, since $\langle I, \pi\rangle : \langle \mathcal{A}, T\rangle \to \langle \mathcal{A}/\Omega^{\mathcal{A}}(T), \pi(T)\rangle$ is a strict surjective morphism,, we get that

$$\mathcal{I}^{\langle \mathcal{A}, T\rangle} = \mathcal{I}^{\langle \mathcal{A}/\Omega^{\mathcal{A}}(T), \pi(T)\rangle} = \mathcal{I}^{\langle \mathcal{A}/\Omega^{\mathcal{A}}(T), \pi(T^*)\rangle}$$

and $\langle \mathcal{A}/\Omega^{\mathcal{A}}(T), \pi(T^*)\rangle \in \mathsf{M}^{\mathcal{I},m}$. Putting things together, we finally obtain

$$\begin{array}{rcl} \mathcal{I}^{\mathsf{M}^{\mathcal{I},m}} & \leq & \bigcap \{\mathcal{I}^{\langle \mathcal{A}/\Omega^{\mathcal{A}}(T), \pi(T^*)\rangle} : T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})\} \\ & = & \bigcap \{\mathcal{I}^{\langle \mathcal{A}, T\rangle} : T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})\} \\ & = & \mathcal{I}^{\mathsf{M}^{\mathcal{I}*}}. \end{array}$$

Therefore, $\mathcal{I}^{\mathsf{M}^{\mathcal{I}*}} = \mathcal{I}^{\mathsf{M}^{\mathcal{I},m}}$. ∎

Proposition 1662 enables us to show that $\mathsf{M}^{\mathcal{I}*}$ and $\mathsf{M}^{\mathcal{I},\mathsf{Su}}$ also generate the same closure system on $\mathbf{F}$.

Corollary 1663 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Then $\mathcal{I}^{\mathsf{M}^{\mathcal{I}*}} = \mathcal{I}^{\mathsf{M}^{\mathcal{I},\mathsf{Su}}}$.*

Proof: By Lemma 1583, $\mathsf{M}^{\mathcal{I},\mathsf{Su}} \subseteq \mathsf{M}^{\mathcal{I}*}$. Also by Lemma 1583, $\mathsf{M}^{\mathcal{I},m} \subseteq \mathsf{M}^{\mathcal{I},\mathsf{Su}}$. So we get $\mathcal{I}^{\mathsf{M}^{\mathcal{I}*}} \leq \mathcal{I}^{\mathsf{M}^{\mathcal{I},\mathsf{Su}}} \leq \mathcal{I}^{\mathsf{M}^{\mathcal{I},m}}$. Therefore, by Proposition 1662, $\mathcal{I}^{\mathsf{M}^{\mathcal{I}*}} = \mathcal{I}^{\mathsf{M}^{\mathcal{I},\mathsf{Su}}}$. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Taking into account Proposition 1662 and Corollary 1663, we define the **strong version of** $\mathcal{I}$, denoted by $\mathcal{I}^+ = \langle \mathbf{F}, C^+\rangle$, by

$$\mathcal{I}^+ := \mathcal{I}^{\mathsf{M}^{\mathcal{I}*}} = \mathcal{I}^{\mathsf{M}^{\mathcal{I},\mathsf{Su}}} = \mathcal{I}^{\mathsf{M}^{\mathcal{I},m}}.$$

There are even more ways to characterize the π-institution $\mathcal{I}^+$. Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Given a class K of $\mathbf{F}$-algebraic systems, we define

$$\begin{array}{rcl}\mathsf{M}_\mathsf{K}^{\mathcal{I},m} & = & \{\langle \mathcal{A}, T \rangle : \mathcal{A} \in \mathsf{K}, T = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\};\\ \mathsf{M}_\mathsf{K}^{\mathcal{I}*} & = & \{\langle \mathcal{A}, T \rangle : \mathcal{A} \in \mathsf{K}, T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})\};\\ \mathsf{M}_\mathsf{K}^{\mathcal{I},\mathsf{Su}} & = & \{\langle \mathcal{A}, T \rangle : \mathcal{A} \in \mathsf{K}, T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})\}.\end{array}$$

Proposition 1664 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$ and $\mathsf{K} = \mathrm{AlgSys}^*(\mathcal{I})$ or $\mathsf{K} = \mathrm{AlgSys}(\mathcal{I})$. Then*

$$\mathcal{I}^+ = \mathcal{I}^{\mathsf{M}_\mathsf{K}^{\mathcal{I},m}} = \mathcal{I}^{\mathsf{M}_\mathsf{K}^{\mathcal{I}*}} = \mathcal{I}^{\mathsf{M}_\mathsf{K}^{\mathcal{I},\mathsf{Su}}}.$$

Proof: By definition and Lemma 1583, we have

$$\mathsf{M}_\mathsf{K}^{\mathcal{I},m} \subseteq \mathsf{M}_\mathsf{K}^{\mathcal{I},\mathsf{Su}} \subseteq \mathsf{M}_\mathsf{K}^{\mathcal{I}*} \subseteq \mathsf{M}^{\mathcal{I}*}.$$

Therefore, we get

$$\mathcal{I}^+ \le \mathcal{I}^{\mathsf{M}_\mathsf{K}^{\mathcal{I}*}} \le \mathcal{I}^{\mathsf{M}_\mathsf{K}^{\mathcal{I},\mathsf{Su}}} \le \mathcal{I}^{\mathsf{M}_\mathsf{K}^{\mathcal{I},m}}.$$

For the converse, suppose $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$ and $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. By Proposition 1572, $T/\Omega^{\mathcal{A}}(T)$ is the least $\mathcal{I}$-filter family of $\mathcal{A}/\Omega^{\mathcal{A}}(T) \in \mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$. Therefore, we get

$$\begin{array}{rcl}\mathcal{I}^{\mathsf{M}_\mathsf{K}^{\mathcal{I},m}} & \le & \bigcap\{\mathcal{I}^{\langle \mathcal{A}/\Omega^{\mathcal{A}}(T), T/\Omega^{\mathcal{A}}(T)\rangle} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}), T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})\}\\ & = & \bigcap\{\mathcal{I}^{\langle \mathcal{A}, T\rangle} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}), T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})\}\\ & = & \mathcal{I}^+.\end{array}$$

We conclude that $\mathcal{I}^+ = \mathcal{I}^{\mathsf{M}_\mathsf{K}^{\mathcal{I},m}} = \mathcal{I}^{\mathsf{M}_\mathsf{K}^{\mathcal{I}*}} = \mathcal{I}^{\mathsf{M}_\mathsf{K}^{\mathcal{I},\mathsf{Su}}}$. ■

The following proposition lists some of the properties of the strong version $\mathcal{I}^+$ of a π-institution $\mathcal{I}$.

Proposition 1665 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.*

(a) $\mathcal{I} \le \mathcal{I}^+$;

(b) $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, *for every $\mathbf{F}$-algebraic system $\mathcal{A}$;*

(c) $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$, *for every $\mathbf{F}$-algebraic system $\mathcal{A}$;*

(d) If $\mathcal{I}$ is family reflective, then $\mathcal{I}^+ = \mathcal{I}$.

Proof:

(a) Since $\mathsf{M}^{\mathcal{I},m} \subseteq \mathrm{MatFam}(\mathcal{I})$, we get $\mathcal{I} = \mathcal{I}^{\mathrm{MatFam}(\mathcal{I})} \le \mathcal{I}^{\mathsf{M}^{\mathcal{I},m}} = \mathcal{I}^+$.

(b) Since, by Part (a), $\mathcal{I} \leq \mathcal{I}^+$, we get that $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$.

(c) By definition of $\mathcal{I}^+$, we have, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, all $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$ and all $T' \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$, $C^+ \leq C^{\langle \mathcal{A},T\rangle}$ and $C^+ \leq C^{\langle \mathcal{A},T'\rangle}$. Moreover, by Lemma 1583, every Suszko filter family is a Leibniz filter family. We conclude that $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$.

(d) By the hypothesis and Proposition 1573, $\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, for every $\mathbf{F}$-algebraic system $\mathcal{A}$. Therefore, $\mathcal{I}^+ = \mathcal{I}^{\mathsf{M}^{\mathcal{I}*}} = \mathcal{I}^{\mathrm{MatFam}(\mathcal{I})} = \mathcal{I}$. ■

It turns out that the strong version $\mathcal{I}^+$ is mostly interesting when $\mathcal{I}$ itself has theorems. In the absence of theorems $\mathcal{I}^+$ has only trivial theory families.

Proposition 1666 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *does not have theorems, then* $\mathcal{I}$ *is almost inconsistent.*

Proof: Assume that $\mathcal{I}$ does not have theorems. Then, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\varnothing \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Therefore, by definition $\mathcal{I}^+ = \bigcap\{\mathcal{I}^{\langle \mathcal{A},\varnothing\rangle} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})\}$. This implies that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, we have, vacuously, for all $\psi \in \mathrm{SEN}^\flat(\Sigma)$, $\psi \in C^+_\Sigma(\phi)$. Therefore, the only Σ-theory families of $\mathcal{I}^+$ are $\varnothing$ and $\mathrm{SEN}^\flat(\Sigma)$. So $\mathcal{I}^+$ is almost inconsistent. ■

The least $\mathcal{I}$-filter family on every algebraic system $\mathcal{A}$ coincides with the least $\mathcal{I}^+$-filter family. As a consequence $\mathcal{I}$ and $\mathcal{I}^+$ share the same theorems.

Lemma 1667 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,*

$$\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \bigcap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}).$$

In particular, $\mathrm{ThFam}(\mathcal{I}^+) = \mathrm{ThFam}(\mathcal{I})$.

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system. By Proposition 1665, $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Thus, we have $\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \leq \bigcap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. On the other hand, by Lemma 1568, $\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$, whence, by Proposition 1665, $\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. Therefore, $\bigcap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \leq \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Equality now follows. ■

Lemma 1667 implies the idempotency of the strong version operator on π-institutions.

Corollary 1668 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *Then* $(\mathcal{I}^+)^+ = \mathcal{I}^+$.

Proof: We have

$$\begin{array}{rcl}(\mathcal{I}^+)^+ & = & \bigcap\{\mathcal{I}^{\langle\mathcal{A},T\rangle} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}), T = \bigcap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{I})\}\\ & = & \bigcap\{\mathcal{I}^{\langle\mathcal{A},T\rangle} : \mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}), T = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{I})\}\\ & = & \mathcal{I}^+.\end{array}$$

The first and last equalities follow by the definition of $^+$, and the main equality is due to Lemma 1667. ∎

The next proposition provides sufficient conditions for recognizing that a given π-institution is the strong version of another π-institution based on the same algebraic system.

Proposition 1669 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F}, C\rangle$, $\mathcal{I}' = \langle\mathbf{F}, C'\rangle$ *π-institutions based on* $\mathbf{F}$*, such that*

1. *$\mathcal{I}'$ is family reflective;*
2. $\mathrm{AlgSys}(\mathcal{I}') = \mathrm{AlgSys}(\mathcal{I})$*;*
3. *For all* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I}')$, $\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \bigcap \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A})$.

Then $\mathcal{I}' = \mathcal{I}^+$.

Proof: We have

$$\begin{array}{rcl}\mathcal{I}' & = & \mathcal{I}'^+ \quad \text{(by 1 and Proposition 1665)}\\ & = & \bigcap\{\mathcal{I}^{\langle\mathcal{A},T\rangle} : \mathcal{A} \in \mathrm{AlgSys}(\mathcal{I}'), T = \bigcap \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A})\}\\ & & \text{(by Proposition 1664)}\\ & = & \bigcap\{\mathcal{I}^{\langle\mathcal{A},T\rangle} : \mathcal{A} \in \mathrm{AlgSys}(\mathcal{I}), T = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}\\ & & \text{(by 2 and 3)}\\ & = & \mathcal{I}^+. \quad \text{(by Proposition 1664)}\end{array}$$

This proves the claim. ∎

We now show that Suszko and Leibniz $\mathcal{I}$-filter families form subclasses, respectively, of the classes of Suszko and Leibniz $\mathcal{I}^+$-filter families on every $\mathbf{F}$-algebraic system.

Proposition 1670 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *a π-institution based on* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,*

$$\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}^+,\mathsf{Su}}(\mathcal{A}) \quad \text{and} \quad \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A}).$$

Proof: By Proposition 1665, $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \mathrm{FFam}^{\mathcal{I}}(\mathcal{A})$. Thus, for all $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$, $[\![T]\!]^{\mathcal{I}^+*} \subseteq [\![T]\!]^{\mathcal{I}*}$ and $[\![T]\!]^{\mathcal{I}^+,\mathsf{Su}} \subseteq [\![T]\!]^{\mathcal{I},\mathsf{Su}}$.

Suppose that $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. Then, by Proposition 1665, $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$ and, moreover, $T = \bigcap [\![T]\!]^{\mathcal{I},\mathsf{Su}} \leq \bigcap [\![T]\!]^{\mathcal{I}^+,\mathsf{Su}}$. Thus, since $T \in [\![T]\!]^{\mathcal{I}^+,\mathsf{Su}}$, we get that $T = \bigcap [\![T]\!]^{\mathcal{I}^+,\mathsf{Su}} \in \mathrm{FiFam}^{\mathcal{I}^+,\mathsf{Su}}(\mathcal{A})$.

The second inclusion may be shown similarly. ∎

But the Leibniz counterpart of an $\mathcal{I}^+$-filter family is identical whether it be considered with respect to $\mathcal{I}$ or with respect to $\mathcal{I}^+$.

Lemma 1671 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, and all* $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$*,* $T^{\mathcal{I}*} = T^{\mathcal{I}^+*}$*.*

Proof: By Proposition 1665, $[\![T]\!]^{\mathcal{I}^+*} \subseteq [\![T]\!]^{\mathcal{I}*}$. Therefore, $T^{\mathcal{I}*} \leq T^{\mathcal{I}^+*}$. On the other hand,

$$\begin{array}{lll} T^{\mathcal{I}*} & \in & \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \quad \text{(by Proposition 1570)} \\ & \subseteq & \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \quad \text{(by Proposition 1670)} \end{array}$$

and, since $T^{\mathcal{I}*} \in [\![T]\!]^{\mathcal{I}*}$, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T^{\mathcal{I}*})$. Thus, $T^{\mathcal{I}*} \subseteq [\![T]\!]^{\mathcal{I}^+*}$, which gives $T^{\mathcal{I}^+*} \leq T^{\mathcal{I}*}$. We conclude that $T^{\mathcal{I}*} = T^{\mathcal{I}^+*}$. ■

And this implies that the Leibniz $\mathcal{I}$-filter families and the Leibniz $\mathcal{I}^+$-filter families coincide on every $\mathbf{F}$-algebraic system.

Corollary 1672 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,*

$$\mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: The right-to-left inclusion was shown in Proposition 1670. For the reverse, assume that $T \in \mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A})$. Then, by Proposition 1665, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and, by Lemma 1671, $T = T^{\mathcal{I}^+*} = T^{\mathcal{I}*}$. Therefore, $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. ■

22.2 Leibniz and Suszko $\mathcal{I}^+$-Filter Families

There is a relation between the $\mathcal{I}^+$-filter families on algebraic systems and the Leibniz and Suszko $\mathcal{I}$-filter families on the same algebraic systems. The following proposition shows how these relations interplay with family c-reflectivity.

Proposition 1673 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

- *(a) If, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A}$*,* $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$*, then* $\mathcal{I}^+$ *is family c-reflective.*
- *(b) If* $\mathcal{I}^+$ *is family c-reflective, then* $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$*, for all* $\mathbf{F}$*-algebraic systems* $\mathcal{A}$*.*

Proof:

(a) Suppose, for all $\mathbf{F}$-algebraic systems $\mathcal{A}$, $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system. By Proposition 1670, $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}^+,\mathsf{Su}}(\mathcal{A})$. Hence, by hypothesis, $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}^+,\mathsf{Su}}(\mathcal{A})$. Thus, $\mathrm{FiFam}^{\mathcal{I}^+,\mathsf{Su}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. By Theorem 1590, $\mathcal{I}^+$ is family c-reflective.

(b) Suppose $\mathcal{I}^+$ is family c-reflective and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system. By Theorem 1590, $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}^+,\mathsf{Su}}(\mathcal{A})$. Since, by Lemma 1583 and Corollary 1672, $\mathrm{FiFam}^{\mathcal{I}^+,\mathsf{Su}}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$, we get that $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. The reverse inclusion holds by Proposition 1665.

■

A necessary and sufficient condition for the $\mathcal{I}^+$-filter families to coincide with the Leibniz $\mathcal{I}$-filter families is the universal reflectivity of the Leibniz operator on $\mathcal{I}^+$-filter families.

Proposition 1674 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,*

$$\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$$

if and only if, for every $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\Omega^{\mathcal{A}}$ *is order reflecting on* $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$*.*

Proof: By Corollary 1672, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. By Proposition 1573, $\Omega^{\mathcal{A}}$ is reflective on $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$, for all $\mathcal{A}$, if and only if $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A})$, for all $\mathcal{A}$. Thus, we get that $\Omega^{\mathcal{A}}$ is reflective on $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$, for all $\mathcal{A}$, if and only if $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$, for all $\mathcal{A}$. ■

Under the stipulation that the strong version of $\mathcal{I}$ be protoalgebraic, the identification of $\mathcal{I}^+$-filter families with the Leibniz $\mathcal{I}$-families have several characterizations.

Proposition 1675 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, such that* $\mathcal{I}^+$ *is protoalgebraic. The following conditions are equivalent:*

(i) $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$*, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*;*

(ii) $\mathrm{ThFam}(\mathcal{I}^+) = \mathrm{ThFam}^*(\mathcal{I})$*;*

(iii) $\mathcal{I}^+$ *is weakly family algebraizable;*

(iv) $\mathcal{I}^+$ *is family c-reflective;*

Proof:

(i)⇒(ii) Trivial.

(ii)⇒(iii) Suppose that $\mathrm{ThFam}(\mathcal{I}^+) = \mathrm{ThFam}^*(\mathcal{I})$. By Proposition 1528, Ω is injective on $\mathrm{ThFam}^*(\mathcal{I})$. By definition it is onto $\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{F})$. Thus, by hypothesis and Corollary 1672, $\Omega : \mathrm{FiFam}(\mathcal{I}^+) \to \mathrm{ConSys}^{\mathcal{I}^+*}(\mathcal{F})$ is a bijection. By hypothesis it is monotone and, by Proposition 1528, it is order reflecting. Therefore, it is an order isomorphism. By Theorem 296, $\mathcal{I}^+$ is weakly family algebraizable.

(iii)⇒(iv) Every weakly family algebraizable π-institution is a fortiori family c-reflective.

(iv)⇒(i) By hypothesis, $\mathcal{I}^+$ is protoalgebraic, whence, by Proposition 1601 and Corollary 1672,

$$\mathrm{FiFam}^{\mathcal{I}^+,\mathsf{Su}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}).$$

By hypothesis and Theorem 1590, $\mathrm{FiFam}^{\mathcal{I}^+,\mathsf{Su}}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. Therefore, we get that $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. ■

We close the section by looking at various consequences of the condition imposed on a π-institution $\mathcal{I}$ that $\Omega^{\mathcal{A}}$ be an order isomorphism from the Leibniz $\mathcal{I}$-filter families of $\mathcal{A}$ onto the $\mathcal{I}^*$-congruence systems on $\mathcal{A}$, for every $\mathcal{I}$-algebraic system $\mathcal{A}$. First, we show that this condition ensures that $\mathcal{I}$-algebraic systems, $\mathcal{I}^*$-algebraic systems, $\mathcal{I}^+$-algebraic systems and $(\mathcal{I}^+)^*$-algebraic systems all coincide.

Lemma 1676 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathcal{I}$*, such that, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{AlgSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism. Then

$$\mathrm{AlgSys}(\mathcal{I}^+) = \mathrm{AlgSys}^*(\mathcal{I}^+) = \mathrm{AlgSys}^*(\mathcal{I}) = \mathrm{AlgSys}(\mathcal{I}).$$

Proof: We show, first, that $\mathrm{AlgSys}^*(\mathcal{I}^+) = \mathrm{AlgSys}^*(\mathcal{I})$. The left-to-right inclusion holds because, by Proposition 1665, $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, for every $\mathbf{F}$-algebraic system $\mathcal{I}$. Assume, conversely, that $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I})$. Then $\Delta^{\mathcal{A}} \in \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$. By hypothesis, then, there exists $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. By Proposition 1665 again, $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. Hence, $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I}^+)$.

Now we have

$$\begin{array}{rcll}
\mathrm{AlgSys}(\mathcal{I}) & = & \mathrm{AlgSys}^*(\mathcal{I}) & \text{(by Lemma 1623)}\\
& = & \mathrm{AlgSys}^*(\mathcal{I}^+) & \text{(shown above)}\\
& \subseteq & \mathrm{AlgSys}(\mathcal{I}^+) & \text{(by Proposition 65)}\\
& \subseteq & \mathrm{AlgSys}(\mathcal{I}). & \text{(by Proposition 1665).}
\end{array}$$

We conclude that all four classes of algebraic system coincide. ∎

Next we show that, under the same hypothesis the Leibniz congruence systems of a filter family and its Leibniz counterpart coincide and that the Suszko congruence system of a filter family coincides with the Leibniz congruence system of its Suszko counterpart.

Proposition 1677 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathcal{I}$*, such that, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{AlgSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism. Then, for every $\mathbf{F}$*-algebraic system and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T^*) \quad \text{and} \quad \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T^{\mathcal{I},\mathsf{Su}}).$$

Proof: By Proposition 1622, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ is an order isomorphism.

Let $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Since $\Omega^{\mathcal{A}}(T) \in \mathrm{ConSys}^{\mathcal{I}^*}(\mathcal{A})$, there exists $T' \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T') = \Omega^{\mathcal{A}}(T)$. Hence, $[\![T]\!]^* = [\![T']\!]^*$, which gives $T^* = T'^* = T'$. Thus, we get $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T') = \Omega^{\mathcal{A}}(T^*)$.

By hypothesis and Lemma 1623, $\mathrm{AlgSys}^*(\mathcal{I}) = \mathrm{AlgSys}(\mathcal{I})$. Since we have $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$, there exists $T'' \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T'') = \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)$. Thus, we get $[\![T]\!]^{\mathsf{Su}} = [\![T'']\!]^*$ and, therefore, $T^{\mathcal{I},\mathsf{Su}} = T''^* = T''$. This gives $\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T'') = \Omega^{\mathcal{A}}(T^{\mathcal{I},\mathsf{Su}})$. ∎

Under the same hypothesis, it turns out that the coincidence of the class of Leibniz filter families with Suszko filter families on every algebraic system characterizes protoalgebraicity.

Corollary 1678 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathcal{I}$*, such that, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{AlgSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism. $\mathcal{I}$ *is protoalgebraic if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$*.*

Proof: If $\mathcal{I}$ is protoalgebraic, then, by Proposition 1601, Leibniz and Suszko classes coincide and, therefore, $\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$.

Suppose, conversely, that, for all $\mathbf{F}$-algebraic systems $\mathcal{A}$, $\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. Let $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. By Lemma 1583, $T^{\mathcal{I},\mathsf{Su}} \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. By the hypothesis and Lemma 1586, $T^{\mathcal{I},\mathsf{Su}}$ is the largest Leibniz $\mathcal{I}$-filter family included in T. Since, by Lemma

1583, $T^{\mathcal{I},\mathsf{Su}} \le T^* \le T$ and, by Proposition 1570, T^* is a Leibniz $\mathcal{I}$-filter family, we get $T^{\mathcal{I},\mathsf{Su}} = T^*$. Therefore, using Proposition 1570, we get

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T^{\mathcal{I},\mathsf{Su}}) = \Omega^{\mathcal{A}}(T^*) = \Omega^{\mathcal{A}}(T).$$

Thus, on every $\mathbf{F}$-algebraic system $\mathcal{A}$, the Suszko and the Leibniz operators coincide and, therefore, by Lemma 1518, $\mathcal{I}$ is protoalgebraic. ■

We already have the tools to show that the property that $\Omega^{\mathcal{A}}$ be an isomorphism between Leibniz filter families and reduced algebraic systems is bequeathed by a π-institution $\mathcal{I}$ to its strong version $\mathcal{I}^+$.

Lemma 1679 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathcal{I}$*, such that, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{AlgSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism. Then, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I}^+)$*,* $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^+*}(\mathcal{A})$ *is also an order isomorphism.*

Proof: By Corollary 1672, we have $\mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. By Lemma 1676, $\mathrm{AlgSys}^*(\mathcal{I})) = \mathrm{AlgSys}^*(\mathcal{I}^+)$. Now, taking into account the hypothesis, we get the conclusion. ■

In a proposition analogous to Proposition 1675, we provide under our working hypothesis, of the Leibniz operator being an order isomorphism, a characterization of the property of $\mathcal{I}^+$ being weakly family algebraizable.

Proposition 1680 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathcal{I}$*, such that, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*,*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{AlgSys}^{\mathcal{I}*}(\mathcal{A})$$

is an order isomorphism. The following conditions are equivalent:

(i) $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$*, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*;*

(ii) $\mathrm{ThFam}(\mathcal{I}^+) = \mathrm{ThFam}^*(\mathcal{I})$*;*

(iii) $\mathcal{I}^+$ *is weakly family algebraizable;*

(iv) $\mathcal{I}^+$ *is family c-reflective;*

(v) Ω *is injective on the collection of reduced* $\mathcal{I}^+$*-filter families.*

Proof:

(i)⇒(ii) Trivial.

(ii)⇒(iii) By hypothesis and Lemma 1676, $\Omega : \mathrm{ThFam}(\mathcal{I}^+) \to \mathrm{ConSys}^{\mathcal{I}^{+*}}(\mathcal{F})$ is an order isomorphism. Thus Ω is both monotone and family c-reflective, whence $\mathcal{I}^+$ is weakly family algebraizable.

(iii)⇒(iv) Weak family algebraizability implies family c-reflectivity.

(iv)⇒(v) If $\mathcal{I}^+$ is family c-reflective, then it is a fortiori injective. Therefore, by Theorem 214, $\Omega^{\mathcal{A}}$ is injective on the $\mathcal{I}$-filter families of every $\mathbf{F}$-algebraic system $\mathcal{A}$.

(v)⇒(i) Suppose (v) holds and let $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$. By Proposition 1665, we have $\mathrm{FiFam}^{\mathcal{I}^*}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. So it suffices to prove the reverse inclusion. To this end, suppose $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. Consider the quotient morphism

$$\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T).$$

$\mathrm{Ker}(\langle I, \pi \rangle) = \Omega^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T^*)$, the last inclusion, since, by Proposition 1525, $T^* \in [\![T]\!]^{\mathcal{I}^*}$. Hence, by Corollary 56,

$$\pi(T), \pi(T^*) \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}/\Omega^{\mathcal{A}}(T))$$

and, by compatibility, $\pi^{-1}(\pi(T)) = T$ and $\pi^{-1}(\pi(T^*)) = T^*$. By Corollary 1554, $\pi(T^*) = \pi(T)^*$. Now we get

$$\begin{array}{rcll} \Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)} & = & \Omega^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\pi(T)) & \text{(by Lemma 1557)} \\ & = & \Omega^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\pi(T)^*) & \text{(by Proposition 1677)} \\ & = & \Omega^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\pi(T^*)). & \end{array}$$

This, both $\pi(T)$ and $\pi(T^*)$ are reduced $\mathcal{I}^+$-filter families and, therefore, by the injectivity hypothesis, $\pi(T) = \pi(T^*)$. Now we conclude that $T = \pi^{-1}(\pi(T)) = \pi^{-1}(\pi(T^*)) = T^*$. This proves that, for all $\mathcal{A}$, $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}^*}(\mathcal{A})$. Equality now follows.

■

22.3 Full $\mathcal{I}^+$-Structures

We now explore the relation between full $\mathcal{I}$-structures and full $\mathcal{I}^+$-structures.

Proposition 1681 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\mathcal{A}$ *an* $\mathbf{F}$*-algebraic system.* $\langle \mathcal{A}, \mathcal{D} \rangle \in \mathrm{FStr}^{\mathcal{I}^+}(\mathcal{A})$ *if and only if, there exists* $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, such that* $\langle \mathcal{A}, \mathcal{T} \rangle \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$ *and* $\mathcal{D} = \mathcal{T} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$*, i.e.,*

$$\mathrm{FStr}(\mathcal{I}^+) = \{\langle \mathcal{A}, \mathcal{T} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \rangle : \langle \mathcal{A}, \mathcal{T} \rangle \in \mathrm{FStr}(\mathcal{I})\}.$$

Proof:

($\Rightarrow$) Suppose that $\langle\mathcal{A},\mathcal{D}\rangle \in \mathrm{FStr}(\mathcal{I}^+)$. Set

$$\mathcal{T} = \{T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) \le \Omega^{\mathcal{A}}(T)\}.$$

If $T \in \mathcal{D}$, then $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) \le \Omega^{\mathcal{A}}(T)$ and $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(T)$. Thus, $T \in \mathcal{T}\cap\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. On the other hand, let $T \in \mathcal{T}\cap\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. Then $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) \le \Omega^{\mathcal{A}}(T)$ and, since $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$ and $\langle\mathcal{A},\mathcal{D}\rangle \in \mathrm{FStr}(\mathcal{I}^+)$, we must have, by Theorem 1395, $T \in \mathcal{D}$. We conclude that $\mathcal{D} = \mathcal{T}\cap\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. Thus, it only remains to show that $\langle\mathcal{A},\mathcal{T}\rangle \in \mathrm{FStr}(\mathcal{I})$.

To this end, let $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \le \Omega^{\mathcal{A}}(T)$. Then, we get $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \le \bigcap_{T'\in\mathcal{D}}\Omega^{\mathcal{A}}(T') = \widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) \le \Omega^{\mathcal{A}}(T)$. Thus, by definition, $T \in \mathcal{T}$. We conclude, using Theorem 1395, that $\langle\mathcal{A},\mathcal{T}\rangle \in \mathrm{FStr}(\mathcal{I})$.

($\Leftarrow$) Suppose, now, that $\langle\mathcal{A},\mathcal{T}\rangle \in \mathrm{FStr}(\mathcal{I})$ and $\mathcal{D} = \mathcal{T}\cap\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. Since, by Proposition 1563, the least element of a full $\mathcal{I}$-structure is a Leibniz $\mathcal{I}$-filter family, we get that $\bigcap\mathcal{T} \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. To see that $\langle\mathcal{A},\mathcal{D}\rangle$ is a dull $\mathcal{I}^+$-structure, let $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) \le \Omega^{\mathcal{A}}(T)$. Then, we infer

$$\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \le \widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) \le \Omega^{\mathcal{A}}(T).$$

Since $\langle\mathcal{A},\mathcal{T}\rangle \in \mathrm{FStr}(\mathcal{I})$, then, by Theorem 1395, $T \in \mathcal{T}$. Since, in addition, by hypothesis, $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$, we get $T \in \mathcal{D}$. Thus, again by Theorem1395, $\langle\mathcal{A},\mathcal{D}\rangle \in \mathrm{FStr}(\mathcal{I}^+)$.

■

Next, we show that the association

$$\langle\mathcal{A},\mathcal{T}\rangle \mapsto \langle\mathcal{A},\mathcal{T}\cap\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})\rangle$$

of full $\mathcal{I}^+$-structures to full $\mathcal{I}$-structures, given in Proposition 1681, is one-to-one, provided that $\mathcal{I}$- and $\mathcal{I}^+$-algebraic systems coincide.

Proposition 1682 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*, such that* $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}(\mathcal{I}^+)$*, and* $\mathcal{A}$ *an* $\mathbf{F}$*-algebraic system. For all* $\langle\mathcal{A},\mathcal{T}\rangle, \langle\mathcal{A},\mathcal{T}'\rangle \in \mathrm{FStr}(\mathcal{I})$*,*

$$\mathcal{T}\cap\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathcal{T}'\cap\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \quad \textit{implies} \quad \mathcal{T} = \mathcal{T}'.$$

Proof: We start with some preparatory remarks. Suppose $\mathcal{A}$ is an $\mathbf{F}$-algebraic system. Since, by hypothesis, $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}(\mathcal{I}^+)$, we get that $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}) = \mathrm{ConSys}^{\mathcal{I}^+}(\mathcal{A})$. Now, using Theorem 1408 (or, alternatively, Corollary 1565), we have that $\mathbf{FStr}^{\mathcal{I}}(\mathcal{A}) \cong \mathrm{FStr}^{\mathcal{I}^+}(\mathcal{A})$, through

$$\mathcal{T} \mapsto \overline{\mathcal{T}} = \{T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \le \Omega^{\mathcal{A}}(T)\}.$$

This is obtained, by applying Theorem 1408 to get an isomorphism

$$\begin{array}{rcl}\gamma : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) & \to & \mathrm{ConSys}^{\mathcal{I}}(\mathcal{A}); \\ \mathcal{T} & \overset{\gamma}{\mapsto} & \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}),\end{array}$$

then, applying Theorem 1408 to get an isomorphism

$$\begin{array}{rcl}\delta : \mathrm{ConSys}^{\mathcal{I}^+}(\mathcal{A}) & \to & \mathrm{FStr}^{\mathcal{I}^+}(\mathcal{A}); \\ \theta & \overset{\delta}{\mapsto} & \{T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) : \theta \le \Omega^{\mathcal{A}}(T)\}\end{array}$$

and, finally, composing these two, taking into account the hypothesis.

Now let $\mathcal{T}, \mathcal{T}' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\langle \mathcal{A}, \mathcal{T}\rangle$, $\langle \mathcal{A}, \mathcal{T}'\rangle \in \mathrm{FStr}^{\mathcal{I}}(\mathcal{A})$, and suppose that $\mathcal{T} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathcal{T}' \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$.

Claim 1: $\overline{\mathcal{T}} = \mathcal{T} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$ and $\overline{\mathcal{T}'} = \mathcal{T}' \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$.

We show the first equality. The second one is shown in exactly the same way. First, if $T \in \overline{\mathcal{T}}$, then $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$ and $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \le \Omega^{\mathcal{A}}(T)$. Since $\langle \mathcal{A}, \mathcal{T}\rangle$ is a full $\mathcal{I}$-structure, by Theorem 1395, $T \in \mathcal{T}$. Thus, $T \in \mathcal{Y} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. If, on the other hand, $T \in \mathcal{T} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$, then $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$ and $T \in \mathcal{T}$. Thus, $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$ and $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \le \Omega^{\mathcal{A}}(T)$. Therefore, $T \in \overline{\mathcal{T}}$.

Claim 2: $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \widetilde{\Omega}^{\mathcal{A}}(\overline{\mathcal{T}})$ and $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}') = \widetilde{\Omega}^{\mathcal{A}}(\overline{\mathcal{T}'})$.

Again, it suffices to show the first equality, since the second is proven in exactly the same way. By Claim 1 and Proposition 1681, $\langle \mathcal{A}, \overline{\mathcal{T}}\rangle \in \mathrm{FStr}^{\mathcal{I}^+}(\mathcal{A})$. Therefore, by Theorem 1395, $\overline{\mathcal{T}} = \{T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{A}}(\overline{\mathcal{T}}) \le \Omega^{\mathcal{A}}(T)\}$. Thus, we get $\delta(\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})) = \delta(\gamma(\mathcal{T})) = \overline{\mathcal{T}} = \delta(\widetilde{\Omega}^{\mathcal{A}}(\overline{\mathcal{T}}))$. Since δ is an isomorphism, we get that $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \widetilde{\Omega}^{\mathcal{A}}(\overline{\mathcal{T}})$.

To finish the proof, we get $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \widetilde{\Omega}^{\mathcal{A}}(\overline{\mathcal{T}}) = \widetilde{\Omega}^{\mathcal{A}}(\overline{\mathcal{T}'}) = \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}')$. Therefore, by Theorem 1408, $\mathcal{T} = \mathcal{T}'$. ■

Now we can formulate an order isomorphism between full $\mathcal{I}$- and full $\mathcal{I}^+$-structures, subject to the condition that $\mathcal{I}$- and $\mathcal{I}^+$-algebraic systems coincide.

Corollary 1683 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*, such that* $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}(\mathcal{I}^+)$*, and* $\mathcal{A}$ *an* $\mathbf{F}$*-algebraic system.*

$$\begin{array}{rcl}h : \mathrm{FStr}^{\mathcal{I}}(\mathcal{A}) & \to & \mathrm{FStr}^{\mathcal{I}^+}(\mathcal{A}); \\ \langle \mathcal{A}, \mathcal{T}\rangle & \overset{h}{\mapsto} & \langle \mathcal{A}, \mathcal{T} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})\rangle\end{array}$$

is an order isomorphism.

Proof: By Propositions 1681 and 1682. ■

We turn next to relationships between full classes of filter families with respect to a π-institution $\mathcal{I}$ and its strong version $\mathcal{I}^+$. Recall that, given any

$\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, we have $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. So we get immediately the following inclusions, for all $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$.

$$\begin{array}{rcl}[\![T]\!]^{\mathcal{I}^+*} & = & \{T' \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) : \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\} \\ & \subseteq & \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\} \\ & = & [\![T]\!]^{\mathcal{I}*}.\end{array}$$

Moreover, taking into account

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T) \leq \widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})^T) = \widetilde{\Omega}^{\mathcal{I}^+,\mathcal{A}}(T),$$

we infer

$$\begin{array}{rcl}[\![T]\!]^{\mathcal{I}^+,\mathsf{Su}} & = & \{T' \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{I}^+,\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\} \\ & \subseteq & \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\} \\ & = & [\![T]\!]^{\mathcal{I},\mathsf{Su}}.\end{array}$$

These relationships may be strengthened to apply to all extensions to a π-institution rather that only its strong version. More precisely, we obtain

Lemma 1684 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *and* $\mathcal{I}' = \langle \mathbf{F}, C' \rangle$ *be* π*-institutions based on* $\mathbf{F}$*, such that* $\mathcal{I} \leq \mathcal{I}'$*,* $\mathcal{A}$ *be an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A})$*. Then*

$$[\![T]\!]^{\mathcal{I}'*} = [\![T]\!]^{\mathcal{I}*} \cap \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A}) \quad \text{and} \quad [\![T]\!]^{\mathcal{I}',\mathsf{Su}} \subseteq [\![T]\!]^{\mathcal{I},\mathsf{Su}} \cap \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A}).$$

Proof: We have, mimicking the process preceding the statement, applied to the extension $\mathcal{I}'$ rather than specifically $\mathcal{I}^+$:

$$\begin{array}{rcl}[\![T]\!]^{\mathcal{I}'*} & = & \{T' \in \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A}) : \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\} \\ & = & \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\} \cap \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A}) \\ & = & [\![T]\!]^{\mathcal{I}*} \cap \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A}).\end{array}$$

Moreover, taking into account

$$\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) = \widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T) \leq \widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A})^T) = \widetilde{\Omega}^{\mathcal{I}',\mathcal{A}}(T),$$

we infer

$$\begin{array}{rcl}[\![T]\!]^{\mathcal{I}',\mathsf{Su}} & = & \{T' \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{I}',\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\} \\ & \subseteq & \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\} \cap \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A}) \\ & = & [\![T]\!]^{\mathcal{I},\mathsf{Su}} \cap \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A}).\end{array}$$

Thus, we have the equality and the inclusion claimed. ■

Since $\mathcal{I}^+$ is an extension of $\mathcal{I}$, then we immediately deduce

Corollary 1685 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A}$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$*. Then*

$$[\![T]\!]^{\mathcal{I}^+*} = [\![T]\!]^{\mathcal{I}*} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \quad \text{and} \quad [\![T]\!]^{\mathcal{I}^+,\mathsf{Su}} \subseteq [\![T]\!]^{\mathcal{I},\mathsf{Su}} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}).$$

Proof: By Lemma 1684, since $\mathcal{I} \leq \mathcal{I}^+$. ■

Finally, we strengthen the preceding relation between Suszko classes to an equality, in the special case, where T happens to be a Suszko $\mathcal{I}$-filter family of $\mathcal{I}$ (recalling that $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{I}) \subseteq \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$).

Lemma 1686 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*,* $\mathcal{A}$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$*. Then* $[\![T]\!]^{\mathcal{I}^+,\mathsf{Su}} = [\![T]\!]^{\mathcal{I},\mathsf{Su}} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$*.*

Proof: Let $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. Then, by Lemma 1583, $[\![T]\!]^{\mathcal{I},\mathsf{Su}} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$. Since $T = \bigcap [\![T]\!]^{\mathcal{I},\mathsf{Su}}$, $[\![T]\!]^{\mathcal{I}^+,\mathsf{Su}} \subseteq [\![T]\!]^{\mathcal{I},\mathsf{Su}}$ and $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$, we get $T = \bigcap [\![T]\!]^{\mathcal{I}^+,\mathsf{Su}}$. Hence $T \in \mathrm{FiFam}^{\mathcal{I}^+,\mathsf{Su}}(\mathcal{A})$. Again, using Lemma 1583, we get $[\![T]\!]^{\mathcal{I}^+,\mathsf{Su}} = \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})^T$. Therefore, we conclude that

$$\begin{array}{rcl} [\![T]\!]^{\mathcal{I}^+,\mathsf{Su}} & = & \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})^T \\ & = & \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \\ & = & [\![T]\!]^{\mathcal{I},\mathsf{Su}} \cap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}). \end{array}$$

■

22.4 Leibniz Truth Equationality

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is **Leibniz truth equational** if there exists $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, such that, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$T^* = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T)),$$

i.e., for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T^*_\Sigma \quad \text{iff} \quad \tau^{\mathcal{A}}_\Sigma[\phi] \leq \Omega^{\mathcal{A}}(T).$$

It follows directly by the definition that, if $\mathcal{I}$ is Leibniz truth equational, then, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \quad \text{iff} \quad T = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T)).$$

Moreover, we can easily see that family truth equationality implies Leibniz truth equationality.

Lemma 1687 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is family truth equational, then* $\mathcal{I}$ *is Leibniz truth equational.*

Proof: Suppose that $\mathcal{I}$ is family truth equational, with witnessing transformations $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$. Thus, by Theorem 848, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$. Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. We have

$$\begin{array}{rcll} \phi \in T_\Sigma & \text{iff} & \tau^{\mathcal{A}}_\Sigma[\phi] \leq \Omega^{\mathcal{A}}(T) & (\mathcal{I} \text{ truth equational}) \\ & \text{implies} & \tau^{\mathcal{A}}[\phi] \leq \Omega^{\mathcal{A}}(T^*) & (T^* \in [\![T]\!]^*) \\ & \text{iff} & \phi \in T^*_\Sigma. & (\mathcal{I} \text{ truth equational}) \end{array}$$

Thus, we get $T \leq T^*$. On the other hand, by Lemma 1568, $T^* \leq T$, whence $T - T^*$. This gives $T^* = T$ and, hence $T^* = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$, showing that $\mathcal{I}$ is Leibniz truth equational. ■

If $\mathcal{I}$ is Leibniz truth equational, then the collection of all its Leibniz filters on every algebraic system forms a closure family.

Proposition 1688 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Leibniz truth equational* π*-institution based on* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$ *is closed under signature-wise intersections and, hence, forms a closure family on* $\mathcal{A}$*.*

Proof: Suppose $\mathcal{I}$ is Leibniz truth-equational, with witnssing transformations $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$. Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $\{T^i : i \in I\} \subseteq \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$ be a collection of Leibniz $\mathcal{I}$-filter families. Then

$$\begin{array}{rcll} \bigcap_{i\in I} T^i & = & \bigcap_{i\in I} (T^i)^* & (T^i \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})) \\ & = & \bigcap_{i\in I} \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T^i)) & (\mathcal{I} \text{ Leibniz truth equational}) \\ & \leq & \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(\bigcap_{i\in I} T^i)) & (\bigcap_{i\in I} \Omega^{\mathcal{A}}(T^i) \leq \Omega^{\mathcal{A}}(\bigcap_{i\in I} T^i)) \\ & = & (\bigcap_{i\in I} T^i)^*. & (\mathcal{I} \text{ Leibniz truth equational}) \end{array}$$

Since, by Lemma 1568, $(\bigcap_{i\in I} T^i)^* \leq \bigcap_{i\in I} T^i$, we get that $(\bigcap_{i\in I} T^i)^* = \bigcap_{i\in I} T^i$ and, therefore, $\bigcap_{i\in I} T^i \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. ■

The next proposition shows that to check that a given π-institution $\mathcal{I}$ is Leibniz truth equational, it is sufficient to work with $\mathcal{I}^*$-algebraic systems only. That is, if the defining property holds for all Leibniz filters of $\mathcal{I}^*$-algebraic systems, then it extends to Leibniz filters over arbitrary $\mathbf{F}$-algebraic systems.

Proposition 1689 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is Leibniz truth equational if and only if, there exists* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*, such that, for all* $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I})$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $T^* = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$*.*

Proof: The implication left-to-right follows from the definition of Leibniz truth equationality. Suppose, conversely, that there exists $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, such that, for all $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I})$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T^* = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$. Let $\mathcal{A}$ be an arbitrary **F**-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, and consider the quotient morphism $\langle I, \pi\rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T)$. Then, by Corollary 1554, $\pi(T^*) = \pi(T)^*$ and, by Proposition 1530, $\pi(T)^*$ is the least $\mathcal{I}$-filter family on $\mathcal{A}/\Omega^{\mathcal{A}}(T)$. Since $\mathcal{A}/\Omega^{\mathcal{A}}(T) \in \mathrm{AlgSys}^*(\mathcal{I})$, we get, by hypothesis,

$$\pi(T)^* = \tau^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(T/\Omega^{\mathcal{A}}(T)) = \tau^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}).$$

Hence, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll} \phi \in T^*_\Sigma & \text{iff} & \phi/\Omega^{\mathcal{A}}_\Sigma(T) \in \pi_\Sigma(T^*_\Sigma) \quad (\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T^*)) \\ & \text{iff} & \phi/\Omega^{\mathcal{A}}_\Sigma(T) \in \pi(T)^*_\Sigma \\ & \text{iff} & \phi/\Omega^{\mathcal{A}}_\Sigma(T) \in \tau_\Sigma^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}) \\ & \text{iff} & \phi \in \tau^{\mathcal{A}}_\Sigma(\Omega^{\mathcal{A}}(T)). \end{array}$$

Thus, $\mathcal{I}$ is Leibniz truth equational. ∎

A fortiori, it suffices to show that the condition in the statement of Proposition 1689 holds for all $\mathcal{I}$-algebraic systems, since this class encompasses all $\mathcal{I}^*$-algebraic systems.

Corollary 1690 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is Leibniz truth equational if and only if, there exists* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*, such that, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T^* = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$.

Proof: The conclusion follows from Proposition 1689, taking into account the fact that $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$. ∎

Next, we provide another characterization of Leibniz truth equationality by showing that it is equivalent to $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}})$ being the least $\mathcal{I}$-filter family on every $\mathcal{I}$- (or $\mathcal{I}\star$-)algebraic system.

Proposition 1691 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*. The following conditions are equivalent.*

(i) $\mathcal{I}$ *is Leibniz truth equational, with witnessing transformations* $\tau^\flat$*;*

(ii) For all $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I})$, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*;*

(iii) For all $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

Proof:

(i)⇒(iii) Suppose $\mathcal{I}$ is Leibniz truth equational, with witnessing transformations $\tau^\flat$. Let $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$ and $T^m = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, by Lemma 1568, $T^m \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. Since $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \leq \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T^m))$, we get, by hypothesis, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) \leq T^m$. On the other hand, since $T^m = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we have, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T^m \leq T^*$, whence, by hypothesis, $T^m \leq \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$. Since, this holds for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get, taking into account that $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$,

$$T^m \leq \tau^{\mathcal{A}}(\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))) = \tau^{\mathcal{A}}(\Delta^{\mathcal{A}}).$$

Therefore, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) = T^m$.

(iii)⇒(ii) Trivial, since $\mathrm{AlgSys}^*(\mathcal{I}) \subseteq \mathrm{AlgSys}(\mathcal{I})$.

(ii)⇒(i) Suppose, for all $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I})$, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and consider the quotient morphism

$$\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T).$$

Then, $\mathcal{A}/\Omega^{\mathcal{A}}(T) \in \mathrm{AlgSys}^*(\mathcal{I})$ and, by Corollary 1554, $\pi(T^*) = \pi(T)^*$ and, by Proposition 1530, $\pi(T)^* = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T))$. Thus, by hypothesis, $\pi(T^*) = \tau^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)})$. Therefore, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll} \phi \in T^*_\Sigma & \text{iff} & \phi/\Omega^{\mathcal{A}}_\Sigma(T) \in \pi_\Sigma(T^*_\Sigma) \\ & \text{iff} & \phi/\Omega^{\mathcal{A}}_\Sigma(T) \in \tau_\Sigma^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}) \\ & \text{iff} & \phi \in \tau^{\mathcal{A}}_\Sigma(\Omega^{\mathcal{A}}(T)). \end{array}$$

Hence, $\tau^\flat$ witnesses the Leibniz truth equationality of $\mathcal{I}$.

■

If $\mathcal{I}$-algebraic systems and $\mathcal{I}^+$-algebraic systems coincide, then truth equationality of $\mathcal{I}^+$ guarantees the Leibniz truth equationality of $\mathcal{I}$.

Proposition 1692 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$ *and* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*. If* $\mathcal{I}^+$ *is family truth equational, with witnessing transformations* $\tau^\flat$ *and* $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}(\mathcal{I}^+)$*, then* $\mathcal{I}$ *is Leibniz truth equational, with witnessing transformations* $\tau^\flat$*.*

Proof: We use Proposition 1691. Suppose $\mathcal{I}^+$ is family truth equational via $\tau^\flat$ and $\mathrm{AlgSys}(\mathcal{I}) = \mathrm{AlgSys}(\mathcal{I}^+)$. Let $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$. Since, by hypothesis $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I}^+)$, we get, by hypothesis, Lemma 1687 and Proposition 1691, $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) = \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. By Lemma 1667, $\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \bigcap \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. Hence, we get $\tau^{\mathcal{A}}(\Delta^{\mathcal{A}}) = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, whence, by Proposition 1691, $\mathcal{I}$ is Leibniz truth equational via $\tau^\flat$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ and K a class of $\mathbf{F}$-algebraic systems. We define, as before, on $\mathbf{F}$ the closure system $C^{\mathsf{K},\tau} = \{C^{\mathsf{K},\tau}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, where, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $C^{\mathsf{K},\tau}_\Sigma : \mathcal{P}(\mathrm{SEN}^\flat(\Sigma)) \to \mathcal{P}(\mathrm{SEN}^\flat(\Sigma))$ is given, for all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, by

$$\phi \in C^{\mathsf{K},\tau}_\Sigma(\Phi) \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq C^{\mathsf{K}}(\tau^\flat_\Sigma[\Phi]).$$

Then we say that K is a **$\tau^\flat$-algebraic semantics for** $\mathcal{I}$ if $C = C^{\mathsf{K},\tau}$.

We show that, if a π-institution $\mathcal{I}$ is Leibniz truth equational, with witnessing transformations $\tau^\flat$, then any of the four classes $\mathrm{AlgSys}^*(\mathcal{I}^+)$, $\mathrm{AlgSys}(\mathcal{I}^+)$, $\mathrm{AlgSys}^*(\mathcal{I})$ or $\mathrm{AlgSys}(\mathcal{I})$ serves as a $\tau^\flat$-algebraic semantics for $\mathcal{I}^+$.

Theorem 1693 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a Leibniz truth equational π-institution based on $\mathbf{F}$, with witnessing transformations $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$. Set $\mathsf{K} = \mathrm{AlgSys}^*(\mathcal{I}^+)$ or $\mathrm{AlgSys}(\mathcal{I}^+)$ or $\mathrm{AlgSys}^*(\mathcal{I})$ or $\mathrm{AlgSys}(\mathcal{I})$. Then K is a $\tau^\flat$-algebraic semantics for $\mathcal{I}^+$.*

Proof: Let, first, $K = \mathrm{AlgSys}^*(\mathcal{I})$ or $\mathrm{AlgSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. Then, we have $\phi \in C^+_\Sigma(\Phi)$ if and only if, by Proposition 1664, $\phi \in C^{\mathsf{M}^{\mathcal{I},m}_{\mathsf{K}}}_\Sigma(\Phi)$ if and only if, for all $\mathcal{A} \in \mathsf{K}$,

$$\alpha_\Sigma(\Phi) \subseteq C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\varnothing) \quad \text{implies} \quad \alpha_\Sigma(\phi) \in C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\varnothing).$$

if and only if, by hypothesis and Proposition 1691,

$$\alpha_\Sigma(\Phi) \subseteq \tau^{\mathcal{A}}_{F(\Sigma)}(\Delta^{\mathcal{A}}) \quad \text{implies} \quad \alpha_\Sigma(\phi) \in \tau^{\mathcal{A}}_{F(\Sigma)}(\Delta^{\mathcal{A}}),$$

if and only if

$$\tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\Phi)] \leq \Delta^{\mathcal{A}} \quad \text{implies} \quad \tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi)] \leq \Delta^{\mathcal{A}},$$

if and only if

$$\alpha(\tau^\flat_\Sigma[\Phi]) \leq \Delta^{\mathcal{A}} \quad \text{implies} \quad \alpha(\tau^\flat_\Sigma[\phi]) \leq \Delta^{\mathcal{A}},$$

if and only if $\tau^\flat_\Sigma[\phi] \leq C^{\mathsf{K}}(\tau^\flat_\Sigma[\Phi])$ if and only if $\phi \in C^{\mathsf{K},\tau}_\Sigma(\Phi)$. Thus, K is a $\tau^\flat$-algebraic semantics of $\mathcal{I}^+$.

Finally, note that, by hypothesis and Lemma 1671, $\mathcal{I}^+$ is Leibniz truth equational via $\tau^\flat$, as well. Moreover, by Corollary 1668, $(\mathcal{I}^+)^+ = \mathcal{I}^+$. Applying, therefore, what was shown above to $\mathcal{I}^+$, we get the result for $\mathsf{K} = \mathrm{AlgSys}^*(\mathcal{I}^+)$ or $\mathrm{AlgSys}(\mathcal{I}^+)$. ∎

Theorem 1693 implies that for $\mathrm{AlgSys}(\mathcal{I})$ to be a $\tau^\flat$-algebraic semantics of a Leibniz truth equational π-institution $\mathcal{I}$, where $\tau^\flat$ is a set of witnessing transformations, $\mathcal{I}$ and $\mathcal{I}^+$ must be identical.

Corollary 1694 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Leibniz truth equational* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$. $\mathrm{AlgSys}(\mathcal{I})$ *is a* $\tau^\flat$*-algebraic semantics for* $\mathcal{I}$ *if and only if* $\mathcal{I} = \mathcal{I}^+$.

Proof: By Theorem 1693, $C^+ = C^{\mathrm{AlgSys}(\mathcal{I}),\tau}$. Therefore, we get that $\mathrm{AlgSys}(\mathcal{I})$ is a $\tau^\flat$-algebraic semantics of $\mathcal{I}$ if and only if, by definition $C = C^{\mathrm{AlgSys}(\mathcal{I}),\tau}$ if and only if $C = C^+$. ■

Moreover, we can show that Leibniz truth equationality of $\mathcal{I}$ implies the family truth equationality of $\mathcal{I}^+$.

Corollary 1695 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is Leibniz truth equational, with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$*, then* $\mathcal{I}^+$ *is family truth equational via* $\tau^\flat$.

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A})$. By hypothesis and Theorem 1693, $\mathcal{I}^+$ has a $\tau^\flat$-algebraic semantics. Therefore, by Corollary 824, $T = \tau^{\mathcal{A}}(\widetilde{\Omega}^{\mathcal{I}^+,\mathcal{A}}(T)) \leq \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$. Conversely, by hypothesis and the fact that, by Proposition 1665, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get, using Lemma 1568, $\tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T)) = T^* \leq T$. We now conclude that $T = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T))$. Thus, $\mathcal{I}^+$ is family truth equational, with witnessing transformations $\tau^\flat$. ■

As another consequence, we get that, under Leibniz truth equationality, $\mathcal{I}^+$ filter families coincide with Leibniz $\mathcal{I}$-filter families on any algebraic system.

Corollary 1696 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is Leibniz truth equational, then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$, $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$.

Proof: Suppose $\mathcal{I}$ is Leibniz truth equational. Then, by Corollary 1695, $\mathcal{I}^+$ is family truth equational. Thus, by Proposition 1673, $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$, for every $\mathbf{F}$-algebraic system $\mathcal{A}$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a Leibniz truth equational π-institution, with witnessing transformations $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$. Let, also, $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, by definition $T^{\mathcal{I},\mathsf{Su}} = \bigcap [\![T]\!]^{\mathcal{I},\mathsf{Su}}$ and, by Proposition 1584, $\langle \mathcal{A}, [\![T]\!]^{\mathcal{I},\mathsf{Su}} \rangle \in \mathrm{FStr}(\mathcal{I})$. Thus, by Proposition 1584, $T^{\mathcal{I},\mathsf{Su}} \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. Now it follows, by hypothesis, that

$$T^{\mathcal{I},\mathsf{Su}} = \tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T^{\mathcal{I},\mathsf{Su}})).$$

There is also an additional characterization of the Suszko filter family, using the Suszko operator.

Proposition 1697 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Leibniz truth equational* π*-institution, with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$T^{\mathcal{I},\mathsf{Su}} = \tau^{\mathcal{A}}(\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)).$$

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and consider the quotient morphism

$$\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T).$$

Then $\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) \in \mathrm{AlgSys}(\mathcal{I})$. Moreover, by Lemma 1557, $\pi(T^{\mathcal{I},\mathsf{Su}}) = \pi(T)^{\mathcal{I},\mathsf{Su}}$ and, by Proposition 1587, $\pi(T)^{\mathcal{I},\mathsf{Su}} = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T))$. Thus, by Proposition 1691,

$$\pi(T^{\mathcal{I},\mathsf{Su}}) = \tau^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(\Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}).$$

Now we get

$$\begin{array}{rcl}
T^{\mathcal{I},\mathsf{Su}} & = & \pi^{-1}(\pi(T^{\mathcal{I},\mathsf{Su}})) \\
& = & \pi^{-1}(\tau^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)}(\Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)})) \\
& = & \tau^{\mathcal{A}}(\pi^{-1}(\Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)})) \\
& = & \tau^{\mathcal{A}}(\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)).
\end{array}$$

This proves the statement. ■

Proposition 1697 enables us to characterize the Suszko filter counterpart $T^{\mathcal{I},\mathsf{Su}}$ of a given filter family T as the intersection of all Leibniz filter family companions of filter families in the upset of T.

Corollary 1698 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Leibniz truth equational* π*-institution, with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$T^{\mathcal{I},\mathsf{Su}} = \bigcap \{T'^* : T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}.$$

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then we have

$$\begin{array}{rcl}
T^{\mathcal{I},\mathsf{Su}} & = & \tau^{\mathcal{A}}(\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)) \quad \text{(by Proposition 1697)} \\
& = & \tau^{\mathcal{A}}(\bigcap\{\Omega^{\mathcal{A}}(T') : T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}) \\
& & \text{(definition of } \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)) \\
& = & \bigcap\{\tau^{\mathcal{A}}(\Omega^{\mathcal{A}}(T')) : T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}) \\
& = & \bigcap\{T'^* : T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}. \\
& & \text{(Leibniz truth equationality)}
\end{array}$$

This proves the corollary. ■

We now get immediately

Corollary 1699 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Leibniz truth equational* π*-institution, with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{I}) \quad \textit{iff} \quad T \le T'^*, \textit{ for all } T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T.$$

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then we have $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$ if and only if, by definition, $T = T^{\mathcal{I},\mathsf{Su}}$ if and only if, by Corollary 1698, $T = \bigcap\{T'^* : T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T\}$, if and only if, taking into account that $T^* \le T$, $T \le T'^*$, for all $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})^T$. ■

We close the section with a characterization of weak family algebraizability of the strong version of $\mathcal{I}$ among those π-institutions that are Leibniz truth equational.

Proposition 1700 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Leibniz truth equational* π*-institution, with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*.* $\mathcal{I}^+$ *is weakly family algebraizable if and only if, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I}^+)$*,* $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^+*}(\mathcal{A})$ *is an order isomorphism.*

Proof: If $\mathcal{I}^+$ is weakly family algebraizable, then it is, a fortiori, protoalgebraic. Therefore, by Proposition 1621, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I}^+)$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^+*}(\mathcal{A})$ is an order isomorphism.

Assume, conversely, that the condition in the statement holds. Then, for every $\mathbf{F}$-algebraic system $\mathcal{A}$,

$$\begin{array}{rcll} \mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A}) & = & \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) & \text{(by Corollary 1672)} \\ & = & \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}). & \text{(by Corollary 1696)} \end{array}$$

Thus, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I}^+)$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^+*}(\mathcal{A})$ is an order isomorphism. Hence, by Theorem 296, $\mathcal{I}^+$ is weakly family algebraizable. ■

Proposition 1700 gives a sufficient condition for the weak family algbebraizability of $\mathcal{I}^+$ that involves only $\mathcal{I}$-algebraic systems.

Corollary 1701 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Leibniz truth equational* π*-institution, with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*. If, for every* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*,* $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is an order isomorphism, then* $\mathcal{I}^+$ *is weakly family algebraizable.*

Proof: By hypothesis and Lemma 1679, for every $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I}^+)$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^+*}(\mathcal{A})$ is an order isomorphism. Hence, by Proposition 1700, $\mathcal{I}^+$ is weakly family algebraizable. ■

22.5 Leibniz Definability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is **Leibniz definable** if, there exists $\mu^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, such that, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$T^* = \mu^{\mathcal{A}}(T),$$

i.e., for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\phi \in T^*_\Sigma \quad \text{iff} \quad \mu^{\mathcal{A}}_\Sigma[\phi] \leq T.$$

We show that it suffices to consider only $\mathcal{I}^*$-algebraic systems to establish Leibniz definability.

Proposition 1702 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ is Leibniz definable if and only if, there exists $\mu^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, such that, for all $\mathcal{A} \in \mathrm{AlgSys}^*(\mathcal{I})$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T^* = \mu^{\mathcal{A}}(T)$.*

Proof: The "only if" is trivial. For the "if", suppose the stated condition holds and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Consider the quotient morphism

$$\langle I, \pi \rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T).$$

Then $\mathcal{A}/\Omega^{\mathcal{A}}(T) \in \mathrm{AlgSys}^*(\mathcal{I})$ and, moreover, $\mathrm{Ker}(\langle I, \pi \rangle) = \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T^*)$, since $T^* \in [\![T]\!]^*$. Now we have

$$\begin{array}{rcll} T^* & = & \pi^{-1}(\pi(T^*)) & (\mathrm{Ker}(\langle I, \pi \rangle) \text{ compatible with } T^*) \\ & = & \pi^{-1}(\pi(T)^*) & (\text{by Lemma 1557}) \\ & = & \pi^{-1}(\mu^{\mathcal{A}/\Omega^{\mathcal{A}}(T)}(\pi(T))) & (\text{by hypothesis}) \\ & = & \mu^{\mathcal{A}}(\pi^{-1}(\pi(T))) & (\text{algebra and surjectivity of } \langle I, \pi \rangle) \\ & = & \mu^{\mathcal{A}}(T). & (\mathrm{Ker}(\langle I, \pi \rangle) \text{ compatible with } T) \end{array}$$

Therefore, $\mathcal{I}$ is Leibniz definable via $\mu^\flat$. ∎

Leibniz definability ensures that the mapping sending a filter family to it Leibniz counterpart is monotone and this, in turn, implies that T^* is the largest Leibniz filter family below T.

Lemma 1703 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a Leibniz definable π-institution based on $\mathbf{F}$, with witnessing transformations $\mu^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$. For every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,*

$$T \leq T' \quad \textit{implies} \quad T^* \leq T'^*.$$

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \leq T'$. Then $T^* = \mu^{\mathcal{A}}(T) \leq \mu^{\mathcal{A}}(T') = T'^*$. ∎

Corollary 1704 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Leibniz definable* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $\mu^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* T^* *is the largest Leibniz filter family below* T*.*

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Suppose $T' \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$, such that $T' \leq T$. Then we have $T' = T'^* \leq T^*$, where the last inclusion is due to Lemma 1703. ■

Under Leibniz definability, the condition that $\Omega^{\mathcal{A}}$ be an order isomorphism from Leibniz filter families of $\mathcal{A}$ onto $\mathcal{I}^*$-congruence systems on $\mathcal{A}$, for every $\mathcal{I}$-algebraic system yields protoalgebraicity.

Proposition 1705 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Leibniz definable* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $\mu^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*. If, for every* $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$*,* $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is an order isomorphism, then* $\mathcal{I}$ *is protoalgebraic.*

Proof: Suppose the stated condition holds and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then we have

$$\begin{aligned}
\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T) &= \bigcap\{\Omega^{\mathcal{A}}(T') : T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \\
&\quad \text{(definition of } \widetilde{\Omega}^{\mathcal{I},\mathcal{A}}(T)) \\
&= \bigcap\{\Omega^{\mathcal{A}}(T'^*) : T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \\
&\quad \text{(by Proposition 1677)} \\
&= \Omega^{\mathcal{A}}(\bigcap\{T'^* : T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}) \\
&\quad \text{(by the hypothesis)} \\
&= \Omega^{\mathcal{A}}(T^*) \quad \text{(by Lemma 1703)} \\
&= \Omega^{\mathcal{A}}(T). \quad \text{(by Proposition 1677)}
\end{aligned}$$

Hence, the Leibniz and Suszko operators on every $\mathbf{F}$-algebraic system coincide, whence, by Lemma 1518, $\mathcal{I}$ is protoalgebraic. ■

We show, next, that, under Leibniz definability, the collection of Leibniz $\mathcal{I}$-filter families on every $\mathbf{F}$-algebraic system is closed under morphic images and preimages and under intersections.

Proposition 1706 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Leibniz definable* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $\mu^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*.*

(a) $\mathbb{M}(\mathsf{M}^{\mathcal{I}*}) \subseteq \mathsf{M}^{\mathcal{I}^*}$ *and* $\mathbb{M}^{-1}(\mathsf{M}^{\mathcal{I}*}) \subseteq \mathsf{M}^{\mathcal{I}^*}$*;*

(b) $\mathbb{II}(\mathsf{M}^{\mathcal{I}*}) \subseteq \mathsf{M}^{\mathcal{I}^*}$.

Proof:

(a) Let $\mathcal{A}$, $\mathcal{B}$ be $\mathbf{F}$-algebraic systems, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$ and $\langle H, \gamma \rangle : \langle \mathcal{A}, T \rangle \to \langle \mathcal{B}, T' \rangle$ a strict surjective morphism. We then have

$$\begin{array}{rll} T = T^* & \text{iff} & T = \mu^{\mathcal{A}}(T) \\ & \text{iff} & \gamma^{-1}(T') = \mu^{\mathcal{A}}(\gamma^{-1}(T')) \\ & \text{iff} & \gamma^{-1}(T') = \gamma^{-1}(\mu^{\mathcal{B}}(T')) \\ & \text{iff} & T' = \mu^{\mathcal{B}}(T') \\ & \text{iff} & T' = T'^*. \end{array}$$

Thus, $\langle \mathcal{A}, T \rangle \in \mathsf{M}^{\mathcal{I}*}$ if and only if $\langle \mathcal{B}, T' \rangle \in \mathsf{M}^{\mathcal{I}*}$.

(b) Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $\{T^i : i \in I\} \subseteq \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. Then we have

$$\begin{array}{rcl} \bigcap_{i \in I} T^i & = & \bigcap_{i \in I} (T^i)^* \\ & = & \bigcap_{i \in I} \mu^{\mathcal{A}}(T^i) \\ & = & \mu^{\mathcal{A}}(\bigcap_{i \in I} T^i) \\ & = & (\bigcap_{i \in I} T^i)^*. \end{array}$$

Therefore $\bigcap_{i \in I} T^i \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. Thus, if $\langle \mathcal{A}, T^i \rangle \in \mathsf{M}^{\mathcal{I}*}$, for all $i \in I$, then $\langle \mathcal{A}, \bigcap_{i \in I} T^i \rangle \in \mathsf{M}^{\mathcal{I}*}$.

■

Proposition 1706, in conjunction with the characterization Theorem 1787 of the $\mathcal{I}^{\mathsf{M}}$-matrix families for a class M of $\mathbf{F}$-matrix families, allow us to prove that, under Leibniz definability, $\mathcal{I}^+$-filter families and Leibniz $\mathcal{I}$-filter families on any $\mathbf{F}$-algebraic system coincide.

Theorem 1707 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Leibniz definable* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* $\mu^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,*

$$\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}).$$

Proof: We have

$$\begin{array}{rcl} \mathrm{MatFam}(\mathcal{I}^+) & = & \mathrm{MatFam}(\mathcal{I}^{\mathsf{M}^{\mathcal{I}*}}) \quad (\mathcal{I}^+ = \mathcal{I}^{\mathsf{M}^{\mathcal{I}*}}, \text{ by definition}) \\ & = & \mathbb{M}\mathbb{I}\mathbb{P}\mathbb{I}\mathbb{M}^{-1}(\mathsf{M}^{\mathcal{I}*}) \quad (\text{by Theorem 1787}) \\ & \subseteq & \mathsf{M}^{\mathcal{I}*}. \quad (\text{by Proposition 1706}) \end{array}$$

This shows that $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. But, by Proposition 1665, the reverse inclusion always holds. Therefore, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. ■

We give several conditions involving the strong version of $\mathcal{I}$ that turn out to characterize both the protoalebraicity of $\mathcal{I}$ and the protoalgebraicity of $\mathcal{I}^+$, under the proviso that $\mathcal{I}$ be Leibniz definable.

Corollary 1708 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Leibniz definable* π*-institution based on* $\mathbf{F}$*. The following conditions are equivalent:*

(i) $\mathcal{I}^+$ *is protoalgebraic;*

(ii) $\mathcal{I}$ *is protoalgebraic;*

(iii) For every $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}*}(\mathcal{A})$ *is an order isomorphism;*

(iv) For every $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I}^+)$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^+*}(\mathcal{A})$ *is an order isomorphism;*

(v) $\mathcal{I}^+$ *is weakly family algebraizable.*

Proof:

(i)⇒(ii) Suppose $\mathcal{I}^+$ is protoalgebraic. Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \leq T'$. By Lemma 1703, $T^* \leq T'^*$. Hence, by Proposition 1665 and the hypothesis, $\Omega^{\mathcal{A}}(T^*) \leq \Omega^{\mathcal{A}}(T'^*)$. By hypothesis, Proposition 1621 and Proposition 1677, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. Thus, the Leibniz operator is monotone on the $\mathcal{I}$-filter families of every $\mathbf{F}$-algebraic system and, therefore, $\mathcal{I}$ is protoalgebraic.

(ii)⇒(iii) By Proposition 1621.

(iii)⇒(iv) By Lemma 1679.

(iv)⇒(v) We have, for every $\mathbf{F}$-algebraic system $\mathcal{A}$,

$$\begin{array}{rcll} \mathrm{FiFam}^{\mathcal{I}^+*}(\mathcal{A}) & = & \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) & \text{(by Corollary 1672)} \\ & = & \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}). & \text{(by Theorem 1707)} \end{array}$$

Therefore, by hypothesis, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}^+}(\mathcal{A}) \to \mathrm{ConSys}^{\mathcal{I}^+*}(\mathcal{A})$ is an order isomorphism. By Theorem 296, $\mathcal{I}^+$ is weakly family algebraizable.

(v)⇒(i) If $\mathcal{I}^+$ is weakly family algebraizable, then it is, a fortiori, protoalgebraic. ■

Finally, we give some consequences of imposing both Leibniz definability and Leibniz truth equationality. The combination is strong enough to guarantee that Leibniz filter families and Suszko filter families coincide.

Proposition 1709 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Leibniz definable and Leibniz truth equational* π*-institution based on* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$T^* = T^{\mathcal{I},\mathsf{Su}}.$$

Proof: Let $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then

$$\begin{array}{lll} T^{\mathcal{I},\mathsf{Su}} & = & \bigcap\{T'^* : T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\} \quad \text{(by Corollary 1698)} \\ & = & T^*. \quad \text{(by Lemma 1703)} \end{array}$$

This proves the statement. ∎

Corollary 1710 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Leibniz definable and Leibniz truth equational* π*-institution based on* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,*

$$\mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A}) = \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}).$$

Proof: Let $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$. By Lemma 1583, $\mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A}) \subseteq \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$. On the other hand, if $T \in \mathrm{FiFam}^{\mathcal{I}*}(\mathcal{A})$, then, by Proposition 1709, $T = T^* = T^{\mathcal{I},\mathsf{Su}}$. Thus, $T \in \mathrm{FiFam}^{\mathcal{I},\mathsf{Su}}(\mathcal{A})$. ∎

Chapter 23

The Frege Hierarchy

23.1 The Frege Hierarchy

23.2 Self Extensionality and Implication

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ a binary natural transformation in $N^\flat$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

We say that $\to^\flat$ has the **Deduction Detachment Property in** $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi, \psi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\psi \in C_\Sigma(\Phi, \phi) \quad \text{iff} \quad \phi \to^\flat_\Sigma \psi \in C_\Sigma(\Phi).$$

$\mathcal{I}$ has the **Uniterm Deduction Detachment Property with respect to** $\to^\flat$ if $\to^\flat$ has the Deduction Detachment Property in $\mathcal{I}$. $\mathcal{I}$ has the **Uniterm Deduction Detachment Property** if it has the Uniterm Deduction Detachment Property with respect to some $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$.

If a π-institution has the Uniterm Deduction Detachment Theorem with respect to two different binary natural transformations in $N^\flat$, then the two must be interderivable in the following precise sense.

Lemma 1711 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\to^\flat, \to'^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *has the Uniterm Deduction Detachment Property with respect to both* $\to^\flat$ *and* $\to'^\flat$*, then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$C_\Sigma(\phi \to^\flat_\Sigma \psi) = C_\Sigma(\phi \to'^\flat_\Sigma \psi).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. We have $\phi \to^\flat_\Sigma \psi \in C_\Sigma(\phi \to^\flat_\Sigma \psi)$. By the Uniterm Deduction Detachment Property with respect to $\to^\flat$, we get $\psi \in C_\Sigma(\phi, \phi \to^\flat_\Sigma \psi)$. By the Uniterm Deduction Detachment Property with respect to $\to'^\flat$, we get $\phi \to'^\flat_\Sigma \psi \in C_\Sigma(\phi \to^\flat_\Sigma \psi)$. Using symmetry, we obtain that $C_\Sigma(\phi \to^\flat_\Sigma \psi) = C_\Sigma(\phi \to'^\flat_\Sigma \psi)$. ■

Thus, for self extensional π-institutions, we get immediately

Corollary 1712 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\to^\flat, \to'^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a self extensional* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *has the Uniterm Deduction Detachment Property with respect to both* $\to^\flat$ *and* $\to'^\flat$*, then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\langle \phi \to^\flat_\Sigma \psi, \phi \to'^\flat_\Sigma \psi \rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I}).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. By Lemma 1711, $\langle \phi \to^\flat_\Sigma \psi, \phi \to'^\flat_\Sigma \psi \rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I})$. But, by self extensionality, $\widetilde{\lambda}(\mathcal{I}) = \widetilde{\Omega}(\mathcal{I})$. This yields the conclusion. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a class of $\mathbf{F}$-algebraic systems. The class K is said to be **Hilbert based with respect to** $\to^\flat$ if, for all $\mathcal{A} \in \mathsf{K}$, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi, \chi \in \mathrm{SEN}(\Sigma)$,

H1. $\phi \to^{\mathcal{A}}_\Sigma \phi = \psi \to^{\mathcal{A}}_\Sigma \psi$;

H2. $(\phi \to^{\mathcal{A}}_\Sigma \phi) \to^{\mathcal{A}}_\Sigma \phi = \phi$;

H3. $\phi \to^{\mathcal{A}}_\Sigma (\psi \to^{\mathcal{A}}_\Sigma \chi) = (\phi \to^{\mathcal{A}}_\Sigma \psi) \to^{\mathcal{A}}_\Sigma (\phi \to^{\mathcal{A}}_\Sigma \chi)$;

H4. $(\phi \to^{\mathcal{A}}_\Sigma \psi) \to^{\mathcal{A}}_\Sigma ((\psi \to^{\mathcal{A}}_\Sigma \phi) \to^{\mathcal{A}}_\Sigma \psi) = (\psi \to^{\mathcal{A}}_\Sigma \phi) \to^{\mathcal{A}}_\Sigma ((\phi \to^{\mathcal{A}}_\Sigma \psi) \to^{\mathcal{A}}_\Sigma \phi)$.

These equations are commonly referred to as the **Hilbert equations**. The class K is **Hilbert based** if it is Hilbert based with respect to $\to^\flat$, for some $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$.

A class K of $\mathbf{F}$-algebraic systems is called **pointed** if there exists $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, such that, for all $\mathcal{A} \in \mathsf{K}$, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)$,

$$\top^{\mathcal{A}}_\Sigma(\vec{\phi}) = \top^{\mathcal{A}}_\Sigma(\vec{\psi}).$$

$\top^\flat$ is then called a **constant in** K and we sometimes write $\top^{\mathcal{A}}_\Sigma$ for $\top^{\mathcal{A}}_\Sigma(\vec{\phi})$, since this value is independent of the argument $\vec{\phi} \in \mathrm{SEN}(\Sigma)$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a Hilbert based class with respect to $\to^\flat$. Then, by the Hilbert equation H1, the natural transformation $\top^\flat : \mathrm{SEN}^\flat \to \mathrm{SEN}^\flat$ in $N^\flat$ defined by

$$\top^\flat := \to^\flat \circ \langle p^{1,0}, p^{1,0} \rangle$$

(in abbreviated more readable form $\top^\flat(x) := x \to^\flat x$) is a constant in K. So in this case, it makes sense to write $\top^{\mathcal{A}}_\Sigma$ for the constant defined by this natural transformation in $\mathcal{A} \in \mathsf{K}$, for $\Sigma \in |\mathbf{Sign}|$.

Moreover, for $\mathcal{A} \in \mathsf{K}$, we define the relation family $\leq^{\mathcal{A}} = \{\leq^{\mathcal{A}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ on $\mathcal{A}$ by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\phi \leq^{\mathcal{A}}_\Sigma \psi \quad \text{iff} \quad \phi \to^{\mathcal{A}}_\Sigma \psi = \top^{\mathcal{A}}_\Sigma.$$

It is not difficult to see that this is actually a partial order system on $\mathcal{A}$.

Lemma 1713 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with a binary* $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* K *a Hilbert based class with respect to* $\to^\flat$*. For all* $\mathcal{A} \in \mathsf{K}$*,* $\leq^{\mathcal{A}}$ *is a posystem on* $\mathcal{A}$*.*

Proof: We show, first, that, for all $\Sigma \in |\mathbf{Sign}|$, $\leq^{\mathcal{A}}_\Sigma$ is a partial order on $\mathrm{SEN}(\Sigma)$. Let $\phi, \psi, \chi \in \mathrm{SEN}(\Sigma)$.

- By definition $\phi \to^{\mathcal{A}}_\Sigma \phi = \top^{\mathcal{A}}_\Sigma$, whence $\phi \leq^{\mathcal{A}}_\Sigma \phi$ and $\leq^{\mathcal{A}}_\Sigma$ is reflexive;

- Suppose $\phi \le^{\mathcal{A}}_\Sigma \psi$ and $\psi \le^{\mathcal{A}}_\Sigma \phi$. Then, we get $\phi \to^{\mathcal{A}}_\Sigma \psi = \psi \to^{\mathcal{A}}_\Sigma \phi = \top^{\mathcal{A}}_\Sigma$. Thus, we get

$$\begin{array}{rcl}\phi & = & \top^{\mathcal{A}}_\Sigma \to^{\mathcal{A}}_\Sigma \phi \quad \text{(by H2)}\\ & = & \top^{\mathcal{A}}_\Sigma \to^{\mathcal{A}}_\Sigma (\top^{\mathcal{A}}_\Sigma \to^{\mathcal{A}}_\Sigma \phi) \quad \text{(by H2)}\\ & = & \top^{\mathcal{A}}_\Sigma \to^{\mathcal{A}}_\Sigma (\top^{\mathcal{A}}_\Sigma \to^{\mathcal{A}}_\Sigma \psi) \quad \text{(by H4)}\\ & = & \top^{\mathcal{A}}_\Sigma \to^{\mathcal{A}}_\Sigma \psi \quad \text{(by H2)}\\ & = & \psi. \quad \text{(by H2)}\end{array}$$

  Hence, $\le^{\mathcal{A}}_\Sigma$ is antisymmetric;

- Suppose $\phi \le^{\mathcal{A}}_\Sigma \psi$ and $\psi \le^{\mathcal{A}}_\Sigma \chi$. Then $\phi \to^{\mathcal{A}}_\Sigma \psi = \psi \to^{\mathcal{A}}_\Sigma \chi = \top^{\mathcal{A}}_\Sigma$. Thus, we get

$$\begin{array}{rcl}\phi \to^{\mathcal{A}}_\Sigma \chi & = & \top^{\mathcal{A}}_\Sigma \to^{\mathcal{A}}_\Sigma (\phi \to^{\mathcal{A}}_\Sigma \chi) \quad \text{(by H2)}\\ & = & (\phi \to^{\mathcal{A}}_\Sigma \psi) \to^{\mathcal{A}}_\Sigma (\phi \to^{\mathcal{A}}_\Sigma \chi) \quad \text{(hypothesis)}\\ & = & \phi \to^{\mathcal{A}}_\Sigma (\psi \to^{\mathcal{A}}_\Sigma \chi) \quad \text{(by H3)}\\ & = & \phi \to^{\mathcal{A}}_\Sigma \top^{\mathcal{A}}_\Sigma \quad \text{(hypothesis)}\\ & = & \phi \to^{\mathcal{A}}_\Sigma (\phi \to^{\mathcal{A}}_\Sigma \phi) \quad \text{(definition)}\\ & = & (\phi \to^{\mathcal{A}}_\Sigma \phi) \to^{\mathcal{A}}_\Sigma (\phi \to^{\mathcal{A}}_\Sigma \phi) \quad \text{(by H3)}\\ & = & \top^{\mathcal{A}}_\Sigma. \quad \text{(definition)}\end{array}$$

  So $\le^{\mathcal{A}}_\Sigma$ is also transitive.

Thus, $\le^{\mathcal{A}}$ is a partial order family on $\mathcal{A}$. We show that, in addition, it is a system, i.e., it is invariant under signature morphisms. To this end, suppose $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\phi \le^{\mathcal{A}}_\Sigma \psi$. Then $\phi \to^{\mathcal{A}}_\Sigma \psi = \top^{\mathcal{A}}_\Sigma$. Hence, $\mathrm{SEN}(f)(\phi \to^{\mathcal{A}}_\Sigma \psi) = \mathrm{SEN}(f)(\top^{\mathcal{A}}_\Sigma)$. This gives

$$\mathrm{SEN}(f)(\phi) \to^{\mathcal{A}}_{\Sigma'} \mathrm{SEN}(f)(\psi) = \top^{\mathcal{A}}_{\Sigma'}.$$

We conclude that $\mathrm{SEN}(f)(\phi) \le^{\mathcal{A}}_{\Sigma'} \mathrm{SEN}(f)(\psi)$. Therefore, $\le^{\mathcal{A}}$ is indeed a posystem on $\mathcal{A}$. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, K a Hilbert based class with respect to $\to^\flat$, $\mathcal{A} \in \mathsf{K}$ and $T \in \mathrm{SenFam}(\mathcal{A})$. We say that T is an **$\to^\flat$-implicative filter family of $\mathcal{A}$** if

- $\top^{\mathcal{A}}_\Sigma \in T_\Sigma$, for all $\Sigma \in |\mathbf{Sign}|$;
- $\phi \to^{\mathcal{A}}_\Sigma \psi \in T_\Sigma$ and $\phi \in T_\Sigma$ imply $\psi \in T_\Sigma$, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$.

We write $\mathrm{FiFam}^{\to}(\mathcal{A})$ for the collection of all $\to^\flat$-implicative filter families on $\mathcal{A}$.

Next, we show that in any $\mathbf{F}$-algebraic system $\mathcal{A}$ in a Hilbert based class K, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi_0, \dots, \phi_{n-1}, \phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{l}\phi_0 \to^{\mathcal{A}}_\Sigma (\phi_1 \to^{\mathcal{A}}_\Sigma \cdots \to^{\mathcal{A}}_\Sigma (\phi_{n-1} \to^{\mathcal{A}}_\Sigma \phi)\cdots) = \top^{\mathcal{A}}_\Sigma\\ \qquad \text{iff} \quad \phi_{\pi(0)} \to^{\mathcal{A}}_\Sigma (\phi_{\pi(1)} \to^{\mathcal{A}}_\Sigma \cdots \to^{\mathcal{A}}_\Sigma (\phi_{\pi(n-1)} \to^{\mathcal{A}}_\Sigma \phi)\cdots) = \top^{\mathcal{A}}_\Sigma,\end{array}$$

where π is any permutation of $\{0, \dots, n-1\}$.

Lemma 1714 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with a binary* $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* K *a Hilbert based class of* $\mathbf{F}$*-algebraic systems with respect to* $\to^\flat$*. For all* $\mathcal{A} \in \mathsf{K}$*, all* $\Sigma \in |\mathbf{Sign}|$*, all* $\phi_0, \phi_1, \ldots, \phi_{n-1}, \phi \in \mathrm{SEN}(\Sigma)$ *and every permutation* π *of* $\{0, 1, \ldots, n-1\}$*,*

$$\begin{aligned}&\phi_0 \to^{\mathcal{A}}_\Sigma (\phi_1 \to^{\mathcal{A}}_\Sigma \cdots \to^{\mathcal{A}}_\Sigma (\phi_{n-1} \to^{\mathcal{A}}_\Sigma \phi)\cdots) = \top^{\mathcal{A}}_\Sigma \\ &\quad \textit{iff} \quad \phi_{\pi(0)} \to^{\mathcal{A}}_\Sigma (\phi_{\pi(1)} \to^{\mathcal{A}}_\Sigma \cdots \to^{\mathcal{A}}_\Sigma (\phi_{\pi(n-1)} \to^{\mathcal{A}}_\Sigma \phi)\cdots) = \top^{\mathcal{A}}_\Sigma.\end{aligned}$$

Proof:

■

Lemma 1714 allows us to write

$$\vec{\vec{\Phi}} \to^{\mathcal{A}}_\Sigma \phi = \top^{\mathcal{A}}_\Sigma$$

for $\phi_0 \to^{\mathcal{A}}_\Sigma (\phi_1 \to^{\mathcal{A}}_\Sigma \cdots \to^{\mathcal{A}}_\Sigma (\phi_{n-1} \to^{\mathcal{A}}_\Sigma \phi)\cdots) = \top^{\mathcal{A}}_\Sigma$, where $\Phi = \{\phi_0, \ldots, \phi_{n-1}\}$, when appropriate, since the equation does not depend on the order in which the elements of Φ are arranged in the implication expression. Moreover, for convenience, if $\Phi = \varnothing$, we take

$$\vec{\vec{\Phi}} \to^{\mathcal{A}}_\Sigma \phi := \phi.$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, K a Hilbert based class with respect to $\to^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a *finitary* π-institution based on $\mathbf{F}$. $\mathcal{I}$ is called **Hilbert based with respect to** K **and** $\to^\flat$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in C_\Sigma(\Phi) \quad \text{iff} \quad \text{for all } \mathcal{A} \in \mathsf{K},\ \alpha_\Sigma(\vec{\vec{\Phi}} \to^\flat_\Sigma \phi) = \top^{\mathcal{A}}_{F(\Sigma)}.$$

We say that $\mathcal{I}$ is **Hilbert based** if there exists $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$ and a Hilbert based class K of $\mathbf{F}$-algebraic systems with respect to $\to^\flat$, such that $\mathcal{I}$ is Hilbert based with respect to K and $\to^\flat$.

Corollary 1715 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with a binary* $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*,* K *a Hilbert based class with respect to* $\to^\flat$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Hilbert based* π*-institution with respect to* K *and* $\to^\flat$*. For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$C_\Sigma(\phi) = C_\Sigma(\psi) \quad \textit{iff} \quad \langle \phi, \psi \rangle \in \mathrm{Ker}_\Sigma(\mathsf{K}).$$

Proof: Suppose $\mathcal{I}$ is Hilbert based with respect to K and $\to^\flat$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $C_\Sigma(\phi) = C_\Sigma(\psi)$ if and only if, by definition, for all $\mathcal{A} \in \mathsf{K}$, $\alpha_\Sigma(\phi \to^\flat_\Sigma \psi) = \alpha_\Sigma(\psi \to^\flat_\Sigma \phi) = \top^{\mathcal{A}}_{F(\Sigma)}$, if and only if, by Lemma 1713, for all $\mathcal{A} \in \mathsf{K}$, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$, if and only if, $\langle \phi, \psi \rangle \in \mathrm{Ker}_\Sigma(\mathsf{K})$. ■

It is not difficult to see that if a π-institution is Hilbert based with respect to a Hilbert based class K, then it is also Hilbert based with respect to the semantic variety generated by K.

Lemma 1716 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with a binary* $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*,* K *a Hilbert based class with respect to* $\to^\flat$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Hilbert based* π*-institution with respect to* K *and* $\to^\flat$*. Then* $\mathcal{I}$ *is also Hilbert based with respect to* $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$ *and* $\to^\flat$*.*

Proof: Assume that $\mathcal{I}$ is Hilbert based with respect K and $\to^\flat$. First, note that, since, for all $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$, $\mathrm{Ker}(\mathsf{K}) \le \mathrm{Ker}(\mathcal{A})$, all $\mathbf{F}$-algebraic systems in $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$ satisfy the Hilbert equations and, hence, $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$ is a Hilbert based class with respect to $\to^\flat$.

Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$.

Suppose $\phi \in C_\Sigma(\Phi)$ and let $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. By hypothesis $\langle \vec{\vec{\Phi}} \to^\flat_\Sigma \phi, \top^\flat_\Sigma \rangle \in \mathrm{Ker}_\Sigma(\mathsf{K})$. Since $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$, $\mathrm{Ker}(\mathsf{K}) \le \mathrm{Ker}(\mathcal{A})$. Therefore, $\langle \vec{\vec{\Phi}} \to^\flat_\Sigma \phi, \top^\flat_\Sigma \rangle \in \mathrm{Ker}_\Sigma(\mathcal{A})$. This shows that $\alpha_\Sigma(\vec{\vec{\Phi}} \to^\flat_\Sigma \phi) = \top^{\mathcal{A}}_{F(\Sigma)}$. Conversely, if, for all $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$, $\alpha_\Sigma(\vec{\vec{\Phi}} \to^\flat_\Sigma \phi) = \top^{\mathcal{A}}_{F(\Sigma)}$, then this holds, a fortiori, for all $\mathcal{A} \in \mathsf{K}$ and, hence, by the hypothesis $\phi \in C_\Sigma(\Phi)$.

Thus, $\mathcal{I}$ is Hilbert based both with respect to $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$ and $\to^\flat$. ∎

Corollary 1717 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with a binary* $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*,* K *a Hilbert based class with respect to* $\to^\flat$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Hilbert based* π*-institution with respect to* K *and* $\to^\flat$*. For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$C_\Sigma(\phi) = C_\Sigma(\psi) \quad \textit{iff} \quad \langle \phi, \psi \rangle \in \mathrm{Ker}_\Sigma(\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})).$$

Proof: By Corollary 1715 and Lemma 1716. ∎

We can also show that, if K and K' are two Hilbert based classes of $\mathbf{F}$-algebraic systems with respect to binary transformations $\to^\flat$ and $\to'^\flat$ in $N^\flat$, respectively, and a π-institution $\mathcal{I}$ happens to be Hilbert based with respect to both K and $\to^\flat$ and K' and $\to'^\flat$, then, the two classes K and K' generate the same semantic variety of $\mathbf{F}$-algebraic systems.

Proposition 1718 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\to^\flat, \to'^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*,* K *a Hilbert class with respect to* $\to^\flat$ *and* K' *a Hilbert class with respect to* $\to'^\flat$*. If* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *is a* π*-institution that is Hilbert based with respect to* K *and* $\to^\flat$ *and Hilbert based with respect to* K' *and* $\to'^\flat$*, then* $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathbb{V}^{\mathsf{Sem}}(\mathsf{K}')$*.*

Proof: We show that $\mathsf{K}' \subseteq \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. Then the conclusion will follow by symmetry. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in \mathrm{Ker}_\Sigma(\mathsf{K})$, and $\mathcal{A}' \in \mathsf{K}'$. By hypothesis, for all $\mathcal{A}$, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. Hence, for all $\mathcal{A} \in \mathsf{K}$, $\alpha_\Sigma(\phi \to^\flat_\Sigma \psi) = \alpha_\Sigma(\psi \to^\flat_\Sigma \phi) = \top^{\mathcal{A}}_{F(\Sigma)}$. Thus, since $\mathcal{I}$ is Hilbert based with respect to K and $\to^\flat$, we get $C_\Sigma(\phi) = C_\Sigma(\psi)$. But, by hypothesis, $\mathcal{I}$ is also Hilbert based with respect to K' and $\to'^\flat$, whence $\alpha'_\Sigma(\phi \to'^\flat_\Sigma \psi) = \alpha'_\Sigma(\psi \to'^\flat_\Sigma \phi) = \top^{\mathcal{A}'}_{F'(\Sigma)}$. Hence, by Lemma 1713, $\alpha'_\Sigma(\phi) = \alpha'_\Sigma(\psi)$ or, equivalently, $\langle\phi,\psi\rangle \in \mathrm{Ker}_\Sigma(\mathcal{A}')$. This shows that $\mathcal{A}' \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. Thus, $\mathsf{K}' \subseteq \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. ■

We conclude that, if $\mathcal{I} = \langle\mathbf{F}, C\rangle$ is a Hilbert based π-institution, there is a unique semantic variety of $\mathbf{F}$-algebraic systems, with respect to which it is Hilbert based. We denote this semantic variety by $\mathbb{V}^{\mathsf{Sem}}(\mathcal{I})$ and call it the **semantic variety of** $\mathcal{I}$.

A key result is that every Hilbert based π-institution is self extensional and has the Deduction Detachment Property. We also show that the semantic variety of $\mathcal{I}$ coincides with the class $\mathsf{K}^{\mathcal{I}}$, the semantic variety of $\mathcal{I}$.

Proposition 1719 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$, K *a Hilbert based class with respect to* $\to^\flat$, *and* $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *a Hilbert based* π*-institution with respect to* K *and* $\to^\flat$.

(a) $\mathcal{I}$ *is self extensional;*

(b) $\mathcal{I}$ *has the Deduction Detachment Property with respect to* $\to^\flat$*;*

(c) $\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}) = \mathsf{K}^{\mathcal{I}}$*; Thus,* $\mathcal{I}$ *is Hilbert based with respect to* $\mathsf{K}^{\mathcal{I}}$ *and* $\to^\flat$.

Proof:

(a) We must show that $\widetilde{\Lambda}(\mathcal{I}) = \widetilde{\Omega}(\mathcal{I})$. Since $\widetilde{\Omega}(\mathcal{I})$ is the largest congruence system on $\mathbf{F}$ that is included in $\widetilde{\lambda}(\mathcal{I})$, it suffices to show that $\widetilde{\lambda}(\mathcal{I})$ is a congruence system. To this end, let $\sigma^\flat$ be a natural transformation in $N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle\phi_i,\psi_i\rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I})$, for all $i < k$. Hence, by definition, $C_\Sigma(\phi_i) = C_\Sigma(\psi_i)$, for all $i \in I$. Since $\mathcal{I}$ is Hilbert based with respect to K and $\to^\flat$, we get, by Corollary 1715, $\langle\phi_i,\psi_i\rangle \in \mathrm{Ker}_\Sigma(\mathsf{K})$, for all $i < k$. But $\mathrm{Ker}(\mathsf{K})$ is a congruence system on $\mathbf{F}$, whence $\langle\sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi})\rangle \in \mathrm{Ker}_\Sigma(\mathsf{K})$. Again, by Corollary 1715, $C_\Sigma(\sigma^\flat_\Sigma(\vec{\phi})) = C_\Sigma(\sigma^\flat_\Sigma(\vec{\psi}))$ and, therefore, $\langle\sigma^\flat_\Sigma(\vec{\phi}), \sigma^\flat_\Sigma(\vec{\psi})\rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I})$. We conclude that $\widetilde{\lambda}(\mathcal{I}) = \widetilde{\Omega}(\mathcal{I})$ and, hence, $\mathcal{I}$ is self extensional.

(b) Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi,\psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$.

- Suppose that $\psi \in C_\Sigma(\Phi,\phi)$. Since $\mathcal{I}$ is finitary, there exists $\Phi' \subseteq_f \Phi$, such that $\psi \in C_\Sigma(\Phi',\phi)$. Since $\mathcal{I}$ is Hilbert based with respect to K and $\to^\flat$, we get $\langle\vec{\Phi}' \to^\flat_\Sigma (\phi \to^\flat_\Sigma \psi), \top^\flat_\Sigma\rangle \in \mathrm{Ker}_\Sigma(\mathsf{K})$. Thus, again, since $\mathcal{I}$ is Hilbert based with respect to K and $\to^\flat$, $\phi \to^\flat_\Sigma \psi \in C_\Sigma(\Phi') \subseteq C_\Sigma(\Phi)$.
- Suppose, conversely, that $\phi \to^\flat_\Sigma \psi \in C_\Sigma(\Phi)$. Again, by finitarity, there exists $\Phi' \subseteq_f \Phi$, such that $\phi \to^\flat_\Sigma \psi \in C_\Sigma(\Phi')$. Hence, since $\mathcal{I}$ is Hilbert based with respect to K and $\to^\flat$, $\langle\vec{\Phi}' \to^\flat_\Sigma (\phi \to^\flat_\Sigma \psi), \top^\flat_\Sigma\rangle \in$

$\mathrm{Ker}_\Sigma(\mathsf{K})$. But, again by the fact that $\mathcal{I}$ is Hilbert based with respect to K and $\to^\flat$, we get that $\psi \in C_\Sigma(\Phi', \phi) \subseteq C_\Sigma(\Phi, \phi)$.

Hence $\mathcal{I}$ has the Deduction Detachment Property with respect to $\to^\flat$.

(c) Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. We have

$$\begin{array}{llll} \langle \phi, \psi \rangle \in \mathrm{Ker}_\Sigma(\mathbb{V}^{\mathsf{Sem}}(\mathcal{I})) & \text{iff} & C_\Sigma(\phi) = C_\Sigma(\psi) & \text{(by Corollary 1717)} \\ & \text{iff} & \langle \phi, \psi \rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}) & \text{(by definition)} \\ & \text{iff} & \langle \phi, \psi \rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I}) & \text{(by Part (a))} \\ & \text{iff} & \langle \phi, \psi \rangle \in \mathrm{Ker}_\Sigma(\mathsf{K}^{\mathcal{I}}). & \text{(by definition)} \end{array}$$

Therefore, since $\mathsf{K}^{\mathcal{I}}$ is a semantic variety by definition, we get that $\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}) = \mathsf{K}^{\mathcal{I}}$. The last statement follows now by Lemma 1716. ■

If $\mathcal{I}$ is Hilbert based, not only is the semantic variety with respect to which it is Hilbert based unique, but, in addition, any two binary natural transformations that serve as the Hilbert implications are in a sense interderivable and, hence, indistinguishable modeulo the Tarski congruence system of $\mathcal{I}$.

Corollary 1720 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\to^\flat$, $\to'^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$, *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Hilbert based* π*-institution with respect to* $\to^\flat$ *and with respect to* $\to'^\flat$. *Then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

(a) $C_\Sigma(\phi \to^\flat_\Sigma \psi) = C_\Sigma(\phi \to'^\flat_\Sigma \psi)$;

(b) $\langle \phi \to^\flat_\Sigma \psi, \phi \to'^\flat_\Sigma \psi \rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I})$.

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{llll} \phi \to^\flat_\Sigma \psi \in C_\Sigma(\phi \to^\flat_\Sigma \psi) & \text{iff} & \psi \in C_\Sigma(\phi, \phi \to^\flat_\Sigma \psi) & \text{(Proposition 1719)} \\ & \text{iff} & \phi \to'^\flat_\Sigma \psi \in C_\Sigma(\phi \to^\flat_\Sigma \psi). & \text{(Proposition 1719)} \end{array}$$

Therefore, $\phi \to'^\flat_\Sigma \psi \in C_\Sigma(\phi \to^\flat_\Sigma \psi)$. By symmetry, we get the conclusion of Part (a). Part (b) follows by Proposition 1719, which asserts that $\mathcal{I}$ is self extensional. ■

If $\mathcal{I}$ is self extensional and has the Deduction Detachment Property with respect to $\to^\flat$, it turns out that the singleton class $\mathsf{K} = \{\mathcal{F}/\widetilde{\Omega}(\mathcal{I})\}$, consisting of the Lindenbaum-Tarski $\mathbf{F}$-algebraic system of $\mathcal{I}$, is Hilbert based with respect to $\to^\flat$.

Lemma 1721 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with a binary* $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$, *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a finitary self extensional* π*-institution, having the Deduction Detachment Property with respect to* $\to^\flat$. *The class* $\mathsf{K} = \{\mathcal{F}/\widetilde{\Omega}(\mathcal{I})\}$ *is Hilbert based with respect to* $\to^\flat$.

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$.

(H1) By the Deduction Detachment Property with respect to $\to^\flat$, we get $C_\Sigma(\phi \to^\flat_\Sigma \phi) = C_\Sigma(\psi \to^\flat_\Sigma \psi) = C_\Sigma(\varnothing)$. Therefore, by self extensionality, $\langle \phi \to^\flat_\Sigma \phi, \psi \to^\flat_\Sigma \psi\rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I})$;

(H2) We have $\phi \to^\flat_\Sigma \phi \in C_\Sigma(\phi \to^\flat_\Sigma \phi)$, whence $\phi \in C_\Sigma(\phi, \phi \to^\flat_\Sigma \phi)$ and, hence, $(\phi \to^\flat_\Sigma \phi) \to^\flat_\Sigma \phi \in C_\Sigma(\phi)$.

On the other hand, $(\phi \to^\flat_\Sigma \phi) \to^\flat_\Sigma \phi \in C_\Sigma((\phi \to^\flat_\Sigma \phi) \to^\flat_\Sigma \phi)$, whence $\phi \in C_\Sigma((\phi \to^\flat_\Sigma \phi) \to^\flat_\Sigma \phi, \phi \to^\flat_\Sigma \phi)$ and, since $\phi \to^\flat_\Sigma \phi \in C_\Sigma(\varnothing)$, $\phi \in C_\Sigma((\phi \to^\flat_\Sigma \phi) \to^\flat_\Sigma \phi)$.

This shows that $C_\Sigma((\phi \to^\flat_\Sigma \phi) \to^\flat_\Sigma \phi) = C_\Sigma(\phi)$ and, hence, by self extensionality, $\langle (\phi \to^\flat_\Sigma \phi) \to^\flat_\Sigma \phi, \phi\rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I})$.

(H3) By the Deduction Detachment Property with respect to $\to^\flat$, we get $\chi \in C_\Sigma(\phi, \phi \to^\flat_\Sigma \psi, (\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma (\phi \to^\flat_\Sigma \chi))$. Thus, since $\phi \to^\flat_\Sigma \psi \in C_\Sigma(\psi)$, we get $\chi \in C_\Sigma(\phi, \psi, (\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma (\phi \to^\flat_\Sigma \chi))$. This gives $\psi \to^\flat_\Sigma \chi \in C_\Sigma(\phi, (\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma (\phi \to^\flat_\Sigma \chi))$ and, hence, $\phi \to^\flat_\Sigma (\psi \to^\flat_\Sigma \chi) \in C_\Sigma((\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma (\phi \to^\flat_\Sigma \chi))$.

On the other hand, $\chi \in C_\Sigma(\phi, \phi \to^\flat_\Sigma \psi, \phi \to^\flat_\Sigma (\psi \to^\flat_\Sigma \chi))$. Therefore, $\phi \to^\flat_\Sigma \chi \in C_\Sigma(\phi \to^\flat_\Sigma \psi, \phi \to^\flat_\Sigma (\psi \to^\flat_\Sigma \chi))$, whence $(\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma (\phi \to^\flat_\Sigma \chi) \in C_\Sigma(\phi \to^\flat_\Sigma (\psi \to^\flat_\Sigma \chi))$.

We conclude that $C_\Sigma(\phi \to^\flat_\Sigma (\psi \to^\flat_\Sigma \chi)) = C_\Sigma((\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma (\phi \to^\flat_\Sigma \chi))$. Thus, by self extensionality, $\langle \phi \to^\flat_\Sigma (\psi \to^\flat_\Sigma \chi), (\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma (\phi \to^\flat_\Sigma \chi)\rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I})$.

(H4) By the Deduction Detachment Property with respect to $\to^\flat$,

$$\psi \in C_\Sigma(\psi \to^\flat_\Sigma \phi, \phi \to^\flat_\Sigma \psi, (\psi \to^\flat_\Sigma \phi) \to^\flat_\Sigma ((\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma \phi)).$$

Therefore, $(\psi \to^\flat_\Sigma \phi) \to^\flat_\Sigma \psi \in C_\Sigma(\phi \to^\flat_\Sigma \psi, (\psi \to^\flat_\Sigma \phi) \to^\flat_\Sigma ((\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma \phi))$, whence $(\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma ((\psi \to^\flat_\Sigma \phi) \to^\flat_\Sigma \psi) \in C_\Sigma((\psi \to^\flat_\Sigma \phi) \to^\flat_\Sigma ((\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma \phi))$. The other inclusion follows similarly and, then, we get the conclusion by self extensionality.

Therefore, $\{\mathcal{F}/\widetilde{\Omega}(\mathcal{I})\}$ is Hilbert based with respect to $\to^\flat$. ■

Now we can fully characterize those finitary π-institutions which are Hilbert based.

Theorem 1722 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat$ be an algebraic system and $\mathcal{I}$ a finitary π-institution based on $\mathbf{F}$. $\mathcal{I}$ is Hilbert based if and only if it is self extensional and has the Uniterm Deduction Detachment Property.*

Proof: The left-to-right implication is given by Proposition 1719. Assume, conversely, that $\mathcal{I}$ is self extensional and has the Uniterm Deduction Detachment Property with respect to some $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$. Consider $\mathsf{K} = \{\mathcal{F}/\widetilde{\Omega}(\mathcal{I})\}$. By Lemma 1721, K is Hilbert based with respect to $\to^\flat$. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{llr}\phi \in C_\Sigma(\Phi) & \text{iff} \;\; \vec{\vec{\Phi}} \to^\flat_\Sigma \phi \in C_\Sigma(\varnothing) & \text{(Deduction Detachment)}\\ & \text{iff} \;\; \langle \vec{\vec{\Phi}} \to^\flat_\Sigma \phi, \top^\flat_\Sigma\rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}) & \text{(definition of } \widetilde{\lambda}(\mathcal{I}))\\ & \text{iff} \;\; \langle \vec{\vec{\Phi}} \to^\flat_\Sigma \phi, \top^\flat_\Sigma\rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I}). & \text{(self extensionality)}\end{array}$$

Therefore, $\mathcal{I}$ is Hilbert based with respect to K and, hence, by Lemma 1716, with respect to $\mathsf{K}^{\mathcal{I}} = \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$, and $\to^\flat$. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat$ be an algebraic system, with $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$ and K a Hilbert based semantic variety with respect to $\to^\flat$. We define the *finitary* π-**institution**

$$\mathcal{I}^{\mathsf{K},\to} = \langle \mathbf{F}, C^{\mathsf{K},\to}\rangle,$$

associated with K and $\to^\flat$, by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in C^{\mathsf{K},\to}_\Sigma(\Phi) \quad \text{iff} \quad \langle \vec{\vec{\Phi}} \to^\flat_\Sigma \phi, \top^\flat_\Sigma\rangle \in \mathrm{Ker}_\Sigma(\mathsf{K}).$$

We can see easily from the definition that $\mathcal{I}^{\mathsf{K},\to}$ is Hilbert based with respect to K and $\to^\flat$.

Corollary 1723 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a Hilbert based semantic variety with respect to $\to^\flat$. Then $\mathcal{I}^{\mathsf{K},\to}$ is Hilbert based with respect to K and $\to^\flat$ and, moreover, $\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}^{\mathsf{K},\to}) = \mathsf{K}$.*

Proof: This follows directly from the definition of $\mathcal{I}^{\mathsf{K},\to}$ and by taking into account Lemma 1716 and the definition of $\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}^{\mathsf{K},\to})$. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat$ be an algebraic system, with $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$. Our next goal is to establish that the two mappings

$$\begin{array}{ccc}\mathcal{I} & \longmapsto & \mathbb{V}^{\mathsf{Sem}}(\mathcal{I})\\ \mathcal{I}^{\mathsf{K},\to} & \longleftarrow\!\!\!\dashv & \mathsf{K}\end{array}$$

form a dual order isomorphism from the collection of Hilbert based π-institutions with respect to $\to^\flat$, under $\leq$, and Hilbert based semantic varieties with respect to $\to^\flat$, under $\subseteq$.

We first show that the Frege operator is both monotone and order reflecting on Hilbert based π-institutions with respect to $\to^\flat$.

Proposition 1724 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$*,* $\mathcal{I}' = \langle \mathbf{F}, C' \rangle$ *be Hilbert based* π*-institutions with respect to* $\to^\flat$*. Then*

$$\mathcal{I} \le \mathcal{I}' \quad \textit{iff} \quad \widetilde{\lambda}(\mathcal{I}) \le \widetilde{\lambda}(\mathcal{I}').$$

Proof: The left-to-right implication (monotonicity) is given by Lemma 1416. For the right-to-left implication, suppose $\widetilde{\lambda}(\mathcal{I}) \le \widetilde{\lambda}(\mathcal{I}')$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{lll} \phi \in C_\Sigma(\Phi) & \text{iff} & C_\Sigma(\overset{\Rightarrow}{\Phi} \to^\flat_\Sigma \phi) = C_\Sigma(\varnothing) \quad \text{(by Theorem 1722)} \\ & \text{iff} & \langle \overset{\Rightarrow}{\Phi} \to^\flat_\Sigma \phi, \top^\flat_\Sigma \rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}) \quad \text{(definition)} \\ & \text{implies} & \langle \overset{\Rightarrow}{\Phi} \to^\flat_\Sigma \phi, \top^\flat_\Sigma \rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}') \quad \text{(hypothesis)} \\ & \text{iff} & C'_\Sigma(\overset{\Rightarrow}{\Phi} \to^\flat_\Sigma \phi) = C'_\Sigma(\varnothing) \quad \text{(definition)} \\ & \text{iff} & \phi \in C'_\Sigma(\Phi). \quad \text{(by Theorem 1722)} \end{array}$$

We conclude that $\mathcal{I} \le \mathcal{I}'$ and, hence $\widetilde{\lambda}$ is also order reflecting. ∎

Now we present the preannounced order isomorphism theorem. For an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$, with $\to^\flat$ a binary natural transformation in $N^\flat$, we let

$$\mathsf{K}^{\mathbf{F},\to}$$

be the semantic variety consisting of all $\mathbf{F}$-algebraic systems satisfying the Hilbert equations with respect to $\to^\flat$.

Theorem 1725 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, There exists a dual order isomorphism between the collection of all Hilbert based* π*-institutions with respect to* $\to^\flat$*, ordered under* $\le$*, and the collection of all semantic subvarieties of the semantic variety* $\mathsf{K}^{\mathbf{F},\to}$*, ordered under* $\subseteq$*, given by* $\mathcal{I} \mapsto \mathsf{K}^{\mathcal{I}}$*.*

Proof: The given mapping is onto, since, by Corollary 1723, for $\mathsf{K} \subseteq \mathsf{K}^{\mathbf{F},\to}$ a semantic subvariety of $\mathsf{K}^{\mathbf{F},\to}$, $\mathsf{K} = \mathsf{K}^{\mathcal{I}^{\mathsf{K},\to}}$. Moreover, it is 1-1, since $\mathsf{K}^{\mathcal{I}} = \mathsf{K}^{\mathcal{I}'}$ implies that $\widetilde{\lambda}(\mathcal{I}) = \widetilde{\Omega}(\mathcal{I}) = \widetilde{\Omega}(\mathcal{I}') = \widetilde{\lambda}(\mathcal{I}')$ and, hence, by Proposition 1724, $\mathcal{I} = \mathcal{I}'$. Finally, monotonicity and order reflectivity are both given by

$$\begin{array}{lll} \mathcal{I} \le \mathcal{I}' & \text{iff} & \widetilde{\lambda}(\mathcal{I}) \le \widetilde{\lambda}(\mathcal{I}') \quad \text{(Proposition 1724)} \\ & \text{iff} & \widetilde{\Omega}(\mathcal{I}) \le \widetilde{\Omega}(\mathcal{I}') \quad \text{(Theorem 1722)} \\ & \text{iff} & \mathcal{F}/\widetilde{\Omega}(\mathcal{I}') \in \mathbb{V}^{\mathsf{Sem}}(\mathcal{F}/\widetilde{\Omega}(\mathcal{I})) \\ & \text{iff} & \mathsf{K}^{\mathcal{I}'} \subseteq \mathsf{K}^{\mathcal{I}}. \end{array}$$

This establishes the order isomorphism. ∎

Our next goal is to show that Hilbert based π-institutions, i.e., finitary self extensional π-institutions that have the Uniterm Deduction Detachment Property (by Theorem 1722) are fully self extensional.

We start by proving that on every $\mathbf{F}$-algebraic system in the Hilbert class $\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}) = \mathsf{K}^{\mathcal{I}}$, $\mathcal{I}$-filter families and implicative filter families coincide.

Lemma 1726 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with a binary* $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Hilbert based* π*-institution with respect to* $\to^\flat$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$*,*

$$\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \mathrm{FiFam}^{\to}(\mathcal{A}).$$

Proof: Let $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$.

Suppose that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$.

- We have $\phi \in C_\Sigma(\phi)$, whence, by Proposition 1719, $\phi \to^\flat_\Sigma \phi \in C_\Sigma(\varnothing)$. Hence, $\top^{\mathcal{A}}_{F(\Sigma)} \in T_{F(\Sigma)}$;
- Assume $\alpha_\Sigma(\phi \to^\flat_\Sigma \psi) \in T_{F(\Sigma)}$ and $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$. Since, again by Proposition 1719, $\psi \in C_\Sigma(\phi, \phi \to^\flat_\Sigma \psi)$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get $\alpha_\Sigma(\psi) \in T_{F(\Sigma)}$.

Therefore, taking into account the surjectivity of $\langle F, \alpha \rangle$, $T \in \mathrm{FiFam}^{\to}(\mathcal{A})$.

