Probabilistic Threshold Agent Networks

Introducing Randomization in Threshold Agent Networks*

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Abstract

Threshold agent networks (TANs) were introduced in [11] and constitute a finitary modification of threshold (neural) networks appropriate for modeling computer simulations. In this paper a generalization of TANs, that was proposed in [10], is introduced and some of its properties explored. The new model, called *probabilistic threshold agent network* (PTAN) is not a finite dynamical system in the classical sense [6], since succession of states is not deterministic but rather probabilistic. We show how known finite dynamical systems may be represented as special cases of this new class of models and present arguments to the effect that, in many applications, PTANs are more realistic than finite dynamical systems.

1 Introduction

Discrete computer models modelling a wide variety of physical systems and physical phenomena have been used repeatedly in the literature. Among these, cellular automata and threshold networks or neural networks have played a prominent role. The books [3, 5, 13] are excellent sources of information regarding these systems.

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Several of the applications for which these models have been used are listed in these references and many more references may be found especially in [5] and [13].

However, despite the use of all these models for computer simulations of several physical phenomena, there has not been a well-developed coherent theory of computer simulations. A recent attempt at this has been made in [1, 2], where sequential dynamical systems (SDSs) were introduced for this purpose. An alternative approach, using a finite discrete version of threshold networks, termed threshold agent networks (TANs), was initiated in [11]. In [12], the study of TANs has been continued with the introduction of morphisms between TANs and the investigation of the categorical properties that the category of TANs, thus formed, possesses. Further expanding the study of the relations between these finite dynamical systems, in [8] functors connecting the category of generalized sequential dynamical systems, a generalization of SDSs introduced in [7], and the category of TANs have been defined and some of their properties investigated.

In this paper, a generalization of TANs to a probabilistic version, the probabilistic TAN (PTAN), that was proposed in [10], is introduced and some of its properties and several applications are explored. This version is more suitable for applications where actions do not trigger reactions with certainty but rather probabilistically. This is done, for instance, in most physical systems. Some systems from biology, physics and business are proposed as possible candidates for applications of this probabilistic model. More precisely, in Section 2, the new probabilistic model is introduced and the way it generates a discrete dynamics is discussed. An example from biology is presented. In Section 3, a theorem is proved that characterizes the class of TANs as a subclass of PTANs. Section 4 describes how finitary versions of cellular automata and neural networks may be seen as PTANs. In Section 5 two further applications of PTANs are proposed. Generalized PTANs (GPTANs) allow for a more flexible interactive structure between the agents and time-dependent PTANs (TDPTANs) constitute a stochastic version in which the probability distributions are allowed to vary with time. Section 6 studies in some detail three special cases of PTANs which are guaranteed to eventually reach a fixed-point, i.e., a one-element limit cycle. Finally, Sections 7, 8 and 9 contain three applications of PTANs to specific disciplines. Section 7 contains an application in modelling of a very simple protocol of coordinated communication between agents. Section 8 contains a numerical simulation of a conduction heat transfer phenomenon. The results are successfully compared to analytical results obtained through a solution of the heat equation. Finally, Section 9, addressing future work, discusses briefly how a model could potentially be set up to simulate the success of an advertising campaign that has been set to achieve a specific goal. The reader is advised to view the examples presented as indicatory of the potential power of the proposed model in different application areas and not as innovative contributions per se. As such, the weight has been placed on their simplicity rather than on their realism.

2 The Probabilistic Model

Denote by k the two element set $\{0,1\}$, by $\mathbf{n}$ the set $\{1,2,\ldots,n\}$ and by $\mathbb{I}$ the closed unit interval $[0,1]$. In what follows, quite often, a subset of $\mathbf{n}$ will be represented interchangeably by its characteristic function and vice-versa.

A *probabilistic threshold agent network* (PTAN) [10] consists of a finite collection $P = \{p_i\}_{1\le i\le n}$ of functions

$$p_i : k^{\mathbf{n}} \to \mathbb{I}, \quad 1 \le i \le n.$$

The indices $1,2,\ldots,n$ are sometimes referred to as the *agents* of the network. The local dynamics of such a model is determined by a collection $\{f_i\}_{1\le i\le n}$ of $0-1$ random variables on the sample space $k^{\mathbf{n}}$, where

$$f_i(x) = \begin{cases} 1, & \text{with probability } p_i(x) \\ 0, & \text{with probability } 1-p_i(x) \end{cases}, \quad \text{for all } x \in k^{\mathbf{n}}, \quad 1 \le i \le n.$$

The global dynamics is then given by

$$f(x) = \langle f_1(x),\ldots,f_n(x)\rangle, \quad \text{for all} \quad x \in k^{\mathbf{n}}.$$

As an example consider the following PTAN on 7 agents $1,2,\ldots,7$. The agents $1,\ldots,6$ are meant to represent parts of the biological system of an organism and agent 7 represents an external source that stimulates the organism. Agents 1 and 2 are supposed to represent two sensors of the organism. They are sensing the stimulant 7 and give directions to their "commander" cell 3. 3, in turn, forwards the stimulation to the primitive brain 4, which sends further directions to the nerves 5 and 6 on how to handle the stimulant 7. All these actions create a biological cycle that is probabilistic in nature in the sense that, depending on the health condition and robustness of the organism and the external environmental conditions for both the organism and the stimulant, the diverse interactions are not taking place deterministically, but rather probabilistically. In the model below, it is assumed that each is taking place with a predetermined probability which stays fixed during the occurrence of this event. A sketch that illustrates the interaction is shown in Figure 1. A formal PTAN $P = \{p_i\}_{1\le i\le 7}$ that could represent this interaction is described by the 7 functions

$$p_i : k^{\mathbf{7}} \to \mathbb{I}, \quad 1 \le i \le 7,$$

given by

$$p_1(X) = \begin{cases} 0.6, & \text{if } 7 \in X \\ 0, & \text{otherwise} \end{cases}, \quad p_2(X) = \begin{cases} 0.4, & \text{if } 7 \in X \\ 0, & \text{otherwise} \end{cases}$$

