Universal Bialgebra: Unifying Universal Algebra and Coalgebra

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Abstract

The concept of bialgebra is introduced as a platform to unify results in universal algebra and coalgebra under one theory. Examples of bialgebras also include universal multialgebras and partial algebras. This generalization clarifies the common points, in many cases reduces two almost identical proofs to one and has the potential of application in other settings as well. Several of the elementary results in universal algebra and coalgebra up to the isomorphism theorems are proven here for bialgebras and hints are given as to how they reduce to these two special cases. Since universal algebra and coalgebra have been extensively used in theoretical computer science as a specification tool for the most common data structures, this unified framework may also lead to useful new applications in this domain.

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1 Inroduction

Universal algebra came into being in the mid 1930's, when Garrett Birkhoff [3] introduced the concept of a *general algebra,* i.e., a set with an arbitrary collection of finitary operations on it, in an attempt to unify many similar results that had appeared in different branches of algebra and continue their study under one roof. At around the same time in the east (and later in the west) Alfred Tarski was laying the foundations of what is now known as first-order model theory of structures. His *structures* or *models* were sets with an arbitrary collection of finitary relations on them. Included in this framework as special instances are, together with universal algebras, *partial algebras* and *multialgebras,* which were also studied separately later (see, e.g., [8] and [10], Chapter 2, respectively). Also in the east, in the former Soviet Union, Anatoliĭ Mal'cev started a tradition very similar to the one founded by Birkhoff and Tarski in the west, by advocating the study of logical properties and advancing the application of techniques from logic in the domain of arbitrary algebras and structures. The field of universal algebra has enjoyed a tremendous development since and several books have appeared that are dedicated to laying the foundations and exploring elementary and more intricate aspects of universal algebras. Mentioned here are many of them [5, 6, 10, 13, 15, 17, 20]. There is quite a big intersection of the material that will be presented in this paper with every one of these but in the sequel most references will be made to [5] and [17] because these are the most recent among them and, probably, the most accessible with the most modern notation (although notation has not changed much in the field).

More recently, many of the key ideas in universal algebra have found a very fruitful application in theoretical computer science in the domain of *denotational* or *initial semantics* of data structures and programming languages. The free or initial universal algebra of a class is used to model the allowable operations or *constructors* on the structure and its initiality is then exploited to obtain a unique morphism into any other algebra in the class, which is perceived as a semantical interpretation of these operations. This process also allows for the formulation of an *induction principle* that provides the tool for the proof of many properties that are relevant and desirable for the specified structures (see, e.g., [21, 24, 26] for further details and examples).

Although a lot of interesting structures may be specified by the use of this universal algebraic framework, there are many that require a different setting. Their specification has to be given via their "observable" properties. Two structures should be considered identical under this approach not when they are the same but rather when their observable properties are the same. This approach leads to the *operational* or *final semantics* of data structures and languages. It turned out that the appropriate framework in which this kind of semantics may be studied is the framework of *coalgebras,* which are, in some sense, dual to algebras. Their general theory, as parallels the theory of universal algebra, has been developed and studied recently in [22, 23, 24, 11] and many other papers have also appeared in the literature on some more specialized aspects and results relating coalgebras to logic and universal algebra. In the operational semantics, the

cofree or *final coalgebra* is used to model the observational properties or *destructors* of a given structure. Its finality is then put into action to give a unique morphism from any other coalgebra into it. This morphism is also interpreted as appropriately describing properties of the specified structure. The process provides now a *coinduction principle* that is used as a proof technique on properties of the behaviour of the structure.

Since the techniques of category theory play a rather major role in the development of universal coalgebra, many categorical definitions and elementary results will be considered known. However, the reader should not be discouraged since, apart from the well-known sources [1, 2, 4, 14, 16], sufficient short and excellent introductions for our purposes are given in [23, 11]. Both expositions deal exclusively with the category **Set** of sets and functions and the same will be done here, although many of the results (in fact most of them, if adequately translated) will be easily seen to hold in arbitrary categories.

Roughly speaking, given a functor $F : \mathbf{Set} \to \mathbf{Set}$, an *$F$-algebra* consists of a set A together with a mapping $\alpha : F(A) \to A$. An *F-coalgebra,* on the other hand, consists of a set A together with a mapping $\alpha : A \to F(A)$. A *homomorphism* from an algebra $\langle A, \alpha \rangle$ to an algebra $\langle B, \beta \rangle$ is a mapping $f : A \to B$ that makes the following diagram commute

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow[f]{} & B \end{array}$$

Similarly, a homomorphism from a coalgebra $\langle A, \alpha \rangle$ to a coalgebra $\langle B, \beta \rangle$ is a mapping $g : A \to B$, such that the following rectangle commutes

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ F(A) & \xrightarrow[F(g)]{} & F(B) \end{array}$$

With those morphisms between them, the classes of F-algebras and F-coalgebras become categories. Although they are not dual to each other, the two categories have a lot in common. Many meaningful and interesting definitions may be formulated for both and a great deal of the elementary propositions in the field of universal coalgebra parallel their prototypes in the algebraic domain. It is only natural then to wonder whether there is a common underlying formalism that may be served as the framework in which both theories may be developed and more general results that specialize to the known ones in each domain proven. Such a framework is suggested in this paper. The notion of *bialgebra* is introduced to serve this purpose. Given two functors $F, G : \mathbf{Set} \to \mathbf{Set}$ an $\langle F, G \rangle$-bialgebra is a set A together with a mapping $\alpha : F(A) \to G(A)$ and a *bialgebra homomorphism* from a bialgebra $\langle A, \alpha \rangle$ to a bialgebra $\langle B, \beta \rangle$

is a mapping $f : A \to B$, such that the following diagram is commutative:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ G(A) & \xrightarrow[G(f)]{} & G(B) \end{array}$$

These basic notions together with many examples of special bialgebras for specific functors that have appeared in the literature will be presented in Section 2. In Section 3, the notion of *homomorphism* that parallels its homonyms from algebra and coalgebra is introduced. Also the notion of *bisimulation* is generalized from coalgebra to this context. Bisimulations correspond exactly to substitutive relations in universal algebra. Thus, bisimulations that are equivalences correspond to congruence relations. In Section 4, the existence of limits and colimits in the category of $\langle F, G\rangle$-coalgebras is investigated and its relation to the preservation of limits and colimits and weak limits and colimits by the two functors F and G is explored in some detail. More precisely, it is proven in Theorem 9 that the underlying set forgetful functor from bialgebras to sets creates and preserves all types of limits that are preserved by G and, in Theorem 13, that it creates and preserves all types of colimits that are preserved by F. Section 5 gives some properties of monos and epis in the category of coalgebras. Preservation of weak pushouts by F amounts to the coincidence of the notions of surjective and epi, whereas preservation of weak pullbacks by G amounts to the coincidence of the notions of injective and mono. Several properties of bisimulations are provided in Section 6 together with a description of the lattice of bisimulations $\mathbf{Bis}(\mathbf{A}, \mathbf{B})$ from a bialgebra $\mathbf{A}$ to a bialgebra $\mathbf{B}$. Section 7 deals with subbialgebras, which generalize the concepts of subalgebras and subcoalgebras. The lattice of subbialgebras of a given bialgebra is also described. Section 8 takes up the formulation of the classical isomorphism theorems from universal algebra and their counterparts in coalgebra and reformulates them in the bialgebraic context. Section 9 briefly mentions the obvious analogs of the concepts of a simple algebra or coalgebra, initial algebra and final coalgebra for the bialgebraic framework, but we rush to remark that they do not seem to give anything interesting in general in the current context. It is desirable that further research be done towards a useful generalization of these concepts that would allow the extraction of some principle that would possibly abstract and generalize induction and coinduction. Finally, in Section 10, it is shown how a pair of natural transformations $\mu : M \to F$ and $\nu : G \to N$, where M, N, F, G are all endofunctors on **Set**, gives a nicely behaved functor from the category of $\langle F, G\rangle$-bialgebras to the category of $\langle M, N\rangle$-bialgebras.

We would like to mention here the debt that this work has to the work of Rutten [23]. It is easy to see that the order of our presentation has been greatly influenced by his and, hopefully, the same has happened with the clarity and firmness of his exposition. Besides this work, his other expository papers [22, 24, 25] together with the excellent paper [11] of Gumm have also been of the utmost help in introducing the author, who comes from the universal algebraic side and is a novice in coalgebraic studies, to the main concepts of coalgebra and the underlying motivation.

Further, it is hoped that bialgebras will shed some light on the similarities and the differences between algebras and coalgebras and that they will provide the framework needed by other

applications for which neither universal algebra nor universal coalgebra are suitable. Many results, then, might be available from this exposition. Finally, there are many more problems that are arising from this work but have not been addressed here. For example, we do not know if and how the concepts of initial algebra and final coalgebra may be unified under a common bialgebraic concept that has some significance and usefulness, similar to the merit each enjoys in its respective domain. The same holds for the induction and coinduction principles. Further work is definitely needed towards this direction but better understanding of these relationships is bound to accompany the development of universal coalgebra along the lines of universal algebra.

2 Basic Notions and Examples

Let $F, G : \mathbf{Set} \to \mathbf{Set}$ be two endomorphisms on the category **Set** of small sets. An $\langle F, G\rangle$-**bialgebra** $\mathbf{A} = \langle A, \alpha\rangle$ is a pair consisting of a set A together with a mapping $\alpha : F(A) \to G(A)$.

$$\begin{array}{c} F(A) \\ \Big\downarrow{\scriptstyle \alpha} \\ G(A) \end{array}$$

Let $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ be two $\langle F, G\rangle$-bialgebras. An $\langle F, G\rangle$-**bialgebra homomorphism** $h : \mathbf{A} \to \mathbf{B}$ is a mapping $h : A \to B$, such that the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(h)} & F(B) \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \beta} \\ G(A) & \xrightarrow[G(h)]{} & G(B) \end{array}$$

Identity morphisms are $\langle F, G\rangle$-bialgebra homomorphisms and, given two bialgebra homomorphisms $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{B} \to \mathbf{C}$, the composition $g \circ f : A \to C$ is also a bialgebra homomorphism $h : \mathbf{A} \to \mathbf{C}$.

$$\begin{array}{ccccc} F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{F(g)} & F(C) \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \gamma} \\ G(A) & \xrightarrow[G(f)]{} & G(B) & \xrightarrow[G(g)]{} & G(C) \end{array}$$

Thus, $\langle F, G\rangle$-bialgebras together with $\langle F, G\rangle$-bialgebra homomorphisms between them form a category, called the **category of** $\langle F, G\rangle$-**bialgebras** and denoted by $\mathbf{Set}^F_G$. To simplify notation for special cases, define $\mathbf{Set}^F \stackrel{\text{def}}{=} \mathbf{Set}^F_{I_{\mathbf{Set}}}$ and $\mathbf{Set}_G \stackrel{\text{def}}{=} \mathbf{Set}^{I_{\mathbf{Set}}}_G$, where by $I_{\mathbf{Set}} : \mathbf{Set} \to \mathbf{Set}$ is denoted the identity functor on **Set**.

Next, some of the main examples that motivate the introduction of bialgebras are summarised, together with some pointers to the literature where the interested reader will find more detailed accounts and many more examples.

1 (a) (See also [21], Section 2, [22], Section 11, [23] Section 13, and [24], Section 5.) If $G = I_{\mathbf{Set}}$, then the category $\mathbf{Set}^F$ is the category of *F-algebras* and *F-homomorphisms.* Its objects are pairs $\mathbf{A} = \langle A, \alpha \rangle$, where A is a set and $\alpha : F(A) \to A$ is a mapping.

$$\begin{array}{c} F(A) \\ \alpha \big\downarrow \\ A \end{array}$$

Given two F-algebras $\mathbf{A} = \langle A, \alpha \rangle$ and $\mathbf{B} = \langle B, \beta \rangle$, an F-algebra homomorphism $h : \mathbf{A} \to \mathbf{B}$ is a mapping $h : A \to B$, such that the following diagram commutes

$$\begin{array}{ccc} F(A) & \xrightarrow{F(h)} & F(B) \\ \alpha \big\downarrow & & \big\downarrow \beta \\ A & \xrightarrow[h]{} & B \end{array}$$

(b) (See also [14], Chapter VI, [16], Chapter 1, [2], Section 14.3, and [1], Section 5.4.) Let $\mathbf{T} = \langle T, \eta, \mu \rangle$ be an algebraic theory in monoid form. I.e., $T : \mathbf{Set} \to \mathbf{Set}$ is a functor and $\eta : I_{\mathbf{Set}} \to T$ and $\mu : TT \to T$ are natural transformations, such that the following diagrams commute, for every set X,

$$\begin{array}{ccccc} T(X) & \xrightarrow{\eta_{T(X)}} & T(T(X)) & \xleftarrow{T(\eta_X)} & T(X) \\ & {\scriptstyle i_{T(X)}} \searrow & \big\downarrow {\scriptstyle \mu_X} & \swarrow {\scriptstyle i_{T(X)}} & \\ & & T(X) & & \end{array} \qquad \begin{array}{ccc} T(T(T(X))) & \xrightarrow{T(\mu_X)} & T(T(X)) \\ \big\downarrow {\scriptstyle \mu_{T(X)}} & & \big\downarrow {\scriptstyle \mu_X} \\ T(T(X)) & \xrightarrow[\mu_X]{} & T(X) \end{array}$$

A $\mathbf{T}$-*algebra* is a pair $\mathbf{A} = \langle A, \alpha \rangle$, where A is a set and $\alpha : T(A) \to A$ is a mapping that makes the following diagrams commute

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & T(A) \\ & {\scriptstyle i_A} \searrow & \big\downarrow {\scriptstyle \alpha} \\ & & A \end{array} \qquad \begin{array}{ccc} T(T(A)) & \xrightarrow{T(\alpha)} & T(A) \\ \big\downarrow {\scriptstyle \mu_A} & & \big\downarrow {\scriptstyle \alpha} \\ T(A) & \xrightarrow[\alpha]{} & A \end{array}$$

Given two $\mathbf{T}$-algebras $\mathbf{A} = \langle A, \alpha \rangle$ and $\mathbf{B} = \langle B, \beta \rangle$, a $\mathbf{T}$-*algebra homomorphism* $h : \mathbf{A} \to \mathbf{B}$ is a mapping $h : A \to B$, such that the following rectangle commutes

$$\begin{array}{ccc} T(A) & \xrightarrow{T(h)} & T(B) \\ \alpha \big\downarrow & & \big\downarrow \beta \\ A & \xrightarrow[h]{} & B \end{array}$$

T-algebras together with **T**-algebra homomorphisms between them form a category, called the *Eilenberg-Moore category* of the algebraic theory **T** in **Set** and denoted by $\mathbf{Set}^{\mathbf{T}}$. This is a subcategory of the category $\mathbf{Set}^{T}$ of $\langle T, I_{\mathbf{Set}}\rangle$-bialgebras.

