Quasi- and Guasi-Equational Theories and Quasi- and Guasi-Varieties of Algebraic Systems

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December 26, 2021

Abstract

Algebraic systems arise in categorical abstract algebraic logic and form a generalization of universal algebras. They allow multiple signatures and accommodate changes between signatures in the form of signature morphisms as well as natural transformations on signatures, which correspond to term operations in the universal algebraic context. In a way similar to ordinary quasi-equational logic and quasivarieties of universal algebras, one may define quasi-equations and natural quasi-equations and the relation of satisfaction between algebraic systems, on the one hand, and quasi-equations or natural quasi-equations, on the other. They give rise, in the former case, to quasi-equational theories and semantic quasi-varieties, and, in the latter, to meta-quasi-equational theories and syntactic quasi-varieties. More generally, one can treat in an analogous way generalized quasi-equations, i.e., those with infinitely many hypotheses, which will be referred to as guasi-equations, and corresponding classes of algebraic systems, termed guasi-varieties. We study, and provide characterizations of, these theories and these classes of algebraic systems.

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[0]*Keywords:* algebraic systems, quasiequational theories, metaquasiequational theories, quasicongruence systems, semantic quasivarieties, syntactic quasivarieties, Galois connections, Mal'cev's Theorem, class operators
2010 AMS Subject Classification: 03G27

1 Introduction

Abstract algebraic logic is the area of mathematical logic that studies the interaction between logical systems, on the one hand, and classes of algebraic structures on the other. These studies incorporate three very closely related but distinct directions. In the first, which constitutes the backbone and unifying theme of the field, the process by which classes of algebraic structures are associated with given logical systems or classes of logical systems sharing some common properties is studied. In the second, the focus is shifted on the classes of algebraic structures and their properties, which are studied and analyzed by algebraic techniques or, sometimes, using model theoretic techniques, typically drawing on both logical and algebraic background and properties. The third direction establishes connections between properties of logical systems and corresponding algebraic properties of the classes of structures used for their algebraization, according to the general algebraization process. All three directions are expounded upon in greater or lesser detail in recent and relatively recent surveys, monographs and books on the field, e.g., [4, 14, 8, 15, 13].

The main underlying logical structure that is used to formalize logical systems in the classical (or universal algebraic) approach to the field is that of a sentential logic or deductive system. One fixes a logical (or algebraic, depending on the point of view) signature $\mathcal{L}$ and considers the free algebra of formulas (or terms, respectively) $\mathbf{Fm}_{\mathcal{L}}(V)$ ($\mathbf{Tm}_{\mathcal{L}}(V)$, respectively), generated by a countably infinite set V of variables. A *sentential logic* or *deductive system* over $\mathcal{L}$ is a pair $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$, where $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)) \times \mathrm{Fm}_{\mathcal{L}}(V)$ is a structural consequence relation on the set of $\mathcal{L}$-formulas, i.e., it satisfies, for all $\Gamma \cup \Delta \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$ and every substitution (endomorphism) $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$,

- $\Gamma \vdash_{\mathcal{S}} \gamma$, for all $\gamma \in \Gamma$;
- $\Gamma \vdash_{\mathcal{S}} \varphi$ implies $\Delta \vdash_{\mathcal{S}} \varphi$, if $\Gamma \subseteq \Delta$;
- $\Gamma \vdash_{\mathcal{S}} \varphi$ and $\Delta \vdash_{\mathcal{S}} \gamma$, for all $\gamma \in \Gamma$, imply $\Delta \vdash_{\mathcal{S}} \varphi$;
- $\Gamma \vdash_{\mathcal{S}} \varphi$ implies $\sigma(\Gamma) \vdash_{\mathcal{S}} \sigma(\varphi)$.

When the algebraization process is applied on a given deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$, a class of $\mathcal{L}$-algebras, in the sense of universal algebra [5, 22, 1], is obtained as its corresponding algebraic counterpart. As pointed out in [4] (see also [13]), in general, this class is a generalized quasivariety of $\mathcal{L}$-algebraic systems, but, very often, it turns out that it is a variety. Depending on the case, the theories of varieties or of generalized quasivarieties of universal algebra can be brought to bear in the study of the original sentential logic or class of sentential logics. This fact underlies the importance of both theories in the study of logical systems.

From the early days of development, it became clear that the sentential framework was not well suited in handling logical systems that encompass multiple signatures and quantifiers. To deal with such logical systems one would have to first recast them as sentential systems, as was done in Appendix C of [4] and then use, e.g., in the case of first-order logic, cylindric [18] or polyadic [17] algebras to algebraize the sentential version of the system. This unappealing process had led Diskin (unpublished notes, but see, also, [11]) to consider using a categorical framework to incorporate changing of signatures and substitutions in the object language, rather than delegating their handling to the metalanguage. At around the same time, in the computer science domain of formal specification of data structures and programming languages, Goguen and Burstall [16] introduced the structure of an institution with a similar goal in mind, i.e., formalize multi-signature logics with quantifiers in an abstract way. For an extensive and thorough study of institutions from the model theoretic point of view, see [10]. Pigozzi, having pointed out in [4] the artificiality of using sentential logics in the handling of multi-signature systems, and being acquainted with both Diskin's and Goguen and Burstall's work, suggested using institutions, instead of sentential logics, as the underlying formal logical structure on which to base and develop the algebraization process. Since the inspiration came from the sentential framework, it was natural to take the simpler step of incorporating signature changing morphisms and substitutions in the object language, but leaving the manipulation of the models (be it logical or algebraic structures) in the metalanguage. The appropriate structures that facilitated this transition were π-institutions [12], structures constituting modifications of institutions, that incorporate multiple signatures, but, instead of determining consequence model theoretically, adopt, as in deductive systems, an axiomatic viewpoint. Later, under the influence of Font and Jansana's work [14], an enriched version of π-institutions, where, in addition to signature changing morphisms, clones of operations were also incorporated in the object language, was considered in [30].[1]

According to current understanding [32], the categorical side of abstract algebraic logic uses as its underlying structures these enhanced versions of π-institutions, which are based (as sentential logics are based on an algebraic signature and the free algebra of formulas) on algebraic systems, structures that capture both the logical and and algebraic fundamentals underlying the logical system under consideration.

An *algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ consists of a category $\mathbf{Sign}^\flat$ of signatures, a sentence functor $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$, giving, for each signature $\Sigma \in |\mathbf{Sign}^\flat|$, the set $\mathrm{SEN}^\flat(\Sigma)$ of Σ-sentences (and specifying how the signature changing morphisms in $\mathbf{Sign}^\flat$ transform sentences) and a category $N^\flat$ of natural transformations on $\mathrm{SEN}^\flat$, which formalizes the clone of algebraic operations and satisfies certain closure properties (contains all projections, is closed under generalized compositions and is closed under the formation of tuples).

A *π-institution* is a pair $\mathcal{I} = \langle \mathbf{F}, C \rangle$, where $\mathbf{F}$ is an algebraic system (called the base algebraic system of $\mathcal{I}$) and $C = \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ is a family of closure operators, one for each signature Σ, that, in addition to the standard axioms of closure operators (inflationarity, monotonicity and idempotence), satisfy the so-called structurality rule, which stipulates that, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\mathrm{SEN}^\flat(f)(C_\Sigma(\Phi)) \subseteq C_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)).$$

If a process analogous to the one applied in the sentential logic framework, suitably modified, is now applied to π-institutions, one obtains a class or classes of algebraic structures that form the algebraic counterpart of the π-institution under consideration. In the same way that the ties between a sentential logic and the corresponding class of algebraic structures classifies logics into appropriate classes of an algebraic hierarchy, called the *Leibniz hierarchy* (see, e.g., [8] or Chapter 6 of [13]), a similar analysis classifies π-institutions into various classes depending on the strength of these ties (see [32]). The main or core classes in the Leibniz hierarchy of sentential logics are the protoalgebraic logics [3], the equivalential logics [6, 7], the truth equational logics [24], the weakly algebraizable [9] and the algebraizable logics

[1] Even though [30] is historically the first work written using this framework, it was published much later than, e.g., [26], which is the first work using the same framework that appeared in print.

[4, 19]. These classes are surrounded by various weakenings and strengthenings that contribute to the hierarchy pictured, e.g., in page 316 of [13] or page xviii of [23]. Corresponding classes have also been introduced in the hierarchy pertaining to logics formalized as π-institutions [27, 28, 29, 31, 25].[2]

But what are the algebraic structures that one considers in the π-institution framework in lieu of universal algebras, which are used in the algebraization of sentential logics? These are the so-called **F**-algebraic systems, the study of whose classes forms the main object of the present work. An **F**-*algebraic system* is a pair $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, where $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ is an algebraic system, such that there exists a surjective functor $N^\flat \to N$ preserving all projection morphisms, and $\langle F, \alpha\rangle : \mathbf{F} \to \mathbf{A}$ is a surjective morphism, meaning that $F : \mathbf{Sign}^\flat \to \mathbf{Sign}$ is surjective on objects and full, and $\alpha_\Sigma : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SEN}(F(\Sigma))$ is surjective, for all $\Sigma \in |\mathbf{Sign}^\flat|$. The class of all **F**-algebraic systems is denoted $\mathrm{AlgSys}(\mathbf{F})$.

In the first installment of the work detailed here, we focused on analogs of equational theories and varieties [33]. In this second part, we focus instead on the theory of quasi-equations and generalized quasi-equations, referred to as guasi-equations, on the one hand, and quasi-varieties and guasi-varieties of algebraic systems, respectively, on the other.

When one wishes to study classes of **F**-algebraic systems defined by objects playing the role of equations in the universal algebraic context, there are two possible choices. the first is to use pairs of Σ-sentences. These form the family of **F**-*equations* defined by $\mathrm{Eq}(\mathbf{F}) = \{\mathrm{Eq}_\Sigma(\mathbf{F})\}_{\Sigma\in|\mathbf{Sign}^\flat|}$, where, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{Eq}_\Sigma(\mathbf{F}) = \mathrm{SEN}^\flat(\Sigma)^2 = \{\phi \approx \psi : \phi, \psi \in \mathrm{SEN}^\flat(\Sigma)\}.$$

Here, the notation $\phi \approx \psi$ is considered interchangeable with $\langle \phi, \psi\rangle$. The second choice is to use pairs of natural transformations σ, τ in $N^\flat$. These are referred to as *natural* **F**-*equations* and we define

$$\mathrm{NEq}(\mathbf{F}) = \{\sigma \approx \tau : \sigma, \tau \in N^\flat\}.$$

By analogy, when one wishes to study classes o f **F**-algebraic systems defined by objects playing the role of (generalized) quasi-equations in the

[2]The entire hierarchy constitutes the main subject of [32], in which many more classes are introduced, based on refinements of the various properties used to define the core classes.

universal algebraic context, there are two similar choices. The first is to form families of (*generalized*) $\mathbf{F}$-*quasi-equations* defined by, respectively, $\mathrm{GEq}(\mathbf{F}) = \{\mathrm{GEq}_\Sigma(\mathbf{F})\}_{\Sigma\in|\mathbf{Sign}^\flat|}$ and $\mathrm{QEq}(\mathbf{F}) = \{\mathrm{QEq}_\Sigma(\mathbf{F})\}_{\Sigma\in|\mathbf{Sign}^\flat|}$, where, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\begin{array}{lcl}\mathrm{GEq}_\Sigma(\mathbf{F}) & = & \{\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi : \vec{\phi}, \vec{\psi}, \phi, \psi \in \mathrm{SEN}^\flat(\Sigma)\};\\ \mathrm{QEq}_\Sigma(\mathbf{F}) & = & \{\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi : \vec{\phi}, \vec{\psi}, \phi, \psi \in \mathrm{SEN}^\flat(\Sigma), |\vec{\phi}| = |\vec{\psi}| < \omega\}.\end{array}$$

Here, the notation $\vec{\phi} \approx \vec{\psi}$ represents the sequence $\langle \phi_i \approx \psi_i : i \in I\rangle$ of $\mathbf{F}$-equations. The second choice is to use natural $\mathbf{F}$-equations instead of equations and we define

$$\begin{array}{lcl}\mathrm{NGEq}(\mathbf{F}) & = & \{\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau : \vec{\sigma}, \vec{\tau}, \sigma, \tau \in N^\flat\};\\ \mathrm{NQEq}(\mathbf{F}) & = & \{\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau : \vec{\sigma}, \vec{\tau}, \sigma, \tau \in N^\flat, |\vec{\sigma}| = |\vec{\tau}| < \omega\}.\end{array}$$

Having provided some motivation for studying classes of algebraic systems as a necessary component in the process of algebraization of logical systems and of its consequences, we now outline the contents of the present work.

In Section 2, we introduce the satisfaction relation by an $\mathbf{F}$-algebraic system of an $\mathbf{F}$-guasi-equation and the restriction of that relation to $\mathbf{F}$-quasi-equations. These establish in the ordinary way a Galois connection and define closure operators C and C^*, respectively, on the logical side, and $\mathbb{G}^{\mathsf{Sem}}$ and $\mathbb{Q}^{\mathsf{Sem}}$, respectively, on the algebraic side. The former associate with a given collection X of $\mathbf{F}$-guasi-equations ($\mathbf{F}$-quasi-equations, respectively) the guasi- (quasi-)equational theory consisting of all $\mathbf{F}$-guasi- (quasi-)equations that are satisfied by all $\mathbf{F}$-algebraic systems satisfying all $\mathbf{F}$-guasi- (quasi-)equations in X. The latter associate in a corresponding way with a given class K of $\mathbf{F}$-algebraic systems the so-called semantic guasi- (quasi-)variety generated by K.

The notion of a congruence system on an algebraic system is well known. It consists of a family of equivalence relations, indexed by the signatures of the algebraic system, that is invariant both under signature morphisms and under the natural transformations of the algebraic system. It corresponds to the notion of congruence and, among other things, it is possible to consider quotients, which inherit many of the properties they possess in universal algebra [30]. In the context of quasi- and guasi-equations, the role of congruence systems is subsumed by that of *quasi-* and *guasi-congruence systems*, respectively. These are collections of $\mathbf{F}$-quasi- or guasi-equations, respectively, whose equational reduct forms a congruence system and which, in addition, satisfy the property of modus ponens. Two special classes of quasi- and guasi-congruence systems play a special role in obtaining characterizations of theories. *Complete quasi-* and *guasi-congruence systems* are those satisfying a completeness condition with respect to their equational reducts and (*completely*) *coverable* ones are those that can be written as intersections of all complete quasi- or guasi-congruence systems, respectively, that contain them. It is shown in Proposition 8 that a class of quasi- or guasi-equations forms a quasi- or guasi-equational theory, respectively, if it is a coverable quasi- or guasi-congruence system, respectively.

Next, but still staying in Section 2, we turn to the characterization of semantic quasi- and guasi-varieties of **F**-algebraic systems. We use operators on classes of **F**-algebraic systems so as to capture in this framework the spirit of the well-known characterizations of varieties of universal algebras of Birkhoff [2] (see, e.g., Theorem 11.9 of [5] or Theorem 4.41 of [1]) and that of quasivaieties of universal algebras of Mal'cev [20, 21] (see, also, Theorems 0.4.4 and 0.4.5 in [8]). We use the operator $\overset{\triangleleft}{\mathbb{\Pi}}$ of closing under subdirect intersections, which has already been introduced and used in [33] and two new operators. The operator $\mathbb{C}$ closes under certifications and may be viewed as an abstraction operator. Roughly speaking, an algebraic system each of whose components satisfies the same equations as that of a (possibly signature dependent) "witness" algebraic system, already known to be in a class K, is included in $\mathbb{C}(\mathsf{K})$. For the case of quasi-equations, to take into account the underlying finitarity, we modify the certification operator to an operator $\mathbb{C}^*$ that involves directed unions. For details, see the definitions and related discussion following Lemma 10. It is shown in Propositions 15 and 16 that a class K of **F**-algebraic systems forms a semantic guasi-variety if and only if it is closed under the operators $\mathbb{C}$ and $\overset{\triangleleft}{\mathbb{\Pi}}$ and a semantic quasi-variety if and only if it is closed under $\mathbb{C}^*$ and $\overset{\triangleleft}{\mathbb{\Pi}}$.

In Section 3, we shift focus on the relation of satisfaction between **F**-algebraic systems and natural **F**-quasi- and guasi-equations. These also establish Galois connections and give rise, each, to two closure operators; the first to N^* and $\mathbb{Q}^{\mathsf{Syn}}$ and the second to N and $\mathbb{G}^{\mathsf{Syn}}$. The operators N^* and N associate with a given collection R of natural **F**-quasi- or guasi-equations, respectively, the equational metatheory consisting of all natural **F**-quasi- or guasi-equations that satisfy all **F**-algebraic systems satisfying all quasi- or guasi-equations in R. On the other hand, $\mathbb{Q}^{\mathsf{Syn}}$ and $\mathbb{G}^{\mathsf{Syn}}$ associate with a given class K of **F**-algebraic systems the syntactic quasi- and guasi-variety, respectively, generated by K, i.e., the class of all $\mathbf{F}$-algebraic systems satisfying all natural $\mathbf{F}$-quasi- or guasi-equations satisfied by all systems in K.

In the context of natural $\mathbf{F}$-quasi- and guasi-equations, the place of quasi- and guasi-congruence systems is assumed by *meta-quasi* and *meta-guasi-congruences*, abbreviated to mqcongruences and mgcongruences, respectively. These are collections of natural $\mathbf{F}$-quasi- or guasi-equations, whose reducts to natural equations form metacongruences and, moreover, satisfy a modus ponens property. An mqcongruence or mgcongruence is called *feasible* if it arises in a natural way from a coverable quasi- or guasi-congruence system, respectively. Quasi-equational and guasi-equational metatheories are characterized in Proposition 22 as being exactly the feasible mqcongruences and mgcongruences, respectively, on $\mathbf{F}$.

To characterize syntactic quasi- and guasi-varieties, we establish a relationship with semantic quasi- and guasi-varieties, respectively. We say that a given class K of $\mathbf{F}$-algebraic systems is *quasi-* or *guasi-natural* if the family of $\mathbf{F}$-quasi- or guasi-equations, respectively, that it satisfies is coextensive (i.e., has the same models as) the corresponding family induced by the natural $\mathbf{F}$-quasi- or guasi-equations that it satisfies. It is then shown in Proposition 30 that a class K constitutes a syntactic quasi- or guasi-variety if and only if it is a quasi-natural semantic quasi-variety or a guasi-natural semantic guasi-variety, respectively.

In Section 4, we focus on the closure operators C^* and N^*, generating quasi-equational and meta-quasi-equational theories, respectively, and the underlying quasi-equational and meta-quasi-equational logics. We show how, starting from a collection X of $\mathbf{F}$-quasi-equations, checking whether a given quasi-equation belongs to the quasi-equational theory $C^*(X)$ can be reduced to checking whether the conclusion belongs to the theory generated in a structured step-wise fashion by X and the hypotheses. Similarly, starting from a collection R of natural $\mathbf{F}$-quasi-equations, we show how membership in $N^*(R)$ can be checked in an analogous way.

