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Categorical Abstract Algebraic Logic
Weakly Algebraizable π-Institutions

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Abstract Weakly algebraizable sentential logics were introduced by Czelakowski and Jansana and constitute a class in the abstract algebraic hierarchy of logics lying between the protoalgebraic logics of Blok and Pigozzi and the algebraizable logics, in the sense of Czelakowski's and Hermann's generalization of the original notion introduced by Blok and Pigozzi. Very recently protoalgebraic π-institutions were introduced by the author in order to abstract the algebraic hierarchy to the categorical level. The present work continues this program by introducing the class of weakly algebraizable π-institutions, a proper superclass of protoalgebraic π-institutions, sharing many of the properties of the weakly algebraizable sentential logics of Czelakowski and Jansana.

Keywords Algebraic logic · Equivalent deductive systems · Algebraizable logics · Lattice of theories · Leibniz operator · Protoalgebraic logics · Institutions · Equivalent institutions · Algebraizable institutions · Leibniz congruence systems · Protoalgebraic π-institutions

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1 Introduction

In [2] Blok and Pigozzi introduced the class of protoalgebraic sentential logics. Given a logic $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ and a set of $\mathcal{L}$-formulas Γ, two formulas α and β are *Γ-equivalent* if, for every formula $\gamma(p, \mathbf{q})$,

$$\Gamma \vdash_{\mathcal{S}} \gamma(\alpha, \mathbf{q}) \quad \text{iff} \quad \Gamma \vdash_{\mathcal{S}} \gamma(\beta, \mathbf{q}).$$

On the other hand, α and β are *Γ-interderivable* if

$$\Gamma, \alpha \vdash_{\mathcal{S}} \beta \quad \text{iff} \quad \Gamma, \beta \vdash_{\mathcal{S}} \alpha.$$

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According to the original definition in [2], a logic $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ is *protoalgebraic* if, for all $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, any two formulas that are Γ-equivalent are also Γ-interderivable. This definition was shown to be equivalent to the monotonicity of the Leibniz operator, that associates to every theory of the logic $\mathcal{S}$ the greatest congruence on the formula algebra that is compatible with the theory. It is now widely accepted in abstract algebraic logic that the class of protoalgebraic logics is the widest class of logics that are amenable to a study via universal algebraic techniques.

Later, in [3], Blok and Pigozzi introduced the class of algebraizable sentential logics. With updated terminology, these are now better known as the finitely algebraizable logics and this new term will be used here in reference to them. The original definition stipulates that a finitary logic $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ is *finitely algebraizable* if there exists a class K of $\mathcal{L}$-algebras, such that $\mathcal{S}$ is interpretable in the equational logic $\mathcal{S}_K = \langle \mathcal{L}, \models_K \rangle$ of the class K, $\mathcal{S}_K$ is interpretable in $\mathcal{S}$ and the two interpretations are inverses of one another in a specific technical sense. It was shown in [3] that finite algebraizability is equivalent to the Leibniz operator being an isomorphism between the complete lattice of theories of $\mathcal{S}$ and the complete lattice of K-congruences on the formula algebra $\mathbf{Fm}_{\mathcal{L}}(V)$. This, in turn is tantamount to the Leibniz operator being injective and continuous on the theories of the logic (see, e.g., Theorem 4.6.2 of [7]).

Taking after the work of Blok and Pigozzi, Czelakowski [6] and, independently, Herrmann [14], [15], [16] generalized the notion to encompass infinitary deductive systems, thus obtaining what are now known as the algebraizable logics. These are characterized by the property of the Leibniz operator being monotone, injective and commuting with inverse substitutions on the theories of the logic. For more details, see Theorem 4.5.5 of [7]. The reader may notice that passing from the protoalgebraic to the algebraizable logics in this sense, besides monotonicity, requires that the Leibniz operator satisfy the two extra properties of injectivity and commutativity with inverse substitutions.

A question now naturally arises concerning the classes that are obtained as intermediate classes between the protoalgebraic and the algebraizable logics if one adds to monotonicity, either commutativity of the Leibniz operator with inverse substitutions alone, or injectivity of the Leibniz operator alone.

The logics that are characterized by the monotonicity of the Leibniz operator on their theories plus its commutativity with inverse substitutions are the equivalential logics, that were introduced by Prucnal and Wroński [18] and, later, studied in detail by Czelakowski [5]. According to the original definition, a logic $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ is *equivalential* if there exists a set $E(p,q)$ of $\mathcal{L}$-formulas in two variables p and q, such that, for all formulas α, β, γ, all operation symbols $\lambda \in \mathcal{L}$ and all tuples $\boldsymbol{\alpha}, \boldsymbol{\beta}$ of formulas of length the arity of λ, the following conditions hold:

1. $\vdash_{\mathcal{S}} E(\alpha, \alpha)$;
2. $E(\alpha, \beta) \vdash_{\mathcal{S}} E(\beta, \alpha)$;
3. $E(\alpha, \beta) \cup E(\beta, \gamma) \vdash_{\mathcal{S}} E(\alpha, \gamma)$;
4. $E(\alpha_0, \beta_0) \cup \ldots \cup E(\alpha_{n-1}, \beta_{n-1}) \vdash_{\mathcal{S}} E(\lambda(\boldsymbol{\alpha}), \lambda(\boldsymbol{\beta}))$;
5. $E(\alpha, \beta) \cup \{\alpha\} \vdash_{\mathcal{S}} \beta$.

In recent work, Czelakowski and Jansana [8] introduced the class of sentential logics that falls between the protoalgebraic and the algebraizable logics and are characterized by the monotonicity of the Leibniz operator on the theories of the logic plus its injectivity. These are the *weakly algebraizable logics*. Since monotonicity of the Leibniz operator characterizes protoalgebraic logics and these had been studied extensively before, [8] focuses on its best part in giving several characterizations of the injectivity of the Leibniz operator in terms of the definability, both implicit and explicit, of the filters of classes of matrices for the logic. These results are then used to obtain several alternative characterizations of weak algebraizability. Czelakowski's and Jansana's work is completed by the presentation of several examples showing that all classes, discussed so far, and summarized in the following diagram, are dinstinct from each other. In the diagram arrows denote inclusion of classes.

$$\begin{tikzcd} & \text{algebraizable} \arrow[dl] \arrow[dr] & \\ \text{equivalential} \arrow[dr] & & \text{weakly algebraizable} \arrow[dl] \\ & \text{protoalgebraic} & \end{tikzcd}$$

In recent work [24]-[31], the theory of the Leibniz operator was adapted to cover logics that are presented as π-institutions. The framework of π-institutions covers logics with multiple signatures and quantifiers and has the special feature that it incorporates substitutions of terms of one signature for basic symbols of another in the object language, rather than relegating them to the metalanguage. It thus provides a more efficient setting for handling non sentential logics (see, e.g., [19], [20], [21] for more explanations). The motivating examples for developing this more abstract framework have been the algebraization of equational and of first-order logic [22], [23] in a way more natural than that employed by more traditional treatments.

As a result of the development of this theory, in more recent work [30], [31] the notion of a *protoalgebraic π-institution* was introduced. In analogy with sentential logics, protoalgebraic π-institutions are π-institutions whose Leibniz operator is monotone on theory families. Several properties of protoalgebraic π-institutions were studied in [30], [31] and the reader is referred to these works for further information. It is remarked here that the generality of this framework allows, in many cases, only partial analogs of results pertaining to sentential logics to be carried over to π-institutions. Nevertheless, the exploration of the limits to which results known to hold in the more concrete framework may be abstracted to π-institutions is necessary for successfully carrying out this abstraction process.

In the present paper, the introduction and study of properties of weakly algebraizable π-institutions, corresponding to the class of weakly algebraizable sentential logics, is initiated. Again an effort is made to establish analogs at the level of π-institutions of as many of the properties obtained by Czelakowski and Jansana for sentential logics as possible. The knowledgeable reader will discover that here, as was the case with protoalgebraic π-institutions, a certain distance may be covered, but full analogs of some results are more difficult to establish.

In Section 2, parameterized equivalence systems, that were previously introduced for π-institutions in [31], are reviewed and an interesting new result characterizing them is obtained. In Section 3, implicit and explicit definability of theory systems is studied. It is shown that implicit definability is equivalent to injectivity of the Leibniz operator on theory families but that explicit definability seems to be a stronger property, unlike the situation encountered in the sentential framework, where all three conditions are shown to be equivalent. Finally, in Section 3, weakly algebraizable π-institutions are introduced and a characterization theorem together with a sufficient condition for weak algebraizability are provided.

For all unexplained categorical terminology and notation the reader is referred to any of [1], [4], [17]. For the definitions pertaining to institutions see [12], [13], whereas π-institutions were introduced in [9]. For background on the theory of abstract algebraic logic and discussion of the classes of the abstract algebraic hierarchy, some of which were mentioned in this introduction, the reader is referred to the review article [11], the monograph [10] and the comprehensive treatise [7].