(c) It is well-known ([16], Section 1.4) that all varieties of universal algebras viewed as categories are Eilenberg-Moore categories: Let $\mathcal{L}$ be an algebraic signature and $\mathcal{V}$ a variety of $\mathcal{L}$-algebras. By $\vec{\mathcal{V}}$ will be denoted the category corresponding to $\mathcal{V}$, i.e., it is the category with objects all algebras in $\mathcal{V}$ and with morphisms all $\mathcal{L}$-algebra homomorphisms between them. Then, there exists an algebraic theory $\mathbf{T}_{\mathcal{V}} = \langle T_{\mathcal{V}}, \eta_{\mathcal{V}}, \mu_{\mathcal{V}}\rangle$ in monoid form in **Set** such that $\mathbf{Set}^{\mathbf{T}_{\mathcal{V}}} = \vec{\mathcal{V}}$. Roughly speaking, the functor $T_{\mathcal{V}} : \mathbf{Set} \to \mathbf{Set}$ sends a set X to the carrier $F_{\mathcal{V}}(X)$ of the free algebra $\mathbf{F}_{\mathcal{V}}(X)$ in $\mathcal{V}$ generated by X, $\eta_{\mathcal{V}_X} : X \to F_{\mathcal{V}}(X)$ is the insertion-of-variables map and $\mu_{\mathcal{V}_X} : F_{\mathcal{V}}(F_{\mathcal{V}}(X)) \to F_{\mathcal{V}}(X)$ is the mapping that combines terms over terms over X, by "performing" the free algebra operations, to simple terms over X.

(d) (See also [11], Section 2.3, [23], Section 2, and [24], Section 5.) An alternative way to view the class of all $\mathcal{L}$-algebras as the class $\mathbf{Set}^{F_{\mathcal{L}}}$ for some functor $F_{\mathcal{L}} : \mathbf{Set} \to \mathbf{Set}$ is as follows:
Let $\mathcal{L} = \langle \Lambda, r\rangle$ be a *type of algebras,* i.e., Λ a set of *operation symbols* and $r : \Lambda \to \omega$ a *rank function* on Λ. Define the functor $F_{\mathcal{L}} : \mathbf{Set} \to \mathbf{Set}$ by

$$F_{\mathcal{L}}(A) = \sum_{\lambda\in\Lambda} A^{r(\lambda)}, \quad \text{for every set} \quad A,$$

and, given $f : A \to B$ in **Set**, let $F_{\mathcal{L}}(f) : \sum_{\lambda\in\Lambda} A^{r(\lambda)} \to \sum_{\lambda\in\Lambda} B^{r(\lambda)}$ be defined by

$$F_{\mathcal{L}}(x) = f^{r(\lambda)}(x), \quad \text{if} \quad x \in A^{r(\lambda)}, \quad \text{for all} \quad x \in \sum_{\lambda\in\Lambda} A^{r(\lambda)}.$$

For this functor $F_{\mathcal{L}}$, the category $\mathbf{Set}^{F_{\mathcal{L}}}$ consists of all $\mathcal{L}$-algebras with $\mathcal{L}$-algebra homomorphisms between them.

2 (a) (See, e.g., [11] and [23].) Let $F = I_{\mathbf{Set}}$. Then the category $\mathbf{Set}_G$ is the category of *G-coalgebras* and *G-coalgebra homomorphisms.* Its objects are pairs $\mathbf{A} = \langle A, \alpha\rangle$ where A is a set and $\alpha : A \to G(A)$ is a mapping.

$$\begin{array}{c} A \\ \alpha \downarrow \\ G(A) \end{array}$$

Given two G-coalgebras $\mathbf{A} = \langle A, \alpha\rangle$ and $\mathbf{B} = \langle B, \beta\rangle$, a G-coalgebra homomorphism $h : \mathbf{A} \to \mathbf{B}$ is a mapping $h : A \to B$ such that the following rectangle commutes

$$\begin{array}{ccc} A & \xrightarrow{\ h\ } & B \\ \alpha \downarrow & & \downarrow \beta \\ G(A) & \xrightarrow[G(h)]{} & G(B) \end{array}$$

Coalgebras have been used in computer science as a formalism in which, among other things, a theory of transition systems may be developed [22] and well-known data structures may be precisely specified via final coalgebra constructions ([21], Section 4, [22], [23], Section 11, [26] and [11], Section 2).

(b) (See, e.g., [22].) The special case of 2(a), where $G = \mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ is the powerset functor, gives the category $\mathbf{Set}_{\mathcal{P}}$ of (nondeterministic, unlabeled) transition systems. A *transition system* $\mathbf{A} = \langle A, \alpha \rangle$ is a pair consisting of a set A of *states* and a *transition function* $\alpha : A \to \mathcal{P}(A)$ which gives, for each state $a \in A$, the set $\alpha(a)$ of possible states which $\mathbf{A}$ may enter in a single transition step from state a. For instance, the transition system given by $A = \{0, 1, 2, 3\}$ and

a	0	1	2	3
$\alpha(a)$	$\{0, 1, 2\}$	$\{1, 2\}$	$\{1, 3\}$	$\emptyset$

may go from state 0 in a single transition step to any of the states $0, 1, 2$, from state 2 to any of the states 1 or 3 and cannot make any legal transition if found in state 3.

3 (a) (See also [20], Section 1.1, [13], Section 4.5, [15], Sections I 2.2 and III 6.1, [10], Chapter 2 and [6], Section II 2.) If $F = F_{\mathcal{L}} : \mathbf{Set} \to \mathbf{Set}$ and $G = I_{\mathbf{Set}} + 1$, where $1 = \{\emptyset\}$ is the terminal object in $\mathbf{Set}$, then the category $\mathbf{Set}^{F_{\mathcal{L}}}_{I_{\mathbf{Set}}+1}$ is the category of *partial $\mathcal{L}$-algebras* with *partial homomorphisms* between them. A partial algebra is a pair $\mathbf{A} = \langle A, \alpha \rangle$, where A is a set and $\alpha : \sum_{\lambda \in \Lambda} A^{r(\lambda)} \to A + 1$ a map that determines the partial fundamental operations of $\mathbf{A}$. Intuitively, $\mathbf{A}$ is a partial algebra, such that the fundamental partial operation $\lambda^{\mathbf{A}} : A^{r(\lambda)} \to A$ is defined on the $r(\lambda)$-tuple $\langle a_0, \ldots, a_{r(\lambda)-1} \rangle \in A^{r(\lambda)}$ if and only if $\alpha(a_0, \ldots, a_{r(\lambda)-1}) \in A$ and is undefined if and only if $\alpha(a_0, \ldots, a_{r(\lambda)-1}) = \emptyset$. Given two partial algebras $\mathbf{A} = \langle A, \alpha \rangle, \mathbf{B} = \langle B, \beta \rangle$, a homomorphism $h : \mathbf{A} \to \mathbf{B}$ is a function $h : A \to B$, such that, if $\lambda^{\mathbf{A}}(a_0, \ldots, a_{r(\lambda)-1})$ is defined, then $\lambda^{\mathbf{B}}(h(a_0), \ldots, h(a_{r(\lambda)-1}))$ is also defined and

$$h(\lambda^{\mathbf{A}}(a_0, \ldots, a_{r(\lambda)-1})) = \lambda^{\mathbf{B}}(h(a_0), \ldots, h(a_{r(\lambda)-1})).$$

(b) If $F = F_{\mathcal{L}}$ and $G = \mathcal{P}$, then $\mathbf{Set}^{F_{\mathcal{L}}}_{\mathcal{P}}$ is the category of *partial $\mathcal{L}$-multialgebras* with *partial multialgebra homomorphisms* between them. Here a fundamental operation $\lambda^{\mathbf{A}}$ of a partial multialgebra $\mathbf{A} = \langle A, \alpha \rangle$ maps a $r(\lambda)$-tuple $\vec{a} \in A^{r(\lambda)}$ to a set of values in A if $\alpha(\vec{a}) \neq \emptyset$, and it is undefined if $\alpha(\vec{a}) = \emptyset$.

(c) (See also [8].) If (b) is altered so that $G = \mathcal{P}^*$, where $\mathcal{P}^*$ sends a set to the set of all its nonempty subsets, one obtains the category $\mathbf{Set}^{F_{\mathcal{L}}}_{\mathcal{P}^*}$ of *$\mathcal{L}$-multialgebras* with *multialgebra homomorphisms* between them. We note that this definition has not been universally used. Sometimes, instead of the commutativity of the diagram

$$\begin{array}{ccc} F_{\mathcal{L}}(A) & \xrightarrow{F_{\mathcal{L}}(h)} & F_{\mathcal{L}}(B) \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \beta} \\ \mathcal{P}^*(A) & \xrightarrow[\mathcal{P}^*(h)]{} & \mathcal{P}^*(B) \end{array}$$

only the relaxed condition that, for all $\lambda \in \Lambda, \vec{a} \in A^{r(\lambda)}$,

$$\mathcal{P}^*(h)(\alpha(\vec{a})) \subseteq \beta(F_{\mathcal{L}}(h)(\vec{a}))$$

is required from a multialgebra homomorphism [7]. But it has been used before in the literature, e.g., in [9], Section 6.

3 Bialgebra Homomorphisms and Bisimulations

From now on, whenever the functors F and G are fixed, the term **bialgebra** will be used in place of the more cumbersome $\langle F, G\rangle$-bialgebra.

Let $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ be two bialgebras. A bialgebra homomorphism $h : \mathbf{A} \to \mathbf{B}$ is said to be an **isomorphism** if h has an inverse $h^{-1} : B \to A$, such that $h^{-1} : \mathbf{B} \to \mathbf{A}$ is also a bialgebra homomorphism. h is said to be a **monomorphism** if it is a mono in the category $\mathbf{Set}_G^F$, i.e., if, for every bialgebra $\mathbf{C} = \langle C, \gamma\rangle$ and all bialgebra homomorphisms $f, g : \mathbf{C} \to \mathbf{A}$,

$$h \circ f = h \circ g \quad \text{implies} \quad f = g.$$

In terms of commutative diagrams we have

$$\mathbf{C} \underset{g}{\overset{f}{\rightrightarrows}} \mathbf{A} \xrightarrow{\ h\ } \mathbf{B} \qquad \text{implies} \qquad \mathbf{A} \underset{g}{\overset{f}{\rightrightarrows}} \mathbf{B}$$

$h : \mathbf{A} \to \mathbf{B}$ is called **injective** if $h : A \rightarrowtail B$ is injective in **Set**. If h is injective, then it is a monomorphism. $h : \mathbf{A} \to \mathbf{B}$ is said to be F-**injective,** G-**injective, biinjective,** if $F(h), G(h)$, both, respectively, are injective in **Set**.

A bialgebra homomorphism $h : \mathbf{A} \to \mathbf{B}$ is said to be an **epimorphism** if it is an epi in the category $\mathbf{Set}_G^F$ of $\langle F, G\rangle$-bialgebras, i.e., if, for every bialgebra $\mathbf{C} = \langle C, \gamma\rangle$ and for all bialgebra homomorphisms $f, g : \mathbf{B} \to \mathbf{C}$,

$$f \circ h = g \circ h \quad \text{implies} \quad f = g.$$

In diagrammatic form

$$\mathbf{A} \xrightarrow{\ h\ } \mathbf{B} \underset{g}{\overset{f}{\rightrightarrows}} \mathbf{C} \qquad \text{implies} \qquad \mathbf{B} \underset{g}{\overset{f}{\rightrightarrows}} \mathbf{C}$$

$h : \mathbf{A} \to \mathbf{B}$ is called **surjective** if $h : A \twoheadrightarrow B$ is surjective in **Set**. If h is surjective, then it is certainly an epimorphism. h is called F-**surjective**, G-**surjective**, **bisurjective,** if $F(h), G(h)$, both, respectively, are surjective.

Recall, from basic category theory, that a functor F : **Set** $\to$ **Set** is said to **preserve injectives**, respectively **surjectives**, if, whenever $h : A \to B$ is an injective, respectively surjective, then $F(h) : F(A) \to F(B)$ is also an injective, respectively surjective. In particular, if $h : A \rightarrowtail B$ is injective and $A \neq \emptyset$. then there exists $h' : B \to A$, such that $h'h = i_A$. But then $F(h')F(h) = i_{F(A)}$ and $F(h)$ is injective. Thus every endofunctor on **Set** preserves injectives with nonempty domains. Similarly, every endofunctor on **Set** preserves surjectives. Note that if h is injective, respectively surjective, and F (G, both) preserves injectives, respectively surjectives, then h is F- (G-, bi-)injective, respectively surjective.

1 Suppose $\mathcal{L}$ is an algebraic signature and $\mathcal{V}$ a variety of $\mathcal{L}$-algebras. Then, the functor $T_{\mathcal{V}} : \mathbf{Set} \to \mathbf{Set}$ preserves injectives and surjectives. Thus, every injective, respectively surjective, $\mathbf{T}_{\mathcal{V}}$-algebra homomorphism is biinjective, respectively bisurjective.

2 All most commonly used coalgebra functors, e.g., polynomial functors, the finite powerset functor and the powerset functor, preserve injectives and surjectives. Thus, every injective, respectively surjective, G-coalgebra homomorphism for one of these functors G is biinjective, respectively bisurjective.

A bialgebra $\mathbf{A} = \langle A, \alpha \rangle$ is a **subbialgebra** of a bialgebra $\mathbf{B} = \langle B, \beta \rangle$ if $A \subseteq B$ and the inclusion map $i : A \hookrightarrow B$ is a homomorphism $i : \mathbf{A} \to \mathbf{B}$. A bialgebra $\mathbf{A}$ is a **homomorphic image** of a bialgebra $\mathbf{B}$ if there exists a surjective homomorphism $h : \mathbf{B} \twoheadrightarrow \mathbf{A}$.

1 $\langle F_{\mathcal{L}}, I_{\mathbf{Set}} \rangle$-subbialgebras in $\mathbf{Set}^{F_{\mathcal{L}}}$ are exactly the subalgebras in the universal algebraic sense.

2 $\langle I_{\mathbf{Set}}, G \rangle$-subbialgebras are exactly the G-subcoalgebras in the usual coalgebraic sense.

Analogous statements hold for homomorphic images of $\langle F_{\mathcal{L}}, I_{\mathbf{Set}} \rangle$-bialgebras and $\langle I_{\mathbf{Set}}, G \rangle$-bialgebras.

Proposition 1 *Every bijective bialgebra homomorphism is an isomorphism.*

Proof:

Suppose $h : \mathbf{A} \to \mathbf{B}$ is a bialgebra homomorphism, such that $h : A \to B$ has an inverse map $h^{-1} : B \to A$. It suffices to show that $h^{-1} : \mathbf{B} \to \mathbf{A}$ is a bialgebra homomorphism.

$$\begin{array}{ccccc} F(B) & \xrightarrow{F(h^{-1})} & F(A) & \xrightarrow{F(h)} & F(B) \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ G(B) & \xrightarrow[G(h^{-1})]{} & G(A) & \xrightarrow[G(h)]{} & G(B) \end{array}$$

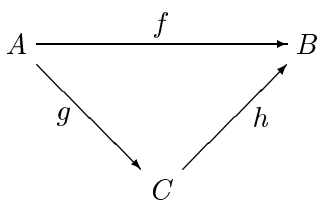
We have

$$\begin{aligned} \alpha F(h^{-1}) &= G(h^{-1})G(h)\alpha F(h^{-1}) \\ &= G(h^{-1})\beta F(h)F(h^{-1}) \\ &= G(h^{-1})\beta. \end{aligned}$$

■

As a corollary of Proposition 1 the corresponding universal algebraic result, Lemma 3.6 of [6], and the corresponding coalgebraic result, Proposition 2.3 of [23], may be obtained.

