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CATEGORICAL ABSTRACT ALGEBRAIC LOGIC ON ADMISSIBLE EQUIVALENCE SYSTEMS

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Abstract

Given a sentential logic $\mathcal{S}$ there exists a least sentential logic $\mathcal{S}^{\mathrm{ad}}$ associated with the set $\mathrm{Thm}\mathcal{S}$ of the theorems of $\mathcal{S}$. If the logic $\mathcal{S}$ is equivalential, then the behavioral theorems of $\mathrm{Thm}\mathcal{S}$ can be determined by an equivalence system for $\mathcal{S}$, but, possibly, they may also be determined by any admissible equivalence system, i.e., an equivalence system for $\mathcal{S}^{\mathrm{ad}}$. Babenyshev and Martins studied the relationship between these two equivalence systems for a given sentential logic $\mathcal{S}$. We extend their study to the case of logics formalized as π-institutions. We introduce the basic notions and show how their results can be applied to provide some similar results in the categorical framework.

1. Introduction

A **language type** $\mathcal{L} = \langle \Lambda, \rho \rangle$ consists of a set Λ of **logical connectives** and an **arity function** $\rho : \Lambda \to \omega$ that gives the arity of each of the connectives in Λ. Given a language type $\mathcal{L}$ and a fixed denumerable set V of propositional variables, the set $\mathrm{Fm}_{\mathcal{L}}(V)$ of $\mathcal{L}$ - **formulas** with variables in V may be constructed in the ordinary recursive way, starting from the variables in V and using the connectives in Λ, respecting the arities. The set $\mathrm{Fm}_{\mathcal{L}}(V)$ is the universe of the absolutely free $\mathcal{L}$- algebra with generators V, denoted $\mathbf{Fm}_{\mathcal{L}}(V)$. A **deductive system** or **sentential logic** $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ consists of

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a language type $\mathcal{L}$ and a structural consequence relation $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)) \times \mathrm{Fm}_{\mathcal{L}}(V)$ on the set $\mathrm{Fm}_{\mathcal{L}}(V)$. A formula $\phi \in \mathrm{Fm}_{\mathcal{L}}(V)$ is an $\mathcal{S}$-**theorem** if $\emptyset \vdash_{\mathcal{S}} \phi$, sometimes also denoted by $\vdash_{\mathcal{S}} \phi$. Define $\mathrm{Thm}\mathcal{S} = \{\phi \in \mathrm{Fm}_{\mathcal{L}}(V) : \vdash_{\mathcal{S}} \phi\}$, the set of all $\mathcal{S}$-theorems. We follow [1] in using the term **theory** for a set of $\mathcal{L}$-formulas and the term **logic** for a theory that is invariant under all substitutions $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$, i.e., all endomorphisms on the formula algebra $\mathbf{Fm}_{\mathcal{L}}(V)$. Note that, because of structurality, the set $\mathrm{Thm}\mathcal{S}$ is a logic. Conversely, a theory T is a logic only if it is the set of theorems of some deductive system $\mathcal{S}$. In fact, given T, the relation

$$\Phi \vdash_{\mathcal{S}T} \phi \text{ iff } \Phi \not\subseteq T \text{ or } \phi \in T$$

defines a deductive system $\mathcal{S}_T = \langle \mathcal{L}, \vdash_{\mathcal{S}_T} \rangle$, such that $\mathrm{Thm}\,\mathcal{S}_T = T$.

It is well-known by the theory of abstract algebraic logic (see, e.g., [3, 7, 10, 11]) that, given a theory T, there exists a largest congruence $\Omega(T)$ on the formula algebra $\mathbf{Fm}_{\mathcal{L}}(V)$, that is compatible with T, in the sense that, for all $\phi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$, if $\langle \phi, \psi \rangle \in \Omega(T)$ and $\phi \in T$, then $\psi \in T$, which amounts to the property that T is a union of $\Omega(T)$-equivalence classes. The congruence $\Omega(T)$ is called the *Leibniz congruence of T*. This congruence may also be characterized by semantic indistinguishability [3]. Namely, for all $\phi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$, $\langle \phi, \psi \rangle \in \Omega(T)$ if and only if, for every formula $\alpha(v, \vec{u}) \in \mathrm{Fm}_{\mathcal{L}}(V)$, where $v \in V$ is a variable appearing in α and $\vec{u}$ is a vector incorporating all other variables in α, we have

$$\alpha(\phi, \vec{u}) \in T \text{ iff } \alpha(\psi, \vec{u}) \in T.$$

Following [1], we call a pair $\langle \phi, \psi \rangle \in \mathrm{Fm}^2_{\mathcal{L}}(V)$, such that $\langle \phi, \psi \rangle \in \Omega(T)$ a **behavioral theorem** of T.

A deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ is called (*finitely*) *equivalential* [5-7], if there exists a set of formulas $\Delta(v, u)$ in two variables, such that

- $\vdash_{\mathcal{S}} \Delta(v, v)$
- $v, \Delta(v, u) \vdash_{\mathcal{S}} u$;

- $\Delta(v, u) \vdash_{\mathcal{S}} \Delta(u, v)$;

- $\Delta(v, u), \Delta(u, w) \vdash_{\mathcal{S}} \Delta(v, w)$;

- $\{\Delta(v_i, u_i) : i < n\} \vdash_{\mathcal{S}} \Delta(\lambda(v_0, \dots, v_{n-1}), \lambda(u_0, \dots, u_{n-1}))$, for all $\lambda \in \Lambda$, with $\rho(\lambda) = n$.

In that case the set Δ is called a *set of equivalence formulas*, or an *equivalence system*, for $\mathcal{S}$.

It turns out (see, e.g., [1, 7]) that, if a deductive system $\mathcal{S}$ is equivalential with equivalence system Δ, then Δ defines the behavioral theorems of any theory T of $\mathcal{S}$, in the sense that

$$\Omega(T) = \{\langle \phi, \psi \rangle \in \mathrm{Fm}_{\mathcal{L}}^2(V) : \Delta(\phi, \psi) \subseteq T\}.$$

A **rule** is a pair $\langle \Gamma, \phi \rangle$, with $\Gamma \cup \{\phi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$. If Γ is finite, the rule is called *finitary*. A rule $\langle \Gamma, \phi \rangle$ is said to be a *rule of a theory* T or **compatible** with T if, for every substitution $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$,

$$h(\Gamma) \subseteq T \text{ implies } h(\phi) \in T.$$

In that case T is said to be **closed under** $\langle \Gamma, \phi \rangle$. A rule of a logic T is called *admissible for* T. Given a logic T, the set of all its admissible rules is denoted by $\mathrm{Adm}(T)$. The finitary deductive system whose theories are all theories that are closed under $\mathrm{Adm}(T)$ is denoted by $\mathcal{S}^{\mathrm{Adm}(T)}$. Given a deductive system $\mathcal{S}$, we set $\mathcal{S}^{\mathrm{ad}} := \mathcal{S}^{\mathrm{Adm}(\mathrm{Thm}\mathcal{S})}$. If the deductive system $\mathcal{S}$ is equivalential, then the behavioral theorems of $\mathrm{Thm}\mathcal{S}$ can be determined by an equivalence system Δ for $\mathcal{S}$. It is possible, however, that they can also be determined by any admissible equivalence system, i.e., an equivalence system for $\mathcal{S}^{\mathrm{ad}}$. Babenyshev and Martins [1] studied the relationship between these two equivalence systems for a given sentential logic $\mathcal{S}$. They showed that various possibilities regarding this relationship may arise:

- First, there are deductive systems that have admissible equivalence systems which are not, however, equivalence systems.

