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Categorical Abstract Algebraic Logic: Behavioral π-Institutions

Abstract. Recently, Caleiro, Gonçalves and Martins introduced the notion of behaviorally algebraizable logic. The main idea behind their work is to replace, in the traditional theory of algebraizability of Blok and Pigozzi, unsorted equational logic with multi-sorted behavioral logic. The new notion accommodates logics over many-sorted languages and with non-truth-functional connectives. Moreover, it treats logics that are not algebraizable in the traditional sense while, at the same time, shedding new light to the equivalent algebraic semantics of logics that are algebraizable according to the original theory. In this paper, the notion of an abstract multi-sorted π-institution is introduced so as to transfer elements of the theory of behavioral algebraizability to the categorical setting. Institutions formalize a wider variety of logics than deductive systems, including logics involving multiple signatures and quantifiers. The framework developed has the same relation to behavioral algebraizability as the classical categorical abstract algebraic logic framework has to the original theory of algebraizability of Blok and Pigozzi.

Keywords: Algebraic Logic, Multi-sorted Behavioral Logic, Behavioral Algebraizability, Behavioral Leibniz Operator, Behavioral Leibniz Hierarchy, Multi-sorted π-Institutions, Behavioral Leibniz Congruence Systems, Behavioral Categorical Leibniz Hierarchy.

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1. Introduction

In [8] Caleiro, Gonçalves and Martins, based on previous work of Caleiro and Gonçalves on the algebraization of multi-sorted logics [6, 7] and of Martins on the behavioral equivalence of k-deductive systems [21, 22], introduced the notion of *behaviorally algebraizable logic*. They noticed that many logics that fail to be algebraizable in the traditional sense of Blok and Pigozzi [4] (or any of its refinements and extensions, e.g., [19, 20, 1]) do so even though they include an algebraizable fragment. This happens because the language includes connectives that are non-truth-functional, i.e., fail to have the congruence property with respect to the equivalence of the algebraizable fragment. Motivated by this observation, they replaced in the traditional

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framework of algebraization unsorted equational logic by many-sorted behavioral logic with one distinguished sort standing for the sort of formulas. In this way, they obtained behaviorally algebraizable logics. Besides extending the scope of the theory to multi-sorted logics and to logics with non-truth-functional connectives, the new theory offers algebraic semantics to logics that were not algebraizable in the traditional sense. Moreover, it sheds new light to the algebraic counterparts of logics that were Blok-Pigozzi (or Herrmann) algebraizable, but whose semantics were not perfectly understood. In addition to introducing the new notion, Caleiro, Gonçalves and Martins also introduce a behavioral Leibniz operator, which provides, given a theory of a logic, the largest behavioral congruence, termed the *behavioral Leibniz congruence*, associated with the theory. In this way, they are able to classify and characterize logics in a behavioral algebraic hierarchy much in the same way as the traditional theory, initiated by Blok and Pigozzi [4, 3, 5] and extended in [9, 10, 12], gave rise to the well-known Leibniz hierarchy (see [14, 11, 15] for overviews). In fact the theory of [8] gives rise to the hierarchy depicted below, which resides inside the class of protoalgebraic logics in the traditional sense.

$$
\begin{tikzcd}
 & \text{Algebraizable} \arrow[dl] \arrow[dd] \arrow[dr] & \\
\text{Equivalential} \arrow[dd] & & \text{Weakly Algebraizable} \arrow[dd] \\
 & \text{Behav. Algebraizable} \arrow[dl] \arrow[dr] & \\
\text{Behav. Equivalential} \arrow[dr] & & \text{Behav. Weakly Algble} \arrow[dl] \\
 & \text{Protoalgebraic} &
\end{tikzcd}
$$

As is pointed out in [8], the term "*behavioral*" originates in Computer Science, and more precisely, in the area of specification and verification of complex, object-oriented systems (see, e.g., [25, 2, 23]). Abstract data types and object classes are specified by properties of their associated operations. In particular, data are classified as *hidden* or *visible*, where the former are intended to capture the internal states of an abstract machine on which programs are run and the latter are the input/output data that are directly accessible and observable by the user. Inferences about the hidden data can only be made indirectly by observing the visible data through "experiments". The notion of *behavioral equivalence* captures the property of two values being indistinguishable based on available experiments. For details about behavioral or hidden equational logic the reader may consult [26, 27, 28].

The key in the treatment of logics in [8] with respect to behavioral properties is to consider multi-sorted signatures with one distinguished sort ϕ, called the *sort of formulas*. Thus, although a corresponding absolutely free term algebra has terms of each sort, only those of sort ϕ will be taken to be formulas of the logic and all the remaining have behaviors that are only observable through their indirect influence as components of terms of sort ϕ. We transfer these treatment to the categorical level by considering logics formalized as π-institutions, in which signatures may be many-sorted but one of them is singled out to represent the sort of formulas of interest in an analogous way that directly abstracts the treatment in [8]. The goal here is to fill in the upper-right-hand side of the following rectangle, where edges are thought of as representing "analogies" and point from the less to the more abstract setting:

$$
\begin{tikzcd}
\text{Algebraizable } \pi\text{-Institutions} \arrow[r] & \text{Behaviorally Algebraizable } \pi\text{-Institutions} \\
\text{Algebraizable Logics} \arrow[u] \arrow[r] & \text{Behaviorally Algebraizable Logics} \arrow[u]
\end{tikzcd}
$$

We now provide an overview of the contents of the paper. In Section 2, several of the notions that have proven invaluable in building the categorical theory of abstract algebraic logic are reviewed. The notion of a π-institution is recalled, which provides the underlying structure in which logical systems are formalized. Moreover, categories of natural transformations on set-valued functors are reviewed. They provide an analog of the clone of algebraic operations in the categorical setting and have played a key role in lifting universal algebraic properties of logics to the categorical level. These categories help in formulating the notion of congruence system over a set-valued functor, which forms an analog of the notion of congruence in universal algebra.

Our new material begins in Section 3, that starts with the introduction of the notion of a multi-sorted sentence functor and that of a category of multi-sorted natural transformations on a multi-sorted functor. These categories are also referred to as transformation signatures. They are both key components in formulating the notion of an abstract multi-sorted π-institution, which generalizes the notion of a π-institution and aims at providing a framework in which a categorical study of behavioral logical systems may take place.

Section 4 introduces the notion of behavioral equivalence system on a given multi-sorted sentence functor. Roughly speaking, two elements are identified in this equivalence system if, regardless of how they are translated across signatures and of which contexts they are used in, they cannot be distinguished by experiments over the given sentence functor. It is shown that the behavioral equivalence system is indeed an equivalence system in the sense of categorical abstract algebraic logic and that it is a congruence system if one restricts attention to those operations in the clone that have output of visible sort.

The remaining three sections of the paper initiate the development of a categorical theory of behavioral algebraic logic providing key analogs of corresponding results from the theory of categorical abstract algebraic logic along the lines of [8]. The feasibility of this endeavor, as well as the question of the precise connections between the work presented in [6, 8] and the categorical framework were raised in [6]

In Section 5, we introduce the notion of a theory family of a multi-sorted π-institution and define the behavioral Leibniz congruence system associated with the family. The operator that maps a theory family to its associated behavioral Leibniz congruence system plays a key role in the behavioral theory of algebraizability. It abstracts, at the same time, the ordinary Leibniz operator of categorical abstract algebraic logic and the behavioral Leibniz operator of [8].

Section 6 introduces the notion of a behaviorally N-protoalgebraic multi-sorted π-institution. It characterizes this class of multi-sorted π-institutions as those on whose lattices of theory families the behavioral Leibniz operator is monotone. It also provides a sufficient condition based on the existence of a protoequivalence system. Both results have been known to hold for ordinary deductive systems and π-institutions and were lifted to the behavioral context in [8].

Finally, the paper concludes with Section 7, which introduces the notion of behaviorally N-equivalential multi-sorted π-institution. These institutions form a subclass of the class of behaviorally N-protoalgebraic multi-sorted π-institutions. Here, as samples of the variety of possible results that may be obtained, we provide an analog of the classical result on the definability of the behavioral Leibniz congruence via equivalence systems as well as an analog of Herrmann's Test, that characterizes those abstract multi-sorted π-institutions that are behaviorally N-equivalential inside the class of all behaviorally N-protoalgebraic multi-sorted π-institutions.

2. π-Institutions and Natural Transformations

To generalize the framework of [8] to accommodate logics with multiple signatures and quantifiers we will use as our underlying structures, instead of sentential logics, π-institutions [13] (see also [17, 18]).

A **π-institution** $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}\rangle$ consists of

(i) A category **Sign** whose objects are called **signatures**;

(ii) A functor SEN : **Sign** $\to$ **Set**, from the category **Sign** of signatures into the category **Set** of small sets, called the **sentence functor** and giving, for each signature Σ, a set whose elements are called **sentences over** that signature Σ or Σ-**sentences**;

(iii) A mapping $C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \to \mathcal{P}(\mathrm{SEN}(\Sigma))$, for each $\Sigma \in |\mathbf{Sign}|$, called Σ-**closure,** such that

  (a) $A \subseteq C_\Sigma(A)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq \mathrm{SEN}(\Sigma)$,

  (b) $C_\Sigma(A) \subseteq C_\Sigma(B)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq B \subseteq \mathrm{SEN}(\Sigma)$,

  (c) $C_\Sigma(C_\Sigma(A)) = C_\Sigma(A)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq \mathrm{SEN}(\Sigma)$,

  (d) $\mathrm{SEN}(f)(C_{\Sigma_1}(A)) \subseteq C_{\Sigma_2}(\mathrm{SEN}(f)(A))$, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2), A \subseteq \mathrm{SEN}(\Sigma_1)$.

For simplicity, we write $C = \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ and call C a **closure (operator) system**. A Σ-**theory** of $\mathcal{I}$ is a set $T_\Sigma \subseteq \mathrm{SEN}(\Sigma)$, such that $C_\Sigma(T_\Sigma) = T_\Sigma$. A **theory family** of $\mathcal{I}$ is a $|\mathbf{Sign}|$-indexed collection $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, such that, for all $\Sigma \in |\mathbf{Sign}|$, T_Σ is a Σ-theory. A **theory system** of $\mathcal{I}$ is a theory family $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, such that, for all $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, $\mathrm{SEN}(f)(T_{\Sigma_1}) \subseteq T_{\Sigma_2}$. In other words, as is customary in categorical abstract algebraic logic, whenever the word "family" is replaced by the word "system", invariance under signature morphisms is also assumed. The collection of all theory families of a π-institution $\mathcal{I}$ is denoted by $\mathrm{ThFam}(\mathcal{I})$ and the collection of all theory systems by $\mathrm{ThSys}(\mathcal{I})$. Endowed with signature-wise inclusion $\leq$, $\mathrm{ThFam}(\mathcal{I})$ becomes a complete lattice, denoted $\mathbf{ThFam}(\mathcal{I}) = \langle \mathrm{ThFam}(\mathcal{I}), \leq\rangle$ and the same holds for $\mathbf{ThSys}(\mathcal{I}) = \langle \mathrm{ThSys}(\mathcal{I}), \leq\rangle$.

As an illustration of the definition, let us indicate how a deductive system may be recast as a π-institution. The reader should keep in mind that there are other possible alternatives and each may have advantages and disadvantages over the others. Given a language type $\mathcal{L}$ and a fixed denumerable set V of variables, the set of all $\mathcal{L}$-formulas with variables in V is denoted by $\mathrm{Fm}_\mathcal{L}(V)$. The corresponding absolutely free $\mathcal{L}$-algebra of formulas is denoted by $\mathbf{Fm}_\mathcal{L}(V)$. A substitution σ is an endomorphism of the formula algebra $\mathbf{Fm}_{\mathcal{L}}(V)$. The set of all substitutions will be denoted by $\mathrm{End}(\mathbf{Fm}_{\mathcal{L}}(V))$. By a (not necessarily finitary) deductive system over $\mathcal{L}$ we mean a pair $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$, where $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)) \times \mathrm{Fm}_{\mathcal{L}}(V)$ satisfies, for all $\Phi \cup \Psi \cup \{\phi, \psi, \chi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

1. $\Phi \vdash_{\mathcal{S}} \phi$, if $\phi \in \Phi$;
2. $\Phi \vdash_{\mathcal{S}} \psi$, for all $\psi \in \Psi$, and $\Psi \vdash_{\mathcal{S}} \chi$ imply $\Phi \vdash_{\mathcal{S}} \chi$;
3. $\Phi \vdash_{\mathcal{S}} \phi$ implies $\sigma(\Phi) \vdash_{\mathcal{S}} \sigma(\phi)$, for all substitutions σ.