Suppose, conversely, that $T \in \mathrm{FiFam}^{\to}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$ and $\alpha_\Sigma(\Phi) \subseteq T_{F(\Sigma)}$. By Proposition 1719, $\vec{\vec{\Phi}} \to^\flat_\Sigma \phi \in C_\Sigma(\varnothing)$. Therefore, $C_\Sigma(\vec{\vec{\Phi}} \to^\flat_\Sigma \phi) = C_\Sigma(\top^\flat_\Sigma)$. Thus, by Proposition 1719 and the fact that $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$, we get that $\alpha_\Sigma(\vec{\vec{\Phi}} \to^\flat_\Sigma \phi) = \top^{\mathcal{A}}_{F(\Sigma)}$. Since, by hypothesis, $T \in \mathrm{FiFam}^{\to}(\mathcal{A})$, $\alpha_\Sigma(\vec{\vec{\Phi}} \to^\flat_\Sigma \phi) \in T_{F(\Sigma)}$. Thus, by the fact that $\alpha_\Sigma(\Phi) \subseteq T_{F(\Sigma)}$ and $T \in \mathrm{FiFam}^{\to}(\mathcal{A})$, we get that $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$. We conclude that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$, with $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, be an algebraic system, K be a Hilbert based class of $\mathbf{F}$-algebraic systems with respect to $\to^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a Hilbert based π-institution with respect to K and $\to^\flat$. For all $\mathcal{A} \in \mathsf{K}$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, define $T^{\langle \Sigma, \phi \rangle} = \{T^{\langle \Sigma, \phi \rangle}_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}|}$ by setting, for all $\Sigma' \in |\mathbf{Sign}|$,

$$T^{\langle \Sigma, \phi \rangle}_\Sigma = \{\chi \in \mathrm{SEN}(\Sigma) : \phi \to^{\mathcal{A}}_\Sigma \chi = \top^{\mathcal{A}}_\Sigma\}$$

and

$$T^{\langle \Sigma, \phi \rangle}_{\Sigma'} = \{\top^{\mathcal{A}}_{\Sigma'}\}, \quad \text{for all } \Sigma' \neq \Sigma.$$

It is not difficult to see that $T^{\langle \Sigma, \phi \rangle} \in \mathrm{FiFam}^{\to}(\mathcal{A})$.

Lemma 1727 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$*, with* $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, be an algebraic system,* K *be a Hilbert based class of* $\mathbf{F}$*-algebraic systems with respect to* $\to^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a Hilbert based* π*-institution with respect to* K *and* $\to^\flat$*. For all* $\mathcal{A} \in \mathsf{K}$*, all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi \in \mathrm{SEN}(\Sigma)$*,* $T^{\langle \Sigma, \phi \rangle} \in \mathrm{FiFam}^{\to}(\mathcal{A})$*.*

Proof: Consider, first, $\Sigma' \neq \Sigma$. By definition, $\top^{\mathcal{A}}_{\Sigma'} \in T^{\langle\Sigma,\phi}_{\Sigma'}$. Moreover, if $\top^{\mathcal{A}}_{\Sigma'} \to^{\mathcal{A}}_{\Sigma'} \in T^{\langle\Sigma,\phi\rangle}_{\Sigma'}$, then, by H2, $\phi \in T^{\langle\Sigma,\phi\rangle}_{\Sigma'}$.

Consider, next, $\Sigma' = \Sigma$. Note that we have

$$\begin{array}{rcl}\phi \to^{\mathcal{A}}_{\Sigma} \top^{\mathcal{A}}_{\Sigma} & = & \phi \to^{\mathcal{A}}_{\Sigma} (\phi \to^{\mathcal{A}}_{\Sigma} \phi) \quad \text{(definition)}\\ & = & (\phi \to^{\mathcal{A}}_{\Sigma} \phi) \to^{\mathcal{A}}_{\Sigma} (\phi \to^{\mathcal{A}}_{\Sigma} \phi) \quad \text{(by H3)}\\ & = & \top^{\mathcal{A}}_{\Sigma} \to^{\mathcal{A}}_{\Sigma} \top^{\mathcal{A}}_{\Sigma} \quad \text{(definition)}\\ & = & \top^{\mathcal{A}}_{\Sigma}. \quad \text{(by H2)}\end{array}$$

Hence, $\top^{\mathcal{A}}_{\Sigma} \in T^{\langle\Sigma,\phi\rangle}_{\Sigma}$. Moreover, if $\psi, \psi \to^{\mathcal{A}}_{\Sigma} \chi \in T^{\langle\Sigma,\phi\rangle}_{\Sigma}$, then, we get, by definition, $\phi \to^{\mathcal{A}}_{\Sigma} \psi = \top^{\mathcal{A}}_{\Sigma}$ and $\phi \to^{\mathcal{A}}_{\Sigma} (\psi \to^{\mathcal{A}}_{\Sigma} \chi) \in \top^{\mathcal{A}}_{\Sigma}$. Therefore, we get

$$\begin{array}{rcl}\phi \to^{\mathcal{A}}_{\Sigma} \chi & = & \top^{\mathcal{A}}_{\Sigma} \to^{\mathcal{A}}_{\Sigma} (\phi \to^{\mathcal{A}}_{\Sigma} \chi) \quad \text{(by H2)}\\ & = & (\phi \to^{\mathcal{A}}_{\Sigma} \psi) \to^{\mathcal{A}}_{\Sigma} (\phi \to^{\mathcal{A}}_{\Sigma} \chi) \quad \text{(hypothesis)}\\ & = & \phi \to^{\mathcal{A}}_{\Sigma} (\psi \to^{\mathcal{A}}_{\Sigma} \chi) \quad \text{(by H3)}\\ & = & \top^{\mathcal{A}}_{\Sigma}, \quad \text{(hypothesis)}\end{array}$$

We conclude that $\chi \in T^{\langle\Sigma,\phi\rangle}_{\Sigma}$. Thus, $T^{\langle\Sigma,\phi\rangle} \in \mathrm{FiFam}^{\to}(\mathcal{A})$. ∎

We can now prove that the Frege equivalence system of every full $\mathcal{I}$-structure of the form $\langle\mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$, with $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$, is the identity congruence system. This shows, in particular that every **F**-algebraic system in $\mathsf{K}^{\mathcal{I}}$ is an $\mathcal{I}$-algebraic system and, moreover, satisfies the Congruence Property.

Lemma 1728 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *a Hilbert based* π*-institution based on* $\mathbf{F}$*. For every* $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$*,* $\widetilde{\lambda}^{\mathcal{I}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}$*. Thus,* $\langle\mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ *is reduced and satisfies the Congruence Property.*

Proof: Let $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$, $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle \notin \Delta^{\mathcal{A}}_{\Sigma}$. Thus, $\phi \neq \psi$. Taking into account Lemma 1727, consider $T^{\langle\Sigma,\phi\rangle}, T^{\langle\Sigma,\psi\rangle} \in \mathrm{FiFam}^{\to}(\mathcal{A})$. By Lemma 1726, $T^{\langle\Sigma,\phi\rangle}, T^{\langle\Sigma,\psi\rangle} \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Moreover, by definition, $\phi \in T^{\langle\Sigma,\phi\rangle}_{\Sigma}$ and $\psi \in T^{\langle\Sigma,\psi\rangle}_{\Sigma}$. On the other hand, if it was the case that $\psi \in T^{\langle\Sigma,\phi\rangle}_{\Sigma}$ and $\phi \in T^{\langle\Sigma,\psi\rangle}_{\Sigma}$, then, by definition, $\phi \to^{\mathcal{A}}_{\Sigma} \psi = \psi \to^{\mathcal{A}}_{\Sigma} \phi = \top^{\mathcal{A}}_{\Sigma}$, whence, by Lemma 1713, $\phi = \psi$, contrary to hypothesis. Thus, it must be the case that $\psi \notin T^{\langle\Sigma,\phi\rangle}_{\Sigma}$ or $\phi \notin T^{\langle\Sigma,\psi\rangle}_{\Sigma}$. We can now conclude that $\langle\phi,\psi\rangle \notin \widetilde{\lambda}^{\mathcal{A}}_{\Sigma}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$. This shows that $\widetilde{\lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}$. Since $\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \leq \widetilde{\lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$, we get that $\langle\mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ is a reduced $\mathcal{I}$-structure and that it satisfies the Congruence Property. ∎

Lemma 1728 allows us to conclude that, for Hilbert based π-institutions $\mathcal{I}$, the semantic variety of $\mathcal{I}$ coincides with the class of all $\mathcal{I}$-algebraic systems.

Theorem 1729 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *a Hilbert based* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathrm{AlgSys}(\mathcal{I}) = \mathsf{K}^{\mathcal{I}} = \mathbb{V}^{\mathsf{Sem}}(\mathcal{I})$*;*

(b) AlgSys($\mathcal{I}$) *is a semantic variety;*

(c) $\mathcal{I}$ *is Hilbert based with respect to* AlgSys($\mathcal{I}$).

Proof: By Proposition 65, we have $\mathrm{AlgSys}(\mathcal{I}) \subseteq \mathsf{K}^{\mathcal{I}}$. On the other hand, Lemma 1728 gives $\mathsf{K}^{\mathcal{I}} \subseteq \mathrm{AlgSys}(\mathcal{I})$. Therefore, $\mathrm{AlgSys}(\mathcal{I}) = \mathsf{K}^{\mathcal{I}}$. Since $\mathsf{K}^{\mathcal{I}}$ is a semantic variety, we conclude that AlgSys($\mathcal{I}$) is also a semantic variety. Finally, since, by Proposition 1719, $\mathcal{I}$ is Hilbert based with respect to $\mathsf{K}^{\mathcal{I}}$, we conclude that $\mathcal{I}$ is Hilbert based with respect to AlgSys($\mathcal{I}$). ∎

In one of the main theorems of the section, we show that a finitary self extensional π-institution with the Uniterm Deduction Detachment Property is necessarily fully self extensional.

Theorem 1730 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a finitary self extensional π-institution with the Uniterm Deduction Detachment Property. Then $\mathcal{I}$ is fully self extensional.*

Proof: Suppose that $\mathcal{I}$ is finitary self extensional and that it has the Uniterm Deduction Detachment Property with respect to $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$. By Theorem 1722 and Proposition 1719, $\mathcal{I}$ is Hilbert based with respect to $\mathsf{K}^{\mathcal{I}}$ and $\to^\flat$. By Theorem 1729, $\mathsf{K}^{\mathcal{I}} = \mathrm{AlgSys}(\mathcal{I})$, whence, by Lemma 1728, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$,

$$\widetilde{\lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}.$$

Now to prove full self extensionality, we use Proposition 1428. To this end, assume $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ is a full $\mathcal{I}$-structure. Then $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \in \mathrm{AlgSys}(\mathcal{I})$ and, by definition,

$$\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})).$$

Thus, by what was shown above,

$$\widetilde{\lambda}^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))) = \Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))}.$$

By Proposition 1426, we infer that $\langle \mathcal{A}.\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ also has the Congruence Property. We now conclude, by Proposition 1428, that $\mathcal{I}$ is fully self extensional. ∎

We finish the section by looking at some connections with the theory of Gentzen π-institutions, that is presented in another chapter.

Recall that, given an algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ and a finitary π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ based on $\mathbf{F}$, a finitary Gentzen π-institution $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ is said to be **adequate for** $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in C_\Sigma(\Phi) \quad \text{iff} \quad \Phi \vdash_\Sigma \phi \in G_\Sigma(\varnothing).$$

We say that the Gentzen π-institution $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ has the **Congruence Property** if, for all $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\sigma^\flat_\Sigma(\vec{\phi}) \vdash \sigma^\flat_\Sigma(\vec{\psi}) \in G_\Sigma(\{\phi_i \vdash_\Sigma \psi_i,\ \psi_i \vdash_\Sigma \phi_i : i \in I\}).$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a finitary Gentzen π-institution based on $\mathbf{F}$. We say that:

- $\mathfrak{G}$ has the **Deduction Rule with respect to** $\to^\flat$, if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi, \psi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$,

$$\Phi \vdash_\Sigma \phi \to^\flat_\Sigma \psi \in G_\Sigma(\Phi, \phi \vdash_\Sigma \psi);$$

- $\mathfrak{G}$ has the **Detachment Rule with respect to** $\to^\flat$, if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi, \psi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$,

$$\Phi, \phi \vdash_\Sigma \psi \in G_\Sigma(\Phi \vdash_\Sigma \phi \to^\flat_\Sigma \psi);$$

- $\mathfrak{G}$ has the **Deduction Detachment Rule with respect to** $\to^\flat$, if it has both the Deduction and the Detachment Property with respect to $\to^\flat$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a finitary self extensional π-institution, having the Deduction Detachment Property with respect to $\to^\flat$.

- Define $\mathrm{Ax}^{\mathcal{I}} = \{\mathrm{Ax}^{\mathcal{I}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{Ax}^{\mathcal{I}}_\Sigma = \{\Phi \vdash_\Sigma \phi : \phi \in C_\Sigma(\Phi)\};$$

- Define $\mathrm{Ir}^{\mathcal{I}} = \{\mathrm{Ir}^{\mathcal{I}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\begin{aligned}\mathrm{Ir}^{\mathcal{I}}_\Sigma \;=\; & \{\langle \{\phi_i \vdash_\Sigma \psi_i, \psi_i \vdash_\Sigma \phi_i : i \in I\}, \sigma^\flat_\Sigma(\vec{\phi}) \vdash_\Sigma \sigma^\flat_\Sigma(\vec{\psi}) \rangle : \\ & \qquad\qquad \sigma^\flat \in N^\flat, \vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)\} \\ & \cup \{\langle \{\Phi, \phi \vdash_\Sigma \psi\}, \Phi \vdash_\Sigma \phi \to^\flat_\Sigma \psi \rangle : \Phi \cup \{\phi, \psi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)\} \\ & \cup \{\langle \{\Phi \vdash_\Sigma \phi \to^\flat_\Sigma \psi\}, \Phi, \phi \vdash_\Sigma \psi \rangle : \Phi \cup \{\phi, \psi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)\}.\end{aligned}$$

- $R^{\mathcal{I}} := \mathrm{Ax}^{\mathcal{I}} \cup \mathrm{Ir}^{\mathcal{I}}$.

Finally, define $\mathfrak{G}^{\mathcal{I}} = \langle \mathbf{F}, C^{\mathcal{I}} \rangle := \mathfrak{G}^{R^{\mathcal{I}}}$ be the Gentzen π-institution generated by the system $R^{\mathcal{I}}$ of Gentzen rules. Recall, by Proposition 1482, that $G^{\mathcal{I}} = \Xi^{R^{\mathcal{I}}}$.

We are almost ready to establish the existence of a fully adequate Gentzen π-institution for any given Hilbert based π-institution. Recall, again, from work in a different chapter, that a Gentzen π-institution $\mathfrak{G}$ is **fully adequate**

for a π-institution $\mathcal{I}$ (with theorems) if, for every $\mathbf{F}$-structure $\mathbb{L} = \langle \mathcal{A}, D \rangle$, $\mathbb{L}$ is a full $\mathcal{I}$-structure if and only if it is a $\mathfrak{G}$-structure.

For Hilbert based π-institutions, it turns out that any $\mathcal{I}$-structure whose Frege equivalence system is the identity is of the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$, for some $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$.

Lemma 1731 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a Hilbert based π-institution based on $\mathbf{F}$. If $\langle \mathcal{A}, D \rangle \in \mathrm{Str}(\mathcal{I})$, such that $\widetilde{\lambda}^{\mathcal{A}}(D) = \Delta^{\mathcal{A}}$, then $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$.*

Proof: Let $\langle \mathcal{A}, D \rangle \in \mathrm{Str}(\mathcal{I})$, such that $\widetilde{\lambda}^{\mathcal{A}}(D) = \Delta^{\mathcal{A}}$. Then, we have

$$\widetilde{\Omega}^{\mathcal{A}}(D) \leq \widetilde{\lambda}^{\mathcal{A}}(D) = \Delta^{\mathcal{A}},$$

i.e., $\widetilde{\Omega}^{\mathcal{A}}(D) = \Delta^{\mathcal{A}}$ and, therefore, $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I}) \subseteq \mathsf{K}^{\mathcal{I}}$.

Suppose, next, that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. By Lemma 1726, $T \in \mathrm{FiFam}^{\rightarrow}(\mathcal{A})$, where $\rightarrow^\flat$ in $N^\flat$ is the binary transformation with respect to which $\mathcal{I}$ is Hilbert based. Let $\Sigma \in |\mathbf{Sign}|$, $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in D_\Sigma(T_\Sigma)$. By finitarity and Proposition 114, there exists $\Phi \subseteq_f T_\Sigma$, such that $\phi \in D_\Sigma(\Phi)$. Thus, by the hypothesis, Proposition 1719 and Corollary 1440, $\vec{\vec{\Phi}} \rightarrow^{\mathcal{A}}_\Sigma \phi \in D_\Sigma(\top^{\mathcal{A}}_\Sigma)$, i.e., $D_\Sigma(\vec{\vec{\Phi}} \rightarrow^{\mathcal{A}}_\Sigma \phi) = D_\Sigma(\top^{\mathcal{A}}_\Sigma)$. By hypothesis, $\vec{\vec{\Phi}} \rightarrow^{\mathcal{A}}_\Sigma \phi = \top^{\mathcal{A}}_\Sigma$. Since $T \in \mathrm{FiFam}^{\rightarrow}(\mathcal{A})$, we have $\vec{\vec{\Phi}} \rightarrow^{\mathcal{A}}_\Sigma \phi \in T_\Sigma$ and, since, also, $\Phi \subseteq T_\Sigma$, we infer that $\phi \in T_\Sigma$. Therefore, $T = D(T)$, showing that $T \in \mathcal{D}$ and, hence, $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. ■

We finally show that any Hilbert based π-institution $\mathcal{I}$ has a fully adequate Gentzen π-institution, namely, the π-institution $\mathfrak{G}^{\mathcal{I}}$, generated by the Gentzen rules $R^{\mathcal{I}}$, which encode the rules of $\mathcal{I}$, the Congruence Property and the Deduction Detachment Property.

Theorem 1732 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a finitary π-institution based on $\mathbf{F}$, having the Uniterm Deduction Detachment Property. $\mathcal{I}$ is self extensional if and only if the Gentzen π-institution $\mathfrak{G} = \langle \mathbf{F}, G^{\mathcal{I}} \rangle$ if fully adequate for $\mathcal{I}$.*

Proof: Suppose that $\mathfrak{G}^{\mathcal{I}}$ is fully adequate for $\mathcal{I}$. We know that $\langle \mathcal{F}, C \rangle$ is a full $\mathcal{I}$-structure. Thus, by full adequacy, $\langle \mathcal{F}, C \rangle \in \mathrm{Str}(\mathfrak{G}^{\mathcal{I}})$. Therefore, $\langle \mathcal{F}, C \rangle$ satisfies all the Gentzen rules that hold in $\mathfrak{G}^{\mathcal{I}}$. In particular, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \subseteq \mathrm{SEN}^\flat(\Sigma)$, $C_\Sigma(\phi_i) = C_\Sigma(\psi_i)$, for all $i \in I$, imply $C_\Sigma(\sigma^\flat_\Sigma(\vec{\phi})) = C_\Sigma(\sigma^\flat_\Sigma(\vec{\psi}))$. Thus, $\widetilde{\lambda}(\mathcal{I})$ is a congruence system and, hence $\mathcal{I}$ is self extensional.

Suppose, conversely, that $\mathcal{I}$ is self extensional. By Theorem 1730, it is fully self extensional. Let $\langle \mathcal{A}, D \rangle \in \mathrm{FStr}(\mathcal{I})$. Then, by Theorem 1444 and the definition of $\mathfrak{G}^{\mathcal{I}}$, we get that $\langle \mathcal{A}, D \rangle \in \mathrm{Str}(\mathfrak{G}^{\mathcal{I}})$. Assume, conversely, that $\langle \mathcal{A}, D \rangle \in \mathrm{Str}(\mathfrak{G}^{\mathcal{I}})$. By considering, if necessary, $\langle \mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(D), \mathcal{D}/\widetilde{\Omega}^{\mathcal{A}}(D) \rangle$, we may assume that $\widetilde{\Omega}^{\mathcal{A}}(D) = \Delta^{\mathcal{A}}$ and must show that $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. By the definition of $\mathfrak{G}^{\mathcal{I}}$, we get $\langle\mathcal{A}, D\rangle \in \mathrm{Str}(\mathcal{I})$. For the same reason, $\langle\mathcal{A}, D\rangle$ satisfies the Congruence Property. Therefore, $\widetilde{\lambda}^{\mathcal{A}}(D) = \widetilde{\Omega}^{\mathcal{A}}(D) = \Delta^{\mathcal{A}}$. Since, by hypothesis, it has the Deduction Detachment Property, we get, by Theorem 1722, that it is Hilbert based. Now, applying Lemma 1731, we conclude that $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Therefore, $\langle\mathcal{A}, \mathcal{D}\rangle \in \mathrm{FStr}(\mathcal{I})$.

This shows that $\mathrm{FStr}(\mathcal{I}) = \mathrm{Str}(\mathfrak{G}^{\mathcal{I}})$ and, hence, $\mathfrak{G}^{\mathcal{I}}$ is, indeed, fully adequate for $\mathcal{I}$. ∎

23.3 Self Extensionality and Conjunction

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle\mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. We say that $\mathcal{I}$ has the **Conjunction Property with respect to** $\wedge^\flat$ and that $\wedge^\flat$ is a **conjunction for** $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

- $\phi \wedge^\flat_\Sigma \psi \in C_\Sigma(\phi, \psi)$;
- $\phi, \psi \in C_\Sigma(\phi \wedge^\flat_\Sigma \psi)$.

Equivalently, $\wedge^\flat$ is a conjunction for $\mathcal{I}$ if and only if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C_\Sigma(\phi \wedge^\flat_\Sigma \psi) = C_\Sigma(\phi, \psi).$$

We say $\mathcal{I}$ is **conjunctive** if it has the Conjunction Property with respect to some $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$.

If a π-institution has the Conjunction Property with respect to two different binary natural transformations in $N^\flat$, then the two concjunctions must be interderivable in an obvious sense.

Lemma 1733 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\wedge^\flat, \wedge'^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle\mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ has the Conjunction Property with respect to both $\wedge^\flat$ and $\wedge'^\flat$, then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,*

$$C_\Sigma(\phi \wedge^\flat_\Sigma \psi) = C_\Sigma(\phi \wedge'^\flat_\Sigma \psi).$$

Proof: Suppose that $\wedge^\flat$ and $\wedge'^\flat$ are both conjunctions for $\mathcal{I}$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\psi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{rcll} C_\Sigma(\phi \wedge^\flat_\Sigma \psi) & = & C_\Sigma(\phi, \psi) & (\wedge^\flat \text{ a conjunction}) \\ & = & C_\Sigma(\phi \wedge'^\flat_\Sigma \psi). & (\wedge'^\flat \text{ a conjunction}) \end{array}$$

This proves the statement. ∎

Corollary 1734 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\wedge^\flat, \wedge'^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is self extensional and has the Conjunction Property with respect to both* $\wedge^\flat$ *and* $\wedge'^\flat$*, then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\langle \phi \wedge^\flat_\Sigma \psi, \phi \wedge'^\flat_\Sigma \psi \rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I}).$$

Proof: Suppose that $\mathcal{I}$ is self extensional and $\wedge^\flat$ and $\wedge'^\flat$ are both conjunctions for $\mathcal{I}$. Then, if $\Sigma \in |\mathbf{Sign}^\flat|$, $\psi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, by Lemma 1733, $\langle \phi \wedge^\flat_\Sigma \psi, \phi \wedge'^\flat_\Sigma \psi \rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I})$, whence, by self extensionality, $\langle \phi \wedge^\flat_\Sigma \psi, \phi \wedge'^\flat_\Sigma \psi \rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I})$. ∎

We also know, by Proposition 1434 that the Conjunction Property transfers from a π-institution $\mathcal{I}$ to all $\mathcal{I}$-structures.

Corollary 1735 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, having the Conjunction Property with respect to* $\wedge^\flat$*. For every* $\mathcal{I}$*-structure* $\langle \mathcal{A}, D \rangle$*, all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,*

$$D_\Sigma(\phi \wedge^{\mathcal{A}}_\Sigma \psi) = D_\Sigma(\phi, \psi).$$

Proof: This is simply a restatement of Proposition 1434. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a class of $\mathbf{F}$-algebraic systems. K is **semilattice based with respect to** $\wedge^\flat$ if, for all $\mathcal{A} \in \mathsf{K}$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi, \chi \in \mathrm{SEN}(\Sigma)$,

- L1. $\phi \wedge^{\mathcal{A}}_\Sigma \phi = \phi$;
- L2. $\phi \wedge^{\mathcal{A}}_\Sigma \psi = \psi \wedge^{\mathcal{A}}_\Sigma \phi$;
- L3. $(\phi \wedge^{\mathcal{A}}_\Sigma \psi) \wedge^{\mathcal{A}}_\Sigma \chi = \phi \wedge^{\mathcal{A}}_\Sigma (\psi \wedge^{\mathcal{A}}_\Sigma \chi)$.

L1-L3 are referred to as the **semilattice equations**. We say that K is **semilattice based** if it is semilattice based with respect to some $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$.

If a class of $\mathbf{F}$-algebraic systems is semilattices based, then the semantic variety generated by the class is also semilattice based with respect to the same binary natural transformation.

Lemma 1736 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* K *a class of* $\mathbf{F}$*-algebraic systems. If* K *is semilattice based with respect to* $\wedge^\flat$*, then* $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$ *is also semilattice based with respect to* $\wedge^\flat$*.*

Proof: Let $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. We show $\mathcal{A}$ satisfies L2. The work for L1 and L3 follows along the same lines. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Since K is semilattice based with respect to $\wedge^\flat$, $\langle \phi \wedge^\flat_\Sigma \psi, \psi \wedge^\flat_\Sigma \phi \rangle \in \mathrm{Ker}_\Sigma(\mathsf{K})$. Since, by hypothesis, $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$, we get $\mathrm{Ker}(\mathsf{K}) \leq \mathrm{Ker}(\mathcal{A})$, whence $\langle \phi \wedge^\flat_\Sigma \psi, \psi \wedge^\flat_\Sigma \phi \rangle \in \mathrm{Ker}_\Sigma(\mathcal{A})$. This shows that $\alpha_\Sigma(\phi \wedge^\flat_\Sigma \psi) = \alpha_\Sigma(\psi \wedge^\flat_\Sigma \phi)$, i.e., $\alpha_\Sigma(\phi) \wedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\psi) = \alpha_\Sigma(\psi) \wedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi)$. Thus, by the surjectivity of $\langle F, \alpha \rangle$, we conclude that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $\phi \wedge^{\mathcal{A}}_\Sigma \psi = \psi \wedge^{\mathcal{A}}_\Sigma \phi$. Therefore, $\mathcal{A}$ satisfies L2. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a semilattice based class of $\mathbf{F}$-algebraic systems with respect to $\wedge^\flat$. For every $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$, define the relation family $\leq^{\mathcal{A}} = \{\leq^{\mathcal{A}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ on $\mathcal{A}$ by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\phi \leq^{\mathcal{A}}_\Sigma \psi \quad \text{iff} \quad \phi \wedge^{\mathcal{A}}_\Sigma \psi = \phi.$$

It is easily shown that $\leq^{\mathcal{A}}$ is a partial order system on $\mathcal{A}$.

Lemma 1737 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a semilattice based class with respect to $\wedge^\flat$. For all $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$, $\leq^{\mathcal{A}}$ is a posystem on $\mathcal{A}$.*

Proof: First, fix $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$, $\Sigma \in |\mathbf{Sign}|$. We show that $\leq^{\mathcal{A}}_\Sigma$ is a partial order on $\mathrm{SEN}(\Sigma)$. To this end, let $\phi, \psi, \chi \in \mathrm{SEN}(\Sigma)$.

- By L1, $\phi = \phi \wedge^{\mathcal{A}}_\Sigma \phi$, whence, by definition, $\phi \leq^{\mathcal{A}}_\Sigma \phi$ and $\leq^{\mathcal{A}}_\Sigma$ is reflexive;
- If $\phi \leq^{\mathcal{A}}_\Sigma \psi$ and $\psi \leq^{\mathcal{A}}_\Sigma \phi$, then, we get

$$\begin{array}{lll} \phi & = & \phi \wedge^{\mathcal{A}}_\Sigma \psi \quad (\phi \leq^{\mathcal{A}}_\Sigma \psi) \\ & = & \psi \wedge^{\mathcal{A}}_\Sigma \phi \quad (\text{by L2}) \\ & = & \psi. \quad (\psi \leq^{\mathcal{A}}_\Sigma \phi) \end{array}$$

  Thus, $\leq^{\mathcal{A}}_\Sigma$ is antisymmetric.
- If $\phi \leq^{\mathcal{A}}_\Sigma \psi$ and $\psi \leq^{\mathcal{A}}_\Sigma \chi$, then

$$\begin{array}{lll} \phi & = & \phi \wedge^{\mathcal{A}}_\Sigma \psi \quad (\phi \leq^{\mathcal{A}}_\Sigma \psi) \\ & = & \phi \wedge^{\mathcal{A}}_\Sigma (\psi \wedge^{\mathcal{A}}_\Sigma \chi) \quad (\psi \leq^{\mathcal{A}}_\Sigma \chi) \\ & = & (\phi \wedge^{\mathcal{A}}_\Sigma \psi) \wedge^{\mathcal{A}}_\Sigma \chi \quad (\text{by L3}) \\ & = & \phi \wedge^{\mathcal{A}}_\Sigma \chi. \quad (\phi \leq^{\mathcal{A}}_\Sigma \psi) \end{array}$$

  Hence $\phi \leq^{\mathcal{A}}_\Sigma \chi$ and $\leq^{\mathcal{A}}_\Sigma$ is also transitive.

Thus, $\leq^{\mathcal{A}}$ is a partial order on $\mathrm{SEN}(\Sigma)$. Suppose, now, that $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\phi \leq^{\mathcal{A}}_\Sigma \psi$. Then, by definition, $\phi = \phi \wedge^{\mathcal{A}}_\Sigma \psi$. Thus, $\mathrm{SEN}^\flat(f)(\phi) = \mathrm{SEN}^\flat(f)(\phi \wedge^{\mathcal{A}}_\Sigma \psi) = \mathrm{SEN}^\flat(f)(\phi) \wedge^{\mathcal{A}}_{\Sigma'} \mathrm{SEN}^\flat(f)(\psi)$.

This shows that $\mathrm{SEN}^\flat(f)(\phi) \leq^{\mathcal{A}}_{\Sigma'} \mathrm{SEN}^\flat(f)(\psi)$. Thus, $\leq^{\mathcal{A}}$ is a partial order system on $\mathcal{A}$. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, K a semilattice based class of $\mathbf{F}$-algebraic systems with respect to $\wedge^\flat$, $\mathcal{A} \in \mathsf{K}$ and $T \in \mathrm{SenFam}(\mathcal{A})$. We say that T is a **semilattice filter family of** $\mathcal{A}$ if the following conditions hold, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$:

- $T_\Sigma \neq \varnothing$;
- $\phi, \psi \in T_\Sigma$ implies $\phi \wedge^{\mathcal{A}}_\Sigma \psi \in T_\Sigma$;
- $\phi \in T_\Sigma$ and $\phi \leq^{\mathcal{A}}_\Sigma \psi$ imply $\psi \in T_\Sigma$.

We denote by $\mathrm{FiFam}^\wedge(\mathcal{A})$ the collection of all semilattice filter families on $\mathcal{A}$. Moreover, we write $\mathrm{FiFam}^{\wedge,\varnothing}(\mathcal{A})$ for the same collection augmented by those sentence families resulting from semilattice filter families after one or more components are replaced by the empty set.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, K a semilattice based class of $\mathbf{F}$-algebraic systems with respect to $\wedge^\flat$, $\mathcal{A} \in \mathsf{K}$, $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. Define

$$T^{\langle \Sigma, \phi \rangle} = \{T^{\langle \Sigma, \phi \rangle}_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}|}$$

by setting,

- $T^{\langle \Sigma, \phi \rangle}_\Sigma = \{\chi \in \mathrm{SEN}(\Sigma) : \phi \leq^{\mathcal{A}}_\Sigma \chi\}$;
- $T^{\langle \Sigma, \phi \rangle}_{\Sigma'} = \begin{cases} \{\chi \in \mathrm{SEN}(\Sigma) : 1_\Sigma \leq^{\mathcal{A}}_\Sigma \chi\}, & \text{if } 1_\Sigma \text{ is a maximum in } \leq^{\mathcal{A}}_\Sigma \\ \varnothing, & \text{if } \leq^{\mathcal{A}}_\Sigma \text{ has no maximum} \end{cases}$, for all $\Sigma \neq \Sigma' \in |\mathbf{Sign}|$,

We show that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $T^{\langle \Sigma, \phi \rangle} \in \mathrm{FiFam}^\wedge(\mathcal{A})$ or $T^{\langle \Sigma, \phi \rangle} \in \mathrm{FiFam}^{\wedge,\varnothing}(\mathcal{A})$.

Lemma 1738 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, K a semilattice based class of $\mathbf{F}$-algebraic systems with respect to $\wedge^\flat$, $\mathcal{A} \in \mathsf{K}$, $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. Then, $T^{\langle \Sigma, \phi \rangle} \in \mathrm{FiFam}^\wedge(\mathcal{A})$ or $T^{\langle \Sigma, \phi \rangle} \in \mathrm{FiFam}^{\wedge,\varnothing}(\mathcal{A})$.*

Proof: It suffices to show that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, the collection $\{\chi \in \mathrm{SEN}(\Sigma) : \phi \leq^{\mathcal{A}}_\Sigma \chi\}$ is an upset under $\leq^{\mathcal{A}}_\Sigma$, closed under $\wedge^{\mathcal{A}}_\Sigma$. Let $\psi, \chi \in \mathrm{SEN}(\Sigma)$.

- If $\psi, \chi \in T^{\langle \Sigma, \phi \rangle}_\Sigma$, then, by definition, $\phi \leq^{\mathcal{A}}_\Sigma \psi$ and $\phi \leq^{\mathcal{A}}_\Sigma \chi$. Thus, we get

$$\begin{array}{rcll} \phi & = & \phi \wedge^{\mathcal{A}}_\Sigma \chi & (\phi \leq^{\mathcal{A}}_\Sigma \chi) \\ & = & (\phi \wedge^{\mathcal{A}}_\Sigma \psi) \wedge^{\mathcal{A}}_\Sigma \chi & (\phi \leq^{\mathcal{A}}_\Sigma \psi) \\ & = & \phi \wedge^{\mathcal{A}}_\Sigma (\psi \wedge^{\mathcal{A}}_\Sigma \chi). & (\text{by L3}) \end{array}$$

Therefore, $\phi \leq^{\mathcal{A}}_\Sigma \psi \wedge^{\mathcal{A}}_\Sigma \chi$ and, hence $\psi \wedge^{\mathcal{A}}_\Sigma \chi \in T^{\langle \Sigma, \phi \rangle}_\Sigma$.

- If $\psi \in T_\Sigma^{\langle\Sigma,\phi\rangle}$ and $\psi \leq_\Sigma^{\mathcal{A}} \chi$, then $\phi \leq_\Sigma^{\mathcal{A}} \psi$ and $\psi \leq_\Sigma^{\mathcal{A}} \chi$, whence, by Lemma 1737, $\phi \leq_\Sigma^{\mathcal{A}} \chi$, i.e., $\chi \in T_\Sigma^{\langle\Sigma,\phi\rangle}$.

Thus, $T^{\langle\Sigma,\phi\rangle} \in \mathrm{FiFam}^{\wedge}(\mathcal{A})$ or $T^{\langle\Sigma,\phi\rangle} \in \mathrm{FiFam}^{\wedge,\varnothing}(\mathcal{A})$. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, K a semilattice based class of $\mathbf{F}$-algebraic systems with respect to $\wedge^\flat$ and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a *finitary* π-institution based on $\mathbf{F}$. We say $\mathcal{I}$ is **semilattice based with respect to** K and $\wedge^\flat$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$, with $\Phi \neq \varnothing$,

$$\phi \in C_\Sigma(\Phi) \quad \text{iff} \quad \text{for all } \mathcal{A} \in \mathsf{K},\ \textstyle\bigwedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\Phi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi).$$

We say that $\mathcal{I}$ is **semilattice based** if it is semilattice based with respect to K and $\wedge^\flat$, for some semilattice based class K with respect to some $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$.

We get immediately from the definition that interderivability in $\mathcal{I}$ is reflected into equality in all algebraic systems in the defining class K.

Lemma 1739 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, K a semilattice based class of $\mathbf{F}$-algebraic systems with respect to $\wedge^\flat$ and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a semilattice based π-institution with respect to K and $\wedge^\flat$. For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,*

$$C_\Sigma(\phi) = C_\Sigma(\psi) \quad \textit{iff} \quad \textit{for all } \mathcal{A} \in \mathsf{K},\ \alpha_\Sigma(\phi) = \alpha_\Sigma(\psi).$$

Proof: By the definition, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $C_\Sigma(\phi) = C_\Sigma(\psi)$ iff, for all $\mathcal{A} \in \mathsf{K}$, $\alpha_\Sigma(\phi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\psi)$ and $\alpha_\Sigma(\psi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi)$ iff, by Lemma 1737, for all $\mathcal{A} \in \mathsf{K}$, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. ∎

Moreover, in case $\mathcal{I}$ is semilattice based with respect to a class K, then it is also semilattice based with respect to the semantic variety generated by K, with respect to the same binary transformation.

Lemma 1740 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, K a semilattice based class of $\mathbf{F}$-algebraic systems with respect to $\wedge^\flat$ and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a semilattice based π-institution with respect to K and $\wedge^\flat$. Then $\mathcal{I}$ is semilattice based with respect to $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$ and $\wedge^\flat$.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$ and $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. Since $\mathcal{I}$ is semilattice based with respect to K and $\wedge^\flat$, $\langle \bigwedge^\flat_\Sigma \Phi \wedge^\flat_\Sigma \phi, \bigwedge^\flat_\Sigma \Phi\rangle \in \mathrm{Ker}_\Sigma(\mathsf{K})$. Since $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$, we get $\mathrm{Ker}(\mathsf{K}) \leq \mathrm{Ker}(\mathcal{A})$. This gives $\bigwedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\Phi) \wedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi) = \bigwedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\Phi)$, and, therefore, $\bigwedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\Phi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi)$.

Conversely, if, for all $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$, $\bigvee^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\Phi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi)$, then, a fortiori, for all $\mathcal{A} \in \mathsf{K}$, we have $\bigvee^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\Phi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi)$. Therefore, by hypothesis, $\phi \in C_\Sigma(\Phi)$. We conclude that $\mathcal{I}$ is semilattice based with respect to $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$ and $\wedge^\flat$. ■

We can now show that, if a π-institution is semilattice based with respect to two different classes of semilattice based $\mathbf{F}$-algebraic systems, then, they both have to generate the same semantic variety.

Lemma 1741 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\wedge^\flat, \wedge'^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*,* K *a semilattice based class with respect to* $\wedge^\flat$ *and* K' *a semilattice based class with respect to* $\wedge'^\flat$*. If* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *is a semilattice based* π*-institution both with respect to* K *and* $\wedge^\flat$ *and with respect to* K' *and* $\wedge'^\flat$*, then* $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathbb{V}^{\mathsf{Sem}}(\mathsf{K}')$*.*

Proof: Suppose $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is a semilattice based π-institution both with respect to K and $\wedge^\flat$ and with respect to K' and $\wedge'^\flat$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in \mathrm{Ker}_\Sigma(\mathsf{K})$ and $\mathcal{A} \in \mathsf{K}'$. Then, since $\mathcal{I}$ is semilattice based with respect to K, by Lemma 1739, $C_\Sigma(\phi) = C_\Sigma(\psi)$. Thus, since $\mathcal{I}$ is semilattice based with respect to K', again, by Lemma 1739, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. Hence, $\mathrm{Ker}(\mathsf{K}) \leq \mathrm{Ker}(\mathcal{A})$, which gives that $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$. We conclude that $\mathsf{K}' \subseteq \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$ and, by symmetry, $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathbb{V}^{\mathsf{Sem}}(\mathsf{K}')$. ■

Thus, if $\mathcal{I}$ is semilattice based with respect to some K and $\wedge^\flat$, it makes sense, based on Lemma 1741 and Lemma 1740, to denote by $\mathbb{V}^{\mathsf{Sem}}(\mathcal{I})$ the unique semantic variety of $\mathbf{F}$-algebraic systems with respect to which it is semilattice based.

In one of the cornerstone results of the section, we show that, if a π-institution is semilattice based, then it is self extensional and has the Conjunction Property, and, in addition, its semantic variety coincides with the semantic variety $\mathsf{K}^{\mathcal{I}}$ generated by the Lindenbaum-Tarski algebraic system of $\mathcal{I}$.

Proposition 1742 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\wedge^\flat$ *in* $N^\flat$*,* K *a semilattice based class with respect to* $\wedge^\flat$*,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a semilattice based* π*-institution with respect to* K *and* $\wedge^\flat$*.*

(a) $\mathcal{I}$ *is self extensional;*

(b) $\mathcal{I}$ *has the Conjunction Property with respect to* $\wedge^\flat$*;*

(c) $\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}) = \mathsf{K}^{\mathcal{I}}$*; Hence* $\mathcal{I}$ *is semilattice based with respect to* $\mathsf{K}^{\mathcal{I}}$*.*

Proof:

(a) For self extensionality, it suffices to show that the Frege equivalence system $\widetilde{\lambda}(\mathcal{I})$ is a congruence system. To this end, let $\sigma^\flat$ be in $N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi_i, \psi_i \rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I})$, for all $i < k$. By definition, we get $C_\Sigma(\phi_i) = C_\Sigma(\psi_i)$, for all $i < k$. Hence, since

$\mathcal{I}$ is semilattice based with respect to K and $\wedge^\flat$, $\langle\phi_i,\psi_i\rangle \in \mathrm{Ker}_\Sigma(\mathsf{K})$, for all $i < k$. But $\mathrm{Ker}(\mathsf{K})$ is a congruence system on $\mathbf{F}$, whence, $\langle\sigma^\flat_\Sigma(\vec{\phi}),\sigma^\flat_\Sigma(\vec{\psi})\rangle \in \mathrm{Ker}_\Sigma(\mathsf{K})$. By Lemma 1739, $C_\Sigma(\sigma^\flat_\Sigma(\vec{\phi})) = C_\Sigma(\sigma^\flat_\Sigma(\vec{\psi}))$, whence $\langle\sigma^\flat_\Sigma(\vec{\phi}),\sigma^\flat_\Sigma(\vec{\psi})\rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I})$. Therefore, $\widetilde{\lambda}(\mathcal{I})$ is a congruence system on $\mathbf{F}$ and $\mathcal{I}$ is self extensional.

(b) Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $\mathcal{A} \in \mathsf{K}$,

$$\alpha_\Sigma(\phi \wedge^\flat_\Sigma \psi) = \alpha_\Sigma(\phi) \wedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\psi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi),\alpha_\Sigma(\psi).$$

Thus, since $\mathcal{I}$ is semilattice based with respect to K and $\wedge^\flat$, we get that $\phi \wedge^\flat_\Sigma \psi \in C_\Sigma(\phi,\psi)$ and $\phi,\psi \in C_\Sigma(\phi \wedge^\flat_\Sigma \psi)$. Hence, $\wedge^\flat$ is a conjunction for $\mathcal{I}$.

(c) We have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}\langle\phi,\psi\rangle \in \mathrm{Ker}_\Sigma(\mathbb{V}^{\mathsf{Sem}}(\mathcal{I})) & \text{iff} & \langle\phi,\psi\rangle \in \mathrm{Ker}_\Sigma(\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})) \\ & & \text{(definition of } \mathbb{V}^{\mathsf{Sem}}(\mathcal{I})) \\ & \text{iff} & \langle\phi,\psi\rangle \in \mathrm{Ker}_\Sigma(\mathsf{K}) \\ & & \text{(definition of } \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})) \\ & \text{iff} & C_\Sigma(\phi) = C_\Sigma(\psi) \\ & & \text{(Lemma 1739)} \\ & \text{iff} & \langle\phi,\psi\rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}) \\ & & \text{(definition of } \widetilde{\lambda}(\mathcal{I})) \\ & \text{iff} & \langle\phi,\psi\rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I}) \\ & & \text{(Part (a))} \\ & \text{iff} & \langle\phi,\psi\rangle \in \mathrm{Ker}_\Sigma(\mathsf{K}^{\mathcal{I}}). \\ & & \text{(definition of } \mathsf{K}^{\mathcal{I}})\end{array}$$

Therefore, since both classes are semantic varieties, we conclude that $\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}) = \mathsf{K}^{\mathcal{I}}$.

■

If $\mathcal{I}$ is a finitary self extensional π-institution $\mathcal{I}$, having the Conjunction Property with respect to $\wedge^\flat$, then the singleton class $\mathsf{K} = \{\mathcal{F}/\widetilde{\Omega}(\mathcal{I})\}$, consisting of its Lindenbaum-Tarski $\mathbf{F}$-algebraic system $\mathcal{F}/\widetilde{\Omega}(\mathcal{I})$, is semilattice based with respect to $\wedge^\flat$.

Lemma 1743 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle\mathbf{F},C\rangle$ a finitary self extensional π-institution, having the Conjunction Property with respect to $\wedge^\flat$. Then the class $\mathsf{K} = \{\mathcal{F}/\widetilde{\Omega}(\mathcal{I})\}$ is semilattice based with respect to $\wedge^\flat$.*

Proof: We have to verify that the class K satisfies the semilattice identities. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi,\psi,\chi \in \mathrm{SEN}^\flat(\Sigma)$.

- By the Conjunction Property, $C_\Sigma(\phi \wedge^\flat_\Sigma \phi) = C_\Sigma(\phi)$. Thus, using self extensionality, $\langle\phi \wedge^\flat_\Sigma \phi,\phi\rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}) = \widetilde{\Omega}_\Sigma(\mathcal{I})$. Hence K satisfies L1.

- By the Conjunction Property, $C_\Sigma(\phi \wedge^\flat_\Sigma \psi) = C_\Sigma(\phi,\psi) = C_\Sigma(\psi \wedge^\flat_\Sigma \phi)$. Thus, again using self extensionality, $\langle \phi \wedge^\flat_\Sigma \psi, \psi \wedge^\flat_\Sigma \phi\rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}) = \widetilde{\Omega}_\Sigma(\mathcal{I})$. Hence K satisfies L2.

- Finally, we have
$$\begin{array}{rcl} C_\Sigma((\phi \wedge^\flat_\Sigma \psi) \wedge^\flat_\Sigma \chi) & = & C_\Sigma(\phi \wedge^\flat_\Sigma \psi, \chi) \\ & = & C_\Sigma(\phi,\psi,\chi) \\ & = & C_\Sigma(\phi, \psi \wedge^\flat_\Sigma \chi) \\ & = & C_\Sigma(\phi \wedge^\flat_\Sigma (\psi \wedge^\flat_\Sigma \chi)). \end{array}$$
Thus, again using self extensionality,
$$\langle (\phi \wedge^\flat_\Sigma \psi) \wedge^\flat_\Sigma \chi, \phi \wedge^\flat_\Sigma (\psi \wedge^\flat_\Sigma \chi)\rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}) = \widetilde{\Omega}_\Sigma(\mathcal{I})$$
and K also satisfies L3.

Thus, K is semilattice based with respect to $\wedge^\flat$. ■

In one of our main theorems, we characterize semilattice based π-institutions as those that are self extensional and conjunctive.

Theorem 1744 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a finitary* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is semilattice based if and only if it is self extensional and conjunctive.*

Proof: The left-to-right implication is by Proposition 1742. Suppose, conversely, that $\mathcal{I}$ is self extensional and conjunctive, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$ a conjunction for $\mathcal{I}$. By Lemma 1743, $\mathsf{K} = \{\mathcal{F}/\widetilde{\Omega}(\mathcal{I})\}$ is semilattice based with respect to $\wedge^\flat$. Moreover, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$,
$$\begin{array}{rll} \phi \in C_\Sigma(\Phi) & \text{iff} & \phi \in C_\Sigma(\bigwedge^\flat_\Sigma \Phi) \quad \text{(Conjunction Property)} \\ & \text{iff} & C_\Sigma(\bigwedge^\flat_\Sigma \Phi \wedge^\flat_\Sigma \phi) = C_\Sigma(\bigwedge^\flat_\Sigma \Phi) \quad \text{(Conjunction Property)} \\ & \text{iff} & \langle \bigwedge^\flat_\Sigma \Phi \wedge^\flat_\Sigma \phi, \bigwedge^\flat_\Sigma \Phi\rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}) \quad \text{(definition of } \widetilde{\lambda}(\mathcal{I})) \\ & \text{iff} & \langle \bigwedge^\flat_\Sigma \Phi \wedge^\flat_\Sigma \phi, \bigwedge^\flat_\Sigma \Phi\rangle \in \widetilde{\Omega}_\Sigma(\mathcal{I}) \quad \text{(self extensionality)} \\ & \text{iff} & \bigwedge_\Sigma^{\mathcal{F}/\widetilde{\Omega}(\mathcal{I})} \Phi/\widetilde{\Omega}_\Sigma(\mathcal{I}) \leq_\Sigma^{\mathcal{F}/\widetilde{\Omega}(\mathcal{I})} \phi/\widetilde{\Omega}_\Sigma(\mathcal{I}). \quad \text{(def. of } \leq^{\mathcal{F}/\widetilde{\Omega}(\mathcal{I})}) \end{array}$$
Therefore, $\mathcal{I}$ is indeed semilattice based with respect to K and $\wedge^\flat$. ■

In some contexts it is desirable to have a specification of the theorems of a π-institution under discussion. However, the hypothesis that $\mathcal{I}$ is semilattice based by itself does not provide information about the theorems of $\mathcal{I}$, since it only specifies, based on properties of the defining class K, entailments with non empty sets of hypotheses. We discuss the *property of being non psudo-axiomatic*, which serves to streamline this ambiguity concerning theorems.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a finitary π-institution based on $\mathbf{F}$. We say that $\mathcal{I}$ is **non pseudo-axiomatic** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{Thm}_\Sigma(\mathcal{I}) = \bigcap\{C_\Sigma(\phi) : \phi \in \mathrm{SEN}^\flat(\Sigma)\}.$$

The property may be equivalently expressed by the condition

$$\mathrm{Thm}(\mathcal{I}) = \bigcap\{T \in \mathrm{ThFam}(\mathcal{I}) : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(T_\Sigma \neq \varnothing)\}.$$

Non pseudo-axiomatic semilattice based π-institutions form a generalization of π-institutions with theorems.

Lemma 1745 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a semilattice based π-institution with theorems, based on $\mathbf{F}$.*

(a) $\mathcal{I}$ is non pseudo-axiomatic;

(b) For all $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$, all $\Sigma \in |\mathbf{Sign}^\flat|$, all $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\alpha_\Sigma(\phi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(t).$$

Proof:

(a) Let $\Sigma \in |\mathbf{Sign}^\flat|$. On the one hand, $\mathrm{Thm}_\Sigma(\mathcal{I}) \subseteq \bigcap\{C_\Sigma(\phi) : \phi \in \mathrm{SEN}^\flat(\Sigma)\}$. On the other, $\bigcap\{C_\Sigma(\phi) : \phi \in \mathrm{SEN}^\flat(\Sigma)\} \subseteq \bigcap\{C_\Sigma(\phi) : \phi \in \mathrm{Thm}_\Sigma(\mathcal{I})\} = \mathrm{Thm}_\Sigma(\mathcal{I})$. Hence, $\mathcal{I}$ is non pseudo-axiomatic.

(b) Suppose $\Sigma \in |\mathbf{Sign}^\flat|$ and $t \in \mathrm{Thm}_\Sigma(\mathcal{I})$. Then, for all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $t \in C_\Sigma(\phi)$. Since, by Proposition 1742, $\mathcal{I}$ is semilattice based with respect to $\mathsf{K}^{\mathcal{I}}$, we get that, for all $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$, $\alpha_\Sigma(\phi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(t)$. ■

By Lemma 1739, for all $\Sigma \in |\mathbf{Sign}^\flat|$, and all $t, t' \in \mathrm{Thm}_\Sigma(\mathcal{I})$, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$, for all $\mathcal{A} \in \mathsf{K}$. Furthermore, the common value of all theorems in $\mathrm{SEN}(\Sigma)$ is, by Lemma 1745, a maximum element under $\leq^{\mathcal{A}}_\Sigma$. This element will be denoted by $1^{\mathcal{A}}_\Sigma$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a semilattice based semantic variety with respect to $\wedge^\flat$. We assume that K is **conditionally max natural**, i.e., for all $\mathcal{A} \in \mathsf{K}$, either for no $\Sigma \in |\mathbf{Sign}|$ is there a maximum under $\leq^{\mathcal{A}}_\Sigma$ or, for every $\Sigma \in |\mathbf{Sign}|$, there exists a maximum $1^{\mathcal{A}}_\Sigma$ under $\leq^{\mathcal{A}}_\Sigma$, and moreover, $1^{\mathcal{A}} = \{1^{\mathcal{A}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ is natural, i.e., it satisfies, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}(f)(1^{\mathcal{A}}_\Sigma) = 1^{\mathcal{A}}_{\Sigma'}.$$

Define a finitary closure system

$$C^{\mathsf{K},\wedge} : \mathcal{P}\mathrm{SEN}^\flat \to \mathcal{P}\mathrm{SEN}^\flat$$

on $\mathbf{F}$ by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$,

- $\phi \in C^{\mathsf{K},\wedge}_\Sigma(\varnothing)$ if and only if, for all $\mathcal{A} \in \mathsf{K}$ and all $\chi \in \mathrm{SEN}(F(\Sigma))$, $\chi \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi)$;
- $\phi \in C^{\mathsf{K},\wedge}_\Sigma(\Phi)$ if and only if, for all $\mathcal{A} \in \mathsf{K}$, $\bigwedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\Phi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi)$.

We note that conditional max naturality is essential in guaranteeing the structurality of $C^{\mathsf{K},\wedge}$.

Lemma 1746 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a conditionally max natural semilattice based semantic variety with respect to $\wedge^\flat$. Then $\mathcal{I}^{\mathsf{K},\wedge} = \langle \mathbf{F}, C^{\mathsf{K},\wedge}\rangle$ is a non pseudo-axiomatic π-institution.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \bigcap\{C_\Sigma(\psi) : \psi \in \mathrm{SEN}^\flat(\Sigma)\}$. Then, by definition, for all $\mathcal{A} \in \mathsf{K}$, $\alpha_\Sigma(\psi) \leq \alpha_\Sigma(\phi)$, for all $\psi \in \mathrm{SEN}^\flat(\Sigma)$. By the surjectivity of $\langle F, \alpha\rangle$, we get that $\phi \in \mathrm{Thm}_\Sigma(\mathcal{I}^{\mathsf{K},\wedge})$. Therefore, $\mathcal{I}^{\mathsf{K},\wedge}$ is non pseudo-axiomatic. ∎

The semantic variety of $\mathcal{I}^{\mathsf{K},\wedge}$ turns out to be the class K.

Proposition 1747 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a conditionally max natural semilattice based semantic variety with respect to $\wedge^\flat$. Then $\mathcal{I}^{\mathsf{K},\wedge} = \langle \mathbf{F}, C^{\mathsf{K},\wedge}\rangle$ is semilattice based with respect to K and $\wedge^\flat$ and $\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}^{\mathsf{K},\wedge}) = \mathsf{K}$.*

Proof: By the second condition in the definition of $C^{\mathsf{K},\wedge}$, we conclude that $\mathcal{I}^{\mathsf{K},\wedge}$ is semilattice based with respect to K and $\wedge^\flat$. Then, by definition $\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}^{\mathsf{K},\wedge}) = \mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$, since, by hypothesis, K is a semantic variety. ∎

Proposition 1748 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a semilattice based non pseudo-axiomatic π-institution. Then $\mathcal{I}^{\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}),\wedge} = \mathcal{I}$.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, with $\Phi \neq \varnothing$. Then, we have:

$$\begin{array}{lll}
\phi \in C^{\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}),\wedge}_\Sigma(\varnothing) & \text{iff} & \text{for all } \mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathcal{I}), \chi \in \mathrm{SEN}(F(\Sigma)),\\
& & \chi \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi),\\
& \text{iff} & \text{for all } \mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathcal{I}), \psi \in \mathrm{SEN}^\flat(\Sigma),\\
& & \alpha_\Sigma(\psi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi),\\
& \text{iff} & \text{for all } \mathcal{A} \in \mathsf{K}, \psi \in \mathrm{SEN}^\flat(\Sigma),\\
& & \alpha_\Sigma(\psi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi),\\
& \text{iff} & \phi \in \bigcap\{C_\Sigma(\psi) : \psi \in \mathrm{SEN}^\flat(\Sigma)\}\\
& \text{iff} & \phi \in C_\Sigma(\varnothing).
\end{array}$$

Moreover,

$$\begin{array}{lll}\phi \in C_\Sigma^{\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}),\wedge}(\Phi) & \text{iff} & \text{for all } \mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathcal{I}),\\ & & \bigwedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\Phi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi)\\ & \text{iff} & \text{for all } \mathcal{A} \in \mathsf{K},\\ & & \bigwedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\Phi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi)\\ & \text{iff} & \phi \in C_\Sigma(\Phi).\end{array}$$

Thus, we get $\mathcal{I}^{\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}),\wedge} = \mathcal{I}$. ∎

For non pseudo-axiomatic semilattice based π-institutions on the same algebraic system, the Frege relations reflect the $\leq$ ordering on their closure systems.

Proposition 1749 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$, *and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$, $\mathcal{I}' = \langle \mathbf{F}, C'\rangle$ *non pseudo-axiomatic semilattice based* π*-institutions with respect to* $\wedge^\flat$. *Then*

$$\mathcal{I} \leq \mathcal{I}' \quad \textit{iff} \quad \widetilde{\lambda}(\mathcal{I}) \leq \widetilde{\lambda}(\mathcal{I}').$$

Proof: The left-to-right implication is by Lemma 1416. Assume, conversely, that $\mathcal{I}$, $\mathcal{I}'$ are non pseudo-axiomatic semilattice based with respect to $\wedge^\flat$, such that $\widetilde{\lambda}(\mathcal{I}) \leq \widetilde{\lambda}(\mathcal{I}')$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$, with $\Phi \neq \varnothing$,

$$\begin{array}{lll}\phi \in C_\Sigma(\Phi) & \text{iff} & \phi \in C_\Sigma(\bigwedge^\flat_\Sigma \Phi)\\ & \text{iff} & C_\Sigma(\bigwedge^\flat_\Sigma \Phi \wedge^\flat_\Sigma \phi) = C_\Sigma(\bigwedge^\flat_\Sigma \Phi)\\ & \text{iff} & \langle \bigwedge^\flat_\Sigma \Phi \wedge^\flat_\Sigma \phi, \bigwedge^\flat_\Sigma \Phi\rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I})\\ & \text{implies} & \langle \bigwedge^\flat_\Sigma \Phi \wedge^\flat_\Sigma \phi, \bigwedge^\flat_\Sigma \Phi\rangle \in \widetilde{\lambda}_\Sigma(\mathcal{I}')\\ & \text{iff} & C'_\Sigma(\bigwedge^\flat_\Sigma \Phi \wedge^\flat_\Sigma \phi) = C'_\Sigma(\bigwedge^\flat_\Sigma \Phi)\\ & \text{iff} & \phi \in C'_\Sigma(\bigwedge^\flat_\Sigma \Phi)\\ & \text{iff} & \phi \in C'_\Sigma(\Phi).\end{array}$$

Moreover, taking into account what was just demonstrated,

$$\begin{array}{lll}\phi \in C_\Sigma(\varnothing) & \text{iff} & \phi \in \bigcap\{C_\Sigma(\psi) : \psi \in \mathrm{SEN}^\flat(\Sigma)\}\\ & \text{implies} & \phi \in \bigcap\{C'_\Sigma(\psi) : \psi \in \mathrm{SEN}^\flat(\Sigma)\}\\ & \text{iff} & \phi \in C'_\Sigma(\varnothing).\end{array}$$

We conclude that $\mathcal{I} \leq \mathcal{I}'$. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$. Denote by $\mathsf{K}^{\mathbf{F},\wedge}$ the semantic variety of $\mathbf{F}$-algebraic systems generated by the semilattice equations L1-L4.

Theorem 1750 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$. *There exists a dual isomorphism between the collection of semilattice based non pseudo-axiomatic* π*-institutions with respect to* $\wedge^f lat$, *ordered under* $\leq$, *and the collection of all conditionally max natural semantic subvarieties of* $\mathsf{K}^{\mathbf{F},\wedge}$, *ordered under* $\subseteq$, *given by* $\mathcal{I} \mapsto \mathsf{K}^{\mathcal{I}}$.

Proof: Consider the mapping $\mathcal{I} \mapsto \mathsf{K}^{\mathcal{I}}$.

Suppose, first, that $\mathcal{I}, \mathcal{I}'$ are non pseudo-axiomatic and semilattice based with respect to $\wedge^\flat$, such that $\mathsf{K}^{\mathcal{I}} = \mathsf{K}^{\mathcal{I}'}$. Then $\mathcal{I}^{\mathsf{K}^{\mathcal{I}},\wedge} = \mathcal{I}^{\mathsf{K}^{\mathcal{I}'},\wedge}$. By Proposition 1742 and Proposition 1748, we get $\mathcal{I} = \mathcal{I}'$. Therefore, the mapping is one-to-one.

Assume, now, that K is conditionally max natural and semilattice based with respect to $\wedge^\flat$. Then, by Lemma 1746 and Proposition 1747, $\mathcal{I}^{\mathsf{K},\wedge}$ is a non pseudo-axiomatic and semilattice based π-institution with respect to K and $\wedge^\flat$, such that $\mathbb{V}^{\mathsf{Sem}}(\mathcal{I}^{\mathsf{K},\wedge}) = \mathsf{K}$. Therefore, by Proposition 1742, $\mathsf{K}^{\mathcal{I}^{\mathsf{K},\wedge}} = \mathsf{K}$ and the mapping is also onto. Thus, it is a bijection from the collection of semilattice based non pseudo-axiomatic π-institutions with respect to $\wedge^f lat$ onto the collection of all conditionally max natural semantic subvarieties of $\mathsf{K}^{\mathbf{F},\wedge}$.

Finally, for all non pseudo-axiomatic and semilattice based π-institutions $\mathcal{I}, \mathcal{I}'$, with respect to $\wedge^\flat$, we have

$$\begin{array}{lll}\mathcal{I} \leq \mathcal{I}' & \text{iff} & \widetilde{\lambda}(\mathcal{I}) \leq \widetilde{\lambda}(\mathcal{I}') \quad \text{(by Proposition 1749)}\\ & \text{iff} & \widetilde{\Omega}(\mathcal{I}) \leq \widetilde{\Omega}(\mathcal{I}') \quad \text{(by Proposition 1742)}\\ & \text{iff} & \mathsf{K}^{\mathcal{I}'} \leq \mathsf{K}^{\mathcal{I}}. \quad \text{(definition of } \mathsf{K}^{\mathcal{I}}, \mathsf{K}^{\mathcal{I}'})\end{array}$$

Thus, the bijection is also order reversing and dual order reflecting and, therefore, it is a dual order isomorphism, as claimed. ■

Our next goal is to show that for semilattice based π-institutions, their semantic variety coincides with the class of all $\mathcal{I}$-algebraic systems. We start by showing that, for such a π-institution, the $\mathcal{I}$-filter families on any algebraic system in their semantic variety coincides (roughly) with the collection of all semilattice filter families.

Lemma 1751 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a semilattice based* π*-institution based on* $\mathbf{F}$*. For all* $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$*,*

$$\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \begin{cases} \mathrm{FiFam}^{\wedge}(\mathcal{A}), & \textit{if } \mathcal{I} \textit{ has theorems}\\ \mathrm{FiFam}^{\wedge,\varnothing}(\mathcal{A}), & \textit{otherwise}\end{cases}$$

Proof: It suffices to show that, for all $T \in \mathrm{SenFam}(\mathcal{A})$, such that $T_\Sigma \neq \varnothing$, for all $\Sigma \in |\mathbf{Sign}|$, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ if and only if $T \in \mathrm{FiFam}^{\wedge}(\mathcal{A})$.

Suppose, first, that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$.

- Suppose that $\alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \in T_{F(\Sigma)}$. Then, since $\phi \wedge^\flat_\Sigma \psi \in C_\Sigma(\phi, \psi)$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get $\alpha_\Sigma(\phi) \wedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\psi) = \alpha_\Sigma(\phi \wedge^\flat_\Sigma \psi) \in T_{F(\Sigma)}$;
- If $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$ and $\alpha_\Sigma(\phi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\psi)$, then $\alpha_\Sigma(\phi \wedge^\flat_\Sigma \psi) = \alpha_\Sigma(\phi) \wedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\psi) = \alpha_\Sigma(\phi) \in T_{F(\Sigma)}$. Since $\psi \in C_\Sigma(\phi \wedge^\flat_\Sigma \psi)$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get $\alpha_\Sigma(\psi) \in T_{F(\Sigma)}$.

Taking into account the surjectivity of $\langle F,\alpha\rangle$, we get $T \in \mathrm{FiFam}^{\wedge}(\mathcal{A})$.

Assume, conversely, that $T \in \mathrm{FiFam}^{\wedge}(\mathcal{A})$ and let $\Sigma \in |\mathbf{Sign}^{\flat}|$, $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^{\flat}(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$ and $\alpha_\Sigma(\Phi) \subseteq T_{F(\Sigma)}$. Since, by Proposition 1742, $\mathcal{I}$ is semilattice based with respect to $\mathsf{K}^{\mathcal{I}}$, we get that $\bigwedge^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\Phi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi)$ and $\alpha_\Sigma(\Phi) \subseteq T_{F(\Sigma)}$. Since, by hypothesis, $T \in \mathrm{FiFam}^{\wedge}(\mathcal{A})$, $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$.

Finally, if $\phi \in C_\Sigma(\varnothing)$, then $\phi \in C_\Sigma(\psi)$, for all $\psi \in \mathrm{SEN}^{\flat}(\Sigma)$. Since $\mathcal{I}$ has theorems, $T_{F(\Sigma)} \neq \varnothing$, whence, by the surjectivity of $\langle F,\alpha\rangle$, for some $\psi \in \mathrm{SEN}^{\flat}(\Sigma)$, $\alpha_\Sigma(\psi) \in T_{F(\Sigma)}$. For this chosen ψ, we also have, since $\mathcal{I}$ is semilattice based with respect to $\mathsf{K}^{\mathcal{I}}$, that $\alpha_\Sigma(\psi) \leq^{\mathcal{A}}_{F(\Sigma)} \alpha_\Sigma(\phi)$. Thus, since $T \in \mathrm{FiFam}^{\wedge}(\mathcal{A})$, we conclude that $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$.

Since in all cases $\phi \in C_\Sigma(\Phi)$ and $\alpha_\Sigma(\Phi) \subseteq T_{F(\Sigma)}$ imply $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. ∎

In the next step, we show that, for a semilattice based π-institution, the Frege congruence system of any $\mathcal{I}$-structure of the form $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$, with $\mathcal{A}$ in the semantic variety of $\mathcal{I}$, is reduced.

Lemma 1752 *Let $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat}\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a semilattice based π-institution based on $\mathbf{F}$. For every $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$,*

$$\widetilde{\lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}.$$

Hence, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ is reduced.

Proof: Let $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$, $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi,\psi\rangle \notin \Delta^{\mathcal{A}}_\Sigma$. Then, by definition, $\phi \neq \psi$ and, by Lemma 1737 and Proposition 1742, we get $\psi \notin T^{\langle \Sigma,\phi\rangle}_\Sigma$ or $\phi \in T^{\langle \Sigma,\psi\rangle}_\Sigma$. Since, by Lemmas 1738 and 1751, $T^{\langle \Sigma,\phi\rangle}, T^{\langle \Sigma,\psi\rangle} \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we get that $\langle \phi,\psi\rangle \notin \widetilde{\lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$. Thus, $\widetilde{\lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}$.

Finally, $\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \leq \widetilde{\lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$, whence, $\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}$ and, therefore, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ is a reduced $\mathcal{I}$-structure. ∎

In the last step before the main theorem, we show that for a semilattice based π-institution $\mathcal{I}$, if $\langle \mathcal{A}, D\rangle$ is any $\mathcal{I}$-structure, such that $\widetilde{\lambda}^{\mathcal{A}}(D) = \Delta^{\mathcal{A}}$, then $\mathcal{D}$ is either $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, if $\mathcal{I}$ has theorems, or $\mathcal{D}^{\varnothing}$ is $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, if $\mathcal{I}$ does not have theorems, where $\mathcal{D}^{\varnothing}$ consists of the filter families in $\mathcal{D}$, (potentially) augmented by filter families in $\mathcal{D}$, in which one or more components have been replaced by $\varnothing$.

Lemma 1753 *Let $\mathbf{F} = \langle \mathbf{Sign}^{\flat}, \mathrm{SEN}^{\flat}, N^{\flat}\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a semilattice based π-institution based on $\mathbf{F}$, and $\langle \mathcal{A}, D\rangle \in \mathrm{Str}(\mathcal{I})$, such that $\widetilde{\lambda}^{\mathcal{A}}(D) = \Delta^{\mathcal{A}}$. If $\mathcal{I}$ has theorems, then $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. If $\mathcal{I}$ does not have theorems, then $\mathcal{D}^{\varnothing} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.*

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and suppose $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, with $T_\Sigma \neq \varnothing$, for all $\Sigma \in |\mathbf{Sign}|$. Let $\Sigma \in |\mathbf{Sign}|$, $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in D_\Sigma(T_\Sigma)$. By Proposition 114, there exists $\Phi \subseteq_f T_\Sigma$, such that $\phi \in D_\Sigma(\Phi)$. Since $\langle \mathcal{A}, D\rangle \in \mathrm{Str}(\mathcal{I})$, by Corollary 1735, $D_\Sigma(\bigwedge_\Sigma^{\mathcal{A}} \Phi \wedge_\Sigma^{\mathcal{A}} \phi) = D_\Sigma(\bigwedge_\Sigma^{\mathcal{A}} \Phi)$. Hence, by hypothesis, $\bigwedge_\Sigma^{\mathcal{A}} \Phi \wedge_\Sigma^{\mathcal{A}} \phi = \bigwedge_\Sigma^{\mathcal{A}} \Phi$. Since $\Phi \subseteq_f T_\Sigma$ and, by Lemma 1751, $T \in \mathrm{FiFam}^\wedge(\mathcal{A})$, $\bigwedge_\Sigma^{\mathcal{A}} \Phi \in T_\Sigma$. By the preceding equation, $\bigwedge_\Sigma^{\mathcal{A}} \Phi \wedge_\Sigma^{\mathcal{A}} \phi \in T_\Sigma$ and, therefore, $\phi \in T_\Sigma$. We conclude that $T = D(T)$ and, hence, $T \in \mathcal{D}$.

If $\mathcal{I}$ has theorems, then, for every $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T_\Sigma \neq \varnothing$, for all $\Sigma \in |\mathbf{Sign}|$. By what was proven above, $\mathcal{D} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. On the other hand, if $\mathcal{I}$ does not have theorems, then any of the components of $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is allowed to be empty and, therefore, $\mathcal{D}^\varnothing = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. ■

In one of the main theorems, we show that, for a semilattice based π-institution $\mathcal{I}$, the semantic variety $\mathsf{K}^{\mathcal{I}}$ of $\mathcal{I}$ coincides with the class $\mathrm{AlgSys}(\mathcal{I})$ of all $\mathcal{I}$-algebraic systems.

Theorem 1754 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a semilattice based* π*-institution based on* $\mathbf{F}$.

(a) $\mathrm{AlgSys}(\mathcal{I}) = \mathsf{K}^{\mathcal{I}}$;

(b) $\mathrm{AlgSys}(\mathcal{I})$ *is a semantic variety;*

(c) $\mathcal{I}$ *is semilattice based with respect to* $\mathrm{AlgSys}(\mathcal{I})$.

Proof: We have, by Proposition 65, that, in general, $\mathrm{AlgSys}(\mathcal{I}) \subseteq \mathsf{K}^{\mathcal{I}}$. Suppose, conversely, that $\mathcal{A} \in \mathsf{K}^{\mathcal{I}}$. Then, by Lema 1752, $\langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$ is reduced. Therefore, $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$. We conclude that $\mathrm{AlgSys}(\mathcal{I}) = \mathsf{K}^{\mathcal{I}}$. Since $\mathsf{K}^{\mathcal{I}}$ is, by definition, a semantic variety, then so is $\mathrm{AlgSys}(\mathcal{I})$. Finally, since, by Proposition 1742, $\mathcal{I}$ is semilattice based with respect to $\mathsf{K}^{\mathcal{I}}$, it is semilattice based with respect to $\mathrm{AlgSys}(\mathcal{I})$. ■

In another main theorem, it is shown that a finitary self extensional and conjunctive π-institution $\mathcal{I}$ is necessarily fully self extensional, i.e., that every $\langle \mathcal{A}, D\rangle \in \mathrm{FStr}(\mathcal{I})$ satisfies the Congruence Property.

Theorem 1755 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a finitary* π*-institution based on* $\mathbf{F}$. *If* $\mathcal{I}$ *is self extensional and conjunctive, then it is fully self extensional.*

Proof: Suppose that $\mathcal{I}$ is finitary, self extensional and has the Conjunction Property with respect to $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$. By Theorem 1744 and Proposition 1742, $\mathcal{I}$ is semilattice based with respect to $\mathsf{K}^{\mathcal{I}}$ and $\wedge^\flat$. By Proposition 65, $\mathrm{AlgSys}(\mathcal{I}) \subseteq \mathsf{K}^{\mathcal{I}}$, whence, by Lemma 1752, if $\mathcal{A} \in \mathrm{AlgSys}(\mathcal{I})$, then $\widetilde{\lambda}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathcal{A}}$.

Suppose, now, that $\langle \mathcal{A}, D\rangle \in \mathrm{FStr}(\mathcal{I})$. Then, by definition,

$$\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(D)) = \mathcal{D}/\widetilde{\Omega}^{\mathcal{A}}(D).$$

Since $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{D}) \in \mathrm{AlgSys}(\mathcal{I})$, by what was shown in the preceding paragraph,

$$\widetilde{\lambda}^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(D)}(\mathcal{D}/\widetilde{\Omega}^{\mathcal{A}}(D)) = \Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(D)}.$$

Therefore, we get that $\langle \mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(D), \mathcal{D}/\widetilde{\Omega}^{\mathcal{A}}(D)\rangle$ has the Congruence Property. Therefore, by Proposition 1426, $\langle \mathcal{A}, D\rangle$ also has the Congruence Property. We conclude that $\mathcal{I}$ is fully self extensional. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ a finitary Gentzen π-institution based on $\mathbf{F}$.

$\mathfrak{G}$ **has congruence** if, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\sigma^\flat_\Sigma(\vec{\phi}) \vdash_\Sigma \sigma^\flat_\Sigma(\vec{\psi}) \in G_\Sigma(\{\phi_i \vdash_\Sigma \psi_i, \psi_i \vdash_\Sigma \phi_i : i < k\}).$$

Moreover, $\mathfrak{G}$ **has conjunction with respect to** $\wedge^\flat$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi, \psi \vdash_\Sigma \phi \wedge^\flat_\Sigma \psi,\ \ \phi \wedge^\flat_\Sigma \psi \vdash_\Sigma \phi,\ \ \phi \wedge^\flat_\Sigma \psi \vdash_\Sigma \psi \in G_\Sigma(\varnothing).$$

Let, also, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a finitary π-institutuon based on $\mathbf{F}$. Recall that:

- If $\mathcal{I}$ has theorems, $\mathfrak{G}$ is **fully adequate for** $\mathcal{I}$ if $\mathrm{Str}(\mathfrak{G}) = \mathrm{FStr}(\mathcal{I})$;
- If $\mathcal{I}$ does not have theorems, then $\mathfrak{G}$ is **fully adequate for** $\mathcal{I}$ if $\mathrm{Str}(\mathfrak{G})^\varnothing = \mathrm{FStr}(\mathcal{I})$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a finitary self extensional π-institution, having the Conjunction Property with respect to $\wedge^\flat$.

- Define $\mathrm{Ax}^{\mathcal{I}} = \{\mathrm{Ax}^{\mathcal{I}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,
$$\mathrm{Ax}^{\mathcal{I}}_\Sigma = \{\Phi \vdash_\Sigma \phi : \phi \in C_\Sigma(\Phi)\};$$
- Define $\mathrm{Ir}^{\mathcal{I}} = \{\mathrm{Ir}^{\mathcal{I}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,
$$\begin{aligned}\mathrm{Ir}^{\mathcal{I}}_\Sigma \ = \ \{\langle\{\phi_i \vdash_\Sigma \psi_i, \psi_i \vdash_\Sigma \phi_i : i \in I\}, \sigma^\flat_\Sigma(\vec{\phi}) \vdash_\Sigma \sigma^\flat_\Sigma(\vec{\psi})\rangle :\\ \sigma^\flat \in N^\flat, \vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)\}.\end{aligned}$$
- $R^{\mathcal{I}} \coloneqq \mathrm{Ax}^{\mathcal{I}} \cup \mathrm{Ir}^{\mathcal{I}}$.

Finally, define $\mathfrak{G}^{\mathcal{I}} = \langle \mathbf{F}, C^{\mathcal{I}}\rangle \coloneqq \mathfrak{G}^{R^{\mathcal{I}}}$ as the Gentzen π-institution generated by the system $R^{\mathcal{I}}$ of Gentzen rules. Recall, by Proposition 1482, that $G^{\mathcal{I}} = \Xi^{R^{\mathcal{I}}}$.

This Gentzen π-institution turns out to be fully adequate for the π-institution $\mathcal{I}$:

Theorem 1756 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a finitary conjunctive* π*-institution.* $\mathcal{I}$ *is self extensional if and only if* $\mathfrak{G}^{\mathcal{I}} = \langle \mathbf{F}, G^{\mathcal{I}} \rangle$ *is fully adequate for* $\mathcal{I}$.

Proof: Suppose, first, that $\mathfrak{G}$ is fully adequate for $\mathcal{I}$. Since $\langle \mathcal{F}, C \rangle \in \mathrm{FStr}(\mathcal{I})$, we get that $\langle \mathcal{F}, C \rangle \in \mathrm{Str}(\mathfrak{G}^{\mathcal{I}})$, if $\mathcal{I}$ has theorems, and that $\langle \mathcal{F}, C \rangle \in \mathrm{Str}(\mathfrak{G}^{\mathcal{I}})^\varnothing$, otherwise. Since, by definition, $\mathfrak{G}^{\mathcal{I}}$ has congruence, we get that $\langle \mathcal{F}, C \rangle$ has the Congruence Property, which amounts to $\mathcal{I}$ having the Congruence Property. Thus, $\mathcal{I}$ is self extensional.

Assume, conversely, that $\mathcal{I}$ is finitary, self extensional and has the Conjunction Property with respect to $\wedge^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$.

- Suppose, first, that $\langle \mathcal{A}, D \rangle \in \mathrm{FStr}(\mathcal{I})$. Then, by Theorem 1755, $\langle \mathcal{A}, D \rangle$ has the Congruence Property. Moreover, by definition, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)$, if $\phi \in C_\Sigma(\Phi)$, then, $\alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi))$. We conclude that, if $\mathcal{I}$ has theorems, then $\langle \mathcal{A}, D \rangle \in \mathrm{Str}(\mathfrak{G}^{\mathcal{I}})$ and that, otherwise, $\langle \mathcal{A}, D \rangle \in \mathrm{Str}(\mathfrak{G}^{\mathcal{I}})^\varnothing$.

- Suppose, conversely, that $\langle \mathcal{A}, D \rangle \in \mathrm{Str}(\mathfrak{G}^{\mathcal{I}})$, if $\mathcal{I}$ has theorems, or $\langle \mathcal{A}, D \rangle \in \mathrm{Str}(\mathfrak{G}^{\mathcal{I}})^\varnothing$, otherwise. Consider the reduction

  $$\langle \mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})), \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))\rangle.$$

  This reduction is an $\mathcal{I}$-structure and, by hypothesis and Proposition 1426, it has the Congruence Property. Thus, we get

  $$\widetilde{\lambda}^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))) = \Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))}.$$

  Now Lemma 1753 allows us to conclude that

  $$\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))).$$

  Therefore, $\langle \mathcal{A}, D \rangle \in \mathrm{FStr}(\mathcal{I})$.

We conclude that $\mathfrak{G}^{\mathcal{I}}$ is fully adequate for $\mathcal{I}$. ■

23.4 Fregeanity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with a binary natural transformation $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

Recall that $\mathcal{I}$ is called:

- **strongly Fregean** if, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\lambda}(T) = \widetilde{\Omega}(T)$, i.e., if and only if the strong Frege equivalence family $\widetilde{\lambda}(T)$ is a congruence system;

- **congruential** if, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\lambda}(T)$ satisfies the congruence property, i.e., $\widetilde{\lambda}(T)$ is a congruence family (but not necessarily a system);

- **Fregean** if, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $\widetilde{\Lambda}(T) = \widetilde{\Omega}(T)$, i.e., if its Frege equivalence system $\widetilde{\Lambda}$ is a congruence system.

Strong Fregeanity implies congruentiality, which, in turn, implies Fregeanity.

Recall, also, that $\mathcal{I}$ is said to have the **Deduction Detachment Theorem with respect to** $\to^\flat$ if, for every $\Sigma \in |\mathbf{Sign}^\flat|$, all $\Phi \cup \{\phi, \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\psi \in C_\Sigma(\Phi, \phi) \quad \text{iff} \quad \phi \to^\flat_\Sigma \psi \in C_\Sigma(\Phi).$$

In the following proposition, it is shown that every strongly Fregean π-institution with the Deduction Detachment Theorem satisfies certain axioms and the rule of Modus Ponens.

Proposition 1757 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with a binary natural transformation $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a congruential π-institution having the Deduction Detachment Theorem with respect to $\to^\flat$. Then, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi, \vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,*

(a) $\phi \to^\flat_\Sigma (\psi \to^\flat_\Sigma \phi) \in C_\Sigma(\varnothing)$;

(b) $(\phi \to^\flat_\Sigma (\psi \to^\flat_\Sigma \chi)) \to^\flat_\Sigma ((\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma (\phi \to^\flat_\Sigma \chi)) \in C_\Sigma(\varnothing)$;

(c) $(\phi_0 \to^\flat_\Sigma \psi_0) \to^\flat_\Sigma ((\psi_0 \to^\flat_\Sigma \phi_0) \to^\flat_\Sigma (\cdots((\phi_{k-1} \to^\flat_\Sigma \psi_{k-1}) \to^\flat_\Sigma ((\psi_{k-1} \to^\flat_\Sigma \phi_{k-1}) \to^\flat_\Sigma (\sigma^\flat_\Sigma(\vec{\phi}) \to^\flat_\Sigma \sigma^\flat_\Sigma(\vec{\psi}))))\cdots)) \in C_\Sigma(\varnothing)$;

(d) $\psi \in C_\Sigma(\phi, \phi \to^\flat_\Sigma \psi)$.

Proof:

(a) We have, by inflationarity, $\phi \in C_\Sigma(\phi, \psi)$, whence, by two applications of the Deduction Theorem, $\phi \to^\flat_\Sigma (\psi \to^\flat_\Sigma \phi) \in C_\Sigma(\varnothing)$.

(b) We have, using the Detachment Theorem,

$$\chi \in C_\Sigma(\psi, \psi \to^\flat_\Sigma \chi) \subseteq C_\Sigma(\phi, \phi \to^\flat_\Sigma \psi, \phi \to^\flat_\Sigma (\psi \to^\flat_\Sigma \chi)).$$

Thus, using the Deduction Theorem, we get

$$(\phi \to^\flat_\Sigma (\psi \to^\flat_\Sigma \chi)) \to^\flat_\Sigma ((\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma (\phi \to^\flat_\Sigma \chi)) \in C_\Sigma(\varnothing).$$

(c) Since $\mathcal{I}$ is congruential, we get, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$C_\Sigma(T_\Sigma, \phi_i) = C_\Sigma(T_\Sigma, \psi_i),\ i < k, \\ \text{imply} \quad C_\Sigma(T_\Sigma, \sigma^\flat_\Sigma(\vec{\phi})) = C_\Sigma(T_\Sigma, \sigma^\flat_\Sigma(\vec{\psi})).$$

But $C_\Sigma(T_\Sigma, \phi_i) = C_\Sigma(T_\Sigma, \psi_i)$ is equivalent, by the Deduction Detachment Theorem, to $\phi \to^\flat_\Sigma \psi$, $\psi \to^\flat_\Sigma \phi \in C_\Sigma(T_\Sigma) = T_\Sigma$. Similarly,

$$C_\Sigma(T_\Sigma, \sigma^\flat_\Sigma(\vec{\phi})) = C_\Sigma(T_\Sigma, \sigma^\flat_\Sigma(\vec{\psi}))$$

is equivalent to $\sigma^\flat_\Sigma(\vec{\phi}) \to^\flat_\Sigma \sigma^\flat_\Sigma(\vec{\psi})$, $\sigma^\flat_\Sigma(\vec{\psi}) \to^\flat_\Sigma \sigma^\flat_\Sigma(\vec{\phi}) \in C_\Sigma(T_\Sigma) = T_\Sigma$. Therefore, we get

$$\sigma^\flat_\Sigma(\vec{\phi}) \to^\flat_\Sigma \sigma^\flat_\Sigma(\vec{\psi}) \in C_\Sigma(\{\phi_i \to^\flat_\Sigma \psi_i, \psi_i \to^\flat_\Sigma \phi_i : i < k\}).$$

Now (c) follows by several applications of the Deduction Theorem.

(d) By inflationarity, $\phi \to^\flat_\Sigma \psi \in C_\Sigma(\phi \to^\flat_\Sigma \psi)$, whence by the Detachment Theorem, $\psi \in C_\Sigma(\phi, \phi \to^\flat_\Sigma \psi)$.

■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$. Define $\mathrm{Ax}^0 = \{\mathrm{Ax}^0_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$, Ax^0_Σ is the set consisting, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi, \vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,

- $\phi \to^\flat_\Sigma (\psi \to^\flat_\Sigma \phi)$;
- $(\phi \to^\flat_\Sigma (\psi \to^\flat_\Sigma \chi)) \to^\flat_\Sigma ((\phi \to^\flat_\Sigma \psi) \to^\flat_\Sigma (\phi \to^\flat_\Sigma \chi))$;
- $(\phi_0 \to^\flat_\Sigma \psi_0) \to^\flat_\Sigma ((\psi_0 \to^\flat_\Sigma \phi_0) \to^\flat_\Sigma (\cdots((\phi_{k-1} \to^\flat_\Sigma \psi_{k-1}) \to^\flat_\Sigma ((\psi_{k-1} \to^\flat_\Sigma \phi_{k-1}) \to^\flat_\Sigma (\sigma^\flat_\Sigma(\vec{\phi}) \to^\flat_\Sigma \sigma^\flat_\Sigma(\vec{\psi}))))\cdots))$.

Furthermore, define $\mathrm{Ir}^0 = \{\mathrm{Ir}^0_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{Ir}^0_\Sigma = \{\langle \{\phi, \phi \to^\flat_\Sigma \psi\}, \psi \rangle : \phi, \psi \in \mathrm{SEN}^\flat(\Sigma)\}.$$

Finally, let $R^0 = \mathrm{Ax}^0 \cup \mathrm{Ir}^0$. Set $\mathcal{I}^0 = \langle \mathbf{F}, C^0 \rangle$ be the finitary π-institution, based on $\mathbf{F}$, with $C^0 = C^{R^0}$ the closure system on $\mathbf{F}$ generated by the collection R^0 of $\mathbf{F}$-axioms and $\mathbf{F}$-rules of inference.

Our work in Proposition 1757 allows us to formalize the fact that a congruential finitary π-institution having the Deduction Detachment Theorem with respect to $\to^\flat$ is an axiomatic extension of $\mathcal{I}^0$. Recall that $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is an axiomatic extension of $\mathcal{I}^0$ if there exists an axiom family Ax', such that $C = C^{R^0 \cup \mathrm{Ax}'}$.

Theorem 1758 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with a binary natural transformation $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a congruential finitary π-institution having the Deduction Detachment Theorem with respect to $\to^\flat$. Then $\mathcal{I}$ is an axiomatic extension of $\mathcal{I}^0$.*

Proof: Since $\mathcal{I}$ is finitary, its closure system is specified by a collection $R = \mathrm{Ax} \cup \mathrm{Ir}$ of $\mathbf{F}$-axioms and $\mathbf{F}$-rules. We define $R' = \mathrm{Ax}' \cup \mathrm{Ir}'$, where, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

- $\mathrm{Ax}'_\Sigma = \mathrm{Ax}_\Sigma \cup \{\phi_0 \to^\flat_\Sigma (\phi_i \to^\flat_\Sigma \cdots \to^\flat_\Sigma (\phi_{n-1} \to^\flat_\Sigma \phi)\cdots) : \langle\{\phi_0,\ldots,\phi_{n-1}\},\phi\rangle \in \mathrm{Ir}_\Sigma\}$;
- $\mathrm{Ir}'_\Sigma = \{\langle\{\langle\phi, \phi \to^\flat_\Sigma \psi\},\psi\rangle : \phi,\psi \in \mathrm{SEN}^\flat(\Sigma)\}$.

Note that, by the Deduction Theorem of $\mathcal{I}$, for every $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathrm{Ax}'_\Sigma \subseteq C_\Sigma(\varnothing)$. Moreover, by the Detachment Theorem for $\mathcal{I}$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, $\psi \in C_\Sigma(\phi, \phi \to^\flat_\Sigma \psi)$. Therefore, we conclude that $C^{\mathrm{R}'} \leq C$.

Conversely, note that, by definition, $\mathrm{Ax} \leq \mathrm{Ax}'$. Moreover, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\langle\{\phi_0,\ldots,\phi_{n-1}\},\phi\rangle \in \mathrm{Ir}_\Sigma$,

$$\phi \in C^{R'}_\Sigma(\Phi, \phi_0 \to^\flat_\Sigma (\phi_1 \to^\flat_\Sigma \cdots \to^\flat_\Sigma (\phi_{n-1} \to^\flat_\Sigma \phi)\cdots)) \subseteq C^{R'}_\Sigma(\Phi).$$

Hence, $C = C^R \leq C^{R'}$. We now conclude that $C = C^{R'}$. ■

Next, we show that, if $\mathcal{I}$ is an axiomatic extension of $\mathcal{I}^0$, then it has the Deduction Detachment Theorem with respect to $\to^\flat$.

Proposition 1759 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with a binary natural transformation $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ an axiomatic extension of $\mathcal{I}^0$. Then $\mathcal{I}$ has the Deduction Detachment Theorem with respect to $\to^\flat$.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi,\psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \to^\flat_\Sigma \psi \in C_\Sigma(\Phi)$. Then, since, by hypothesis, $C^{R^0} \leq C$, we get

$$\psi \in C_\Sigma(\phi, \phi \to^\flat_\Sigma \psi) \subseteq C_\Sigma(\Phi, \phi).$$

Suppose, conversely, that $\psi \in C_\Sigma(\Phi,\phi)$. Then, there exists in $\mathcal{I}$ a proof $\phi_0,\phi_1,\ldots,\phi_n = \psi$ of ψ from premises $\Phi \cup \{\phi\}$. We show by induction on $k \leq n$ that there exists a proof in $\mathcal{I}$ of $\phi \to^\flat_\Sigma \psi$ from premises Φ in $\mathcal{I}$.

- If $k = 0$, then $\phi_0 \in \mathrm{Ax}_\Sigma$ or $\phi_0 \in \Phi \cup \{\phi\}$.
  - If $\phi_0 \in \mathrm{Ax}_\Sigma$, then $\phi_0 \to^\flat_\Sigma (\phi \to^\flat_\Sigma \phi_0), \phi_0, \phi \to^\flat_\Sigma \phi_0$ is a proof in $\mathcal{I}$ of $\phi \to^\flat_\Sigma \phi_0$ from Φ;
  - If $\phi_0 \in \Phi$, then $\phi_0 \to^\flat_\Sigma (\phi \to^\flat_\Sigma \phi_0), \phi_0, \phi \to^\flat_\Sigma \phi_0$ is a proof in $\mathcal{I}$ of $\phi \to^\flat_\Sigma \phi_0$ from premises Φ.
  - If $\phi_0 = \phi$, then

$$\begin{array}{l}(\phi_0 \to^\flat_\Sigma ((\phi_0 \to^\flat_\Sigma \phi_0) \to^\flat_\Sigma \phi_0)) \\ \qquad\qquad \to^\flat_\Sigma ((\phi_0 \to^\flat_\Sigma (\phi_0 \to^\flat_\Sigma \phi_0)) \to^\flat_\Sigma (\phi_0 \to^\flat_\Sigma \phi_0)) \\ \phi_0 \to^\flat_\Sigma ((\phi_0 \to^\flat_\Sigma \phi_0) \to^\flat_\Sigma \phi_0) \\ (\phi_0 \to^\flat_\Sigma (\phi_0 \to^\flat_\Sigma \phi_0)) \to^\flat_\Sigma (\phi_0 \to^\flat_\Sigma \phi_0) \\ \phi_0 \to^\flat_\Sigma (\phi_0 \to^\flat_\Sigma \phi_0) \\ \phi_0 \to^\flat_\Sigma \phi_0\end{array}$$

    is a proof in $\mathcal{I}$ of $\phi_0 \to^\flat_\Sigma \phi_0$ from Φ

- If $k > 0$, assume that, for all $\ell < k$, $\phi \to^\flat_\Sigma \phi_\ell \in C_\Sigma(\Phi)$. If ϕ_k is either an axiom or in $\Phi \cup \{\phi\}$, then the treatment is the same as in the Induction Basis. So assume, for the final case, that ϕ_k follows from preceding Σ-sentences in the sequel by an application of the only **F**-rule available, i.e., that, for some $i, j < k$, $\phi_i = \phi_j \to^\flat_\Sigma \phi_k$. Then, by the Induction Hypothesis, $\phi \to^\flat_\Sigma (\phi_j \to^\flat_\Sigma \phi_k) \in C_\Sigma(\Phi)$ and $\phi \to^\flat_\Sigma \phi_j \in C_\Sigma(\Phi)$. Then by adjoining the following Σ-sentences to the sequence consisting of the proofs in $\mathcal{I}$ from Φ of $\phi \to^\flat_\Sigma (\phi_j \to^\flat_\Sigma \phi_k)$ and $\phi \to^\flat_\Sigma \phi_j$, we obtain a proof in $\mathcal{I}$ from Φ of $\phi \to^\flat_\Sigma \phi_k$:

$$\begin{array}{l} \vdots \\ \phi \to^\flat_\Sigma (\phi_j \to^\flat_\Sigma \phi_k) \\ \vdots \\ \phi \to^\flat_\Sigma \phi_j \\ (\phi \to^\flat_\Sigma (\phi_j \to^\flat_\Sigma \phi_k)) \to^\flat_\Sigma ((\phi \to^\flat_\Sigma \phi_j) \to^\flat_\Sigma (\phi \to^\flat_\Sigma \phi_k)) \\ (\phi \to^\flat_\Sigma \phi_j) \to^\flat_\Sigma (\phi \to^\flat_\Sigma \phi_k) \\ \phi \to^\flat_\Sigma \phi_k \end{array}$$

  This completes the Induction Step.

Thus, we conclude that $\phi \to^\flat_\Sigma \psi \in C_\Sigma(\Phi)$ and, therefore, $\mathcal{I}$ has the Deduction Detachment Theorem with respect to $\to^\flat$. ■

Moreover, under the same hypotheses, $\mathcal{I}$ turns out to be strongly Fregean.

Proposition 1760 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with a binary natural transformation* $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *an axiomatic extension of* $\mathcal{I}^0$*. Then* $\mathcal{I}$ *is congruential.*

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\sigma^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ be in $N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$, such that $C_\Sigma(T_\Sigma, \phi_i) = C_\Sigma(T_\Sigma, \psi_i)$, for all $i < k$. Then, by Proposition 1759, we get that

$$\phi_i \to^\flat_\Sigma \psi_i, \psi_i \to^\flat_\Sigma \phi_i \in C_\Sigma(T_\Sigma) = T_\Sigma,\ i < k.$$

Since $C^{R^0} \le C$, we get, by multiple applications of the Detachment Theorem, $\sigma^\flat_\Sigma(\vec{\phi}) \to^\flat_\Sigma \sigma^\flat_\Sigma(\vec{\psi}), \sigma^\flat_\Sigma(\vec{\psi}) \to^\flat_\Sigma \sigma^\flat_\Sigma(\vec{\phi}) \in C_\Sigma(T_\Sigma)$. Hence, $C_\Sigma(T_\Sigma, \sigma^\flat_\Sigma(\vec{\phi})) = C_\Sigma(T_\Sigma, \sigma^\flat_\Sigma(\vec{\psi}))$. Thus, $\widetilde{\lambda}(T)$ is a congruence family on $\mathbf{F}$ and, therefore, $\mathcal{I}$ is congruential. ■

Thus, we have obtained an exact characterization of those π-institutions that are congruential and possess the Deduction Detachment Property with respect to a binary natural transformation $\to^\flat$.

Theorem 1761 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with a binary natural transformation* $\to^\flat: (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a finitary* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is congruential and has the Deduction Detachment Theorem with respect to* $\to^\flat$ *if and only if it is an axiomatic extension of* $\mathcal{I}^0$*.*

Proof: The implication left-to-right is by Theorem 1758. The converse is given by Propositions 1759 and 1760. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. Recall that a Σ-**sequent** is an expression of the form $\Phi \vdash_\Sigma \phi$, where $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. It is **finite** if Φ is a finite set. Moreover, a **Gentzen F-rule** is an expression of the form

$$\langle \{\Phi_i \vdash_\Sigma \phi_i : i \in I\}, \Phi \vdash_\Sigma \phi \rangle,$$

where $\Phi_i \vdash_\Sigma \phi_i$, $i \in I$, and $\Phi \vdash_\Sigma \phi$ are Σ-sequents. We say the rule is **finitary** if I is finite and all sequents in the rule are finite.

Let $\mathbb{L} = \langle \mathcal{A}, D \rangle$ be an **F**-structure, $\Sigma \in |\mathbf{Sign}^\flat|$, $s = \Phi \vdash_\Sigma \phi$ a Σ-sequent and $r = \langle \{\Phi_i \vdash_\Sigma \phi_i : i \in I\}, \Phi \vdash_\Sigma \phi \rangle$ a Gentzen **F**-rule.

- $\mathbb{L}$ **satisfies** s or s is **true** or **valid** or **holds in** $\mathbb{L}$, written $\mathbb{L} \vDash_\Sigma s$, if $\alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi))$;
- $\mathbb{L}$ **satisfies** r or r is **true** or **valid** or **holds in** $\mathbb{L}$, written $\mathbb{L} \vDash_\Sigma r$, if

$$\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi_i)),\ i \in I, \quad \text{imply} \quad \alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

These definitions are extended in the ordinary way to sets of rules and sets of structures.

Let M be a class of **F**-structures. We say that M is a (**finitary**) **Gentzen class** if it is specified by a collection $R = \{R_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ of (finitary) Gentzen **F**-rules (including sequents, viewed as rules with empty sets of premises).

The following examples illustrate the definition.

- The class of all finitary **F**-structures having the Deduction Detachment Theorem with respect to a binary natural transformation $\to^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ in $N^\flat$ is a finitary Gentzen class specified by $R = \{R_\Sigma\}_{\Sigma \in |\mathbf{Sign}|^\flat|}$, where

$$\begin{array}{rcl} R_\Sigma & = & \{\phi, \phi \to^\flat_\Sigma \psi \vdash_\Sigma \psi : \phi, \psi \in \mathrm{SEN}^\flat(\Sigma)\} \\ & & \cup \{\langle \{\Phi, \phi \vdash_\Sigma \psi\}, \Phi \vdash_\Sigma \phi \to^\flat_\Sigma \psi \rangle : \Phi \cup \{\phi, \psi\} \subseteq_f \mathrm{SEN}^\flat(\Sigma)\}. \end{array}$$

- The class of all finitary self extensional **F**-structures is also a finitary Gentzen class specified by $R = \{R_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, with

$$\begin{array}{r} R_\Sigma = \{\langle \{\phi_i \vdash_\Sigma \psi_i,\ \psi_i \vdash_\Sigma \phi_i : i < k\}, \sigma^\flat_\Sigma(\vec{\phi}) \vdash_\Sigma \sigma^\flat_\Sigma(\vec{\psi}) \rangle : \\ \sigma^\flat \in N^\flat, \vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)\}. \end{array}$$

- The class of all finitary congruential **F**-structures is also a finitary Gentzen class specified by $R = \{R_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, with

$$\begin{array}{r} R_\Sigma = \{\langle \{\Phi, \phi_i \vdash_\Sigma \psi_i,\ \Phi, \psi_i \vdash_\Sigma \phi_i : i < k\},\ \Phi, \sigma^\flat_\Sigma(\vec{\phi}) \vdash_\Sigma \sigma^\flat_\Sigma(\vec{\psi}) \rangle : \\ \sigma^\flat \in N^\flat, \Phi \subseteq_f \mathrm{SEN}^\flat(\Sigma), \vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)\}. \end{array}$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\Sigma \in |\mathbf{Sign}^\flat|$ and $r = \langle \{\Phi_i \vdash_\Sigma \phi_i : i \in I\}, \Phi \vdash_\Sigma \phi \rangle$ a finitary Gentzen $\mathbf{F}$-rule. The **accumulation of** r, denoted $\mathrm{acm}(r)$, is the collection of finitary Gentzen $\mathbf{F}$-rules

$$\mathrm{acm}(r) = \{\langle \{X, \Phi_i \vdash_\Sigma \phi_i : i \in I\},\ X, \Phi \vdash_\Sigma \phi \rangle : X \subseteq_f \mathrm{SEN}^\flat(\Sigma)\}.$$

We say that a collection R of Gentzen rules is **accumulative** if it is the union of accumulations. We say that a class M of $\mathbf{F}$-structures is an **accumulative class** if it is a Gentzen class specified by an accumulative collection of Gentzen $\mathbf{F}$-rules.

Note, e.g., that both the class of all finitary $\mathbf{F}$-structures having the Deduction Detachment Theorem with respect to $\rightarrow^\flat$ and the class of congruential finitary $\mathbf{F}$-structures are accumulative classes. On the other hand, the class of of all self extensional finitary $\mathbf{F}$-structures is not accumulative.

It is not difficult to see that satisfaction of Gentzen $\mathbf{F}$-rules is preserved under bilogical morphisms between $\mathbf{F}$-structures.

Proposition 1762 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathbb{L} = \langle \mathcal{A}, D \rangle$*,* $\mathbb{L}' = \langle \mathcal{A}', D' \rangle$ *two* $\mathbf{F}$*-structures,* $\langle H, \gamma \rangle : \mathbb{L} \vdash \mathbb{L}'$ *a bilogical morphism,* $\Sigma \in |\mathbf{Sign}^\flat|$ *and* $r = \langle \{\Phi_i \vdash \phi_i : i \in I\}, \Phi \vdash_\Sigma \phi \rangle$ *a Gentzen* $\mathbf{F}$*-rule. Then*

$$\mathbb{L} \vDash_\Sigma r \quad \textit{iff} \quad \mathbb{L}' \vDash_\Sigma r.$$

Proof: We have, by the definition of satisfaction and that of bilogical morphism, $\mathbb{L} \vDash_\Sigma r$ if and only if

$$\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi_i)),\ i \in I, \quad \text{imply} \quad \alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi)),$$

if and only if

$$\begin{aligned} &\gamma_{F(\Sigma)}(\alpha_\Sigma(\phi_i)) \in D'_{H(F(\Sigma))}(\gamma_{F(\Sigma)}(\alpha_\Sigma(\Phi_i))),\ i \in I, \\ &\qquad\qquad \text{imply} \quad \gamma_{F(\Sigma)}(\alpha_\Sigma(\phi)) \in D'_{H(F(\Sigma))}(\gamma_{F(\Sigma)}(\alpha_\Sigma(\Phi))), \end{aligned}$$

if and only if

$$\alpha'_\Sigma(\phi_i) \in D'_{F'(\Sigma)}(\alpha'_\Sigma(\Phi_i)),\ i \in I, \quad \text{imply} \quad \alpha'_\Sigma(\phi) \in D'_{F'(\Sigma)}(\alpha'_\Sigma(\Phi)),$$

if and only if $\mathbb{L}' \vDash_\Sigma r$. ∎

Additionally, we can show that the accumulation of a Gentzen rule holding in a finitary $\mathbf{F}$-structure $\mathbb{L} = \langle \mathcal{A}, D \rangle$ necessarily holds in all structures of the form $\mathbb{L}^T = \langle \mathcal{A}, D^T \rangle$, where, for all $T \in \mathrm{ThFam}(\mathbb{L})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in D^T_\Sigma(\Phi) \quad \text{iff} \quad \phi \in D_\Sigma(T_\Sigma, \Phi).$$

Lemma 1763 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\Sigma \in |\mathbf{Sign}^\flat|$, $r = \langle \{\Phi_i \vdash_\Sigma \phi_i : i \in I\}, \Phi \vdash_\Sigma \phi \rangle$ *be an* $\mathbf{F}$*-rule and* $\mathbb{L} = \langle \mathcal{A}, D \rangle$ *a finitary* $\mathbf{F}$*-structure. If* $\mathbb{L} \vDash_\Sigma \mathrm{acm}(r)$*, then, for all* $T \in \mathrm{ThFam}(\mathbb{L})$*,* $\mathbb{L}^T \vDash_\Sigma \mathrm{acm}(r)$*.*

Proof: Let $X \subseteq_f \mathrm{SEN}^\flat(\Sigma)$ and assume $\alpha_\Sigma(\phi_i) \in D^T_{F(\Sigma)}(\alpha_\Sigma(X), \alpha_\Sigma(\Phi_i))$, for all $i \in I$. By definition,

$$\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(T_\Sigma, \alpha_\Sigma(X), \alpha_\Sigma(\Phi_i)),\ i \in I.$$

But $\langle F, \alpha \rangle$ is surjective, whence there exists $\Psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\alpha_\Sigma(\Psi) = T_\Sigma$. Therefore, we get

$$\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Psi), \alpha_\Sigma(X), \alpha_\Sigma(\Phi_i)),\ i \in I.$$

By finitarity of $\mathbb{L}$, we get that there exists $\Psi' \subseteq_f \Psi$, such that

$$\alpha_\Sigma(\phi_i) \in D_{F(\Sigma)}(\alpha_\Sigma(\Psi'), \alpha_\Sigma(X), \alpha_\Sigma(\Phi_i)),\ i \in I.$$

By the hypothesis, $\alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Psi'), \alpha_\Sigma(X), \alpha_\Sigma(\Phi))$. Since $\Psi' \subseteq \Psi$, we get $\alpha_\Sigma(\phi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Psi), \alpha_\Sigma(X), \alpha_\Sigma(\Phi))$. Thus,

$$\alpha_\Sigma(\phi) \in D_{F(\Sigma)}(T_\Sigma, \alpha_\Sigma(X), \alpha_\Sigma(\Phi)),$$

i.e., $\alpha_\Sigma(\phi) \in D^T_{F(\Sigma)}(\alpha_\Sigma(X), \alpha_\Sigma(\Phi)$. We conclude that $\mathbb{L}^T \vDash_\Sigma \mathrm{acm}(r)$. ■

Proposition 1764 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* M *an accumulative class of finitary* $\mathbf{F}$*-structures. If* $\mathbb{L} = \langle \mathcal{A}, D \rangle \in \mathsf{M}$*, then, for all* $T \in \mathrm{ThFam}(\mathbb{L})$*,* $\mathbb{L}^T = \langle \mathcal{A}, D^T \rangle \in \mathsf{M}$*.*

Proof: Directly by Lemma 1763. ■

Next, we show that, if $\mathcal{I}$ is an accumulative protoalgebraic finitary π-institution, then all full $\mathcal{I}$-structures satisfy the defining Gentzen $\mathbf{F}$-rules of $\mathcal{I}$.

Theorem 1765 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* M *an accumulative class of finitary* $\mathbf{F}$*-structures. If* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *is a protoalgebraic* π*-institution in* M*, then the full* $\mathcal{I}$*-structures of the form* $\mathbb{L} = \langle \mathcal{A}, D \rangle$*, where* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* F *an isomorphism, are in* M*.*

Proof: Suppose $\mathcal{I} = \langle \mathbf{F}, C \rangle$ is a protoalgebraic π-institution in M. By Proposition 1762, it suffices to show that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with F an isomorphism, $\mathbb{L} = \langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle \in \mathsf{M}$. By protoalgebraicity and Theorem 1577, we get

$$\alpha^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \mathrm{ThFam}(\mathcal{I})^T,$$

where $T = \alpha^{-1}(C^{\mathcal{I},\mathcal{A}}(\varnothing))$. But

$$\langle F,\alpha\rangle : \langle \mathcal{F}, \alpha^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))\rangle \to \langle \mathcal{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\rangle$$

is a bilogical morphism. Since, by the hypothesis and Proposition 1764, we have $\langle \mathcal{F}, \mathrm{ThFam}(\mathcal{I})^T\rangle \in \mathsf{M}$, we get, by Proposition 1762. $\mathbb{L} \in \mathsf{M}$. ∎

Now we get easily the following results concerning the Deduction Detachment Theorem and congruentiality, respectively.

Corollary 1766 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\to^\flat$: $(\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ a binary natural transformations in $N^\flat$, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a finitary π-institution based on $\mathbf{F}$ that has the Deduction Detachment Theorem with respect to $\to^\flat$. Then, every full $\mathcal{I}$-structure of the form $\mathbb{L} = \langle \mathcal{A}, D\rangle$, where $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with F an isomorphism, has the Deduction Detachment Theorem with respect to $\to^\flat$.*

Proof: This follows from Theorem 1765 once it is show that if $\mathcal{I}$ has the Deduction Detachment Property with respect to $\to^\flat$, then it is protoalgebraic. Suppose $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \le T'$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi,\psi\rangle \in \Omega_\Sigma(T)$. Then, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$, we have

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}) \in T_{\Sigma'} \quad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}) \in T_{\Sigma'}.$$

But, by the Deduction Detachment Theorem, this holds if and only if,

$$\begin{array}{l}\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}) \to^\flat_{\Sigma'} \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}),\\ \qquad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}) \to^\flat_{\Sigma'} \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}) \in T_{\Sigma'}.\end{array}$$

Since $T \le T'$, we get that

$$\begin{array}{l}\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}) \to^\flat_{\Sigma'} \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}),\\ \qquad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}) \to^\flat_{\Sigma'} \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}) \in T'_{\Sigma'}.\end{array}$$

Hence, again by the Deduction Detachment Theorem,

$$\sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi),\vec{\chi}) \in T'_{\Sigma'} \quad \text{iff} \quad \sigma^\flat_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi),\vec{\chi}) \in T'_{\Sigma'}.$$

This gives $\langle \phi,\psi\rangle \in \Omega_\Sigma(T')$. Therefore, $\Omega(T) \le \Omega(T')$ and, hence, $\mathcal{I}$ is protoalgebraic. ∎

Corollary 1767 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a protoalgebraic finitary π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is congruential, then the full $\mathcal{I}$-structures of the form $\mathbb{L} = \langle \mathcal{A}, D\rangle$, where $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with F an isomorphism, are also congruential.*

Proof: This follows from Theorem 1765 and the fact that the class of all congruential finitary **F**-structures is accumulative. ∎

Now we look at the converse, in a certain sense, of the inheritance problem of properties specified by Gentzen **F**-rules. Namely, we identify a type of properties that are bequeathed to the π-institution specified by classes of **F**-structures, when all structures in the class satisfy the property.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and M be a class of **F**-structures. Recall that the π-institution $\mathcal{I}^\mathsf{M} = \langle \mathbf{F}, C^\mathsf{M} \rangle$ determined by, or specified by or generated by, M is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}\phi \in C^\mathsf{M}_\Sigma(\Phi) & \text{iff} & \text{for all } \langle \mathcal{A}, D \rangle \in \mathsf{M}, \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma, \Sigma'),\\ & & \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \in D_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))).\end{array}$$

Let, now, $\Sigma \in |\mathbf{Sign}^\flat|$ and $r = \langle \{\Phi_i \vdash_\Sigma \phi_i : i \in I\}, \Phi \vdash_\Sigma \phi \rangle$ be a Gentzen **F**-rule. The **structure of** r, denoted $\mathrm{str}(r)$ is the family of all Gentzen **F**-rules of the form

$$\begin{array}{rl}\mathrm{SEN}^\flat(f)(r) := \langle \{ & \!\!\!\mathrm{SEN}^\flat(f)(\Phi_i) \vdash_{\Sigma'} \mathrm{SEN}^\flat(f)(\phi_i) : i \in I\},\\ & \mathrm{SEN}^\flat(f)(\Phi) \vdash_{\Sigma'} \mathrm{SEN}^\flat(f)(\phi)\rangle,\end{array}$$

where $\Sigma' \in |\mathbf{Sign}^\flat|$ and $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$. We say that a collection R of Gentzen **F**-rules is **structural** if it is the union of structures. We say that a class M of **F**-structures is a **structural class** if it is a Gentzen class specified by a structural collection of Gentzen **F**-rules.

Lemma 1768 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\Sigma \in |\mathbf{Sign}^\flat|$, $r = \langle \{\Phi_i \vdash_\Sigma \phi_i : i \in I\}, \Phi \vdash_\Sigma \phi \rangle$ *a Gentzen* **F***-rule and* M *a class* **F***-structures. If* $\mathsf{M} \vDash \mathrm{str}(r)$*, then* $\mathrm{str}(r)$ *holds in* $\mathcal{I}^\mathsf{M}$.