- Second, there are sentential logics that are not finitely equivalential, but, nevertheless, have finite admissible equivalence systems.

• Finally, there are non-protoalgebraic deductive systems that possess admissible equivalence systems.

The goal of the present work is to extend their study to the case of logics formalized as π-institutions. We introduce the basic notions, reveal some basic relationships between the corresponding equivalence systems and show how the results of [1] can be applied to provide some similar results in the categorical framework.

2. Admissible Rules and Admissible Counterparts

For all unexplained basic categorical terminology and notation the reader is encouraged to consult any of the standard introductory references on the subject, e.g., [2, 4, 14].

Recall that a sentence functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is an arbitrary **Set**-valued functor, where **Set** denotes the category of all small sets. To abstract the concept of an algebraic signature (or logical language) from the level of deductive systems to the categorical level, we consider the notion of the category of natural transformations on a given sentence functor. Let **Sign** be a category and $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ a functor. The **clone of all natural transformations** on SEN is defined to be the locally small category with collection of objects $\{\mathrm{SEN}^\alpha : \alpha \text{ an ordinal}\}$ and collection of morphisms $\tau : \mathrm{SEN}^\alpha \to \mathrm{SEN}^\beta$ β-sequences of natural transformations $\tau_i : \mathrm{SEN}^\alpha \to \mathrm{SEN}$. Composition

$$\mathrm{SEN}^\alpha \xrightarrow{\langle \tau_i : i < \beta \rangle} \mathrm{SEN}^\beta \xrightarrow{\langle \sigma_j : j < \gamma \rangle} \mathrm{SEN}^\gamma$$

is defined by

$$\langle \sigma_j : j < \gamma \rangle \circ \langle \tau_i : i < \beta \rangle = \langle \sigma_j(\langle \tau_i : i < \beta \rangle) : j < \gamma \rangle.$$

A subcategory N of this category containing all objects of the form SEN^k for $k < \omega$, and all projection morphisms $p^{k,i} : \mathrm{SEN}^k \to \mathrm{SEN}$, $i < k$, $k < \omega$, with $p_\Sigma^{k,i} : \mathrm{SEN}(\Sigma)^k \to \mathrm{SEN}(\Sigma)$ given by

$$p_\Sigma^{k,i}(\vec{\phi}) = \phi_i, \text{ for all } \vec{\phi} \in \mathrm{SEN}(\Sigma)^k,$$

and such that, for every family $\{\tau_i : \mathrm{SEN}^k \to \mathrm{SEN} : i < l\}$ of natural transformations in N, the sequence $\langle \tau_i : i < l \rangle : \mathrm{SEN}^k \to \mathrm{SEN}^l$ is also in N, is referred to as a **category of natural transformations on** SEN.

A π-institution ([9]; see also [12 and 13]) $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ is a triple consisting of an arbitrary category **Sign**, a sentence functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ and a collection $C = \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ of closure operators $C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \to \mathcal{P}(\mathrm{SEN}(\Sigma))$, $\Sigma \in |\mathbf{Sign}|$, such that, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$,

$$\mathrm{SEN}(f)(C_{\Sigma_1}(\Phi)) \subseteq C_{\Sigma_2}(\mathrm{SEN}(f)(\Phi)). \tag{1}$$

(The map $C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \to \mathcal{P}(\mathrm{SEN}(\Sigma))$ is a closure operator if it satisfies, for all $\Phi \subseteq \Psi \subseteq \mathrm{SEN}(\Sigma)$,

- $\Phi \subseteq C_\Sigma(\Phi)$; (Reflexivity)
- $C_\Sigma(\Phi) \subseteq C_\Sigma(\Psi)$ (Monotonicity)
- $C_\Sigma(C_\Sigma(\Phi)) = \mathrm{C}_\Sigma(\Phi)$. (Idempotency)

Moreover C is termed a closure system on SEN if, in addition, condition (1) holds).

The structure of a π-institution abstracts that of a deductive system, which is used as the underlying structure supporting the concept of a logical system in universal abstract algebraic logic. Categorical abstract algebraic logic aspires to abstract the methods and results of the universal treatment to a wider class of logical systems and, as a result, broaden their applicability. To achieve this goal, it uses π-institutions as the underlying supporting structures representing logical systems, because π-institutions can readily accommodate logical systems with multiple signatures and quantifiers which are more difficult to deal with using deductive systems (see, e.g., Appendix C in [3] and relevant discussions in both [16] and [17]).

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a sentence functor. An **axiom family** of SEN is a collection $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, with $T_\Sigma \subseteq \mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$. An

axiom system, which in the present work will also be referred to as a logic (to match similar terminology of [1]), is an axiom family $L = \{L_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ of SEN, such that, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$,

$$\mathrm{SEN}(f)(L_{\Sigma_1}) \subseteq L_{\Sigma_2}.$$

Recall that, given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, its theorem system $\mathrm{Thm}(\mathcal{I}) = \{\mathrm{Thm}_\Sigma(\mathcal{I})\}_{\Sigma \in |\mathbf{Sign}|}$ is the family consisting of $\mathrm{Thm}_\Sigma(\mathcal{I}) = C_\Sigma(\emptyset)$, the Σ-theorems of the π-institution $\mathcal{I}$, for every $\Sigma \in |\mathbf{Sign}|$. The following proposition states that this collection is a logic on SEN, for every π-institution $\mathcal{I}$, with sentence functor SEN.

Proposition 1. *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *be a* π-*institution. Then its theorem system* $\mathrm{Thm}(\mathcal{I}) = \{\mathrm{Thm}_\Sigma(\mathcal{I})\}_{\Sigma \in |\mathbf{Sign}|}$, *is a logic on* SEN.

Proof. For all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, we have

$$\mathrm{SEN}(f)(C_{\Sigma_1}(\emptyset)) \subseteq C_{\Sigma_2}(\mathrm{SEN}(f)(\emptyset)) = C_{\Sigma_2}(\emptyset)$$

(the inclusion holding by Property (1)).