These conditions also imply that, for all $\Phi \cup \Psi \cup \{\phi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

4. $\Phi \vdash_{\mathcal{S}} \phi$ implies $\Psi \vdash_{\mathcal{S}} \phi$, if $\Phi \subseteq \Psi$.

Define $C_{\mathcal{S}} : \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)) \to \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V))$ by setting, for all $\Phi \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

$$C_{\mathcal{S}}(\Phi) = \{\phi \in \mathrm{Fm}_{\mathcal{L}}(V) : \Phi \vdash_{\mathcal{S}} \phi\}. \tag{1}$$

Let $\mathcal{I}_{\mathcal{S}} = \langle \mathbf{Sign}_{\mathcal{L}}, \mathrm{SEN}_{\mathcal{L}}, C_{\mathcal{S}} \rangle$, where

- $\mathbf{Sign}_{\mathcal{L}}$ is a single object category, with object, say, V, and $\mathbf{Sign}_{\mathcal{L}}(V, V) = \mathrm{End}(\mathbf{Fm}_{\mathcal{L}}(V))$. Composition and identities are the usual ones in the monoid of endomorphisms.
- $\mathrm{SEN}_{\mathcal{L}}(V) = \mathrm{Fm}_{\mathcal{L}}(V)$ and, given $\sigma \in \mathbf{Sign}_{\mathcal{L}}(V, V)$, we get $\mathrm{SEN}_{\mathcal{L}}(\sigma)(\phi) = \sigma(\phi)$, for all $\phi \in \mathrm{Fm}_{\mathcal{L}}(V)$. This defines a functor $\mathrm{SEN}_{\mathcal{L}} : \mathbf{Sign}_{\mathcal{L}} \to \mathbf{Set}$.
- Finally, $C_{\mathcal{S}} : \mathcal{P}(\mathrm{SEN}_{\mathcal{L}}(V)) \to \mathcal{P}(\mathrm{SEN}_{\mathcal{L}}(V))$ is the function defined in (1).

It is not difficult to verify that $\mathcal{I}_{\mathcal{S}}$ is a π-institution. It is called the π-institution **associated with** the deductive system $\mathcal{S}$.

The notion of a category of natural transformations on a sentence functor was introduced in [35] and was updated in [29, 31]. It is intended to capture in the categorical level the counterpart of the universal algebraic notion of clone of algebraic operations generated by the fundamental operations of a universal algebra. Before introducing the notion formally, note that in the context of the sentence functor $\mathrm{SEN}_{\mathcal{L}}$ of the π-institution $\mathcal{I}_{\mathcal{S}}$ associated with a given deductive system $\mathcal{S}$, n-ary term operations $t(v_0, \ldots, v_{n-1})$ may be identified with natural transformations $\tau : \mathrm{SEN}_{\mathcal{L}}^n \to \mathrm{SEN}_{\mathcal{L}}$. In fact, the mappings $t(\vec{v}) \mapsto \tau$, with $\tau_V(\vec{\phi}) = t(\vec{\phi})$, for all $\vec{\phi} \in \mathrm{SEN}_{\mathcal{L}}^n(V)$, and $\tau \mapsto \tau_V(\vec{v})$, where $\vec{v} = \langle v_0, \ldots, v_{n-1} \rangle$ if τ is n-ary, establish a bijection between n-ary term operations and n-ary natural transformations on $\mathrm{SEN}_{\mathcal{L}}$. Motivated by this paradigm, we make the following definition.

Let **Sign** be a category and SEN : **Sign** $\to$ **Set** a functor. The **clone of all natural transformations on** SEN is defined to be the locally small category with collection of objects $\{\mathrm{SEN}^\alpha : \alpha \text{ an ordinal}\}$ and collection of morphisms $\tau : \mathrm{SEN}^\alpha \to \mathrm{SEN}^\beta$ β-sequences of natural transformations $\tau_i : \mathrm{SEN}^\alpha \to \mathrm{SEN}$. Composition of

$$\mathrm{SEN}^\alpha \xrightarrow{\langle \tau_i : i < \beta \rangle} \mathrm{SEN}^\beta \xrightarrow{\langle \sigma_j : j < \gamma \rangle} \mathrm{SEN}^\gamma$$

is defined by

$$\langle \sigma_j : j < \gamma \rangle \circ \langle \tau_i : i < \beta \rangle = \langle \sigma_j(\langle \tau_i : i < \beta \rangle) : j < \gamma \rangle.$$

A subcategory N of this category containing *all* objects of the form SEN^k for $k < \omega$, and all projection morphisms $p^{k,i} : \mathrm{SEN}^k \to \mathrm{SEN}, i < k, k < \omega$, with $p^{k,i}_\Sigma : \mathrm{SEN}(\Sigma)^k \to \mathrm{SEN}(\Sigma)$ given by

$$p^{k,i}_\Sigma(\vec{\phi}) = \phi_i, \quad \text{for all} \quad \vec{\phi} \in \mathrm{SEN}(\Sigma)^k,$$

and such that, for every family $\{\tau_i : \mathrm{SEN}^k \to \mathrm{SEN} : i < l\}$ of natural transformations in N, the sequence $\langle \tau_i : i < l \rangle : \mathrm{SEN}^k \to \mathrm{SEN}^l$ is also in N, is referred to as a **category of natural transformations on** SEN.

Motivated by the fact that congruences in universal algebra are equivalence relations that are preserved by the fundamental operations and, therefore, as a result by all derived operations on the clone generated by the fundamental operations, we make the following definition of congruence systems on set-valued functors. Let **Sign** be a category, SEN : **Sign** $\to$ **Set** be a functor and N be a category of natural transformations on SEN. Given $\Sigma \in |\mathbf{Sign}|$, an equivalence relation θ_Σ on $\mathrm{SEN}(\Sigma)$ is said to be an N-**congruence** if, for all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)^k$,

$$\vec{\phi}\ \theta^k_\Sigma\ \vec{\psi} \quad \text{imply} \quad \sigma_\Sigma(\vec{\phi})\ \theta_\Sigma\ \sigma_\Sigma(\vec{\psi}).$$

A collection $\theta = \{\theta_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ is called an **equivalence system** of SEN if

- θ_Σ is an equivalence relation on $\mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$,
- $\mathrm{SEN}(f)^2(\theta_{\Sigma_1}) \subseteq \theta_{\Sigma_2}$, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$.

If, in addition, θ_Σ is an N-congruence, for all $\Sigma \in |\mathbf{Sign}|$, then θ is said to be an N-**congruence system** of SEN. By $\mathrm{Con}^N(\mathrm{SEN})$ is denoted the collection of all N-congruence systems of SEN.

Let now $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} \rangle$ be a π-institution. An equivalence system θ of SEN is called a **logical equivalence system** of $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi, \psi \rangle \in \theta_\Sigma \quad \text{implies} \quad C_\Sigma(\phi) = C_\Sigma(\psi).$$

An N-congruence system of SEN is a **logical N-congruence system** of $\mathcal{I}$ if it is logical as an equivalence system of $\mathcal{I}$. This notion abstracts the notion of a logical congruence, which was defined in the context of abstract logics, i.e., generalized matrices, in [14].

3. Abstract Multi-Sorted π-Institutions

In this section, based on the notion of π-institution, we introduce the notion of an (abstract) multi-sorted π-institution. These institutions are used to transfer elements of the theory of algebraization of multi-sorted deductive systems [6] and of behavioral abstract algebraic logic [8] to the categorical setting, where they have the potential of wider applicability.

To motivate the definition of a multi-sorted sentence functor, we look at the case of a multi-sorted deductive system and how it can be recast as a π-institution. A multi-sorted signature $\Sigma = \langle S, F \rangle$ consists of a set S of sorts together with an indexed family $F = \{F_{ws}\}_{w \in S^*, s \in S}$ of sets of operation symbols. Given Σ and a fixed indexed collection $X = \{X_s\}_{s \in S}$ of denumerable sets of variables, one for each sort, the set of all Σ-formulas with variables in X is denoted by $\mathrm{Fm}_\Sigma(X) = \{\mathrm{Fm}_{\Sigma,s}(X)\}_{s \in S}$. The corresponding absolutely free Σ-algebra of formulas is denoted by $\mathbf{Fm}_\Sigma(X)$. A substitution σ is an endomorphism of the formula algebra $\mathbf{Fm}_\Sigma(X)$, which, due to freeness, exactly corresponds to a family of functions $\sigma = \{\sigma_s : X_s \to \mathrm{Fm}_{\Sigma,s}(X)\}_{s \in S}$, indexed by the set of sorts. The set of all substitutions will be denoted by $\mathrm{End}(\mathbf{Fm}_\Sigma(X))$. In [6] (modulo some technical details), a (not necessarily finitary) multi-sorted deductive system over Σ is defined to be a pair $\mathcal{S} = \langle \Sigma, \vdash_\mathcal{S} \rangle$, where $\vdash_\mathcal{S} \subseteq \mathcal{P}(\mathrm{Fm}_\Sigma(X)) \times \mathrm{Fm}_\Sigma(X)$ satisfies, for all $\Phi \cup \Psi \cup \{\phi, \psi, \chi\} \subseteq \mathrm{Fm}_\Sigma(X)$,

1. $\Phi \vdash_\mathcal{S} \phi$, if $\phi \in \Phi$;
2. $\Phi \vdash_\mathcal{S} \psi$, for all $\psi \in \Psi$, and $\Psi \vdash_\mathcal{S} \chi$ imply $\Phi \vdash_\mathcal{S} \chi$;
3. $\Phi \vdash_\mathcal{S} \phi$ implies $\sigma(\Phi) \vdash_\mathcal{S} \sigma(\phi)$, for all substitutions σ.

In [8], where the focus is on behavioral algebraization, a special sort $\pounds \in S$ of formulas is singled out and a multi-sorted deductive system is postulated to be one of the form $\mathcal{S} = \langle \Sigma, \vdash_{\mathcal{S}} \rangle$, where $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(\mathrm{Fm}_{\Sigma,\pounds}(X)) \times \mathrm{Fm}_{\Sigma,\pounds}(X)$ satisfies, for all $\Phi \cup \Psi \cup \{\phi, \psi, \chi\} \subseteq \mathrm{Fm}_{\Sigma,\pounds}(X)$, the conditions listed above.

We formulate the definition of the π-institution $\mathcal{I}_{\mathcal{S}}$ associated with the multi-sorted deductive system $\mathcal{S}$ in such a way as to accommodate both definitions (and also others). Define $C_{\mathcal{S}}$ to be the closure operator associated with the consequence relation $\vdash_{\mathcal{S}}$ exactly as in the case of a single-sorted logic $\mathcal{S}$. This is either a function $C_{\mathcal{S}} : \mathcal{P}(\mathrm{Fm}_{\Sigma}(X)) \to \mathcal{P}(\mathrm{Fm}_{\Sigma}(X))$ or a function $C_{\mathcal{S}} : \mathcal{P}(\mathrm{Fm}_{\Sigma,\pounds}(X)) \to \mathcal{P}(\mathrm{Fm}_{\Sigma,\pounds}(X))$, depending on which of the two definitions of a multi-sorted deductive system is adopted.

Let $\mathcal{I}_{\mathcal{S}} = \langle \mathbf{Sign}_{\Sigma}, \mathrm{SEN}_{\Sigma}, C_{\mathcal{S}} \rangle$, where

- $\mathbf{Sign}_{\Sigma}$ is a single object category, with object, say, X, and $\mathbf{Sign}_{\Sigma}(X, X) = \mathrm{End}(\mathbf{Fm}_{\Sigma}(X))$. Composition and identities are the usual ones in the monoid of endomorphisms.
- $\mathrm{SEN}_{\Sigma}(X) = \mathrm{Fm}_{\Sigma}(X)$ (or $\mathrm{SEN}_{\Sigma}(X) = \mathrm{Fm}_{\Sigma,\pounds}(X)$) and, given $\sigma \in \mathbf{Sign}_{\Sigma}(X, X)$, we get $\mathrm{SEN}_{\Sigma}(\sigma)(\phi) = \sigma(\phi)$, for all $\phi \in \mathrm{SEN}_{\Sigma}(X)$. This defines a functor $\mathrm{SEN}_{\Sigma} : \mathbf{Sign}_{\Sigma} \to \mathbf{Set}$.
- Finally, $C_{\mathcal{S}}$ is the closure operator defined above.