2 Parameterized Equivalence Systems

Some of the elements of the theory of N-parameterized equivalence systems, presented in Section 4 of [31], that are needed for the part of the theory developed here, will be reviewed in this section.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ be a π-institution and N a category of natural transformations on SEN. Suppose that E is a set of natural transformations of the form $\epsilon : \mathrm{SEN}^{k+2} \to \mathrm{SEN}$ in N, i.e.,

$$E = \{\epsilon^i : \mathrm{SEN}^{k_i+2} \to \mathrm{SEN} : i \in I\},$$

where ϵ^i is in N, for all $i \in I$. The following notation, borrowed from [7], will prove convenient.

$$E_\Sigma(\phi, \psi, \boldsymbol{\chi}) = \{\epsilon_\Sigma(\phi, \psi, \chi_0, \ldots, \chi_{k-1}) : \epsilon \in E \text{ with } \epsilon : \mathrm{SEN}^{k+2} \to \mathrm{SEN}\},$$

for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$ and $\boldsymbol{\chi} \in \mathrm{SEN}(\Sigma)^\omega$. Also, following [7], Section 1.2, denote, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$E_\Sigma(\langle \phi, \psi\rangle) = \bigcup_{\boldsymbol{\chi}\in \mathrm{SEN}(\Sigma)^\omega} E_\Sigma(\phi, \psi, \boldsymbol{\chi}).$$

Moreover, given $\Sigma \in |\mathbf{Sign}|$ and $\Delta \subseteq \mathrm{SEN}(\Sigma)^2$, let

$$E_\Sigma(\langle \Delta\rangle) = \bigcup_{\langle\phi,\psi\rangle\in\Delta} E_\Sigma(\langle \phi, \psi\rangle).$$

Finally, for all theory families T of $\mathcal{I}$, define the family of binary relations $E(T) = \{E_\Sigma(T)\}_{\Sigma\in|\mathbf{Sign}|}$ by letting, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{r}\langle \phi, \psi\rangle \in E_\Sigma(T) \text{ iff } E_{\Sigma'}(\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)\rangle) \subseteq T_{\Sigma'}, \\ \text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma').\end{array} \quad (1)$$

It is easy to see that, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$,

$$\langle \phi, \psi\rangle \in E_{\Sigma_1}(T) \quad \text{implies} \quad \langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)\rangle \in E_{\Sigma_2}(T),$$

i.e., $E(T)$ is a relation system on SEN.[1]

[1] *System*, as opposed to *family*, is used in the categorical theory to describe a collection indexed by signature objects and preserved by signature morphisms.

The right-hand side of the Equivalence (1) will sometimes be abbreviated to

$$(\forall f)(E_{\Sigma'}(\langle \mathrm{SEN}(f)^2(\phi, \psi)\rangle) \subseteq T_{\Sigma'}).$$

In Proposition 4.1 of [31], it is shown that, if $E(T)$ is a reflexive relation system, i.e., $E_\Sigma(T), \Sigma \in |\mathbf{Sign}|$, is a reflexive binary relation on $\mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$, then $E(T)$ contains the Leibniz N-congruence system $\Omega^N(T)$. Thus, by the maximality property of $\Omega^N(T)$, if $E(T)$ is an N-congruence system of $\mathcal{I}$ compatible with the theory family T, then it has to coincide with $\Omega^N(T)$.

Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, a subset E of N, as above, will be said to be an N-**parameterized equivalence system** for $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$, $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, and all $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{SEN}(\Sigma)^n$,

(R) $E_\Sigma(\langle \phi, \phi\rangle) \subseteq C_\Sigma(\emptyset)$,

(MP) for every theory family T of $\mathcal{I}$,
- $\phi \in T_\Sigma$ and
- $E_{\Sigma'}(\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)\rangle) \subseteq T_{\Sigma'}$, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

imply $\psi \in T_\Sigma$,

(RP) for every theory family T of $\mathcal{I}$, $E_{\Sigma'}(\langle \mathrm{SEN}(f)(\phi_i), \mathrm{SEN}(f)(\psi_i)\rangle) \subseteq T_{\Sigma'}$, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma'), i < n$, imply

$$E_{\Sigma'}(\langle \mathrm{SEN}(f)(\sigma_\Sigma(\boldsymbol{\phi})), \mathrm{SEN}(f)(\sigma_\Sigma(\boldsymbol{\psi}))\rangle) \subseteq T_{\Sigma'},$$

for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$.

(R) stands for reflexivity, (MP) for modus ponens and (RP) for replacement. The added complexity of these three conditions, as compared to the corresponding conditions of [7], is due to the additional effort needed in the present framework to make the collection of relations $E(T)$ structural, i.e., a relation system.

Note that the three conditions have the following abbreviated forms, according to the convention introduced after Equivalence (1):

(R) $E_\Sigma(\langle \phi, \phi\rangle) \subseteq C_\Sigma(\emptyset)$,

(MP) for every theory family T of $\mathcal{I}$, if $\phi \in T_\Sigma$ and $(\forall f)(E_{\Sigma'}(\langle \mathrm{SEN}(f)^2(\phi, \psi)\rangle) \subseteq T_{\Sigma'})$, then $\psi \in T_\Sigma$,

(RP) for every theory family T of $\mathcal{I}$, if $(\forall f)(E_{\Sigma'}(\langle \mathrm{SEN}(f)^2(\phi_i, \psi_i)\rangle) \subseteq T_{\Sigma'})$, for all $i < n$, then $(\forall f)(E_{\Sigma'}(\langle \mathrm{SEN}(f)^2(\sigma_\Sigma(\boldsymbol{\phi}), \sigma_\Sigma(\boldsymbol{\psi}))\rangle) \subseteq T_{\Sigma'})$.

Lemma 4.3 of [31] asserts that an N-parameterized equivalence system E of a π-institution $\mathcal{I}$ gives rise to a reflexive, symmetric and N-invariant relation system $E(T)$ that is compatible with T, for every theory family T of $\mathcal{I}$. More precisely, if E is an N-parameterized equivalence system for $\mathcal{I}$ and T a theory family of $\mathcal{I}$, then

1. $E(T)$ is reflexive,
2. $E(T)$ is compatible with T and
3. for all $\Sigma \in |\mathbf{Sign}|, \sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{SEN}(\Sigma)^n$,

$$\langle \phi_i, \psi_i\rangle \in E_\Sigma(T),\ i < n, \quad \text{imply} \quad \langle \sigma_\Sigma(\boldsymbol{\phi}), \sigma_\Sigma(\boldsymbol{\psi})\rangle \in E_\Sigma(T).$$

Furthermore, in Lemma 4.4 of [31], it is shown that the relation $E(T)$, associated with an N-parameterized equivalence system E of a π-institution $\mathcal{I}$ and a theory family T of $\mathcal{I}$, satisfies a property satisfied by all N-congruence systems of $\mathcal{I}$. More precisely, using the notational convention of Equation (2) of [30], if T a theory family of $\mathcal{I}$, then, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$, if $\langle \phi, \psi \rangle \in E_\Sigma(T)$ then, for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$, and all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, $\boldsymbol{\chi} \in \mathrm{SEN}(\Sigma')^{n-1}$,

$$\langle \sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi), \boldsymbol{\chi}), \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \boldsymbol{\chi})\rangle \in E_{\Sigma'}(T).$$

This is followed by Lemma 4.5, which shows that $E(T)$ is a symmetric relation system of $\mathcal{I}$, for every N-parameterized equivalence system E and every theory family T of $\mathcal{I}$, and this chain of results culminates in the characterization Theorem 4.6 of [31], which shows that a collection E of natural transformations in N is an N-parameterized equivalence system for a π-institution $\mathcal{I}$ if and only if $E(T) = \Omega^N(T)$, for every theory family T of $\mathcal{I}$.

Finally, in the main Proposition 4.8 of [31], it is shown that a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is N-protoalgebraic if it possesses an N-parameterized equivalence system.

Proposition 1 [Proposition 4.8 of [31]] *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution and N a category of natural transformations on* SEN. *If $\mathcal{I}$ possesses an N-parameterized equivalence system, then it is N-protoalgebraic.*

An alternative characterization of an N-parameterized equivalence system for a π-institution $\mathcal{I}$ is given in the following theorem, which is an analog of Theorem 4.1 of [8].

Theorem 1 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution and N a category of natural transformations on* SEN. *Then a subcollection $E = \{\epsilon^i : \mathrm{SEN}^{2+k_i} \to \mathrm{SEN} : i \in I\}$ of natural transformations in N is an N-parameterized equivalence system for $\mathcal{I}$ if and only if*

1. *For all $\Sigma \in |\mathbf{Sign}|, \Delta \cup \{\langle \phi, \psi\rangle\} \subseteq \mathrm{SEN}(\Sigma)^2$ and all $T \in \mathrm{ThFam}(\mathcal{I})$,*

$$\begin{aligned}(\Delta \subseteq \Omega^N(T) \Rightarrow \langle \phi, \psi\rangle \in \Omega^N(T)) \quad &\textit{implies}\\ ((\forall f)(E_{\Sigma'}(\langle \mathrm{SEN}(f)^2(\Delta)\rangle) \subseteq T_{\Sigma'}) &\Rightarrow\\ &(\forall f)(E_{\Sigma'}(\langle \mathrm{SEN}(f)^2(\phi, \psi)\rangle) \subseteq T_{\Sigma'})).\end{aligned}$$

2. *For all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$ and all $T \in \mathrm{ThFam}(\mathcal{I})$, if $\phi \in T_\Sigma$ and $(\forall f)(E_{\Sigma'}(\langle \mathrm{SEN}(f)^2(\phi, \psi)\rangle) \subseteq T_{\Sigma'})$, then $\psi \in T_\Sigma$.*

Proof Suppose, first, that E is an N-parameterized equivalence system for $\mathcal{I}$. Then Condition 2 is satisfied by the definition, whence it suffices to prove Condition 1. To this end, let $\Sigma \in |\mathbf{Sign}|, \Delta \cup \{\langle \phi, \psi\rangle\} \subseteq \mathrm{SEN}(\Sigma)^2$ and $T \in \mathrm{ThFam}(\mathcal{I})$. Then

$$\begin{aligned}&\Delta \subseteq \Omega^N_\Sigma(T) \Rightarrow \langle \phi, \psi\rangle \in \Omega^N_\Sigma(T) \quad \text{iff}\\ &\Delta \subseteq E_\Sigma(T) \Rightarrow \langle \phi, \psi\rangle \in E_\Sigma(T) \quad \text{iff}\\ &(\forall f)(E_{\Sigma'}(\langle \mathrm{SEN}(f)^2(\Delta)\rangle) \subseteq T_{\Sigma'}) \Rightarrow (\forall f)(E_{\Sigma'}(\langle \mathrm{SEN}(f)^2(\phi, \psi)\rangle) \subseteq T_{\Sigma'}).\end{aligned}$$

Suppose, conversely, that Conditions 1 and 2 of the statement hold for E. Because of Condition 2, to show that E is an N-parameterized equivalence system for $\mathcal{I}$, it suffices to prove that (R) and (RP) hold for E. Now Condition 1 gives, for all $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi, \phi \rangle \in \Omega^N_\Sigma(C_\Sigma(\emptyset)) \quad \text{implies} \quad (\forall f)(E_{\Sigma'}(\langle \mathrm{SEN}(f)^2(\phi, \phi)\rangle) \subseteq C_{\Sigma'}(\emptyset)).$$

Since the hypothesis always holds, we get, using $f = i_\Sigma$, that $E_\Sigma(\langle \phi, \phi \rangle) \subseteq C_\Sigma(\emptyset)$. Hence (R) holds.