Given a sentence functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, with N a category of natural transformations on SEN, an N-**rule** (also a **rule**, if N is clear from context) $r = \langle \sigma^0, \ldots, \sigma^{n-1}, \tau\rangle$ is a tuple of natural transformations $\sigma^0, \ldots, \sigma^{n-1}, \tau : \mathrm{SEN}^k : \mathrm{SEN}$ in N. An N-rule $r = \langle \sigma^0, \ldots, \sigma^{n-1}, \tau\rangle$ is **compatible with** an axiom family T of SEN if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$,

$$\sigma^0_\Sigma(\vec{\phi}), \ldots, \sigma^{n-1}_\Sigma(\vec{\phi}) \in T_\Sigma \text{ implies } \tau_\Sigma(\vec{\phi}) \in T_\Sigma.$$

If a rule r, as above, is compatible with T, then T is said to be *closed under* r and r to be a **rule of** T. If r is compatible with a logic L, then r is said to be *admissible for* L. By $\mathrm{Adm}(L)$ will be denoted the collection of all rules that are admissible for a logic L. An N-rule $r = \langle \sigma^0, \ldots, \sigma^{n-1}, \tau\rangle$ is a **rule of** a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, $\tau_\Sigma(\vec{\chi}) \in C_\Sigma(\sigma^0_\Sigma(\vec{\chi}), \ldots, \sigma^{n-1}_\Sigma(\vec{\chi}))$.

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a sentence functor, with N a category of natural transformations on SEN, and R a collection of N-rules. Given $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, we say that ϕ follows from Φ via R, if there exists a rule $\langle \sigma^0, \ldots, \sigma^{n-1}, \tau \rangle$ in R and $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, such that $\sigma^i_\Sigma(\vec{\chi}) \in \Phi$, for all $i < n$, and $\tau^i_\Sigma(\vec{\chi}) = \phi$. A **proof of** ϕ **from** Φ **via** R is a finite sequence $\phi_0, \phi_1, \ldots, \phi_n$ in $\mathrm{SEN}(\Sigma)$, such that $\phi_n = \phi$ and, for all $i \leq n$, $\phi_i \in \Phi$ or ϕ_i follows from $\{\phi_0, \ldots, \phi_{i-1}\}$ via R. If such a proof exists, then we say that Φ is R-**provable** or R-**derivable from** Φ. Define $C^R = \{C^R_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ by letting, for all $\Sigma \in |\mathbf{Sign}|$, $C^R_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \to \mathcal{P}(\mathrm{SEN}(\Sigma))$ be defined as follows: For all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)$,

$$C^R_\Sigma(\Phi) = \{\phi \in \mathrm{SEN}(\Sigma) : \phi \text{ is } R\text{-provable from } \Phi\}.$$

This was shown in [19] to be a closure system on SEN. Thus, the tuple $\mathcal{I}^R = \langle \mathbf{Sign}, \mathrm{SEN}, C^R \rangle$ is a π-institution.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on SEN. The closure system C and the π-institution $\mathcal{I}$ are said to be **finitary**, if, for all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$, there exists a finite $\Psi \subseteq \Phi$, such that $\phi \in C_\Sigma(\Psi)$. The sentence functor SEN is said to be N-**rule based with respect to** a logic L on SEN, if, for all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, such that if $\Phi \subseteq L_\Sigma$, then $\phi \in L_\Sigma$, there exists an N-rule $\langle \sigma^0, \ldots, \sigma^{n-1}, \tau \rangle$ admissible for L, and a $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, such that $\sigma^i_\Sigma(\vec{\chi}) \in \Phi$, for all $i < n$, and $\tau_\Sigma(\vec{\chi}) = \phi$. The finitary π-institution $\mathcal{I}$, on the other hand, is called *N-rule based* if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi_0, \ldots, \phi_{n-1}, \psi \in \mathrm{SEN}(\Sigma)$, such that $\psi \in C_\Sigma(\phi_0, \ldots, \phi_{n-1})$ there exists an N-rule $\langle \sigma^0, \ldots, \sigma^{n-1}, \tau \rangle$ of $\mathcal{I}$, and a $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, such that $\sigma^i_\Sigma(\vec{\chi}) = \phi_i$, for all $i < n$, and $\tau_\Sigma(\vec{\chi}) = \psi$.

Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{C} \rangle$, with N a category of natural transformations on SEN, define

$$\mathcal{I}^{\mathrm{ad}} = \langle \mathrm{Sign}, \mathrm{SEN}, C^{\mathrm{ad}} \rangle := \mathcal{I}^{\mathrm{Adm}(\mathrm{Thm}(\mathcal{I}))},$$

i.e., the π-institution with closure system the closure system $C^{\mathrm{Adm}(\mathrm{Thm}(\mathcal{I}))}$ on SEN, that is specified by all rules admissible for the theorem system of $\mathcal{I}$.

Recall that a **theory family** $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ of a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ is an axiom family, such that T_Σ is C_Σ-closed, for all $\Sigma \in |\mathbf{Sign}|$, i.e., such that $C_\Sigma(T_\Sigma) = T_\Sigma$. The collection of all C_Σ-closed sets T_Σ is denoted by $\mathrm{Th}_\Sigma(\mathcal{I})$, whereas the collection of all theory families of $\mathcal{I}$ is denoted by $\mathrm{ThFam}(\mathcal{I})$. In the next lemma, it is shown that a logic L is always a theory family of the π-institution $\mathcal{I}^{\mathrm{Adm}(L)}$, induced by the set $\mathrm{Adm}(L)$ of all admissible rules of inference for L.

Lemma 2. *Let* SEN : **Sign** $\to$ **Set** *be a functor, with N a category of natural transformations on* SEN, *and L a logic on* SEN. *Then* $L \in \mathrm{ThFam}(\mathcal{I}^{\mathrm{Adm}(L)})$, *i.e., for all* $\Sigma \in |\mathbf{Sign}|$, $C_\Sigma^{\mathrm{Adm}(L)}(L_\Sigma) = L_\Sigma$.

Proof. It suffices to show that, for all $\phi \in \mathrm{SEN}(\Sigma)$, if $\phi \in C_\Sigma^{\mathrm{Adm}(L)}(L_\Sigma)$, then $\phi \in L_\Sigma$. If $\phi \in C_\Sigma^{\mathrm{Adm}(L)}(L_\Sigma)$, there exists an $\mathrm{Adm}(L)$-proof $\phi_0, \ldots \phi_n$ of ϕ from premises L_Σ. Since all hypotheses in the proof are in L_Σ and all rules in $\mathrm{Adm}(L)$ are admissible for L, we conclude (by straightforward induction on $i \le n$) that, for all $i \le n$, $\phi_i \in L_\Sigma$. Thus, $\phi = \phi_n \in L_\Sigma$.

The following lemma forms an analog of Lemma 1 of [1] for π-institutions. It provides a characterization of the closure system $C^{\mathrm{Adm}(L)}$ of the π-institution $\mathcal{I}^{\mathrm{Adm}(L)}$, induced by the set $\mathrm{Adm}(L)$ of all admissible rules of inference for L, in terms of the logic L. Recall that, given two closure systems C and C' on the same sentence functor SEN : **Sign** $\to$ **Set** we write $C \le C'$ to signify that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)$, $C_\Sigma(\Phi) \subseteq C'_\Sigma(\Phi)$.