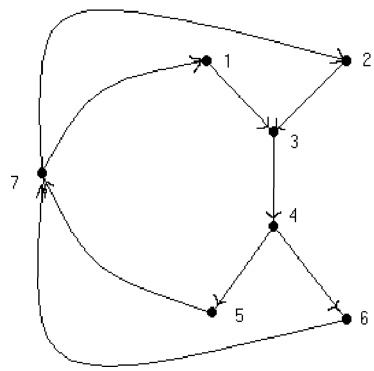

Figure 1: The organism on the right interacts with the external stimulant on the left.

$$p_3(X) = \begin{cases} 0.8, & \text{if } \{1,2\} \subseteq X \\ 0.5, & \text{if either } 1 \in X \text{ or } 2 \in X \\ 0, & \text{otherwise} \end{cases}, \quad p_4(X) = \begin{cases} 0.5, & \text{if } 3 \in X \\ 0, & \text{otherwise} \end{cases},$$

$$p_5(X) = \begin{cases} 0.6, & \text{if } 4 \in X \\ 0, & \text{otherwise} \end{cases}, \quad p_6(X) = \begin{cases} 0.4, & \text{if } 4 \in X \\ 0, & \text{otherwise} \end{cases},$$

$$p_7(X) = \begin{cases} 0.6, & \text{if } \{5,6\} \subseteq X \\ 0.5, & \text{if either } 5 \in X \text{ or } 6 \in X \\ 0, & \text{otherwise} \end{cases}.$$

As an illustration consider the values of p_3. We have, for instance $p_3(\{1,2,5,7\}) = 0.8$ and $p_3(\{1,2\}) = 0.8$. Similarly, $p_3(\{1,4,6,7\}) = 0.5$, $p_3(\{1\}) = 0.5$ and $p_3(\{3,4,5,6,7\}) = 0$.

3 TANs and PTANs

A *threshold agent network* (TAN), introduced in [11], consists of a collection $A = \{A_1, \ldots, A_n\}$ of *agents*, where each agent A_i is formally an ordered pair $A_i = \langle k_i, P_i \rangle$, where k_i is an integer and $P_i \subseteq \{1, \ldots, n\}$. $k_i, 1 \leq i \leq n$, is the *threshold* of agent i and P_i is his *output set*. The dynamics of the TAN is generated by stipulating that agent i be active at time j if at least k_i agents that have i in their output sets are active at time $j-1$. Note that if k_i is negative, then agent i will always be active at time j except if at least $-k_i$ agents that have i in their output sets are active at time $j-1$. $P_i, 1 \leq i \leq n$, is the set of agents that will be affected by agent i at the end of each time step in case agent i's threshold has been attained at the end of the previous time step. This dynamical behavior is formally expressed by the function $h^A : k^{\mathbf{n}} \to k^{\mathbf{n}}$ defined as follows: First, given a condition c that an n-tuple $\langle x_1, \ldots, x_n \rangle \in k^{\mathbf{n}}$ may or may not satisfy, let $\chi_c : k^{\mathbf{n}} \to k$ be the characteristic function of c, i.e.,

$$\chi_c(\langle x_1, \ldots, x_n \rangle) = \begin{cases} 1, & \text{if } \langle x_1, \ldots, x_n \rangle \text{ satisfies } c \\ 0, & \text{otherwise} \end{cases}.$$

Then define the functions $h_i^A : k^{\mathbf{n}} \to k, 1 \leq i \leq n$, by

$$h_i^A(x) = \begin{cases} \chi_{|\{j : x_j = 1 \text{ and } i \in P_j\}| \geq k_i}(x), & \text{if } k_i \geq 0 \\ \chi_{|\{j : x_j = 1 \text{ and } i \in P_j\}| < k_i}(x), & \text{if } k_i < 0 \end{cases}, \quad \text{for all } x \in k^{\mathbf{n}}.$$

Finally, set

$$h^A(x) = \langle h_1^A(x), \ldots, h_n^A(x) \rangle, \quad \text{for all } x \in k^{\mathbf{n}}.$$

h^A is called the *dynamics* of the TAN A.

TANs are special cases of PTANs. More specifically, the following theorem provides a characterization of TANs inside the class of PTANs. In order to facilitate the formulation of the theorem, the following terminological convention is introduced.

A function $f : k^{\mathbf{n}} \to \mathbb{I}$ is called *levelled* if it is a 0-1 monotone or antimonotone function, such that all minimal or maximal, respectively, elements mapped to 1 have the same number of 1's. Functions of the first kind are called *positively levelled* and functions of the second kind *negatively levelled*.

For instance the function $f : k^{\mathbf{3}} \to \mathbb{I}$, given by the table below is levelled, whereas $g : k^{\mathbf{3}} \to \mathbb{I}$, given by the same table, is not.

X	$f(X)$	$g(X)$
$\emptyset$	0	0
$\{1\}$	1	1
$\{2\}$	0	0
$\{3\}$	0	0
$\{1,2\}$	1	1
$\{1,3\}$	1	1
$\{2,3\}$	0	1
$\{1,2,3\}$	1	1

An illustration of these two functions is given in Figure 2.