It is not difficult to verify that $\mathcal{I}_{\mathcal{S}}$ is a π-institution, called the π-institution **associated with** the multi-sorted deductive system $\mathcal{S}$. As can be easily seen by abstracting from this example, we could have taken any subset of the sorts in S as relevant for the consequence relation, rather than just the sort $\pounds$, and the corresponding multi-sorted deductive system as well as the associated π-institution would have been built similarly.

We now proceed to define the abstract notion of a multi-sorted sentence functor. We define it in such a way so as to be able to accommodate later the notion of a category of multi-sorted natural transformations that will generalize the clone of multi-sorted operations generated by the basic operations of a multi-sorted universal algebra.

A sentence functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is said to be **multi-sorted** if there exists a set S of **sorts** and set-valued functors $\mathrm{SEN}_s : \mathbf{Sign} \to \mathbf{Set}, s \in S$, such that $\mathrm{SEN} = \prod_{s \in S} \mathrm{SEN}_s$. A multi-sorted sentence functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ over a set of sorts S is said to be **hidden** if a subset $V \subseteq S$ of the sorts, called the set of **visible sorts**, has been singled out. The set $H = S \backslash V$ is called the set of **hidden sorts**.

The notion of a category of natural transformations on the sentence functor of an abstract multi-sorted π-institution is a generalization of the corresponding notion for a π-institution, presented in the previous section. Note again that, given a multi-sorted signature $\Sigma = \langle S, F \rangle$, as before, the derived operations of the form $t(x_1 : s_1, \dots, x_n : s_n)$ of sort s correspond to natural transformations $\tau : \mathrm{SEN}_{s_1} \times \cdots \times \mathrm{SEN}_{s_n} \to \mathrm{SEN}_s$. This motivates the following definition of the category of multi-sorted natural transformations on SEN, which is intended to abstract the entire clone of derived operations on a multi-sorted universal algebra.

Let **Sign** be a category and SEN : **Sign** $\to$ **Set** a multi-sorted sentence functor. The **clone of all natural transformations on** SEN is defined to be the locally small category with collection of objects $\{\prod_{\kappa<\alpha} \mathrm{SEN}_{s_\kappa} : s_\kappa \in S, \alpha \text{ an ordinal}\}$ and collection of morphisms $\tau : \prod_{\kappa<\alpha} \mathrm{SEN}_{s_\kappa} \to \prod_{\lambda<\beta} \mathrm{SEN}_{s'_\lambda}$ β-sequences of natural transformations $\tau_i : \prod_{\kappa<\alpha} \mathrm{SEN}_{s_\kappa} \to \mathrm{SEN}_{s'_\lambda}$, $\lambda < \beta$. Composition of

$$\prod_{\kappa<\alpha} \mathrm{SEN}_{s_\kappa} \xrightarrow{\langle \tau_i : i < \beta \rangle} \prod_{\lambda<\beta} \mathrm{SEN}_{s'_\lambda} \xrightarrow{\langle \sigma_j : j < \gamma \rangle} \prod_{\mu<\gamma} \mathrm{SEN}_{s''_\mu}$$

is defined by

$$\langle \sigma_j : j < \gamma \rangle \circ \langle \tau_i : i < \beta \rangle = \langle \sigma_j(\langle \tau_i : i < \beta \rangle) : j < \gamma \rangle.$$

A subcategory N of this category containing *all* objects of form $\prod_{i=1}^k \mathrm{SEN}_{s_i}$ for $k < \omega$, and all projection morphisms $p^{s_1 \dots s_k \to s_i} : \prod_{i=1}^k \mathrm{SEN}_{s_i} \to \mathrm{SEN}_{s_i}$, $i < k, k < \omega$, with $p_\Sigma^{s_1 \dots s_k \to s_i} : \prod_{i=1}^k \mathrm{SEN}_{s_i}(\Sigma) \to \mathrm{SEN}_{s_i}(\Sigma)$ given by

$$p_\Sigma^{s_1 \dots s_k \to s_i}(\vec{\phi}) = \phi_i, \quad \text{for all} \quad \vec{\phi} \in \prod_{i=1}^k \mathrm{SEN}_{s_i}(\Sigma),$$

and such that, for every family $\{\tau_i : \prod_{i=1}^k \mathrm{SEN}_{s_k} \to \mathrm{SEN}_{s'_i} : i < l\}$ of natural transformations in N, the sequence $\langle \tau_i : i < l \rangle : \prod_{i=1}^k \mathrm{SEN}_{s_k} \to \prod_{i=1}^l \mathrm{SEN}_{s'_i}$ is also in N, is referred to as a **category of (multi-sorted) natural transformations on** SEN. We will refer to such a category also as a **transformation signature on** SEN. A natural transformation $\sigma : \prod_{i=1}^m \mathrm{SEN}_{s_i} \to \mathrm{SEN}_s$ in N will be said to be **of type** $s_1 \dots s_m \to s$ or **of sort** s, if only the output sort is relevant.

Recall that a subcategory of a given category is **wide** if it contains all objects of the original category. On some occasions in the remainder of the paper, we will be considering, given a multi-sorted functor SEN : **Sign** $\to$ **Set** and a category N of multi-sorted natural transformations on SEN, a wide subcategory N' of N, that is also a category of natural transformations on SEN on its own right, i.e. it contains the projection natural transformations and is closed under formation of tuples. Such a subcategory will be referred to as a **transformation subsignature** of N.

Following [8] (see, also, [21, 22] and [23]), we present a definition of a multi-sorted π-institution that captures the case of a multi-sorted deductive system, presented above, in which a single sort $\pounds$ of formulas is singled out and an entailment relation is introduced only on sentences of sort $\pounds$. However, as was indicated in the preceding example, this definition may be generalized further to accommodate entailment relations over sentences of sorts belonging to arbitrary subsets of the set S of sorts.

Let **Sign** be a category and SEN : **Sign** $\to$ **Set** be a multi-sorted sentence functor over a set S of sorts containing a distinguished sort $\pounds$ of **formulas**. A **multi-sorted π-institution over** SEN is a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}_\pounds, C\rangle$.

Recall that a subcategory of a given category is **full** if, for any pair of objects that it contains, it also contains all morphisms in the original category between the two objects. A category of natural transformations N on $\mathrm{SEN}_\pounds$ is said to be a **transformation signature of $\mathcal{I}$ over** SEN if it is the full subcategory of a transformation signature N' on SEN with objects $\mathrm{SEN}_\pounds^k$, for all $k < \omega$.

Let SEN : **Sign** $\to$ **Set** be a hidden functor and N a transformation signature on SEN. An N-**context for** sort $s \in S$ (and **of type** $ss_1 \ldots s_m \to s'$ or, putting the emphasis on the output sort, **of sort** s') is a natural transformation

$$\sigma : \mathrm{SEN}_s \times \prod_{i=1}^{m} \mathrm{SEN}_{s_i} \to \mathrm{SEN}_{s'} \text{ in } N, \text{ for some } s_1, \ldots, s_m, s' \in S. \tag{2}$$

Important Notational Convention: In writing (2), we follow a common convention in categorical abstract algebraic logic, by which, although the specified argument of sort s appears, for simplicity, in the first position, the implied meaning is that it may appear in any position. Thus, (2) should be viewed as a shorthand for

$$\sigma : \prod_{i=1}^{k-1} \mathrm{SEN}_{s_i} \times \mathrm{SEN}_s \times \prod_{i=k}^{m} \mathrm{SEN}_{s_i} \to \mathrm{SEN}_{s'} \text{ in } N,$$

for some $s_1, \ldots, s_m, s' \in S$ and some $1 \leq k \leq m$. This notational convention will be used in multiple places throughout the paper without being explicitly mentioned. Hopefully, it will not cause any confusion.

An N-context for sort s, as above, is called an N-**experiment**, if $s' \in V$. We denote by $\mathcal{C}^N(s)$ and $\mathcal{E}^N(s)$ the collections of all N-contexts for sort s and of all N-experiments for sort s, respectively. Moreover, we let $\mathcal{C}^N_{s'}(s)$

and $\mathcal{E}^N_{s'}(s)$ be, respectively, the collections of all N-contexts and of all N-experiments of (output) sort s' for sort s.

Observe that, given $\sigma : \mathrm{SEN}_s \times \prod_{i=1}^m \mathrm{SEN}_{s_i} \to \mathrm{SEN}_{s'} \in \mathcal{C}^N_{s'}(s)$ and $\phi \in \mathrm{SEN}_s(\Sigma)$, we obtain a function $\sigma_\Sigma(\phi, \vec{x}) : \prod_{i=1}^m \mathrm{SEN}_{s_i}(\Sigma) \to \mathrm{SEN}_{s'}(\Sigma)$.

4. Behavioral Equivalence Systems

In this section, given a hidden sentence functor with a multi-sorted transformation signature on it, we define the notion of a behavioral equivalence system on the functor. This is an equivalence system in the sense of categorical abstract algebraic logic that identifies all those sentences over a given signature that, informally speaking, are behaving identically with respect to all experiments. This follows the pioneering work of Martins [21, 22] and Martins and Pigozzi [23] in the context of sentential logics.

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a multi-sorted sentence functor over a set of sorts S, with N a transformation signature on SEN. An **equivalence family on** SEN is a $|\mathbf{Sign}|$-indexed family $\theta = \{\theta_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, where, for all $\Sigma \in |\mathbf{Sign}|$, θ_Σ is an S-indexed family $\theta_\Sigma = \{\theta^s_\Sigma\}_{s \in S}$ of equivalence relations on $\mathrm{SEN}_s(\Sigma)$. θ will be said to be an **equivalence system on** SEN if, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$ and all $s \in S$, we have

$$\mathrm{SEN}_s(f)(\theta^s_{\Sigma_1}) \subseteq \theta^s_{\Sigma_2}. \tag{3}$$

An equivalence family θ on SEN is said to be an N-**congruence family** if, for all $\sigma : \prod_{i=1}^m \mathrm{SEN}_{s_i} \to \mathrm{SEN}_{s'}$ in N, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi}, \vec{\psi} \in \prod_{i=1}^m \mathrm{SEN}_{s_i}(\Sigma)$,

$$\vec{\phi} \prod_{i=1}^m \theta^{s_i}_\Sigma \ \vec{\psi} \quad \text{implies} \quad \sigma_\Sigma(\vec{\phi}) \ \theta^{s'}_\Sigma \ \sigma_\Sigma(\vec{\psi}).$$

An N-congruence family is an N-**congruence system on** SEN if it satisfies Condition (3). By $\mathrm{Con}^N(\mathrm{SEN})$ is denoted the collection of all N-congruence systems of SEN.

Let **Sign** be a category, $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a hidden functor over a set S of sorts, with visible set of sorts V, and N a transformation signature on SEN. Define an equivalence family $\equiv = \{\equiv_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ on SEN, by letting, for all $\Sigma \in |\mathbf{Sign}|$, $\equiv_\Sigma = \{\equiv^s_\Sigma\}_{s \in S}$, be given, for all $s \in S$ and all $\phi, \psi \in \mathrm{SEN}_s(\Sigma)$, by

$$\phi \equiv^s_\Sigma \psi \quad \text{iff} \quad \sigma_{\Sigma'}(\mathrm{SEN}_s(f)(\phi), \vec{\chi}) = \sigma_{\Sigma'}(\mathrm{SEN}_s(f)(\psi), \vec{\chi}),$$

for all $\sigma : \mathrm{SEN}_s \times \prod_{i=1}^m \mathrm{SEN}_{s_i} \to \mathrm{SEN}_{s'} \in \mathcal{E}^N(s)$, all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\vec{\chi} \in \prod_{i=1}^m \mathrm{SEN}_{s_i}(\Sigma')$. Note that, in this definition, the important notational convention (2) for σ applies.

It is shown in Proposition 1 that $\equiv$ is an equivalence system.