Condition 1 also gives, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, all $\Sigma \in |\mathbf{Sign}|$ and all $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{SEN}(\Sigma)^n$,

$$\begin{array}{l}(\forall i < n)(\langle \phi_i, \psi_i \rangle \in \Omega^N_\Sigma(T)) \Rightarrow \langle \sigma_\Sigma(\boldsymbol{\phi}), \sigma_\Sigma(\boldsymbol{\psi})\rangle \in \Omega^N_\Sigma(T) \quad \text{implies}\\ (\forall i < n)(\forall f)(E_{\Sigma'}(\langle \mathrm{SEN}(f)^2(\phi_i, \psi_i)\rangle) \subseteq T_{\Sigma'}) \Rightarrow \\ \qquad\qquad\qquad\qquad (\forall f)(E_{\Sigma'}(\langle \mathrm{SEN}(f)^2(\sigma_\Sigma(\boldsymbol{\phi}), \sigma_\Sigma(\boldsymbol{\psi}))\rangle) \subseteq T_{\Sigma'}).\end{array}$$

Since the hypothesis again holds in general, we obtain

$$\begin{array}{l}(\forall i < n)(\forall f)(E_{\Sigma'}(\langle \mathrm{SEN}(f)^2(\phi_i, \psi_i)\rangle) \subseteq T_{\Sigma'}) \Rightarrow \\ \qquad\qquad\qquad\qquad (\forall f)(E_{\Sigma'}(\langle \mathrm{SEN}(f)^2(\sigma_\Sigma(\boldsymbol{\phi}), \sigma_\Sigma(\boldsymbol{\psi}))\rangle) \subseteq T_{\Sigma'}),\end{array}$$

which is the rule (RP).

Therefore, E is an N-parameterized equivalence system for $\mathcal{I}$.

3 Injectivity of the Leibniz Operator on Theory Systems

Recall from [31], Section 5, the relation $\sim$ between two theory systems T^1, T^2 in the collection $\mathrm{ThSys}^{\langle F, \alpha\rangle}_{\mathcal{I}}(\mathrm{SEN}')$ of all theory systems of an $\langle F, \alpha\rangle$-min (N, N')-model $\mathcal{I}'^{\min} = \langle \mathbf{Sign}', \mathrm{SEN}', C'^{\min}\rangle$ of $\mathcal{I}$ on SEN', via a surjective (N, N')-logical morphism $\langle F, \alpha\rangle : \mathcal{I} \rangle\!\!-^{se} \mathcal{I}'$, which was defined by

$$T^1 \sim T^2 \quad \text{iff} \quad \Omega^{N'}(T^1) = \Omega^{N'}(T^2),$$

i.e., $\sim$ is the kernel of the N'-Leibniz operator as applied on the collection of theory systems $\mathrm{ThSys}^{\langle F, \alpha\rangle}_{\mathcal{I}}(\mathrm{SEN}')$. Proposition 5.3 of [31] showed that, for $\mathcal{I}$ N-protoalgebraic, at most one of the theory systems in each $\sim$-equivalence class is a member of $\mathrm{ThSys}^{\langle F, \alpha\rangle^\star}_{\mathcal{I}}(\mathrm{SEN}')$, the collection of all those theory systems T' of $\mathcal{I}'^{\min}$ that are such that $\mathcal{I}'^{\min^{T'}} = \langle \mathbf{Sign}', \mathrm{SEN}', C'^{\min^{T'}}\rangle$ is an $\langle F, \alpha\rangle$-full (N, N')-model of $\mathcal{I}$.

If $\mathcal{I}$ is N-protoalgebraic, then, denoting by $T/{\sim}$ the $\sim$-equivalence class of a theory system T, we get that

$$\Omega^N(\bigcap T/{\sim}) = \bigcap_{T' \in T/\sim} \Omega^N(T') = \bigcap_{T' \in T/\sim} \Omega^N(T) = \Omega^N(T),$$

whence the theory system $\bigcap T/{\sim}$ is in the same $\sim$-class with T. It was shown in Proposition 5.4 of [31] that $\bigcap T/{\sim}$ is in $\mathrm{ThSys}^{\langle F, \alpha\rangle^\star}_{\mathcal{I}}(\mathrm{SEN}')$, i.e., that $\bigcap T/{\sim}$ is the representative of $T/{\sim}$ in $\mathrm{ThSys}^{\langle F, \alpha\rangle^\star}_{\mathcal{I}}(\mathrm{SEN}')$. More precisely, we had

Proposition 2 [Proposition 5.4 of [31]] *Suppose $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution and N a category of natural transformations on* SEN. *Let $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ be a functor, N' a category of natural transformations on SEN' and $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ a surjective (N, N')-epimorphic translation. If $\mathcal{I}$ is N-protoalgebraic and T is a theory system of the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN', then the following statements are equivalent:*

1. $T \in \mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle^\bigstar}(\mathrm{SEN}')$.
2. T *is the least element in the class* $T/{\sim}$.
3. $T/\Omega^{N'}(T)$ *is the least element in* $\mathrm{ThSys}_{\mathcal{I}}^{\langle F,\pi_F^{N'}\alpha\rangle}(\mathrm{SEN}'^{N'})$.

Proposition 2 yielded the following corollary:

Proposition 3 [Proposition 5.5 of [31]] *Suppose* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *is a* π*-institution and* N *a category of natural transformations on* SEN. *Let* $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ *be a functor,* N' *a category of natural transformations on* SEN' *and* $\langle F,\alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ *a surjective* (N,N')*-epimorphic translation. If* $\mathcal{I}$ *is* N*-protoalgebraic, then we have that* $\mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle^\bigstar}(\mathrm{SEN}') = \mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$ *if and only if* $\Omega_{\mathrm{SEN}'}^{\langle F,\alpha\rangle}$ *is injective on* $\mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$.

Now recall from Proposition 4.8 of [31] (repeated in the present paper as Proposition 1) that the existence of an N-parameterized equivalence system for a π-institution $\mathcal{I}$ implies that $\mathcal{I}$ is N-protoalgebraic. Therefore, by the preceding remarks, if $\mathcal{I}$ has an N-parameterized equivalence system, then every $\sim$-equivalence class of theory systems has a minimum member. This minimum element is now characterized based on the existence of an N-parameterized equivalence system. This result forms an analog in the π-institution framework of Lemma 3.4 of [8] for sentential logics.

Lemma 1 *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *is a* π*-institution,* N *a category of natural transformations on* SEN *and* E *an* N*-parameterized equivalence system for* $\mathcal{I}$*. Then, for every theory system* T *of* $\mathcal{I}$*, the collection* $T^* = \{T^*_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$*, defined, for all* $\Sigma \in |\mathbf{Sign}|$*, by*

$$T^*_\Sigma = C_\Sigma\left(\bigcup\{E_\Sigma(\langle\phi,\psi\rangle) : \langle\phi,\psi\rangle \in \Omega^N_\Sigma(T)\}\right),$$

is a theory system of $\mathcal{I}$ *and it is the minimum element of* $T/{\sim}$*.*

Proof It is shown, first, that $T^* = \{T^*_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ is indeed a theory system of $\mathcal{I}$. It is clearly a theory family, whence it suffices to show that, for all $\Sigma_1,\Sigma_2 \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma_1,\Sigma_2)$, $\mathrm{SEN}(f)(T^*_{\Sigma_1}) \subseteq T^*_{\Sigma_2}$. We have

$$\begin{aligned}
\mathrm{SEN}(f)(T^*_{\Sigma_1}) &= \mathrm{SEN}(f)(C_{\Sigma_1}(\textstyle\bigcup\{E_{\Sigma_1}(\langle\phi,\psi\rangle) : \langle\phi,\psi\rangle \in \Omega^N_{\Sigma_1}(T)\}))\\
&\subseteq C_{\Sigma_2}(\mathrm{SEN}(f)(\textstyle\bigcup\{E_{\Sigma_1}(\langle\phi,\psi\rangle) : \langle\phi,\psi\rangle \in \Omega^N_{\Sigma_1}(T)\}))\\
&= C_{\Sigma_2}(\textstyle\bigcup\{\mathrm{SEN}(f)(E_{\Sigma_1}(\langle\phi,\psi\rangle)) : \langle\phi,\psi\rangle \in \Omega^N_{\Sigma_1}(T)\})\\
&\subseteq C_{\Sigma_2}(\textstyle\bigcup\{E_{\Sigma_2}(\langle\mathrm{SEN}(f)(\phi),\mathrm{SEN}(f)(\psi)\rangle) :\\
&\qquad\qquad \langle\mathrm{SEN}(f)(\phi),\mathrm{SEN}(f)(\psi)\rangle \in \Omega^N_{\Sigma_2}(T)\})\\
&\subseteq C_{\Sigma_2}(\textstyle\bigcup\{E_{\Sigma_2}(\langle\phi,\psi\rangle) : \langle\phi,\psi\rangle \in \Omega^N_{\Sigma_2}(T)\})\\
&= T^*_{\Sigma_2}.
\end{aligned}$$

For every $\langle\phi,\psi\rangle \in \Omega^N_\Sigma(T) = E_\Sigma(T)$, we have that $E_\Sigma(\langle\phi,\psi\rangle) \subseteq T_\Sigma$, whence

$$\bigcup\{E_\Sigma(\langle\phi,\psi\rangle) : \langle\phi,\psi\rangle \in \Omega^N_\Sigma(T)\} \subseteq T_\Sigma,$$

for every $\Sigma \in |\mathbf{Sign}|$. This immediately yields that $T^*_\Sigma \subseteq T_\Sigma$, for all $\Sigma \in |\mathbf{Sign}|$, i.e., $T^* \leq T$. Therefore, using Proposition 1, we obtain, by the N-protoalgebraicity of $\mathcal{I}$, that $\Omega^N(T^*) \leq \Omega^N(T)$.