Theorem 1 *Let $P = \{p_i\}_{1 \leq i \leq n}$ be a PTAN. P is a TAN if and only if, for all $1 \leq i \leq n$, p_i is levelled.*

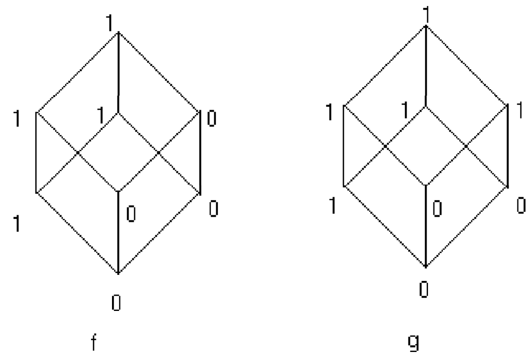

Figure 2: The levelled function f and the non-levelled function g.

Another interesting class of models that has been considered in the literature of finite dynamical systems is that of sequential dynamical systems (SDSs for short). The definition is recalled below and then, based on a theorem of [11] and Theorem 1, a simulation result is obtained, relating SDSs with PTANs.

Let $G = \langle V, E\rangle$ be a simple graph with vertex set $V = \{v_1, \ldots, v_n\}$. For each $i \in V$, suppose that we are given a function $F_i : k^{\mathbf{n}} \to k^{\mathbf{n}}$, that only changes the value of the i-th position and only depends on the i-th position and those positions j, such that $(i, j) \in E$. The F_i's are referred to as the *local update functions*. Now let $\pi \in S_n$ be a permutation of V. π is called an *update schedule*. The functions F_i are composed in the order prescribed by π to obtain the function $F(G, \pi) = F_{\pi(n)} \circ F_{\pi(n-1)} \circ \ldots \circ F_{\pi(1)} : k^{\mathbf{n}} \to k^{\mathbf{n}}$. We call the function $F(G, \pi)$ the *sequential dynamical system* (SDS) determined by G, the local update functions F_i and the update schedule $\pi \in S_n$.

In [11] the following was proved about the possibility of simulating SDSs with TANs.

Theorem 2 *For every SDS $F(G, \pi)$ of dimension n, there exist a TAN with set of agents $A = \{A_i\}_{i\in I}$, a subset $J \subset I$ and a positive integer d, such that, for every initial condition x^0 for F, there exists an initial condition y^0 for A, such that*

$$\langle\langle y_j^{ds} : j \in J\rangle, s \in \omega\rangle = \langle x^s, s \in \omega\rangle.$$

This, combined with Theorem 1, gives the following result about the possibility of simulating SDSs with PTANs.

Corollary 3 *For every SDS $F(G,\pi)$ of dimension n, there exist a PTAN $P = \{p_i\}_{i\in I}$, a subset $J \subset I$ and a positive integer d, such that, for every initial condition x^0 for F, there exists an initial condition y^0 for P, such that*

$$\langle\langle y_j^{ds} : j \in J\rangle, s \in \omega\rangle = \langle x^s, s \in \omega\rangle.$$

4 CAs and Neural Networks as PTANs

Two of the best known models that have been used in the past for modelling physical systems are *cellular automata* [13, 3, 5] and *threshold* or *neural networks* [4, 5]. Both models are potentially infinite. Here, a finite modification will be described and a brief account on how these finite versions of the original models may be modelled as PTANs will be given.

Let I be an index set (finite for our purposes). A *finite automata network* defined on I is a triple $\mathcal{A} = \langle G, Q, (g_i)_{i\in I}\rangle$, such that

- $G = \langle I, E\rangle$ is a digraph with vertex set I,
- Q is a finite set of states ($Q = \{0, 1\}$ for our purposes) and
- $g_i : Q^{|E_i|} \to Q$, for all $i \in I$, where $E_i = \{j \in I : (j,i) \in E\}$. The g_i's are the *local update functions.*

It is assumed that synchronous updating is used for obtaining the global update function $G : Q^I \to Q^I$ from the local ones.

A PTAN $P = \{p_i\}_{i\in I}$ that captures this finite automata network is defined by

$$p_i(x) = \begin{cases} 1, & \text{if } g_i(x \upharpoonright_{E_i}) = 1 \\ 0, & \text{otherwise} \end{cases}, \quad \text{for all } x \in Q^I.$$

A *finite cellular space* consists of a regular graph $G = \langle I, E\rangle$ together with a transitive group H of automorphisms of G. *Finite cellular automata* are finite automata networks that are defined on finite cellular spaces, all of whose local update functions are invariant under the automorphisms in H. Therefore, a finite cellular automaton is a special case of a finite automata network and, thus, it can be modelled as a PTAN exactly as a finite automata network was modelled above.

Finite threshold or *neural networks* form a subclass of finite automata networks as well. The graph $G = \langle I, E\rangle$ possesses weighted structures $t : I \to \mathbb{R}$ giving, for each vertex $i \in I$, its *threshold* $t(i)$, and $w : E \to \mathbb{R}$, giving, for each edge (i, j), its *weight* $w(i, j)$. Setting $w(i, j) = 0$, whenever $(i, j) \notin E$, we may view w as a function

$w : I \times I \to \mathbb{R}$. The local update functions are then defined with the help of these weighted structures by

$$g_i(x_j : j \in E_i) = T(\sum_{j \in E_i} w(j,i)x_j - t(i)), \quad \text{for all } x = \langle x_i : i \in I \rangle,$$

where the function $T : \mathbb{R} \to \{0,1\}$ is given by

$$T(u) = \begin{cases} 1, & \text{if } u \geq 0 \\ 0, & \text{if } u < 0 \end{cases}.$$

The finite neural network $\mathcal{N} = \langle I, E, w, t \rangle$, being a finite automata network, may also be modelled as a PTAN as before. Thus PTANs are significantly more general than many natural finite versions of several well-known models that one may want to consider as appropriate for simulating natural phenomena or for other applications.