Lemma 3. *Let* SEN : **Sign** $\to$ **Set** *be a functor, with N a category of natural transformations on* SEN, *and L a logic on* SEN. *If* SEN *is N-rule based with respect to L, then* $C^{\mathrm{Adm}(L)}$ *is the largest closure system C on* SEN *in the $\le$-order i.e., the one with the smallest collection of theory families, such* *that, for all* $\Sigma \in |\mathbf{Sign}|$,

$$\{\mathrm{SEN}(f)^{-1}(L_{\Sigma'}) : f = \mathbf{Sign}(\Sigma, \Sigma'),\ \Sigma \in |\mathbf{Sign}|\} \subseteq \mathrm{Th}_{\Sigma}(C).$$

Proof. Let us denote by $P(C)$ the property of a closure system C on SEN, satisfied iff, for all $\Sigma \in |\mathbf{Sign}|$, $\{\mathrm{SEN}(f)^{-1}(L_{\Sigma'}) : f = \mathbf{Sign}(\Sigma, \Sigma'), \Sigma' \in |\mathbf{Sign}|\} \subseteq \mathrm{Th}_{\Sigma}(C)$. Then, if C^* is the largest closure system in the family $\{C : P(C)\}$, to prove the lemma, we must show that $C^{\mathrm{Adm}(L)} = C^*$.

$\leq$: For this part, it suffices to show $P(C^{\mathrm{Adm}(L)})$. To this end, let us fix $\Sigma, \Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma, \Sigma')$. We must show

$$C_{\Sigma}^{\mathrm{Adm}(L)}(\mathrm{SEN}(f)^{-1}(L_{\Sigma'})) = \mathrm{SEN}(f)^{-1}(L_{\Sigma'}).$$

Since the right-to-left inclusion is obvious, it suffices to show that, for all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\phi \in C_{\Sigma}^{\mathrm{Adm}(L)}(\mathrm{SEN}(f)^{-1}(L_{\Sigma'})) \text{ implies } \mathrm{SEN}(f)(\phi) \in L_{\Sigma'}.$$

The hypothesis means that there exists an $\mathrm{Adm}(L)$-proof $\phi_0, \ldots, \phi_n$ of ϕ from premises $\mathrm{SEN}^{-1}(f)(L_{\Sigma'})$. Then, $\mathrm{SEN}(f)(\phi_0), \ldots \mathrm{SEN}(f)(\phi_n)$ is an $\mathrm{Adm}(L)$-proof of $\mathrm{SEN}(f)(\phi)$ from $L_{\Sigma'}$. Thus, $\mathrm{SEN}(f)(\phi) \in C_{\Sigma'}^{\mathrm{Adm}(L)}(L_{\Sigma'}) = L_{\Sigma'}$, the last equality holding by Lemma 2.

$\geq$: For this part, it suffices to show that, if $P(C)$ holds and $\phi \in C_{\Sigma}(\Phi)$, for some $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, then ϕ is $\mathrm{Adm}(L)$-provable from Φ. By following contraposition, we show that, if ϕ is not $\mathrm{Adm}(L)$-provable from Φ and $P(C)$ holds, then $\phi \notin C_{\Sigma}(\Phi)$. If ϕ is not $\mathrm{Adm}(L)$-provable from Φ, then, for every admissible N-rule $\langle \sigma^0, \ldots, \sigma^{n-1}, \tau \rangle$ for L, and all $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, we have $\{\sigma_{\Sigma}^0(\vec{\chi}), \ldots, \sigma_{\Sigma}^{n-1}(\vec{\chi})\} \not\subseteq \Phi$ or $\tau_{\Sigma}(\vec{\chi}) \neq \phi$. But, then, since SEN is N-rule based with respect to L, we have that $\Phi \subseteq L_{\Sigma}$ and $\phi \notin L_{\Sigma}$. Thus, since $P(C)$ holds, we get that $\phi \notin C_{\Sigma}(\Phi)$.

3. Weak Equivalence Systems

In this section the main notions of an admissible equivalence system and of a weak admissible equivalence system will be defined. These parallel corresponding notions introduced for sentential logics in [1].

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a functor and N a category of natural transformations on SEN. Following [1], we call a pair $\langle \phi, \psi \rangle \in \mathrm{SEN}(\Sigma)^2$ a **behavioral Σ-theorem** of an axiom family $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ of SEN if, for all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')^k$,

$$\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi}) \in T_{\Sigma'} \text{ iff } \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}) \in T_{\Sigma'}.$$

In writing down this equivalence, an established convention in categorical abstract algebraic logic has been followed, according to which the two expressions displayed are shorthands for the corresponding expressions in which $\mathrm{SEN}(f)(\phi)$ and $\mathrm{SEN}(f)(\psi)$ may appear in any of the k positions of $\sigma_{\Sigma'}$ and not just the first, as long as they appear in the same position in both expressions.

We denote by $\Omega^N(T) = \{\Omega^N_\Sigma(T)\}_{\Sigma \in |\mathbf{Sign}|}$ the collection of all behavioral Σ-theorems of T. It is well-known in categorical abstract algebraic logic that $\Omega^N(T)$ is the Leibniz N-congruence system on SEN associated with the axiom family T, i.e., the largest N-congruence system that is compatible with T (see, e.g., [18]).

Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on SEN, a set of binary natural transformations $\Delta : \mathrm{SEN}^2 \to \mathrm{SEN}$ in N is called an *N-equivalence system* for $\mathcal{I}$ if

1. $\Delta_\Sigma(\phi, \phi) \subseteq C_\Sigma(\emptyset)$, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$;

2. $\phi \in T_\Sigma$ and $\Delta_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)) \subseteq T_{\Sigma'}$ for all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, imply $\psi \in T_\Sigma$, for all $T \in \mathrm{ThFam}(\mathcal{I})$;

3. $\Delta_\Sigma(\psi, \phi) \subseteq C_\Sigma(\Delta_\Sigma(\phi, \psi))$, for all $\Sigma \in |\mathrm{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$;

4. $\Delta_\Sigma(\phi, \chi) \subseteq C_\Sigma(\Delta_\Sigma(\phi, \psi), \Delta_\Sigma(\psi, \chi))$, for all $\Sigma \in |\mathrm{Sign}|$ and all $\phi, \psi, \chi \in \mathrm{SEN}(\Sigma)$;

5. $\Delta_\Sigma(\sigma_\Sigma(\vec{\phi}), \sigma_\Sigma(\vec{\psi})) \subseteq C_\Sigma(\Delta_\Sigma(\phi_0, \psi_0), \ldots, \Delta_\Sigma(\phi_{n-1}, \psi_{n-1}))$, for all n-ary σ in N, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)^n$.

If such a (finite) N-equivalence system exists, then $\mathcal{I}$ is termed (**finitely**) **syntactically N-equivalential**.

In the next lemma, an analog of Lemma 4 of [1] (see also [7]), it is shown that an N-behavioral theorem of a theory family T of a π-institution $\mathcal{I}$ can be determined by using an N-equivalence system Δ for $\mathcal{I}$, in case $\mathcal{I}$ is syntactically N-equivalential.