Suppose, now, that $\langle \phi, \psi \rangle \in \Omega^N_\Sigma(T)$. Since $\Omega^N(T)$ is, by definition, an N-congruence system, we obtain that, for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi) \rangle \in \Omega^N_{\Sigma'}(T),$$

and, therefore, by the definition of T^*, that $E_{\Sigma'}(\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi) \rangle) \subseteq T^*_{\Sigma'}$. Thus, by the definition of $E_\Sigma(T)$, we get that $\langle \phi, \psi \rangle \in E_\Sigma(T^*) = \Omega^N_\Sigma(T^*)$. Thus $\Omega^N(T) \leq \Omega^N(T^*)$.

Therefore $\Omega^N(T) = \Omega^N(T^*)$, i.e., $T \sim T^*$. If, T' is a theory system of $\mathcal{I}$, such that $T' \sim T$, then, since $\Omega^N(T') = \Omega^N(T)$, we obtain that $T'^* = T^*$, whence $T^* \subseteq T'$, which shows that T^* is the least theory system of $\mathcal{I}$ in $T/\!\sim$.

Recall, from [30] the definition of the least theory system $T^{\langle \Sigma_0, \Phi_0 \rangle}$ of a π-institution $\mathcal{I}$ that contains the theory system T of $\mathcal{I}$ and a fixed set Φ_0 of Σ_0-sentences over some signature $\Sigma_0 \in |\mathbf{Sign}|$. Using this notion, a theorem is provided that characterizes the injectivity of the N-Leibniz operator on theory systems of a π-institution $\mathcal{I}$ in terms of the $\sim$-equivalence class of the theory system $T^{\langle \Sigma_0, \phi_0 \rangle}$, for an arbitrary Σ_0-sentence ϕ_0, for arbitrary $\Sigma_0 \in |\mathbf{Sign}|$. This result forms an analog of Theorem 3.5 of [8] for π-institutions.

Recall that, given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, by

$$\mathrm{Thm} = \{\mathrm{Thm}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} = \{C_\Sigma(\emptyset)\}_{\Sigma \in |\mathbf{Sign}|}$$

is denoted the *theorem system* of $\mathcal{I}$, which is always the smallest theory system and the smallest theory family of $\mathcal{I}$.

Theorem 2 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ is a π-institution, N a category of natural transformations on* SEN *and E an N-parameterized equivalence system for $\mathcal{I}$. Then, the following statements are equivalent:*

1. *Ω^N is injective on theory systems.*
2. *For all $\Sigma_0 \in |\mathbf{Sign}|, \phi_0 \in \mathrm{SEN}(\Sigma_0)$, $\mathrm{Thm}^{\langle \Sigma_0, \phi_0 \rangle}/\!\sim$ is a singleton.*
3. *For all $\Sigma, \Sigma_0 \in |\mathbf{Sign}|, \phi_0 \in \mathrm{SEN}(\Sigma_0)$,*

$$\mathrm{Thm}^{\langle \Sigma_0, \phi_0 \rangle}_\Sigma = C_\Sigma(\bigcup\{E_\Sigma(\langle \psi, \chi \rangle) : \langle \psi, \chi \rangle \in \Omega^N_\Sigma(\mathrm{Thm}^{\langle \Sigma_0, \phi_0 \rangle})\}). \qquad (2)$$

Proof. 1 → 2 This part is obvious.

2 → 3 Suppose, next, that, for all $\Sigma_0 \in |\mathbf{Sign}|, \phi_0 \in \mathrm{SEN}(\Sigma_0)$, $\mathrm{Thm}^{\langle \Sigma_0, \phi_0 \rangle}/\!\sim$ is a singleton. Thus, we must have $\mathrm{Thm}^{\langle \Sigma_0, \phi_0 \rangle^*} = \mathrm{Thm}^{\langle \Sigma_0, \phi_0 \rangle}$, whence, by Lemma 1, for all $\Sigma \in |\mathbf{Sign}|$,

$$\mathrm{Thm}^{\langle \Sigma_0, \phi_0 \rangle}_\Sigma = C_\Sigma(\bigcup\{E_\Sigma(\langle \psi, \chi \rangle) : \langle \psi, \chi \rangle \in \Omega^N_\Sigma(\mathrm{Thm}^{\langle \Sigma_0, \phi_0 \rangle})\}).$$

3 → 1 Suppose, now, that, for all $\Sigma, \Sigma_0 \in |\mathbf{Sign}|, \phi_0 \in \mathrm{SEN}(\Sigma_0)$, Equation (2) holds. The goal is to show that, for every theory system T of $\mathcal{I}$, $T^* = T$, since this will yield that $T/\!\sim$ is a singleton, for every theory system T, and, therefore, that Ω^N is injective on theory systems.

As it was already seen in the proof of Lemma 1, $T^* \leq T$, for all theory systems T of $\mathcal{I}$. Now let $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in T_\Sigma$. We have, by the minimality of $\mathrm{Thm}^{\langle \Sigma,\phi\rangle}$ (Lemma 3.6 of [30]), that $\mathrm{Thm}^{\langle \Sigma,\phi\rangle} \leq T$. Hence, by N-protoalgebraicity, we obtain that $\Omega^N(\mathrm{Thm}^{\langle \Sigma,\phi\rangle}) \leq \Omega^N(T)$. Thus, using Equation (2), we get that

$$\begin{aligned}\phi &\in \mathrm{Thm}_\Sigma^{\langle \Sigma,\phi\rangle}\\ &= C_\Sigma(\bigcup\{E_\Sigma(\langle\psi,\chi\rangle) : \langle\psi,\chi\rangle \in \Omega_\Sigma^N(\mathrm{Thm}^{\langle \Sigma,\phi\rangle})\})\\ &\subseteq C_\Sigma(\bigcup\{E_\Sigma(\langle\psi,\chi\rangle) : \langle\psi,\chi\rangle \in \Omega_\Sigma^N(T)\})\\ &= T_\Sigma^*.\end{aligned}$$

Thus $\phi \in T_\Sigma^*$, which yields $T \leq T^*$. Therefore $T = T^*$, as was to be shown.

The next main result (Theorem 3) is a restricted "transfer theorem". It says that the injectivity of the N-Leibniz operator on the theory systems of a given π-institution implies the injectivity of the N'-Leibniz operator on the theory systems of any (N,N')-model $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ of $\mathcal{I}$ via a surjective (N,N')-logical morphism $\langle F,\alpha\rangle : \mathcal{I}\rangle\!\!-^{se}\mathcal{I}'$. Theorem 3 forms an analog of Theorem 3.6 of [8] for π-institutions. For its proof, however, a chain of technical lemmas are needed, which will now be formulated and presented.

The first lemma asserts that if both a theory system T and a specific Σ_0-sentence ϕ_0 of a π-institution $\mathcal{I}$ are mapped inside a theory system T' of another π-institution $\mathcal{I}'$ via a surjective singleton semi-interpretation $\langle F,\alpha\rangle : \mathcal{I}\rangle\!\!-^{s}\mathcal{I}'$, then the least theory system $T^{\langle \Sigma_0,\phi_0\rangle}$ is also mapped inside T'.

Lemma 2 *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle, \mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ *are* π*-institutions,* $\langle F,\alpha\rangle : \mathcal{I}\rangle\!\!-^{s}\mathcal{I}'$ *a surjective singleton semi-interpretation,* T *a theory system of* $\mathcal{I}$*,* T' *a theory system of* $\mathcal{I}'$*,* $\Sigma_0 \in |\mathbf{Sign}|$ *and* $\phi_0 \in \mathrm{SEN}(\Sigma_0)$*. If, for all* $\Sigma \in |\mathbf{Sign}|, \alpha_\Sigma(T_\Sigma) \subseteq T'_{F(\Sigma)}$ *and* $\alpha_{\Sigma_0}(\phi_0) \in T'_{F(\Sigma_0)}$*, then, for all* $\Sigma \in |\mathbf{Sign}|$*,*

$$\alpha_\Sigma(T_\Sigma^{\langle \Sigma_0,\phi_0\rangle}) \subseteq T'_{F(\Sigma)}.$$

Proof To show the required inclusions, it suffices, by Lemma 5 of [30], to show that $T \leq \alpha^{-1}(T')$ and $\phi_0 \in \alpha_{\Sigma_0}^{-1}(T'_{F(\Sigma_0)})$.