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$, $r = \langle \{\Phi_i \vdash_\Sigma \phi_i : i \in I\}, \Phi \vdash_\Sigma \phi \rangle$ and suppose $\mathsf{M} \vDash \mathrm{str}(r)$ and $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, such that

$$\mathrm{SEN}^\flat(f)(\phi_i) \in C^\mathsf{M}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi_i)), \ i \in I.$$

Then, by definition of $\mathcal{I}^\mathsf{M}$, for all $\langle \mathcal{A}, D \rangle \in \mathsf{M}$, all $\Sigma'' \in |\mathbf{Sign}^\flat|$ and all $g \in \mathbf{Sign}^\flat(\Sigma', \Sigma'')$,

$$\Sigma \xrightarrow{\ f\ } \Sigma' \xrightarrow{\ g\ } \Sigma''$$

$$\alpha_{\Sigma''}(\mathrm{SEN}^\flat(g)(\mathrm{SEN}^\flat(f)(\phi_i))) \in D_{F(\Sigma'')}(\alpha_{\Sigma''}(\mathrm{SEN}^\flat(g)(\mathrm{SEN}^\flat(f)(\Phi_i)))), \ i \in I,$$

i.e.,

$$\alpha_{\Sigma''}(\mathrm{SEN}^\flat(gf)(\phi_i)) \in D_{F(\Sigma'')}(\alpha_{\Sigma''}(\mathrm{SEN}^\flat(gf)(\Phi_i))), \ i \in I.$$

Since, by hypothesis, $\mathsf{M} \vDash \mathrm{str}(r)$ and $\langle \mathcal{A}, D\rangle \in \mathsf{M}$, we get

$$\alpha_{\Sigma''}(\mathrm{SEN}^\flat(gf)(\phi)) \in D_{F(\Sigma'')}(\alpha_{\Sigma''}(\mathrm{SEN}^\flat(gf)(\Phi)))$$

and, thus,

$$\alpha_{\Sigma''}(\mathrm{SEN}^\flat(g)(\mathrm{SEN}^\flat(f)(\phi))) \in D_{F(\Sigma'')}(\alpha_{\Sigma''}(\mathrm{SEN}^\flat(g)(\mathrm{SEN}^\flat(f)(\Phi)))).$$

By the definition of $\mathcal{I}^{\mathsf{M}}$, we conclude that $\mathrm{SEN}^\flat(f)(\phi) \in C^{\mathsf{M}}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi))$. Therefore, $\mathcal{I}^{\mathsf{M}} \vDash \mathrm{str}(r)$. ■

Theorem 1769 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* P *a structural class* $\mathbf{F}$*-structures. If* $\mathsf{M} \subseteq \mathsf{P}$*, then* $\mathcal{I}^{\mathsf{M}} \in \mathsf{P}$*.*

Proof: Suppose that $\mathrm{str}(r)$ is a rule of P. Since $\mathsf{M} \subseteq \mathsf{P}$, $\mathrm{str}(r)$ is a rule of M. Therefore, by Lemma 1768, $\mathrm{str}(r)$ is a rule of $\mathcal{I}^{\mathsf{M}}$. Thus, $\mathcal{I}^{\mathsf{M}}$ satisfies all Gentzen $\mathbf{F}$-rules determining P (since all of them are, by hypothesis, structural) and, therefore, $\mathcal{I}^{\mathsf{M}} \in \mathsf{P}$. ■

An application of Theorem 1769 gives that, if all $\mathbf{F}$-structures in a class M are congruential, then the π-institution determined by the class is also congruential.

Corollary 1770 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* M *a class of congruential* $\mathbf{F}$*-structures. Then* $\mathcal{I}^{\mathsf{M}}$ *is congruential.*

Proof: It suffices, by Theorem 1769 to show that the class of congruential $\mathbf{F}$-structures is a structural class. This is easily seen by observing that it is the class of $\mathbf{F}$-structures specified by $R = \{R_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, with

$$\begin{aligned} R_\Sigma = \{\langle \{\Phi, \phi_i \vdash_\Sigma \psi_i,\ \Phi, \psi_i \vdash_\Sigma \phi_i : i < k\},\ \Phi, \sigma^\flat_\Sigma(\vec{\phi}) \vdash_\Sigma \sigma^\flat_\Sigma(\vec{\psi})\rangle :\\ \sigma^\flat \in N^\flat, \Phi \subseteq \mathrm{SEN}^\flat(\Sigma), \vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)\}.\end{aligned}$$

It is easy to check that R is a structural class of Gentzen $\mathbf{F}$-rules, whence the class of all congruential $\mathbf{F}$-structures is a structural class. ■

23.5 Fregeanity and Congruence Orderability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a $\top^\flat$-pointed guasivariety of $\mathbf{F}$-algebraic systems.

We say that K is **congruence orderable** if, for all $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle \in \mathsf{K}$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{aligned} \phi = \psi \quad \text{if} \quad & \text{for all } \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma, \Sigma'),\\ & \Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}^\flat(f)(\phi), \top^{\mathcal{A}}_{\Sigma'}) = \Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}^\flat(f)(\psi), \top^{\mathcal{A}}_{\Sigma'}).\end{aligned}$$

Moreover, we say that K is **Fregean** if it is both relatively point regular and congruence orderable.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, and K a $\top^\flat$-pointed guasivariety of $\mathbf{F}$-algebraic systems. For every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, define the relation family

$$\leq^{\mathsf{K},\mathcal{A}} = \{\leq^{\mathsf{K},\mathcal{A}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$$

on $\mathcal{A}$ by letting, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll} \phi \leq^{\mathsf{K},\mathcal{A}}_\Sigma \psi & \text{iff} & \text{for all } \Sigma' \in |\mathbf{Sign}, f \in \mathbf{Sign}(\Sigma, \Sigma'), \\ & & \Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}(f)(\phi), \top^{\mathcal{A}}_{\Sigma'}) \geq \Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}(f)(\psi), \top^{\mathcal{A}}_{\Sigma'}). \end{array}$$

We show that $\leq^{\mathsf{K},\mathcal{A}}$ is in fact a quasiordering system (**qosystem**, for short) on $\mathcal{A}$.

Proposition 1771 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* K *a* $\top^\flat$*-pointed guasivariety of* $\mathbf{F}$*-algebraic systems. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\leq^{\mathsf{K},\mathcal{A}}$ *is a quasiordering system on* $\mathcal{A}$.

Proof: Let $\Sigma \in |\mathbf{Sign}|$. Since, for all $\phi \in \mathrm{SEN}(\Sigma)$, all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}(f)(\phi), \top^{\mathcal{A}}_{\Sigma'}) = \Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}(f)(\phi), \top^{\mathcal{A}}_{\Sigma'})$, we get that $\phi \leq^{\mathsf{K},\mathcal{A}}_\Sigma \phi$ and $\leq^{\mathsf{K},\mathcal{A}}$ is reflexive. Since, for all $\phi, \psi, \chi \in \mathrm{SEN}(\Sigma)$, if $\phi \leq^{\mathsf{K},\mathcal{A}}_\Sigma \psi$ and $\psi \leq^{\mathsf{K},\mathcal{A}}_\Sigma \chi$ imply, by definition, that, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

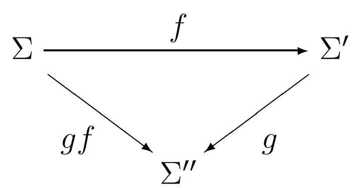
$$\Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}(f)(\phi), \top^{\mathcal{A}}_{\Sigma'}) \geq \Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}(f)(\psi), \top^{\mathcal{A}}_{\Sigma'}) \geq \Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}(f)(\chi), \top^{\mathcal{A}}_{\Sigma'}),$$

we, get, again by definition, $\phi \leq^{\mathsf{K},\mathcal{A}}_\Sigma \chi$. Thus, $\leq^{\mathsf{K},\mathcal{A}}$ is also transitive.

Finally, suppose $\phi \leq^{\mathsf{K},\mathcal{A}}_\Sigma \psi$ and let $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$. Then, by definition, for all $\Sigma'' \in |\mathbf{Sign}|$ and all $h \in \mathbf{Sign}(\Sigma, \Sigma'')$, we get

$$\Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}(h)(\phi), \top^{\mathcal{A}}_{\Sigma''}) \geq \Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}(h)(\psi), \top^{\mathcal{A}}_{\Sigma''}).$$

In particular, for all $\Sigma'' \in |\mathbf{Sign}|$ and all $g \in \mathbf{Sign}(\Sigma', \Sigma'')$,



$$\Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}(g)(\mathrm{SEN}(f)(\phi)), \top^{\mathcal{A}}_{\Sigma''}) \geq \Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}(g)(\mathrm{SEN}(f)(\psi)), \top^{\mathcal{A}}_{\Sigma''}),$$

i.e., $\mathrm{SEN}(f)(\phi) \leq^{\mathsf{K},\mathcal{A}}_{\Sigma'} \mathrm{SEN}(f)(\psi)$ and $\leq^{\mathsf{K},\mathcal{A}}$ is a system. ■

Corollary 1772 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* K *a* $\top^\flat$*-pointed guasivariety of* $\mathbf{F}$*-algebraic systems. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, the qosystem* $\leq^{\mathsf{K},\mathcal{A}}$ *is a posystem if and only if* K *is congruence orderable.*

Proof: Clear, by Proposition 1772 and the definitions of $\leq^{\mathsf{K},\mathcal{A}}$ and of congruence orderability. ■

Recall the assertional π-institution $\mathcal{I}^{\mathsf{K},\top}$ associated with a $\top^\flat$-pointed guasivariety K of $\mathbf{F}$-algebraic systems. Recall, also, that, if $\mathcal{I}^{\mathsf{K},\top}$ is family regular, protoalgebraic, with $\top^\flat$ a natural theorem, then the guasivariety K is relatively point regular.

We show, next, that, if $\mathcal{I}^{\mathsf{K},\top}$ is strongly Fregean, protoalgebraic, with $\top^\flat$ a natural theorem, then it is also family regular. Thus, the property of being strongly Freagean, protoalgebraic, with $\top^\flat$ a natural theorem is stronger than being family regular, protoalgebraic, with $\top^\flat$ a natural theorem. In terms of the $\top^\flat$-pointed guasivariety K, this is reflected, as we shall see in the following theorem, in the fact that, in additional to being relatively point regular, it is also congruence orderable, i.e., it is a Fregean guasivariety of $\mathbf{F}$-algebraic systems.

Lemma 1773 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* K *a* $\top^\flat$*-pointed guasivariety of* $\mathbf{F}$*-algebraic systems. If* $\mathcal{I}^{\mathsf{K},\top} = \langle \mathbf{F}, C^{\mathsf{K},\top} \rangle$ *is Fregean, then it is also family regular.*

Proof: Suppose $\mathcal{I}^{\mathsf{K},\top}$ is Fregean. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$ and consider the theory family $C^{\mathsf{K},\top}(\phi, \psi)$. We have, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$C^{\mathsf{K},\top}_{\Sigma'}(C^{\mathsf{K},\top}_{\Sigma'}(\phi,\psi), \mathrm{SEN}^\flat(f)(\phi)) = C^{\mathsf{K},\top}_{\Sigma'}(\phi,\psi) = C^{\mathsf{K},\top}_{\Sigma'}(C^{\mathsf{K},\top}_{\Sigma'}(\phi,\psi), \mathrm{SEN}^\flat(f)(\psi)).$$

Therefore, we get

$$\begin{array}{rcll}\langle \phi, \psi \rangle & \in & \widetilde{\Lambda}_\Sigma(C^{\mathsf{K},\top}(\phi,\psi)) & \\ & = & \widetilde{\Omega}_\Sigma(C^{\mathsf{K},\top}(\phi,\psi)) & \text{(by Fregeanity)} \\ & \subseteq & \Omega_\Sigma(C^{\mathsf{K},\top}(\phi,\psi)). & \end{array}$$

This shows that $\mathcal{I}^{\mathsf{K},\top}$ is family regular. ■

Theorem 1774 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\top^\flat : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, and* K *a* $\top^\flat$*-pointed guasivariety of* $\mathbf{F}$*-algebraic systems. If* $\mathcal{I}^{\mathsf{K},\top} = \langle \mathbf{F}, C^{\mathsf{K},\top} \rangle$ *is Fregean, protoalgebraic, with* $\top^\flat$ *a natural theorem, then* K *is Fregean.*

Proof: Since $\mathcal{I}^{\mathsf{K},\top}$ is Fregean, by Lemma 1774, it is family regular. Since $\mathcal{I}^{\mathsf{K},\top}$ is family regular, protoalgebraic, with $\top^\flat$ a natural theorem, by Theorem 1356, K is a relatively point regular guasivariety of **F**-algebraic systems. Thus, to show that K ir Fregean, it suffices, by definition, to show that it is also congruence orderable.

To this end, assume that $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle \in \mathsf{K}$, $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}(f)(\phi), \top^{\mathcal{A}}_{\Sigma'}) = \Theta^{\mathsf{K},\mathcal{A}}(\mathrm{SEN}(f)(\psi), \top^{\mathcal{A}}_{\Sigma'}).$$

This is equivalent to asserting that

$$C_\Sigma^{\mathcal{I}^{\mathsf{K},\top},\mathcal{A}}(\phi) = C_\Sigma^{\mathcal{I}^{\mathsf{K},\top},\mathcal{A}}(\psi).$$

Thus, we obtain

$$\begin{array}{lll}\langle \phi, \psi\rangle & \in & \widetilde{\Lambda}^{\mathcal{A}}_\Sigma(\mathrm{FiFam}^{\mathcal{I}^{\mathsf{K},\top}}(\mathcal{A})) \quad \text{(definition of Frege relation)}\\ & = & \widetilde{\Omega}^{\mathcal{A}}_\Sigma(\mathrm{FiFam}^{\mathcal{I}^{\mathsf{K},\top}}(\mathcal{A})) \quad \text{(Fregeanity)}\\ & = & \Omega^{\mathcal{A}}_\Sigma(\{\top^{\mathcal{A}}\}) \quad \text{(protoalgebraicity)}\\ & = & \Delta^{\mathcal{A}}_\Sigma.\end{array}$$

We conclude that $\phi = \psi$ and, therefore, K is also congruence orderable. ∎

To conclude the section, we would like to prove the converse of Theorem 1774, i.e., that, if K is a Fregean class of **F**-algebraic systems, then the assertional π-institution $\mathcal{I}^{\mathsf{K},top}$ of K is a Fregean, protoalgebraic π-institution with $\top^\flat$ a natural theorem. Parts of the conclusion, we have already obtained in Theorem 1356. To obtain the full conclusion, we work towards the only remaining subgoal, expressed in the following proposition.

Proposition 1775 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* K *a* $\top^\flat$*-pointed class of* **F***-algebraic systems. If* K *is Fregean, then the assertional* π*-institution* $\mathcal{I}^{\mathsf{K},\top} = \langle \mathbf{F}, C^{\mathsf{K},\top}\rangle$ *of* K *is Fregean.*

Proof: Suppose K is a Fregean guasivariety of **F**-algebraic systems, i.e., relatively point regular and congruence orderable. We must show that, for all $T \in \mathrm{ThFam}(\mathcal{I}^{\mathsf{K},\top})$, $\widetilde{\Lambda}^{\mathcal{I}^{\mathsf{K},\top}}(T) = \widetilde{\Omega}^{\mathcal{I}^{\mathsf{K},\top}}(T)$. Let $T \in \mathrm{ThFam}(\mathcal{I}^{\mathsf{K},\top})$. Since $\widetilde{\Omega}^{\mathcal{I}^{\mathsf{K},\top}}(T) \leq \widetilde{\Lambda}^{\mathcal{I}^{\mathsf{K},\top}}(T)$ always holds, it suffices to show the reverse inclusion, i.e., that $\widetilde{\Lambda}^{\mathcal{I}^{\mathsf{K},\top}}(T) \leq \widetilde{\Omega}^{\mathcal{I}^{\mathsf{K},\top}}(T)$. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi\rangle \notin \widetilde{\Omega}^{\mathcal{I}^{\mathsf{K},\top}}_\Sigma(T)$. Equivalently, $\phi/\widetilde{\Omega}^{\mathcal{I}^{\mathsf{K},\top}}_\Sigma(T) \neq \psi/\widetilde{\Omega}^{\mathcal{I}^{\mathsf{K},\top}}_\Sigma(T)$. Let us denote, for the sake of brevity $\theta := \widetilde{\Omega}^{\mathcal{I}^{\mathsf{K},\top}}(T)$. Then, by Lemma 1351 and Proposition 1352, $\mathcal{F}/\theta \in \mathsf{K}$. Thus,y congruence orderability, there exists $\Sigma' \in |\mathbf{Sign}^\flat|$ and $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, such that

$$\Theta^{\mathsf{K},\mathcal{F}/\theta}(\mathrm{SEN}^\flat(f)(\phi)/\theta_{\Sigma'}, \top^{\mathcal{F}/\theta}_{\Sigma'}) \neq \Theta^{\mathsf{K},\mathcal{F}/\theta}(\mathrm{SEN}^\flat(f)(\psi)/\theta_{\Sigma'}, \top^{\mathcal{F}/\theta}_{\Sigma'}).$$

Thus, by relative point regularity, we must have

$$\top^{\mathcal{F}/\theta}/\Theta^{\mathsf{K},\mathcal{F}/\theta}(\mathrm{SEN}^\flat(f)(\phi)/\theta_{\Sigma'},\top_{\Sigma'}^{\mathcal{F}/\theta}) \neq \top^{\mathcal{F}/\theta}/\Theta^{\mathsf{K},\mathcal{F}/\theta}(\mathrm{SEN}^\flat(f)(\psi)/\theta_{\Sigma'},\top_{\Sigma'}^{\mathcal{F}/\theta}).$$

This gives that

$$C_\Sigma^{\mathsf{K},\top}(\mathrm{SEN}^\flat(f)(\phi),\top^\flat_{\Sigma'}/\Omega_{\Sigma'}(T)) \neq C_\Sigma^{\mathsf{K},\top}(\mathrm{SEN}^\flat(f)(\phi),\top^\flat_{\Sigma'}/\Omega_{\Sigma'}(T)),$$

which translates to $\langle\phi,\psi\rangle \notin \widetilde{\Lambda}_\Sigma^{\mathcal{I}^{\mathsf{K},\top}}(T)$. We conclude that $\mathcal{I}^{\mathsf{K},\top}$ is Fregean. ■

Finally, putting this together, we get

Theorem 1776 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and K a $\top^\flat$-pointed class of $\mathbf{F}$-algebraic systems. If K is Fregean, then $\mathcal{I}^{\mathsf{K},\top} = \langle\mathbf{F},C^{\mathsf{K},\top}\rangle$ is a Fregean, protoalgebraic π-institution, with $\top^\flat$ a natural theorem.*

Proof: By Proposition 1348, $\top^\flat$ is a natural theorem of $\mathcal{I}^{\mathsf{K},\top}$. By Proposition 1352, $\mathcal{I}^{\mathsf{K},\top}$ is protoalgebraic. Finally, by Proposition 1775, $\mathcal{I}^{\mathsf{K},\top}$ is Fregean. ■

The main result proven in this section is summarized in

Theorem 1777 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and K a $\top^\flat$-pointed class of $\mathbf{F}$-algebraic systems. K is Fregean if and only if $\mathcal{I}^{\mathsf{K},\top} = \langle\mathbf{F},C^{\mathsf{K},\top}\rangle$ is a Fregean, protoalgebraic π-institution, with $\top^\flat$ a natural theorem.*

Proof: The "if" by Theorem 1774. The "only if" by Theorem 1776. ■

Chapter 24

Special Topics

24.1 Rule Based π-Institutions

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system.

An **F-rule** is a pair $\langle P, \rho \rangle$, where $P \cup \{\rho\} : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ is a finite set of natural transformations in $N^\flat$. If $P = \varnothing$, then $\langle \varnothing, \rho \rangle$ is called an **F-axiom** and it is ordinarily identified with ρ.

Let $R = \langle P, \rho \rangle$ be an **F**-rule, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. We say ϕ R-**follows from** Φ, written $\Phi \to^R_\Sigma \phi$, if there exists $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$P_\Sigma(\vec{\chi}) \subseteq \Phi \quad \text{and} \quad \rho_\Sigma(\vec{\chi}) = \phi.$$

Consider, now, a set $\mathcal{R}$ of **F**-rules. For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, we say ϕ is $\mathcal{R}$-**provable from** Φ, written $\phi \in C^{\mathcal{R}}_\Sigma(\Phi)$ or $\Phi \vdash^{\mathcal{R}}_\Sigma \phi$, if there exists a sequence

$$\phi_0, \phi_1, \phi_2, \dots, \phi_{n-1}, \phi_n$$

in $\mathrm{SEN}^\flat(\Sigma)$, such that $\phi_n = \phi$ and, for all $i \le n$,

- $\phi_i \in \Phi$ or
- ϕ_i R-follows from $\{\phi_0, \phi_1, \dots, \phi_{i-1}\}$, for some $R \in \mathcal{R}$.

A sequence $\phi_0, \phi_1, \dots, \phi_n$ witnessing $\Phi \vdash^{\mathcal{R}}_\Sigma \phi$ is called an $\mathcal{R}$-**proof** of ϕ from Φ.

We show that $C^{\mathcal{R}}$, as defined here, is indeed a closure system on the base algebraic system **F**.

Proposition 1778 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{R}$ *a collection of* **F***-rules. Then* $C^{\mathcal{R}} = \{C^{\mathcal{R}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ *is a closure system on* **F**.

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \Psi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$.

(i) If $\phi \in \Phi$, then ϕ is an $\mathcal{R}$-proof of ϕ from Φ. So $\phi \in C^{\mathcal{R}}_\Sigma(\Phi)$ and $C^{\mathcal{R}}$ is inflationary.

(ii) If $\Phi \subseteq \Psi$ and $\phi \in C^{\mathcal{R}}_\Sigma(\Phi)$, then, there exists an $\mathcal{R}$-proof of ϕ from Φ. The same sequence is then an $\mathcal{R}$-proof of ϕ from Ψ. So $\phi \in C^{\mathcal{R}}_\Sigma(\Psi)$ and $C^{\mathcal{R}}$ is monotone.

(iii) Suppose $\phi \in C^{\mathcal{R}}_\Sigma(C^{\mathcal{R}}_\Sigma(\Phi))$. Then, there exists an $\mathcal{R}$-proof of ϕ from $C^{\mathcal{R}}_\Sigma(\Phi)$, say

$$\phi_0, \phi_1, \dots, \phi_{n-1}, \phi_n = \phi.$$

Then, for each $\phi_i \in C^{\mathcal{R}}_\Sigma(\Phi)$, there exists an $\mathcal{R}$-proof of ϕ_i from Φ. For each such ϕ_i, we insert its $\mathcal{R}$-proof from Φ in its place in the sequence. The new sequence is an $\mathcal{R}$-proof of ϕ from Φ. Thus, we get that $\phi \in C^{\mathcal{R}}_\Sigma(\Phi)$ and $C^{\mathcal{R}}$ is also idempotent.

(iv) Finally, it remains to show structurality. Let $\Sigma,\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C^{\mathcal{R}}_\Sigma(\Phi)$. Let $\phi_0,\phi_1,\ldots,\phi_{n-1},\phi_n = \phi$ be an $\mathcal{R}$-proof of ϕ from Φ. We consider the sequence

$$\mathrm{SEN}^\flat(f)(\phi_0),\mathrm{SEN}^\flat(f)(\phi_1),\ldots,\mathrm{SEN}^\flat(f)(\phi_{n-1}),\mathrm{SEN}^\flat(f)(\phi_n).$$

Then $\mathrm{SEN}^\flat(f)(\phi_n) = \mathrm{SEN}^\flat(f)(\phi)$ and, moreover, for all $i \le n$, if $\phi_i \in \Phi$, then $\mathrm{SEN}^\flat(f)(\phi_i) \in \mathrm{SEN}^\flat(f)(\Phi)$, and, if ϕ_i R-follows from $\{\phi_0,\phi_1,\ldots,\phi_{i-1}\}$, for some $R \in \mathcal{R}$, then $\mathrm{SEN}^\flat(f)(\phi_i)$ R-follows from $\{\mathrm{SEN}^\flat(f)(\phi_0),\mathrm{SEN}^\flat(f)(\phi_1),\ldots,\mathrm{SEN}^\flat(f)(\phi_{i-1})\}$ because of the naturality of R. So, the displayed sequence is an $\mathcal{R}$-proof of $\mathrm{SEN}^\flat(f)(\phi)$ from $\mathrm{SEN}^\flat(f)(\Phi)$ and $C^{\mathcal{R}}$ is also structural.

We conclude that $C^{\mathcal{R}}$ is a closure system on $\mathbf{F}$. ■

We denote by $\mathcal{I}^{\mathcal{R}} = \langle \mathbf{F}, C^{\mathcal{R}}\rangle$ the π-institution corresponding to $C^{\mathcal{R}}$.

In general, given a π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$, we say that $\mathcal{I}$ is **rule based** if there exists a collection $\mathcal{R}$ of $\mathbf{F}$-rules, such that $C = C^{\mathcal{R}}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{R}$ a collection of $\mathbf{F}$-rules, $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, an $\mathbf{F}$-algebraic system and $T \in \mathrm{SenFam}(\mathcal{A})$. We say that T is **closed under** $\mathcal{R}$ or is $\mathcal{R}$-**closed** if, for all $R = \langle P,\rho\rangle \in \mathcal{R}$, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma)$,

$$P^{\mathcal{A}}_\Sigma(\vec{\chi}) \subseteq T_\Sigma \quad \text{implies} \quad \rho^{\mathcal{A}}_\Sigma(\vec{\chi}) \in T_\Sigma.$$

This terminology allows the following elegant characterization of $\mathcal{I}^{\mathcal{R}}$-filter families of $\mathcal{A}$.

Proposition 1779 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{R}$ *a collection of* $\mathbf{F}$*-rules,* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*, an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{SenFam}(\mathcal{A})$*. Then* $T \in \mathrm{FiFam}^{\mathcal{I}^{\mathcal{R}}}(\mathcal{A})$ *if and only if* T *is* $\mathcal{R}$*-closed.*

Proof: Assume, first, that $T \in \mathrm{FiFam}^{\mathcal{I}^{\mathcal{R}}}(\mathcal{A})$, $R = \langle P,\rho\rangle \in \mathcal{R}$ and, using surjectivity of $\langle F,\alpha\rangle$, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$P^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi})) \subseteq T_{F(\Sigma)}.$$

Then we get $\alpha_\Sigma(P_\Sigma(\vec{\chi})) \subseteq T_{F(\Sigma)}$. Since, by the definition of $C^{\mathcal{I}^{\mathcal{R}}}$, $\rho_\Sigma(\vec{\chi}) \in C^{\mathcal{I}^{\mathcal{R}}}_\Sigma(P_\Sigma(\vec{\chi}))$ and, by hypothesis, $T \in \mathrm{FiFam}^{\mathcal{I}^{\mathcal{R}}}(\mathcal{A})$, we get $\alpha_\Sigma(\rho_\Sigma(\vec{\chi})) \in T_{F(\Sigma)}$ or, equivalently, $\rho^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\chi})) \in T_{F(\Sigma)}$. Thus, T is $\mathcal{R}$-closed.

Suppose, conversely, that $T \in \mathrm{SenFam}(\mathcal{A})$ is $\mathcal{R}$-closed. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in C^{\mathcal{I}^{\mathcal{R}}}_\Sigma(\Phi)$ and consider $\Sigma' \in |\mathbf{Sign}^\flat|$ and $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$, such that

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq T_{F(\Sigma')}.$$

Since $\phi \in C_\Sigma^{\mathcal{I}^\mathcal{R}}(\Phi)$, there exists an $\mathcal{R}$-proof of ϕ from Φ, say

$$\phi_0, \phi_1, \dots, \phi_{n-1}, \phi_n = \phi.$$

We prove by induction on $i \le n$ that, every member of the sequence

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi_0)), \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi_1)), \dots, \\ \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi_{n-1})), \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi_n))$$

belongs to $T_{F(\Sigma')}$. The case $i = n$, will yield the desired conclusion.

First, if $\phi_i \in \Phi$, then $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi_i)) \in \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq T_{F(\Sigma')}$, where the latter inclusion holds by hypothesis.

Suppose, on the other hand, that ϕ_i R-follows from $\{\phi_0, \phi_1, \dots, \phi_{i-1}\}$, for some $R = \langle P, \rho\rangle \in \mathcal{R}$. Thus, there exists $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$P_\Sigma(\vec{\chi}) \subseteq \{\phi_0, \phi_1, \dots, \phi_{i-1}\} \quad \text{and} \quad \rho_\Sigma(\vec{\chi}) = \phi_i.$$

But then

$$\begin{array}{rcl} P^\mathcal{A}_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\chi}))) & = & \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(P_\Sigma(\vec{\chi}))) \\ & \subseteq & \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\{\phi_0, \dots, \phi_{i-1}\})) \\ & \subseteq & T_{F(\Sigma')}, \end{array}$$

where the last inclusion follows by the induction hypothesis, and, hence, since T is $\mathcal{R}$-closed, we get that $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi_i)) = \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\rho_\Sigma(\vec{\chi}))) = \rho^\mathcal{A}_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\chi}))) \in T_{F(\Sigma')}$. This concludes the induction step and shows that, for all $i \le n$, $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi_i)) \in T_{F(\Sigma')}$. ■

In addition, we can characterize $\mathcal{I}^\mathcal{R}$-filter families generated by a given sentence family as follows.

Proposition 1780 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $\mathcal{R}$ *a collection of* $\mathbf{F}$*-rules,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$*, an* $\mathbf{F}$*-algebraic system and* $X \in \mathrm{SenFam}(\mathcal{A})$*. Then, for all* $\Sigma \in |\mathbf{Sign}|$*,*

$$C_\Sigma^{\mathcal{I}^\mathcal{R}, \mathcal{A}}(X) = \{\phi \in \mathrm{SEN}(\Sigma) : X_\Sigma \vdash_\Sigma^\mathcal{R} \phi\}.$$

Proof: Define $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, by letting, for all $\Sigma \in |\mathbf{Sign}|$,

$$T_\Sigma = \{\phi \in \mathrm{SEN}(\Sigma) : X_\Sigma \vdash_\Sigma^\mathcal{R} \phi\}.$$

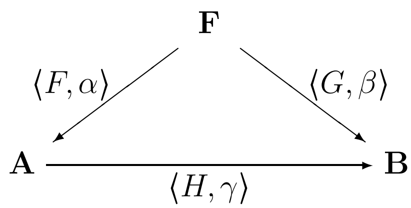
It is not difficult to see that $X \le T$ and T is $\mathcal{R}$-closed. Thus, by Proposition 1779, $C^{\mathcal{I}^\mathcal{R}, \mathcal{A}}(X) \le T$. On the other hand, if $T' \in \mathrm{SenFam}(\mathcal{I})$ contains X and is $\mathcal{R}$-closed, then $T \le T'$. Therefore, we conclude that $T \le C^{\mathcal{I}^\mathcal{R}, \mathcal{A}}(X)$. Equality now follows. ■

24.2 Operators on Classes of Matrix Families

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system. Recall that an $\mathbf{F}$-algebraic system is a pair $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, where $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ is an $N^\flat$-algebraic system and $\langle F, \alpha \rangle : \mathbf{F} \to \mathbf{A}$ is a surjective $N^\flat$-algebraic system morphism. Recall, also, that an $\mathbf{F}$-matrix family is a pair $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, where $\mathcal{A}$ is an $\mathbf{F}$-algebraic system and $T \in \mathrm{SenFam}(\mathbf{A})$ is a sentence family on $\mathbf{A}$.

We define now some class operators on classes of $\mathbf{F}$-matrix families, i.e., operators that, given, as input a class of $\mathbf{F}$-matrix families, produce a new class of $\mathbf{F}$-matrix families.

Given $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$, and $\mathbf{F}$-matrix families $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ and $\mathfrak{B} = \langle \mathcal{B}, T' \rangle$, we say that $\mathfrak{B}$ is a **morphic image** of $\mathfrak{A}$ and write $\mathfrak{B} \in \mathbb{M}(\mathfrak{A})$, if there exists a surjective morphism $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ (that is, such that $\langle G, \beta \rangle = \langle H, \gamma \rangle \circ \langle F, \alpha \rangle$)



such that

$$\gamma^{-1}(T') = T.$$

In this case, we call $\mathfrak{A}$ an **inverse morphic image** or a **morphic preimage** of $\mathfrak{B}$ and write $\mathfrak{A} \in \mathbb{M}^{-1}(\mathfrak{B})$.

Given a class M of $\mathbf{F}$-matrix families, we write $\mathfrak{B} \in \mathbb{M}(\mathsf{M})$ if there exists $\mathfrak{A} \in \mathsf{M}$, such that $\mathfrak{B} \in \mathbb{M}(\mathfrak{A})$.

Similarly, we write $\mathfrak{A} \in \mathbb{M}^{-1}(\mathsf{M})$ if there exists $\mathfrak{B} \in \mathsf{M}$, such that $\mathfrak{A} \in \mathbb{M}^{-1}(\mathfrak{B})$.

It is not difficult to show that both $\mathbb{M}$ and $\mathbb{M}^{-1}$ are closure operators on the collection of all $\mathbf{F}$-matrix families.

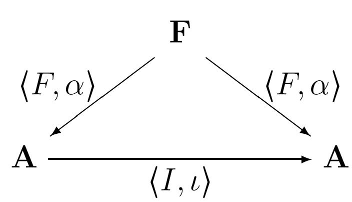
Lemma 1781 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system. Then*

$$\mathbb{M}, \mathbb{M}^{-1} : \mathcal{P}(\mathrm{MatFam}(\mathbf{F})) \to \mathcal{P}(\mathrm{MatFam}(\mathbf{F}))$$

are closure operators on $\mathrm{MatFam}(\mathbf{F})$.

Proof: We prove the statement for $\mathbb{M}$ in detail. The proof for $\mathbb{M}^{-1}$ is similar.

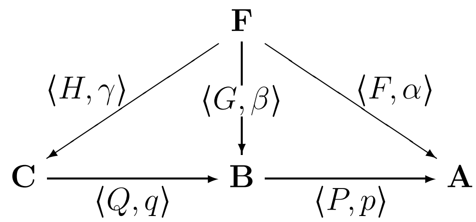
Suppose, first, that M is a class of $\mathbf{F}$-matrix families and $\mathfrak{A} \in \mathsf{M}$. Then, the diagram



where $\langle I, \iota\rangle : \mathcal{A} \to \mathcal{A}$ is the identity morphism, shows that $\mathfrak{A} \in \mathbb{M}(\mathsf{M})$. Therefore, $\mathbb{M}$ is inflationary.

Monotonicity is obvious, since, if M, N are classes of $\mathbf{F}$-matrix families, such that $\mathsf{M} \subseteq \mathsf{N}$, and $\mathfrak{A} \in \mathbb{M}(\mathsf{M})$, then, by definition, $\mathfrak{A} \in \mathbb{M}(\mathfrak{B})$, with $\mathcal{B} \in \mathsf{M}$. But then, since $\mathsf{M} \subseteq \mathsf{N}$, $\mathfrak{A} \in \mathbb{M}(\mathfrak{B})$, with $\mathcal{B} \in \mathsf{N}$ and, again, by definition, $\mathfrak{A} \in \mathbb{M}(\mathsf{N})$. Thus, we have $\mathbb{M}(\mathsf{M}) \subseteq \mathbb{M}(\mathsf{N})$.

Finally, assume that M is a class of $\mathbf{F}$-matrix families and $\mathfrak{A} \in \mathbb{M}(\mathbb{M}(\mathsf{M}))$. Then, there exists $\mathfrak{B} \in \mathbb{M}(\mathsf{M})$, such that $\mathfrak{A} \in \mathbb{M}(\mathfrak{B})$. Furthermore, there exists $\mathfrak{C} \in \mathsf{M}$, such that $\mathfrak{B} \in \mathbb{M}(\mathfrak{C})$. But these two statements combined reveal the existence of the following diagram, in which the two small triangles commute.



As a result, the big triangle also commutes and this ensures that $\mathfrak{A} \in \mathbb{M}(\mathfrak{C})$, which yields $\mathfrak{A} \in \mathbb{M}(\mathsf{M})$. ■

Next, we introduce another class operator on classes of $\mathbf{F}$-matrix families.

Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $\mathfrak{A}^i = \langle \mathcal{A}, T^i\rangle$, $i \in I$, a collection of $\mathbf{F}$-matrix families, all over $\mathcal{A}$. Define the **intersection** of the $\mathfrak{A}^i$, $i \in I$, as the $\mathbf{F}$-matrix family, with the same underlying $\mathbf{F}$-algebraic system $\mathcal{A}$ and with filter family the intersection of the T^i's; more formally

$$\bigcap_{i\in I} \mathfrak{A}^i = \langle \mathcal{A}, \bigcap_{i\in I} T^i\rangle.$$

Given a class M of $\mathbf{F}$-matrix families and an $\mathbf{F}$-matrix family $\mathfrak{B}$, we write $\mathfrak{B} \in \mathbb{I}(\mathsf{M})$ if $\mathfrak{B}$ is the intersection of members of M, i.e., $\mathfrak{B} = \bigcap_{i\in I} \mathfrak{A}^i$, with $\mathfrak{A}^i \in \mathsf{M}$, for all $i \in I$.

Again, it is not difficult to show that $\mathbb{I}$ is a closure operator on the collection of $\mathbf{F}$-matrix families.

Lemma 1782 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system. Then*

$$\mathbb{I} : \mathcal{P}(\mathrm{MatFam}(\mathbf{F})) \to \mathcal{P}(\mathrm{MatFam}(\mathbf{F}))$$

is a closure operator on $\mathrm{MatFam}(\mathbf{F})$.

Proof: To show inflationarity, notice that, trivially, for all $\mathfrak{A} \in \mathsf{M}$, $\mathfrak{A} = \bigcap\{\mathfrak{A}\}$, whence $\mathfrak{A} \in \mathbb{I}(\mathsf{M})$.

Monotonicity is straightforward, since, if $\mathsf{M} \subseteq \mathsf{N}$ and $\mathfrak{A} \in \mathbb{I}(\mathsf{M})$, then $\mathfrak{A} = \bigcap_{i\in I} \mathfrak{A}^i$, with $\mathfrak{A}^i \in \mathsf{M}$, for all $i \in I$, and, hence, $\mathfrak{A} = \bigcap_{i\in I} \mathfrak{A}^i$, with $\mathfrak{A}^i \in \mathsf{N}$, for all $i \in I$. So $\mathfrak{A} \in \mathbb{I}(\mathsf{N})$.

Finally, for transitivity, if $\mathfrak{A} \in \mathbb{I}\mathbb{I}(\mathbb{I}\mathbb{I}(\mathsf{M}))$, then $\mathfrak{A} = \bigcap_{i\in I}\mathfrak{A}^i$, where $\mathfrak{A}^i \in \mathbb{I}\mathbb{I}(\mathsf{M})$, for all $i \in I$. Thus, for all $i \in I$, $\mathfrak{A}^i = \bigcap_{j\in J_i}\mathfrak{A}^{ij}$, where $\mathfrak{A}^{ij} \in \mathsf{M}$, for all $j \in J_i$. Therefore, we get

$$\mathfrak{A} = \bigcap_{i\in I}\mathfrak{A}^i = \bigcap_{i\in I}\bigcap_{j\in J_i}\mathfrak{A}^{ij},$$

where $\mathfrak{A}^{ij} \in \mathsf{M}$, for all $i \in I$, $j \in J_i$, and, hence, $\mathfrak{A} \in \mathbb{I}\mathbb{I}(\mathsf{M})$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and M a class of $\mathbf{F}$-matrix families. Recall the closure system $C^{\mathsf{M}} : \mathcal{P}\mathrm{SEN}^\flat \to \mathcal{P}\mathrm{SEN}^\flat$ on $\mathbf{F}$ generated by M. It is defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, by $\phi \in C^{\mathsf{M}}_\Sigma(\Phi)$ if and only if, for all $\mathfrak{A} = \langle \mathcal{A}, T\rangle \in \mathsf{M}$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)) \subseteq T_{F(\Sigma')} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \in T_{F(\Sigma')}.$$

$\mathcal{I}^{\mathsf{M}} = \langle \mathbf{F}, C^{\mathsf{M}}\rangle$ denotes the corresponding π-institution generated by M.

Now, given a π-institution $\mathcal{I}$, one can consider its matrix family models, i.e., those $\mathbf{F}$-matrix families $\mathfrak{A}$, such that

$$\mathcal{I} \le \mathcal{I}^{\mathfrak{A}}.$$

Doing this for the specific π-institution $\mathcal{I}^{\mathsf{M}}$, generated by the class M of $\mathbf{F}$-matrix families, we consider the class $\mathrm{MatFam}(\mathcal{I}^{\mathsf{M}})$ of $\mathcal{I}^{\mathsf{M}}$-matrix families. Clearly, since, for every $\mathfrak{A} \in \mathsf{M}$, $C^{\mathsf{M}} \le C^{\mathfrak{A}}$,

$$\mathsf{M} \subseteq \mathrm{MatFam}(\mathcal{I}^{\mathsf{M}}).$$

In the spirit of many classical problems in universal algebraic logic, the following question naturally arises:

> Characterize $\mathrm{MathFam}(\mathcal{I}^{\mathsf{M}})$, i.e., find a list of operators on classes of $\mathbf{F}$-matrix families so that, when applied to M consecutively, they generate the class $\mathrm{MatFam}(\mathcal{I}^{\mathsf{M}})$.

Our goal here is to show that the list of operators that are needed consists of $\mathbb{M}\mathbb{I}\mathbb{I}\mathbb{M}^{-1}$, i.e., that, given any class M of $\mathbf{F}$-matrix families, we have

$$\mathrm{MatFam}(\mathcal{I}^{\mathsf{M}}) = \mathbb{M}\mathbb{I}\mathbb{I}\mathbb{M}^{-1}(\mathsf{M}).$$

We start by showing that applying each of the three operators to classes of matrix family models of a π-institution $\mathcal{I}$ always results in classes of the same character.

Proposition 1783 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

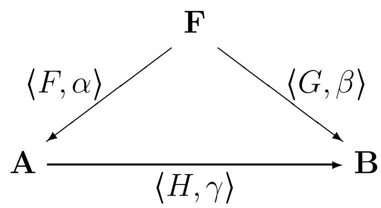
(a) $\mathbb{I}\mathbb{M}(\text{MatFam}(\mathcal{I})) \subseteq \text{MatFam}(\mathcal{I})$*;*

(b) $\mathbb{I}\mathbb{I}(\text{MatFam}(\mathcal{I})) \subseteq \text{MatFam}(\mathcal{I})$*;*

(c) $\mathbb{I}\mathbb{M}^{-1}(\text{MatFam}(\mathcal{I})) \subseteq \text{MatFam}(\mathcal{I})$*.*

Proof:

(a) Let $\mathfrak{A} = \langle \mathcal{A}, T \rangle \in \text{MatFam}(\mathcal{I})$ and consider a surjective morphism $\langle H, \gamma \rangle : \mathfrak{A} \to \mathfrak{B}$, where $\mathfrak{B} = \langle \mathcal{B}, T' \rangle$, as in the diagram.



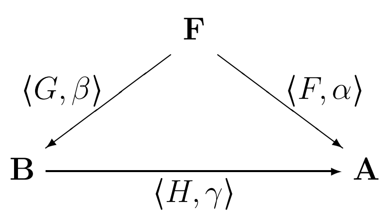
We now have

$$\beta^{-1}(T') = \alpha^{-1}(\gamma^{-1}(T')) = \alpha^{-1}(T) \in \text{ThFam}(\mathcal{I}),$$

where the last membership follows by the hypothesis and Lemma 51. Thus, again by Lemma 51, we get that $T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{B})$ and, hence, $\mathfrak{B} \in \text{MatFam}(\mathcal{I})$.

(b) Suppose, next, that $\mathfrak{A}^i = \langle \mathcal{A}, T^i \rangle$, $i \in I$, are $\mathcal{I}$-matrix families. Then $T^i \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$, for all $i \in I$. Since the collection $\text{FiFam}^{\mathcal{I}}(\mathcal{A})$ forms a closure system on $\mathcal{A}$, it follows that $\bigcap_{i \in I} T^i \in \text{FiFam}^{\mathcal{I}}(\mathcal{A})$. Thus, we get that $\bigcap_{i \in I} \mathfrak{A}^i \in \text{MatFam}(\mathcal{I})$. So $\text{MatFam}(\mathcal{I})$ is closed under $\mathbb{I}\mathbb{I}$.

(c) Let $\mathfrak{A} = \langle \mathcal{A}, T \rangle \in \text{MatFam}(\mathcal{I})$ and consider a surjective morphism $\langle H, \gamma \rangle : \mathfrak{B} \to \mathfrak{A}$, where $\mathfrak{B} = \langle \mathcal{B}, T' \rangle$, as in the diagram.



We now have

$$\beta^{-1}(T') = \beta^{-1}(\gamma^{-1}(T)) = \alpha^{-1}(T) \in \text{ThFam}(\mathcal{I}),$$

where the last membership follows by the hypothesis and Lemma 51. Thus, again by Lemma 51, we get that $T' \in \text{FiFam}^{\mathcal{I}}(\mathcal{B})$ and, hence, $\mathfrak{B} \in \text{MatFam}(\mathcal{I})$.

■

Proposition 1783 gives

Corollary 1784 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then*

$$\mathbb{M}\mathbb{\Pi}\mathbb{I}\mathbb{M}^{-1}(\mathrm{MatFam}(\mathcal{I})) \subseteq \mathrm{MatFam}(\mathcal{I}).$$

Proof: We have, using Proposition 1783,

$$\begin{array}{rcl}\mathbb{M}\mathbb{\Pi}\mathbb{I}\mathbb{M}^{-1}(\mathrm{MatFam}(\mathcal{I})) & \subseteq & \mathbb{M}\mathbb{\Pi}(\mathrm{MatFam}(\mathcal{I})) \\ & \subseteq & \mathbb{M}(\mathrm{MatFam}(\mathcal{I})) \\ & \subseteq & \mathrm{MatFam}(\mathcal{I}).\end{array}$$

■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall that a Lindenbaum $\mathcal{I}$-matrix family is an $\mathcal{I}$-matrix family of the form $\langle \mathcal{F}, T \rangle$, where $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and $T \in \mathrm{ThFam}(\mathcal{I})$. We show, next, that the class of all $\mathcal{I}$-matrix families can be obtained by applying the $\mathbb{M}$ operator on the class of all Lindenbaum matrix families, i.e., $\mathrm{MatFam}(\mathcal{I}) = \mathbb{M}(\mathrm{LMatFam}(\mathcal{I}))$.

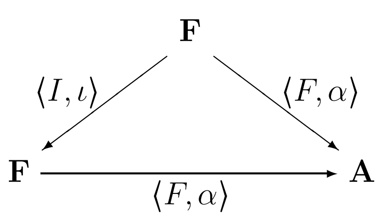
Lemma 1785 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. Then*

$$\mathrm{MatFam}(\mathcal{I}) = \mathbb{M}(\mathrm{LMatFam}(\mathcal{I})).$$

Proof: First, observe that, since $\mathrm{LMatFam}(\mathcal{I}) \subseteq \mathrm{MatFam}(\mathcal{I})$, we have, by Proposition 1783,

$$\mathbb{M}(\mathrm{LMatFam}(\mathcal{I})) \subseteq \mathbb{M}(\mathrm{MatFam}(\mathcal{I})) \subseteq \mathrm{MatFam}(\mathcal{I}).$$

Suppose, conversely, that $\mathfrak{A} = \langle \mathcal{A}, T \rangle \in \mathrm{MatFam}(\mathcal{I})$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$. Then, we have, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$. Hence, $\langle \mathcal{F}, \alpha^{-1}(T) \rangle \in \mathrm{LMatFam}(\mathcal{I})$. Now, it suffices to consider the surjective morphism $\langle F, \alpha \rangle : \langle \mathcal{F}, \alpha^{-1}(T) \rangle \to \mathfrak{A}$



to conclude that $\mathfrak{A} \in \mathbb{M}(\mathrm{LMatFam}(\mathcal{I}))$. Therefore, we obtain $\mathrm{MatFam}(\mathcal{I}) \subseteq \mathbb{M}(\mathrm{LMatFam}(\mathcal{I}))$. ■

Now, to complete our task, we turn again to the specific π-institution $\mathcal{I}^{\mathsf{M}}$, generated by a given class M of $\mathbf{F}$-matrix families. We show that all its Lindenbaum matrix families, i.e., all matrix families of the form $\langle \mathcal{F}, T \rangle$, where $T \in \mathrm{ThFam}(\mathcal{I}^{\mathsf{M}})$, can be obtained by applying the operator $\mathbb{\Pi}\mathbb{I}\mathbb{M}^{-1}$ on the class M.

Lemma 1786 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* M *a collection of* $\mathbf{F}$*-matrix families. Then*

$$\mathrm{LMatFam}(\mathcal{I}^{\mathsf{M}}) \subseteq \mathbb{\Pi}\mathbb{M}^{-1}(\mathsf{M}).$$

Proof: Let $\mathfrak{F} = \langle \mathcal{F}, T \rangle \in \mathrm{LMatFam}(\mathcal{I}^{\mathsf{M}})$, i.e., $T \in \mathrm{ThFam}(\mathcal{I}^{\mathsf{M}})$. Thus, there exist $\mathfrak{A}^i = \langle \mathcal{A}^i, T^i \rangle \in \mathsf{M}$, with $\mathcal{A} = \langle \mathbf{A}^i, \langle F^i, \alpha^i \rangle \rangle$, $i \in I$, such that

$$T = \bigcap_{i \in I} (\alpha^i)^{-1}(T^i).$$

Consider the collection $\mathfrak{F}^i = \langle \mathcal{F}, (\alpha^i)^{-1}(T^i) \rangle$, $i \in I$. Taking into account the surjective morphisms $\langle F^i, \alpha^i \rangle : \mathfrak{F}^i \to \mathfrak{A}^i$, $i \in I$, and the fact that $\mathfrak{A}^i \in \mathsf{M}$, we conclude that $\mathfrak{F}^i \in \mathbb{M}^{-1}(\mathsf{M})$, for all $i \in I$. Finally, observing that $\mathfrak{F} = \bigcap_{i \in I} \mathfrak{F}^i$, we get that $\mathfrak{F} \in \mathbb{\Pi}\mathbb{M}^{-1}(\mathsf{M})$. Therefore, $\mathrm{LMatFam}(\mathcal{I}^{\mathsf{M}}) \subseteq \mathbb{\Pi}\mathbb{M}^{-1}(\mathsf{M})$. ∎

Now we are ready to provide the promised characterization of $\mathrm{MatFam}(\mathcal{I}^{\mathsf{M}})$ in terms of M and the class operators $\mathbb{M}$, $\mathbb{\Pi}$ and $\mathbb{M}^{-1}$, introduced in this section.

Theorem 1787 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* M *a collection of* $\mathbf{F}$*-matrix families. Then*

$$\mathrm{MatFam}(\mathcal{I}^{\mathsf{M}}) = \mathbb{M}\mathbb{\Pi}\mathbb{M}^{-1}(\mathsf{M}).$$

Proof: First, since $\mathsf{M} \subseteq \mathrm{MatFam}(\mathcal{I}^{\mathsf{M}})$, we have, using Corollary 1784,

$$\mathbb{M}\mathbb{\Pi}\mathbb{M}^{-1}(\mathsf{M}) \subseteq \mathbb{M}\mathbb{\Pi}\mathbb{M}^{-1}(\mathrm{MatFam}(\mathcal{I}^{\mathsf{M}})) \subseteq \mathrm{MatFam}(\mathcal{I}^{\mathsf{M}}).$$

Conversely, let $\mathfrak{A} \in \mathrm{MatFam}(\mathcal{I}^{\mathsf{M}})$. Then, by Lemmas 1785 and 1786,

$$\mathfrak{A} \in \mathbb{M}(\mathrm{LMatFam}(\mathcal{I}^{\mathsf{M}})) \subseteq \mathbb{M}\mathbb{\Pi}\mathbb{M}^{-1}(\mathsf{M}).$$

Therefore, $\mathrm{MatFam}(\mathcal{I}^{\mathsf{M}}) \subseteq \mathbb{M}\mathbb{\Pi}\mathbb{M}^{-1}(\mathsf{M})$. ∎

As a consequence of this characterization, we can also show that the operator $\mathbb{M}\mathbb{\Pi}\mathbb{M}^{-1}$ is a closure operator on classes of $\mathbf{F}$-matrix families and, moreover, given any such class M, applying the operator to the class results in the smallest class of $\mathbf{F}$-matrix systems that contains M and is closed under the operations $\mathbb{M}$, $\mathbb{\Pi}$ and $\mathbb{M}^{-1}$.

Theorem 1788 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* M *a collection of* $\mathbf{F}$*-matrix families.*

(a) $\mathbb{M}\mathbb{\Pi}\mathbb{M}^{-1} : \mathcal{P}(\mathrm{MatFam}(\mathbf{F})) \to \mathcal{P}(\mathrm{MatFam}(\mathbf{F}))$ *is a closure operator;*

(b) $\mathbb{M}\mathbb{\Pi}\mathbb{M}^{-1}(\mathsf{M})$ *is the smallest class of* $\mathbf{F}$*-matrix families containing* M *and closed under the operators* $\mathbb{M}$*,* $\mathbb{\Pi}$ *and* $\mathbb{M}^{-1}$*.*

Proof:

(a) Inflationarity and monotonicity follow from the corresponding properties of the three operators, which were established in Lemmas 1781 and 1782. For idempotency, we have

$$\begin{array}{rcl}\mathbb{M}\mathbb{\Pi}\mathbb{M}^{-1}(\mathbb{M}\mathbb{\Pi}\mathbb{M}^{-1}(\mathsf{M})) & = & \mathbb{M}\mathbb{\Pi}\mathbb{M}^{-1}(\mathrm{MatFam}(\mathcal{I}^{\mathsf{M}})) \\ & & \text{(by Theorem 1787)} \\ & \subseteq & \mathrm{MatFam}(\mathcal{I}^{\mathsf{M}}) \\ & & \text{(by Corollary 1784)} \\ & = & \mathbb{M}\mathbb{\Pi}\mathbb{M}^{-1}(\mathsf{M}). \\ & & \text{(again by Theorem 1787)}\end{array}$$

(b) By Part (a), $\mathsf{M} \subseteq \mathbb{M}\mathbb{\Pi}\mathbb{M}^{-1}(\mathsf{M})$. Moreover, if $\mathbb{O} \in \{\mathbb{M}, \mathbb{\Pi}, \mathbb{M}^{-1}\}$, then

$$\begin{array}{rcll}\mathbb{O}(\mathbb{M}\mathbb{\Pi}\mathbb{M}^{-1}(\mathsf{M})) & = & \mathbb{O}(\mathrm{MatFam}(\mathcal{I}^{\mathsf{M}})) & \text{(by Theorem 1787)} \\ & \subseteq & \mathrm{MatFam}(\mathcal{I}^{\mathsf{M}}) & \text{(by Corollary 1784)} \\ & = & \mathbb{M}\mathbb{\Pi}\mathbb{M}^{-1}(\mathsf{M}). & \text{(by Theorem 1787)}\end{array}$$

Hence, $\mathbb{M}\mathbb{\Pi}\mathbb{M}^{-1}(\mathsf{M})$ is closed under all three operators. If N is a class of $\mathbf{F}$-matrix families such that $\mathsf{M} \subseteq \mathsf{N}$ and N closed under the three operators, then, clearly, $\mathbb{M}\mathbb{\Pi}\mathbb{M}^{-1}(\mathsf{M}) \subseteq \mathbb{M}\mathbb{\Pi}\mathbb{M}^{-1}(\mathsf{N}) = \mathsf{N}$. Therefore, $\mathbb{M}\mathbb{\Pi}\mathbb{M}^{-1}(\mathsf{M})$ is the smallest class satisfying these properties. ∎

24.3 Classes of Reduced Matrix Families

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Recall that $\mathrm{LMatFam}^*(\mathcal{I})$ is the class of all reduced Lindenbaum $\mathcal{I}$-matrix families, i.e., all $\mathbf{F}$-matrix families of the form $\langle \mathcal{F}^{\Omega(T)}, T/\Omega(T) \rangle$, where $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$ and $T \in \mathrm{ThFam}(\mathcal{I})$, and that $\mathcal{I}$ is complete with respect to $\mathrm{LMatFam}^*(\mathcal{I})$.

Recall, also, that $\mathrm{MatFam}^*(\mathcal{I})$ is the collection of all reduced $\mathcal{I}$-matrix families, i.e., $\mathbf{F}$-matrix families of the form $\langle \mathcal{A}, T \rangle$, where $\mathcal{A}$ is an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. Moreover, $\mathcal{I}$ is also complete with respect to $\mathrm{MatFam}^*(\mathcal{I})$.

Our first goal is to show that the class $\mathrm{MatFam}^*(\mathcal{I})$ is, in fact, the class generated by applying the morphic image operator $\mathbb{M}$, introduced in the previous section, on the class $\mathrm{LMatFam}^*(\mathcal{I})$.

We prove, first, that the operator

$$\mathbb{M} : \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F})) \to \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F})),$$

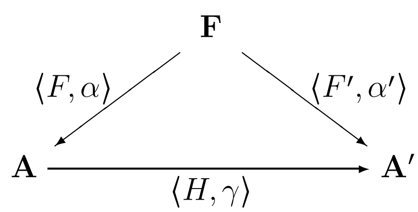
i.e., the operator $\mathbb{M}$, introduced in Section 24.2, restricted to reduced $\mathbf{F}$-matrix families, is also a closure operator.

Proposition 1789 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system. Then*

$$\mathbb{M} : \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F})) \to \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F}))$$

is a closure operator on $\mathrm{MatFam}^*(\mathbf{F})$.

Proof: Since we know, by Lemma 1781, that $\mathbb{M}$ is inflationary, monotone and idempotent, it suffices to show that it is well-defined, i.e., that, when applied to collections of reduced $\mathbf{F}$-matrix families, it produces collections of the same kind. In turn, it suffices to show that, given a reduced $\mathbf{F}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, an $\mathbf{F}$-matrix family $\mathfrak{A}' = \langle \mathcal{A}', T' \rangle$, with $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha' \rangle \rangle$, and a strict surjective morphism $\langle H, \gamma \rangle : \mathfrak{A} \to \mathfrak{A}'$, then $\mathfrak{A}'$ is also reduced.



Taking into account the surjectivity of $\langle F', \alpha' \rangle$, we reason as follows. For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, we have

$$\begin{array}{rl} \multicolumn{2}{l}{\langle \alpha'_\Sigma(\phi), \alpha'_\Sigma(\psi) \rangle \in \Omega^{\mathcal{A}'}_{F'(\Sigma)}(T')} \\ \text{iff} & \langle \gamma_{F(\Sigma)}(\alpha_\Sigma(\phi)), \gamma_{F(\Sigma)}(\alpha_\Sigma(\psi)) \rangle \in \Omega^{\mathcal{A}'}_{G(F(\Sigma))}(T') \\ \text{iff} & \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \gamma^{-1}_{F(\Sigma)}(\Omega^{\mathcal{A}'}_{G(F(\Sigma))}(T')) \\ \text{iff} & \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \Omega^{\mathcal{A}}_{F(\Sigma)}(\gamma^{-1}(T')) \\ \text{iff} & \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \Omega^{\mathcal{A}}_{F(\Sigma)}(T) \\ \text{iff} & \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \Delta^{\mathcal{A}}_{F(\Sigma)}(T) \\ \text{iff} & \alpha_\Sigma(\phi) = \alpha_\Sigma(\psi) \\ \text{implies} & \gamma_{F(\Sigma)}(\alpha_\Sigma(\phi)) = \gamma_{F(\Sigma)}(\alpha_\Sigma(\psi)) \\ \text{iff} & \alpha'_\Sigma(\phi) = \alpha'_\Sigma(\psi). \end{array}$$

Therefore $\Omega^{\mathcal{A}'}(T') = \Delta^{\mathcal{A}'}$ and, hence $\mathfrak{A}'$ is also reduced. ■

Next, we show that, given π-institution $\mathcal{I}$, the class $\mathrm{MatFam}^*(\mathcal{I})$ is obtained by applying the operator $\mathbb{M}$ on the class $\mathrm{LMatFam}^*(\mathcal{I})$.

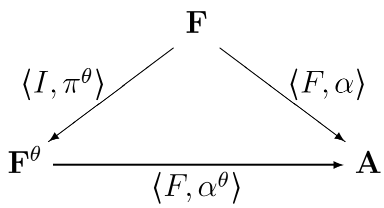
Proposition 1790 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *Then*

$$\mathrm{MatFam}^*(\mathcal{I}) = \mathbb{M}(\mathrm{LMatFam}^*(\mathcal{I})).$$

Proof: The inclusion $\mathbb{M}(\mathrm{LMatFam}^*(\mathcal{I})) \subseteq \mathrm{MatFam}^*(\mathcal{I})$ is obtained by observing that $\mathrm{LMatFam}^*(\mathcal{I}) \subseteq \mathrm{MatFam}^*(\mathcal{I})$ and applying $\mathbb{M}$:

$$\begin{array}{rcll} \mathbb{M}(\mathrm{LMatFam}^*(\mathcal{I})) & \subseteq & \mathbb{M}(\mathrm{MatFam}^*(\mathcal{I})) & \text{(Lemma 1781)} \\ & \subseteq & \mathrm{MatFam}^*(\mathcal{I}). & \text{(Propositions 1783 and 1789)} \end{array}$$

Suppose, conversely, that $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, is a reduced $\mathcal{I}$-matrix family. Let $\theta = \mathrm{Ker}(\langle F, \alpha \rangle)$ and consider the commutative diagram



where, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\alpha^\theta_\Sigma(\phi/\theta_\Sigma) = \alpha_\Sigma(\phi).$$

It now suffices to show that $\mathfrak{F} := \langle \mathcal{F}^\theta, \alpha^{-1}(T)/\theta \rangle \in \mathrm{LMatFam}^*(\mathcal{I})$. First, note that since $\mathfrak{A} \in \mathrm{MatFam}^*(\mathcal{I}) \subseteq \mathrm{MatFam}(\mathcal{I})$, then

$$\mathfrak{F} \in \mathbb{M}^{-1}(\mathrm{MatFam}(\mathcal{I})) \subseteq \mathrm{MatFam}(\mathcal{I}),$$

by Proposition 1783. So it suffices to show that $\Omega^{\mathcal{F}^\theta}(\alpha^{-1}(T)/\theta) = \Delta^{\mathcal{F}^\theta}$. We have

$$\begin{array}{rcl}\Omega^{\mathcal{F}^\theta}(\alpha^{-1}(T)/\theta) & = & \Omega^{\mathcal{F}^\theta}((\alpha^\theta)^{-1}(T)) \\ & = & (\alpha^\theta)^{-1}(\Omega^{\mathcal{A}}(T)) \\ & = & (\alpha^\theta)^{-1}(\Delta^{\mathcal{A}}) \\ & = & \mathrm{Ker}(\langle F, \alpha^\theta \rangle) = \Delta^{\mathcal{F}^\theta}.\end{array}$$

Now we conclude that $\mathfrak{A} \in \mathbb{M}(\mathrm{LMatFam}^*(\mathcal{I}))$. ∎

Consider, again, a base algebraic system $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ and a collection M of reduce $\mathbf{F}$-matrix families. We pose now a problem similar to that posed in Section 24.2, but for classes of reduced matrix families.

> Characterize the class $\mathrm{MatFam}^*(\mathcal{I})$, i.e., find a list of operators on classes of reduced $\mathbf{F}$-matrix families so that, when applied to M consecutively, they generate the class $\mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}})$.

Unlike the operator $\mathbb{M}$ that, when applied to reduced matrix families yields reduced matrix families, the other two operators that we considered in Section 24.2, namely $\mathbb{\Pi}$ and $\mathbb{M}^{-1}$, do not share this property. So to "localize" them to reduced matrix families, we must take the output classes of $\mathbf{F}$-matrix families that they produce and "reduce" them so that the output produced becomes a collection of reduced $\mathbf{F}$-matrix families. According to this scheme, we consider the following operators, induced by the operators $\mathbb{\Pi}$ and $\mathbb{M}^{-1}$ on class of matrix families, introduced in Section 24.2.

- $\mathbb{\Pi}^* : \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F})) \to \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F}))$ is given, by setting, for all $\mathsf{M} \subseteq \mathrm{MatFam}^*(\mathbf{F})$,

$$\mathbb{\Pi}^*(\mathsf{M}) = \{\mathfrak{A}^* : \mathfrak{A} \in \mathbb{\Pi}(\mathsf{M})\};$$

- $\mathbb{M}^{-1*} : \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F})) \to \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F}))$ is given, by setting, for all $\mathsf{M} \subseteq \mathrm{MatFam}^*(\mathbf{F})$,
$$\mathbb{M}^{-1*}(\mathsf{M}) = \{\mathfrak{A}^* : \mathfrak{A} \in \mathbb{M}^{-1}(\mathsf{M})\}.$$

It is not very difficult to prove that both $\mathbb{I}\mathbb{I}^*$ and $\mathbb{M}^{-1*}$ are closure operators on the class of reduced $\mathbf{F}$-matrix families.

Proposition 1791 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system. Then*
$$\mathbb{I}\mathbb{I}^* : \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F})) \to \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F}))$$
is a closure operator on $\mathrm{MatFam}^*(\mathbf{F})$.

Proof: Let $\mathsf{M} \subseteq \mathrm{MatFam}^*(\mathbf{F})$ and $\mathfrak{A} \in \mathsf{M}$. Then, by Proposition 1782, $\mathfrak{A} \in \mathbb{I}\mathbb{I}(\mathsf{M})$ and, as $\mathfrak{A}$ is reduced, we get $\mathfrak{A} \in \mathbb{I}\mathbb{I}^*(\mathsf{M})$. Thus, $\mathbb{I}\mathbb{I}^*$ is inflationary.

Suppose, next, that $\mathsf{M} \subseteq \mathsf{N} \subseteq \mathrm{MatFam}^*(\mathbf{F})$ and $\mathfrak{A} \in \mathbb{I}\mathbb{I}^*(\mathsf{M})$. Then $\mathfrak{A} = (\bigcap_{i\in I} \mathfrak{A}^i)^*$, with $\mathfrak{A}^i \in \mathsf{M}$, for all $i \in I$. But then, since $\mathsf{M} \subseteq \mathsf{N}$, $\mathfrak{A} = (\bigcap_{i\in I} \mathfrak{A}^i)^*$, with $\mathfrak{A}^i \in \mathsf{N}$, for all $i \in I$, and, hence, $\mathfrak{A} \in \mathbb{I}\mathbb{I}^*(\mathsf{N})$. Therefore $\mathbb{I}\mathbb{I}^*$ is also monotone.

Suppose, finally, that $\mathsf{M} \subseteq \mathrm{MatFam}^*(\mathbf{F})$ and that $\mathfrak{A} \in \mathbb{I}\mathbb{I}^*(\mathbb{I}\mathbb{I}^*(\mathsf{M}))$. Then $\mathfrak{A} = (\bigcap_{i\in I} \mathfrak{A}^i)^*$, where $\mathfrak{A}^i \in \mathbb{I}\mathbb{I}^*(\mathsf{M})$. Hence, for all $i \in I$, $\mathfrak{A}^i = (\bigcap_{j\in J_i} \mathfrak{A}^{ij})^*$, where $\mathfrak{A}^{ij} \in \mathsf{M}$, for all $i \in I$ and all $j \in J_i$. Now note the following:

- For every $i \in I$, for $\bigcap_{j\in J_i} \mathfrak{A}^{ij}$ to be defined, we must have $\mathfrak{A}^{ij} = \langle \mathcal{A}^i, T^{ij} \rangle$, for all $j \in J_i$.
- For $\bigcap_{i\in I} \mathcal{A}^i = \bigcap_{i\in I} (\bigcap_{j\in J_i} \mathfrak{A}^{ij})^*$ to be defined, we must have, for all $i \in I$, $\mathcal{A}^i = \mathcal{A}$, for some $\mathbf{F}$-algebraic system $\mathcal{A}$, and, moreover, for all $i \in I$, $\Omega^{\mathcal{A}}(\bigcap_{j\in J_i} T^{ij}) = \theta$, for some $\theta \in \mathrm{ConSys}(\mathcal{A})$.

Under these restrictions, it is easy to show that
$$\langle I, \pi \rangle : \langle \mathcal{A}, \bigcap_{i\in I} \bigcap_{j\in J_i} T^{ij} \rangle \to \mathcal{A}^\theta / \Omega^{\mathcal{A}^\theta}(\bigcap_{i\in I} ((\bigcap_{j\in J_i} T^{ij})/\theta))$$
defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, by
$$\pi_\Sigma(\phi) = (\phi/\theta_\Sigma)/\Omega^{\mathcal{A}^\theta}_\Sigma(\bigcap_{i\in I} ((\bigcap_{j\in J_i} T^{ij})/\theta)),$$
is a strict surjective matrix morphism, with kernel
$$\mathrm{Ker}(\langle I, \pi \rangle) = \Omega^{\mathcal{A}}(\bigcap_{i\in I} \bigcap_{j\in J_i} T^{ij}).$$
Therefore, we get an isomorphism
$$\mathfrak{A}/\Omega^{\mathcal{A}}(\bigcap_{i\in I} \bigcap_{j\in J_i} T^{ij}) \cong (\mathfrak{A}^\theta)(\Omega^{\mathcal{A}^\theta}(\bigcap_{i\in I} ((\bigcap_{j\in J_i} T^{ij})/\theta))).$$
We conclude that $\mathfrak{A} \in \mathbb{I}\mathbb{I}^*(\mathsf{M})$ and, therefore, $\mathbb{I}\mathbb{I}^*$ is also idempotent. ■

To show that $\mathbb{M}^{-1*}$ is a closure operator, we employ a lemma to the effect that, given a class M of reduced $\mathbf{F}$-matrix families, $\mathbb{M}^{-1*}(\mathsf{M}) \subseteq \mathbb{M}^{-1}(\mathsf{M})$.

Lemma 1792 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system. For every* $\mathbf{F}$*-matrix family* $\mathfrak{A} = \langle \mathcal{A}, T \rangle$*, every reduced* $\mathbf{F}$*-matrix family* $\mathfrak{A}' = \langle \mathcal{A}', T' \rangle$ *and strict surjective morphism* $\langle H, \gamma \rangle : \mathfrak{A} \to \mathfrak{A}'$*, there exists a strict surjective morphism* $\langle H, \gamma^* \rangle : \mathfrak{A}^* \to \mathfrak{A}'$*, such that the following triangle commutes,*

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\langle H, \gamma \rangle} & \mathfrak{A}' \\ & {\scriptstyle \langle I, \pi \rangle} \searrow \quad \nearrow {\scriptstyle \langle H, \gamma^* \rangle} & \\ & \mathfrak{A}^* & \end{array}$$

where $\langle I, \pi \rangle : \mathfrak{A} \to \mathfrak{A}^*$ *is the quotient morphism.*

Proof: We define $\gamma^* : \mathrm{SEN}^* \to \mathrm{SEN}' \circ H$ by setting, for all $\Sigma \in |\mathbf{Sign}|$, and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\gamma^*_\Sigma(\phi^*) = \gamma_\Sigma(\phi).$$

This makes sense, since, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\phi^* = \psi^*$, we have $\langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_\Sigma(T) = \Omega^{\mathcal{A}}_\Sigma(\gamma^{-1}(T'))$, whence $\langle \phi, \psi \rangle \in \gamma^{-1}_\Sigma(\Omega^{\mathcal{A}'}_{H(\Sigma)}(T'))$ and, hence, $\langle \gamma_\Sigma(\phi), \gamma_\Sigma(\psi) \rangle \in \Delta^{\mathcal{A}'}_{H(\Sigma)}$, i.e., $\gamma_\Sigma(\phi) = \gamma_\Sigma(\psi)$.

Moreover, $\gamma : \mathrm{SEN}^* \to \mathrm{SEN} \circ H$ is a natural transformation, since, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{ccc} \mathrm{SEN}^*(\Sigma) & \xrightarrow{\gamma^*_\Sigma} & \mathrm{SEN}'(H(\Sigma)) \\ {\scriptstyle \mathrm{SEN}^*(f)} \downarrow & & \downarrow {\scriptstyle \mathrm{SEN}'(H(f))} \\ \mathrm{SEN}^*(\Sigma') & \xrightarrow[\gamma^*_{\Sigma'}]{} & \mathrm{SEN}'(H(\Sigma')) \end{array}$$

$$\begin{array}{rcl} \mathrm{SEN}'(H(f))(\gamma^*_\Sigma(\phi^*)) & = & \mathrm{SEN}'(H(f))(\gamma_\Sigma(\phi)) \\ & = & \gamma_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \\ & = & \gamma^*_{\Sigma'}(\mathrm{SEN}(f)(\phi)^*) \\ & = & \gamma^*_{\Sigma'}(\mathrm{SEN}^*(g)(\phi^*)). \end{array}$$

Further, the triangle commutes, by the definition of $\langle H, \gamma^* \rangle$ and, finally, $\langle H, \gamma^* \rangle : \mathfrak{A}^* \to \mathfrak{A}'$ is strict since $\pi^{-1}((\gamma^*)^{-1}(T')) = \gamma^{-1}(T') = T$ and, therefore, $(\gamma^*)^{-1}(T') = \pi(T) = T^*$. ■

Now, we show $\mathbb{M}^{-1*}$ is a closure operator.

Proposition 1793 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system. Then*

$$\mathbb{M}^{-1*} : \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F})) \to \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F}))$$

is a closure operator on $\mathrm{MatFam}^*(\mathbf{F})$*.*

Proof: Suppose, first, that $\mathsf{M} \subseteq \mathrm{MatFam}^*(\mathbf{F})$ and $\mathfrak{A} \in \mathsf{M}$. Then, we have, by Proposition 1781, $\mathfrak{A} \in \mathbb{M}^{-1}(\mathsf{M})$ and, since $\mathfrak{A}$ is reduced, we get $\mathfrak{A} \in \mathbb{M}^{-1*}(\mathsf{M})$. So $\mathbb{M}^{-1*}$ is inflationary.

Suppose, next, that $\mathsf{M} \subseteq \mathsf{N} \subseteq \mathrm{MatFam}^*(\mathbf{F})$ and $\mathfrak{A} \in \mathbb{M}^{-1*}(\mathsf{M})$. Then $\mathfrak{A} = \mathfrak{B}^*$, with $\mathfrak{B} \in \mathbb{M}^{-1}(\mathsf{M})$. Thus, by Proposition 1781, we get $\mathfrak{A} = \mathfrak{B}^*$, with $\mathfrak{B} \in \mathbb{M}^{-1}(\mathsf{N})$. We conclude that $\mathfrak{A} \in \mathbb{M}^{-1*}(\mathsf{N})$ and, therefore, $\mathbb{M}^{-1*}$ is also monotone.

Finally, suppose that $\mathsf{M} \subseteq \mathrm{MatFam}^*(\mathbf{F})$ and that $\mathfrak{A} \in \mathbb{M}^{-1*}(\mathbb{M}^{-1*}(\mathsf{M}))$. Then, using Lemma 1792, we get

$$\mathfrak{A} \in \mathbb{M}^{-1*}(\mathbb{M}^{-1*}(\mathsf{M})) \subseteq \mathbb{M}^{-1}(\mathbb{M}^{-1*}(\mathsf{M})) \subseteq \mathbb{M}^{-1}(\mathbb{M}^{-1}(\mathsf{M})) \subseteq \mathbb{M}^{-1}(\mathsf{M}),$$

and, since $\mathfrak{A}$ is reduced, we get $\mathfrak{A} \in \mathbb{M}^{-1*}(\mathsf{M})$. Therefore $\mathbb{M}^{-1*}$ is also idempotent. ■

We need one more operator on reduced classes of $\mathbf{F}$-matrix families.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. We define

$$\overleftarrow{\mathbb{P}}^* : \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F})) \to \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F}))$$

by setting, for all $\mathsf{M} \subseteq \mathrm{MatFam}^*(\mathbf{F})$,

$$\overleftarrow{\mathbb{P}}^*(\mathsf{M}) = (\mathbb{P}\mathbb{M}^{-1}(\mathsf{M}))^*.$$

Note that this operator dominates both $\mathbb{P}^*$ and $\mathbb{M}^{-1*}$.

Proposition 1794 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. Then, for all $\mathsf{M} \subseteq \mathrm{MatFam}^*(\mathbf{F})$,*

$$\mathbb{P}^*(\mathsf{M}) \subseteq \overleftarrow{\mathbb{P}}^*(\mathsf{M}) \quad \textit{and} \quad \mathbb{M}^{-1*}(\mathsf{M}) \subseteq \overleftarrow{\mathbb{P}}^*(\mathsf{M}).$$

Proof: The proofs of both statements are parallel. We have

$$\begin{aligned} \mathbb{P}^*(\mathsf{M}) &= (\mathbb{P}(\mathsf{M}))^* & \qquad \mathbb{M}^{-1}(\mathsf{M}) &= (\mathbb{M}^{-1}(\mathsf{M}))^* \\ &\subseteq (\mathbb{P}\mathbb{M}^{-1}(\mathsf{M}))^* & &\subseteq (\mathbb{P}\mathbb{M}^{-1}(\mathsf{M}))^* \\ &= \overleftarrow{\mathbb{P}}^*(\mathsf{M}) & &= \overleftarrow{\mathbb{P}}^*(\mathsf{M}) \end{aligned}$$

where the inclusions follow from Lemmas 1781 and 1782, respectively. ■

Our next goal is to show that the list of operators that are needed to obtain the class of all reduced $\mathcal{I}^{\mathsf{M}}$-matrix families from a class M of reduced $\mathbf{F}$-matrix families generating a closure operator C^{M} (of a π-institution $\mathcal{I}^{\mathsf{M}} = \langle \mathbf{F}, C^{\mathsf{M}} \rangle$) consists of $\mathbb{M}\overleftarrow{\mathbb{P}}^*$, i.e., that, given any class M of reduced $\mathbf{F}$-matrix families, we have

$$\mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}}) = \mathbb{M}\overleftarrow{\mathbb{P}}^*(\mathsf{M}).$$

We start by showing that applying each of these operators to classes of reduced matrix family models of a π-institution $\mathcal{I}$ always results in classes of the same character. This forms an analog of Proposition 1783.

Proposition 1795 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$.

(a) $\mathbb{M}(\mathrm{MatFam}^*(\mathcal{I})) \subseteq \mathrm{MatFam}^*(\mathcal{I})$;

(b) $\overleftarrow{\mathbb{IP}}^*(\mathrm{MatFam}^*(\mathcal{I})) \subseteq \mathrm{MatFam}^*(\mathcal{I})$.

Proof:

(a) We have

$$\begin{array}{rcl}\mathbb{M}(\mathrm{MatFam}^*(\mathcal{I})) & \subseteq & \mathbb{M}(\mathrm{MatFam}(\mathcal{I})) \cap \mathbb{M}(\mathrm{MatFam}^*(\mathbf{F}))\\ & & \text{(Proposition 1781)}\\ & \subseteq & \mathrm{MatFam}(\mathcal{I}) \cap \mathrm{MatFam}^*(\mathbf{F})\\ & & \text{(Propositions 1783 and 1789)}\\ & = & \mathrm{MatFam}^*(\mathcal{I}). \quad \text{(Definition)}\end{array}$$

(b) Similarly,

$$\begin{array}{rcl}\overleftarrow{\mathbb{IP}}^*(\mathrm{MatFam}^*(\mathcal{I})) & = & (\mathbb{IPM}^{-1}(\mathrm{MatFam}^*(\mathcal{I})))^*\\ & \subseteq & (\mathbb{IPM}^{-1}(\mathrm{MatFam}(\mathcal{I})))^*\\ & & \text{(Lemmas 1781 and 1782)}\\ & \subseteq & (\mathrm{MatFam}(\mathcal{I}))^*\\ & & \text{(Proposition 1783)}\\ & = & \mathrm{MatFam}^*(\mathcal{I}).\end{array}$$

■

Proposition 1795, together with Proposition 1783, give the following

Corollary 1796 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$. *Then*

$$\mathbb{M}\overleftarrow{\mathbb{IP}}^*(\mathrm{MatFam}^*(\mathcal{I})) \subseteq \mathrm{MatFam}^*(\mathcal{I}).$$

Proof: We have, using Propositions 1783 and 1795,

$$\begin{array}{rcl}\mathbb{M}\overleftarrow{\mathbb{IP}}^*(\mathrm{MatFam}^*(\mathcal{I})) & \subseteq & \mathbb{M}(\mathrm{MatFam}^*(\mathcal{I}))\\ & \subseteq & \mathrm{MatFam}^*(\mathcal{I}).\end{array}$$

■

In order to establish our final result, we must show that, given a class M of reduced $\mathbf{F}$-matrix families, all reduced Lindenbaum matrix families of the π-institution $\mathcal{I}^{\mathsf{M}}$, i.e., all matrix families of the form $\langle \mathcal{F}/\Omega(T), T/\Omega(T) \rangle$, where $T \in \mathrm{ThFam}(\mathcal{I}^{\mathsf{M}})$, can be obtained by applying the operator $\overleftarrow{\mathbb{IP}}^*$ on the class M.

Lemma 1797 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* M *a collection of reduced* $\mathbf{F}$*-matrix families. Then*

$$\mathrm{LMatFam}^*(\mathcal{I}^{\mathsf{M}}) \subseteq \overleftarrow{\mathbb{\Pi}}^*(\mathsf{M}).$$

Proof: we have

$$\begin{array}{rcl}\mathrm{LMatFam}^*(\mathcal{I}^{\mathsf{M}}) & = & (\mathrm{LMatFam}(\mathcal{I}^{\mathsf{M}}))^* \quad \text{(Definition)}\\ & \subseteq & (\mathbb{\Pi}\mathbb{M}^{-1}(M))^* \quad \text{(Lemma 1786)}\\ & = & \overleftarrow{\mathbb{\Pi}}^*(\mathsf{M}). \quad \text{(Definition)}\end{array}$$

■

Now we provide the promised characterization of $\mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}})$ in terms of the class M of reduced $\mathbf{F}$-matrix families and the class operators $\mathbb{M}$ and $\overleftarrow{\mathbb{\Pi}}^*$.

Theorem 1798 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* M *a collection of reduced* $\mathbf{F}$*-matrix families. Then*

$$\mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}}) = \mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathsf{M}).$$

Proof: First, since $\mathsf{M} \subseteq \mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}})$, we have, using Corollary 1796,

$$\mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathsf{M}) \subseteq \mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}})) \subseteq \mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}}).$$

Conversely, let $\mathfrak{A} \in \mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}})$. Then, by Proposition 1790 and Lemma 1797,

$$\mathfrak{A} \in \mathbb{M}(\mathrm{LMatFam}^*(\mathcal{I}^{\mathsf{M}})) \subseteq \mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathsf{M}).$$

Therefore, $\mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}}) \subseteq \mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathsf{M})$, and equality follows. ■

As a consequence of this characterization, we can also show that the operator $\mathbb{M}\overleftarrow{\mathbb{\Pi}}^*$ is a closure operator on classes of reduced $\mathbf{F}$-matrix families and, moreover, given any such class M, applying the operator to the class results in the smallest class of reduced $\mathbf{F}$-matrix systems that contains M and is closed under the operations $\mathbb{M}$, $\overleftarrow{\mathbb{\Pi}}^*$.

Theorem 1799 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* M *a collection of reduced* $\mathbf{F}$*-matrix families.*

(a) $\mathbb{M}\overleftarrow{\mathbb{\Pi}}^* : \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F})) \to \mathcal{P}(\mathrm{MatFam}^*(\mathbf{F}))$ *is a closure operator;*

(b) $\mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathsf{M})$ *is the smallest class of* $\mathbf{F}$*-matrix families containing* M *and closed under the operators* $\mathbb{M}$ *and* $\overleftarrow{\mathbb{\Pi}}^*$*.*

Proof:

(a) Inflationarity and monotonicity follow from the corresponding properties of the operators $\mathbb{M}$ and $\mathbb{\Pi}$, which were established in Lemmas 1781 and 1782. For idempotency, we have

$$\begin{array}{rcl}\mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathsf{M})) & = & \mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}})) \\ & & \text{(by Theorem 1798)} \\ & \subseteq & \mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}}) \\ & & \text{(by Corollary 1796)} \\ & = & \mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathsf{M}). \\ & & \text{(again by Theorem 1798)}\end{array}$$

(b) By Part (a), $\mathsf{M} \subseteq \mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathsf{M})$. Moreover, if $\mathbb{O} \in \{\mathbb{M}, \overleftarrow{\mathbb{\Pi}}^*\}$, then

$$\begin{array}{rcl}\mathbb{O}(\mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathsf{M})) & = & \mathbb{O}(\mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}})) \quad \text{(by Theorem 1798)} \\ & \subseteq & \mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}}) \quad \text{(by Corollary 1796)} \\ & = & \mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathsf{M}). \quad \text{(by Theorem 1798)}\end{array}$$

Hence, $\mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathsf{M})$ is closed under both operators. If N is a class of reduced **F**-matrix families such that $\mathsf{M} \subseteq \mathsf{N}$ and N closed under both operators, then, clearly, $\mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathsf{M}) \subseteq \mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathsf{N}) = \mathsf{N}$. Therefore, $\mathbb{M}\overleftarrow{\mathbb{\Pi}}^*(\mathsf{M})$ is the smallest class satisfying these properties.

■

24.4 Protoclasses of Matrix Families

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. A class of **F**-matrix families M is called a **protoclass** if it is the class of all reduced $\mathcal{I}$-matrix families for a protoalgebraic π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ based on **F**.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{A}^i = \langle \mathbf{A}^i, \langle F^i, \alpha^i \rangle \rangle$, $i \in I$, a collection of **F**-algebraic systems and $\mathfrak{A}^i = \langle \mathcal{A}^i, T^i \rangle$ a collection of **F**-matrix families. We say that an **F**-matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, is a **subdirect intersection** of the collection $\mathfrak{A}^i$, $i \in I$, if there exist surjective morphisms

$$\langle H^i, \gamma^i \rangle : \mathfrak{A} \to \mathfrak{A}^i, \quad i \in I,$$

such that:

- $T = \bigcap_{i \in I} (\gamma^i)^{-1}(T^i)$;
- $\bigcap_{i \in I} \mathrm{Ker}(\langle H^i, \gamma^i \rangle) = \Delta^{\mathcal{A}}$.

Let M be a class of $\mathbf{F}$-matrix families. Given an $\mathbf{F}$-matrix family $\mathfrak{A}$, we write $\mathfrak{A} \in \overset{\triangleleft}{\mathbb{I}\mathrm{I}}(\mathsf{M})$ to denote the fact that $\mathfrak{A}$ is a subdirect intersection of members of M.

It is not difficult to see that $\overset{\triangleleft}{\mathbb{I}\mathrm{I}}$ is a closure operator on classes of $\mathbf{F}$-matrix families.

Lemma 1800 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system. Then*

$$\overset{\triangleleft}{\mathbb{I}\mathrm{I}} : \mathcal{P}(\mathrm{MatFam}(\mathbf{F})) \to \mathcal{P}(\mathrm{MatFam}(\mathbf{F}))$$

is a closure operator on $\mathrm{MatFam}(\mathbf{F})$.

Proof: Assume, first, that $\mathsf{M} \subseteq \mathrm{MatFam}(\mathbf{F})$ and $\mathfrak{A} \in \mathsf{M}$. Then $\langle I, \iota \rangle : \mathfrak{A} \to \mathfrak{A}$ is a subdirect intersection morphism and, therefore, since $\mathfrak{A} \in \mathsf{M}$, we get $\mathfrak{A} \in \overset{\triangleleft}{\mathbb{I}\mathrm{I}}(\mathsf{M})$. Therefore $\overset{\triangleleft}{\mathbb{I}\mathrm{I}}$ is inflationary.

Suppose, next, that $\mathsf{M} \subseteq \mathsf{N} \subseteq \mathrm{MatFam}(\mathbf{F})$. Let $\langle H^i, \gamma^i \rangle : \mathfrak{A} \to \mathfrak{A}'^i$, $i \in I$, be a collection of subdirect intersection morphisms, with $\mathfrak{A}'^i \in \mathsf{M}$, for all $i \in I$. Since, then, $\mathfrak{A}'^i \in \mathsf{N}$, for all $i \in I$, the same collection of morphisms witnesses that $\mathfrak{A} \in \overset{\triangleleft}{\mathbb{I}\mathrm{I}}(\mathsf{N})$. Therefore, $\overset{\triangleleft}{\mathbb{I}\mathrm{I}}$ is also monotone.

Finally, assume that $\mathfrak{A} \in \overset{\triangleleft}{\mathbb{I}\mathrm{I}}(\overset{\triangleleft}{\mathbb{I}\mathrm{I}}(\mathsf{M}))$, where $\mathsf{M} \subseteq \mathrm{MatFam}(\mathbf{F})$. Thus, there exists a collection of subdirect intersection morphisms

$$\langle H^i, \gamma^i \rangle : \mathfrak{A} \to \mathfrak{A}^i, \quad i \in I,$$

where $\mathfrak{A}^i \in \overset{\triangleleft}{\mathbb{I}\mathrm{I}}(\mathsf{M})$, for all $i \in I$. It now follows that, for each $i \in I$, there exists a collection of subdirect intersection morphisms

$$\langle H^{ij}, \gamma^{ij} \rangle : \mathfrak{A}^i \to \mathfrak{A}^{ij}, \quad j \in J_i,$$

where $\mathfrak{A}^{ij} \in \mathsf{M}$, for all $i \in I$ and all $j \in J_i$. We look at the collection

$$\langle H^{ij}, \gamma^{ij} \rangle \circ \langle H^i, \gamma^i \rangle : \mathfrak{A} \to \mathfrak{A}^{ij}, \quad i \in I, j \in J_i,$$

with $\mathfrak{A}^{ij} \in \mathsf{M}$, for all $i \in I, j \in J_i$. We have

- For filter family intersections,

$$\begin{array}{rcl} \bigcap_{i \in I} \bigcap_{j \in J_i} (\gamma^i)^{-1}((\gamma^{ij})^{-1}(T^{ij})) & = & \bigcap_{i \in I} (\gamma^i)^{-1}(\bigcap_{j \in J_i} (\gamma^{ij})^{-1}(T^{ij})) \\ & = & \bigcap_{i \in I} (\gamma^i)^{-1}(T^i) \\ & = & T. \end{array}$$

- Similarly, for kernels,

$$\begin{array}{l} \bigcap_{i \in I} \bigcap_{j \in J_i} \mathrm{Ker}(\langle H^{ij}, \gamma^{ij} \rangle \circ \langle H^i, \gamma^i \rangle) \\ \quad = \bigcap_{i \in I} \bigcap_{j \in J_i} (\gamma^i)^{-1}((\gamma^{ij})^{-1}(\Delta^{\mathcal{A}^{ij}})) \\ \quad = \bigcap_{i \in I} (\gamma^i)^{-1}(\bigcap_{j \in J_i} \mathrm{Ker}(\langle H^{ij}, \gamma^{ij} \rangle)) \\ \quad = \bigcap_{i \in I} (\gamma^i)^{-1}(\Delta^{\mathcal{A}^i}) \\ \quad = \bigcap_{i \in I} \mathrm{Ker}(\langle H^i, \gamma^i \rangle) \\ \quad = \Delta^{\mathcal{A}}. \end{array}$$

Therefore, $\langle H^{ij},\gamma^{ij}\rangle\circ\langle H^i,\gamma^i\rangle$, $i\in I$, $j\in J_i$, is also a collection of subdirect intersection morphisms, and, hence $\mathfrak{A}\in\overset{\triangleleft}{\mathbb{IP}}(\mathsf{M})$. We conclude that $\overset{\triangleleft}{\mathbb{IP}}$ is also idempotent. ■

In general, given a class M of reduced $\mathbf{F}$-matrix families, its closures under both operators $\mathbb{IP}$ and $\mathbb{M}^{-1*}$ are included in its closure under $\overset{\triangleleft}{\mathbb{IP}}$.

Proposition 1801 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* M *a class of reduced* $\mathbf{F}$*-matrix families. Then*

$$\mathbb{IP}(\mathsf{M})\subseteq\overset{\triangleleft}{\mathbb{IP}}(\mathsf{M})\quad\textit{and}\quad\mathbb{M}^{-1*}(\mathsf{M})\subseteq\overset{\triangleleft}{\mathbb{IP}}(\mathsf{M}).$$

Proof: Assume, first, that $\mathfrak{A}=\langle\mathcal{A},T\rangle\in\mathbb{IP}(\mathsf{M})$. Thus, there exists a collection $\mathfrak{A}^i=\langle\mathcal{A},T^i\rangle\in\mathsf{M}$, such that

$$T=\bigcap_{i\in I}T^i.$$

Consider the family of surjective morphisms

$$\langle I,\iota\rangle:\langle\mathcal{A},T\rangle\to\langle\mathcal{A},T^i\rangle,\quad i\in I.$$

We have

- $T=\bigcap_{i\in I}T^i=\bigcap_{i\in I}\iota^{-1}(T^i)$, by hypothesis;
- $\bigcap_{i\in I}\mathrm{Ker}(\langle I,\iota\rangle)=\bigcap_{i\in I}\Delta^{\mathcal{A}}=\Delta^{\mathcal{A}}$.

Therefore, since $\mathfrak{A}^i\in\mathsf{M}$, for all $i\in I$, $\mathfrak{A}\in\overset{\triangleleft}{\mathbb{IP}}(\mathsf{M})$.

Assume, next, that $\mathfrak{A}^*=\langle\mathcal{A}/\Omega^{\mathcal{A}}(T),T/\Omega^{\mathcal{A}}(T)\rangle\in\mathbb{M}^{-1*}(\mathsf{M})$, where $\langle H,\gamma\rangle:\mathfrak{A}\to\mathfrak{A}'$ is a strict surjective morphism, with $\mathfrak{A}'=\langle\mathcal{A}',T'\rangle\in\mathsf{M}$. Since $\mathsf{M}\subseteq\mathrm{MatFam}^*(\mathbf{F})$, there exists a factorization

$$\begin{array}{ccc}\mathfrak{A} & \xrightarrow{\langle H,\gamma\rangle} & \mathfrak{A}'\\ {\scriptstyle\langle I,\pi\rangle}\searrow & & \swarrow{\scriptstyle\langle H,\gamma^*\rangle}\\ & \mathfrak{A}^* & \end{array}$$

Moreover, we have

- $\pi^{-1}(\gamma^{*-1}(T'))=\gamma^{-1}(T')=T$, whence $\gamma^{*-1}(T')=T/\Omega^{\mathcal{A}}(T)$;
- $\mathrm{Ker}(\langle H,\gamma^*\rangle)=\Delta^{\mathcal{A}^*}$ holds, since, for all $\Sigma\in|\mathbf{Sign}|$ and all $\phi,\psi\in\mathrm{SEN}(\Sigma)$,

$$\begin{array}{rll}\langle\phi,\psi\rangle\in\mathrm{Ker}_\Sigma(\langle H,\gamma^*\rangle) & \text{iff} & \gamma^*_\Sigma(\phi/\Omega^{\mathcal{A}}_\Sigma(T))=\gamma^*_\Sigma(\psi/\Omega^{\mathcal{A}}_\Sigma(T))\\ & \text{iff} & \gamma_\Sigma(\phi)=\gamma_\Sigma(\psi)\\ & \text{iff} & \langle\phi,\psi\rangle\in\gamma^{-1}_\Sigma(\Omega^{\mathcal{A}'}_{H(\Sigma)}(T'))\\ & \text{iff} & \langle\phi,\psi\rangle\in\Omega^{\mathcal{A}}_\Sigma(\gamma^{-1}(T'))\\ & \text{iff} & \langle\phi,\psi\rangle\in\Omega^{\mathcal{A}}_\Sigma(T).\end{array}$$

Therefore, $\mathfrak{A}^* \in \overset{\triangleleft}{\mathbb{P}}(\mathsf{M})$. ∎

Another useful feature of the operator $\overset{\triangleleft}{\mathbb{P}}$ is that among model classes of matrix families, it characterizes those that are protoclasses.

Theorem 1802 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathsf{M} = \mathrm{MatFam}^*(\mathcal{I})$ *a class of reduced* $\mathbf{F}$*-matrix families. Then* M *is a protoclass if and only if* $\overset{\triangleleft}{\mathbb{P}}(\mathsf{M}) \subseteq \mathsf{M}$.

Proof: Suppose, first, that $\mathsf{M} = \mathrm{MatFam}^*(\mathcal{I})$, with $\mathcal{I}$ protoalgebraic and let $\langle H^i, \gamma^i\rangle : \mathfrak{A} \to \mathfrak{A}'^i$, $i \in I$, be a collection of subdirect intersection morphisms. Then, clearly,

$$\begin{array}{rcll}\mathfrak{A} & \in & \mathbb{P}\mathbb{M}^{-1}(\mathsf{M}) & (\text{Definition of } \overset{\triangleleft}{\mathbb{P}})\\ & \subseteq & \mathbb{P}\mathbb{M}^{-1}(\mathrm{MatFam}^*(\mathcal{I})) & (\text{Lemmas 1781 and 1782})\\ & \subseteq & \mathrm{MatFam}(\mathcal{I}). & (\text{Proposition 1783})\end{array}$$

It suffices now to show that $\mathcal{A}$ is reduced. We have

$$\begin{array}{rcll}\Omega^{\mathcal{A}}(T) & = & \Omega^{\mathcal{A}}(\bigcap_{i\in I}(\gamma^i)^{-1}(T'^i)) & (\text{Subdirect Intersection})\\ & = & \bigcap_{i\in I}\Omega^{\mathcal{A}}((\gamma^i)^{-1}(T'^i)) & (\mathcal{I}\text{ protoalgebraic})\\ & = & \bigcap_{i\in I}(\gamma^i)^{-1}(\Omega^{\mathcal{A}'^i}(T'^i)) & (\langle H^i,\gamma^i\rangle\text{ surjective})\\ & = & \bigcap_{i\in I}(\gamma^i)^{-1}(\Delta^{\mathcal{A}'^i}) & (\mathfrak{A}'^i\text{ reduced})\\ & = & \bigcap_{i\in I}\mathrm{Ker}(\langle H^i,\gamma^i\rangle) & \\ & = & \Delta^{\mathcal{A}}. & (\text{Subdirect Intersection})\end{array}$$

Since $\mathfrak{A} \in \mathrm{MatFam}(\mathcal{I})$ and $\mathfrak{A}$ is reduced, we conclude that $\mathfrak{A} \in \mathrm{MatFam}^*(\mathcal{I})$. So $\overset{\triangleleft}{\mathbb{P}}(\mathrm{MatFam}^*(\mathcal{I})) \subseteq \mathrm{MatFam}^*(\mathcal{I})$.

Suppose, conversely, that $\overset{\triangleleft}{\mathbb{P}}(\mathrm{MatFam}^*(\mathcal{I})) \subseteq \mathrm{MatFam}^*(\mathcal{I})$ and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, with $T \leq T'$. We set

$$\mathfrak{F} \coloneqq \langle \mathcal{F}/(\Omega(T) \cap \Omega(T')), (T \cap T')/(\Omega(T) \cap \Omega(T'))\rangle$$

and consider the surjective natural projection morphisms

$$\begin{array}{rl}\langle I, \pi\rangle : & \mathfrak{F} \to \langle \mathcal{F}/\Omega(T), T/\Omega(T)\rangle\\ \langle I, \pi'\rangle : & \mathfrak{F} \to \langle \mathcal{F}/\Omega(T'), T'/\Omega(T')\rangle.\end{array}$$

We observe that

- As far as filter families, we have

$$\begin{array}{l}(T \cap T')/(\Omega(T) \cap \Omega(T'))\\ = T/(\Omega(T) \cap \Omega(T')) \cap T'/(\Omega(T) \cap \Omega(T'))\\ = \pi^{-1}(T/\Omega(T')) \cap \pi'^{-1}(T'/\Omega(T'));\end{array}$$

- As far as kernels, we get

$$\begin{aligned}&\mathrm{Ker}(\langle I,\pi\rangle)\cap\mathrm{Ker}(\langle I,\pi'\rangle)\\&=\Omega(T)/(\Omega(T)\cap\Omega(T'))\cap\Omega(T')/(\Omega(T)\cap\Omega(T'))\\&=(\Omega(T)\cap\Omega(T'))/(\Omega(T)\cap\Omega(T'))=\Delta^{\mathfrak{F}}.\end{aligned}$$

Therefore,

$$\mathfrak{F}\in\overset{\triangleleft}{\mathbb{P}}(\mathrm{MatFam}^*(\mathcal{I}))\subseteq\mathrm{MatFam}^*(\mathcal{I}).$$

Hence $\Omega(T)=\Omega(T\cap T')=\Omega(T)\cap\Omega(T')$, which implies that $\Omega(T)\leq\Omega(T')$. Thus, Ω is monotone on theory families and, hence, $\mathcal{I}$ is protoalgebraic. ■

Finally, we work to obtain expressions for the protoclass $\mathrm{MatFam}^*(\mathcal{I})$ based on a reduced class M of generating $\mathbf{F}$-matrix families for $\mathcal{I}$.

We show, first, that if M is a class of reduced models of a protoalgebraic π-institution, then its closure under $\overset{\triangleleft}{\mathbb{P}}$ is included in its closure under $\overset{\leftarrow}{\mathbb{P}}^*$.

Proposition 1803 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a protoalgebraic π-institution based on* $\mathbf{F}$ *and* $\mathsf{M}\subseteq\mathrm{MatFam}^*(\mathcal{I})$ *a class of reduced $\mathcal{I}$-matrix families. Then*

$$\overset{\triangleleft}{\mathbb{P}}(\mathsf{M})\subseteq\overset{\leftarrow}{\mathbb{P}}^*(\mathsf{M}).$$

Proof: Let $\mathfrak{A}=\langle\mathcal{A},T\rangle\in\overset{\triangleleft}{\mathbb{P}}(\mathsf{M})$. Then, there exists a collection of subdirect intersection morphisms

$$\langle H^i,\gamma^i\rangle:\langle\mathcal{A},T\rangle\to\langle\mathcal{A}'^i,T'^i\rangle,\quad i\in I,$$

where $\mathfrak{A}'^i=\langle\mathcal{A}'^i,T'^i\rangle\in\mathsf{M}$, for all $i\in I$. By using the same morphisms,

$$\langle H^i,\gamma^i\rangle:\langle\mathcal{A},(\gamma^i)^{-1}(T'^i)\rangle\to\mathfrak{A}'^i,\quad i\in I,$$

which have now become strict and surjective, we get that, for all $i\in I$, $\langle\mathcal{A},(\gamma^i)^{-1}(T'^i)\rangle\in\mathbb{M}^{-1}(\mathsf{M})$. Moreover, since, by the definition of a subdirect intersection, $\mathfrak{A}=\langle\mathcal{A},T\rangle=\langle\mathcal{A},\bigcap_{i\in I}(\gamma^i)^{-1}(T'^i)\rangle$, we get that $\mathfrak{A}\in\mathbb{P}\mathbb{M}^{-1}(\mathsf{M})$. Now, by Theorem 1802, $\mathfrak{A}$ is reduced, whence $\mathfrak{A}\in(\mathbb{P}\mathbb{M}^{-1}(\mathsf{M}))^*=\overset{\leftarrow}{\mathbb{P}}^*(\mathsf{M})$. ■

Next, it is shown that, if M is a class of reduced models of a protoalgebraic π-institution, then its closure under $\overset{\leftarrow}{\mathbb{P}}^*$ is included in its closure under the operator $\overset{\triangleleft}{\mathbb{P}}\mathbb{M}^{-1*}$.

Proposition 1804 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a protoalgebraic π-institution based on* $\mathbf{F}$ *and* $\mathsf{M}\subseteq\mathrm{MatFam}^*(\mathcal{I})$ *a class of reduced $\mathcal{I}$-matrix families. Then*

$$\overset{\leftarrow}{\mathbb{P}}^*(\mathsf{M})\subseteq\overset{\triangleleft}{\mathbb{P}}\mathbb{M}^{-1*}(\mathsf{M}).$$

Proof: Suppose that $\mathsf{M} \subseteq \mathrm{MatFam}^*(\mathcal{I})$, for a protoalgebraic π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$ and let $\mathfrak{A}^* = \langle \mathcal{A}/\Omega^{\mathcal{A}}(T), T/\Omega^{\mathcal{A}}(T)\rangle \in \overleftarrow{\mathbb{P}}^*(\mathsf{M})$, where $\mathfrak{A} = \langle \mathcal{A}, T\rangle = \langle \mathcal{A}, \bigcap_{i\in I} T^i\rangle$ is such that there exist strict surjective morphisms

$$\langle H^i, \gamma^i\rangle : \langle \mathcal{A}, T^i\rangle \to \langle \mathcal{A}'^i, T'^i\rangle, \quad i \in I,$$

with $\mathfrak{A}'^i = \langle \mathcal{A}'^i, T'^i\rangle \in \mathsf{M}$, for all $i \in I$. The key now is to look at the collection of the projection morphisms

$$\langle I, \pi^i\rangle : \mathfrak{A}^* \to \langle \mathcal{A}/\Omega^{\mathcal{A}}(T^i), T^i/\Omega^{\mathcal{A}}(T^i)\rangle, \quad i \in I,$$

where, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\pi^i_\Sigma(\phi/\Omega^{\mathcal{A}}_\Sigma(\bigcap_{i\in I} T^i)) = \phi/\Omega^{\mathcal{A}}_\Sigma(T^i).$$

Since $\langle H^i, \gamma^i\rangle$ is strict and surjective, we have that $\langle \mathcal{A}, T^i\rangle \in \mathbb{M}^{-1}(\mathsf{M})$, for all $i \in I$. Thus, $\langle \mathcal{A}/\Omega^{\mathcal{A}}(T^i), T^i/\Omega^{\mathcal{A}}(T^i)\rangle \in \mathbb{M}^{-1*}(\mathsf{M})$. Therefore, to complete the proof, it suffices to show that the collection $\langle I, \pi^i\rangle$, $i \in I$, constitutes a collection of subdirect intersection morphisms. This is not difficult to verify. We have

- $\bigcap_{i\in I}(\pi^i)^{-1}(T^i/\Omega^{\mathcal{A}}(T^i)) = \bigcap_{i\in I} T^i/\Omega^{\mathcal{A}}(\bigcap_{i\in I} T^i) = (\bigcap_{i\in I} T^i)/\Omega^{\mathcal{A}}(\bigcap_{i\in I} T^i)$;
- For kernels,

$$\begin{array}{lcll}\bigcap_{i\in I}\mathrm{Ker}(\langle I, \pi^i\rangle) & = & \bigcap_{i\in I}\Omega^{\mathcal{A}}(T^i)/\Omega^{\mathcal{A}}(\bigcap_{i\in I} T^i) & \\ & = & \Omega^{\mathcal{A}}(\bigcap_{i\in I} T^i)/\Omega^{\mathcal{A}}(\bigcap_{i\in I} T^i) & (\mathcal{I} \text{ protoalgebraic}) \\ & = & \Delta^{\mathcal{A}/\Omega^{\mathcal{A}}(\bigcap_{i\in I} T^i)}. & \end{array}$$

Now we have $\mathfrak{A}^* \in \overset{\triangleleft}{\mathbb{P}}\mathbb{M}^{-1}(\mathsf{M})$. ■

We are now able to obtain, under protoalgebraicity, some equivalent expressions for the operator $\overleftarrow{\mathbb{P}}^*$, which, based on Theorem 1798, will allow us to provide characterizations for the class $\mathrm{MatFam}^*(\mathcal{I})$, in case $\mathcal{I}$ is protoalgebraic.

Theorem 1805 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathsf{M} \subseteq \mathrm{MatFam}^*(\mathcal{I})$*, for a protoalgebraic* π*-institution* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *based on* $\mathbf{F}$*. Then*

$$\overset{\triangleleft}{\mathbb{P}}(\mathsf{M}) = \overleftarrow{\mathbb{P}}^*(\mathsf{M}) = \overset{\triangleleft}{\mathbb{P}}\mathbb{M}^{-1*}(\mathsf{M}).$$

Proof: Suppose $\mathsf{M} \subseteq \mathrm{MatFam}^*(\mathcal{I})$, for a protoalgebraic π-institution $\mathcal{I} = \langle \mathbf{F}, C\rangle$. Then, we have

$$\begin{array}{lcll}\overset{\triangleleft}{\mathbb{P}}(\mathsf{M}) & \subseteq & \overleftarrow{\mathbb{P}}^*(\mathsf{M}) & (\text{Proposition 1803}) \\ & \subseteq & \overset{\triangleleft}{\mathbb{P}}\mathbb{M}^{-1*}(\mathsf{M}) & (\text{Proposition 1804}) \\ & \subseteq & \overset{\triangleleft}{\mathbb{P}}(\overset{\triangleleft}{\mathbb{P}}(\mathsf{M})) & (\text{Proposition 1801}) \\ & = & \overset{\triangleleft}{\mathbb{P}}(\mathsf{M}). & (\text{Lemma 1800})\end{array}$$

The conclusion follows. ∎

Finally, we get the following characterization of $\mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}})$ in terms of closure operators on M, under the hypothesis that M is a subclass of a protoclass of $\mathbf{F}$-matrix families.

Theorem 1806 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathsf{M} \subseteq \mathrm{MatFam}^*(\mathcal{I})$*, for a protoalgebraic* π*-institution* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *based on* $\mathbf{F}$*. Then*

$$\mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}}) = \mathbb{M}\overset{\triangleleft}{\mathbb{III}}(\mathsf{M}).$$

Proof: Suppose $\mathsf{M} \subseteq \mathrm{MatFam}^*(\mathcal{I})$, for a protoalgebraic π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$. Then,

$$\begin{array}{rcll} \mathrm{MatFam}^*(\mathcal{I}^{\mathsf{M}}) & = & \mathbb{M}\overset{\leftarrow}{\mathbb{III}}^*(\mathsf{M}) & \text{(Theorem 1798)} \\ & = & \mathbb{M}\overset{\triangleleft}{\mathbb{III}}(\mathsf{M}). & \text{(Theorem 1805)} \end{array}$$

∎

24.5 Irreducibility

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, an $\mathbf{F}$-algebraic system and $\mathfrak{A} = \langle \mathcal{A}, X \rangle \in \mathrm{MatFam}(\mathcal{I})$.

An $\mathcal{I}$-filter family $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$ is **completely meet irreducible in** $\mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$ if, for all $\{T^i : i \in I\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathfrak{A})$,

$$T = \bigcap_{i \in I} T^i \quad \text{implies} \quad T = T^i, \text{ for some } i \in I.$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathcal{A}^i = \langle \mathbf{A}^i, \langle F^i, \alpha^i \rangle \rangle$, $i \in I$, a collection of $\mathbf{F}$-algebraic systems and $\mathfrak{A}^i = \langle \mathcal{A}^i, T^i \rangle$ a collection of $\mathbf{F}$-matrix families. Recall that an $\mathbf{F}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, T \rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, is a **subdirect intersection** of the collection $\mathfrak{A}^i$, $i \in I$, if there exist surjective morphisms

$$\langle H^i, \gamma^i \rangle : \mathfrak{A} \to \mathfrak{A}^i, \quad i \in I,$$

such that $T = \bigcap_{i \in I} (\gamma^i)^{-1}(T^i)$ and $\bigcap_{i \in I} \mathrm{Ker}(\langle H^i, \gamma^i \rangle) = \Delta^{\mathcal{A}}$. This subdirect intersection is called a **special subdirect intersection** if $H^i : \mathbf{Sign} \to \mathbf{Sign}^i$ is an isomorphism, for all $i \in I$.

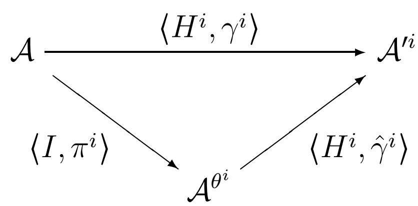
It turns out that $\mathbf{F}$-matrix families are representable as subdirect intersections if and only they are representable as special subdirect intersections.

Proposition 1807 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, with* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*, an* $\mathbf{F}$*-algebraic system and* $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ *an* $\mathbf{F}$*-matrix family. Then* $\langle H^i, \gamma^i \rangle : \mathfrak{A} \to \mathfrak{A}'^i$*,* $i \in I$*, is a collection of subdirect intersection morphisms if and only if*

$$\langle I, \pi^i \rangle : \mathfrak{A} \to \langle \mathcal{A}/\mathrm{Ker}(\langle H^i, \gamma^i \rangle), (\gamma^i)^{-1}(T'^i)/\mathrm{Ker}(\langle H^i, \gamma^i \rangle) \rangle, \quad i \in I,$$

is a collection of special subdirect intersection morphisms.

Proof: Suppose, first, that $\langle H^i, \gamma^i \rangle : \mathfrak{A} \to \mathcal{A}'^i$, $i \in I$, is a sibdirect intersection representation of $\mathfrak{A}$. For convenience, denote $\theta^i = \mathrm{Ker}(\langle H^i, \gamma^i \rangle)$, $i \in I$. Note that there exist algebraic system morphisms $\langle H^i, \hat{\gamma}^i \rangle : \mathcal{A}^{\theta^i} \to \mathcal{A}'^i$, such that



$$\langle H^i, \gamma^i \rangle = \langle H^i, \hat{\gamma}^i \rangle \circ \langle I, \pi^i \rangle,$$

where $\langle I, \pi^i \rangle : \mathcal{A} \to \mathcal{A}^{\theta^i}$, $i \in I$, are the quotient morphisms. Moreover, these morphisms are well-defined $\mathbf{F}$-matrix family morphisms, since, for all $i \in I$, we have, on the one hand, $T \le (\gamma^i)^{-1}(T'^i) = (\pi^i)^{-1}((\gamma^i)^{-1}(T'^i)/\theta^i)$, and, on the other, $(\pi^i)^{-1}((\gamma^i)^{-1}(T'^i)/\theta^i) = (\gamma^i)^{-1}(T'^i) = (\pi^i)^{-1}((\hat{\gamma}^i)^{-1}(T'^i))$ and, hence, by the surjectivity of $\langle I, \pi^i \rangle$, $(\gamma^i)^{-1}(T'^i)/\theta^i = (\hat{\gamma}^i)^{-1}(T'^i)$. Now we compute:

- For the filter families:

$$\begin{array}{l} \bigcap_{i \in I} (\pi^i)^{-1}((\gamma^i)^{-1}(T'^i)/\theta^i) \\ \quad = \bigcap_{i \in I} (\pi^i)^{-1}(\pi^i((\gamma^i)^{-1}(T'^i))) \\ \quad = \bigcap_{i \in I} (\gamma^i)^{-1}(T'^i) \\ \quad\quad (\theta^i \text{ compatible with } (\gamma^i)^{-1}(T'^i)) \\ \quad = T. \quad \text{(by hypothesis)} \end{array}$$

- For the kernels

$$\begin{array}{rcl} \bigcap_{i \in I} \mathrm{Ker}(\langle I, \pi^i \rangle) & = & \bigcap_{i \in I} \theta^i \\ & = & \Delta^{\mathcal{A}}. \quad \text{(by hypothesis)} \end{array}$$

Therefore,

$$\langle I, \pi^i \rangle : \mathfrak{A} \to \langle \mathcal{A}/\mathrm{Ker}(\langle H^i, \gamma^i \rangle), (\gamma^i)^{-1}(T'^i)/\mathrm{Ker}(\langle H^i, \gamma^i \rangle) \rangle, \quad i \in I,$$

is a collection of special subdirect intersection morphisms. ■

Special subdirect intersections of reduced matrix families have a characterization similar to the one applicable for subdirect products of reduced matriced.