PROPOSITION 1. *Let* SEN : **Sign** $\to$ **Set** *be a hidden sentence functor over a set of sorts S, with set of visible sorts V, and N a transformation signature on* SEN*. The equivalence family $\equiv$ is an equivalence system on* SEN*.*

PROOF. This is a standard proof in categorical abstract algebraic logic. Let $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$ and assume that $\phi, \psi \in \mathrm{SEN}_s(\Sigma_1)$, such that $\phi \equiv^s_{\Sigma_1} \psi$. Thus, for all $\sigma : \mathrm{SEN}_s \times \prod_{i=1}^m \mathrm{SEN}_{s_i} \to \mathrm{SEN}_{s'} \in \mathcal{E}^N(s)$, all $\Sigma' \in |\mathbf{Sign}|, g \in \mathbf{Sign}(\Sigma_1, \Sigma')$ and all $\vec{\chi} \in \prod_{i=1}^m \mathrm{SEN}_{s_i}(\Sigma')$,

$$\sigma_{\Sigma'}(\mathrm{SEN}_s(g)(\phi), \vec{\chi}) = \sigma_{\Sigma'}(\mathrm{SEN}_s(g)(\psi), \vec{\chi}).$$

This implies that, for all $\sigma : \mathrm{SEN}_s \times \prod_{i=1}^m \mathrm{SEN}_{s_i} \to \mathrm{SEN}_{s'} \in \mathcal{E}^N(s)$, all $\Sigma' \in |\mathbf{Sign}|, h \in \mathbf{Sign}(\Sigma_2, \Sigma')$

$$\begin{tikzcd}
\Sigma_1 \arrow[rr, "f"] \arrow[dr, "g"'] & & \Sigma_2 \arrow[dl, "h"] \\
& \Sigma' &
\end{tikzcd}$$

and all $\vec{\chi} \in \prod_{i=1}^m \mathrm{SEN}_{s_i}(\Sigma')$,

$$\sigma_{\Sigma'}(\mathrm{SEN}_s(h \circ f)(\phi), \vec{\chi}) = \sigma_{\Sigma'}(\mathrm{SEN}_s(h \circ f)(\psi), \vec{\chi}),$$

whence, for all $\sigma : \mathrm{SEN}_s \times \prod_{i=1}^m \mathrm{SEN}_{s_i} \to \mathrm{SEN}_{s'} \in \mathcal{E}^N(s)$, all $\Sigma' \in |\mathbf{Sign}|, h \in \mathbf{Sign}(\Sigma_2, \Sigma')$ and all $\vec{\chi} \in \prod_{i=1}^m \mathrm{SEN}_{s_i}(\Sigma')$, $\sigma_{\Sigma'}(\mathrm{SEN}_s(h)(\mathrm{SEN}_s(f)(\phi)), \vec{\chi}) = \sigma_{\Sigma'}(\mathrm{SEN}_s(h)(\mathrm{SEN}_s(f)(\psi)), \vec{\chi})$, showing that $\mathrm{SEN}_s(f)(\phi) \equiv^s_{\Sigma_2} \mathrm{SEN}_s(f)(\psi)$, i.e., that $\equiv$ is an equivalence system on SEN. ■

The equivalence system $\equiv$ on SEN will be called the **N-behavioral equivalence system on** SEN. Note that it does depend on N due to the use of the collection $\mathcal{E}^N(s)$ of N-experiments for sort s in its definition.

By way of illustrating the definition of $\equiv$ and making it more transparent to readers familiar with the universal algebraic framework, we look at the special case of the π-institution $\mathcal{I}_{\mathcal{S}}$ associated with a hidden multisorted deductive system $\mathcal{S}$. In fact, in that special case, there exists only one signature object X, the signature morphisms correspond to substitutions $\sigma : \mathbf{Fm}_\Sigma(X) \to \mathbf{Fm}_\Sigma(X)$ and the N-experiments correspond to derived term operations of visible sort associated with terms $t(x : s, \vec{x} : \vec{s})$.

Now observe that the condition defining the behavioral equivalence between $\phi, \psi \in \mathrm{Fm}_{\Sigma,s}(X)$ on the hidden sentence functor SEN_Σ of the multi-sorted π-institution $\mathcal{I}_\mathcal{S}$ takes the form $t(\sigma(\phi), \vec{\chi}) = t(\sigma(\psi), \vec{\chi})$, for every substitution σ, every experiment t for sort s and all parameters $\vec{\chi}$ of appropriate sorts. But this condition can be easily seen to be equivalent in that context to the condition $t(\phi, \vec{\chi}) = t(\psi, \vec{\chi})$, for all experiments t for sort s and all parameters $\vec{\chi}$ of appropriate sorts. This, in turn, is the condition that defines the behavioral equivalence relation between two formulas in the universal algebraic case (see Definition 4 of [8]).

Although, in general, the N-behavioral equivalence system on SEN is not an N-congruence system in the sense of categorical abstract algebraic logic, it is an N'-congruence system for a suitably selected transformation subsignature N' of N. This fact is detailed below.

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a hidden sentence functor over a set of sorts S, with set of visible sorts V, and N a transformation signature on SEN. Let N_V be the transformation subsignature of N, that apart from the projection natural transformations, contains only those natural transformations in N of type $s_1 \dots s_n \to s$, with $s \in V$.

Proposition 2. *Let* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a hidden sentence functor over a set of sorts* S*, with set of visible sorts* V*, and* N *a transformation signature on* SEN*. Then, the* N*-behavioral equivalence system* $\equiv$ *on* SEN *is an* N_V*-congruence system on* SEN*.*

Proof. During some of the steps in this proof the reader is advised to keep in mind the notational convention (2). Let τ be in N_V of type $s_1 \dots s_n \to s$, $\Sigma \in |\mathbf{Sign}|$ and $\phi_i, \psi_i \in \mathrm{SEN}_{s_i}(\Sigma)$, for all $i = 1, \dots, n$, such that $\phi_i \equiv_\Sigma^{s_i} \psi_i$, $i = 1, \dots, n$. The goal is to show that $\tau_\Sigma(\vec{\phi}) \equiv_\Sigma^s \tau_\Sigma(\vec{\psi})$. Since $\phi_i \equiv_\Sigma^{s_i} \psi_i$, we have, for all $\sigma : \mathrm{SEN}_{s_i} \times \prod_{i=1}^m \mathrm{SEN}_{s'_i} \to \mathrm{SEN}_{s'} \in \mathcal{E}^N(s_i)$, all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\vec{\chi} \in \prod_{i=1}^m \mathrm{SEN}_{s'_i}(\Sigma')$,

$$\sigma_{\Sigma'}(\mathrm{SEN}_s(f)(\phi_i), \vec{\chi}) = \sigma_{\Sigma'}(\mathrm{SEN}_s(f)(\psi_i), \vec{\chi}),$$

for all $i = 1, \dots, n$. Thus, for all $\sigma : \mathrm{SEN}_s \times \prod_{i=1}^m \mathrm{SEN}_{s'_i} \to \mathrm{SEN}_{s'} \in \mathcal{E}^N(s)$, all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\vec{\chi} \in \prod_{i=1}^m \mathrm{SEN}_{s'_i}(\Sigma')$, we have

$$\begin{aligned}
&\sigma_{\Sigma'}(\mathrm{SEN}_s(f)(\tau_\Sigma(\vec{\phi})), \vec{\chi}) \\
&\qquad = \sigma_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}_{s_1}(f)(\phi_1), \mathrm{SEN}_{s_2}(f)(\phi_2), \dots, \mathrm{SEN}_{s_n}(f)(\phi_n)), \vec{\chi}) \\
&\qquad = \sigma_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}_{s_1}(f)(\psi_1), \mathrm{SEN}_{s_2}(f)(\phi_2), \dots, \mathrm{SEN}_{s_n}(f)(\phi_n)), \vec{\chi}) \\
&\qquad = \cdots \\
&\qquad = \sigma_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}_{s_1}(f)(\psi_1), \mathrm{SEN}_{s_2}(f)(\psi_2), \dots, \mathrm{SEN}_{s_n}(f)(\psi_n)), \vec{\chi}) \\
&\qquad = \sigma_{\Sigma'}(\mathrm{SEN}_s(f)(\tau_\Sigma(\vec{\psi})), \vec{\chi}),
\end{aligned}$$

which shows that $\tau_\Sigma(\vec{\phi}) \equiv_\Sigma^s \tau_\Sigma(\vec{\psi})$. ∎

5. N- and $N_{\pounds}$-Congruence Systems

Congruence systems of algebraic systems have been introduced in categorical abstract algebraic logic to provide suitable analogs of the notion of a congruence on an algebra. Given a congruence system, the quotient of an algebraic system by the congruence system may be considered. For the particular case of algebraic systems consisting of functors that are underlying sentence functors of π-institutions, if attention is restricted to logical congruence systems, then the quotient π-institution may also be considered. If the logical congruence system happens to be a Tarski congruence system, i.e., the largest logical congruence system that is compatible with the closure system of the π-institution under consideration, the associated quotient is Tarski-reduced, in the sense that its own Tarski N-congruence system is the signature-wise identity congruence system. Considering congruences, congruence systems and reductions has had a deep influence in the development of both the traditional and the categorical sides of abstract algebraic logic [4, 14, 35]. In this section, we provide the foundations for carrying over aspects of these studies to the behavioral π-institution framework. We are still modeling our work after the corresponding ideas from the universal algebraic treatment, presented in [8].

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a multi-sorted sentence functor over a set of sorts S, with distinguished sort $\pounds$ of formulas, and N a transformation signature on SEN. Let $\mathrm{Con}^N(\mathrm{SEN})$ denote the collection of all N-congruence systems on SEN. An $N^{\pounds}$**-congruence system** $\theta^{\pounds}$ on SEN is a $\pounds$-reduct of an N-congruence system on SEN, i.e., it is the sub-collection $\theta^{\pounds} = \{\theta^{\pounds}_{\Sigma}\}_{\Sigma \in |\mathbf{Sign}|}$ of an N-congruence system $\theta = \{\theta_{\Sigma}\}_{\Sigma \in |\mathbf{Sign}|}$, with $\theta_{\Sigma} = \{\theta^s_{\Sigma}\}_{s \in S}$. The collection of all $N^{\pounds}$-congruence systems on SEN, will be denoted by $\mathrm{Con}^N_{\pounds}(\mathrm{SEN})$.

There is another way to create an equivalence system on $\mathrm{SEN}_{\pounds}$, that we are going to explore next. Let $N_{\pounds}$ be the full subcategory of transformation signature N with objects only those objects of the form $\mathrm{SEN}^k_{\pounds}, k < \omega$. One may then consider the collection $\mathrm{Con}^{N_{\pounds}}(\mathrm{SEN}_{\pounds})$ of the $N_{\pounds}$-congruence systems on the sentence functor $\mathrm{SEN}_{\pounds} : \mathbf{Sign} \to \mathbf{Set}$.

It turns out that the $N^{\pounds}$-congruence systems on SEN form a subclass of the class of all $N_{\pounds}$-congruence systems on $\mathrm{SEN}_{\pounds}$. This is proven in the following proposition.

Proposition 3. *Let* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a multi-sorted sentence functor over a set of sorts S, with distinguished sort $\pounds$ of formulas, and N a transformation signature on* SEN. *If $\theta^{\pounds}$ is an $N^{\pounds}$-congruence system on*

SEN, *then it is also an* $N_{\pounds}$*-congruence system on* $\mathrm{SEN}_{\pounds}$*, i.e.,* $\mathrm{Con}^N_{\pounds}(\mathrm{SEN}) \subseteq \mathrm{Con}^{N_{\pounds}}(\mathrm{SEN}_{\pounds})$.