For the first inclusion, if $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in T_\Sigma$, then $\alpha_\Sigma(\phi) \in \alpha_\Sigma(T_\Sigma) \subseteq T'_{F(\Sigma)}$, whence $\phi \in \alpha_\Sigma^{-1}(T'_{F(\Sigma)})$. Therefore, for all $\Sigma \in |\mathbf{Sign}|$, $T_\Sigma \subseteq \alpha_\Sigma^{-1}(T'_{F(\Sigma)})$, which verifies that $T \leq \alpha^{-1}(T')$.

The second inclusion follows from the hypothesis $\alpha_{\Sigma_0}(\phi_0) \in T'_{F(\Sigma_0)}$.

The second lemma in the series asserts that, if a theory system T of a π-institution $\mathcal{I}$ is mapped inside a theory system T' of a π-institution $\mathcal{I}'$ via a surjective (N,N')-logical morphism $\langle F,\alpha\rangle : \mathcal{I}\rangle\!\!-^{se}\mathcal{I}'$, then the N-Leibniz congruence system of T is also mapped inside the N'-Leibniz congruence system of T'.

Lemma 3 *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle, \mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ *are* π*-institutions, with* N, N' *categories of natural transformations on* $\mathrm{SEN}, \mathrm{SEN}'$*, respectively, and* $\langle F,\alpha\rangle : \mathcal{I}\rangle\!\!-^{se}\mathcal{I}'$ *a surjective* (N,N')*-logical morphism,* T *a theory system of* $\mathcal{I}$*,* T' *a theory system of* $\mathcal{I}'$*, such that* $\alpha_\Sigma(T_\Sigma) \subseteq T'_{F(\Sigma)}$*, for all* $\Sigma \in |\mathbf{Sign}|$*. If* $\mathcal{I}$ *is* N*-protoalgebraic, then*

$$\alpha_\Sigma(\Omega_\Sigma^N(T)) \subseteq \Omega_{F(\Sigma)}^{N'}(T'), \quad \textit{for all } \Sigma \in |\mathbf{Sign}|.$$

Proof By Lemma 5.21 of [30], we have that

$$\Omega^N_\Sigma(\alpha^{-1}(T')) = \alpha^{-1}_\Sigma(\Omega^{N'}_{F(\Sigma)}(T')), \tag{3}$$

for all $\Sigma \in |\mathbf{Sign}|$. But, by the hypothesis, we have $T \leq \alpha^{-1}(T')$, whence, by N-protoalgebraicity and Equation (3), we get, for all $\Sigma \in |\mathbf{Sign}|$,

$$\Omega^N_\Sigma(T) \subseteq \Omega^N_\Sigma(\alpha^{-1}(T')) = \alpha^{-1}_\Sigma(\Omega^{N'}_{F(\Sigma)}(T')),$$

which yields $\alpha_\Sigma(\Omega^N_\Sigma(T)) \subseteq \Omega^{N'}_{F(\Sigma)}(T')$.

Finally, the last lemma needed in order to proceed with the proof of Theorem 3 verifies that, if $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$ is a surjective (N, N')-logical morphism and E is an N-parameterized equivalence system for $\mathcal{I}$, then E', the collection of natural transformations in N' corresponding to the collection E via the (N, N')-epimorphic property, is an N'-parameterized equivalence system for $\mathcal{I}'$.

Lemma 4 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle, \mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ are π-institutions, with N, N' categories of natural transformations on $\mathrm{SEN}, \mathrm{SEN}'$, respectively, $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$ a surjective (N, N')-logical morphism and E an N-parameterized equivalence system for $\mathcal{I}$. Then E', the collection corresponding to E via the (N, N')-epimorphic property, is an N'-parameterized equivalence system for $\mathcal{I}'$.*

Proof It suffices, by Theorem 4.6 of [31] and surjectivity, to show that, for every theory family T' of $\mathcal{I}'$ and all $\Sigma \in |\mathbf{Sign}|$, $E'_{F(\Sigma)}(T') = \Omega^{N'}_{F(\Sigma)}(T')$. We have, in fact,

$$\begin{aligned}
\Omega^{N'}_{F(\Sigma)}(T') &= \alpha_\Sigma(\Omega^N_\Sigma(\alpha^{-1}(T'))) && \text{(by Lemma 5.21 of [30])}\\
&= \alpha_\Sigma(E_\Sigma(\alpha^{-1}(T'))) && \text{(by hypothesis and Thm 4.6 of [31])}\\
&= E'_{F(\Sigma)}(T'),
\end{aligned}$$

where the last equality follows from the following string of equalities

$$\begin{aligned}
&\alpha_\Sigma(E_\Sigma(\alpha^{-1}(T')))\\
&= \alpha_\Sigma(\{\langle\phi, \psi\rangle : (\forall f)(E_{\Sigma'}(\langle \mathrm{SEN}(f)^2(\phi, \psi)\rangle) \subseteq \alpha^{-1}_{\Sigma'}(T'_{F(\Sigma')}))\})\\
&= \{\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle : (\forall f)(\alpha_{\Sigma'}(E_{\Sigma'}(\langle \mathrm{SEN}(f)^2(\phi, \psi)\rangle)) \subseteq T'_{F(\Sigma')})\}\\
&= \{\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle : (\forall f)(E'_{F(\Sigma')}(\alpha_{\Sigma'}(\langle \mathrm{SEN}(f)^2(\phi, \psi)\rangle)) \subseteq T'_{F(\Sigma')})\}\\
&= \{\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle :\\
&\qquad\qquad (\forall f)(E'_{F(\Sigma')}(\langle \mathrm{SEN}'(F(f))^2(\alpha_\Sigma(\phi), \alpha_\Sigma(\psi))\rangle)) \subseteq T'_{F(\Sigma')})\}\\
&= \{\langle \phi', \psi'\rangle : (\forall f')(E'_{\Sigma''}(\langle \mathrm{SEN}'(f')^2(\phi', \psi')\rangle) \subseteq T'_{\Sigma''})\}\\
&= E'_{F(\Sigma)}(T'),
\end{aligned}$$

where in many of the steps of the above chain of equalities the surjectivity of $\langle F, \alpha\rangle$ played a crucial role.

We are now ready to proceed with formulating and proving the "transfer theorem" asserting that the property of injectivity of the Leibniz operator on theory systems of a π-institution transfers to the injectivity of the Leibniz operator on the theory systems of all its models via surjective logical morphisms.

Theorem 3 *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *is a* π*-institution,* N *a category of natural transformations on* SEN *and* E *an* N*-parameterized equivalence system for* $\mathcal{I}$*. Then,* Ω^N *is injective on theory systems if and only if, for every* (N, N')*-model* $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ *of* $\mathcal{I}$ *via a surjective* (N, N')*-logical morphism* $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$, $\Omega^{\langle F,\alpha\rangle}_{\mathcal{I}} := \Omega^{N'}$ *is injective on theory systems.*

Proof The "if" direction is obvious, since $\langle \mathrm{I}_{\mathbf{Sign}}, \iota\rangle : \mathcal{I} \vdash^{se} \mathcal{I}$ is a surjective (N, N)-logical morphism from $\mathcal{I}$ to itself.

For the "only if" direction, suppose that Ω^N is injective on the theory systems of $\mathcal{I}$ and let $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ be an (N, N')-model of $\mathcal{I}$ via a surjective (N, N')-logical morphism $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$. By Theorem 2 and the fact that $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$ is a semi-interpretation, we have, for all $\Sigma, \Sigma_0 \in |\mathbf{Sign}|$ and all $\phi_0 \in \mathrm{SEN}(\Sigma_0)$, that

$$C'_{F(\Sigma)}(\alpha_\Sigma(\mathrm{Thm}^{\langle \Sigma_0,\phi_0\rangle}_\Sigma)) = \\ C'_{F(\Sigma)}(\alpha_\Sigma(\bigcup\{E_\Sigma(\langle \psi, \chi\rangle) : \langle \psi, \chi\rangle \in \Omega^N_\Sigma(\mathrm{Thm}^{\langle \Sigma_0,\phi_0\rangle})\})).$$

This, combined with the fact that $\langle F, \alpha\rangle$ is (N, N')-epimorphic and surjective, yields that

$$C'_{F(\Sigma)}(\alpha_\Sigma(\mathrm{Thm}^{\langle \Sigma_0,\phi_0\rangle}_\Sigma)) = \\ C'_{F(\Sigma)}(\bigcup\{E'_{F(\Sigma)}(\langle \alpha_\Sigma(\psi), \alpha_\Sigma(\chi)\rangle) : \langle \psi, \chi\rangle \in \Omega^N_\Sigma(\mathrm{Thm}^{\langle \Sigma_0,\phi_0\rangle})\}). \tag{4}$$

Consider two theory systems T, T' of $\mathcal{I}'$ and suppose that $\Omega^{N'}(T) = \Omega^{N'}(T')$. It will be shown that $T \leq T'$. By symmetry, then, it is obtained that $T' \leq T$, whence it will follow that $T = T'$ and, hence, that $\Omega^{N'}$ is injective on theory systems.