5 0-, 1- and 0-1-PTANs

In this section, three special classes of PTANs are studied in more detail. One is the class of all these networks all of whose probability functions take values in the semi-closed interval $[0,1)$ and have the zero vector as a fixed point. It is shown that systems of this kind will end up at the zero vector state with probability 1 as time goes to infinity. The second class consists of all these networks all of whose probability functions take values in the semi-closed interval $(0,1]$ and have the all one vector as a fixed point. Systems of these kind will be shown to end up at the all one vector with probability 1 as time goes to infinity. Finally, the third class is the class of all networks all of whose probability functions are in the open interval $(0,1)$ except that they have both the zero and the all one vector as fixed points. These systems will end up either at the zero or at the all one vector with probability 1 as time goes to infinity. The formal definitions of these classes of systems together with the theorems predicting their asymptotic behavior will now be given.

A PTAN $P = \{p_i\}_{1 \leq i \leq n}$ is said to be a *zero PTAN*, written 0-PTAN, if, for all $1 \leq i \leq n$ and all $x \in k^{\mathbf{n}}$,

$$p_i(x) < 1 \quad \text{and} \quad p_i(\langle 0, \ldots, 0 \rangle) = 0.$$

Theorem 4 *Let $P = \{p_i\}_{1 \leq i \leq n}$ be a 0-PTAN. Then, for all initial vector conditions $p^0 \in k^{\mathbf{n}}$, there exists a stochastic $N \geq 0$, such that*

$$p_i^j = 0, \quad \textit{for all } 1 \leq i \leq n, j \geq N.$$

Proof:

Since, for all $1 \leq i \leq n, x \in k^{\mathbf{n}}, p_i(x) < 1$, there exists a maximum $p < 1$, such that $p_i(x) \leq p$, for all $1 \leq i \leq n$, $x \in k^{\mathbf{n}}$. Thus, from every state, the probability that, in the next time step, the system will be at the zero state is at least $q^n > 0$, where $q = 1 - p$. Therefore the system will eventually pass from the zero state with probability 1. But since the zero state is a fixed point, the system will end up at the zero state with probability 1. ■

A PTAN $P = \{p_i\}_{1 \leq i \leq n}$ is said to be a *one PTAN*, written 1-PTAN, if, for all $1 \leq i \leq n$ and all $x \in k^{\mathbf{n}}$,

$$p_i(x) > 0 \quad \text{and} \quad p_i(\langle 1, \ldots, 1 \rangle) = 1.$$

Theorem 5 *Let $P = \{p_i\}_{1 \leq i \leq n}$ be a 1-PTAN. Then, for all initial vector conditions $p^0 \in k^{\mathbf{n}}$, there exists a stochastic $N \geq 0$, such that*

$$p_i^j = 1, \quad \textit{for all } 1 \leq i \leq n, j \geq N.$$

The proof of Theorem 5 is very similar to the proof of Theorem 4 and will, therefore, be omitted.

Finally, a PTAN $P = \{p_i\}_{1 \leq i \leq n}$ is said to be a *zero-one PTAN*, written 0-1-PTAN, if, for all $1 \leq i \leq n$ and all $x \in k^{\mathbf{n}} - \{\langle 0, \ldots, 0 \rangle, \langle 1, \ldots, 1 \rangle\}$,

$$0 < p_i(x) < 1, \quad p_i(\langle 0, \ldots, 0 \rangle) = 0 \quad \text{and} \quad p_i(\langle 1, \ldots, 1 \rangle) = 1.$$

Theorem 6 *Let $P = \{p_i\}_{1 \leq i \leq n}$ be a 0-1-PTAN. Then, for all initial vector conditions $p^0 \in k^{\mathbf{n}}$, there exists a stochastic $N \geq 0$, such that ($p_i^j = 0$, for all $1 \leq i \leq n$ and all $j \geq N$) or ($p_i^j = 1$, for all $1 \leq i \leq n$ and all $j \geq N$.)*

6 Coordinating Access to a Resource

A simple but interesting application of a special case of PTANs is their use to model a protocol for coordinating communication of a collection of agents with a central processing unit with limited input capability or for coordinating requests of agents to access a single resource with limited output capability. More precisely, we are dealing with the following setting which is illustrated in Figure 3. Agent 1, which is a central processing unit, is supposed to collect information for processing from a collection of peripheral sensor agents $2, 3, \ldots, N$. The communication channels through which this information is to be transmitted are the solid lines in Figure 3. However, the central processing unit has a limited input capability, i.e., it can only receive input from a limited number of agents at the same time step. For simplicity, in this example it