Finally, in Section 5, we concentrate on the operators $\mathbb{Q}^{\mathsf{Sem}}$ and $\mathbb{G}^{\mathsf{Sem}}$ and provide for each a Birkhoff HSP-style (or a Mal'cev SP-style) characterization. Recalling that a class K of $\mathbf{F}$-algebraic systems is a semantic (quasi-) guasi-variety if and only if it is closed under (directed) certifications and subdirect intersections, we show in Proposition 40 that $\mathbb{Q}^{\mathsf{Sem}} = \mathbb{C}^*\overset{\triangleleft}{\mathbb{I}\Pi}$ and, correspondingly, that $\mathbb{G}^{\mathsf{Sem}} = \mathbb{C}\overset{\triangleleft}{\mathbb{I}\Pi}$.

2 Quasi- and Guasi-Equations and Semantic Quasi- and Guasi-Varieties

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system. Define a binary relation

$$\vDash^* \subseteq \mathrm{AlgSys}(\mathbf{F}) \times \mathrm{QEq}(\mathbf{F})$$

by setting, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi}, \phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$ (with $|\vec{\phi}| = |\vec{\psi}| = n < \omega$, so that $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in \mathrm{QEq}_\Sigma(\mathbf{F})$),

$$\begin{aligned}\mathcal{A} \vDash^*_\Sigma \vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \text{ iff} \\ \alpha_\Sigma(\phi_i) = \alpha_\Sigma(\psi_i),\ i < n,\ \text{imply } \alpha_\Sigma(\phi) = \alpha_\Sigma(\psi).\end{aligned}$$

More generally, we define a binary relation

$$\vDash \subseteq \mathrm{AlgSys}(\mathbf{F}) \times \mathrm{GEq}(\mathbf{F})$$

by setting, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi}, \phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{aligned}\mathcal{A} \vDash_\Sigma \vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \text{ iff} \\ \alpha_\Sigma(\phi_i) = \alpha_\Sigma(\psi_i),\ i \in I,\ \text{imply } \alpha_\Sigma(\phi) = \alpha_\Sigma(\psi).\end{aligned}$$

We shall use notation such as $\alpha_\Sigma(\vec{\phi}) = \alpha_\Sigma(\vec{\psi})$ to denote the set of equalities $\alpha_\Sigma(\phi_i) = \alpha_\Sigma(\psi_i)$, $i < n$, where $\vec{\phi} = \langle \phi_0, \phi_1, \ldots, \phi_{n-1} \rangle$ and $\vec{\psi} = \langle \psi_0, \psi_1, \ldots, \psi_{n-1} \rangle$, and, similarly, for tuples over arbitrary index sets. Similar self-explanatory abbreviations will be used throughout, hopefully without causing confusion.

The notation is extended to apply to collections of $\mathbf{F}$-algebraic systems and families of $\mathbf{F}$-quasiequations by setting, for all $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$ and all $X \leq \mathrm{QEq}(\mathbf{F})$,

$$\begin{aligned}\mathsf{K} \vDash^* X \quad \text{iff,} \quad & \text{for all } \mathcal{A} \in \mathsf{K}, \text{ all } \Sigma \in |\mathbf{Sign}^\flat| \text{ and all } q \in X_\Sigma, \\ & \mathcal{A} \vDash^*_\Sigma q.\end{aligned}$$

It is also extended to apply to collections of $\mathbf{F}$-algebraic systems and families of $\mathbf{F}$-guasiequations by setting, for all $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$ and all $X \leq \mathrm{GEq}(\mathbf{F})$,

$$\begin{aligned}\mathsf{K} \vDash X \quad \text{iff,} \quad & \text{for all } \mathcal{A} \in \mathsf{K}, \text{ all } \Sigma \in |\mathbf{Sign}^\flat| \text{ and all } g \in X_\Sigma, \\ & \mathcal{A} \vDash_\Sigma g.\end{aligned}$$

It is clear that $\models^*$ determines a Galois connection between $\mathcal{P}(\mathrm{AlgSys}(\mathbf{F}))$ and $\mathcal{P}(\mathrm{QEq}(\mathbf{F}))$ and that, similarly, $\models$ determines a Galois connection between $\mathcal{P}(\mathrm{AlgSys}(\mathbf{F}))$ and $\mathcal{P}(\mathrm{GEq}(\mathbf{F}))$. Related to these Galois connections, we use the following notational conventions.

First, given a class K of $\mathbf{F}$-algebraic systems, we define the collection $\mathrm{QEq}(\mathsf{K}) = \{\mathrm{QEq}_\Sigma(\mathsf{K})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, where, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{QEq}_\Sigma(\mathsf{K}) = \{q \in \mathrm{QEq}_\Sigma(\mathbf{F}) : \mathsf{K} \models^*_\Sigma q\}.$$

Analogously, the collection $\mathrm{GEq}(\mathsf{K}) = \{\mathrm{GEq}_\Sigma(\mathsf{K})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ is defined by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{GEq}_\Sigma(\mathsf{K}) = \{g \in \mathrm{GEq}_\Sigma(\mathbf{F}) : \mathsf{K} \models_\Sigma g\}.$$

Next, given a family $X = \{X_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ of $\mathbf{F}$-quasiequations, we define

$$\mathrm{Mod}^*(X) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : \mathcal{A} \models^* X\}.$$

Similarly, given a family $X = \{X_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ of $\mathbf{F}$-guasiequations, we define

$$\mathrm{Mod}(X) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : \mathcal{A} \models X\}.$$

Finally, for the closure operators associated with the Galois connection $\models^*$, we set, for all $X \leq \mathrm{QEq}(\mathbf{F})$ and all $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$,

$$\begin{array}{rcl} C^*(X) & = & \mathrm{QEq}(\mathrm{Mod}^*(X)); \\ \mathbb{Q}^{\mathsf{Sem}}(\mathsf{K}) & = & \mathrm{Mod}^*(\mathrm{QEq}(\mathsf{K})). \end{array}$$

Moreover, for the closure operators associated with the Galois connection $\models$, we set, for all $X \leq \mathrm{GEq}(\mathbf{F})$ and all $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$,

$$\begin{array}{rcl} C(X) & = & \mathrm{GEq}(\mathrm{Mod}(X)); \\ \mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) & = & \mathrm{Mod}(\mathrm{GEq}(\mathsf{K})). \end{array}$$

By the general theory of Galois connections, we know that the closed sets of the closure operator C^* are the ones of the form $\mathrm{QEq}(\mathsf{K})$ for a class K of $\mathbf{F}$-algebraic systems and those of the closure operator $\mathbb{Q}^{\mathsf{Sem}}$ are those of the form $\mathrm{Mod}^*(X)$ for a family X of $\mathbf{F}$-quasiequations. Similarly, the closed sets of the closure operator C are the ones of the form $\mathrm{GEq}(\mathsf{K})$ for a class K of $\mathbf{F}$-algebraic systems and those of the closure operator $\mathbb{G}^{\mathsf{Sem}}$ are those of the form $\mathrm{Mod}(X)$ for a family X of $\mathbf{F}$-guasiequations.

In the following section, we set out to provide intrinsic characterizations of those closed sets.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system and X a collection of $\mathbf{F}$-quasiequations or, more generally, a collection of $\mathbf{F}$-guasiequations. We let $\Theta(X) = \{\Theta_\Sigma(X)\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ be the smallest congruence system on $\mathbf{F}$ satisfying, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in X_\Sigma$,

$$\vec{\phi} \approx \vec{\psi} \subseteq \Theta_\Sigma(X) \quad \text{imply} \quad \phi \approx \psi \in \Theta_\Sigma(X).$$

We use the informal expression "$\Theta(X)$ *satisfies* $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi$" to refer to the displayed condition above.

This congruence system is well-defined, since $\nabla^{\mathbf{F}}$ has this property and the collection of all congruence systems having this property is closed under (signature-wise) intersections.

A key property of the congruence system $\Theta(X)$ is that the quotient $\mathcal{F}/\Theta(X)$ satisfies the $\mathbf{F}$-quasiequations (or $\mathbf{F}$-guasiequations) in X.

Lemma 1 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$*, with* $\langle I, \iota \rangle : \mathbf{F} \to \mathbf{F}$ *the identity morphism.*

(a) If $X \leq \mathrm{QEq}(\mathbf{F})$*, then* $X \leq \mathrm{QEq}(\mathcal{F}/\Theta(X))$*;*

(b) If $X \leq \mathrm{GEq}(\mathbf{F})$*, then* $X \leq \mathrm{GEq}(\mathcal{F}/\Theta(X))$*.*

Proof: We only prove Part (a). Part (b) is proven similarly. Suppose $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in \mathrm{QEq}_\Sigma(\mathbf{F})$, such that $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in X_\Sigma$ and $\mathcal{F}/\Theta(X) \vDash^*_\Sigma \vec{\phi} \approx \vec{\psi}$. Then $\vec{\phi} \approx \vec{\psi} \subseteq \Theta_\Sigma(X)$, whence, by the definition of $\Theta(X)$ and the fact that $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in X_\Sigma$, we get that $\phi \approx \psi \in \Theta_\Sigma(X)$. Hence $\mathcal{F}/\Theta(X) \vDash^*_\Sigma \phi \approx \psi$. Therefore, $\mathcal{F}/\Theta(X) \vDash^*_\Sigma \vec{\phi} \approx \vec{\psi} \to \phi \approx \psi$, i.e., $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in \mathrm{QEq}_\Sigma(\mathcal{F}/\Theta(X))$. We conclude that $X \leq \mathrm{QEq}(\mathcal{F}/\Theta(X))$. ∎

Next we define the key concept of *quasicongruence* (or *guasicongruence* in case we are dealing with guasiequations), which parallels in the context of quasiequational theories (guasiequational theories, respectively) the notion of congruence system in the context of equational theories.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system and X a collection of $\mathbf{F}$-quasiequations ($\mathbf{F}$-guasiequations, respectively). First, define

$$\dot{X} := X \cap \mathrm{Eq}(\mathbf{F}),$$

i.e., $\dot{X}$ is the family of **F**-equations contained in X. We say that X is a **quasicongruence system** (**guasicongruence system**, respectively) or, more simply, a **quasicongruence** (**guasicongruence**, respectively) **on F** if $\dot{X}$ is a congruence system on $\mathbf{F}$ that, in addition, satisfies, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi}, \phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\vec{\phi} \approx \vec{\psi} \subseteq X_\Sigma \quad \text{and} \quad \vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in X_\Sigma \quad \text{imply} \quad \phi \approx \psi \in X_\Sigma.$$

Equivalently, X is a quasicongruence (guasicongruence, respectively) on $\mathbf{F}$ if the following conditions are satisfied, for all σ in $N^\flat$, all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\vec{\phi}, \vec{\psi}, \phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$:

Reflexivity $\phi \approx \phi \in X_\Sigma$;

Symmetry $\phi \approx \psi \in X_\Sigma$ implies $\psi \approx \phi \in X_\Sigma$;

Transitivity $\phi \approx \psi, \psi \approx \chi \in X_\Sigma$ imply $\phi \approx \chi \in X_\Sigma$;

Congruence $\vec{\phi} \approx \vec{\psi} \subseteq X_\Sigma$ implies $\sigma_\Sigma(\vec{\phi}) \approx \sigma_\Sigma(\vec{\psi}) \in X_\Sigma$;

Invariance $\phi \approx \psi \in X_\Sigma$ implies $\mathrm{SEN}^\flat(f)(\phi \approx \psi) \in X_{\Sigma'}$;

Modus Ponens $\vec{\phi} \approx \vec{\psi} \subseteq X_\Sigma$ and $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in X_\Sigma$ imply $\phi \approx \psi \in X_\Sigma$.

A few remarks on quasicongruences (and guasicongruences) follow. First, note that every congruence system on $\mathbf{F}$ is a quasicongruence (and a guasicongruence) system, since it contains no proper **F**-guasiequations. Let us denote by

- $\overset{*}{\nabla}{}^{\mathbf{F}}$ the family $\mathrm{QEq}(\mathbf{F})$ of all **F**-quasiequations;
- $\nabla^{\mathbf{F}}$ the family $\mathrm{GEq}(\mathbf{F})$ of all **F**-guasiequations; and
- $\dot{\nabla}^{\mathbf{F}}$ the family $\mathrm{Eq}(\mathbf{F})$ of all **F**-equations, which, as is appropriate respecting previously introduced notation, satisfies

$$\dot{\nabla}^{\mathbf{F}} = \nabla^{\mathbf{F}} \cap \mathrm{Eq}(\mathbf{F}) = \overset{*}{\nabla}{}^{\mathbf{F}} \cap \mathrm{Eq}(\mathbf{F}).$$

We usually write $\overset{*}{\nabla}$, ∇ and $\dot{\nabla}$, respectively, omitting the superscript $^{\mathbf{F}}$, to simplify notation.

Note that $\overset{*}{\nabla}$ constitutes a quasicongruence system on $\mathbf{F}$. Moreover, the signature-wise intersection of any family of quasicongruence systems is also a quasicongruence system, whence the family of all quasicongruence systems on $\mathbf{F}$ forms a complete lattice under signature-wise inclusion $\leq$, which is denoted by $\mathbf{QonSys(F)} = \langle \mathrm{QonSys}(\mathbf{F}), \leq \rangle$. Analogously, ∇ constitutes a guasicongruence system on $\mathbf{F}$ and the family of all guasicongruence systems is clsed under arbitrary intersections, whence it forms a complete lattice under signature-wise inclusion $\leq$, which is denoted by $\mathbf{GonSys(F)} = \langle \mathrm{GonSys}(\mathbf{F}), \leq \rangle$.

We say that a quasicongruence (guasicongruence, respectively) X is **complete** if, in addition, it satisfies, for all $\Sigma \in |\mathbf{Sign}^\flat|$, and all $\vec{\phi}, \vec{\psi}, \phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$

Completeness $\vec{\phi} \approx \vec{\psi} \notin X_\Sigma$ or $\phi \approx \psi \in X_\Sigma$ imply $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in X_\Sigma$.

Again, note:

- First, $\overset{*}{\nabla}$ is a complete quasicongruence and ∇ is a complete guasicongruence;
- Second, if a quasicongruence (guasicongruence, respectively) X is complete, then it is completely determined by the congruence system $\dot{X}$ in the sense that, if $X, Y \in \mathrm{QonSys}(\mathbf{F})$ (or $X, Y \in \mathrm{GonSys}(\mathbf{F})$, respectively) are complete, such that $\dot{X} = \dot{Y}$, then $X = Y$.

Let $\mathrm{QonSys}^\approx(\mathbf{F})$ ($\mathrm{GonSys}^\approx(\mathbf{F})$, respectively) denote the collection of all complete quasicongruence (guasicongruence, respectively) systems on $\mathbf{F}$. Apart from the two basic properties pointed out above, quasicongruences and guasicongruences also satisfy a certain maximality property as detailed in

Lemma 2 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and X, Y be complete $\mathbf{F}$-quasicongruence (or $\mathbf{F}$-guasicongruence, respectively) systems. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$,*

$$Y_\Sigma = X_\Sigma \quad \textit{or} \quad Y_\Sigma = \overset{*}{\nabla}_\Sigma \quad (\nabla_\Sigma, \ \textit{respectively}).$$

Proof: We prove the statement for quasicongruence systems. A similar proof applies to guasicongruence systems. Suppose $X, Y \in \mathrm{QonSys}^\approx(\mathbf{F})$, such that $X \leq Y$, and let $\Sigma \in |\mathbf{Sign}^\flat|$, such that $X_\Sigma \subsetneq Y_\Sigma \neq \overset{*}{\nabla}_\Sigma$. Since X, Y and $\overset{*}{\nabla}$ are complete, they are determined by the corresponding families of equations. Thus, there exist

$$\phi \approx \psi \in Y_\Sigma \backslash X_\Sigma \quad \text{and} \quad \phi' \approx \psi' \in \overset{*}{\nabla}_\Sigma \backslash Y_\Sigma.$$

But, by completeness and modus ponens, these imply that

$$\phi \approx \psi \to \phi' \approx \psi' \in X_\Sigma \backslash Y_\Sigma.$$

The latter, however, contradicts the hypothesis that $X \leq Y$. ∎

Finally, we say that a quasicongruence (guasicongruence, respectively) system X on $\mathbf{F}$ is (**completely**) **coverable** if, for every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $g \notin X_\Sigma$, there exists a complete $Y \in \mathrm{QonSys}^\approx(\mathbf{F})$ ($Y \in \mathrm{GonSys}^\approx(\mathbf{F})$, respectively), such that

$$X \leq Y \quad \text{and} \quad g \notin Y_\Sigma.$$

The following observation follows immediately:

Lemma 3 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* X *a quasicongruence or guasicongruence system on* $\mathbf{F}$*.* X *is coverable if and only if*

$$X = \bigcap \{Y \in \mathrm{QonSys}^\approx(\mathbf{F}) : X \leq Y\},$$

or $X = \bigcap\{Y \in \mathrm{GonSys}^\approx(\mathbf{F}) : X \leq Y\}$*, respectively.*

Proof: Again, we prove the equivalence for quasicongruence systems only, but a very similar proof applies in the case of guasicongruences.

Suppose, first, that $X \in \mathrm{QonSys}(\mathbf{F})$ is coverable. Clearly, $X \leq \bigcap\{Y \in \mathrm{QonSys}^\approx(\mathbf{F}) : X \leq Y\}$. On the other hand, for all $\Sigma \in |\mathbf{Sign}^\flat|$, if $q \notin X_\Sigma$, there exists $Y \in \mathrm{QonSys}^\approx(\mathbf{F})$, such that $X \leq Y$ and $q \notin Y_\Sigma$. Hence, $q \notin \bigcap\{Y_\Sigma : X \leq Y \in \mathrm{QonSys}^\approx(\mathbf{F})\}$. This shows that $\bigcap\{Y \in \mathrm{QonSys}^\approx(\mathbf{F}) : X \leq Y\} \leq X$.

Suppose, conversely, that $X = \bigcap\{Y \in \mathrm{QonSys}^\approx(\mathbf{F}) : X \leq Y\}$ and let $\Sigma \in |\mathbf{Sign}^\flat|$, $q \notin X_\Sigma$. Then, by hypothesis, $q \notin \bigcap\{Y_\Sigma : X \leq Y \in \mathrm{QonSys}^\approx(\mathbf{F})\}$, whence, there exists $Y \in \mathrm{QonSys}^\approx(\mathbf{F})$, such that $X \leq Y$ and $q \notin Y_\Sigma$, showing that X is a coverable quasicongruence system on $\mathbf{F}$. ∎

Using Lemma 3, one may show that the class of all coverable quasicongruence systems on $\mathbf{F}$ forms a complete lattice. A similar assertion holds for the class of all coverable guasicongruence systems on $\mathbf{F}$.