To show that $T \leq T'$, let $\Sigma' \in |\mathbf{Sign}'|$ and $\phi' \in T_{\Sigma'}$. By surjectivity of $\langle F, \alpha\rangle$, there exists $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$, such that $F(\Sigma) = \Sigma'$ and $\alpha_\Sigma(\phi) = \phi'$. Then we have

$$\begin{aligned} \phi' &\in C'_{F(\Sigma)}(\alpha_\Sigma(\mathrm{Thm}^{\langle \Sigma,\phi\rangle}_\Sigma)) \quad (\text{since } \phi' = \alpha_\Sigma(\phi)) \\ &= C'_{F(\Sigma)}(\bigcup\{E'_{F(\Sigma)}(\langle \alpha_\Sigma(\psi), \alpha_\Sigma(\chi)\rangle) : \langle \psi, \chi\rangle \in \Omega^N_\Sigma(\mathrm{Thm}^{\langle \Sigma,\phi\rangle})\}) \\ &\quad (\text{by Equation (4)}) \\ &\subseteq C'_{F(\Sigma)}(\bigcup\{E'_{F(\Sigma)}(\langle \alpha_\Sigma(\psi), \alpha_\Sigma(\chi)\rangle) : \langle \alpha_\Sigma(\psi), \alpha_\Sigma(\chi)\rangle \in \Omega^{N'}_{F(\Sigma)}(T)\}) \\ &\quad (\text{by Lemmas 2 and 3}) \\ &\subseteq C'_{F(\Sigma)}(\bigcup\{E'_{F(\Sigma)}(\langle \alpha_\Sigma(\psi), \alpha_\Sigma(\chi)\rangle) : \langle \alpha_\Sigma(\psi), \alpha_\Sigma(\chi)\rangle \in \Omega^{N'}_{F(\Sigma)}(T')\}) \\ &\quad (\text{by hypothesis, since } \Omega^{N'}(T) = \Omega^{N'}(T')) \\ &\subseteq T'_{F(\Sigma)} \quad (\text{by Lemma 4 and } N'\text{-protoalgebraicity}). \end{aligned}$$

Therefore $\phi' \in T'_{F(\Sigma)}$, yielding that $T \leq T'$, as was to be shown.

The following result is borrowed from [31].

Proposition 4 [Proposition 5.5 of [31]] *Suppose* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *is a* π*-institution and* N *a category of natural transformations on* SEN*. Let* $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ *be a functor,* N' *a category of natural transformations on* SEN' *and* $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ *a surjective singleton* (N, N')*-epimorphic translation. If* $\mathcal{I}$ *is* N*-protoalgebraic, then* $\mathrm{ThSys}^{\langle F,\alpha\rangle^\bigstar}_{\mathcal{I}}(\mathrm{SEN}') = \mathrm{ThSys}^{\langle F,\alpha\rangle}_{\mathcal{I}}(\mathrm{SEN}')$ *if and only if* $\Omega^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}$ *is injective on* $\mathrm{ThSys}^{\langle F,\alpha\rangle}_{\mathcal{I}}(\mathrm{SEN}')$.

Using this result, a necessary condition and a sufficient condition for those π-institutions whose full models via surjective (N,N')-logical morphisms $\langle F,\alpha\rangle$ can be identified with the theory systems of their $\langle F,\alpha\rangle$-min (N,N')-models is given. This is a partial analog of Theorem 3.8 of [10] and paves the way for introducing the notion of a *weakly algebraizable π-institution*, analogous, in the π-institution context, to the notion of a weakly algebraizable sentential logic.

Theorem 4 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution and N a category of natural transformations on* SEN*. Then the following statements are related by $(1 \leftrightarrow 2) \to 3 \to 4 \to 5 \to 6$:*

1. *$\mathcal{I}$ is N-protoalgebraic and, for every functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, with N' a category of natural transformations on SEN', and $\langle F,\alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$ a surjective singleton (N,N')-epimorphic translation, if $T \in \mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$, then $T/\Omega^{N'}(T)$ is the least theory system in the collection $\mathrm{ThSys}_{\mathcal{I}}^{\langle F,\pi_F^{\Omega^{N'}(T)}\alpha\rangle}(\mathrm{SEN}'^{\Omega^{N'}(T)})$.*
2. *For every functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, with N' a category of natural transformations on SEN', and $\langle F,\alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$ a surjective singleton (N,N')-epimorphic translation, the Leibniz operator $\Omega^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}$ is monotone on theory families and injective on theory systems.*
3. *For every functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, with N' a category of natural transformations on SEN', and $\langle F,\alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$ a surjective singleton (N,N')-epimorphic translation, the mapping $T \mapsto C'^{\min^T}$ is a bijection from $\mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$ to $\mathrm{FMod}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$ and, as a consequence, a complete lattice isomorphism between $\mathbf{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$ and $\mathbf{FMod}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$.*
4. *For every functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, with N' a category of natural transformations on SEN', and $\langle F,\alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$ a surjective singleton (N,N')-epimorphic translation, the Leibniz operator $\Omega^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}$ is a lattice isomorphism from $\mathbf{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$ to $\mathbf{Con}^{\langle F,\alpha\rangle}_{\mathrm{Alg}^N(\mathcal{I})}(\mathrm{SEN}')$.*
5. *For every functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, with N' a category of natural transformations on SEN', and $\langle F,\alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$ a surjective singleton (N,N')-epimorphic translation, the Leibniz operator $\Omega^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}$ is a lattice isomorphism from $\mathbf{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$ to $\mathbf{Con}^{\langle F,\alpha\rangle}_{\mathrm{Alg}^N(\mathcal{I})^*}(\mathrm{SEN}')$.*
6. *For every functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, with N' a category of natural transformations on SEN', and $\langle F,\alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$ a surjective singleton (N,N')-epimorphic translation, the Leibniz operator $\Omega^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}$ is monotone and injective on theory systems.*

Proof.
$1 \leftrightarrow 2$ N-protoalgebraicity is equivalent to the N'-Leibniz operator being monotone on theory families and, by Proposition 5.4 of [31], the property in Part 1 is equivalent to $T \in \mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle^\star}(\mathrm{SEN}')$. Therefore, by Proposition 4, the property in Part 1 is equivalent to the injectivity of the N'-Leibniz operator on theory systems.

$1 \to 3$ The given mapping is always injective and, by Proposition 5.4 of [31], the image lies in $\mathrm{FMod}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$. By Theorem 5.2 of [31], it is a surjective mapping, whence it is a bijection. Since, both itself and its inverse are obviously order-preserving, this mapping is a lattice isomorphism as claimed.

$3 \to 4$ Using again Theorem 5.2 of [31], we conclude that $\mathcal{I}$ is an N-protoalgebraic π-institution. Now compose the mapping of Part 3 with the mapping provided by the

Isomorphism Theorem 13 of [27] to obtain an isomorphism from $\mathbf{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$ to $\mathbf{Con}_{\mathrm{Alg}^N(\mathcal{I})}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$. This composite mapping is the mapping

$$T \mapsto \widetilde{\Omega}_{\mathrm{SEN}'}^{\langle F,\alpha\rangle}(C'^{\min^T}) = \Omega^{N'}(T),$$

the second equality following from Part 3 of Proposition 17 of [30]. We conclude that the N'-Leibniz operator is a lattice isomorphism from $\mathbf{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$ to $\mathbf{Con}_{\mathrm{Alg}^N(\mathcal{I})}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$.

$4 \to 5$ It suffices to show that $\mathrm{Alg}^N(\mathcal{I}) = \mathrm{Alg}^N(\mathcal{I})^*$. It was shown in Proposition 18 of [30] that $\mathrm{Alg}^N(\mathcal{I})^* \subseteq \mathrm{Alg}^N(\mathcal{I})$, whence, it suffices to show that $\mathrm{Alg}^N(\mathcal{I}) \subseteq \mathrm{Alg}^N(\mathcal{I})^*$. Every element of $\mathrm{Con}_{\mathrm{Alg}^N(\mathcal{I})}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$ is, by the hypothesis, of the form $\Omega^{N'}(T)$, for some $T \in \mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$. But, for every $T \in \mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$, $\Omega^{N'}(T) \in \mathrm{Alg}^N(\mathcal{I})^*$, whence, we obtain $\mathrm{Alg}^N(\mathcal{I}) \subseteq \mathrm{Alg}^N(\mathcal{I})^*$.

$5 \to 6$ This implication is trivial.

4 Definability of Theory Systems

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, N a category of natural transformations on SEN. An N-**matrix system** for $\mathcal{I}$ is a pair $\langle\langle \mathrm{SEN}', \langle F,\alpha\rangle\rangle, T'\rangle$, where $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ is a functor, with N' a category of natural transformations on SEN', $\langle F,\alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ is an (N,N')-epimorphic translation and $T' \in \mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$ is a theory system of the $\langle F,\alpha\rangle$-min (N,N')-model of $\mathcal{I}$ on SEN'. In this case T' will be said to be the **designated theory system** of the N-matrix system. An N-**matrix system class for** $\mathcal{I}$ is a class of N-matrix systems for $\mathcal{I}$.

An N-matrix system class for $\mathcal{I}$

$$\mathfrak{M} = \{\langle\langle \mathrm{SEN}^i, \langle F^i, \alpha^i\rangle\rangle, T^i\rangle : i \in I\}$$

will be said to **have designated theory systems implicitly definable** if, for all $i,j \in I$, such that $\langle \mathrm{SEN}^i, \langle F^i,\alpha^i\rangle\rangle = \langle \mathrm{SEN}^j, \langle F^j,\alpha^j\rangle\rangle$, we necessarily have $T^i = T^j$.