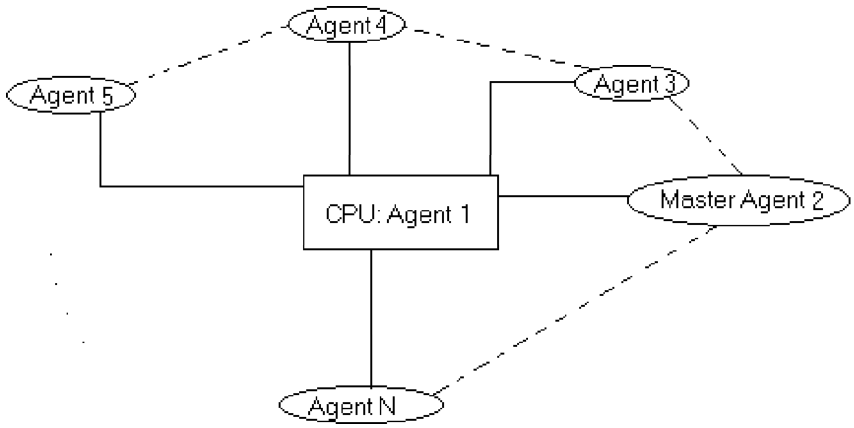

Figure 3: Agents competing to send information to a central processing unit.

is assumed that it can only receive input from one of the other agents at a time. A protocol, thus, has to be established so that not more than one agents try to contact the central unit at each time step. The implementation of this protocol is to be achieved by the use of both the transmission and some additional, communication lines that connect the sensor agents cyclically and are depicted as dashed lines in Figure 3. Here, the following protocol is implemented. When the central processing unit is started, many sensor agents may be trying to access it simultaneously. The unit is detecting conflicting requests and sends a signal out to all agents. A signal received by the central unit causes all but a distinguished agent, termed the master agent and taken to be agent 2 in Figure 3, to stop transmission. All the sensor agents then wait for their predecessor in the circle to send its data to the central unit and then transmit their own data.

The PTAN that implements this configuration, together with the communication protocol, is a very simple one in the sense that all probability functions $p_i : k^{\mathbf{N}} \rightarrow \mathbb{I}, 1 \leq i \leq N$, are binary valued functions $p_i : k^{\mathbf{N}} \rightarrow k$. However, it is not a TAN because the lattice-theoretic condition of Theorem 1 is not satisfied. It consists of N agents $1, 2, \ldots, N$. Agent 1 is the central processing unit and agent 2 is the master sensor agent. The probability functions $p_i, 1 \leq i \leq N$, are given by

$$p_1(x) = \begin{cases} 0, & \text{if } \sum_{i=2}^{N} x_i < 2 \text{ or } x_1 = 1 \\ 1, & \text{otherwise} \end{cases},$$

$$p_2(x) = \begin{cases} 1, & \text{if } x_1 = 1 \text{ or } (x_1 = 0 \text{ and } x_N = 1) \\ 0, & \text{otherwise} \end{cases}$$

and, for $3 \leq i \leq N$,

$$p_i(x) = \begin{cases} 1, & \text{if } x_1 = 0 \text{ and } x_{i-1} = 1 \\ 0, & \text{otherwise} \end{cases} .$$

The following table gives the output of a run of this model with $N = 7$ and initial state 0110101 for 8 time steps.

Time Step	1	2	3	4	5	6	7
0	0	1	1	0	1	0	1
1	1	1	1	1	0	1	0
2	0	1	0	0	0	0	0
3	0	0	1	0	0	0	0
4	0	0	0	1	0	0	0
5	0	0	0	0	1	0	0
6	0	0	0	0	0	1	0
7	0	0	0	0	0	0	1
8	1	0	0	0	0	0	0
⋮	⋮						

7 Diffusion of Energy

Another very interesting application of PTANs is their use in simulating physical systems. In this section it will be shown how a PTAN may be used to simulate diffusion of energy or diffusion of a substance through a medium.

The physical setting consists of a thin square layer of material of dimension $n \times n$ whose left hand side has been heated and whose right hand side is cold. The square is shown in Figure 4, where the heated side has been shaded. The square is then insulated so that heat cannot escape into the environment and the diffusion of heat from its left hand side to the entire surface is observed.

The PTAN that models this diffusion phenomenon is constructed as follows. Assume that the given square is of dimensions 10×10 and that one element per unit is sufficient to model the material contained per square unit. Thus a 10×10 network of agents will be used to model heat conduction. The collection of these agents together with the way they interact with each other is shown in Figure 5. The probability that an agent is active at a certain time step is exactly proportional to the number of agents in its neighborhood (including himself) that were active during the previous time step. This reflects the fact that an element of the material will be hot at a

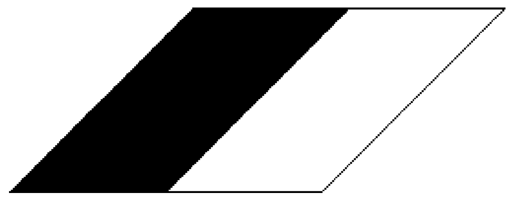

Figure 4: The half-heated square.

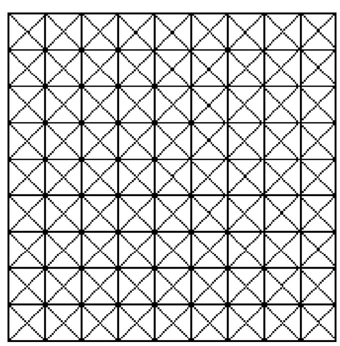

Figure 5: The grid of 100 agents used in the PTAN modelling heat diffusion.