Proposition 4 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\{X^i : i \in I\}$ *a collection of coverable quasicongruence (guasicongruence, respectively) systems on* $\mathbf{F}$. *Then* $\bigcap_{i\in I} X^i$ *is also a coverable quasicongruence (guasicongruence, respectively) system on* $\mathbf{F}$.

Proof: We focus on quasicongruence systems. The case of guasicongruence systems is handled similarly.

By hypothesis, we have, for all $i \in I$,

$$X^i = \bigcap \{Y \in \mathrm{QonSys}^{\approx}(\mathbf{F}) : X^i \leq Y\}.$$

Our aim is to show that $\bigcap_{i\in I} X^i = \bigcap \{Y \in \mathrm{QonSys}^{\approx}(\mathbf{F}) : \bigcap_{i\in I} X^i \leq Y\}$.

First, note that, for all $i \in I$,

$$\{Y \in \mathrm{QonSys}^{\approx}(\mathbf{F}) : X^i \leq Y\} \subseteq \{Y \in \mathrm{QonSys}^{\approx}(\mathbf{F}) : \bigcap_{i\in I} X^i \leq Y\}.$$

This gives

$$\bigcap \{Y \in \mathrm{QonSys}^{\approx}(\mathbf{F}) : \bigcap_{i\in I} X^i \leq Y\} \leq \bigcap \{Y \in \mathrm{QonSys}^{\approx}(\mathbf{F}) : X^i \leq Y\}.$$

So we obtain

$$\begin{aligned}
\bigcap \{Y \in \mathrm{QonSys}^{\approx}(\mathbf{F}) &: \textstyle\bigcap_{i\in I} X^i \leq Y\} \\
&\leq \textstyle\bigcap_{i\in I} \bigcap \{Y \in \mathrm{QonSys}^{\approx}(\mathbf{F}) : X^i \leq Y\} \\
&= \textstyle\bigcap_{i\in I} X^i \\
&\leq \textstyle\bigcap \{Y \in \mathrm{QonSys}^{\approx}(\mathbf{F}) : \bigcap_{i\in I} X^i \leq Y\}.
\end{aligned}$$

Thus, $\bigcap_{i\in I} X^i = \bigcap \{Y \in \mathrm{QonSys}^{\approx}(\mathbf{F}) : \bigcap_{i\in I} X^i \leq Y\}$. ■

We denote by $\mathbf{QonSys}^{\wedge}(\mathbf{F}) = \langle \mathrm{QonSys}^{\wedge}(\mathbf{F}), \leq \rangle$ the complete lattice of coverable quasicongruence systems on $\mathbf{F}$ and by $\mathbf{GonSys}^{\wedge}(\mathbf{F}) = \langle \mathrm{GonSys}^{\wedge}(\mathbf{F}), \leq \rangle$ the complete lattice of coverable guasicongruence systems on $\mathbf{F}$. Note the following obvious inclusions

$$\begin{aligned}
\mathrm{QonSys}^{\approx}(\mathbf{F}) \subseteq \mathrm{QonSys}^{\wedge}(\mathbf{F}) \subseteq \mathrm{QonSys}(\mathbf{F}); \\
\mathrm{GonSys}^{\approx}(\mathbf{F}) \subseteq \mathrm{GonSys}^{\wedge}(\mathbf{F}) \subseteq \mathrm{GonSys}(\mathbf{F}).
\end{aligned}$$

Recall that, by the definition of a quasicongruence and of a guasicongruence, if X is a quasicongruence (guasicongruence, respectively) on $\mathbf{F}$, then $\dot{X}$ is a congruence system that satisfies all quasiequations (guasiequations, respectively) in X. Based on this, it is not difficult to show that, in case X is a quasicongruence (guasicongruence, respectively) on $\mathbf{F}$, the relation $\Theta(X)$ actually coincides with $\dot{X}$.

Lemma 5 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* X *be a quasicongruence or guasicongruence system on* $\mathbf{F}$. *Then* $\Theta(X) = \dot{X}$.

Proof: The inclusion from right to left follows from the fact that $\dot{X} \leq X$ and $\Theta(X)$ satisfies all guasiequations in X, which, in particular, implies that it contains all equations in $\dot{X}$. On the other hand, by the definition of guasicongruence system, $\dot{X}$ is a congruence system on $\mathbf{F}$ that, because of Modus Ponens, satisfies all guasiequations in X. Therefore, by the minimality of $\Theta(X)$, we get $\Theta(X) \leq \dot{X}$. ∎

We now formulate two lemmas that will play a crucial role in the subsequent characterization of the closure operators C^* and C.

Lemma 6 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota \rangle \rangle$, *with* $\langle I, \iota \rangle : \mathbf{F} \to \mathbf{F}$ *the identity morphism.*

(a) If $X \in \mathrm{QonSys}^{\approx}(\mathbf{F})$, *then* $X = \mathrm{QEq}(\mathcal{F}/\dot{X})$;

(b) If $X \in \mathrm{GonSys}^{\approx}(\mathbf{F})$, *then* $X = \mathrm{GEq}(\mathcal{F}/\dot{X})$.

Proof: Let us focus again on Part (a), Part (b) being treated similarly. The inclusion from left to right follows from Lemmas 1 and 5. For the reverse inclusion, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in \mathrm{QEq}_\Sigma(\mathcal{F}/\dot{X})$. By definition, this means that $\mathcal{F}/\dot{X} \not\vDash^*_\Sigma \vec{\phi} \approx \vec{\psi}$ or $\mathcal{F}/\dot{X} \vDash^*_\Sigma \phi \approx \psi$. Equivalently, $\vec{\phi} \approx \vec{\psi} \notin \dot{X}_\Sigma$ or $\phi \approx \psi \in \dot{X}_\Sigma$. Since, by hypothesis, $X \in \mathrm{QonSys}^{\approx}(\mathbf{F})$, we get, by Completeness, that $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in X_\Sigma$. ∎

Lemma 7 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$. *Then*

$$\mathrm{QEq}(\mathcal{A}) \in \mathrm{QonSys}^{\approx}(\mathbf{F}) \quad \textit{and} \quad \mathrm{GEq}(\mathcal{A}) \in \mathrm{GonSys}^{\approx}(\mathbf{F}).$$

Proof: Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system, where $\langle F, \alpha \rangle : \mathbf{F} \to \mathbf{A}$. We verify that $\mathrm{QEq}(\mathcal{A})$ satisfies the three conditions required of a collection of $\mathbf{F}$-quasiequations to qualify as a complete quasicongruence on $\mathbf{F}$. To this end, assume σ in $N^\flat$, $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and $\vec{\phi}, \vec{\psi}, \phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$.

- First, we have $\mathrm{Eq}(\mathbf{A}) \in \mathrm{ConSys}(\mathbf{F})$ and, since $\mathrm{Eq}(\mathcal{A}) = \mathrm{QEq}(\mathcal{A}) \cap \mathrm{Eq}(\mathbf{F})$, the first condition is satisfied;

- Modus Ponens clearly holds: If $\Sigma \in |\mathbf{Sign}^\flat|$, $\vec{\phi} \approx \vec{\psi} \subseteq \mathrm{Eq}_\Sigma(\mathcal{A})$ and $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in \mathrm{QEq}_\Sigma(\mathcal{A})$, then,

$$\mathcal{A} \vDash^*_\Sigma \vec{\phi} \approx \vec{\psi} \quad \text{and} \quad \mathcal{A} \vDash^*_\Sigma \vec{\phi} \approx \vec{\psi} \to \phi \approx \psi,$$

  whence, by the interpretation of the implication connective, we get $\mathcal{A} \vDash^*_\Sigma \phi \approx \psi$ and, therefore, $\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathcal{A})$;

- Finally, for Completeness, the trick is again played by the interpretation of implication. If $\Sigma \in |\mathbf{Sign}^\flat|$, $\vec{\phi} \approx \vec{\psi} \notin \mathrm{Eq}_\Sigma(\mathcal{A})$ or $\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathcal{A})$, then

$$\mathcal{A} \nvDash^*_\Sigma \vec{\phi} \approx \vec{\psi} \quad \text{or} \quad \mathcal{A} \vDash^*_\Sigma \phi \approx \psi,$$

  whence, $\mathcal{A} \vDash^*_\Sigma \vec{\phi} \approx \vec{\psi} \to \phi \approx \psi$, i.e., $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in \mathrm{QEq}_\Sigma(\mathcal{A})$.

All necessary properties being satisfied, we conclude that $\mathrm{QEq}(\mathcal{A})$ is a complete quasicongruence system on $\mathbf{F}$.

More generally, an identical reasoning involving guasiequations shows that $\mathrm{GEq}(\mathcal{A})$ is a complete guasicongruence system on $\mathbf{F}$. ■

Now, we are now ready to characterize the closed sets of C^* in $\mathcal{P}(\mathrm{QEq}(\mathbf{F}))$ and of C in $\mathcal{P}(\mathrm{GEq}(\mathbf{F}))$. Those turn out to be exactly the coverable quasicongruences and coverable guasicongruences, respectively, on $\mathbf{F}$.

Proposition 8 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system. Suppose* $X \leq \mathrm{QEq}(\mathbf{F})$ *and* $Y \leq \mathrm{GEq}(\mathbf{F})$.

(a) $C^*(X) = X$ *if and only if* $X \in \mathrm{QonSys}^\wedge(\mathbf{F})$*;*

(b) $C(Y) = Y$ *if and only if* $Y \in \mathrm{GonSys}^\wedge(\mathbf{F})$.

Proof: We prove Part (a). Part (b) is proven similarly.

Let $X \leq \mathrm{QEq}(\mathbf{F})$, such that $C^*(X) = X$. Then, by the theory of Galois connections, there exists $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$, such that $X = \mathrm{QEq}(\mathsf{K})$. Now we get

$$\begin{array}{rcll} X & = & \mathrm{QEq}(\mathsf{K}) & \\ & = & \bigcap_{\mathcal{A} \in \mathsf{K}} \mathrm{QEq}(\mathcal{A}) & \text{(definition of QEq)} \\ & \in & \mathrm{QonSys}^\wedge(\mathbf{F}). & \text{(Lemma 7 and Definition of } \mathrm{QonSys}^\wedge(\mathbf{F})) \end{array}$$

Suppose, conversely, that $X \in \mathrm{QonSys}^{\wedge}(\mathbf{F})$. Then, there exists, by Lemma 3, a collection $\{X^i : i \in I\} \subseteq \mathrm{QonSys}^{\approx}(\mathbf{F})$, such that $X = \bigcap_{I \in I} X^i$. But, by Lemma 6, for all $i \in I$, we have $X^i = \mathrm{QEq}(\mathcal{F}/\dot{X}^i)$. Hence,

$$X = \bigcap_{i \in I} X^i = \bigcap_{i \in I} \mathrm{QEq}(\mathcal{F}/\dot{X}^i) = \mathrm{QEq}(\{\mathcal{F}/\dot{X}^i : i \in I\}).$$

Since X is in the image of QEq, by the theory of Galois connections, we get $C^*(X) = X$. ■

Next, we characterize the closed sets in $\mathcal{P}(\mathrm{AlgSys}(\mathbf{F}))$ both under $\mathbb{Q}^{\mathsf{Sem}}$ and $\mathbb{G}^{\mathsf{Sem}}$, i.e., the semantic quasivarieties and guasivarieties of $\mathbf{F}$-algebraic systems, respectively. Semantic quasivarieties are those classes that are closed under *directed certifications*, *subdirect intersections*. Semantic guasivarieties, on the other hand, turn out to be those classes closed under *certifications* and subdirect intersections. We note that, as will become clear shortly, certifications are special cases of directed certifications. Before we provide the relevant characterizations, we define those operations and provide some lemmas concerning their functionality and properties.

First, as far as subdirect intersections are concerned, given an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathbf{F}$-algebraic systems $\mathcal{A}^i = \langle \mathbf{A}^i, \langle F^i, \alpha^i \rangle \rangle$ and morphisms $\langle H^i, \gamma^i \rangle : \mathcal{A} \to \mathcal{A}^i$, $i \in I$, we say that the collection

$$\langle H^i, \gamma^i \rangle : \mathcal{A} \to \mathcal{A}^i, \quad i \in I,$$

is a **subdirect intersection** if

$$\bigcap_{i \in I} \mathrm{Ker}(\langle H^i, \gamma^i \rangle) = \Delta^{\mathcal{A}}.$$

Given a class K of $\mathbf{F}$-algebraic systems, we denote by $\overset{\triangleleft}{\mathbb{I}\mathbb{I}}(\mathsf{K})$ the class of all $\mathbf{F}$-algebraic systems $\mathcal{A}$, such that there exists a subdirect intersection $\{\langle H^i, \gamma^i \rangle : \mathcal{A} \to \mathcal{A}^i : i \in I\}$, with $A^i \in \mathsf{K}$, for all $i \in I$. Moreover, we say that the class K is **closed under subdirect intersections** if $\overset{\triangleleft}{\mathbb{I}\mathbb{I}}(\mathsf{K}) \subseteq \mathsf{K}$.

A useful lemma characterizes subdirect intersections in terms of relations between the kernels of the algebraic systems involved.

Lemma 9 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system,* $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, $\mathcal{A}^i = \langle \mathbf{A}^i, \langle F^i, \alpha^i \rangle \rangle$, $i \in I$, *$\mathbf{F}$-algebraic systems and* $\{\langle H^i, \gamma^i \rangle : \mathcal{A} \to \mathcal{A}^i : i \in I\}$ *algebraic morphisms. The collection* $\{\langle H^i, \gamma^i \rangle : i \in I\}$ *is a subdirect intersection if and only if* $\mathrm{Ker}(\langle F, \alpha \rangle) = \bigcap_{i \in I} \mathrm{Ker}(\langle F^i, \alpha^i \rangle)$.

Proof: Suppose, first, that $\{\langle H^i,\gamma^i\rangle : i\in I\}$ is a subdirect intersection and let $\Sigma\in|\mathbf{Sign}^\flat|$, $\phi,\psi\in\mathrm{SEN}^\flat(\Sigma)$. Then

$$\begin{array}{lll}\langle\phi,\psi\rangle\in\mathrm{Ker}_\Sigma(\langle F,\alpha\rangle) & \text{iff} & \alpha_\Sigma(\phi)=\alpha_\Sigma(\psi)\\ & \text{iff} & \langle\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\rangle\in\Delta^{\mathcal{A}}_{F(\Sigma)}\\ & \text{iff} & \langle\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\rangle\in\bigcap_{i\in I}\mathrm{Ker}_\Sigma(\langle H^i,\gamma^i\rangle)\\ & \text{iff} & \gamma^i_{F(\Sigma)}(\alpha_\Sigma(\phi))=\gamma^i_{F(\Sigma)}(\alpha_\Sigma(\psi)), i\in I\\ & \text{iff} & \alpha^i_\Sigma(\phi)=\alpha^i_\Sigma(\psi), i\in I\\ & \text{iff} & \langle\phi,\psi\rangle\in\bigcap_{i\in I}\mathrm{Ker}_\Sigma(\langle F^i,\alpha^i\rangle).\end{array}$$

The reverse relies on the surjectivity of $\langle F,\alpha\rangle$. Suppose $\Sigma\in|\mathbf{Sign}^\flat|$ and $\phi,\psi\in\mathrm{SEN}^\flat(\Sigma)$. Then we get

$$\begin{array}{lll}\langle\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\rangle\in\Delta^{\mathcal{A}}_{F(\Sigma)} & \text{iff} & \langle\phi,\psi\rangle\in\mathrm{Ker}_\Sigma(\langle F,\alpha\rangle)\\ & \text{iff} & \langle\phi,\psi\rangle\in\bigcap_{i\in I}\mathrm{Ker}_\Sigma(\langle F^i,\alpha^i\rangle)\\ & \text{iff} & \alpha^i_\Sigma(\phi)=\alpha^i_\Sigma(\psi), i\in I\\ & \text{iff} & \gamma^i_{F(\Sigma)}(\alpha_\Sigma(\phi))=\gamma^i_{F(\Sigma)}(\alpha_\Sigma(\psi)), i\in I\\ & \text{iff} & \langle\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\rangle\in\bigcap_{i\in I}\mathrm{Ker}_{F(\Sigma)}(\langle H^i,\gamma^i\rangle).\end{array}$$

Thus, by the surjectivity of $\langle F,\alpha\rangle$ we get that $\Delta^{\mathcal{A}}=\bigcap_{i\in I}\mathrm{Ker}(\langle H^i,\gamma^i\rangle)$. ∎

The following is a key lemma concerning a property of subdirect intersections.

Lemma 10 *Let* $\mathbf{F}=\langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ *be an algebraic system and consider a class* $\mathsf{K}\subseteq\mathrm{AlgSys}(\mathbf{F})$*. The class of morphisms*

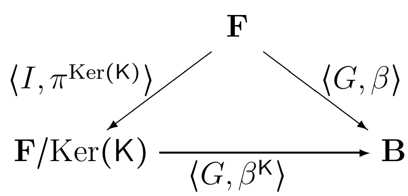
$$\langle G,\beta^{\mathsf{K}}\rangle:\mathcal{F}/\bigcap_{\mathcal{B}\in\mathsf{K}}\mathrm{Ker}(\langle G,\beta\rangle)\to\mathcal{B},\quad \mathcal{B}=\langle\mathbf{B},\langle G,\beta\rangle\rangle\in\mathsf{K},$$

where, for all $\Sigma\in|\mathbf{Sign}^\flat|$ *and all* $\phi\in\mathrm{SEN}^\flat(\Sigma)$*,*

$$\beta^{\mathsf{K}}_\Sigma(\phi/\bigcap_{\mathcal{B}\in\mathsf{K}}\mathrm{Ker}_\Sigma(\langle G,\beta\rangle))=\beta_\Sigma(\phi),$$

forms a subdirect intersection.

Proof: It is not difficult to see that β^{K} is well defined and forms a natural transformation. Moreover, $\langle G,\beta^{\mathsf{K}}\rangle$ is an $\mathbf{F}$-morphism. Letting $\mathrm{Ker}(\mathsf{K})=\bigcap_{\mathcal{B}\in\mathsf{K}}\mathrm{Ker}(\langle G,\beta\rangle)$, we have, by definition, the following commutative triangle.