Proof. Suppose that $\theta \in \mathrm{Con}^N(\mathrm{SEN})$ and consider $\theta^{\pounds}$. Clearly, $\theta^{\pounds}$ is a family of equivalence relations on $\mathrm{SEN}_{\pounds}$. Moreover, since θ is a congruence system on SEN, $\theta^{\pounds}$ is preserved by all **Sign**-morphisms, showing that $\theta^{\pounds}$ is an equivalence system on $\mathrm{SEN}_{\pounds}$. Finally, it is an $N_{\pounds}$-congruence system because θ itself is an N-congruence system and all natural transformations in $N_{\pounds}$ are natural transformations in N. ∎

We illustrate the fact that the inclusion of Proposition 3 may be a proper inclusion with a concrete example that may also help illuminate the definition of $N^{\pounds}$-congruence system on SEN and of $N_{\pounds}$-congruence system on $\mathrm{SEN}_{\pounds}$ and showcase their differences. Consider a set of sorts $S = \{s, \pounds\}$, a trivial signature category **Sign**, with object, say, $\star$, and a functor SEN, defined by $\mathrm{SEN}_s(\star) = \mathrm{SEN}_{\pounds}(\star) = \{0, \frac{1}{2}, 1\}$. Suppose that the category N of natural transformations on SEN is the category generated by the two natural transformations $\ominus : \mathrm{SEN}_s \times \mathrm{SEN}_{\pounds} \to \mathrm{SEN}_{\pounds}$ and $\neg : \mathrm{SEN}_{\pounds} \to \mathrm{SEN}_{\pounds}$, given by the following tables:

$$\begin{array}{c|ccc} \ominus & 0 & \frac{1}{2} & 1 \\ \hline 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & 1 & \frac{1}{2} \\ 1 & 0 & 1 & \frac{1}{2} \end{array} \qquad \begin{array}{c|c} & \neg \\ \hline 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{array} \tag{4}$$

Then, it is clear that $\theta^\neg = \{\theta^\neg_\star\}$, with $\theta^\neg_\star \subseteq \mathrm{SEN}_{\pounds}(\star) \times \mathrm{SEN}_{\pounds}(\star)$, given by the partition $\{\{0, 1\}, \{\frac{1}{2}\}\}$ is an $N_{\pounds}$-congruence system on $\mathrm{SEN}_{\pounds}$, i.e., $\theta^\neg \in \mathrm{Con}^{N_{\pounds}}(\mathrm{SEN}_{\pounds})$. However, it is not an $N^{\pounds}$-congruence system on SEN, i.e., there does not exist any N-congruence system $\theta = \{\theta^s, \theta^{\pounds}\} \in \mathrm{Con}^N(\mathrm{SEN})$, such that $\theta^{\pounds} = \theta^\neg$. This can be easily seen by assuming to the contrary that such a $\theta = \{\theta^s, \theta^{\pounds}\}$ exists. Then, from the fact that $\langle 0, 1\rangle \in \theta^\neg_\star = \theta^{\pounds}_\star$, we obtain $\langle \ominus(0, 0), \ominus(0, 1)\rangle \in \theta^{\pounds}_\star$, i.e., $\langle 0, \frac{1}{2}\rangle \in \theta^{\pounds}_\star = \theta^\neg_\star$, a contradiction.

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a multi-sorted sentence functor over set of sorts S, with distinguished sort $\pounds$, and N a transformation signature on SEN. A **$\pounds$-sentence family on** SEN is a family $F = \{F_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, where $F_\Sigma \subseteq \mathrm{SEN}_{\pounds}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$. A $\pounds$-sentence family is said to be a **$\pounds$-sentence system on** SEN if, for every $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, $\mathrm{SEN}_{\pounds}(f)(F_{\Sigma_1}) \subseteq F_{\Sigma_2}$. A $\pounds$-sentence family, in other words, is a collection of sets of sentences of sort $\pounds$, whereas a $\pounds$-sentence system is a family that is invariant under the action of signature morphisms.

An N-congruence system θ on SEN is said to be **compatible with** a $\pounds$-sentence family F if, for every $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}_\pounds(\Sigma)$, $\langle \phi, \psi \rangle \in \theta^\pounds_\Sigma$ and $\phi \in F_\Sigma$ imply $\psi \in F_\Sigma$. We show that, for every $\pounds$-sentence family, there exists a largest N-congruence system on SEN that is compatible with the family.

Proposition 4. *Let* SEN : **Sign** $\to$ **Set** *be a multi-sorted sentence functor over a set of sorts S, with distinguished sort $\pounds$, and N a transformation signature on* SEN. *Let, also, F be a $\pounds$-sentence family on* SEN. *Then, there exists a largest N-congruence system θ on* SEN *that is compatible with F.*

Proof. Let $\theta = \{\theta_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, with $\theta_\Sigma = \{\theta^s_\Sigma\}_{s \in S}$, be defined, for all $\Sigma \in |\mathbf{Sign}|$, all $s \in S$ and all $\phi, \psi \in \mathrm{SEN}_s(\Sigma)$, by $\langle \phi, \psi \rangle \in \theta^s_\Sigma$ iff, for all σ in N of type $ss_1 \dots s_m \to \pounds$, all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\vec{\chi} \in \prod_{i=1}^m \mathrm{SEN}_{s_i}(\Sigma')$,

$$\sigma_{\Sigma'}(\mathrm{SEN}_s(f)(\phi), \vec{\chi}) \in F_{\Sigma'} \quad \text{iff} \quad \sigma_{\Sigma'}(\mathrm{SEN}_s(f)(\psi), \vec{\chi}) \in F_{\Sigma'}.$$

Note that in defining θ, we used the notational convention (2) in the quantification over σ. In other words, we are allowing the argument of sort s in the sequence of arguments of σ to appear in any-not just in the first-argument position. We need to show that θ is an N-congruence system on SEN that is compatible with F and, moreover, that every other congruence system on SEN compatible with F is included in θ.

That θ is an equivalence family on SEN is straightforward. It is an equivalence system, since, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, if $\langle \phi, \psi \rangle \in \theta^s_{\Sigma_1}$, then for all σ in N of type $ss_1 \dots s_m \to \pounds$, all $\Sigma' \in |\mathbf{Sign}|$, all $g \in \mathbf{Sign}(\Sigma_1, \Sigma')$ and all $\vec{\chi} \in \prod_{i=1}^m \mathrm{SEN}_{s_i}(\Sigma')$,

$$\sigma_{\Sigma'}(\mathrm{SEN}_s(g)(\phi), \vec{\chi}) \in F_{\Sigma'} \quad \text{iff} \quad \sigma_{\Sigma'}(\mathrm{SEN}_s(g)(\psi), \vec{\chi}) \in F_{\Sigma'}.$$

$$\begin{tikzcd}
\Sigma_1 \arrow[rr, "f"] \arrow[dr, "g"'] & & \Sigma_2 \arrow[dl, "h"] \\
& \Sigma' &
\end{tikzcd}$$

Thus, for all σ in N of type $ss_1 \dots s_m \to \pounds$, all $\Sigma' \in |\mathbf{Sign}|$, all $h \in \mathbf{Sign}(\Sigma_2, \Sigma')$ and all $\vec{\chi} \in \prod_{i=1}^m \mathrm{SEN}_{s_i}(\Sigma')$,

$$\sigma_{\Sigma'}(\mathrm{SEN}_s(hf)(\phi), \vec{\chi}) \in F_{\Sigma'} \quad \text{iff} \quad \sigma_{\Sigma'}(\mathrm{SEN}_s(hf)(\psi), \vec{\chi}) \in F_{\Sigma'},$$

which shows that

$$\sigma_{\Sigma'}(\mathrm{SEN}_s(h)(\mathrm{SEN}_s(f)(\phi)), \vec{\chi}) \in F_{\Sigma'} \quad \text{iff} \quad \sigma_{\Sigma'}(\mathrm{SEN}_s(h)(\mathrm{SEN}_s(f)(\psi)), \vec{\chi}) \in F_{\Sigma'},$$

proving that $\langle \mathrm{SEN}_s(f)(\phi), \mathrm{SEN}_s(f)(\psi)\rangle \in \theta^s_{\Sigma_2}$. Finally, θ is a congruence system on SEN, since, for every τ in N of type $t_1 \dots t_k \to s$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi_i, \psi_i \in \mathrm{SEN}_{t_i}(\Sigma)$, $i = 1, \dots, k$, if $\langle \phi_i, \psi_i\rangle \in \theta^{t_i}_\Sigma$, we have that, for all σ in N of type $ss_1 \dots s_m \to £$, all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\vec{\chi} \in \prod_{i=1}^m \mathrm{SEN}_{s_i}(\Sigma')$,

$$\begin{array}{ll} \multicolumn{2}{l}{\sigma_{\Sigma'}(\mathrm{SEN}_s(f)(\tau_\Sigma(\phi_1, \dots, \phi_k)), \vec{\chi}) \in F_{\Sigma'}} \\ \text{iff} & \sigma_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}_{t_1}(f)(\phi_1), \dots, \mathrm{SEN}_{t_k}(f)(\phi_k)), \vec{\chi}) \in F_{\Sigma'} \\ \text{iff} & \sigma_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}_{t_1}(f)(\psi_1), \dots, \mathrm{SEN}_{t_k}(f)(\phi_k)), \vec{\chi}) \in F_{\Sigma'} \\ \dots & \\ \text{iff} & \sigma_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}_{t_1}(f)(\psi_1), \dots, \mathrm{SEN}_{t_k}(f)(\psi_k)), \vec{\chi}) \in F_{\Sigma'} \\ \text{iff} & \sigma_{\Sigma'}(\mathrm{SEN}_s(f)(\tau_\Sigma(\psi_1, \dots, \psi_k)), \vec{\chi}) \in F_{\Sigma'}, \end{array}$$

showing that $\langle \tau_\Sigma(\vec{\phi}), \tau_\Sigma(\vec{\psi})\rangle \in \theta^s_\Sigma$. Compatibility with F is straightforward by considering the identity natural transformation $\iota : \mathrm{SEN}_£ \to \mathrm{SEN}_£$ and the identity signature morphism $i_\Sigma : \Sigma \to \Sigma$.

Suppose, next, that η is an N-congruence system on SEN compatible with the $£$-sentence family F. Let $s \in S$, $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}_s(\Sigma)$, such that $\langle \phi, \psi\rangle \in \eta^s_\Sigma$. Consider σ in N of type $ss_1 \dots s_m \to £$, $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\vec{\chi} \in \prod_{i=1}^m \mathrm{SEN}_{s_i}(\Sigma')$. Since η is an N-congruence system, we get that $\langle \mathrm{SEN}_s(f)(\phi), \mathrm{SEN}_s(f)(\psi)\rangle \in \eta^s_{\Sigma'}$. Moreover, by the congruence property of η, $\langle \sigma_{\Sigma'}(\mathrm{SEN}_s(f)(\phi), \vec{\chi}), \sigma_{\Sigma'}(\mathrm{SEN}_s(f)(\psi), \vec{\chi})\rangle \in \eta^£_{\Sigma'}$. Therefore, by the compatibility property of η,

$$\sigma_{\Sigma'}(\mathrm{SEN}_s(f)(\phi), \vec{\chi}) \in F_{\Sigma'} \quad \text{iff} \quad \sigma_{\Sigma'}(\mathrm{SEN}_s(f)(\psi), \vec{\chi}) \in F_{\Sigma'}.$$

Thus, $\langle \phi, \psi\rangle \in \theta^s_\Sigma$, showing that $\eta \le \theta$. Hence, θ is the largest N-congruence system on SEN, that is compatible with F. ∎

The largest N-congruence system that is compatible with a $£$-sentence family F on SEN is called the **behavioral Leibniz N-congruence system associated with** F and is denoted by $\Omega^{N,b}(F) = \{\Omega^{N,b}_\Sigma(F)\}_{\Sigma \in |\mathbf{Sign}|}$, where, for all $\Sigma \in |\mathbf{Sign}|$, $\Omega^{N,b}_\Sigma(F) = \{\Omega^{N,b,s}_\Sigma(F)\}_{s \in S}$. To avoid carrying around the triple superscript, when N and b are clear from context, we will simply write $\Omega^s_\Sigma(F)$ in place of the more accurate $\Omega^{N,b,s}_\Sigma(F)$.