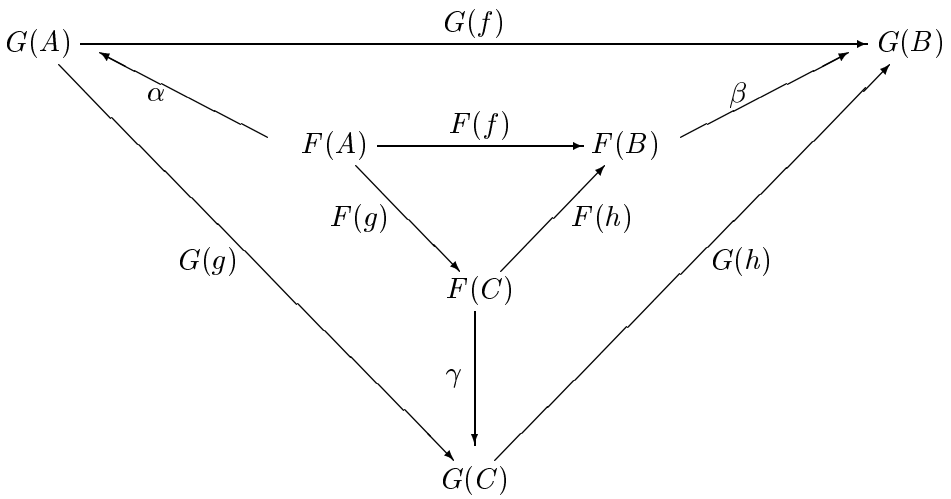
Proposition 2 *Let* $\mathbf{A} = \langle A, \alpha \rangle, \mathbf{B} = \langle B, \beta \rangle$ *and* $\mathbf{C} = \langle C, \gamma \rangle$ *be bialgebras and* $f : A \to B, g : A \to C$ *and* $h : C \to B$ *mappings such that* $f = h \circ g$.



1 If g is F-surjective and f, g are homomorphisms, then h is a homomorphism.

2 If h is G-injective and f, h are homomorphisms, then g is a homomorphism.

Proof:



1 Let $c \in F(C)$. Then, since g is F-surjective, there exists $a \in F(A)$, such that $c = F(g)(a)$. Thus

$$\begin{aligned} G(h)\gamma(c) &= G(h)\gamma F(g)(a) \\ &= G(h)G(g)\alpha(a) \\ &= G(hg)\alpha(a) \\ &= G(f)\alpha(a) \\ &= \beta F(f)(a) \\ &= \beta F(h)F(g)(a) \\ &= \beta F(h)(c). \end{aligned}$$

2

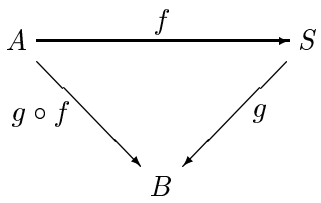
$$\begin{aligned} G(h)\gamma F(g) &= \beta F(h)F(g) \\ &= \beta F(hg) \\ &= \beta F(f) \\ &= G(f)\alpha \\ &= G(hg)\alpha \\ &= G(h)G(g)\alpha. \end{aligned}$$

Thus, since h is G-injective, i.e., $G(h)$ is injective and, therefore, mono, $\gamma F(g) = G(g)\alpha$.

■

Corresponding results from universal algebra and coalgebra may again be immediately obtained as corollaries. Next, the image and preimage constructions of [11], Section 3, and some related results are proven for bialgebras.

Lemma 3 *Let $\mathbf{A} = \langle A, \alpha \rangle$ be a bialgebra, S a set and $f : A \twoheadrightarrow S$ a surjective function. Then, there exists a bialgebra structure σ on S, such that, for every bialgebra $\mathbf{B} = \langle B, \beta \rangle$ and all $g : S \to B$, if $g \circ f$ is a homomorphism, so is g.*



Proof:

Since $f : A \twoheadrightarrow S$ is surjective, there exists $f' : S \to A$, such that $f \circ f' = i_S$. Define $\sigma : F(S) \to G(S)$ by

$$F(S) \xrightarrow{F(f')} F(A) \xrightarrow{\alpha} G(A) \xrightarrow{G(f)} G(s)$$

$$\sigma = G(f)\alpha F(f').$$

Now we have

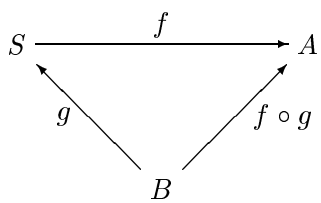
$$\begin{array}{ccccc} F(A) & \xrightarrow{F(f)} & F(S) & \xrightarrow{F(g)} & F(B) \\ & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \beta} \\ & & G(S) & \xrightarrow[G(g)]{} & G(B) \end{array}$$

$$\begin{aligned} G(g)\sigma F(f) &= G(g)G(f)\alpha F(f')F(f) \\ &= G(gf)\alpha F(f')F(f) \\ &= \beta F(gf)F(f')F(f) \\ &= \beta F(g)F(f)F(f')F(f) \\ &= \beta F(g)F(f) \end{aligned}$$

But f is surjective, whence $F(f)$ is surjective and, therefore, $G(g)\sigma = \beta F(g)$ and g is a homomorphism. ■

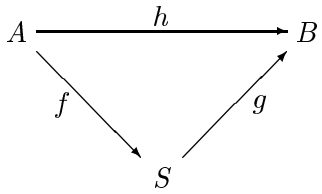
Similarly, we may show the following

Lemma 4 *Let $\mathbf{A} = \langle A, \alpha \rangle$ be a bialgebra, $S \neq \emptyset$ a set and $f : S \rightarrowtail A$ an injective mapping. Then, there is a bialgebra structure σ on S, such that, for all bialgebras $\mathbf{B} = \langle B, \beta \rangle$ and all mappings $g : B \to S$, if $f \circ g$ is a homomorphism, the so is g.*



If the functor F preserves the empty set, then the assumption that $S \neq \emptyset$ in Lemma 4 may be dropped.

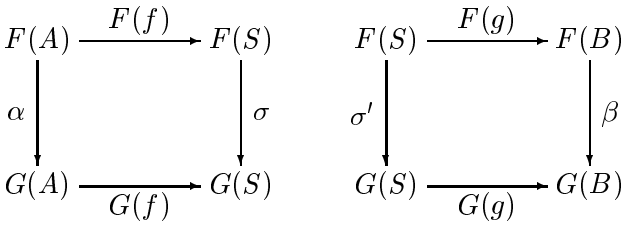
Lemma 5 *Let* $\mathbf{A} = \langle A, \alpha \rangle, \mathbf{B} = \langle B, \beta \rangle$ *be bialgebras, with* $A \neq \emptyset$, *and* $h : \mathbf{A} \to \mathbf{B}$ *a homomorphism. Suppose that* $h = g \circ f$ *is an epi-mono factorization of* h *in* **Set**, *i.e.,* $f : A \twoheadrightarrow S$ *is an epimorphism,* $g : S \rightarrowtail B$ *is a monomorphism and the following commutes.*



Then, there is a unique bialgebra structure σ *on* S, *such that* f *and* g *become homomorphisms.*

Proof:

Clearly, $S \neq \emptyset$. Thus, by Lemmas 3 and 4, there exist bialgebra structures $\sigma, \sigma' : F(S) \to G(S)$, such that the following rectangles commute

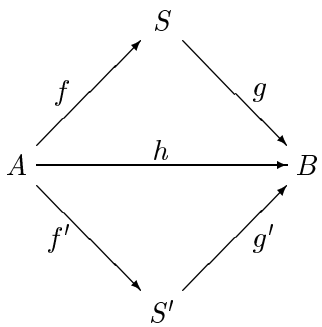


Now it suffices to show that $\sigma = \sigma'$. We have

$$\begin{aligned} G(g)\sigma F(f) &= G(g)G(f)\alpha \\ &= G(gf)\alpha \\ &= \beta F(gf) \\ &= \beta F(g)F(f) \\ &= G(g)\sigma' F(f). \end{aligned}$$

But f is surjective, whence $F(f)$ is also surjective and g is injective, whence $G(g)$ is also injective and, therefore, $\sigma = \sigma'$. ■

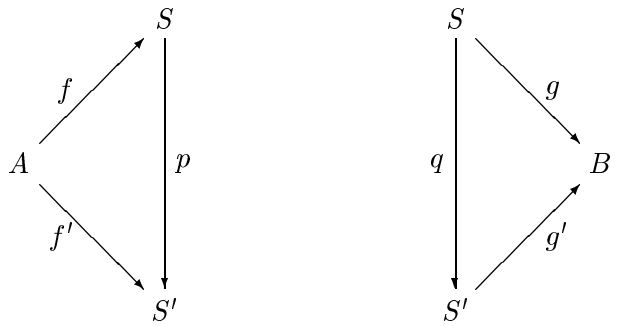
Lemma 6 *Let* $\mathbf{A} = \langle A, \alpha \rangle, \mathbf{B} = \langle B, \beta \rangle$ *be bialgebras, with* $A \neq \emptyset$, *and* $h : \mathbf{A} \to \mathbf{B}$ *a homomorphism. Suppose that* $h = g \circ f = g' \circ f'$ *are epi-mono factorizations of* h *in* **Set**.



Then, the coalgebras $\mathbf{S} = \langle S, \sigma \rangle$ *and* $\mathbf{S}' = \langle S', \sigma' \rangle$ *defined by Lemma 5 are isomorphic.*

Proof:

It is easy to see that the kernel of f must be contained in the kernel of f'. But then, there exists in **Set** a unique (since f is surjective) mapping $p : S \to S'$, such that $pf = f'$. Now, by Proposition 2, we get that $p : \mathbf{S} \to \mathbf{S}'$ is a homomorphism, which is surjective since f' is surjective.



Similarly, it is easy to check that $g(S) \subseteq g'(S')$, whence there exists in **Set** a unique (since g' is injective) mapping $q : S \to S'$, such that $g'q = g$. Now, again by Proposition 2, we get that $q : \mathbf{S} \to \mathbf{S}'$ is a homomorphism and it is injective since g is. Now it suffices to show that $p = q$. We have

$$\begin{aligned} g'pf &= g'f' \\ &= gf \\ &= g'qf. \end{aligned}$$

But f is an epimorphism and g' a monomorphism whence $p = q$. ■

Lemmas 5 and 6 have the following easy corollary

Theorem 7 *Let* $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ *be* $\langle F, G\rangle$*-bialgebras, with* $A \neq \emptyset$*, and* $h : \mathbf{A} \to \mathbf{B}$ *a bialgebra homomorphism.* h *has a unique epi-mono factorization in* $\mathbf{Set}_G^F$ *as* $\mathbf{A} \twoheadrightarrow h(\mathbf{A}) \hookrightarrow \mathbf{B}$.

Let $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ be two $\langle F, G\rangle$-bialgebras. A subset $R \subseteq A \times B$ is called an $\langle F, G\rangle$-**bisimulation** between $\mathbf{A}$ and $\mathbf{B}$ if there exists an $\langle F, G\rangle$-bialgebra structure $\rho : F(R) \to G(R)$ on R that makes the following diagram commute

$$\begin{array}{ccccc} F(A) & \xleftarrow{F(\pi_1)} & F(R) & \xrightarrow{F(\pi_2)} & F(B) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \beta} \\ G(A) & \xleftarrow[G(\pi_1)]{} & G(R) & \xrightarrow[G(\pi_2)]{} & G(B) \end{array}$$

where $\pi_1 : R \to A$ and $\pi_2 : R \to B$ are the coordinate-wise projections. I.e., $R \subseteq A \times B$ is a bisimulation between $\mathbf{A}$ and $\mathbf{B}$ if there exists an $\langle F, G\rangle$-bialgebra structure ρ on R, such that the two projections become $\langle F, G\rangle$-bialgebra homomorphisms. Again, if $G = I_{\mathbf{Set}}$ and no confusion is possible, the term F-**bisimulation** is used in place of $\langle F, I_{\mathbf{Set}}\rangle$-bisimulation. Similarly, if $F = I_{\mathbf{Set}}$, the term G-**bisimulation** is used in place of $\langle I_{\mathbf{Set}}, G\rangle$-bisimulation. This terminology

introduces ambiguity which will hopefully be dispersed by the context in which the terms will appear. An $\langle F, G\rangle$-**bisimulation on A** is an $\langle F, G\rangle$-bisimulation from **A** to itself. An $\langle F, G\rangle$-**bisimulation equivalence** on **A** is a bisimulation on **A** that is also an equivalence relation on A.

1 (See also [21], Section 2.) The example of varieties of universal algebras is revisited. A subset $R \subseteq A \times B$ of the cartesian product of the carriers of two $\mathcal{L}$-algebras $\mathbf{A}, \mathbf{B}$ is an $F_{\mathcal{L}}$-bisimulation in $\mathbf{Set}^{F_{\mathcal{L}}}$ if and only if R is a substitutive relation from **A** to **B**, i.e., for all $\lambda \in \Lambda$ and $a_0, \ldots, a_{r(\lambda)-1} \in A, b_0, \ldots, b_{r(\lambda)-1} \in B$,

$$\langle a_0, b_0\rangle, \ldots, \langle a_{r(\lambda)-1}, b_{r(\lambda)-1}\rangle \in R \text{ implies } \langle \lambda^{\mathbf{A}}(a_0, \ldots, a_{r(\lambda)-1}), \lambda^{\mathbf{B}}(b_0, \ldots, b_{r(\lambda)-1})\rangle \in R.$$

To see this, suppose, first, that $R \subseteq A \times B$ is an $F_{\mathcal{L}}$-bisimulation in $\mathbf{Set}^{F_{\mathcal{L}}}$, i.e., there exists a $F_{\mathcal{L}}$-algebra structure $\rho : F_{\mathcal{L}}(R) \to R$, such that the following diagram commutes

$$\begin{array}{ccccc} F_{\mathcal{L}}(A) & \xleftarrow{F_{\mathcal{L}}(\pi_1)} & F_{\mathcal{L}}(R) & \xrightarrow{F_{\mathcal{L}}(\pi_2)} & F_{\mathcal{L}}(B) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \beta} \\ A & \xleftarrow[\pi_1]{} & R & \xrightarrow[\pi_2]{} & B \end{array}$$

Now let $\lambda \in \Lambda, \langle a_0, b_0\rangle, \ldots, \langle a_{r(\lambda)-1}, b_{r(\lambda)-1}\rangle \in R$. Then by commutativity of the left rectangle above,

$$\pi_1(\rho(\lambda(\langle a_0, b_0\rangle, \ldots, \langle a_{r(\lambda)-1}, b_{r(\lambda)-1}\rangle))) = \alpha(F_{\mathcal{L}}(\pi_1)(\lambda(\langle a_0, b_0\rangle, \ldots, \langle a_{r(\lambda)-1}, b_{r(\lambda)-1}\rangle)))$$

or, equivalently,

$$\pi_1(\rho(\lambda(\langle a_0, b_0\rangle, \ldots, \langle a_{r(\lambda)-1}, b_{r(\lambda)-1}\rangle))) = \lambda^{\mathbf{A}}(a_0, \ldots, a_{r(\lambda)-1}).$$

Similarly, by commutativity of the second rectangle,

$$\pi_2(\rho(\lambda(\langle a_0, b_0\rangle, \ldots, \langle a_{r(\lambda)-1}, b_{r(\lambda)-1}\rangle))) = \lambda^{\mathbf{B}}(b_0, \ldots, b_{r(\lambda)-1}).$$

Thus,

$$\rho(\lambda(\langle a_0, b_0\rangle, \ldots, \langle a_{r(\lambda)-1}, b_{r(\lambda)-1}\rangle)) = \langle \lambda^{\mathbf{A}}(a_0, \ldots, a_{r(\lambda)-1}), \lambda^{\mathbf{B}}(b_0, \ldots, b_{r(\lambda)-1})\rangle.$$

Therefore $\langle \lambda^{\mathbf{A}}(a_0, \ldots, a_{r(\lambda)-1}), \lambda^{\mathbf{B}}(b_0, \ldots, b_{r(\lambda)-1})\rangle \in R$, as claimed. The converse is easier and is left to the reader.