To show that the displayed family forms a subdirect intersection, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then, we get

$$\begin{array}{ll} \langle \phi/\mathrm{Ker}_\Sigma(\mathsf{K}), \psi/\mathrm{Ker}_\Sigma(\mathsf{K})\rangle \in \bigcap_{\mathcal{B}\in\mathsf{K}} \mathrm{Ker}_\Sigma(\langle G, \beta^\mathsf{K}\rangle) & \\ \quad \text{iff} \quad \beta^\mathsf{K}_\Sigma(\phi/\mathrm{Ker}_\Sigma(\mathsf{K})) = \beta^\mathsf{K}_\Sigma(\psi/\mathrm{Ker}_\Sigma(\mathsf{K})), \quad \mathcal{B} \in \mathsf{K}, & \\ \quad \text{iff} \quad \beta_\Sigma(\phi) = \beta_\Sigma(\psi), \quad \mathcal{B} \in \mathsf{K}, & \\ \quad \text{iff} \quad \phi/\mathrm{Ker}_\Sigma(\mathsf{K}) = \psi/\mathrm{Ker}_\Sigma(\mathsf{K}). & \end{array}$$

Thus, $\bigcap_{\mathcal{B}\in\mathsf{K}} \mathrm{Ker}(\langle G, \beta^\mathsf{K}\rangle) = \Delta^{\mathcal{F}/\mathrm{Ker}(\mathsf{K})}$, showing that

$$\langle G, \beta^\mathsf{K}\rangle : \mathcal{F}/\bigcap_{\mathcal{B}\in\mathsf{K}} \mathrm{Ker}(\langle G, \beta\rangle) \to \mathcal{B}, \quad \mathcal{B} = \langle \mathbf{B}, \langle G, \beta\rangle\rangle \in \mathsf{K},$$

constitutes indeed a subdirect intersection. ■

As far as certifications are concerned, given a class K of $\mathbf{F}$-algebraic systems, we say that an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ is **K-certified** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, there exists an $\mathbf{F}$-algebraic system $\mathcal{A}^\Sigma \in \mathsf{K}$, such that

$$\mathrm{Ker}_\Sigma(\mathcal{A}) = \mathrm{Ker}_\Sigma(\mathcal{A}^\Sigma).$$

We denote by $\mathbb{C}(\mathsf{K})$ the class of all $\mathbf{F}$-algebraic systems that are K-certified. Moreover, we say that the class K is **closed under certifications** if $\mathbb{C}(\mathsf{K}) \subseteq \mathsf{K}$.

More generally, given a class K of $\mathbf{F}$-algebraic systems, we say that an $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ is **directed K-certified** if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, there exists a collection of $\mathbf{F}$-algebraic systems $\{\mathcal{A}^{\Sigma,i} : i \in I_\Sigma\} \subseteq \mathsf{K}$, such that

- $\bigcup_{i\in I_\Sigma} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i})$ is directed, where, for all $i \in I_\Sigma$, $\mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i})$ denotes the collection of all finite subsets of $\mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,i})$, and
- $\mathrm{Ker}_\Sigma(\mathcal{A}) = \bigcup_{i\in I_\Sigma} \mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,i})$.

We denote by $\mathbb{C}^*(\mathsf{K})$ the class of all $\mathbf{F}$-algebraic systems that are directed K-certified. Moreover, we say that the class K is **closed under directed certifications** if $\mathbb{C}^*(\mathsf{K}) \subseteq \mathsf{K}$.

We show next, that both closure under subdirect intersections and closure under certifications are necessary conditions for a class of $\mathbf{F}$-algebraic systems to form a semantic quasivariety and the same applies to semantic guasivarieties. On the other hand, closure under directed certifications is a necessary requirement for semantic quasivarieties, but not so, in general, for semantic guasivarieties.

First we show that if a class K of $\mathbf{F}$-algebraic systems is in the image of Mod^* or of Mod, then it is closed under subdirect intersections.

Proposition 11 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$. *If* $\mathsf{K} = \mathbb{G}^{\mathsf{Sem}}(\mathsf{K})$, *then* $\overset{\triangleleft}{\mathbb{II}}(\mathsf{K}) \subseteq \mathsf{K}$.

Proof: Assume that $\mathsf{K} = \mathbb{G}^{\mathsf{Sem}}(\mathsf{K})$. Let $X = \mathrm{GEq}(\mathsf{K})$. Assume that $\mathcal{A} \in \overset{\triangleleft}{\mathbb{II}}(\mathsf{K})$ and $\Sigma \in |\mathbf{Sign}^\flat|$, $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in X_\Sigma$, such that $\mathcal{A} \vDash_\Sigma \vec{\phi} \approx \vec{\psi}$, i.e., $\vec{\phi} \approx \vec{\psi} \subseteq \mathrm{Eq}_\Sigma(\mathcal{A})$. Since $\mathcal{A} \in \overset{\triangleleft}{\mathbb{II}}(\mathsf{K})$, there exists a subdirect intersection

$$\langle H^i, \gamma^i \rangle : \mathcal{A} \to \mathcal{A}^i, \quad i \in I,$$

such that $\mathcal{A}^i \in \mathsf{K}$, for all $i \in I$. Hence, we get $\vec{\phi} \approx \vec{\psi} \subseteq \mathrm{Eq}_\Sigma(\mathcal{A}^i)$, $i \in I$. Now, since $\mathcal{A}^i \in \mathsf{K}$ and $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in X_\Sigma = \mathrm{GEq}_\Sigma(\mathsf{K})$, we conclude that $\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathcal{A}^i)$, for all $i \in I$. Therefore, $\phi \approx \psi \in \bigcap_{i \in I} \mathrm{Eq}_\Sigma(\mathcal{A}^i) = \mathrm{Eq}_\Sigma(\mathcal{A})$, the latter by the definition of subdirect intersection and Lemma 7. Therefore, $\mathcal{A} \vDash_\Sigma \vec{\phi} \approx \vec{\psi} \to \phi \approx \psi$. This shows that $\mathcal{A} \in \mathrm{Mod}(X) = \mathrm{Mod}(\mathrm{GEq}(\mathsf{K})) = \mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$. We conclude that $\overset{\triangleleft}{\mathbb{II}}(\mathsf{K}) \subseteq \mathsf{K}$, i.e., K is closed under subdirect intersections. ■

For the operator $\mathbb{C}$, we prove two properties. The first asserts that, if a class K of $\mathbf{F}$-algebraic systems is a guasiequational class, i.e., in the image of Mod (or, a fortiori, in the image of Mod^*), then it is closed under certifications.

Proposition 12 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$. *If* $\mathsf{K} = \mathbb{G}^{\mathsf{Sem}}(\mathsf{K})$, *then* $\mathbb{C}(\mathsf{K}) \subseteq \mathsf{K}$.

Proof: Let $X = \mathrm{GEq}(\mathsf{K})$. Assume that $\mathcal{A} \in \mathbb{C}(\mathsf{K})$ and $\Sigma \in |\mathbf{Sign}^\flat|$, $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in X_\Sigma$, such that $\mathcal{A} \vDash_\Sigma \vec{\phi} \approx \vec{\psi}$, i.e., $\vec{\phi} \approx \vec{\psi} \subseteq \mathrm{Eq}_\Sigma(\mathcal{A})$. Since $\mathcal{A} \in \mathbb{C}(\mathsf{K})$, there exists $\mathcal{A}^\Sigma \in \mathsf{K}$, such that $\mathrm{Eq}_\Sigma(\mathcal{A}) = \mathrm{Eq}_\Sigma(\mathcal{A}^\Sigma)$. Hence, $\vec{\phi} \approx \vec{\psi} \subseteq \mathrm{Eq}_\Sigma(\mathcal{A}^\Sigma)$. Now, since $\mathcal{A}^\Sigma \in \mathsf{K}$ and $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in X_\Sigma = \mathrm{GEq}_\Sigma(\mathsf{K})$, we conclude that $\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathcal{A}^\Sigma) = \mathrm{Eq}_\Sigma(\mathcal{A})$. Therefore, $\mathcal{A} \vDash_\Sigma \vec{\phi} \approx \vec{\psi} \to \phi \approx \psi$. This shows that $\mathcal{A} \in \mathrm{Mod}(X) = \mathrm{Mod}(\mathrm{GEq}(\mathsf{K})) = \mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$. We conclude that $\mathbb{C}(\mathsf{K}) \subseteq \mathsf{K}$, i.e., K is closed under certifications. ■

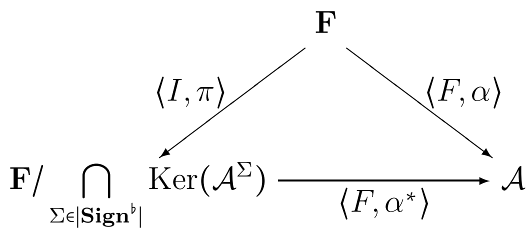
The second lemma regarding $\mathbb{C}$ shows that it is dominated by the operator $\mathbb{H}\overset{\triangleleft}{\mathbb{II}}$. Recall that, given a class K of $\mathbf{F}$-algebraic systems and an $\mathbf{F}$-algebraic system $\mathcal{A}$, $\mathcal{A} \in \mathbb{H}(\mathsf{K})$ if there exists a morphism $\langle H, \gamma \rangle : \mathcal{B} \to \mathcal{A}$, with $\mathcal{B} \in \mathsf{K}$.

Lemma 13 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems. Then* $\mathbb{C}(\mathsf{K}) \subseteq \mathbb{H}(\overset{\triangleleft}{\mathbb{II}}(\mathsf{K}))$.

Proof: Suppose $\mathcal{A} \in \mathbb{C}(\mathsf{K})$. Then, by definition, for all $\Sigma \in |\mathbf{Sign}^\flat|$, there exists $\mathcal{A}^\Sigma \in \mathsf{K}$, such that $\mathrm{Eq}_\Sigma(\mathcal{A}) = \mathrm{Eq}_\Sigma(\mathcal{A}^\Sigma)$. Consider the family of morphisms

$$\langle H^\Sigma, \gamma^\Sigma \rangle : \mathcal{F}/\bigcap_{\Sigma \in |\mathbf{Sign}^\flat|} \mathrm{Ker}(\mathcal{A}^\Sigma) \to \mathcal{A}^\Sigma, \quad \Sigma \in |\mathbf{Sign}^\flat|.$$

By Lemma 10, it constitutes a subdirect intersection, whence, since $\mathcal{A}^\Sigma \in \mathsf{K}$, for all Σ, we infer that $\mathcal{F}/\bigcap_{\Sigma \in |\mathbf{Sign}^\flat|} \mathrm{Ker}(\mathcal{A}^\Sigma) \in \overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$. Now it is not difficult to see that there exists a morphism $\langle F, \alpha^* \rangle : \mathcal{F}/\bigcap_{\Sigma \in |\mathbf{Sign}^\flat|} \mathrm{Ker}(\mathcal{A}^\Sigma) \to \mathcal{A}$, such that the following diagram commutes



The natural transformation α^* is defined, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma')$, by

$$\alpha^*_{\Sigma'}(\phi/\bigcap_{\Sigma \in |\mathbf{Sign}^\flat|} \mathrm{Ker}_{\Sigma'}(\mathcal{A}^\Sigma)) = \alpha_{\Sigma'}(\phi),$$

and it is well-defined, since, for all $\Sigma' \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma')$, we have

$$\begin{array}{lll} \langle \phi, \psi \rangle \in \bigcap_{\Sigma \in |\mathbf{Sign}^\flat|} \mathrm{Ker}_{\Sigma'}(\mathcal{A}^\Sigma) & \text{implies} & \langle \phi, \psi \rangle \in \mathrm{Ker}_{\Sigma'}(\mathcal{A}^{\Sigma'}) \\ & \text{implies} & \langle \phi, \psi \rangle \in \mathrm{Ker}_{\Sigma'}(\mathcal{A}). \end{array}$$

We conclude that $\mathcal{A} \in \mathbb{H}(\overset{\triangleleft}{\mathbb{P}}(\mathsf{K}))$. Therefore, $\mathbb{C}(\mathsf{K}) \subseteq \mathbb{H}(\overset{\triangleleft}{\mathbb{P}}(\mathsf{K}))$. ∎

The last closure proposition is the one asserting that a semantic quasivariety of $\mathbf{F}$-algebraic systems is closed under directed certifications.

Proposition 14 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$. *If* $\mathsf{K} = \mathbb{Q}^{\mathsf{Sem}}(\mathsf{K})$*, then* $\mathbb{C}^*(\mathsf{K}) \subseteq \mathsf{K}$.

Proof: Assume that $\mathsf{K} = \mathbb{Q}^{\mathsf{Sem}}(\mathsf{K})$. Let $X = \mathrm{QEq}(\mathsf{K})$. Assume that $\mathcal{A} \in \mathbb{C}^*(\mathsf{K})$ and $\Sigma \in |\mathbf{Sign}^\flat|$, $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in X_\Sigma$, such that $\mathcal{A} \vDash^*_\Sigma \vec{\phi} \approx \vec{\psi}$, i.e.,

$\vec{\phi} \approx \vec{\psi} \subseteq \mathrm{Eq}_\Sigma(\mathcal{A})$. Since $\mathcal{A} \in \mathbb{C}^*(\mathsf{K})$, there exists a collection $\{\mathcal{A}^{\Sigma,i} : i \in I_\Sigma\} \subseteq \mathsf{K}$, such that $\bigcup\{\mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i}) : i \in I_\Sigma\}$ is directed and

$$\mathrm{Eq}_\Sigma(\mathcal{A}) = \bigcup_{i \in I_\Sigma} \mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,i}).$$

Since $\vec{\phi} \approx \vec{\psi} \subseteq \mathrm{Eq}_\Sigma(\mathcal{A})$, for all $i < n$ (recall $\vec{\phi} \approx \vec{\psi}$ is finite), there exists, $\mathcal{A}^{\Sigma,i}$, such that $\phi_i \approx \psi_i \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i})$. Since $\bigcup\{\mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i}) : i \in I_\Sigma\}$ is directed, we get that there exists $k \in I_\Sigma$, such that $\vec{\phi} \approx \vec{\psi} \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,k})$. But $\mathcal{A}^{\Sigma,k} \in \mathsf{K}$ and $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in X_\Sigma = \mathrm{QEq}_\Sigma(\mathsf{K})$. Hence, $\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,k}) \subseteq \bigcup_{i \in I_\Sigma} \mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,i}) = \mathrm{Eq}_\Sigma(\mathcal{A})$. This shows that $\mathcal{A} \in \mathrm{Mod}^*(X) = \mathrm{Mod}^*(\mathrm{QEq}(\mathsf{K})) = \mathbb{Q}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$. We conclude $\mathbb{C}^*(\mathsf{K}) \subseteq \mathsf{K}$, i.e., K is closed under directed certifications. ■

We characterize semantic guasivarieties as those classes of algebraic systems that are closed under subdirect intersections and certifications.

Proposition 15 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$. *Then* $\mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$ *if and only if* K *is closed under* $\mathbb{C}$ *and* $\overset{\triangleleft}{\mathbb{II}}$.

Proof: Let $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$. Suppose, first, that $\mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$. Then, by Proposition 11, K is closed under $\overset{\triangleleft}{\mathbb{II}}$ and, by Proposition 12, it is closed under $\mathbb{C}$.

Conversely, suppose that $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$, such that $\overset{\triangleleft}{\mathbb{II}}(\mathsf{K}) \subseteq \mathsf{K}$ and $\mathbb{C}(\mathsf{K}) \subseteq \mathsf{K}$. It suffices to show that $\mathsf{K} = \mathrm{Mod}(\mathrm{GEq}(\mathsf{K}))$. The left to right inclusion is obvious. For the converse, consider $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle \in \mathrm{Mod}(\mathrm{GEq}(\mathsf{K}))$. For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \approx \psi \notin \mathrm{Eq}_\Sigma(\mathcal{A})$, we consider the $\mathbf{F}$-guasiequation

$$g^{\Sigma,\phi\approx\psi} := \mathrm{Eq}_\Sigma(\mathcal{A}) \to \phi \approx \psi.$$

(Strictly speaking, in place of $\mathrm{Eq}_\Sigma(\mathcal{A})$, we are supposed to have a possibly infinite vector; so we assume given a default ordering of all Σ-sentences, which induces a default ordering of all Σ-equations.) Since $\mathcal{A} \vDash_\Sigma \mathrm{Eq}_\Sigma(\mathcal{A})$ and $\mathcal{A} \nvDash_\Sigma \phi \approx \psi$, we get that $g^{\Sigma,\phi,\approx\psi} \notin \mathrm{GEq}_\Sigma(\mathcal{A})$. Thus, since $\mathcal{A} \in \mathrm{Mod}(\mathrm{GEq}(\mathsf{K}))$, we infer that $g^{\Sigma,\psi\approx\psi} \notin \mathrm{GEq}_\Sigma(\mathsf{K})$. Therefore, there exists $\mathcal{A}^{\Sigma,\phi\approx\psi} \in \mathsf{K}$, such that $\mathcal{A}^{\Sigma,\phi\approx\psi} \nvDash_\Sigma g^{\Sigma,\phi\approx\psi}$, i.e.,

$$\mathcal{A}^{\Sigma,\phi\approx\psi} \vDash_\Sigma \mathrm{Eq}_\Sigma(\mathcal{A}) \quad \text{and} \quad \mathcal{A}^{\Sigma,\phi\approx\psi} \nvDash_\Sigma \phi \approx \psi.$$

Let

$$\mathsf{A}^\Sigma = \{\mathcal{A}^{\Sigma,\phi\approx\psi} : \phi \approx \psi \notin \mathrm{Eq}_\Sigma(\mathcal{A})\}.$$

By Proposition 10,

$$\mathcal{F}/\mathrm{Ker}(\mathsf{A}^\Sigma) = \mathcal{F}/\bigcap_{\phi\approx\psi\notin\mathrm{Eq}_\Sigma(\mathcal{A})}\mathrm{Ker}(\mathcal{A}^{\Sigma,\phi\approx\psi}) \in \overset{\triangleleft}{\mathbb{P}}\mathbb{I}(\mathsf{K}) = \mathsf{K}.$$

But note that, by construction, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{Eq}_\Sigma(\mathcal{A}) = \mathrm{Eq}_\Sigma(\mathcal{F}/\mathrm{Ker}(\mathsf{A}^\Sigma)).$$

Indeed, for all $\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathbf{F})$,

- if $\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathcal{A})$, then, for all $\mathcal{B} \in \mathsf{A}^\Sigma$, $\mathcal{B} \vDash_\Sigma \phi \approx \psi$, i.e., $\phi \approx \psi \in \mathrm{Ker}_\Sigma(\mathcal{B})$. Therefore, $\phi \approx \psi \in \mathrm{Ker}_\Sigma(\mathsf{A}^\Sigma)$, showing that $\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathcal{F}/\mathrm{Ker}(\mathsf{A}^\Sigma))$;
- if $\phi \approx \psi \notin \mathrm{Eq}_\Sigma(\mathcal{A})$, then $\mathcal{A}^{\Sigma,\phi\approx\psi} \nvDash_\Sigma \phi \approx \psi$, whence $\phi \approx \psi \notin \mathrm{Ker}_\Sigma(\mathsf{A}^\Sigma)$, showing that $\phi \approx \psi \notin \mathrm{Eq}_\Sigma(\mathcal{F}/\mathrm{Ker}(\mathsf{A}^\Sigma))$.

Thus, since, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathcal{F}/\mathrm{Ker}(\mathsf{A}^\Sigma) \in \mathsf{K}$, we get, by the definition of $\mathbb{C}$ and the equality just proven, that $\mathcal{A} \in \mathbb{C}(\mathsf{K}) = \mathsf{K}$. We conclude that $\mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$. ■

Similarly, we characterize semantic quasivarieties as those classes of algebraic systems that are closed under subdirect intersections and directed certifications.