Lemma 4. *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *be a* π-*institution, with* N *a category of natural transformations on* SEN, *and* Δ *an* N-*equivalence system for* $\mathcal{I}$. *Then, for all* $T \in \mathrm{ThFam}(\mathcal{I})$ *and all* $\Sigma \in |\mathbf{Sign}|$,

$$\Omega^N_\Sigma(T) = \{\langle \phi, \psi\rangle \in \mathrm{SEN}(\Sigma)^2 : \Delta_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)) \subseteq T_{\Sigma'} \textit{ for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')\}.$$

Proof. For all $\Sigma \in |\mathbf{Sign}|$, define θ_Σ as the set on the right hand side of the displayed equation above and let $\theta = \{\theta_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$. The goal is to show that $\theta = \Omega^N(T)$.

$\leq$: For this inclusion, it suffices to show that θ is an N-congruence system on SEN that is compatible with T. In fact, Properties 1, 3 and 4 of an equivalence system ensure that θ_Σ is an equivalence relation on $\mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$. Moreover, Property 5 shows that it is an N-congruence relation on $\mathrm{SEN}(\Sigma)$. The family θ is a system, i.e., invariant under signature morphisms, because, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, if $\langle \phi, \psi\rangle \in \theta_{\Sigma_1}$, then, for all $\Sigma' \in |\mathbf{Sign}|$ and all $g \in \mathbf{Sign}(\Sigma_1, \Sigma')$,

$$\begin{tikzcd}
\Sigma_1 \arrow[rr, "f"] \arrow[dr, "g"'] & & \Sigma_2 \arrow[dl, "h"] \\
& \Sigma' &
\end{tikzcd}$$

$\Delta_{\Sigma'}(\mathrm{SEN}(g)(\phi), \mathrm{SEN}(g)(\psi)) \subseteq T_{\Sigma'}$. Thus, for all $h \in \mathbf{Sign}(\Sigma_2, \Sigma')$, we have $\Delta_{\Sigma'}(\mathrm{SEN}(hf)(\phi), \mathrm{SEN}(hf)(\psi)) \subseteq T_{\Sigma'}$. and, hence, we obtain $\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)\rangle \in \theta_{\Sigma_2}$. Finally, by Property 2 of an equivalence system, θ is compatible with T. This proves that $\theta \leq \Omega^N(T)$, since the latter is the largest N-congruence system on SEN compatible with T.

$\geq$: For this inclusion, it suffices to show that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, if $\langle \phi, \psi\rangle \in \Omega^N_\Sigma(T)$, then, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\Delta_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)) \subseteq T_{\Sigma'}$. Notice, first, that, since $\Omega^N(T)$ is an N-congruence system, we get that $\langle \phi, \psi\rangle \in \Omega^N_\Sigma(T)$ implies that $\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)\rangle \in \Omega^N_{\Sigma'}(T)$. Since Δ is a collection of natural transformations in N and $\Omega^N(T)$ is an N-congruence system, we have that

$$\langle \delta_{\Sigma'}(\mathrm{SEN}(f)^2(\phi, \phi)), \delta_{\Sigma'}(\mathrm{SEN}(f)^2(\phi, \psi))\rangle \in \Omega^N_{\Sigma'}(T),$$

for all $\delta \in \Delta$. But, since Δ is an N-equivalence system, we obtain $\Delta_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\phi)) \subseteq T_{\Sigma'}$, whence, by the compatibility property of $\Omega^N(T)$ with T, we obtain $\Delta_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)) \subseteq T_{\Sigma'}$.

A collection $\Delta : \mathrm{SEN}^2 \to \mathrm{SEN}$ in N is an **admissible N-equivalence system for** $\mathcal{I}$ if it is an N-equivalence system for $\mathcal{I}^{\mathrm{ad}}$.

Proposition 5. *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *be a* π-*institution, with* N *a category of natural transformations on* SEN, *and assume that* SEN *is N-rule based with respect to* $\mathrm{Thm}(\mathcal{I})$. *Then, every N-equivalence system* Δ *for* $\mathcal{I}$ *is an admissible N-equivalence system for* $\mathcal{I}$.

Proof. Lemma 3 shows that $C \leq C^{\mathrm{ad}}$, whence, it also holds $\mathrm{ThFam}(\mathcal{I}^{\mathrm{ad}}) \subseteq \mathrm{ThFam}(\mathcal{I})$, and the conclusion follows from the definition of an equivalence system.

An admissible N-equivalence system for a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is called *weak* if it is an N-equivalence system for $\mathcal{I}^{\mathrm{ad}}$, but not an N-equivalence system for $\mathcal{I}$.

4. Characterization of $\mathrm{ThFam}(\mathcal{I}^{\mathrm{ad}})$

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a sentence functor and N a category of natural transformations on SEN. Given a collection $\mathcal{T} \subseteq \mathrm{AxFam}(\mathrm{SEN})$ of axiom families on SEN, define

- $\mathbf{X}\mathcal{T}$: the collection of all signature-wise intersections of arbitrary subfamilies of $\mathcal{T}$;

- $\hat{\mathbf{U}}\mathcal{T}$: the collection of all signature-wise unions of $\leq$-upward directed subfamilies of $\mathcal{T}$.

Given $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ an axiom family of SEN, an inverse signature image of T is an axiom family $T' = \{T'_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, such that, for every $\Sigma \in |\mathbf{Sign}|$, there exists a $\Sigma' \in |\mathbf{Sign}|$ and an $f \in \mathbf{Sign}(\Sigma, \Sigma')$, such that $T'_\Sigma = \mathrm{SEN}(f)^{-1}(T_{\Sigma'})$. Define, also,

- $\mathbf{S}^{-1}\mathcal{T}$: the collection of all inverse signature images of axiom families in $\mathcal{T}$.

The following lemma contains a few easy observations concerning some of the axiom families that are contained in the collections of axiom families obtained by applying these three operators on arbitrary axiom families of a sentence functor. The proofs are straightforward and, therefore, omitted.

Lemma 6. *Let* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a sentence functor, with* N *a category of natural transformations on* SEN, *and* $\mathcal{T}$ *a collection of axiom families on* SEN. *Then*

(i) $\mathrm{SEN} \in \mathbf{X}\mathcal{T}$;

(ii) $\mathcal{T} \subseteq \mathbf{X}\mathcal{T}$;

(iii) $\mathcal{T} \subseteq \hat{\mathbf{U}}\mathcal{T}$.

The next theorem, an analog of Theorem 7 of [1], characterizes the collection of theory families of the π-institution $\mathcal{I}^{\mathrm{Adm}(L)}$, generated by the admissible rules for a logic L on a sentence functor SEN, that is N-rule based with respect to L, in terms of the closure operators introduced above.