2 If $F = I_{\mathbf{Set}}$, then a G-bisimulation between two $\langle I_{\mathbf{Set}}, G\rangle$-bialgebras is the usual notion of G-bisimulation of coalgebras (see, e.g., [23]). The same holds for G-bisimulation equivalence.

These notions restricted to transition systems give restrictions of the prototypical notions of bisimulation that first appeared in the context of concurrency theory [18, 19].

Theorem 8 *Let* $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ *be two* $\langle F, G\rangle$*-bialgebras. A function* $f : A \to B$ *is a bialgebra homomorphism if and only if its graph* $\Gamma(f)$ *is an* $\langle F, G\rangle$*-bisimulation.*

Proof:

Suppose $f : A \to B$ is such that $\Gamma(f) \subseteq A \times B$ is a bisimulation between $\mathbf{A}$ and $\mathbf{B}$ with structure map $\gamma : F(\Gamma(f)) \to G(\Gamma(f))$, i.e., the following diagram commutes

$$\begin{array}{ccccc} F(A) & \xleftarrow{F(\pi_1)} & F(\Gamma(f)) & \xrightarrow{F(\pi_2)} & F(B) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} \\ G(A) & \xleftarrow[G(\pi_1)]{} & G(\Gamma(f)) & \xrightarrow[G(\pi_2)]{} & G(B) \end{array}$$

Since f is a function, $\pi_1 : \Gamma(f) \to A$ is bijective. Thus $\pi_1^{-1} : A \to \Gamma(f)$ exists and is a bialgebra homomorphism by Proposition 1. Hence $f = \pi_2 \circ \pi_1^{-1}$ is also a bialgebra homomorphism.

Conversely, suppose that $f : \mathbf{A} \to \mathbf{B}$ is a bialgebra homomorphism.

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ G(A) & \xrightarrow[G(f)]{} & G(B) \end{array}$$

Endow $\Gamma(f) \subseteq A \times B$ with the bialgebra structure $\gamma = G(\pi_1^{-1})\alpha F(\pi_1) : F(\Gamma(f)) \to G(\Gamma(f))$.

$$F(\Gamma(f)) \xrightarrow{F(\pi_1)} F(A) \xrightarrow{\alpha} G(A) \xrightarrow{G(\pi_1^{-1})} G(\Gamma(f))$$

Then

$$G(\pi_1)\gamma = G(\pi_1)G(\pi_1^{-1})\alpha F(\pi_1) = \alpha F(\pi_1)$$

and

$$\begin{aligned} G(\pi_2)\gamma &= G(\pi_2)G(\pi_1^{-1})\alpha F(\pi_1) \\ &= G(\pi_2\pi_1^{-1})\alpha F(\pi_1) \\ &= G(f)\alpha F(\pi_1) \\ &= \beta F(f)F(\pi_1) \\ &= \beta F(f\pi_1) \\ &= \beta F(\pi_2). \end{aligned}$$

■

Theorem 8 yields as corollaries Proposition 2.7 of [21] and Theorem 2.7 of [23].

4 Limits and Colimits of Bialgebras

For the categorical notions of *limit* and *colimit* of a given diagram $d : D \to \mathcal{C}$ with base graph $D = \langle V(D), E(D)\rangle$ in a category $\mathcal{C}$ the reader is referred to the general references [14], [2], [1] and [4]. The notions of *product, equalizer, pullback* and the duals of *coproduct, coequalizer, pushout* will also be considered known and the same references may be consulted for their definitions and several illuminating examples. By the **type** of a limit or colimit we mean the isomorphism class, in the category **Grph** of graphs and graph homomorphisms, of the base graph of the limit or colimit, respectively. A functor F : **Set** $\to$ **Set** is said to **preserve** a type D of limit, respectively colimit, if, for every diagram $d : D \to$ **Set** with limit, respectively colimit, $\langle L, \{f_d : d \in V(D)\}\rangle$, $\langle F(L), \{F(f_d) : d \in V(D)\}\rangle$ is the limit, respectively colimit, of the diagram $F \circ d : D \to$ **Set**. A **weak limit** (**weak colimit**) of a diagram in a category is defined similarly to a limit (colimit) except that uniqueness of the fill-in morphisms in the corresponding universal mapping properties is not required. A functor F : **Set** $\to$ **Set** is said to **preserve** a certain type of weak limit (colimit) if a weak limiting (colimiting) cone $\langle L, \{f_d : d \in V(D)\}\rangle$ of a diagram $d : D \to$ **Set** is sent to a weak limiting (colimiting) cone $\langle F(L), \{F(f_d) : d \in V(D)\}\rangle$ of the diagram $F \circ d : D \to$ **Set**.

In this section a general study of limits and colimits in the category $\mathbf{Set}^F_G$ of $\langle F, G\rangle$-bialgebras is undertaken. The ultimate goal is to prove that $\mathbf{Set}^F_G$ has all limits that are preserved by G and all colimits that are preserved by F. Moreover, if G preserves a certain type of weak limit, then, for every diagram of that type in $\mathbf{Set}^F_G$, the corresponding diagram in **Set** has a limit that may be endowed with a bialgebra structure so that the resulting diagram consists of bialgebra homomorphisms that commute in $\mathbf{Set}^F_G$, as do in **Set** their underlying maps. Analogously, if F preserves a certain type of weak colimit, then, for every diagram of that type in $\mathbf{Set}^F_G$, the corresponding diagram in **Set** has a colimit that may be endowed with a bialgebra structure so that the resulting diagram consists of bialgebra homomorphisms that commute in $\mathbf{Set}^F_G$, as do in **Set** their underlying maps.

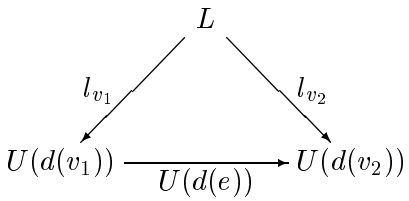
To this end, let $U : \mathbf{Set}^F_G \to$ **Set** be the underlying set forgetful functor from the category of $\langle F, G\rangle$-bialgebras to the category of sets. For every bialgebra $\mathbf{A} = \langle A, \alpha\rangle$, $U(\mathbf{A}) = A$ and, for every bialgebra homomorphism $h : \mathbf{A} \to \mathbf{B}$, $U(h) = h : A \to B$. U is said to **create a type** D of **limit** (**colimit**) if, for every diagram $d : D \to \mathbf{Set}^F_G$, its limit (colimit) is constructed by first taking the limit (colimit) of $U \circ d : D \to$ **Set** in **Set** and then supplying it in a unique way with an $\langle F, G\rangle$-bialgebra structure.

Theorem 9 U : $\mathbf{Set}^F_G \to$ **Set** *creates and preserves all types of limits that the functor* G : **Set** $\to$ **Set** *preserves.*

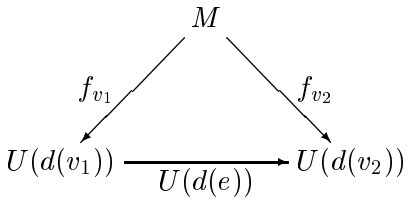
Proof:

Let $D = \langle V(D), E(D)\rangle$ be a graph and $d : D \to \mathbf{Set}^F_G$ a diagram in $\mathbf{Set}^F_G$. Suppose that G preserves limits of type D. Consider the diagram $U \circ d : D \to$ **Set** in **Set**. Since **Set** is complete, $U \circ d$ has a limit $\langle L, \{l_v : v \in V(D)\}\rangle$ in **Set**, i.e., L is a set and, for all $v \in V(D)$, $l_v : L \to U(d(v))$ is a mapping, such that, for all $e : v_1 \to v_2$ in $E(D)$, the following diagram

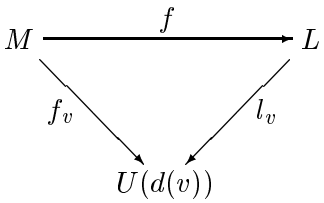
commutes



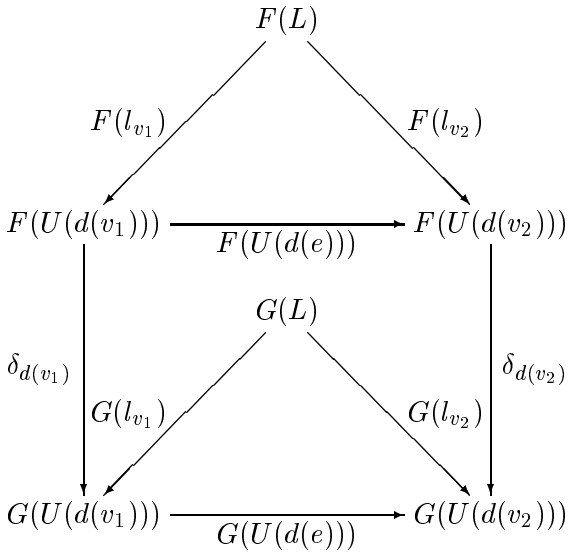
and such that, for all other cones $\langle M, \{f_v : v \in V(D)\}\rangle, f_v : M \to U(d(v)), v \in V(D)$, such that, for all $e : v_1 \to v_2$ in $E(D)$, the following triangle commutes



there exists a unique mapping $f : M \to L$, such that the following triangle commutes, for all $v \in V(D)$,



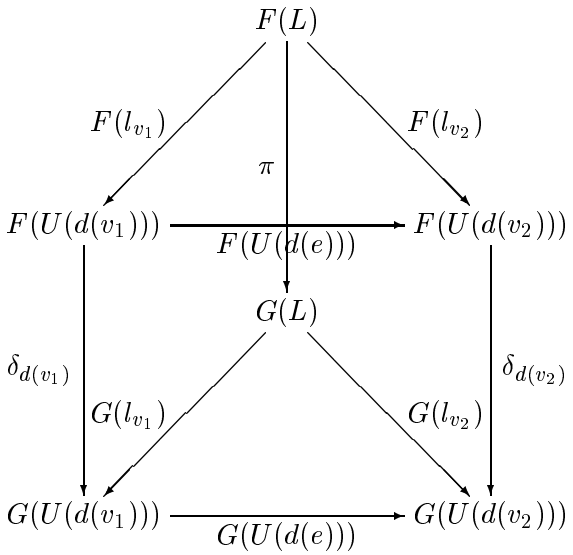
Now consider the diagram



Since G preserves limits of type D, The cone $\langle G(L), \{G(l_v) : v \in V(D)\}\rangle$ is a limiting cone in **Set**. Therefore, since

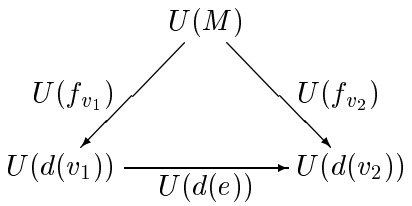
$$\begin{aligned} G(U(d(e)))\delta_{d(v_1)}F(l_{v_1}) &= \delta_{d(v_2)}F(U(d(e)))F(l_{v_1}) \\ &= \delta_{d(v_2)}F(l_{v_2}), \end{aligned}$$

there exists a unique map $\pi : F(L) \to G(L)$, such that the following diagram commutes in **Set** :

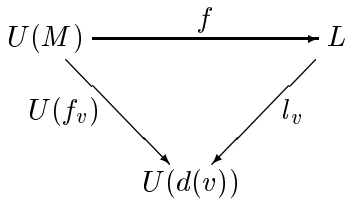


Clearly, $\mathbf{L} = \langle L, \pi \rangle$ is a bialgebra and, for all $v \in V(d)$, $l_v : \mathbf{L} \to d(v)$ is a bialgebra homomorphism.

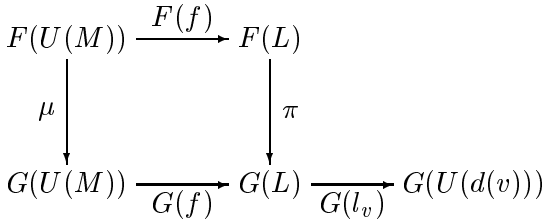
To show that $\langle \mathbf{L}, \{l_v : v \in V(D)\} \rangle$ is a limiting cone in $\mathbf{Set}^F_G$, consider any other cone $\langle \mathbf{M}, \{f_v : v \in V(D)\} \rangle$ in $\mathbf{Set}^F_G$:



Since $\langle L, \{l_v : v \in V(D)\} \rangle$ is a limiting cone in **Set**, there exists a unique mapping $f : U(M) \to L$ in **Set**, such that, for all $v \in V(D)$, the following triangle commutes



It suffices to show that $f : U(M) \to L$ is a bialgebra homomorphism $f : \mathbf{M} \to \mathbf{L}$. We have, for all $v \in V(D)$,



$$\begin{aligned} G(l_v)\pi F(f) &= \delta_{d(v)} F(l_v)F(f) \\ &= \delta_{d(v)} F(l_v f) \\ &= \delta_{d(v)} F(U(f_v)) \\ &= G(U(f_v))\mu \\ &= G(l_v f)\mu \\ &= G(l_v)G(f)\mu, \end{aligned}$$

whence, by the universal mapping property of $\langle G(L), \{G(l_v) : v \in V(D)\}\rangle$, it now follows that $\pi F(f) = G(f)\mu$, as was to be shown. ■

This yields the following

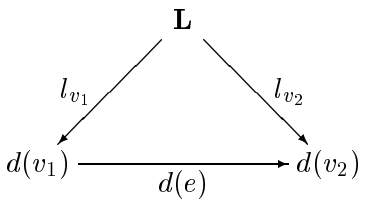
Corollary 10 *The categories* $\mathbf{Set}^F$ *of* F*-algebras, for any functor* $F : \mathbf{Set} \to \mathbf{Set}$, $\mathbf{Set}^{F_\mathcal{L}}$ *of* $\mathcal{L}$*-algebras, and* $\vec{\mathcal{V}}$*, of any variety* $\mathcal{V}$*, are complete, i.e., have all small limits.*

Also, Theorem 4.6 of [23]

Corollary 11 *The category* $\mathbf{Set}_G$ *of* G*-coalgebras has all limits that* $G : \mathbf{Set} \to \mathbf{Set}$ *preserves. Moreover the forgetful functor* $U : \mathbf{Set}_G \to \mathbf{Set}$ *creates all these limits.*

Following the lines of the proof of Theorem 9, the following may also be proved.

Theorem 12 *Let* D *be a graph,* $d : D \to \mathbf{Set}^F_G$ *be a diagram in* $\mathbf{Set}^F_G$ *and suppose that the functor* $G : \mathbf{Set} \to \mathbf{Set}$ *preserves weak limits of type* D*. Then the limit* $\langle L, \{l_v : v \in V(D)\}\rangle$ *of* $U \circ d : D \to \mathbf{Set}$ *in* $\mathbf{Set}$ *may be endowed with a bialgebra structure* $\mathbf{L} = \langle L, \pi\rangle$*, such that the following diagram commutes in* $\mathbf{Set}^F_G$*, for all* $e : v_1 \to v_2$ *in* $E(D)$*,*



The following results are the analogs of Theorems 9 and 12. Their proofs are also very similar to the proof of Theorem 9 and will, therefore, be omitted. First, the analog of Theorem 9.

Theorem 13 $U : \mathbf{Set}^F_G \to \mathbf{Set}$ *creates and preserves all types of colimits that the functor* $F : \mathbf{Set} \to \mathbf{Set}$ *preserves.*

This yields Theorem 4.5 of [23]. (See also Theorem 4.2 of [11].)