Proposition 16 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$. *Then* $\mathbb{Q}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$ *if and only if* K *is closed under* $\mathbb{C}^*$ *and* $\overset{\triangleleft}{\mathbb{P}}\mathbb{I}$.

Proof: Let $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$. Suppose, first, that $\mathbb{Q}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$. Then, since K is a semantic guasivariety, we get, by Proposition 11, that K is closed under $\overset{\triangleleft}{\mathbb{P}}\mathbb{I}$ and, by Proposition 14, K is also closed under $\mathbb{C}^*$.

Conversely, suppose that $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$, such that $\overset{\triangleleft}{\mathbb{P}}\mathbb{I}(\mathsf{K}) \subseteq \mathsf{K}$ and $\mathbb{C}^*(\mathsf{K}) \subseteq \mathsf{K}$. It suffices to show that $\mathsf{K} = \mathrm{Mod}^*(\mathrm{QEq}(\mathsf{K}))$. The left to right inclusion is obvious. For the converse, consider $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle \in \mathrm{Mod}^*(\mathrm{QEq}(\mathsf{K}))$. For all $\Sigma \in |\mathbf{Sign}^\flat|$, all $X \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A})$ and all $\phi \approx \psi \notin \mathrm{Eq}_\Sigma(\mathcal{A})$, we consider the $\mathbf{F}$-quasiequation

$$q^{\Sigma,X,\phi\approx\psi} := \vec{X} \to \phi \approx \psi,$$

where, again, the hypotheses $\vec{X}$ may be taken as an arbitrary ordering of the finite set X. Since $\mathcal{A} \vDash^*_\Sigma \mathrm{Eq}_\Sigma(\mathcal{A})$ and $\mathcal{A} \nvDash^*_\Sigma \phi \approx \psi$, we get that $q^{\Sigma,X,\phi,\approx\psi} \notin$

$\mathrm{QEq}_\Sigma(\mathcal{A})$. Thus, since $\mathcal{A} \in \mathrm{Mod}^*(\mathrm{QEq}(\mathsf{K}))$, we infer that $q^{\Sigma,X,\psi\approx\psi} \notin \mathrm{QEq}_\Sigma(\mathsf{K})$. Therefore, there exists $\mathcal{A}^{\Sigma,X,\phi\approx\psi} \in \mathsf{K}$, such that $\mathcal{A}^{\Sigma,X,\phi\approx\psi} \nvDash^*_\Sigma q^{\Sigma,X,\phi\approx\psi}$, i.e.,

$$\mathcal{A}^{\Sigma,X,\phi\approx\psi} \vDash^*_\Sigma X \quad \text{and} \quad \mathcal{A}^{\Sigma,X,\phi\approx\psi} \nvDash^*_\Sigma \phi \approx \psi.$$

Let, for all $X \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A})$,

$$\mathsf{A}^{\Sigma,X} = \{\mathcal{A}^{\Sigma,X,\phi\approx\psi} : \phi \approx \psi \notin \mathrm{Eq}_\Sigma(\mathcal{A})\}.$$

Define, for all $X \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A})$,

$$\mathcal{A}^{\Sigma,X} := \mathcal{F}/\mathrm{Ker}(\mathsf{A}^{\Sigma,X}) = \mathcal{F}/\bigcap_{\phi\approx\psi\notin\mathrm{Eq}_\Sigma(\mathcal{A})} \mathrm{Ker}(\mathcal{A}^{\Sigma,X,\phi\approx\psi}).$$

By Proposition 10, for all $X \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A})$, $\mathcal{A}^{\Sigma,X} \in \overset{\triangleleft}{\mathbb{IP}}(\mathsf{K}) = \mathsf{K}$. It suffices now to show the following:

- $\bigcup_{X\in\mathrm{Eq}^\omega_\Sigma(\mathcal{A})} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,X})$ is directed;
- $\mathrm{Ker}_\Sigma(\mathcal{A}) = \bigcup_{X\in\mathrm{Eq}^\omega_\Sigma(\mathcal{A})} \mathrm{Ker}_\Sigma(\mathcal{A}^{\Sigma,X})$.

Suppose, first, that $E \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,X})$ and $E' \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,X'})$, for some $X, X' \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A})$. Then, by construction of $\mathcal{A}^{\Sigma,X}$ and $\mathcal{A}^{\Sigma,X'}$, we get that $E, E' \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A})$. Therefore, $E \cup E' \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,E\cup E'})$ and, hence, $\bigcup_{X\in\mathrm{Eq}^\omega_\Sigma(\mathcal{A})} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,X})$ is indeed directed.

Finally, note that, by construction, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{Eq}_\Sigma(\mathcal{A}) = \bigcup_{X\in\mathrm{Eq}^\omega_\Sigma(\mathcal{A})} \mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,X}).$$

Indeed, for all $\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathbf{F})$,

- if $\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathcal{A})$, then, $\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,\{\phi\approx\psi\}})$, whence $\phi \approx \psi \in \bigcup_{X\in\mathrm{Eq}^\omega_\Sigma(\mathcal{A})} \mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,X})$.
- if $\phi \approx \psi \notin \mathrm{Eq}_\Sigma(\mathcal{A})$, then, by construction, for all $X \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A})$, $\phi \approx \psi \notin \mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,X})$. Therefore, $\phi \approx \psi \notin \bigcup_{X\in\mathrm{Eq}^\omega_\Sigma(\mathcal{A})} \mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,X})$.

Since, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $X \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A})$, $\mathcal{A}^{\Sigma,X} \in \mathsf{K}$, we get, by the definition of $\mathbb{C}^*$ and the two properties just proven, that $\mathcal{A} \in \mathbb{C}^*(\mathsf{K}) = \mathsf{K}$. We conclude that $\mathbb{Q}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$. ∎

3 Natural Quasi- and Guasi-Equations and Syntactic Quasi- and Guasi-Varieties

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system. Define a binary relation

$$\vDash^* \subseteq \mathrm{AlgSys}(\mathbf{F}) \times \mathrm{NQEq}(\mathbf{F})$$

by setting, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, and every $\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in \mathrm{NQEq}(\mathbf{F})$ (i.e., such that $\vec{\sigma} \approx \vec{\tau}$ is a finite vector of natural $\mathbf{F}$-equations),

$$\begin{array}{lll}\mathcal{A} \vDash^* \vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau & \text{iff} & \text{for all } \Sigma \in |\mathbf{Sign}| \text{ and all } \vec{\phi} \in \mathrm{SEN}(\Sigma), \\ & & \vec{\sigma}_\Sigma^\mathcal{A}(\vec{\phi}) = \vec{\tau}_\Sigma^\mathcal{A}(\vec{\phi}) \text{ implies } \sigma_\Sigma^\mathcal{A}(\vec{\phi}) = \tau_\Sigma^\mathcal{A}(\vec{\phi}).\end{array}$$

Note that, because of the surjectivity of $\langle F, \alpha \rangle : \mathbf{F} \to \mathbf{A}$, the condition above may be equivalently expressed by saying that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\vec{\sigma}_{F(\Sigma)}^\mathcal{A}(\alpha_\Sigma(\vec{\phi})) = \vec{\tau}_{F(\Sigma)}^\mathcal{A}(\alpha_\Sigma(\vec{\phi})) \text{ implies } \sigma_{F(\Sigma)}^\mathcal{A}(\alpha_\Sigma(\vec{\phi})) = \tau_{F(\Sigma)}^\mathcal{A}(\alpha_\Sigma(\vec{\phi})).$$

More generally, we define a binary relation

$$\vDash \subseteq \mathrm{AlgSys}(\mathbf{F}) \times \mathrm{NGEq}(\mathbf{F})$$

by setting, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, and every $\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in \mathrm{NGEq}(\mathbf{F})$ (here $\vec{\sigma} \approx \vec{\tau}$ may be of arbitrary length),

$$\begin{array}{lll}\mathcal{A} \vDash \vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau & \text{iff} & \text{for all } \Sigma \in |\mathbf{Sign}| \text{ and all } \vec{\phi} \in \mathrm{SEN}(\Sigma), \\ & & \vec{\sigma}_\Sigma^\mathcal{A}(\vec{\phi}) = \vec{\tau}_\Sigma^\mathcal{A}(\vec{\phi}) \text{ implies } \sigma_\Sigma^\mathcal{A}(\vec{\phi}) = \tau_\Sigma^\mathcal{A}(\vec{\phi}).\end{array}$$

Again relying on the surjectivity of $\langle F, \alpha \rangle$, the condition can be equivalently expressed by saying that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\vec{\sigma}_{F(\Sigma)}^\mathcal{A}(\alpha_\Sigma(\vec{\phi})) = \vec{\tau}_{F(\Sigma)}^\mathcal{A}(\alpha_\Sigma(\vec{\phi})) \text{ implies } \sigma_{F(\Sigma)}^\mathcal{A}(\alpha_\Sigma(\vec{\phi})) = \tau_{F(\Sigma)}^\mathcal{A}(\alpha_\Sigma(\vec{\phi})).$$

We extend the notation to apply it to collections of $\mathbf{F}$-algebraic systems and families of natural $\mathbf{F}$-quasiequations by setting, for all $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$ and all $Q \subseteq \mathrm{NQEq}(\mathbf{F})$,

$$\begin{array}{lll}\mathsf{K} \vDash^* Q & \text{iff} & \text{for all } \mathcal{A} \in \mathsf{K} \text{ and all } q \in Q, \\ & & \mathcal{A} \vDash q.\end{array}$$

Analogously, for collections of $\mathbf{F}$-algebraic systems and families of natural $\mathbf{F}$-guasiequations, we set, for all $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$ and all $G \subseteq \mathrm{NGEq}(\mathbf{F})$,

$$\mathsf{K} \vDash G \quad \text{iff} \quad \text{for all } \mathcal{A} \in \mathsf{K} \text{ and all } g \in G, \; \mathcal{A} \vDash g.$$

It is clear that $\vDash^*$ determines a Galois connection between $\mathcal{P}(\mathrm{AlgSys}(\mathbf{F}))$ and $\mathcal{P}(\mathrm{NQEq}(\mathbf{F}))$ and $\vDash$ determines a Galois connection between $\mathcal{P}(\mathrm{AlgSys}(\mathbf{F}))$ and $\mathcal{P}(\mathrm{NGEq}(\mathbf{F}))$. Related to these Galois connections, we use the following notational conventions.

First, given a class K of $\mathbf{F}$-algebraic systems, we define the collection

$$\mathrm{NQEq}(\mathsf{K}) = \{q \in \mathrm{NQEq}(\mathbf{F}) : \mathsf{K} \vDash^* q\}.$$

Similarly, we define the collection

$$\mathrm{NGEq}(\mathsf{K}) = \{g \in \mathrm{NGEq}(\mathbf{F}) : \mathsf{K} \vDash g\}.$$

Next, given a collection Q of natural $\mathbf{F}$-quasiequations, we define

$$\mathrm{NMod}^*(Q) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : \mathcal{A} \vDash^* Q\}$$

and, given a collection G of natural $\mathbf{F}$-guasiequations, we define

$$\mathrm{NMod}(G) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : \mathcal{A} \vDash G\}.$$

Finally, for the closure operators associated with the Galois connection $\vDash^*$, we set, for all $Q \subseteq \mathrm{NQEq}(\mathbf{F})$ and all $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$,

$$\begin{array}{rcl} N^*(Q) & = & \mathrm{NQEq}(\mathrm{NMod}^*(Q)); \\ \mathbb{Q}^{\mathsf{Syn}}(\mathsf{K}) & = & \mathrm{NMod}^*(\mathrm{NQEq}(\mathsf{K})). \end{array}$$

Analogously, for the closure operators associated with the Galois connection $\vDash$, we set, for all $G \subseteq \mathrm{NGEq}(\mathbf{F})$ and all $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$,

$$\begin{array}{rcl} N(G) & = & \mathrm{NGEq}(\mathrm{NMod}(G)); \\ \mathbb{G}^{\mathsf{Syn}}(\mathsf{K}) & = & \mathrm{NMod}(\mathrm{NGEq}(\mathsf{K})). \end{array}$$

By the general theory of Galois connections, we know that the closed sets of the closure operator N^* are the ones of the form $\mathrm{NQEq}(\mathsf{K})$ for a class K of $\mathbf{F}$-algebraic systems and those of the closure operator $\mathbb{Q}^{\mathsf{Syn}}$ are those of the form $\mathrm{NMod}^*(Q)$ for a collection Q of natural $\mathbf{F}$-quasiequations. Moreover, the closed sets of the closure operator N are the ones of the form $\mathrm{NGEq}(\mathsf{K})$ for a class K of $\mathbf{F}$-algebraic systems and those of the closure operator $\mathbb{G}^{\mathsf{Syn}}$ are those of the form $\mathrm{NMod}(G)$ for a collection G of natural $\mathbf{F}$-guasiequations.

We set out to provide intrinsic characterizations of these closed sets.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and let $R \subseteq \mathrm{NGeq}(\mathbf{F})$ be a set of $\mathbf{F}$-guasiequations. First, define

$$\dot{R} \coloneqq R \cap \mathrm{NEq}(\mathbf{F}),$$

i.e., $\dot{R}$ is the set of natural equations included in R, which, of course, constitute special cases of $\mathbf{F}$-guasiequations. Moreover, recall that a binary relation $Q \subseteq \mathrm{NEq}(\mathbf{F})$ is called a **metacongruence on F** if it is an equivalence relation on $N^\flat$ and, in addition, satisfies the **property of substitution**, i.e.,

> For all $o, \rho : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$ and all $\sigma^i, \tau^i : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, $i < \omega$,
>
> $$\langle o, \rho \rangle \in Q \quad \text{and} \quad \langle \sigma^i, \tau^i \rangle \in Q,\ i < \omega, \quad \text{imply} \quad \langle o \circ \vec{\sigma}, \rho \circ \vec{\tau} \rangle \in Q.$$

We say that $R \subseteq \mathrm{NGEq}(\mathbf{F})$ is a **metaguasicongruence on F**, or **mgcongruence** for short, if

- $\dot{R}$ is a metacongruence on $\mathbf{F}$;
- R satisfies the **modus ponens**, i.e., for all $\vec{\sigma}, \vec{\tau}, \sigma, \tau$ in $N^\flat$,

$$\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in R \quad \text{and} \quad \vec{\sigma} \approx \vec{\tau} \subseteq \dot{R} \quad \text{imply} \quad \sigma \approx \tau \in R.$$

The special case in which R consists entirely of natural $\mathbf{F}$-quasiequations is termed a **metaquasicongruence on F**, or **mqcongruence** for short.

Let $\mathrm{MetGon}(\mathbf{F})$ and $\mathrm{MetQon}(\mathbf{F})$, respectively, stand for the collection of all mgcongruences and the collection of all mqcongruences on $\mathbf{F}$. It is clear that both form complete lattices under ordinary inclusion, which are denoted by $\mathbf{MetGon}(\mathbf{F}) = \langle \mathrm{MetGon}(\mathbf{F}), \subseteq \rangle$ and $\mathbf{MetQon}(\mathbf{F}) = \langle \mathrm{MetQon}(\mathbf{F}), \subseteq \rangle$, respectively.

As was, perhaps, to be expected, mgcongruences (and mqcongruences on $\mathbf{F}$, in particular), on the one hand, and metacongruence systems on $\mathbf{F}$, on the other, are very closely related.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system and $R \in \mathrm{MetGon}(\mathbf{F})$. Define the family $X^R = \{X^R_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ on $\mathbf{F}$ by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi}, \phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in X^R_\Sigma & \text{iff} & \text{there exist } \vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in R, \\ & & \text{and } \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma), \text{ with} \\ & & \vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \to \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi}) \\ & & \qquad = \vec{\phi} \approx \vec{\psi} \to \phi \approx \psi.\end{array}$$

I.e., we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$X^R_\Sigma = \{\vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \to \sigma_\Sigma(\vec{\chi}), \tau_\Sigma(\vec{\chi}) : \\ \vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in R, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)\}.$$

Among other properties of this construction, we shall have the chance to exploit the following:

Lemma 17 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system.*

(a) If $R \subseteq \mathrm{NQEq}(\mathbf{F})$, then $\mathrm{Mod}^*(X^R) = \mathrm{NMod}^*(R)$.

(a) If $R \subseteq \mathrm{NGEq}(\mathbf{F})$, then $\mathrm{Mod}(X^R) = \mathrm{NMod}(R)$.

Proof: We only prove Part (b), since Part (a) is a special case. We have, for all $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, $\mathcal{A} \in \mathrm{Mod}(X^R)$ iff, by the definition of Mod and X^R, for all $\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in R$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\mathcal{A} \vDash_\Sigma \vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \to \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi}),$$

iff, by the definition of $\vDash$, for all $\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in R$,

$$\mathcal{A} \vDash \vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau,$$

iff, by the definition of NMod, $\mathcal{A} \in \mathrm{NMod}(R)$. ■

On the other hand, let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system and $X \in \mathrm{GonSys}(\mathbf{F})$. Define $R^X \subseteq \mathrm{NGEq}(\mathbf{F})$ by setting, for all $\vec{\sigma}, \vec{\tau}, \sigma, \tau$ in $N^\flat$,

$$\begin{array}{lll}\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in R^X & \text{iff} & \text{for all } \Sigma \in |\mathbf{Sign}^\flat| \text{ and all } \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma), \\ & & \vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \to \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi}) \in X_\Sigma.\end{array}$$

In other words,

$$R^X = \{\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau : (\forall \Sigma)(\forall \vec{\chi})(\vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \to \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi}) \in X_\Sigma)\}.$$

We characterize the closed sets in $\mathcal{P}(\mathrm{NGEq}(\mathbf{F}))$ under N. They turn out to be those mgcongruences on $\mathbf{F}$ satisfying an additional property. We also obtain a similar characterization of the closed sets in $\mathcal{P}(\mathrm{NQEq}(\mathbf{F}))$ under N^*.

Lemma 18 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* R *a collection of natural* $\mathbf{F}$*-guasiequations. If* $N(R) = R$*, then* $R \in \mathrm{MetGon}(\mathbf{F})$.

Proof: Let $R \subseteq \mathrm{NGEq}(\mathbf{F})$ and $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$, such that $R = \mathrm{NGEq}(\mathsf{K})$. First, note that $\dot{R} = \mathrm{NEq}(\mathsf{K}) \in \mathrm{MetCon}(\mathbf{F})$. So it suffices to show that R satisfies modus ponens. To this end, let $\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in R$ and $\vec{\sigma} \approx \vec{\tau} \subseteq \dot{R}$. Then, by hypothesis, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\mathsf{K} \vDash_\Sigma \vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \to \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi}) \quad \text{and} \quad \mathsf{K} \vDash_\Sigma \vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}).$$

Thus, taking into account the meaning of implication, for all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $\mathsf{K} \vDash_\Sigma \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi})$. This shows that $\mathsf{K} \vDash \sigma \approx \tau$ and, therefore, $\sigma \approx \tau \in \dot{R}$. Thus, R satisfies modus ponens and, hence, $R \in \mathrm{MetGon}(\mathbf{F})$. ■

Following along the lines of the proof of Lemma 18, we obtain

Lemma 19 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* R *a collection of natural* $\mathbf{F}$*-quasiequations. If* $N^*(R) = R$*, then* $R \in \mathrm{MetQon}(\mathbf{F})$.