To provide an example of the definition, let us consider again the trivial one element category **Sign**, with object $\star$, the two-sorted sentence functor SEN : **Sign** $\to$ **Set**, with sorts s and $£$, such that $\mathrm{SEN}_s(\star) = \mathrm{SEN}_£(\star) = \{0, \frac{1}{2}, 1\}$ and the transformation signature N on SEN generated by $\ominus$ of type $s\pounds \to \pounds$ and $\neg$ of type $\pounds \to \pounds$, that were defined by Tables (4). Consider, for instance, the $\pounds$-sentence family $F = \{F_\star\}$, with $F_\star = \{\frac{1}{2}, 1\}$. It is relatively easy to see that $\Omega_\star^{N,b}(F) = \{\Omega_\star^{N,b,s}(F), \Omega_\star^{N,b,\pounds}(F)\}$, with $\Omega_\star^{N,b,s}(F) = \nabla_\star^{\mathrm{SEN}_s}$ and $\Omega_\star^{N,b,\pounds}(F) = \Delta_\star^{\mathrm{SEN}_\pounds}$. For the latter, notice that $\langle 0, \frac{1}{2}\rangle, \langle 0, 1\rangle \notin \Omega_\star^{N,b,\pounds}(F)$, because they consist of elements one of which is inside and the other outside of $F_\star$. Moreover $\langle \frac{1}{2}, 1\rangle \notin \Omega_\star^{N,b,\pounds}(F)$ either, since $\neg\frac{1}{2} \in F_\star$ but $\neg 1 \notin F_\star$. Thus, no pair outside the diagonal $\Delta_\star^{\mathrm{SEN}_\pounds}$ can be in $\Omega_\star^{N,b,\pounds}(F)$. On the other hand, the only nontrivial natural transformations with an input of sort s and output of sort $\pounds$ are generated by $\ominus$, whose output is independent of the input of sort s, once the other parameter is fixed. Thus, all pairs of elements in $\mathrm{SEN}_s(\star)$ are in $\Omega_\star^{N,b,s}(F)$.

6. Behavioral Protoalgebraicity

In this section, we introduce behaviorally protoalgebraic multi-sorted π-institutions, taking after the corresponding notion for deductive systems based on multi-sorted languages, introduced in Section 4 (see Definition 30) of [8]. This notion generalizes the well-known protoalgebraic deductive systems of Blok and Pigozzi [3]. Our notion also extends the notion of a protoalgebraic π-institution, that was introduced in [31].

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a multi-sorted sentence functor, as before, over a set S of sorts, with a distinguished sort $\pounds$, and N a transformation signature on SEN. Let, also, $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}_\pounds, C\rangle$ be a multi-sorted π-institution over SEN. The **behavioral N-Leibniz operator of $\mathcal{I}$** is the function $\Omega^{N,b} : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{Con}^N(\mathrm{SEN})$ defined, for all $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, by letting $\Omega^{N,b}(T)$ be the behavioral Leibniz N-congruence system on SEN associated with T (see Section 5 for the definition).

A many-sorted π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}_\pounds, C\rangle$ over a multi-sorted sentence functor SEN, with set of sorts S and a transformation signature N on SEN, is called **behaviorally (semantically) N-protoalgebraic** if, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}_\pounds(\Sigma)$,

$$\langle \phi, \psi\rangle \in \Omega_\Sigma^{N,b,\pounds}(T) \quad \text{implies} \quad C_\Sigma(T_\Sigma \cup \{\phi\}) = C_\Sigma(T_\Sigma \cup \{\psi\}).$$

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a multi-sorted sentence functor over set of sorts S, with distinguished sort $\pounds$, and N a transformation signature on SEN. Let Δ be a collection of natural transformations $\delta : \mathrm{SEN}_\pounds \times \mathrm{SEN}_\pounds \times \prod_{i=1}^m \mathrm{SEN}_{s_i} \to \mathrm{SEN}_\pounds$ in N. We use the following notation that is borrowed from [11] and used in a similar categorical context in [32]. For all $\Sigma \in |\mathbf{Sign}|$

and all $\phi, \psi \in \mathrm{SEN}_{\pounds}(\Sigma), \vec{\chi} \in \prod_{i=1}^{m} \mathrm{SEN}_{s_i}(\Sigma)$,

$$\Delta_\Sigma(\phi, \psi, \vec{\chi}) = \{\delta_\Sigma(\phi, \psi, \vec{\chi}) : \delta \in \Delta\},$$

$$\Delta_\Sigma(\langle \phi, \psi \rangle) = \bigcup_{\vec{\chi} \in \prod_{i=1}^{m} \mathrm{SEN}_{s_i}(\Sigma)} \Delta_\Sigma(\phi, \psi, \vec{\chi}).$$

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}_{\pounds}, C \rangle$ be a multi-sorted π institution over a multi-sorted sentence functor SEN, as above. A collection Δ of natural transformations of the form $\delta : \mathrm{SEN}_{\pounds} \times \mathrm{SEN}_{\pounds} \times \prod_{i=1}^{m} \mathrm{SEN}_{s_i} \to \mathrm{SEN}_{\pounds}$, where $s_i \neq \pounds$ and $s_i \neq s_j$, for all $i, j = 1, \ldots, m$, $i \neq j$, is an N-**protoequivalence system for** $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}|$, and all $\phi, \psi \in \mathrm{SEN}_{\pounds}(\Sigma)$,

(R) $\Delta_\Sigma(\langle \phi, \phi \rangle) \subseteq C_\Sigma(\emptyset)$;

(MP) for every theory family T of $\mathcal{I}$, if

- $\phi \in T_\Sigma$ and
- $\Delta_{\Sigma'}(\langle \mathrm{SEN}_{\pounds}(f)(\phi), \mathrm{SEN}_{\pounds}(f)(\psi) \rangle) \subseteq T_{\Sigma'}$, for every $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

then $\psi \in T_\Sigma$.

The following theorem abstracts to the behavioral setting Lemma 3.3 of [31] and to the categorical setting Theorem 33 of [8]. All these results take after the original characterization result of Blok and Pigozzi [3] (see also Theorem 1.1.3 of [11]).

Theorem 5. *A many-sorted π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}_{\pounds}, C \rangle$ over a multi-sorted sentence functor* SEN*, with set of sorts S with a distinguished sort $\pounds$, and with a transformation signature N on* SEN*, is behaviorally N-protoalgebraic iff the behavioral N-Leibniz operator $\Omega^{N,b}$ is monotone on $\mathbf{ThFam}(\mathcal{I})$ iff $\Omega^{N,b,\pounds}$ is monotone on $\mathbf{ThFam}(\mathcal{I})$.*

Proof. We only prove the first equivalence of the conclusion. The reader is invited to check that the proof is still valid with $\Omega^{N,b}$ replaced by $\Omega^{N,b,\pounds}$.

Suppose, first, that $\mathcal{I}$ is behaviorally N-protoalgebraic. Consider T^1, $T^2 \in \mathrm{ThFam}(\mathcal{I})$, with $T^1 \leq T^2$. To show that $\Omega^{N,b}(T^1) \leq \Omega^{N,b}(T^2)$, it suffices to show that $\Omega^{N,b}(T^1)$ is compatible with T^2. To this end, let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}_{\pounds}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \Omega^{N,b,\pounds}_\Sigma(T^1)$ and $\phi \in T^2_\Sigma$. Then, we have

$$\begin{array}{rcll} \psi & \in & C_\Sigma(T^1_\Sigma \cup \{\psi\}) & \\ & = & C_\Sigma(T^1_\Sigma \cup \{\phi\}) & (\text{since } \langle \phi, \psi \rangle \in \Omega^{N,b,\pounds}_\Sigma(T^1)) \\ & \subseteq & C_\Sigma(T^2_\Sigma \cup \{\phi\}) & \\ & = & T^2_\Sigma & (\text{since } \phi \in T^2_\Sigma). \end{array}$$

Thus, $\Omega^{N,b}(T^1)$ is compatible with T^2, showing that $\Omega^{N,b}(T_1) \leq \Omega^{N,b}(T^2)$.

Before engaging with the converse, recall from [31] that, given a theory family T, $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}_{\pounds}(\Sigma)$, by $T^{[\langle\Sigma,\phi\rangle]}$ is denoted the least theory family T' of $\mathcal{I}$, such that $T \leq T'$ and $\phi \in T'_\Sigma$. Assume that $\Omega^{N,b}$ is monotone on $\mathbf{ThFam}(\mathcal{I})$ and consider $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}_{\pounds}(\Sigma)$, such that $\langle\phi,\psi\rangle \in \Omega^{N,b,\pounds}_\Sigma(T)$. Then, by monotonicity, we obtain that $\langle\phi,\psi\rangle \in \Omega^{N,b,\pounds}_\Sigma(T^{[\langle\Sigma,\phi\rangle]})$, whence, since $\phi \in T^{[\langle\Sigma,\phi\rangle]}_\Sigma$, we get, by compatibility, $\psi \in T^{[\langle\Sigma,\phi\rangle]}_\Sigma = C_\Sigma(T_\Sigma \cup \{\phi\})$. By symmetry, we obtain $C_\Sigma(T_\Sigma \cup \{\phi\}) = C_\Sigma(T_\Sigma \cup \{\psi\})$. This proves that $\mathcal{I}$ is behaviorally N-protoalgebraic. ■

Finally, we show that the existence of an N-protoequivalence system for a multi-sorted π-institution $\mathcal{I}$ implies the behavioral N-protoalgebraicity of $\mathcal{I}$. It is actually the case for sentential logics that protoalgebraicity is equivalent to the existence of a universal algebraic version of protoequivalence systems (see Theorem 1.2.7 of [11]). This result was extended in Theorem 33 of [8] to characterize behavioral protoalgebraicity. It is well-known that in the categorical context, the existence of a protoequivalence system implies protoalgebraicity of a given π-institution, but it has been conjectured that the converse does not hold in general [31]. Therefore, in the next proposition, we will extend this one direction to the case of behavioral protoalgebraicity of multi-sorted π-institutions. We leave the proof of the converse or, more likely, the discovery of a counterexample for the converse as an open problem.

Proposition 6. *Let $\mathcal{I} = \langle\mathbf{Sign}, \mathrm{SEN}_{\pounds}, C\rangle$ be a multi-sorted π institution over a multi-sorted sentence functor* SEN*, with set of sorts S with a distinguished sort $\pounds$, and N a transformation signature on* SEN*. If there exists an N-protoequivalence system Δ for $\mathcal{I}$, then $\mathcal{I}$ is behaviorally N-protoalgebraic.*

Proof. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi \in \mathrm{SEN}_{\pounds}(\Sigma)$, such that $\langle\phi,\psi\rangle \in \Omega^{N,b,\pounds}_\Sigma(T)$. By the congruence property, for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma')$, $\delta : \mathrm{SEN}_{\pounds} \times \mathrm{SEN}_{\pounds} \times \prod_{i=1}^m \mathrm{SEN}_{s_i} \to \mathrm{SEN}_{\pounds}$ in Δ and $\vec{\chi} \in \prod_{i=1}^m \mathrm{SEN}_{s_i}(\Sigma')$,

$$\begin{aligned}\langle\delta_{\Sigma'}(&\mathrm{SEN}_{\pounds}(f)(\phi), \mathrm{SEN}_{\pounds}(f)(\psi), \vec{\chi}),\\ &\delta_{\Sigma'}(\mathrm{SEN}_{\pounds}(f)(\phi), \mathrm{SEN}_{\pounds}(f)(\phi), \vec{\chi})\rangle \in \Omega^{N,b,\pounds}_{\Sigma'}(T).\end{aligned}$$

By Property (R) of Δ, $\Delta_{\Sigma'}(\langle\mathrm{SEN}_{\pounds}(f)(\phi), \mathrm{SEN}_{\pounds}(f)(\phi)\rangle) \subseteq C_{\Sigma'}(\emptyset) \subseteq T_{\Sigma'}$. Therefore, by compatibility, $\Delta_{\Sigma'}(\langle\mathrm{SEN}_{\pounds}(f)(\phi), \mathrm{SEN}_{\pounds}(f)(\psi)\rangle) \subseteq T_{\Sigma'}$. Now, applying Property (MP) of Δ yields $\psi \in C_\Sigma(T_\Sigma \cup \{\phi\})$. By symmetry $C_\Sigma(T_\Sigma \cup \{\phi\}) = C_\Sigma(T_\Sigma \cup \{\psi\})$, showing that $\mathcal{I}$ is behaviorally N-protoalgebraic. ■

As mentioned before Proposition 6, it has been conjectured, in the case of the theory of protoalgebraic π-institutions, that the existence of an N-protoequivalence system is stronger than N-protoalgebraicity. This contrasts with the theory of deductive systems, where existence of protoequivalence systems is equivalent to protoalgebraicity. In fact, in the context of behavioral protoalgebraic deductive systems, Theorem 33 of [8] asserts that, besides the monotonicity of the behavioral Leibniz operator, behavioral protoalgebraicity may also be characterized by the existence of a protoequivalence system for the deductive system under consideration. As is pointed out in [8], this equivalence implies that the class of behaviorally protoalgebraic deductive systems lies inside the class of protoalgebraic deductive systems. Since, if a π-institution $\mathcal{I}$ is behaviorally N-protoalgebraic, it is not necessarily the case that there exists an N-protoequivalence system for $\mathcal{I}$, in general the first link in the chain of implications

$$\begin{array}{rcl}\text{Behavioral } N\text{-protoalegbraic} & \rightarrow & \text{Existence of } N\text{-protoequivalence} \\ & \rightarrow & N\text{-protoalegbraic}\end{array}$$

fails in the categorical context, whence the method of proof of [8] cannot be used in the categorical framework to show that every behaviorally N-protoalgebraic π-institution is N-protoalgebraic. In fact, at the present time, we neither have a proof of this statement nor do we know of any counterexamples.