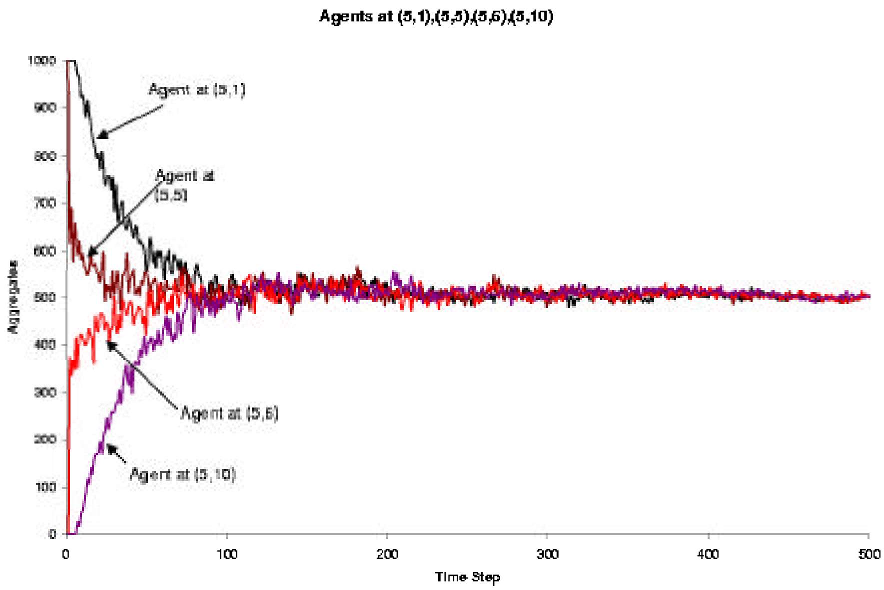

Figure 6: The agent energy in terms of time for four of the agents in the first experiment.

certain time step depending on how hot it was at the previous time step and how hot the other elements next to it had been during the previous time step. To model the initial condition the left half of the agents are set to 1 and the right half is set to 0 at the beginning of the simulation. Then the model is left to run for 1000 time steps for a total of 1000 times and the states of the agents at corresponding positions and time steps are added over all 1000 random runs. The sum over 1000 shows how likely it is that the specific element will be hot or cold at that time step. The diagram in Figure 6 show the aggregates obtained for a horizontal cross section of the agents over time starting from the agent at the 5th row and 1st column and ending at the agent at the 5th row and 10th column. Figure 7 shows the spatial distribution of heat over specific time steps.

These diagrams show that the PTAN approximation of the transient heat model is very close to the results obtained by analytically solving the heat equation (see, e.g., [9], Section 2.4)

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, \quad 0 \leq x \leq 1, \quad 0 < t,$$

that describes the heat flow subject to the boundary value conditions

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(1,t) = 0, \quad 0 < t,$$

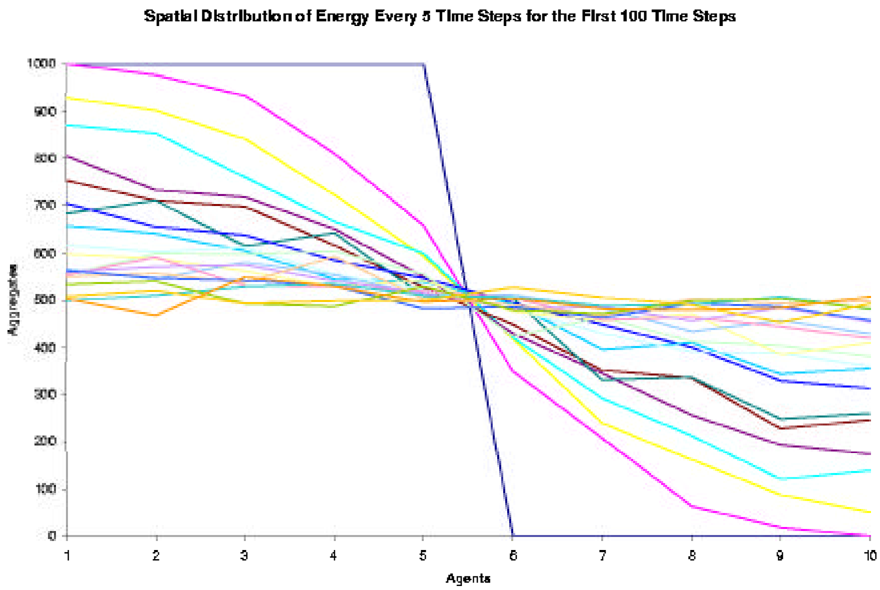

Figure 7: The distribution of energy along the square every 5 time steps until time step 100.

and the initial value condition

$$u(x,0) = f(x) = \begin{cases} 1, & \text{if } 0 < x < \frac{1}{2} \\ 0, & \text{if } \frac{1}{2} < x < 1 \end{cases} .$$

Here, it is assumed that heat transfer is taking place uniformly across a cross section of the material flowing from the heated to the colder side. So $u(x,t)$ is the heat function in terms of the distance x from the heated edge and the time t. k is the diffusivity constant. The boundary value conditions assure that the material is insulated so that energy is preserved during the experiment and the initial value condition makes explicit the original heating configuration.

This boundary value-initial value problem may be solved analytically to obtain the explicit solution

$$u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin \frac{n\pi}{2} \cos(n\pi x) e^{-(n\pi)^2 kt}, \quad 0 \leq x \leq 1, \quad 0 < t.$$

A graph for an approximation of this solution for $x = 0$ in terms of time is given in Figure 8 and for $x = 1$ in terms of time in Figure 9. A graph for an approximation of the solution for $t = 0.1$ in terms of x is given in Figure 10.