There is one additional property, however, that, by necessity, all mgcongruences on $\mathbf{F}$ of the form $\mathrm{NGEq}(\mathsf{K})$, for some class K of $\mathbf{F}$-algebraic systems, must satisfy. And the same applies to all mqcongruences on $\mathbf{F}$ of the form $\mathrm{NQEq}(\mathsf{K})$. Recall that, given $X \in \mathrm{GonSys}(\mathbf{F})$ (and, in particular, $X \in \mathrm{QonSys}(\mathbf{F})$), we defined

$$\begin{aligned} R^X \;=\; & \{\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)) \\ & (\vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \to \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi}) \in X_\Sigma)\}. \end{aligned}$$

Definition 20 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system.*

- *An mgcongruence* $R \in \mathrm{MetGon}(\mathbf{F})$ *is called* **feasible** *if there exists a coverable guasicongruence system* $X \in \mathrm{GonSys}^\wedge(\mathbf{F})$*, such that* $R = R^X$.

- *An mqcongruence* $R \in \mathrm{MetQon}(\mathbf{F})$ *is called* **feasible** *if there exists a coverable quasicongruence system* $X \in \mathrm{QonSys}^{\wedge}(\mathbf{F})$*, such that* $R = R^X$*.*

Lemma 21 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* $R \subseteq \mathrm{NGEq}(\mathbf{F})$*. If* $N(R) = R$*, then* R *is a feasible mgcongruence on* $\mathbf{F}$*.*

Proof: Let $R \subseteq \mathrm{NGEq}(\mathbf{F})$, such that $R = N(R)$. By Lemma 18, $R \in \mathrm{MetGon}(\mathbf{F})$. To see that R is feasible, let $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$, such that $R = \mathrm{NGEq}(\mathsf{K})$ and set $X = \mathrm{GEq}(\mathsf{K})$. We know that $X \in \mathrm{GonSys}^{\wedge}(\mathbf{F})$. So, it suffices to show that $R = R^X$.

Suppose, first, that $\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in R = \mathrm{NGEq}(\mathsf{K})$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\mathsf{K} \models_\Sigma \vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \to \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi}).$$

Equivalently, we get $\vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \to \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi}) \in \mathrm{GEq}_\Sigma(\mathsf{K}) = X_\Sigma$. Hence, $\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in R^X$, showing that $R \subseteq R^X$.

Suppose, conversely, that $\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in R^X$. Thus, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \to \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi}) \in X_\Sigma = \mathrm{GEq}_\Sigma(\mathsf{K}).$$

Thus, $\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in \mathrm{NGEq}(\mathsf{K}) = R$, showing that $R^X \subseteq R$. ■

Similarly, we may demonstrate the following necessary condition for a subset of $\mathrm{NQEq}(\mathbf{F})$ to be closed under N^*.

Lemma 22 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* $R \subseteq \mathrm{NQEq}(\mathbf{F})$*. If* $N^*(R) = R$*, then* R *is a feasible mqcongruence on* $\mathbf{F}$*.*

We are now ready for the promised characterization of the closed sets of natural quasiequations under N^* and of natural guasiequations under N.

Proposition 23 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system.*

(a) If $R \subseteq \mathrm{NQEq}(\mathbf{F})$*, then* $N^*(R) = R$ *if and only if* R *is a feasible mqcongruence on* $\mathbf{F}$*;*

(b) If $R \subseteq \mathrm{NGEq}(\mathbf{F})$*, then* $N(R) = R$ *if and only if* R *is a feasible mgcongruence on* $\mathbf{F}$*.*

Proof: We only prove Part (b). Part (a) may be handled similarly.

If $R = N(R)$, then, by Lemma 21, R is a feasible mgcongruence on $\mathbf{F}$. Suppose, conversely, that R is a feasible mgcongruence on $\mathbf{F}$. Then, by definition, there exists $X \in \mathrm{GonSys}^{\wedge}(\mathbf{F})$, such that $R = R^X$. By the definition of coverable guasicongruence systems, there exists a collection $\{X^i : i \in I\} \subseteq \mathrm{GonSys}^{\approx}(\mathbf{F})$, such that $X = \bigcap_{i\in I} X^i$. By the theory of Galois connections, to show that R is closed under N, it suffices to show that it is in the image of NGEq, i.e., that $R = \mathrm{NGEq}(\mathsf{K})$, for some class $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$. We aim to show that $R = \mathrm{NGEq}(\{\mathcal{F}/\dot{X}^i : i \in I\})$ or, equivalently,

$$R^X = \mathrm{NGEq}(\{\mathcal{F}/\dot{X}^i : i \in I\}).$$

Suppose, first, that $\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in R^X$. Thus, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \to \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi}) \in X_\Sigma = \bigcap_{i\in I} X^i_\Sigma.$$

Thus, for all $i \in I$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \to \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi}) \in X^i_\Sigma.$$

Since X^i satisfies modus ponens, we get that, for all $i \in I$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \notin \dot{X}^i_\Sigma \quad \text{or} \quad \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi}) \in \dot{X}^i_\Sigma.$$

This shows that for all $i \in I$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\mathcal{F}/\dot{X}^i \vDash_\Sigma \vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \to \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi}).$$

Therefore, for all $i \in I$, $\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in \mathrm{NGEq}(\mathcal{F}/\dot{X}^i)$. So, we finally get that $\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in \bigcap_{i\in I} \mathrm{NGEq}(\mathcal{F}/\dot{X}^i) = \mathrm{NGEq}(\{\mathcal{F}/\dot{X}^i : i \in I\})$.

Suppose, conversely, that $\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in \mathrm{NGEq}(\{\mathcal{F}/\dot{X}^i : i \in I\})$. Thus, for all $i \in I$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\mathcal{F}/\dot{X}^i \vDash_\Sigma \vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \to \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi}).$$

Thus, by the definition of $\vDash$, for all $i \in I$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \notin \dot{X}^i_\Sigma \quad \text{or} \quad \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi}) \in \dot{X}^i_\Sigma.$$

Since, $X^i \in \mathrm{GonSys}^{\approx}(\mathbf{F})$, we get by completeness, for all $i \in I$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \to \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi}) \in X^i_\Sigma.$$

Hence, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \to \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi}) \in \bigcap_{i\in I} X^i_\Sigma = X_\Sigma.$$

This shows that $\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in R^X$ and, therefore, $\mathrm{NGEq}(\{\mathcal{F}/\dot{X}^i : i \in I\}) \subseteq R^X$.

Since $R = \mathrm{NGEq}(\{\mathcal{F}/\dot{X}^i : i \in I\})$, we conclude that $R = N(R)$. ∎

Finally, we characterize the closed sets in $\mathcal{P}(\mathrm{AlgSys}(\mathbf{F}))$ under $\mathbb{Q}^{\mathsf{Syn}}$ and those closed under $\mathbb{G}^{\mathsf{Syn}}$, i.e., those that constitute syntactic quasivarieties and syntactic guasivarieties, respectively, of $\mathbf{F}$-algebraic systems. Similarly to the case of semantic quasivarieties and semantic guasivarieties, syntactic quasivarieties turn out to be those classes of $\mathbf{F}$-algebraic systems that are closed under subdirect intersections and directed certifications and, in addition, are related in a specific way to semantics. Analogously, syntactic guasivarieties are those classes that are closed under subdirect intersections and certifications and are related in a similar way to semantics.

Proposition 24 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system. For all* $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$*,*

(a) $\mathbb{Q}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{Q}^{\mathsf{Syn}}(\mathsf{K})$*;*

(b) $\mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{G}^{\mathsf{Syn}}(\mathsf{K})$.

Proof: We only prove Part (b), since Part (a) is very similar.

Let K be a class of $\mathbf{F}$-algebraic systems and $\mathcal{A}$ an $\mathbf{F}$-algebraic system. Suppose $\mathcal{A} \in \mathbb{G}^{\mathsf{Sem}}(\mathsf{K})$ and let $\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in \mathrm{NGEq}(\mathsf{K})$. This means that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \to \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi}) \in \mathrm{GEq}_\Sigma(\mathsf{K}).$$

Since $\mathcal{A} \in \mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) = \mathrm{Mod}(\mathrm{GEq}(\mathsf{K}))$, we get that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $\mathcal{A} \vDash_\Sigma \vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \to \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi})$. Since this holds for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, we get that $\mathcal{A} \vDash \vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau$. But $\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in \mathrm{NGEq}(\mathsf{K})$ was arbitrary, whence $\mathcal{A} \in \mathrm{NMod}(\mathrm{NGEq}(\mathsf{K})) = \mathbb{G}^{\mathsf{Syn}}(\mathsf{K})$. We conclude that $\mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{G}^{\mathsf{Syn}}(\mathsf{K})$. ∎

Corollary 25 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$.

(a) If $\mathbb{Q}^{\mathsf{Syn}}(\mathsf{K}) = \mathsf{K}$*, then* $\mathbb{Q}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$*;*

(b) If $\mathbb{G}^{\mathsf{Syn}}(\mathsf{K}) = \mathsf{K}$*, then* $\mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$*.*

Proof: We again focus on the case of guasivarieties. If $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$, such that $\mathbb{G}^{\mathsf{Syn}}(\mathsf{K}) = \mathsf{K}$, then

$$\begin{array}{rcll} \mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) & \subseteq & \mathbb{G}^{\mathsf{Syn}}(\mathsf{K}) & \text{(by Proposition 24)} \\ & = & \mathsf{K}. & \text{(by hypothesis)} \end{array}$$

Since the reverse inclusion always holds, we get the conclusion. ■

But, if K is a syntactic quasi- or guasivariety, it has to satisfy an additional condition. Recalling the definition of X^R, for $R \in \mathrm{MetQon}(\mathbf{F})$, given a class $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$, we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\begin{array}{rcl} X_\Sigma^{\mathrm{NQEq}(\mathsf{K})} & = & \{\vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \to \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi}) : \\ & & \quad \vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in \mathrm{NQEq}(\mathsf{K}), \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)\}, \end{array}$$

and, similarly,

$$\begin{array}{rcl} X_\Sigma^{\mathrm{NGEq}(\mathsf{K})} & = & \{\vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \to \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi}) : \\ & & \quad \vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in \mathrm{NGEq}(\mathsf{K}), \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)\}, \end{array}$$

Definition 26 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$ *a class of* $\mathbf{F}$*-algebraic systems. The class* K *is called*

(a) **quasi-natural** *if* $\mathrm{QEq}(\mathsf{K})$ *and* $X^{\mathrm{NQEq}(\mathsf{K})}$ *have identical models, i.e., if*

$$\mathrm{Mod}^*(\mathrm{QEq}(\mathsf{K})) = \mathrm{Mod}^*(X^{\mathrm{NQEq}(\mathsf{K})});$$

(b) **guasi-natural** *if* $\mathrm{GEq}(\mathsf{K})$ *and* $X^{\mathrm{NGEq}(\mathsf{K})}$ *have identical models, i.e., if*

$$\mathrm{Mod}(\mathrm{GEq}(\mathsf{K})) = \mathrm{Mod}(X^{\mathrm{NGEq}(\mathsf{K})}).$$

Note that an alternative way to express the two conditions defining quasi-naturality and guasi-naturality, respectively, is obtained by recalling that the left had sides constitute the classes $\mathbb{Q}^{\mathsf{Sem}}(\mathsf{K})$ and $\mathbb{G}^{\mathsf{Sem}}(\mathsf{K})$, respectively, by the definition of the operators $\mathbb{Q}^{\mathsf{Sem}}$ and $\mathbb{G}^{\mathsf{Sem}}$. Thus, we may rewrite the corresponding equations as

$$\mathbb{Q}^{\mathsf{Sem}}(\mathsf{K}) = \mathrm{Mod}^*(X^{\mathrm{NQEq}(\mathsf{K})}) \text{ and } \mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) = \mathrm{Mod}(X^{\mathrm{NGEq}(\mathsf{K})}).$$

Note, also, that, as follows from the upcoming lemma, the left to right inclusions in the conditions of Definition 26 always hold. Therefore, the definition essentially relies on the two opposite inclusions.

Lemma 27 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, and* $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$ *a class of* $\mathbf{F}$*-algebraic systems. Then*

$$X^{\mathrm{NQEq}(\mathsf{K})} \leq \mathrm{QEq}(\mathsf{K}) \quad \textit{and} \quad X^{\mathrm{NGEq}(\mathsf{K})} \leq \mathrm{GEq}(\mathsf{K}).$$

Proof: We show the second inclusion. The first follows along similar lines. Suppose $\Sigma \in |\mathbf{Sign}^\flat|$, $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in X_\Sigma^{\mathrm{NGEq}(\mathsf{K})}$ and let $\mathcal{A} \in \mathsf{K}$, such that $\mathcal{A} \vDash_\Sigma \vec{\phi} \approx \vec{\psi}$. Then, by definition of $X^{\mathrm{NGEq}(\mathsf{K})}$, there exist $\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in \mathrm{NGEq}(\mathsf{K})$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, such that

$$\vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) = \vec{\phi} \approx \vec{\psi} \quad \text{and} \quad \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi}) = \phi \approx \psi.$$

Since $\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in \mathrm{NGEq}(\mathsf{K})$ and $\mathcal{A} \in \mathsf{K}$, we get $\mathcal{A} \vDash_\Sigma \vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \to \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi})$, i.e., $\mathcal{A} \vDash_\Sigma \vec{\phi} \approx \vec{\psi} \to \phi \approx \psi$. Since, by assumption, $\mathcal{A} \vDash_\Sigma \vec{\phi} \approx \vec{\psi}$, we conclude that $\mathcal{A} \vDash_\Sigma \phi \approx \psi$. This shows that $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in \mathrm{GEq}_\Sigma(\mathsf{K})$. We conclude that $X^{\mathrm{NGEq}(\mathsf{K})} \leq \mathrm{GEq}(\mathsf{K})$. ∎

Corollary 28 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system, and* $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$ *a class of* $\mathbf{F}$*-algebraic systems. Then*

$$\mathrm{Mod}^*(\mathrm{QEq}(\mathsf{K})) \subseteq \mathrm{Mod}^*(X^{\mathrm{NQEq}(\mathsf{K})}) \textit{ and } \mathrm{Mod}(\mathrm{GEq}(\mathsf{K})) \subseteq \mathrm{Mod}(X^{\mathrm{NGEq}(\mathsf{K})}).$$

Proof: By Lemma 27. ∎

Equivalently, we always have

$$\mathbb{Q}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathrm{Mod}^*(X^{\mathrm{NQEq}(\mathsf{K})}) \text{ and } \mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathrm{Mod}(X^{\mathrm{NGEq}(\mathsf{K})}).$$

We show next that, for a class of $\mathbf{F}$-algebraic systems, quasi-naturality is a necessary condition for constituting a syntactic quasivariety and guasi-naturality a necessary condition for being a syntactic guasivariety.

Proposition 29 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$.

(a) If $\mathbb{Q}^{\mathsf{Syn}}(\mathsf{K}) = \mathsf{K}$*, then* K *is a quasi-natural class;*

(b) If $\mathbb{G}^{\mathsf{Syn}}(\mathsf{K}) = \mathsf{K}$*, then* K *is a guasi-natural class.*

Proof: We again prove only Part (b). Let $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$, such that $\mathbb{G}^{\mathsf{Syn}}(\mathsf{K}) = \mathsf{K}$. We must show that $\mathrm{Mod}(\mathrm{GEq}(\mathsf{K})) = \mathrm{Mod}(X^{\mathrm{NGEq}(\mathsf{K})})$. We have:

$$\begin{array}{rcll} \mathrm{Mod}(\mathrm{GEq}(\mathsf{K})) & = & \mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) & (\text{definition of } \mathbb{G}^{\mathsf{Sem}}) \\ & = & \mathsf{K} & (\text{Corollary 25}) \\ & = & \mathbb{G}^{\mathsf{Syn}}(\mathsf{K}) & (\text{hypothesis}) \\ & = & \mathrm{NMod}(\mathrm{NGEq}(\mathsf{K})) & (\text{definition of } \mathbb{G}^{\mathsf{Syn}}) \\ & = & \mathrm{Mod}(X^{\mathrm{NGEq}(\mathsf{K})}). & (\text{Lemma 17}) \end{array}$$

Therefore, K is a natural class. ■

Now, we are ready to provide a characterization of syntactic quasivarieties and syntactic guasivarieties. The first are exactly those semantic quasivarieties that are quasi-natural and the second those semantic guasivarieties that are guasi-natural.

Proposition 30 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system.*

(a) A class K *of* $\mathbf{F}$*-algebraic systems is a syntactic quasivariety if and only if it is a quasi-natural semantic quasivariety.*

(b) A class K *of* $\mathbf{F}$*-algebraic systems is a syntactic guasivariety if and only if it is a guasi-natural semantic guasivariety.*

Proof: We prove Part (b). Part (a) may be handled similarly.

If K is a syntactic guasivariety, then, by Corollary 25, it is a semantic guasivariety and, moreover, by Proposition 29, it is a guasi-natural class.

Suppose, conversely, that K is a guasi-natural semantic guasivariety. We have

$$\begin{array}{rcll} \mathsf{K} & = & \mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) & (\text{hypothesis}) \\ & = & \mathrm{Mod}(\mathrm{GEq}(\mathsf{K})) & (\text{definition of } \mathbb{G}^{\mathsf{Sem}}) \\ & = & \mathrm{Mod}(X^{\mathrm{NGEq}(\mathsf{K})}) & (\text{guasi-naturality}) \\ & = & \mathrm{NMod}(\mathrm{NGEq}(\mathsf{K})) & (\text{Lemma 17}) \\ & = & \mathbb{G}^{\mathsf{Syn}}(\mathsf{K}). & (\text{definition of } \mathbb{G}^{\mathsf{Syn}}) \end{array}$$

Hence, K is a syntactic guasivariety. ■

Corollary 31 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system.*

(a) A class K *of* $\mathbf{F}$*-algebraic systems is a syntactic quasivariety if and only if* K *is quasi-natural and closed under* $\mathbb{C}^*$ *and* $\overset{\triangleleft}{\mathbb{IP}}$*;*

(b) A class K *of* $\mathbf{F}$*-algebraic systems is a syntactic guasivariety if and only if* K *is guasi-natural and closed under* $\mathbb{C}$ *and* $\overset{\triangleleft}{\mathbb{IP}}$*.*

Proof: We have K is a syntactic quasivariety (guasivariety, respectively) if and only if, by Proposition 30, it is a quasi- (guasi-, respectively) natural semantic quasivariety (guasivariety, respectively) if and only if, by Proposition 16 (Proposition 15, respectively), it is a natural class closed under $\mathbb{C}^*$ ($\mathbb{C}$, respectively) and $\overset{\triangleleft}{\mathbb{IP}}$. ∎

4 The Closures C^* and N^*

In this section, we characterize

$$\begin{array}{rl} C^*: & \mathcal{P}\mathrm{QEq}(\mathbf{F}) \to \mathcal{P}\mathrm{QEq}(\mathbf{F}) \\ N^*: & \mathcal{P}\mathrm{NQEq}(\mathbf{F}) \to \mathcal{P}\mathrm{NQEq}(\mathbf{F}) \end{array}$$

as closure operators by showing how to obtain the closure of a given $X \leq \mathrm{QEq}(\mathbf{F})$ or a given $R \subseteq \mathrm{NQEq}(\mathbf{F})$, respectively, in a step-wise fashion. Similar results may be established for C and N, but, since these two operators are not finitary, one has to use transfinite, in place of finite, induction. So we focus upon the finitary cases.