Proposition 1808 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle$ *be an* $\mathbf{F}$*-algebraic system and* $\mathfrak{A} = \langle \mathcal{A}, T \rangle$ *an* $\mathbf{F}$*-matrix family.* $\mathfrak{A}$ *is a special subdirect intersection of the system* $\{\mathfrak{A}'^i = \langle \mathcal{A}'^i, T'^i \rangle : i \in I\}$ *of reduced* $\mathbf{F}$*-matrix families if and only if, there exists a corresponding system of sentence families* $\{T^i : i \in I\} \subseteq \mathrm{SenFam}(\mathcal{A})$*, such that:*

(i) $\bigcap_{i \in I} T^i = T$*;*

(ii) $\mathfrak{A}/T^i \cong \mathfrak{A}'^i$*, for all* $i \in I$*.*

Proof: Suppose, first, that $\langle H^i, \gamma^i \rangle : \mathfrak{A} \to \mathfrak{A}'^i$, $i \in I$, is a collection of special subdirect intersection morphisms. Define $T^i = (\gamma^i)^{-1}(T'^i)$, $i \in I$. Then, we have

- $\bigcap_{i \in I} T^i = \bigcap_{i \in I} (\gamma^i)^{-1}(T'^i) = T$;
- Noting that

$$\begin{array}{rcl}\Omega^{\mathcal{A}}(T^i) & = & \Omega^{\mathcal{A}}((\gamma^i)^{-1}(T'^i)) \quad (\text{definition of } T^i)\\ & = & (\gamma^i)^{-1}(\Omega^{\mathcal{A}'^i}(T'^i)) \quad (\text{Proposition 24})\\ & = & (\gamma^i)^{-1}(\Delta^{\mathcal{A}'^i}) \quad (\mathfrak{A}'^i \text{ reduced})\\ & = & \mathrm{Ker}(\langle H^i, \gamma^i \rangle), \quad (\text{set theory})\end{array}$$

  we obtain

$$\begin{array}{rcl}\mathfrak{A}/T^i & = & \langle \mathcal{A}/\Omega^{\mathcal{A}}(T^i), T^i/\Omega^{\mathcal{A}}(T^i) \rangle\\ & = & \langle \mathcal{A}/\mathrm{Ker}(\langle H^i, \gamma^i \rangle), (\gamma^i)^{-1}(T'^i)/\mathrm{Ker}(\langle H^i, \gamma^i \rangle) \rangle\\ & \cong & \mathfrak{A}'^i,\end{array}$$

  where the last isomorphism is established by the morphism $\langle H^i, \hat{\gamma}^i \rangle : \mathcal{A}/\mathrm{Ker}(\langle H^i, \gamma^i \rangle) \to \mathcal{A}'^i$, given in Proposition 1807.

Thus, (i) and (ii) of the statement hold.

Assume, conversely, that there exists a system $\{T^i : i \in I\} \subseteq \mathrm{SenFam}(\mathcal{A})$ satisfying (i) and (ii). Consider $\langle I, \pi^i \rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T^i)$, $i \in I$. This forms a well-defined system of $\mathbf{F}$-matrix family morphisms

$$\langle I, \pi^i \rangle : \mathfrak{A} \to \mathfrak{A}/T^i, \quad i \in I.$$

Since, by hypothesis, $\mathfrak{A}/T^i \cong \mathfrak{A}'^i$, for all $i \in I$, it suffices to show that the above system of morphisms constitutes a subdirect intersection. We indeed have

- $\bigcap_{i \in I} (\pi^i)^{-1}(T^i/\Omega^{\mathcal{A}}(T^i)) = \bigcap_{i \in I} T^i = T$;
- $\bigcap_{i \in I} \mathrm{Ker}(\langle I, \pi^i \rangle) = \bigcap_{i \in I} \Omega^{\mathcal{A}}(T^i) \leq \Omega^{\mathcal{A}}(\bigcap_{i \in I} T^i) = \Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}$.

So $\{\langle I,\pi^i\rangle : \mathfrak{A} \to \mathfrak{A}/T^i : i \in I\}$ is a system of special subdirect intersection morphisms. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, M a class of reduced $\mathbf{F}$-matrix families and $\mathfrak{A} = \langle \mathcal{A}, T\rangle \in \mathsf{M}$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$.

The $\mathbf{F}$-matrix family $\mathfrak{A} \in \mathsf{M}$ is called **subdirectly irreducible relative to** M if, for every subdirect intersection

$$\langle H^i, \gamma^i\rangle : \mathfrak{A} \to \mathfrak{A}'^i, \quad i \in I,$$

with $\mathfrak{A}'^i \in \mathsf{M}$, for all $i \in I$, there exists $i \in I$, such that

(i) $T = (\gamma^i)^{-1}(T'^i)$ and

(ii) $\mathrm{Ker}(\langle H^i, \gamma^i\rangle) = \Delta^{\mathcal{A}}$.

We write $\mathsf{M}^\S$ for the class of all relatively subdirectly irreducible members of M.

If $\mathcal{I} = \langle \mathbf{F}, C\rangle$ is a π-institution based on $\mathbf{F}$ and $\mathsf{M} = \mathrm{MatFam}^*(\mathcal{I})$ is the class of all reduced $\mathcal{I}$-matrix families, then a subdirectly irreducible $\mathfrak{A}$ relative to M is also called **subdirectly irreducible relative to** $\mathcal{I}$.

It turns out that relative subdirect irreducibility and complete meet irreducibility have a close relationship. To detail the relationship, we need an additional operator on classes of $\mathbf{F}$-matrix families.

Proposition 1809 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, \mathbb{N}^\flat\rangle$ *be an algebraic system and* M *a class of reduced* $\mathbf{F}$*-matrix families closed under reduced inverse morphic images, i.e., such that* $\mathbb{M}^{-1*}(\mathsf{M}) \subseteq \mathsf{M}$*. Then an* $\mathbf{F}$*-matrix family* $\mathfrak{A} = \langle \mathcal{A}, T\rangle \in \mathsf{M}$ *is subdirectly irreducible relative to* M *if and only if* T *is completely meet irreducible in* $\mathcal{X} = \{X \in \mathrm{SenFam}(\mathcal{A}) : \mathfrak{A}/X \in \mathsf{M}\}$.

Proof: Suppose, first, that $\mathfrak{A} = \langle \mathcal{A}, T\rangle \in \mathsf{M}^\S$ and let $\{X^i : i \in I\} \subseteq \mathcal{X}$, such that $T = \bigcap_{i\in I} X^i$. Then, by Proposition 1807,

$$\langle I, \pi^i\rangle : \mathfrak{A} \to \mathfrak{A}/X^i, \quad i \in I,$$

constitutes a special subdirect intersection. Moreover, since $X^i \in \mathcal{X}$, for all $i \in I$, we have that $\mathfrak{A}/X^i \in \mathsf{M}$, for all $i \in I$. By hypothesis, there exists an $i \in I$, such that $T = (\pi^i)^{-1}(X^i/\Omega^{\mathcal{A}}(X^i)) = X^i$. We conclude that T is completely meet irreducible in $\mathcal{X}$.

Assume, conversely, that T is completely meet irreducible in $\mathcal{X}$ and let

$$\langle H^i, \gamma^i\rangle : \mathfrak{A} \to \mathfrak{A}'^i, \quad i \in I,$$

be a system of subdirect intersection morphisms, with $\mathfrak{A}'^i \in \mathsf{M}$, for all $i \in I$. By Proposition 1807,

$$\langle I, \pi^i\rangle : \mathfrak{A} \to \mathfrak{A}/(\gamma^i)^{-1}(T'^i), \quad i \in I,$$

is a collection of special subdirect intersection morphisms. Moreover, $\langle H^i, \hat{\gamma}^i\rangle : \mathfrak{A}/(\gamma^i)^{-1}(T'^i) \to \mathfrak{A}'^i$, $i \in I$, are strict surjective morphisms and $\mathfrak{A}/(\gamma^i)^{-1}(T'^i)$ is reduced. Thus, since $\mathfrak{A}'^i \in \mathsf{M}$, for all $i \in I$ and $\mathbb{M}^{-1*}(\mathsf{M}) \subseteq \mathsf{M}$, we get that $\mathfrak{A}/(\gamma^i)^{-1}(T'^i) \in \mathsf{M}$, for all $i \in I$. This shows that $(\gamma^i)^{-1}(T'^i) \in \mathcal{X}$, for all $i \in I$. But, by the subdirect intersection property, $T = \bigcap_{i\in I}(\gamma^i)^{-1}(T'^i)$, whence, by hypothesis, there exists $i \in I$, such that $T = (\gamma^i)^{-1}(T'^i)$. Moreover, $\mathrm{Ker}(\langle H^i, \gamma^i\rangle) = (\gamma^i)^{-1}(\Delta^{\mathcal{A}'^i}) = (\gamma^i)^{-1}(\Omega^{\mathcal{A}'^i}(T'^i)) = \Omega^{\mathcal{A}}((\gamma^i)^{-1}(T'^i)) = \Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. Therefore, $\mathfrak{A}$ is subdirectly irreducible relative to M. ∎

Chapter 25

Order Algebraizability

25.1 Algebraic PoSystems

Let **Sign** be a category and $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a sentence functor. A **qofamily** $\leq = \{\leq_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ on SEN is a relation family on SEN, such that, for all $\Sigma \in |\mathbf{Sign}|$, $\leq_\Sigma \subseteq \mathrm{SEN}(\Sigma)^2$ is a quasi-order on $\mathrm{SEN}(\Sigma)$. A **pofamily** $\leq = \{\leq_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ on SEN is a relation family on SEN, such that, for all $\Sigma \in |\mathbf{Sign}|$, $\leq_\Sigma \subseteq \mathrm{SEN}(\Sigma)^2$ is a partial order on $\mathrm{SEN}(\Sigma)$. A **qosystem** $\leq$ on SEN is a qofamily that is also a relation system. i.e., invariant under **Sign**-morphisms, that is, such that, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}(f)(\leq_\Sigma) \subseteq \leq_{\Sigma'}.$$

Similarly, a **posystem** $\leq$ on SEN is a pofamily that is also a relation system.

Let $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an algebraic system. A **qosystem** (**posystem**) on $\mathbf{A}$ is a qosystem (posystem, respectively) on SEN. The pair $\langle \mathbf{A}, \leq \rangle$ is then called an **algebraic qosystem** (**algebraic posystem**, respectively).

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system. A **qosystem** (**posystem**) on $\mathcal{A}$ is a qosystem (posystem, respectively) on $\mathbf{A}$. We then term the pair $\langle \mathcal{A}, \leq \rangle$ an **F-algebraic qosystem** (**F-algebraic posystem**, respectively).

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system.

- The family of **F-inequations** $\mathrm{In}(\mathbf{F}) = \{\mathrm{In}_\Sigma(\mathbf{F})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{In}_\Sigma(\mathbf{F}) = \{\phi \preccurlyeq \psi : \phi, \psi \in \mathrm{SEN}^\flat(\Sigma)\};$$

- The family of **F-quasi inequations** $\mathrm{QIn}(\mathbf{F}) = \{\mathrm{QIn}_\Sigma(\mathbf{F})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{In}_\Sigma(\mathbf{F}) = \{\langle \{\phi_i \preccurlyeq \psi_i : i < k\}, \phi \preccurlyeq \psi \rangle : \vec{\phi}, \vec{\psi}, \phi, \psi \in \mathrm{SEN}^\flat(\Sigma)\};$$

- The family of **F-guasi inequations** $\mathrm{GIn}(\mathbf{F}) = \{\mathrm{GIn}_\Sigma(\mathbf{F})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{In}_\Sigma(\mathbf{F}) = \{\langle \{\phi_i \preccurlyeq \psi_i : i \in I\}, \phi \preccurlyeq \psi \rangle : \vec{\phi}, \vec{\psi}, \phi, \psi \in \mathrm{SEN}^\flat(\Sigma)\}.$$

As done previously, we sometimes abbreviate a guasi inequation $\langle \{\phi_i \preccurlyeq \psi_i : i \in I\}, \phi \preccurlyeq \psi \rangle$ by writing $\langle \vec{\phi} \preccurlyeq \vec{\psi}, \phi \preccurlyeq \psi \rangle$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and K a class of $\mathbf{F}$-algebraic posystems. We define the family $C^{\mathsf{K},\leq} : \mathcal{P}\mathrm{In}(\mathbf{F}) \to \mathcal{P}\mathrm{In}(\mathbf{F})$ by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $I \cup \{\phi \preccurlyeq \psi\} \subseteq \mathrm{In}_\Sigma(\mathbf{F})$,

$$\begin{array}{l} \phi \preccurlyeq \psi \in C^{\mathsf{K},\leq}_\Sigma(I) \text{ iff, for all } \langle \mathcal{A}, \leq^{\mathcal{A}} \rangle \in \mathsf{K}, \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma, \Sigma'), \\ \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(I)) \subseteq \leq^{\mathcal{A}}_{F(\Sigma')} \\ \qquad\qquad \text{implies } \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \leq^{\mathcal{A}}_{F(\Sigma')} \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi)). \end{array}$$

It is not difficult to see that $C^{\mathsf{K},\leq}$ is a closure system on $\mathrm{In}(\mathbf{F})$.

Lemma 1810 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic posystems. Then* $C^{\mathsf{K},\leq} : \mathcal{P}\mathrm{In}(\mathbf{F}) \to \mathcal{P}\mathrm{In}(\mathbf{F})$ *is a closure system on* $\mathrm{In}(\mathbf{F})$*.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$. It is straightforward from the definition of $C^{\mathsf{K},\leq}$ that $C^{\mathsf{K},\leq}_\Sigma$ is inflationary and monotone. We show that it is also idempotent. To this end, let $I \cup \{\phi \preccurlyeq \psi\} \subseteq \mathrm{In}_\Sigma(\mathbf{F})$, be such that $\phi \preccurlyeq \psi \in C^{\mathsf{K},\leq}_\Sigma(C^{\mathsf{K},\leq}_\Sigma(I))$. Then, for all$\langle \mathcal{A}, \leq^{\mathcal{A}} \rangle \in \mathsf{K}$, all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, we have $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(I)) \subseteq \leq^{\mathcal{A}}_{F(\Sigma')}$ implies, by definition,

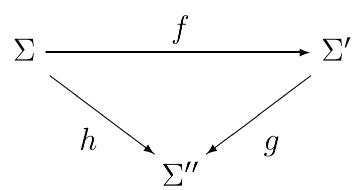
$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(C^{\mathsf{K},\leq}_\Sigma(I))) \subseteq \leq^{\mathcal{A}}_{F(\Sigma')},$$

whence, by the hypothesis and the definition,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \leq^{\mathcal{A}}_{F(\Sigma')} \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi)).$$

We now get $\phi \preccurlyeq \psi \in C^{\mathsf{K},\leq}_\Sigma(I)$. Therefore, $C^{\mathsf{K},\leq}_\Sigma$ is also idempotent.

Finally, it only remains to show structurality. To this end, let $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $I \cup \{\phi \preccurlyeq \psi\} \subseteq \mathrm{In}_\Sigma(\mathbf{F})$, such that $\phi \preccurlyeq \psi \in C^{\mathsf{K},\leq}_\Sigma(I)$. Then, by definition, for every $\langle \mathcal{A}, \leq^{\mathcal{A}} \rangle \in \mathsf{K}$, all $\Sigma'' \in |\mathbf{Sign}^\flat|$ and all $h \in \mathbf{Sign}^\flat(\Sigma, \Sigma'')$, $\alpha_{\Sigma''}(\mathrm{SEN}^\flat(h)(I)) \subseteq \leq^{\mathcal{A}}_{F(\Sigma'')}$ implies $\alpha_{\Sigma''}(\mathrm{SEN}^\flat(h)(\phi)) \leq^{\mathcal{A}}_{F(\Sigma'')} \alpha_{\Sigma''}(\mathrm{SEN}^\flat(h)(\psi))$.



In particular, for all $\Sigma'' \in |\mathbf{Sign}^\flat|$ and all $g \in \mathbf{Sign}^\flat(\Sigma', \Sigma'')$,

$$\alpha_{\Sigma''}(\mathrm{SEN}^\flat(g)(\mathrm{SEN}^\flat(f)(I))) \subseteq \leq^{\mathcal{A}}_{F(\Sigma'')}$$

implies

$$\alpha_{\Sigma''}(\mathrm{SEN}^\flat(g)(\mathrm{SEN}^\flat(f)(\phi))) \leq^{\mathcal{A}}_{F(\Sigma'')} \alpha_{\Sigma''}(\mathrm{SEN}^\flat(g)(\mathrm{SEN}^\flat(f)(\psi))).$$

Therefore, $\mathrm{SEN}^\flat(f)(\phi) \preccurlyeq \mathrm{SEN}^\flat(f)(\psi) \in C^{\mathsf{K},\leq}_{\Sigma'}(\mathrm{SEN}^\flat(f)(I))$. We conclude that $C^{\mathsf{K},\leq}$ is also structural and, hence, a closure system on $\mathrm{In}(\mathbf{F})$. ∎

As a result of Lemma 1810, it makes sense to define the **inequational** π**-institution** $\mathcal{I}^{\mathsf{K},\leq} = \langle \mathbf{F}, C^{\mathsf{K},\leq} \rangle$ **associated with** a class K of $\mathbf{F}$-algebraic posystems.

We can show that the π-institution $\mathcal{I}^{\mathsf{K},\leq}$ associated with the class K satisfies a reflexivity and transitivity property. On the other hand, antisymmetry is not expressible in the language under consideration, since it is a language without equality.

Lemma 1811 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic posystems. For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$*,*

(a) $\phi \preccurlyeq \phi \in C_\Sigma^{\mathsf{K},\leq}(\varnothing)$*;*

(b) $\phi \preccurlyeq \chi \in C_\Sigma^{\mathsf{K},\leq}(\phi \preccurlyeq \psi, \psi \preccurlyeq \chi)$*.*

Proof: Clearly, for all $\langle \mathcal{A}, \leq^{\mathcal{A}} \rangle \in \mathsf{K}$, all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, we get, by the reflexivity of $\leq^{\mathcal{A}}$, that

$$\mathrm{SEN}^\flat(f)(\phi) \leq^{\mathcal{A}}_{F(\Sigma')} \mathrm{SEN}^\flat(f)(\phi).$$

Thus, by definition, $\phi \preccurlyeq \phi \in C_\Sigma^{\mathsf{K},\leq}(\varnothing)$.

Similarly, for all $\langle \mathcal{A}, \leq^{\mathcal{A}} \rangle \in \mathsf{K}$, all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, by the transitivity of $\leq^{\mathcal{A}}$, we get that

$$\mathrm{SEN}^\flat(f)(\phi) \leq^{\mathcal{A}}_{F(\Sigma')} \mathrm{SEN}^\flat(f)(\psi) \text{ and } \mathrm{SEN}^\flat(f)(\psi) \leq^{\mathcal{A}}_{F(\Sigma')} \mathrm{SEN}^\flat(f)(\chi)$$

imply $\mathrm{SEN}^\flat(f)(\phi) \leq^{\mathcal{A}}_{F(\Sigma')} \mathrm{SEN}^\flat(f)(\chi)$. Therefore, by definition $\phi \preccurlyeq \chi \in C_\Sigma^{\mathsf{K},\leq}(\phi \preccurlyeq \psi, \psi \preccurlyeq \chi)$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\langle \mathcal{A}, \leq^{\mathcal{A}} \rangle$ an $\mathbf{F}$-algebraic posystem and $g = \langle \vec{\phi} \preccurlyeq \vec{\psi}, \phi \preccurlyeq \psi \rangle \in \mathrm{GIn}_\Sigma(\mathbf{F})$. We say $\langle \mathcal{A}, \leq^{\mathcal{A}} \rangle$ **satisfies** g or that g **holds** or **is valid in** $\langle \mathcal{A}, \leq^{\mathcal{A}} \rangle$, written

$$\langle \mathcal{A}, \leq^{\mathcal{A}} \rangle \vDash_\Sigma g,$$

if, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi_i)) \leq^{\mathcal{A}}_{F(\Sigma')} \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi_i)), \text{ for all } i \in I,$$

imply $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \leq^{\mathcal{A}}_{F(\Sigma')} \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi))$. Equivalently, $\langle \mathcal{A}, \leq^{\mathcal{A}} \rangle$ satisfies g if $\phi \preccurlyeq \psi \in C_\Sigma^{\{\langle \mathcal{A}, \leq^{\mathcal{A}} \rangle\},\leq}(\vec{\phi} \preccurlyeq \vec{\psi})$.

Given a class K of $\mathbf{F}$-algebraic posystems and a class G of $\mathbf{F}$-guasi inequations, we write $\mathrm{GIn}(\mathsf{K})$ for the class of all $\mathbf{F}$-guasi inequations satisfied by every $\mathbf{F}$-algebraic posystem in K and $\mathrm{PAlgSys}(G)$ for the class of all $\mathbf{F}$-algebraic posystems that satisfy all $\mathbf{F}$-guasi inequations in G.

We now turn to examining some operations on classes of $\mathbf{F}$-algebraic posystems. We first show that the inverse image of a posystem under an $\mathbf{F}$-algebraic system morphism is also a posystem.

Lemma 1812 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*,* $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$ *be two* $\mathbf{F}$*-algebraic systems and* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism.*

(a) If $\leq^{\mathcal{B}}$ *is a posystem on* $\mathcal{B}$*, then* $\gamma^{-1}(\leq^{\mathcal{B}})$ *is a qosystem on* $\mathcal{A}$*;*

(b) If $\leq^{\mathcal{B}}$ is a posystem on $\mathcal{B}$ and $\mathrm{Ker}(\langle H,\gamma\rangle) = \Delta^{\mathcal{A}}$*, then* $\gamma^{-1}(\leq^{\mathcal{B}})$ *is a posystem on* $\mathcal{A}$.

Proof: Let $\Sigma \in |\mathbf{Sign}|$, $\phi,\psi,\chi \in \mathrm{SEN}(\Sigma)$.

By the reflexivity of $\leq^{\mathcal{B}}$, we have $\gamma_\Sigma(\phi) \leq^{\mathcal{B}}_{H(\Sigma)} \gamma_\Sigma(\phi)$. Thus, $\phi\gamma_\Sigma^{-1}(\leq^{\mathcal{B}})\phi$ and, hence $\gamma_\Sigma^{-1}(\leq^{\mathcal{B}})$ is reflexive.

Suppose, next, that $\phi\gamma_\Sigma^{-1}(\leq^{\mathcal{B}})\psi$ and $\psi\gamma_\Sigma^{-1}(\leq^{\mathcal{B}})\chi$. Then, $\gamma_\Sigma(\phi) \leq^{\mathcal{B}}_{H(\Sigma)} \gamma_\Sigma(\psi)$ and $\gamma_\Sigma(\psi) \leq^{\mathcal{B}}_{H(\Sigma)} \gamma_\Sigma(\chi)$. Thus, by the transitivity of $\leq^{\mathcal{B}}$, we get $\gamma_\Sigma(\phi) \leq^{\mathcal{B}}_{H(\Sigma)} \gamma_\Sigma(\chi)$. Therefore, $\phi\gamma_\Sigma^{-1}(\leq^{\mathcal{B}})\chi$ and $\gamma_\Sigma^{-1}(\leq^{\mathcal{B}})$ is also transitive.

If $\phi\gamma_\Sigma^{-1}(\leq^{\mathcal{B}})\psi$, $\Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma,\Sigma')$, then we get $\gamma_\Sigma(\phi) \leq^{\mathcal{B}}_{H(\Sigma)} \gamma_\Sigma(\psi)$, whence $\mathrm{SEN}'(H(f))(\gamma_\Sigma(\phi)) \leq^{\mathcal{B}}_{H(\Sigma')} \mathrm{SEN}'(H(f))(\gamma_\Sigma(\psi))$. Thus,

$$\gamma_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \leq^{\mathcal{B}}_{H(\Sigma')} \gamma_{\Sigma'}(\mathrm{SEN}(f)(\psi)).$$

So we obtain $\mathrm{SEN}(f)(\phi)\gamma_{\Sigma'}^{-1}(\leq^{\mathcal{B}})\mathrm{SEN}(f)(\psi)$. This shows that $\gamma^{-1}(\leq^{\mathcal{B}})$ is a qosystem on $\mathcal{A}$.

Suppose, finally, for the sake of proving Part (b), that $\phi\gamma_\Sigma^{-1}(\leq^{\mathcal{B}})\psi$ and $\psi\gamma_\Sigma^{-1}(\leq^{\mathcal{B}})\phi$. Then, $\gamma_\Sigma(\phi) \leq^{\mathcal{B}}_{H(\Sigma)} \gamma_\Sigma(\psi)$ and $\gamma_\Sigma(\psi) \leq^{\mathcal{B}}_{H(\Sigma)} \gamma_\Sigma(\phi)$. Thus, by the antisymmetry of $\leq^{\mathcal{B}}$, we get $\gamma_\Sigma(\phi) = \gamma_\Sigma(\psi)$. Since, by hypothesis, $\mathrm{Ker}(\langle H,\gamma\rangle) = \Delta^{\mathcal{A}}$, we get $\phi = \psi$ and, hence, $\gamma^{-1}(\leq^{\mathcal{B}})$ is a posystem on $\mathcal{A}$ in this case. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, K a class of $\mathbf{F}$-algebraic posystems and $\langle \mathcal{A}, \leq^{\mathcal{A}}\rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, an $\mathbf{F}$-algebraic posystem.

- Given $\Sigma \in |\mathbf{Sign}^\flat|$, we say that $\langle \mathcal{A}, \leq\rangle$ is Σ-K-**order certified** if there exists $\langle \mathcal{A}^\Sigma, \leq^\Sigma\rangle \in \mathsf{K}$, such that $\mathrm{In}_\Sigma(\langle \mathcal{A}, \leq\rangle) = \mathrm{In}_\Sigma(\langle \mathcal{A}^\Sigma, \leq^\Sigma\rangle)$. In this case $\langle \mathcal{A}^\Sigma, \leq^\Sigma\rangle$ will be referred to as the Σ-K-**order certificate** of $\langle \mathcal{A}, \leq\rangle$.

- We say that $\langle \mathcal{A}, \leq\rangle$ is K-**order certified** if it is Σ-K-order certified, for all $\Sigma \in |\mathbf{Sign}^\flat|$. This, of course, means that

$$(\forall \Sigma \in |\mathbf{Sign}^\flat|)(\exists \langle \mathcal{A}^\Sigma, \leq^\Sigma\rangle \in \mathsf{K})(\mathrm{In}_\Sigma(\langle \mathcal{A}, \leq\rangle) = \mathrm{In}_\Sigma(\langle \mathcal{A}^\Sigma, \leq^\Sigma\rangle)).$$

We write $\mathbb{C}(\mathsf{K})$ for the class of all $\mathbf{F}$-algebraic posystems that are K-order certified. We say that K is an **abstract order class** whenever every K-order certified $\mathbf{F}$-algebraic posystem belongs to K, i.e., when $\mathbb{C}(\mathsf{K}) = \mathsf{K}$.

It is not difficult to show that $\mathbb{C}$ is a closure operator on classes of $\mathbf{F}$-algebraic systems.

Proposition 1813 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system. Then the operator* $\mathbb{C}$ *on classes of* $\mathbf{F}$*-algebraic posystems is a closure operator.*

Proof: Suppose K is a class of $\mathbf{F}$-algebraic posystems.

- Let $\langle\mathcal{A},\leq\rangle\in\mathsf{K}$. Then, since for all $\Sigma\in|\mathbf{Sign}^\flat|$, $\langle\mathcal{A}^\Sigma,\leq^\Sigma\rangle=\langle\mathcal{A},\leq\rangle\in\mathsf{K}$ is a Σ-K-order certificate for $\langle\mathcal{A},\leq\rangle$, we get that $\langle\mathcal{A},\leq\rangle\in\mathbb{C}(\mathsf{K})$. Thus, $\mathsf{K}\subseteq\mathbb{C}(\mathsf{K})$ and $\mathbb{C}$ is inflationary.

- If $\mathsf{K}\subseteq\mathsf{K}'$ and $\langle\mathcal{A},\leq\rangle\in\mathbb{C}(\mathsf{K})$, then, by definition, for every $\Sigma\in|\mathbf{Sign}^\flat|$, there exists a Σ-K-order certificate $\langle\mathcal{A}^\Sigma,\leq^\Sigma\rangle$. Since $\mathsf{K}\subseteq\mathsf{K}'$, $\langle\mathcal{A}^\Sigma,\leq^\Sigma\rangle\in\mathsf{K}'$ is also a Σ-K'-order certificate. Thus, $\langle\mathcal{A},\leq\rangle\in\mathbb{C}(\mathsf{K}')$ and $\mathbb{C}$ is also monotone.

- Finally, suppose that $\langle\mathcal{A},\leq\rangle\in\mathbb{C}(\mathbb{C}(\mathsf{K}))$. Then, there exists, for all $\Sigma\in|\mathbf{Sign}^\flat|$, a Σ-$\mathbb{C}(\mathsf{K})$-order certificate $\langle\mathcal{A}^\Sigma,\leq^\Sigma\rangle$ for $\mathcal{A}$. Therefore, for every $\Sigma'\in|\mathbf{Sign}^\flat|$, there exists a Σ'-K-order certificate $\langle\mathcal{A}^{\langle\Sigma,\Sigma'\rangle},\leq^{\langle\Sigma,\Sigma'\rangle}\rangle$ for $\langle\mathcal{A}^\Sigma,\leq^\Sigma\rangle$. Thus, for every $\Sigma\in|\mathbf{Sign}^\flat|$, there exists a Σ-K-order certificate $\langle\mathcal{A}^{\langle\Sigma,\Sigma\rangle},\leq^{\langle\Sigma,\Sigma\rangle}\rangle$ for $\langle\mathcal{A},\leq\rangle$, since, by hypothesis,
$$\mathrm{In}_\Sigma(\langle\mathcal{A},\leq\rangle)=\mathrm{In}_\Sigma(\langle\mathcal{A}^\Sigma,\leq^\Sigma\rangle)=\mathrm{In}_\Sigma(\langle\mathcal{A}^{\langle\Sigma,\Sigma\rangle},\leq^{\langle\Sigma,\Sigma\rangle}\rangle).$$

Thus $\mathbb{C}$ is a closure operator on classes of $\mathbf{F}$-algebraic posystems. ∎

The importance of abstract classes of $\mathbf{F}$-algebraic posystems rests on the fact that the validity of an $\mathbf{F}$-guasi inequation transfers from K-order certificates to an $\mathbf{F}$-algebraic posystem itself.

Lemma 1814 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* K *a class of* $\mathbf{F}$*-algebraic posystems and* $\langle\mathcal{A},\leq\rangle$, $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$*, an* $\mathbf{F}$*-algebraic posystem. If* $\mathcal{A}\in\mathbb{C}(\mathsf{K})$*, then* $\mathrm{GIn}(\mathsf{K})\leq\mathrm{GIn}(\mathcal{A})$.

Proof: Suppose $\langle\mathcal{A},\leq\rangle\in\mathbb{C}(\mathsf{K})$, $\Sigma\in|\mathbf{Sign}^\flat|$ and $\langle\vec{\phi}\preccurlyeq\vec{\psi},\phi\preccurlyeq\psi\rangle\in\mathrm{GIn}_\Sigma(\mathsf{K})$, such that $\vec{\phi}\preccurlyeq\vec{\psi}\subseteq\mathrm{In}_\Sigma(\langle\mathcal{A},\leq\rangle)$. Let $\langle\mathcal{A}^\Sigma,\leq^\Sigma\rangle\in\mathsf{K}$ be a Σ-K-order certificate for $\mathcal{A}$. Then, by definition $\vec{\phi}\preccurlyeq\vec{\psi}\subseteq\mathrm{In}_\Sigma(\langle\mathcal{A}^\Sigma,\leq^\Sigma\rangle)$. Since $\langle\mathcal{A}^\Sigma,\leq^\Sigma\rangle\in\mathsf{K}$ and $\langle\vec{\phi}\preccurlyeq\vec{\psi},\phi\preccurlyeq\psi\rangle\in\mathrm{GIn}_\Sigma(\mathsf{K})$, we get $\phi\preccurlyeq\psi\in\mathrm{In}_\Sigma(\langle\mathcal{A}^\Sigma,\leq^\Sigma\rangle)=\mathrm{In}_\Sigma(\langle\mathcal{A},\leq\rangle)$. Therefore, $\langle\vec{\phi}\preccurlyeq\vec{\psi},\phi\preccurlyeq\psi\rangle\in\mathrm{GIn}_\Sigma(\langle\mathcal{A},\leq\rangle)$. We conclude that $\mathrm{GIn}(\mathsf{K})\leq\mathrm{GIn}(\langle\mathcal{A},\leq\rangle)$. ∎

Let $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and K a class of $\mathbf{F}$-algebraic posystems.

- K is called an **inequational class** if there exists $I\leq\mathrm{In}(\mathbf{F})$, such that $\mathsf{K}=\mathrm{PAlgSys}(I)$;

- K is called a **quasi inequational class** if there exists $Q\leq\mathrm{QIn}(\mathbf{F})$, such that $\mathsf{K}=\mathrm{PAlgSys}(Q)$;

- K is called a **guasi inequational class** if there exists $G\leq\mathrm{GIn}(\mathbf{F})$, such that $\mathsf{K}=\mathrm{PAlgSys}(G)$.

Clearly, by definition, if K is an inequational class, then it is a quasi inequational class and, if it is a quasi inequational class, then it is a guasi inequational class.

Directly by these definitions and Lemma 1814, we get

Corollary 1815 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic posystems. If* K *is a guasi inequational class (and, hence, a fortiori, if it is a quasi inequational class or an inequational class), then it is abstract.*

Proof: Suppose K is a guasi inequational class defined by the $\mathbf{F}$-guasi inequations $G \leq \mathrm{GIn}(\mathbf{F})$ and let $\langle \mathcal{A}, \leq \rangle \in \mathbb{C}(\mathsf{K})$. Then, by Lemma 1814, $\mathrm{GIn}(\mathsf{K}) \leq \mathrm{GIn}(\langle \mathcal{A}, \leq \rangle)$, whence

$$\begin{array}{rcl} \langle \mathcal{A}, \leq \rangle & \in & \mathrm{PAlgSys}(\mathrm{GIn}(\mathcal{A})) \\ & \subseteq & \mathrm{PAlgSys}(\mathrm{GIn}(\mathsf{K})) \\ & = & \mathrm{PAlgSys}(\mathrm{GIn}(\mathrm{PAlgSys}(G))) \\ & = & \mathrm{PAlgSys}(G) \\ & = & \mathsf{K}. \end{array}$$

Thus, K is an abstract class. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and K a class of $\mathbf{F}$-algebraic posystems. We define:

- The **semantic order variety generated by** K
$$\mathbb{VO}^{\mathsf{Sem}}(\mathsf{K}) = \{\langle \mathcal{A}, \leq \rangle \in \mathrm{PAlgSys}(\mathbf{F}) : \mathrm{In}(\mathsf{K}) \leq \mathrm{In}(\langle \mathcal{A}), \leq \rangle\};$$
- The **semantic order quasivariety generated by** K
$$\mathbb{QO}^{\mathsf{Sem}}(\mathsf{K}) = \{\langle \mathcal{A}, \leq \rangle \in \mathrm{PAlgSys}(\mathbf{F}) : \mathrm{QIn}(\mathsf{K}) \leq \mathrm{QIn}(\langle \mathcal{A}, \leq \rangle)\};$$
- The **semantic order guasivariety generated by** K
$$\mathbb{GO}^{\mathsf{Sem}}(\mathsf{K}) = \{\langle \mathcal{A}, \leq \rangle \in \mathrm{PAlgSys}(\mathbf{F}) : \mathrm{GIn}(\mathsf{K}) \leq \mathrm{GIn}(\langle \mathcal{A}, \leq \rangle)\}.$$

We have the following obvious relations between these classes.

Lemma 1816 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic posystems. Then*

$$\mathsf{K} \subseteq \mathbb{GO}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{QO}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{VO}^{\mathsf{Sem}}(\mathsf{K}).$$

Proof: The essential observation is that

$$\mathrm{In}(\mathsf{K}) \leq \mathrm{QIn}(\mathsf{K}) \leq \mathrm{GIn}(\mathsf{K}).$$

Thus, we get

$$\begin{array}{l} \{\langle \mathcal{A}, \leq \rangle \in \mathrm{PAlgSys}(\mathbf{F}) : (\forall g \in \mathrm{GIn}(\mathsf{K}))(\langle \mathcal{A}, \leq \rangle \vDash g)\} \\ \quad \subseteq \{\langle \mathcal{A}, \leq \rangle \in \mathrm{PAlgSys}(\mathbf{F}) : (\forall q \in \mathrm{QIn}(\mathsf{K}))(\langle \mathcal{A}, \leq \rangle \vDash q)\} \\ \quad \subseteq \{\langle \mathcal{A}, \leq \rangle \in \mathrm{PAlgSys}(\mathbf{F}) : (\forall e \in \mathrm{In}(\mathsf{K}))(\langle \mathcal{A}, \leq \rangle \vDash e)\}. \end{array}$$

In other words, $\mathsf{K} \subseteq \mathbb{GO}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{QO}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{VO}^{\mathsf{Sem}}(\mathsf{K})$. ■

Given a class K of $\mathbf{F}$-algebraic posystems

- K is a **semantic order variety** if $\mathbb{VO}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$;
- K is a **semantic order quasivariety** if $\mathbb{QO}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$;
- K is a **semantic order guasivariety** if $\mathbb{GO}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$.

We have the following result identifying inequational classes with semantic order varieties, quasi inequational classes with semantic order quasivarieties and guasi inequational classes with semantic order guasivarieties.

Proposition 1817 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic posystems.*

(a) K *is an inequational class iff it is a semantic order variety;*

(b) K *is a quasi inequational class iff it is a semantic order quasivariety;*

(c) K *is a guasi inequational class iff it is a semantic order guasivariety.*

Proof:

(a) Suppose, first, that K is an inequational class. Then, there exists $I \leq \mathrm{In}(\mathbf{F})$, such that $\mathsf{K} = \mathrm{PAlgSys}(I)$. Let $\langle \mathcal{A}, \leq \rangle \in \mathrm{PAlgSys}(\mathbf{F})$, such that $\mathrm{In}(\mathsf{K}) \leq \mathrm{In}(\langle \mathcal{A}, \leq \rangle)$. Then we have $\langle \mathcal{A}, \leq \rangle \in \mathrm{PAlgSys}(\mathrm{In}(\langle \mathcal{A}, \leq \rangle)) \subseteq \mathrm{PAlgSys}(\mathrm{In}(\mathsf{K})) = \mathrm{PAlgSys}(\mathrm{In}(\mathrm{PAlgSys}(I))) = \mathrm{PAlgSys}(I) = \mathsf{K}$. Therefore, K is a semantic order variety.

Suppose, conversely, that K is a semantic order variety. Set $I = \mathrm{In}(\mathsf{K})$. Then $\mathsf{K} \subseteq \mathrm{PAlgSys}(\mathrm{In}(\mathsf{K})) = \mathrm{PAlgSys}(I)$. On the other hand, if $\langle \mathcal{A}, \leq \rangle \in \mathrm{PAlgSys}(I)$, then

$$\mathrm{In}(\mathsf{K}) = \mathrm{In}(\mathrm{PAlgSys}(\mathrm{In}(\mathsf{K}))) = \mathrm{In}(\mathrm{PAlgSys}(I)) \leq \mathrm{In}(\langle \mathcal{A}, \leq \rangle),$$

whence, by hypothesis, $\langle \mathcal{A}, \leq \rangle \in \mathsf{K}$. Therefore, $\mathsf{K} = \mathrm{PAlgSys}(I)$ and K is an inequational class.

(b) Suppose, first, that K is a quasi inequational class. Then, there exists $Q \leq \mathrm{QIn}(\mathbf{F})$, such that $\mathsf{K} = \mathrm{PAlgSys}(Q)$. Let $\langle \mathcal{A}, \leq \rangle \in \mathrm{PAlgSys}(\mathbf{F})$, such that $\mathrm{QIn}(\mathsf{K}) \leq \mathrm{QIn}(\langle \mathcal{A}, \leq \rangle)$. Then we have

$$\begin{array}{rcl}\langle \mathcal{A}, \leq \rangle & \in & \mathrm{PAlgSys}(\mathrm{QIn}(\langle \mathcal{A}, \leq \rangle)) \\ & \subseteq & \mathrm{PAlgSys}(\mathrm{QIn}(\mathsf{K})) \\ & = & \mathrm{PAlgSys}(\mathrm{QIn}(\mathrm{PAlgSys}(Q))) \\ & = & \mathrm{PAlgSys}(Q) \\ & = & \mathsf{K}.\end{array}$$

Therefore, K is a semantic order quasivariety.

Suppose, conversely, that K is a semantic order quasivariety. Set $Q = \mathrm{QIn}(\mathsf{K})$. Then $\mathsf{K} \subseteq \mathrm{PAlgSys}(\mathrm{QIn}(\mathsf{K})) = \mathrm{PAlgSys}(Q)$. On the other hand, if $\langle \mathcal{A}, \leq\rangle \in \mathrm{PAlgSys}(Q)$, then

$$\mathrm{QIn}(\mathsf{K}) = \mathrm{QIn}(\mathrm{PAlgSys}(\mathrm{QIn}(\mathsf{K}))) = \mathrm{QIn}(\mathrm{PAlgSys}(Q)) \leq \mathrm{QIn}(\langle \mathcal{A}, \leq\rangle),$$

whence, by hypothesis, $\langle \mathcal{A}, \leq\rangle \in \mathsf{K}$. Therefore, $\mathsf{K} = \mathrm{PAlgSys}(Q)$ and K is a quasi inequational class.

(c) Very similar to Part (b).

■

We introduce, next, some operators on classes of **F**-algebraic posystems, paralleling those introduced previously for classes of **F**-algebraic systems, that will serve to provide different characterizations to the inequational, quasi inequational and guasi inequational classes of **F**-algebraic posystems.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\langle \mathcal{A}, \leq\rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, $\langle \mathcal{A}^i, \leq^i\rangle$, with $\mathcal{A}^i = \langle \mathbf{A}^i, \langle F^i, \alpha^i\rangle\rangle$, $i \in I$, **F**-algebraic posystems and $\langle H^i, \gamma^i\rangle : \langle \mathcal{A}, \leq\rangle \to \langle \mathcal{A}^i, \leq^i\rangle$, $i \in I$, surjective morphisms. We say the collection

$$\langle H^i, \gamma^i\rangle : \langle \mathcal{A}, \leq\rangle \to \langle \mathcal{A}^i, \leq^i\rangle, \quad i \in I,$$

is a **subdirect intersection** if

$$\bigcap_{i\in I} (\gamma^i)^{-1}(\leq^i) = \leq.$$

Note that this implies that

$$\bigcap_{i\in I} \mathrm{Ker}(\langle H^i, \gamma^i\rangle) = \Delta^{\mathcal{A}}.$$

Indeed, we have

$$\begin{array}{rcl}\bigcap_{i\in I}\mathrm{Ker}(\langle H^i,\gamma^i\rangle) & = & \bigcap_{i\in I}((\gamma^i)^{-1}(\leq^i) \cap (\gamma^i)^{-1}(\leq^i)^{-1})\\ & = & \bigcap_{i\in I}(\gamma^i)^{-1}(\leq^i) \cap \bigcap_{i\in I}(\gamma^i)^{-1}(\leq^i)^{-1}\\ & = & \bigcap_{i\in I}(\gamma^i)^{-1}(\leq^i) \cap (\bigcap_{i\in I}(\gamma^i)^{-1}(\leq^i))^{-1}\\ & = & \leq \cap (\leq)^{-1}\\ & = & \Delta^{\mathcal{A}}.\end{array}$$

Given a class K of **F**-algebraic posystems, we write $\langle \mathcal{A}, \leq\rangle \in \overset{\triangleleft}{\mathbb{II}}(\mathsf{K})$ in case there exists a subdirect intersection $\{\langle H^i, \gamma^i\rangle : \langle \mathcal{A}, \leq\rangle \to \langle \mathcal{A}^i, \leq^i\rangle, i \in I\}$, with $\langle \mathcal{A}^i, \leq^i\rangle \in \mathsf{K}$, for all $i \in I$. If $\overset{\triangleleft}{\mathbb{II}}(\mathsf{K}) = \mathsf{K}$, we say that K is **closed under subdirect intersections**.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ an **F**-algebraic system and $\leq$ a posystem on $\mathcal{A}$. A congruence system $\theta \in$

$\mathrm{ConSys}(\mathcal{A})$ is said to be **compatible with** $\le$ if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \phi', \psi, \psi' \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{ccc} \phi & \overset{\le_\Sigma}{\rule{2cm}{0.4pt}} & \psi \\ \theta_\Sigma \Big| & & \Big| \theta_\Sigma \\ \phi' & \underset{\le_\Sigma}{\rule{2cm}{0.4pt}} & \psi' \end{array}$$

$$\phi \le_\Sigma \psi, \quad \phi \mathrel{\theta_\Sigma} \phi' \quad \text{and} \quad \psi \mathrel{\theta_\Sigma} \psi' \quad \text{imply} \quad \phi' \le_\Sigma \psi'.$$

A congruence system $\theta \in \mathrm{ConSys}(\mathcal{A})$ is said to be a **congruence system on the F**-algebraic posystem $\langle \mathcal{A}, \le^{\mathcal{A}} \rangle$ if it is compatible with $\le^{\mathcal{A}}$. We write $\mathrm{ConSys}(\langle \mathcal{A}, \le^{\mathcal{A}} \rangle)$ for the collection of all congruence systems on $\langle \mathcal{A}, \le^{\mathcal{A}} \rangle$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\langle \mathcal{A}, \le \rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, an $\mathbf{F}$-algebraic posystem and $\{\theta^i : i \in I\} \subseteq \mathrm{ConSys}(\langle \mathcal{A}, \le \rangle)$ a (upward) directed collection of congruence systems on $\langle \mathcal{A}, \le \rangle$. It is not difficult to show that $\bigcup_{i \in I} \theta^i \in \mathrm{ConSys}(\langle \mathcal{A}, \le \rangle)$.

Lemma 1818 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\langle \mathcal{A}, \le \rangle$ *an* $\mathbf{F}$*-algebraic posystem and* $\{\theta^i : i \in I\} \subseteq \mathrm{ConSys}(\langle \mathcal{A}, \le \rangle)$ *a directed collection of congruence systems on* $\langle \mathcal{A}, \le \rangle$*. Then* $\bigcup_{i \in I} \theta^i$ *is a congruence system on* $\langle \mathcal{A}, \le \rangle$*.*

Proof: We know, by Lemma **??** that $\bigcup_{i \in I} \theta^i_\Sigma$ is a congruence system on $\mathcal{A}$. Thus, it suffices to show that it is compatible with $\le$. To this end, suppose $\Sigma \in |\mathbf{Sign}|$, $\phi, \phi', \psi, \psi' \in \mathrm{SEN}(\Sigma)$, such that $\phi \le_\Sigma \psi$, $\phi \mathrel{\bigcup_{i \in I} \theta^i_\Sigma} \phi'$ and $\psi \mathrel{\bigcup_{i \in I} \theta^i_\Sigma} \psi'$. Thus, there exist $j \in I$ and $j' \in I$, such that $\phi \mathrel{\theta^j_\Sigma} \phi'$ and $\psi \mathrel{\theta^{j'}_\Sigma} \psi'$. But $\{\theta^i\}_{i \in I}$ is directed, whence, there exists $k \in I$, such that $\phi \mathrel{\theta^k_\Sigma} \phi'$ and $\psi \mathrel{\theta^k_\Sigma} \psi'$. Therefore, since $\theta^k \in \mathrm{ConSys}(\langle \mathcal{A}, \le \rangle)$, we get $\phi' \le_\Sigma \psi'$. We conclude that $\bigcup_{i \in I} \theta^i \in \mathrm{ConSys}(\langle \mathcal{A}, \le \rangle)$. ∎

Due to Lemma 1818, it makes sense to consider the quotient $\langle \mathcal{A}, \le \rangle / \bigcup_{i \in I} \theta^i$. This $\mathbf{F}$-algebraic posystem is called the **directed union** of the collection $\langle \mathcal{A}, \le \rangle / \theta^i$. Given a class K of $\mathbf{F}$-algebraic posystems, we write $\langle \mathcal{A}, \le \rangle / \bigcup_{i \in I} \theta^i \in \mathbb{U}(\mathsf{K})$ in case $\langle \mathcal{A}, \le \rangle / \theta^i \in \mathsf{K}$, for all $i \in I$. If $\mathbb{U}(\mathsf{K}) = \mathsf{K}$, we say that K is **closed under directed unions**.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\langle \mathcal{A}, \le^{\mathcal{A}} \rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\langle \mathcal{B}, \le^{\mathcal{B}} \rangle$, with $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$, $\mathbf{F}$-algebraic posystems and

$$\langle H, \gamma \rangle : \langle \mathcal{A}, \le^{\mathcal{A}} \rangle \to \langle \mathcal{B}, \le^{\mathcal{B}} \rangle$$

a surjective morphism. In this case we say $\langle \mathcal{B}, \le^{\mathcal{B}} \rangle$ is an **morphic image** of $\langle \mathcal{A}, \le^{\mathcal{A}} \rangle$. Given a class K of $\mathbf{F}$-algebraic posystems, we write $\langle \mathcal{B}, \le^{\mathcal{B}} \rangle \in \mathbb{H}(\mathsf{K})$ in case there exists a surjective morphism

$$\langle H, \gamma \rangle : \langle \mathcal{A}, \le^{\mathcal{A}} \rangle \to \langle \mathcal{B}, \le^{\mathcal{B}} \rangle,$$

with $\langle \mathcal{A}, \leq^{\mathcal{A}} \rangle \in \mathsf{K}$. If $\mathbb{H}(\mathsf{K}) = \mathsf{K}$, we say that K is **closed under morphic images**.

It is not difficult to verify that all three operators are closure operators on classes of **F**-algebraic posystems.

Proposition 1819 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system. Then the operators* $\overset{\triangleleft}{\mathbb{P}}$*,* $\mathbb{U}$ *and* $\mathbb{H}$ *on classes of* **F***-algebraic posystems are closure operators.*

Proof: We first look at $\overset{\triangleleft}{\mathbb{P}}$. Suppose K is a class of **F**-algebraic posystems.

- If $\langle \mathcal{A}, \leq \rangle \in \mathsf{K}$, then $\{\langle I, \iota \rangle : \langle \mathcal{A}, \leq \rangle \to \langle \mathcal{A}, \leq \rangle\}$, where $\langle I, \iota \rangle : \mathcal{A} \to \mathcal{A}$ is the identity morphism, is a subdirect intersection family. Thus, we get that $\langle \mathcal{A}, \leq \rangle \in \overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$. Hence $\mathsf{K} \subseteq \overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$ and $\overset{\triangleleft}{\mathbb{P}}$ is inflationary;
- It is obvious that $\overset{\triangleleft}{\mathbb{P}}$ is monotonic;
- Suppose that $\langle \mathcal{A}, \leq \rangle \in \overset{\triangleleft}{\mathbb{P}}(\overset{\triangleleft}{\mathbb{P}}(\mathsf{K}))$. Then, there exists a subdirect intersection family

  $$\{\langle H^i, \gamma^i \rangle : \langle \mathcal{A}, \leq \rangle \to \langle \mathcal{A}^i, \leq^i \rangle, \ i \in I\},$$

  with $\langle \mathcal{A}^i, \leq^i \rangle \in \overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$, for all $i \in I$. Therefore, for each $i \in I$, there exists a sibdirect intersection family

  $$\{\langle H^{ij}, \gamma^{ij} \rangle : \langle \mathcal{A}^i, \leq^i \rangle \to \langle \mathcal{A}^{ij}, \leq^{ij} \rangle, \ j \in J_i\},$$

  with $\langle \mathcal{A}^{ij}, \leq^{ij} \rangle \in \mathsf{K}$, for all $i \in I$ and all $j \in J_i$. Consider

  $$\{\langle H^{ij}, \gamma^{ij} \rangle \circ \langle H^i, \gamma^i \rangle : \langle \mathcal{A}, \leq \rangle \to \langle \mathcal{A}^{ij}, \leq^{ij} \rangle, \ i \in I, j \in J_i\}.$$

  It is a subdirect intersection family, since

  $$\begin{array}{rcl} \bigcap_{i \in I, j \in J_i} (\gamma^{ij} \circ \gamma^i)^{-1}(\leq^{ij}) & = & \bigcap_{i \in I, j \in J_i} (\gamma^i)^{-1}((\gamma^{ij})^{-1}(\leq^{ij})) \\ & = & \bigcap_{i \in I} (\gamma^i)^{-1}(\bigcap_{j \in J_i} (\gamma^{ij})^{-1}(\leq^{ij})) \\ & = & \bigcap_{i \in I} (\gamma^i)^{-1}(\leq^i) \\ & = & \leq. \end{array}$$

  Since $\langle \mathcal{A}^{ij}, \leq^{ij} \rangle \in \mathsf{K}$, for all $i \in I$, $j \in J_i$, we get that $\overset{\triangleleft}{\mathbb{P}}(\overset{\triangleleft}{\mathbb{P}}(\mathsf{K})) \subseteq \overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$ and $\overset{\triangleleft}{\mathbb{P}}$ is idempotent.

Thus, $\overset{\triangleleft}{\mathbb{P}}$ is a closure operator.

Now we turn to $\mathbb{U}$. Suppose, again, that K is a class of **F**-algebraic posystems.

- If $\langle \mathcal{A}, \leq \rangle \in \mathsf{K}$, we look at the singleton family $\{\Delta^{\mathcal{A}}\}$, consisting of the identity congruence system on $\langle \mathcal{A}, \leq \rangle$. Clearly, it is directed and its union is $\Delta^{\mathcal{A}}$. Therefore, $\langle \mathcal{A}, \leq \rangle \cong \langle \mathcal{A}, \leq \rangle / \Delta^{\mathcal{A}} \in \mathbb{U}(\mathsf{K})$;

- Monotonicity is obvious in this case as well;

- Suppose that $\langle \mathcal{A}, \leq \rangle / \theta \in \mathbb{U}(\mathbb{U}(\mathsf{K}))$. Then $\theta = \bigcup_{i \in I} \theta^i$ for a directed family $\{\theta^i : i \in I\} \subseteq \mathrm{ConSys}(\langle \mathcal{A}, \leq \rangle)$, such that $\langle \mathcal{A}, \leq \rangle / \theta^i \in \mathbb{U}(\mathsf{K})$, for all $i \in I$. Thus, for all $i \in I$, $\theta^i = \bigcup_{j \in J_i} \theta^{ij}$ for a directed family $\{\theta^{ij} : j \in J_i\} \subseteq \mathrm{ConSys}(\langle \mathcal{A}, \leq \rangle)$, such that $\langle \mathcal{A}, \leq \rangle / \theta^{ij} \in \mathsf{K}$, for all $j \in J_i$. Now, let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \theta^{ij_i}_\Sigma \cup \theta^{i'j_{i'}}_\Sigma$. By hypothesis, $\langle \phi, \psi \rangle \in \theta^i_\Sigma \cup \theta^{i'}_\Sigma$. Hence, since $\{\theta^i : i \in I\}$ is directed, there exists $k \in I$, such that $\langle \phi, \psi \rangle \in \theta^k_\Sigma$. Thus, again, by the hypothesis, there exists, $j_k \in J_k$, such that $\langle \phi, \psi \rangle \in \theta^{kj_k}$. We conclude that the collection $\{\theta^{ij} : j \in J_i, i \in I\}$ is directed, such that $\langle \mathcal{A}, \leq \rangle / \theta^{ij} \in \mathsf{K}$, for all $i \in I, j \in J_i$, and, moreover, $\theta = \bigcup_{i \in I} \theta^i = \bigcup_{i \in I} \bigcup_{j \in J_i} \theta^{ij} = \bigcup_{\substack{j \in J_i \\ i \in I}} \theta^{ij}$. Thus, $\langle \mathcal{A}, \leq \rangle / \theta \in \mathbb{U}(\mathsf{K})$ and $\mathbb{U}$ is also idempotent.

Therefore, $\mathbb{U}$ is also a closure operator.

Finally, we deal with $\mathbb{H}$, which is the easiest case. Let K be a class of $\mathbf{F}$-algebraic posystems. If $\langle \mathcal{A}, \leq \rangle \in \mathsf{K}$, then, using again the identity $\langle I, \iota \rangle : \langle \mathcal{A}, \leq \rangle \to \langle \mathcal{A}, \leq \rangle$, we see that $\langle \mathcal{A}, \leq \rangle \in \mathbb{H}(\mathsf{K})$, and, hence, $\mathbb{H}$ is inflationary. It is again obvious that it is monotonic. Finally, if $\mathcal{A} \in \mathbb{H}(\mathbb{H}(\mathsf{K}))$, then, there exists a surjective morphism $\langle G, \beta \rangle : \langle \mathcal{A}', \leq' \rangle \to \langle \mathcal{A}, \leq \rangle$, with $\langle \mathcal{A}', \leq' \rangle \in \mathbb{H}(\mathsf{K})$, whence, there also exists a surjective morphism $\langle H, \gamma \rangle : \langle \mathcal{A}'', \leq'' \rangle \to \langle \mathcal{A}', \leq' \rangle$, with $\langle \mathcal{A}'', \leq'' \rangle \in \mathsf{K}$. Now the surjective morphism

$$\langle G, \beta \rangle \circ \langle H, \gamma \rangle : \langle \mathcal{A}'', \leq'' \rangle \to \langle \mathcal{A}, \leq \rangle$$

witnesses the fact that $\langle \mathcal{A}, \leq \rangle \in \mathbb{H}(\mathsf{K})$. Therefore, $\mathbb{H}(\mathbb{H}(\mathsf{K})) \subseteq \mathbb{H}(\mathsf{K})$, and $\mathbb{H}$ is idempotent. Thus, $\mathbb{H}$ is a closure operator. ■

We show next that, if a class K of $\mathbf{F}$-algebraic posystems is closed under morphic images, then it is also closed under directed unions.

Proposition 1820 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *be a class of* $\mathbf{F}$*-algebraic posystems. If* K *is closed under morphic images, then it is closed under directed unions.*

Proof: Suppose K is closed under $\overset{\triangleleft}{\mathbb{P}}$ and $\mathbb{H}$ and let $\langle \mathcal{A}. \leq \rangle$, with $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, be an $\mathbf{F}$-algebraic posystem and $\{\theta^i : i \in I\} \subseteq \mathrm{ConSys}(\langle \mathcal{A}, \leq \rangle)$, a directed family of congruence systems, such that $\langle \mathcal{A}, \leq \rangle / \theta^i \in \mathsf{K}$. Consider a morphism

$$\langle I, \pi^i \rangle : \langle \mathcal{A}, \leq \rangle / \theta^i \to \langle \mathcal{A}, \leq \rangle / \bigcup_{i \in I} \theta^i,$$

given, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, by

$$\pi^i_\Sigma(\phi/\theta^i_\Sigma) = \phi/\bigcup_{i\in I}\theta^i_\Sigma.$$

It is well defined, since, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi,\psi \in \mathrm{SEN}(\Sigma)$, if $\langle\phi,\psi\rangle \in \theta^i_\Sigma$, then, automatically, $\langle\phi,\psi\rangle \in \bigcup_{i\in I}\theta^i_\Sigma$. Therefore, since $\langle\mathcal{A},\leq\rangle/\theta^i \in \mathsf{K}$, we get, by hypothesis, $\langle\mathcal{A},\leq\rangle/\bigcup_{i\in I}\theta^i \in \mathbb{H}(\mathsf{K}) = \mathsf{K}$. We conclude that $\mathbb{U}(\mathsf{K}) \subseteq \mathsf{K}$ and, hence, K is closed under directed unions. ■

We are now ready to provide alternative characterizations of inequational, quasi inequational and guasi inequational classes of **F**-algebraic posystems. Namely, we show that an abstract class of **F**-algebraic posystems is a guasi inequational class if and only if it is closed under subdirect intersections, that it is a quasi inequational class if and only if it is closed under subdirect intersections and directed unions and that it is an inequational class if and only if it is closed under subdirect intersections and morphic images.

Theorem 1821 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* K *an abstract class of* **F***-algebraic posystems.* K *is a guasi inequational class if and only if it is closed under subdirect intersections.*

Proof: Suppose, first, that K is a guasi inequational class and consider a subdirect intersection

$$\{\langle H^i,\gamma^i\rangle : \langle\mathcal{A},\leq\rangle \to \langle\mathcal{A}^i,\leq^i\rangle,\ i\in I\},$$

with $\langle\mathcal{A}^i,\leq^i\rangle \in \mathsf{K}$. Let G be the set of guasi inequations defining K and $\Sigma \in |\mathbf{Sign}^\flat|$, $\langle\vec{\phi} \preccurlyeq \vec{\psi},\phi \preccurlyeq \psi\rangle \in G_\Sigma$, such that, for some $\Sigma' \in |\mathbf{Sign}^\flat|$ and $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi_j)) \leq_{F(\Sigma')} \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi_j)),\ \text{for all } j \in J.$$

Then we get $\gamma^i_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi_j))) \leq^i_{H^i(F(\Sigma'))} \gamma^i_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi_j)))$, for all $i \in I$, $j \in J$. This gives $\alpha^i_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi_j)) \leq^i_{F^i(\Sigma')} \alpha^i_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi_j))$, for all $i \in I$, $j \in J$. Since $\langle\mathcal{A}^i,\leq^i\rangle \in \mathsf{K}$, for all $i \in I$, and $\langle\vec{\phi} \preccurlyeq \vec{\psi},\phi \preccurlyeq \psi\rangle \in G_\Sigma$, we get that $\alpha^i_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \leq^i_{F^i(\Sigma')} \alpha^i_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi))$, for all $i \in I$. Equivalently, $\gamma^i_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi))) \leq^i_{H^i(F(\Sigma'))} \gamma^i_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi)))$, for all $i \in I$, i.e.,

$$\langle\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)),\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi))\rangle \in \bigcap_{i\in I}(\gamma^i_{F(\Sigma')})^{-1}(\leq^i).$$

Since $\{\langle H^i,\gamma^i\rangle : i \in I\}$ is a subdirect intersection, we get

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \leq_{F(\Sigma')} \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi)).$$

We conclude that $\langle\mathcal{A},\le\rangle\in\mathrm{PAlgSys}(G)=\mathsf{K}$. Hence, K is closed under subdirect intersections.

Assume, conversely, that K is closed under subdirect intersections and set $G=\mathrm{GIn}(\mathsf{K})$. Let $\langle\mathcal{A},\le\rangle\in\mathrm{PAlgSys}(\mathbf{F})$, such that $G\le\mathrm{GIn}(\langle\mathcal{A},\le\rangle)$. Let $\Sigma\in|\mathbf{Sign}^\flat|$, such that $\phi\preccurlyeq\psi\notin\mathrm{In}_\Sigma(\langle\mathcal{A},\le\rangle)$, i.e., for some $\Sigma'\in|\mathbf{Sign}^\flat|$ and some $f\in\mathbf{Sign}^\flat(\Sigma,\Sigma')$, $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi))\not\le_{F(\Sigma')}\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi))$. Thus, by definition, the guasi inequation $\langle\mathrm{In}_\Sigma(\langle\mathcal{A},\le\rangle),\phi\preccurlyeq\psi\rangle\notin\mathrm{GIn}_\Sigma(\langle\mathcal{A},\le\rangle)$. Therefore, since $G\le\mathrm{GIn}(\langle\mathcal{A},\le\rangle)$, $\langle\mathrm{In}_\Sigma(\mathcal{A}),\phi\preccurlyeq\psi\rangle\notin\mathrm{GIn}_\Sigma(\mathsf{K})$. Hence, for every $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\phi\preccurlyeq\psi\notin\mathrm{In}_\Sigma(\langle\mathcal{A},\le\rangle)$, there exists $\langle\mathcal{K}^{\langle\Sigma,\phi,\psi\rangle},\le^{\langle\Sigma,\phi,\psi\rangle}\rangle\in\mathsf{K}$, such that

$$\begin{gathered}\mathrm{In}_\Sigma(\langle\mathcal{A},\le\rangle)\subseteq\mathrm{In}_\Sigma(\langle\mathcal{K}^{\langle\Sigma,\phi,\psi\rangle},\le^{\langle\Sigma,\phi,\psi\rangle}\rangle),\\ \phi\preccurlyeq\psi\notin\mathrm{In}_\Sigma(\langle\mathcal{K}^{\langle\Sigma,\phi,\psi\rangle},\le^{\langle\Sigma,\phi,\psi\rangle}\rangle).\end{gathered}$$

We conclude that

$$\mathrm{In}_\Sigma(\langle\mathcal{A},\le\rangle)=\bigcap\{\mathrm{In}_\Sigma(\langle\mathcal{K}^{\langle\Sigma,\phi,\psi\rangle},\le^{\langle\Sigma,\phi,\psi\rangle}\rangle):\phi\preccurlyeq\psi\notin\mathrm{In}_\Sigma(\langle\mathcal{A},\le\rangle)\}.$$

Denote, for all $\Sigma\in|\mathbf{Sign}^\flat|$, $\mathsf{K}^\Sigma=\{\langle\mathcal{K}^{\langle\Sigma,\phi,\psi\rangle},\le^{\langle\Sigma,\phi,\psi\rangle}\rangle:\phi\preccurlyeq\psi\notin\mathrm{In}_\Sigma(\langle\mathcal{A},\le\rangle)\}$, for brevity.

- Since K is closed under subdirect intersections, and

  $$\{\langle F^{\mathcal{K}},\alpha^{\mathcal{K}}\rangle:\langle\mathcal{F},\bigcap_{\langle\mathcal{K},\le^{\mathcal{K}}\rangle\in\mathsf{K}^\Sigma}(\alpha^{\mathcal{K}})^{-1}(\le^{\mathcal{K}})\rangle/\mathrm{Eq}(\mathsf{K}^\Sigma)\to\langle\mathcal{K},\le^{\mathcal{K}}\rangle,\langle\mathcal{K},\le^{\mathcal{K}}\rangle\in\mathsf{K}^\Sigma\}$$

  is a subdirect intersection, we get that

  $$\langle\mathcal{F},\bigcap_{\langle\mathcal{K},\le^{\mathcal{K}}\rangle\in\mathsf{K}^\Sigma}(\alpha^{\mathcal{K}})^{-1}(\le^{\mathcal{K}})\rangle/\mathrm{Eq}(\mathsf{K}^\Sigma)\in\mathsf{K}.$$

- Since, for all $\Sigma\in|\mathbf{Sign}^\flat|$,

  $$\begin{array}{rcl}\mathrm{In}_\Sigma(\langle\mathcal{A},\le\rangle)&=&\mathrm{In}_\Sigma(\mathsf{K}^\Sigma)\\ &=&\mathrm{In}_\Sigma(\langle\mathcal{F},\bigcap_{\langle\mathcal{K},\le^{\mathcal{K}}\rangle\in\mathsf{K}^\Sigma}(\alpha^{\mathcal{K}})^{-1}(\le^{\mathcal{K}})\rangle/\mathrm{Eq}(\mathsf{K}^\Sigma))\end{array}$$

  and $\langle\mathcal{F},\bigcap_{\langle\mathcal{K},\le^{\mathcal{K}}\rangle\in\mathsf{K}^\Sigma}(\alpha^{\mathcal{K}})^{-1}(\le^{\mathcal{K}})\rangle/\mathrm{Eq}(\mathsf{K}^\Sigma)\in\mathsf{K}$, $\langle\mathcal{A},\le\rangle\in\mathbb{C}(\mathsf{K})$. Since K is abstract, we conclude that $\langle\mathcal{A},\le\rangle\in\mathsf{K}$.

Hence, K is indeed a guasi inequational class of $\mathbf{F}$-algebraic posystems. ■

Theorem 1822 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and* K *an abstract class of* $\mathbf{F}$*-algebraic posystems.* K *is a quasi inequational class if and only if it is closed under subdirect intersections and directed unions.*

Proof: Suppose, first, that K is a quasi inequational class, defined by a collection Q of $\mathbf{F}$-quasi inequations. Then it is a guasi inequational class and, therefore, by Theorem 1821, closed under subdirect intersections. Let

$\langle\mathcal{A},\le\rangle$ be an $\mathbf{F}$-algebraic posystem and $\{\theta^i : i\in I\}\subseteq \mathrm{ConSys}(\langle\mathcal{A},\le\rangle)$ a directed union of congruence systems on $\langle\mathcal{A},\le\rangle$, such that $\langle\mathcal{A},\le\rangle/\theta^i\in\mathsf{K}$, for all $i\in I$. Let $\Sigma\in|\mathbf{Sign}^\flat|$, $\langle\vec{\psi}\preccurlyeq\vec{\psi},\phi\preccurlyeq\psi\rangle\in Q_\Sigma$, such that, for some $\Sigma'\in|\mathbf{Sign}^\flat|$ and $f\in\mathbf{Sign}^\flat(\Sigma,\Sigma')$,

$$\alpha_{\Sigma'}^{\bigcup_{i\in I}\theta^i}(\mathrm{SEN}^\flat(f)(\phi_j))\le_{F(\Sigma')}^{\bigcup_{i\in I}\theta^i}\alpha_{\Sigma'}^{\bigcup_{i\in I}\theta^i}(\mathrm{SEN}^\flat(f)(\psi_j)),\quad j<n.$$

Thus, for every $j<n$, there exists $k_j\in I$, such that

$$\alpha_{\Sigma'}^{\theta^{k_j}}(\mathrm{SEN}^\flat(f)(\phi_j))\le_{F(\Sigma')}^{\theta^{k_j}}\alpha_{\Sigma'}^{\theta^{k_j}}(\mathrm{SEN}^\flat(f)(\psi_j)).$$

Since $\{\theta^i : i\in I\}$ is directed, there exists $k\in I$, such that

$$\alpha_{\Sigma'}^{\theta^k}(\mathrm{SEN}^\flat(f)(\phi_j))\le_{F(\Sigma')}^{\theta^k}\alpha_{\Sigma'}^{\theta^k}(\mathrm{SEN}^\flat(f)(\psi_j)),\quad j<n.$$

Since $\langle\mathcal{A},\le\rangle/\theta^k\in\mathsf{K}$ and $\langle\vec{\psi}\preccurlyeq\vec{\psi},\phi\preccurlyeq\psi\rangle\in Q_\Sigma$, we get that

$$\alpha_{\Sigma'}^{\bigcup_{i\in I}\theta^i}(\mathrm{SEN}^\flat(f)(\phi))\le_{F(\Sigma')}^{\bigcup_{i\in I}\theta^i}\alpha_{\Sigma'}^{\bigcup_{i\in I}\theta^i}(\mathrm{SEN}^\flat(f)(\psi)).$$

Therefore, $\langle\mathcal{A},\le\rangle/\bigcup_{i\in I}\theta^i\in\mathrm{PAlgSys}(Q)=\mathsf{K}$ and K is closed under directed unions.

Suppose, conversely, that K is an abstract class of $\mathbf{F}$-algebraic posystems closed under subdirect intersections and directed unions. Set $Q=\mathrm{QIn}(\mathsf{K})$ and let $\langle\mathcal{A},\le\rangle\in\mathrm{PAlgSys}(\mathbf{F})$, such that $Q\le\mathrm{QIn}(\mathcal{A})$. Let $\Sigma\in|\mathbf{Sign}^\flat|$, such that $\phi\preccurlyeq\psi\notin\mathrm{In}_\Sigma(\langle\mathcal{A},\le\rangle)$, i.e., for some $\Sigma'\in|\mathbf{Sign}^\flat|$ and some $f\in\mathbf{Sign}^\flat(\Sigma,\Sigma')$, $\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi))\nleq_{F(\Sigma')}\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi))$. Thus, by definition, for every finite $I\le\mathrm{In}_\Sigma(\mathcal{A})$ the quasi inequation $\langle I,\phi\preccurlyeq\psi\rangle\notin\mathrm{QIn}_\Sigma(\langle\mathcal{A},\le\rangle)$. Therefore, since $Q\le\mathrm{QIn}(\langle\mathcal{A},\le\rangle)$, $\langle I,\phi\preccurlyeq\psi\rangle\notin\mathrm{QIn}_\Sigma(\mathsf{K})$. Hence, for every $\Sigma\in|\mathbf{Sign}^\flat|$, all $I\subseteq_f\mathrm{In}_\Sigma(\langle\mathcal{A},\le\rangle)$ and all $\phi\preccurlyeq\psi\notin\mathrm{In}_\Sigma(\langle\mathcal{A},\le\rangle)$, there exists

$$\langle\mathcal{K}^{\langle\Sigma,I,\phi,\psi\rangle},\le^{\langle\Sigma,I,\phi,\psi\rangle}\rangle\in\mathsf{K},$$

such that

- $I\subseteq\mathrm{In}_\Sigma(\langle\mathcal{K}^{\langle\Sigma,I,\phi,\psi\rangle},\le^{\langle\Sigma,I,\phi,\psi\rangle}\rangle)$;
- $\phi\preccurlyeq\psi\notin\mathrm{In}_\Sigma(\langle\mathcal{K}^{\langle\Sigma,I,\phi,\psi\rangle},\le^{\langle\Sigma,I,\phi,\psi\rangle}\rangle)$.

We conclude that, for all $\Sigma\in|\mathbf{Sign}^\flat|$,

$$\begin{aligned}\mathrm{In}_\Sigma(\langle\mathcal{A},\le\rangle)=\bigcap\{\bigcup\{&\mathrm{In}_\Sigma(\mathcal{K}^{\langle\Sigma,I,\phi,\psi\rangle},\le^{\langle\Sigma,I,\phi,\psi\rangle}\rangle):\\ &I\subseteq_f\mathrm{In}_\Sigma(\langle\mathcal{A},\le\rangle)\}:\phi\preccurlyeq\psi\notin\mathrm{In}_\Sigma(\langle\mathcal{A},\le\rangle)\}.\end{aligned}$$

Denote, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\phi\preccurlyeq\psi\notin\mathrm{In}_\Sigma(\langle\mathcal{A},\le\rangle)$,

$$\mathsf{K}^{\langle\Sigma,\phi,\psi\rangle}=\{\langle\mathcal{F},(\alpha^{\langle\Sigma,I,\phi,\psi\rangle})^{-1}(\le^{\langle\Sigma,I,\phi,\psi\rangle})\rangle/\mathrm{Eq}(\mathcal{K}^{\langle\Sigma,I,\phi,\psi\rangle}):I\subseteq_f\mathrm{In}_\Sigma(\langle\mathcal{A},\le\rangle)\},$$

and, for all $\Sigma\in|\mathbf{Sign}^\flat|$,

$$\begin{aligned}\mathsf{K}^\Sigma=\{\langle\mathcal{F},\textstyle\bigcup_{\mathcal{K}\in\mathsf{K}^{\langle\Sigma,\phi,\psi\rangle}}(\alpha^{\mathcal{K}})^{-1}(\le^{\mathcal{K}})\rangle/\textstyle\bigcup_{\mathcal{K}\in\mathsf{K}^{\langle\Sigma,\phi,\psi\rangle}}\mathrm{Eq}(\mathcal{K}):&\\ \phi\preccurlyeq\psi\notin\mathrm{In}_\Sigma(\langle\mathcal{A},\le\rangle)\},&\end{aligned}$$

for brevity.

- Since K is closed under directed unions, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \leqslant \psi \notin \mathrm{In}_\Sigma(\mathcal{A})$, we have $\langle \mathcal{F}, \bigcup_{\mathcal{K} \in \mathsf{K}\langle\Sigma,\phi,\psi\rangle} (\alpha^{\mathcal{K}})^{-1}(\leq^{\mathcal{K}})\rangle / \bigcup_{\mathcal{K} \in \mathsf{K}\langle\Sigma,\phi,\psi\rangle} \mathrm{Eq}(\mathcal{K}) \in \mathsf{K}$.
- Since K is closed under subdirect intersections,
$$\langle \mathcal{F}, \bigcap_{\mathcal{K} \in \mathsf{K}^\Sigma} (\alpha^{\mathcal{K}})^{-1}(\leq^{\mathcal{K}})\rangle / \mathrm{Eq}(\mathsf{K}^\Sigma) \in \mathsf{K},$$
for all $\Sigma \in |\mathbf{Sign}^\flat|$.
- Finally, noting that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\langle \mathcal{F}, \bigcap_{\mathcal{K} \in \mathsf{K}^\Sigma} (\alpha^{\mathcal{K}})^{-1}(\leq^{\mathcal{K}})\rangle / \mathrm{Eq}(\mathsf{K}^\Sigma)$ is a Σ-K-certificate for $\mathcal{A}$, and, taking into account that K is abstract, we conclude that $\mathcal{A} \in \mathsf{K}$.

Therefore K is indeed a quasiequational class of $\mathbf{F}$-algebraic systems. ■

25.2 Syntactic Order Algebraizability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$ and K a class of $\mathbf{F}$-algebraic posystems. If $\mathcal{I}$ is equivalent to $\mathcal{I}^{\mathsf{K},\leq}$ via a conjugate pair $(\alpha, \beta) : \mathcal{I} \rightleftarrows \mathcal{I}^{\mathsf{K},\leq}$, we say that the class K **β-order algebraizes** the π-institution $\mathcal{I}$. Recall, in more detail, that this means that there exist a collection $\alpha : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with a single distinguished argument and a collection $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with two distinguished arguments, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$ and $I \subseteq \mathrm{In}_\Sigma(\mathbf{F})$,

1. $\phi \in C_\Sigma(\Phi)$ if and only if $\alpha_\Sigma[\phi] \leq C_\Sigma^{\mathsf{K},\leq}(\alpha_\Sigma[\Phi])$;
2. $\phi \leqslant \psi \in C_\Sigma^{\mathsf{K},\leq}(I)$ if and only if $\beta_\Sigma[\phi, \psi] \leq C(\beta_\Sigma[I])$;
3. $C^{\mathsf{K},\leq}(\phi \leqslant \psi) = C^{\mathsf{K},\leq}(\alpha[\beta_\Sigma[\phi, \psi]])$;
4. $C(\phi) = C(\beta[\alpha_\Sigma[\phi]])$.

Moreover, we say that $\mathcal{I}$ is **β-order algebraizable** if there exists a class K of $\mathbf{F}$-algebraic posystems, such that K β-order algebraizes $\mathcal{I}$. In this case, we call the least order guasivariety including K the **β-ordered class of $\mathcal{I}$** and denote it by $\mathrm{PAlgSys}(\mathcal{I})$.

Lemma 1823 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$ having two distinguished arguments, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$ and K, K' two classes of $\mathbf{F}$-algebraic posystems. If both K and K' β-order algebraize $\mathcal{I}$, then $\mathcal{I}^{\mathsf{K},\leq} = \mathcal{I}^{\mathsf{K}',\leq}$. Therefore, $\mathbb{GO}^{\mathsf{Sem}}(\mathsf{K}) = \mathbb{GO}^{\mathsf{Sem}}(\mathsf{K}')$.*

Proof: Suppose both K and K' β-order algebraize $\mathcal{I}$. Then, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $I \cup \{\phi \preccurlyeq \psi\} \in \mathrm{In}_\Sigma(\mathbf{F})$,

$$\begin{array}{lll}\phi \preccurlyeq \psi \in C_\Sigma^{\mathsf{K},\leq}(I) & \text{iff} & \beta_\Sigma[\phi,\psi] \leq C(\beta_\Sigma[I]) \\ & \text{iff} & \phi \preccurlyeq \psi \in C_\Sigma^{\mathsf{K}',\leq}(I).\end{array}$$

We conclude that $\mathcal{I}^{\mathsf{K},\leq} = \mathcal{I}^{\mathsf{K}',\leq}$, whence the semantic order guasivarieties generated by K and K' coincide. ■

We call the unique semantic order guasivariety that β-order algebraizes $\mathcal{I}$ the β-**order class of** $\mathcal{I}$ and denote it by $\mathrm{PAlgSys}^\beta(\mathcal{I})$.

Next we show that if two families $\beta, \beta' : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$ are deductively equivalent, in the sense that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $\beta_\Sigma[\phi,\psi]$ and $\beta'_\Sigma[\phi,\psi]$ are interderivable, then $\mathcal{I}$ is β-order algebraizable if and only if it is β'-order algebraizable and, in fact, in that case, the corresponding order classes of $\mathcal{I}$ coincide.

Lemma 1824 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\beta, \beta' : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with two distinguished arguments, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$. If, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $C(\beta_\Sigma[\phi,\psi]) = C(\beta'_\Sigma[\phi,\psi])$, then $\mathcal{I}$ is β-order algebraizable if and only if $\mathcal{I}$ is β'-order algebraizable. In that case, the β- and β'-order classes of $\mathcal{I}$ coincide, i.e.,* $\mathrm{PAlgSys}^\beta(\mathcal{I}) = \mathrm{PAlgSys}^{\beta'}(\mathcal{I})$.

Proof: Suppose that $\mathcal{I}$ is β-order algebraizable. Then, there exists a conjugate pair $(\alpha,\beta) : \mathcal{I} \rightleftarrows \mathcal{I}^{\mathsf{K},\leq}$. We show that $\mathcal{I}$ is also β'-order algebraizable via the conjugate pair $(\alpha,\beta') : \mathcal{I} \rightleftarrows \mathcal{I}^{\mathsf{K},\leq}$.

- We have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $I \cup \{\phi \preccurlyeq \psi\} \subseteq \mathrm{In}_\Sigma(\mathbf{F})$,

$$\begin{array}{lll}\phi \preccurlyeq \psi \in C_\Sigma^{\mathsf{K},\leq}(I) & \text{iff} & \beta_\Sigma[\phi,\psi] \leq C(\beta_\Sigma[I]) \\ & \text{iff} & \beta'_\Sigma[\phi,\psi] \leq C(\beta'_\Sigma[I]).\end{array}$$

- $C(\phi) = C(\beta[\alpha_\Sigma[\phi]]) = C(\beta'[\alpha_\Sigma[\phi]])$.

Thus, by Proposition 898, $\mathcal{I}$ and $\mathcal{I}^{\mathsf{K},\leq}$ are equivalent via (α,β'). By symmetry, we infer the first statement of the lemma. The second conclusion now follows directly from Lemma 1823, since the same class K both β- and β'-order algebraizes $\mathcal{I}$. ■

Moreover, we can show that the conjugate transformation α in a β-order algebraization is essentially unique, in the sense that any two of them are deductively equivalent modulo inequational derivability.

Lemma 1825 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$ having two distinguished arguments, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$ and K a class of $\mathbf{F}$-algebraic posystems. If K β-order* *algebraizes* $\mathcal{I}$ *via a conjugate pair* $(\alpha,\beta):\mathcal{I}\rightleftarrows\mathcal{I}^{\mathsf{K},\le}$ *and via a conjugate pair* $(\alpha',\beta):\mathcal{I}\rightleftarrows\mathcal{I}^{\mathsf{K},\le}$*, then, for all* $\Sigma\in|\mathbf{Sign}^\flat|$ *and all* $\phi\in\mathrm{SEN}^\flat(\Sigma)$*,*

$$C^{\mathsf{K},\le}(\alpha_\Sigma[\phi])=C^{\mathsf{K},\le}(\alpha'_\Sigma[\phi]).$$

Proof: We have, for all $\Sigma\in|\mathbf{Sign}^\flat|$ and all $\phi\in\mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}\alpha_\Sigma[\phi]\le C^{\mathsf{K},\le}(\alpha'_\Sigma[\phi]) & \text{iff} & \beta[\alpha_\Sigma[\phi]]\le C(\beta[\alpha'_\Sigma[\phi]])\\ & \text{iff} & \phi\in C_\Sigma(\phi).\end{array}$$

Therefore, $\alpha_\Sigma[\phi]\le C^{\mathsf{K},\le}(\alpha'_\Sigma[\phi])$ and, hence, by symmetry, $C^{\mathsf{K},\le}(\alpha_\Sigma[\phi])=C^{\mathsf{K},\le}(\alpha'_\Sigma[\phi])$. ■

We give next some conditions that are equivalent to $\overset{\leftrightarrow}{\beta}$ defining Leibniz congruence systems of theory families of $\mathcal{I}$. Recall that this is tantamount to $\mathcal{I}$ being syntactically protoalgebraic.

Theorem 1826 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system, with* $\beta:(\mathrm{SEN}^\flat)^\omega\to\mathrm{SEN}^\flat$ *in* $N^\flat$ *having two distinguished arguments and* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following conditions are equivalent:*

(i) *For all* $\langle\mathcal{A},T\rangle\in\mathrm{MatFam}^*(\mathcal{I})$*,* $\beta^{\mathcal{A}}(T)$ *is reflexive and antisymmetric;*

(ii) *For all* $\sigma^\flat$ *in* $N^\flat$*, all* $\Sigma\in|\mathbf{Sign}^\flat|$ *and all* $\phi,\psi,\vec{\chi}\in\mathrm{SEN}^\flat(\Sigma)$*,*

- $\beta_\Sigma[\phi,\phi]\le\mathrm{Thm}(\mathcal{I})$;
- $\sigma^\flat_\Sigma(\psi,\vec{\chi})\in C_\Sigma(\beta_\Sigma[\phi,\psi],\beta_\Sigma[\psi,\phi],\sigma^\flat_\Sigma(\phi,\vec{\chi}))$;

(iii) *For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}=\langle\mathbf{A},\langle F,\alpha\rangle\rangle$ *and every* $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $\Omega^{\mathcal{A}}(T)=\overset{\leftrightarrow}{\beta}{}^{\mathcal{A}}(T)$*.*

Proof:

(i)⇒(ii) Suppose that, for all $\langle\mathcal{A},T\rangle\in\mathrm{MatFam}^*(\mathcal{I})$, $\beta^{\mathcal{A}}(T)$ is reflexive and antisymmetric and let $\sigma^\flat\in N^\flat$, $\Sigma\in|\mathbf{Sign}^\flat|$, and $\phi,\psi,\vec{\chi}\in\mathrm{SEN}^\flat(\Sigma)$. Consider $\langle\mathcal{F}/\Omega(\mathrm{Thm}(\mathcal{I})),\mathrm{Thm}(\mathcal{I})/\Omega(\mathrm{Thm}(\mathcal{I}))\rangle\in\mathrm{MatFam}^*(\mathcal{I})$. Then, by hypothesis, $\langle\phi,\psi\rangle\in\beta_\Sigma^{\mathcal{F}/\Omega(\mathrm{Thm}(\mathcal{I}))}(\mathrm{Thm}(\mathcal{I}))$, i.e.,

$$\beta_\Sigma^{\mathcal{F}/\Omega(\mathrm{Thm}(\mathcal{I}))}[\phi/\Omega_\Sigma(\mathrm{Thm}(\mathcal{I})),\phi/\Omega_\Sigma(\mathrm{Thm}(\mathcal{I}))]\le\mathrm{Thm}(\mathcal{I})/\Omega(\mathrm{Thm}(\mathcal{I})).$$

This is equivalent to $\beta_\Sigma[\phi,\phi]\le\mathrm{Thm}(\mathcal{I})$.

Assume, next, that for some $T\in\mathrm{ThFam}(\mathcal{I})$, $\beta_\Sigma[\phi,\psi],\beta_\Sigma[\psi,\phi]\le T$ and $\sigma^\flat_\Sigma(\phi,\vec{\chi})\in T_\Sigma$. Then, we get

$$\begin{array}{l}\beta_\Sigma^{\mathcal{F}/\Omega(T)}[\phi/\Omega_\Sigma(T),\psi/\Omega_\Sigma(T)]\le T/\Omega(T),\\ \beta_\Sigma^{\mathcal{F}/\Omega(T)}[\psi/\Omega_\Sigma(T),\phi/\Omega_\Sigma(T)]\le T/\Omega(T),\end{array}$$

i.e.,

$$\langle\phi/\Omega_\Sigma(T),\psi/\Omega_\Sigma(T)\rangle\in\beta_\Sigma^{\mathcal{F}/\Omega(T)}(T/\Omega(T)),$$
$$\langle\psi/\Omega_\Sigma(T),\phi/\Omega_\Sigma(T)\rangle\in\beta_\Sigma^{\mathcal{F}/\Omega(T)}(T/\Omega(T)).$$

But on $\langle\mathcal{F}/\Omega(T),T/\Omega(T)\rangle\in\mathrm{MatFam}^*(\mathcal{I})$, $\beta^{\mathcal{F}/\Omega(T)}(T/\Omega(T))$ is, by hypothesis, antisymmetric, whence we get $\langle\phi,\psi\rangle\in\Omega_\Sigma(T)$. Therefore, since $\sigma_\Sigma^\flat(\phi,\vec{\chi})\in T_\Sigma$, we have, by compatibility, that $\sigma_\Sigma^\flat(\psi,\vec{\chi})\in T_\Sigma$.

(ii)⇒(iii) Suppose that Condition (ii) holds and let $\mathcal{A}$ be an **F**-algebraic system and $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Let $\Sigma\in|\mathbf{Sign}|$ and $\phi,\psi\in\mathrm{SEN}(\Sigma)$.

- If $\langle\phi,\psi\rangle\in\Omega_\Sigma^{\mathcal{A}}(T)$, then, for all $\sigma^\flat\in\beta$ and all $\Sigma'\in|\mathbf{Sign}|$, $f\in\mathbf{Sign}(\Sigma,\Sigma')$, $\vec{\chi}\in\mathrm{SEN}(\Sigma')$,

$$\begin{aligned}&\langle\sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}(f)(\phi),\mathrm{SEN}(f)(\psi),\vec{\chi}),\\ &\qquad\sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}(f)(\phi),\mathrm{SEN}(f)(\phi),\vec{\chi})\rangle\in\Omega_{\Sigma'}^{\mathcal{A}}(T),\\ &\langle\sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}(f)(\psi),\mathrm{SEN}(f)(\phi),\vec{\chi}),\\ &\qquad\sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}(f)(\phi),\mathrm{SEN}(f)(\phi),\vec{\chi})\rangle\in\Omega_{\Sigma'}^{\mathcal{A}}(T).\end{aligned}$$

  Thus, by compatibility, $\beta_\Sigma^{\mathcal{A}}[\phi,\psi]\le T$ and $\beta_\Sigma^{\mathcal{A}}[\psi,\phi]\le T$, i.e., $\overset{\leftrightarrow\mathcal{A}}{\beta}_\Sigma[\phi,\psi]\le T$. Thus, $\langle\phi,\psi\rangle\in\overset{\leftrightarrow}{\beta}_\Sigma(T)$.

- Suppose, conversely, that $\langle\phi,\psi\rangle\in\overset{\leftrightarrow\mathcal{A}}{\beta}_\Sigma(T)$, i.e., $\beta_\Sigma^{\mathcal{A}}[\phi,\psi]\le T$ and $\beta_\Sigma^{\mathcal{A}}[\psi,\phi]\le T$. Then, by hypothesis, we have, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma'\in|\mathbf{Sign}|$, all $f\in\mathbf{Sign}(\Sigma,\Sigma')$ and all $\vec{\chi}\in\mathrm{SEN}(\Sigma')$,

$$\sigma\mathcal{A}_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi})\in T_{\Sigma'}\quad\text{iff}\quad\sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}(f)(\psi),\vec{\chi})\in T_{\Sigma'}.$$

  Therefore, $\langle\phi,\psi\rangle\in\Omega^{\mathcal{A}}(T)$.

we conclude that $\overset{\leftrightarrow\mathcal{A}}{\beta}(T)=\Omega^{\mathcal{A}}(T)$.

(iii)⇒(i) Finally, suppose that, for every **F**-algebraic system $\mathcal{A}$ and all $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\overset{\leftrightarrow\mathcal{A}}{\beta}(T)=\Omega^{\mathcal{A}}(T)$ and let $\langle\mathcal{A},T\rangle\in\mathrm{MatFam}^*(\mathcal{I})$. Then we get that $\overset{\leftrightarrow\mathcal{A}}{\beta}(T)=\Omega^{\mathcal{A}}(T)=\Delta^{\mathcal{A}}$. Thus, clearly, $\beta^{\mathcal{A}}(T)$ is reflexive and antisymmetric. ■

We obtain as a corollary characterizing those collection of natural transformations with two distinguished arguments that define posystems on the class $\mathrm{AlgSys}^*(\mathcal{I})$, i.e., on the algebraic system reducts of the reduced matrix families of a π-institution $\mathcal{I}$.

Corollary 1827 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system, with* $\beta:(\mathrm{SEN}^\flat)^\omega\to\mathrm{SEN}^\flat$ *in* $N^\flat$ *having two distinguished arguments, and* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$. *For all* $\langle\mathcal{A},T\rangle\in\mathrm{MatFam}^*(\mathcal{I})$, $\beta^{\mathcal{A}}(T)$ *is a posystem on* $\mathcal{A}$ *if and only if the following conditions hold, for all* $\sigma^\flat$ *in* $N^\flat$, $\Sigma\in|\mathbf{Sign}^\flat|$, $\phi,\psi,\chi,\vec{\chi}\in\mathrm{SEN}^\flat(\Sigma)$*:*

1. $\beta_\Sigma[\phi,\phi] \leq \mathrm{Thm}(\mathcal{I})$;

2. $\beta_\Sigma[\phi,\chi] \leq C(\beta_\Sigma[\phi,\psi],\beta_\Sigma[\psi,\chi])$;

3. $\sigma^\flat_\Sigma(\psi,\vec{\chi}) \in C_\Sigma(\overleftrightarrow{\beta}_\Sigma[\phi,\psi],\sigma^\flat(\phi,\vec{\chi}))$.

Proof: Suppose that, for all $\langle\mathcal{A},T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$, $\beta^{\mathcal{A}}(T)$ is a posystem on $\mathcal{A}$. Then, by Theorem 1826, Conditions 1 and 3 hold. By considering all reduced matrix families $\langle/\Omega(T),T/\Omega(T)\rangle$, with $t \in \mathrm{ThFam}(\mathcal{I})$, we get, by the transitivity of $\beta^{\mathcal{F}/\Omega(T)}(T/\Omega(T))$, $\beta^{\mathcal{F}/\Omega(T)}_\Sigma[\phi/\Omega_\Sigma(T),\psi/\Omega_\Sigma(T)] \leq T/\Omega(T)$ and $\beta^{\mathcal{F}/\Omega(T)}_\Sigma[\psi/\Omega_\Sigma(T),\chi/\Omega_\Sigma(T)] \leq T/\Omega(T)$ imply $\beta^{\mathcal{F}/\Omega(T)}_\Sigma[\phi/\Omega_\Sigma(T),\chi/\Omega_\Sigma(T)] \leq T/\Omega(T)$, i.e., $\beta_\Sigma[\phi,\psi] \leq T$ and $\beta_\Sigma[\psi,\chi] \leq T$ imply $\beta_\Sigma[\phi,\chi] \leq T$. This proves that $\beta_\Sigma[\phi,\chi] \leq C(\beta_\Sigma[\phi,\psi],\beta_\Sigma[\psi,\chi])$, i.e., that Condition 2 also holds.

Conversely, suppose Conditions 1-3 hold and let $\langle\mathcal{A},T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$. Then, by Theorem 1826, the relation system $\beta^{\mathcal{A}}(T)$ is reflexive and antisymmetric. But, by Condition 2 of the hypothesis, it is also transitive and, therefore, it is a posystem on $\mathcal{A}$. ∎

Given a π-institution $\mathcal{I} - \langle\mathbf{F},C\rangle$, we term any collection $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$, with two distinguished arguments, that satisfies 1-3 of Corollary 1827 a **semi-equivalence system for** $\mathcal{I}$.

Now we are in a position to provide a characterization of syntactic order algebraizability.

Theorem 1828 *Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system and $\mathcal{I} = \langle\mathbf{F},C\rangle$ a π-institution based on $\mathbf{F}$. The following conditions are equivalent:*

(i) $\mathcal{I}$ is syntactically order algebraizable;

(ii) There exists $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$ having two distinguishes arguments and $\alpha : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ with a single distinguished argument, such that, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi,\chi,\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

- $\beta_\Sigma[\phi,\phi] \leq \mathrm{Thm}(\mathcal{I})$;
- $\beta_\Sigma[\phi,\chi] \leq C(\beta_\Sigma[\phi,\psi],\beta_\Sigma[\psi,\chi])$;
- $\beta_\Sigma[\sigma^\flat_\Sigma(\psi,\vec{\chi}),\tau^\flat_\Sigma(\psi,\vec{\chi})] \leq C(\overleftrightarrow{\beta}_\Sigma[\phi,\psi],\beta_\Sigma[\sigma^\flat_\Sigma(\phi,\vec{\chi}),\tau^\flat_\Sigma(\phi,\vec{\chi})])$;
- $C(\phi) = C(\beta[\alpha_\Sigma[\phi]])$;

(iii) $\mathcal{I}$ has a semi-equivalence system β and there exists $\alpha : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ with a single distinguished argument, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $C(\phi) = C(\beta[\alpha_\Sigma[\phi]])$.

If any of Conditions (i)-(iii) holds, then $\mathcal{I}$ is β-order algebraizable, with β in $N^\flat$ any collection satisfying Condition (ii) or (iii).

Proof:

(i)⇒(ii) Suppose $\mathcal{I}$ is syntactically order algebraizable. Then, by definition, it is equivalent to the inequational π-institution $\mathcal{I}^{\mathsf{K},\leq} = \langle \mathbf{F}, C^{\mathsf{K},\leq}\rangle$ associated with some class K of $\mathbf{F}$-algebraic posystems, via a conjugate pair $(\alpha,\beta) : \mathcal{I} \rightleftarrows \mathcal{I}^{\mathsf{K},\leq}$. Let $\sigma^\flat$ in $N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi,\chi,\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$.

- We have, by Lemma 1811, $\phi \preccurlyeq \phi \in C_\Sigma^{\mathsf{K},\leq}(\varnothing)$. Thus, we get $\beta_\Sigma[\phi,\phi] \leq C(\varnothing)$.
- Similarly, by Lemma 1811, $\phi \preccurlyeq \chi \in C_\Sigma^{\mathsf{K},\leq}(\phi \preccurlyeq \psi, \psi \preccurlyeq \chi)$. Therefore, $\beta_\Sigma[\phi,\chi] \leq C(\beta_\Sigma[\phi,\psi],\beta_\Sigma[\psi,\chi])$.
- Since K is a class of $\mathbf{F}$-algebraic posystems, we have

  $$\sigma_\Sigma^\flat(\psi,\vec{\chi}) \preccurlyeq \tau_\Sigma^\flat(\psi,\vec{\chi}) \in C_\Sigma^{\mathsf{K},\leq}(\phi \preccurlyeq \psi, \psi \preccurlyeq \phi, \sigma_\Sigma^\flat(\phi,\vec{\chi}) \preccurlyeq \tau_\Sigma^\flat(\phi,\vec{\chi})).$$

  From this, we get

  $$\beta_\Sigma[\sigma_\Sigma^\flat(\psi,\vec{\chi}),\tau_\Sigma^\flat(\psi,\vec{\chi})] \leq C(\overleftrightarrow{\beta}_\Sigma[\phi,\psi],\beta_\Sigma[\sigma_\Sigma^\flat(\phi,\vec{\chi}),\tau_\Sigma^\flat(\phi,\vec{\chi})]).$$
- $C(\phi) = C(\beta[\alpha_\Sigma[\phi]])$ holds by the definition of equivalence.

(ii)⇒(iii) Assume that $\alpha : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ with one distinguished argument and $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ with two distinguished arguments satisfy the Conditions in (ii). According to the definition of a semi-equivalence system, it suffices to show that, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi,\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\sigma_\Sigma^\flat(\psi,\vec{\chi}) \in C_\Sigma(\overleftrightarrow{\beta}_\Sigma[\phi,\psi],\sigma_\Sigma^\flat(\phi,\vec{\chi})).$$

By the last condition in the hypothesis, we have

$$\beta[\alpha_\Sigma[\sigma_\Sigma^\flat(\phi,\vec{\chi})]] \leq C(\sigma_\Sigma^\flat(\phi,\vec{\chi})).$$

By the third condition in the hypothesis, we get

$$\beta[\alpha_\Sigma[\sigma_\Sigma^\flat(\psi,\vec{\chi})]] \leq C(\overleftrightarrow{\beta}_\Sigma[\phi,\psi],\beta[\alpha_\Sigma[\sigma_\Sigma^\flat(\phi,\vec{\chi})]]).$$

Again, using the last condition in the hypothesis, we get

$$\sigma_\Sigma^\flat(\psi,\vec{\chi}) \in C_\Sigma(\beta[\alpha_\Sigma[\sigma_\Sigma^\flat(\psi,\vec{\chi})]]).$$

Combining these, we get $\sigma_\Sigma^\flat(\psi,\vec{\chi}) \in C_\Sigma(\overleftrightarrow{\beta}_\Sigma[\phi,\psi],\sigma_\Sigma^\flat(\phi,\vec{\chi}))$.

(iii)⇒(i) Suppose $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with two distinguished arguments, is a semi-equivalence system for $\mathcal{I}$ and $\alpha : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$ satisfies the condition in (iii). We have to construct a class of

$\mathbf{F}$-algebraic posystems that will serve as the basis for the syntactic order algebraization of $\mathcal{I}$. Consider $\langle \mathcal{A}, T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$. Define on $\mathcal{A}$, $\leq^{\mathcal{A},T} = \{\leq^{\mathcal{A},T}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ by setting, for all $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\phi \leq^{\mathcal{A},T}_\Sigma \psi \quad \text{iff} \quad \beta^{\mathcal{A}}_\Sigma[\phi, \psi] \leq T.$$

By Corollary 1827, $\leq^{\mathcal{A},T}$ is a posystem on $\mathcal{A}$. Set

$$\mathsf{K} = \{\langle \mathcal{A}, \leq^{\mathcal{A},T}\rangle : \langle \mathcal{A}, T\rangle \in \mathrm{MatFam}^*(\mathcal{I})\}.$$

It now suffices to show that $\mathcal{I}$ is equivalent to $\mathcal{I}^{\mathsf{K},\leq}$ via the conjugate pair $(\alpha, \beta) : \mathcal{I} \rightleftarrows \mathcal{I}^{\mathsf{K},\leq}$. One of the two requirements demanded by Proposition 898 is fulfilled by the hypothesis. It suffices, therefore, to show that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $I \cup \{\phi \preccurlyeq \psi\} \subseteq \mathrm{In}_\Sigma(\mathbf{F})$,

$$\phi \preccurlyeq \psi \in C^{\mathsf{K},\leq}_\Sigma(I) \quad \text{iff} \quad \beta_\Sigma[\phi, \psi] \leq C(\beta_\Sigma[I]).$$

We have $\phi \preccurlyeq \psi \in C^{\mathsf{K},\leq}_\Sigma(I)$ if and only if, for all $\langle \mathcal{A}, \leq^{\mathcal{A},T}\rangle \in \mathsf{K}$, $\Sigma' \in |\mathbf{Sign}^\flat|$ $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\begin{array}{l} \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(I)) \subseteq \leq^{\mathcal{A},T}_{F(\Sigma')} \\ \qquad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \leq^{\mathcal{A},T}_{F(\Sigma')} \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi)) \end{array}$$

if and only if, for all $\langle \mathcal{A}, T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$, all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\begin{array}{l} \beta[\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(I))] \leq T \\ \qquad \text{implies} \quad \beta[\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)), \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi))] \leq T \end{array}$$

if and only if, for all $\langle \mathcal{A}, T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$, all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\begin{array}{l} \alpha(\beta_{\Sigma'}[\mathrm{SEN}^\flat(f)(I)]) \leq T \\ \qquad \text{implies} \quad \alpha(\beta_{\Sigma'}[\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi)]) \leq T \end{array}$$

iff, by the completeness of $\mathcal{I}$ with respect to $\mathrm{MatFam}^*(\mathcal{I})$, $\beta_\Sigma[\phi, \psi] \leq C(\beta_\Sigma[I])$. ■

This characterization allows us to obtain several properties pertaining to syntactic order algebraizability.

Theorem 1829 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, having two distinguished arguments, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a β-order algebraizable π-institution based on $\mathbf{F}$.*

(a) For every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$\Omega^{\mathcal{A}}(T) = \overset{\leftrightarrow}{\beta}{}^{\mathcal{A}}(T);$$

(b) *The β-order class of $\mathcal{I}$ is the semantic order guasivariety generated by* $\mathsf{K} = \{\langle\mathcal{A}, \leq^{\mathcal{A},T}\rangle : \langle\mathcal{A},T\rangle \in \mathrm{MatFam}^*(\mathcal{I})\}$*, where, for all* $\langle\mathcal{A},T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$*,* $\Sigma \in |\mathbf{Sign}|$*,* $\phi,\psi \in \mathrm{SEN}(\Sigma)$*,*

$$\phi \leq_\Sigma^{\mathcal{A},T} \psi \quad \textit{iff} \quad \beta_\Sigma^{\mathcal{A}}[\phi,\psi] \leq T;$$

(c) *For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, the mapping* $T \mapsto \beta^{\mathcal{A}}(T)$ *in injective on* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*.*

Proof:

(a) The conclusion follows from Theorems 1828 and 1826.

(b) This also follows from Theorem 1828.

(c) Suppose $\alpha : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, having one distinguished argument, be as in Theorem 1828 and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system, $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$. Then we have

$$\begin{array}{lll} \phi \in T_\Sigma & \text{iff} & \beta^{\mathcal{A}}[\alpha_\Sigma^{\mathcal{A}}[\phi]] \leq T \\ & \text{iff} & \alpha_\Sigma^{\mathcal{A}}[\phi] \leq \beta^{\mathcal{A}}(T), \end{array}$$

and, similarly, $\phi \in T'_\Sigma$ if and only if $\alpha_\Sigma^{\mathcal{A}}[\phi] \leq \beta^{\mathcal{A}}(T')$. We conclude that, if $\beta^{\mathcal{A}}(T) = \beta^{\mathcal{A}}(T')$, then $T = T'$ and, hence, $T \mapsto \beta^{\mathcal{A}}(T)$ is injective.

■

We can now establish some connections between syntactic order algebraizability and syntactic protoalgebraicity.

Proposition 1830 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathcal{I} = \langle\mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.*

(a) $\mathcal{I}$ *has a semi-equivalence system if and only if it is syntactically protoalgebraic;*

(b) *If* $\mathcal{I}$ *is syntactically order algebraizable, then it is syntactically protoalgebraic;*

(c) *If* $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, with two distinguished arguments, witnesses the syntactic protoalgebraicity of* $\mathcal{I}$ *and* $\mathcal{I}$ *is* β*-order algebraizable, then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$C(\overset{\leftrightarrow}{I}{}^\flat_\Sigma[\phi,\psi]) = C(\overset{\leftrightarrow}{\beta}_\Sigma[\phi,\psi]).$$

Proof:

(a) $\mathcal{I}$ has a semi-equivalence system if and only if, by Corollary 1827, for all $\langle \mathcal{A}, T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$, $\beta^{\mathcal{A}}(T)$ is a posystem on $\mathcal{A}$, implies that, for all $\langle \mathcal{A}, T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$, $\beta^{\mathcal{A}}(T)$ is reflexive and antisymmetric, if and only if, by Theorem 1826, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Omega^{\mathcal{A}}(T) = \overrightarrow{\beta}^{\mathcal{A}}(T)$, if and only if $\mathcal{I}$ is syntactically protoalgebraic.

On the other hand, if $\mathcal{I}$ is syntactically protoalgebraic, with witnessing transformations $I^\flat : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$, having two distinguished arguments, then, for all $\langle \mathcal{A}, T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$, $\overleftrightarrow{I}^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}$, whence for all $\langle \mathcal{A}, T\rangle \in \mathrm{MatFam}^*(\mathcal{I})$, $\overleftrightarrow{I}^{\mathcal{A}}(T)$ is a posystem on $\mathcal{A}$ and, hence, by Corollary 1827, $\overleftrightarrow{I}^\flat$ is a semi-equivalence system for $\mathcal{I}$.

(b) By Part (a) of Theorem 1829.

(c) This follows from the fact that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $\overleftrightarrow{I}^\flat_\Sigma[\phi, \psi] \le T$ if and only if $\langle \phi, \psi\rangle \in \Omega(T)$ if and only if $\overleftrightarrow{\beta}_\Sigma[\phi, \psi] \le T$.

■

Theorem 1830 reveals two important properties. First, that syntactic order algebraizability implies syntactic protoalgebraicity and, second, that the latter is equivalent to the existence of a semi-equivalence system for $\mathcal{I}$. Recall that syntactic protoalgebraicity is one component in syntactic WF algebraizability. We turn now to investigating how far syntactic WF algebraizability is from syntactic order algebraizability.

Theorem 1831 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, having two distinguished arguments, $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a β-order algebraizable π-institution, based on $\mathbf{F}$, and K the β-order class of $\mathcal{I}$. Then, the following conditions are equivalent:*

(i) $\mathcal{I}$ is syntactically WF algebraizable;

(ii) There exists $\gamma : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with two distinguished arguments, such that, for all $\langle \mathcal{A}, \le^{\mathcal{A}}\rangle \in \mathsf{K}$, $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\phi \preccurlyeq^{\mathcal{A}}_\Sigma \psi \quad \text{iff} \quad \gamma^{\mathcal{A}}_\Sigma[\phi, \psi] \le \Delta^{\mathcal{A}};$$

(iii) There exists $\gamma : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with two distinguished arguments, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C(\beta_\Sigma[\phi, \psi]) = C(\overleftrightarrow{\beta}[\gamma_\Sigma[\phi, \psi]]);$$

(iv) $\mathcal{I}$ is $S^{\mathcal{I}}$-fortified and for every $\mathbf{F}$-algebraic system $\mathcal{A}$, $\Omega^{\mathcal{A}}$ is injective on $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$;

(v) $\mathcal{I}$ is $S^{\mathcal{I}}$-fortified and Ω is injective on ThFam($\mathcal{I}$).

Proof:

(iv)⇔(v) We know that injectivity of the Leibniz operator transfers from Theory families to all filter families over arbitrary algebraic systems.

(i)⇔(iv) If $\mathcal{I}$ is syntactically WF algebraizable, then it is $R^{\mathcal{I}}S^{\mathcal{I}}$-fortified, protoalgebraic and family injective. Suppose, conversely, that $\mathcal{I}$ is $S^{\mathcal{I}}$-fortified and family injective. This implies that $\mathcal{I}$ is family truth equational. Together with the syntactic protoalgebraicity following from the hypothesis and Proposition 1830, we get that $\mathcal{I}$ is syntactically WF algebraizable.

(ii)⇔(iii) Let $\gamma : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with two distinguished arguments. Suppose, first, that, for all $\langle\mathcal{A},\leq^{\mathcal{A}}\rangle \in \mathsf{K}$, $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi \in \mathrm{SEN}(\Sigma)$, $\phi \leq^{\mathcal{A}}_\Sigma \psi$ iff $\gamma^{\mathcal{A}}_\Sigma[\phi,\psi] \leq \Delta^{\mathcal{A}}$. Then, we have, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\beta_\Sigma[\phi,\psi] \leq T \quad \text{iff} \quad \overleftrightarrow{\beta}[\gamma_\Sigma[\phi,\psi]] \leq T,$$

which yields the conclusion. Conversely, if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, $C(\beta_\Sigma[\phi,\psi]) = C(\overleftrightarrow{\beta}[\gamma_\Sigma[\phi,\psi]])$, then, we get

$$C^{\mathsf{K},\leq}(\alpha[\beta_\Sigma[\phi,\psi]]) = C^{\mathsf{K},\leq}(\alpha[\overleftrightarrow{\beta}[\gamma_\Sigma[\phi,\psi]]]).$$

Thus, $C^{\mathsf{K},\leq}(\phi \preccurlyeq \psi) = C^{\mathsf{K},\leq}(\gamma_\Sigma[\phi,\psi] \cup \gamma_\Sigma[\phi,\psi]^{-1})$. This yields the conclusion if we take into account that K consists of $\mathbf{F}$-algebraic posystems.

(i)⇒(ii) Suppose that $\mathcal{I}$ is syntactically WF algebraizable via the conjugate pair $(\tau, I) : \mathcal{I} \rightleftarrows \mathcal{Q}^{\mathrm{AlgSys}^*(\mathcal{I})}$. Then, by Proposition 1830, we get that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, $C(\overleftrightarrow{I}^\flat_\Sigma[\phi,\psi]) = C(\overleftrightarrow{\beta}_\Sigma[\phi,\psi])$. Thus, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, $C(\phi) = C(\overleftrightarrow{I}^\flat[\tau^\flat_\Sigma[\phi]]) = C(\overleftrightarrow{\beta}[\tau^\flat_\Sigma[\phi]])$. Therefore, in particular, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C(\beta_\Sigma[\phi,\psi]) = C(\overleftrightarrow{\beta}[\tau^\flat[\beta_\Sigma[\phi,\psi]]]).$$

Now consider any $\langle\mathcal{A},\leq^{\mathcal{A},T}\rangle$, where $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is such that $\Omega^{\mathcal{A}}(T) = \Delta^{\mathcal{A}}$. Then, we get, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi,\psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{aligned} \phi \leq^{\mathcal{A},T}_\Sigma \psi \quad &\text{iff} \quad \beta^{\mathcal{A}}_\Sigma[\phi,\psi] \leq T \\ &\text{iff} \quad \overleftrightarrow{\beta}^{\mathcal{A}}[\tau^{\mathcal{A}}[\beta^{\mathcal{A}}_\Sigma[\phi,\psi]]] \leq T \\ &\text{iff} \quad \tau^{\mathcal{A}}[\beta^{\mathcal{A}}_\Sigma[\phi,\psi]] \leq \Delta^{\mathcal{A}}. \end{aligned}$$

Thus, taking into account the fact that K is the semantic order guasivariety generated by the class $\{\langle\mathcal{A},\leq^{\mathcal{A},T}\rangle : \langle\mathcal{A},T\rangle \in \mathrm{MatFam}^*(\mathcal{I})\}$, we conclude that $\gamma := \tau \circ \beta$ is witnessing the property asserted in Part (ii).