The experiment is then repeated over the initial condition depicted in Figure 11. The plots depicting the energy in terms of time for the agents in positions 1,4,5 and

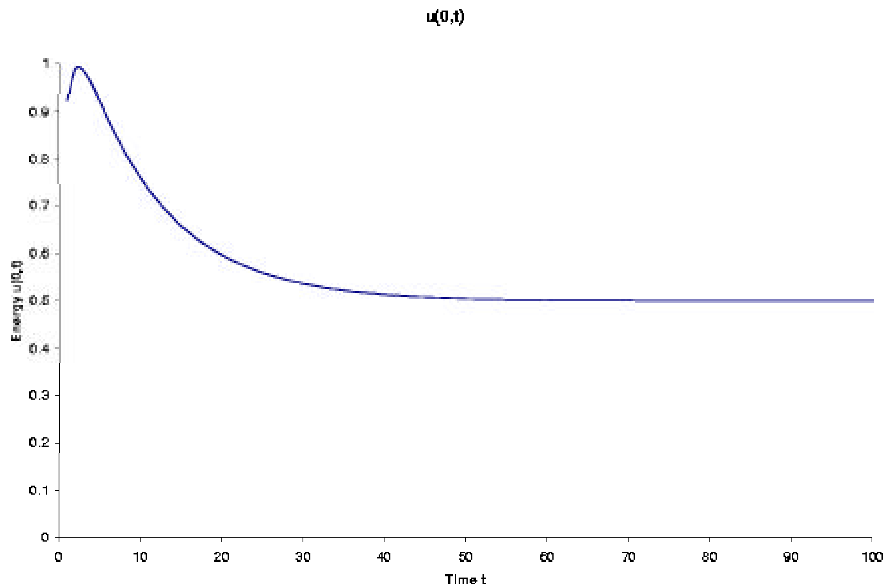

Figure 8: Approximating the analytical solution $u(0, t)$.

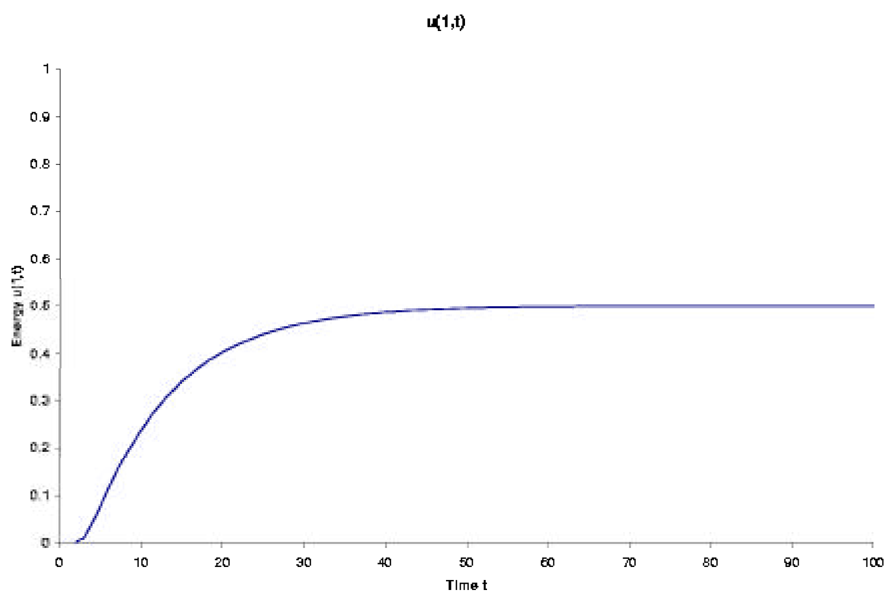

Figure 9: Approximating the analytical solution $u(1, t)$.

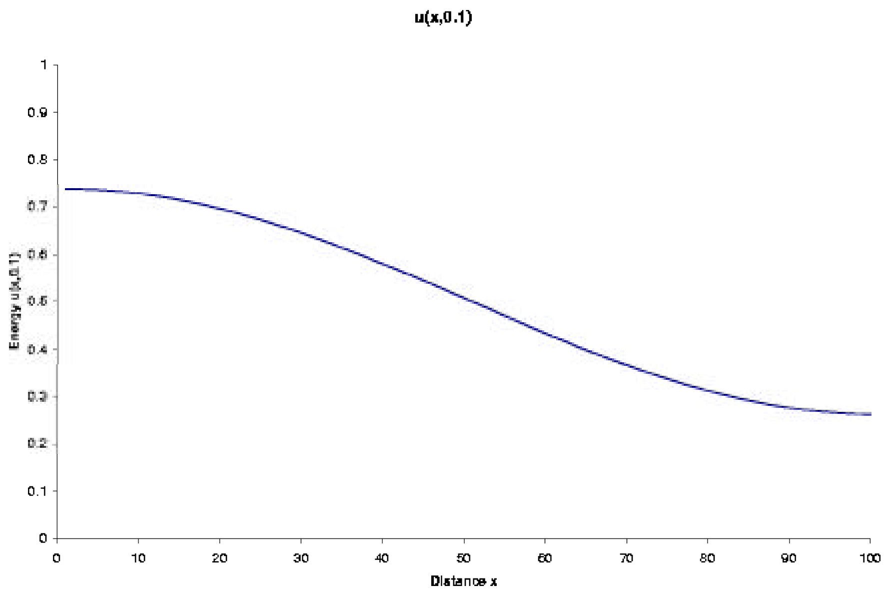

Figure 10: Approximating the analytical solution $u(x, 0.1)$.

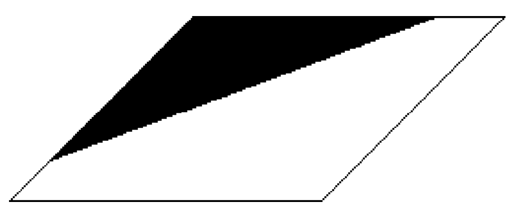

Figure 11: The half-heated square of the second experiment.