For the semantic case of C^*, the key observation is that membership of $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in C^*_\Sigma(X)$ is equivalent to membership of $\phi \approx \psi$ in the congruence system $\Theta(X \cup \{\vec{\phi} \approx \vec{\psi}\})$.

Lemma 32 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $X \leq \mathrm{QEq}(\mathbf{F})$*,* $\Sigma \in |\mathbf{Sign}^\flat|$ *and* $\vec{\phi} \approx \vec{\psi} \in \mathrm{QEq}_\Sigma(\mathbf{F})$*. Then*

$$\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in C^*_\Sigma(X) \quad \textit{iff} \quad \phi \approx \psi \in \Theta_\Sigma(X \cup \{\vec{\phi} \approx \vec{\psi}\}).$$

Proof: Suppose, first, that $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in \mathrm{QEq}_\Sigma(\mathrm{Mod}^*(X))$. Let θ be a congruence system on $\mathbf{F}$ satisfying $X \cup \{\vec{\phi} \approx \vec{\psi}\}$. Consider $\mathcal{F}/\theta$. Since θ satisfies X, we certainly have $\mathcal{F}/\theta \vDash^* X$. Thus $\mathcal{F}/\theta \in \mathrm{Mod}^*(X)$. So, by hypothesis, $\mathcal{F}/\theta \vDash^* \vec{\phi} \approx \vec{\psi} \to \phi \approx \psi$. But $\vec{\phi} \approx \vec{\psi} \subseteq \theta_\Sigma$, since θ satisfies $X \cup \{\vec{\phi} \approx \vec{\psi}\}$. Thus, by the definition of $\vDash^*$, $\phi \approx \psi \in \theta_\Sigma$. Since every congruence system on $\mathbf{F}$ satisfying $X \cup \{\vec{\phi} \approx \vec{\psi}\}$ contains $\phi \approx \psi$, so does $\Theta(X \cup \{\vec{\phi} \approx \vec{\psi}\})$.

Suppose, conversely, that $\phi \approx \psi \in \Theta_\Sigma(X \cup \{\vec{\phi} \approx \vec{\psi}\})$. Let $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, such that $\mathcal{A} \vDash^*_\Sigma \vec{\phi} \approx \vec{\psi}$. Then $\mathrm{Ker}(\mathcal{A})$ is a congruence system on $\mathbf{F}$ that satisfies $X \cup \{\vec{\phi} \approx \vec{\psi}\}$. By hypothesis, $\phi \approx \psi \in \mathrm{Ker}_\Sigma(\mathcal{A})$. But this gives $\mathcal{A} \vDash^*_\Sigma \phi \approx \psi$. Hence, $\mathcal{A} \vDash^*_\Sigma \vec{\phi} \approx \vec{\psi} \to \phi \approx \psi$. Since $\mathcal{A} \in \mathrm{Mod}^*(X)$ was arbitrary, we conclude that $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in \mathrm{QEq}_\Sigma(\mathrm{Mod}^*(X)) = C^*_\Sigma(X)$. ■

By Lemma 32, testing whether $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in C^*_\Sigma(X)$ amounts to testing whether $\phi \approx \psi \in \Theta_\Sigma(X \cup \{\vec{\phi} \approx \vec{\psi}\})$. Motivated by this result, we turn to a characterization of $\Theta(X)$ for a given $X \leq \mathrm{QEq}(\mathbf{F})$.

Let $X \leq \mathrm{QEq}(\mathbf{F})$. We define, for all $k < \omega$, by induction on k, the family $\Theta^k(X) = \{\Theta^k_\Sigma(X)\}_{\Sigma \in |\mathbf{Sign}^\flat|} \leq \mathrm{Eq}(\mathbf{F})$ by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$, X^k_Σ be given by

$$\begin{array}{rcl}
\Theta^0_\Sigma(X) & = & \dot{X}_\Sigma \cup \{\phi \approx \phi : \phi \in \mathrm{SEN}^\flat(\Sigma)\};\\
\Theta^{k+1}_\Sigma(X) & = & \Theta^k_\Sigma(X) \cup \{\psi \approx \phi : \phi \approx \psi \in \Theta^k_\Sigma(X)\}\\
& & \cup \{\phi \approx \chi : \phi \approx \psi, \psi \approx \chi \in \Theta^k_\Sigma(X)\}\\
& & \cup \{\sigma_\Sigma(\vec{\phi}) \approx \sigma_\Sigma(\vec{\psi}) : \vec{\phi} \approx \vec{\psi} \subseteq \Theta^k_\Sigma(X) \text{ and } \sigma \in N^\flat\}\\
& & \cup \{\mathrm{SEN}^\flat(f)(\phi \approx \psi) : \phi \approx \psi \in \Theta^k_{\Sigma'}(X)\\
& & \qquad \text{and } f \in \mathbf{Sign}^\flat(\Sigma', \Sigma)\}\\
& & \cup \{\phi \approx \psi : \vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in X_\Sigma \text{ and } \vec{\phi} \approx \vec{\psi} \subseteq \Theta^k_\Sigma(X)\}.
\end{array}$$

Finally, set

$$\Theta(X) = \bigcup_{k<\omega} \Theta^k(X).$$

Our goal is to show that $\Theta(X)$ coincides with the least congruence system on $\mathbf{F}$ that satisfies X (for which, we recall, the notation $\Theta(X)$ was originally introduced). This result will alleviate any concerns about overloading or introducing ambiguity in the meaning of $\Theta(X)$.

Proposition 33 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system and $X \leq \mathrm{QEq}(\mathbf{F})$. $\Theta(X)$, as constructed above, coincides with the least congruence system on $\mathbf{F}$ that satisfies X.*

Proof: We must show that $\Theta(X) \in \mathrm{ConSys}(\mathbf{F})$, $\Theta(X)$ satisfies X and, finally, that, if $\theta \in \mathrm{ConSys}(\mathbf{F})$ satisfies X, then $\Theta(X) \leq \theta$.

First, note that, by definition, $\Theta(X)$ is reflexive, symmetric and transitive, that it satisfies the congruence property and is, also, invariant under $\mathbf{Sign}^\flat$-morphisms. These properties together guarantee that $\Theta(X)$ is a congruence system on $\mathbf{F}$.

Suppose, next, that $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in X_\Sigma$, such that $\vec{\phi} \approx \vec{\psi} \subseteq \Theta_\Sigma(X)$. Since $\vec{\phi} \approx \vec{\psi}$ consists of a finite collection of $\mathbf{F}$-equations, there exists $k < \omega$, such that $\vec{\phi} \approx \vec{\psi} \subseteq \Theta^k_\Sigma(X)$. But, then, by definition, $\phi \approx \psi \in \Theta^{k+1}_\Sigma(X) \subseteq \Theta_\Sigma(X)$. Thus, $\Theta(X)$ does satisfy all $\mathbf{F}$-quasiequations in X.

Finally, using the hypotheses $\theta \in \mathrm{ConSys}(\mathbf{F})$ and θ satisfies X, it is not difficult to show, using induction on $k < \omega$, that $\Theta^k(X) \leq \theta$.

- For $k = 0$, since θ satisfies X, $\dot{X} \leq \theta$ and, since θ is reflexive, $\{\phi \approx \phi : \phi \in \mathrm{SEN}^\flat(\Sigma)\} \subseteq \theta_\Sigma$, for all $\Sigma \in |\mathbf{Sign}^\flat|$. Thus, we get $\Theta^0(X) \leq \theta$.

- Suppose, next, that $\Theta^k(X) \leq \theta$, for some $k < \omega$. Then, since θ is a congruence system (symmetric, transitive, satisfies the congruence property and is $\mathbf{Sign}^\flat$-invariant), we get, using the induction hypothesis, that for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\begin{array}{ll} & \{\psi \approx \phi : \phi \approx \psi \in \Theta^k_\Sigma(X)\} \\ \cup & \{\phi \approx \chi : \phi \approx \psi, \psi \approx \chi \in \Theta^k_\Sigma(X)\} \\ \cup & \{\sigma_\Sigma(\vec{\phi}) \approx \sigma_\Sigma(\vec{\psi}) : \vec{\phi} \approx \vec{\psi} \subseteq \Theta^k_\Sigma(X)\} \\ \cup & \{\mathrm{SEN}^\flat(f)(\phi \approx \psi) : f \in \mathbf{Sign}^\flat(\Sigma', \Sigma), \phi \approx \psi \in \Theta^k_{\Sigma'}(X)\} \quad \subseteq \theta_\Sigma. \end{array}$$

  Further, since θ satisfies X, we get, again using the induction hypothesis, that, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\{\phi \approx \psi : \vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in X_\Sigma \text{ and } \vec{\phi} \approx \vec{\psi} \subseteq \Theta^k_\Sigma(X)\} \subseteq \theta_\Sigma.$$

  Thus, we have $\Theta^{k+1}(X) \leq \theta$.

Now we obtain $\Theta^k(X) \leq \theta$, for all $k < \omega$. Hence, $\Theta(X) = \bigcup_{k<\omega} \Theta^k(X) \leq \theta$. ∎

Recall that, given a sentence family X of $\mathbf{F}$, we say that it is locally finite if, for all $\Sigma \in |\mathbf{Sign}^\flat|$, X_Σ is finite. Based on the inductive definition of $\Theta(X)$, we can now show that $\Theta(X)$ is a finitary closure operator in the following sense.

Lemma 34 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system,* $X \leq \mathrm{QEq}(\mathbf{F})$, $\Sigma \in |\mathbf{Sign}^\flat|$ *and* $\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathbf{F})$. *If* $\phi \approx \psi \in \Theta^k_\Sigma(X)$, *then, there exists a locally finite* $Y \leq X$, *such that* $\phi \approx \psi \in \Theta^k_\Sigma(Y)$.

Proof: We prove the statement by induction on $k < \omega$.

If $k = 0$, $\phi \approx \psi \in \dot{X}_\Sigma$ or $\phi = \psi$. In the first case, $\phi \approx \psi \in \Theta^0_\Sigma(\{\phi \approx \psi\})$ and, in the second, $\phi \approx \psi \in \Theta^0_\Sigma(\varnothing)$.

Suppose that the statement holds for some $k < \omega$. Consider $\phi \approx \psi \in \Theta^{k+1}_\Sigma(X)$. We look at the various possibilities that may occur.

- If $\phi \approx \psi \in \Theta^k_\Sigma(X)$, then the conclusion follows directly by the induction hypothesis.
- If $\psi \approx \phi \in \Theta^k_\Sigma(X)$, then, by the induction hypothesis, there exists a locally finite $Y \le X$, such that $\psi \approx \phi \in \Theta^k_\Sigma(Y)$, whence, by definition, $\phi \approx \psi \in \Theta^{k+1}_\Sigma(Y)$.
- If $\phi \approx \chi, \chi \approx \psi \in \Theta^k_\Sigma(X)$, then, by te induction hypothesis, there exist locally finite $Y, Z \le X$, such that $\phi \approx \chi \in \Theta^k_\Sigma(Y)$ and $\chi \approx \psi \in \Theta^k_\Sigma(Z)$. Therefore, $\phi \approx \psi \in \Theta^{k+1}_\Sigma(Y \cup Z)$.
- If $\vec{\phi} \approx \vec{\psi} \subseteq \Theta^k_\Sigma(X)$ (where $\phi \approx \psi \equiv \sigma_\Sigma(\vec{\phi}) \approx \sigma_\Sigma(\vec{\psi})$, for some σ in $N^\flat$), then, by the induction hypothesis, $\phi_i \approx \psi_i \in \Theta^k_\Sigma(Y^i)$, for some locally finite $Y^i \le X$, $i < n$. Now we have $\phi \approx \psi \equiv \sigma_\Sigma(\vec{\phi}) \approx \sigma_\Sigma(\vec{\psi}) \in \Theta^{k+1}_\Sigma(\bigcup_{i<n} Y^i)$, with $\bigcup_{i<n} Y^i$ being a locally finite subfamily of X.
- If $\phi \approx \psi \equiv \mathrm{SEN}^\flat(f)(\phi' \approx \psi')$, for some $f \in \mathbf{Sign}^\flat(\Sigma', \Sigma)$ and some $\phi' \approx \psi' \in \Theta^k_{\Sigma'}(X)$, then, by the induction hypothesis, there exists locally finite $Y \le X$, such that $\phi' \approx \psi' \in \Theta^k_{\Sigma'}(Y)$. But this yields that $\phi \approx \psi \in \Theta^{k+1}_\Sigma(Y)$.
- Finally, assume that $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in X_\Sigma$ and $\vec{\phi} \approx \vec{\psi} \subseteq \Theta^k_\Sigma(X)$. Again, using the induction hypothesis, we get a locally finite $Y^i \le X$, $i < n$, such that $\phi_i \approx \psi_i \in \Theta^k_\Sigma(Y^i)$. Therefore, $\phi \approx \psi \in \Theta^{k+1}_\Sigma(\bigcup_{i<n} Y^i)$, where $\bigcup_{i<n} Y^i \le X$ is also locally finite.

Thus, since all possible cases have been treated successfully, the proof is complete. ■

Proposition 35 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system,* $X \le \mathrm{DEq}(\mathbf{F})$*,* $\Sigma \in |\mathbf{Sign}^\flat|$ *and* $\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathbf{F})$*. If* $\phi \approx \psi \in \Theta_\Sigma(X)$*, then, there exists a locally finite* $Y \le X$*, such that* $\phi \approx \psi \in \Theta_\Sigma(Y)$*.*

Proof: If $\phi \approx \psi \in \Theta_\Sigma(X)$, then, by Proposition 33, there exists $k < \omega$, such that $\phi \approx \psi \in \Theta^k_\Sigma(X)$, whence, by Lemma 34, there exists a locally finite $Y \leq X$, such that $\phi \approx \psi \in \Theta^k_\Sigma(Y)$. Thus, again by Proposition 33, $\phi \approx \psi \in \Theta_\Sigma(Y)$. ■

Corollary 36 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system. The closure operator* $C^* : \mathcal{P}\mathrm{QEq}(\mathbf{F}) \to \mathcal{P}\mathrm{QEq}(\mathbf{F})$ *is finitary.*

Proof: Let $X \leq \mathrm{QEq}(\mathbf{F})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in \mathrm{QEq}_\Sigma(\mathbf{F})$, such that $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in C^*_\Sigma(X)$. By Lemma 32, this holds if and only if $\phi \approx \psi \in \Theta_\Sigma(X \cup \{\vec{\phi} \approx \vec{\psi}\})$. By Proposition 35, this holds if and only if there exists $Y \leq X$ locally finite, such that $\phi \approx \psi \in \Theta_\Sigma(Y \cup \{\vec{\phi} \approx \vec{\psi}\})$. Therefore, by one more application of Lemma 32, we get that $\vec{\phi} \approx \vec{\psi} \to \phi \approx \psi \in C^*_\Sigma(Y)$. This shows that C^* is a finitary closure operator. ■

Next, we provide, based on C^*, a necessary and sufficient condition for a natural quasiequation to be in the closure N^* of a given collection R of natural quasiequations.

Lemma 37 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system,* $R \subseteq \mathrm{NQEq}(\mathbf{F})$ *and* $\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in \mathrm{NQEq}(\mathbf{F})$. *Then*

$$\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in N^*(R) \quad \textit{iff} \quad \textit{for all } \Sigma \in |\mathbf{Sign}^\flat|, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma), \quad \vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \to \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi}) \in C^*_\Sigma(X^R).$$

Proof: Suppose that $\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in N^*(R)$ and let $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, such that $\mathcal{A} \vDash^* X^R$ and $\mathcal{A} \vDash^*_\Sigma \vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi})$. The hypothesis $\mathcal{A} \vDash^* X^R$ implies that $\mathcal{A} \vDash^* R$. Hence, the hypothesis $\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in N^*(R)$ implies that $\mathcal{A} \vDash^* \vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau$. Thus, since $\mathcal{A} \vDash^*_\Sigma \vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi})$, we get that $\mathcal{A} \vDash^*_\Sigma \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi})$. We now conclude that $\mathcal{A} \vDash^*_\Sigma \vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \to \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi})$. This proves that $\vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \to \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi}) \in C^*_\Sigma(X^R)$.

Assume, conversely, that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, $\vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \to \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi}) \in C^*_\Sigma(X^R)$. Let $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$, such that $\mathcal{A} \vDash^* R$. This implies that $\mathcal{A} \vDash^* X^R$, whence, by the hypothesis, $\mathcal{A} \vDash^*_\Sigma \vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi}) \to \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi})$. Since this holds for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, we get, by definition, $\mathcal{A} \vDash^* \vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau$. Thus, by the definition of N^*, $\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in N^*(R)$. ■

Lemma 37, combined with Lemma 32, provides a way to check whether $\vec{\sigma} \approx \vec{\tau} \to \sigma \approx \tau \in N^*(R)$. One has to check whether, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi}) \in \Theta_\Sigma(X^R \cup \{\vec{\sigma}_\Sigma(\vec{\chi}) \approx \vec{\tau}_\Sigma(\vec{\chi})\}).$$

This is, admittedly, a rather inefficient and complicated procedure and seems more likely to be useful for falsification, rather than for verification, purposes.

5 The Closures $\mathbb{Q}^{\mathsf{Sem}}$ and $\mathbb{G}^{\mathsf{Sem}}$

In this section we show how to obtain the semantic and the syntactic quasivarieties and guasivarieties generated by a given class K of $\mathbf{F}$-algebraic systems by applying on K a series of class operators.

First, it is fairly easy to verify that all three operators $\mathbb{C}$, $\mathbb{C}^*$ and $\overset{\triangleleft}{\mathbb{\Pi}}$, introduced previously, are closure operators on classes of $\mathbf{F}$-algebraic systems.

Lemma 38 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system. Then the operators*

$$\mathbb{C}, \mathbb{C}^*, \overset{\triangleleft}{\mathbb{\Pi}} : \mathcal{P}(\mathrm{AlgSys}(\mathbf{F})) \to \mathcal{P}(\mathrm{AlgSys}(\mathbf{F}))$$

are closure operators.