(iii)⇒(v) Finally, assume $(\alpha, \beta) : \mathcal{I} \rightleftarrows \mathcal{I}^{\mathsf{K},\le}$ witnesses the β-order algebraizability of $\mathcal{I}$ and that $\gamma : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$, with two distinguished arguments satisfies the property in Condition (iii). Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, we have

$$\begin{array}{rll} \phi \in T_\Sigma & \text{iff} & \beta[\alpha_\Sigma[\phi]] \le T \\ & \text{iff} & \overleftrightarrow{\beta}[\gamma[\alpha_\Sigma[\phi]]] \le T \\ & \text{iff} & \gamma[\alpha_\Sigma[\phi]] \le \Omega^{\mathcal{A}}(T). \end{array}$$

This shows that $\mathcal{I}$ is truth equational, which implies that it is $S^{\mathcal{I}}$-fortified and family injective.

■

25.3 Polarities

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. A **polarity for F** is a pair $M = (M^+, M^-)$, where M^+ and M^- are subsets of $N^\flat$.

The intuition behind the definition is that

- if $\sigma^\flat \in M^+$, then it is monotone in the first argument and
- if $\sigma^\flat \in M^-$, then it is antimonotone in the first argument.

Why only referring to the first argument? The reason is that it suffices to refer to the first argument to cover all arguments. Suppose, e.g., that $\sigma^\flat : (\mathrm{SEN}^\flat)^2 \to \mathrm{SEN}^\flat$ is in $N^\flat$. Then $\sigma^\flat \circ \langle p^{2,1}, p^{2,0} \rangle$ is also in $N^\flat$. If we denote $\sigma^\flat$ informally by $\sigma^\flat(x, y)$, then we may denote $\sigma^\flat \circ \langle p^{2,1}, p^{2,0} \rangle$ by $\sigma^\flat(y, x)$. Since both transformations are in $N^\flat$, if we wanted to declare that $\sigma^\flat$ is, say, antimonotone in the second argument, then we would assign $\sigma^\flat(y, x)$ in M^-, getting away with referring only to the first argument of some natural transformation in $N^\flat$. The same trick may be used for any argument position and, hence, the expression "$\sigma^\flat$ **has positive** (or **negative polarity**) **in the k-th argument**" should come as no surprise, even though the formal assignment is done only by classifying leading arguments.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with two distinguished arguments, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. Define the **polarity** $B = (B^+, B^-)$ **induced by** β (the letter B here is chosen to correspond to the transformation β) by setting, for all $\sigma^\flat$ in $N^\flat$:

(+) $\sigma^\flat \in B^+$ if and only if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\sigma^\flat_\Sigma(\psi, \vec{\chi}) \in C_\Sigma(\beta_\Sigma[\phi, \psi], \sigma^\flat_\Sigma(\phi, \vec{\chi}));$$

(−) $\sigma^\flat \in B^-$ if and only if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\sigma^\flat_\Sigma(\phi, \vec{\chi}) \in C_\Sigma(\beta_\Sigma[\phi, \psi], \sigma^\flat_\Sigma(\psi, \vec{\chi})).$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system, $\le$ a relation system on $\mathcal{A}$ and $T \in \mathrm{SenFam}(\mathcal{A})$. We say that $\le$ is M-**compatible with** T if, for all $\sigma^\flat$ in $N^\flat$, $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi, \vec{\chi} \in \mathrm{SEN}(\Sigma)$,

- if $\sigma^\flat \in M^+$, $\phi \le_\Sigma \psi$, then $\sigma^{\mathcal{A}}_\Sigma(\phi, \vec{\chi}) \in T_\Sigma$ imply $\sigma^{\mathcal{A}}_\Sigma(\psi, \vec{\chi}) \in T_\Sigma$;
- if $\sigma^\flat \in M^-$, $\phi \le_\Sigma \psi$, then $\sigma^{\mathcal{A}}_\Sigma(\psi, \vec{\chi}) \in T_\Sigma$ imply $\sigma^{\mathcal{A}}_\Sigma(\phi, \vec{\chi}) \in T_\Sigma$.

Proposition 1832 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T \in \mathrm{SenFam}(\mathcal{A})$*, there exists a largest qosystem on* $\mathcal{A}$ *that is* M*-compatible with* T*.*

Proof: We consider the class $\mathrm{QoSys}^{\mathcal{A}}(T)$ of all qosystems on $\mathcal{A}$ that are M-compatible with T. We take the transitive closure of the union of all qosystems in $\mathrm{QoSys}^{\mathcal{A}}(T)$,

$$\mathrm{tc}(\bigcup \mathrm{QoSys}^{\mathcal{A}}(T)) = \{\mathrm{tc}_\Sigma(\bigcup \mathrm{QoSys}^{\mathcal{A}}(T))\}_{\Sigma \in |\mathbf{Sign}|}.$$

It suffices to show that this is also a qosystem on $\mathcal{A}$ M-compatible with T, i.e., it is itself a member of $\mathrm{QoSys}^{\mathcal{A}}(T)$. It will then follow that it is its largest member. It is clear by the definition that $\mathrm{tr}(\bigcup \mathrm{QoSys}^{\mathcal{A}}(T))$ is a qosystem on $\mathcal{A}$. So it suffices to show that it is M-compatible with T. Suppose $\sigma^\flat$ in M^+, $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi, \vec{\chi} \in \mathrm{SEN}(\Sigma)$, such that $\phi \ \mathrm{tr}_\Sigma(\bigcup \mathrm{QoSys}^{\mathcal{A}}(T)) \ \psi$ and $\sigma^\flat_\Sigma(\phi, \vec{\chi}) \in T_\Sigma$. Then, there exist $q^0, \ldots, q^k \in \mathrm{QoSys}^{\mathcal{A}}(T)$ and $\xi_1, \ldots, \xi_k \in \mathrm{SEN}(\Sigma)$, such that

$$\phi \ q^0_\Sigma \ \xi_1 \ q^1_\Sigma \ \xi_2 \ q^2_\Sigma \ \cdots \ q^{k-1}_\Sigma \ \xi_k \ q^k_\Sigma \ \psi.$$

Since $\phi \ q^0_\Sigma \ \xi_1$ and $\sigma^\flat_\Sigma(\phi, \vec{\chi}) \in T_\Sigma$, we get $\sigma^\flat_\Sigma(\xi_1, \vec{\chi}) \in T_\Sigma$. Similarly, since $\xi_1 \ q^1_\Sigma \ \xi^2$ and $\sigma^\flat_\Sigma(\xi_1, \vec{\chi}) \in T_\Sigma$, we get $\sigma^\flat_\Sigma(\xi_2, \vec{\chi}) \in T_\Sigma$. We move one step to the right at a time in a similar fashion until we obtain $\sigma^\flat_\Sigma(\psi, \vec{\chi}) \in T_\Sigma$. A similar argument is used to handle the case of negative polarity for $\sigma^\flat$. This proves that $\mathrm{tr}(\bigcup \mathrm{QoSys}^{\mathcal{A}}(T)) \in \mathrm{QoSys}^{\mathcal{A}}(T)$ and, therefore, that it is its largest member. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system and $T \in \mathrm{SenFam}(\mathcal{A})$. The M-**Leibniz order of** T **on** $\mathcal{A}$ is the largest qosystem $\preccurlyeq^{M,\mathcal{A}}(T)$ on $\mathcal{A}$ that is M-compatible with T.

The next theorem provides a characterization of the M-Leibniz order of a sentence family.

Theorem 1833 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *be a polarity for* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system and* $T \in \mathrm{SenFam}(\mathcal{A})$*. For all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,* $\phi \preccurlyeq^{M,\mathcal{A}}_\Sigma(T) \ \psi$ *if and only if, for all* $\sigma^\flat$ *in* $N^\flat$*,* $\Sigma' \in |\mathbf{Sign}|$*,* $f \in \mathbf{Sign}(\Sigma, \Sigma')$*,* $\vec{\chi} \in \mathrm{SEN}(\Sigma')$*,*

- $\sigma^\flat \in M^+$ *and* $\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi}) \in T_{\Sigma'}$ *imply* $\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}) \in T_{\Sigma'}$*;*
- $\sigma^\flat \in M^-$ *and* $\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}) \in T_{\Sigma'}$ *imply* $\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi}) \in T_{\Sigma'}$*.*

Proof: We let $\preccurlyeq^{\mathcal{A}} = \{\preccurlyeq^{\mathcal{A}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ be defined by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $\phi \preccurlyeq^{\mathcal{A}}_\Sigma \psi$ if and only if, for all $\sigma^\flat$ in $N^\flat$, $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

- $\sigma^\flat \in M^+$ and $\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi}) \in T_{\Sigma'}$ imply $\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}) \in T_{\Sigma'}$;
- $\sigma^\flat \in M^-$ and $\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}) \in T_{\Sigma'}$ imply $\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi}) \in T_{\Sigma'}$.

Then it it clear that $\preccurlyeq^{\mathcal{A}}$ is a qosystem on $\mathcal{A}$. Moreover, by its definition, it is compatible with T. Hence, by the maximality of the M-Leibniz order of T on $\mathcal{A}$, $\preccurlyeq^{\mathcal{A}} \leq \preccurlyeq^{M,\mathcal{A}}(T)$. On the other hand, if $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\phi \preccurlyeq^{M,\mathcal{A}}_\Sigma(T)\psi$, then, since $\preccurlyeq^{M,\mathcal{A}}(T)$ is a qosystem, we get for all $\Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\mathrm{SEN}^\flat(f)(\phi) \preccurlyeq^{M,\mathcal{A}}_{\Sigma'}(T)\mathrm{SEN}(f)(\psi)$. Thus, since $\preccurlyeq^{M,\mathcal{A}}(T)$ is M-compatible with T, we get $\phi \preccurlyeq^{\mathcal{A}}_\Sigma \psi$. Therefore, $\preccurlyeq^{M,\mathcal{A}}(T) \leq \preccurlyeq^{\mathcal{A}}$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The pair $\langle \mathcal{I}, M \rangle$ is called a **polar π-institution**.

Given an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, the qosystem $\preccurlyeq^{M,\mathcal{A}}(T)$ is called the **M-Leibniz order of T on $\mathcal{A}$**. The collection of maps

$$T \mapsto \preccurlyeq^{M,\mathcal{A}}(T), \quad T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}),$$

for all $\mathcal{A}$, constitute the **M-Leibniz order operator** $\preccurlyeq^M$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. We denote by $O = (O^+, O^-)$ the **total polarity for F**, i.e., the polarity consisting of

$$O^+ = O^- = N^\flat.$$

Corollary 1834 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $O = (O^+, O^-)$ *the total polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$*. The O-Leibniz order operator* $\preccurlyeq^O$ *of* $\mathcal{I}$ *coincides with the Leibniz operator* Ω *of* $\mathcal{I}$*.*

Proof: This follows directly from the definition of O, Theorem 1833 and Theorem 19. ■

Next we give two properties of the operator $\preccurlyeq^M$. The first is commutativity with inverse surjective morphisms and the second is a characterization of monotonicity. Both properties take after similar properties of the Leibniz operator that were established in previous chapters.

Lemma 1835 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a π-institution based on* $\mathbf{F}$*. For all* $\mathbf{F}$*-algebraic systems* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle$*, all surjective morphisms* $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$*,*

$$\gamma^{-1}(\preccurlyeq^{M,\mathcal{B}}(T)) = \preccurlyeq^{M,\mathcal{A}}(\gamma^{-1}(T)).$$

Proof: Let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$. We have $\phi \preccurlyeq_\Sigma^{M,\mathcal{A}}(\gamma^{-1}(T)\psi$ if and only if, for all $\sigma^\flat$ in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

- if $\sigma^\flat \in M^+$, then $\sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}(f)(\phi), \vec{\chi}) \in \gamma_{\Sigma'}^{-1}(T_{H(\Sigma')})$ implies $\sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}(f)(\psi), \vec{\chi}) \in \gamma_{\Sigma'}^{-1}(T_{H(\Sigma')})$;
- if $\sigma^\flat \in M^-$, then $\sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}(f)(\psi), \vec{\chi}) \in \gamma_{\Sigma'}^{-1}(T_{H(\Sigma')})$ implies $\sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}(f)(\phi), \vec{\chi}) \in \gamma_{\Sigma'}^{-1}(T_{H(\Sigma')})$;

if and only if for all $\sigma^\flat$ in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

- if $\sigma^\flat \in M^+$, then $\gamma_{\Sigma'}(\sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}(f)(\phi), \vec{\chi})) \in T_{H(\Sigma')}$ implies $\gamma_{\Sigma'}(\sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}(f)(\psi), \vec{\chi}) \in T_{H(\Sigma')}$;
- if $\sigma^\flat \in M^-$, then $\gamma_{\Sigma'}(\sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}(f)(\psi), \vec{\chi})) \in T_{H(\Sigma')}$ implies $\gamma_{\Sigma'}(\sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}(f)(\phi), \vec{\chi}) \in T_{H(\Sigma')}$;

if and only if for all $\sigma^\flat$ in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

- if $\sigma^\flat \in M^+$, then $\sigma_{H(\Sigma')}^{\mathcal{B}}(\mathrm{SEN}'(H(f))(\gamma_\Sigma(\phi)), \gamma_{\Sigma'}(\vec{\chi})) \in T_{H(\Sigma')}$ implies $\sigma_{H(\Sigma')}^{\mathcal{B}}(\mathrm{SEN}'(H(f))(\gamma_\Sigma(\psi)), \gamma_{\Sigma'}(\vec{\chi})) \in T_{H(\Sigma')}$;
- if $\sigma^\flat \in M^-$, then $\sigma_{H(\Sigma')}^{\mathcal{B}}(\mathrm{SEN}'(H(f))(\gamma_\Sigma(\psi)), \gamma_{\Sigma'}(\vec{\chi})) \in T_{H(\Sigma')}$ implies $\sigma_{H(\Sigma')}^{\mathcal{B}}(\mathrm{SEN}'(H(f))(\gamma_\Sigma(\phi)), \gamma_{\Sigma'}(\vec{\chi})) \in T_{H(\Sigma')}$;

if and only, by the surjectivity of $\langle H, \gamma\rangle$, $\gamma_\Sigma(\phi) \preccurlyeq_{H(\Sigma)}^{M,\mathcal{B}} \gamma_\Sigma(\psi)$ if and only if $\phi\gamma_\Sigma^{-1}(\preccurlyeq_{H(\Sigma)}^{M,\mathcal{B}}(T))\psi$. ■

Lemma 1836 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. For every $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, $\preccurlyeq^{M,\mathcal{A}}$ is monotone if and only if it commutes with arbitrary intersections.*

Proof: Suppose, first, that $\preccurlyeq^{M,\mathcal{A}}$ is monotone and let $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, by monotonicity, $\preccurlyeq^{M,\mathcal{A}}(\bigcap_{T\in\mathcal{T}} T) \leq \bigcap_{T\in\mathcal{T}} \preccurlyeq^{M,\mathcal{A}}(T)$. On the other hand, $\bigcap_{T\in\mathcal{T}} \preccurlyeq^{M,\mathcal{A}}(T)$ is a qosystem on $\mathcal{A}$, which can be easily seen to be M-compatible with $\bigcap\mathcal{T}$. Thus, by the maximality property of $\preccurlyeq^{M,\mathcal{A}}(\bigcap\mathcal{T})$, we get $\bigcap_{T\in\mathcal{T}} \preccurlyeq^{M,\mathcal{A}}(T) \leq \preccurlyeq^{M,\mathcal{A}}(\bigcap_{T\in\mathcal{T}} T)$. Therefore, the two qosystems are equal and $\preccurlyeq^{M,\mathcal{A}}$ commutes with arbitrary intersections.

Suppose, conversely, $\preccurlyeq^{M,\mathcal{A}}$ commutes with arbitrary intersections and let $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \leq T'$. Then, we have

$$\preccurlyeq^{M,\mathcal{A}}(T) = \preccurlyeq^{M,\mathcal{A}}(T \cap T') = \preccurlyeq^{M,\mathcal{A}}(T) \cap \preccurlyeq^{M,\mathcal{A}}(T'),$$

whence, we get $\preccurlyeq^{M,\mathcal{A}}(T) \leq \preccurlyeq^{M,\mathcal{A}}(T')$ and, therefore, $\preccurlyeq^{M,\mathcal{A}}$ is monotone. ■

25.4 Directional Systems

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The polar π-institution $\langle \mathcal{I}, M \rangle$ is called **directional** and the π-institution $\mathcal{I}$ is called M-**directional** if there exists $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with two distinguished arguments, such that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\phi \preccurlyeq_\Sigma^{M,\mathcal{A}}(T)\ \psi \quad \text{iff} \quad \beta_\Sigma^{\mathcal{A}}[\phi, \psi] \le T,$$

The collection β in $N^\flat$ will be called a family of **witnessing transformations for the M-directionality of $\mathcal{I}$**.

Here are a couple of direct consequences of the definition. The first asserts that any two set of witnessing transformations for the M-directionality of a given π-institution are deductively equivalent. The second asserts that M-directionality is preserved under extensions.

Lemma 1837 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\beta, \beta' : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, having two distinguished arguments, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is M-directional with witnessing transformations β and β', then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,*

$$C(\beta_\Sigma[\phi, \psi]) = C(\beta'_\Sigma[\phi, \psi]).$$

Proof: We have, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$ and all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\begin{array}{rl} \beta_\Sigma[\phi, \psi] \le T & \text{iff} \quad \phi \preccurlyeq_\Sigma^{M,\mathcal{F}}(T)\ \psi \\ & \text{iff} \quad \beta'_\Sigma[\phi, \psi] \le T. \end{array}$$

Therefore, $C(\beta_\Sigma[\phi, \psi]) = C(\beta'_\Sigma[\phi, \psi])$. ■

Lemma 1838 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, having two distinguished arguments, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$, $\mathcal{I}' = \langle \mathbf{F}, C' \rangle$ two π-institutions based on $\mathbf{F}$. If $\mathcal{I}$ is M-directional with witnessing transformations β and $\mathcal{I} \le \mathcal{I}'$, then $\mathcal{I}'$ is also M-directional with witnessing transformations β.*

Proof: Suppose $\mathcal{I}$ is M-directional with witnessing transformations β and $\mathcal{I} \le \mathcal{I}'$. Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}'}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$. Then, since every $\mathcal{I}'$-filter family of $\mathcal{A}$ is also an $\mathcal{I}$-filter family, we have, by hypothesis, $\phi \preccurlyeq_\Sigma^{M,\mathcal{A}}(T)\ \psi$ iff $\beta_\Sigma^{\mathcal{A}}[\phi, \psi] \le T$. We conclude that $\mathcal{I}'$ is also M-directional, with witnessing transformations β. ■

We give, next, sufficient conditions for the M-directionality of a given π-institution.

Theorem 1839 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$ *having two distinguished arguments, and* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$*, satisfying the following conditions:*

1. $\beta_\Sigma[\phi, \phi] \le \mathrm{Thm}(\mathcal{I})$*, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*;*
2. $M \le B$*, where* $B = (B^+, B^-)$ *is the polarity induced by* β*;*
3. *For all* $\sigma^\flat \in \beta$*,* $\sigma^\flat(x, y, \vec{z}) \in M^-$ *or* $\sigma^\flat(y, x, \vec{z}) \in M^+$*.*

Then $\mathcal{I}$ *is* M*-directional, with witnessing transformations* β*.*

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$.

Suppose, first, that $\phi \preccurlyeq_\Sigma^{M,\mathcal{A}}(T)\ \psi$ and $\sigma^\flat \in \beta$. Then, by Condition 3, either $\sigma^\flat$ is of negative M-polarity in the first argument or of positive M-polarity in the second argument.

- Assume $\sigma^\flat$ has negative polarity in the first argument. By Condition 1, we have $\sigma_\Sigma^{\mathcal{A}}[\psi, \psi] \le T$. Therefore, by Condition 2 and the hypothesis, we get $\sigma_\Sigma^{\mathcal{A}}[\phi, \psi] \le T$.
- Assume $\sigma^\flat$ has positive polarity in the second argument. By Condition 1, we have $\sigma_\Sigma^{\mathcal{A}}[\phi, \phi] \le T$. Therefore, by Condition 2 and the hypothesis, we get $\sigma_\Sigma^{\mathcal{A}}[\phi, \psi] \le T$.

In either case $\sigma_\Sigma^{\mathcal{A}}[\phi, \psi] \le T$, whence, $\beta_\Sigma^{\mathcal{A}}[\phi, \psi] \le T$.

Assume, conversely, that $\beta_\Sigma^{\mathcal{A}}[\phi, \psi] \le T$. Let $\sigma^\flat$ in $N^\flat$, viewed as having one distinguished argument.

- If $\sigma^\flat \in M^+$, then, by Condition 2, $\sigma^\flat \in B^+$. Hence, by definition of B and the hypothesis, $\sigma_\Sigma^{\mathcal{A}}[\phi] \le T$ implies $\sigma_\Sigma^{\mathcal{A}}[\psi] \le T$.
- If $\sigma^\flat \in M^-$, then, by Condition 2, $\sigma^\flat \in B^-$. Hence, by definition of B and the hypothesis, $\sigma_\Sigma^{\mathcal{A}}[\psi] \le T$ implies $\sigma_\Sigma^{\mathcal{A}}[\phi] \le T$.

Therefore, by Theorem 1833, we conclude that $\phi \preccurlyeq_\Sigma^{M,\mathcal{A}}(T)\ \psi$. Hence, $\mathcal{I}$ is M-directional with witnessing transformations β. ■

Now we look at some properties of M-directional π-institutions.

Theorem 1840 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$ *having two distinguished arguments,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is* M*-directional, with witnessing transformations* β*, then the following properties hold:*

(a) *For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\beta_\Sigma[\phi, \phi] \le \mathrm{Thm}(\mathcal{I}) \quad \textit{and} \quad \beta_\Sigma[\phi, \chi] \le C(\beta_\Sigma[\phi, \psi], \beta_\Sigma[\psi, \chi]);$$

(b) $M \leq B$, *where* B *is the polarity for* $\mathbf{F}$ *induced by* β*;*

(c) *For all* $\sigma^\flat \in \beta$,

$$\sigma^\flat(x,y,\vec{z}) \in B^- \quad \textit{and} \quad \sigma^\flat(y,x,\vec{z}) \in B^+;$$

(d) $\mathcal{I}$ *is* B*-directional, with witnessing transformations* β*;*

(e) *For every* $\mathbf{F}$*-algebraic system* $\mathcal{A}$, $\preccurlyeq^{M,\mathcal{A}} = \preccurlyeq^{B,\mathcal{A}}$*;*

(f) B *is the largest polarity* M' *for* $\mathbf{F}$, *such that* $\preccurlyeq^{M'} = \preccurlyeq^M$.

Proof:

(a) Since $\preccurlyeq^M$ is a qosystem, it is reflexive and transitive. Thus, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi,\chi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \preccurlyeq^{M,\mathcal{F}}_\Sigma (\mathrm{Thm}(\mathcal{I}))\ \phi,$$
$$\phi \preccurlyeq^{M,\mathcal{F}}_\Sigma (T)\ \psi \text{ and } \psi \preccurlyeq^{M,\mathcal{F}}_\Sigma (T)\ \chi \text{ imply } \phi \preccurlyeq^{M,\mathcal{F}}_\Sigma (T)\ \chi.$$

Hence, by M-directionality, we get $\beta_\Sigma[\phi,\phi] \leq \mathrm{Thm}(\mathcal{I})$ and

$$\beta_\Sigma[\phi,\psi] \leq T \text{ and } \beta_\Sigma[\psi,\chi] \leq T \text{ imply } \beta_\Sigma[\phi,\chi] \leq T.$$

The latter gives $\beta_\Sigma[\phi,\chi] \leq C(\beta_\Sigma[\phi,\psi],\beta_\Sigma[\psi,\chi])$.

(b) Suppose $\sigma^\flat \in M^+$ and let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi,\vec{\chi} \in \mathrm{SEN}(\Sigma)$, such that

$$\beta^{\mathcal{A}}_\Sigma[\phi,\psi] \leq T \quad \text{and} \quad \sigma^{\mathcal{A}}_\Sigma(\phi,\vec{\chi}) \in T_\Sigma.$$

By M-directionality, we get $\phi \preccurlyeq^{M,\mathcal{A}}_\Sigma (T)\ \psi$ and $\sigma^{\mathcal{A}}_\Sigma(\phi,\vec{\chi}) \in T_\Sigma$. Thus, since $\sigma^\flat \in M^+$, we get $\sigma^{\mathcal{A}}_\Sigma(\psi,\vec{\chi}) \in T_\Sigma$. We conclude that $\sigma^\flat \in B^+$ and, hence, $M^+ \subseteq B^+$. Similarly, we get that $M^- \subseteq B^-$ and, therefore, $M \leq B$.

(c) This follows directly by the second assertion of Part (a) and the definition of B.

(d) This follows from Parts (a), (c) and Theorem 1839.

(e) By the hypothesis and Part (d), we have, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and all $\phi,\psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{aligned}\phi \preccurlyeq^{M,\mathcal{A}}_\Sigma (T)\ \psi \quad &\text{iff} \quad \beta^{\mathcal{A}}_\Sigma[\phi,\psi] \leq T\\ &\text{iff} \quad \phi \preccurlyeq^{B,\mathcal{A}}_\Sigma (T)\ \psi.\end{aligned}$$

Therefore, $\preccurlyeq^{M,\mathcal{A}} = \preccurlyeq^{B,\mathcal{A}}$.

(f) We have that $\preccurlyeq^{M'} = \preccurlyeq^M$ if and only β witnesses the M'-directionality of $\mathcal{I}$. This implies, by Part (b), that $M' \leq B$.

■

We now obtain the following characterization of the existence of a polarity M for which $\mathcal{I}$ is M-directional with a predetermined set $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ of natural transformations in $N^\flat$ as its witnessing set.

Corollary 1841 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$ *having two distinguished arguments, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following conditions are equivalent:*

(i) There exists a polarity $M = (M^+, M^-)$ *for* $\mathbf{F}$*, such that* $\mathcal{I}$ *is* M*-directional with witnessing transformations* β*;*

(ii) For all $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\beta_\Sigma[\phi,\phi] \leq \mathrm{Thm}(\mathcal{I}) \quad \textit{and} \quad \beta_\Sigma[\phi,\chi] \leq C(\beta_\Sigma[\phi,\psi], \beta_\Sigma[\psi,\chi]);$$

(iii) $\mathcal{I}$ *is* B*-directional with witnessing transformations* β*.*

Proof: If Condition (i) holds, then Part (a) of Theorem 1840 ensures that Condition (ii) holds. If Condition (ii) holds, then, we get Part (c) of Theorem 1840 and, from Part (a) (our hypothesis) and Part (c) of Theorem 1840, we get, using Theorem 1839, Part (d) of Theorem 1840, which is Condition (iii). Finally, if (iii) holds, then B is a polarity on $\mathbf{F}$, such that $\mathcal{I}$ is B-directional, with witnessing transformations β and, thus, Condition (i) holds. ■

Our results allow us to show that families of collections of natural transformations in $N^\flat$ with two distinguished arguments, satisfying Condition (ii) of Corollary 1841 and polarities on $\mathbf{F}$ are in correspondence under appropriate identifications of deductively equivalent collections of transformations and of polarities giving rise to the same Leibniz order operators.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$.

- Let $\mathfrak{B}(\mathcal{I})$ be the collection of all families $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, with two distinguished arguments, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$,

  $$\beta_\Sigma[\phi,\phi] \leq \mathrm{Thm}(\mathcal{I}) \quad \text{and} \quad \beta_\Sigma[\phi,\chi] \leq C(\beta_\Sigma[\phi,\psi], \beta_\Sigma[\psi,\chi]).$$

  Moreover, we declare two collections $\beta, \beta' \in \mathfrak{B}(\mathcal{I})$ to be equivalent, written $\beta \equiv^{\mathcal{I}} \beta'$ if and only if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

  $$C(\beta_\Sigma[\phi,\psi]) = C(\beta'_\Sigma[\phi,\psi]).$$

- Let $\mathfrak{M}(\mathcal{I})$ be the collection of all polarities for $\mathbf{F}$, such that $\mathcal{I}$ is M-directional.

  Moreover, we declare two polarities M, M' in $\mathfrak{M}(\mathcal{I})$ to be equivalent, written $M \sim^{\mathcal{I}} M'$, if and only if $\preccurlyeq^M = \preccurlyeq^{M'}$.

Then we have the following correspondence.

Theorem 1842 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. There exists a bijection from* $\mathcal{B}(\mathcal{I})/{\equiv^{\mathcal{I}}}$ *onto* $\mathfrak{M}(\mathcal{I})/{\sim^{\mathcal{I}}}$*, such that*

$$\beta/{\equiv^{\mathcal{I}}} \mapsto B/{\sim^{\mathcal{I}}}, \quad \beta \in \mathfrak{B}(\mathcal{I}),$$

and such that every $\beta' \in \mathfrak{B}(\mathcal{I})$*, such that* $\beta' \equiv^{\mathcal{I}} \beta$*, witnesses the* M*-directionality of* $\mathcal{I}$*, for all* $M \in \mathfrak{M}(\mathcal{I})$*, such that* $M \sim^{\mathcal{I}} B$*.*

Proof: Let $\beta, \beta' \in \mathfrak{B}(\mathcal{I})$, such that $\beta \equiv^{\mathcal{I}} \beta'$. Then, for every $\mathbf{F}$-algebraic system, all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{rl}\phi \preccurlyeq^{B,\mathcal{A}}_\Sigma(T)\ \psi & \text{iff} \quad \beta_\Sigma[\phi,\psi] \leq T\\ & \text{iff} \quad \beta'_\Sigma[\phi,\psi] \leq T\\ & \text{iff} \quad \phi \preccurlyeq^{B',\mathcal{A}}_\Sigma\ \psi.\end{array}$$

Thus, $B \sim^{\mathcal{I}} B'$ and the mapping in the statement of the theorem is well-defined.

By definition of $\mathfrak{M}(\mathcal{I})$ and Theorem 1840, it is onto.

Finally, if $\beta, \beta' \in \mathfrak{B}(\mathcal{I})$, such that $B \sim^{\mathcal{I}} B'$, then, by definition, $\preccurlyeq^B = \preccurlyeq^{B'}$. Thus, for all $\langle \mathcal{A}, T \rangle \in \mathrm{MatFam}^*(\mathcal{I})$, we get, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{rl}\beta_\Sigma[\phi,\psi] \leq T & \text{iff} \quad \phi \preccurlyeq^{B,\mathcal{A}}_\Sigma(T)\ \psi\\ & \text{iff} \quad \phi \preccurlyeq^{B',\mathcal{A}}_\Sigma(T)\ \psi\\ & \text{iff} \quad \beta'_\Sigma[\phi,\psi] \leq T.\end{array}$$

Therefore, by the completeness of $\mathcal{I}$ with respect to $\mathrm{MatFam}^*(\mathcal{I})$, we get that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, $C(\beta_\Sigma[\phi,\psi]) = C(\beta'_\Sigma[\phi,\psi])$, i.e., $\beta \equiv^{\mathcal{I}} \beta'$ and, therefore, the map in the statement of the theorem is also injective. ■

Corollary 1843 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$ *having two distinguished arguments, and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* β *is a semi-equivalence system for* $\mathcal{I}$ *(in particular, if* $\mathcal{I}$ *is* β*-order algebraizable), then* $\mathcal{I}$ *is* B*-directional, with witnessing transformations* β*.*

Proof: By Theorem 1828 and Corollary 1841. ■

25.5 Monotonicity and Directionaility

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I}$ be a π-institution based on $\mathbf{F}$.

- We say that $\mathcal{I}$ is M-**order monotone** if the M-Leibniz order operator $\preccurlyeq^M$ is monotone.
- We say that $\mathcal{I}$ is M-**directional** if there exists $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, having two distinguished arguments, such that, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,
$$\phi \preccurlyeq^{M,\mathcal{A}}_\Sigma(T)\ \psi \quad \text{iff} \quad \beta^{\mathcal{A}}_\Sigma[\phi, \psi] \leq T.$$

Our goal is to connect these two notions.

We have the following obvious relationship.

Theorem 1844 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I}$ be a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is M-directional, then $\mathcal{I}$ is M-order monotone.*

Proof: Suppose $\mathcal{I}$ is M-directional, with witnessing transformations β. Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \leq T'$. Then, we get, by the M-directionality of $\mathcal{I}$, we get
$$\preccurlyeq^{M,\mathcal{A}}(T) = \beta(T) \leq \beta(T') = \preccurlyeq^{M,\mathcal{A}}(T').$$
Therefore, $\mathcal{I}$ is M-order monotone. ∎

We introduce a collection of natural transformations associated with $\mathcal{I}$ that play in the present context a role analog to the role that the reflexive core $R^{\mathcal{I}}$ played in the case of syntactic protoalgebraicity. In fact the collection we introduce is a subcollection of the reflexive core of a π-institution $\mathcal{I}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The M-**quasicore** $Q^{\mathcal{I},M}$ of $\mathcal{I}$ is the collection
$$\begin{array}{rcl} Q^{\mathcal{I},M} & = & \{\kappa^\flat \in N^\flat : \kappa^\flat(x, y, \vec{z}) \in M^- \text{ and } \kappa^\flat(y, x, \vec{z}) \in M^+ \text{ and} \\ & & (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \phi \in \mathrm{SEN}^\flat(\Sigma))(\kappa^\flat_\Sigma[\phi, \phi] \leq \mathrm{Thm}(\mathcal{I}))\}. \end{array}$$

It turns out that, if $\mathcal{I}$ is M-directional with witnessing transformations β, then $\beta \subseteq Q^{\mathcal{I},M}$.

Lemma 1845 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$ having two distinguished arguments, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is M-directional, with witnessing transformations β, then $\beta \subseteq Q^{\mathcal{I},M}$.*

Proof: The conclusion follows directly from Parts (a) and (c) of Theorem 1840 and the definition of $Q^{\mathcal{I},M}$. ■

The M-directionality of a π-institution $\mathcal{I}$ guarantees that the M-quasicore of $\mathcal{I}$ has the global family modus ponens.

Theorem 1846 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is* M*-directional, then* $Q^{\mathcal{I},M}$ *has the global family modus ponens.*

Proof: Suppose $\mathcal{I}$ is M-directional with witnessing transformations β and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in T_\Sigma$ and $Q^{\mathcal{I},M}_\Sigma[\phi,\psi] \le T$. Then, by Lemma 1845, $\phi \in T_\Sigma$ and $\beta_\Sigma[\phi,\psi] \le T$. By M-directionality, $\phi \in T_\Sigma$ and $\phi \preccurlyeq^{M,\mathcal{F}}_\Sigma (T)\ \psi$. Therefore, by the definition of $\preccurlyeq^{M,\mathcal{F}}(T)$, $\psi \in T_\Sigma$. We conclude that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\psi \in C_\Sigma(Q^{\mathcal{I},M}_\Sigma[\phi,\psi], \phi),$$

i.e., $Q^{\mathcal{I},M}$ has the global family modus ponens in $\mathcal{I}$. ■

Conversely, it turns out that, if the M-quasicore $Q^{\mathcal{I},M}$ of $\mathcal{I}$ has the global family modus ponens, then $\mathcal{I}$ is M-directional, with $Q^{\mathcal{I},M}$ as its set of witnessing transformations.

Theorem 1847 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $Q^{\mathcal{I},M}$ *has the global family modus ponens in* $\mathcal{I}$*, then* $\mathcal{I}$ *is* M*-directional with witnessing transformations* $Q^{\mathcal{I},M}$*.*

Proof: We must show that, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\preccurlyeq^{M,\mathcal{A}}(T) = Q^{\mathcal{I},M,\mathcal{A}}(T)$.

Let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \preccurlyeq^{M,\mathcal{A}}_\Sigma (T)\ \psi$ and $\sigma \in Q^{\mathcal{I},M}$. Then, by the definition of the M-quasicore, $\sigma^{\mathcal{A}}_\Sigma[\psi,\psi] \le T$. Since $\phi \preccurlyeq^{M,\mathcal{A}}_\Sigma (T)\ \psi$ and $\sigma^\flat \in M^-$, we get that $\sigma^{\mathcal{A}}_\Sigma[\phi,\psi] \le T$. Therefore, $Q^{\mathcal{I},M,\mathcal{A}}_\Sigma[\phi,\psi] \le T$, which gives $\langle \phi,\psi\rangle \in Q^{\mathcal{I},M,\mathcal{A}}_\Sigma(T)$. Thus, $\preccurlyeq^{M,\mathcal{A}}(T) \le Q^{\mathcal{I},M,\mathcal{A}}(T)$.

Conversely, to see that $Q^{\mathcal{I},M,\mathcal{A}}(T) \le \preccurlyeq^{M,\mathcal{A}}(T)$, it suffices to show that $Q^{\mathcal{I},M,\mathcal{A}}(T)$ is a qosystem on $\mathcal{A}$ that is M-compatible with T.

- By definition of the M-quasicore, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $Q^{\mathcal{I},M,\mathcal{A}}_\Sigma[\phi,\phi] \le T$, whence $Q^{\mathcal{I},M,\mathcal{A}}(T)$ is reflexive.
- Next let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi, \chi \in \mathrm{SEN}(\Sigma)$, such that $Q^{\mathcal{I},M,\mathcal{A}}_\Sigma[\phi,\psi] \le T$ and $Q^{\mathcal{I},M,\mathcal{A}}_\Sigma[\psi,\chi] \le T$. For $\sigma^\flat, \tau^\flat \in Q^{\mathcal{I},M}$, note that, by the definition of the M-quasicore, the transformation $\tau^\flat(\sigma^\flat(z,x,\vec{p}), \sigma^\flat(z,y,\vec{p}), \vec{q}) \in$

$Q^{\mathcal{I},M}$. Hence, using modus ponens, we get, for all $\sigma^\flat \in Q^{\mathcal{I},M}$ and all $\vec{\xi} \in \mathrm{SEN}(\Sigma')$,

$$\begin{array}{l}\sigma^{\mathcal{A}}_\Sigma(\phi,\chi,\vec{\xi})\\ \in C^{\mathcal{I},\mathcal{A}}_\Sigma(Q^{\mathcal{I},M,\mathcal{A}}_{\Sigma'}[\sigma^{\mathcal{A}}_\Sigma(\phi,\psi,\vec{\xi}),\sigma^{\mathcal{A}}_\Sigma(\phi,\chi,\vec{\xi})],\sigma^{\mathcal{A}}_\Sigma(\phi,\psi,\vec{\xi}))\\ \le C^{\mathcal{I},\mathcal{A}}_{\Sigma'}(Q^{\mathcal{I},M,\mathcal{A}}_{\Sigma'}[\psi,\chi],\sigma^{\mathcal{A}}_\Sigma(\phi,\psi,\vec{\xi})).\end{array}$$

We conclude that $Q^{\mathcal{I},M,\mathcal{A}}_\Sigma[\phi,\chi] \le C^{\mathcal{I},\mathcal{A}}(Q^{\mathcal{I},M,\mathcal{A}}_\Sigma[\phi,\psi],Q^{\mathcal{I},M,\mathcal{A}}_\Sigma[\psi,\chi])$ and, therefore, $Q^{\mathcal{I},M,\mathcal{A}}(T)$ is also transitive.

- Suppose, next, that $\sigma^\flat \in M^+$, $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi,\vec{\chi} \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle \in Q^{\mathcal{I},M,\mathcal{A}}_\Sigma(T)$ and $\sigma^\flat_\Sigma(\phi,\vec{\chi}) \in T_\Sigma$. Then $Q^{\mathcal{I},M,\mathcal{A}}_\Sigma[\phi,\psi] \le T$ and $\sigma^{\mathcal{A}}(\phi,\vec{\chi}) \in T_\Sigma$. So we get

$$\begin{array}{rcl}\sigma^{\mathcal{A}}_\Sigma(\psi,\vec{\chi}) & \in & C^{\mathcal{I},\mathcal{A}}_\Sigma(Q^{\mathcal{I},M,\mathcal{A}}_\Sigma[\sigma^{\mathcal{A}}_\Sigma(\phi,\vec{\chi}),\sigma^{\mathcal{A}}_\Sigma(\psi,\vec{\chi})],\sigma^{\mathcal{A}}_\Sigma(\phi,\vec{\chi}))\\ & \subseteq & C^{\mathcal{I},\mathcal{A}}(Q^{\mathcal{I},M,\mathcal{A}}_\Sigma[\phi,\psi],\sigma^{\mathcal{A}}_\Sigma(\phi,\vec{\chi}))\\ & \subseteq & T_\Sigma.\end{array}$$

  Similarly, consider $\sigma^\flat \in M^-$, $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi,\vec{\chi} \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle \in Q^{\mathcal{I},M,\mathcal{A}}_\Sigma(T)$ and $\sigma^\flat_\Sigma(\psi,\vec{\chi}) \in T_\Sigma$. Then $Q^{\mathcal{I},M,\mathcal{A}}_\Sigma[\phi,\psi] \le T$ and $\sigma^{\mathcal{A}}(\psi,\vec{\chi}) \in T_\Sigma$. So we get

$$\begin{array}{rcl}\sigma^{\mathcal{A}}_\Sigma(\phi,\vec{\chi}) & \in & C^{\mathcal{I},\mathcal{A}}_\Sigma(Q^{\mathcal{I},M,\mathcal{A}}_\Sigma[\sigma^{\mathcal{A}}_\Sigma(\psi,\vec{\chi}),\sigma^{\mathcal{A}}_\Sigma(\phi,\vec{\chi})],\sigma^{\mathcal{A}}_\Sigma(\psi,\vec{\chi}))\\ & \subseteq & C^{\mathcal{I},\mathcal{A}}(Q^{\mathcal{I},M,\mathcal{A}}_\Sigma[\phi,\psi],\sigma^{\mathcal{A}}_\Sigma(\psi,\vec{\chi}))\\ & \subseteq & T_\Sigma.\end{array}$$

  Thus, $Q^{\mathcal{I},M,\mathcal{A}(T)}$ is M-compatible with T.

We conclude that $Q^{\mathcal{I},M,\mathcal{A}}(T)$ is a qosystem on $\mathcal{A}$ that is M-compatible with T, whence, by the maximality of $\preccurlyeq^{M,\mathcal{A}}(T)$, we get $Q^{\mathcal{I},M,\mathcal{A}}(T) \le \preccurlyeq^{M,\mathcal{A}}(T)$. ■

We now have a characterization of M-directionality in terms of the property of modus ponens of the M-quasicore $Q^{\mathcal{I},M}$ of the π-institution $\mathcal{I}$.

$$\mathcal{I} \text{ is } M\text{-directional} \longleftrightarrow Q^{\mathcal{I},M} \text{ has Global Family MP.}$$

Theorem 1848 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* $M = (M^+,\ M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle\mathbf{F},C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is* M*-directional if and only if* $Q^{\mathcal{I},M}$ *has the global family modus ponens in* $\mathcal{I}$*.*

Proof: Theorem 1846 gives the "only if" and the "if" is by Theorem 1847. ■

As a corollary, we obtain

Corollary 1849 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$ *having two distinguished arguments,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is* M*-directional with witnessing transformations* β*, then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$C(Q^{\mathcal{I},M}_\Sigma[\phi,\psi]) = C(\beta_{\Sigma|}[\phi,\psi]).$$

Proof: If $\mathcal{I}$ is M-directional, with witnessing transformations β, then, by Theorems 1846 and 1847, both β and $Q^{\mathcal{I},M}$ are families of witnessing transformations for the M-directionality of $\mathcal{I}$. Therefore, by Lemma 1837, we get the conclusion. ■

We get relatively easily another related characterization of M-directionality.

$$\mathcal{I} \text{ is } M\text{-directional} \longleftrightarrow Q^{\mathcal{I},M} \text{ Defines } M\text{-Leibniz QoSystems.}$$

Theorem 1850 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is* M*-directional if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$\leqslant^{M,\mathcal{A}}(T) = Q^{\mathcal{I},M,\mathcal{A}}(T).$$

Proof: If $\mathcal{I}$ is M-directional, then, by Theorem 1846 and Theorem 1847, $Q^{\mathcal{I},M}$ constitutes a collection of witnessing transformations, whence, for every $\mathbf{F}$-algebraic system $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ $\leqslant^{M,\mathcal{A}}(T) = Q^{\mathcal{I},M,\mathcal{A}}(T)$.

The converse follows by the definition of M-directionality, since, in that case, $Q^{\mathcal{I},M}$ forms a collection of witnessing transformations. ■

We finally show that the property that separates M-order monotonicity from M-directionality is the M-order compatibility property with respect to the theory family generated by the M-quasicore.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. In analogy with the property of the reflexive core being Leibniz, we say that the M-quasicore $Q^{\mathcal{I},M}$ is **order Leibniz** if, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\phi \leqslant^{M,\mathcal{A}}_\Sigma (C^{\mathcal{I},\mathcal{A}}(Q^{\mathcal{I},M,\mathcal{A}}_\Sigma[\phi,\psi]))\ \psi.$$

This property is weaker than $Q^{\mathcal{I},M}$ having the global family modus ponens, i.e., if $Q^{\mathcal{I},M}$ has the global family modus ponens, then it is order Leibniz.

Proposition 1851 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. If* $Q^{\mathcal{I},M}$ *has the global family modus ponens, then it is order Leibniz.*

Proof: If $Q^{\mathcal{I},M}$ has the global family modus ponens, then, by Theorem 1847, we get, for every **F**-algebraic system $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$\preccurlyeq^{M,\mathcal{A}}(T) = Q^{\mathcal{I},M,\mathcal{A}}(T).$$

Therefore, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, by considering, in particular, $T = C^{\mathcal{I},\mathcal{A}}(Q_\Sigma^{\mathcal{I},M,\mathcal{A}}[\phi,\psi])$, and taking into account that $Q_\Sigma^{\mathcal{I},M,\mathcal{A}}[\phi,\psi] \leq C^{\mathcal{I},\mathcal{A}}(Q_\Sigma^{\mathcal{I},M,\mathcal{A}}[\phi,\psi])$, we get that $\phi \preccurlyeq_\Sigma^{M,\mathcal{A}}(C^{\mathcal{I},\mathcal{A}}(Q_\Sigma^{\mathcal{I},M,\mathcal{A}}[\phi,\psi]))\ \psi$. Thus, $Q^{\mathcal{I},M}$ is order Leibniz. ■

In the opposite direction, in an M-order monotone π-institution $\mathcal{I}$, if the M-quasicore is order Leibniz, then it has the global family modus ponens in $\mathcal{I}$.

Proposition 1852 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* **F** *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be an* M*-order monotone* π*-institution based on* **F**. *If* $Q^{\mathcal{I},M}$ *is order Leibniz, then it has the global family modus ponens in* $\mathcal{I}$.

Proof: Suppose that $\mathcal{I}$ is M-order monotone and that $Q^{\mathcal{I},M}$ is order Leibniz. Let $\mathcal{A}$ be an **F**-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in T_\Sigma$ and $Q_\Sigma^{\mathcal{I},M,\mathcal{A}}[\phi,\psi] \leq T$. Since $Q^{\mathcal{I},M}$ is order Leibniz, we have

$$\phi \preccurlyeq_\Sigma^{M,\mathcal{A}}(C(Q_\Sigma^{\mathcal{I},M,\mathcal{A}}[\phi,\psi]))\ \psi,$$

whence, since $\mathcal{I}$ is M-order monotone and $Q_\Sigma^{\mathcal{I},M,\mathcal{A}}[\phi,\psi] \leq T$,

$$\phi \preccurlyeq_\Sigma^{M,\mathcal{A}}(T)\ \psi.$$

Therefore, since $\phi \in T_\Sigma$, we get, by M-compatibility of $\preccurlyeq^{M,\mathcal{A}}(T)$ with T, that $\psi \in T_\Sigma$. We conclude that $Q^{\mathcal{I},M}$ has the global family modus ponens in $\mathcal{I}$. ■

We now show that a π-institution is M-directional if and only if it is M-order monotone and it has an order Leibniz M-quasicore.

M-Directionality	=	$Q^{\mathcal{I},M}$ has Global Family MP
	=	$Q^{\mathcal{I},M}$ Defines Leibniz QoSystems
	=	M-Order Monotonicity + $Q^{\mathcal{I},M}$ Order Leibniz

Theorem 1853 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+,\ M^-)$ *be a polarity for* **F** *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* **F**. $\mathcal{I}$ *is* M*-directional if and only if it is* M*-order monotone and has an order Leibniz* M*-quasicore.*

Proof: Suppose, first, that $\mathcal{I}$ is M-directional. Then it is M-order monotone by Theorem 1844. Moreover, its M-quasicore has the global family modus ponens by Theorem 1846 and, hence, by Proposition 1851, its M-quasicore is order Leibniz.

Suppose, conversely, that $\mathcal{I}$ is M-order monotone with an order Leibniz M-quasicore. Then, by Proposition 1852, its M-quasicore has the global family modus ponens and, therefore, by Theorem 1848, $\mathcal{I}$ is M-directional. ∎

25.6 c-Reflectivity and Truth Inequationality

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. The polar π-institution $\langle \mathcal{I}, M \rangle$ is called **truth inequational** and the π-institution $\mathcal{I}$ is called M-**truth inequational** if there exists a collection $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ in $N^\flat$, with a single distinguished argument, such that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T).$$

In this case $\tau^\flat$ is called a family of **witnessing transformations for the** M-**truth inequationality of** $\mathcal{I}$.

We can show, based on preceding work, that every β-order algebraizable π-institution $\mathcal{I}$ is B-truth inequational, where B is the polarity induced by β.

Proposition 1854 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, with $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, having two distinguished arguments, and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is β-order algebraizable, then $\mathcal{I}$ is B-truth inequational.*

Proof: Suppose $\mathcal{I}$ is β-order algebraizable. Then, by Corollary 1843, it is B-directional, with witnessing transformations β. Thus, by Theorem 1828, there exists $\alpha : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$, with a single distinguished argument, such that, for every $\mathbf{F}$-algebraic system $\mathcal{A}$, all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll} \phi \in T_\Sigma & \text{iff} & \beta^{\mathcal{A}}[\alpha^{\mathcal{A}}_\Sigma[\phi]] \leq T \\ & \text{iff} & \alpha^{\mathcal{A}}_\Sigma[\phi] \leq \preccurlyeq^{B,\mathcal{A}}(T). \end{array}$$

Thus, $\mathcal{I}$ is B-truth inequational, with witnessing transformations α. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ be a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. We say that

$\preccurlyeq^M$ is **completely order reflecting** or **c-reflecting**, for short, if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,

$$\bigcap_{T\in\mathcal{T}} \preccurlyeq^{M,\mathcal{F}}(T) \leq \preccurlyeq^{M,\mathcal{F}}(T') \quad \text{implies} \quad \bigcap_{T\in\mathcal{T}} T \leq T'.$$

If this is the case, we call $\mathcal{I}$ M-**c-reflective**.

We formulate an equivalent condition to M-c-reflectivity.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Given an $\mathbf{F}$-algebraic system $\mathcal{A}$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we define the qosystem

$$\widetilde{\preccurlyeq}^{M,\mathcal{A}}(T) = \bigcap\{\preccurlyeq^{M,\mathcal{A}}(T') : T \leq T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}.$$

By analogy with the Suszko congruence system, we call $\widetilde{\preccurlyeq}^{M,\mathcal{A}}(T)$ the M-**Suszko qosystem of** T.

Lemma 1855 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is* M*-c-reflective if and only if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\widetilde{\preccurlyeq}^{M,\mathcal{F}}(T) \leq \preccurlyeq^{M,\mathcal{F}}(T') \quad \textit{implies} \quad T \leq T'.$$

Proof: Assume, first, that $\mathcal{I}$ is M-c-reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{\preccurlyeq}^{M,\mathcal{F}}(T) \leq \preccurlyeq^{M,\mathcal{F}}(T')$. Then, we have

$$\bigcap\{\preccurlyeq^{M,\mathcal{F}}(T'') : T \leq T'' \in \mathrm{ThFam}(\mathcal{I})\} \leq \preccurlyeq^{M,\mathcal{F}}(T').$$

Therefore, by M-c-reflectivity, $\bigcap\{T'' : T \leq T'' \in \mathrm{ThFam}(\mathcal{I})\} \leq T'$, i.e., $T \leq T'$.

Suppose, conversely, that the displayed condition holds and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \preccurlyeq^{M,\mathcal{F}}(T) \leq \preccurlyeq^{M,\mathcal{F}}(T')$. Then, we get

$$\begin{array}{rcl}\widetilde{\preccurlyeq}^{M,\mathcal{F}}(\bigcap\mathcal{T}) & = & \bigcap\{\preccurlyeq^{M,\mathcal{F}}(T) : \bigcap\mathcal{T} \leq T \in \mathrm{ThFam}(\mathcal{I})\} \\ & \leq & \bigcap\{\preccurlyeq^{M,\mathcal{F}}(T) : T \in \mathcal{T}\} \\ & \leq & \preccurlyeq^{M,\mathcal{F}}(T').\end{array}$$

Thus, by hypothesis, $\bigcap\mathcal{T} \leq T'$ and, therefore, $\mathcal{I}$ is M-c-reflective. ∎

Furthermore, under M-order monotonicity, it turns out that M-c-reflectivity is equivalent to the injectivity of the M-Leibniz order operator.

Lemma 1856 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is* M*-order monotone, then* $\mathcal{I}$ *is* M*-c-reflective if and only if* $\preccurlyeq^{M,\mathcal{F}}$ *is injective on theory families.*

Proof: Suppose that $\mathcal{I}$ is M-order monotone.

Assume, first, that $\mathcal{I}$ is M-c-reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\preccurlyeq^{M,\mathcal{F}}(T) = \preccurlyeq^{M,\mathcal{F}}(T')$. Then, we have

$$\preccurlyeq^{M,\mathcal{F}}(T) = \preccurlyeq^{M,\mathcal{F}}(T) \cap \preccurlyeq^{M,\mathcal{F}}(T') \leq \preccurlyeq^{M,\mathcal{F}}(T'),$$

whence, by M-c-reflectivity, $T \cap T' \leq T'$, i.e., $T \leq T'$. By symmetry, we get $T = T'$ and, therefore, $\preccurlyeq^{M,\mathcal{F}}$ is injective on theory families.

Assume, conversely, that $\preccurlyeq^{M,\mathcal{F}}$ is injective on theory families and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \preccurlyeq^{M,\mathcal{F}}(T) \leq \preccurlyeq^{M,\mathcal{F}}(T')$. Then we get

$$\begin{array}{rcl}\preccurlyeq^{M,\mathcal{F}}(\bigcap_{T\in\mathcal{T}} T) & = & \bigcap_{T\in\mathcal{T}} \preccurlyeq^{M,\mathcal{F}}(T) \quad \text{(monotonicity)}\\ & = & \bigcap_{T\in\mathcal{T}} \preccurlyeq^{M,\mathcal{F}}(T) \cap \preccurlyeq^{M,\mathcal{F}}(T') \quad \text{(hypothesis)}\\ & = & \preccurlyeq^{M,\mathcal{F}}(\bigcap \mathcal{T} \cap T'). \quad \text{(monotonicity)}\end{array}$$

Thus, by injectivity, $\bigcap \mathcal{T} = \bigcap \mathcal{T} \cap T'$, whence $\bigcap \mathcal{T} \leq T'$ and, therefore, $\mathcal{I}$ is M-c-reflective. ∎

It is always the case that truth inequationality implies c-reflectivity.

Theorem 1857 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ be a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is M-truth inequational, then it is M-c-reflective.*

Proof: Suppose that $\mathcal{I}$ is M-truth inequational, with witnessing transformations $\tau^\flat$, and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \preccurlyeq^{M,\mathcal{F}}(T) \leq \preccurlyeq^{M,\mathcal{F}}(T')$. Then

$$\begin{array}{rcl}\bigcap_{T\in\mathcal{T}} T & = & \bigcap_{T\in\mathcal{T}} \tau^\flat(\preccurlyeq^{M,\mathcal{F}}(T)) \quad \text{(Truth Inequationality)}\\ & = & \tau^\flat(\bigcap_{T\in\mathcal{T}} \preccurlyeq^{M,\mathcal{F}}(T)) \quad \text{(Set Theory)}\\ & \leq & \tau^\flat(\preccurlyeq^{M,\mathcal{F}}(T')) \quad \text{(Hypothesis and Lemma 94)}\\ & = & T'. \quad \text{(Truth Inequationality)}\end{array}$$

Thus, $\mathcal{I}$ is M-c-reflective. ∎

Recall the characterization of truth equationality in terms of the solubility property of the Suszko core of the π-institution. We now work to establish an analog for truth inequationaility. More precisely, we provide a characterization of truth inequationality in terms of the order solubility property of the order core of a π-institution. Then, we provide an exact description of those M-c-reflective π-institutions which are M-truth inequational.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. We define the M-**order (Suszko) core** of $\mathcal{I}$ to be the collection

$$O^{\mathcal{I},M} = \{\sigma^\flat \in N^\flat : (\forall T \in \mathrm{ThFam}(\mathcal{I}))(\sigma^\flat[T] \leq \widetilde{\preccurlyeq}^{M,\mathcal{F}}(T))\}.$$

Lemma 1858 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. For all* $\sigma^\flat$ *in* $N^\flat$*, the following conditions are equivalent:*

(i) For every $T \in \mathrm{ThFam}(\mathcal{I})$*,* $\sigma^\flat[T] \le \precsim^{M,\mathcal{F}}(T)$*;*

(ii) For every $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,* $\sigma^\flat_\Sigma[\phi] \le \precsim^{M,\mathcal{F}}(C(\phi))$*.*

Proof: Suppose Condition (i) holds and let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, setting $T = C(\phi)$ in (i), we obtain $\sigma^\flat[C(\phi)] \le \precsim^{M,\mathcal{F}}(C(\phi))$, whence, a fortiori, $\sigma^\flat_\Sigma[\phi] \le \precsim^{M,\mathcal{F}}(C(\phi))$. Assume, conversely, that Condition (ii) holds and let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, we get

$$\begin{array}{rcll}\sigma^\flat[T] & = & \bigcup\{\sigma^\flat_\Sigma[\phi] : \phi \in T_\Sigma, \Sigma \in |\mathbf{Sign}^\flat|\} & \text{(definition)}\\ & \le & \bigcup\{\precsim^{M,\mathcal{F}}(C(\phi)) : \phi \in T_\Sigma, \Sigma \in |\mathbf{Sign}^\flat|\} & \text{(Condition (ii))}\\ & \le & \bigcup\{\precsim^{M,\mathcal{F}}(T) : \phi \in T_\Sigma, \Sigma \in |\mathbf{Sign}^\flat|\} & (\text{monotonicity of } \precsim^{M,\mathcal{F}})\\ & = & \precsim^{M,\mathcal{F}}(T). & \end{array}$$

Thus shows that Condition (i) holds and, therefore, that the two conditions are equivalent. ■

By Lemma 1858, this definition is equivalent to setting

$$\begin{array}{c}O^{\mathcal{I},M} = \{\sigma^\flat \in N^\flat : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \phi \in \mathrm{SEN}^\flat(\Sigma))\\ (\sigma^\flat_\Sigma[\phi] \le \precsim^{M,\mathcal{F}}(C(\phi)))\}.\end{array}$$

It is clear, by definition that the M-order core of a π-institution satisfies the following property:

Proposition 1859 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. For every* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$T \le O^{\mathcal{I},M}(\preccurlyeq^{M,\mathcal{F}}(T)).$$

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rll}\phi \in T_\Sigma & \text{implies} & O^{\mathcal{I},M}_\Sigma[\phi] \le \precsim^{M,\mathcal{F}}(T) \quad (\text{definition of } O^{\mathcal{I},M})\\ & \text{implies} & O^{\mathcal{I},M}_\Sigma[\phi] \le \preccurlyeq^{M,\mathcal{F}}(T). \quad (\precsim^{M,\mathcal{F}}(T) \le \preccurlyeq^{M,\mathcal{F}}(T))\end{array}$$

Thus, we get that $T \le O^{\mathcal{I},M}(\preccurlyeq^{M,\mathcal{F}}(T))$. ■

It is possible, but not necessary, that the M-order core of a π-institution satisfies the reverse inclusion. We call this property order solubility.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. We say that the M-order core of $\mathcal{I}$ is **order soluble** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$O^{\mathcal{I},M}(\preccurlyeq^{M,\mathcal{F}}(T)) \le T.$$

In other words $O^{\mathcal{I},M}$ is order soluble if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$O^{\mathcal{I},M}_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T) \quad \text{implies} \quad \phi \in T_\Sigma.$$

It turns out that possession of the order solubility property by the M-order core intrinsically characterizes M-truth inequationality. We show, first, that the M-order core being order soluble is necessary for M-truth inequationality. To see this, observe that, in case a π-institution is M-truth inequational, the witnessing transformations form a subset of the M-order core.

Lemma 1860 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is M-truth inequational, with witnessing transformations $\tau^\flat \subseteq N^\flat$, then $\tau^\flat \subseteq O^{\mathcal{I},M}$.*

Proof: By truth inequationality, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T).$$

Thus, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}\phi \in T_\Sigma & \text{iff} & (\forall T \leq T' \in \mathrm{ThFam}(\mathcal{I}))(\phi \in T'_\Sigma)\\ & \text{iff} & (\forall T \leq T' \in \mathrm{ThFam}(\mathcal{I}))(\tau^\flat_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T'))\\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \bigcap\{\preccurlyeq^{M,\mathcal{F}}(T') : T \leq T' \in \mathrm{ThFam}(\mathcal{I})\}\\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \widetilde{\preccurlyeq}^{M,\mathcal{F}}(T).\end{array}$$

We conclude, by the definition of $O^{\mathcal{I},M}$, that $\tau^\flat \subseteq O^{\mathcal{I},M}$. ■

Now we prove the necessity of order solubility for truth inequationality.

Theorem 1861 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$. If $\mathcal{I}$ is M-truth inequational, then $O^{\mathcal{I},M}$ is order soluble.*

Proof: Suppose that $\mathcal{I}$ is M-truth equational, with witnessing equations $\tau^\flat$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{llll}O^{\mathcal{I},M}_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T) & \text{implies} & \tau^\flat_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T) & \text{(Lemma 1860)}\\ & \text{iff} & \phi \in T_\Sigma. & \text{(truth inequationality)}\end{array}$$

Thus, $O^{\mathcal{I},M}$ is order soluble. ■

The reverse implication, which also holds and completes the promised characterization of M-truth inequationality in terms of the M-order core, is presented in the following result.

Theorem 1862 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ be a π-institution based on $\mathbf{F}$. If $O^{\mathcal{I},M}$ is order soluble, then $\mathcal{I}$ is M-truth inequational, with witnessing equations $O^{\mathcal{I},M}$.*

Proof: It suffices to show that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad O^{\mathcal{I},M}_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T).$$

The left-to-right implication is given in Proposition 1859, whereas the converse is ensured by the postulated order solubility of $O^{\mathcal{I},M}$. ∎

Theorems 1861 and 1862 provide the promised characterization of M-truth inequationality in terms of the order solubility of the M-order core.

$$\mathcal{I} \text{ is } M\text{-Truth Inequational} \longleftrightarrow O^{\mathcal{I},M} \text{ is Soluble.}$$

Theorem 1863 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is* M*-truth inequational if and only if* $O^{\mathcal{I},M}$ *is order soluble.*

Proof: Theorem 1861 gives the "only if" and the "if" is by Theorem 1862. ∎

If $\mathcal{I}$ is M-truth inequational, then the M-order core defines theory families in $\mathcal{I}$ in terms of their M-Leibniz qosystems.

Proposition 1864 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. If* $O^{\mathcal{I},M}$ *is order soluble, then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$T = O^{\mathcal{I},M}(\preccurlyeq^{M,\mathcal{F}}(T)).$$

Proof: If $O^{\mathcal{I},M}$ is order soluble, then, by Theorem 1862, $O^{\mathcal{I},M}$ forms a set of witnessing transformations for the M-truth inequationality of $\mathcal{I}$. Therefore, by definition, we get that, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $T = O^{\mathcal{I},M}(\preccurlyeq^{M,\mathcal{F}}(T))$. ∎

In fact, this property may also be restated as another characterization of truth inequationality. Let us say that $O^{\mathcal{I},M}$ **defines theory families** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T = O^{\mathcal{I},M}(\preccurlyeq^{M,\mathcal{F}}(T))$. Then we have:

$$\mathcal{I} \text{ is } M\text{-Truth Equational} \longleftrightarrow O^{\mathcal{I},M} \text{ Defines Theory Families.}$$

Theorem 1865 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is* M*-truth inequational if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$T = O^{\mathcal{I},M}(\preccurlyeq^{M,\mathcal{F}}(T)).$$

Proof: If $\mathcal{I}$ is truth equational, then, by Theorem 1861, $O^{\mathcal{I},M}$ is order soluble. Thus, by Proposition 1864, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T = O^{\mathcal{I},M}(\preccurlyeq^{M,\mathcal{F}}(T))$.

Conversely, if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $T = O^{\mathcal{I},M}(\preccurlyeq^{M,\mathcal{F}}(T))$, then, $O^{\mathcal{I},M}$ is order soluble. Thus, again by Theorem 1863, $O^{\mathcal{I},M}$ is a set of witnessing equations and $\mathcal{I}$ is M-truth inequational. ∎

We finally show that the property that separates M-complete reflectivity from M-truth inequationality is exactly the adequacy property of the M-order core. Roughly speaking, this property ensures that the M-order core is rich enough to define M-Suszko qosystems in terms of the M-Leibniz qosystems of theory families that it selects via inclusion.

We have the following relationship connecting the M-order core with both M-Leibniz quosystems and M-Suszko qosystems of enveloping theory families.

Proposition 1866 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *be a* π*-institution based on* $\mathbf{F}$*. For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\bigcap \{ \preccurlyeq^{M,\mathcal{F}}(T) : O^{\mathcal{I},M}_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T) \} \leq \precapprox^{M,\mathcal{F}}(C(\phi)).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\begin{array}{lll} \phi \in T_\Sigma & \text{implies} & O^{\mathcal{I},M}_\Sigma[\phi] \leq \precapprox^{M,\mathcal{F}}(T) \quad (M\text{-order core}) \\ & \text{implies} & O^{\mathcal{I},M}_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T). \end{array}$$

Therefore, we have

$$\begin{array}{lll} \bigcap \{ \preccurlyeq^{M,\mathcal{F}}(T) : O^{\mathcal{I},M}_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T) \} & \leq & \bigcap \{ \preccurlyeq^{M,\mathcal{F}}(T) : O^{\mathcal{I},M}_\Sigma[\phi] \leq \precapprox^{M,\mathcal{F}}(T) \} \\ & \leq & \bigcap \{ \preccurlyeq^{M,\mathcal{F}}(T) : \phi \in T_\Sigma \} \\ & = & \precapprox^{M,\mathcal{F}}(C(\phi)). \end{array}$$

■

It is possible, but not necessary, that the M-order core of a π-institution satisfies, for every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, the reverse inclusion of that given in Proposition 1866:

$$\precapprox^{M,\mathcal{F}}(C(\phi)) \leq \bigcap \{ \preccurlyeq^{M,\mathcal{F}}(T) : O^{\mathcal{I},M}_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T) \}.$$

Intuitively speaking, this means that the M-order core $O^{\mathcal{I},M}$ is rich enough to allow, for every Σ-sentence ϕ, the determination of those theory families whose M-Leibniz qosystems form a covering of the M-Suszko qosystem of $C(\phi)$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$ and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution based on $\mathbf{F}$. We say that the M-order core $O^{\mathcal{I},M}$ of $\mathcal{I}$ is **order adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\precapprox^{M,\mathcal{F}}(C(\phi)) = \bigcap \{ \preccurlyeq^{M,\mathcal{F}}(T) : O^{\mathcal{I},M}_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T) \}.$$

It is not difficult to see that, if $O^{\mathcal{I},M}$ is order soluble, then it is order adequate.

Corollary 1867 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. If* $O^{\mathcal{I},M}$ *is order soluble, then it is order adequate.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{rcl}\widetilde{\preccurlyeq}^{M,\mathcal{F}}(C(\phi)) & = & \bigcap\{\preccurlyeq^{M,\mathcal{F}}(T) : \phi \in T_\Sigma\} \quad (\text{definition of } \widetilde{\preccurlyeq}^{M,\mathcal{F}}(C(\phi))) \\ & = & \bigcap\{\preccurlyeq^{M,\mathcal{F}}(T) : O^{\mathcal{I},M}_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T)\}. \\ & & (\text{order solubility of } S^{\mathcal{I}} \text{ and Proposition 1864})\end{array}$$

We conclude that $O^{\mathcal{I},M}$ is order adequate. ■

In the opposite direction, in an M-c-reflective π-institution $\mathcal{I}$, if the M-order core is order adequate, then it is also order soluble.

Proposition 1868 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *an* M*-c-reflective* π*-institution based on* $\mathbf{F}$*. If* $O^{\mathcal{I},M}$ *is order adequate, then it is order soluble.*

Proof: Suppose that $\mathcal{I}$ is M-c-reflective and that $O^{\mathcal{I},M}$ is order adequate. We must show that, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$

$$\phi \in T_\Sigma \quad \text{iff} \quad O^{\mathcal{I},M}_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T).$$

The implication left-to-right is always satisfied by Proposition 1859. For the converse, assume that $O^{\mathcal{I},M}_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}(T)$. Then, by the adequacy of $O^{\mathcal{I},M}$, we get that $\widetilde{\preccurlyeq}^{M,\mathcal{F}}(C(\phi)) \leq \preccurlyeq^{M,\mathcal{F}}(T)$. Thus, by M-c-reflectivity, we conclude that $C(\phi) \leq T$, which gives $\phi \in T_\Sigma$. ■

We finally show that a π-institution is M-truth inequational if and only if it is M-c-reflective and it has an order adequate M-order core.

$$\begin{array}{rcl}M\text{-Truth Inequationality} & = & O^{\mathcal{I},M} \text{ Order Soluble} \\ & = & O^{\mathcal{I},M} \text{ Defines Theory Families} \\ & = & M\text{-c-Reflectivity} + O^{\mathcal{I},M} \text{ Order Adequate}\end{array}$$

Theorem 1869 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is* M*-truth inequational if and only if it is* M*-c-reflective and has an order adequate* M*-order core.*

Proof: Suppose, first, that $\mathcal{I}$ is M-truth inequational. Then it is M-c-reflective by Theorem 1857. Moreover, its M-order core is order soluble by Theorem 1861 and, hence, by Corollary 1867, its M-order core is order adequate.

Suppose, conversely, that $\mathcal{I}$ is M-c-reflective with an order adequate M-order core. Then, by Proposition 1868, its M-order core is order soluble and, therefore, by Theorem 1863, $\mathcal{I}$ is M-truth inequational. ■

Taking into account Lemma 1856 we obtain the following

Corollary 1870 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *an* M*-order monotone* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is* M*-truth inequational if and only if it is* M*-order injective and has an order adequate* M*-order core.*

Proof: By Theorem 1869 and Lemma 1856. ■

Finally, it is not difficult to see that M-truth inequationality transfers from a π-institution to all $\mathcal{I}$-matrix families.

Theorem 1871 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $M = (M^+, M^-)$ *a polarity for* $\mathcal{I}$ *and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *be a* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is* M*-truth inequational, with witnessing transformations* $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ *in* $N^\flat$*, if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$*, and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,* $T = \tau^{\mathcal{A}}(\preccurlyeq^{M,\mathcal{A}}(T))$*.*

Proof: Suppose $\mathcal{I}$ is truth equational, with witnessing transformations $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, by Lemma 51, $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, whence, by hypothesis, $\alpha^{-1}(T) = \tau^\flat(\preccurlyeq^{M,\mathcal{F}}(\alpha^{-1}(T)))$. Hence, by Lemma 1835, $\alpha^{-1}(T) = \tau^\flat(\alpha^{-1}(\preccurlyeq^{M,\mathcal{A}}(T)))$. Therefore, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi \in \mathrm{SEN}^\flat(\Sigma)$, we get

$$\begin{array}{lll}\alpha_\Sigma(\phi) \in T_{F(\Sigma)} & \text{iff} & \phi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})\\ & \text{iff} & \tau^\flat_\Sigma[\phi] \leq \alpha^{-1}(\preccurlyeq^{M,\mathcal{A}}(T))\\ & \text{iff} & \alpha(\tau^\flat_\Sigma[\phi]) \leq \preccurlyeq^{M,\mathcal{A}}(T)\\ & \text{iff} & \tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\phi)] \leq \preccurlyeq^{M,\mathcal{A}}(T). \quad (\langle F, \alpha\rangle \text{ surjective})\end{array}$$

Taking again into account the surjectivity of $\langle F, \alpha\rangle$, we conclude that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $\phi \in T_\Sigma$ if and only if $\tau^{\mathcal{A}}_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{A}}(T)$, i.e., $T = \tau^{\mathcal{A}}(\preccurlyeq^{M,\mathcal{A}}(T))$. ■

25.7 Order Algebraizability

Theorem 1872 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system, with* $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$ *having two distinguished arguments,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *an* M*-directional* π*-institution based on* $\mathbf{F}$*, with witnessing transformations* β*, such that, for all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$*, all* $\sigma^\flat, \tau^\flat$ *in* $N^\flat$*, and all* $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\beta_\Sigma[\sigma^\flat_\Sigma(\psi, \vec{\chi}), \tau^\flat_\Sigma(\psi, \vec{\chi})] \leq C(\overleftrightarrow{\beta}_\Sigma[\phi, \psi], \beta_\Sigma[\sigma^\flat_\Sigma(\phi, \vec{\chi}), \tau^\flat_\Sigma(\phi, \vec{\chi})]).$$

Then the following conditions are equivalent:

(i) $\mathcal{I}$ *is* β*-order algebraized by* $\{\langle \mathcal{A}, \leq^{\mathcal{A},T}(T)\rangle : \langle \mathcal{A}, T\rangle \in \mathrm{MatFam}^*(\mathcal{I})\}$*;*

(ii) $\mathcal{I}$ *is* β*-order algebraizable;*

(iii) $\mathcal{I}$ *is* M*-truth inequational;*

(iv) $\mathcal{I}$ *is* M*-order injective and has an order adequate* M*-order core.*

Proof:

(i)⇒(ii) This implication is trivial.

(ii)⇒(iii) By hypothesis, β witnesses the M-directionality of $\mathcal{I}$. Therefore, by Theorem 1840, $\preccurlyeq^M = \preccurlyeq^B$. Thus, since, by hypothesis, $\mathcal{I}$ is β-order algebraizable, by Proposition 1854, $\mathcal{I}$ is M-truth equational, with witnessing transformations β.

(iii)⇒(i) Suppose $\mathcal{I}$ is M-truth inequational, with witnessing transformations $\tau^\flat : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$, having a single distinguished argument. Thus, we have, for every $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \tau^\flat_\Sigma[\phi] \leq \preccurlyeq^{M,\mathcal{F}}_\Sigma(T).$$

Thus, by M-directionality, $\phi \in T_\Sigma$ if and only if $\beta[\tau^\flat_\Sigma[\phi]] \leq T$. Thus, we get that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$C(\phi) = C(\beta[\tau^\flat_\Sigma[\phi]]). \tag{25.1}$$

Since, by hypothesis, $\mathcal{I}$ is M-directional, we have, by Theorem 1840, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{l}\beta_\Sigma[\phi,\phi] \leq \mathrm{Thm}(\mathcal{I});\\ \beta_\Sigma[\phi,\chi] \leq C(\beta_\Sigma[\phi,\psi], \beta_\Sigma[\psi,\chi]).\end{array} \tag{25.2}$$

Given the hypothesis, Conditions (25.2) and Condition (25.1), we get, by Theorem 1828, that $\mathcal{I}$ is β-order algebraizable. Therefore, again by Theorem 1828, $\mathcal{I}$ is β-order algebraized by the class $\{\langle \mathcal{A}, \leq^{\mathcal{A},T}(T)\rangle : \langle \mathcal{A}, T\rangle \in \mathrm{MatFam}^*(\mathcal{I})\}$.

(iii)⇒(iv) Since $\mathcal{I}$ is M-truth inequational, by Theorem 1869, it is M-c-reflective and has an order adequate M-order core. Since $\mathcal{I}$ is M-directional, by Theorem 1853, it is M-order monotone. Hence, since it is M-c-reflective, by Lemma 1856, it is M-order injective, Thus, $\mathcal{I}$ is M-order injective and has an order adequate M-order core.

(iv)⇒(iii) Suppose $\mathcal{I}$ is M-order injective, with an order adequate M-order core. Since, by hypothesis, $\mathcal{I}$ is M-directional, it is, by Theorem 1853, M-order monotone. Thus, since it is, by hypothesis, M-order injective, it is by Lemma 1856, M-c-reflective. Being M-c-reflective with an M-order adequate M-order core, it is, by Theorem 1869, M-truth inequational.

■

Theorem 1873 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*. The following conditions are equivalent:*

(i) There exists a polarity $M = (M^+, M^-)$ *for* $\mathbf{F}$*, such that* $\mathcal{I}$ *is* M*-order monotone,* M*-order injective,* $\preccurlyeq^M$ *is antisymmetric on* $\{T : \langle \mathcal{A}, T \rangle \in \mathrm{MatFam}^*(\mathcal{I})\}$*, with an order Leibniz* M*-quasicore and an order adequate* M*-order core;*

(ii) $\mathcal{I}$ *is order algebraizable, i.e., it is* β*-order algebraizable, for some* $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$ *having two distinguished arguments.*

If Condition (i) holds, then β *can be chosen so that* $\preccurlyeq^M = \preccurlyeq^B$ *and*

$$\{\langle \mathcal{A}, \leq^{\mathcal{A},T} \rangle : \langle \mathcal{A}, T \rangle \in \mathrm{MatFam}^*(\mathcal{I})\}$$

generates the β*-order class of* $\mathcal{I}$*.*

If Condition (ii) holds, then Condition (i) holds with $M = B$*.*

Proof:

(i)⇒(ii) Suppose Condition (i) holds. Since, by hypothesis, $\mathcal{I}$ is M-order monotone and has an order Leibniz M-quasicore, we get, by Theorem 1853, that $\mathcal{I}$ is M-directional, with some family β of witnessing transformations. Thus, by Theorem 1840, $\preccurlyeq^M = \preccurlyeq^B$. By hypothesis and Theorem 1869, we get that $\mathcal{I}$ is M-truth inequational. Therefore, by hypothesis, Theorem 1826 and Theorem 1872, we get that $\mathcal{I}$ is β-order algebraizable and that its β-order class is generated by $\{\langle \mathcal{A}, \leq^{\mathcal{A},T} \rangle : \langle \mathcal{A}, T \rangle \in \mathrm{MatFam}^*(\mathcal{I})\}$.

(ii)⇒(i) Suppose Condition (ii) holds. Then, by Corollary 1843, $\mathcal{I}$ is B-directional, with witnessing transformations β. Thus, by Theorem 1853, it is B-order monotone and has an order Leibniz B-quasicore. Moreover, by Proposition 1854, $\mathcal{I}$ is B-truth inequational and, therefore, by Theorem 1869, it is B-c-reflective and has on order adequate B-order core. Finally, taking into account Theorem 1828, we may apply Theorem 1872 to establish that $\preccurlyeq^B$ is antisymmetric on $\{T : \langle \mathcal{A}, T \rangle \in \mathrm{MatFam}^*(\mathcal{I})\}$. ■

Corollary 1874 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, with* $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ *in* $N^\flat$*, having two distinguished arguments,* $M = (M^+, M^-)$ *a polarity for* $\mathbf{F}$ *and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a* π*-institution based on* $\mathbf{F}$*, such that, for all* $\sigma^\flat, \tau^\flat$ *in* $N^\flat$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $\phi, \psi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$*:*

1. $\beta_\Sigma[\phi, \phi] \leq \mathrm{Thm}(\mathcal{I})$*;*

2. $\sigma_\Sigma(\psi,\vec{\chi}) \in C_\Sigma(\beta_\Sigma[\phi,\psi],\sigma^\flat_\Sigma(\phi,\vec{\chi}))$, *if* $\sigma^\flat \in M^+$;

3. $\sigma_\Sigma(\phi,\vec{\chi}) \in C_\Sigma(\beta_\Sigma[\phi,\psi],\sigma^\flat_\Sigma(\psi,\vec{\chi}))$, *if* $\sigma^\flat \in M^-$;

4. $\beta_\Sigma[\sigma^\flat_\Sigma(\psi,\vec{\chi}),\tau^\flat_\Sigma(\psi,\vec{\chi})] \leq C(\overleftrightarrow{\beta}_\Sigma[\phi,\psi],\beta_\Sigma[\sigma^\flat_\Sigma(\phi,\vec{\chi}),\tau^\flat_\Sigma(\phi,\vec{\chi})])$.

If, for all $\sigma^\flat \in \beta$, $\sigma^\flat(x,y,\vec{z}) \in M^-$ *or* $\sigma^\flat(y,x,\vec{z}) \in M^+$, *then* $\mathcal{I}$ *is* β*-order algbebraizable if and only if it is* M*-order injective and has an order adequate* M*-order core.*

Proof: By Theorem 1872, it suffices to show that $\mathcal{I}$ is M-directional. But this follows from Theorem 1839. ■

25.8 Tonicity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, $M = (M^+,M^-)$ a polarity for $\mathbf{F}$, $\mathcal{A} = \langle \mathbf{A},\langle F,\alpha\rangle\rangle$ an $\mathbf{F}$-algebraic system and $\leq^\mathcal{A}$ a qosystem on $\mathcal{A}$. $\leq^\mathcal{A}$ is called an M-**order** if, for all $\sigma^\flat$ in $N^\flat$, $\Sigma \in |\mathbf{Sign}|$, $\phi,\psi,\vec{\chi} \in \mathrm{SEN}(\Sigma)$,

- if $\sigma^\flat \in M^+$, then $\phi \leq^\mathcal{A}_\Sigma \psi$ implies $\sigma^\mathcal{A}_\Sigma(\phi,\vec{\chi}) \leq^\mathcal{A}_\Sigma \sigma^\mathcal{A}_\Sigma(\psi,\vec{\chi})$;
- if $\sigma^\flat \in M^-$, then $\phi \leq^\mathcal{A}_\Sigma \psi$ implies $\sigma^\mathcal{A}_\Sigma(\psi,\vec{\chi}) \leq^\mathcal{A}_\Sigma \sigma^\mathcal{A}_\Sigma(\phi,\vec{\chi})$.

In a way similar to the proof of the existence of $\preccurlyeq^{M,\mathcal{A}}(T)$ in Proposition 1832, we can also show that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A},\langle F,\alpha\rangle\rangle$ and all $T \in \mathrm{SenFam}(\mathcal{A})$, there always exists a largest M-order on $\mathcal{A}$, such that T is upward closed, i.e., for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi,\psi \in \mathrm{SEN}(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{and} \quad \phi \leq^{M,\mathcal{A}}_\Sigma \psi \quad \text{imply} \quad \psi \in T_\Sigma.$$

Proposition 1875 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* $M = (M^+,M^-)$ *a polarity for* $\mathbf{F}$. *For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A},\langle F,\alpha\rangle\rangle$ *and all* $T \in \mathrm{SenFam}(\mathcal{A})$*, there exists a largest* M*-order on* $\mathcal{A}$*, such that* T *is upward closed.*

Proof: We consider the class $\mathrm{MOrd}^\mathcal{A}(T)$ of all M-orders on $\mathcal{A}$ with respect to which T is upward closed. We take the transitive closure of the union of all qosystems in $\mathrm{MOrd}^\mathcal{A}(T)$,

$$\mathrm{tc}(\bigcup \mathrm{MOrd}^\mathcal{A}(T)) = \{\mathrm{tc}_\Sigma(\bigcup \mathrm{MOrd}^\mathcal{A}(T)\}_{\Sigma\in|\mathbf{Sign}|}.$$

It suffices to show that this is also an M-order on $\mathcal{A}$ with respect to which T is upward closed. i.e., it is itself a member of $\mathrm{MOrd}^\mathcal{A}(T)$. It will then follow that it is its largest member.

It is clear by the definition that $\mathrm{tr}(\bigcup \mathrm{MOrd}^{\mathcal{A}}(T))$ is a qosystem on $\mathcal{A}$. So it suffices to show that it is an M-order with respect to which T is upward closed.

Suppose $\sigma^\flat$ in M^+, $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi, \vec{\chi} \in \mathrm{SEN}(\Sigma)$, such that

$$\phi \ \mathrm{tr}_\Sigma(\bigcup \mathrm{MOrd}^{\mathcal{A}}(T)) \ \psi.$$

Then, there exist $q^0, \dots, q^k \in \mathrm{MOrd}^{\mathcal{A}}(T)$ and $\xi_1, \dots, \xi_k \in \mathrm{SEN}(\Sigma)$, such that

$$\phi \ q^0_\Sigma \ \xi_1 \ q^1_\Sigma \ \xi_2 \ q^2_\Sigma \cdots q^{k-1}_\Sigma \ \xi_k \ q^k_\Sigma \ \psi.$$

Since $\phi \ q^0_\Sigma \ \xi_1$ and $q^0 \in \mathrm{MOrd}^{\mathcal{A}}(T)$, we get $\sigma^{\mathcal{A}}_\Sigma(\phi, \vec{\chi}) \ q^0_\Sigma \ \sigma^{\mathcal{A}}_\Sigma(\xi_1, \vec{\chi})$. Since $\xi_1 \ q^1_\Sigma \ \xi_2$ and $q^1 \in \mathrm{MOrd}^{\mathcal{A}}(T)$, we get $\sigma^{\mathcal{A}}_\Sigma(\xi_1, \vec{\chi}) \ q^1_\Sigma \ \sigma^{\mathcal{A}}_\Sigma(\xi_2, \vec{\chi})$. We move one step to the right at a time in a similar fashion until we obtain $\sigma^{\mathcal{A}}_\Sigma(\xi_k, \vec{\chi}) \ q^k_\Sigma \ \sigma^{\mathcal{A}}_\Sigma(\psi, \vec{\chi})$. Thus, we obtain

$$\sigma^{\mathcal{A}}_\Sigma(\phi, \vec{\chi}) \ \mathrm{tr}_\Sigma(\bigcup \mathrm{MOrd}^{\mathcal{A}}(T)) \ \sigma^{\mathcal{A}}_\Sigma(\psi, \vec{\chi}).$$

A similar argument is used to handle the case of negative polarity for $\sigma^\flat$. This proves that $\mathrm{tr}(\bigcup \mathrm{MOrd}^{\mathcal{A}}(T))$ is also an M-order on $\mathcal{A}$.

Finally, suppose $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in T_\Sigma$ and

$$\phi \ \mathrm{tr}_\Sigma(\bigcup \mathrm{MOrd}^{\mathcal{A}}(T)) \ \psi.$$

Then, there exist $q^0, \dots, q^k \in \mathrm{MOrd}^{\mathcal{A}}(T)$ and $\xi_1, \dots, \xi_k \in \mathrm{SEN}(\Sigma)$, such that

$$\phi \ q^0_\Sigma \ \xi_1 \ q^1_\Sigma \ \xi_2 \ q^2_\Sigma \cdots q^{k-1}_\Sigma \ \xi_k \ q^k_\Sigma \ \psi.$$

Since T is upward closed with respect to all elements in $\mathrm{MOrd}^{\mathcal{A}}(T)$ and $\phi \in T_\Sigma$, we get $\xi_1 \in T_\Sigma$, then $\xi_2 \in T_\Sigma$, then $\dots$, until, in the last step, $\xi_k \in T_\Sigma$ implies $\psi \in T_\Sigma$. Therefore, T is also upward closed with respect to $\mathrm{tr}_\Sigma(\bigcup \mathrm{MOrd}^{\mathcal{A}}(T))$, showing that $\mathrm{tr}_\Sigma(\bigcup \mathrm{MOrd}^{\mathcal{A}}(T)) \in \mathrm{MOrd}^{\mathcal{A}}(T)$, whence it is its largest element. ■

Given an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and $T \in \mathrm{SenFam}(\mathcal{A})$, the **Leibniz M-order $\leq^{M,\mathcal{A}}(T)$ of $\langle \mathcal{A}, T \rangle$** is the largest M-order on $\mathcal{A}$, such that T is upward closed, whose existence is assured by Proposition 1875.

It turns out that the Leibniz M-order $\leq^{M,\mathcal{A}}(T)$ is included in the M-Leibniz qosystem $\preccurlyeq^{M,\mathcal{A}}(T)$ of T on $\mathcal{A}$.

Proposition 1876 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$. For every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T \in \mathrm{SenFam}(\mathcal{A})$,*

$$\leq^{M,\mathcal{A}}(T) \leq \preccurlyeq^{M,\mathcal{A}}(T).$$

Proof: It suffices to show that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T \in \mathrm{SenFam}(\mathcal{A})$, $\leq^{M,\mathcal{A}}(T)$ is M-compatible with T. To this end, let $\sigma^\flat$ in $N^\flat$, $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi, \vec{\chi} \in \mathrm{SEN}(\Sigma)$.

- Suppose $\sigma^\flat \in M^+$, $\phi \leq_\Sigma^{M,\mathcal{A}}(T)\psi$ and $\sigma_\Sigma^{\mathcal{A}}(\phi,\vec{\chi}) \in T_\Sigma$. Since $\phi \leq_\Sigma^{M,\mathcal{A}}(T)\psi$ and $\leq^{M,\mathcal{A}}(T)$ is an M-order, we get $\sigma_\Sigma^{\mathcal{A}}(\phi,\vec{\chi}) \leq \sigma_\Sigma^{\mathcal{A}}(\psi,\vec{\chi})$. Hence, since $\sigma_\Sigma^{\mathcal{A}}(\phi,\vec{\chi}) \in T_\Sigma$ and T is upward closed with respect to $\leq^{M,\mathcal{A}}(T)$, we get $\sigma_\Sigma^{\mathcal{A}}(\psi,\vec{\chi}) \in T_\Sigma$.

- Suppose $\sigma^\flat \in M^-$, $\phi \leq_\Sigma^{M,\mathcal{A}}(T)\psi$ and $\sigma_\Sigma^{\mathcal{A}}(\psi,\vec{\chi}) \in T_\Sigma$. Since $\phi \leq_\Sigma^{M,\mathcal{A}}(T)\psi$ and $\leq^{M,\mathcal{A}}(T)$ is an M-order, we get $\sigma_\Sigma^{\mathcal{A}}(\psi,\vec{\chi}) \leq \sigma_\Sigma^{\mathcal{A}}(\phi,\vec{\chi})$. Hence, since $\sigma_\Sigma^{\mathcal{A}}(\psi,\vec{\chi}) \in T_\Sigma$ and T is upward closed with respect to $\leq^{M,\mathcal{A}}(T)$, we get $\sigma_\Sigma^{\mathcal{A}}(\phi,\vec{\chi}) \in T_\Sigma$.

Thus, $\leq^{M,\mathcal{A}}(T)$ is M-compatible with T and, hence, by the maximality of $\preccurlyeq^{M,\mathcal{A}}(T)$, $\leq^{M,\mathcal{A}}(T) \leq \preccurlyeq^{M,\mathcal{A}}(T)$. ■

We finally provide sufficient conditions ensuring that the two orders on $\mathcal{A}$ associated with $\mathcal{I}$-filter families T of a π-institution $\mathcal{I}$, $\leq^{M,\mathcal{A}}(T)$ and $\preccurlyeq^{M,\mathcal{A}}(T)$, coincide.

Proposition 1877 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, with $\beta : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$ having two distinguished arguments, $M = (M^+, M^-)$ a polarity for $\mathbf{F}$, such that $p^{1,0} \in M^+$, and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a π-institution based on $\mathbf{F}$. Suppose $\mathcal{I}$ is M-directional, with witnessing transformations β, and that, for all σ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi,\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,*

- *if $\sigma^\flat \in M^+$, $\beta_\Sigma[\sigma_\Sigma^\flat(\phi,\vec{\chi}), \sigma_\Sigma^\flat(\psi,\vec{\chi})] \leq C(\beta_\Sigma[\phi,\psi])$;*

- *if $\sigma^\flat \in M^-$, $\beta_\Sigma[\sigma_\Sigma^\flat(\psi,\vec{\chi}), \sigma_\Sigma^\flat(\phi,\vec{\chi})] \leq C(\beta_\Sigma[\phi,\psi])$.*

Then, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\preccurlyeq^{M,\mathcal{A}}(T)$ is the largest M-order on $\mathcal{A}$ with respect to which T is upward closed, i.e., $\preccurlyeq^{M,\mathcal{A}}(T) = \leq^{M,\mathcal{A}}(T)$.

Proof: Let $\mathcal{A}$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. We show that $\preccurlyeq^{M,\mathcal{A}}(T)$ is an M-order on $\mathcal{A}$, with respect to which T is upward closed. Then, it will follow, by the maximality property of $\leq^{M,\mathcal{A}}(T)$, that $\preccurlyeq^{M,\mathcal{A}}(T) \leq \leq^{M,\mathcal{A}}(T)$.

Let $\sigma^\flat$ in $N^\flat$, $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi,\vec{\chi} \in \mathrm{SEN}(\Sigma)$.

- Suppose $\sigma^\flat \in M^+$ and $\phi \preccurlyeq_\Sigma^{M,\mathcal{A}}(T)\ \psi$. Thus, by M-directionality of $\mathcal{I}$, $\beta_\Sigma^{\mathcal{A}}[\phi,\psi] \leq T$, whence, by hypothesis, $\beta_\Sigma^{\mathcal{A}}[\sigma_\Sigma^{\mathcal{A}}(\phi,\vec{\chi}), \sigma_\Sigma^{\mathcal{A}}(\psi,\vec{\chi})] \leq T$. Thus, again by M-directionaility, $\sigma^{\mathcal{A}}(\phi,\vec{\chi}) \preccurlyeq_\Sigma^{M,\mathcal{A}}(T)\ \sigma_\Sigma^{\mathcal{A}}(\psi,\vec{\chi})$.

- Suppose $\sigma^\flat \in M^-$ and $\phi \preccurlyeq_\Sigma^{M,\mathcal{A}}(T)\ \psi$. Thus, by M-directionality of $\mathcal{I}$, $\beta_\Sigma^{\mathcal{A}}[\phi,\psi] \leq T$, whence, by hypothesis, $\beta_\Sigma^{\mathcal{A}}[\sigma_\Sigma^{\mathcal{A}}(\psi,\vec{\chi}), \sigma_\Sigma^{\mathcal{A}}(\phi,\vec{\chi})] \leq T$. Thus, again by M-directionaility, $\sigma^{\mathcal{A}}(\psi,\vec{\chi}) \preccurlyeq_\Sigma^{M,\mathcal{A}}(T)\ \sigma_\Sigma^{\mathcal{A}}(\phi,\vec{\chi})$.

Thus, $\preccurlyeq^{M,\mathcal{A}}(T)$ is an M-order on $\mathcal{A}$.

Finally, suppose $\phi \preccurlyeq_\Sigma^{M,\mathcal{A}}(T)\ \psi$ and $\phi \in T_\Sigma$. Then, since, by hypothesis, $p^{1,0} \in M^+$ and $\preccurlyeq^{M,\mathcal{A}}(T)$ is M-compatible with T, we get $\psi \in T_\Sigma$. Therefore, $\preccurlyeq^{M,\mathcal{A}}(T)$ is an M-order with respect to which T is upward closed. It now follows by the maximality of $\leq^{M,\mathcal{A}}(T)$, that $\preccurlyeq^{M,\mathcal{A}}(T) \leq \leq^{M,\mathcal{A}}(T)$ and, hence, by Proposition 1876, that $\preccurlyeq^{M,\mathcal{A}}(T) = \leq^{M,\mathcal{A}}(T)$. ■

Chapter 26

Gentzen π-Institutions

26.1 Gentzen π-Institutions Revisited

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be an $N^\flat$-algebraic system, $\Sigma \in |\mathbf{Sign}|$ and $m, n \in \omega$. An $\langle m,n \rangle$**-**Σ**-sequent of A** is an expression

$$\phi_0, \ldots, \phi_{m-1} \rhd_\Sigma \psi_0, \ldots, \psi_{n-1},$$

abbreviated $\vec{\phi} \rhd_\Sigma \vec{\psi}$, consisting of two finite (possibly empty) sequences $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)$. A $\langle 0,n \rangle$-Σ-sequent $\varnothing \rhd_\Sigma \vec{\psi}$ is abbreviated $\rhd_\Sigma \vec{\psi}$.

Given $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and an $\langle m,n \rangle$-Σ-sequent $\vec{\phi} \rhd_\Sigma \vec{\psi}$, we write

$$\mathrm{SEN}(f)(\vec{\phi} \rhd_\Sigma \vec{\psi}) \coloneqq \mathrm{SEN}(f)(\vec{\phi}) \rhd_{\Sigma'} \mathrm{SEN}(f)(\vec{\psi}),$$

where, as usual,

$$\begin{array}{rcl}\mathrm{SEN}(f)(\vec{\phi}) & \coloneqq & \langle \mathrm{SEN}(f)(\phi_0), \ldots, \mathrm{SEN}(f)(\phi_{m-1}) \rangle, \\ \mathrm{SEN}(f)(\vec{\psi}) & \coloneqq & \langle \mathrm{SEN}(f)(\psi_0), \ldots, \mathrm{SEN}(f)(\psi_{n-1}) \rangle.\end{array}$$

Sometimes, we denote a Σ-sequent by $\boldsymbol{\phi} \coloneqq \vec{\phi}^0 \rhd_\Sigma \vec{\phi}^1$ and a set of Σ-sequents by $\mathbf{\Phi}$. The notation for images under morphisms is then extended to sets of Σ-sequents by writing

$$\mathrm{SEN}(f)(\mathbf{\Phi}) = \{\mathrm{SEN}(f)(\boldsymbol{\phi}) : \boldsymbol{\phi} \in \mathbf{\Phi}\}.$$

A **trace** tr is a nonempty subset of $\omega \times \omega$. An $\langle m,n \rangle$-Σ-sequent is a **tr-**Σ**-sequent** if $\langle m,n \rangle \in \mathrm{tr}$. The collection of all tr-Σ-sequents of $\mathbf{A}$ is denoted by $\mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{A})$ and we set

$$\mathrm{Seq}^{\mathrm{tr}}(\mathbf{A}) = \{\mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{A})\}_{\Sigma \in |\mathbf{Sign}|}.$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and tr be a given trace. A **Gentzen** π**-institution** $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ **of trace** tr based on $\mathbf{F}$ consists of a closure system

$$G : \mathcal{P}\mathrm{Seq}^{\mathrm{tr}}(\mathbf{F}) \to \mathcal{P}\mathrm{Seq}^{\mathrm{tr}}(\mathbf{F}),$$

i.e., a collection of closure operators

$$G_\Sigma : \mathcal{P}\mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F}) \to \mathcal{P}\mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F}), \quad \Sigma \in |\mathbf{Sign}^\flat|,$$

that also satisfy structurality, that is, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, and all $\mathbf{\Phi} \subseteq \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$,

$$\mathrm{SEN}(f)(G_\Sigma(\mathbf{\Phi})) \subseteq G_{\Sigma'}(\mathrm{SEN}(f)(\mathbf{\Phi})).$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$. If, for some $\Sigma \in |\mathbf{Sign}^\flat|$,

$\mathbf{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$, such that $\boldsymbol{\phi} \in G_\Sigma(\mathbf{\Phi})$, we say that $\langle \mathbf{\Phi}, \boldsymbol{\phi}\rangle$ is a Σ-**rule of** $\mathfrak{G}$ or a Σ-**derivable rule of** $\mathfrak{G}$, sometimes denoted

$$\frac{\mathbf{\Phi}}{\boldsymbol{\phi}}.$$

A Σ-rule of form $\langle \varnothing, \boldsymbol{\phi}\rangle$ is called a Σ-**derivable sequent** or a Σ-**theorem of** $\mathfrak{G}$.

$\mathfrak{G}$ is **inconsistent** if all elements in $\mathrm{Seq}^{\mathrm{tr}}(\mathbf{F})$ are derivable sequents in $\mathfrak{G}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, tr a trace and $\mathfrak{G}^i = \langle \mathbf{F}, G^i\rangle$, $i \in I$, a collection of Gentzen π-institutions, all of trace tr, based on $\mathbf{F}$. Then

$$\bigcap_{i\in I} \mathfrak{G}^i = \langle \mathbf{F}, \bigcap_{i\in I} G^i\rangle,$$

defined, by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathbf{\Phi} \subseteq \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$,

$$(\bigcap_{i\in I} G^i)_\Sigma(\mathbf{\Phi}) = \bigcap_{i\in I} G^i_\Sigma(\mathbf{\Phi}),$$

is also a Gentzen π-institution.

Therefore, given a family $\mathfrak{X} = \{\mathfrak{X}_\Sigma\}_{\Sigma\in|\mathbf{Sign}^\flat|}$ of rules, there is a smallest Gentzen π-institution $\mathfrak{G}^{\mathfrak{X}} = \langle \mathbf{F}, G^{\mathfrak{X}}\rangle$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\langle \mathbf{\Phi}, \boldsymbol{\phi}\rangle \in \mathfrak{X}_\Sigma$,

$$\boldsymbol{\phi} \in G^{\mathfrak{X}}_\Sigma(\mathbf{\Phi}).$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ a Gentzen π-institution based on $\mathbf{F}$, with $\langle 0, 1\rangle \in \mathrm{tr}$. Consider

$$G^0 : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}$$

defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, by

$$\phi \in G^0_\Sigma(\Phi) \quad \text{iff} \quad \rhd_\Sigma \phi \in G_\Sigma(\{\rhd_\Sigma \psi : \psi \in \Phi\}).$$

Lemma 1878 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen* π*-institution based on* $\mathbf{F}$*, with* $\langle 0, 1\rangle \in \mathrm{tr}$*.* $G^0 : \mathcal{P}\mathrm{SEN}^\flat \to \mathcal{P}\mathrm{SEN}^\flat$ *is a closure system on* $\mathbf{F}$*.*

Proof: Suppose, first, that $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in \Phi$. Then, by the inflationarity of G, $\rhd_\Sigma \phi \in G_\Sigma(\{\rhd_\Sigma \psi : \psi \in \Phi\})$ and, hence, by definition of G^0, $\phi \in G^0_\Sigma(\Phi)$. Suppose, next, that $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \Psi \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\Phi \subseteq \Psi$. Then, by monotonicity of G, $G_\Sigma(\{\rhd_\Sigma \phi : \phi \in \Phi\}) \subseteq G_\Sigma(\{\rhd_\Sigma \psi : \psi \in \Psi\})$, whence, by the definition of G^0, $G^0_\Sigma(\Phi) \subseteq G^0_\Sigma(\Psi)$. Now assume that $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \in G^0_\Sigma(G^0_\Sigma(\Phi))$. Then, taking into account the idempotency of G, we get

$$\begin{array}{rcl}\rhd_\Sigma \phi & \in & G_\Sigma(\{\rhd_\Sigma \psi : \ \psi \in G^0_\Sigma(\Phi)\}) \\ & \subseteq & G_\Sigma(G_\Sigma(\{\rhd_\Sigma \phi : \phi \in \Phi\})) \\ & \subseteq & G_\Sigma(\{\rhd_\Sigma \phi : \phi \in \Phi\}),\end{array}$$

whence $\phi \in G^0_\Sigma(\Phi)$. Finally, the structurality property of G^0 follows directly by the structurality property of G. ■

According to Lemma 1878, the structure $\mathcal{G}^0 = \langle \mathbf{F}, G^0 \rangle$ is a π-institution, called the **π-institution reduct of** the Gentzen π-institution $\mathfrak{G}$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and K a class of $\mathbf{F}$-algebraic systems. Recall the closure system $C^\mathsf{K} : \mathcal{P}(\mathrm{SEN}^\flat)^2 \to \mathcal{P}(\mathrm{SEN}^\flat)^2$ defined, by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $E \cup \{\phi \approx \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)^2$,

$$\begin{array}{lll}\phi \approx \psi \in C^\mathsf{K}_\Sigma(E) & \text{iff} & \text{for all } \mathcal{A} \in \mathsf{K}, \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}(\Sigma, \Sigma'),\\ & & \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(E)) \subseteq \Delta^\mathcal{A}_{F(\Sigma')}\\ & & \text{implies } \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) = \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi)).\end{array}$$

The π-institution $\mathcal{I}^\mathsf{K} = \langle \mathbf{F}, C^\mathsf{K} \rangle$ was called the **equational π-institution associated with** the class K. This π-institution may be recast as a Gentzen π-institution of trace $\{\langle 1, 1 \rangle\}$. More precisely, we define the Gentzen π-institution $\mathfrak{G}^\mathsf{K} = \langle \mathbf{F}, G^\mathsf{K} \rangle$ by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\{\phi_i, \psi_i : i \in I\} \cup \{\phi, \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \rhd_\Sigma \psi \in G^\mathsf{K}_\Sigma(\{\phi_i \rhd_\Sigma \psi_i : i \in I\}) \quad \text{iff} \quad \phi \approx \psi \in C^\mathsf{K}_\Sigma(\{\phi_i \approx \psi_i : i \in I\}).$$

We call $\mathfrak{G}^\mathsf{K}$ the **Gentzen π-institution associated with** the class K.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a π-institution based on $\mathbf{F}$. $\mathcal{I}$ may also be recast as a Gentzen π-institution of trace $\{\langle 0, 1 \rangle\}$. More precisely, given $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$, denote by

$$\rhd_\Sigma \Phi = \{\rhd_\Sigma \phi : \phi \in \Phi\}$$

and, similarly, given $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|} \in \mathrm{SenFam}(\mathbf{F})$, let

$$\rhd T = \{\rhd_\Sigma T_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}.$$

We define $\mathfrak{G}^\mathcal{I} = \langle \mathbf{F}, G^\mathcal{I} \rangle$ by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\rhd_\Sigma \phi \in G^\mathcal{I}_\Sigma(\rhd_\Sigma \Phi) \quad \text{iff} \quad \phi \in C_\Sigma(\Phi).$$

We call $\mathfrak{G}^\mathcal{I}$ the **Hilbert π-institution associated with** $\mathcal{I}$. In this terminology, a Hilbert π-institution is a Gentzen π-institution of trace $\{\langle 0, 1 \rangle\}$.

Given a Gentzen π-institution $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ of trace tr, such that $\langle 0, 1 \rangle \in \mathrm{tr}$, we call the Hilbert π-institution $\mathfrak{G}^{\mathcal{G}^0}$ associated with the π-institution reduct $\mathcal{G}^0$ of $\mathfrak{G}$ the **Hilbert π-institution reduct of** $\mathfrak{G}$ and we denote it by $\mathfrak{G}^0 = \langle \mathbf{F}, G^0 \rangle$ (note the overloading of notation for G^0, used both for the closure system of the π-institution $\mathcal{G}^0$ and for the closure system of $\mathfrak{G}^{\mathcal{G}^0}$; hopefully, this will not result into any confusion, since it should be resolvable based on context).

26.2 Equivalence of Gentzen π-Institutions

Let $\mathbf{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, $\mathbf{F}' = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ be algebraic systems and tr, tr′ be traces. A tr-tr′-**translation** is a collection of functions

$$\alpha = \{\alpha^{m,n} : \langle m,n\rangle \in \mathrm{tr}\},$$

where, for all $\langle m,n\rangle \in \mathrm{tr}$,

$$\alpha^{m,n} = \{\alpha^{m,n}_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$$

is such that, for all $\Sigma \in |\mathbf{Sign}|$,

$$\alpha^{m,n}_\Sigma : \mathrm{SEN}(\Sigma)^{m,n} \to \mathcal{P}(\mathrm{Seq}^{\mathrm{tr}'}_\Sigma(\mathbf{F}'))$$

assigns to each $\langle m,n\rangle$-Σ-sequent $\vec{\phi} \rhd_\Sigma \vec{\psi}$ of $\mathbf{F}$ a set of tr′-Σ-sequents of $\mathbf{F}'$

$$\alpha^{m,n}_\Sigma[\vec{\phi};\vec{\psi}].$$

We extend the notation in a natural way in order to write expressions more concisely. Thus, given $\Sigma \in |\mathbf{Sign}|$ and $\mathbf{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$, we set

$$\alpha_\Sigma[\boldsymbol{\phi}] = \alpha_\Sigma[\vec{\phi};\vec{\psi}],$$

if $\boldsymbol{\phi} = \vec{\phi} \rhd_\Sigma \vec{\psi}$, and

$$\alpha_\Sigma[\mathbf{\Phi}] = \bigcup\{\alpha^{m,n}_\Sigma[\boldsymbol{\phi}] : \boldsymbol{\phi} \in \mathbf{\Phi}^{m,n},\ \langle m,n\rangle \in \mathrm{tr}\}.$$

Finally, if $\mathbf{\Phi} = \{\mathbf{\Phi}_\Sigma\}_{\Sigma\in|\mathbf{Sign}|} \leq \mathrm{Seq}^{\mathrm{tr}}(\mathbf{F})$, we set

$$\alpha[\mathbf{\Phi}] = \bigcup\{\alpha_\Sigma[\mathbf{\Phi}_\Sigma] : \Sigma \in |\mathbf{Sign}|\}.$$

Even though we defined translations in a very general way, we will deal almost exclusively with a special kind of translation, called a transformation. To introduce those, we fix $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ and two traces tr and tr′. A tr-tr′-translation $\alpha = \{\alpha^{m,n} : \langle m,n\rangle \in \mathrm{tr}\}$ is called a tr-tr′-**transformation** if there exists a family

$$\tau = \{\tau^{m,n} : \langle m,n\rangle \in \mathrm{tr}\},$$

such that, for all $\langle m,n\rangle \in \mathrm{tr}$,

$$\tau^{m,n} : \mathrm{SEN}^\omega \to \bigcup\{\mathrm{SEN}^{k+\ell} : \langle k,\ell\rangle \in \mathrm{tr}'\}$$

is a collection of natural transformations in $N^\flat$, with $m+n$ distinguished arguments, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \rhd_\Sigma \vec{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m,n\rangle$,

$$\alpha^{m,n}_\Sigma[\vec{\phi};\vec{\psi}] = \tau^{m,n}_\Sigma[\vec{\phi};\vec{\psi}],$$

where, we let $\tau^{m,n}_\Sigma[\vec{\phi};\vec{\psi}]$ be defined, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, by

$$\tau^{m,n}_\Sigma[\vec{\phi};\vec{\psi}] \quad = \quad \bigcup\{\tau^{m,n}_\Sigma(\vec{\phi},\vec{\psi},\vec{\chi}) : \vec{\chi}' \in \mathrm{SEN}^\flat(\Sigma)\}.$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, tr and tr′ two traces and $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ and $\mathfrak{G}' = \langle \mathbf{F}, G'\rangle$ two Gentzen π-institutions of traces tr and tr′, respectively, both based on $\mathbf{F}$. A tr-tr′-transformation τ is an **interpretation from $\mathfrak{G}$ to $\mathfrak{G}'$**, written $\tau : \mathfrak{G} \to \mathfrak{G}'$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\mathbf{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$,

$$\boldsymbol{\phi} \in G_\Sigma(\mathbf{\Phi}) \quad \text{iff} \quad \tau_\Sigma[\boldsymbol{\phi}] \subseteq G'_\Sigma(\tau_\Sigma[\mathbf{\Phi}]).$$

The two π-institutions $\mathfrak{G}$ and $\mathfrak{G}'$ are **equivalent** if there exist a tr-tr′-transformation τ and a tr′-tr-transformation ρ, such that:

- $\tau : \mathfrak{G} \to \mathfrak{G}'$ is an interpretation;
- $\rho : \mathfrak{G}' \to \mathfrak{G}$ is an interpretation;
- for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$,
$$G_\Sigma(\boldsymbol{\phi}) = G_\Sigma(\rho_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]]);$$
- for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi}' \in \mathrm{Seq}^{\mathrm{tr}'}_\Sigma(\mathbf{F})$,
$$G'_\Sigma(\boldsymbol{\phi}') = G'_\Sigma(\tau_\Sigma[\rho_\Sigma[\boldsymbol{\phi}']]).$$

In this case the pair (τ,ρ) is called a **conjugate pair** of transformations and denoted by $(\tau,\rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$.

As in Lemma 889, it suffices to check only the first and last conditions, or, equivalently, the middle two conditions to ensure that two Gentzen π-institutions are equivalent.

Lemma 1879 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr, tr′ *be traces,* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$, $\mathfrak{G}' = \langle \mathbf{F}, G'\rangle$ *two Gentzen π-institutions of traces* tr, tr′, *respectively, based on* $\mathbf{F}$, τ *a* tr-tr′*-transformation and* ρ *a* tr′-tr-*transformation. The following are equivalent:*

(i) $\tau : \mathfrak{G} \to \mathfrak{G}'$ *is an interpretation and, for all* $\Sigma \in |\mathbf{Sign}^\flat|$, $\boldsymbol{\phi}' \in \mathrm{Seq}^{\mathrm{tr}'}_\Sigma(\mathbf{F})$, $G'_\Sigma(\boldsymbol{\phi}') = G'_\Sigma(\tau_\Sigma[\rho_\Sigma[\boldsymbol{\phi}']])$;

(ii) $\rho : \mathfrak{G}' \to \mathfrak{G}$ *is an interpretation and, for all* $\Sigma \in |\mathbf{Sign}^\flat|$, $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$, $G_\Sigma(\boldsymbol{\phi}) = G_\Sigma(\rho_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]])$.