If $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution and N a category of natural transformations on SEN, we use the abbreviated notation $\langle \mathrm{SEN}, T\rangle$ to denote the N-matrix system $\langle\langle \mathrm{SEN}, \langle I_{\mathbf{Sign}}, \iota\rangle\rangle, T\rangle$, for all $T \in \mathrm{ThSys}(\mathcal{I})$. Similarly, slightly overloading notation, the notation $\langle \mathrm{SEN}^\theta, T/\theta\rangle$ will be used for $\langle\langle \mathrm{SEN}^\theta, \langle I_{\mathbf{Sign}}, \pi^\theta\rangle\rangle, T/\theta\rangle$ for an N-congruence system θ on SEN, where, of course, $\langle I_{\mathbf{Sign}}, \pi^\theta\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}^\theta$ denotes the natural (N, N^θ)-epimorphic projection.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, N a category of natural transformations on SEN and T a theory system of $\mathcal{I}$. T is said to **be explicitly** N-**definable** by a set of pairs of natural transformations $\Delta = \{\langle \gamma^j, \delta^j\rangle\}_{j\in J}$, with $\gamma^j, \delta^j : \mathrm{SEN} \to \mathrm{SEN}$ in N, for all $j \in J$, if, for all $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$,

$$\phi \in T_\Sigma \quad \text{iff} \quad (\forall j \in J)(\langle \gamma_\Sigma^j(\phi), \delta_\Sigma^j(\phi)\rangle \in \Omega_\Sigma^N(T)).$$

In the following proposition, a characterization of the injectivity of the N-Leibniz operator on theory systems is given in terms of the implicit definability of designated theory systems of given matrix system classes. Proposition 5 forms an analog of Proposition 3.7 of [8].

Proposition 5 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, N a category of natural transformations on* SEN *and E an N-parameterized equivalence system for $\mathcal{I}$. Then, the following are equivalent:*

1. *Ω^N is injective on theory systems.*
2. *The matrix system class $\{\langle \mathrm{SEN}^{\Omega^N(T)}, T/\Omega^N(T)\rangle : T \in \mathrm{ThSys}(\mathcal{I})\}$ has designated theory systems implicitly definable.*
3. *The matrix system class*

$$\{\langle\langle \mathrm{SEN}'^{\Omega^{N'}(T)}, \langle F, \pi_F^{\Omega^{N'}(T)}\alpha\rangle\rangle, T/\Omega^{N'}(T)\rangle : \langle F,\alpha\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}' \\ \textit{surjective and } T \in \mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')\}$$

has designated theory systems implicitly definable.

Proof $1 \rightarrow 3$ If Ω^N is injective on theory systems, then, by Theorem 3, $\Omega^{N'}$ is injective on the theory systems of the $\langle F,\alpha\rangle$-min (N,N')-model $\mathcal{I}'$ of $\mathcal{I}$ via the surjective (N,N')-logical morphism $\langle F,\alpha\rangle$. Thus, if

$$\langle \mathrm{SEN}'^{\Omega^{N'}(T)}, \langle F, \pi_F^{\Omega^{N'}(T)}\alpha\rangle\rangle = \langle \mathrm{SEN}'^{\Omega^{N'}(T')}, \langle F, \pi_F^{\Omega^{N'}(T')}\alpha\rangle\rangle,$$

then we must have $\mathrm{SEN}'^{\Omega^{N'}(T)} = \mathrm{SEN}'^{\Omega^{N'}(T')}$, whence $\Omega^{N'}(T) = \Omega^{N'}(T')$, whence $T = T'$ and, therefore, Condition 3 holds.

$3 \rightarrow 2$ This is obvious, since $\mathcal{I}$ is an $\langle \mathrm{I}_{\mathbf{Sign}}, \iota\rangle$-min (N,N)-model of $\mathcal{I}$ via the surjective (N,N)-logical morphism $\langle \mathrm{I}_{\mathbf{Sign}}, \iota\rangle : \mathcal{I} \vdash^{se} \mathcal{I}$.

$2 \rightarrow 1$ Suppose that T, T' are theory systems of $\mathcal{I}$ with $\Omega^N(T) = \Omega^N(T')$. Then, obviously, $\mathrm{SEN}^{\Omega^N(T)} = \mathrm{SEN}^{\Omega^N(T')}$. Consider the matrix systems

$$\langle \mathrm{SEN}^{\Omega^N(T)}, T/\Omega^N(T)\rangle \quad \text{and} \quad \langle \mathrm{SEN}^{\Omega^N(T')}, T'/\Omega^N(T')\rangle.$$

These two have the same functor component, whence, by Condition 2, we obtain $T/\Omega^N(T) = T'/\Omega^N(T')$. Therefore, since $\Omega^N(T)$ is compatible with T' and $\Omega^N(T')$ is compatible with T, we obtain $T = T'$.

Finally, in the following lemma, a characterization is provided of the explicit definability of designated theory systems of a given matrix system class. Lemma 5 forms a partial analog of Theorem 3.8 of [8].

Lemma 5 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, N a category of natural transformations on* SEN *and E an N-parameterized equivalence system for $\mathcal{I}$. Let $\Delta = \{\langle \gamma^j, \delta^j\rangle : j \in J\}$ be a set of pairs of natural transformations $\gamma^j, \delta^j : \mathrm{SEN} \rightarrow \mathrm{SEN}$ in N. Then $\mathcal{I}$ has its theory systems explicitly definable by Δ iff, for all $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,*

$$\phi \in T_\Sigma \textit{ iff } \bigcup\{E_{\Sigma'}(\langle \mathrm{SEN}(f)(\psi), \mathrm{SEN}(f)(\chi)\rangle) : \langle \psi,\chi\rangle \in \Delta_\Sigma(\phi)\} \subseteq T_{\Sigma'} \\ \textit{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma').$$

Proof $\mathcal{I}$ has its theory systems explicitly definable by Δ iff, for all $T \in \mathrm{ThSys}(\mathcal{I})$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $\phi \in T_\Sigma$ iff, for all $j \in J$, $\langle \gamma^j_\Sigma(\phi), \delta^j_\Sigma(\phi)\rangle \in \Omega^N_\Sigma(T)$ iff, since E is an N-parameterized equivalence system for $\mathcal{I}$, $\phi \in T_\Sigma$ iff, for all $j \in J$, $\langle \gamma^j_\Sigma(\phi), \delta^j_\Sigma(\phi)\rangle \in E_\Sigma(T)$ iff $\phi \in T_\Sigma$ iff, for all $j \in J$,

$$E_{\Sigma'}(\langle \mathrm{SEN}(f)(\gamma^j_\Sigma(\phi)), \mathrm{SEN}(f)(\delta^j_\Sigma(\phi))\rangle) \subseteq T_{\Sigma'},$$

for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma')$.

Finally, a partial analog of Theorem 3.9 of [8] is presented that ties the implicit definability with the explicit definability of designated theory systems of matrix systems for a given π-institution.

Theorem 5 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, N a category of natural transformations on* SEN *and E an N-parameterized equivalence system for $\mathcal{I}$. Then, the following are related by $1 \to (2 \leftrightarrow 3)$:*

1. *$\mathcal{I}$ has its theory systems explicitly definable by a set $\Delta = \{\langle \gamma^j, \delta^j\rangle\}_{j\in J}$ of pairs on natural transformations $\gamma^j, \delta^j : \mathrm{SEN} \to \mathrm{SEN}, j \in J$, in N.*
2. *Ω^N is injective on theory systems.*
3. *The N-matrix system class $\{\langle \mathrm{SEN}^{\Omega^N(T)}, T/\Omega^N(T)\rangle : T \in \mathrm{ThSys}(\mathcal{I})\}$ has designated theory systems implicitly definable.*

Proof The equivalence between 2 and 3 was shown in Proposition 5. That 1 implies 2 is easy to see. Let $T, T' \in \mathrm{ThSys}(\mathcal{I})$, such that $\Omega^N(T) = \Omega^N(T')$. We have, for all $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{aligned}\phi \in T_\Sigma \ &\text{iff}\ \langle \gamma^j_\Sigma(\phi), \delta^j_\Sigma(\phi)\rangle \in \Omega^N(T),\ \text{for all } j \in J,\\ &\text{iff}\ \langle \gamma^j_\Sigma(\phi), \delta^j_\Sigma(\phi)\rangle \in \Omega^N(T'),\ \text{for all } j \in J,\\ &\text{iff}\ \phi \in T'_\Sigma.\end{aligned}$$

Hence $T = T'$ and Ω^N is injective on theory systems.

We were unable to show, and think it unlikely that it holds in general, that implicit definability implies explicit definability, as is the case for sentential logics (see, e.g., Theorem 3.9 of [8]).

5 Weakly Algebraizable π-Institutions

A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is said to be N-**weakly algebraizable** if the Leibniz operator Ω^N is monotone on the collection of all theory families and injective on the collection of all theory systems of $\mathcal{I}$.

By combining the characterization of N-protoalgebraicity provided in Lemma 3.8 of [30] in terms of the monotonicity of the N-Leibniz operator on theory families and the characterization of the injectivity of the N-Leibniz operator on theory systems provided in Theorem 5, we obtain the following characterization of N-weak algebraizability.

Theorem 6 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution and N a category of natural transformations on* SEN. *Then $\mathcal{I}$ is N-weakly algebraizable iff $\mathcal{I}$ is N-protoalgebraic and the N-matrix system class $\{\langle \mathrm{SEN}^{\Omega^N(T)}, T/\Omega^N(T)\rangle : T \in \mathrm{ThSys}(\mathcal{I})\}$ has designated theory systems implicitly definable.*

Proof $\mathcal{I}$ is N-weakly algebraizable iff, by definition, Ω^N is monotone on theory families and injective on theory systems iff, by the characterization of N-protoalgebraicity (Lemma 3.8 of [30]) and by Theorem 5, $\mathcal{I}$ is N-protoalgebraic and the matrix system class $\{\langle \mathrm{SEN}^{\Omega^N(T)}, T/\Omega^N(T)\rangle : T \in \mathrm{ThSys}(\mathcal{I})\}$ has designated theory systems implicitly definable.

A sufficient condition for N-weak algebraizability will be presented next. A technical lemma is needed first.