Proof: Inflationarity is easy to verify for all three operators. Suppose $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$ and $\mathcal{A} \in \mathsf{K}$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathcal{A}^\Sigma := \mathcal{A} \in \mathsf{K}$ K-certifies $\mathcal{A}$, whence $\mathsf{K} \subseteq \mathbb{C}(\mathsf{K}) \subseteq \mathbb{C}^*(\mathsf{K})$. Moreover, $\{\langle I, \iota \rangle : \mathcal{A} \to \mathcal{A}\}$ is a subdirect intersection, with $\mathcal{A} \in \mathsf{K}$, whence $\mathsf{K} \subseteq \overset{\triangleleft}{\mathbb{\Pi}}(\mathsf{K})$.

Monotonicity is equally straightforward. If $\mathsf{K} \subseteq \mathsf{L} \subseteq \mathrm{AlgSys}(\mathbf{F})$, then any collection of K-certificates constitutes a collection of L-certificates and any subdirect intersection with targets in K is one with targets in L.

Slightly more involved is the task of showing the idempotency of all three operators. Let $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$.

First, suppose that $\mathcal{A} \in \mathbb{C}(\mathbb{C}(\mathsf{K}))$. Thus, for all $\Sigma \in |\mathbf{Sign}^\flat|$, there exists $\mathcal{A}^\Sigma \in \mathbb{C}(\mathsf{K})$, such that $\mathrm{Eq}_\Sigma(\mathcal{A}) = \mathrm{Eq}_\Sigma(\mathcal{A}^\Sigma)$. Hence, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, there exists $\mathcal{A}^{\Sigma,\Sigma'} \in \mathsf{K}$, such that $\mathrm{Eq}_{\Sigma'}(\mathcal{A}^\Sigma) = \mathrm{Eq}_{\Sigma'}(\mathcal{A}^{\Sigma,\Sigma'})$. Thus, for all $\Sigma \in |\mathbf{Sign}^\flat|$, there exists $\mathcal{A}^{\Sigma,\Sigma} \in \mathsf{K}$, such that $\mathrm{Eq}_\Sigma(\mathcal{A}) = \mathrm{Eq}_\Sigma(\mathcal{A}^\Sigma) = \mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,\Sigma})$, which shows that $\mathcal{A} \in \mathbb{C}(\mathsf{K})$. Thus, $\mathbb{C}$ is idempotent.

Suppose, next, that $\mathcal{A} \in \overset{\triangleleft}{\mathbb{I}\Pi}(\overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K}))$. Then, there exists a subdirect intersection $\{\langle F^i, \alpha^i \rangle : \mathcal{A} \to \mathcal{A}^i : i \in I\}$, with $\mathcal{A}^i \in \overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K})$, for all $i \in I$. Therefore, for all $i \in I$, there exists a subdirect intersection $\{\langle F^{ij}, \alpha^{ij} \rangle : \mathcal{A}^i \to \mathcal{A}^{ij} : j \in J_i\}$, with $\mathcal{A}^{ij} \in \mathsf{K}$, for all $j \in J_i$. Considering the collection

$$\{\langle F^{ij}, \alpha^{ij} \rangle \circ \langle F^i, \alpha^i \rangle : \mathcal{A} \to \mathcal{A}^{ij} : i \in I, j \in J_i\},$$

we have that $\mathcal{A}^{ij} \in \mathsf{K}$, for all $i \in I$ and all $j \in J_i$ and, also $\mathrm{Ker}(\mathcal{A}) = \bigcap_{i \in I} \mathrm{Ker}(\mathcal{A}^i) = \bigcap_{i \in I} \bigcap_{j \in J_i} \mathrm{Ker}(\mathcal{A}^{ij})$, i.e., that it constitutes a subdirect intersection. This shows that $\mathcal{A} \in \overset{\triangleleft}{\mathbb{I}\Pi}(\mathsf{K})$ and, hence, $\overset{\triangleleft}{\mathbb{I}\Pi}$ is idempotent.

Perhaps the most involved case is the idempotency of $\mathbb{C}^*$. So suppose that $\mathcal{A} \in \mathbb{C}^*(\mathbb{C}^*(\mathsf{K}))$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$, there exists $\{\mathcal{A}^{\Sigma,i} : i \in I_\Sigma\} \subseteq \mathbb{C}^*(\mathsf{K})$, such that $\bigcup_{i \in I_\Sigma} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i})$ is directed and, moreover, $\mathrm{Ker}_\Sigma(\mathcal{A}) = \bigcup_{i \in I_\Sigma} \mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,i})$. Thus, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$ and all $i \in I_\Sigma$, there exists $\{\mathcal{A}^{\Sigma,i,\Sigma',j} : j \in J^{\Sigma,i}_{\Sigma'}\} \subseteq \mathsf{K}$, such that $\bigcup_{j \in J^{\Sigma,i}_{\Sigma'}} \mathrm{Eq}^\omega_{\Sigma'}(\mathcal{A}^{\Sigma,i,\Sigma,j})$ is directed and, moreover, $\mathrm{Ker}_{\Sigma'}(\mathcal{A}^{\Sigma,i}) = \bigcup_{j \in J^{\Sigma,i}_{\Sigma'}} \mathrm{Eq}_{\Sigma'}(\mathcal{A}^{\Sigma,i,\Sigma,j})$. Now notice that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\{\mathcal{A}^{\Sigma,i,\Sigma,j} : i \in I_\Sigma, j \in J^{\Sigma,i}_\Sigma\} \subseteq \mathsf{K}$, such that

$$\mathrm{Eq}_\Sigma(\mathcal{A}) = \bigcup_{i \in I_\Sigma} \mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,i}) = \bigcup_{i \in I_\Sigma} \bigcup_{j \in J^{I,\Sigma}_\Sigma} \mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,i,\Sigma,j}).$$

Thus, it suffices to show that the collection

$$\bigcup_{i \in I_\Sigma} \bigcup_{j \in J^{\Sigma,i}_\Sigma} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i,\Sigma,j})$$

is directed. Consider $X \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i,\Sigma,j})$ and $X' \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i',\Sigma,j'})$. Then, as $\mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,i}) = \bigcup_{j \in J^{\Sigma,i}_\Sigma} \mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,i,\Sigma,j})$ and $\mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,i'}) = \bigcup_{j \in J^{\Sigma,i'}_\Sigma} \mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,i',\Sigma,j})$, we get that $X \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i})$ and $X' \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i'})$. As $\bigcup_{i \in I_\Sigma} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i})$ is directed, there exists $k \in I_\Sigma$ and $Y \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,k})$, such that $X, X' \subseteq Y$. But then it follows from the fact that $\mathrm{Eq}_\Sigma(\mathcal{A}^{\Sigma,k}) = \bigcup_{j \in J^{\Sigma,k}_\Sigma} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,k,\Sigma,j})$, the finiteness of Y and the fact that the union is directed, that there exists $\ell \in J^{\Sigma,k}_\Sigma$, such that $Y \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,k,\Sigma,\ell})$. This establishes the directedness of the collection $\bigcup_{i \in I_\Sigma} \bigcup_{j \in J^{\Sigma,i}_\Sigma} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{\Sigma,i,\Sigma,j})$.

We have now shown that the three class operators $\mathbb{C}$, $\mathbb{C}^*$ and $\overset{\triangleleft}{\mathbb{I}\Pi}$ are inflationary, monotone and idempotent, whence they constitute closure operators on $\mathrm{AlgSys}(\mathbf{F})$. ∎

It is not difficult to see that, for every class K of $\mathbf{F}$-algebraic systems, we have, besides the obvious inclusion $\mathbb{C}(\mathsf{K}) \subseteq \mathbb{C}^*(\mathsf{K})$, two additional inclusions governing the mode of interaction of $\mathbb{C}$ and $\mathbb{C}^*$ with $\overset{\triangleleft}{\mathbb{I}\mathbb{I}}$.

Lemma 39 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system. Then, for all* $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$*, we have*

$$\overset{\triangleleft}{\mathbb{I}\mathbb{I}}(\mathbb{C}(\mathsf{K})) \subseteq \mathbb{C}(\overset{\triangleleft}{\mathbb{I}\mathbb{I}}(\mathsf{K})) \quad \textit{and} \quad \overset{\triangleleft}{\mathbb{I}\mathbb{I}}(\mathbb{C}^*(\mathsf{K})) \subseteq \mathbb{C}^*(\overset{\triangleleft}{\mathbb{I}\mathbb{I}}(\mathsf{K})).$$

Proof: Let K be a class of $\mathbf{F}$-algebraic systems and $\mathcal{A} \in \overset{\triangleleft}{\mathbb{I}\mathbb{I}}\mathbb{C}(\mathsf{K})$. Then, there exists a subdirect intersection

$$\langle H^i, \gamma^i \rangle : \mathcal{A} \to \mathcal{A}^i, \quad i \in I,$$

where, for all $i \in I$, $\mathcal{A}^i \in \mathbb{C}(\mathsf{K})$. Thus, for all $i \in I$ and for all $\Sigma \in |\mathbf{Sign}^\flat|$, there exists $\mathcal{A}^{i,\Sigma} \in \mathsf{K}$, such that

$$\mathrm{Eq}_\Sigma(\mathcal{A}^i) = \mathrm{Eq}_\Sigma(\mathcal{A}^{i,\Sigma}). \tag{1}$$

Fix $\Sigma \in |\mathbf{Sign}^\flat|$. Define $\mathcal{A}^\Sigma := \mathcal{F}/\bigcap_{i \in I} \mathrm{Eq}(\mathcal{A}^{i,\Sigma})$. Then, since $\mathcal{A}^{i,\Sigma} \in \mathsf{K}$, for all $i \in I$, we get, by Lemma 10, $\mathcal{A}^\Sigma \in \overset{\triangleleft}{\mathbb{I}\mathbb{I}}(\mathsf{K})$. So to conclude the proof, it suffices to show that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $\mathrm{Eq}_\Sigma(\mathcal{A}) = \mathrm{Eq}_\Sigma(\mathcal{A}^\Sigma)$. Indeed, we have,

$$\begin{array}{lcl}
\mathrm{Eq}_\Sigma(\mathcal{A}) & = & \bigcap_{i \in I} \mathrm{Eq}_\Sigma(\mathcal{A}^i) \quad (\{\langle H^i, \gamma^i \rangle\} \text{ subdirect intersetion}) \\
 & = & \bigcap_{i \in I} \mathrm{Eq}_\Sigma(\mathcal{A}^{i,\Sigma}) \quad (\text{by Equation (1)}) \\
 & = & \mathrm{Eq}_\Sigma(\mathcal{A}^\Sigma). \quad (\text{definition of } \mathcal{A}^\Sigma)
\end{array}$$

We now conclude that $\mathcal{A} \in \mathbb{C}\overset{\triangleleft}{\mathbb{I}\mathbb{I}}(\mathsf{K})$.

Suppose, next, that $\mathcal{A} \in \overset{\triangleleft}{\mathbb{I}\mathbb{I}}\mathbb{C}^*(\mathsf{K})$. Then, there exists a subdirect intersection

$$\langle H^i, \gamma^i \rangle : \mathcal{A} \to \mathcal{A}^i, \quad i \in I,$$

where, for all $i \in I$, $\mathcal{A}^i \in \mathbb{C}^*(\mathsf{K})$. Thus, for all $i \in I$ and all $\Sigma \in |\mathbf{Sign}^\flat|$, there exists $\{\mathcal{A}^{i,\Sigma,j} : j \in J^{i,\Sigma}\} \subseteq \mathsf{K}$, such that $\bigcup_{j \in J^{i,\Sigma}} \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{i,\Sigma,j})$ is directed and, moreover,

$$\mathrm{Eq}_\Sigma(\mathcal{A}^i) = \bigcup_{j \in J^{i,\Sigma}} \mathrm{Eq}_\Sigma(\mathcal{A}^{i,\Sigma,j}). \tag{2}$$

Now we consider, for all $\Sigma \in |\mathbf{Sign}^\flat|$, the following collection of $\mathbf{F}$-algebraic systems:

$$\{\mathcal{F}/\bigcap_{i\in I}\mathrm{Eq}(\mathcal{A}^{i,\Sigma,j_i}) : \langle j_i : i \in I\rangle \in \prod_{i\in I} J^{i,\Sigma}\}.$$

Then, since $\mathcal{A}^{i,\Sigma,j_i} \in \mathsf{K}$, for all $i \in I$ and all $j \in J^{i,\Sigma}$, we get, by Lemma 10, $\mathcal{F}/\bigcap_{i\in I}\mathrm{Eq}(\mathcal{A}^{i,\Sigma,j_i}) \in \overset{\triangleleft}{\mathbb{IP}}(\mathsf{K})$, for all $\langle j_i : i \in I\rangle \in \prod_{i\in I} J^{i,\Sigma}$. Moreover, we have

$$\begin{aligned}
&\textstyle\bigcup_{\langle j_i\rangle\in\prod J^{i,\Sigma}} \mathrm{Eq}_\Sigma(\mathcal{F}/\bigcap_{i\in I}\mathrm{Eq}(\mathcal{A}^{i,\Sigma,j_i}))\\
&\quad= \textstyle\bigcup_{\langle j_i\rangle\in\prod J^{i,\Sigma}} \bigcap_{i\in I}\mathrm{Eq}_\Sigma(\mathcal{A}^{i,\Sigma,j_i})\\
&\quad= \textstyle\bigcap_{i\in I}\bigcup_{j_i\in J^{i,\Sigma}}\mathrm{Eq}_\Sigma(\mathcal{A}^{i,\Sigma,j_i})\\
&\quad= \textstyle\bigcap_{i\in I}\mathrm{Eq}_\Sigma(\mathcal{A}^i)\\
&\quad= \mathrm{Eq}_\Sigma(\mathcal{A}).
\end{aligned}$$

So to conclude the proof, it suffices to show that, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\bigcup\{\bigcap_{i\in I}\mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{i,\Sigma,j_i}) : \langle j_i : i \in I\rangle \in \prod_{i\in I} J^{i,\Sigma}\}$$

is directed. This is not difficult to see, since, if $X \in \bigcap_{i\in I}\mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{i,\Sigma,j_i})$ and $X' \in \bigcap_{i\in I}\mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{i,\Sigma,j'_i})$, then, by the hypothesis that $\bigcup_{j\in J^{i,\Sigma}}\mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{i,\Sigma,j})$ is directed for all $i \in I$, we get that, there exists $k_i \in J^{i,\Sigma}$, such that $X, X' \subseteq Y \in \mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{i,\Sigma,k_i})$. Therefore, $X, X' \subseteq Y \in \bigcap_{i\in I}\mathrm{Eq}^\omega_\Sigma(\mathcal{A}^{i,\Sigma,k_i})$, where $\langle k_i : i \in I\rangle \in \prod_{i\in I} J^{i,\Sigma}$. ■

We now prove the main result of this section which expresses the closure operators $\mathbb{Q}^{\mathsf{Sem}}$ and $\mathbb{G}^{\mathsf{Sem}}$ in terms of the operators $\mathbb{C}$, $\mathbb{C}^*$ and $\overset{\triangleleft}{\mathbb{IP}}$.

Proposition 40 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system and K a class of $\mathbf{F}$-algebraic systems.*

(a) $\mathbb{Q}^{\mathsf{Sem}}(\mathsf{K}) = \mathbb{C}^*\overset{\triangleleft}{\mathbb{IP}}(\mathsf{K})$*;*

(b) $\mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) = \mathbb{C}\overset{\triangleleft}{\mathbb{IP}}(\mathsf{K})$*.*

Proof: We start with Part (a). Since $\mathsf{K} \subseteq \mathbb{Q}^{\mathsf{Sem}}(\mathsf{K})$ and, by Proposition 16, $\mathbb{G}^{\mathsf{Sem}}(\mathsf{K})$ is closed under $\mathbb{C}^*$ and $\overset{\triangleleft}{\mathbb{IP}}$, we get that $\mathbb{C}^*\overset{\triangleleft}{\mathbb{IP}}(\mathsf{K}) \subseteq \mathbb{Q}^{\mathsf{Sem}}(\mathsf{K})$.

For the converse, we can either appeal directly to the construction presented in the proof of Proposition 16 (which shows that $\mathbb{Q}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{C}^*(\overset{\triangleleft}{\mathbb{IP}}(\mathsf{K})))$ or, alternatively, notice that, by Lemma 38, $\mathsf{K} \subseteq \mathbb{C}^*\overset{\triangleleft}{\mathbb{IP}}(\mathsf{K})$, and observe that

$\mathbb{C}^*\overset{\triangleleft}{\mathbb{II}}(\mathsf{K})$ is obviously closed under $\mathbb{C}^*$ and, by Lemma 39, is also closed under $\overset{\triangleleft}{\mathbb{II}}$. Hence, it forms, by Proposition 16, a semantic quasivariety, and, therefore, by the minimality property underlying the definition of $\mathbb{Q}^{\mathsf{Sem}}$, $\mathbb{Q}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{C}^*\overset{\triangleleft}{\mathbb{II}}(\mathsf{K})$.

For Part (b), on the one hand, since $\mathsf{K} \subseteq \mathbb{G}^{\mathsf{Sem}}(\mathsf{K})$ and, by Proposition 15, $\mathbb{G}^{\mathsf{Sem}}(\mathsf{K})$ is closed under $\mathbb{C}$ and $\overset{\triangleleft}{\mathbb{II}}$, we get that $\mathbb{C}\overset{\triangleleft}{\mathbb{II}}(\mathsf{K}) \subseteq \mathbb{G}^{\mathsf{Sem}}(\mathsf{K})$.

For the converse, we can either appeal directly to the construction presented in the proof of Proposition 15 (which shows that $\mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{C}(\overset{\triangleleft}{\mathbb{II}}(\mathsf{K})))$ or, alternatively, notice that, by Lemma 38, $\mathsf{K} \subseteq \mathbb{C}\overset{\triangleleft}{\mathbb{II}}(\mathsf{K})$, and observe that $\mathbb{C}\overset{\triangleleft}{\mathbb{II}}(\mathsf{K})$ is obviously closed under $\mathbb{C}$ and, by Lemma 39, is also closed under $\overset{\triangleleft}{\mathbb{II}}$. Hence, it forms, by Proposition 15, a semantic guasivariety, and, therefore, by the minimality of $\mathbb{G}^{\mathsf{Sem}}$, $\mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{C}\overset{\triangleleft}{\mathbb{II}}(\mathsf{K})$. ■

Proposition 30 allows obtaining similar characterization of the syntactic quasi- and guasi-variety operators.

Proposition 41 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system.*

(a) A class K *of* $\mathbf{F}$*-algebraic systems is a syntactic quasivariety if and only if it is quasi-natural and* $\mathbb{Q}^{\mathsf{Sem}}(\mathsf{K}) = \mathbb{C}^*\overset{\triangleleft}{\mathbb{II}}(\mathsf{K})$*;*

(b) A class K *of* $\mathbf{F}$*-algebraic systems is a syntactic guasivariety if and only if it is guasi-natural and* $\mathbb{G}^{\mathsf{Sem}}(\mathsf{K}) = \mathbb{C}\overset{\triangleleft}{\mathbb{II}}(\mathsf{K})$*.*

Proof: Directly by combining Propositions 30 and 40. ■

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