Proof: Similar to the proof of Lemma 889. Suppose that the conditions in (i) hold. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\mathbf{\Phi}' \cup \{\boldsymbol{\phi}'\} \subseteq \mathrm{Seq}^{\mathrm{tr}'}_\Sigma(\mathbf{F})$, we have

$$\begin{array}{lll}\boldsymbol{\phi}' \in G'_\Sigma(\mathbf{\Phi}') & \text{iff} & \tau_\Sigma[\rho_\Sigma[\boldsymbol{\phi}']] \subseteq G'_\Sigma(\tau_\Sigma[\rho_\Sigma[\mathbf{\Phi}']]) \\ & \text{iff} & \rho_\Sigma[\boldsymbol{\phi}'] \subseteq G_\Sigma(\rho_\Sigma[\mathbf{\Phi}']).\end{array}$$

Hence, $\rho : \mathfrak{G}' \to \mathfrak{G}$ is also an interpretation. Finally, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$, we get, for all $\boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$,

$$\begin{array}{lll}\boldsymbol{\psi} \in G_\Sigma(\boldsymbol{\phi}) & \text{iff} & \tau_\Sigma[\boldsymbol{\psi}] \subseteq G'_\Sigma(\tau_\Sigma[\boldsymbol{\phi}]) \\ & \text{iff} & \tau_\Sigma[\boldsymbol{\psi}] \subseteq G_\Sigma(\tau_\Sigma[\rho_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]]]) \\ & \text{iff} & \boldsymbol{\psi} \in G_\Sigma(\rho_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]]).\end{array}$$

Thus, the second condition of (ii) is also satisfied. Thus (i) implies (ii) holds and, by symmetry, we conclude that (i) and (ii) are equivalent. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr, tr′ traces and τ a tr-tr′-transformation. Define

$$\tau^* : \mathrm{SenFam}(\mathrm{Seq}^{\mathrm{tr}'}(\mathbf{F})) \to \mathrm{SenFam}(\mathrm{Seq}^{\mathrm{tr}}(\mathbf{F}))$$

by setting, for all $\mathbf{\Phi}' \in \mathrm{SenFam}(\mathrm{Seq}^{\mathrm{tr}'}(\mathbf{F}))$,

$$\tau^*(\mathbf{\Phi}') = \{\tau^*_\Sigma(\mathbf{\Phi}')\}_{\Sigma \in |\mathbf{Sign}^\flat|}$$

be given, for all $\Sigma \in |\mathbf{Sign}^\flat|$, by

$$\tau^*_\Sigma(\mathbf{\Phi}') = \{\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F}) : \tau_\Sigma[\boldsymbol{\phi}] \subseteq \mathbf{\Phi}'_\Sigma\}.$$

Analogously to Theorem 893, we can show that, if $\mathfrak{G}$ and $\mathfrak{G}'$ are equivalent Gentzen π-institutions via a conjugate pair $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$, then $\rho^* : \mathbf{ThFam}(\mathfrak{G}) \to \mathbf{ThFam}(\mathfrak{G}')$ and $\tau^* : \mathbf{ThFam}(\mathfrak{G}') \to \mathbf{ThFam}(\mathfrak{G})$ form a pair of mutually inverse order isomorphisms between the complete lattices of the corresponding theory families.

Theorem 1880 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr*,* tr′ *traces,* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$*,* $\mathfrak{G}' = \langle \mathbf{F}, G' \rangle$ *Gentzen* π*-institutions of traces* tr*,* tr′*, respectively, and* $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ *a conjugate pair of transformations. Then*

$$\rho^* : \mathbf{ThFam}(\mathfrak{G}) \to \mathbf{ThFam}(\mathfrak{G}') \quad \textit{and} \quad \tau^* : \mathbf{ThFam}(\mathfrak{G}') \to \mathbf{ThFam}(\mathfrak{G})$$

are mutually inverse order isomorphisms.

Proof: Similar to the proof of Theorem 893. Let $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$, we get

$$\begin{array}{lll}\boldsymbol{\phi} \in \tau^*_\Sigma(\rho^*(\boldsymbol{T})) & \text{iff} & \tau_\Sigma[\boldsymbol{\phi}] \subseteq \rho^*_\Sigma(\boldsymbol{T}) \\ & \text{iff} & \rho_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]] \subseteq \boldsymbol{T}_\Sigma \\ & \text{iff} & \boldsymbol{\phi} \in \boldsymbol{T}_\Sigma.\end{array}$$

Thus, $\tau^*(\rho^*(\boldsymbol{T})) = \boldsymbol{T}$. By symmetry, for all $\boldsymbol{T}' \in \mathrm{ThFam}(\mathfrak{G}')$, $\rho^*(\tau^*(\boldsymbol{T}')) = \boldsymbol{T}'$. Thus, ρ^* and τ^* are mutually inverse bijections and, since they are both order preserving, they form a pair of mutually inverse order isomorphisms between $\mathbf{ThFam}(\mathfrak{G})$ and $\mathbf{ThFam}(\mathfrak{G}')$. ■

Conversely, it is true that, under ceratin hypotheses, given mutually inverse order isomorphisms between the complete lattices of two Gentzen π-institutions, one may define a conjugate pair between the two that establishes the order-isomorphism via the process that was described above.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr, tr′ be traces, and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$, $\mathfrak{G}' = \langle \mathbf{F}, G' \rangle$ be Gentzen π-institutions of traces tr, tr′, respectively, based on $\mathbf{F}$. Consider an order isomorphism

$$h : \mathbf{ThFam}(\mathfrak{G}') \to \mathbf{ThFam}(\mathfrak{G})$$

between the corresponding complete lattices of theory families.

Define $\overrightarrow{h} = \{\overrightarrow{h}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by letting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\overrightarrow{h}_\Sigma : \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F}) \to \mathcal{P}(\mathrm{Seq}_\Sigma^{\mathrm{tr}'}(\mathbf{F}))$$

be given, for all $\boldsymbol{\phi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$, by

$$\overrightarrow{h}_\Sigma[\boldsymbol{\phi}] = h_\Sigma^{-1}(G(\boldsymbol{\phi})).$$

Further, define $\overleftarrow{h} = \{\overleftarrow{h}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\overleftarrow{h}_\Sigma : \mathrm{Seq}_\Sigma^{\mathrm{tr}'}(\mathbf{F}) \to \mathcal{P}(\mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F}))$$

be given, for all $\boldsymbol{\phi}' \in \mathrm{Seq}_\Sigma^{\mathrm{tr}'}(\mathbf{F})$, by

$$\overleftarrow{h}_\Sigma[\boldsymbol{\phi}'] = h_\Sigma(G'(\boldsymbol{\phi}')).$$

The order isomorphism $h : \mathbf{ThFam}(\mathfrak{G}') \to \mathbf{ThFam}(\mathfrak{G})$ is called **transformational** if there exist

- a tr-tr′-translation τ,
- a tr′-tr-translation ρ,

such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\boldsymbol{\phi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ and all $\boldsymbol{\phi}' \in \mathrm{Seq}_\Sigma^{\mathrm{tr}'}(\mathbf{F})$,

$$\overrightarrow{h}_\Sigma[\boldsymbol{\phi}] = G'_\Sigma(\tau_\Sigma[\boldsymbol{\phi}]) \quad \text{and} \quad \overleftarrow{h}_\Sigma[\boldsymbol{\phi}'] = G_\Sigma(\rho_\Sigma[\boldsymbol{\phi}']),$$

i.e., by definition of $\overrightarrow{h}$ and $\overleftarrow{h}$, if and only if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\boldsymbol{\phi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ and all $\boldsymbol{\phi}' \in \mathrm{Seq}_\Sigma^{\mathrm{tr}'}(\mathbf{F})$,

$$h_\Sigma^{-1}(G(\boldsymbol{\phi})) = G'_\Sigma(\tau_\Sigma[\boldsymbol{\phi}]) \quad \text{and} \quad h_\Sigma(G'(\boldsymbol{\phi}')) = G_\Sigma(\rho_\Sigma[\boldsymbol{\phi}']).$$

Here $G(\phi)$ and $G'(\phi')$ denote the theory families of $\mathfrak{G}$ and $\mathfrak{G}$' generated by the Σ-sequents ϕ and ϕ', respectively. Since all components of these theory families other than the Σ-components consist of sets of theorems, we sometimes write by a slight abuse of notation

$$h_\Sigma^{-1}(G_\Sigma(\phi)) = G'_\Sigma(\tau_\Sigma[\phi]) \quad \text{and} \quad h_\Sigma(G'_\Sigma(\phi')) = G_\Sigma(\rho_\Sigma[\phi']).$$

In this case, we say that h is **induced by** the pair of translations (τ, ρ).

We can show that the properties defining transformationality of an order isomorphism extend to sets of Σ-sequents.

Lemma 1881 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr, tr′ *be traces and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$*,* $\mathfrak{G}' = \langle \mathbf{F}, G' \rangle$ *Gentzen π-institutions of traces* tr*,* tr′*, respectively, based on* $\mathbf{F}$*. If* $h : \mathbf{ThFam}(\mathfrak{G}') \to \mathbf{ThFam}(\mathfrak{G})$ *a transformational order isomorphism induced by the pair* (τ, ρ) *of translations, then, for all for all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $\mathbf{\Phi} \subseteq \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ *and all* $\mathbf{\Phi}' \subseteq \mathrm{Seq}_\Sigma^{\mathrm{tr}'}(\mathbf{F})$*,*

$$h_\Sigma^{-1}(G(\mathbf{\Phi})) = G'_\Sigma(\tau_\Sigma[\mathbf{\Phi}]) \quad \text{and} \quad h_\Sigma(G'(\mathbf{\Phi}')) = G_\Sigma(\rho_\Sigma[\mathbf{\Phi}']).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$, and $\mathbf{\Phi} \subseteq \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$. Then, taking into account that both $\mathbf{ThFam}(\mathfrak{G})$ and $\mathbf{ThFam}(\mathfrak{G}')$ are ordered signature-wise, we have

$$\begin{array}{rcl} h_\Sigma^{-1}(G(\mathbf{\Phi})) & = & h_\Sigma^{-1}(\bigvee_{\phi \in \mathbf{\Phi}} G(\phi)) \\ & = & \bigvee_{\phi \in \mathbf{\Phi}} h_\Sigma^{-1}(G(\phi)) \\ & = & \bigvee_{\phi \in \mathbf{\Phi}} G'_\Sigma(\tau_\Sigma[\phi]) \\ & = & G'_\Sigma(\bigcup_{\phi \in \mathbf{\Phi}} \tau_\Sigma[\phi]) \\ & = & G'_\Sigma(\tau_\Sigma[\mathbf{\Phi}]). \end{array}$$

The second equality holds by symmetry. ∎

Then the following result forms an analog of Theorem 900.

Theorem 1882 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr*,* tr′ *be traces and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$*,* $\mathfrak{G}' = \langle \mathbf{F}, G' \rangle$ *Gentzen π-institutions of traces* tr*,* tr′*, respectively, based on* $\mathbf{F}$*. If* $h : \mathbf{ThFam}(\mathfrak{G}') \to \mathbf{ThFam}(\mathfrak{G})$ *a transformational order isomorphism induced by the pair* (τ, ρ) *of translations, then* $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ *is a conjugate pair of transformations.*

Proof: Similar to the proof of Theorem 900. Suppose $h : \mathbf{ThFam}(\mathfrak{G}') \to \mathbf{ThFam}(\mathfrak{G})$ is an order isomorphism and let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\mathbf{\Phi}' \cup \{\phi'\} \subseteq \mathrm{Seq}_\Sigma^{\mathrm{tr}'}(\mathfrak{G}')$. Then we have

$$\begin{array}{rll} \phi' \in G'_\Sigma(\mathbf{\Phi}') & \text{iff} & G'_\Sigma(\phi') \subseteq G'_\Sigma(\mathbf{\Phi}') \\ & \text{iff} & h_\Sigma(G'(\phi')) \subseteq h_\Sigma(G'(\mathbf{\Phi}')) \\ & \text{iff} & G_\Sigma(\rho_\Sigma[\phi]) \subseteq G_\Sigma(\rho_\Sigma[\mathbf{\Phi}]) \\ & \text{iff} & \rho_\Sigma[\phi] \subseteq G_\Sigma(\rho_\Sigma[\mathbf{\Phi}]). \end{array}$$

Thus, $\rho : \mathfrak{G}' \to \mathfrak{G}$ is an interpretation. Furthermore, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$, we have

$$\begin{array}{rcl} G_\Sigma(\rho_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]]) & = & h_\Sigma(G'_\Sigma(\tau_\Sigma[\boldsymbol{\phi}])) \\ & = & h_\Sigma(h_\Sigma^{-1}(G_\Sigma(\boldsymbol{\phi}))) \\ & = & G_\Sigma(\boldsymbol{\phi}). \end{array}$$

We conclude that $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ is a conjugate pair. ■

Finally, we show that interpretations compose and the same holds for equivalences of Gentzen π-institutions.

Lemma 1883 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr, tr$'$, tr$''$ *be traces and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$, $\mathfrak{G}' = \langle \mathbf{F}, G' \rangle$, $\mathfrak{G}'' = \langle \mathbf{F}, G'' \rangle$ *be Gentzen π-institutions of traces* tr, tr$'$, tr$''$*, respectively, based on* $\mathbf{F}$.

(a) If $\tau : \mathfrak{G} \to \mathfrak{G}'$ and $\tau' : \mathfrak{G}' \to \mathfrak{G}''$ are interpretations, then $\tau' \circ \tau : \mathfrak{G} \to \mathfrak{G}''$ is also an interpretation;

(b) If $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ and $(\tau', \rho') : \mathfrak{G}' \rightleftarrows \mathfrak{G}''$ are conjugate pairs, then $(\tau' \circ \tau, \rho \circ \rho') : \mathfrak{G} \rightleftarrows \mathfrak{G}''$ is also a conjugate pair.

Proof:

(a) Suppose $\tau : \mathfrak{G} \to \mathfrak{G}'$ and $\tau' : \mathfrak{G}' \to \mathfrak{G}''$ are interpretations. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$, we get

$$\begin{array}{rll} \boldsymbol{\phi} \in G_\Sigma(\boldsymbol{\Phi}) & \text{iff} & \tau_\Sigma[\boldsymbol{\phi}] \subseteq G'_\Sigma(\tau_\Sigma[\boldsymbol{\Phi}]) \\ & \text{iff} & \tau'_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]] \subseteq G''_\Sigma(\tau'_\Sigma[\tau_\Sigma[\Phi]]). \end{array}$$

hence, $\tau' \circ \tau : \mathfrak{G} \to \mathfrak{G}''$ is also an interpretation.

(b) Now suppose that $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ and $(\tau', \rho') : \mathfrak{G}' \rightleftarrows \mathfrak{G}''$ are conjugate pairs. Then, by Part (a), $\tau' \circ \tau : \mathfrak{G} \to \mathfrak{G}''$ is an interpretation. Moreover, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi}'', \boldsymbol{\psi}'' \in \mathrm{Seq}_\Sigma^{\mathrm{tr}''}(\mathbf{F})$, we have $\boldsymbol{\psi}'' \in G''_\Sigma(\boldsymbol{\phi}'')$ if and only if

$$\rho'_\Sigma[\boldsymbol{\psi}''] \subseteq G'_\Sigma(\rho'_\Sigma[\boldsymbol{\phi}'']) = G'_\Sigma(\tau_\Sigma[\rho_\Sigma[\rho'_\Sigma[\boldsymbol{\phi}'']]]).$$

This holds if and only if

$$\tau'_\Sigma[\rho'_\Sigma[\boldsymbol{\psi}'']] \subseteq G''_\Sigma(\tau'_\Sigma[\tau_\Sigma[\rho_\Sigma[\rho'_\Sigma[\boldsymbol{\phi}'']]]]).$$

Equivalently,

$$\boldsymbol{\psi}'' \in G''_\Sigma(\tau'_\Sigma[\tau_\Sigma[\rho_\Sigma[\rho'_\Sigma[\boldsymbol{\phi}'']]]]).$$

We conclude that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi}'' \in \mathrm{Seq}_\Sigma^{\mathrm{tr}''}(\mathbf{F})$,

$$G''_\Sigma(\boldsymbol{\phi}'') = G''_\Sigma(\tau'_\Sigma[\tau_\Sigma[\rho_\Sigma[\rho'_\Sigma[\boldsymbol{\phi}'']]]]).$$

Therefore, by Lemma 1879, $(\tau' \circ \tau, \rho \circ \rho') : \mathfrak{G} \rightleftarrows \mathfrak{G}''$ is also a conjugate pair.

■

26.3 Hilbertizability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution based on $\mathbf{F}$. $\mathfrak{G}$ is **Hilbertizable** if it is equivalent to a Hilbert π-institution based on $\mathbf{F}$. In other words, $\mathfrak{G}$ is Hilbertizable if there exists a Hilbert π-institution $\mathfrak{H} = \langle \mathbf{F}, H \rangle$, based on $\mathbf{F}$, and a conjugate pair of transformations $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{H}$.

We have the following proposition that follows directly from the relevant definitions.

Proposition 1884 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system,* tr *a trace and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace* tr *based on $\mathbf{F}$. $\mathfrak{G}$ is Hilbertizable if and only if there exist:*

(1) A Hilbert π-institution $\mathfrak{H} = \langle \mathbf{F}, H \rangle$;

(2) A collection $\rho : (\mathrm{SEN}^\flat)^\omega \to \bigcup_{\langle m,n \rangle \in \mathrm{tr}} \mathrm{SEN}^{m+m}$ in $N^\flat$ with a single distinguished argument;

(3) A family $\tau = \{\tau^{m,n} : \langle m, n \rangle \in \mathrm{tr}\}$, where, for all $\langle m, n \rangle \in \mathrm{tr}$, the collection $\tau^{m,n} : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}$ in $N^\flat$ has $m + n$ distinguished arguments;

such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\mathbf{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

(a) $\boldsymbol{\phi} \in G_\Sigma(\mathbf{\Phi})$ iff $\tau_\Sigma[\boldsymbol{\phi}] \subseteq H_\Sigma(\tau_\Sigma[\mathbf{\Phi}])$;

(b) $H_\Sigma(\phi) = H_\Sigma(\tau_\Sigma[\rho_\Sigma[\phi]])$;

or, equivalently, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$ and all $\boldsymbol{\phi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$,

(c) $\rhd_\Sigma \phi \in H_\Sigma(\rhd_\Sigma \Phi)$ iff $\rho_\Sigma[\phi] \subseteq G_\Sigma(\rho_\Sigma[\Phi])$;

(d) $G_\Sigma(\boldsymbol{\phi}) = G_\Sigma(\rho_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]])$.

Proof: This is a rephrasing of the definition of Hilbertizability using the conditions establishing an equivalence between two Gentzen π-institutions and taking into account Lemma 1879. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution and $\mathfrak{H} = \langle \mathbf{F}, H \rangle$ a Hilbert π-institution both based on $\mathbf{F}$. Define the $\{\langle 0, 1 \rangle\}$-tr-transformation ρ^0 by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\rho_\Sigma^0[\phi] = \{\rhd_\Sigma \phi\}.$$

We say that $\mathfrak{G}$ and $\mathfrak{H}$ are **simply equivalent** if they are equivalent via a conjugate pair of the form $(\tau, \rho^0) : \mathfrak{G} \rightleftarrows \mathfrak{H}$. The Gentzen π-institution $\mathfrak{G}$ is **simply Hilbertizable** if it is simply equivalent to some Hilbert π-institution $\mathfrak{H} = \langle \mathbf{F}, H \rangle$.

If $\mathfrak{G}$ is simply Hilbertizable, it turns out that there is a unique Hilbert π-institution simply equivalent to $\mathfrak{G}$, namely, the Hilbert π-institution reduct $\mathfrak{G}^0$ of $\mathfrak{G}$.

Proposition 1885 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *is simply Hilbertizable, then it is simply equivalent to a unique Hilbert* π*-institution, namely, the Hilbert* π*-institution reduct* $\mathfrak{G}^0 = \langle \mathbf{F}, G^0 \rangle$ *of* $\mathfrak{G}$*.*

Proof: Suppose that $\mathfrak{G}$ is simply Hilbertizable via the conjugate pair $(\tau, \rho^0) : \mathfrak{G} \to \mathfrak{H}$, with $\mathfrak{H} = \langle \mathbf{F}, H \rangle$. It suffices to show that $H = G^0$. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{lll} \rhd_\Sigma \phi \in H_\Sigma(\rhd_\Sigma \Phi) & \text{iff} & \rho^0_\Sigma[\phi] \subseteq G_\Sigma(\rho^0_\Sigma[\Phi]) \quad \text{(by hypothesis)} \\ & \text{iff} & \rhd_\Sigma \phi \in G_\Sigma(\rhd_\Sigma \Phi) \quad \text{(definition of } \rho^0) \\ & \text{iff} & \rhd_\Sigma \phi \in G^0_\Sigma(\rhd_\Sigma \Phi). \quad \text{(definition of } G^0) \end{array}$$

Therefore $\mathfrak{H} = \mathfrak{G}^0$, whence it follows that $\mathfrak{G}$ is simply Hilbertizable via a simple equivalence involving the Hilbert π-institution reduct $\mathfrak{G}^0$ of $\mathfrak{G}$ ■

We have, further, the following simpler characterization of simple Hilbertizability, due to the fact that the interpretation in one of the two directions is required to be a fixed one.

Proposition 1886 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$*, be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*.* $\mathfrak{G}$ *is simply Hilbertizable if and only if there exists a* tr*-*$\{\langle 0,1 \rangle\}$*-transformation* $\tau = \{\tau^{m,n} : \langle m,n \rangle \in \mathrm{tr}\}$*, such that, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$*,*

$$G_\Sigma(\boldsymbol{\phi}) = G_\Sigma(\rhd\, \tau_\Sigma[\boldsymbol{\phi}]).$$

Proof: If $\mathfrak{G}$ is simply Hilbertizable, then, by Proposition 1885, it is equivalent to the Hilbert π-institution reduct $\mathfrak{G}^0$ of $\mathfrak{G}$ via some conjugate pair $(\tau, \rho^0) : \mathfrak{G} \rightleftarrows \mathfrak{G}^0$. Thus, by the definition of equivalence, we get, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$,

$$\begin{array}{lll} G_\Sigma(\boldsymbol{\phi}) & = & G_\Sigma(\rho^0_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]]) \\ & = & G_\Sigma(\rhd_\Sigma \tau_\Sigma[\boldsymbol{\phi}]). \end{array}$$

Assume, conversely, that there exists a tr-$\{\langle 0,1 \rangle\}$-transformation τ, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$, $G_\Sigma(\boldsymbol{\phi}) = G_\Sigma(\rhd\, \tau_\Sigma[\boldsymbol{\phi}])$. To show that $\mathfrak{G}$ is simply Hilbertizable, it suffices, by Proposition 1885 and Proposition 1884, to show that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\rhd_\Sigma \phi \in G^0_\Sigma(\rhd_\Sigma \Phi) \quad \text{iff} \quad \rhd_\Sigma \phi \in G_\Sigma(\rhd_\Sigma \Phi).$$

This equivalence, however, holds by the definition of G^0. ■

26.4 Syntactic WF Algebraizability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$. $\mathfrak{G}$ is (**syntactically WF**) **algebraizable** if it is equivalent to the Gentzen π-institution $\mathfrak{G}^{\mathsf{K}} = \langle \mathbf{F}, G^{\mathsf{K}} \rangle$ associated with some class K of $\mathbf{F}$-algebraic systems.

Explicitly, using the definition of equivalence, this means that there exists a class K of $\mathbf{F}$-algebraic systems, a tr-$\{\langle 1,1 \rangle\}$-transformation τ and a $\{\langle 1,1 \rangle\}$-tr-transformation ρ, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\mathbf{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

(a) $\boldsymbol{\phi} \in G_\Sigma(\mathbf{\Phi})$ iff $\tau_\Sigma[\boldsymbol{\phi}] \subseteq G^{\mathsf{K}}_\Sigma(\tau_\Sigma[\mathbf{\Phi}])$;

(b) $G^{\mathsf{K}}_\Sigma(\phi \rhd_\Sigma \psi) = G^{\mathsf{K}}_\Sigma(\tau_\Sigma[\rho_\Sigma[\phi;\psi]])$;

or, equivalently, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$,

(c) $\phi \rhd_\Sigma \psi \in G^{\mathsf{K}}_\Sigma(E)$ iff $\rho_\Sigma[\phi;\psi] \subseteq G_\Sigma(\rho_\Sigma[E])$;

(d) $G_\Sigma(\boldsymbol{\phi}) = G_\Sigma(\rho_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]])$.

Recall that, given a class K of $\mathbf{F}$-algebraic systems, we denote by $\mathbb{G}(\mathsf{K})$, the guasivariety of $\mathbf{F}$-algebraic systems generated by K, i.e., the collection of all $\mathbf{F}$-algebraic systems that satisfy the $\mathbf{F}$-guasiequations that are satisfied by all $\mathcal{A} \in \mathsf{K}$.

It turns out that, when a Gentzen π-institution $\mathfrak{G}$ is algebraizable via two different classes K and K' of $\mathbf{F}$-algebraic systems, then both classes K and K' generate the same guasivariety and, hence, that there exists a unique guasivariety of $\mathbf{F}$-algebraic systems that serves as the algebraizing class of $\mathfrak{G}$. This is proven in Proposition 1888, following a needed lemma.

Lemma 1887 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system,* tr *a trace and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace* tr *based on $\mathbf{F}$. If $\mathfrak{G}$ is algebraizable via $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}}$, then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m,n \rangle \in \mathrm{tr}$,*

$$\boldsymbol{\psi} \in G_\Sigma(\{\boldsymbol{\phi}\} \cup \bigcup\{\rho_\Sigma[\boldsymbol{\phi}_i;\boldsymbol{\psi}_i] : i < m+n\}).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m,n \rangle \in \mathrm{tr}$. By the definition of an equational Gentzen π-institution, we get

$$\tau_\Sigma[\boldsymbol{\psi}] \subseteq G^{\mathsf{K}}_\Sigma(\tau_\Sigma[\boldsymbol{\phi}] \cup \{\boldsymbol{\phi}_i \rhd_\Sigma \boldsymbol{\psi}_i : i < m+n\}).$$

Thus, since, by the definition of equivalence

$$G^{\mathsf{K}}_\Sigma(\boldsymbol{\phi}_i \rhd_\Sigma \boldsymbol{\psi}_i) = G^{\mathsf{K}}_\Sigma(\tau_\Sigma[\rho_\Sigma[\boldsymbol{\phi}_i;\boldsymbol{\psi}_i]]),$$

we get that

$$\tau_\Sigma[\boldsymbol{\psi}] \subseteq G^{\mathsf{K}}_\Sigma(\tau_\Sigma[\boldsymbol{\phi}] \cup \bigcup\{\tau_\Sigma[\rho_\Sigma[\boldsymbol{\phi}_i;\boldsymbol{\psi}_i]] : i < m+n\}).$$

Therefore, since τ is an interpretation,

$$\boldsymbol{\psi} \in G_\Sigma(\{\boldsymbol{\phi}\} \cup \bigcup\{\rho_\Sigma[\boldsymbol{\phi}_i;\boldsymbol{\psi}_i] : i < m+n\}).$$

This establishes the conclusion. ∎

Proposition 1888 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$. *If* $\mathfrak{G}$ *is algebraizable via both the conjugate pair* $(\tau,\rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}}$ *of transformations and the conjugate pair* $(\tau',\rho') : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}'}$ *of transformations, then* $\mathbb{G}(\mathsf{K}) = \mathbb{G}(\mathsf{K}')$.

Proof: Suppose that $\mathfrak{G}$ is algebraizable via both the conjugate pair $(\tau,\rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}}$ of transformations and the conjugate pair $(\tau',\rho') : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}'}$ of transformations.

We show, first, that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$G_\Sigma(\rho_\Sigma[\phi;\psi]) = G_\Sigma(\rho'_\Sigma[\phi;\psi]).$$

Note that $\rho'_\Sigma[\phi;\phi] \subseteq G_\Sigma(\varnothing)$, since $\phi \rhd_\Sigma \phi \in G^{\mathsf{K}'}_\Sigma(\varnothing)$ and $\rho' : \mathfrak{G}^{\mathsf{K}'} \to \mathfrak{G}$ is an interpretation. Moreover, for all $\boldsymbol{\sigma} \in \rho'$ of trace $\langle m,n\rangle \in \mathrm{tr}$, all $i < m+n$,all $\Sigma \in |\mathbf{Sign}^\flat|$, and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\rho_\Sigma[\boldsymbol{\sigma}^i_\Sigma(\phi,\phi,\vec{\chi});\boldsymbol{\sigma}^i_\Sigma(\phi,\psi,\vec{\chi})] \subseteq G_\Sigma(\rho_\Sigma[\phi;\psi]).$$

Since, by Lemma 1887, ρ has the modus ponens in $\mathfrak{G}$, we get that $\rho'_\Sigma[\phi;\psi] \subseteq G_\Sigma(\rho_\Sigma[\phi;\psi])$. By symmetry, we conclude that $G_\Sigma(\rho_\Sigma[\phi;\psi]) = G_\Sigma(\rho'_\Sigma[\phi;\psi])$.

Finally, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll} \phi \rhd_\Sigma \psi \in G^{\mathsf{K}}_\Sigma(E) & \text{iff} & \rho_\Sigma[\phi;\psi] \subseteq G_\Sigma(\rho_\Sigma[E]) \\ & \text{iff} & \rho'_\Sigma[\phi;\psi] \subseteq G_\Sigma(\rho'_\Sigma[E]) \\ & \text{iff} & \phi \rhd_\Sigma \psi \in G^{\mathsf{K}'}_\Sigma(E). \end{array}$$

Thus, we get that $\mathbb{G}(\mathsf{K}) = \mathbb{G}(\mathsf{K}')$. ∎

Given an algebraizable Gentzen π-institution $\mathfrak{G}$, there exists, by Proposition 1888, a unique guasivariety K that serves as the algebraic counterpart of $\mathfrak{G}$. It is called the **equivalent algebraic semantics of** $\mathfrak{G}$.

The next result asserts that equivalent Gentzen systems have the same status vis-à-vis algebraizability and, in case they are algebraizable, they share a common algebraic semantics. Moreover, they share the same Hilbertizability status and, in case they are Hilbertizable, they share the same Hilbertizations (which, however, are not unique).

Proposition 1889 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr, tr′ *traces and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$, $\mathfrak{G} = \langle \mathbf{F}, G' \rangle$ *two equivalent Gentzen* π*-institutions of traces* tr, tr′*, respectively, based on* $\mathbf{F}$.

(a) $\mathfrak{G}$ *is algebraizable if and only if* $\mathfrak{G}'$ *is algebraizable. If this is the case,* $\mathfrak{G}$ *and* $\mathfrak{G}'$ *have the same algebraic semantics.*

(b) $\mathfrak{G}$ *is Hilbertizable if and only if* $\mathfrak{G}'$ *is Hilbertizable. If this is the case, every Hilbertization of* $\mathfrak{G}$ *is one of* $\mathfrak{G}'$ *also.*

Proof:

(a) Suppose $\mathfrak{G}$ and $\mathfrak{G}'$ are equivalent via $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ and that $\mathfrak{G}'$ is algebraizable via $(\tau', \rho') : \mathfrak{G}' \rightleftarrows \mathfrak{G}^{\mathsf{K}'}$, for some class K' of $\mathbf{F}$-algebraic systems. Then, by Lemma 1883,

$$\mathfrak{G} \underset{\rho}{\overset{\tau}{\rightleftarrows}} \mathfrak{G}' \underset{\rho'}{\overset{\tau'}{\rightleftarrows}} \mathfrak{G}^{\mathsf{K}'}$$

$(\tau' \circ \tau, \rho \circ \rho') : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}'}$ is witnessing the algebraizability of $\mathfrak{G}$. By symmetry $\mathfrak{G}$ is algebraizable if and only if $\mathfrak{G}'$ is algebraizable. Since any algebraizing class K' for $\mathfrak{G}'$ is also an algebraizing class for $\mathfrak{G}$, and vice versa, we get that $\mathfrak{G}$ and $\mathfrak{G}'$ have the same equivalent algebraic semantics.

(b) Suppose $\mathfrak{G}$ and $\mathfrak{G}'$ are equivalent via $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ and that $\mathfrak{G}'$ is Hilbertizable via $(\tau', \rho') : \mathfrak{G}' \rightleftarrows \mathfrak{H}'$, for some Hilbert π-institution $\mathfrak{H}'$. Then, again by Lemma 1883,

$$\mathfrak{G} \underset{\rho}{\overset{\tau}{\rightleftarrows}} \mathfrak{G}' \underset{\rho'}{\overset{\tau'}{\rightleftarrows}} \mathfrak{H}'$$

$(\tau' \circ \tau, \rho \circ \rho') : \mathfrak{G} \rightleftarrows \mathfrak{H}'$ is witnessing the Hilbertizability of $\mathfrak{G}$. By symmetry, $\mathfrak{G}$ is Hilbertizable if and only if $\mathfrak{G}'$ is. Moreover, any Hilbertization $\mathfrak{H}'$ for $\mathfrak{G}'$ serves also as one for $\mathfrak{G}$, and vice versa, i.e., $\mathfrak{G}$ and $\mathfrak{G}'$ share the same Hilbertizations.

■

Suppose that a Gentzen π-institution $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ of trace tr, together with a trace tr′, are given. We give, next, a characterization of the existence of an equivalence $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ of $\mathfrak{G}$ with some Gentzen π-institution $\mathfrak{G}'$, having the given trace tr′.

Theorem 1890 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr, tr′ *be traces and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$. $\mathfrak{G}$ *is equivalent to a Gentzen* π*-institution* $\mathfrak{G}' = \langle \mathbf{F}, G' \rangle$ *of trace* tr′ *based on* $\mathbf{F}$ *if and only if there exist a* tr-tr′*-transformation* τ *and a* tr′-tr*-transformation* ρ*, such that:*

(1) $\rho^* : \mathrm{ThFam}(\mathfrak{G}) \to \mathrm{SenFam}(\mathrm{Seq}^{\mathrm{tr}'}(\mathbf{F}))$ *is injective on* $\mathrm{ThFam}(\mathfrak{G})$*;*

(2) For all $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$*,* $\rho^*_\Sigma(G(\boldsymbol{\phi})) = G'_\Sigma(\tau_\Sigma[\boldsymbol{\phi}])$*, where* G' *is the closure system induced by* $\rho^*(\mathrm{ThFam}(\mathfrak{G}))$*.*

Proof: Suppose, first, that there exists an equivalence $(\tau,\rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$, where $\mathfrak{G}' = \langle \mathbf{F}, G'\rangle$ is a Gentzen π-institution of trace tr′ based on $\mathbf{F}$. By Theorem 1880, we know that $\rho^* : \mathbf{ThFam}(\mathfrak{G}) \to \mathbf{ThFam}(\mathfrak{G}')$ is an order isomorphism, whence, in particular, it is injective on $\mathrm{ThFam}(\mathfrak{G})$. Moreover, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$, we have

$$\begin{array}{lll}\boldsymbol{\psi} \in \rho^*_\Sigma(G(\boldsymbol{\phi})) & \text{iff} & \rho_\Sigma[\boldsymbol{\psi}] \subseteq G_\Sigma(\boldsymbol{\phi}) \\ & \text{iff} & \tau_\Sigma[\rho_\Sigma[\boldsymbol{\psi}]] \subseteq G'_\Sigma(\tau_\Sigma[\boldsymbol{\phi}]) \\ & \text{iff} & \boldsymbol{\psi} \in G'_\Sigma(\tau_\Sigma[\boldsymbol{\phi}]).\end{array}$$

Therefore, $\rho^*_\Sigma(G(\boldsymbol{\phi})) = G'_\Sigma(\tau_\Sigma[\boldsymbol{\phi}])$.

Suppose, conversely, that there exist a tr-tr′-transformation τ and a tr′-tr-transformation ρ, such that Conditions (1) and (2) of the statement hold. Since $\rho^*(\mathrm{ThFam}(\mathfrak{G}))$ is closed under intersection, it defines a closure system on $\mathrm{Seq}^{\mathrm{tr}'}(\mathbf{F})$, which we denote by G', writing $\mathfrak{G}' = \langle \mathbf{F}, G'\rangle$ for the corresponding Gentzen π-institution of trace tr′. It suffices now, by Theorem 1882, to show that $\rho^* : \mathbf{ThFam}(\mathfrak{G}) \to \mathbf{ThFam}(\mathfrak{G}')$ is a transformational order isomorphism induced by (τ,ρ). We know, by hypothesis, that ρ^* is injective. By definition of $\mathfrak{G}'$, it is surjective. By definition of ρ^*, it is order preserving. Finally, it is order reflecting, since, for all $\boldsymbol{T}, \boldsymbol{T}' \in \mathrm{ThFam}(\mathfrak{G})$,

$$\begin{array}{lll}\rho^*(\boldsymbol{T}) \le \rho^*(\boldsymbol{T}') & \text{iff} & \rho^*(\boldsymbol{T}) \cap \rho^*(\boldsymbol{T}') = \rho^*(\boldsymbol{T}) \\ & \text{iff} & \rho^*(\boldsymbol{T} \cap \boldsymbol{T}') = \rho^*(\boldsymbol{T}) \\ & \text{iff} & \boldsymbol{T} \cap \boldsymbol{T}' = \boldsymbol{T} \\ & \text{iff} & \boldsymbol{T} \le \boldsymbol{T}'.\end{array}$$

Therefore, $\rho^* : \mathbf{ThFam}(\mathfrak{G}) \to \mathbf{ThFam}(\mathfrak{G}')$ is, indeed, an order isomorphism. To show that $\rho^* : \mathbf{ThFam}(\mathfrak{G}) \to \mathbf{ThFam}(\mathfrak{G}')$ is transformational, it suffices to show that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ and all $\boldsymbol{\phi}' \in \mathrm{Seq}^{\mathrm{tr}'}_\Sigma(\mathbf{F})$,

$$\rho^*_\Sigma(G(\boldsymbol{\phi})) = G'_\Sigma(\tau_\Sigma[\boldsymbol{\phi}]) \quad \text{and} \quad (\rho^*)^{-1}_\Sigma(G'(\boldsymbol{\phi}')) = G_\Sigma(\rho_\Sigma[\boldsymbol{\phi}']).$$

The first holds by hypothesis and the second holds by the definition of G', since $\rho^*_\Sigma(G(\rho_\Sigma[\boldsymbol{\phi}']))$ is the least theory family of $\mathfrak{G}'$ containing $\boldsymbol{\phi}'$. ∎

26.5 Matrix Families and Algebraic Semantics

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, tr a trace and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, an $\mathbf{F}$-algebraic system. A tr-**filter family**

of $\mathcal{A}$ is a family $\boldsymbol{T} \le \mathrm{Seq}^{\mathrm{tr}}(\mathcal{A})$. The pair $\mathfrak{A} = \langle \mathcal{A}, \boldsymbol{T}\rangle$ is called a tr-**matrix family**. It defines a closure family $G^{\mathfrak{A}}$ of trace tr on $\mathbf{F}$ as follows: For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$,

$$\boldsymbol{\phi} \in G^{\mathfrak{A}}_\Sigma(\boldsymbol{\Phi}) \quad \text{iff} \quad \text{for all } \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma,\Sigma'),$$
$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\boldsymbol{\Phi})) \subseteq \boldsymbol{T}_{F(\Sigma')} \text{ implies } \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\boldsymbol{\phi})) \in \boldsymbol{T}_{F(\Sigma')}.$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$. Let $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, be an $\mathbf{F}$-algebraic system and $\boldsymbol{T}$ a tr-filter family of $\mathcal{A}$. $\boldsymbol{T}$ is called a $\mathfrak{G}$-**filter family of** $\mathcal{A}$ if $G \le G^{\mathfrak{A}}$, i.e., if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$,

$$\boldsymbol{\phi} \in G_\Sigma(\boldsymbol{\Phi}) \quad \text{implies} \quad \boldsymbol{\phi} \in G^{\mathfrak{A}}_\Sigma(\boldsymbol{\Phi}).$$

Note that, as was pointed out previously, because of the structurality of G, it suffices to check that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$, such that $\boldsymbol{\phi} \in G_\Sigma(\boldsymbol{\Phi})$, we have

$$\alpha_\Sigma(\boldsymbol{\Phi}) \subseteq \boldsymbol{T}_{F(\Sigma)} \quad \text{implies} \quad \alpha_\Sigma(\boldsymbol{\phi}) \in \boldsymbol{T}_{F(\Sigma)}.$$

If $\boldsymbol{T}$ is a $\mathfrak{G}$-filter family of $\mathcal{A}$, then the pair $\mathfrak{A} = \langle \mathcal{A}, \boldsymbol{T}\rangle$ is called a $\mathfrak{G}$-**matrix family of** $\mathcal{A}$. We denote by $\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$ the collection of all $\mathfrak{G}$-filter families of $\mathcal{A}$ and by $\mathrm{MatFam}(\mathfrak{G})$ the collection of all $\mathfrak{G}$-matrix families.

Many facts, introduced previously in this work, that hold for $\mathcal{I}$-filter families and $\mathcal{I}$-matrix families, for a π-institution $\mathcal{I}$, have analogs for $\mathfrak{G}$-filter and $\mathfrak{G}$-matrix families, respectively. We list some of those that will be needed in the sequel.

Lemma 1891 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace,* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$ *and* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G,\beta\rangle\rangle$ $\mathbf{F}$*-algebraic systems.*

(a) The collection $\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$ *forms a complete lattice*

$$\mathbf{FiFam}^{\mathfrak{G}}(\mathcal{A}) = \langle \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}), \le\rangle$$

under signature-wise inclusion $\le$*;*

(b) $\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{F}) = \mathrm{ThFam}(\mathfrak{G})$*;*

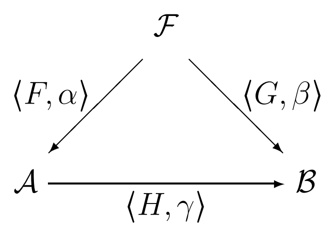
(c) If $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$, $\mathcal{B} = \langle \mathbf{B}, \langle G,\beta\rangle\rangle$ *are* $\mathbf{F}$*-algebraic systems and* $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$ *a surjective morphism, then* $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{B})$ *if and only if* $\gamma^{-1}(\boldsymbol{T}) \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$*.*

Proof:

(a) Let $\{\boldsymbol{T}^i : i \in I\} \subseteq \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$, such that $\boldsymbol{\phi} \subseteq G_\Sigma(\boldsymbol{\Phi})$. Then, if $\alpha_\Sigma(\boldsymbol{\Phi}) \subseteq \bigcap_{i\in I} \boldsymbol{T}^i_{F(\Sigma)}$, we get $\alpha_\Sigma(\boldsymbol{\Phi}) \subseteq \boldsymbol{T}^i_{F(\Sigma)}$, for all $i \in I$, whence, since $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, $\alpha_\Sigma(\boldsymbol{\phi}) \in \boldsymbol{T}^i_{F(\Sigma)}$, for all $i \in I$, i.e., $\alpha_\Sigma(\boldsymbol{\phi}) \in \bigcap_{i\in I} \boldsymbol{T}^i_{F(\Sigma)}$. We conclude that $\bigcap_{i\in I} \boldsymbol{T}^i \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$.

(b) For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$, such that $\boldsymbol{\phi} \in G_\Sigma(\boldsymbol{\Phi})$, we get that, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, $\boldsymbol{\Phi} \subseteq \boldsymbol{T}_\Sigma$ implies $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$. Therefore, $\mathrm{ThFam}(\mathfrak{G}) \subseteq \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{F})$. On the other hand, if $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{F})$, then, if $\boldsymbol{\phi} \in G_\Sigma(\boldsymbol{T}_\Sigma)$, then $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$, i.e., $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$. Therefore, $\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{F}) = \mathrm{ThFam}(\mathfrak{G})$.

(c) Assume, first, that $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{B})$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\boldsymbol{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$, such that $\boldsymbol{\phi} \in G_\Sigma(\boldsymbol{\Phi})$ and $\alpha_\Sigma(\boldsymbol{\Phi}) \subseteq \gamma^{-1}_{F(\Sigma)}(\boldsymbol{T}_{H(F(\Sigma))})$. Then



$\gamma_{F(\Sigma)}(\alpha_\Sigma(\boldsymbol{\Phi})) \subseteq \boldsymbol{T}_{H(F(\Sigma))}$, i.e., $\beta_\Sigma(\boldsymbol{\Phi}) \subseteq \boldsymbol{T}_{G(\Sigma)}$. Since $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{B})$, we now get $\beta_\Sigma(\boldsymbol{\phi}) \in \boldsymbol{T}_{G(\Sigma)}$. Reversing the steps above, we conclude that $\alpha_\Sigma(\boldsymbol{\phi}) \in \gamma^{-1}_{F(\Sigma)}(\boldsymbol{T}_{H(F(\Sigma))})$. Therefore, $\gamma^{-1}(\boldsymbol{T}) \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$.

Assume, conversely, that $\gamma^{-1}(\boldsymbol{T}) \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\boldsymbol{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$, such that $\boldsymbol{\phi} \in G_\Sigma(\boldsymbol{\Phi})$ and $\beta_\Sigma(\boldsymbol{\Phi}) \subseteq \boldsymbol{T}_{G(\Sigma)}$. Then $\gamma_{F(\Sigma)}(\alpha_\Sigma(\boldsymbol{\Phi})) \subseteq \boldsymbol{T}_{H(F(\Sigma))}$, whence $\alpha_\Sigma(\boldsymbol{\Phi}) \subseteq \gamma^{-1}_{F(\Sigma)}(\boldsymbol{T}_{H(F(\Sigma))})$. Since $\gamma^{-1}(\boldsymbol{T}) \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, we get $\alpha_\Sigma(\boldsymbol{\phi}) \in \gamma^{-1}_{F(\Sigma)}(\boldsymbol{T}_{H(F(\Sigma))})$. Reversing, once more, the preceding steps, we get that $\beta_\Sigma(\boldsymbol{\phi}) \in \boldsymbol{T}_{G(\Sigma)}$. Therefore, $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{B})$. ■

The isomorphism between the complete lattices of theory families induced by an equivalence extends to corresponding order isomorphisms between the complete lattices of filter families of the equivalent Gentzen π-institutions on the same algebraic system.

Proposition 1892 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr, tr′ *be traces, and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$, $\mathfrak{G}' = \langle \mathbf{F}, G'\rangle$ *two Gentzen π-institutions of traces* tr, tr′*, respectively, based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *and* $\mathfrak{G}'$ *are equivalent via the conjugate pair* $(\tau,\rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$*, then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$*, the mappings*

$$T \longmapsto \rho^{\mathcal{A}*}(T), \quad T \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}),$$

$$\tau^{\mathcal{A}*}(T') \longleftarrow T', \quad T' \in \mathrm{FiFam}^{\mathfrak{G}'}(\mathcal{A}),$$

are mutually inverse isomorphisms from $\mathbf{FiFam}^{\mathfrak{G}}(\mathcal{A})$ *onto* $\mathbf{FiFam}^{\mathfrak{G}'}(\mathcal{A})$*.*

Proof: We show, first, that, for all $\boldsymbol{T} \in \mathrm{FiFam}^{\mathcal{G}}(\mathcal{A})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$,

$$\rho^{\mathcal{A}}_{F(\Sigma)}[\tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\boldsymbol{\phi})]] \subseteq \boldsymbol{T}_{F(\Sigma)} \quad \text{iff} \quad \alpha_\Sigma(\boldsymbol{\phi}) \in \boldsymbol{T}_{F(\Sigma)}.$$

Indeed, taking into account the surjectivity of $\langle F, \alpha\rangle$, we obtain

$$\begin{array}{lll}\rho^{\mathcal{A}}_{F(\Sigma)}[\tau^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\boldsymbol{\phi})]] \subseteq \boldsymbol{T}_{F(\Sigma)} & \text{iff} & \rho^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\tau_\Sigma[\boldsymbol{\phi}])] \subseteq \boldsymbol{T}_{F(\Sigma)}\\ & \text{iff} & \alpha_\Sigma(\rho_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]]) \subseteq \boldsymbol{T}_{F(\Sigma)}\\ & \text{iff} & \rho_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]] \subseteq \alpha^{-1}_\Sigma(\boldsymbol{T}_{F(\Sigma)})\\ & \text{iff} & \boldsymbol{\phi} \in \alpha^{-1}_\Sigma(\boldsymbol{T}_{F(\Sigma)})\\ & \text{iff} & \alpha_\Sigma(\boldsymbol{\phi}) \in \boldsymbol{T}_{F(\Sigma)}.\end{array}$$

By symmetry, we also have, for all $\boldsymbol{T}' \in \mathrm{FiFam}^{\mathcal{G}'}(\mathcal{A})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi}' \in \mathrm{Seq}^{\mathrm{tr}'}_\Sigma(\mathbf{F})$,

$$\tau^{\mathcal{A}}_{F(\Sigma)}[\rho^{\mathcal{A}}_{F(\Sigma)}[\alpha_\Sigma(\boldsymbol{\phi}')]] \subseteq \boldsymbol{T}'_{F(\Sigma)} \quad \text{iff} \quad \alpha_\Sigma(\boldsymbol{\phi}') \in \boldsymbol{T}'_{F(\Sigma)}.$$

Using the first of these equivalences and, once again, taking into account the surjectivity of $\langle F, \alpha\rangle$, we get, for all $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\boldsymbol{\phi} \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll}\boldsymbol{\phi} \in \tau^{\mathcal{A}*}_\Sigma(\rho^{\mathcal{A}*}(\boldsymbol{T})) & \text{iff} & \tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}] \subseteq \rho^{\mathcal{A}*}_\Sigma(\boldsymbol{T})\\ & \text{iff} & \rho^{\mathcal{A}}_\Sigma[\tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}]] \subseteq \boldsymbol{T}_\Sigma\\ & \text{iff} & \boldsymbol{\phi} \in \boldsymbol{T}_\Sigma.\end{array}$$

Thus, $\tau^{\mathcal{A}*}(\rho^{\mathcal{A}*}(\boldsymbol{T})) = \boldsymbol{T}$, for all $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$ and, by symmetry, we also have $\rho^{\mathcal{A}*}(\tau^{\mathcal{A}*}(\boldsymbol{T}')) = \boldsymbol{T}'$, for all $\boldsymbol{T}' \in \mathrm{FiFam}^{\mathfrak{G}'}(\mathcal{A})$. Therefore, $\rho^{\mathcal{A}*}$ and $\tau^{\mathcal{A}*}$ are mutually inverse bijections and reflect component-wise inclusion, since they are obviously order preserving undel $\leq$. We conclude that

$$\rho^{\mathcal{A}*} : \mathbf{FiFam}^{\mathfrak{G}}(\mathcal{A}) \rightleftarrows \mathbf{FiFam}^{\mathfrak{G}'}(\mathcal{A}) : \tau^{\mathcal{A}*}$$

are mutually inverse order isomorphisms. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, tr a trace, $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ an $N^\flat$-algebraic system and $\theta \in \mathrm{ConSys}(\mathbf{A})$.

Given $\boldsymbol{\phi} = \vec{\phi} \rhd_\Sigma \vec{\psi}$, $\boldsymbol{\phi}' = \vec{\phi}' \rhd_\Sigma \vec{\psi}' \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{A})$ of the same trace $\langle m, n\rangle$, we say that $\boldsymbol{\phi}$ is **θ-equivalent to $\boldsymbol{\phi}'$**, denoted $\boldsymbol{\phi}\ \theta_\Sigma\ \boldsymbol{\phi}'$, if, for all $i < m$ and all $j < n$,

$$\phi_i\ \theta_\Sigma\ \phi'_i \quad \text{and} \quad \psi_j\ \theta_\Sigma\ \psi'_j.$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, tr a trace, $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ an $N^\flat$-algebraic system, $\boldsymbol{T} \leq \mathrm{Seq}^{\mathrm{tr}}(\mathbf{A})$ and $\theta \in \mathrm{ConSys}(\mathbf{A})$.

We say that θ is **compatible with $\boldsymbol{T}$** if, for all $\Sigma \in |\mathbf{Sign}|$, and all $\boldsymbol{\phi}, \boldsymbol{\phi}' \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{A})$ (of the same trace),

$$\boldsymbol{\phi}\ \theta_\Sigma\ \boldsymbol{\phi}' \quad \text{and} \quad \boldsymbol{\phi} \in \boldsymbol{T}_\Sigma \quad \text{imply} \quad \boldsymbol{\phi}' \in \boldsymbol{T}_\Sigma.$$

An alternative characterization of compatibility is given in the following lemma.

Lemma 1893 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *an* $\mathbf{F}$*-algebraic system,* $\boldsymbol{T} \le \mathrm{Seq}^{\mathrm{tr}}(\mathcal{A})$ *and* $\theta \in \mathrm{ConSys}(\mathcal{A})$*.* θ *is compatible with* $\boldsymbol{T}$ *if and only if the quotient morphism* $\langle I, \pi^\theta \rangle : \mathcal{A} \to \mathcal{A}^\theta$ *induces a strict matrix family morphism*

$$\langle I, \pi^\theta \rangle : \langle \mathcal{A}, \boldsymbol{T} \rangle \to \langle \mathcal{A}^\theta, \pi^\theta(\boldsymbol{T}) \rangle,$$

i.e., if and only if $(\pi^\theta)^{-1}(\pi^\theta(\boldsymbol{T})) = \boldsymbol{T}$*.*

Proof: Suppose, first, that θ is compatible with $\boldsymbol{T}$ and let $\Sigma \in |\mathbf{Sign}|$, $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathcal{A})$, such that $\boldsymbol{\phi} \in (\pi^\theta_\Sigma)^{-1}(\pi^\theta_\Sigma(\boldsymbol{T}_\Sigma))$. Then, we get $\pi^\theta_\Sigma(\boldsymbol{\phi}) \in \pi^\theta_\Sigma(\boldsymbol{T}_\Sigma)$. Hence, there exists $\boldsymbol{\phi}' \in \boldsymbol{T}_\Sigma$, such that $\boldsymbol{\phi} \; \theta_\Sigma \; \boldsymbol{\phi}'$. Therefore, by the compatibility of θ with $\boldsymbol{T}$, we get that $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$. Thus, $(\pi^\theta)^{-1}(\pi^\theta(\boldsymbol{T})) \le \boldsymbol{T}$ and, since the reverse inclusion always holds, we conclude that $(\pi^\theta)^{-1}(\pi^\theta(\boldsymbol{T})) = \boldsymbol{T}$.

Conversely, assume that $(\pi^\theta)^{-1}(\pi^\theta(\boldsymbol{T})) = \boldsymbol{T}$. Let $\Sigma \in |\mathbf{Sign}|$, $\boldsymbol{\phi}, \boldsymbol{\phi}' \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathcal{A})$, such that $\boldsymbol{\phi} \; \theta_\Sigma \; \boldsymbol{\phi}'$ and $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$. Then $\boldsymbol{\phi}' \in (\pi^\theta_\Sigma)^{-1}(\pi^\theta_\Sigma(\boldsymbol{T}_\Sigma)) = \boldsymbol{T}_\Sigma$ and, hence, θ is compatible with $\boldsymbol{T}$. ■

Given a Gentzen π-institution, taking the quotient of any filter family by a compatible congruence system results in a filter family on the quotient algebraic system.

Lemma 1894 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace,* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*,* $\langle \mathcal{A}, \boldsymbol{T} \rangle$ *an* $\mathbf{F}$*-matrix family and* $\theta \in \mathrm{ConSys}(\mathcal{A})$*. If* θ *is compatible with* $\boldsymbol{T}$*, then*

$$\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}) \quad \textit{iff} \quad \boldsymbol{T}/\theta \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}^\theta).$$

Proof: Suppose that θ is compatible with $\boldsymbol{T}$. Then, using Lemmas 1891 and 1893, we have the following equivalences:

$$\begin{array}{lll} \boldsymbol{T}/\theta \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}/\theta) & \text{iff} & (\pi^\theta \circ \alpha)^{-1}(\boldsymbol{T}/\theta) \in \mathrm{ThFam}(\mathfrak{G}) \\ & \text{iff} & \alpha^{-1}((\pi^\theta)^{-1}(\boldsymbol{T}/\theta)) \in \mathrm{ThFam}(\mathfrak{G}) \\ & \text{iff} & \alpha^{-1}(\boldsymbol{T}) \in \mathrm{ThFam}(\mathfrak{G}) \\ & \text{iff} & \boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}). \end{array}$$

Hence $\boldsymbol{T}/\theta$ is a $\mathfrak{G}$-filter family of $\mathcal{A}^\theta$ iff $\boldsymbol{T}$ is a $\mathfrak{G}$-filter family of $\mathcal{A}$. ■

The following lemma forms an analog of the characterization of the Leibniz congruence system of a filter family on a given $\mathbf{F}$-algebraic system. It will also give rise to a corresponding operator, also termed the Leibniz operator, for theory families of Gentzen π-institutions.

Lemma 1895 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ *an* $N^\flat$*-algebraic system,* $\boldsymbol{T} \le \mathrm{Seq}^{\mathrm{tr}}(\mathbf{A})$ *and* $\theta \in \mathrm{ConSys}(\mathbf{A})$*.* θ *is compatible with* $\boldsymbol{T}$ *if and only if, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,*

$\langle\phi,\psi\rangle\in\theta_\Sigma$ *implies, for all* $\langle m,n\rangle\in\mathrm{tr}$*, all* $\vec{\sigma}=\langle\sigma^0,\dots,\sigma^{m-1}\rangle$*, and all* $\vec{\tau}=\langle\tau^0,\dots,\tau^{n-1}\rangle$ *in* $N^\flat$*, all* $\Sigma'\in|\mathbf{Sign}|$*, all* $f\in\mathbf{Sign}(\Sigma,\Sigma')$ *and all* $\vec{\chi}\in\mathrm{SEN}(\Sigma')$*,*

$$\vec{\sigma}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi})\rhd_{\Sigma'}\vec{\tau}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi})\in\boldsymbol{T}_{\Sigma'}$$
$$\textit{iff}\quad\vec{\sigma}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi})\rhd_{\Sigma'}\vec{\tau}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi})\in\boldsymbol{T}_{\Sigma'}.$$

Proof: Suppose that $\Sigma\in|\mathbf{Sign}|$ and $\langle\phi,\psi\rangle\in\mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle\in\theta_\Sigma$. Since θ is a congruence *system*, for all $\Sigma'\in|\mathbf{Sign}|$ and all $f\in\mathbf{Sign}(\Sigma,\Sigma')$ $\langle\mathrm{SEN}(f)(\phi),\mathrm{SEN}(f)(\psi)\rangle\in\theta_{\Sigma'}$. Since θ is a *congruence* system, we get, for all $i<m$, all $j<n$ and all $\vec{\chi}\in\mathrm{SEN}(\Sigma')$,

$$\langle\sigma^i_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}),\sigma^i_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi})\rangle\in\theta_{\Sigma'}$$
$$\text{and}\quad\langle\tau^j_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}),\tau^j_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi})\rangle\in\theta_{\Sigma'}.$$

The conclusion follows immediately by the assumption of compatibility of θ with $\boldsymbol{T}$. ■

Lemma 1895 serves to show that, given an algebraic system $\mathbf{A}$ and $\boldsymbol{T}\le\mathrm{Seq}^{\mathrm{tr}}(\mathbf{A})$, there exists a largest congruence system on $\mathbf{A}$ that is compatible with $\boldsymbol{T}$.

Corollary 1896 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* tr *a trace,* $\mathbf{A}=\langle\mathbf{Sign},\mathrm{SEN},N\rangle$ *an* $N^\flat$*-algebraic system and* $\boldsymbol{T}\le\mathrm{Seq}^{\mathrm{tr}}(\mathbf{A})$*. There exists a largest congruence system on* $\mathbf{A}$ *compatible with* $\boldsymbol{T}$*.*

Proof: Define $\theta=\{\theta_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ as follows: For all $\Sigma\in|\mathbf{Sign}|$ and all $\phi,\psi\in\mathrm{SEN}(\Sigma)$, $\langle\phi,\psi\rangle\in\theta_\Sigma$ iff, for all $\langle m,n\rangle\in\mathrm{tr}$, all $\vec{\sigma}=\langle\sigma^0,\dots,\sigma^{m-1}\rangle$, and all $\vec{\tau}=\langle\tau^0,\dots,\tau^{n-1}\rangle$ in $N^\flat$, all $\Sigma'\in|\mathbf{Sign}|$, all $f\in\mathbf{Sign}(\Sigma,\Sigma')$ and all $\vec{\chi}\in\mathrm{SEN}(\Sigma')$,

$$\vec{\sigma}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi})\rhd_{\Sigma'}\vec{\tau}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi})\in\boldsymbol{T}_{\Sigma'}$$
$$\text{iff}\quad\vec{\sigma}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi})\rhd_{\Sigma'}\vec{\tau}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi})\in\boldsymbol{T}_{\Sigma'}.$$

It is easy to see that θ, thus defined, is a congruence system on $\mathbf{A}$ compatible with $\boldsymbol{T}$. By Lemma 1895, it is the largest one compatible with $\boldsymbol{T}$. ■

The largest congruence system on $\mathbf{A}$ compatible with $\boldsymbol{T}$ is denoted by $\Omega^{\mathbf{A}}(\boldsymbol{T})$ and called the **Leibniz congruence system of $\boldsymbol{T}$ on $\mathbf{A}$**.

As a consequence of the definition of the Leibniz congruence system, given $\boldsymbol{T}\in\mathrm{Seq}^{\mathrm{tr}}(\mathbf{A})$ and $\theta\in\mathrm{ConSys}(\mathbf{A})$,

$$\theta\text{ is compatible with }\boldsymbol{T}\text{ if and only if }\theta\le\Omega^{\mathbf{A}}(\boldsymbol{T}).$$

Given an algebraic system $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$, a trace tr, a Gentzen π-institution $\mathfrak{G}=\langle\mathbf{F},G\rangle$ of trace tr based on $\mathbf{F}$ and an $\mathbf{F}$-algebraic system $\mathcal{A}$, the operator

$$\Omega^{\mathbf{A}}:\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\to\mathrm{ConSys}(\mathcal{A})$$

is called the **Leibniz operator of $\mathfrak{G}$ on $\mathcal{A}$**.

Recall from Proposition 1892 that, given two equivalent Gentzen π-institutions, the conjugate transformations establishing the equivalence induce an order isomorphism between the corresponding filter families of the gentzen π-institutions involved on arbitrary algebraic systems. It turns out that, under this isomorphism, corresponding filter families have identical Leibniz congruence systems.

Proposition 1897 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr, tr′ *traces and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$, $\mathfrak{G}' = \langle \mathbf{F}, G' \rangle$ *Gentzen π-institutions of traces* tr, tr′*, respectively, based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *and* $\mathfrak{G}'$ *are equivalent via a conjugate pair* $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ *of transformations, then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$*,*

$$\Omega^{\mathcal{A}}(\boldsymbol{T}) = \Omega^{\mathcal{A}}(\rho^{\mathcal{A}*}(\boldsymbol{T})).$$

Proof: Let $\Sigma \in |\mathbf{Sign}|$, $\boldsymbol{\phi}, \boldsymbol{\phi}' \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathcal{A})$, such that $\boldsymbol{\phi} \ \Omega^{\mathcal{A}}_\Sigma(\boldsymbol{T}) \ \boldsymbol{\phi}'$ and suppose that $\boldsymbol{\phi} \in \rho^{\mathcal{A}*}_\Sigma(\boldsymbol{T})$. Then, we obtain $\rho^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}] \subseteq \boldsymbol{T}_\Sigma$. Thus, since, by definition, $\Omega^{\mathcal{A}}(\boldsymbol{T})$ is a congruence system compatible with $\boldsymbol{T}$, we get that $\rho^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}'] \subseteq \boldsymbol{T}_\Sigma$ and, therefore, $\boldsymbol{\phi}' \in \rho^{\mathcal{A}*}_\Sigma(\boldsymbol{T})$. Hence $\Omega^{\mathcal{A}}(\boldsymbol{T})$ is compatible with $\rho^{\mathcal{A}*}(\boldsymbol{T})$, showing that $\Omega^{\mathcal{A}}(\boldsymbol{T}) \leq \Omega^{\mathcal{A}}(\rho^{\mathcal{A}*}(\boldsymbol{T}))$.

Assume, conversely, that $\Sigma \in |\mathbf{Sign}|$, $\boldsymbol{\phi}, \boldsymbol{\phi}' \in \mathrm{Seq}^{\mathrm{tr}'}_\Sigma(\mathcal{A})$, such that

$$\boldsymbol{\phi} \ \Omega^{\mathcal{A}}_\Sigma(\rho^{\mathcal{A}*}(\boldsymbol{T})) \ \boldsymbol{\phi}'$$

and suppose that $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$. Then, we obtain $\rho^{\mathcal{A}}_\Sigma[\tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}]] \subseteq \boldsymbol{T}_\Sigma$, i.e., $\tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}] \subseteq \rho^{\mathcal{A}*}_\Sigma(\boldsymbol{T})$. Thus, since, by definition, $\Omega^{\mathcal{A}}(\rho^{\mathcal{A}*}(\boldsymbol{T}))$ is a congruence system compatible with $\rho^{\mathcal{A}*}(\boldsymbol{T})$, we get that $\tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}'] \subseteq \rho^{\mathcal{A}*}(\boldsymbol{T})$. Therefore, $\rho^{\mathcal{A}}_\Sigma[\tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}']] \subseteq \boldsymbol{T}_\Sigma$. So $\boldsymbol{\phi}' \in \boldsymbol{T}_\Sigma$ and, hence, $\Omega^{\mathcal{A}}(\rho^{\mathcal{A}*}(\boldsymbol{T}))$ is compatible with $\boldsymbol{T}$, showing that $\Omega^{\mathcal{A}}(\rho^{\mathcal{A}*}(\boldsymbol{T})) \leq \Omega^{\mathcal{A}}(\boldsymbol{T})$. ∎

As was the case with ordinary π-institutions, the Suszko operator is a very useful tool in the study of the algebraization of Gentzen π-institutions.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr a trace, $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$ and $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ an $\mathbf{F}$-algebraic system. The **Suszko operator** $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}$ **of** $\mathfrak{G}$ **on** $\mathcal{A}$ is the operator

$$\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}} : \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}) \to \mathrm{ConSys}(\mathcal{A})$$

defined, for all $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, by

$$\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) = \bigcap \{\Omega^{\mathcal{A}}(\boldsymbol{T}') : \boldsymbol{T} \leq \boldsymbol{T}' \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\}.$$

Since, obviously, for all $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$,

$$\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) \leq \Omega^{\mathcal{A}}(\boldsymbol{T}),$$

$\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$ is also a congruence system on $\mathcal{A}$ compatible with $\boldsymbol{T}$. Moreover, the operator $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}$ is monotone on $\mathbf{FiFam}^{\mathfrak{G}}(\mathcal{A})$, for every $\mathbf{F}$-algebraic system $\mathcal{A}$.

Using the definition of the Suszko congruence system and Corollary 1896, it is not difficult to see that the following characterization of the Suszko congruence system of a filter family holds:

Proposition 1898 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace,* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$*,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system and* $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$*. For all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,* $\langle \phi, \psi\rangle \in \widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}_\Sigma(\boldsymbol{T})$ *if and only if, for all* $\langle m, n\rangle \in \mathrm{tr}$*, all* $\sigma^0, \dots, \sigma^{m-1}$*,* $\tau^0, \dots, \tau^{n-1}$ *in* $N^\flat$*, all* $\Sigma' \in |\mathbf{Sign}|$*, all* $f \in \mathbf{Sign}(\Sigma, \Sigma')$ *and all* $\vec{\chi} \in \mathrm{SEN}(\Sigma')$*,*

$$\begin{aligned} &G^{\mathfrak{G},\mathcal{A}}_{\Sigma'}(\boldsymbol{T}_{\Sigma'}, \vec{\sigma}^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \rhd_{\Sigma'} \vec{\tau}^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}))\\ &\quad = G^{\mathfrak{G},\mathcal{A}}_{\Sigma'}(\boldsymbol{T}_{\Sigma'}, \vec{\sigma}^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}) \rhd_{\Sigma'} \vec{\tau}^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi})).\end{aligned}$$

Proof: The statement follows directly by combining the definition of the Suszko congruence system of $\boldsymbol{T}$ on $\mathcal{A}$ with the characterization of the Leibniz operator of each $\boldsymbol{T}'$, with $\boldsymbol{T} \le \boldsymbol{T}'$, given in the proof of Corollary 1896. ■

Moreover, as is clear from the definition, we have

Lemma 1899 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *be a trace,* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$*, and* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *an* $\mathbf{F}$*-algebraic system. The Suszko and the Leibniz operators on* $\mathcal{A}$ *coincide, i.e.,* $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}} = \Omega^{\mathcal{A}}$*, if and only if* $\Omega^{\mathcal{A}}$ *is monotone on* $\mathbf{FiFam}^{\mathfrak{G}}(\mathcal{A})$*.*

Proof: Since $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}$ is monotone on $\mathbf{FiFam}^{\mathfrak{G}}(\mathcal{A})$, if the two operators coincide, $\Omega^{\mathcal{A}}$ is also monotone on $\mathbf{FiFam}^{\mathfrak{G}}(\mathcal{A})$.

On the other hand, if $\Omega^{\mathcal{A}}$ is monotone on $\mathbf{FiFam}^{\mathfrak{G}}(\mathcal{A})$, then, for all $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, we get

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) & = & \bigcap\{\Omega^{\mathcal{A}}(\boldsymbol{T}') : \boldsymbol{T} \le \boldsymbol{T}' \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\}\\ & = & \Omega^{\mathcal{A}}(\boldsymbol{T}).\end{array}$$

Therefore, $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}} = \Omega^{\mathcal{A}}$. ■

An analog of Proposition 1897 holds also for the Suszko operator. That is, under the isomorphism between the corresponding filter families of two gentzen π-institutions that are equivalent, corresponding filter families have identical Suszko congruence systems.

Proposition 1900 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr*,* tr′ *traces and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$*,* $\mathfrak{G}' = \langle \mathbf{F}, G'\rangle$ *Gentzen π-institutions of traces* tr*,* tr′*, respectively, based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *and* $\mathfrak{G}'$ *are equivalent via the conjugate pair* $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$*, then, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ *and all* $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$*,*

$$\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) = \widetilde{\Omega}^{\mathfrak{G}',\mathcal{A}}(\rho^{\mathcal{A}*}(\boldsymbol{T})).$$

Proof: Since $\rho^{\mathcal{A}*} : \mathbf{FiFam}^{\mathfrak{G}}(\mathcal{A}) \to \mathbf{FiFam}^{\mathfrak{G}'}(\mathcal{A})$ is an order isomorphism, and taking into account Proposition 1897, we obtain, for all $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$,

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) & = & \bigcap\{\Omega^{\mathcal{A}}(\boldsymbol{T}') : \boldsymbol{T} \leq \boldsymbol{T}' \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\} \\ & = & \bigcap\{\Omega^{\mathcal{A}}(\rho^{\mathcal{A}*}(\boldsymbol{T}')) : \rho^{\mathcal{A}*}(\boldsymbol{T}) \leq \rho^{\mathcal{A}*}(\boldsymbol{T}') \in \mathrm{FiFam}^{\mathfrak{G}'}(\mathcal{A})\} \\ & = & \bigcap\{\Omega^{\mathcal{A}}(\boldsymbol{T}'') : \rho^{\mathcal{A}*}(\boldsymbol{T}) \leq \boldsymbol{T}'' \in \mathrm{FiFam}^{\mathfrak{G}'}(\mathcal{A})\} \\ & = & \widetilde{\Omega}^{\mathfrak{G}',\mathcal{A}}(\rho^{\mathcal{A}*}(\boldsymbol{T})).\end{array}$$

Thus, the conclusion holds. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$. A $\mathfrak{G}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, \boldsymbol{T} \rangle$ is called **Suszko reduced** if

$$\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) = \Delta^{\mathcal{A}}.$$

We denote by $\mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$ the class of all Suszko reduced $\mathfrak{G}$-matrix families.

Foe every $\mathfrak{G}$-matrix family $\mathfrak{A} = \langle \mathcal{A}, \boldsymbol{T} \rangle$, the quotient structure

$$\langle \mathcal{A}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}), \boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) \rangle$$

is also a $\mathfrak{G}$-matrix family and it is Suszko reduced. Moreover, if a $\mathfrak{G}$-matrix family $\langle \mathcal{A}, \boldsymbol{T} \rangle$ is Suszko reduced, it is obviously isomorphic to a $\mathfrak{G}$-matrix family of this form.

Among other things, Suszko reduced $\mathfrak{G}$-matrix families are important because they form a class of structures with respect to which $\mathfrak{G}$ enjoys a completeness property.

Theorem 1901 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *be a trace, and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$. *For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\boldsymbol{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$, $\boldsymbol{\phi} \in G_\Sigma(\boldsymbol{\Phi})$ *if and only if, for all* $\langle \mathcal{A}, \boldsymbol{T} \rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$, *all* $\Sigma' \in |\mathbf{Sign}^\flat|$ *and all* $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\boldsymbol{\Phi})) \subseteq \boldsymbol{T}_{F(\Sigma')} \quad \textit{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\boldsymbol{\phi})) \in \boldsymbol{T}_{F(\Sigma')}.$$

Proof: Suppose $\boldsymbol{\phi} \in G_\Sigma(\boldsymbol{\Phi})$ and let $\langle \mathcal{A}, \boldsymbol{T} \rangle$ be a Suszko reduced $\mathfrak{G}$-matrix family. Then $\langle \mathcal{A}, \boldsymbol{T} \rangle$ is, in particular, a $\mathfrak{G}$-matrix family, whence the conclusion holds by applying the definition of a $\mathfrak{G}$-filter family to the $\mathfrak{G}$-filter family $\boldsymbol{T}$. Suppose, conversely, that, for all $\langle \mathcal{A}, \boldsymbol{T} \rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$, all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\boldsymbol{\Phi})) \subseteq \boldsymbol{T}_{F(\Sigma')} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\boldsymbol{\phi})) \in \boldsymbol{T}_{F(\Sigma')}.$$

Let $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$ and consider the Suszko reduced $\mathfrak{G}$-matrix family

$$\langle \mathcal{F}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}), \boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}) \rangle.$$

Then, we have, by hypothesis, taking $\Sigma' = \Sigma$ and $f = i_\Sigma$,

$$\mathbf{\Phi}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}_\Sigma(\boldsymbol{T}) \subseteq \boldsymbol{T}_\Sigma/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}_\Sigma(\boldsymbol{T}) \quad \text{implies} \quad \boldsymbol{\phi}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}_\Sigma(\boldsymbol{T}) \in \boldsymbol{T}_\Sigma/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}_\Sigma(\boldsymbol{T}),$$

i.e., using the compatibility of $\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})$ with $\boldsymbol{T}$, $\mathbf{\Phi} \subseteq \boldsymbol{T}_\Sigma$ implies $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$. Equivalently, since $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$ was arbitrary, $\boldsymbol{\phi} \in G_\Sigma(\mathbf{\Phi})$. ∎

If two Gentzen π-institutions are equivalent, then the classes of algebraic system reducts of their Suszko reduced matrix families coincide.

Theorem 1902 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr*,* tr′ *traces and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$*,* $\mathfrak{G}' = \langle \mathbf{F}, G'\rangle$ *Gentzen* π*-institutions of traces* tr*,* tr′*, respectively, based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *and* $\mathfrak{G}'$ *are equivalent via the conjugate pair* $(\tau,\rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$*, then* $\mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$ *and* $\mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G}')$ *have the same class of* $\mathbf{F}$*-algebraic system reducts.*

Proof: Suppose that $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ is the $\mathbf{F}$-algebraic system reduct of $\langle \mathcal{A}, \boldsymbol{T}\rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$. Then, by definition, we have $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) = \Delta^{\mathcal{A}}$. Therefore, by Proposition 1900, we obtain $\widetilde{\Omega}^{\mathfrak{G}',\mathcal{A}}(\rho^{\mathcal{A}*}(\boldsymbol{T})) = \Delta^{\mathcal{A}}$. Since, $\rho^{\mathcal{A}*}(\boldsymbol{T}) \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, we conclude that $\langle \mathcal{A}, \rho^{\mathcal{A}*}(\boldsymbol{T})\rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G}')$ and, hence, $\mathcal{A}$ is also the $\mathbf{F}$-algebraic system reduct of a Suszko reduced $\mathfrak{G}'$-matrix family. By symmetry of equivalence, every $\mathbf{F}$-algebraic system reduct of a Suszko reduced $\mathfrak{G}'$-matrix family is also one of a Suszko reduced $\mathfrak{G}$-matrix family. Therefore, the two classes of $\mathbf{F}$-algebraic system reducts coincide. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, tr be a trace, and $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$. The class of all $\mathbf{F}$-algebraic system reducts of Suszko reduced $\mathfrak{G}$-matrix families is denoted by $\mathrm{AlgSys}(\mathfrak{G})$, i.e., we have, by definition,

$$\begin{array}{rcl}\mathrm{AlgSys}(\mathfrak{G}) & = & \{\mathcal{A} : (\exists \boldsymbol{T} \leq \mathrm{Seq}^{\mathrm{tr}}(\mathcal{A}))(\langle \mathcal{A}, \boldsymbol{T}\rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G}))\} \\ & = & \{\mathcal{A} : (\exists \boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}))(\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) = \Delta^{\mathcal{A}})\}.\end{array}$$

It is not difficult to show that the class $\mathrm{AlgSys}(\mathfrak{G})$ is closed under $\overset{\triangleleft}{\mathbb{I}\mathbb{P}}$ and, thence, conclude that the class of all $\mathrm{AlgSys}(\mathfrak{G})$-congruence systems on every $\mathbf{F}$-algebraic system $\mathcal{A}$ forms a complete lattice under signature-wise inclusion.

Proposition 1903 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. Then* $\mathrm{AlgSys}(\mathfrak{G})$ *is closed under subdirect intersections, i.e.,*

$$\overset{\triangleleft}{\mathbb{I}\mathbb{P}}(\mathrm{AlgSys}(\mathfrak{G})) \subseteq \mathrm{AlgSys}(\mathfrak{G}).$$

Proof: Suppose that $\mathcal{A}^i = \langle \mathbf{A}^i, \langle F^i, \alpha^i\rangle\rangle \in \mathrm{AlgSys}(\mathfrak{G})$, for all $i \in I$, and let

$$\langle H^i, \gamma^i\rangle : \mathfrak{A} \to \mathcal{A}^i, \quad i \in I,$$

be a subdirect intersection, i.e., such that $\bigcap_{i\in I} \mathrm{Ker}(\langle H^i, \gamma^i\rangle) = \Delta^{\mathcal{A}}$. Then, for all $i \in I$, there exists $\boldsymbol{T}^i \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}^i)$, such that $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}^i}(\boldsymbol{T}^i) = \Delta^{\mathcal{A}^i}$. We consider the least $\mathfrak{G}$-filter family on $\mathcal{A}$, namely $\bigcap \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$. We have

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\bigcap \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})) & = & \bigcap\{\Omega^{\mathcal{A}}(\boldsymbol{X}) : \boldsymbol{X} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\}\\ & \le & \bigcap_{i\in I}\bigcap_{\boldsymbol{X}^i\in\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}^i)}(\gamma^i)^{-1}(\Omega^{\mathcal{A}^i}(\boldsymbol{X}^i))\\ & = & \bigcap_{i\in I}(\gamma^i)^{-1}(\bigcap_{\boldsymbol{X}^i\in\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}^i)}\Omega^{\mathcal{A}^i}(\boldsymbol{X}^i))\\ & \le & \bigcap_{i\in I}(\gamma^i)^{-1}(\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}^i}(\boldsymbol{T}^i))\\ & = & \bigcap_{i\in I}(\gamma^i)^{-1}(\Delta^{\mathcal{A}^i})\\ & = & \Delta^{\mathcal{A}}.\end{array}$$

Hence, we get that $\mathcal{A} \in \mathrm{AlgSys}(\mathfrak{G})$. Therefore, $\mathrm{AlgSys}(\mathfrak{G})$ is indeed closed under subdirect intersections. ∎

26.6 Equivalence and Algebraic Counterpart

In Theorem 1890, given a π-institution $\mathfrak{G}$ and a trace tr', we gave a characterization of the existence of an equivalence $(\tau,\rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ of $\mathfrak{G}$ with some Gentzen π-institution $\mathfrak{G}'$, having the given trace tr'. We strengthen this result here, by considering only $\mathfrak{G}$-filter families on algebraic systems belonging to $\mathrm{AlgSys}(\mathfrak{G})$.

Theorem 1904 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr*,* tr' *be traces and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*.* $\mathfrak{G}$ *is equivalent to a Gentzen* π*-institution* $\mathfrak{G}' = \langle \mathbf{F}, G'\rangle$ *of trace* tr' *based on* $\mathbf{F}$ *if and only if there exist a* tr-tr'*-transformation* τ *and a* tr'-tr*-transformation* ρ*, such that, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathfrak{G})$*:*

(1) $\rho^{\mathcal{A}*} : \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}) \to \mathrm{SenFam}(\mathrm{Seq}^{\mathrm{tr}'}(\mathcal{A}))$ *is injective on* $\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$*;*

(2) For all $\Sigma \in |\mathbf{Sign}|$ *and all* $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathcal{A})$*,* $\rho^{\mathcal{A}*}_\Sigma(G^{\mathfrak{G},\mathcal{A}}(\boldsymbol{\phi})) = G'_\Sigma(\tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}])$*, where* G' *is the closure system on* $\mathcal{A}$ *induced by* $\rho^{\mathcal{A}*}(\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}))$*.*

Proof: Suppose, first, that there exists an equivalence $(\tau,\rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$, where $\mathfrak{G}' = \langle \mathbf{F}, G'\rangle$ is a Gentzen π-institution of trace tr' based on $\mathbf{F}$. By Proposition 1892, we know that $\rho^{\mathcal{A}*} : \mathbf{FiFam}^{\mathfrak{G}}(\mathcal{A}) \to \mathbf{FiFam}^{\mathfrak{G}'}(\mathcal{A})$ is an order isomorphism, whence, in particular, it is injective on $\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$. Moreover, for all $\Sigma \in |\mathbf{Sign}|$ and all $\boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathcal{A})$, we have

$$\begin{array}{rll}\boldsymbol{\psi} \in \rho^{\mathcal{A}*}_\Sigma(G^{\mathfrak{G},\mathcal{A}}(\boldsymbol{\phi})) & \text{iff} & \rho^{\mathcal{A}}_\Sigma[\boldsymbol{\psi}] \subseteq G^{\mathfrak{G},\mathcal{A}}_\Sigma(\boldsymbol{\phi})\\ & \text{iff} & \tau^{\mathcal{A}}_\Sigma[\rho^{\mathcal{A}}_\Sigma[\boldsymbol{\psi}]] \subseteq G^{\mathfrak{G}',\mathcal{A}}_\Sigma(\tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}])\\ & \text{iff} & \boldsymbol{\psi} \in G^{\mathfrak{G}',\mathcal{A}}_\Sigma(\tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}]).\end{array}$$

Therefore, $\rho^{\mathcal{A}*}_\Sigma(G^{\mathfrak{G},\mathcal{A}}(\boldsymbol{\phi})) = G^{\mathfrak{G}',\mathcal{A}}_\Sigma(\tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}])$ and, again by Proposition 1892, $G^{\mathfrak{G}',\mathcal{A}}$ is the closure system on $\mathcal{A}$ induced by $\rho^{\mathcal{A}*}(\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}))$.

Suppose, conversely, that there exist a tr-tr′-transformation τ and a tr′-tr-transformation ρ, such that Conditions (1) and (2) of the statement hold. The function $\rho^{\mathcal{A}*}$ commutes with intersections of $\mathfrak{G}$-filter families on $\mathcal{A}$. As a consequence, we obtain, on the one hand, that $\rho^{\mathcal{A}*}$ is order reflecting on $\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$ and, on the other, that $\rho^{\mathcal{A}*}(\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}))$ is closed under intersection, and, hence, defines a closure system on $\mathrm{Seq}^{\mathrm{tr}'}(\mathcal{A})$, which we denote by G'. It suffices now, to prove the two conditions of Theorem 1890.

Assume, first, that $\boldsymbol{T},\boldsymbol{T}' \in \mathrm{ThFam}(\mathfrak{G})$, such that $\rho^*(\boldsymbol{T}) \le \rho^*(\boldsymbol{T}')$. Let $\boldsymbol{X} = G(\rho[\rho^*(\boldsymbol{T})])$ and $\mathcal{A} = \mathcal{F}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X})$. Since $\boldsymbol{X}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X}) \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, we get that $\langle \mathcal{A}, \boldsymbol{X}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X})\rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$. Therefore, $\mathcal{A} \in \mathrm{AlgSys}(\mathfrak{G})$. Further, $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$ and $\rho[\rho^*(\boldsymbol{T})] \le \boldsymbol{T}$, which give $\boldsymbol{X} \le \boldsymbol{T}$. Moreover, $\rho[\rho^*(\boldsymbol{T})] \le \rho[\rho^*(\boldsymbol{T}')] \le \boldsymbol{T}'$. Hence, $\boldsymbol{X} \le \boldsymbol{T}'$. Thus, by the monotonicity of the Suszko operator, $\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X}) \le \widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})$ and $\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X}) \le \widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}')$. These imply that $\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X})$ is compatible with both $\boldsymbol{T}$ and $\boldsymbol{T}'$. This, in turn, gives that both $\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X})$ and $\boldsymbol{T}'/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X})$ are $\mathfrak{G}$-filter families on $\mathcal{A}$ and, furthermore, that

$$\rho^{\mathcal{A}*}(\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X})) = \rho^*(\boldsymbol{T})/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X})$$

and, similarly, $\rho^{\mathcal{A}*}(\boldsymbol{T}'/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X})) = \rho^*(\boldsymbol{T}')/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X})$. Since, by hypothesis, $\rho^*(\boldsymbol{T}) \le \rho^*(\boldsymbol{T}')$, we get

$$\rho^{\mathcal{A}*}(\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X})) \le \rho^{\mathcal{A}*}(\boldsymbol{T}'/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X})).$$

Thus, by Condition (1) in the hypothesis, we get $\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X}) \le \boldsymbol{T}'/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X})$, whence, using again the compatibility of $\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{X})$ with both $\boldsymbol{T}$ and $\boldsymbol{T}'$, we obtain $\boldsymbol{T} \le \boldsymbol{T}'$. We conclude that ρ^* is order reflecting and, therefore, a fortiori, injective on $\mathrm{ThFam}(\mathfrak{G})$.

Finally, let $\Sigma \in |\mathbf{Sign}^\flat|$, $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ and consider $\theta = \widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\mathrm{Thm}(\mathfrak{G}))$. Then $\mathcal{F}/\theta \in \mathrm{AlgSys}(\mathfrak{G})$, whence, by hypothesis,

$$\rho^{(\mathcal{F}/\theta)*}(G^{\mathfrak{G},\mathcal{F}/\theta}(\boldsymbol{\phi}/\theta_\Sigma)) = G'_\Sigma(\tau^{\mathcal{F}/\theta}_\Sigma[\boldsymbol{\phi}/\theta_\Sigma]),$$

where G' is the closure system on $\mathcal{F}/\theta$ generated by

$$\begin{array}{rcl}\rho^{(\mathcal{F}/\theta)*}(\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{F}/\theta)) & = & \rho^{(\mathcal{F}/\theta)*}(\mathrm{ThFam}(\mathfrak{G})/\theta)\\ & = & \rho^*(\mathrm{ThFam}(\mathfrak{G}))/\theta.\end{array}$$

Thus, we get $\rho^*_\Sigma(G(\boldsymbol{\phi}))/\theta = G'_\Sigma(\tau_\Sigma[\boldsymbol{\phi}]/\theta_\Sigma)$, whence

$$\begin{array}{rcl}\rho^{(\mathcal{F}/\theta)*}_\Sigma(G(\boldsymbol{\phi})/\theta) & = & \bigcap\{\rho^*(\boldsymbol{X})/\theta : \tau_\Sigma[\boldsymbol{\phi}]/\theta \subseteq \rho^*(\boldsymbol{X})/\theta\}\\ & = & \bigcap\{\rho^*(\boldsymbol{X}) : \tau_\Sigma[\boldsymbol{\phi}] \subseteq \rho^*(\boldsymbol{X})\}/\theta.\end{array}$$

Therefore, $\rho^*_\Sigma(G(\boldsymbol{\phi})) = \bigcap\{\rho^*(\boldsymbol{X}) : \tau_\Sigma[\boldsymbol{\phi}] \subseteq \rho^*_\Sigma(\boldsymbol{X})\} = G''(\tau_\Sigma[\boldsymbol{\phi}])$, where G'' is the closure system on $\mathcal{F}$ generated by $\rho^*(\mathrm{ThFam}(\mathfrak{G}))$. ■

Theorem 1904 may be used to provide a characterization of equivalence based on the coincidence of the algebraic counterparts of two Gentzen π-institutions.

Theorem 1905 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr, tr′ *traces and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$, $\mathfrak{G}' = \langle \mathbf{F}, G' \rangle$ *Gentzen π-institutions of traces* tr*,* tr′*, respectively, based on* $\mathbf{F}$*.* $\mathfrak{G}$ *and* $\mathfrak{G}'$ *are equivalent if and only if*

- $\mathrm{AlgSys}(\mathfrak{G}) = \mathrm{AlgSys}(\mathfrak{G}')$ *and*
- *there exist a* tr-tr′*-transformation* τ *and a* tr′-tr*-transformation* ρ*, such that, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathfrak{G})$*,*
  - $\rho^{\mathcal{A}*} : \mathbf{FiFam}^{\mathfrak{G}}(\mathcal{A}) \to \mathbf{FiFam}^{\mathfrak{G}'}(\mathcal{A})$ *is an order isomorphism and*
  - *for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}(\mathcal{A})$*,*
$$\rho^{\mathcal{A}*}_\Sigma(G^{\mathfrak{G},\mathcal{A}}(\boldsymbol{\phi})) = G'^{\mathcal{A}}_\Sigma(\tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}]),$$
where $G'^{\mathcal{A}}$ *is the closure system on* $\mathcal{A}$ *induced by* $\rho^{\mathcal{A}*}(\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}))$*.*

Proof: Suppose that $\mathfrak{G}$ and $\mathfrak{G}'$ are equivalent via the conjugate pair $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$. Then, by Theorem 1902, $\mathrm{AlgSys}(\mathfrak{G}) = \mathrm{AlgSys}(\mathfrak{G}')$. By Proposition 1892, $\rho^{\mathcal{A}*}$ is an order isomorphism and, finally, for all $\Sigma \in |\mathbf{Sign}|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathcal{A})$ and all $\boldsymbol{\phi}' \in \mathrm{Seq}^{\mathrm{tr}'}_\Sigma(\mathcal{A})$,

$$\begin{array}{rll} \boldsymbol{\phi}' \in \rho^{\mathcal{A}*}_\Sigma(G^{\mathfrak{G},\mathcal{A}}(\boldsymbol{\phi})) & \text{iff} & \rho^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}'] \subseteq G^{\mathfrak{G},\mathcal{A}}_\Sigma(\boldsymbol{\phi}) \\ & \text{iff} & \tau^{\mathcal{A}}_\Sigma[\rho^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}']] \subseteq G^{\mathfrak{G}',\mathcal{A}}_\Sigma(\tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}]) \\ & \text{iff} & \boldsymbol{\phi}' \in G^{\mathfrak{G}',\mathcal{A}}_\Sigma(\tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}]), \end{array}$$

i.e., for all $\Sigma \in |\mathbf{Sign}|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}(\mathcal{A})$, $\rho^{\mathcal{A}*}_\Sigma(G^{\mathfrak{G},\mathcal{A}}(\boldsymbol{\phi})) = G^{\mathfrak{G}',\mathcal{A}}_\Sigma(\tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}])$.

Conversely, assume that the conditions in the claimed characterization of equivalence hold. Then, by Theorem 1904, there exists a Gentzen π-institution $\mathfrak{X}'$ of trace tr′ to which $\mathfrak{G}$ is equivalent, such that $\rho^* : \mathbf{ThFam}(\mathfrak{G}) \to \mathbf{ThFam}(\mathfrak{X})$ is an order isomorphism. Thus, ρ^* is both order preserving and order reflecting and, hence, injective, on $\mathrm{ThFam}(\mathfrak{G})$. Thus, it suffices to show that it is onto $\mathrm{ThFam}(\mathfrak{G}')$.

Suppose $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$. Set $\mathcal{A} = \mathcal{F}/\Omega(\boldsymbol{T})$ and let $\langle I, \pi \rangle : \mathcal{F} \to \mathcal{A}$ be the quotient morphism. Then, since $\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}) \leq \Omega(\boldsymbol{T})$, we get, by the definition of $\mathrm{AlgSys}(\mathfrak{G})$ and the hypothesis, $\mathcal{A} \in \mathrm{AlgSys}(\mathfrak{G}) = \mathrm{AlgSys}(\mathfrak{G}')$. By the compatibility of $\Omega(\boldsymbol{T})$ with $\boldsymbol{T}$, we get that $\boldsymbol{T}/\Omega(\boldsymbol{T}) \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$ and $\pi^{-1}(\boldsymbol{T}/\Omega(\boldsymbol{T})) = \boldsymbol{T}$. By hypothesis, $\rho^{\mathcal{A}*}(\boldsymbol{T}/\Omega(\boldsymbol{T})) \in \mathrm{FiFam}^{\mathfrak{G}'}(\mathcal{A})$, whence $\pi^{-1}(\rho^{\mathcal{A}*}(\boldsymbol{T}/\Omega(\boldsymbol{T}))) \in \mathrm{ThFam}(\mathfrak{G}')$. On the other hand, we have

$$\rho^*(\boldsymbol{T}) = \rho^*(\pi^{-1}(\boldsymbol{T}/\Omega(\boldsymbol{T}))) = \pi^{-1}(\rho^{\mathcal{A}*}(\boldsymbol{T}/\Omega(\boldsymbol{T}))).$$

Hence, we obtain $\rho^*(\boldsymbol{T}) \in \mathrm{ThFam}(\mathfrak{G}')$.

Finally, consider $\boldsymbol{T}' \in \text{ThFam}(\mathfrak{G}')$. Set $\mathfrak{B} = \mathcal{F}/\Omega(\boldsymbol{T}') \in \text{AlgSys}(\mathfrak{G}') = \text{AlgSys}(\mathfrak{G})$ and let $\langle I, \pi' \rangle : \mathcal{F} \to \mathcal{B}$ be the quotient morphism. Then we have $\boldsymbol{T}'/\Omega(\boldsymbol{T}') \in \text{FiFam}^{\mathfrak{G}'}(\mathcal{B})$ and, by compatibility, $\pi'^{-1}(\boldsymbol{T}'/\Omega(\boldsymbol{T}')) = \boldsymbol{T}'$. By hypothesis, there exists $\boldsymbol{T} \in \text{FiFam}^{\mathfrak{G}}(\mathcal{B})$, such that $\boldsymbol{T}'/\Omega(\boldsymbol{T}') = \rho^{\mathcal{B}*}(\boldsymbol{T})$. On the other hand, $\pi'^{-1}(\boldsymbol{T}) \in \text{ThFam}(\mathfrak{G})$ and

$$\rho^*(\pi'^{-1}(\boldsymbol{T})) = \pi'^{-1}(\rho^{\mathcal{B}*}(\boldsymbol{T})) = \pi'^{-1}(\boldsymbol{T}'/\Omega(\boldsymbol{T}')) = \boldsymbol{T}'.$$

thus, ρ^* maps $\text{ThFam}(\mathfrak{G})$ onto $\text{ThFam}(\mathfrak{G}')$ and, hence, it is an order isomorphism from $\mathbf{ThFam}(\mathfrak{G})$ onto $\mathbf{ThFam}(\mathfrak{G}')$. Therefore, $\mathfrak{G}' = \mathfrak{X}$ and $\mathfrak{G}'$ is equivalent to $\mathfrak{G}$. ■

Directly from Theorem 1904, we get the following

Corollary 1906 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$. $\mathfrak{G}$ *is Hilbertizable if and only if there exist a* tr-$\{\langle 0,1 \rangle\}$*-transformation* τ *and a* $\{\langle 0,1 \rangle\}$-tr*-transformation* ρ*, such that, for all* $\mathcal{A} \in \text{AlgSys}(\mathfrak{G})$*:*

(1) $\rho^{\mathcal{A}*} : \text{FiFam}^{\mathfrak{G}}(\mathcal{A}) \to \text{SenFam}(\mathcal{A})$ *is injective on* $\text{FiFam}^{\mathfrak{G}}(\mathcal{A})$*;*

(2) For all $\Sigma \in |\mathbf{Sign}|$ *and all* $\boldsymbol{\phi} \in \text{Seq}^{\text{tr}}_\Sigma(\mathcal{A})$*,*

$$\rho^{\mathcal{A}*}_\Sigma(G^{\mathfrak{G},\mathcal{A}}(\boldsymbol{\phi})) = G'^{\mathcal{A}}_\Sigma(\tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}]),$$

where $G'^{\mathcal{A}}$ *is the closure system on* $\mathcal{A}$ *induced by* $\rho^{\mathcal{A}*}(\text{FiFam}^{\mathfrak{G}}(\mathcal{A}))$.

Proof: This is a special case of Theorem 1904. ■

Specializing further, we get the following result characterizing simple Hilbertizability.

Corollary 1907 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$. $\mathfrak{G}$ *is simply Hilbertizable if and only if there exists a* tr-$\{\langle 0,1 \rangle\}$*-transformation* τ*, such that:*

(1) For all $\mathcal{A} \in \text{AlgSys}(\mathfrak{G})$ *and all* $\boldsymbol{T}, \boldsymbol{T}' \in \text{FiFam}^{\mathfrak{G}}(\mathcal{A})$*,*

$$\boldsymbol{T} \cap \triangleright \mathcal{A} = \boldsymbol{T}' \cap \triangleright \mathcal{A} \quad \textit{implies} \quad \boldsymbol{T} = \boldsymbol{T}';$$

(2) For all $\Sigma \in |\mathbf{Sign}|$ *and all* $\boldsymbol{\phi} \in \text{Seq}^{\text{tr}}_\Sigma(\mathcal{A})$*,*

$$G^{\mathfrak{G},\mathcal{A}}_\Sigma(\boldsymbol{\phi}) \cap \triangleright \mathcal{A} = \bigcap \{\triangleright \boldsymbol{T}_\Sigma : \tau^{\mathcal{A}}_\Sigma[\boldsymbol{\phi}] \subseteq \boldsymbol{T}_\Sigma, \boldsymbol{T} \in \text{FiFam}^{\mathfrak{G}}(\mathcal{A})\}.$$

Proof: It suffices to see that Conditions (1) and (2) in the statement reflect exactly Conditions (1) and (2) in the statement of Corollary 1906, where the role of ρ is assumed by the special $\{\langle 0,1 \rangle\}$-tr-transformation ρ^0. ■

Finally, we obtain a characterization of those algebraic Gentzen π-institutions, i.e., Gentzen π-institutions associated with guasivarieties of algebraic systems, which are equivalent to some Hilbert π-institution.

Corollary 1908 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a quasivariety of* $\mathbf{F}$*-algebraic systems.* $\mathfrak{G}^\mathsf{K} = \langle \mathbf{F}, G^\mathsf{K} \rangle$ *is Hilbertizable if and only if there exists a* $\{\langle 1,1\rangle\}$-$\{\langle 0,1\rangle\}$*-transformation* τ *and a* $\{\langle 0,1\rangle\}$-$\{\langle 1,1\rangle\}$*-transformation* ρ*, such that, for all* $\mathcal{A} \in \mathsf{K}$*:*

(1) $\rho^{\mathcal{A}*} : \mathrm{ConSys}^\mathsf{K}(\mathcal{A}) \to \mathrm{SenFam}(\underset{\sim}{\mathcal{A}})$ *is injective on* $\mathrm{ConSys}^\mathsf{K}(\mathcal{A})$*;*

(2) For all $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,*

$$\rho_\Sigma^{\mathcal{A}*}(\Theta^{\mathsf{K},\mathcal{A}}(\phi \approx \psi)) = G'^{\mathcal{A}}(\tau_\Sigma^{\mathcal{A}}[\phi;\psi]),$$

where $G'^{\mathcal{A}}$ *is the closure system on* $\mathcal{A}$ *induced by* $\rho^{\mathcal{A}*}(\mathrm{ConSys}^\mathsf{K}(\mathcal{A}))$*.*

Proof: This is again a specialization of Theorem 1904 for $\mathfrak{G} = \mathfrak{G}^\mathsf{K}$, where we take into account the facts $\mathrm{FiFam}^{\mathfrak{G}^\mathsf{K}}(\mathcal{A}) = \mathrm{ConSys}^\mathsf{K}(\mathcal{A})$, $\mathrm{AlgSys}(\mathfrak{G}^\mathsf{K}) = \mathsf{K}$ and, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, we have, under appropriate identifications, $G^{\mathfrak{G}^\mathsf{K},\mathcal{A}}(\phi \rhd_\Sigma \psi) = \Theta^{\mathsf{K},\mathcal{A}}(\phi \approx \psi)$. ■

26.7 Protoalgebraicity

We now start a relatively brief tour of analogs of some of the classes in the algebraic hierarchy of π-institutions that were introduced in the earlier chapters of this work, as adapted and generalized for Gentzen π-institutions. Even though we revisit and recast only very few of the classes considered previously for π-institutions, the observant reader would realize that all other classes have similarly adapted analogs that have analogous properties.

In this section, we define protoalgebraic and syntactically protoalgebraic Gentzen π-institutions and study some of their properties. In the following section, we shall take a look at order algebraizable Gentzen π-institutions, which parallel the order algebraizable π-institutions of Chapter 25. In the last section, we look at completely reflective and truth equational Gentzen π-institutions.

We look, first, at some properties of the Leibniz operator, whose analogs for π-institutions have been established in Chapter 2.

Lemma 1909 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$*,* $\mathbf{B} = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ *be* $N^\flat$*-algebraic systems and* $\langle H, \gamma \rangle : \mathbf{A} \to \mathbf{B}$ *a morphism. For every trace* tr *and all* $\boldsymbol{T} \leq \mathrm{Seq}^{\mathrm{tr}}(\mathbf{B})$*,*

(a) $\gamma^{-1}(\Omega^\mathbf{B}(\boldsymbol{T})) \leq \Omega^\mathbf{A}(\gamma^{-1}(\boldsymbol{T}))$*;*

(b) $\gamma^{-1}(\Omega^\mathbf{B}(\boldsymbol{T})) = \Omega^\mathbf{A}(\gamma^{-1}(\boldsymbol{T}))$*, if* $\langle H, \gamma \rangle$ *is surjective.*

Proof:

(a) It is straightforward to check that $\gamma^{-1}(\Omega^{\mathbf{B}}(\boldsymbol{T}))$ is a congruence system on $\mathbf{A}$ compatible with $\gamma^{-1}(\boldsymbol{T})$. Hence, by the maximality property of $\Omega^{\mathbf{A}}(\gamma^{-1}(\boldsymbol{T}))$, we get that $\gamma^{-1}(\Omega^{\mathbf{B}}(\boldsymbol{T})) \leq \Omega^{\mathbf{A}}(\gamma^{-1}(\boldsymbol{T}))$.

(b) Suppose, now, that $\langle H,\gamma\rangle$ is surjective and let $\Sigma \in |\mathbf{Sign}|$, $\phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi,\psi\rangle \in \Omega^{\mathbf{A}}_\Sigma(\gamma^{-1}(\boldsymbol{T}))$. Then, by Lemma 1895, we get that, for all $\langle m,n\rangle \in \mathrm{tr}$, all $\vec{\sigma} = \langle\sigma^0,\dots,\sigma^{m-1}\rangle$, and all $\vec{\tau} = \langle\tau^0,\dots,\tau^{n-1}\rangle$ in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma,\Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

$$\begin{array}{l}\vec{\sigma}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}) \rhd_{\Sigma'} \vec{\tau}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}) \in \gamma^{-1}_{\Sigma'}(\boldsymbol{T}_{H(\Sigma')})\\ \quad\text{iff}\quad \vec{\sigma}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi}) \rhd_{\Sigma'} \vec{\tau}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi}) \in \gamma^{-1}_{\Sigma'}(\boldsymbol{T}_{H(\Sigma')}).\end{array}$$

Equivalently,

$$\begin{array}{l}\gamma_{\Sigma'}(\vec{\sigma}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}) \rhd_{\Sigma'} \vec{\tau}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi})) \in \boldsymbol{T}_{H(\Sigma')}\\ \quad\text{iff}\quad \gamma_{\Sigma'}(\vec{\sigma}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi}) \rhd_{\Sigma'} \vec{\tau}^{\mathbf{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi})) \in \boldsymbol{T}_{H(\Sigma')}.\end{array}$$

This holds if and only if, by the morphism property,

$$\begin{array}{l}\vec{\sigma}^{\mathbf{B}}_{H(\Sigma')}(\gamma_{\Sigma'}(\mathrm{SEN}(f)(\phi)),\gamma_{\Sigma'}(\vec{\chi}))\\ \qquad \rhd_{H(\Sigma')} \vec{\tau}^{\mathbf{B}}_{H(\Sigma')}(\gamma_{\Sigma'}(\mathrm{SEN}(f)(\phi)),\gamma_{\Sigma'}(\vec{\chi})) \in \boldsymbol{T}_{H(\Sigma')}\\ \quad\text{iff}\quad \vec{\sigma}^{\mathbf{B}}_{H(\Sigma')}(\gamma_{\Sigma'}(\mathrm{SEN}(f)(\psi)),\gamma_{\Sigma'}(\vec{\chi}))\\ \qquad \rhd_{H(\Sigma')} \vec{\tau}^{\mathbf{B}}_{H(\Sigma')}(\gamma_{\Sigma'}(\mathrm{SEN}(f)(\psi)),\gamma_{\Sigma'}(\vec{\chi})) \in \boldsymbol{T}_{H(\Sigma')}.\end{array}$$

Equivalently, by the naturality of γ,

$$\begin{array}{l}\vec{\sigma}^{\mathbf{B}}_{H(\Sigma')}(\mathrm{SEN}'(H(f))(\gamma_\Sigma(\phi)),\gamma_{\Sigma'}(\vec{\chi}))\\ \qquad \rhd_{H(\Sigma')} \vec{\tau}^{\mathbf{B}}_{H(\Sigma')}(\mathrm{SEN}'(H(f))(\gamma_\Sigma(\phi)),\gamma_{\Sigma'}(\vec{\chi})) \in \boldsymbol{T}_{H(\Sigma')}\\ \quad\text{iff}\quad \vec{\sigma}^{\mathbf{B}}_{H(\Sigma')}(\mathrm{SEN}'(H(f))(\gamma_\Sigma(\psi)),\gamma_{\Sigma'}(\vec{\chi}))\\ \qquad \rhd_{H(\Sigma')} \vec{\tau}^{\mathbf{B}}_{H(\Sigma')}(\mathrm{SEN}'(H(f))(\gamma_\Sigma(\psi)),\gamma_{\Sigma'}(\vec{\chi})) \in \boldsymbol{T}_{H(\Sigma')}.\end{array}$$

Hence, taking into account the surjectivity of $\langle H,\gamma\rangle$, by Lemma 1895, we get $\langle\gamma_\Sigma(\phi),\gamma_\Sigma(\psi)\rangle \in \Omega^{\mathbf{B}}_{H(\Sigma)}(\boldsymbol{T})$, i.e., $\gamma_\Sigma(\Omega^{\mathbf{A}}_\Sigma(\gamma^{-1}(\boldsymbol{T}))) \subseteq \Omega^{\mathbf{B}}_{H(\Sigma)}(\boldsymbol{T})$. We conclude that $\Omega^{\mathbf{A}}(\gamma^{-1}(\boldsymbol{T})) \leq \gamma^{-1}(\Omega^{\mathbf{B}}(\boldsymbol{T}))$.

■

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle\mathbf{F},G\rangle$ a Gentzen π-institution based on $\mathbf{F}$.

- We say $\mathfrak{G}$ is **protoalgebraic** if the Leibniz operator $\Omega : \mathrm{ThFam}(\mathfrak{G}) \to \mathrm{ConSys}(\mathcal{F})$ is monotone on $\mathrm{ThFam}(\mathfrak{G})$;
- We say $\mathfrak{G}$ is **syntactically protoalgebraic** if, for all $\langle m,n\rangle \in \mathrm{tr}$, there exists $I^{\langle m,n\rangle} : (\mathrm{SEN}^\flat)^\omega \to \bigcup_{\langle k,\ell\rangle\in\mathrm{tr}}(\mathrm{SEN}^\flat)^{k+\ell}$ in $N^\flat$ with $(m+n)+(m+n)$ distinguished arguments, such that, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi},\boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}(\mathbf{F})$ of trace $\langle m,n\rangle$,

$$\langle\boldsymbol{\phi},\boldsymbol{\psi}\rangle \in \Omega_\Sigma(\boldsymbol{T}) \quad\text{iff}\quad I^{\langle m,n\rangle}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma.$$

In this case the collection $I = \{I^{\langle m,n\rangle} : \langle m,n\rangle \in \mathrm{tr}\}$ is called a collection of **witnessing transformations** of the syntactic protoalgebraicity of $\mathfrak{G}$.

We give an alternative characterization of syntactic protoalgebraicity that comes handy in what follows.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and tr a trace. Given $\langle m,n\rangle \in \mathrm{tr}$, we say that a collection $I : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^k$ of natural transformations in $N^\flat$, with $(m+n)+(m+n)$ distinguished variables is **(pairwise) permutable** if and only if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$ and all $\{i_1,\dots,i_{m+n}\} = \{0,\dots,m+n-1\}$,

$$I_\Sigma[\phi_{i_1},\dots,\phi_{i_{m+n}},\psi_{i_1},\dots,\psi_{i_{m+n}}] = I_\Sigma[\phi_0,\dots,\phi_{(m+n)-1},\psi_0,\dots,\psi_{(n+m)-1}].$$

When we want to refer to an arbitrary pairwise permutation of two sequences $\vec{\phi}, \vec{\psi}$ of the same length as above, we write $\vec{\phi}^\pi, \vec{\psi}^\pi$, the meaning being that $\vec{\phi}, \vec{\psi}$ have the same length and that in $\vec{\phi}^\pi, \vec{\psi}^\pi$, their elements have been permuted both by applying the same arbitrary permutation π.

Theorem 1910 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen π-institution based on* $\mathbf{F}$. $\mathfrak{G}$ *is syntactically protoalgebraic if and only if, for all* $\langle m,n\rangle \in \mathrm{tr}$, *there exists* $\hat{I}^{\langle m,n\rangle} : (\mathrm{SEN}^\flat)^\omega \to \bigcup_{\langle k,\ell\rangle \in \mathrm{tr}} (\mathrm{SEN}^\flat)^{k+\ell}$ *in* $N^\flat$, *with* $(m+n)+(m+n)$ *distinguished arguments, which is permutable, such that, for all* $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, *all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi, \psi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \phi,\psi\rangle \in \Omega_\Sigma(\boldsymbol{T}) \quad \textit{iff} \quad \hat{I}^{\langle m,n\rangle}_\Sigma[\langle \phi,\vec{\chi}\rangle, \langle \psi,\vec{\chi}\rangle] \subseteq \boldsymbol{T}_\Sigma.$$

Proof: Suppose, first, that $\mathfrak{G}$ is syntactically protoalgebraic, with witnessing transformations $I = \{I^{\langle m,n\rangle} : \langle m,n\rangle \in \mathrm{tr}\}$. For all $\langle m,n\rangle \in \mathrm{tr}$, we symmetrize $I^{\langle m,n\rangle}$ by defining $\hat{I}^{\langle m,n\rangle}$ in $N^\flat$, with $(m+n)+(m+n)$ distinguished arguments, by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m,n\rangle$,

$$\hat{I}^{\langle m,n\rangle}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}] = \bigcup\{I^{\langle m,n\rangle}_\Sigma[\boldsymbol{\phi}^\pi,\boldsymbol{\psi}^\pi] : \pi \text{ a permutation}\}.$$

Then, we have, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of the same trace $\langle m,n\rangle$,

$$\begin{array}{rll} I^{\langle m,n\rangle}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma & \text{iff} & \langle \boldsymbol{\phi},\boldsymbol{\psi}\rangle \in \Omega_\Sigma(\boldsymbol{T}) \\ & \text{iff} & \langle \boldsymbol{\phi}^\pi,\boldsymbol{\psi}^\pi\rangle \in \Omega_\Sigma(\boldsymbol{T}), \text{ for all } \pi, \\ & \text{iff} & \hat{I}^{\langle m,n\rangle}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}] \subseteq_\Sigma \boldsymbol{T}_\Sigma. \end{array}$$

Therefore, we obtain, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{rll} \langle \phi,\psi\rangle \in \Omega_\Sigma(\boldsymbol{T}) & \text{iff} & I^{\langle m,n\rangle}_\Sigma[\langle \phi,\vec{\chi}\rangle, \langle \psi,\vec{\chi}\rangle] \subseteq \boldsymbol{T}_\Sigma \\ & \text{iff} & \hat{I}^{\langle m,n\rangle}_\Sigma[\langle \phi,\vec{\chi}\rangle, \langle \psi,\vec{\chi}\rangle] \subseteq \boldsymbol{T}_\Sigma. \end{array}$$

Suppose, conversely, that there exists a permutable $I = \{I^{\langle m,n\rangle} : \langle m,n\rangle \in \text{tr}\}$ that satisfies the condition in the statement of the theorem. Define a collection $\check{I} = \{\check{I}^{\langle m,n\rangle} : \langle m,n\rangle \in \text{tr}\}$ in $N^\flat$ having $(m+n)+(m+n)$ distinguished arguments by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi},\boldsymbol{\psi} \in \text{Seq}^{\text{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m,n\rangle$,

$$\check{I}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}] = \bigcup\{I^{\langle m,n\rangle}_\Sigma[(\boldsymbol{\phi}\boldsymbol{\psi})^{i+1},(\boldsymbol{\phi}\boldsymbol{\psi})^i] : i < m+n-1\},$$

where

$$(\boldsymbol{\phi}\boldsymbol{\psi})^i \coloneqq \langle \phi_0,\dots,\phi_{i-1},\psi_i,\dots,\psi_{m+n-1}\rangle.$$

Then we have, for all $\boldsymbol{T} \in \text{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi},\boldsymbol{\psi} \in \text{Seq}^{\text{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m,n\rangle$,

$$\begin{array}{lll}\langle\boldsymbol{\phi},\boldsymbol{\psi}\rangle \in \Omega_\Sigma(\boldsymbol{T}) & \text{iff} & \langle\phi_i,\psi_i\rangle \in \Omega_\Sigma(\boldsymbol{T}),\ i < m+n-1,\\ & \text{iff} & I^{\langle m,n\rangle}[(\boldsymbol{\phi}\boldsymbol{\psi})^{i+1},(\boldsymbol{\phi}\boldsymbol{\psi})^i] \subseteq \boldsymbol{T}_\Sigma,\ i < m+n-1,\\ & \text{iff} & \check{I}^{\langle m,n\rangle}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma.\end{array}$$

Therefore, $\mathfrak{G}$ is syntactically protoalgebraic with witnessing transformations $\check{I}$. ∎

Before embarking on a characterization of the exact relationship between syntactic protoalgebraicity and protoalgebraicity, we look at some properties related to notions that have been studied in this chapter, namely, the algebraic counterpart of a Gentzen π-institution and equivalence between Gentzen π-institutions.

The first property states that it suffices to check monotonicity of the Leibniz operator only on the filter families of algebraic systems belonging to the algebraic counterpart.