Lemma 6 *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *is a* π*-institution,* N *a category of natural transformations on* SEN, $\Delta = \{\langle \gamma^j, \delta^j\rangle\}_{j\in J}$ *a collection of pairs of natural transformations* $\gamma^j, \delta^j : \mathrm{SEN} \to \mathrm{SEN}$ *in* N *and* $E = \{\epsilon^i : i \in I\}$ *a collection of natural transformations* $\epsilon^i : \mathrm{SEN}^{2+k_i} \to \mathrm{SEN}, i \in I$, *in* N. *If*

– *for all* $\Sigma \in |\mathbf{Sign}|, \Theta \cup \{\langle \phi, \psi\rangle\} \subseteq \mathrm{SEN}(\Sigma)^2$ *and all* $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\begin{aligned}(\Theta \subseteq \Omega^N_\Sigma(T) \Rightarrow \langle \phi, \psi\rangle \in \Omega^N_\Sigma(T)) \quad &\textit{implies}\\ (\forall f)(E_{\Sigma'}(\langle \mathrm{SEN}(f)^2(\Theta)\rangle) \subseteq T_{\Sigma'}) &\Rightarrow\\ &(\forall f)(E_{\Sigma'}(\langle \mathrm{SEN}(f)^2(\phi, \psi)\rangle) \subseteq T_{\Sigma'})\end{aligned}$$

– *for all* $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$, *and all theory families* $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\phi \in T_\Sigma \quad \textit{iff} \quad (\forall f)(E_{\Sigma'}(\langle \mathrm{SEN}(f)^2(\Delta_\Sigma(\phi))\rangle) \subseteq T_{\Sigma'}),$$

then, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$ *and all* $T \in \mathrm{ThFam}(\mathcal{I})$, *if* $\phi \in T_\Sigma$ *and* $(\forall f)(E_{\Sigma'}(\langle \mathrm{SEN}(f)^2(\phi, \psi)\rangle) \subseteq T_{\Sigma'})$, *then* $\psi \in T_\Sigma$.

Proof The proof is rather long, but not very difficult. It is modeled after the proof of Lemma 4.2 of [8]. To compactify the notation inside the proof the abbreviation $f(\phi) := \mathrm{SEN}(f)(\phi)$, for $f \in \mathbf{Sign}(\Sigma, \Sigma'), \phi \in \mathrm{SEN}(\Sigma)$, will be used.

By the first hypothesis, we have, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, all $\phi, \psi, \chi \in \mathrm{SEN}(\Sigma)$ and all $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{SEN}(\Sigma)^n$,

$$\begin{aligned}&(\forall f)(E_{\Sigma'}(\langle f(\phi), f(\psi)\rangle) \subseteq T_{\Sigma'}) \Rightarrow (\forall f)(E_{\Sigma'}(\langle f(\psi), f(\phi)\rangle) \subseteq T_{\Sigma'})\\ &(\forall i < n)(\forall f)(E_{\Sigma'}(\langle f(\phi_i), f(\psi_i)\rangle) \subseteq T_{\Sigma'}) \Rightarrow\\ &\qquad\qquad (\forall f)(E_{\Sigma'}(\langle f(\sigma_\Sigma(\boldsymbol{\phi})), f(\sigma_\Sigma(\boldsymbol{\psi}))\rangle) \subseteq T_{\Sigma'})\\ &(\forall f)(E_{\Sigma'}(\langle f(\phi), f(\psi)\rangle) \subseteq T_{\Sigma'}) \text{ and } (\forall f)(E_{\Sigma'}(\langle f(\psi), f(\chi)\rangle) \subseteq T_{\Sigma'}) \Rightarrow\\ &\qquad\qquad (\forall f)(E_{\Sigma'}(\langle f(\phi), f(\chi)\rangle) \subseteq T_{\Sigma'})\end{aligned}$$

Now, fix $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$ and $T \in \mathrm{ThFam}(\mathcal{I})$ and assume that $\phi \in T_\Sigma$ and $(\forall f)(E_{\Sigma'}(\langle f(\phi), f(\psi)\rangle) \subseteq T_{\Sigma'}$. Now, by the second condition in the hypothesis,

$$\phi \in T_\Sigma \quad \text{iff} \quad (\forall f)(E_{\Sigma'}(\langle f^2(\Delta_\Sigma(\phi))\rangle) \subseteq T_{\Sigma'}).$$

Combining the above we have, for all $j \in J$,

$$\begin{aligned}&(\forall f)(E_{\Sigma'}(\langle f(\phi), f(\psi)\rangle) \subseteq T_{\Sigma'}) \Rightarrow (\forall f)(E_{\Sigma'}(\langle f(\gamma^j_\Sigma(\phi)), f(\gamma^j_\Sigma(\psi))\rangle) \subseteq T_{\Sigma'})\\ &(\forall f)(E_{\Sigma'}(\langle f(\phi), f(\psi)\rangle) \subseteq T_{\Sigma'}) \Rightarrow (\forall f)(E_{\Sigma'}(\langle f(\delta^j_\Sigma(\phi)), f(\delta^j_\Sigma(\psi))\rangle) \subseteq T_{\Sigma'})\\ &(\forall f)(E_{\Sigma'}(\langle f(\gamma^j_\Sigma(\phi)), f(\gamma^j_\Sigma(\psi))\rangle) \subseteq T_{\Sigma'}) \Rightarrow\\ &\qquad\qquad (\forall f)(E_{\Sigma'}(\langle f(\gamma^j_\Sigma(\psi)), f(\gamma^j_\Sigma(\phi))\rangle) \subseteq T_{\Sigma'})\end{aligned}$$

We also have

$$\begin{aligned}&(\forall f)(E_{\Sigma'}(\langle f^2(\Delta_\Sigma(\phi))\rangle) \subseteq T_{\Sigma'}) \text{ and}\\ &\qquad (\forall f)(\forall j \in J)(E_{\Sigma'}(\langle f(\delta^j_\Sigma(\phi)), f(\delta^j_\Sigma(\psi))\rangle) \subseteq T_{\Sigma'})\\ &\qquad\quad \Rightarrow (\forall f)(\forall j \in J)(E_{\Sigma'}(\langle f(\gamma^j_\Sigma(\phi)), f(\delta^j_\Sigma(\psi))\rangle) \subseteq T_{\Sigma'})\end{aligned}$$

and

$$(\forall f)(\forall j \in J)(E_{\Sigma'}(\langle f(\gamma^j_\Sigma(\psi)), f(\gamma^j_\Sigma(\phi))\rangle) \subseteq T_{\Sigma'}) \text{ and}$$
$$(\forall f)(\forall j \in J)(E_{\Sigma'}(\langle f(\gamma^j_\Sigma(\phi)), f(\delta^j_\Sigma(\psi))\rangle) \subseteq T_{\Sigma'})$$
$$\Rightarrow (\forall f)(\forall j \in J)(E_{\Sigma'}(\langle f(\gamma^j_\Sigma(\psi)), f(\delta^j_\Sigma(\psi))\rangle) \subseteq T_{\Sigma'})$$

Therefore, we end up with $(\forall f)(E_{\Sigma'}(\langle f^2(\Delta_\Sigma(\psi))\rangle) \subseteq T_{\Sigma'})$, which is, by the second hypothesis, equivalent to $\psi \in T_\Sigma$.

Theorem 7 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, N a category of natural transformations on* SEN*, $\Delta = \{\langle \gamma^j, \delta^j\rangle\}_{j\in J}$ a collection of pairs of natural transformations $\gamma^j, \delta^j : \mathrm{SEN} \to \mathrm{SEN}, j \in J$, in N and $E = \{\epsilon^i : i \in I\}$ a collection of natural transformations $\epsilon^i : \mathrm{SEN}^{2+k_i} \to \mathrm{SEN}, i \in I$, in N. If*

- *for all $\Sigma \in |\mathbf{Sign}|, \Theta \cup \{\langle \phi, \psi\rangle\} \subseteq \mathrm{SEN}(\Sigma)^2$ and all $T \in \mathrm{ThFam}(\mathcal{I})$,*

$$(\Theta \subseteq \Omega^N_\Sigma(T) \Rightarrow \langle \phi, \psi\rangle \in \Omega^N_\Sigma(T)) \quad \textit{implies}$$
$$(\forall f)(E_{\Sigma'}(\langle \mathrm{SEN}(f)^2(\Theta)\rangle) \subseteq T_{\Sigma'}) \Rightarrow$$
$$(\forall f)(E_{\Sigma'}(\langle \mathrm{SEN}(f)^2(\phi, \psi)\rangle) \subseteq T_{\Sigma'})$$

- *for all $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$, and all theory families $T \in \mathrm{ThFam}(\mathcal{I})$,*

$$\phi \in T_\Sigma \quad \textit{iff} \quad (\forall f)(E_{\Sigma'}(\langle \mathrm{SEN}(f)^2(\Delta_\Sigma(\phi))\rangle) \subseteq T_{\Sigma'}),$$

then $\mathcal{I}$ is N-weakly algebraizable.

Proof Note that by Lemma 6 and Theorem 1, E is an N-parameterized equivalence system for $\mathcal{I}$. Therefore, by Proposition 1, $\mathcal{I}$ is N-protoalgebraic. So it suffices to show that Ω^N is injective on theory systems. This is done by using the $1 \to 2$ implication of Theorem 5. We have, for all $T \in \mathrm{ThSys}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{aligned} \phi \in T_\Sigma &\text{ iff } (\forall f)(E_{\Sigma'}(\langle \mathrm{SEN}(f)^2(\Delta_\Sigma(\phi))\rangle) \subseteq T_{\Sigma'}) \\ &\text{ iff } \Delta_\Sigma(\phi) \subseteq E_\Sigma(T) \\ &\text{ iff } \Delta_\Sigma(\phi) \subseteq \Omega^N_\Sigma(T). \end{aligned}$$

Therefore Δ explicitly defines theory systems of $\mathcal{I}$, and, by Theorem 5, Ω^N is injective on theory systems.

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