Theorem 7. *Let* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a sentence functor, with* N *a category of natural transformations on* SEN, *and* $L = \{L_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ *a logic on* SEN. *If* SEN *is* N*-rule based with respect to* L, *then*

$$\mathrm{ThFam}(\mathcal{I}^{\mathrm{Adm}(L)}) = \hat{\mathbf{U}}\mathbf{X}\mathbf{S}^{-1}(\{L\}).$$

Proof. We show, first, that $\hat{\mathbf{U}}\mathbf{X}\mathbf{S}^{-1}(\{L\}) \subseteq \mathrm{ThFam}(\mathcal{I}^{\mathrm{Adm}(L)})$. Since L is closed under the admissible N-rules for L, to attain this, it suffices to show that application of the three operators on families of axiom systems closed under these rules also results to collections of axiom systems for which the rules are still admissible.

• First, for inverse signature images, suppose that T' is the inverse image of T, which is closed under $\mathrm{Adm}(L)$, and $\langle \sigma^0, \ldots, \sigma^{n-1}, \tau \rangle$ an N-rule in $\mathrm{Adm}(L)$. Let $\Sigma \in |\mathbf{Sign}|$, $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, such that $\sigma_\Sigma^0(\vec{\chi}), \ldots, \sigma_\Sigma^{n-1}(\vec{\chi}) \in T'_\Sigma$. Since T is the inverse signature image of T', there exists $\Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma, \Sigma')$, such that $T'_\Sigma = \mathrm{SEN}(f)^{-1}(T_{\Sigma'})$. Thus, we obtain that $\mathrm{SEN}(f)(\sigma_\Sigma^i(\vec{\chi})) \in T_{\Sigma'}$, for all $i < n$, which yields that $\sigma_{\Sigma'}^i(\mathrm{SEN}(f)^k(\vec{\chi})) \in T_{\Sigma'}$, for all $i < n$, Since T is closed under $\mathrm{Adm}(L)$, we get $\tau_{\Sigma'}(\mathrm{SEN}(f)^k(\vec{\chi})) \in T_{\Sigma'}$, whence $\mathrm{SEN}(f)(\tau_\Sigma(\vec{\chi})) \in T_{\Sigma'}$, showing that $\tau_\Sigma(\vec{\chi}) \in \mathrm{SEN}(f)^{-1}(T_{\Sigma'}) = T'_\Sigma$. This concludes the proof that T' is also closed under $\mathrm{Adm}(L)$.

• Suppose, next, that $T = \bigcap_{i \in I} T_i$, where all T_i's are closed under $\mathrm{Adm}(L)$, and let $\langle \sigma^0, \ldots, \sigma^{n-1}, \tau \rangle$ be in $\mathrm{Adm}(L)$, $\Sigma \in |\mathbf{Sign}|$ and $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, such that $\sigma_\Sigma^j(\vec{\chi}) \in T_\Sigma$, for all $j < n$. Then $\sigma_\Sigma^j(\vec{\chi}) \in T_\Sigma^i$, for all $j < n$ and all $i \in I$, whence, since T^i is closed under $\mathrm{Adm}(L)$, we get that $\tau_\Sigma(\vec{\chi}) \in T_\Sigma^i$, for all $i \in I$, proving that $\tau_\Sigma(\vec{\chi}) \in T_\Sigma$ and, therefore, that signature-wise intersection preserves closure under the admissible rules of L.

• Suppose, finally, that $T = \bigcup_{i \in I} T^i$, where $\{T^i : i \in I\}$ is an upward directed collection of axiom systems, each of which is closed under $\mathrm{Adm}(L)$. Let $\langle \sigma^0, \ldots, \sigma^{n-1}, \tau\rangle$ be in $\mathrm{Adm}(L)$, $\Sigma \in |\mathbf{Sign}|$ and $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, such that $\sigma^i_\Sigma(\vec{\chi}) \in \bigcup_{i \in I} T^i_\Sigma$. Thus, for all $j < n$, there exists $i_j \in I$, such that $\sigma^j_\Sigma(\vec{\chi}) \in T^{i_j}_\Sigma$. By the directedness of $\{T_i\}_{i \in I}$, there exists $m \in I$, such that $T^{i_j} \leq T^m$, for all $j < n$. Therefore $\sigma^j_\Sigma(\vec{\chi}) \in T^m_\Sigma$, for all $j < n$, and, since T^m is closed under $\mathrm{Adm}(L)$, we get that $\tau_\Sigma(\vec{\chi}) \in T^m_\Sigma \subseteq \bigcup_{i \in I} T^i_\Sigma = T_\Sigma$. Thus union of upward directed families of axiom systems also preserves closure under $\mathrm{Adm}(L)$.

For the reverse inclusion $\mathrm{ThFam}(\mathcal{I}^{\mathrm{Adm}(L)}) \subseteq \hat{\mathbf{U}}\mathbf{XS}^{-1}(\{L\})$, it suffices to show that $\hat{\mathbf{U}}\mathcal{C}$ is closed under intersections and inverse signature images, whenever $\mathcal{C}$ is a closure system on SEN. This would imply that $\hat{\mathbf{U}}\mathbf{XS}^{-1}(\{L\})$ is a closure set system on SEN. Thus, since $\mathbf{S}^{-1}(L) \subseteq \hat{\mathbf{U}}\mathbf{XS}^{-1}(\{L\})$, we can use Lemma 3 to conclude that $\mathrm{ThFam}(\mathcal{I}^{\mathrm{Adm}(L)}) \subseteq \hat{\mathbf{U}}\mathbf{XS}^{-1}(\{L\})$.

• Suppose that $\mathcal{C}$ is a closure system on SEN and let $\{T^i : i \in I\} \subseteq \hat{\mathbf{U}}\mathcal{C}$, i.e., that, for all $i \in I$, there exists an upward directed collection $\{T^{ij} : j \in J_i\} \subseteq \mathcal{C}$, such that $T^i = \bigcup_{j \in J_i} T^{ij}$, for all $i \in I$. We must show that $T = \bigcap_{i \in I} T^i = \bigcap_{i \in I} \bigcup_{j \in J_i} T^{ij} \in \hat{\mathbf{U}}\mathcal{C}$. To accomplish this, set $K = \prod_{i \in I} J_i$ and, for all $k = \langle j_i : i \in I\rangle \in K$, define $T^k = \bigcap_{i \in I} T^{ij_i}$. Consider $\bigcup_{k \in K} T^k = \bigcup_{\langle j_i : i \in I\rangle \in \prod_{i \in I} J_i} \bigcap_{i \in I} T^{ij_i}$. Obviously, $T^k \in \mathcal{C}$, for all $k \in K$, since $\mathcal{C}$ is a closure system. Moreover, $\{T^k : k \in K\}$ is upward directed: in fact, let $k, k' \in K$, with $T^k = \bigcap_{i \in I} T^{ij_i}$ and $T^{k'} = \bigcap_{i \in I} T^{ij'_i}$. Then, since $\{T^{ij} : j \in J_i\}$ is directed, for all $i \in I$, we get that, for all $i \in I$, there exists $l_i \in J_i$, such that $T^{ij_i}, T^{ij'_i} \leq T^{il_i}$. Thus, $T^k, T^{k'} \leq \bigcap_{i \in I} T^{il_i} \in \{T^k : k \in K\}$. Thus, to conclude this part of the proof, it suffices to show that

$$\bigcap_{i\in I}\bigcup_{j\in J_i} T^{ij} = \bigcup_{\langle j_i : i\in I\rangle\in\prod_{i\in I} J_i}\ \bigcap_{i\in I} T^{ij_i}.$$