Lemma 1911 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \text{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$. *If, for all* $\mathcal{A} = \langle \mathbf{A},\langle F,\alpha\rangle\rangle \in \text{AlgSys}(\mathfrak{G})$, $\Omega^{\mathcal{A}}$ *is monotone, then* $\mathfrak{G}$ *is protoalgebraic.*

Proof: Suppose that $\Omega^{\mathcal{A}}$ is monotone, for all $\mathcal{A} \in \text{AlgSys}(\mathfrak{G})$ and let $\boldsymbol{T},\boldsymbol{T}' \in \text{ThFam}(\mathfrak{G})$, such that $\boldsymbol{T} \le \boldsymbol{T}'$. Then, by the monotonicity of the Suszko operator, $\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}) \le \widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}')$. Thus, the congruence system $\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})$ is compatible with both $\boldsymbol{T}$ and $\boldsymbol{T}'$. Hence, both $\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})$ and $\boldsymbol{T}'/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})$ are $\mathfrak{G}$-filter families of $\mathcal{F}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})$, such that $\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}) \le \boldsymbol{T}'/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})$. By hypothesis, since $\mathcal{F}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}) \in \text{AlgSys}(\mathfrak{G})$,

$$\Omega^{\mathcal{F}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})}(\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})) \le \Omega^{\mathcal{F}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})}(\boldsymbol{T}'/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})).$$

Thus, applying the inverse of the quotient morphism $\langle I,\pi\rangle : \mathcal{F} \to \mathcal{F}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})$, we get that

$$\pi^{-1}(\Omega^{\mathcal{F}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})}(\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}))) \le \pi^{-1}(\Omega^{\mathcal{F}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})}(\boldsymbol{T}'/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}))),$$

whence, by Lemma 1909,

$$\Omega(\pi^{-1}(\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}))) \leq \Omega(\pi^{-1}(\boldsymbol{T}'/\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}))).$$

Thus, since $\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T})$ is compatible with both $\boldsymbol{T}$ and $\boldsymbol{T}'$, we get that $\Omega(\boldsymbol{T}) \leq \Omega(\boldsymbol{T}')$. Therefore, $\mathfrak{G}$ is protoalgebraic. ■

Now we prove that protoalgebraicity is preserved under equivalence.

Theorem 1912 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr*,* tr′ *traces and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$*,* $\mathfrak{G}' = \langle \mathbf{F}, G' \rangle$ *Gentzen* π*-institutions of traces* tr*,* tr′*, respectively, based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *and* $\mathfrak{G}'$ *are equivalent, then* $\mathfrak{G}$ *is protoalgebraic if and only if* $\mathfrak{G}'$ *is also.*

Proof: Suppose $\mathfrak{G}$ and $\mathfrak{G}'$ are equivalent via the conjugate pair $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ and that $\mathfrak{G}'$ is protoalgebraic. Let $\boldsymbol{T}, \boldsymbol{T}' \in \mathrm{ThFam}(\mathfrak{G})$, such that $\boldsymbol{T} \leq \boldsymbol{T}'$. Then, by Theorem 1880, $\rho^*(\boldsymbol{T}) \leq \rho^*(\boldsymbol{T}')$. Thus, by hypothesis, $\Omega(\rho^*(\boldsymbol{T})) \leq \Omega(\rho^*(\boldsymbol{T}'))$. Hence, by Proposition 1897, $\Omega(\boldsymbol{T}) \leq \Omega(\boldsymbol{T}')$. Therefore, $\mathfrak{G}$ is also protoalgebraic. The converse follows by the symmetry of equivalence. ■

Finally, it is shown that the same applies to syntactic protoalgebraicity, i.e., if two Gentzen π-institutions are equivalent, then one is syntactically protoalgebraic if and only if the other is also.

Theorem 1913 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr*,* tr′ *traces and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$*,* $\mathfrak{G}' = \langle \mathbf{F}, G' \rangle$ *Gentzen* π*-institutions of traces* tr*,* tr′*, respectively, based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *and* $\mathfrak{G}'$ *are equivalent, then* $\mathfrak{G}$ *is syntactically protoalgebraic if and only if* $\mathfrak{G}'$ *is also.*

Proof: Suppose that $\mathfrak{G}$ and $\mathfrak{G}'$ are equivalent via a conjugate pair $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ and that $\mathfrak{G}'$ is syntactically protoalgebraic, with witnessing transformations $I := \{I^{\langle m,m \rangle} : \langle m, n \rangle \in \mathrm{tr}'\}$. Then, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$, we get, setting, according to Theorem 1880, $\boldsymbol{T}' \in \mathrm{ThFam}(\mathfrak{G}')$ be such that $\boldsymbol{T} \underset{\tau^*}{\overset{\rho^*}{\rightleftarrows}} \boldsymbol{T}'$, and taking into account Theorem 1919,

$$\begin{array}{rll}\langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle \in \Omega_\Sigma(\boldsymbol{T}) & \text{iff} & \langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle \in \Omega_\Sigma(\rho^*(\boldsymbol{T})) \\ & \text{iff} & \langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle \in \Omega_\Sigma(\boldsymbol{T}') \\ & \text{iff} & \hat{I}_\Sigma[\langle \phi_i, \vec{\chi} \rangle, \langle \psi_i, \vec{\chi} \rangle] \subseteq \boldsymbol{T}'_\Sigma, i < m + n, \\ & \text{iff} & \hat{I}_\Sigma[\langle \phi_i, \vec{\chi} \rangle, \langle \psi_i, \vec{\chi} \rangle] \subseteq \tau^*_\Sigma(\boldsymbol{T}), i < m + n, \\ & \text{iff} & \tau_\Sigma[\hat{I}_\Sigma[\langle \phi_i, \vec{\chi} \rangle, \langle \psi_i, \vec{\chi} \rangle]] \subseteq \boldsymbol{T}_\Sigma, i < m + n.\end{array}$$

Therefore, $(\tau \circ \hat{I})^\smile$ witnesses the syntactic protoalgebraicity of $\mathfrak{G}$. The converse follows by the symmetry of equivalence. ■

It is relatively easy to see that, if a Gentzen π-institution $\mathfrak{G}$ is syntactically protoalgebraic, then it is protoalgebraic.

Theorem 1914 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *is syntactically protoalgebraic, then it is protoalgebraic.*

Proof: Suppose $\mathfrak{G}$ is syntactically protoalgebraic, with witnessing transformations $I = \{I^{\langle m,n \rangle} : \langle m,n \rangle \in \mathrm{tr}\}$ in $N^\flat$, and let $\boldsymbol{T}, \boldsymbol{T}' \in \mathrm{ThFam}(\mathfrak{G})$, such that $\boldsymbol{T} \le \boldsymbol{T}'$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ of the same trace $\langle m,n \rangle$, we have

$$\begin{array}{rcl} \langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle \in \Omega_\Sigma(\boldsymbol{T}) & \text{iff} & I_\Sigma^{\langle m,n \rangle}[\boldsymbol{\phi}, \boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma \\ & \text{implies} & I_\Sigma^{\langle m,n \rangle}[\boldsymbol{\phi}, \boldsymbol{\psi}] \subseteq \boldsymbol{T}'_\Sigma \\ & \text{iff} & \langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle \in \Omega_\Sigma(\boldsymbol{T}'). \end{array}$$

Hence $\Omega(\boldsymbol{T}) \le \Omega(\boldsymbol{T}')$ and $\mathfrak{G}$ is protoalgebraic. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$. The **reflexive core** $R^{\mathfrak{G}}$ **of** $\mathfrak{G}$ is the collection

$$R^{\mathfrak{G}} = \{R^{\mathfrak{G},\langle m,n \rangle} : \langle m,n \rangle \in \mathrm{tr}\},$$

where, for all $\langle m,n \rangle \in \mathrm{tr}$, $R^{\mathfrak{G},\langle m,n \rangle}$ consists of all natural transformations $\rho : (\mathrm{SEN}^\flat)^\omega \to \bigcup_{\langle k,\ell \rangle \in \mathrm{tr}} (\mathrm{SEN}^\flat)^{k+\ell}$ in $N^\flat$ with $(m+n)+(m+n)$ distinguished arguments that satisfy:

1. For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,
$$\rho_\Sigma[\langle \phi, \vec{\chi} \rangle, \langle \phi, \vec{\chi} \rangle] \subseteq \mathrm{Thm}_\Sigma(\mathfrak{G});$$
2. For all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ of trace $\langle m,n \rangle$,
$$\rho_{\Sigma'}[\mathrm{SEN}^\flat(f)(\boldsymbol{\phi}), \mathrm{SEN}^\flat(f)(\boldsymbol{\psi})] \subseteq G_{\Sigma'}(\rho_\Sigma[\boldsymbol{\phi}, \boldsymbol{\psi}]).$$

Using the notation in the proof of Theorem 1919, we observe that, $\hat{R}^{\mathfrak{G}} \subseteq R^{\mathfrak{G}}$ and that $\check{R}^{\mathfrak{G}} \subseteq R^{\mathfrak{G}}$:

- If $\rho \in R^{\mathfrak{G}}$, then, for
$$\sigma_\Sigma(\boldsymbol{\phi}, \boldsymbol{\psi}, \vec{\chi}) \coloneqq \rho_\Sigma(\boldsymbol{\phi}^\pi, \boldsymbol{\psi}^\pi, \vec{\chi}),$$
we get $\sigma_\Sigma[\boldsymbol{\phi}, \boldsymbol{\phi}] = \rho_\Sigma[\boldsymbol{\phi}^\pi, \boldsymbol{\phi}^\pi] \subseteq \mathrm{Thm}_\Sigma(\mathfrak{G})$;
- If $\rho \in R^{\mathfrak{G}}$, then, for
$$\sigma_\Sigma(\boldsymbol{\phi}, \boldsymbol{\psi}, \vec{\chi}) \coloneqq \rho_\Sigma((\boldsymbol{\phi}\boldsymbol{\psi})^{i+1}, (\boldsymbol{\phi}\boldsymbol{\psi})^i, \vec{\chi}),$$
we get $\sigma_\Sigma[\boldsymbol{\phi}, \boldsymbol{\phi}] = \rho_\Sigma[\boldsymbol{\phi}, \boldsymbol{\phi}] \subseteq \mathrm{Thm}_\Sigma(\mathfrak{G})$.

If a Gentzen π-institution $\mathfrak{G}$ of trace tr is syntactically protoalgebraic with witnessing transformations I, then $I^{\langle m,n\rangle} \subseteq R^{\mathfrak{G},\langle m,n\rangle}$, for all $\langle m,n\rangle \in \mathrm{tr}$.

Lemma 1915 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *is syntactically protoalgebraic with witnessing transformations* $I = \{I^{\langle m,n\rangle} : \langle m,n\rangle \in \mathrm{tr}\}$*, then* $I \subseteq R^{\mathfrak{G}}$*.*

Proof: Suppose that $\mathfrak{G}$ is syntactically protoalgebraic, with witnessing transformations I.

- Since, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$, $\langle \boldsymbol{\phi}, \boldsymbol{\phi}\rangle \in \Omega_\Sigma(\mathrm{Thm}(\mathfrak{G}))$, we get that $I_\Sigma[\boldsymbol{\phi}, \boldsymbol{\phi}] \subseteq \mathrm{Thm}_\Sigma(\mathfrak{G})$.
- If, for some $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$, we have $I_\Sigma[\boldsymbol{\phi}, \boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma$, then we get $\langle \boldsymbol{\phi}, \boldsymbol{\psi}\rangle \in \Omega_\Sigma(\boldsymbol{T})$, whence, since $\Omega(\boldsymbol{T})$ is a congruence system on $\mathbf{F}$, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, we get $\langle \mathrm{SEN}^\flat(f)(\boldsymbol{\phi}), \mathrm{SEN}^\flat(f)(\boldsymbol{\psi})\rangle \in \Omega_{\Sigma'}(\boldsymbol{T})$, showing that
$$I_{\Sigma'}[\mathrm{SEN}^\flat(f)(\boldsymbol{\phi}), \mathrm{SEN}^\flat(f)(\boldsymbol{\psi})] \subseteq \boldsymbol{T}_{\Sigma'}.$$

Thus, by definition of $R^{\mathfrak{G}}$, we get that $I \subseteq R^{\mathfrak{G}}$. ■

Another important property of syntactic protoalgebraicity is that it guarantees that the reflexive core of $\mathfrak{G}$ possesses a modus ponens property in $\mathfrak{G}$.

Theorem 1916 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *is syntactically protoalgebraic, then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ *of the same trace,*
$$\boldsymbol{\psi} \in G_\Sigma(\boldsymbol{\phi}, R^{\mathfrak{G}}_\Sigma[\boldsymbol{\phi}, \boldsymbol{\psi}]).$$

Proof: Suppose $\mathfrak{G}$ is syntactically protoalgebraic, with witnessing transformations I and let $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$, such that $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$ and $R^{\mathfrak{G}}_\Sigma[\boldsymbol{\phi}, \boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma$. Then, by Lemma 1915, we get $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$ and $I_\Sigma[\boldsymbol{\phi}, \boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma$, that is, by syntactic protoalgebraicity, $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$ and $\langle \boldsymbol{\phi}, \boldsymbol{\psi}\rangle \in \Omega_\Sigma(\boldsymbol{T})$. Therefore, by compatibility, we get $\boldsymbol{\psi} \in \boldsymbol{T}_\Sigma$, showing that $\boldsymbol{\psi} \in G_\Sigma(\boldsymbol{\phi}, R^{\mathfrak{G}}_\Sigma[\boldsymbol{\phi}, \boldsymbol{\psi}])$. ■

Conversely, if the reflexive core $R^{\mathfrak{G}}$ of a Gentzen π-institution $\mathfrak{G}$ has the modus ponens property in $\mathfrak{G}$, then $\mathfrak{G}$ is syntactically protoalgebraic, with witnessing transformations $R^{\mathfrak{G}}$. First, a lemma of a technical nature. For a Gentzen π-institution $\mathfrak{G}$ and $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, we set
$$R^{\mathfrak{G}}(\boldsymbol{T}) = \{R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})\}_{\Sigma \in |\mathbf{Sign}^\flat|},$$

where, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$R^{\mathfrak{G}}_\Sigma(\boldsymbol{T}) = \{\langle\phi,\psi\rangle \in \mathrm{SEN}^\flat(\Sigma) : (\forall\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma))(R^{\mathfrak{G}}_\Sigma[\langle\phi,\vec{\chi}\rangle,\langle\psi,\vec{\chi}\rangle] \subseteq \boldsymbol{T}_\Sigma)\}.$$

Of course, by the symmetry of the transformations in $N^\flat$, in this definition, ϕ and ψ may appear, equivalently, in any position of the sequents on the right, as long as they appear in the same position in both of the first sequent arguments of $R^{\mathfrak{G}}$.

Lemma 1917 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle\mathbf{F}, G\rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$. *If, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\boldsymbol{\phi},\boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ *of the same trace,*

$$\boldsymbol{\psi} \in G_\Sigma(\boldsymbol{\phi}, R^{\mathfrak{G}}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}]),$$

the $R^{\mathfrak{G}}(\boldsymbol{T})$ *is a congruence family on* $\mathbf{F}$ *compatible with* $\boldsymbol{T}$.

Proof: We start by showing that $R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$ is an equivalence family on $\mathbf{F}$.

- By the definition of $R^{\mathfrak{G}}$, we get, for all $\phi,\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

  $$R^{\mathfrak{G}}_\Sigma[\langle\phi,\vec{\chi}\rangle,\langle\phi,\vec{\chi}\rangle] \subseteq \mathrm{Thm}_\Sigma(\mathfrak{G}) \subseteq \boldsymbol{T}_\Sigma.$$

  Thus, $\langle\phi,\phi\rangle \in R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$ and $R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$ is reflexive.

- Suppose $\langle\phi,\psi\rangle \in R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$. Then, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, we have

  $$R^{\mathfrak{G}}_\Sigma[\langle\phi,\vec{\chi}\rangle,\langle\psi,\vec{\chi}\rangle] \subseteq \boldsymbol{T}_\Sigma.$$

  But then, by the definition of $R^{\mathfrak{G}}$ and the symmetry of $N^\flat$, we get

  $$R^{\mathfrak{G}}_\Sigma[\langle\psi,\vec{\chi}\rangle,\langle\phi,\vec{\chi}\rangle] \subseteq R^{\mathfrak{G}}_\Sigma[\langle\phi,\vec{\chi}\rangle,\langle\psi,\vec{\chi}\rangle] \subseteq \boldsymbol{T}_\Sigma.$$

  Therefore, $\langle\psi,\phi\rangle \in R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$ and $R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$ is also symmetric.

- Suppose, now, that $\langle\phi,\psi\rangle,\langle\psi,\chi\rangle \in R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$. Thus, we get, for all $\vec{\chi} \in \mathrm{SEN}(\Sigma)$,

  $$R^{\mathfrak{G}}_\Sigma[\langle\phi,\vec{\chi}\rangle,\langle\psi,\vec{\chi}\rangle] \subseteq \boldsymbol{T}_\Sigma \text{ and } R^{\mathfrak{G}}_\Sigma[\langle\psi,\vec{\chi}\rangle,\langle\chi,\vec{\chi}\rangle] \subseteq \boldsymbol{T}_\Sigma.$$

  By hypothesis, we have, for all $\rho \in R^{\mathfrak{G}}$ and all $\vec{\xi} \in \mathrm{SEN}^\flat(\Sigma)$,

  $$\begin{aligned}\rho_\Sigma(\langle\phi,\vec{\chi}\rangle,\langle\chi,\vec{\chi}\rangle,\vec{\xi}) \subseteq G_\Sigma(&\rho_\Sigma(\langle\phi,\vec{\chi}\rangle,\langle\psi,\vec{\chi}\rangle,\vec{\xi}),\\ &R^{\mathfrak{G}}_\Sigma[\rho_\Sigma(\langle\phi,\vec{\chi}\rangle,\langle\psi,\vec{\chi}\rangle,\vec{\xi}),\rho_\Sigma(\langle\phi,\vec{\chi}\rangle,\langle\chi,\vec{\chi}\rangle,\vec{\xi})])\\ \subseteq G_\Sigma(&R^{\mathfrak{G}}_\Sigma[\langle\phi,\vec{\chi}\rangle,\langle\psi,\vec{\chi}\rangle],R^{\mathfrak{G}}_\Sigma[\langle\psi,\vec{\chi}\rangle,\langle\chi,\vec{\chi}\rangle])\\ \subseteq G_\Sigma(&\boldsymbol{T}_\Sigma) = \boldsymbol{T}_\Sigma.\end{aligned}$$

  Therefore, $R^{\mathfrak{G}}_\Sigma[\langle\phi,\vec{\chi}\rangle,\langle\chi,\vec{\chi}\rangle] \subseteq \boldsymbol{T}_\Sigma$, showing that $\langle\phi,\chi\rangle \in R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$ and, hence, $R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$ is also transitive.

We show, next, that $R^{\mathfrak{G}}(\boldsymbol{T})$ is a congruence family. Let σ be in $N^\flat$, $\vec{\phi},\vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$, such that, for all $i < k$, $\langle \phi_i,\psi_i\rangle \in R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$. Then, for all $i < k$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $R^{\mathfrak{G}}_\Sigma[\langle \phi_i,\vec{\chi}\rangle,\langle \psi_i,\vec{\chi}\rangle] \subseteq \boldsymbol{T}_\Sigma$. But, then, for all $i < k$,

$$R^{\mathfrak{G}}_\Sigma[\langle \sigma_\Sigma((\vec{\phi}\vec{\psi})^{i+1}),\vec{\chi}\rangle,\langle \sigma_\Sigma((\vec{\phi}\vec{\psi})^{i}),\vec{\chi}\rangle] \subseteq R^{\mathfrak{G}}_\Sigma[\langle \phi_i,\vec{\chi}\rangle,\langle \psi_i,\vec{\chi}\rangle] \subseteq \boldsymbol{T}_\Sigma,$$

i.e., $\langle \sigma_\Sigma((\vec{\phi}\vec{\psi})^{i+1}),\sigma_\Sigma((\vec{\phi}\vec{\psi})^{i})\rangle \in R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$. Since this holds for all $i < k$, we get by the transitivity of $R^{\mathfrak{G}}(\boldsymbol{T})$ proven above, that $\langle \sigma_\Sigma(\vec{\phi}),\sigma_\Sigma(\vec{\psi})\rangle \in R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$ and, therefore, $R^{\mathfrak{G}}(\boldsymbol{T})$ is also a congruence family.

Finally, $R^{\mathfrak{G}}(\boldsymbol{T})$ is a congruence system by the definition of $R^{\mathfrak{G}}$. Compatibility of $R^{\mathfrak{G}}(\boldsymbol{T})$ with $\boldsymbol{T}$ is also readily obtainable by the hypothesis, since $\langle \boldsymbol{\phi},\boldsymbol{\psi}\rangle \in R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$ implies $R^{\mathfrak{G}}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma$. Therefore, if $\boldsymbol{\phi} \in T_\Sigma$ and $\langle \boldsymbol{\phi},\boldsymbol{\psi}\rangle \in R^{\mathfrak{G}}_\Sigma(\boldsymbol{T})$, we get

$$\boldsymbol{\psi} \in G_\Sigma(\boldsymbol{\phi},R^{\mathfrak{G}}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}]) \subseteq G_\Sigma(\boldsymbol{T}_\Sigma) = \boldsymbol{T}_\Sigma.$$

Hence, $R^{\mathfrak{G}}(\boldsymbol{T})$ is a congruence system on $\mathbf{F}$ compatible with $\boldsymbol{T}$. ■

Now we are ready for the promised theorem.

Theorem 1918 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F},G\rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$. *If, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\boldsymbol{\phi},\boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ *of the same trace,*

$$\boldsymbol{\psi} \in G_\Sigma(\boldsymbol{\phi},R^{\mathfrak{G}}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}]),$$

then $\mathfrak{G}$ *is syntactically protoalgebraic, with witnessing transformations* $R^{\mathfrak{G}}$.

Proof: Suppose that $R^{\mathfrak{G}}$ satisfies the displayed condition. We must show that, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi},\boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$,

$$\langle \boldsymbol{\phi},\boldsymbol{\psi}\rangle \in \Omega_\Sigma(\boldsymbol{T}) \quad \text{iff} \quad R^{\mathfrak{G}}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma.$$

Suppose, first, that $\langle \boldsymbol{\phi},\boldsymbol{\psi}\rangle \in \Omega_\Sigma(\boldsymbol{T})$. Then,, since $\Omega(\boldsymbol{T})$ is a congruence system on $\mathbf{F}$, we get, for all $\rho \in R^{\mathfrak{G}}$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \rho_\Sigma(\boldsymbol{\phi},\boldsymbol{\phi},\vec{\chi}),\rho_\Sigma(\boldsymbol{\phi},\boldsymbol{\psi},\vec{\chi})\rangle \in \Omega_\Sigma(\boldsymbol{T}).$$

Moreover, $R^{\mathfrak{G}}_\Sigma[\boldsymbol{\phi},\boldsymbol{\phi}] \subseteq \mathrm{Thm}_\Sigma(\mathfrak{G}) \subseteq \boldsymbol{T}_\Sigma$, by the definition of the reflexive core. Therefore, by the compatibility of $\Omega(\boldsymbol{T})$, with $\boldsymbol{T}$, we get that, for all $\rho \in R^{\mathfrak{G}}$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $\rho_\Sigma(\boldsymbol{\phi},\boldsymbol{\psi},\vec{\chi}) \in \boldsymbol{T}_\Sigma$. We conclude that $R^{\mathfrak{G}}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma$.

Assume, conversely, that $R^{\mathfrak{G}}_\Sigma[\boldsymbol{\phi},\boldsymbol{\psi}] \subseteq T_\Sigma$. Since, by Lemma 1917, $R^{\mathfrak{G}}(\boldsymbol{T})$ is a congruence system on $\mathbf{F}$ compatible with $\boldsymbol{T}$, we get, by the maximality of $\Omega(\boldsymbol{T})$, that $R^{\mathfrak{G}}(\boldsymbol{T}) \leq \Omega(\boldsymbol{T})$. But the hypothesis implies that $\langle \boldsymbol{\phi},\boldsymbol{\psi}\rangle \in R^{\mathfrak{G}}_\Sigma(T)$. Therefore, we conclude that $\langle \boldsymbol{\phi},\boldsymbol{\psi}\rangle \in \Omega_\Sigma(\boldsymbol{T})$. ■

We now have a characterization of syntactic protoalgebraicity in terms of the property of modus ponens of the reflexive core $R^{\mathfrak{G}}$ of the Gentzen π-institution $\mathfrak{G}$.

$$\mathfrak{G} \text{ is syntactically protoalgebraic} \longleftrightarrow R^{\mathfrak{G}} \text{ has the MP}.$$

Theorem 1919 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*.* $\mathfrak{G}$ *is syntactically protoalgebraic if and only if* $R^{\mathfrak{G}}$ *has the modus ponens in* $\mathfrak{G}$*.*

Proof: Theorem 1916 gives the "only if" and the "if" is by Theorem 1918. ∎

As a corollary, we obtain

Corollary 1920 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *is syntactically protoalgebraic with witnessing transformations* $I = \{I^{\langle m,n \rangle} : \langle m,n \rangle \in \mathrm{tr}\}$*, then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ *of trace* $\langle m,n \rangle$*,*

$$G_\Sigma(R_\Sigma^{\mathfrak{G},\langle m,n \rangle}[\boldsymbol{\phi}, \boldsymbol{\psi}]) = G_\Sigma(I_\Sigma^{\langle m,n \rangle}[\boldsymbol{\phi}, \boldsymbol{\psi}]).$$

Proof: If $\mathfrak{G}$ is syntactically protoalgebraic, with witnessing transformations I, then, by Theorems 1919 and 1918, both I and $R^{\mathfrak{G}}$ are families of witnessing transformations for the syntactic protoalgebraicity of $\mathfrak{G}$. Therefore, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ of trace $\langle m,n \rangle$,

$$\begin{array}{rll} R_\Sigma^{\mathfrak{G},\langle m,n \rangle}[\boldsymbol{\phi}, \boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma & \text{iff} & \langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle \in \Omega_\Sigma(\boldsymbol{T}) \\ & \text{iff} & I_\Sigma^{\langle m,n \rangle}[\boldsymbol{\phi}, \boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma. \end{array}$$

Therefore, $G_\Sigma(R_\Sigma^{\mathfrak{G},\langle m,n \rangle}[\boldsymbol{\phi}, \boldsymbol{\psi}]) = G_\Sigma(I_\Sigma^{\langle m,n \rangle}[\boldsymbol{\phi}, \boldsymbol{\psi}])$. ∎

We get relatively easily another related characterization of syntactic protoalgberaicity.

$\mathfrak{G}$ is syntactically protoalgebraic
$\longleftrightarrow$ $R^{\mathfrak{G}}$ Defines Leibniz Congruence Systems.

Theorem 1921 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*.* $\mathfrak{G}$ *is syntactically protoalgebraic if and only if, for every* $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$*,*

$$\Omega(\boldsymbol{T}) = R^{\mathfrak{G}}(\boldsymbol{T}).$$

Proof: If $\mathfrak{G}$ is syntactically protoalgebraic, then, by Theorems 1919 and 1918, $R^{\mathfrak{G}}$ constitutes a collection of witnessing transformations, whence, for every $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$ $\Omega(\boldsymbol{T}) = \hat{R}^{\mathfrak{G}}(\boldsymbol{T}) = R^{\mathfrak{G}}(\boldsymbol{T})$.

The converse follows by the definition of syntactic protoalgberaicity, since, in that case, $\check{R}^{\mathfrak{G}} = R^{\mathfrak{G}}$ forms a collection of witnessing transformations. ∎

We finally show that the property that separates protoalgebraicity from syntactic protoalgebraicity is the compatibility property with respect to the theory family generated by the reflexive core.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$. We say that the reflexive core $R^{\mathfrak{G}}$ is **Leibniz** if, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$,

$$\boldsymbol{\phi} \; \Omega_\Sigma(G(R_\Sigma^{\mathfrak{G}}[\boldsymbol{\phi}, \boldsymbol{\psi}])) \; \boldsymbol{\psi}.$$

This property is weaker than $R^{\mathfrak{G}}$ having the modus ponens, i.e., if $R^{\mathfrak{G}}$ has the modus ponens, then it is Leibniz.

Proposition 1922 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system,* tr *a trace and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace* tr *based on $\mathbf{F}$. If $R^{\mathfrak{G}}$ has the modus ponens, then it is Leibniz.*

Proof: If $R^{\mathfrak{G}}$ has the modus ponens, then, by Theorem 1919, we get, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ of trace $\langle m, n \rangle \in \mathrm{tr}$,

$$\boldsymbol{\phi} \; \Omega_\Sigma(\boldsymbol{T}) \; \boldsymbol{\psi} \quad \text{iff} \quad R_\Sigma^{\mathfrak{G}}[\boldsymbol{\phi}, \boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma.$$

Therefore, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$, by considering, in particular, $\boldsymbol{T} = G(R_\Sigma^{\mathfrak{G}}[\boldsymbol{\phi}, \boldsymbol{\psi}])$, and taking into account that

$$R_\Sigma^{\mathfrak{G}}[\boldsymbol{\phi}, \boldsymbol{\psi}] \subseteq G_\Sigma(R_\Sigma^{\mathfrak{G}}[\boldsymbol{\phi}, \boldsymbol{\psi}]),$$

we get that $\boldsymbol{\phi} \; \Omega_\Sigma(G(R_\Sigma^{\mathfrak{G}}[\boldsymbol{\phi}, \boldsymbol{\psi}])) \; \boldsymbol{\psi}$. Thus, $R^{\mathfrak{G}}$ is Leibniz. ∎

In the opposite direction, in a protoalgebraic Gentzen π-institution $\mathfrak{G}$, if the reflexive core $R^{\mathfrak{G}}$ is Leibniz, then it has the modus ponens in $\mathfrak{G}$.

Proposition 1923 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system,* tr *a trace and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a protoalgebraic Gentzen π-institution of trace* tr *based on $\mathbf{F}$. If $R^{\mathfrak{G}}$ is Leibniz, then it has the modus ponens in $\mathfrak{G}$.*

Proof: Suppose that $\mathfrak{G}$ is protoalgebraic and that $R^{\mathfrak{G}}$ is Leibniz. Let $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ of trace $\langle m, n \rangle \in \mathrm{tr}$, such that $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$ and $R_\Sigma^{\mathfrak{G}}[\boldsymbol{\phi}, \boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma$. Since $R^{\mathfrak{G}}$ is Leibniz, we have

$$\boldsymbol{\phi} \; \Omega_\Sigma(G(R_\Sigma^{\mathfrak{G}}[\boldsymbol{\phi}, \boldsymbol{\psi}])) \; \boldsymbol{\psi},$$

whence, since $\mathfrak{G}$ is protoalgebraic and $R_\Sigma^{\mathfrak{G}}[\boldsymbol{\phi}, \boldsymbol{\psi}] \subseteq \boldsymbol{T}_\Sigma$, we get $\boldsymbol{\phi} \; \Omega_\Sigma(\boldsymbol{T}) \; \boldsymbol{\psi}$. Therefore, since $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$, we get, by the compatibility of $\Omega(\boldsymbol{T})$ with $\boldsymbol{T}$, that $\boldsymbol{\psi} \in \boldsymbol{T}_\Sigma$. We conclude that $R^{\mathfrak{G}}$ has the modus ponens in $\mathfrak{G}$. ∎

We now show that a Gentzen π-institution is syntactically protoalgebraic if and only if it is protoalgebraic and it has a Leibniz reflexive core.

Syntactic Protoalgebraicity	=	$R^{\mathfrak{G}}$ has the Modus Ponens
	=	$R^{\mathfrak{G}}$ Defines Leibniz Congruence Systems
	=	Protoalgebraicity + $R^{\mathfrak{G}}$ Leibniz

Theorem 1924 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$. $\mathcal{I}$ *is syntactically protoalgebraic if and only if it is protoalgebraic and has a Leibniz reflexive core.*

Proof: Suppose, first, that $\mathfrak{G}$ is syntactically protoalgebraic. Then it is protoalgebraic by Theorem 1914. Moreover, its reflexive core has the modus ponens by Theorem 1916 and, hence, by Proposition 1922, its reflexive core is Leibniz.

Suppose, conversely, that $\mathfrak{G}$ is protoalgebraic with a Leibniz reflexive core. Then, by Proposition 1923, its reflexive core has the modus ponens and, therefore, by Theorem 1919, $\mathfrak{G}$ is syntactically protoalgebraic. ■

26.8 Order Algebraizability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and K a class of $\mathbf{F}$-algebraic posystems. Recall the inequational π-institution $\mathcal{I}^{\mathsf{K}} = \langle \mathbf{F}, C^{\mathsf{K}} \rangle$ associated with the class K, i.e., in which, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $I \cup \{\phi \preccurlyeq \psi\} \subseteq \mathrm{In}_\Sigma(\mathbf{F})$,

$$\begin{array}{lll}\phi \preccurlyeq \psi \in C^{\mathsf{K}}_\Sigma(I) & \text{iff} & \text{for all } \langle \mathcal{A}, \leq \rangle \in \mathsf{K}, \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma, \Sigma'),\\ & & \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(I)) \subseteq \leq_{F(\Sigma')} \text{ implies}\\ & & \qquad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi)) \leq_{F(\Sigma')} \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi)).\end{array}$$

To $\mathcal{I}^{\mathsf{K}}$ we associate the Gentzen π-institution $\mathfrak{G}^{\mathsf{K}} = \langle \mathbf{F}, G^{\mathsf{K}} \rangle$ of trace $\{\langle 1, 1 \rangle\}$ defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\{\phi_i, \psi_i : i \in I\} \cup \{\phi, \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \rhd_\Sigma \psi \in G^{\mathsf{K}}_\Sigma(\{\phi_i \rhd_\Sigma \psi_i : i \in I\}) \quad \text{iff} \quad \phi \preccurlyeq \psi \in C^{\mathsf{K}}_\Sigma(\{\phi_i \preccurlyeq \psi_i : i \in I\}).$$

We call $\mathfrak{G}^{\mathsf{K}}$ the **inequational Gentzen π-institution associated with K**.

It turns out that, for every class K of $\mathbf{F}$-algebraic posystems, the associated inequational Gentzen π-institution $\mathfrak{G}^{\mathsf{K}}$ is syntactically protoalgebraic.

Theorem 1925 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic posystems. Then* $\mathfrak{G}^{\mathsf{K}} = \langle \mathbf{F}, G^{\mathsf{K}} \rangle$ *is syntactically protoalgebraic.*

Proof: Consider $I = \{I^{\langle 1,1 \rangle}\}$, where $I^{\langle 1,1 \rangle} : (\mathrm{SEN}^\flat)^4 \to (\mathrm{SEN}^\flat)^2$ is given, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \phi', \psi' \in \mathrm{SEN}^\flat(\Sigma)$, by

$$I^{\langle 1,1 \rangle}_\Sigma[\langle \phi, \psi \rangle, \langle \phi', \psi' \rangle] = \{\phi \rhd_\Sigma \phi',\ \phi' \rhd_\Sigma \phi\ \psi \rhd_\Sigma \psi',\ \psi' \rhd_\Sigma \psi\}.$$

Then, we have, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G}^{\mathsf{K}}$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \rhd_\Sigma \psi \in \mathrm{Seq}^{\{\langle 1,1 \rangle\}}_\Sigma(\mathbf{F})$,

$$\begin{array}{lll}\langle \phi, \phi' \rangle, \langle \psi, \psi' \rangle \in \Omega_\Sigma(\boldsymbol{T}) & \text{iff} & \{\phi \rhd_\Sigma \phi',\ \phi' \rhd_\Sigma \phi,\ \psi \rhd_\Sigma \psi',\ \psi' \rhd_\Sigma \psi\} \subseteq \boldsymbol{T}_\Sigma\\ & \text{iff} & I^{\langle 1,1 \rangle}_\Sigma[\phi \rhd_\Sigma \psi, \phi' \rhd_\Sigma \psi'] \subseteq \boldsymbol{T}_\Sigma.\end{array}$$

Therefore, $\mathfrak{G}^{\mathsf{K}}$ is syntactically protoalgebraic, with witnessing transformations I. ■

Note, also, how $I^{\langle 1,1\rangle}$ satisfies the modus ponens property in $\mathfrak{G}^{\mathsf{K}}$, i.e., for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \phi', \psi' \in \mathrm{SEN}^\flat(\Sigma)$,

$$\phi' \rhd_\Sigma \psi' \in G^{\mathsf{K}}_\Sigma(\phi \rhd_\Sigma \psi, I^{\langle 1,1\rangle}_\Sigma[\phi \rhd_\Sigma \psi, \phi' \rhd_\Sigma \psi']).$$

We now show that, if the class K happens to be an order guasivariety of $\mathbf{F}$-algebraic posystems, then the Leibniz reduced $\mathfrak{G}^{\mathsf{K}}$-matrix families coincide with the class K.

Proposition 1926 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* K *an ordered guasivariety of* $\mathbf{F}$*-algebraic posystems. Then* $\mathrm{MatFam}^*(\mathfrak{G}^{\mathsf{K}}) = \mathsf{K}$.

Proof: Suppose $\langle \mathcal{A}, \leq\rangle \in \mathsf{K}$ and let $\Sigma \in |\mathbf{Sign}|$ $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi\rangle \in \Omega^{\mathcal{A}}_\Sigma(\leq)$. Since $\phi \leq_\Sigma \phi$, we get, by compatibility of $\Omega^{\mathcal{A}}(\leq)$ with $\leq$, that $\phi \leq_\Sigma \psi$ and $\psi \leq_\Sigma \phi$. Thus, since $\leq$ is a posystem on $\mathcal{A}$ and, therefore, antisymmetric, we get that $\phi = \psi$. Hence, $\Omega^{\mathcal{A}}(\leq) = \Delta^{\mathcal{A}}$. We conclude that $\langle \mathcal{A}, \leq\rangle \in \mathrm{MatFam}^*(\mathfrak{G}^{\mathsf{K}})$. Thus, $\mathsf{K} \subseteq \mathrm{MatFam}^*(\mathfrak{G}^{\mathsf{K}})$.

Suppose, conversely, that $\langle \mathcal{A}, \leq\rangle \in \mathrm{MatFam}^*(\mathfrak{G}^{\mathsf{K}})$. Then $\Omega^{\mathcal{A}}(\leq) = \Delta^{\mathcal{A}}$. Since K is a class of $\mathbf{F}$-algebraic posystems, we get that, for all σ, τ in $N^\flat$, all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, all $f \in |\mathbf{Sign}(\Sigma, \Sigma')$ and all $\phi, \psi, \vec{\chi} \in \mathrm{SEN}(\Sigma')$,

$$\begin{aligned}&\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi}) \leq_{\Sigma'} \tau^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi), \vec{\chi})\\ &\in G^{\mathsf{K},\mathcal{A}}_\Sigma(\phi \leq_\Sigma \psi, \psi \leq_\Sigma \phi, \sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi}) \leq_{\Sigma'} \tau^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \vec{\chi})).\end{aligned}$$

Therefore, if $\phi \leq_\Sigma \psi$ and $\psi \leq_\Sigma \phi$, then we get that $\langle \phi, \psi\rangle \in \Omega^{\mathcal{A}}_\Sigma(\leq) = \Delta^{\mathcal{A}}_\Sigma$, i.e., that $\phi = \psi$. Therefore, $\leq$ is antisymmetric, i.e., $\langle \mathcal{A}, \leq\rangle \in \mathbb{GO}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$. We conclude that $\mathrm{MatFam}^*(\mathfrak{G}^{\mathsf{K}}) \subseteq \mathsf{K}$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$. $\mathfrak{G}$ is **order algebraizable** if it is equivalent to the inequational Gentzen π-institution $\mathfrak{G}^{\mathsf{K}}$ associated with some class K of $\mathbf{F}$-algebraic posystems.

Order algebraizability implies protoalgebraicity.

Proposition 1927 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *is order algebraizable, then it is protoalgebraic.*

Proof: Suppose that $\mathfrak{G}$ is equivalent to $\mathfrak{G}^{\mathsf{K}}$, for some class K of $\mathbf{F}$-algebraic posystems. Then, since, by Theorem 1925, $\mathfrak{G}^{\mathsf{K}}$ is syntactically protoalgebraic, it is, by Theorem 1914, protoalgebraic. Therefore, by Theorem 1912, $\mathfrak{G}$ is protoalgebraic as well. ■

The following result provides a characterization of order algebraizability.

Theorem 1928 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$. $\mathfrak{G}$ *is order algebraizable if and only if there exist a* tr-$\{\langle 1,1\rangle\}$*-transformation* τ *and an* $\{\langle 1,1\rangle\}$-tr*-transformation* ρ*, such that, for all* σ, σ' *in* $N^\flat$*, all* $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$*, all* $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$*, all* $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$*, all* $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$*, all* $\{\phi_i, \psi_i : i \in I\} \subseteq \mathrm{SEN}^\flat(\Sigma)$*, and all* $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$*:*

(1) $\rho_\Sigma[\phi, \phi] \subseteq \mathrm{Thm}_\Sigma(\mathfrak{G})$*;*

(2) $\rho_\Sigma[\phi, \chi] \subseteq G_\Sigma(\rho_\Sigma[\phi, \psi], \rho_\Sigma[\psi, \chi])$*;*

(3) $\rho_\Sigma[\sigma_\Sigma(\psi, \vec{\chi}), \sigma'_\Sigma(\psi, \vec{\chi})]$
$\subseteq G_\Sigma(\rho_\Sigma[\phi, \psi], \rho_\Sigma[\psi, \phi], \rho_\Sigma[\sigma_\Sigma(\phi, \vec{\chi}), \sigma'_\Sigma(\phi, \vec{\chi})])$*;*

(4) $\rho_\Sigma[\phi, \psi] \subseteq G_\Sigma(\bigcup_{i \in I} \rho_\Sigma[\phi_i, \psi_i])$ *implies*

$$\rho_{\Sigma'}[\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(\psi)] \subseteq G_{\Sigma'}(\bigcup_{i \in I} \rho_{\Sigma'}[\mathrm{SEN}^\flat(f)(\phi_i), \mathrm{SEN}^\flat(f)(\psi_i)]);$$

(5) $G_\Sigma(\boldsymbol{\phi}) = G_\Sigma(\rho_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]])$.

Proof: Suppose, first, that $\mathfrak{G}$ is order algebraizable. Then there exist τ and ρ as postulated and a class K of $\mathbf{F}$-algebraic posystems, such that $\mathfrak{G}$ is equivalent to $\mathfrak{G}^{\mathsf{K}}$ via the conjugate pair $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}}$. Since, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \rhd_\Sigma \phi \in \mathrm{Thm}_\Sigma(\mathfrak{G}^{\mathsf{K}})$, we get that $\rho_\Sigma[\phi, \phi] \subseteq \mathrm{Thm}_\Sigma(\mathfrak{G})$. So Condition (1) holds. Since, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \rhd_\Sigma \chi \in G^{\mathsf{K}}_\Sigma(\phi \rhd \psi, \psi \rhd_\Sigma \chi)$, we get that

$$\rho_\Sigma[\phi, \chi] \subseteq G_\Sigma(\rho_\Sigma[\phi, \psi], \rho_\Sigma[\psi, \chi]).$$

Hence, Condition (2) is also satisfied. If, for some $\langle \mathcal{A}, \leq \rangle \in \mathsf{K}$, we have, for some $\Sigma \in |\mathbf{Sign}|$ and some $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $\phi \leq_\Sigma \psi$ and $\psi \leq_\Sigma \phi$, then, since K is a class of $\mathbf{F}$-algebraic posystems, we get that $\phi = \psi$. Hence, it follows that, if, for σ, σ' in $N^\flat$, and $\vec{\chi} \in \mathrm{SEN}(\Sigma)$, $\sigma^{\mathcal{A}}_\Sigma(\phi, \vec{\chi}) \leq_\Sigma \sigma'^{\mathcal{A}}_\Sigma(\phi, \vec{\chi})$, then, we will also have $\sigma^{\mathcal{A}}_\Sigma(\psi, \vec{\chi}) \leq_\Sigma \sigma'^{\mathcal{A}}_\Sigma(\psi, \vec{\chi})$. In other words, we get that, for all σ, σ' in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$, and all $\phi, \psi, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\sigma_\Sigma(\psi, \vec{\chi}) \rhd_\Sigma \sigma'_\Sigma(\psi, \vec{\chi}) \in G^{\mathsf{K}}_\Sigma(\phi \rhd_\Sigma \psi, \psi \rhd_\Sigma \phi, \sigma_\Sigma(\phi, \vec{\chi}) \rhd_\Sigma \sigma'_\Sigma(\phi, \vec{\chi})).$$

Again, by applying ρ we get that Condition (3) holds. Suppose, now, that for some $\Sigma \in |\mathbf{Sign}^\flat|$ and $\{\phi_i, \psi_i : i \in I\} \cup \{\phi, \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$, $\rho_\Sigma[\phi, \psi] \subseteq G_\Sigma(\bigcup_{i \in I} \rho_\Sigma[\phi_i, \psi_i])$. Then, we get $\phi \rhd_\Sigma \psi \in G^{\mathsf{K}}_\Sigma(\{\phi_i \rhd_\Sigma \psi_i : i \in I\})$. Therefore, since $\mathfrak{G}^{\mathsf{K}}$ is structural, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$,

$$\mathrm{SEN}^\flat(f)(\phi \rhd_\Sigma \psi) \in G^{\mathsf{K}}_{\Sigma'}(\{\mathrm{SEN}^\flat(f)(\phi_i \rhd_\Sigma \psi_i) : i \in I\}).$$

By applying ρ again, we get that Condition (4) holds. Finally, Condition (5) holds directly by the definition of equivalence.

Assume, conversely, that ρ and τ, as postulated in the statement, exist and that they satisfy Conditions (1)-(5). Define $\mathfrak{G}' = \langle \mathbf{F}, G' \rangle$ of trace $\{\langle 1,1 \rangle\}$ by setting, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $\{\phi_i, \psi_i : i \in I\} \cup \{\phi, \psi\} \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\phi \rhd_\Sigma \psi \in G'_\Sigma(\{\phi_i \rhd_\Sigma \psi_i : i \in I\}) \quad \text{iff} \quad \rho_\Sigma[\phi, \psi] \subseteq G_\Sigma(\bigcup\{\rho_\Sigma[\phi_i, \psi_i] : i \in I\}).$$

Then, by the fact that $\mathfrak{G}$ is a Gentzen π-institution and Property (4), we get that $\mathfrak{G}'$ is also a Gentzen π-institution. Moreover, by its definition and Condition (5), taking into account Lemma 1879, $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ is an equivalence. Thus, it suffices to show that $\mathfrak{G}' = \mathfrak{G}^{\mathsf{K}}$, for some class K of $\mathbf{F}$-algebraic posystems. For this, in turn, it suffices, by Theorem 1901, to show that $\mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G}')$ is a class of $\mathbf{F}$-algebraic posystems.

Note, first, that $I = \{I^{\langle 1,1 \rangle}\}$, defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \phi', \psi' \in \mathrm{SEN}^\flat(\Sigma)$,

$$I^{\langle 1,1 \rangle}_\Sigma[\langle \phi, \psi \rangle, \langle \phi', \psi' \rangle] \coloneqq \{\phi \rhd_\Sigma \phi',\ \phi' \rhd_\Sigma \phi,\ \psi \rhd_\Sigma \psi',\ \psi' \rhd_\Sigma \psi\}$$

is a subset of $R^{\mathfrak{G}'}$, which, by Condition (2) and the definition of $\mathfrak{G}'$ satisfies the Modus Ponens in $\mathfrak{G}'$. Therefore, by Theorem 1918, $\mathfrak{G}'$ is syntactically protoalgebraic and, hence, by Theorem 1914, it is protoalgebraic. Thus, by Lemma 1899, the Leibniz and the Suszko operator coincide. Moreover, by Conditions (1) and (2) and the definition of $\mathfrak{G}'$, for all $\langle \mathcal{A}, \leq \rangle \in \mathrm{MatFam}(\mathfrak{G}')$, the relation family $\leq$ is reflexive and transitive. Also, by Condition (3) and the definition of $\mathfrak{G}'$, we get that, for all $\langle \mathcal{A}, \leq \rangle \in \mathrm{MatFam}(\mathfrak{G}')$, all σ, σ' in $N^\flat$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi, \vec{\chi} \in \mathrm{SEN}(\Sigma)$,

$$\phi \leq_\Sigma \psi,\ \psi \leq_\Sigma \phi,\ \sigma^{\mathcal{A}}_\Sigma(\phi, \vec{\chi}) \leq_\Sigma \sigma'^{\mathcal{A}}_\Sigma(\phi, \vec{\chi}) \text{ imply } \sigma^{\mathcal{A}}_\Sigma(\psi, \vec{\chi}) \leq_\Sigma \sigma'^{\mathcal{A}}_\Sigma(\psi, \vec{\chi}).$$

We finish the proof by showing that, for all $\langle \mathcal{A}, \leq \rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G}')$, $\leq$ is also antisymmetric. To this end, let $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\phi \leq_\Sigma \psi$ and $\psi \leq_\Sigma \phi$. Then, by Property (4) and the definition of $\mathfrak{G}'$, we get that, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}(f)(\phi) \leq_{\Sigma'} \mathrm{SEN}(f)(\psi) \text{ and } \mathrm{SEN}(f)(\psi) \leq_{\Sigma'} \mathrm{SEN}(f)(\phi).$$

Then, by what was shown above, we have, for all σ, σ' in $N^\flat$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')$,

$$\begin{aligned} &\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi}) \leq_{\Sigma'} \sigma'^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi}) \\ &\qquad \text{iff} \quad \sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}) \leq_{\Sigma'} \sigma'^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}). \end{aligned}$$

Therefore, by Corollary 1896, we get $\langle \phi, \psi \rangle \in \Omega^{\mathcal{A}}_\Sigma(\leq) = \widetilde{\Omega}^{\mathfrak{G}', \mathcal{A}}_\Sigma(\leq) = \Delta^{\mathcal{A}}_\Sigma$. We conclude that $\langle \mathcal{A}, \leq \rangle$ is indeed an $\mathbf{F}$-algebraic posystem. Hence, $\mathfrak{G}'$ is an inequational Gentzen π-institution associated with the class $\mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G}')$ of $\mathbf{F}$-algebraic posystems and, as a consequence, the Gentzen π-institution $\mathfrak{G}$ is indeed order algebraizable. ∎

Specializing Theorem 1928 to the case of Hilbert π-institutions, we get

Corollary 1929 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathfrak{H} = \langle \mathbf{F}, H \rangle$ *a Hilbert* π*-institution based on* $\mathbf{F}$*.* $\mathfrak{H}$ *is order algebraizable if and only if there exist a* $\{\langle 0,1\rangle\}$-$\{\langle 1,1\rangle\}$*-transformation* τ *and an* $\{\langle 1,1\rangle\}$-$\{\langle 0,1\rangle\}$*-transformation* ρ*, such that, for all* σ, σ' *in* $N^\flat$*, all* $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$*, all* $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$*, all* $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$ *all* $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$ *and all* $\{\phi_i, \psi_i : i \in I\} \subseteq \mathrm{SEN}^\flat(\Sigma)$*:*

(1) $\rho_\Sigma[\phi, \phi] \subseteq \mathrm{Thm}_\Sigma(\mathfrak{H})$*;*

(2) $\rho_\Sigma[\phi, \chi] \subseteq H_\Sigma(\rho_\Sigma[\phi, \psi], \rho_\Sigma[\psi, \chi])$*;*

(3) $$\rho_\Sigma[\sigma_\Sigma(\psi, \vec{\chi}), \sigma'_\Sigma(\psi, \vec{\chi})] \subseteq H_\Sigma(\rho_\Sigma[\phi, \psi], \rho_\Sigma[\psi, \phi], \rho_\Sigma[\sigma_\Sigma(\phi, \vec{\chi}), \sigma'_\Sigma(\phi, \vec{\chi})]);$$

(4) $\rho_\Sigma[\phi, \psi] \subseteq H_\Sigma(\bigcup_{i \in I} \rho_\Sigma[\phi_i, \psi_i])$ *implies*
$$\rho_{\Sigma'}[\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(\psi)] \subseteq H_{\Sigma'}(\bigcup_{i \in I} \rho_{\Sigma'}[\mathrm{SEN}^\flat(f)(\phi_i), \mathrm{SEN}^\flat(f)(\psi_i)]);$$

(5) $H_\Sigma(\triangleright_\Sigma \phi) = H_\Sigma(\rho_\Sigma[\tau_\Sigma[\phi]])$.

Proof: Directly from Theorem 1928. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$. $\mathfrak{G}$ is **simply order algebraizable** if it is equivalent to the inequational Gentzen π-institution $\mathfrak{G}^{\mathsf{K}}$, associated with some class K of $\mathbf{F}$-algebraic posystems, via a conjugate pair $(\tau, \rho^0) : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}}$, where, as before, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,
$$\rho^0_\Sigma(\phi; \psi) = \phi \triangleright_\Sigma \psi.$$

We have the following analog of Lemma 1823.

Lemma 1930 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *is simply order algebraizable via both* $(\tau, \rho^0) : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}}$ *and* $(\tau', \rho^0) : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}'}$*, then* $\mathbb{GO}^{\mathsf{Sem}}(\mathsf{K}) = \mathbb{GO}^{\mathsf{Sem}}(\mathsf{K}')$*.*

Proof: Suppose $\mathfrak{G}$ is simply order algebraizable via both $(\tau, \rho^0) : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}}$ and $(\tau', \rho^0) : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}'}$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $I \cup \{\phi \leq \psi\} \subseteq \mathrm{In}_\Sigma(\mathbf{F})$, we have
$$\begin{array}{lll} \phi \leq \psi \in G^{\mathsf{K}}_\Sigma(I) & \text{iff} & \rho^0_\Sigma[\phi; \psi] \subseteq G_\Sigma(\rho^0_\Sigma[I]) \\ & \text{iff} & \phi \leq \psi \in G^{\mathsf{K}'}_\Sigma(I). \end{array}$$
Thus, K and K' satisfy exactly the same $\mathbf{F}$-guasiinequations. ■

The unique order guasivariety K that simply order algebraizes a simply order algebraizable Gentzen π-institution $\mathfrak{G}$ is called the **order class of** $\mathfrak{G}$.

Specializing Theorem 1928, we get

Corollary 1931 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace, with* $\langle 1,1 \rangle \in$ tr*, and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*.* $\mathfrak{G}$ *is simply order algebraizable if and only if there exists a* tr*-*$\{\langle 1,1 \rangle\}$*-transformation* τ*, such that, for all* σ, σ' *in* $N^\flat$*, all* $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$*, all* $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$*, all* $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$*, all* $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$ *and all* $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$*:*

(1) $\phi \rhd_\Sigma \phi \in \mathrm{Thm}_\Sigma(\mathfrak{G})$*;*

(2) $\phi \rhd_\Sigma \chi \in G_\Sigma(\phi \rhd_\Sigma \psi, \psi \rhd_\Sigma \chi)$*;*

(3) $\sigma_\Sigma(\psi, \vec{\chi}) \rhd_\Sigma \sigma'_\Sigma(\psi, \vec{\chi}) \in G_\Sigma(\phi \rhd_\Sigma \psi, \psi \rhd_\Sigma \phi, \sigma_\Sigma(\phi, \vec{\chi}) \rhd_\Sigma \sigma'_\Sigma(\phi, \vec{\chi}))$*;*

(4) $G_\Sigma(\boldsymbol{\phi}) = G_\Sigma(\rho^0_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]])$*.*

Proof: Directly by Theorem 1928. ■

26.9 Truth Equationality

By Theorem 1901, the closure system G of a Gentzen π-institution $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ can be recovered by the class $\mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$ of its Suszko reduced matrix families. A related issue is to investigate when G can be recovered just from the class of underlying $\mathbf{F}$-algebraic systems of the class $\mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$, i.e., from the class $\mathrm{AlgSys}(\mathfrak{G})$. The algebraizability property of $\mathfrak{G}$ gives that

$$\mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G}) = \{\langle \mathcal{A}, \tau^{\mathcal{A}*}(\Delta^{\mathcal{A}}) : \mathcal{A} \in \mathrm{AlgSys}(\mathfrak{G})\},$$

where $\tau : \mathfrak{G} \to \mathfrak{G}^{\mathsf{K}}$ is the $\{\langle 1,1 \rangle\}$-tr-transformation witnessing the algebraizability. In this case, the $\mathbf{F}$-algebraic system $\mathcal{A} \in \mathrm{AlgSys}(\mathfrak{G})$ is the $\mathbf{F}$-algebraic system reduct of a unique Suszko reduced $\mathfrak{G}$-matrix family, i.e., the $\mathfrak{G}$-filter family of every Suszko reduced $\mathfrak{G}$-matrix family is uniquely determined by the $\mathbf{F}$-algebraic system $\mathcal{A}$, since it is exactly $\tau^{\mathcal{A}*}(\Delta^{\mathcal{A}})$ and this expression does not depend on the choice of τ witnessing the algebraizability of $\mathfrak{G}$.

Even in the absence of algebraizability, however, if each $\mathbf{F}$-algebraic system in $\mathrm{AlgSys}(\mathfrak{G})$ is the $\mathbf{F}$-algebraic system reduct of e unique Suszko reduced $\mathfrak{G}$-matrix family, then, there exists, modulo a technical condition, analogous to the adequacy of the Suszko core introduced in a preceding chapter, a $\{\langle 1,1 \rangle\}$-tr-transformation τ that determines the unique $\mathfrak{G}$-matrix filter on the $\mathbf{F}$-algebraic system, as described previously.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution os trace tr based on $\mathbf{F}$.

- $\mathfrak{G}$ is **completely reflective**, or **c-reflective** for short, if, for all $\boldsymbol{\mathcal{T}} \cup \{\boldsymbol{T}'\} \subseteq \mathrm{ThFam}(\mathfrak{G})$,

$$\bigcap_{\boldsymbol{T} \in \boldsymbol{\mathcal{T}}} \Omega^{\mathcal{A}}(\boldsymbol{T}) \leq \Omega^{\mathcal{A}}(\boldsymbol{T}') \quad \text{implies} \quad \bigcap \boldsymbol{\mathcal{T}} \leq \boldsymbol{T}';$$

- $\mathfrak{G}$ is **truth equational** if there exists $\tau = \{\tau^{\langle m,n\rangle} : \langle m,n\rangle \in \mathrm{tr}\}$, where $\tau^{\langle m,n\rangle} : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$, with $m+n$ distinguished arguments, such that, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m,n\rangle$,
$$\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma \quad \text{iff} \quad \tau^{\langle m,n\rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T}).$$

First, we provide a characterization of c-reflectivity.

Theorem 1932 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen π-institution os trace* tr *based on* $\mathbf{F}$*. Then, the following statements are equivalent:*

(i) For every $\mathcal{A} \in \mathrm{AlgSys}(\mathfrak{G})$*, there exists unique* $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$*, such that* $\langle \mathcal{A}, \boldsymbol{T}\rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$*;*

(ii) For every $\mathbf{F}$*-algebraic system* $\mathcal{A}$*, and all* $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$*,* $\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$ *is the least $\mathfrak{G}$-filter family on* $\mathcal{A}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$*;*

(iii) For every $\mathbf{F}$*-algebraic system* $\mathcal{A}$*,* $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}$ *is injective on* $\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$*;*

(iv) For every $\mathbf{F}$*-algebraic system* $\mathcal{A}$ *and all* $\boldsymbol{\mathcal{T}} \cup \{\boldsymbol{T}'\} \subseteq \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$*,*
$$\bigcap_{\boldsymbol{T}\in\boldsymbol{\mathcal{T}}} \Omega^{\mathcal{A}}(\boldsymbol{T}) \le \Omega^{\mathcal{A}}(\boldsymbol{T}') \quad \textit{implies} \quad \bigcap \boldsymbol{\mathcal{T}} \le \boldsymbol{T}';$$

(v) For all $\boldsymbol{\mathcal{T}} \cup \{\boldsymbol{T}'\} \subseteq \mathrm{ThFam}(\mathfrak{G})$*,* $\bigcap_{\boldsymbol{T}\in\boldsymbol{\mathcal{T}}} \Omega(\boldsymbol{T}) \le \Omega(\boldsymbol{T}')$ *implies* $\bigcap \boldsymbol{\mathcal{T}} \le \boldsymbol{T}'$*.*

Proof:

(i)⇒(ii) Suppose (i) holds and let $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$. Consider the algebraic system $\mathcal{A}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$ and let $\boldsymbol{T}'$ be the least filter on $\mathcal{A}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$. Then, since $\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}))$, we get that $\boldsymbol{T}' \le \boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$. Thus, by the monotonicity of the Suszko operator,
$$\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})}(\boldsymbol{T}') \le \widetilde{\Omega}^{\mathfrak{G},\mathcal{A}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})}(\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})) = \Delta^{\mathcal{A}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})}.$$
But, noting that $\mathcal{A}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) \in \mathrm{AlgSys}(\mathfrak{G})$, we get, by hypothesis, that $\boldsymbol{T}' = \boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$. Therefore, $\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$ is the least $\mathfrak{G}$-filter family on $\mathcal{A}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$.

(ii)⇒(iii) Suppose that (ii) holds and let $\mathcal{A} = \langle \mathbf{A}, \langle F,\alpha\rangle\rangle$ be an $\mathbf{F}$-algebraic system and $\boldsymbol{T}, \boldsymbol{T}' \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) = \widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}')$. Then $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$ is compatible with both $\boldsymbol{T}$ and $\boldsymbol{T}'$ and, hence, $\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$ and $\boldsymbol{T}'/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$ are both $\mathfrak{G}$-filter families on $\mathcal{A}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$. Thus, by hypothesis, $\boldsymbol{T}/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) \le \boldsymbol{T}'/\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$. Therefore, taking into account the compatibility of $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T})$ with both $\boldsymbol{T}$ and $\boldsymbol{T}'$, we get $\boldsymbol{T} \le \boldsymbol{T}'$. By symmetry, we also have $\boldsymbol{T}' \le \boldsymbol{T}$, whence $\boldsymbol{T} = \boldsymbol{T}'$. Thus, $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}$ is injective on $\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$.

(iii)⇒(iv) Suppose (iii) holds and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $\mathcal{T} \cup \{\boldsymbol{T}'\} \subseteq \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, such that

$$\bigcap_{\boldsymbol{T} \in \mathcal{T}} \Omega^{\mathcal{A}}(\boldsymbol{T}) \leq \Omega^{\mathcal{A}}(\boldsymbol{T}').$$

Then, we have

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\bigcap \mathcal{T} \cap \boldsymbol{T}') & = & \bigcap\{\Omega^{\mathcal{A}}(\boldsymbol{X}) : \mathcal{T} \cap \boldsymbol{T}' \leq \boldsymbol{X} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\} \\ & = & \bigcap\{\Omega^{\mathcal{A}}(\boldsymbol{X}) : \bigcap \mathcal{T} \leq \boldsymbol{X} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\} \\ & = & \widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\bigcap \mathcal{T}).\end{array}$$

By hypothesis, we get $\bigcap \mathcal{T} \cap \boldsymbol{T}' = \bigcap \mathcal{T}$, whence $\bigcap \mathcal{T} \leq \boldsymbol{T}'$.

(iv)⇒(v) Condition (v) is a special case of Condition (iv).

(v)⇒(i) Assume that (v) holds and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle \in \mathrm{AlgSys}(\mathfrak{G})$ and $\boldsymbol{T}, \boldsymbol{T}' \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) = \widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}') = \Delta^{\mathcal{A}}$. By Lemma 1891, $\alpha^{-1}(\boldsymbol{T})$ and $\alpha^{-1}(\boldsymbol{T}')$ are both theory families of $\mathfrak{G}$. Now we have, by hypothesis, $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}) = \widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}(\boldsymbol{T}')$, whence, by the definition of the Suszko operator,

$$\bigcap\{\Omega^{\mathcal{A}}(\boldsymbol{X}) : \boldsymbol{T} \leq \boldsymbol{X} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\} \leq \Omega^{\mathcal{A}}(\boldsymbol{T}').$$

Hence, applying α^{-1} to both sides,

$$\alpha^{-1}(\bigcap\{\Omega^{\mathcal{A}}(\boldsymbol{X}) : \boldsymbol{T} \leq \boldsymbol{X} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\}) \leq \alpha^{-1}(\Omega^{\mathcal{A}}(\boldsymbol{T}')).$$

Equivalently,

$$\bigcap\{\alpha^{-1}(\Omega^{\mathcal{A}}(\boldsymbol{X})) : \boldsymbol{T} \leq \boldsymbol{X} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\} \leq \alpha^{-1}(\Omega^{\mathcal{A}}(\boldsymbol{T}')).$$

By Lemma 1909,

$$\bigcap\{\Omega(\alpha^{-1}(\boldsymbol{X})) : \boldsymbol{T} \leq \boldsymbol{X} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\} \leq \Omega(\alpha^{-1}(\boldsymbol{T}')).$$

By Condition (v),

$$\bigcap\{\alpha^{-1}(\boldsymbol{X}) : \boldsymbol{T} \leq \boldsymbol{X} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\} \leq \alpha^{-1}(\boldsymbol{T}').$$

Hence, $\alpha^{-1}(\boldsymbol{T}) \leq \alpha^{-1}(\boldsymbol{T}')$, which gives, by the surjectivity of $\langle F, \alpha \rangle$, $\boldsymbol{T} \leq \boldsymbol{T}'$. By symmetry, we get that $\boldsymbol{T} = \boldsymbol{T}'$ and, therefore, there exists only one $\mathfrak{G}$-filter family $\boldsymbol{T}$ on $\mathcal{A}$, such that $\langle \mathcal{A}, \boldsymbol{T} \rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$. ■

It also turns out that a sufficient condition for the c-reflectivity of a Gentzen π-institution $\mathfrak{G}$ is the injectivity of the Suszko operator on all $\mathbf{F}$-algebraic systems in $\mathrm{AlgSys}(\mathfrak{G})$.

Lemma 1933 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. If, for all* $\mathcal{A} \in \mathrm{AlgSys}(\mathfrak{G})$*,* $\widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}$ *is injective on* $\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$*, then* $\mathfrak{G}$ *is c-reflective.*

Proof: By the hypothesis, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathfrak{G})$, there exists a unique $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, such that $\langle \mathcal{A}, \boldsymbol{T}\rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$. Therefore, by Theorem 1932, $\mathfrak{G}$ is c-reflective. ∎

Next we provide an alternative characterization of truth equationality, forming an analog of Theorem 818.

Theorem 1934 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*.* $\mathfrak{G}$ *is truth equational if and only if, there exists* $\tau = \{\tau^{\langle m,n\rangle} : \langle m,n\rangle \in \mathrm{tr}\}$*, where* $\tau^{\langle m,n\rangle} : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2$*, with* $m+n$ *distinguished arguments, such that, for all* $\langle \mathcal{A}, \boldsymbol{T}\rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$*,* $\boldsymbol{T} = \tau^{\mathcal{A}*}(\Delta^{\mathcal{A}})$*.*

Proof: Suppose $\mathfrak{G}$ is truth equational, with witnessing transformations $\tau = \{\tau^{\langle m,n\rangle} : \langle m,n\rangle \in \mathrm{tr}\}$. Let $\langle \mathcal{A}, \boldsymbol{T}\rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m,n\rangle$. Then we have

$$\begin{array}{lll}
\alpha_\Sigma(\boldsymbol{\phi}) \in \boldsymbol{T}_{F(\Sigma)} & \text{iff} & \alpha_\Sigma(\boldsymbol{\phi}) \in \boldsymbol{T}'_{F(\Sigma)}, \text{ all } \boldsymbol{T} \le \boldsymbol{T}' \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\\
& \text{iff} & \boldsymbol{\phi} \in \alpha_\Sigma^{-1}(\boldsymbol{T}'_{F(\Sigma)}), \text{ all } \boldsymbol{T} \le \boldsymbol{T}' \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\\
& \text{iff} & \tau_\Sigma^{\langle m,n\rangle}[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\alpha^{-1}(\boldsymbol{T}')), \text{ all } \boldsymbol{T} \le \boldsymbol{T}' \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\\
& \text{iff} & \tau_\Sigma^{\langle m,n\rangle}[\boldsymbol{\phi}] \subseteq \alpha_\Sigma^{-1}(\Omega^{\mathcal{A}}_{F(\Sigma)}(\boldsymbol{T}')), \text{ all } \boldsymbol{T} \le \boldsymbol{T}' \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\\
& \text{iff} & \alpha_\Sigma(\tau_\Sigma^{\langle m,n\rangle}[\boldsymbol{\phi}]) \subseteq \Omega^{\mathcal{A}}_{F(\Sigma)}(\boldsymbol{T}'), \text{ all } \boldsymbol{T} \le \boldsymbol{T}' \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})\\
& \text{iff} & \tau_{F(\Sigma)}^{\mathcal{A},\langle m,n\rangle}[\alpha_\Sigma(\boldsymbol{\phi})] \subseteq \widetilde{\Omega}^{\mathfrak{G},\mathcal{A}}_{F(\Sigma)}(\boldsymbol{T})\\
& \text{iff} & \tau_{F(\Sigma)}^{\mathcal{A},\langle m,n\rangle}[\alpha_\Sigma(\boldsymbol{\phi})] \subseteq \Delta^{\mathcal{A}}_{F(\Sigma)}.
\end{array}$$

The conclusion follows by taking into account the surjectivity of $\langle F, \alpha\rangle$.

Conversely, assume that the condition in the statement holds and let $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m,n\rangle$. Then, since

$$\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}/\Omega(\boldsymbol{T})}(\boldsymbol{T}/\Omega(\boldsymbol{T})) \le \Omega^{\mathcal{F}/\Omega(\boldsymbol{T})}(\boldsymbol{T}/\Omega * \boldsymbol{T}) = \Delta^{\mathcal{F}/\Omega(\boldsymbol{T})},$$

we get that $\langle \mathcal{F}/\Omega(\boldsymbol{T}), \boldsymbol{T}/\Omega(\boldsymbol{T})\rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$. Therefore, by hypothesis,

$$\boldsymbol{\phi}/\Omega_\Sigma(\boldsymbol{T}) \in \boldsymbol{T}_\Sigma/\Omega_\Sigma(\boldsymbol{T}) \quad \text{iff} \quad \tau_\Sigma^{\mathcal{F}/\Omega(\boldsymbol{T}),\langle m,n\rangle}[\boldsymbol{\phi}/\Omega_\Sigma(\boldsymbol{T})] \subseteq \Delta_\Sigma^{\mathcal{F}/\Omega(\boldsymbol{T})},$$

i.e.,

$$\boldsymbol{\phi}/\Omega_\Sigma(\boldsymbol{T}) \in \boldsymbol{T}_\Sigma/\Omega_\Sigma(\boldsymbol{T}) \quad \text{iff} \quad \tau_\Sigma^{\langle m,n\rangle}[\boldsymbol{\phi}]/\Omega_\Sigma(\boldsymbol{T}) \subseteq \Delta_\Sigma^{\mathcal{F}/\Omega(\boldsymbol{T})}.$$

By the compatibility of $\Omega(\boldsymbol{T})$ with $\boldsymbol{T}$, we now get

$$\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma \quad \text{iff} \quad \tau_\Sigma^{\langle m,n\rangle}[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T}).$$

Therefore, $\mathfrak{G}$ is truth equational. ∎

Before turning into a characterization of the exact relationship between c-reflectivity and truth equationality, we prove that both c-reflectivity and truth equationality are preserved under equivalence of Gentzen π-institutions.

Theorem 1935 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr, tr′ *traces and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$, $\mathfrak{G}' = \langle \mathbf{F}, G' \rangle$ *Gentzen π-institutions of traces* tr, tr′*, respectively, based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *and* $\mathfrak{G}'$ *are equivalent, then* $\mathfrak{G}$ *is c-reflective if and only if* $\mathfrak{G}'$ *is also.*

Proof: Suppose that $\mathfrak{G}'$ is c-reflective and let $\boldsymbol{\mathcal{T}} \cup \{\boldsymbol{T}'\} \subseteq \mathrm{ThFam}(\mathfrak{G})$, such that $\bigcap_{\boldsymbol{T} \in \boldsymbol{\mathcal{T}}} \Omega(\boldsymbol{T}) \leq \Omega(\boldsymbol{T}')$. Then, by Proposition 1897, $\bigcap_{\boldsymbol{T} \in \boldsymbol{\mathcal{T}}} \Omega(\rho^*(\boldsymbol{T})) \leq \Omega(\rho^*(\boldsymbol{T}'))$. Thus, by Theorem 1880 and the hypothesis, we get $\bigcap_{\boldsymbol{T} \in \boldsymbol{\mathcal{T}}} \rho^*(\boldsymbol{T}) \leq \rho^*(\boldsymbol{T}')$ and, then, $\rho^*(\bigcap \boldsymbol{\mathcal{T}}) \leq \rho^*(\boldsymbol{T}')$. As ρ^* is order reflecting, we conclude that $\bigcap \boldsymbol{\mathcal{T}} \leq \boldsymbol{T}'$ and, therefore, $\mathfrak{G}$ is c-reflective. The converse follows by the symmetry of the notion of equivalence. ∎

Theorem 1936 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr, tr′ *traces and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$, $\mathfrak{G}' = \langle \mathbf{F}, G' \rangle$ *Gentzen π-institutions of traces* tr, tr′*, respectively, based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *and* $\mathfrak{G}'$ *are equivalent, then* $\mathfrak{G}$ *is truth equational if and only if* $\mathfrak{G}'$ *is also.*

Proof: Suppose that $\mathfrak{G}$ and $\mathfrak{G}'$ are equivalent via a conjugate pair $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}'$ and that $\mathfrak{G}'$ is truth equational, with witnessing transformations $\sigma := \{\sigma^{\langle m,m \rangle} : \langle m, n \rangle \in \mathrm{tr}'\}$. Then, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m, n \rangle \in \mathrm{tr}$, we get, setting, according to Theorem 1880, $\boldsymbol{T}' \in \mathrm{ThFam}(\mathfrak{G}')$ be such that $\boldsymbol{T} \underset{\tau^*}{\overset{\rho^*}{\rightleftarrows}} \boldsymbol{T}'$,

$$\begin{array}{rll}
\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma & \text{iff} & \boldsymbol{\phi} \in \tau^*_\Sigma(\boldsymbol{T}') \quad \text{(definition of } \boldsymbol{T}'\text{)} \\
& \text{iff} & \tau_\Sigma[\boldsymbol{\phi}] \subseteq \boldsymbol{T}'_\Sigma \quad \text{(definition of } \tau^*\text{)} \\
& \text{iff} & \sigma_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]] \subseteq \Omega_\Sigma(\boldsymbol{T}') \quad \text{(hypothesis)} \\
& \text{iff} & \sigma_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]] \subseteq \Omega_\Sigma(\rho^*(\boldsymbol{T})) \quad \text{(definition of } \boldsymbol{T}^*\text{)} \\
& \text{iff} & \sigma_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]] \subseteq \Omega_\Sigma(\boldsymbol{T}). \quad \text{(Proposition 1897)}
\end{array}$$

Therefore, $\sigma \circ \tau$ witnesses the truth equationality of $\mathfrak{G}$. The converse follows by the symmetry of equivalence. ∎

We now turn to the investigation of the exact relationship between complete reflectivity and truth equationality. We will show that for a Gentzen π-institution to be truth equational, it must be c-reflective and, in addition satisfy a technical condition analogous to the adequacy of the Suszko core in the context of π-institutions, that ensures that there are enough natural transformations in its category of natural transformations to specify the Suszko operator in a precise sense.

We start by showing that truth equationality implies c-reflectivity.

Proposition 1937 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$. *if* $\mathfrak{G}$ *is truth equational, then it is c-reflective.*

Proof: Suppose $\mathfrak{G}$ is truth equational, with witnessing transformations $\tau = \{\tau^{\langle m,n\rangle} : \langle m,n\rangle \in \mathrm{tr}\}$, and let $\mathcal{T} \cup \{\boldsymbol{T}'\} \subseteq \mathrm{ThFam}(\mathfrak{G})$, such that $\bigcap_{\boldsymbol{T}\in\mathcal{T}} \Omega(\boldsymbol{T}) \leq \Omega(\boldsymbol{T}')$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m,n\rangle$, we have

$$\begin{array}{rcl} \boldsymbol{\phi} \in \bigcap_{\boldsymbol{T}\in\mathcal{T}} \boldsymbol{T}_\Sigma & \text{iff} & \boldsymbol{\phi} \in \boldsymbol{T}_\Sigma,\ \boldsymbol{T} \in \mathcal{T}, \\ & \text{iff} & \tau^{\langle m,n\rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T}),\ \boldsymbol{T} \in \mathcal{T}, \\ & \text{iff} & \tau^{\langle m,n\rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \bigcap_{\boldsymbol{T}\in\mathcal{T}} \Omega_\Sigma(\boldsymbol{T}) \\ & \text{implies} & \tau^{\langle m,n\rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T}') \\ & \text{iff} & \boldsymbol{\phi} \in \boldsymbol{T}'_\Sigma. \end{array}$$

Thus, $\bigcap \mathcal{T} \leq \boldsymbol{T}'$ and, hence, $\mathfrak{G}$ is c-reflective. ∎

The property of c-reflectivity also has a characterization involving both the Suszko and the Leibniz operator. Namely, we obtain

Lemma 1938 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$. $\mathfrak{G}$ *is c-reflective if and only if, for all* $\boldsymbol{T}, \boldsymbol{T}' \in \mathrm{ThFam}(\mathfrak{G})$,

$$\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}) \leq \Omega(\boldsymbol{T}') \quad \textit{implies} \quad \boldsymbol{T} \leq \boldsymbol{T}'.$$

Proof: Suppose, first, that $\mathfrak{G}$ is c-reflective and let $\boldsymbol{T}, \boldsymbol{T}' \in \mathrm{ThFam}(\mathfrak{G})$, such that $\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}) \leq \Omega(\boldsymbol{T}')$. Then, we get

$$\bigcap\{\Omega(\boldsymbol{X}) : \boldsymbol{T} \leq \boldsymbol{X} \in \mathrm{ThFam}(\mathfrak{G})\} \leq \Omega(\boldsymbol{T}').$$

Hence, by hypothesis, $\bigcap\{\boldsymbol{X} : \boldsymbol{T} \leq \boldsymbol{X} \in \mathrm{ThFam}(\mathfrak{G})\} \leq \boldsymbol{T}'$, i.e., $\boldsymbol{T} \leq \boldsymbol{T}'$.

Assume, conversely, that the condition of the statement holds and let $\mathcal{T} \cup \{\boldsymbol{T}'\} \subseteq \mathrm{ThFam}(\mathfrak{G})$, such that $\bigcap_{\boldsymbol{T}\in\mathcal{T}} \Omega(\boldsymbol{T}) \leq \Omega(\boldsymbol{T}')$. Then we get

$$\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\bigcap \mathcal{T}) \leq \bigcap\{\Omega(\boldsymbol{T}) : \boldsymbol{T} \in \mathcal{T}\} \leq \Omega(\boldsymbol{T}').$$

Thus, by hypothesis, $\bigcap \mathcal{T} \leq \boldsymbol{T}'$ and $\mathfrak{G}$ is c-reflective. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle \mathbf{F}, G\rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$. Define the **Suszko core**

$$S^{\mathfrak{G}} = \{S^{\mathfrak{G},\langle m,n\rangle} : \langle m,n\rangle \in \mathrm{tr}\}$$

of $\mathfrak{G}$, by setting, for all $\langle m,n\rangle \in \mathrm{tr}$,

$$\begin{array}{rl} S^{\mathfrak{G},\langle m,n\rangle} = \{\sigma : (\mathrm{SEN}^\flat)^\omega \to (\mathrm{SEN}^\flat)^2 \in N^\flat : & \\ & (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \boldsymbol{\phi} \in \mathrm{Seq}^{\{\langle m,n\rangle\}}_\Sigma(\mathbf{F})) \\ & \qquad (\sigma_\Sigma[\boldsymbol{\phi}] \subseteq \widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}_\Sigma(G(\boldsymbol{\phi})))\}. \end{array}$$

$S^{\mathfrak{G}}$ is a set of natural candidates from which to seek witnesses for the truth equationality of $\mathfrak{G}$, if such exist, since it satisfies the following property.

Lemma 1939 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *is truth equational, with witnessing transformations* τ*, then* $\tau \subseteq S^{\mathfrak{G}}$*.*

Proof: Suppose $\mathfrak{G}$ is truth equational, with witnessing transformations $\tau = \{\tau^{\langle m,n \rangle} : \langle m,n \rangle \in \mathrm{tr}\}$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ of trace $\langle m,n \rangle$, we have $\boldsymbol{\phi} \in G_\Sigma(\boldsymbol{\phi})$, whence, $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$, for all $G(\boldsymbol{\phi}) \leq \boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$. Thus, by truth equationality, $\tau_\Sigma^{\langle m,n \rangle}[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T})$ and, therefore, $\tau_\Sigma^{\langle m,n \rangle}[\boldsymbol{\phi}] \subseteq \widetilde{\Omega}_\Sigma^{\mathfrak{G},\mathcal{F}}(G(\boldsymbol{\phi}))$. We conclude that $\tau^{\langle m,n \rangle} \subseteq S^{\mathfrak{G},\langle m,n \rangle}$. ■

The Suszko core of $\mathfrak{G}$ always carries a theory family $\boldsymbol{T}$ of $\mathfrak{G}$ into the Leibniz congruence system $\Omega(\boldsymbol{T})$ of the theory family $\boldsymbol{T}$.

Proposition 1940 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. For all* $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$ *of trace* $\langle m,n \rangle \in \mathrm{tr}$*,*

$$S_\Sigma^{\mathfrak{G},\langle m,n \rangle}[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T}).$$

Proof: Let $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$ of trace $\langle m,n \rangle \in \mathrm{tr}$. Then, by the definition of $S^{\mathfrak{G}}$, we get

$$S_\Sigma^{\mathfrak{G},\langle m,n \rangle}[\boldsymbol{\phi}] \subseteq \widetilde{\Omega}_\Sigma^{\mathfrak{G},\mathcal{F}}(G(\boldsymbol{\phi})) \subseteq \widetilde{\Omega}_\Sigma^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}) \subseteq \Omega_\Sigma(\boldsymbol{T}).$$

This establishes the conclusion. ■

The converse property, which does not always hold, is called *solubility of the Suszko core.*

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$. $S^{\mathfrak{G}}$ is **soluble** if, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ of trace $\langle m,n \rangle \in \mathrm{tr}$, we get

$$S_\Sigma^{\mathfrak{G},\langle m,n \rangle}[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T}) \quad \text{implies} \quad \boldsymbol{\phi} \in \boldsymbol{T}_\Sigma.$$

Truth equationality of a Gentzen π-institution guarantees the solubility of its Suszko core.

Theorem 1941 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *is truth equational, then the Suszko core* $S^{\mathfrak{G}}$ *is soluble.*

Proof: Suppose that $\mathfrak{G}$ is truth equational, with witnessing transformations τ, and let $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ of trace $\langle m,n \rangle$, such that $S_\Sigma^{\mathfrak{G},\langle m,n \rangle}[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T})$. Then, by Lemma 1939, $\tau_\Sigma^{\langle m,n \rangle}[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T})$. By truth equationality, $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$. Therefore, $S^{\mathfrak{G}}$ is indeed soluble. ■

Conversely, if the Suszko core of a given Gentzen π-institution $\mathfrak{G}$ is soluble, then it acts as a set of witnessing transformations for the truth equationality of $\mathfrak{G}$.

Theorem 1942 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$. *If* $S^{\mathfrak{G}}$ *is soluble, then* $\mathfrak{G}$ *is truth equational, with witnessing transformations* $S^{\mathfrak{G}}$.

Proof: We must show that, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ of trace $\langle m, n \rangle$,

$$\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma \quad \text{iff} \quad S_\Sigma^{\mathfrak{G},\langle m,n \rangle}[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T}).$$

The left to right implication is by Proposition 1940, whereas the reverse is by the hypothesis of the solubility of the Suszko core. ∎

Theorems 1941 and 1942 allow two characterizations of truth equationality in terms of the solubility of the Suszko core and in terms of the definability of theory families by the Suszko core.

$$\mathfrak{G} \text{ is Truth Equational} \longleftrightarrow S^{\mathfrak{G}} \text{ is Soluble.}$$

Theorem 1943 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$. $\mathfrak{G}$ *is truth equational if and only if its Suszko core* $S^{\mathfrak{G}}$ *is soluble.*

Proof: The "only if" by Theorem 1941. The "if" by Theorem 1942. ∎

We say that $S^{\mathfrak{G}}$ **defines theory families** if, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$ and all $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$,

$$\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma \quad \text{iff} \quad S_\Sigma^{\mathfrak{G},\langle m,n \rangle}[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T}).$$

Then we can show

$$\mathfrak{G} \text{ is Truth Equational} \longleftrightarrow S^{\mathfrak{G}} \text{ Defines Theory Families.}$$

Theorem 1944 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$. $\mathfrak{G}$ *is truth equational if and only if* $S^{\mathfrak{G}}$ *defines theory families.*

Proof: If $\mathfrak{G}$ is truth equational, then, by Theorem 1943, $S^{\mathfrak{G}}$ is soluble, whence it defines theory families. On the other hand, if $S^{\mathfrak{G}}$ defines theory families, then it is soluble and, hence, by Theorem 1943, $\mathfrak{G}$ is truth equational. ∎

We now know that truth equationality of a Gentzen π-institution is equivalent to the solubility property of its Suszko core. The solubility property implies another property, which, in accordance with our previous work on π-institutions, we call adequacy. It says, roughly speaking, that in a Gentzen π-institution the category of natural transformations is rich enough to determine Suszko congruence systems in terms of the Leibniz congruence systems that it selects by inclusion. This property arises in a natural way by considering the following result relating the Suszko core with both Suszko and Leibniz congruence systems of theory families generated by single sequents.

Proposition 1945 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. For all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ *of trace* $\langle m, n \rangle$*,*

$$\bigcap \{\Omega(\boldsymbol{T}) : S^{\mathfrak{G},\langle m,n \rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T})\} \leq \widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(G(\boldsymbol{\phi})).$$

Proof: Note that, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m, n \rangle$, we have

$$\begin{array}{lll} \boldsymbol{\phi} \in \boldsymbol{T}_\Sigma & \Rightarrow & S^{\mathfrak{G},\langle m,n \rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}_\Sigma(\boldsymbol{T}) \quad \text{(Suszko core)} \\ & \Rightarrow & S^{\mathfrak{G},\langle m,n \rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T}). \quad (\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}) \leq \Omega(\boldsymbol{T})) \end{array}$$

Therefore, we get

$$\begin{array}{lll} \bigcap\{\Omega(\boldsymbol{T}) : S^{\mathfrak{G},\langle m,n \rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T})\} & \leq & \bigcap\{\Omega(\boldsymbol{T}) : S^{\mathfrak{G},\langle m,n \rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}_\Sigma(\boldsymbol{T})\} \\ & \leq & \bigcap\{\Omega(\boldsymbol{T}) : \boldsymbol{\phi} \in \boldsymbol{T}_\Sigma\} \\ & = & \widetilde{\Omega}(G(\boldsymbol{\phi})). \end{array}$$

Thus, the displayed inclusion always holds. ∎

The reverse inclusion is not always guaranteed, but, when it holds, we say that the Suszko core of $\mathfrak{G}$ is adequate. As the name suggests, the property somehow conveys the idea that $S^{\mathfrak{G}}[\boldsymbol{\phi}]$ suffices to determine the theory families whose Leibniz congruence systems form a covering of the Suszko congruence system corresponding to the theory family $G(\boldsymbol{\phi})$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$. The Suszko core $S^{\mathfrak{G}}$ is **adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m, n \rangle$,

$$\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}_\Sigma(G(\boldsymbol{\phi})) \leq \bigcap\{\Omega(\boldsymbol{T}) : S^{\mathfrak{G},\langle m,n \rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T})\}.$$

We can prove immediately that solubility implies adequacy.

Proposition 1946 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. If the Suszko core* $S^{\mathfrak{G}}$ *is soluble, then it is adequate.*

Proof: Suppose $S^{\mathfrak{G}}$ is soluble. We have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m, n \rangle$,

$$\begin{array}{lll} \widetilde{\Omega}^{\mathfrak{G},\mathcal{F}}(G(\boldsymbol{\phi})) & = & \bigcap\{\Omega(\boldsymbol{T}) : \boldsymbol{\phi} \in \boldsymbol{T}_\Sigma\} \\ & & \text{(Suszko congrunece system)} \\ & = & \bigcap\{\Omega(\boldsymbol{T}) : S^{\mathfrak{G},\langle m,n \rangle}_\Sigma[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T})\}. \\ & & \text{(solubility of } S^{\mathfrak{G}}) \end{array}$$

Hence, the Suszko core of $\mathfrak{G}$ is adequate. ∎

Conversely, if a Gentzen π-institution is c-reflective, then the adequacy of its Suszko core is sufficient to give its solubility.

Proposition 1947 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *is c-reflective and the Suszko core* $S^{\mathfrak{G}}$ *is adequate, then* $S^{\mathfrak{G}}$ *is soluble.*

Proof: Assume $\mathfrak{G}$ is c-reflective and $S^{\mathfrak{G}}$ is adequate. Let $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ of trace $\langle m, n \rangle$.

If $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$, then, by the definition of the Suszko core, we get

$$S_\Sigma^{\mathfrak{G},\langle m,n \rangle}[\boldsymbol{\phi}] \subseteq \widetilde{\Omega}_\Sigma^{\mathfrak{G},\mathcal{F}}(G(\boldsymbol{\phi})) \subseteq \widetilde{\Omega}_\Sigma^{\mathfrak{G},\mathcal{F}}(\boldsymbol{T}) \subseteq \Omega_\Sigma(\boldsymbol{T}).$$

Assume conversely, that $S_\Sigma^{\mathfrak{G},\langle m,n \rangle}[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T})$. Then, by adequacy of the Suszko core, $\widetilde{\Omega}^{\mathfrak{G},\mathfrak{F}}(G(\boldsymbol{\phi})) \leq \Omega(\boldsymbol{T})$. Hence, by c-reflectivity and Lemma 1938, $G(\boldsymbol{\phi}) \leq \boldsymbol{T}$, i.e., $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$. We conclude that $S^{\mathfrak{G}}$ is soluble. ■

We finally show that a Gentzen π-institution is truth equational if and only if it is c-reflective and has an adequate Suszko core.

Truth Equationality	=	$S^{\mathfrak{G}}$ Soluble
	=	$S^{\mathfrak{G}}$ Defines Theory Families
	=	c-Reflectivity + $S^{\mathfrak{G}}$ Adequate

Theorem 1948 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*.* $\mathfrak{G}$ *is truth equational if and only if it is c-reflective and has an adequate Suszko core.*

Proof: If $\mathfrak{G}$ is truth equational, then, by Proposition 1937, it is c-reflective, by Theorem 1941, its Suszko core is soluble and, by Proposition 1946, its Suszko core is adequate. On the other hand, if $\mathfrak{G}$ is c-reflective with an adequate Suszko core, then, by Proposition 1947, its Suszko core is soluble and, hence, by Theorem 1942, $\mathfrak{G}$ is truth equational. ■

We also obtain immediately

Corollary 1949 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a protoalgebraic Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*.* $\mathfrak{G}$ *is truth equational if and only if its Leibniz operator is injective on theory families and has an adequate Suszko core.*

Proof: If $\mathfrak{G}$ is truth equational, then, by Theorem 1948, it is c-reflective and has an adequate Suszko core, whence, it has, a fortiori, a Leibniz operator injective on theory families and an adequate Suszko core.

Conversely, by Theorem 1948, it suffices to show that monotonicity and injectivity of the Leibniz operator imply its c-reflectivity. In fact, given $\boldsymbol{T}, \boldsymbol{T}' \in \mathrm{ThFam}(\mathfrak{G})$, we have

$$\begin{array}{rcl}
\widetilde{\Omega}^{\mathfrak{G},\mathcal{G}}(\boldsymbol{T}) \leq \Omega(\boldsymbol{T}') & \Rightarrow & \Omega(\boldsymbol{T}) \leq \Omega(\boldsymbol{T}') \quad \text{(Protoalgebraicity)} \\
& \Rightarrow & \Omega(\boldsymbol{T} \cap \boldsymbol{T}') = \Omega(\boldsymbol{T}) \cap \Omega(\boldsymbol{T}') = \Omega(\boldsymbol{T}) \\
& & \text{(Protoalgebraicity)} \\
& \Rightarrow & \boldsymbol{T} \cap \boldsymbol{T}' = \boldsymbol{T} \quad \text{(Injectivity)} \\
& \Rightarrow & \boldsymbol{T} \leq \boldsymbol{T}'.
\end{array}$$

Thus, $\mathfrak{G}$ is c-reflective, by Lemma 1938. ■

We close the section by a result asserting that truth equationality transfers from a Gentzen π-institution $\mathfrak{G}$ to all $\mathfrak{G}$-matrix families.

Theorem 1950 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$. $\mathfrak{G}$ *is truth equational, with witnessing transformations* $\tau = \{\tau^{\langle m,n \rangle} : \langle m,n \rangle \in \mathrm{tr}\}$ *if and only if, for every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, and all* $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$*,* $\Sigma \in |\mathbf{Sign}|$ *and* $\boldsymbol{\phi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathcal{A})$ *of trace* $\langle m,n \rangle$*,*

$$\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma \quad \textit{iff} \quad \tau_\Sigma^{\mathcal{A},\langle m,n \rangle}[\boldsymbol{\phi}] \subseteq \Omega_\Sigma^{\mathcal{A}}(\boldsymbol{T}).$$

Proof: Suppose $\mathfrak{G}$ is truth equational, with witnessing transformations $\tau = \{\tau^{\langle m,n \rangle} : \langle m,n \rangle \in \mathrm{tr}\}$ and let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system, $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi} \in \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$ of trace $\langle m,n \rangle$. Then, we have

$$\begin{array}{rll} \alpha_\Sigma(\boldsymbol{\phi}) \in \boldsymbol{T}_{F(\Sigma)} & \text{iff} & \boldsymbol{\phi} \in \alpha_\Sigma^{-1}(\boldsymbol{T}_{F(\Sigma)}) \\ & \text{iff} & \tau_\Sigma^{\langle m,n \rangle}[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\alpha^{-1}(\boldsymbol{T})) \\ & \text{iff} & \tau_\Sigma^{\langle m,n \rangle}[\boldsymbol{\phi}] \subseteq \alpha_\Sigma^{-1}(\Omega_{F(\Sigma)}^{\mathcal{A}}(\boldsymbol{T})) \\ & \text{iff} & \alpha_\Sigma(\tau_\Sigma^{\langle m,n \rangle}[\boldsymbol{\phi}]) \subseteq \Omega_{F(\Sigma)}^{\mathcal{A}}(\boldsymbol{T}) \\ & \text{iff} & \tau_{F(\Sigma)}^{\mathcal{A},\langle m,n \rangle}[\alpha_\Sigma(\boldsymbol{\phi})] \subseteq \Omega_{F(\Sigma)}^{\mathcal{A}}(\boldsymbol{T}). \end{array}$$

Taking into account the surjectivity of $\langle F, \alpha \rangle$, we have the conclusion. ■

26.10 Weak Algebraizability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, tr a trace and $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ a Gentzen π-institution of trace tr based on $\mathbf{F}$. $\mathfrak{G}$ is called **WF algebraizable** if it is protoalgebraic and c-reflective.

Proposition 1951 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen π-institution of trace* tr *based on* $\mathbf{F}$. $\mathfrak{G}$ *is WF algebraizable if and only if the Leibniz operator is monotone and injective on* $\mathrm{ThFam}(\mathfrak{G})$.

Proof: It suffices to show that, under monotonicity, c-reflectivity and injectivity are equivalent properties. Indeed, c-reflectivity always implies injectivity because it implies order reflectivity. On the other hand, suppose that the Leibniz operator is monotone and injective. Then, we have, by monotonicity, for all $\mathcal{T} \cup \{\boldsymbol{T}\} \subseteq \mathrm{ThFam}(\mathfrak{G})$, such that $\bigcap_{\boldsymbol{T} \in \mathcal{T}} \Omega(\boldsymbol{T}) \leq \Omega(\boldsymbol{T}')$,

$$\Omega(\bigcap \mathcal{T} \cap \boldsymbol{T}') = \bigcap_{\boldsymbol{T} \in \mathcal{T}} \Omega(\boldsymbol{T}) \cap \Omega(\boldsymbol{T}') = \bigcap_{\boldsymbol{T} \in \mathcal{T}} \Omega(\boldsymbol{T}) = \Omega(\bigcap \mathcal{T}).$$

Thus, by injectivity, $\bigcap \mathcal{T} \cap \boldsymbol{T}' = \bigcap \mathcal{T}$ and, hence, $\bigcap \mathcal{T} \leq \boldsymbol{T}'$. Therefore $\mathfrak{G}$ is also c-reflective. ■

The following theorem provides characterations of WF algebraizability.

Theorem 1952 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. Then the following statements are equivalent:*

(i) $\mathfrak{G}$ *is WF algebraizable;*

(ii) The Leibniz operator defines an order isomorphism from $\mathbf{ThFam}(\mathfrak{G})$ *onto the lattice of all* $\mathrm{AlgSys}(\mathfrak{G})$*-congruence families on* $\mathcal{F}$*;*

(iii) For every $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$*, the Leibniz operator defines an order isomorphism from* $\mathbf{FiFam}^{\mathfrak{G}}(\mathcal{A})$ *onto the lattice of all* $\mathrm{AlgSys}(\mathfrak{G})$*-congruence systems on* $\mathcal{A}$*;*

(iv) For every $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle \in \mathrm{AlgSys}(\mathfrak{G})$*, the Leibniz operator defines an order isomorphism from* $\mathrm{FiFam}^{\mathfrak{G}}(\mathcal{A})$ *onto the lattice of all* $\mathrm{AlgSys}(\mathfrak{G})$*-congruence systems on* $\mathcal{A}$*.*

Proof:

(i)⇒(ii) Suppose $\mathfrak{G}$ is WF algebraizable. Denote $\mathrm{ConSys}(\mathfrak{G})$ the collection of all $\mathrm{AlgSys}(\mathfrak{G})$-congruences on $\mathcal{F}$. Then, since, for all $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$,

$$\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}/\Omega(\boldsymbol{T})}(\boldsymbol{T}/\Omega(\boldsymbol{T})) \le \Omega^{\mathcal{F}/\Omega(\boldsymbol{T})}(\boldsymbol{T}/\Omega(\boldsymbol{T})) = \Delta^{\mathcal{F}/\Omega(\boldsymbol{T})},$$

we get that $\langle \mathcal{F}/\Omega(\boldsymbol{T}), \boldsymbol{T}/\Omega(\boldsymbol{T}) \rangle \in \mathrm{MatFam}^{\mathsf{Su}}(\mathfrak{G})$. Thus, $\mathcal{F}/\Omega(\boldsymbol{T}) \in \mathrm{AlgSys}(\mathfrak{G})$ and, therefore, $\Omega(\boldsymbol{T}) \in \mathrm{ConSys}(\mathfrak{G})$. This shows that $\Omega : \mathrm{ThFam}(\mathfrak{G}) \to \mathrm{ConSys}(\mathfrak{G})$ is well defined. By Proposition 1951, it is injective. To see that it is surjective, consider $\theta \in \mathrm{ConSys}(\mathfrak{G})$. Then, by definition, $\mathcal{F}/\theta \in \mathrm{AlgSys}(\mathfrak{G})$, i.e., there exists $\boldsymbol{T} \in \mathrm{FiFam}^{\mathfrak{G}}(\mathcal{F}/\theta)$, such that $\widetilde{\Omega}^{\mathfrak{G},\mathcal{F}/\theta}(\boldsymbol{T}) = \Delta^{\mathcal{F}/\theta}$. However, since the Leibniz operator is monotone, by hypothesis, we get that the Susko operator coincides with the Leibniz operator, whence $\Omega^{\mathcal{F}/\theta}(\boldsymbol{T}) = \Delta^{\mathcal{F}/\theta}$. Denoting by $\langle I, \pi \rangle : \mathcal{F} \to \mathcal{F}/\theta$ the quotient morphism, we now get

$$\Omega(\pi^{-1}(\boldsymbol{T})) = \pi^{-1}(\Omega^{\mathcal{F}/\theta}(\boldsymbol{T})) = \pi^{-1}(\Delta^{\mathcal{F}/\theta}) = \theta.$$

Thus, Ω is indeed surjective. It is monotone by hypothesis and it is order reflecting, since it is c-reflective. Thus, $\Omega : \mathbf{ThFam}(\mathfrak{G}) \to \mathbf{ConSys}(\mathfrak{G})$ is in fact an order isomorphism.

(ii)⇒(iii) It is not difficult to show that $\Omega^{\mathcal{A}}$ is also monotone and c-reflective. Therefore, one can work in the same way as in Part (ii) replacing the mapping Ω by $\Omega^{\mathcal{A}} : \mathbf{FiFam}^{\mathfrak{G}}(\mathcal{A}) \to \mathbf{ConSys}^{\mathfrak{G}}(\mathcal{A})$, where $\mathrm{ConSys}^{\mathfrak{G}}(\mathcal{A})$ denotes the collection of $\mathrm{AlgSys}(\mathfrak{G})$-congruence systems on $\mathcal{A}$.

(iii)⇒(iv) Condition (iv) is a special case of Condition (iii).

(iv)⇒(i) If Condition (iv) holds, the $\mathfrak{G}$ is protoalgebraic, by Lemma 1911. Hence the Leibniz and Suszko operators coincide on the $\mathfrak{G}$-filter families of all $\mathbf{F}$-algebraic systems. Thus, by Theorem 1932, $\mathfrak{G}$ is also truth equational. Therefore, it is WF algebraizable.

■

Finally, based on results of preceding sections, we can also give a relation between algebraizability and WF algebraizability.

We show, first, that, if $\mathfrak{G}$ is algebraizable, then it is both syntactically protoalgebraic and truth equational.

We start by giving a modus ponens property in the case of algebraizability.

Lemma 1953 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$. *If* $\mathfrak{G}$ *is algebraizable via the conjugate pair* $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}}$*, for some class* K *of* $\mathbf{F}$*-algebraic systems, then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\boldsymbol{\phi}, \boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ *of trace* $\langle m, n \rangle$,

$$\boldsymbol{\psi} \in G_\Sigma(\{\boldsymbol{\phi}\} \cup \bigcup_{i < m+n} \rho_\Sigma[\phi_i, \psi_i]).$$

Proof: We have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\boldsymbol{\phi}, \boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m, n \rangle$,

$$\tau_\Sigma[\boldsymbol{\psi}] \subseteq G^{\mathsf{K}}_\Sigma(\tau_\Sigma[\boldsymbol{\phi}] \cup \{\phi_i \rhd_\Sigma \psi_i : i < m+n\}).$$

Thus, we get

$$\rho_\Sigma[\tau_\Sigma[\boldsymbol{\psi}]] \subseteq G_\Sigma(\rho_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]] \cup \bigcup_{i < m+n} \rho_\Sigma[\phi_i, \psi_i]).$$

Therefore, $\boldsymbol{\psi} \in G_\Sigma(\{\boldsymbol{\phi}\} \cup \bigcup_{i<m+n} \rho_\Sigma[\phi_i, \psi_i])$. ■

Moreover, in case of algebraizability, the isomorphism ρ^* from the theory families of the Gentzen π-institution $\mathfrak{G}$ to the K-congruence systems on $\mathcal{F}$ coincides with the Leibniz operator Ω.

Proposition 1954 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$. *If* $\mathfrak{G}$ *is algebraizable via the conjugate pair* $(\tau, \rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}}$*, for some class* K *of* $\mathbf{F}$*-algebraic systems, then, for all* $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$,

$$\rho^*(\boldsymbol{T}) = \Omega(\boldsymbol{T}).$$

Proof: Let $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$.

If $\langle \phi, \psi \rangle \in \Omega_\Sigma(\boldsymbol{T})$, then, since $\Omega(\boldsymbol{T})$ is a congruence system, we get, for all $\sigma \in \rho$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\langle \sigma_\Sigma(\phi, \phi, \vec{\chi}), \sigma_\Sigma(\phi, \psi, \vec{\chi}) \rangle \in \Omega_\Sigma(\boldsymbol{T}).$$

But $\sigma_\Sigma(\phi,\phi,\vec{\chi}) \in \mathrm{Thm}_\Sigma(\mathfrak{G}) \subseteq \boldsymbol{T}_\Sigma$. Therefore, by the compatibility of $\Omega(\boldsymbol{T})$ with $\boldsymbol{T}$, we get that $\sigma_\Sigma(\phi,\psi,\vec{\chi}) \in \boldsymbol{T}_\Sigma$. Therefore, $\rho_\Sigma[\phi,\psi] \subseteq \boldsymbol{T}_\Sigma$, which gives that $\langle\phi,\psi\rangle \in \rho^*_\Sigma(\boldsymbol{T})$.

Conversely, to see that $\rho^*(\boldsymbol{T}) \le \Omega(\boldsymbol{T})$ it suffices, by the maximality property of $\Omega(\boldsymbol{T})$, to show that $\rho^*(\boldsymbol{T})$ is compatible with $\boldsymbol{T}$. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi},\boldsymbol{\psi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m,n\rangle$, such that $\langle\boldsymbol{\phi},\boldsymbol{\psi}\rangle \in \rho^*_\Sigma(\boldsymbol{T})$ and $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$. Then, we have $\rho_\Sigma[\phi_i,\psi_i] \subseteq \boldsymbol{T}_\Sigma$, for all $i < m+n$, and $\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma$, whence, by Lemma 1953, $\boldsymbol{\psi} \in \boldsymbol{T}_\Sigma$. We conclude that $\rho^*(\boldsymbol{T})$ is compatible with $\boldsymbol{T}$, giving $\rho^*(\boldsymbol{T}) \le \Omega(\boldsymbol{T})$. ■

Now, we prove one of the main theorems of the section to the effect that algebraizability implies both syntactic protoalgebraicity and truth equationality.

Theorem 1955 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle\mathbf{F}, G\rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *is algebraizable via the conjugate pair* $(\tau,\rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}}$*, for some class* K *of* $\mathbf{F}$*-algebraic systems, then,* $\mathfrak{G}$ *is syntactically protoalgebraic and truth equational.*

Proof: Suppose $\mathfrak{G}$ is algebraizable via the conjugate pair $(\tau,\rho) : \mathfrak{G} \rightleftarrows \mathfrak{G}^{\mathsf{K}}$, for some class K of $\mathbf{F}$-algebraic systems.

Let, first, $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$. Then we have

$$\begin{array}{llll}\langle\phi,\psi\rangle \in \Omega_\Sigma(\boldsymbol{T}) & \text{iff} & \langle\phi,\psi\rangle \in \rho^*_\Sigma(\boldsymbol{T}) & \text{(Proposition 1954)}\\ & \text{iff} & \rho_\Sigma[\phi,\psi] \subseteq \boldsymbol{T}_\Sigma. & \text{(definition of } \rho^*)\end{array}$$

Therefore, $\mathfrak{G}$ is syntactically protoalgebraic, with witnessing transformations ρ.

Finally, let $\boldsymbol{T} \in \mathrm{ThFam}(\mathfrak{G})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\boldsymbol{\phi} \in \mathrm{Seq}^{\mathrm{tr}}_\Sigma(\mathbf{F})$ of trace $\langle m,n\rangle$. Then, we have

$$\begin{array}{llll}\boldsymbol{\phi} \in \boldsymbol{T}_\Sigma & \text{iff} & \rho_\Sigma[\tau_\Sigma[\boldsymbol{\phi}]] \subseteq \boldsymbol{T}_\Sigma & ((\tau,\rho) \text{ conjugate pair})\\ & \text{iff} & \tau_\Sigma[\boldsymbol{\phi}] \subseteq \rho^*_\Sigma(\boldsymbol{T}) & \text{(definition of } \rho^*)\\ & \text{iff} & \tau_\Sigma[\boldsymbol{\phi}] \subseteq \Omega_\Sigma(\boldsymbol{T}). & \text{((Proposition 1954))}\end{array}$$

Therefore, $\mathfrak{G}$ is truth equational, with witnessing transformations τ. ■

We show, next, that, conversely, syntactic protoalgebraicity and truth equationality guarantee algebraizability.

Theorem 1956 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle\mathbf{F}, G\rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. If* $\mathfrak{G}$ *is syntactically protoalgebraic and truth equational, then it is algebraizable.*

Proof: Suppose that $\mathfrak{G}$ is syntactically protoalgebraic, with witnessing transformations ρ, and truth equational, with witnessing transformations τ. Then, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\mathbf{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Seq}_\Sigma^{\mathrm{tr}}(\mathbf{F})$,

$$\begin{array}{lll} \boldsymbol{\phi} \in G_\Sigma(\mathbf{\Phi}) & \text{iff} & \boldsymbol{\phi} \in \bigcap\{\boldsymbol{T}_\Sigma : \mathbf{\Phi} \subseteq \boldsymbol{T}_\Sigma\} \\ & \text{iff} & \tau_\Sigma[\boldsymbol{\phi}] \subseteq \bigcap\{\Omega_\Sigma(\boldsymbol{T}) : \tau_\Sigma[\mathbf{\Phi}] \subseteq \Omega_\Sigma(\boldsymbol{T})\} \\ & \text{iff} & \tau_\Sigma[\boldsymbol{\phi}] \subseteq G_\Sigma^{\mathsf{K}}(\tau_\Sigma[\mathbf{\Phi}]). \end{array}$$

Moreover, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll} \langle \phi, \psi \rangle \in \Omega_\Sigma(\boldsymbol{T}) & \text{iff} & \rho_\Sigma[\phi, \psi] \subseteq \boldsymbol{T}_\Sigma \\ & \text{iff} & \tau_\Sigma[\rho_\Sigma[\phi, \psi]] \subseteq \Omega_\Sigma(\boldsymbol{T}). \end{array}$$

Hence, we have that $G_\Sigma^{\mathsf{K}}(\phi \rhd_\Sigma \psi) = G_\Sigma^{\mathsf{K}}(\tau_\Sigma[\rho_\Sigma[\phi, \psi]])$.

We conclude, by Lemma 1879, that $\mathfrak{G}$ is equivalent to $\mathfrak{G}^{\mathsf{K}}$ and, therefore, $\mathfrak{G}$ is algebraizable. ■

Now we can formulate the main characterization theorem:

Theorem 1957 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* tr *a trace and* $\mathfrak{G} = \langle \mathbf{F}, G \rangle$ *a Gentzen* π*-institution of trace* tr *based on* $\mathbf{F}$*. The following statements are equivalent:*

(i) $\mathfrak{G}$ *is algebraizable;*

(ii) $\mathfrak{G}$ *is syntactically protoalgebraic and truth equational;*

(iii) $\mathfrak{G}$ *is WF algebraizable (i.e., protoalgebraic and c-reflective) and has both a Leibniz reflexive core and an adequate Suszko core.*

Proof: If $\mathfrak{G}$ is algebraizable, then, by Theorem 1955, it is syntactically protoalgebraic and truth equational. If $\mathfrak{G}$ is syntactically protoalgebraic and truth equational, then, by Theorems 1924 and 1948, it is protoalgebraic, c-reflective and has both a Leibniz reflexive core and an adequate Suszko core. Finally, if $\mathfrak{G}$ is WF algebraizable, with a Leibniz reflexive core and an adequate Suszko core, then, by Theorems 1924 and 1948, it is syntactically protoalgebraic and truth equational, whence, by Theorem 1956, it is algebraizable. ■

Chapter 27

Behavioral Algebraizability

27.1 Behavioral π-Institutions

Let **Sign** be an arbitrary category of **signatures**, S a nonempty set of **sorts** and, for each $s \in S$,

$$\mathrm{SEN}_s : \mathbf{Sign} \to \mathbf{Set}$$

a functor giving, for each signature Σ, a set of Σ-sentences **of sort** s. By a **multi-sorted sentence functor over set of sorts** S we understand the collection

$$\{\mathrm{SEN}_s : s \in S\},$$

where all sets $\mathrm{SEN}_s(\Sigma)$, $s \in S$, are assumed to be disjoint, i.e.,

$$\mathrm{SEN}_s(\Sigma) \cap \mathrm{SEN}_{s'}(\Sigma) = \varnothing, \text{ for all } s, s' \in S, s \neq s'.$$

Because of this condition, given $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\Phi \subseteq \bigcup_{s \in S} \mathrm{SEN}_s(\Sigma)$, we write

$$\mathrm{SEN}(f)(\Phi) = \bigcup_{s \in S} \{\mathrm{SEN}_s(f)(\phi) : \phi \in \Phi \text{ of sort } s\}.$$

A multi-sorted sentence functor over set of sorts S is called **behavioral** if a nonempty subset $V \subseteq S$ of **formula sorts** has been singled out and, moreover, there exists a companion subset $V^* = \{v^* : v \in V\}$ of **visible sorts**. In that case the (perhaps empty) set $H = S - (V \cup V^*)$ is called the set of **hidden sorts**. To denote a behavioral functor, making the set of visible and set of hidden sorts explicit, we write

$$\{\mathrm{SEN}_v, \mathrm{SEN}_{v^*}, \mathrm{SEN}_h : v \in V, h \in H\},$$

or sometimes, for the sake of succinctness,

$$\{\mathrm{SEN}_s\}_H^{V,V^*}.$$

Let **Sign** be a category and $\{\mathrm{SEN}_s\}_{s \in S}$ a multi-sorted sentence functor. The **clone of all natural transformations** on $\{\mathrm{SEN}_s\}_{s \in S}$ is the locally small category with:

- objects $\prod_{\kappa < \alpha} \mathrm{SEN}_{s_\kappa}$, with $s_\kappa \in S$, α an ordinal;
- morphisms $\tau : \prod_{\kappa < \alpha} \mathrm{SEN}_{s_\kappa} \to \prod_{\lambda < \beta} \mathrm{SEN}_{s'_\lambda}$ are β-sequences of natural transformations

$$\tau_\lambda : \prod_{\kappa < \alpha} \mathrm{SEN}_{s_\kappa} \to \mathrm{SEN}_{s'_\lambda}, \ \lambda < \beta.$$

Composition is defined as ordinary composition, i.e., by setting

$$\prod_{\kappa < \alpha} \mathrm{SEN}_{s_\kappa} \xrightarrow{\langle \tau_\lambda : \lambda < \beta \rangle} \prod_{\lambda < \beta} \mathrm{SEN}_{s'_\lambda} \xrightarrow{\langle \sigma_\mu : \mu < \gamma \rangle} \prod_{\mu < \gamma} \mathrm{SEN}_{s''_\mu}$$

$$\langle \sigma_\mu : \mu < \gamma \rangle \circ \langle \tau_\lambda : \lambda < \beta \rangle = \langle \sigma_\mu(\langle \tau_\lambda : \lambda < \beta \rangle) : \mu < \gamma \rangle.$$

A subcategory N of the clone of all natural transformations on $\{\mathrm{SEN}_s\}_{s\in S}$, with objects all objects of the form $\prod_{i=1}^k \mathrm{SEN}_{s_i}$, $k < \omega$, is called a **category of natural transformations on** $\{\mathrm{SEN}_s\}_{s\in S}$ if the following conditions hold:

- It contains all natural projections

$$p^{s_1\cdots s_k \to s_i} : \prod_{i=1}^k \mathrm{SEN}_{s_i} \to \mathrm{SEN}_{s_i}, i < k, k < \omega.$$

- For every collection $\{\tau_i : \prod_{i=1}^k \mathrm{SEN}_{s_i} \to \mathrm{SEN}_{s'_i} : i < \ell\}$ of ℓ natural transformations in N, the tuple

$$\langle \tau_i : i < \ell \rangle : \prod_{i=1}^k \mathrm{SEN}_{s_i} \to \prod_{j=1}^\ell \mathrm{SEN}_{s'_j}$$

  is also a natural transformation in N.