Open Problem: Is every behaviorally N-protoalgebraic π-institution $N_{\pounds}$-protoalgebraic? The same question can be posed, but replacing N-protoalgebraicity by N-prealgebraicity, which requires that conditions involving the Leibniz operator hold only for theory systems of the π-institution, rather than for all theory families (see [31] for more details).

Let us close this section by providing a multi-sorted π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}_{\pounds}, C\rangle$ that is not behaviorally N-protoalgebraic, for some transformation signature N on the multi-sorted sentence functor SEN. Let us consider the language $\mathcal{L}$ with two binary connectives $\wedge, \vee$ and one unary connective $\neg$ and $\mathcal{L}'$ the language with only the two binary connectives $\wedge, \vee$. Let also V be a denumerable set of propositional variables. We recall that, by Proposition 2.8 of [16], the deductive system $\mathcal{S}_{\wedge,\vee}$ over the language type $\mathcal{L}'$, that corresponds to the $\{\wedge, \vee\}$-fragment of classical propositional calculus, is not protoalgebraic. We will base the multi-sorted π-institution $\mathcal{I}$ on $\mathcal{S}_{\wedge,\vee}$ and we will use the fact that $\mathcal{S}_{\wedge,\vee}$ is not protoalgebraic to show that $\mathcal{I}$ is not behaviorally N-protoalgebraic. Let **Sign** be a trivial category with object, say, $\star$. Consider a set of sorts $\{s, \pounds\}$ and let $\mathrm{SEN}_s(\star) = \mathrm{Fm}_{\mathcal{L}}(V)$

and $\mathrm{SEN}_\pounds(\star) = \mathrm{Fm}_{\mathcal{L}'}(V)$. Moreover, let N be the transformation signature that is generated by the two binary operations $\wedge, \vee : \mathrm{SEN}_\pounds^2 \to \mathrm{SEN}_\pounds$, given, for all $\phi, \psi \in \mathrm{SEN}_\pounds(\star)$, by

$$\wedge(\phi, \psi) = \phi \wedge \psi, \quad \vee(\phi, \psi) = \phi \vee \psi,$$

and the unary operation $f : \mathrm{SEN}_s \to \mathrm{SEN}_\pounds$, such that, for all $\phi \in \mathrm{SEN}_s(\star)$, $f(\phi)$ is the formula that results from ϕ by dropping all negations, if any, appearing in ϕ. These are all well-defined natural transformations on SEN, since **Sign** is trivial. Consider the 2-sorted π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}_\pounds, C\rangle$, whose closure system consists of the closure operator corresponding to the entailment of $\mathcal{S}_{\wedge,\vee}$. We show that, given a theory family $T = \{T_\star\}$ of $\mathcal{I}$, $\Omega^{N,b,\pounds}_\star(T) = \Omega(T)$, where the latter denotes the ordinary Leibniz congruence on $\mathrm{Fm}_{\mathcal{L}'}(V)$ corresponding to the $T_\star \in \mathrm{Th}(\mathcal{S}_{\wedge,\vee})$. In fact, note that, by the characterization of the behavioral Leibniz N-congruence system included in the proof of Proposition 4, we obtain that $\Omega^{N,b,\pounds}_\star(T) \leq \Omega(T)$. Moreover, since $\Omega^{N,b}(T)$ is an N-congruence system on SEN, $\Omega^{N,b,s}(T) \subseteq f^{-1}(\Omega^{N,b,\pounds}(T))$. From these two inequalities, it follows that $\Omega^{N,b}(T) = \{\Omega^{N,b,s}(T), \Omega^{N,b,\pounds}(T)\}$, with $\Omega^{N,b,s}_\star(T) = f^{-1}(\Omega(T))$ and $\Omega^{N,b,\pounds}_\star(T) = \Omega(T)$. Thus, the monotonicity of the N-Leibniz operator on the theory families of $\mathcal{I}$ is equivalent to the monotonicity of the Leibniz operator on the theories of $\mathcal{S}_{\wedge,\vee}$. Since the latter is known not to be monotonic, the former is also not monotonic, showing, using Theorem 5, that $\mathcal{I}$ is not behaviorally N-protoalgebraic.

7. Behavioral Equivalentiality; Herrmann's Test

In this final section of the paper we concentrate on providing an analog of the notion of behavioral equivalentiality for multi-sorted π-institutions. This notion will take after the syntactic equivalentiality of π-institutions of [33] and combine it with behavioral equivalentiality [8] to suitably adapt it to cover multi-sorted π-institutions.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}_\pounds, C\rangle$ be a multi-sorted π-institution over a multi-sorted sentence functor SEN, with set of sorts S, and N a transformation signature on SEN. A collection Δ of natural transformations in N of the form $\delta : \mathrm{SEN}_\pounds \times \mathrm{SEN}_\pounds \to \mathrm{SEN}_\pounds$ is said to be an N-**equivalence system for** $\mathcal{I}$ if, for all $\sigma : \mathrm{SEN}_\pounds^n \times \prod_{i=1}^m \mathrm{SEN}_{s_i} \to \mathrm{SEN}_\pounds$ in N, all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}_\pounds(\Sigma), \vec{\phi}, \vec{\psi} \in \mathrm{SEN}_\pounds^n(\Sigma)$,

(R) $\Delta_\Sigma(\phi, \phi) \subseteq C_\Sigma(\emptyset)$;

(RP) If, for all $i = 1, \ldots, n$ and all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\Delta_{\Sigma'}(\mathrm{SEN}_{\pounds}(f)(\phi_i), \mathrm{SEN}_{\pounds}(f)(\psi_i)) \subseteq T_{\Sigma'},$$

then, for all $\Sigma', \Sigma'' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma'), g \in \mathbf{Sign}(\Sigma', \Sigma''), \vec{\chi} \in \prod_{i=1}^m \mathrm{SEN}_{s_i}(\Sigma')$,

$$\begin{aligned}\Delta_{\Sigma''}(&\mathrm{SEN}_{\pounds}(g)(\sigma_{\Sigma'}(\mathrm{SEN}_{\pounds}(f)^n(\vec{\phi}), \vec{\chi})),\\ &\mathrm{SEN}_{\pounds}(g)(\sigma_{\Sigma'}(\mathrm{SEN}_{\pounds}(f)^n(\vec{\psi}), \vec{\chi}))) \subseteq T_{\Sigma''};\end{aligned}$$

(MP) If $\phi \in T_\Sigma$ and, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\Delta_{\Sigma'}(\mathrm{SEN}_{\pounds}(f)(\phi), \mathrm{SEN}_{\pounds}(f)(\psi)) \subseteq T_{\Sigma'},$$

then $\psi \in T_\Sigma$.

A multi-sorted π-institution $\mathcal{I}$, as above, for which there exists an N-equivalence system Δ is said to be **behaviorally (syntactically) N-equivalential**. This notion abstracts the notion of a behaviorally equivalential deductive system, presented in [8], as well as the notion of a syntactically equivalential π-institution, introduced in [33]. Both notions, in turn, are generalizations of equivalential logics, first introduced by Prucnal and Wroński [24], and studied in more detail in the seminal papers of Czelakowski [9, 10] (see also Chapter 3 of [11]).

Given any collection Δ of natural transformations in N, as above, and an sentence family T on $\mathrm{SEN}_{\pounds}$, define $\Delta(T) = \{\Delta_\Sigma(T)\}_{\Sigma \in |\mathbf{Sign}|}$ by

$$\begin{aligned}\Delta_\Sigma(T) = \{\langle \phi, \psi \rangle \in \mathrm{SEN}_{\pounds}^2(\Sigma) :\ &\text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma'),\\ &\Delta_{\Sigma'}(\mathrm{SEN}_{\pounds}(f)(\phi), \mathrm{SEN}_{\pounds}(f)(\psi)) \subseteq T_{\Sigma'}\}.\end{aligned}$$

Whenever reference to Σ' and $\mathbf{Sign}$ is clear from context, the defining condition will sometimes be abbreviated as $(\forall f)(\Delta_{\Sigma'}(\mathrm{SEN}_{\pounds}^2(f)(\phi, \psi)) \subseteq T_{\Sigma'})$.

Using the technique employed in the proof of Proposition 36 of [8], we may show that, for every theory family of a given multi-sorted π-institution $\mathcal{I}$, the N-congruence system $\Delta(T)$ coincides with $\Omega^{N,b,\pounds}(T)$, provided that Δ is an N-equivalence system for $\mathcal{I}$.

Proposition 7. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}_{\pounds}, C \rangle$ be a multi-sorted π-institution over a multi-sorted sentence functor* SEN*, with set of sorts S with distinguished sort $\pounds$, and N a transformation signature on* SEN*. Let, also, Δ be a collection of natural transformations $\mathrm{SEN}_{\pounds} \times \mathrm{SEN}_{\pounds} \to \mathrm{SEN}_{\pounds}$ in N. Then Δ is an N-equivalence system for $\mathcal{I}$ iff, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $\Delta(T) = \Omega^{N,b,\pounds}(T)$.*

PROOF. Let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}_{\pounds}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \Delta_\Sigma(T)$. Then, for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$, we get that

$$\Delta_{\Sigma'}(\mathrm{SEN}_{\pounds}(f)(\phi), \mathrm{SEN}_{\pounds}(f)(\psi)) \subseteq T_{\Sigma'}.$$

Consider $\sigma \in \mathcal{C}^N_{\pounds}(\pounds)$, of type $\pounds s_1 \dots s_m \to \pounds$, $\Sigma', \Sigma'' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma'), g \in \mathbf{Sign}(\Sigma', \Sigma'')$ and $\vec{\chi} \in \prod_{i=1}^m \mathrm{SEN}_{s_i}(\Sigma')$. Then, by (RP),

$$\begin{aligned}\Delta_{\Sigma''}(&\mathrm{SEN}_{\pounds}(g)(\sigma_{\Sigma'}(\mathrm{SEN}_{\pounds}(f)(\phi), \vec{\chi})),\\ &\qquad\qquad \mathrm{SEN}_{\pounds}(g)(\sigma_{\Sigma'}(\mathrm{SEN}_{\pounds}(f)(\psi), \vec{\chi}))) \subseteq T_{\Sigma''}.\end{aligned}$$

Since this holds, for all $\Sigma'' \in |\mathbf{Sign}|$ and all $g \in \mathbf{Sign}(\Sigma', \Sigma'')$, we conclude from (MP) that, for all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\vec{\chi} \in \prod_{i=1}^m \mathrm{SEN}_{s_i}(\Sigma')$,

$$\sigma_{\Sigma'}(\mathrm{SEN}_{\pounds}(f)(\phi), \vec{\chi}) \in T_{\Sigma'} \quad \text{iff} \quad \sigma_{\Sigma'}(\mathrm{SEN}_{\pounds}(f)(\psi), \vec{\chi}) \in T_{\Sigma'}.$$

Therefore, by Proposition 4, $\langle \phi, \psi \rangle \in \Omega^{N,b,\pounds}_\Sigma(T)$. This shows that $\Delta(T) \leq \Omega^{N,b,\pounds}(T)$.