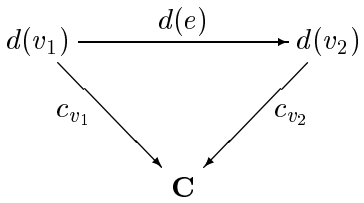
Corollary 14 *The category* $\mathbf{Set}_G$ *of* G*-coalgebras, for any functor* $G : \mathbf{Set} \to \mathbf{Set}$*, is cocomplete, i.e., has all small colimits. Moreover, the forgetful functor* $U : \mathbf{Set}_G \to \mathbf{Set}$ *creates colimits.*

The following result concerning universal algebras may also be obtained

Corollary 15 *The category* $\mathbf{Set}^{F_\mathcal{L}}$ *of* $F_\mathcal{L}$*-algebras has all colimits that* $F_\mathcal{L} : \mathbf{Set} \to \mathbf{Set}$ *preserves. Moreover, the forgetful functor* $U : \mathbf{Set}^{F_\mathcal{L}} \to \mathbf{Set}$ *creates all these colimits.*

Finally, the analog of Theorem 12 for colimits is given.

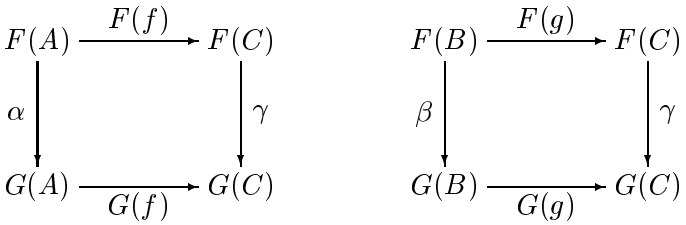
Theorem 16 *Let D be a graph, $d : D \to \mathbf{Set}_G^F$ be a diagram in $\mathbf{Set}_G^F$ and suppose that the functor $F : \mathbf{Set} \to \mathbf{Set}$ preserves weak colimits of type D. Then the colimit $\langle C, \{c_v : v \in V(D)\}\rangle$ of $U \circ d : D \to \mathbf{Set}$ in $\mathbf{Set}$ may be endowed with a bialgebra structure $\mathbf{C} = \langle C, \kappa\rangle$, such that the following diagram commutes in $\mathbf{Set}_G^F$, for all $e : v_1 \to v_2$ in $E(D)$,*



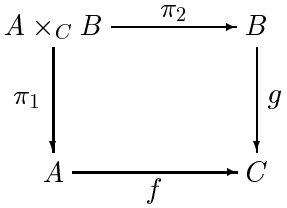
The following Theorem, which is the analog of Theorem 4.3 of [23], exhibits some of the fruitful interaction of the notion of bisimulation with the weak preservation of limits by the functor G.

Theorem 17 *Let $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle, \mathbf{C} = \langle C, \gamma\rangle$ be $\langle F, G\rangle$-bialgebras and $f : \mathbf{A} \to \mathbf{C}, g : \mathbf{B} \to \mathbf{C}$ be bialgebra homomorphisms. If G preserves weak pullbacks, then the pullback of $f : A \to C, g : B \to C$ in $\mathbf{Set}$ is a bisimulation from $\mathbf{A}$ to $\mathbf{B}$.*

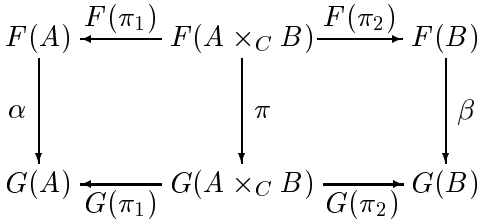
Proof:
Let $f : \mathbf{A} \to \mathbf{C}$ and $g : \mathbf{B} \to \mathbf{C}$ be bialgebra homomorphisms.



Consider the pullback diagram in $\mathbf{Set}$



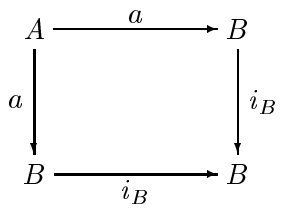
Then, since G preserves weak pullbacks, by Theorem 12, there exists a structure map $\pi : F(A \times_C B) \to G(A \times_C B)$, such that the following diagram commutes in $\mathbf{Set}_G^F$



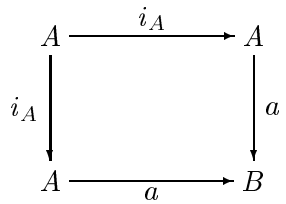
Hence $\mathbf{A} \times_C \mathbf{B} = \langle A \times_C B, \pi\rangle$ is a bialgebra and the projections π_1, π_2 become homomorphisms. ■

5 On Monos and Epis in $\mathbf{Set}_G^F$

Recall from category theory (see, e.g., [14] and [2]) that an arrow $a : A \to B$ in a category $\mathcal{C}$ is an epimorphism or epi if and only if the following is a pushout in $\mathcal{C}$



Dually, $a : A \to B$ in $\mathcal{C}$ is a monomorphism or mono if and only if the following diagram is a pullback in $\mathcal{C}$



Since weak pullbacks of monos are regular pullbacks and weak pushouts of epis are regular pushouts, the following proposition follows:

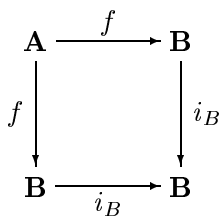
Proposition 18 *Let* $\mathbf{A} = \langle A, \alpha \rangle, \mathbf{B} = \langle B, \beta \rangle$ *be two* $\langle F, G \rangle$*-bialgebras and* $f : \mathbf{A} \to \mathbf{B}$ *an* $\langle F, G \rangle$*-bialgebra homomorphism.*

1 *If* F *preserves weak pushouts, then* f *is surjective if and only if it is epi.*

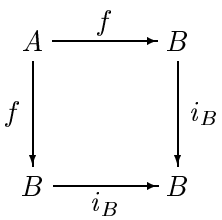
2 *If* G *preserves weak pullbacks, then* f *is injective if and only if it is mono.*

Proof:

1 By Theorem 13, $U : \mathbf{Set}_G^F \to \mathbf{Set}$ creates all types of colimits that F preserves. Thus, in particular, U creates pushouts of epis. Since, in addition, U preserves all colimits that it creates



is a pushout in $\mathbf{Set}_G^F$ iff



is a pushout in **Set**.

Thus, $f : \mathbf{A} \to \mathbf{B}$ is epi iff $f : A \to B$ is epi iff $f : A \to B$ is surjective iff $f : \mathbf{A} \to \mathbf{B}$ is surjective.

2 Note, similarly, that weak pullbacks of monos are ordinary pullbacks and that G preserves these. So U creates and preserves them as well. Now proceed as in 1. ■

As corollaries we obtain

Corollary 19 *In every variety of algebras injectives and monos coincide.*

and also Proposition 4.7 of [23]

Corollary 20 *For every functor $G : \mathbf{Set} \to \mathbf{Set}$, surjectives and epis coincide in $\mathbf{Set}_G$. Moreover, if G preserves weak pullbacks, then injectives and monos also coincide in $\mathbf{Set}_G$.*

6 More on Bisimulations

In this section, we elaborate more on bialgebra bisimulations. The goal is to show that several results concerning substitutive relations and congruences on the universal algebraic side and bisimulation relations and bisimulation equivalences, respectively, on the coalgebraic side may be jointly formulated in the framework of bialgebras.

Proposition 21 *Let $\mathbf{A} = \langle A, \alpha \rangle$ be an $\langle F, G \rangle$-bialgebra. The diagonal Δ_A of A is a bisimulation equivalence on $\mathbf{A}$.*

Proof:
Note that $\Delta_A = \Gamma(i_A)$ and recall Theorem 8 ■

Theorem 22 *Let $\mathbf{A} = \langle A, \alpha \rangle, \mathbf{B} = \langle B, \beta \rangle$ be $\langle F, G \rangle$-bialgebras and $R \subseteq A \times B$ a bisimulation between $\mathbf{A}$ and $\mathbf{B}$. The converse $R^{-1} \subseteq B \times A$ is a bisimulation between $\mathbf{B}$ and $\mathbf{A}$.*

Proof:
Suppose $R \subseteq A \times B$ is a bisimulation between $\mathbf{A}$ and $\mathbf{B}$ with bialgebra structure map $\rho : F(R) \to G(R)$, i.e., the following rectangles commute

$$\begin{array}{ccccc} F(A) & \xleftarrow{F(\pi_1)} & F(R) & \xrightarrow{F(\pi_2)} & F(B) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \beta} \\ G(A) & \xleftarrow[G(\pi_1)]{} & G(R) & \xrightarrow[G(\pi_2)]{} & G(B) \end{array}$$

Denote by $r : R \to R^{-1}$ the map sending $\langle a, b \rangle \in R$ to $\langle b, a \rangle \in R^{-1}$. Endow R^{-1} with the bialgebra structure

$$\begin{array}{ccc} F(R^{-1}) & \xrightarrow{F(r^{-1})} & F(R) \\ \downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho} \\ G(R^{-1}) & \xleftarrow[G(r)]{} & G(R) \end{array}$$

$$\rho' = G(r)\rho F(r^{-1}) : F(R^{-1}) \to G(R^{-1}).$$

We then have

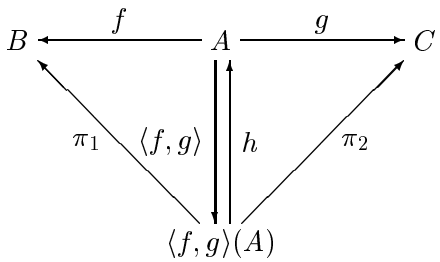
$$\begin{array}{ccccc} F(B) & \xleftarrow{F(\pi_1)} & F(R^{-1}) & \xrightarrow{F(\pi_2)} & F(A) \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \alpha} \\ G(B) & \xleftarrow[G(\pi_1)]{} & G(R^{-1}) & \xrightarrow[G(\pi_2)]{} & G(A) \end{array}$$

$$\begin{aligned} G(\pi_2)\rho' &= G(\pi_2)G(r)\rho F(r^{-1}) \\ &= G(\pi_2 r)\rho F(r^{-1}) \\ &= G(\pi_1)\rho F(r^{-1}) \\ &= \alpha F(\pi_1)F(r^{-1}) \\ &= \alpha F(\pi_1 r^{-1}) \\ &= \alpha F(\pi_2) \end{aligned}$$

and, similarly, $G(\pi_1)\rho' = \beta F(\pi_1)$. ■

Lemma 23 *Let* $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ *and* $\mathbf{C} = \langle C, \gamma\rangle$ *be* $\langle F, G\rangle$*-bialgebras and* $f : \mathbf{A} \to \mathbf{B}, g : \mathbf{A} \to \mathbf{C}$ *bialgebra homomorphisms. Then the image* $\langle f, g\rangle(A) = \{\langle f(a), g(a)\rangle : a \in A\} \subseteq B \times C$ *is a bisimulation from* $\mathbf{B}$ *to* $\mathbf{C}$.

Proof:
Consider the diagram



where $\langle f, g\rangle : A \to \langle f, g\rangle(A); a \mapsto \langle f(a), g(a)\rangle$ is surjective and, therefore, has a right inverse h. Thus $\langle f, g\rangle \circ h = i_A$. Define on $\langle f, g\rangle(A)$ the bialgebra structure

$$F(\langle f, g\rangle(A)) \xrightarrow{F(h)} F(A) \xrightarrow{\alpha} G(A) \xrightarrow{G(\langle f, g\rangle)} G(\langle f, g\rangle(A))$$

$$\delta = G(\langle f, g\rangle)\alpha F(h) : F(\langle f, g\rangle(A)) \to G(\langle f, g\rangle(A)).$$

Then

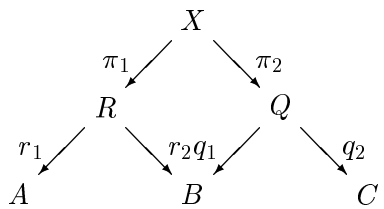
$$\begin{array}{ccccc} F(B) & \xleftarrow{F(\pi_1)} & F(\langle f, g\rangle(A)) & \xrightarrow{F(\pi_2)} & F(C) \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \gamma} \\ G(B) & \xleftarrow[G(\pi_1)]{} & G(\langle f, g\rangle(A)) & \xrightarrow[G(\pi_2)]{} & G(C) \end{array}$$

$$\begin{aligned} G(\pi_2)\delta &= G(\pi_2)G(\langle f,g\rangle)\alpha F(h) \\ &= G(\pi_2\langle f,g\rangle)\alpha F(h) \\ &= G(g)\alpha F(h) \\ &= \gamma F(g)F(h) \\ &= \gamma F(gh) \\ &= \gamma F(\pi_2) \end{aligned}$$

and, similarly, $G(\pi_1)\delta = \beta F(\pi_1)$. ■

Theorem 24 *Let* $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ *and* $\mathbf{C} = \langle C, \gamma\rangle$ *be* $\langle F, G\rangle$*-bialgebras and* $R \subseteq A \times B, Q \subseteq B \times C$ $\langle F, G\rangle$*-bisimulations. If* G *preserves weak pullbacks, then* $R \circ Q \subseteq A \times C$ *is a bisimulation from* $\mathbf{A}$ *to* $\mathbf{C}$.

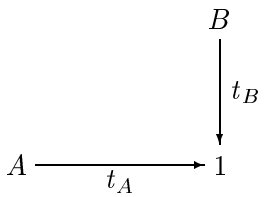
Proof:
The following pullback in **Set**



is such that $R \circ Q = \langle r_1\pi_1, q_2\pi_2\rangle(X)$ ([23], Section 18). Now, by Theorem 12, X may be endowed with a unique bialgebra structure such that the diagram above lifted to $\mathbf{Set}_G^F$ still commutes. Then, Lemma 23 shows that $R \circ Q$ is a bisimulation from $\mathbf{A}$ to $\mathbf{C}$. ■

Theorem 24 shows, in the algebraic side, that the composition of two substitutive relations is a substitutive relation and, in the coalgebraic side, yields as a corollary Theorem 5.4 or [23].

Regarding the assumptions in part *2* of the following theorem, note that if a functor G : **Set** $\to$ **Set** preserves generalized pullbacks, then it also preserves products. This is because preservation of the "empty" pullback yields preservation of the terminal object $1 = \{\emptyset\}$ and because the pullback of the diagram



where $t_A : A \to 1$ and $t_B : B \to 1$ are the unique morphisms into the terminal object, gives the product of A and B.