For the left-to-right inclusion, let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \bigcap_{i\in I}\bigcup_{j\in J_i} T^{ij}_\Sigma$. Thus, $\phi \in \bigcup_{i_j\in J_i} T^{ij}_\Sigma$, for all $i \in I$, which shows that, for all $i \in I$, there exists $j_i \in J_i$, such that $\phi \in T^{ij_i}_\Sigma$. Hence $\phi \in \bigcap_{i\in I} T^{ij_i}_\Sigma \subseteq \bigcup_{\langle j_i : i\in I\rangle\in\prod_{i\in I} J_i}\bigcap_{i\in I} T^{ij_i}_\Sigma$. For the reverse inclusion, let $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \bigcup_{\langle j_i : i\in I\rangle\in\prod_{i\in I} J_i}\bigcap_{i\in I} T^{ij_i}_\Sigma$. Thus, there exists $\langle j_i : i \in I\rangle \in \prod_{i\in I} J_i$, such that $\phi \in \bigcap_{i\in I} T^{ij_i}_\Sigma \subseteq \bigcap_{i\in I}\bigcup_{j\in J_i} T^{ij}_\Sigma$.

• Finally, we must show that $\hat{\mathbf{U}}\mathcal{C}$ is closed under inverse signature images. To this end, let $T = \bigcup_{i\in I} T^i$, with $\{T^i : i \in I\} \subseteq \mathcal{C}$, upward directed and T' an inverse signature image of T. This means, that, for all $\Sigma \in |\mathbf{Sign}|$, there exists $\Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma, \Sigma')$, such that $T'_\Sigma = \mathrm{SEN}(f)^{-1}(T_{\Sigma'})$. Notice that, for all $\Sigma \in |\mathbf{Sign}|$, we have

$$T'_\Sigma = \mathrm{SEN}(f)^{-1}\left(\bigcup_{i\in I} T^i_{\Sigma'}\right) = \bigcup_{i\in I}\mathrm{SEN}(f)^{-1}(T^i_{\Sigma'}).$$

Therefore, to complete the proof, it suffices to show that the family of axiom systems $\{\{\mathrm{SEN}(f)^{-1}(T^i_{\Sigma'})\}_{\Sigma\in|\mathbf{Sign}|} : i \in I\}$ is an upward directed subfamily of $\mathcal{C}$. Membership in $\mathcal{C}$ follows from the fact that $T^i \in \mathcal{C}$, for all $i \in I$, and $\mathcal{C}$ is a closure set system on SEN. Upward directedness follows easily from the fact that $\{T^i : i \in I\}$ is upward directed.

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a functor, with N a category of natural transformations on SEN, and L a logic on SEN. We say that the N-congruence system $\Omega^N(L)$ consisting of the behavioral N-theorems of L is *explicitly N-definable* iff, there exists a set Δ of binary natural transformations in N, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega_{\Sigma}^{N}(L) \text{ iff } \Delta_{\Sigma}(\phi, \psi) \subseteq L_{\Sigma}.$$

Then Δ is called a *defining set* for behavioral theorems of L.

The following proposition, an analog of Proposition 9 of [1] for π-institutions, shows that admissible equivalence systems for a given π-institution $\mathcal{I}$ with respect to a given logic L on its sentence functor, i.e., equivalence systems for $\mathcal{I}^{\mathrm{Adm}(L)}$, coincide with defining sets of behavioral theorems for the same logic.

Proposition 8. *Let* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a functor*, *with* N *a category of natural transformations on* SEN, L *a logic on* SEN *and* Δ *a finite set of binary natural transformations in* N. *If* SEN *is* N*-rule based with respect to* L, *then* Δ *is a defining set for the behavioral* N*-theorems of* L *iff* Δ *is an* N*-equivalence system for* $\mathcal{I}^{\mathrm{Adm}(L)}$.

Proof. Suppose, first, that Δ is an N-equivalence system for $\mathcal{I}^{\mathrm{Adm}(L)}$ This means that, for every $T \in \mathrm{ThFam}(\mathcal{I}^{\mathrm{Adm}(L)})$, every $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $\langle \phi, \psi \rangle \in \Omega_{\Sigma}^{N}(T)$ iff $\Delta_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)) \subseteq T_{\Sigma'}$, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$. Note that the latter condition, in case T happens to be a theory system, reduces to $\Delta_{\Sigma}(\phi, \psi) \subseteq T_{\Sigma}$. Therefore, since $L \in \mathrm{ThSys}(\mathcal{I}^{\mathrm{Adm}(L)})$, we get that for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $\langle \phi, \psi \rangle \in \Omega_{\Sigma}^{N}(L)$ iff $\Delta_{\Sigma}(\phi, \psi) \subseteq L_{\Sigma}$. i.e., that Δ is a defining set for the behavioral N-theorems of L.

Suppose, conversely, that Δ is a defining set for the behavioral N-theorems of L. To show that Δ is an N-equivalence system for $\mathcal{I}^{\mathrm{Adm}(L)}$, we use Theorem 7. Namely, we first show that all five properties defining an N-equivalence system hold for L, then that they hold for every theory that is an inverse signature image of L and, finally, that both $\mathbf{X}$ and $\hat{\mathbf{U}}$ preserve the validity of these properties. We do this in steps, but at each step we will show only Properties 2 and 3 in detail. The remaining Properties 1, 4 and 5 may be handled similarly.

• Suppose, first, that $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in L_\Sigma$ and that, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\Delta_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)) \subseteq \Omega^N_{\Sigma'}(L).$$

Thus, we have $\phi \in L_\Sigma$ and $\langle \phi, \psi \rangle \in \Omega^N_\Sigma(L)$. Therefore, by the compatibility of $\Omega^N(L)$ with L, we get that $\psi \in L_\Sigma$. So Property 2 holds. On the other hand, if $\Delta_\Sigma(\phi, \psi) \subseteq L_\Sigma$, then we have that $\langle \phi, \psi \rangle \in \Omega^N_\Sigma(L)$ and, since $\Omega^N_\Sigma(L)$ is symmetric, we get that $\langle \psi, \phi \rangle \in \Omega^N_\Sigma(L)$, whence $\Delta_\Sigma(\phi, \psi) \subseteq L_\Sigma$. So Property 3 is also satisfied.