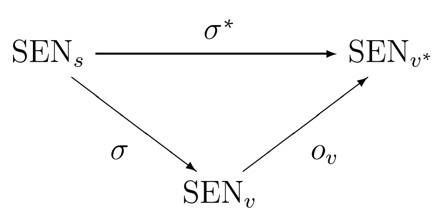
We refer to these conditions by saying that N "includes all projections" and is "closed under combinations" of natural transformations.

Let $\{\mathrm{SEN}_s\}_H^{V,V^*}$ be a behavioral sentence functor. A subcategory N of the clone of all natural transformations on $\{\mathrm{SEN}_s\}_{s\in S}$, with objects all objects of the form $\prod_{i=1}^k \mathrm{SEN}_{s_i}$, $k < \omega$, is called a **category of natural transformations on** $\{\mathrm{SEN}_s\}_H^{V,V^*}$ if, in addition to being a category of natural transformations on $\{\mathrm{SEN}_s\}_{s\in S}$, i.e., to including all projections and being closed under combinations, the following condition also holds:

- For all $v \in V$, there is no outgoing natural transformation from SEN_{v^*}, other than the identity, and there exists a unique surjective natural transformation

$$o_v : \mathrm{SEN}_v \to \mathrm{SEN}_{v^*},$$

  called the v-**observation natural transformation**, or, simply, **observation**, when the formula sort v to which it corresponds is clear from context, such that, every incoming natural transformation $\sigma^* : \mathrm{SEN}_s \to \mathrm{SEN}_{v^*}$ factors through o_v:



We express this condition by saying that N "has observations".

Given a natural transformation $\sigma : \prod_{i=1}^k \mathrm{SEN}_{s_i} \to \mathrm{SEN}_s$ in N, we call $s_1 \cdots s_k \to s$ the **type of** σ and say that σ is **of sort** s (i.e., of the output sort).

A **multi-sorted algebraic system** $\mathbf{F} = \langle \mathbf{Sign}, \{\mathrm{SEN}_s\}_{s\in S}, N\rangle$ consists of a category of signatures, a multi-sorted sentence functor and a category N of natural transformations on $\{\mathrm{SEN}\}_{s\in S}$. It is called **behavioral** if $\{\mathrm{SEN}_s\}_H^{V,V^*}$ is a behavioral sentence functor and N is a category of natural transformations on $\{\mathrm{SEN}_s\}_H^{V,V^*}$ (i.e., has observations), and we then write

$$\mathbf{F} = \langle \mathbf{Sign}, \{\mathrm{SEN}_s\}_H^{V,V^*}, N\rangle.$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}_s^\flat\}_{s\in S}, N^\flat\rangle$ be a multi-sorted algebraic system. An $N^\flat$**-algebraic system** is a multi-sorted algebraic system

$$\mathbf{A} = \langle \mathbf{Sign}, \{\mathrm{SEN}_s\}_{s\in S}, N\rangle,$$

such that there exists a surjective functor $F : N^\flat \to N$ that preserves all natural projections (and, hence, the type of all natural transformations in $N^\flat$). We use $\sigma^{\mathbf{A}}$ to refer to the image of σ in $N^\flat$ under F.

Moreover, given two N-algebraic systems $\mathbf{A} = \langle \mathbf{Sign}, \{\mathrm{SEN}_s\}_{s\in S}, N\rangle$ and $\mathbf{B} = \langle \mathbf{Sign}', \{\mathrm{SEN}'_s\}_{s\in S}, N'\rangle$, a **morphism**

$$\langle F, \alpha\rangle : \mathbf{A} \to \mathbf{B}$$

consists of a functor $F : \mathbf{Sign} \to \mathbf{Sign}'$ and a collection $\alpha = \{\alpha^s\}_{s\in S}$ of natural transformations $\alpha^s : \mathrm{SEN}_s \to \mathrm{SEN}'_s \circ F$, $s \in S$, such that, for every $\sigma : \prod_{i=1}^k \mathrm{SEN}_{s_i}^\flat \to \mathrm{SEN}_s^\flat$ in N, all $\Sigma \in |\mathbf{Sign}|$, and all $\phi_i \in \mathrm{SEN}_{s_i}(\Sigma)$, $i \le k$,

$$\alpha_\Sigma^s(\sigma_\Sigma^{\mathbf{A}}(\phi_1, \dots \phi_k)) = \sigma_{F(\Sigma)}^{\mathbf{B}}(\alpha_\Sigma^{s_1}(\phi_1), \dots, \alpha_\Sigma^{s_k}(\phi_k)).$$

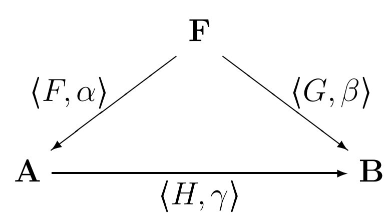
Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}_s^\flat\}_{s\in S}, N^\flat\rangle$ be a multi-sorted algebraic system. An $\mathbf{F}$**-algebraic system** $\mathcal{A}$ is a pair $\langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, where

- $\mathbf{A} = \langle \mathbf{Sign}, \{\mathrm{SEN}_s\}_{s\in S}, N\rangle$ is an $N^\flat$-algebraic system;
- $\langle F, \alpha\rangle : \mathbf{F} \to \mathbf{A}$ is a surjective morphism, i.e., such that $F : \mathbf{Sign} \to \mathbf{Sign}'$ is surjective and full and $\alpha_\Sigma^s : \mathrm{SEN}_s^\flat(\Sigma) \to \mathrm{SEN}_s(F(\Sigma))$ is surjective, for all $\Sigma \in |\mathbf{Sign}|$ and all $s \in S$.

Given two $\mathbf{F}$-algebraic systems $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ and $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \{\mathrm{SEN}_s\}_{s\in S}, N\rangle$ and $\mathbf{B} = \langle \mathbf{Sign}', \{\mathrm{SEN}'_s\}_{s\in S}, N'\rangle$, a **morphism**

$$\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$$

is a morphism $\langle H, \gamma\rangle : \mathbf{A} \to \mathbf{B}$, such that



$$\langle H,\gamma\rangle\circ\langle F,\alpha\rangle=\langle G,\beta\rangle.$$

A **behavioral π-institution** is a pair $\mathcal{I}=\langle\mathbf{F},C\rangle$, where

- $\mathbf{F}=\langle\mathbf{Sign}^\flat,\{\mathrm{SEN}^\flat_s\}^{V,V^*}_H,N^\flat\rangle$ is a behavioral algebraic system;
- $C=\{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}^\flat|}$ is a **closure system on** $\{\mathrm{SEN}^\flat_v\}_{v\in V}$, i.e., for all $\Sigma\in|\mathbf{Sign}^\flat|$,
$$C_\Sigma:\mathcal{P}(\bigcup_{v\in V}\mathrm{SEN}^\flat_v(\Sigma))\to\mathcal{P}(\bigcup_{v\in V}\mathrm{SEN}^\flat_v(\Sigma))$$
is a closure operator and, for all $\Sigma,\Sigma'\in|\mathbf{Sign}^\flat|$, all $f\in\mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\Phi\subseteq\cup_{v\in V}\mathrm{SEN}^\flat_v(\Sigma)$,
$$\mathrm{SEN}^\flat(f)(C_\Sigma(\Phi))\subseteq C_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)).$$

Given a behavioral algebraic system $\mathbf{F}=\langle\mathbf{Sign}^\flat,\{\mathrm{SEN}^\flat_s\}^{V,V^*}_H,N^\flat\rangle$, a **behavioral sentence family $T=\{T_\Sigma\}_{\Sigma\in|\mathbf{Sign}^\flat|}$ of F** consists of subsets

$$T_\Sigma\subseteq\bigcup_{v\in V}\mathrm{SEN}_v(\Sigma),\quad\Sigma\in|\mathbf{Sign}^\flat|.$$

Given a behavioral π-institution $\mathcal{I}=\langle\mathbf{F},C\rangle$, based on $\mathbf{F}$, a **behavioral theory family $T=\{T_\Sigma\}_{\Sigma\in|\mathbf{Sign}^\flat|}$ of $\mathcal{I}$** is a behavioral sentence family of $\mathbf{F}$, such that, for all $\Sigma\in|\mathbf{Sign}^\flat|$,
$$C_\Sigma(T_\Sigma)=T_\Sigma.$$
We write $\mathrm{ThFam}(\mathcal{I})$ for the collection of all behavioral theory families of $\mathcal{I}$.

27.2 Behavioral Algebra

Let $\mathbf{F}=\langle\mathbf{Sign},\{\mathrm{SEN}_s\}_{s\in S},N\rangle$ be a multi-sorted algebraic system. An **equivalence family $\theta=\{\theta_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ on F** is a family, such that, for all $\Sigma\in|\mathbf{Sign}|$, $\theta_\Sigma=\{\theta^s_\Sigma\}_{s\in S}$ consists of equivalence relations $\theta^s_\Sigma\subseteq\mathrm{SEN}_s(\Sigma)^2$. It is called an **equivalence system on F** if it is invariant under **Sign**-morphisms, i.e., such that, for all $\Sigma,\Sigma'\in|\mathbf{Sign}|$, $f\in\mathbf{Sign}(\Sigma,\Sigma')$ and $s\in S$,

$$\mathrm{SEN}_s(f)(\theta^s_\Sigma)\subseteq\theta^s_{\Sigma'}.$$

An equivalence family/system θ on $\mathbf{F}$ is called a **congruence family/system on F** if, for all $\sigma:\prod_{i=1}^k\mathrm{SEN}_{s_i}\to\mathrm{SEN}_s$ in N, all $\Sigma\in|\mathbf{Sign}|$ and all $\vec{\phi},\vec{\psi}\in\prod_{i=1}^k\mathrm{SEN}_{s_i}(\Sigma)$,

$$\vec{\phi}\ \prod_{i=1}^k\theta^{s_i}_\Sigma\ \vec{\psi}\quad\text{implies}\quad\sigma_\Sigma(\vec{\phi})\ \theta^s_\Sigma\ \sigma_\Sigma(\vec{\psi}).$$

The collection of all congruence systems on $\mathbf{F}$ is denoted by ConSys($\mathbf{F}$) and it forms a complete lattice under signature-wise and sort-wise inclusion $\le$:

$$\mathbf{ConSys}(\mathbf{F}) = \langle \mathrm{ConSys}(\mathbf{F}), \le \rangle.$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}_s\}_H^{V,V^*}, N^\flat \rangle$ be a behavioral algebraic system and $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ a behavioral sentence family of $\mathbf{F}$. A congruence family $\theta = \{\theta_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ on $\mathbf{F}$ is **compatible with** T if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi, \psi \in \mathrm{SEN}_v(\Sigma)$,

$$\langle \phi, \psi \rangle \in \theta_\Sigma^v \quad \text{and} \quad \phi \in T_\Sigma \quad \text{imply} \quad \psi \in T_\Sigma.$$

A fundamental result, akin to that allowing us to define Leibniz congruence systems in the context of ordinary π-institutions, is asserting that, given a behavioral sentence family, there exists a largest congruence system on $\mathbf{F}$ compatible with the theory family.

Theorem 1958 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}_s^\flat\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system,* $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ *a behavioral sentence family of* $\mathbf{F}$*. There exists a largest congruence system on* $\mathbf{F}$ *compatible with* T*.*

Proof: We define $\theta = \{\theta_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, where $\theta_\Sigma = \{\theta_\Sigma^s\}_{s \in S}$ by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $s \in S$ and all $\phi, \psi \in \mathrm{SEN}_s^\flat(\Sigma)$, $\langle \phi, \psi \rangle \in \theta_\Sigma^s$ if and only if, for all $\sigma : \mathrm{SEN}_s^\flat \times \prod_{i=1}^k \mathrm{SEN}_{s_i}^\flat \to \mathrm{SEN}_v^\flat$ in $N^\flat$, with $v \in V$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}_{s_i}^\flat(\Sigma')$,

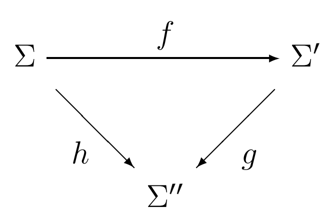
$$\sigma_{\Sigma'}(\mathrm{SEN}_s^\flat(f)(\phi), \vec{\chi}) \in T_{\Sigma'} \quad \text{iff} \quad \sigma_{\Sigma'}(\mathrm{SEN}_s^\flat(f)(\psi), \vec{\chi}) \in T_{\Sigma'}.$$

We show that θ, thus defined, is a congruence system on $\mathbf{F}$ that is compatible with T.

First, it is straightforward by the definition that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $s \in S$, θ_Σ^s is reflexive, symmetric and transitive. So θ is an equivalence family on $\mathbf{F}$. To see that it is a system, let $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, $s \in S$ and $\phi, \psi \in \mathrm{SEN}_s^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in \theta_\Sigma^s$. Then, for all $\sigma : \mathrm{SEN}_s^\flat \times \prod_{i=1}^k \mathrm{SEN}_{s_i}^\flat \to \mathrm{SEN}_v^\flat$ in $N^\flat$, with $v \in V$, all $\Sigma'' \in |\mathbf{Sign}^\flat|$, all $h \in \mathbf{Sign}^\flat(\Sigma, \Sigma'')$ and all $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}_{s_i}^\flat(\Sigma'')$,

$$\sigma_{\Sigma''}(\mathrm{SEN}_s^\flat(h)(\phi), \vec{\chi}) \in T_{\Sigma''} \quad \text{iff} \quad \sigma_{\Sigma''}(\mathrm{SEN}_s^\flat(f)(\psi), \vec{\chi}) \in T_{\Sigma''}.$$

Thus, as fortiori, for all $\sigma : \mathrm{SEN}_s^\flat \times \prod_{i=1}^k \mathrm{SEN}_{s_i}^\flat \to \mathrm{SEN}_v^\flat$ in $N^\flat$, with $v \in V$, all $\Sigma'' \in |\mathbf{Sign}^\flat|$, all $g \in \mathbf{Sign}^\flat(\Sigma', \Sigma'')$ and all $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}_{s_i}^\flat(\Sigma'')$,



$$\sigma_{\Sigma''}(\mathrm{SEN}^\flat_s(g)(\mathrm{SEN}^\flat_s(f)(\phi)),\vec{\chi}) \in T_{\Sigma''}$$
$$\text{iff}\quad \sigma_{\Sigma''}(\mathrm{SEN}^\flat_s(g)(\mathrm{SEN}^\flat_s(f)(\psi)),\vec{\chi}) \in T_{\Sigma''}.$$

This shows that $\langle \mathrm{SEN}^\flat_s(f)(\phi), \mathrm{SEN}^\flat_s(f)(\psi)\rangle \in \theta^s_{\Sigma'}$ and, hence, θ is an equivalence system.

To see that θ is a congruence system, let $\tau : \prod_{j=1}^\ell \mathrm{SEN}^\flat_{s'_j} \to \mathrm{SEN}^\flat_s$ be in $N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi},\vec{\psi} \in \prod_{j=1}^\ell \mathrm{SEN}^\flat_{s'_j}(\Sigma)$, such that $\vec{\phi} \prod_{j=1}^\ell \theta^{s'_j}_\Sigma \vec{\psi}$. Then, we have, for all $\sigma : \mathrm{SEN}^\flat_s \times \prod_{i=1}^k \to \mathrm{SEN}^\flat_v$ in $N^\flat$, with $v \in V$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i}(\Sigma')$,

$$\begin{array}{ll}
& \sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\tau_\Sigma(\vec{\phi})),\vec{\chi}) \in T_{\Sigma'} \\
\text{iff} & \sigma_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}^\flat_{s'_1}(f)(\phi_1), \mathrm{SEN}^\flat_{s'_2}(f)(\phi_2), \mathrm{SEN}^\flat_{s'_3}(f)(\phi_3),\ldots, \\
& \qquad \mathrm{SEN}^\flat_{s'_{\ell-1}}(f)(\phi_{\ell-1}), \mathrm{SEN}^\flat_{s'_\ell}(f)(\phi_\ell)),\vec{\chi}) \in T_{\Sigma'} \\
\text{iff} & \sigma_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}^\flat_{s'_1}(f)(\psi_1), \mathrm{SEN}^\flat_{s'_2}(f)(\phi_2), \mathrm{SEN}^\flat_{s'_3}(f)(\phi_3),\ldots, \\
& \qquad \mathrm{SEN}^\flat_{s'_{\ell-1}}(f)(\phi_{\ell-1}), \mathrm{SEN}^\flat_{s'_\ell}(f)(\phi_\ell)),\vec{\chi}) \in T_{\Sigma'} \\
\text{iff} & \sigma_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}^\flat_{s'_1}(f)(\psi_1), \mathrm{SEN}^\flat_{s'_2}(f)(\psi_2), \mathrm{SEN}^\flat_{s'_3}(f)(\phi_3),\ldots, \\
& \qquad \mathrm{SEN}^\flat_{s'_{\ell-1}}(f)(\phi_{\ell-1}), \mathrm{SEN}^\flat_{s'_\ell}(f)(\phi_\ell)),\vec{\chi}) \in T_{\Sigma'} \\
\text{iff}\cdots & \\
\text{iff} & \sigma_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}^\flat_{s'_1}(f)(\psi_1), \mathrm{SEN}^\flat_{s'_2}(f)(\psi_2), \mathrm{SEN}^\flat_{s'_3}(f)(\psi_3),\ldots, \\
& \qquad \mathrm{SEN}^\flat_{s'_{\ell-1}}(f)(\psi_{\ell-1}), \mathrm{SEN}^\flat_{s'_\ell}(f)(\phi_\ell)),\vec{\chi}) \in T_{\Sigma'} \\
\text{iff} & \sigma_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}_{s'_1}(f)(\psi_1), \mathrm{SEN}^\flat_{s'_2}(f)(\psi_2), \mathrm{SEN}^\flat_{s'_3}(f)(\psi_3),\ldots, \\
& \qquad \mathrm{SEN}^\flat_{s'_{\ell-1}}(f)(\psi_{\ell-1}), \mathrm{SEN}^\flat_{s'_\ell}(f)(\psi_\ell)),\vec{\chi}) \in T_{\Sigma'} \\
\text{iff} & \sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\tau_\Sigma(\vec{\psi})),\vec{\chi}) \in T_{\Sigma'}.
\end{array}$$

Hence, $\langle \tau_\Sigma(\vec{\phi}), \tau_\Sigma(\vec{\psi})\rangle \in \theta^s_\Sigma$, showing that θ is a congruence system on $\mathbf{F}$.

θ is also compatible with T, since, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\psi,\psi \in \mathrm{SEN}^\flat_v(\Sigma)$, such that $\phi \in T_\Sigma$ and $\langle \phi,\psi\rangle \in \theta^v_\Sigma$, we get, as a special instance in the definition by taking $\sigma = \iota : \mathrm{SEN}^\flat_v \to \mathrm{SEN}^\flat_v$ in $N^\flat$, $\Sigma' = \Sigma$ and $f = i_\Sigma$, $\phi \in T_\Sigma$ iff $\psi \in T_\Sigma$. Therefore, $\psi \in T_\Sigma$ and θ is, in fact, a congruence system on $\mathbf{F}$ compatible with T.

Finally, we show that, if θ' is a congruence system on $\mathbf{F}$ compatible with T, then $\theta' \le \theta$. Suppose, to this end, that θ' is a congruence system on $\mathbf{F}$ compatible with T and let $\Sigma \in |\mathbf{Sign}^\flat|$, $s \in S$ and $\phi,\psi \in \mathrm{SEN}^\flat_s(\Sigma)$, such that $\langle \phi,\psi\rangle \in \theta'^s_\Sigma$. Then, since θ' is a congruence system, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$, $\langle \mathrm{SEN}^\flat_s(f)(\phi), \mathrm{SEN}^\flat_s(f)(\psi)\rangle \in \theta'^s_{\Sigma'}$. Thus, since θ' is a congruence system, for all $\sigma : \mathrm{SEN}^\flat_s \times \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i} \to \mathrm{SEN}^\flat_v$ in $N^\flat$, with $v \in V$, and all $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i}(\Sigma')$,

$$\langle \sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\phi),\vec{\chi}), \sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\psi),\vec{\chi})\rangle \in \theta'^v_{\Sigma'}.$$

Therefore, by the compatibility of θ' with T, we get

$$\sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\phi),\vec{\chi}) \in T_{\Sigma'} \quad \text{iff} \quad \sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\psi),\vec{\chi}) \in T_{\Sigma'}.$$

This shows that $\langle\phi,\psi\rangle \in \theta^s_\Sigma$ and, hence, $\theta' \le \theta$. Thus, θ is indeed the largest congruence system on $\mathbf{F}$ compatible with T. ■

The largest congruence system on $\mathbf{F}$ compatible with T is called the **behavioral Leibniz congruence system of** T **on** $\mathbf{F}$ and is denoted by $\Upsilon(T)$. Moreover, given a behavioral π-institution $\mathcal{I} = \langle\mathbf{F}, C\rangle$, and a behavioral theory family $T \in \mathrm{ThFam}(\mathcal{I})$, we define the **behavioral Suszko congruence system of** T **on** $\mathbf{F}$ by

$$\widetilde{\Upsilon}^{\mathcal{I}}(T) = \bigcap\{\Upsilon(T') : T \le T' \in \mathrm{ThFam}(\mathcal{I})\}.$$

Let $\mathbf{F} = \langle\mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}^{V,V^*}_H, N^\flat\rangle$ be a behavioral algebraic system and $\mathcal{A} = \langle\mathbf{A}, \langle F,\alpha\rangle\rangle$, with $\mathbf{A} = \langle\mathbf{Sign}, \{\mathrm{SEN}_s\}_{s\in S}, N\rangle$, an $\mathbf{F}$-algebraic system. We define $\equiv^{\mathcal{A}} = \{\equiv^{\mathcal{A}}_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$, where, for all $\Sigma \in |\mathbf{Sign}|$, $\equiv^{\mathcal{A}}_\Sigma = \{\equiv^{\mathcal{A},s}_\Sigma\}_{s\in S}$ is given, for all $s \in S$, all $\phi,\psi \in \mathrm{SEN}_s(\Sigma)$, by $\phi \equiv^{\mathcal{A},s}_\Sigma \psi$ if and only if, for all $\sigma : \mathrm{SEN}_s \times \prod_{i=1}^k \mathrm{SEN}_{s_i} \to \mathrm{SEN}_{v^*}$, with $v \in V$, all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma,\Sigma')$ and all $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}_{s_i}(\Sigma')$,

$$\sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}_s(f)(\phi),\vec{\chi}) = \sigma^{\mathcal{A}}_{\Sigma'}(\mathrm{SEN}_s(f)(\psi),\vec{\chi}).$$

We show that $\equiv^{\mathcal{A}}$ is a congruence system on $\mathcal{A}$.

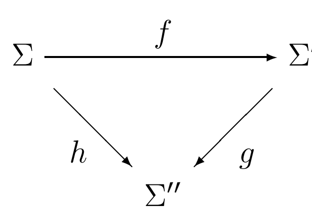
Proposition 1959 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}^{V,V^*}_H, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{A} = \langle\mathbf{A}, \langle F,\alpha\rangle\rangle$*, with* $\mathbf{A} = \langle\mathbf{Sign}, \{\mathrm{SEN}_s\}^{V,V^*}_H, N\rangle$*, an* $\mathbf{F}$*-algebraic system. The relation family* $\equiv^{\mathcal{A}}$ *is a congruence system on* $\mathcal{A}$*.*

Proof: By the definition, it is obvious that, for all $\Sigma \in |\mathbf{Sign}|$ and all $s \in S$, $\equiv^{\mathcal{A},s}_\Sigma$ is an equivalence family on $\mathrm{SEN}_s(\Sigma)$. We show that $\equiv^{\mathcal{A}}$ is a system and that it satisfies the congruence property.

Let $\Sigma,\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma,\Sigma')$, $s \in S$ and $\phi,\psi \in \mathrm{SEN}_s(\Sigma)$, such that $\phi \equiv^{\mathcal{A},s}_\Sigma \psi$. Then, for all $\sigma : \mathrm{SEN}_s \times \prod_{i=1}^k \mathrm{SEN}_{s_i} \to \mathrm{SEN}_{v^*}$, with $v \in V$, all $\Sigma'' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma,\Sigma'')$ and $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}_{s_i}(\Sigma'')$,

$$\sigma^{\mathcal{A}}_{\Sigma''}(\mathrm{SEN}_s(h)(\phi),\vec{\chi}) = \sigma^{\mathcal{A}}_{\Sigma''}(\mathrm{SEN}_s(h)(\psi),\vec{\chi}).$$

In particular, for all $g \in \mathbf{Sign}(\Sigma',\Sigma'')$ and all $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}_{s_i}(\Sigma'')$,



$$\sigma^{\mathcal{A}}_{\Sigma''}(\mathrm{SEN}_s(g)(\mathrm{SEN}_s(f)(\phi)),\vec{\chi}) = \sigma^{\mathcal{A}}_{\Sigma''}(\mathrm{SEN}_s(g)(\mathrm{SEN}_s(f)(\psi)),\vec{\chi}).$$

Thus, by definition, $\mathrm{SEN}_s(f)(\phi) \equiv^{\mathcal{A},s}_{\Sigma'} \mathrm{SEN}_s(f)(\psi)$ and, therefore, $\equiv^{\mathcal{A}}$ is an equivalence system.

Finally, let $\sigma : \prod_{i=1}^k \mathrm{SEN}_{s_i} \to \mathrm{SEN}_s$ be in N, $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi},\vec{\psi} \in \prod_{i=1}^k \mathrm{SEN}_{s_i}(\Sigma)$, such that $\vec{\phi} \prod_{i=1}^k \equiv_\Sigma^{\mathcal{A},s_i} \vec{\psi}$. Then, we have, for all $\tau : \mathrm{SEN}_s \times \prod_{j=1}^\ell \mathrm{SEN}_{s'_j} \to \mathrm{SEN}_{v^*}$, with $v \in V$, all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma,\Sigma')$ and all $\vec{\chi} \in \prod_{j=1}^\ell \mathrm{SEN}_{s'_j}(\Sigma')$,

$$\begin{aligned}
&\tau_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}_s(f)(\sigma_\Sigma^{\mathcal{A}}(\vec{\phi})),\vec{\chi})\\
&= \tau_{\Sigma'}^{\mathcal{A}}(\sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}_{s_1}(f)(\phi_1),\mathrm{SEN}_{s_2}(f)(\phi_2),\mathrm{SEN}_{s_3}(f)(\phi_3),\dots,\\
&\qquad\qquad \mathrm{SEN}_{s_{k-1}}(f)(\phi_{k-1}),\mathrm{SEN}_{s_k}(f)(\phi_k)),\vec{\chi})\\
&= \tau_{\Sigma'}^{\mathcal{A}}(\sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}_{s_1}(f)(\psi_1),\mathrm{SEN}_{s_2}(f)(\phi_2),\mathrm{SEN}_{s_3}(f)(\phi_3),\dots,\\
&\qquad\qquad \mathrm{SEN}_{s_{k-1}}(f)(\phi_{k-1}),\mathrm{SEN}_{s_k}(f)(\phi_k)),\vec{\chi})\\
&= \tau_{\Sigma'}^{\mathcal{A}}(\sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}_{s_1}(f)(\psi_1),\mathrm{SEN}_{s_2}(f)(\psi_2),\mathrm{SEN}_{s_3}(f)(\phi_3),\dots,\\
&\qquad\qquad \mathrm{SEN}_{s_{k-1}}(f)(\phi_{k-1}),\mathrm{SEN}_{s_k}(f)(\phi_k)),\vec{\chi})\\
&= \cdots\\
&= \tau_{\Sigma'}^{\mathcal{A}}(\sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}_{s_1}(f)(\psi_1),\mathrm{SEN}_{s_2}(f)(\psi_2),\mathrm{SEN}_{s_3}(f)(\psi_3),\dots,\\
&\qquad\qquad \mathrm{SEN}_{s_{k-1}}(f)(\psi_{k-1}),\mathrm{SEN}_{s_k}(f)(\phi_k)),\vec{\chi})\\
&= \tau_{\Sigma'}^{\mathcal{A}}(\sigma_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}_{s_1}(f)(\psi_1),\mathrm{SEN}_{s_2}(f)(\psi_2),\mathrm{SEN}_{s_3}(f)(\psi_3),\dots,\\
&\qquad\qquad \mathrm{SEN}_{s_{k-1}}(f)(\psi_{k-1}),\mathrm{SEN}_{s_k}(f)(\psi_k)),\vec{\chi})\\
&= \tau_{\Sigma'}^{\mathcal{A}}(\mathrm{SEN}_s(f)(\sigma_\Sigma^{\mathcal{A}}(\vec{\psi})),\vec{\chi})
\end{aligned}$$

Hence, $\sigma_\Sigma^{\mathcal{A}}(\vec{\phi}) \equiv_\Sigma^{\mathcal{A},s} \sigma_\Sigma^{\mathcal{A}}(\vec{\phi})$ and $\equiv^{\mathcal{A}}$ is a congruence system on $\{\mathrm{SEN}_s\}_H^{V,V^*}$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}_s^\flat\}_H^{V,V^*}, N^\flat\rangle$ be a behavioral algebraic system and K a class of $\mathbf{F}$-algebraic systems. We define the closure system $C^{\mathsf{K}} = \{C_\Sigma^{\mathsf{K}}\}_{\Sigma\in|\mathbf{Sign}^\flat|}$ by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$C_\Sigma^{\mathsf{K}} : \mathcal{P}(\bigcup_{v\in V} \mathrm{SEN}_v^\flat(\Sigma)^2) \to \mathcal{P}(\bigcup_{v\in V} \mathrm{SEN}_v^\flat(\Sigma)^2)$$

be given, for all $E \cup \{\phi \approx \psi\} \subseteq \bigcup_{v\in V} \mathrm{SEN}_v(\Sigma)^2$,

$$\begin{aligned}
\phi \approx \psi \in C_\Sigma^{\mathsf{K}}(E) \quad \text{iff} \quad & \text{for all } \mathcal{A} \in \mathsf{K}, \Sigma' \in |\mathbf{Sign}^\flat|, f \in \mathbf{Sign}^\flat(\Sigma,\Sigma'),\\
& \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(E)) \subseteq \equiv_{F(\Sigma')}^{\mathcal{A}} \text{ implies}\\
& \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi)) \equiv_{F(\Sigma')}^{\mathcal{A}} \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\psi)).
\end{aligned}$$

Proposition 1960 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}_s^\flat\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems. Then* C^{K} *is a closure system on* $\bigcup_{v\in V}(\mathrm{SEN}_v^\flat)^2$.

Proof: It is straightforward to check that C_Σ^{K} is inflationary, monotone and idempotent, for all $\Sigma \in |\mathbf{Sign}^\flat|$. The fact that it is invariant under $\mathbf{Sign}^\flat$-morphisms can be shown in a way similar to that in the proof of Proposition 1959. ■

We call $\mathcal{I}^{\mathsf{K}} = \langle \mathbf{F}, C^{\mathsf{K}}\rangle$ the **behavioral equational π-institution associated with** the class K of $\mathbf{F}$-algebraic systems.

27.3 Behavioral Algebraizability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ be a behavioral algebraic system, K a class of $\mathbf{F}$-algebraic systems and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a behavioral π-institution based on $\mathbf{F}$.

- A **transformation** τ from $\mathcal{I}$ to $\mathcal{I}^\mathsf{K}$ is a collection $\tau = \{\tau^v : v \in V\}$, where, for every $v \in V$, $\tau^v = \{\tau^{v,u} : u \in V\}$ is such that
$$\tau^{v,u} : \mathrm{SEN}^\flat_v \times \prod_{i<\omega} \mathrm{SEN}^\flat_{s_i} \to (\mathrm{SEN}^\flat_u)^2$$
is a collection of natural transformations in $N^\flat$;
- A **transformation** ρ from $\mathcal{I}^\mathsf{K}$ to $\mathcal{I}$ is a collection $\rho = \{\rho^v : v \in V\}$, where, for every $v \in V$, $\rho^v = \{\rho^{v,u} : u \in V\}$ is such that
$$\rho^{v,u} : (\mathrm{SEN}^\flat_v)^2 \times \prod_{i<\omega} \mathrm{SEN}^\flat_{s_i} \to \mathrm{SEN}^\flat_u$$
is a collection of natural transformations in $N^\flat$.

A transformation τ from $\mathcal{I}$ to $\mathcal{I}^\mathsf{K}$ is called an **interpretation**, written $\tau : \mathcal{I} \to \mathcal{I}^\mathsf{K}$, if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\Phi \cup \{\phi\} \subseteq \bigcup_{v\in V} \mathrm{SEN}_v(\Sigma)$,
$$\phi \in C_\Sigma(\Phi) \quad \text{iff} \quad \tau_\Sigma[\phi] \subseteq C^\mathsf{K}_\Sigma(\tau_\Sigma[\Phi]).$$
Similarly, a transformation ρ from $\mathcal{I}^\mathsf{K}$ to $\mathcal{I}$ is called an **interpretation**, written $\rho : \mathcal{I}^\mathsf{K} \to \mathcal{I}$, if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $E \cup \{\phi \approx \psi\} \subseteq \bigcup_{v\in V} \mathrm{SEN}^\flat_v(\Sigma)^2$,
$$\phi \approx \psi \in C^\mathsf{K}_\Sigma(E) \quad \text{iff} \quad \rho_\Sigma[\phi, \psi] \subseteq C_\Sigma(\rho_\Sigma[E]).$$

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a behavioral π-institution based on $\mathbf{F}$. $\mathcal{I}$ is said to be **behaviorally** (**syntactically WF**) **algebraizable** if there exists a class K of $\mathbf{F}$-algebraic systems and interpretations $\tau : \mathcal{I} \to \mathcal{I}^\mathsf{K}$, $\rho : \mathcal{I}^\mathsf{K} \to \mathcal{I}$, that form a **conjugate pair**, i.e., such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$,

- $C_\Sigma(\phi) = C_\Sigma(\rho_\Sigma[\tau_\Sigma[\phi]])$;
- $C^\mathsf{K}_\Sigma(\phi \approx \psi) = C^\mathsf{K}_\Sigma(\tau_\Sigma[\rho_\Sigma[\phi, \psi]])$.

In this case we also say that $\mathcal{I}$ and $\mathcal{I}^\mathsf{K}$ are **equivalent via** (τ, ρ) and we write $(\tau, \rho) : \mathcal{I} \rightleftarrows \mathcal{I}^\mathsf{K}$.

Explicitly, $\mathcal{I}$ is behaviorally algebraizable if and only if, there exists a class K of $\mathbf{F}$-algebraic systems and translations τ from $\mathcal{I}$ to $\mathcal{I}^\mathsf{K}$ and ρ from $\mathcal{I}^\mathsf{K}$ to $\mathcal{I}$, such that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $\Phi \cup \{\phi\} \subseteq \bigcup_{v\in V} \mathrm{SEN}^\flat_v(\Sigma)$ and all $E \cup \{\phi \approx \psi\} \subseteq \bigcup_{v\in V} \mathrm{SEN}^\flat_v(\Sigma)^2$,

(1) $\phi \in C_\Sigma(\Phi)$ if and only if $\tau_\Sigma[\phi] \subseteq C^{\mathsf{K}}_\Sigma(\tau_\Sigma[\Phi])$;

(2) $\phi \approx \psi \in C^{\mathsf{K}}_\Sigma(E)$ if and only if $\rho_\Sigma[\phi,\psi] \subseteq C_\Sigma(\rho_\Sigma[E])$;

(3) $C_\Sigma(\phi) = C_\Sigma(\rho_\Sigma[\tau_\Sigma[\phi]])$;

(4) $C^{\mathsf{K}}_\Sigma(\phi \approx \psi) = C^{\mathsf{K}}_\Sigma(\tau_\Sigma[\rho_\Sigma[\phi,\psi]])$.

As in normal syntactic WF algebraizability, it turns out that, in this case as well, Conditions (1) and (4), or dually, Conditions (2) and (3) suffice to establish behavioral algebraizability.

Proposition 1961 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*,* K *a class of* $\mathbf{F}$*-algebraic systems and* τ*,* ρ *translations from* $\mathcal{I}$ *to* $\mathcal{I}^{\mathsf{K}}$ *and from* $\mathcal{I}^{\mathsf{K}}$ *to* $\mathcal{I}$*, respectively. The following statements are equivalent:*

(i) $\tau : \mathcal{I} \to \mathcal{I}^{\mathsf{K}}$ *is an interpretation and, for all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $v \in V$ *and all* $\phi \approx \psi \in \mathrm{SEN}^\flat_v(\Sigma)$*,* $C^{\mathsf{K}}_\Sigma(\phi \approx \psi) = C^{\mathsf{K}}_\Sigma(\tau_\Sigma[\rho_\Sigma[\phi,\psi]])$*;*

(ii) $\rho : \mathcal{I}^{\mathsf{K}} \to \mathcal{I}$ *is an interpretation and, for all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $v \in V$ *and all* $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$*,* $C_\Sigma(\phi) = C_\Sigma(\rho_\Sigma[\tau_\Sigma[\phi]])$*.*

Proof: We only prove that (i) implies (ii), since the converse then follows by the symmetry of the notion of equivalence. Suppose that (i) holds and let $\Sigma \in |\mathbf{Sign}^\flat|$ and $E \cup \{\phi \approx \psi\} \subseteq \bigcup_{v\in V} \mathrm{SEN}^\flat_v(\Sigma)^2$. Then we have

$$\begin{array}{lll} \phi \approx \psi \in C^{\mathsf{K}}_\Sigma(E) & \text{iff} & \tau_\Sigma[\rho_\Sigma[\phi,\psi]] \subseteq C^{\mathsf{K}}_\Sigma(\tau_\Sigma[\rho_\Sigma[E]]) \\ & \text{iff} & \rho_\Sigma[\phi,\psi] \subseteq C_\Sigma(\rho_\Sigma[E]). \end{array}$$

Moreover, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$, we have, for all $\psi \in \bigcup_{v\in V} \mathrm{SEN}^\flat_v(\Sigma)$,

$$\begin{array}{lll} \psi \in C_\Sigma(\rho_\Sigma[\tau_\Sigma[\phi]]) & \text{iff} & \tau_\Sigma[\psi] \subseteq C^{\mathsf{K}}_\Sigma(\tau_\Sigma[\rho_\Sigma[\tau_\Sigma[\phi]]]) \\ & \text{iff} & \tau_\Sigma[\psi] \subseteq C^{\mathsf{K}}_\Sigma(\tau_\Sigma[\phi]) \\ & \text{iff} & \psi \in C_\Sigma(\phi). \end{array}$$

Hence, $C_\Sigma(\phi) = C_\Sigma(\rho_\Sigma[\tau_\Sigma[\phi]])$. This shows that Condition (ii) holds. ■

We look next at some properties that are entailed by behavioral algebraizability.

Proposition 1962 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$ *and* K *a class of* $\mathbf{F}$*-algebraic systems. If* $(\tau,\rho) : \mathcal{I} \rightleftarrows \mathcal{I}^{\mathsf{K}}$ *is a conjugate pair, then, for all* $v, u \in V$*, all* $\sigma : \mathrm{SEN}_v \times \prod_{i=1}^k \mathrm{SEN}_{s_i} \to \mathrm{SEN}_u$ *in* $N^\flat$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $\phi, \psi, \chi \in \mathrm{SEN}^\flat_v(\Sigma)$ *and all* $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i}(\Sigma)$*,* *(a)* $\rho_\Sigma[\phi,\phi] \subseteq \mathrm{Thm}_\Sigma(\mathcal{I})$;

(b) $\rho_\Sigma[\psi,\phi] \subseteq C_\Sigma(\rho_\Sigma[\phi,\psi])$;

(c) $\rho_\Sigma[\phi,\chi] \subseteq C_\Sigma[\rho_\Sigma[\phi,\psi],\rho_\Sigma[\psi,\chi])$;

(d) $\rho_\Sigma[\sigma_\Sigma(\phi,\vec{\chi}),\sigma_\Sigma(\psi,\vec{\chi})] \subseteq C_\Sigma(\rho_\Sigma[\phi,\psi])$;

(e) $\psi \in C_\Sigma(\phi,\rho_\Sigma[\phi,\psi])$.

Proof:

(a) We have $\phi \approx \phi \in C^{\mathsf{K}}_\Sigma(\varnothing)$, whence, since $\rho : \mathcal{I}^{\mathsf{K}} \to \mathcal{I}$ is an interpretation, $\rho_\Sigma[\phi,\phi] \subseteq C_\Sigma(\varnothing)$.

(b) Since $\psi \approx \phi \in C^{\mathsf{K}}_\Sigma(\phi \approx \psi)$, we get, again by the fact $\rho : \mathcal{I}^{\mathsf{K}} \to \mathcal{I}$ is an interpretation, $\rho_\Sigma[\psi,\phi] \subseteq C_\Sigma(\rho_\Sigma[\phi,\psi])$.

(c) Since $\phi \approx \chi \in C^{\mathsf{K}}_\Sigma(\phi \approx \psi,\psi \approx \chi)$ and $\rho : \mathcal{I}^{\mathsf{K}} \to \mathcal{I}$ is an interpretation, we get that $\rho_\Sigma[\phi,\chi] \subseteq C_\Sigma(\rho_\Sigma[\phi,\psi],\rho_\Sigma[\psi,\chi])$.

(d) By Proposition 1959, we have, for all $\sigma : \mathrm{SEN}_v \times \prod_{i=1}^k \mathrm{SEN}_{s_i} \to \mathrm{SEN}_u$ and all $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i}(\Sigma)$, $\sigma_\Sigma(\phi,\vec{\chi}) \approx \sigma_\Sigma(\psi,\vec{\chi}) \in C^{\mathsf{K}}_\Sigma(\phi \approx \psi)$. Hence, again by the fact that $\rho : \mathcal{I}^{\mathsf{K}} \to \mathcal{I}$ is an interpretation, we get that $\rho_\Sigma[\sigma_\Sigma(\phi,\vec{\chi}),\sigma_\Sigma(\psi,\vec{\chi})] \subseteq C_\Sigma(\rho_\Sigma[\phi,\psi])$.

(e) In $\mathcal{I}^{\mathsf{K}}$, we have $\tau_\Sigma[\psi] \subseteq C^{\mathsf{K}}_\Sigma(\tau_\Sigma[\phi],\phi \approx \psi)$. Hence, by Property (4) of equivalence, $\tau_\Sigma[\psi] \subseteq C^{\mathsf{K}}_\Sigma(\tau_\Sigma[\phi],\tau_\Sigma[\rho_\Sigma[\phi,\psi]])$. Thus, by Property (1) of equivalence, we get that $\psi \in C_\Sigma(\phi,\rho_\Sigma[\phi,\psi])$.

■

We can also prove that, if a behavioral π-institution $\mathcal{I}$ is behaviorally algebraizable in two different ways, then the interpretations are, roughly speaking, interderivable and the classes of **F**-algebraic systems serving as behavioral algebraic semantics generate the same behavioral consequence operators.

Theorem 1963 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally algebraizable via conjugate pairs* $(\tau,\rho) : \mathcal{I} \rightleftarrows \mathcal{I}^{\mathsf{K}}$ *and* $(\tau',\rho') : \mathcal{I} \rightleftarrows \mathcal{I}^{\mathsf{K}'}$*, then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $v \in V$ *and all* $\phi,\psi \in \mathrm{SEN}^\flat_v(\Sigma)$*,*

(a) $C_\Sigma(\rho_\Sigma[\phi,\psi]) = C_\Sigma(\rho'_\Sigma[\phi,\psi])$;

(b) $C^{\mathsf{K}} = C^{\mathsf{K}'}$;

(c) $C^{\mathsf{K}}_\Sigma(\tau_\Sigma[\phi]) = C^{\mathsf{K}}_\Sigma(\tau'_\Sigma[\phi])$.

Proof:

(a) For all $\sigma' \in \rho'$ and all $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i}(\Sigma)$, we get

$$\rho_\Sigma[\sigma'_\Sigma(\phi,\phi,\vec{\chi}),\sigma'_\Sigma(\phi,\psi,\vec{\chi})] \subseteq C_\Sigma(\rho_\Sigma[\phi,\psi]).$$

But we also have $\rho'_\Sigma[\phi,\phi] \subseteq C_\Sigma(\varnothing) \subseteq C_\Sigma(\rho_\Sigma[\phi,\psi])$. Thus, by Proposition 1962, Part (e), $\rho'_\Sigma[\phi,\psi] \subseteq C_\Sigma(\rho_\Sigma[\phi,\psi])$. By symmetry, we now get $C_\Sigma(\rho_\Sigma[\phi,\psi]) = C_\Sigma(\rho'_\Sigma[\phi,\psi])$.

(b) Using Part (a), we get, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $E \cup \{\phi \approx \psi\} \subseteq \bigcup_{v\in V}\mathrm{SEN}^\flat_v(\Sigma)^2$,

$$\begin{array}{rll}\phi \approx \psi \in C^{\mathsf{K}}_\Sigma(E) & \text{iff} & \rho_\Sigma[\phi,\psi] \subseteq C_\Sigma(\rho_\Sigma[E]) \\ & \text{iff} & \rho'_\Sigma[\phi,\psi] \subseteq C_\Sigma(\rho'_\Sigma[E]) \\ & \text{iff} & \phi \approx \psi \in C^{\mathsf{K}'}_\Sigma(E).\end{array}$$

(c) Using Parts (a) and (b), we get

$$\begin{array}{rll}C_\Sigma(\phi) = C_\Sigma(\phi) & \text{iff} & C_\Sigma(\rho_\Sigma[\tau_\Sigma[\phi]]) = C_\Sigma(\rho'_\Sigma[\tau'_\Sigma[\phi]]) \\ & \text{iff} & C_\Sigma(\rho_\Sigma[\tau_\Sigma[\phi]]) = C_\Sigma(\rho_\Sigma[\tau'_\Sigma[\phi]]) \\ & \text{iff} & C^{\mathsf{K}}_\Sigma(\tau_\Sigma[\phi]) = C^{\mathsf{K}}_\Sigma(\tau'_\Sigma[\phi]).\end{array}$$

■

27.4 Behavioral Protoalgebraicity

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a behavioral π-institution based on $\mathbf{F}$.

- $\mathcal{I}$ is **behaviorally protoalgebraic** if the behavioral Leibniz operator $\Upsilon : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ConSys}(\mathbf{F})$ is monotone on the behavioral theory families of $\mathcal{I}$, i.e., for all $T,T' \in \mathrm{ThFam}(\mathcal{I})$,

$$T \leq T' \quad \text{implies} \quad \Upsilon(T) \leq \Upsilon(T').$$

- $\mathcal{I}$ is **behaviorally syntactically protoalgebraic** if there exists a collection $\rho = \{\rho^v : v \in V\}$, where, for all $v \in V$, $\rho^v = \{\rho^{v,u} : u \in V\}$ is such that $\rho^{v,u} : (\mathrm{SEN}^\flat_v)^2 \times \prod_{i<\omega}\mathrm{SEN}^\flat_{s_i} \to \mathrm{SEN}^\flat_u$ in $N^\flat$ is a collection of natural transformations satisfying, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat_v(\Sigma)$,

$$\langle \phi,\psi\rangle \in \Upsilon_\Sigma(T) \quad \text{iff} \quad \rho^v_\Sigma[\phi,\psi] \leq T.$$

  The set ρ is referred to as the set of **witnessing transformations** for the behavioral syntactic protoalgebraicity of $\mathcal{I}$.

We have the following characterization of behavioral protoalgebraicity.

Proposition 1964 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is behaviorally protoalgebraic if and only if, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $v \in V$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$*,*

$$\langle \phi, \psi\rangle \in \Upsilon_\Sigma(T) \quad \textit{implies} \quad C_\Sigma(T_\Sigma, \phi) = C_\Sigma(T_\Sigma, \psi).$$

Proof: Suppose, first, that $\mathcal{I}$ is behaviorally protoalgebraic and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$, such that $\langle \phi, \psi\rangle \in \Upsilon_\Sigma(T)$. Let $T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T_\Sigma \subseteq T'_\Sigma$ and $\psi \in T'_\Sigma$. Then, by behavioral protoalgebraicity, we have $\langle \phi, \psi\rangle \in \Upsilon_\Sigma(T) \subseteq \Upsilon_\Sigma(T')$. Since, by hypothesis, $\psi \in T'_\Sigma$, we get, by compatibility of $\Upsilon(T')$ with T', that $\phi \in T'_\Sigma$. Thus, $\phi \in C_\Sigma(T_\Sigma, \psi)$ and, by symmetry, $C_\Sigma(T_\Sigma, \phi) = C_\Sigma(T_\Sigma, \psi)$.

Suppose, conversely, that the condition in the statement holds and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$ and $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$, such that $\langle \phi, \psi\rangle \in \Upsilon_\Sigma(T)$. Then, since $\Upsilon(T)$ is a congruence system on $\mathbf{F}$, we get, for all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, all $\sigma : \mathrm{SEN}^\flat_v \times \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i} \to \mathrm{SEN}^\flat_u$ in $N^\flat$, with $u \in V$, and all $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i}(\Sigma')$,

$$\langle \sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\phi), \vec{\chi}), \sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\psi), \vec{\chi})\rangle \in \Upsilon_{\Sigma'}(T).$$

By hypothesis,

$$C_{\Sigma'}(T_{\Sigma'}, \sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\phi), \vec{\chi})) = C_{\Sigma'}(T_{\Sigma'}, \sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\psi), \vec{\chi})).$$

Hence, since $T \leq T'$,

$$C_{\Sigma'}(T'_{\Sigma'}, \sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\phi), \vec{\chi})) = C_{\Sigma'}(T'_{\Sigma'}, \sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\psi), \vec{\chi})).$$

We now get

$$\sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\phi), \vec{\chi}) \in T'_{\Sigma'} \quad \text{iff} \quad \sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\psi), \vec{\chi}) \in T'_{\Sigma'}.$$

Therefore, by the characterization in the proof of Theorem 1958, we get that $\langle \phi, \psi\rangle \in \Upsilon_\Sigma(T')$. Hence $\Upsilon(T) \leq \Upsilon(T')$ and it follows that $\mathcal{I}$ is behaviorally protoalgebraic. ∎

Behavioral syntactic protoalgebraicity implies behavioral protoalgebraicity.

Theorem 1965 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally syntactically protoalgebraic, then it is behaviorally protoalgebraic.*

Proof: Suppose $\mathcal{I}$ is behaviorally syntactically protoalgebraic, with witnessing transformations ρ, and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$, we have

$$\begin{array}{rcl}\langle \phi, \psi\rangle \in \Upsilon_\Sigma(T) & \text{iff} & \rho^v_\Sigma[\phi, \psi] \leq T \\ & \text{implies} & \rho^v_\Sigma[\phi, \psi] \leq T' \\ & \text{iff} & \langle \phi, \psi\rangle \in \Upsilon_\Sigma(T').\end{array}$$

Hence, $\Upsilon(T) \leq \Upsilon(T')$ and, therefore, $\mathcal{I}$ is behaviorally protoalgebraic. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a behavioral π-institution based on $\mathbf{F}$. We define the **behavioral reflexive core of** $\mathcal{I}$

$$A^{\mathcal{I}} = \{A^{\mathcal{I},s} : s \in S\},$$

by letting, for all $s \in S$, $A^{\mathcal{I},s}$ be the collection of all natural transformations $\sigma : (\mathrm{SEN}^\flat_s)^2 \times \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i} \to \mathrm{SEN}^\flat_v$ in $N^\flat$, with $v \in V$, such that:

For all $\Sigma \in |\mathbf{Sign}^\flat|$, all $s \in S$, all $\phi \in \mathrm{SEN}^\flat_s(\Sigma)$,

$$\sigma_\Sigma[\phi, \phi] \leq \mathrm{Thm}(\mathcal{I}).$$

The importance of the behavioral reflexive core lies, as in previous cases, in the fact that it forms a pool of candidates for drawing witnessing transformations for the behavioral syntactic protoalgebraicity of $\mathcal{I}$.

Lemma 1966 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally syntactically protoalgebraic, with witnessing transformations* ρ*, then* $\rho \subseteq A^{\mathcal{I}}$*.*

Proof: Suppose $\mathcal{I}$ is behaviorally syntactically protoalgebraic, with witnessing transformations ρ. Let $\sigma \in \rho$, $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$. Since $\langle \phi, \phi \rangle \in \Upsilon_\Sigma(\mathrm{Thm}(\mathcal{I}))$, we get that $\sigma_\Sigma[\phi, \phi] \leq \rho_\Sigma[\phi, \phi] \leq \mathrm{Thm}(\mathcal{I})$. Therefore, we get that $\rho \subseteq A^{\mathcal{I}}$. ■

Moreover, if $\mathcal{I}$ is behaviorally syntactically protoalgebraic, then $A^{\mathcal{I}}$ satisfies a modus ponens property in $\mathcal{I}$.

Theorem 1967 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally syntactically protoalgebraic, then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $v \in V$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$*,*

$$\psi \in C_\Sigma(\phi, A^{\mathcal{I}}_\Sigma[\phi, \psi]).$$

Proof: Assume $\mathcal{I}$ is behaviorally syntactically protoalgebraic, with witnessing transformations ρ and let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$, such that $\phi \in T_\Sigma$ and $A^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T$. Then we have $\phi \in T_\Sigma$ and, by Lemma 1966, $\rho_\Sigma[\phi, \psi] \leq A^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T$, whence $\phi \in T_\Sigma$ and $\langle \phi, \psi \rangle \in \Upsilon_\Sigma(T)$. Thus, by compatibility of $\Upsilon(T)$ with T, we conclude that $\psi \in T_\Sigma$. Therefore, $\psi \in C_\Sigma(\phi, \mathcal{A}^{\mathcal{I}}_\Sigma[\phi, \psi])$. ■

Define, for all $T \in \mathrm{ThFam}(\mathcal{I})$, a relation family $A^{\mathcal{I}}(T) = \{A^{\mathcal{I}}_\Sigma(T)\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ on $\mathbf{F}$, by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $s \in S$ and all $\phi, \psi \in \mathrm{SEN}^\flat_s(\Sigma)$,

$$\langle \phi, \psi \rangle \in A^{\mathcal{I}}_\Sigma(T) \quad \text{iff} \quad A^{\mathcal{I},s}_\Sigma[\phi, \psi] \leq T.$$

Then we have

Lemma 1968 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If, for all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $v \in V$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$*,*

$$\psi \in C_\Sigma(\phi, A^{\mathcal{I}}_\Sigma[\phi, \psi]),$$

then, for all $T \in \mathrm{ThFam}(\mathcal{I})$*,* $A^{\mathcal{I}}(T)$ *is a congruence system on* $\mathbf{F}$ *compatible with* T*.*

Proof: Fix $T \in \mathrm{ThFam}(\mathcal{I})$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $s \in S$ and $\phi \in \mathrm{SEN}^\flat_s(\Sigma)$. By definition of $\mathcal{A}^{\mathcal{I}}$, we have $A^{\mathcal{I}}_\Sigma[\phi, \phi] \leq \mathrm{Thm}(\mathcal{I}) \leq T$. Therefore, $\langle \phi, \phi \rangle \in A^{\mathcal{I}}_\Sigma(T)$ and, hence, $A^{\mathcal{I}}(T)$ is reflexive.

Let $\Sigma \in |\mathbf{Sign}^\flat|$, $s \in S$ and $\phi, \psi \in \mathrm{SEN}^\flat_s(\Sigma)$, such that $\langle \phi, \psi \rangle \in A^{\mathcal{I}}_\Sigma(T)$. Then $A^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T$. Again by the definition of $A^{\mathcal{I}}$, we get that $A^{\mathcal{I}}_\Sigma[\psi, \phi] = A^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T$, whence $\langle \psi, \phi \rangle \in A^{\mathcal{I}}_\Sigma(T)$ and $A^{\mathcal{I}}(T)$ is symmetric.

Let $\Sigma \in |\mathbf{Sign}^\flat|$, $s \in S$ and $\phi, \psi, \chi \in \mathrm{SEN}^\flat_s(\Sigma)$, such that $\langle \phi, \psi \rangle \in A^{\mathcal{I}}_\Sigma(T)$ and $\langle \psi, \chi \rangle \in A^{\mathcal{I}}_\Sigma(T)$. Then, we have $A^{\mathcal{I}}_\Sigma[\phi, \psi] \leq T$ and $A^{\mathcal{I}}_\Sigma[\psi, \chi] \leq T$. Thus, by hypothesis, we have, for all $\alpha \in A^{\mathcal{I}}$, all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma')$ of appropriate sorts,

$$\begin{aligned}
&\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\chi}) \\
&\in C_{\Sigma'}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), \vec{\chi}), \\
&\qquad A^{\mathcal{I}}_{\Sigma'}[\alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\psi), \vec{\chi}), \\
&\qquad\qquad \alpha_{\Sigma'}(\mathrm{SEN}^\flat(f)(\phi), \mathrm{SEN}^\flat(f)(\chi), \vec{\chi})]) \\
&\subseteq C_{\Sigma'}(A^{\mathcal{I}}_\Sigma[\phi, \psi], A^{\mathcal{I}}_\Sigma[\psi, \chi]) \\
&\subseteq T_{\Sigma'}.
\end{aligned}$$

Hence $A^{\mathcal{I}}_\Sigma[\phi, \chi] \leq T$ and, therefore, $\langle \phi, \chi \rangle \in A^{\mathcal{I}}_\Sigma(T)$ and $A^{\mathcal{I}}(T)$ is also transitive. It is, by its definition, a system. To see that it is a congruence system, suppose $\sigma : \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i} \to \mathrm{SEN}_s$ is in N $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi}, \vec{\psi} \in \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i}(\Sigma)$, such that $\vec{\phi} \prod_{i=1}^k A^{\mathcal{I}, s_i}_\Sigma(T) \vec{\psi}$. Then we have

$$\begin{aligned}
A^{\mathcal{I}}_\Sigma[\sigma_\Sigma(\vec{\phi}), \sigma_\Sigma(\vec{\phi})] \leq T \quad &\text{iff} \quad A^{\mathcal{I}}_\Sigma[\sigma_\Sigma(\vec{\phi}), \sigma_\Sigma(\psi_1, \phi_2, \phi_3, \ldots, \phi_{k-1}, \phi_k)] \leq T \\
&\text{iff} \quad A^{\mathcal{I}}_\Sigma[\sigma_\Sigma(\vec{\phi}), \sigma_\Sigma(\psi_1, \psi_2, \phi_3, \ldots, \phi_{k-1}, \phi_k)] \leq T \\
&\text{iff} \quad \cdots \\
&\text{iff} \quad A^{\mathcal{I}}_\Sigma[\sigma_\Sigma(\vec{\phi}), \sigma_\Sigma(\psi_1, \psi_2, \psi_3, \ldots, \psi_{k-1}, \phi_k)] \leq T \\
&\text{iff} \quad A^{\mathcal{I}}_\Sigma[\sigma_\Sigma(\vec{\phi}), \sigma_\Sigma(\vec{\psi})] \leq T.
\end{aligned}$$

Therefore, $A^{\mathcal{I}}(T)$ is a congruence system. Finally, by hypothesis, it is compatible with T. ∎

Lemma 1968 enables us to show that, if the behavioral reflexive core of a behavioral π-institution satisfies the modus ponens property postulated in its hypothesis, then it is behaviorally syntactically protoalgebraic, with witnessing transformations $A^{\mathcal{I},V} = \{A^{\mathcal{I},v} : v \in V\}$.

Theorem 1969 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If, for all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $v \in V$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$*,*

$$\psi \in C_\Sigma(\phi, A^{\mathcal{I}}_\Sigma[\phi, \psi]),$$

then $\mathcal{I}$ *is behaviorally syntactically protoalgebraic, with witnessing transformations* $A^{\mathcal{I},V} = \{A^{\mathcal{I},v} : v \in V\}$*.*

Proof: Suppose that $\mathcal{I}$ satisfies the condition in the hypothesis. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$.

- Assume, first, that $\langle \phi, \psi\rangle \in \Upsilon_\Sigma(T)$. Then, since $\Upsilon(T)$ is a congruence system on $\mathbf{F}$, we have, for all $\sigma : (\mathrm{SEN}^\flat_v)^2 \times \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i} \to \mathrm{SEN}^\flat_u$ in $A^{\mathcal{I},v}$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i}(\Sigma')$,
$$\langle \sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\phi), \mathrm{SEN}^\flat_v(f)(\phi), \vec{\chi}), \sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\phi), \mathrm{SEN}^\flat_v(f)(\psi), \vec{\chi})\rangle \in \Upsilon_{\Sigma'}(T).$$
But, by definition of $A^{\mathcal{I}}$, we also have that
$$\sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\phi), \mathrm{SEN}^\flat_v(f)(\phi), \vec{\chi}) \in \mathrm{Thm}_{\Sigma'}(\mathcal{I}) \subseteq T_{\Sigma'}.$$
Hence, by the compatibility property of $\Upsilon(T)$ with T, we get that $\sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\phi), \mathrm{SEN}^\flat_v(f)(\psi), \vec{\chi}) \in T_{\Sigma'}$. Thus, $A^{\mathcal{I},v}_\Sigma[\phi, \psi] \leq T$.
- Assume, conversely, that $A^{\mathcal{I},v}_\Sigma[\phi, \psi] \leq T$. Then, we get $\langle \phi, \psi\rangle \in A^{\mathcal{I}}_\Sigma(T)$. But, by Lemma 1968 and the hypothesis, $A^{\mathcal{I}}(T)$ is a congruence system on $\mathbf{F}$ compatible with T, whence, by the maximality of $\Upsilon(T)$, $A^{\mathcal{I}}(T) \leq \Upsilon(T)$. Thus, $\langle \phi, \psi\rangle \in \Upsilon_\Sigma(T)$.

We now conclude that $\mathcal{I}$ is behaviorally syntactically protoalgebraic. ■

We have now the essential ingredients for formulating a characterization of behavioral syntactic protoalgebraicity.

$\mathcal{I}$ is behaviorally syntactically protoalgebraic $\longleftrightarrow$ $A^{\mathcal{I},V}$ has the MP.

Theorem 1970 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is behaviorally syntactically protoalgebraic if and only if, for all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $v \in V$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$*,* $\psi \in C_\Sigma(\phi, A^{\mathcal{I},v}_\Sigma[\phi, \psi])$*.*

Proof: The “only if” is by Theorem 1967. The “if” is by Theorem 1969. ■

It is not difficult to show now that, if a behavioral π-institution is behaviorally syntactically protoalgebraic, then any set of witnessing transformations is deductively equivalent to the visible part of the behavioral reflexive core.

Corollary 1971 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally syntactically protoalgebraic, with witnessing transformations* ρ*, then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $v \in V$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$*,*

$$C(A^{\mathcal{I},v}_\Sigma[\phi,\psi]) = C(\rho^v_\Sigma[\phi,\psi]).$$

Proof: Suppose $\mathcal{I}$ is behaviorally syntactically protoalgebraic, with witnessing transformations ρ. Then, by Theorems 1967 and 1969, $A^{\mathcal{I},V}$ is also a collection of witnessing transformations for the behavioral syntactic protoalgebraicity of $\mathcal{I}$. Therefore, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$, we get

$$\begin{array}{lll} A^{\mathcal{I},v}_\Sigma[\phi,\psi] \leq T & \text{iff} & \langle \phi,\psi \rangle \in \Upsilon_\Sigma(T) \\ & \text{iff} & \rho^v_\Sigma[\phi,\psi] \leq T. \end{array}$$

Hence, we get $C(A^{\mathcal{I},v}_\Sigma[\phi,\psi]) = C(\rho^v_\Sigma[\phi,\psi])$. ■

Another characterizing property, therefore, of behavioral syntactic protoalgebraicity is that the behavioral reflexive core define behavioral Leibniz congruence systems in $\mathcal{I}$.

$\mathcal{I}$ is behaviorally syntactically protoalgebraic
$\longleftrightarrow$ $A^{\mathcal{I},V}$ defines behavioral Leibniz congruence systems.

Theorem 1972 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is behaviorally syntactically protoalgebraic if and only if* $A^{\mathcal{I},V}$ *defines behavioral Leibniz congruence systems of theory families in* $\mathcal{I}$*, i.e., for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,* $A^{\mathcal{I},V}(T) = \Upsilon(T)$*.*

Proof: Suppose, first, that $\mathcal{I}$ is behaviorally syntactically protoalgebraic. Then, by Theorems 1967 and 1969, $A^{\mathcal{I},V}$ is a collection of witnessing transformations for the behavioral syntactic protoalgebraicity of $\mathcal{I}$. Therefore, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $A^{\mathcal{I},V}(T) = \Upsilon(T)$. Conversely, if $A^{\mathcal{I},V}(T) = \Upsilon(T)$, for all $T \in \mathrm{ThFam}(\mathcal{I})$, then $\mathcal{I}$ is behaviorally syntactically protoalgebraic, with witnessing transformations $A^{\mathcal{I},V}$. ■

The connection between behavioral syntactic protoalgebraicity and behavioral protoalgebraicity passes through another property of the behavioral Suszko core that we term *Leibniz*.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a behavioral π-institution based on $\mathbf{F}$. We say that the behavioral reflexive core $A^{\mathcal{I}}$ of $\mathcal{I}$ is **Leibniz** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$,

$$\langle \phi,\psi \rangle \in \Upsilon_\Sigma(C(A^{\mathcal{I},v}_\Sigma[\phi,\psi])).$$

It is straightforward to show that, if $A^{\mathcal{I},V}$ has the modus ponens property in $\mathcal{I}$, then it is also Leibniz.

Proposition 1973 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $A^{\mathcal{I}}$ *has the modus ponens in* $\mathcal{I}$*, then it is Leibniz.*

Proof: Suppose that $A^{\mathcal{I}}$ has the modus ponens in $\mathcal{I}$. Then, by Theorem 1969, $\mathcal{I}$ is behaviorally syntactically protoalgebraic, with witnessing transformations $A^{\mathcal{I},V}$. Thus, we obtain, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$,

$$\langle \phi, \psi\rangle \in \Upsilon_\Sigma(C(A^{\mathcal{I},v}_\Sigma[\phi,\psi])) \quad \text{iff} \quad A^{\mathcal{I},v}_\Sigma[\phi,\psi] \le C(A^{\mathcal{I},v}_\Sigma[\phi,\psi]).$$

However, the condition of the right always holds, whence, we get that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$, $\langle \phi, \psi\rangle \in \Upsilon_\Sigma(C(A^{\mathcal{I},v}_\Sigma[\phi,\psi]))$, i.e., $A^{\mathcal{I}}$ is Leibniz. ∎

The opposite implication is not true in general. It holds, however, in behaviorally protoalgebraic behavioral π-institutions.

Proposition 1974 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behaviorally protoalgebraic behavioral* π*-institution based on* $\mathbf{F}$*. If* $A^{\mathcal{I}}$ *is Leibniz, then it has the modus ponens in* $\mathcal{I}$*.*

Proof: Suppose that $\mathcal{I}$ is behaviorally protoalgebraic and $A^{\mathcal{I}}$ is Leibniz. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$, such that $\phi \in T_\Sigma$ and $A^{\mathcal{I},v}_\Sigma[\phi,\psi] \le T$. Since $A^{\mathcal{I}}$ is Leibniz, we get that $\langle \phi, \psi\rangle \in \Upsilon_\Sigma(C(A^{\mathcal{I},v}_\Sigma[\phi,\psi]))$. Since $A^{\mathcal{I},v}_\Sigma[\phi,\psi] \le T$, we get, by the hypothesis of behavioral protoalgebraicity, $\Upsilon(C(A^{\mathcal{I},v}_\Sigma[\phi,\psi])) \le \Upsilon(T)$, whence, $\langle \phi, \psi\rangle \in \Upsilon_\Sigma(T)$. Hence, by the compatibility of $\Upsilon(T)$, with T, we get $\psi \in T_\Sigma$. We conclude that $\psi \in C_\Sigma(\phi, A^{\mathcal{I},v}_\Sigma[\phi,\psi])$ and, thus, $A^{\mathcal{I}}$ has the modus ponens in $\mathcal{I}$. ∎

We close by formulating the exact relation between behavioral syntactic protoalgebraicity and behavioral protoalgebraicity.

Behavioral Syntactic Protoalgebraicity
= $A^{\mathcal{I}}$ has the Modus Ponens
= $A^{\mathcal{I}}$ Defines Behavioral Leibniz Congruence Systems
= Behavioral Protoalgebraicity + $A^{\mathcal{I}}$ Leibniz

Theorem 1975 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is behaviorally syntactically protoalgebraic if and only if it is behaviorally protoalgebraic and has a Leibniz behavioral reflexive core.*

Proof: If $\mathcal{I}$ is behaviorally syntactically protoalgebraic, then, by Theorem 1965, it is behaviorally protoalgebraic, by Theorem 1967, $A^{\mathcal{I}}$ has the modus ponens and, hence, by Proposition 1973, $A^{\mathcal{I}}$ is Leibniz.

Conversely, if $\mathcal{I}$ is behaviorally protoalgebraic and $A^{\mathcal{I}}$ is Leibniz, then, by Proposition 1974, $A^{\mathcal{I}}$ has the modus ponens in $\mathcal{I}$, whence, by Theorem 1969, $\mathcal{I}$ is behaviorally syntactically protoalgebraic. ∎

27.5 Behavioral Truth Equationality

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}_s^\flat\}_H^{V,V^*}, N^\flat \rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a behavioral π-institution based on $\mathbf{F}$.

- $\mathcal{I}$ is **behaviorally completely reflective** (or **behaviorally c-reflective**, for short), if, for all $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$,
$$\bigcap_{T \in \mathcal{T}} \Upsilon(T) \le \Upsilon(T') \quad \text{implies} \quad \bigcap_{T \in \mathcal{T}} T \le T'.$$
- $\mathcal{I}$ is **behaviorally truth equational** if there exists $\tau = \{\tau^v : v \in V\}$, where, for all $v \in V$, $\tau^v = \{\tau^{v,u} : u \in V\}$ is a collection of natural transformations $\tau^{v,u} : \mathrm{SEN}_v^\flat \times \prod_{i<\omega} \mathrm{SEN}_{s_i}^\flat \to (\mathrm{SEN}_u^\flat)^2$ in $N^\flat$, such that, for all $T \in \mathbf{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi \in \mathrm{SEN}_v^\flat(\Sigma)$,
$$\phi \in T_\Sigma \quad \text{iff} \quad \tau_\Sigma^v[\phi] \le \Upsilon(T).$$
In this case, the collection τ forms a set of **witnessing transformations for the behavioral truth equationality of** $\mathcal{I}$.

We have the following alternative characterization of behavioral c-reflectivity, involving both the behavioral Suszko and the behavioral Leibniz operator.

Lemma 1976 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}_s^\flat\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$. $\mathcal{I}$ *is behaviorally c-reflective if and only if, for all* $T, T' \in \mathrm{ThFam}(\mathcal{I})$,
$$\widetilde{\Upsilon}(T) \le \Upsilon(T') \quad \textit{implies} \quad T \le T'.$$

Proof: Suppose, first, that $\mathcal{I}$ is behaviorally c-reflective and let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $\widetilde{\Upsilon}(T) \le \Upsilon(T')$. Then, we have $\bigcap\{\Upsilon(X) : T \le X \in \mathrm{FiFam}(\mathcal{I})\} \le \Upsilon(T')$, whence, by behavioral c-reflectivity, $\bigcap\{X : T \le X \in \mathrm{ThFam}(\mathcal{I})\} \le T'$, i.e., $T \le T'$. Thus, the condition of the statement holds.

Assume, conversely, that the condition of the statement holds and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Upsilon(T) \le \Upsilon(T')$. Then we get
$$\widetilde{\Upsilon}(\bigcap \mathcal{T}) \le \bigcap\{\Upsilon(T) : T \in \mathcal{T}\} \le \Upsilon(T').$$

Therefore, by the hypothesis, $\bigcap \mathcal{T} \leq T'$ and, hence, $\mathcal{I}$ is behaviorally c-reflective. ∎

It is easy to see that behavioral truth equationality implies behavioral c-reflectivity.

Proposition 1977 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally truth equational, then it is behaviorally c-reflective.*

Proof: Suppose that $\mathcal{I}$ is behaviorally truth equational, with witnessing transformations $\tau = \{\tau^v : v \in V\}$, and let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T \in \mathcal{T}} \Upsilon(T) \leq \Upsilon(T')$. Then, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$,

$$\begin{array}{rcl}\phi \in \bigcap_{T \in \mathcal{T}} T_\Sigma & \text{iff} & \phi \in T_\Sigma,\ T \in \mathcal{T},\\ & \text{iff} & \tau^v_\Sigma[\phi] \leq \Upsilon(T),\ T \in \mathcal{T},\\ & \text{iff} & \tau^v_\Sigma[\phi] \leq \bigcap_{T \in \mathcal{T}} \Upsilon(T)\\ & \text{implies} & \tau^v_\Sigma[\phi] \leq \Upsilon(T')\\ & \text{iff} & \phi \in T'_\Sigma.\end{array}$$

Therefore, $\bigcap \mathcal{T} \leq T'$ and $\mathcal{I}$ is indeed behaviorally c-reflective. ∎

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a behavioral π-institution based on $\mathbf{F}$. We define the **behavioral Suszko core** $\Sigma^{\mathcal{I}} = \{\Sigma^{\mathcal{I},v} : v \in V\}$ **of** $\mathcal{I}$ by setting, for all $v \in V$,

$$\begin{array}{l}\Sigma^{\mathcal{I},v} = \{\sigma : \mathrm{SEN}^\flat_v \times \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i} \to (\mathrm{SEN}^\flat_u)^2, u \in V :\\ \qquad (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \phi \in \mathrm{SEN}^\flat_v(\Sigma))(\sigma_\Sigma[\phi] \leq \widetilde{\Upsilon}(C(\phi)))\}\end{array}$$

$\Sigma^{\mathcal{I}}$ is a pool for possible candidates witnessing the potential behavioral truth equationality of $\mathcal{I}$.

Lemma 1978 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally truth equational, with witnessing transformations* τ*, then* $\tau \subseteq \Sigma^{\mathcal{I}}$*.*

Proof: Suppose $\mathcal{I}$ is behaviorally truth equational, with witnessing transformations $\tau = \{\tau^v : v \in V\}$ and let $v \in V$, $\sigma \in \tau^v$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$. Then, we have, for all $T \in \mathrm{ThFam}(\mathcal{I})$, such that $\phi \in T_\Sigma$, $\sigma_\Sigma[\phi] \leq \Upsilon(T)$, whence

$$\sigma_\Sigma[\phi] \leq \bigcap\{\Upsilon(T) : \phi \in T_\Sigma\} = \widetilde{\Upsilon}(C(\phi)).$$

We conclude that $\sigma \in \Sigma^{\mathcal{I},v}$. Therefore, $\tau \subseteq \Sigma^{\mathcal{I}}$. ∎

The behavioral Suszko core $\Sigma^{\mathcal{I}}$ was devised to carry a sentence of visible sort into the behavioral Suszko congruence system of the theory family generated by it. Because of the monotonicity of the behavioral Suszko operator and the fact that the behavioral Suszko operator is universally subsumed by the behavioral Leibniz operator, however, it turns out that the image of any behavioral theory family under the behavioral Suszko core always lies inside the behavioral Leibniz congruence system of the family.

Proposition 1979 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system,* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*,* $T \in \mathrm{ThFam}(\mathcal{I})$*,* $\Sigma \in |\mathbf{Sign}^\flat|$*,* $v \in V$ *and* $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$*. If* $\phi \in T_\Sigma$*, then*

$$\Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \Upsilon(T).$$

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$, such that $\phi \in T_\Sigma$. Then, we have

$$\begin{array}{lll}\Sigma^{\mathcal{I},v}_\Sigma[\phi] & \leq & \widetilde{\Upsilon}(C(\phi)) \quad (\text{definition of } \Sigma^{\mathcal{I}})\\ & \leq & \widetilde{\Upsilon}(T) \quad (\text{monotonicity of } \widetilde{\Upsilon})\\ & \leq & \Upsilon(T). \quad (\widetilde{\Upsilon} \leq \Upsilon)\end{array}$$

This proves the conclusion. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a behavioral π-institution based on $\mathbf{F}$. We say that the behavioral Suszko core $\Sigma^{\mathcal{I}}$ of $\mathcal{I}$ is **soluble** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$,

$$\Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \Upsilon(T) \quad \text{implies} \quad \phi \in T_\Sigma.$$

The solubility of the behavioral Suszko core is a necessary condition for a behavioral π-institution to be behaviorally truth equational.

Theorem 1980 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally truth equational, then* $\Sigma^{\mathcal{I}}$ *is soluble.*

Proof: Suppose $\mathcal{I}$ is behaviorally truth equational, with witnessing transformations $\tau = \{\tau^v : v \in V\}$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$, such that $\Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \Upsilon(T)$, we have, by Lemma 1978,

$$\tau^v_\Sigma[\phi] \leq \Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \Upsilon(T),$$

whence, by the fact that τ witnesses the behavioral truth equationality of $\mathcal{I}$, $\phi \in T_\Sigma$. Therefore, $\Sigma^{\mathcal{I}}$ is indeed soluble. ■

Conversely, the solubility of the behavioral Suszko core ensures that it can serve as a collection of witnessing transformations for the behavioral truth equationality of $\mathcal{I}$.

Theorem 1981 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}_s^\flat\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If the behavioral Suszko core* $\Sigma^{\mathcal{I}}$ *is soluble, then* $\mathcal{I}$ *is behaviorally truth equational, with witnessing transformations* $\Sigma^{\mathcal{I}}$*.*

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi \in \mathrm{SEN}_v^\flat(\Sigma)$. If $\phi \in T_\Sigma$, then, by Proposition 1979, $\Sigma_\Sigma^{\mathcal{I},v}[\phi] \leq \Upsilon(T)$. On the other hand, if $\Sigma_\Sigma^{\mathcal{I},v}[\phi] \leq \Upsilon(T)$, then, by the postulated solubility of $\Sigma^{\mathcal{I}}$, we get that $\phi \in T_\Sigma$. Hence, we have $\phi \in T_\Sigma$ if and only if $\Sigma_\Sigma^{\mathcal{I},v}[\phi] \leq \Upsilon(T)$, showing that $\Sigma^{\mathcal{I}}$ witnesses the behavioral truth equationality of $\mathcal{I}$. ∎

We now have the following characterization of behavioral truth equationality depending on the behavior (in the ordinary sense) of the behavioral Suszko core.

$$\mathcal{I} \text{ is Behaviorally Truth Equational} \longleftrightarrow \Sigma^{\mathcal{I}} \text{ is Soluble.}$$

Theorem 1982 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}_s^\flat\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is behaviorally truth equational if and only if it has a soluble behavioral Suszko core.*

Proof: Necessity is by Theorem 1980, whereas sufficiency is proved in Theorem 1981. ∎

We say that the behavioral Suszko core $\Sigma^{\mathcal{I}}$ of a behavioral π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$ **defines theory families** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi \in \mathrm{SEN}_v^\flat(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad \Sigma_\Sigma^{\mathcal{I},v}[\phi] \leq \Upsilon(T).$$

Then, another characterization of behavioral truth equationality is the following:

$$\mathcal{I} \text{ is Behaviorally Truth Equational} \longleftrightarrow \Sigma^{\mathcal{I}} \text{ Defines Theory Families.}$$

Theorem 1983 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}_s^\flat\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is behaviorally truth equational if and only if its behavioral Suszko core* $\Sigma^{\mathcal{I}}$ *defines theory families.*

Proof: $\mathcal{I}$ is behaviorally truth equational if and only if, by Theorem 1982 $\Sigma^{\mathcal{I}}$ is soluble if and only if, by Proposition 1979 and the definition of solubility, $\Sigma^{\mathcal{I}}$ defines theory families in $\mathcal{I}$. ∎

We have just seen that behavioral truth equationality of a behavioral π-institution is equivalent to the solubility property of its behavioral Suszko core. The solubility property implies another property, which, taking after similar work in preceding chapters, we call *adequacy*. It says, roughly speaking, that in a behavioral π-institution the category of natural transformations is rich enough to determine behavioral Suszko congruence systems in terms of the behavioral Leibniz congruence systems that it selects by inclusion. The property of adequacy is motivated by the following property that holds in every behavioral π-institution.

Proposition 1984 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. For all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $v \in V$ *and all* $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$*,*

$$\bigcap\{\Upsilon(T) : \Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \Upsilon(T)\} \leq \widetilde{\Upsilon}(C(\phi)).$$

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$. Then we have, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\begin{array}{lll}\phi \in T_\Sigma & \text{implies} & \Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \widetilde{\Upsilon}(T)\\ & \text{implies} & \Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \Upsilon(T).\end{array}$$

Thus, we get

$$\begin{array}{lll}\bigcap\{\Upsilon(T) : \Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \Upsilon(T)\} & \leq & \bigcap\{\Upsilon(T) : \Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \widetilde{\Upsilon}(T)\}\\ & \leq & \bigcap\{\Upsilon(T) : \phi \in T_\Sigma\}\\ & = & \widetilde{\Upsilon}(C(\phi)).\end{array}$$

Hence, the inclusion in the statement holds. ■

The reverse inclusion is not always guaranteed, but, when it holds, we say that the behavioral Suszko core of $\mathcal{I}$ is *adequate*. The terminology is intended to convey the idea that $\Sigma^{\mathcal{I},v}_\Sigma[\phi]$ suffices to determine the theory families whose behavioral Leibniz congruence systems form a "covering" of the behavioral Suszko congruence system corresponding to the theory family $C(\phi)$.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a behavioral π-institution based on $\mathbf{F}$. The behavioral Suszko core $\Sigma^{\mathcal{I}}$ of $\mathcal{I}$ is **adequate** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$,

$$\widetilde{\Upsilon}(C(\phi)) \leq \bigcap\{\Upsilon(T) : \Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \Upsilon(T)\}.$$

We can prove immediately that the solubility of the behavioral Suszko core implies its adequacy.

Proposition 1985 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If the behavioral Suszko core* $\Sigma^{\mathcal{I}}$ *is soluble, then it is adequate.*

Proof: Suppose that $\Sigma^{\mathcal{I}}$ is soluble. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$. By solubility, for all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \Upsilon(T) \quad \text{implies} \quad \phi \in T_\Sigma.$$

Hence, we get

$$\widetilde{\Upsilon}(C(\phi)) \leq \widetilde{\Upsilon}(T) \leq \Upsilon(T).$$

Since this holds, for all $T \in \mathrm{ThFam}(\mathcal{I})$, such that $\Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \Upsilon(T)$, we get that

$$\widetilde{\Upsilon}(C(\phi)) \leq \bigcap\{\Upsilon(T) : \Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \Upsilon(T)\}.$$

Therefore, $\Sigma^{\mathcal{I}}$ is adequate. ∎

Conversely, if a behavioral π-institution is behaviorally c-reflective, then the adequacy of its behavioral Suszko core is sufficient to give its solubility.

Proposition 1986 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}^{V,V^*}_H, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally c-reflective and the behavioral Suszko core* $\Sigma^{\mathcal{I}}$ *is adequate, then* $\Sigma^{\mathcal{I}}$ *is soluble.*

Proof: Suppose that $\mathcal{I}$ is behaviorally c-reflective and that the behavioral Suszko core $\Sigma^{\mathcal{I}}$ is adequate. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$, such that $\Sigma^{\mathcal{I},v}_\Sigma[\phi] \leq \Upsilon(T)$. Then, by the adequacy of the Suszko core, we get that $\widetilde{\Upsilon}(C(\phi)) \leq \Upsilon(T)$, whence, by behavioral c-reflectivity and Lemma 1976, we get $C(\phi) \leq T$, i.e., $\phi \in T_\Sigma$. We conclude that $\Sigma^{\mathcal{I}}$ is soluble. ∎

We can now show that a behavioral π-institution is behaviorally truth equational if and only if it is behaviorally c-reflective and has an adequate behavioral Suszko core.

Behavioral Truth Equationality
= $\Sigma^{\mathcal{I}}$ Soluble
= $\Sigma^{\mathcal{I}}$ Defines Theory Families
= Behavioral c-Reflectivity + $\Sigma^{\mathcal{I}}$ Adequate

Theorem 1987 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}^{V,V^*}_H, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is behaviorally truth equational if and only if it is behaviorally c-reflective and has an adequate behavioral Suszko core.*

Proof: If $\mathcal{I}$ is behaviorally truth equational, then, by Proposition 1977, it is behaviorally c-reflective, by Theorem 1980, its behavioral Suszko core is soluble and, by Proposition 1985, its behavioral Suszko core is adequate. Conversely, if $\mathcal{I}$ is behaviorally c-reflective with an adequate behavioral Suszko core, then, by Proposition 1986, its behavioral Suszko core is soluble and, hence, by Theorem 1981, $\mathcal{I}$ is behaviorally truth equational. ∎

27.6 Behavioral Weak Algebraizability

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a behavioral π-institution based on $\mathbf{F}$. $\mathcal{I}$ is **behaviorally WF algebraizable** if it is behaviorally protoalgebraic and behaviorally c-reflective.

Lemma 1988 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally protoalgebraic, then, for all* $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$*,*

$$\Upsilon(\bigcap_{i\in I} T^i) = \bigcap_{i\in I} \Upsilon(T^i).$$

Proof: Suppose $\mathcal{I}$ is behaviorally protoalgebraic and let $\{T^i : i \in I\} \subseteq \mathrm{ThFam}(\mathcal{I})$. Then, by hypothesis, $\Upsilon(\bigcap_{i\in I} T^i) \leq \bigcap_{i\in I} \Upsilon(T^i)$. On the other hand, $\bigcap_{i\in I} \Upsilon(T^i)$ is a congruence system on $\mathbf{F}$. Moreover, it is easy to see that it is compatible with $\bigcap_{i\in I} T^i$. Hence, by the maximality property of the behavioral Leibniz congruence system, we get $\bigcap_{i\in I} \Upsilon(T^i) \leq \Upsilon(\bigcap_{i\in I} T^i)$. ■

Lemma 1989 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally protoalgebraic and the behavioral Leibniz operator is injective, then, for all* $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$*,*

$$\bigcap_{T\in\mathcal{T}} \Upsilon(T) \leq \Upsilon(T') \quad \textit{implies} \quad \bigcap \mathcal{T} \leq T'.$$

Proof: Suppose that $\mathcal{I}$ is behaviorally protoalgebraic and that the behavioral Leibniz operator is injective. Let $\mathcal{T} \cup \{T'\} \subseteq \mathrm{ThFam}(\mathcal{I})$, such that $\bigcap_{T\in\mathcal{T}} \Upsilon(T) \leq \Upsilon(T')$. Then we have

$$\begin{array}{rcll}\Upsilon(\bigcap \mathcal{T} \cap T') & = & \bigcap_{T\in\mathcal{T}} \Upsilon(T) \cap \Upsilon(T') & \text{(Lemma 1988)}\\ & = & \bigcap_{T\in\mathcal{T}} \Upsilon(T) & \text{(hypothesis)}\\ & = & \Upsilon(\bigcap \mathcal{T}). & \text{(Lemma 1988)}\end{array}$$

Hence, by the injectivity of the behavioral Leibniz operator, $\bigcap \mathcal{T} \cap T' = \bigcap \mathcal{T}$, showing that $\bigcap \mathcal{T} \leq T'$. ■

Proposition 1990 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is behaviorally WF algebraizable if and only if the behavioral Leibniz operator is monotone and injective on* $\mathrm{ThFam}(\mathcal{I})$*.*

Proof: Suppose $\mathcal{I}$ is behaviorally WF algebraizable. Then by definition, it is behaviorally protoalgebraic and behaviorally c-reflective. Thus, the behavioral Leibniz operator is monotone and c-reflective on $\mathrm{ThFam}(\mathcal{I})$, whence it is monotone and, a fortiori, injective on $\mathrm{ThFam}(\mathcal{I})$.

If, conversely, Υ is monotone and injective on $\mathrm{ThFam}(\mathcal{I})$, then it is monotone and, by Lemma 1989, c-reflective on $\mathrm{ThFam}(\mathcal{I})$. Hence, $\mathcal{I}$ is behaviorally protoalgebraic and behaviorally c-reflective, i.e., by definition, it is behaviorally WF algebraizable. ∎

Another characterization of behavioral WF algebraizability asserts that it is equivalent to the existence of an isomorphism from the complete lattice of theory families of a behavioral π-institution to the complete lattice of the $\mathcal{I}$-congruence systems on its underlying behavioral algebraic system.

We need the following preparatory definitions.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}_s^\flat\}_H^{V,V^*}, N^\flat \rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a behavioral π-institution based on $\mathbf{F}$.

- Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system. A family $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, with $T_\Sigma \subseteq \bigcup_{v \in V} \mathrm{SEN}_v(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$, is called an **$\mathcal{I}$-filter family** of $\mathcal{A}$ if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, all $v \in V$ and all $\Phi \cup \{\phi\} \subseteq \bigcup_{v \in V} \mathrm{SEN}_v^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$,
$$\alpha_\Sigma(\Phi) \subseteq T_{F(\Sigma)} \quad \text{implies} \quad \alpha_\Sigma(\phi) \in T_{F(\Sigma)}.$$
The collection of all $\mathcal{I}$-filter families of $\mathcal{A}$ is denoted by $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. It is a complete lattice, whose corresponding closure operator will be denoted by $C^{\mathcal{I},\mathcal{A}}$.

- An $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ is an **$\mathcal{I}$-algebraic system** if there exists $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Upsilon}(T) = \Delta^{\mathcal{A}}$. The collection of all $\mathcal{I}$-algebraic systems is denoted by $\mathrm{AlgSys}(\mathcal{I})$.

- Given an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, a congruence system $\theta \in \mathrm{ConSys}(\mathcal{A})$ is a **$\mathcal{I}$-congruence system on $\mathcal{A}$** if $\mathcal{A}/\theta \in \mathrm{AlgSys}(\mathcal{I})$. The collection of all $\mathcal{I}$-congruence systems on $\mathcal{A}$ is denoted by $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{A})$.

Lemma 1991 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}_s^\flat\}_H^{V,V^*}, N^\flat \rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C \rangle$ a behavioral π-institution based on $\mathbf{F}$. For every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ and all $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, such that, for all $\Sigma \in |\mathbf{Sign}|$, $T_\Sigma \subseteq \bigcup_{v \in V} \mathrm{SEN}_v(\Sigma)$,*
$$T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \quad \textit{iff} \quad \alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I}).$$

Proof: Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, such that, for all $\Sigma \in |\mathbf{Sign}|$, $T_\Sigma \subseteq \bigcup_{v \in V} \mathrm{SEN}_v(\Sigma)$.

Assume, first, that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi \in \mathrm{SEN}_v^\flat(\Sigma)$, such that $\phi \in C_\Sigma(\alpha_\Sigma^{-1}(T_{F(\Sigma)}))$. Then, by the definition of $C^{\mathcal{I},\mathcal{A}}$, we get
$$\alpha_\Sigma(\phi) \in C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\alpha_\Sigma^{-1}(T_{F(\Sigma)}))) \subseteq C^{\mathcal{I},\mathcal{A}}_{F(\Sigma)}(T_{F(\Sigma)}) = T_{F(\Sigma)}.$$
Hence, we get $\phi \in \alpha_\Sigma^{-1}(T_{F(\Sigma)})$ and we conclude that $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$.

Suppose, conversely, that $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $\Phi \cup \{\phi\} \subseteq \bigcup_{v \in V} \mathrm{SEN}^\flat_v(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$ and $\alpha_\Sigma(\Phi) \subseteq T_{F(\Sigma)}$. Then $\Phi \subseteq \alpha^{-1}_\Sigma(T_{F(\Sigma)})$, whence, since $\phi \in C_\Sigma(\Phi)$ and $\alpha^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, we get that $\phi \in \alpha^{-1}_\Sigma(T_{F(\Sigma)})$, i.e., $\alpha_\Sigma(\phi) \in T_{F(\Sigma)}$. We conclude that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. ■

Lemma 1992 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}^{V,V^*}_H, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. For every* $\mathbf{F}$*-algebraic system* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ *and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*,*

$$\Upsilon(\alpha^{-1}(T)) = \alpha^{-1}(\Upsilon^{\mathcal{A}}(T)).$$

Proof: Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $s \in S$ and $\phi, \psi \in \mathrm{SEN}^\flat_s(\Sigma)$, we have $\langle \phi, \psi \rangle \in \Upsilon_\Sigma(\alpha^{-1}(T))$ if and only if, for all $\sigma : \mathrm{SEN}^\flat_s \times \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i} \to \mathrm{SEN}^\flat_v$, with $v \in V$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i}(\Sigma')$,

$$\begin{array}{l} \sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\phi), \vec{\chi}) \in \alpha^{-1}_{\Sigma'}(T_{F(\Sigma')}) \\ \qquad \text{iff} \quad \sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\psi), \vec{\chi}) \in \alpha^{-1}_{\Sigma'}(T_{F(\Sigma')}) \end{array}$$

iff

$$\begin{array}{l} \alpha_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\phi), \vec{\chi})) \in T_{F(\Sigma')} \\ \qquad \text{iff} \quad \alpha_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\psi), \vec{\chi})) \in T_{F(\Sigma')} \end{array}$$

iff

$$\begin{array}{l} \sigma^{\mathcal{A}}_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\phi)), \alpha_{\Sigma'}(\vec{\chi})) \in T_{F(\Sigma')} \\ \qquad \text{iff} \quad \sigma^{\mathcal{A}}_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\psi)), \alpha_{\Sigma'}(\vec{\chi})) \in T_{F(\Sigma')} \end{array}$$

iff

$$\begin{array}{l} \sigma^{\mathcal{A}}_{F(\Sigma')}(\mathrm{SEN}_s(F(f))(\alpha_\Sigma(\phi)), \alpha_{\Sigma'}(\vec{\chi})) \in T_{F(\Sigma')} \\ \qquad \text{iff} \quad \sigma^{\mathcal{A}}_{F(\Sigma')}(\mathrm{SEN}_s(F(f))(\alpha_\Sigma(\psi)), \alpha_{\Sigma'}(\vec{\chi})) \in T_{F(\Sigma')} \end{array}$$

if and only if, by the surjectivity of $\langle F, \alpha \rangle$, $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \Upsilon^{\mathcal{A}}_{F(\Sigma)}(T)$ if and only if $\langle \phi, \psi \rangle \in \alpha^{-1}_{F(\Sigma)}(\Upsilon^{\mathcal{A}}_{F(\Sigma)}(T))$. We now conclude that $\Upsilon(\alpha^{-1}(T)) = \alpha^{-1}(\Upsilon^{\mathcal{A}}(T))$. ■

Now we have the following characterization result for behavioral WF algebraizability.

Theorem 1993 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}^{V,V^*}_H, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*.* $\mathcal{I}$ *is behaviorally WF algebraizable if and only if* $\Upsilon : \mathbf{ThFam}(\mathcal{I}) \to \mathbf{ConSys}^{\mathcal{I}}(\mathcal{F})$ *is an order isomorphism.*

Proof: Suppose that $\mathcal{I}$ is behaviorally WF algebraizable. Then, by Proposition 1990, Υ is monotone and injective on $\mathrm{ThFam}(\mathcal{I})$. Moreover, by definition of behavioral WF algebraizability Υ is c-reflective on $\mathrm{ThFam}(\mathcal{I})$ and, therefore, a fortiori, it is order reflecting. Thus, it suffices to show that it is surjective, i.e., onto $\mathrm{ConSys}^{\mathcal{I}}(\mathcal{F})$. To this end, let $\theta \in \mathrm{ConSys}^{\mathcal{I}}(\mathcal{F})$. By definition, $\mathcal{F}/\theta \in \mathrm{AlgSys}(\mathcal{I})$. Thus, there exists $T^\theta \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{F}/\theta)$, such that $\widetilde{\Upsilon}^{\mathcal{F}/\theta}(T^\theta) = \Delta^{\mathcal{F}/\theta}$. Let $\langle I, \pi^\theta\rangle : \mathcal{F} \to \mathcal{F}/\theta$ be the quotient morphism. Now, by Lemma 1991, $\pi^{\theta^{-1}}(T^\theta) \in \mathrm{ThFam}(\mathcal{I})$ and

$$\begin{array}{rcll}\Upsilon(\pi^{\theta^{-1}}(T^\theta)) & = & \pi^{\theta^{-1}}(\Upsilon^{\mathcal{F}/\theta}(T^\theta)) & \text{(Lemma 1992)}\\ & = & \pi^{\theta^{-1}}(\Delta^{\mathcal{F}/\theta}) & \text{(hypothesis and protoalgebraicity)}\\ & = & \theta. & \text{(set theory)}\end{array}$$

Therefore, Υ is surjective and, hence, an order isomorphism from $\mathbf{ThFam}(\mathcal{I})$ onto $\mathbf{ConSys}^{\mathcal{I}}(\mathcal{F})$.

Conversely, if $\Upsilon : \mathbf{ThFam}(\mathcal{I}) \to \mathbf{ConSys}^{\mathcal{I}}(\mathcal{F})$ is an order isomorphism, then it is monotone and injective on $\mathrm{ThFam}(\mathcal{I})$ and, hence, by Proposition 1990, $\mathcal{I}$ is behaviorally WF algebraizable. ■

Finally, we close by providing a relation between behavioral algebraizability and behavioral WF algebraizability. Our first step in this direction is to show that behavioral algebraizability implies both behavioral syntactic protoalgebraicity and behavioral truth equationality. To be able to show this, we start by proving two technical results asserting that the binary relation family induced on the underlying behavioral algebraic system of a given behaviorally algebraizable π-institution by one of the two interpretations witnessing the behavioral algebraizability is a congruence system having a compatibility property.

Lemma 1994 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally algebraizable via a conjugate pair* $(\tau, \rho) : \mathcal{I} \rightleftarrows \mathcal{I}^{\mathsf{K}}$*, for some class* K *of* $\mathbf{F}$*-algebraic systems, then, for all* $\Sigma \in |\mathbf{Sign}^\flat|$*, all* $v \in V$ *and all* $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$*,*

$$\psi \in C_\Sigma(\phi, \rho^v_\Sigma[\phi, \psi]).$$

Proof: Assume $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi, \psi \in \mathrm{SEN}^\flat_v(\Sigma)$, such that $\phi \in T_\Sigma$ and $\rho^v_\Sigma[\phi, \psi] \leq T$. Then we get that

$$\tau^v_\Sigma[\phi] \leq C^{\mathsf{K}}(\tau_\Sigma[T]) \quad \text{and} \quad \langle \phi, \psi\rangle \in C^{\mathsf{K}}(\tau_\Sigma[T]).$$

Hence, by the definition of C^{K}, we get that $\tau^v_\Sigma[\psi] \leq C^{\mathsf{K}}(\tau_\Sigma[T])$ and, therefore, $\psi \in C_\Sigma(T) = T_\Sigma$. We conclude that $\psi \in C_\Sigma(\phi, \rho^v_\Sigma[\phi, \psi])$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ be a behavioral algebraic system and $\mathcal{I} = \langle \mathbf{F}, C\rangle$ a behaviorally algebraizable π-institution, as witnessed by the conjugate pair $(\tau,\rho):\mathcal{I}\rightleftarrows\mathcal{I}^{\mathsf{K}}$, for some class K of $\mathbf{F}$-algebraic systems. We define a class $\rho^+=\{\rho^{+,s}:s\in S\}$ of natural transformations in $N^\flat$ by setting, for all $s\in S$, $\rho^{+,s}$ to be the collection of all natural transformations in $N^\flat$ of the form

$$\sigma^v(\sigma(x,\vec{z}),\sigma(y,\vec{z}),\vec{w}),$$

where

$$\sigma:\mathrm{SEN}^\flat_s\times\prod_{i=1}^k\mathrm{SEN}^\flat_{s_i}\to\mathrm{SEN}^\flat_v,\quad \sigma^v\in\rho^v,\quad v\in V.$$

Moreover, for all $T\in\mathrm{ThFam}(\mathcal{I})$, we define $\rho^{+*}(T)=\{\rho^{+*}_\Sigma(T)\}_{\Sigma\in|\mathbf{Sign}^\flat|}$, where, for all $\Sigma\in|\mathbf{Sign}^\flat|$, we set

$$\rho^{+*}_\Sigma(T)=\{\rho^{+*,s}_\Sigma(T):s\in S\}$$

by letting, for all $\Sigma\in|\mathbf{Sign}^\flat|$, all $s\in S$ and all $\phi,\psi\in\mathrm{SEN}^\flat_s(\Sigma)$,

$$\langle\phi,\psi\rangle\in\rho^{+*,s}_\Sigma(T)\quad\text{iff}\quad\rho^{+,s}_\Sigma[\phi,\psi]\le T.$$

Proposition 1995 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\{\mathrm{SEN}^\flat_s\}_H^{V,V^*},N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I}=\langle\mathbf{F},C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally algebraizable via a conjugate pair* $(\tau,\rho):\mathcal{I}\rightleftarrows\mathcal{I}^{\mathsf{K}}$*, for some class* K *of* $\mathbf{F}$*-algebraic systems, then, for all* $T\in\mathrm{ThFam}(\mathcal{I})$*,* $\rho^{+*}(T)$ *is a congruence system on* $\mathbf{F}$ *compatible with* T*.*

Proof: Let $T\in\mathrm{ThFam}(\mathcal{I})$ and $\Sigma\in|\mathbf{Sign}^\flat|$. Then $\rho^{+*}_\Sigma(T)$ is reflexive, symmetric and transitive, by the definition of $\mathcal{I}^{\mathsf{K}}$, the definition of ρ^+ and the fact that ρ is an interpretation.

E.g., to show symmetry, we let $\Sigma\in|\mathbf{Sign}^\flat|$, $s\in S$ and $\phi,\psi\in\mathrm{SEN}^\flat_s(\Sigma)$, such that $\langle\phi,\psi\rangle\in\rho^{+*,s}_\Sigma(T)$. Then, we have $\rho^{+,s}_\Sigma[\phi,\psi]\le T$ and, thus, for all $\sigma:\mathrm{SEN}^\flat_s\times\prod_{i=1}^k\mathrm{SEN}^\flat_{s_i}\to\mathrm{SEN}^\flat_v$ in $N^\flat$, $\Sigma'\in|\mathbf{Sign}^\flat|$, $f\in\mathbf{Sign}^\flat(\Sigma,\Sigma')$ and $\vec{\chi}\in\prod_{i=1}^k\mathrm{SEN}^\flat_{s_i}(\Sigma')$,

$$\rho^v_{\Sigma'}[\sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\phi),\vec{\chi}),\sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\psi),\vec{\chi})]\le T.$$

This, however, implies that

$$\rho^v_{\Sigma'}[\sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\psi),\vec{\chi}),\sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\phi),\vec{\chi})]\le T.$$

Reversing the steps above, we get that $\langle\psi,\phi\rangle\in\rho^{+*,s}_\Sigma(T)$. Hence, $\rho^{+*}_\Sigma(T)$ is symmetric.

Moreover, it has, by the same considerations, the congruence property. Finally, it is a system by the definition of $\rho^{+*}(T)$. It is compatible with T due to Lemma 1994. ■

Corollary 1996 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally algebraizable via a conjugate pair* $(\tau,\rho) : \mathcal{I} \rightleftarrows \mathcal{I}^{\mathsf{K}}$*, for some class* K *of* $\mathbf{F}$*-algebraic systems, then, for all* $v \in V$*,* $\rho^{+*,v}(T) = \rho^{*,v}(T)$*.*

Proof: Let $v \in V$ and suppose $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi,\psi \in \mathrm{SEN}^\flat_v(\Sigma)$, such that $\langle\phi,\psi\rangle \in \rho^{+*,v}_\Sigma(T)$. Then $\rho^{+,v}_\Sigma[\phi,\psi] \leq T$. But, by definition, $\rho \subseteq \rho^+$, whence, $\rho^v_\Sigma[\phi,\psi] \leq T$, Therefore, $\langle\phi,\psi\rangle \in \rho^{*,v}_\Sigma(T)$.

Suppose, conversely, that $\langle\phi,\psi\rangle \in \rho^{*,v}_\Sigma(T)$. Then $\rho^v_\Sigma[\phi,\psi] \leq T$. But this implies that, for all $\sigma : \mathrm{SEN}^\flat_v \times \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i} \to \mathrm{SEN}^\flat_u$, with $u \in V$, in $N^\flat$, all $\Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and all $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i}(\Sigma')$,

$$\rho^u_{\Sigma'}[\sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\phi),\vec{\chi}),\sigma_{\Sigma'}(\mathrm{SEN}^\flat_v(f)(\psi),\vec{\chi})] \leq T.$$

Therefore, we conclude that $\rho^{+,v}_\Sigma[\phi,\psi] \leq T$, giving that $\langle\phi,\psi\rangle \in \rho^{+*,v}_\Sigma(T)$. ■

Proposition 1995 allows us to establish that the congruence system $\rho^{+*}(T)$ coincides with the behavioral Leibniz congruence system $\Upsilon(T)$ of T in $\mathcal{I}$.

Theorem 1997 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally algebraizable via a conjugate pair* $(\tau,\rho) : \mathcal{I} \rightleftarrows \mathcal{I}^{\mathsf{K}}$*, for some class* K *of* $\mathbf{F}$*-algebraic systems, then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$*,*

$$\rho^{+*}(T) = \Upsilon(T).$$

Proof: Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $s \in S$ and $\phi,\psi \in \mathrm{SEN}^\flat_s(\Sigma)$, such that $\langle\phi,\psi\rangle \in \Upsilon_\Sigma(T)$. Since $\Upsilon(T)$ is a congruence system, we get, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$, $\langle \mathrm{SEN}^\flat_s(f)(\phi), \mathrm{SEN}^\flat_s(f)(\psi)\rangle \in \Upsilon_{\Sigma'}(T)$. Since $\Upsilon(T)$ is a congruence system, we now get, for all $\sigma^v \in \rho^v$, all $\sigma : \mathrm{SEN}^\flat_s \times \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i} \to \mathrm{SEN}^\flat_v$ in $N^\flat$, and all $\vec{\chi} \in \prod_{i=1}^k \mathrm{SEN}^\flat_{s_i}(\Sigma')$, $\vec{\xi} \in \prod_{j<\omega} \mathrm{SEN}^\flat_{s_j}(\Sigma')$,

$$\begin{array}{l}\langle \sigma^v_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\phi),\vec{\chi}),\sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\phi),\vec{\chi}),\vec{\xi}),\\ \qquad \sigma^v_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\phi),\vec{\chi}),\sigma_{\Sigma'}(\mathrm{SEN}^\flat_s(f)(\psi),\vec{\chi}),\vec{\xi})\rangle \in \Upsilon_{\Sigma'}(T).\end{array}$$

On the other hand, we know that $\rho^{+,s}_\Sigma[\phi,\phi] \leq T$, whence, by the compatibility of $\Upsilon(T)$ with T, we get that $\rho^{+,s}_\Sigma[\phi,\psi] \leq T$. Therefore, $\langle\phi,\psi\rangle \in \rho^{+*}_\Sigma(T)$.

Conversely, since, by Proposition 1995, $\rho^{+*}(T)$ is a congruence system on $\mathbf{F}$ that is compatible with T, we get, by the maximality property of the behavioral Leibniz operator, $\rho^{+*}(T) \leq \Upsilon(T)$. ■

Now, we prove that behavioral algebraizability implies both behavioral syntactic protoalgebraicity and behavioral truth equationality.

Theorem 1998 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally algebraizable, then it is both behaviorally syntactically protoalgebraic and behaviorally truth equational.*

Proof: Suppose $\mathcal{I}$ is behaviorally algebraizable via the conjugate pair $(\tau,\rho) : \mathcal{I} \rightleftarrows \mathcal{I}^{\mathsf{K}}$, for some class K of $\mathbf{F}$-algebraic systems.

Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi,\psi \in \mathrm{SEN}^\flat_v(\Sigma)$. Then we have

$$\begin{array}{llll}\langle\phi,\psi\rangle \in \Upsilon_\Sigma(T) & \text{iff} & \langle\phi,\psi\rangle \in \rho^{+*}_\Sigma(T) & \text{(Theorem 1997)}\\ & \text{iff} & \langle\phi,\psi\rangle \in \rho^*_\Sigma(T) & \text{(by Corollary 1996)}\\ & \text{iff} & \rho_\Sigma[\phi,\psi] \le T. & \text{(definition of } \rho^*)\end{array}$$

Therefore, $\mathcal{I}$ is behaviorally syntactically protoalgebraic, with witnessing transformations ρ.

Finally, let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and $\phi \in \mathrm{SEN}^\flat_v(\Sigma)$. Then, we have

$$\begin{array}{llll}\phi \in T_\Sigma & \text{iff} & \rho_\Sigma[\tau_\Sigma[\phi]] \le T & ((\tau,\rho) \text{ conjugate pair})\\ & \text{iff} & \tau_\Sigma[\phi] \subseteq \rho^*_\Sigma(T) & \text{(definition of } \rho^*)\\ & \text{iff} & \tau_\Sigma[\phi] \subseteq \rho^{+*}_\Sigma(T) & \text{(by Corollary 1996)}\\ & \text{iff} & \tau_\Sigma[\phi] \le \Upsilon(T). & \text{(Theorem 1997)}\end{array}$$

Therefore, $\mathcal{I}$ is behaviorally truth equational, with witnessing transformations τ. ■

We show, next, that, conversely, behavioral syntactic protoalgebraicity and behavioral truth equationality guarantee behavioral algebraizability.

Theorem 1999 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}^{V,V^*}_H, N^\flat\rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C\rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. If* $\mathcal{I}$ *is behaviorally syntactically protoalgebraic and behaviorally truth equational, then it is behaviorally algebraizable.*

Proof: Suppose that $\mathcal{I}$ is behaviorally syntactically protoalgebraic, with witnessing transformations ρ, and behaviorally truth equational, with witnessing transformations τ. Then, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\Phi \cup \{\phi\} \subseteq \bigcup_{v\in V}\mathrm{SEN}^\flat_v(\Sigma)$,

$$\begin{array}{lll}\phi \in C_\Sigma(\Phi) & \text{iff} & \phi \in \bigcap\{T_\Sigma : \Phi \subseteq T_\Sigma\}\\ & \text{iff} & \tau_\Sigma[\phi] \le \bigcap\{\Upsilon(T) : \tau_\Sigma[\Phi] \le \Upsilon(T)\}\\ & \text{iff} & \tau_\Sigma[\phi] \le C^{\mathsf{K}}(\tau_\Sigma[\Phi]).\end{array}$$

Moreover, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $v \in V$ and all $\phi,\psi \in \mathrm{SEN}^\flat_v(\Sigma)$,

$$\begin{array}{lll}\langle\phi,\psi\rangle \in \Upsilon_\Sigma(T) & \text{iff} & \rho^v_\Sigma[\phi,\psi] \le T\\ & \text{iff} & \tau[\rho^v_\Sigma[\phi,\psi]] \le \Upsilon(T).\end{array}$$

Hence, we have that $C^{\mathsf{K}}(\phi \approx \psi) = C^{\mathsf{K}}(\tau[\rho^v_\Sigma[\phi,\psi]])$.

We conclude, by Proposition 1961, that $\mathcal{I}$ is equivalent to $\mathcal{I}^{\mathsf{K}}$ and, therefore, $\mathcal{I}$ is behaviorally algebraizable. ■

Now we can formulate the main characterization theorem:

Theorem 2000 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \{\mathrm{SEN}^\flat_s\}_H^{V,V^*}, N^\flat \rangle$ *be a behavioral algebraic system and* $\mathcal{I} = \langle \mathbf{F}, C \rangle$ *a behavioral* π*-institution based on* $\mathbf{F}$*. The following statements are equivalent:*

(i) $\mathcal{I}$ *is behaviorally algebraizable;*

(ii) $\mathcal{I}$ *is behaviorally syntactically protoalgebraic and behaviorally truth equational;*

(iii) $\mathcal{I}$ *is behaviorally WF algebraizable (i.e., behaviorally protoalgebraic and behaviorally c-reflective) and has both a Leibniz behavioral reflexive core and an adequate behavioral Suszko core.*

Proof: If $\mathcal{I}$ is behaviorally algebraizable, then, by Theorem 1998, it is both behaviorally syntactically protoalgebraic and behaviorally truth equational. If $\mathcal{I}$ is behaviorally syntactically protoalgebraic and behaviorally truth equational, then, by Theorems 1975 and 1987, it is behaviorally protoalgebraic, behaviorally c-reflective and has both a Leibniz behavioral reflexive core and an adequate behavioral Suszko core. Finally, if $\mathcal{I}$ is behaviorally WF algebraizable, with a Leibniz behavioral reflexive core and an adequate behavioral Suszko core, then, by Theorems 1975 and 1987, it is behaviorally syntactically protoalgebraic and behaviorally truth equational, whence, by Theorem 1999, it is behaviorally algebraizable. ∎

Chapter 28

List of Problems

A Few Words

We gather some of the problems arising at various places in the main text. They are listed in order of occurrence and no suggestion is made about the degree of their difficulty. In fact, I have not put the same amount of effort in tackling all problems in the list. However, solving them, regardless of difficulty, will definitively inform the theory and the context in which they appear.

Each problem consists of a **task**, which is, in my estimation, the most likely outcome. The **alternative** is also explicitly given. When a task seems a bit involved and I am less certain about its status, I inserted in the list some special subtasks that might be easier to tackle. So, not all problems appearing in the list are independent, and, when aware, I tried to state this explicitly.

As for the format, here is a generic sample, before presenting the list:

Problem X (Chapter 00, Example 00)
This is a special case of Problem Y.
Task: Construct a π-institution in class A but not in class B.
Alternative: Prove that every π-institution in A is also in B.

The List

No problems in the list for the time being.

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