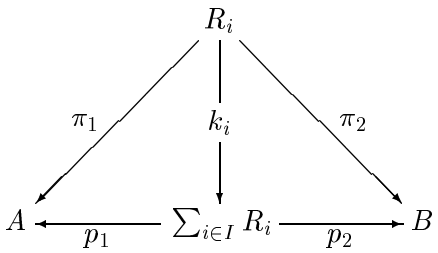
Theorem 25 *Let* $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ *be* $\langle F, G\rangle$*-bialgebras and* $R_i \subseteq A \times B, i \in I$, $\langle F, G\rangle$*-bisimulations with bialgebra structures* $\rho_i : F(R_i) \to G(R_i), i \in I$, *respectively.*

1 If F *preserves weak coproducts, then* $\bigcup_{i\in I} R_i$ *is a bisimulation.*

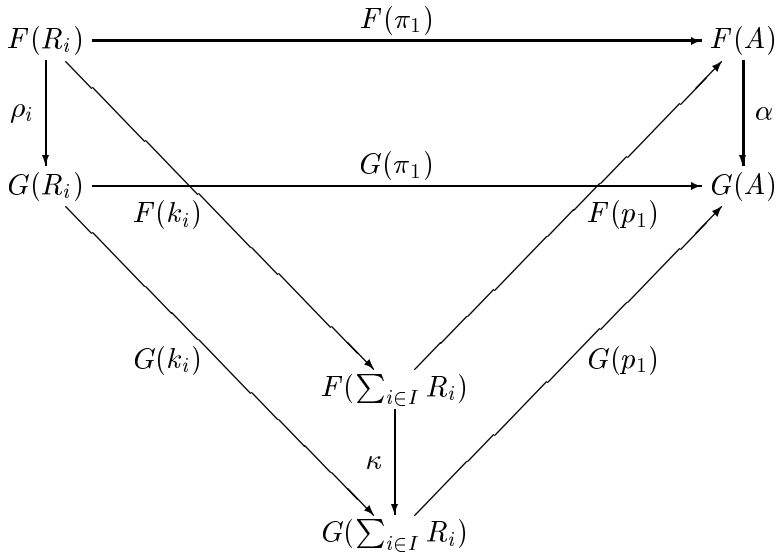
2 If G preserves generalized pullbacks (or products and generalized weak pullbacks), then $\bigcap_{i\in I} R_i$ is a bisimulation.

Proof:

1 Consider the coproduct diagram in **Set**



where $k_i : R_i \to \sum_{i\in I} R_i, i \in I$, are the injections and $\pi_1 : R_i \to A$, $\pi_2 : R_i \to B, p_1 : \sum_{i\in I} R_i \to A$ and $p_2 : \sum_{i\in I} R_i \to B$ are the coordinate-wise projections. Since F preserves weak coproducts, by Theorem 16, there is a bialgebra structure $\kappa : F(\sum_{i\in I} R_i) \to G(\sum_{i\in I} R_i)$, such that $\sum_{i\in I} \mathbf{R}_i = \langle \sum_{i\in I} R_i, \kappa\rangle$ is a bialgebra and $k_i : \mathbf{R}_i \to \sum_{i\in I} \mathbf{R}_i, i \in I$, are homomorphisms. It is not difficult to check by chasing the following diagram that $p_1 : \sum_{i\in I} \mathbf{R}_i \to \mathbf{A}$ and $p_2 : \sum_{i\in I} \mathbf{R}_i \to \mathbf{B}$ are also homomorphisms.



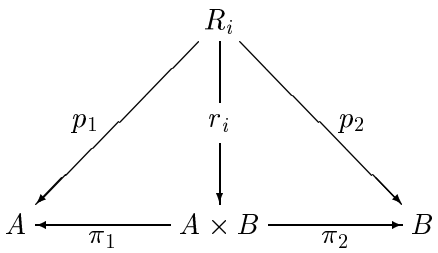
But, note that $\bigcup_{i\in I} R_i = \langle p_1, p_2\rangle(\sum_{i\in I} R_i)$. Thus, by Lemma 23, $\bigcup_{i\in I} R_i$ is also an $\langle F, G\rangle$-bisimulation.

2 Now consider the product diagram

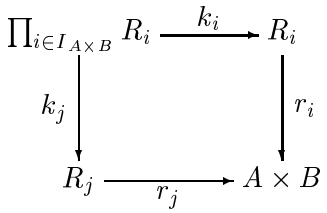
$$A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$$

Since G preserves pullbacks, it also preserves products (or products directly). Thus, by Theorem 9, there exists a bialgebra structure $\pi : F(A \times B) \to G(A \times B)$, such that

$\mathbf{A} \times \mathbf{B} = \langle A \times B, \pi \rangle$ is the product of $\mathbf{A}$ and $\mathbf{B}$ in $\mathbf{Set}_G^F$. Now, by the same theorem, and considering the diagram



the inclusion $r_i : R_i \to A \times B$ is a bialgebra homomorphism from $\mathbf{R}_i$ to $\mathbf{A} \times \mathbf{B}, i \in I$. Finally, consider the generalized pullback diagram in **Set**



Note that $\prod_{i \in I_{A \times B}} R_i = \bigcap_{i \in I} R_i$. Since G preserves generalized weak pullbacks $\prod_{i \in I_{A \times B}} R_i$ may be endowed with a bialgebra structure, such that all projections $k_i, i \in I$, become homomorphisms. But then $\pi_1 r_i k_i : \bigcap_{i \in I} R_i \to A$ and $\pi_2 r_i k_i : \bigcap_{i \in I} R_i \to B$ become homomorphisms and $\bigcap_{i \in I} R_i$ is a bisimulation. ■

Given two $\langle F, G \rangle$-bialgebras $\mathbf{A}$ and $\mathbf{B}$, denote by $\mathrm{Bis}(\mathbf{A}, \mathbf{B})$ the collection of all $\langle F, G \rangle$-bisimulations between $\mathbf{A}$ and $\mathbf{B}$.

The following corollary has as special cases the statements that, in universal algebra, the collection of all substitutive relations between two algebras forms a complete lattice under inclusion with meet given by intersection and, in coalgebra, the collection of all bisimulations between two coalgebras also forms a complete lattice under inclusion with join given by union.

Corollary 26 *Let $\mathbf{A} = \langle A, \alpha \rangle, \mathbf{B} = \langle B, \beta \rangle$ be $\langle F, G \rangle$-bialgebras.*

1 *If F preserves weak coproducts, then the collection of all bisimulations from $\mathbf{A}$ to $\mathbf{B}$ forms a complete lattice, such that, for all bisimulations $R_i, i \in I$,*

$$\bigvee_{i \in I} R_i = \bigcup_{i \in I} R_i \quad \textit{and} \quad \bigwedge_{i \in I} R_i = \bigcup \{R \in \mathrm{Bis}(\mathbf{A}, \mathbf{B}) : R \subseteq \bigcap_{i \in I} R_i\}.$$

2 *If G preserves generalized pullbacks (or products and generalized weak pullbacks), then the collection of all bisimulations from $\mathbf{A}$ to $\mathbf{B}$ forms a complete lattice, such that, for all bisimulations $R_i, i \in I$,*

$$\bigwedge_{i \in I} R_i = \bigcap_{i \in I} R_i \quad \textit{and} \quad \bigvee_{i \in I} R_i = \bigcap \{R \in \mathrm{Bis}(\mathbf{A}, \mathbf{B}) : \bigcup_{i \in I} R_i \subseteq R\}.$$

As corollaries of Corollary 26 the well-known characterizations of joins and meets in the lattice of universal algebraic congruences (see, e.g., [5], Section II.5, and [17], Sections 1.4 and 4.3) and universal coalgebraic bisimulations ([23], Corollary 5.6) may be obtained.

Given two $\langle F, G\rangle$-bialgebras $\mathbf{A}$ and $\mathbf{B}$, denote by $\mathbf{Bis}(\mathbf{A}, \mathbf{B})$ the complete lattice of Corollary 26.*1*, in case F preserves weak coproducts or Corollary 26.*2*, in case F preserves generalized pullbacks (or products and generalized weak pullbacks).

Let $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ be $\langle F, G\rangle$-bialgebras and $f : \mathbf{A} \to \mathbf{B}$ a homomorphism. Denote by $\mathrm{K}(f) \subseteq A^2$ the **kernel** of f, i.e.,

$$\mathrm{K}(f) = \{\langle a, a'\rangle \in A^2 : f(a) = f(a')\}.$$

Proposition 27 *Let $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ be $\langle F, G\rangle$-bialgebras and $f : \mathbf{A} \to \mathbf{B}$ a homomorphism. If G preserves weak pullbacks, then $\mathrm{K}(f)$ is a bisimulation equivalence on $\mathbf{A}$.*

Proof:
Clearly $\mathrm{K}(f)$ is an equivalence relation on A. Note that $\mathrm{K}(f) = \Gamma(f) \circ \Gamma(f)^{-1}$. Now, by Theorem 8, $\Gamma(f)$ is a bisimulation. Thus, by Theorem 22, $\Gamma(f)^{-1}$ is also a bisimulation. Finally, by Theorem 24, $\mathrm{K}(f) = \Gamma(f) \circ \Gamma(f)^{-1}$ is a bisimulation. ■

As a corollary we get that, in general, kernels of homomorphisms in universal algebra are congruences ([5], Theorem II.6.8, also [21], Proposition 2.6) and kernels of G-coalgebra homomorphisms are bisimulation equivalences in case G preserves weak pullbacks ([23], Proposition 5.7).

Proposition 28 *Let $\mathbf{A} = \langle A, \alpha\rangle$ be an $\langle F, G\rangle$-bialgebra and $R \subseteq A^2$ a bisimulation equivalence on $\mathbf{A}$. If F preserves weak coequalizers, then there exists a bialgebra structure $\alpha_R : F(A/R) \to G(A/R)$ on A/R, such that the quotient map $q_R : A \to A/R$ becomes a homomorphism*

$$\begin{array}{ccc} F(A) & \xrightarrow{F(q_R)} & F(A/R) \\ \alpha \downarrow & & \downarrow \alpha_R \\ G(A) & \xrightarrow[G(q_R)]{} & G(A/R) \end{array}$$

and $\mathbf{A}/R = \langle A/R, \alpha_R\rangle$ becomes the coequalizer in $\mathbf{Set}_G^F$ of

$$\mathbf{R} \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \mathbf{A}$$

Proof:
Note that

$$R \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} A \xrightarrow{q_R} A/R$$

is a coequalizer diagram in **Set** with both π_1 and π_2 bialgebra homomorphisms. Now $q_R : A \to A/R$ is surjective, whence $F(q_R) : F(A) \to F(A/R)$ is also surjective. Therefore the weak coequalizer

$$F(R) \underset{F(\pi_2)}{\overset{F(\pi_1)}{\rightrightarrows}} F(A) \xrightarrow{F(q_R)} F(A/R)$$

is actually a coequalizer. Thus F preserves this coequalizer. Hence, by Theorem 13, A/R may be endowed with a bialgebra structure $\alpha_R : F(A/R) \to G(A/R)$, such that $q_R : \mathbf{A} \to \mathbf{A}/R$ is the coequalizer in $\mathbf{Set}_G^F$ of

$$\mathbf{R} \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \mathbf{A}$$

■

We may now obtain the analog of Proposition 5.9 of [23] for bialgebra bisimulations.

Proposition 29 *Let $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ be $\langle F, G\rangle$-bialgebras and $f : \mathbf{A} \to \mathbf{B}$ a bialgebra homomorphism. Suppose that G preserves weak pullbacks.*

1. *If $R \subseteq A^2$ is a bisimulation on $\mathbf{A}$, then $f(R) = \{\langle f(a), f(a')\rangle : \langle a, a'\rangle \in R\}$ is a bisimulation on $\mathbf{B}$.*
2. *If $Q \subseteq B^2$ is a bisimulation on $\mathbf{B}$, then $f^{-1}(Q) = \{\langle a, a'\rangle \in A^2 : \langle f(a), f(a')\rangle \in Q\}$ is a bisimulation on $\mathbf{A}$.*

Proof:
For *1*, note that $f(R) = \Gamma(f)^{-1} \circ R \circ \Gamma(f)$ and use Theorems 8, 22 and 24. For *2*, note that $f^{-1}(Q) = \Gamma(f) \circ Q \circ \Gamma(f)^{-1}$ and proceed as before. ■

7 Subbialgebras

Let $\mathbf{A} = \langle A, \alpha\rangle$ be an $\langle F, G\rangle$-bialgebra and $B \subseteq A$ with inclusion $i : B \hookrightarrow A$. If there exists a bialgebra structure $\beta : F(B) \to G(B)$ on B such that i is a homomorphism

$$\begin{array}{ccc} F(B) & \xrightarrow{F(i)} & F(A) \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\alpha} \\ G(B) & \xrightarrow[G(i)]{} & G(A) \end{array}$$

then $\mathbf{B} = \langle B, \beta\rangle$ is said to be a **subbialgebra** of $\mathbf{A}$. This is denoted by $\mathbf{B} \leq \mathbf{A}$. If $B \neq \emptyset$ and such a bialgebra structure β on B exists, it is unique. To show this, recall that, any endofunctor on **Set** preserves all monomorphisms whose domains are different from the empty set $\emptyset$ ([23], Proposition 18.1).

Proposition 30 *Let* $\mathbf{A} = \langle A, \alpha \rangle$ *be a bialgebra and* $\emptyset \neq B \subseteq A$ *with inclusion* $i : B \hookrightarrow A$. *Let* $\mathbf{B} = \langle B, \beta \rangle, \mathbf{B}' = \langle B, \beta' \rangle$ *be two bialgebras such that* $i : \mathbf{B} \to \mathbf{A}$ *and* $i : \mathbf{B}' \to \mathbf{A}$ *are homomorphisms.*

$$\begin{array}{ccc} F(B) & \xrightarrow{F(i)} & F(A) \\ {\scriptstyle\beta}\,\Big\|\,{\scriptstyle\beta'} & & \Big\downarrow{\scriptstyle\alpha} \\ G(B) & \xrightarrow[G(i)]{} & G(A) \end{array}$$

Then $\beta = \beta'$, *i.e.,* $\mathbf{B} = \mathbf{B}'$.

Proof:

If $B \neq \emptyset$, then $G(i) : G(B) \to G(A)$ is mono. Thus, since $G(i)\beta = \alpha F(i) = G(i)\beta'$, we conclude $\beta = \beta'$. ■

Corollary 31 *Let* $\mathbf{A} = \langle A, \alpha \rangle$ *be an* $\langle F, G \rangle$*-bialgebra. If* G *preserves monos, then, if* $B \subseteq A$ *admits a subbialgebra structure, it is unique. In particular, this holds if* G *preserves weak pullbacks.*

A bialgebra $\mathbf{A} = \langle A, \alpha \rangle$ is called **minimal** if it does not have a subbialgebra $\mathbf{B} = \langle B, \beta \rangle$, with $B \neq \emptyset, A$.

Proposition 32 *Let* $\mathbf{A} = \langle A, \alpha \rangle$ *be a bialgebra and* $B \subseteq A$. B *is the universe of a subbialgebra of* $\mathbf{A}$ *iff* Δ_B *is a bisimulation on* $\mathbf{A}$.

Proof:

Suppose Δ_B is a bisimulation on $\mathbf{A}$ with structure map $\delta : F(\Delta_B) \to G(\Delta_B)$, i.e., such that the following diagram commutes

$$\begin{array}{ccccc} F(A) & \xleftarrow{F(\pi_1)} & F(\Delta_B) & \xrightarrow{F(\pi_2)} & F(A) \\ {\scriptstyle\alpha}\,\Big\downarrow & & \Big\downarrow{\scriptstyle\delta} & & \Big\downarrow{\scriptstyle\beta} \\ G(A) & \xleftarrow[G(\pi_1)]{} & G(\Delta_B) & \xrightarrow[G(\pi_2)]{} & G(A) \end{array}$$

Define a bialgebra structure on B by

$$F(B) \xrightarrow{F(d^{-1})} F(\Delta_B) \xrightarrow{\delta} G(\Delta_B) \xrightarrow{G(d)} G(B)$$

$$\beta = G(d)\delta F(d^{-1}) : F(B) \to G(B),$$

where $d : \Delta_B \to B; \langle b, b\rangle \mapsto b$. Then

$$\begin{array}{ccc} F(B) & \xrightarrow{F(i)} & F(A) \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\alpha} \\ G(B) & \xrightarrow[G(i)]{} & G(A) \end{array}$$

$$\begin{aligned} G(i)\beta &= G(i)G(d)\delta F(d^{-1}) \\ &= G(id)\delta F(d^{-1}) \\ &= G(\pi_1)\delta F(d^{-1}) \\ &= \alpha F(\pi_1)F(d^{-1}) \\ &= \alpha F(\pi_1 d^{-1}) \\ &= \alpha F(i). \end{aligned}$$

Conversely, if $\beta : F(B) \to G(B)$ is a bialgebra structure on B, such that $i : B \hookrightarrow A$ is a homomorphism, then, by Theorem 8, $\Delta_B = \Gamma(i) \subseteq B \times A$ is a bisimulation. Thus $\Delta_B \subseteq A^2$ is also a bisimulation. ■

Proposition 32 has as corollaries the well-known facts that a universal algebra $\mathbf{B}$ is a subalgebra of an algebra $\mathbf{A}$ if and only if $\Delta_B \subseteq A^2$ is a substitutive relation on $\mathbf{A}$ and Proposition 6.2 of [23].