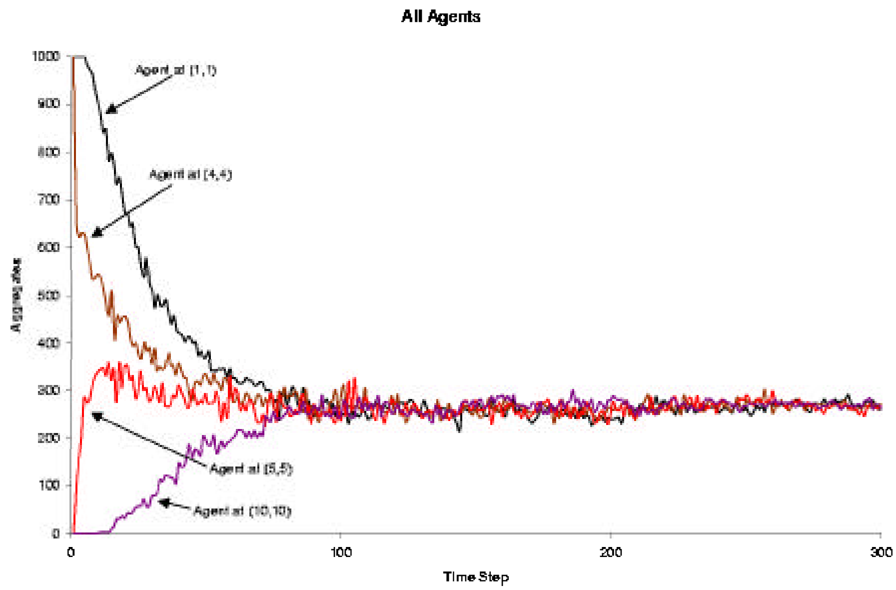

Figure 12: The agent energy in terms of time for four of the agents in the second experiment.

10 across the diagonal starting at the hottest and ending at the coldest corner are shown in Figure 12. Finally, in Figure 13 are shown the spatial distributions of energy among the 10 agents on the same diagonal every 5 time steps for the first 100 time steps of the run.

8 Future Work: A Potential Business Application

The ideas used in the example above may also be successfully applicable in a business related context. More specifically, we are dealing with the following setting. A telecommunications company has a network of telephone lines that are used by its customers. If too many of the customers try to place phone calls at the same time, then the servers get overwhelmed by service requests and the calls cannot go through causing frustration and dissatisfaction among the customers. On the other hand, if too few of the customers place calls at the same time, then the lines are under-used causing the company to lose money because of smaller usage than capacity. So, the ideal situation for the phone company is to have usage always balanced to a volume very close to the capacity of the network it is employing. What is required is to find an advertisement strategy that will encourage users to call in areas where lines are under-used and during low usage time periods and will discourage users from calling in areas where the lines are used to capacity and during times when too many users

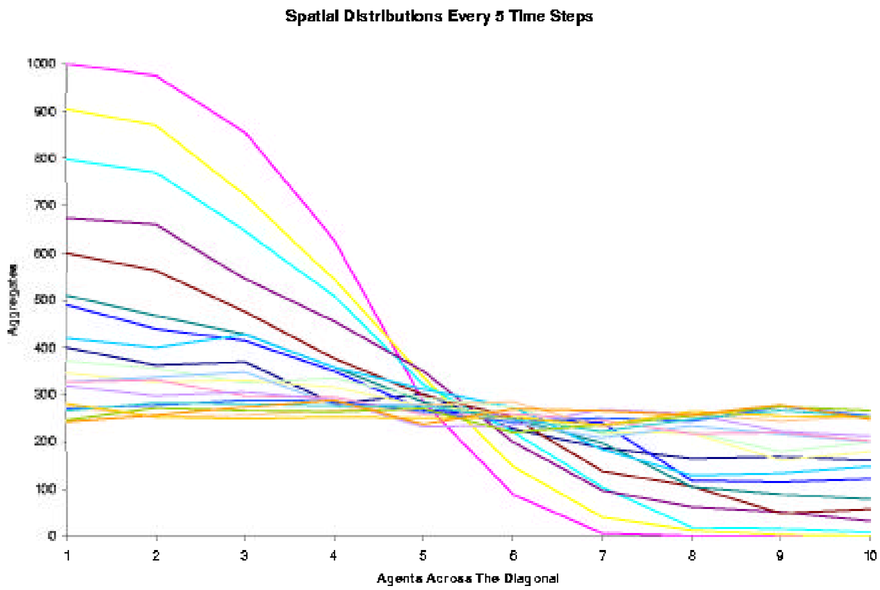

Figure 13: The distribution of energy across the diagonal every 5 time steps until time step 100.

are trying to get through, causing the load in the communications network to be close or over capacity.

A PTAN that would potentially be of use in this setting would consist of users and their interconnections as agents and their state would show whether they are placing or not phone calls during a specific time step. The probabilities will have to do with a statistical survey of how effective a specific advertisement strategy followed by the phone company at a specific geographical area is. Based on the results of the runs that will indicate what the company should expect in terms of loads in its network, the company will have to modify or adjust its strategy to achieve the load that optimizes the usage of the phone lines and, consequently, the company earnings.

A more detailed analysis of the usefulness of PTANs in modelling a realistic business application and other similar applications will be the subject of a future, more applied, research effort in this important simulation area.

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