To see that the reverse system of inclusions holds, suppose that $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}_{\pounds}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \Omega^{N,b,\pounds}_\Sigma(T)$. Then, by the system property of $\Omega^{N,b}(T)$, for every $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, we have that $\langle \mathrm{SEN}_{\pounds}(f)(\phi), \mathrm{SEN}_{\pounds}(f)(\psi) \rangle \in \Omega^{N,b,\pounds}_{\Sigma'}(T)$. By the N-congruence property, for all $\delta \in \Delta$, we obtain

$$\langle \delta_{\Sigma'}(\mathrm{SEN}_{\pounds}(f)^2(\phi, \phi)), \delta_{\Sigma'}(\mathrm{SEN}_{\pounds}(f)^2(\phi, \psi)) \rangle \in \Omega^{N,b,\pounds}_{\Sigma'}(T).$$

By the reflexivity of $\Delta(T)$, we get that $\Delta_{\Sigma'}(\mathrm{SEN}_{\pounds}(f)^2(\phi, \phi)) \subseteq C_{\Sigma'}(\emptyset) \subseteq T_{\Sigma'}$. Hence, by the compatibility property of $\Omega^{N,b}(T)$ with T, we obtain that $\Delta_{\Sigma'}(\mathrm{SEN}_{\pounds}(f)^2(\phi, \psi)) \subseteq T_{\Sigma'}$. Since $\Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma, \Sigma')$ were arbitrary, we obtain that $\langle \phi, \psi \rangle \in \Delta_\Sigma(T)$. Therefore, $\Omega^{N,b,\pounds}(T) \leq \Delta(T)$. ■

Proposition 7 is used in Theorem 8 to prove that behavioral N-equivalentiality implies behavioral N-protoalgebraicity.

THEOREM 8. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}_{\pounds}, C \rangle$ be a multi-sorted π-institution over a multi-sorted sentence functor* SEN*, with set of sorts S with distinguished sort $\pounds$, and N a transformation signature on* SEN*. If $\mathcal{I}$ is behaviorally N-equivalential, then it is behaviorally N-protoalgebraic.*

PROOF. Suppose that $\mathcal{I}$ is behaviorally N-equivalential. Thus, it has an N-equivalence system Δ. Let $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \leq T'$. Then, for all $\Sigma \in |\mathbf{Sign}|$,

$$\begin{array}{lcl}\Omega_\Sigma^{N,b,\pounds}(T) & = & \Delta_\Sigma(T) \quad \text{(by Proposition 7)}\\ & = & \{\langle\phi,\psi\rangle : (\forall f)(\Delta_{\Sigma'}(\mathrm{SEN}_\pounds(f)^2(\phi,\psi)) \subseteq T_{\Sigma'})\}\\ & \subseteq & \{\langle\phi,\psi\rangle : (\forall f)(\Delta_{\Sigma'}(\mathrm{SEN}_\pounds(f)^2(\phi,\psi)) \subseteq T'_{\Sigma'})\}\\ & = & \Delta_\Sigma(T')\\ & = & \Omega_\Sigma^{N,b,\pounds}(T'). \quad \text{(by Proposition 7)}\end{array}$$

Thus, by Theorem 5, $\mathcal{I}$ is behaviorally N-protoalgebraic. ■

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}_\pounds, C\rangle$ be a multi-sorted π-institution over a multi-sorted sentence functor SEN, with set of sorts S, and N a transformation signature on SEN. Let, also, Δ be a collection of natural transformations in N of the form $\delta : \mathrm{SEN}_\pounds \times \mathrm{SEN}_\pounds \to \mathrm{SEN}_\pounds$. Given $\Sigma_0 \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}_\pounds(\Sigma_0)$, define $\Delta^{\langle\Sigma_0,\phi,\psi\rangle} = \{\Delta_\Sigma^{\langle\Sigma_0,\phi,\psi\rangle}\}_{\Sigma\in|\mathbf{Sign}|}$ by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\Delta_\Sigma^{\langle\Sigma_0,\phi,\psi\rangle} = C_\Sigma\Big(\bigcup_{f\in\mathbf{Sign}(\Sigma_0,\Sigma)} \Delta_\Sigma(\mathrm{SEN}_\pounds^2(f)(\phi,\psi))\Big).$$

Using virtually the same proof as that employed in Proposition 14 of [33], it is not difficult to see that the following holds.

PROPOSITION 9. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}_\pounds, C\rangle$ be a multi-sorted π-institution over a multi-sorted sentence functor* SEN*, with set of sorts S with distinguished sort $\pounds$, and N a transformation signature on* SEN*. Let, also, Δ be a collection of natural transformations in N of the form $\delta : \mathrm{SEN}_\pounds \times \mathrm{SEN}_\pounds \to \mathrm{SEN}_\pounds$. Then, for all $\Sigma_0 \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}_\pounds(\Sigma_0)$, $\Delta^{\langle\Sigma_0,\phi,\psi\rangle}$ is a theory system of $\mathcal{I}$.*

We conclude our exposition by proving an analog in the behavioral context of a well-known lemma due to Herrmann [20] (see also Theorem 3.3.3 of [11]), that provides a characterization of equivalential deductive systems inside the broader class of protoalgebraic deductive systems. In the present context, which constitutes a generalization of the one presented in [33] for π-institutions, Herrmann's test characterizes behaviorally N-equivalential multi-sorted π-institutions inside the class of behaviorally N-protoalgebraic multi-sorted π-institutions.

THEOREM 10 (Herrmann's Test). *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}_\pounds, C\rangle$ be a multi-sorted π-institution over a multi-sorted sentence functor* SEN*, with set of sorts S with distinguished sort $\pounds$, and N a transformation signature on* SEN*.*

Assume that $\mathcal{I}$ is behaviorally N-protoalgebraic and let Δ be a collection of natural transformations $\mathrm{SEN}^2_\pounds \to \mathrm{SEN}_\pounds$ in N. Then Δ is an N-equivalence system for $\mathcal{I}$ iff, for all $\Sigma \in |\mathbf{Sign}|$, and all $\phi, \psi \in \mathrm{SEN}_\pounds(\Sigma)$,

$$\Delta_\Sigma(\phi, \phi) \subseteq C_\Sigma(\emptyset) \quad \textit{and} \quad \langle \phi, \psi \rangle \in \Omega^{N,b,\pounds}_\Sigma(\Delta^{\langle \Sigma, \phi, \psi \rangle}).$$

Proof. Suppose that the two conditions of the theorem hold for Δ. By Proposition 7, it suffices to show that, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $\Delta(T) = \Omega^{N,b,\pounds}(T)$.

Suppose, first, that $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}_\pounds(\Sigma)$, such that $\langle \phi, \psi \rangle \in \Delta_\Sigma(T)$. This implies, by the definition of $\Delta(T)$, that

$$(\forall f)(\Delta_{\Sigma'}(\mathrm{SEN}_\pounds(f)^2(\phi, \psi)) \subseteq T_{\Sigma'}).$$

Therefore, $C_{\Sigma'}(\{\Delta_{\Sigma'}(\mathrm{SEN}_\pounds(f)^2(\phi, \psi)) : f \in \mathbf{Sign}(\Sigma, \Sigma')\}) \subseteq T_{\Sigma'}$. This shows that, for all $\Sigma' \in |\mathbf{Sign}|$, $\Delta^{\langle \Sigma, \phi, \psi \rangle}_{\Sigma'} \subseteq T_{\Sigma'}$, i.e., that $\Delta^{\langle \Sigma, \phi, \psi \rangle} \leq T$. Thus, by behavioral N-protoalgebraicity, $\Omega^{N,b}(\Delta^{\langle \Sigma, \phi, \psi \rangle}) \leq \Omega^{N,b}(T)$, and, since, by hypothesis, $\langle \phi, \psi \rangle \in \Omega^{N,b,\pounds}_\Sigma(\Delta^{\langle \Sigma, \phi, \psi \rangle})$, we obtain that $\langle \phi, \psi \rangle \in \Omega^{N,b,\pounds}_\Sigma(T)$. This concludes the proof that $\Delta_\Sigma(T) \subseteq \Omega^{N,b,\pounds}_\Sigma(T)$. This holding for an arbitrary $\Sigma \in |\mathbf{Sign}|$, we have $\Delta(T) \leq \Omega^{N,b,\pounds}(T)$.

Suppose, for the reverse inclusion, that $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}_\pounds(\Sigma)$, such that $\langle \phi, \psi \rangle \in \Omega^{N,b,\pounds}_\Sigma(T)$. Then, since Δ is a subcollection of natural transformations in N and $\Omega^{N,b}(T)$ is an N-congruence system, we get that, for every $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\langle \delta_{\Sigma'}(\mathrm{SEN}_\pounds(f)^2(\phi, \psi)), \delta_{\Sigma'}(\mathrm{SEN}_\pounds(f)^2(\phi, \phi)) \rangle \in \Omega^{N,b,\pounds}_{\Sigma'}(T)$, for all $\delta \in \Delta$. But, by hypothesis,

$$\delta_{\Sigma'}(\mathrm{SEN}_\pounds(f)^2(\phi, \phi)) \in C_{\Sigma'}(\emptyset) \subseteq T_{\Sigma'},$$

for all $\delta \in \Delta$, whence, by the compatibility property of $\Omega^{N,b}(T)$ with T, $\delta_{\Sigma'}(\mathrm{SEN}_\pounds(f)^2(\phi, \psi)) \in T_{\Sigma'}$, for all $\delta \in \Delta$, i.e., $\Delta_{\Sigma'}(\mathrm{SEN}_\pounds(f)^2(\phi, \psi)) \subseteq T_{\Sigma'}$. Since this holds for arbitrary $\Sigma' \in |\mathbf{Sign}|$ and arbitrary $f \in \mathbf{Sign}(\Sigma, \Sigma')$, we obtain that $\langle \phi, \psi \rangle \in \Delta_\Sigma(T)$. Thus, we have $\Omega^{N,b,\pounds}_\Sigma(T) \subseteq \Delta_\Sigma(T)$. Therefore, $\Omega^{N,b,\pounds}(T) \leq \Delta(T)$.

Suppose, conversely, that Δ is an N-equivalence system for $\mathcal{I}$. Since the first condition in the statement of the theorem is part of the definition of an N-equivalence system, it suffices to prove that the second condition is also satisfied. By Proposition 7, we have that $\Delta(T) = \Omega^{N,b,\pounds}(T)$, for every theory family T of $\mathcal{I}$. In particular, we obtain that, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}_\pounds(\Sigma)$, $\Delta_\Sigma(\Delta^{\langle \Sigma, \phi, \psi \rangle}) = \Omega^{N,b,\pounds}_\Sigma(\Delta^{\langle \Sigma, \phi, \psi \rangle})$. Thus, it suffices to show that $\langle \phi, \psi \rangle \in \Delta_\Sigma(\Delta^{\langle \Sigma, \phi, \psi \rangle})$, i.e., that $(\forall f)(\Delta_{\Sigma'}(\mathrm{SEN}_\pounds(f)^2(\phi, \psi)) \subseteq \Delta^{\langle \Sigma, \phi, \psi \rangle}_{\Sigma'})$. This is equivalent to

$$\delta_{\Sigma'}(\mathrm{SEN}_\pounds(f)^2(\phi, \psi)) \in C_{\Sigma'}(\{\Delta_{\Sigma'}(\mathrm{SEN}_\pounds(f)^2(\phi, \psi)) : f \in \mathbf{Sign}(\Sigma, \Sigma')\}),$$

for all $\delta \in \Delta, \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$. But this is obvious because of the reflexivity property of C. ■

We close this section with a comment concerning the dependence of behavioral N-protoalgebraicity, behavioral N-equivalentiality and existence of an N-equivalence system on the transformation signature N on the underlying multi-sorted sentence functor SEN of a multi-sorted π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}_{\mathcal{L}}, C\rangle$. Since N plays in the categorical context the role of the clone of operations generated by the fundamental operations of an algebra and only derived operations in that clone are allowed to participate in determining congruence systems and the natural transformations in an equivalence system, all the preceding notions are sensitive to the choice of the transformation signature N. In other words, it might be the case that a π-institution in the ordinary sense, or a multi-sorted π-institution in the behavioral sense, is N-protoalgebraic but not N'-protoalgebraic, or N-equivalential but not N'-equivalential, for two different choices of transformation signatures N and N' on its multi-sorted sentence functor. Moreover, in the behavioral case, it may happen that for the same transformation signature N on SEN, there exist transformation subsignatures N' and N'' of N, such that $\mathcal{I}$ is N'-equivalential and N''-equivalential without the corresponding equivalence systems being unique up to deductive equivalence. This happens in the universal algebraic framework even for behaviorally algebraizable deductive systems, as is shown in Section 3.2 of [8], and any example from that framework may be lifted to obtain a corresponding one in the categorical framework. Freedom in the choice of the signature introduces this non-uniqueness feature in both frameworks. As is shown in Theorem 10 of [8], in the context of behaviorally algebraizable deductive systems, uniqueness is recovered once the subsignature is fixed.

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