Proposition 33 *Let* $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ *be* $\langle F, G\rangle$*-bialgebras,* $R \subseteq A \times B$ *a bisimulation between* $\mathbf{A}$ *and* $\mathbf{B}$ *with structure map* $\rho : F(R) \to G(R)$ *and* $\mathbf{Q} \leq \mathbf{R}$. *Then* $Q \subseteq A \times B$ *is also a bisimulation between* $\mathbf{A}$ *and* $\mathbf{B}$.

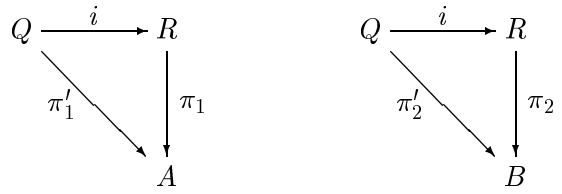
Proof:
By assumption, the following diagram commutes

$$\begin{array}{ccccc} F(A) & \xleftarrow{F(\pi_1)} & F(R) & \xrightarrow{F(\pi_2)} & F(B) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\beta} \\ G(A) & \xleftarrow[G(\pi_1)]{} & G(R) & \xrightarrow[G(\pi_2)]{} & G(B) \end{array}$$

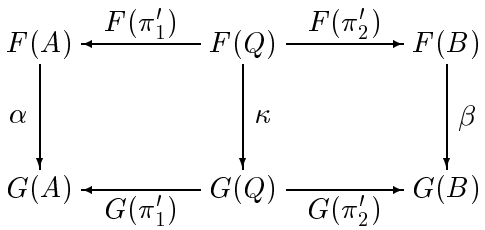
where $\pi_1 : R \to A$ and $\pi_2 : R \to B$ are the two projections. Also, if $i : Q \hookrightarrow R$ is the inclusion, the following diagram commutes

$$\begin{array}{ccc} F(Q) & \xrightarrow{F(i)} & F(R) \\ {\scriptstyle\kappa}\downarrow & & \downarrow{\scriptstyle\rho} \\ G(Q) & \xrightarrow[G(i)]{} & G(R) \end{array}$$

where $\kappa : F(Q) \to G(Q)$ is the structure map of the subbialgebra $\mathbf{Q} \leq \mathbf{R}$. Then for $\pi_1' : Q \to A, \pi_2' : Q \to B$, defined by



we have



$$\begin{aligned} G(\pi_1')\kappa &= G(i\pi_1)\kappa \\ &= G(\pi_1)G(i)\kappa \\ &= G(\pi_1)\rho F(i) \\ &= \alpha F(\pi_1)F(i) \\ &= \alpha F(\pi_1') \end{aligned}$$

and, similarly, $G(\pi_2')\kappa = \beta F(\pi_2')$. ■

Theorem 34 *Let $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ be bialgebras and $f : \mathbf{A} \to \mathbf{B}$ a bialgebra homomorphism. If G preserves weak pullbacks, then*

1 If $C \subseteq A$ admits a subbialgebra structure, so does $f(C) \subseteq B$.

2 If $D \subseteq B$ admits a subbialgebra structure, so does $f^{-1}(D) \subseteq A$.

Proof:

1 If $C \subseteq A$ admits a subbialgebra structure, then, by Proposition 32, Δ_C is a bisimulation on $\mathbf{A}$. Thus, by Proposition 29, $f(\Delta_C) = \Delta_{f(C)}$ is a bisimulation on $\mathbf{B}$. Thus, by Proposition 32 again, $f(C) \subseteq B$ admits a subbialgebra structure.

2 Similarly, if $D \subseteq B$ admits a subbialgebra structure, then, by Proposition 32, Δ_D is a bisimulation on $\mathbf{B}$. Thus, by Proposition 29, $f^{-1}(\Delta_D) = \Delta_{f^{-1}(D)}$ is a bisimulation on $\mathbf{A}$. Thus, by Proposition 32 again, $f^{-1}(D) \subseteq A$ admits a subbialgebra structure. ■

Theorem 34 immediately yields as corollaries Theorem II.6.3 of [5] and Theorem 6.3 of [23]. (See also Lemma 4.6 and Theorem 4.7 of [11] together with its corollaries.)

Theorem 35 *Let $\mathbf{A} = \langle A, \alpha\rangle$ be an $\langle F, G\rangle$-bialgebra.*

1 If F preserves weak coproducts, then the collection of all subbialgebras of $\mathbf{A}$ forms a complete lattice such that, for all $\mathbf{B}_i \leq \mathbf{A}, i \in I$, the universe of $\bigvee_{i\in I} \mathbf{B}_i$ is $\bigcup_{i\in I} B_i$.

2 If G preserves generalized pullbacks (or products and generalized weak pullbacks), then the collection of all subbialgebras of $\mathbf{A}$ forms a complete lattice such that, for all $\mathbf{B}_i \leq \mathbf{A}, i \in I$, the universe of $\bigwedge_{i\in I} \mathbf{B}_i$ is $\bigcap_{i\in I} B_i$.

Proof:

1 Let $\mathbf{B}_i = \langle B_i, \beta_i\rangle, i \in I$, be a collection of subbialgebras of $\mathbf{A}$. Then, by Proposition 32, $\Delta_{B_i}, i \in I$, are bisimulations on $\mathbf{A}$. By Theorem 25, $\Delta_{\cup_{i\in I} B_i} = \bigcup_{i\in I} \Delta_{B_i}$ is a bisimulation on $\mathbf{A}$. Thus, again by Proposition 32, $\bigcup_{i\in I} B_i$ admits a subbialgebra structure.

2 Similar to 1. ■

Part *2* of Theorem 35 has as corollary the fact that, for any universal algebra, the colection of all its subalgebras forms a complete lattice under inclusion with meet given by intersection. On the other hand parts *1* and *2* combined, yield the statement that, given a functor G that preserves generalized pullbacks, the collection of all subcoalgebras of a given G-coalgebra forms a complete lattice with meet given by intersection and join given by union. In [23], this statement (Theorem 6.4) is proven with the relaxed assumption that G preserves generalized weak pullbacks. We were unable to relax the condition of preservetion of generalized ordinary pullbacks here.

Corollary 36 *1 The collection of subalgebras of an F-algebra forms a complete lattice with meet given by intersection.*

2 If G preserves generalized pullbacks (or products and generalized weak pullbacks), then the collection of subcoalgebras of a G-coalgebra forms a complete lattice with join given by union and meet given by intersection.

Corollary 36 justifies the following definitions. Let $\mathbf{A} = \langle A, \alpha\rangle$ be a bialgebra and $X \subseteq A$. In case G preserves generalized pullbacks (or products and generalized weak pullbacks), define the **subsystem $\langle \mathbf{X}\rangle = \langle \langle X\rangle, \hat{\xi}\rangle$ of $\mathbf{A}$ generated by** X by letting

$$\langle \mathbf{X}\rangle = \bigwedge\{\mathbf{B} \leq \mathbf{A} : X \subseteq B\}.$$

If F preserves weak coproducts, define the **greatest subsystem $[\mathbf{X}] = \langle [X], \check{\xi}\rangle$ contained in** X by

$$[\mathbf{X}] = \bigvee\{\mathbf{B} \leq \mathbf{A} : B \subseteq X\}.$$

We close this section with another result about intersections. Inspired by an analogous recent result for coalgebras ([12], Theorem 3.1), we show that, if the underlying sets of two subbialgebras of a given bialgebra have nonempty intersection, then this intersection admits a subbialgebra structure.

Theorem 37 *Let* $\mathbf{A} = \langle A, \alpha \rangle$ *be an* $\langle F, G \rangle$*-bialgebra, and* $\mathbf{B} = \langle B, \beta \rangle, \mathbf{C} = \langle C, \gamma \rangle$ *two subbialgebras of* $\mathbf{A}$*, such that* $B \cap C \neq \emptyset$*. Then the intersection* $B \cap C$ *admits a subbialgebra structure.*

Proof:

By assumption, if $i_B^A : B \hookrightarrow A$ and $i_C^A : C \hookrightarrow A$ are the inclusion maps, the following diagrams commute

$$\begin{array}{ccc} F(B) & \xrightarrow{F(i_B^A)} & F(A) \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\alpha} \\ G(B) & \xrightarrow[G(i_B^A)]{} & G(A) \end{array} \qquad \begin{array}{ccc} F(C) & \xrightarrow{F(i_C^A)} & F(A) \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\alpha} \\ G(C) & \xrightarrow[G(i_C^A)]{} & G(A) \end{array}$$

Since $B \cap C \neq \emptyset$, assume $d \in B \cap C$ and define $p : B \to B \cap C$ and $q : A \to C$ by

$$p(b) = \begin{cases} b, & \text{if } b \in B \cap C \\ d, & \text{otherwise} \end{cases} \quad \text{and} \quad q(a) = \begin{cases} a, & \text{if } a \in C \\ d, & \text{otherwise} \end{cases}$$

for all $b \in B, a \in A$. Then

$$q \circ i_C^A = i_C \quad \text{and} \quad i_{B\cap C}^C \circ p = q \circ i_B^A.$$

Now endow $B \cap C$ with a structure map $\delta : F(B \cap C) \to G(B \cap C)$, with

$$F(B \cap C) \xrightarrow{F(i_{B\cap C}^B)} F(B) \xrightarrow{\beta} G(B) \xrightarrow{G(p)} G(B \cap C)$$

$$\delta = G(p) \circ \beta \circ F(i_{B\cap C}^B).$$

It suffices to show that the following rectangle commutes

$$\begin{array}{ccc} F(B \cap C) & \xrightarrow{F(i_{B\cap C}^A)} & F(A) \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\alpha} \\ G(B \cap C) & \xrightarrow[G(i_{B\cap C}^A)]{} & G(A) \end{array}$$

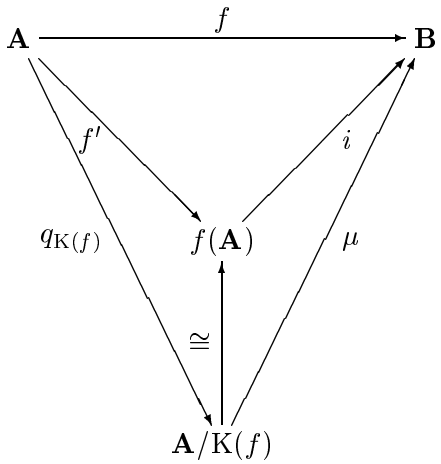
We have

$$\begin{aligned} G(i_{B\cap C}^A)\delta &= G(i_{B\cap C}^A)G(p)\beta F(i_{B\cap C}^B) \\ &= G(i_C^A i_{B\cap C}^C)G(p)\beta F(i_{B\cap C}^B) \\ &= G(i_C^A)G(i_{B\cap C}^C)G(p)\beta F(i_{B\cap C}^B) \\ &= G(i_C^A)G(q)G(i_B^A)\beta F(i_{B\cap C}^B) \\ &= G(i_C^A)G(q)\alpha F(i_B^A)F(i_{B\cap C}^B) \\ &= G(i_C^A)G(q)\alpha F(i_{B\cap C}^A) \\ &= G(i_C^A)G(q)\alpha F(i_C^A)F(i_{B\cap C}^C) \\ &= G(i_C^A)G(q)G(i_C^A)\gamma F(i_{B\cap C}^C) \\ &= G(i_C^A)G(i_C)\gamma F(i_{B\cap C}^C) \\ &= \alpha F(i_C^A)F(i_{B\cap C}^C) \\ &= \alpha F(i_{B\cap C}^A). \end{aligned}$$

■

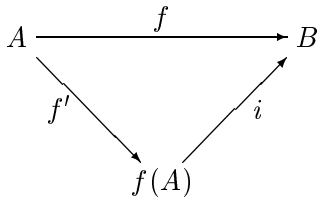
8 Isomorphism Theorems

Theorem 38 *Let* $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ *be* $\langle F, G\rangle$*-bialgebras and* $f : \mathbf{A} \to \mathbf{B}$ *a bialgebra homomorphism. If* F *preserves weak coequalizers and* G *preserves weak pullbacks, then there exists the following commutative diagram of homomorphisms*



where f'*, given by* $a \mapsto f(a)$*, is a surjection,* $i : f(A) \hookrightarrow B$ *is the inclusion,* $q_{\mathrm{K}(f)}$ *is the quotient map of the kernel* $\mathrm{K}(f)$ *and* μ *is an injection.*

Proof:
Clearly, in **Set** there exists a triangle



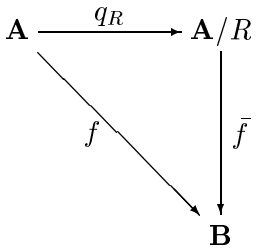
where $f' : A \to f(A); a \mapsto f(a)$. Since G preserves weak pullbacks, by Theorem 34, $f(A) \subseteq B$ admits a subbialgebra structure and $i : f(\mathbf{A}) \to \mathbf{B}$ is a homomorphism. Since i is injective and G preserves weak pullbacks i is G-injective. Hence, by Proposition 2, $f' : \mathbf{A} \to f(\mathbf{A})$ is a homomorphism. Thus, the upper triangle of the diagram in the statement of the Theorem commutes in $\mathbf{Set}^F_G$. Since G preserves weak pullbacks, $\mathrm{K}(f)$ is a bisimulation on $\mathbf{A}$ by Proposition 27. Thus, since F preserves weak coequalizers, by Proposition 28, $\mathbf{A}/\mathrm{K}(f)$ admits a bialgebra structure such that the following diagram

$$\mathrm{K}(f) \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \mathbf{A} \xrightarrow{q_R} \mathbf{A}/\mathrm{K}(f)$$

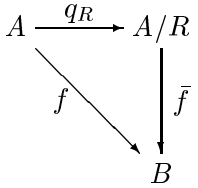
is a coequalizer in $\mathbf{Set}^F_G$. Now the existence of $\cong$: $\mathbf{A}/\mathrm{K}(f) \to f(\mathbf{A})$ and $\mu : \mathbf{A}/\mathrm{K}(f) \to \mathbf{B}$ follows by the universal mapping property of the coequalizer $\mathbf{A}/\mathrm{K}(f)$. Since $\cong$ is bijective, it is an isomorphism by Proposition 1. Hence $\mu = i \circ \cong$ is injective since i is. ■

Theorem 38 is the analog of Theorem II.6.12 of [5] and Theorem 7.1 of [23].