• Suppose that T is an inverse signature image of L and let $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in T_\Sigma$ and $\Delta_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)) \subseteq T_{\Sigma'}$, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$. By the hypothesis, there exists $\Sigma'' \in |\mathbf{Sign}|$ and $g \in \mathbf{Sign}(\Sigma, \Sigma'')$, such that $T_\Sigma = \mathrm{SEN}(g)^{-1}(L_{\Sigma''})$. Therefore, $\mathrm{SEN}(g)(\phi) \in L_{\Sigma''}$. But, we also have, by the hypothesis, that $\Delta_\Sigma(\phi, \psi) \subseteq T_\Sigma = \mathrm{SEN}(g)^{-1}(L_{\Sigma''})$, whence $\Delta_{\Sigma''}(\mathrm{SEN}(g)(\phi), \mathrm{SEN}(g)(\psi)) \subseteq L_{\Sigma''}$. This gives that $\mathrm{SEN}(g)(\psi) \in L_{\Sigma''}$, i.e., that $\psi \in T_\Sigma$ and, thus, Property 2 holds. Finally, $\Delta_\Sigma(\phi, \psi) \subseteq T_\Sigma$ implies $\Delta_\Sigma(\phi, \psi) \subseteq \mathrm{SEN}(g)^{-1}(L_{\Sigma''})$, whence

$$\Delta_{\Sigma''}(\mathrm{SEN}(g)(\phi), \mathrm{SEN}(g)(\psi)) \subseteq L_{\Sigma''},$$

giving $\Delta_{\Sigma''}(\mathrm{SEN}(g)(\psi), \mathrm{SEN}(g)(\phi)) \subseteq L_{\Sigma''}$. This, following the reverse steps, entails that $\Delta_\Sigma(\psi, \phi) \subseteq T_\Sigma$. So Property 3 holds.

• The case of the operator $\mathbf{X}$ satisfying Properties 2 and 3 when applied on axiom families satisfying those properties is fairly easy and the details will be omitted.

• Similarly, for the operator $\hat{\mathbf{U}}$, the work is not very difficult; one has to take into account that Δ is finite and that $\hat{\mathbf{U}}$ is applied to upward directed collections of axiom families that are assumed to satisfy Properties 2 and 3, respectively.

Lemma 9. *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ *be a* π-*institution, with N a category of natural transformations on* SEN, *such that* SEN *is N-rule based with respect to* $\mathrm{Thm}(\mathcal{I})$. *If* Δ *is a finite N-equivalence system for* $\mathcal{I}$ *then* Δ *is a finite N-equivalence system for* $\mathcal{I}^{\mathrm{ad}}$.

Proof. Lemma 3 shows that $C \le C^{\mathrm{ad}}$, i.e., also, $\mathrm{ThFam}(\mathcal{I}^{\mathrm{ad}}) \subseteq \mathrm{ThFam}(\mathcal{I})$, whence the conclusion follows by the definition of an N-equivalence system.

Recall that given a sentence functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ and two closure systems C, C' on SEN, we write $C \le C'$ if, for all $\Sigma \in |\mathrm{Sign}|$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)$, $C_{\Sigma}(\Phi) \subseteq C'_{\Sigma}(\Phi)$. Moreover, if $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, and $\mathcal{I}' = \langle \mathbf{Sign}, \mathrm{SEN}, C' \rangle$ denote the corresponding π-institutions, we write $\mathcal{I} \le \mathcal{I}'$ whenever $C \le C'$ in this sense.

Lemma 10. *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ *be a* π-*institution, with N a category of natural transformations on* SEN, $\Delta : \mathrm{SEN}^2 \to \mathrm{SEN}$ *a finite set of binary natural transformations in N and assume that* SEN *is N-rule based with respect to* $\mathrm{Thm}(\mathcal{I})$. *Then* Δ *is a finite weak N-equivalence system for* $\mathcal{I}$ *iff there exists a* π-*institution* $\mathcal{I}'$, *such that* $\mathcal{I} \le \mathcal{I}'$, *such that*

1. $\mathcal{I} \le \mathcal{I}' \le \mathcal{I}^{\mathrm{ad}}$;

2. Δ is a finite N-equivalence system for $\mathcal{I}'$;

3. Δ is not an N-equivalence system for $\mathcal{I}$.

Proof. ($\Rightarrow$) If Δ is a finite weak N-equivalence system for $\mathcal{I}$, then it is, by definition, an N-equivalence system for $\mathcal{I}^{\mathrm{ad}}$ and $\mathcal{I}^{\mathrm{ad}}$ satisfies all conditions postulated for $\mathcal{I}'$.

($\Leftarrow$) Since $\mathcal{I}^{\mathrm{ad}} \le \mathcal{I}'$, Δ is an N-equivalence system for $\mathcal{I}^{\mathrm{ad}}$. Therefore $\mathcal{I}$ has an admissible N-equivalence system, which is not an N-equivalence system.

5. Concluding Discussion

In this section, we provide a summary of the results of Babenyshev and Martins [1] concerning various possibilities that arise regarding the relationship between equivalence systems and weak equivalence systems of sentential logics. These have obvious consequences in the categorical framework which are explained at the end of the section.

In Example 12 of [1], Babenyshev and Martins show that there exists a non-equivalential deductive system with a finite weak equivalence system. They use the deductive system resulting by considering the least classical modal logic *E* and taking modus ponens as its sole rule of inference versus its admissible counterpart. Malinowski has shown in [15] that the first is not equivalential, whereas the latter is finitely equivalential with equivalence system $\{x \leftrightarrow y\}$.

In Example 13 of [1], Babenyshev and Martins show that there exists a non-finitely equivalential sentential logic with a finite weak equivalence system. They use the deductive system resulting from the least normal modal logic *K*, taking modus ponens as its only rule of inference. Again, Malinowski [15] has shown that this system has the infinite equivalence system $\{\Box^n(x \leftrightarrow y) : n \in \omega\}$, but that it is not finitely equivalential. Moreover, its admissible counterpart is finitely equivalential with finite equivalence $\{x \leftrightarrow y\}$.

Finally, in Example 14 of [1], Babenyshev and Martins show that there exists a deductive system that is not even protoalgebraic, but that it possesses a weak equivalence system. We do not provide the details of this example, for which the reader may consult [1], but we point out that it is based on work of Dziobiak [8] on structurally complete normal modal logics.

These examples provide immediately corresponding examples concerning the relationships that may arise between equivalence systems and weak equivalence systems for logics formalized as π-institutions, since it is very well known that a sentential logic gives rise to a corresponding π-institution (see, e.g., [16, 17]) and the properties of being protoalgebraic, equivalential and finitely equivalential, as well as the property of having a weak equivalence system, as detailed in the present work, all carry over from the original logics to the corresponding associated π-institutions.

Finally, we close by reformulating one of the questions left as open problems in [1] in the categorical language and posing it as an open problem for future investigation (note that, because of the added generality, a solution at this level would also answer the original open problem in [1]):

Open Problem

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on SEN, be a syntactically N-equivalential π-institution. Is any of the two directions of the claim: $\mathcal{I}$ is syntactically N-algebraizable [20] iff it does not possess a weak N-equivalence system true in general? How about when SEN is N-rule based with respect to $\mathrm{Thm}(\mathcal{I})$?

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