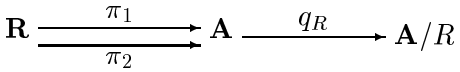
Theorem 39 *Let* $\mathbf{A} = \langle A, \alpha\rangle, \mathbf{B} = \langle B, \beta\rangle$ *be* $\langle F, G\rangle$*-bialgebras,* $f : \mathbf{A} \to \mathbf{B}$ *a homomorphism and* $R \subseteq A^2$ *a bisimulation such that* $R \subseteq \mathrm{K}(f)$*. If* F *preserves weak coequalizers, then there is a unique homomorphism* $\bar{f} : \mathbf{A}/R \to \mathbf{B}$*, such that the following diagram commutes*



Proof:
Since $R \subseteq \mathrm{K}(f)$, there exists in **Set** $\bar{f} : A/R \to B$, such that the following diagram commutes:



Since F preserves weak coequalizers, by Proposition 28, the following



is a coequalizer diagram in $\mathbf{Set}_G^F$. But, since $R \subseteq \mathrm{K}(f)$, $f \circ \pi_1 = f \circ \pi_2$. Thus, the existence of $\bar{f} : \mathbf{A}/R \to \mathbf{B}$ follows. ■

Theorem 39 provides the bialgebraic analog of Lemma II.6.14 of [5] and Theorem 7.2 of [23]. Theorem 40 below is the analog of Theorem II.6.18 of [5] and (modulo weak versus regular pullbacks (see also the remarks after the proof of Theorem 35)) of Theorem 7.3 of [23].

Theorem 40 *Let* $\mathbf{A} = \langle A, \alpha\rangle$ *be a bialgebra,* $\mathbf{B} = \langle B, \beta\rangle$ *a subbialgebra of* $\mathbf{A}$ *and* $R \subseteq A^2$ *a bisimulation equivalence on* $\mathbf{A}$*. Define* $B^R \subseteq A$ *by*

$$B^R = \{a \in A : \exists b \in B : \langle a, b\rangle \in R\}.$$

If F *preserves weak coequalizers and* G *preserves pullbacks, then*

1 $B^R \subseteq A$ *admits a subbialgebra structure.*

2 $Q = R \cap (B \times B)$ *is a bisimulation equivalence on* $\mathbf{B}$*.*

3 $\mathbf{B}/Q \cong \mathbf{B}^R/R$*.*

Proof:

1 Note that

$$\mathbf{A} \xleftarrow{\pi_1} \mathbf{R} \xrightarrow{\pi_2} \mathbf{A}$$

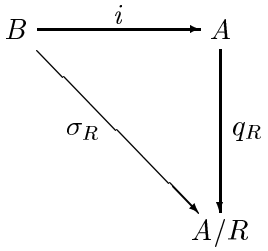
$B^R = \pi_1(\pi_2^{-1}(B))$. Thus, since G preserves weak pullbacks, by Theorem 34, $B^R \subseteq A$ admits a subbialgebra structure.

2 Since G preserves weak pullbacks, $\pi_1^{-1}(B) \subseteq R$ and $\pi_2^{-1}(B) \subseteq R$ admit both subbialgebra structures by Theorem 34. Thus, by Theorem 35, since G preserves pullbacks, $\pi_1^{-1}(B) \cap \pi_2^{-1}(B) = R \cap (B \times B) \subseteq R$ also admits a subbialgebra structure. Now, by Proposition 33, $R \cap (B \times B)$ is a bisimulation on $\mathbf{B}$ and is clearly an equivalence since R is.

3 Since F preserves weak coequalizers, by Proposition 28,

$$\mathbf{R} \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \mathbf{A} \xrightarrow{q_R} \mathbf{A}/R$$

is a coequalizer diagram in $\mathbf{Set}_G^F$. Consider the restriction $\sigma_R : B \to A/R$ of q_R to B in **Set**.



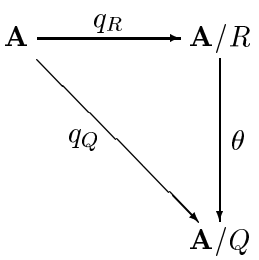
It is a homomorphism since $\sigma_R = q_R \circ i$, where $i : \mathbf{B} \to \mathbf{A}$ is the inclusion.

Now, since F preserves weak coequalizers, G preserves weak pullbacks, and since $\sigma_R(B) = q_r(i(B)) = q_R(B^R) = B^R/R$ and $\mathrm{K}(\sigma_R) = Q$, we obtain, by Theorem 38, $\mathbf{B}/Q \cong \mathbf{B}^R/R$.

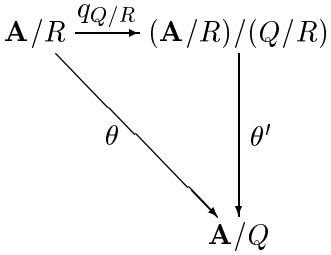
■

The following is the bialgebraic analog of Theorem II.6.15 of [5] and of Theorem 7.4 of [23]. (See also Lemma 4.13 of [11].)

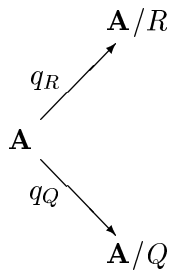
Theorem 41 *Let $\mathbf{A} = \langle A, \alpha \rangle$ be a bialgebra and $R, Q \subseteq A^2$ be bisimulation equivalences on $\mathbf{A}$, such that $R \subseteq Q$. If F preserves weak coequalizers and G preserves weak pullbacks, then there is a unique homomorphism $\theta : \mathbf{A}/R \to \mathbf{A}/Q$, such that $\theta \circ q_R = q_Q$*



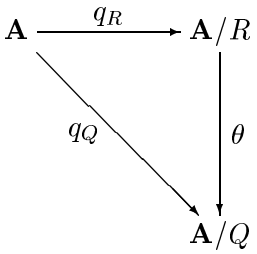
Moreover $Q/R = \mathrm{K}(\theta)$ *is a bisimulation equivalence on* $\mathbf{A}/R$ *and induces an isomorphism* $\theta' : (\mathbf{A}/R)/(Q/R) \to \mathbf{A}/Q$, *such that* $\theta = \theta' \circ q_{Q/R}$.



Proof:
Consider the diagram



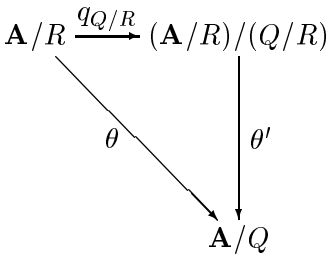
which exists, by Proposition 28, since F preserves weak coequalizers. By Theorem 39, since F preserves weak coequalizers and $R \subseteq Q = \mathrm{K}(q_Q)$, there exists a unique $\theta = \bar{q_Q} : \mathbf{A}/R \to \mathbf{A}/Q$, such that the following diagram commutes



Let $Q/R = \mathrm{K}(\theta)$. Since G preserves weak pullbacks, $\mathrm{K}(\theta)$ is a bisimulation equivalence on $\mathbf{A}/R$ by Theorem 27. Since F preserves weak coequalizers

$$\mathrm{K}(\theta) \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \mathbf{A}/R \xrightarrow{q_{Q/R}} (\mathbf{A}/R)/(Q/R)$$

is a coequalizer diagram, by Proposition 28. Thus, by the universal mapping property of the coequalizer $(\mathbf{A}/R)/(Q/R)$, there exists unique $\theta' : (\mathbf{A}/R)/(Q/R) \to \mathbf{A}/Q$, such that



Now note that θ' is a bijection and use Proposition 1. ■

9 Simple, Initial and Final Bialgebras

In this section, the definitions of the analogs of simple algebras and coalgebras, initial or free algebras and final or cofree coalgebras are given for bialgebras. These play a major role in the development of universal algebra and the theory of coalgebras. Only some very straightforward facts are mentioned about those special bialgebras here and not much is said about the reasons why they enjoy such a central role in the corresponding theories. The reader is encouraged to consult, e.g., [5] or [17] for motivation and many more results on free algebras and [21, 24, 25, 26] for motivation and more results on cofree coalgebras. It is not clear yet if and how these results may be obtained under the umbrella of universal bialgebra.

An $\langle F, G\rangle$-bialgebra is called **simple** if it has no proper homomorphic images, i.e., if every surjection $f : \mathbf{A} \to \mathbf{B}$ is an isomorphism.

Theorem 42 *Let $\mathbf{A} = \langle A, \alpha\rangle$ be an $\langle F, G\rangle$-bialgebra. If F preserves weak coequalizers and G preserves weak pullbacks, then $\mathbf{A}$ is simple iff Δ_A is the only bisimulation equivalence on $\mathbf{A}$.*

Proof:

Suppose $\mathbf{A}$ is simple and let $R \subseteq A \times A$ be a bisimulation equivalence on $\mathbf{A}$. Since F preserves weak coequalizers, by Proposition 28, there is a bialgebra structure on $\mathbf{A}/R$, such that

$$\mathbf{R} \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \mathbf{A} \xrightarrow{q_R} \mathbf{A}/R$$

is a commuting diagram in $\mathbf{Set}^F_G$. Thus, q_R is an isomorphism and therefore $R = \Delta_A$.

Conversely, suppose that the only bisimulation equivalence on $\mathbf{A}$ is Δ_A and let $f : \mathbf{A} \to \mathbf{B}$ be a surjective homomorphism. Since G preserves weak pullbacks, by Proposition 27, $\mathrm{K}(f)$ is a bisimulation equivalence on $\mathbf{A}$. Thus, $\mathrm{K}(f) = \Delta_A$ and f is an isomorphism. ■

An $\langle F, G\rangle$-bialgebra $\mathbf{I} = \langle I, \iota\rangle$ is said to be **initial** if it is initial in the category $\mathbf{Set}^F_G$, i.e., if, for every bialgebra $\mathbf{A} = \langle A, \alpha\rangle$, there exists a unique homomorphism $f : \mathbf{I} \to \mathbf{A}$.

Dually, an $\langle F, G\rangle$-bialgebra $\mathbf{T} = \langle T, \tau\rangle$ is said to be **final** or **termimal** if it is final in the category $\mathbf{Set}^F_G$, i.e., if, for every bialgebra $\mathbf{A} = \langle A, \alpha\rangle$, there exists a unique homomorphism $g : \mathbf{A} \to \mathbf{T}$.

The following is easy to see.

Proposition 43 *If F preserves the initial object $\emptyset$ of* **Set**, *then $\emptyset = \langle \emptyset, \emptyset\rangle$ is initial.*

If G preserves the final object $\{\emptyset\}$ of **Set**, *then $\mathbf{1} = \langle \{\emptyset\}, 1_{F(\{\emptyset\})}\rangle$, where $1_{F(\{\emptyset\})} : F(\{\emptyset\}) \to \{\emptyset\}$ is the unique terminal map, is final.*

10 Comparing Categories of Bialgebras

Let $F, G, M, N :$ **Set** $\to$ **Set** be endofunctors on **Set**. Consider two natural transformations $\mu : M \to F$ and $\nu : G \to N$. These are collections of mappings $\mu_X : M(X) \to F(X)$ and

$\nu_X : G(X) \to N(X)$, for all sets X, respectively, that make the following diagrams commute, for all mappings $f : X \to Y$ in **Set**.

$$\begin{array}{ccc} M(X) & \xrightarrow{\mu_X} & F(X) \\ {\scriptstyle M(f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ M(Y) & \xrightarrow[\mu_Y]{} & F(Y) \end{array} \qquad \begin{array}{ccc} G(X) & \xrightarrow{\nu_X} & N(X) \\ {\scriptstyle G(f)}\downarrow & & \downarrow{\scriptstyle N(f)} \\ G(Y) & \xrightarrow[\nu_Y]{} & N(Y) \end{array}$$

Suppose $\mathbf{A} = \langle A, \alpha \rangle$ is an $\langle F, G \rangle$-bialgebra. Then

$$M(A) \xrightarrow{\mu_A} F(A) \xrightarrow{\alpha} G(A) \xrightarrow{\nu_A} N(A)$$

$\mathbf{A}^{\mu}_{\nu} = \langle A, \nu_A \alpha \mu_A \rangle$ is an $\langle M, N \rangle$-bialgebra. Moreover, given two $\langle F, G \rangle$-bialgebras $\mathbf{A} = \langle A, \alpha \rangle$ and $\mathbf{B} = \langle B, \beta \rangle$ and a homomorphism $h : \mathbf{A} \to \mathbf{B}$,

$$\begin{array}{ccc} M(A) & \xrightarrow{M(h)} & M(B) \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle \mu_B} \\ F(A) & \xrightarrow[F(h)]{} & F(B) \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ G(A) & \xrightarrow[G(h)]{} & G(B) \\ {\scriptstyle \nu_A}\downarrow & & \downarrow{\scriptstyle \nu_B} \\ N(A) & \xrightarrow[N(h)]{} & N(B) \end{array}$$

h is also an $\langle M, N \rangle$-bialgebra homomorphism $h : \mathbf{A}^{\mu}_{\nu} \to \mathbf{B}^{\mu}_{\nu}$.

If $R \subseteq A \times B$ is an $\langle F, G \rangle$-bisimulation with structure map $\rho : F(R) \to G(R)$, then $R \subseteq A \times B$ is also an $\langle M, N \rangle$-bisimulation with structure map $\nu_R \rho \mu_R : M(R) \to N(R)$:

$$\begin{array}{ccccc} M(A) & \xleftarrow{M(\pi_1)} & M(R) & \xrightarrow{M(\pi_2)} & M(B) \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle \mu_R} & & \downarrow{\scriptstyle \mu_B} \\ F(A) & \xleftarrow[F(\pi_1)]{} & F(R) & \xrightarrow[F(\pi_2)]{} & F(B) \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \beta} \\ G(A) & \xleftarrow[G(\pi_1)]{} & G(R) & \xrightarrow[G(\pi_2)]{} & F(B) \\ {\scriptstyle \nu_A}\downarrow & & \downarrow{\scriptstyle \nu_R} & & \downarrow{\scriptstyle \nu_B} \\ N(A) & \xleftarrow[N(\pi_1)]{} & N(R) & \xrightarrow[N(\pi_2)]{} & N(B) \end{array}$$

The following theorem has been established (See [23], Theorem 14.1.)

Theorem 44 *Let $F, G, M, N : \mathbf{Set} \to \mathbf{Set}$ be endomorphisms on $\mathbf{Set}$ and $\mu : M \to F, \nu : G \to N$ be natural transformations. Then μ, ν induce a functor $\nu \circ (-) \circ \mu : \mathbf{Set}_G^F \to \mathbf{Set}_N^M$ which maps an $\langle F, G\rangle$-bialgebra $\mathbf{A} = \langle A, \alpha\rangle$ to the $\langle M, N\rangle$-bialgebra $\mathbf{A}_\nu^\mu = \langle A, \nu_A \alpha \mu_A\rangle$ and an $\langle F, G\rangle$-bialgebra homomorphism $h : \mathbf{A} \to \mathbf{B}$ to the $\langle M, N\rangle$-bialgebra homomorphism $h : \mathbf{A}_\nu^\mu \to \mathbf{B}_\nu^\mu$. Moreover, $\nu \circ (-) \circ \mu$ preserves bisimulations, i.e., if $R \subseteq A \times B$ is an $\langle F, G\rangle$-bisimulation between $\mathbf{A}$ and $\mathbf{B}$, then $R \subseteq A \times B$ is also an $\langle M, N\rangle$-bisimulation between $\mathbf{A}_\nu^\mu$ and $\mathbf{B}_\nu^\mu$.*

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