Applied Categorical Structures **10:** 531–568, 2002.

Categorical Abstract Algebraic Logic: Algebraizable Institutions ⋆

To Don Pigozzi this work is dedicated

GEORGE VOUTSADAKIS
Department of Mathematics, Iowa State University, Ames, IA 50011 U.S.A. and Department of Mathematics, Case Western Reserve University, 10900 Euclid Avenue, Cleveland, OH 44106 U.S.A.

(Received: 6 April 2000; acepted: 26 April 2002)

Abstract. The framework developed by Blok and Pigozzi for the algebraizability of deductive systems is extended to cover the algebraizability of multisignature logics with quantifiers. Institutions are used as the supporting structure in place of deductive systems. In particular, the concept of an algebraic institution and that of an algebraizable institution are made precise using the theory of monads from categorical algebra and the notion of equivalence of institutions introduced by Voutsadakis. Several examples of algebraic and algebraizable institutions are provided.

Mathematics Subject Classifications (2000): Primary: 03Gxx, 18Cxx; Secondary: 08Bxx, 08Cxx, 68N05.

Key words: algebraic logic, equivalent deductive systems, algebraizable logics, institutions, equivalent institutions, algebraic theories, monads, triples, adjunctions, algebraic institutions, equivalent categories, algebraizable institutions, equational logic, clone algebras, substitution algebras, first-order logic, cylindric algebras, polyadic algebras, diagram-based logics.

1. Introduction

"...when a logic is algebraizable, the powerful methods of modern algebra can be used in its investigation, and this has had a profound influence on the development of these logics." (Wim Blok and Don Pigozzi, 1989).

In 1989 Blok and Pigozzi [4], following in the footsteps of Czelakowski [8] and their own previous work [3], made precise for the first time the notion of *algebraizable logic*. A bulk of work has been published since, influenced by this "Memoirs monograph", that has collectively come to be known under the name of *abstract algebraic logic*. Contrasted with the "traditional" algebraic logic, abstract algebraic logic deals with abstract deductive systems, rather than with specific ones, and its main purpose is to study the framework for the algebraization of

⋆ The research that led to the results presented in this paper began with the author's doctoral dissertation, written under the supervision of Professor Don Pigozzi of Iowa State University. While at Iowa State University the author was partially supported by National Science Foundation grant CCR-9593168.

these systems and unveil general conditions under which wide classes of deductive systems can be shown to be algebraizable. Another goal is to exploit algebraizability to study properties of an algebraizable deductive system or class of such systems by studying corresponding properties of algebraizing classes of algebras and vice-versa.

A *deductive system* $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ consists, roughly speaking, of a structural and finitary consequence relation on the set of formulas $\mathrm{Fm}_{\mathcal{L}}(V)$ over some language type $\mathcal{L}$, built using a fixed denumerable set of variables V. Given a class K of $\mathcal{L}$-algebras, an *algebraic deductive system* $\mathcal{S}_K = \langle \mathcal{L}, \models_K \rangle$ may be constructed, whose consequence relation is now a consequence relation on the set of equations $\mathrm{Eq}_{\mathcal{L}}(V)$ over the type $\mathcal{L}$, defined by

$$E \models_K \phi \approx \psi \quad \text{iff}$$

for all $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle \in K, \vec{a} \in A^{\omega}$,

$$e_1^{\mathbf{A}}(\vec{a}) = e_2^{\mathbf{A}}(\vec{a}), \text{ for all } e_1 \approx e_2 \in E, \text{ implies } \phi^{\mathbf{A}}(\vec{a}) = \psi^{\mathbf{A}}(\vec{a}).$$

In [5], the notion of a *k-deductive system* was introduced to unify these two notions. A k-deductive system consists of a consequence relation on k-tuples of $\mathcal{L}$-formulas. Thus, a deductive system in the original sense is a 1-deductive system and an algebraic deductive system is a 2-deductive system.

A deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ is *interpretable* in an algebraic deductive system $\mathcal{S}_K = \langle \mathcal{L}, \models_K \rangle$ if there exists a finite set $\delta(v) \approx \epsilon(v) = \{\delta_i(v) \approx \epsilon_i(v) : i < n\}$ of n equations in one variable v, such that, for all $\Phi \cup \{\psi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Phi \vdash_{\mathcal{S}} \psi \quad \text{iff} \quad \{\delta(\phi) \approx \epsilon(\phi) : \phi \in \Phi\} \models_K \delta(\psi) \approx \epsilon(\psi).$$

In this case the set $\delta \approx \epsilon$ is the set of *defining equations* for $\mathcal{S}$ and K.

On the other hand, the algebraic deductive system $\mathcal{S}_K$ is *interpretable* in the deductive system $\mathcal{S}$ if there exists a finite set $\Delta(v, u) = \{\Delta_j(v, u) : j < m\}$ of m formulas in two variables v, u, such that, for all $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_{\mathcal{L}}(V)$,

$$E \models_K \phi \approx \psi \quad \text{iff} \quad \{\Delta(e_1, e_2) : e_1 \approx e_2 \in E\} \vdash_{\mathcal{S}} \Delta(\phi, \psi).$$

In this case the set Δ is the set of *equivalence formulas* for $\mathcal{S}$ and K.

It is worth noting, parenthetically, that to unify these two notions of interpretability, the notion of *k–l-interpretability* was introduced in [6] applying to a general k-deductive system being interpretable in a general l-deductive system. Then, the first case considered above is the description of 1–2-interpretability and the second of 2–1-interpretability. The generalized notion was, in turn, the main inspiration for the definition of interpretability for institutions in [22].

The k- and the l-deductive system are then said to be *equivalent* if, in addition to being interpretable in one another, the two interpretations are inverses of each other. More precisely, if we start from a k-formula, apply the k–l-interpretation and then apply the l–k-interpretation to the resulting set of l-formulas, we obtain a set of k-formulas that are interderivable with the original k-formula, with respect to the k-consequence relation, and the same holds if we start with an l-formula and apply the interpretations in the reverse order. Specializing to 1- and 2-deductive systems again, we have that the deductive system $\mathcal{S}$ is *equivalent* to the algebraic deductive system $\mathcal{S}_K$ if, for every $\phi \in \mathrm{Fm}_{\mathcal{L}}(V)$ and $\phi \approx \psi \in \mathrm{Eq}_{\mathcal{L}}(V)$, the systems of defining equations $\delta \approx \epsilon$ and equivalence formulas Δ are inverses of each other, i.e., satisfy the following conditions:

$$\phi \dashv\vdash_{\mathcal{S}} \Delta(\delta(\phi), \epsilon(\phi))$$

and

$$\phi \approx \psi =\mid\models_K \delta(\Delta(\phi, \psi)) \approx \epsilon(\Delta(\phi, \psi)).$$

A deductive system $\mathcal{S}$ is *algebraizable* in the sense of Blok and Pigozzi if there exists a class K of $\mathcal{L}$-algebras, such that $\mathcal{S}$ is equivalent to $\mathcal{S}_K$. K is called the *equivalent algebraic semantics* of $\mathcal{S}$. This notion was later generalized to cover infinitary deductive systems [15]. (See also [16] and [17].) In this case the set of equivalence formulas and the set of defining equations are allowed to be infinite. Other generalizations can be found in [12, 1] and [20].

An easy example is that of Classical Propositional Calculus, viewed as a deductive system $\mathcal{CPC} = \langle \mathcal{L}_{\mathrm{CPC}}, \vdash_{\mathrm{CPC}} \rangle$ over the language type $\mathcal{L}_{\mathrm{CPC}}$ consisting of the nullary connectives $\top$, $\bot$, the unary connective $\neg$ and the binary connectives $\wedge$ and $\vee$. This system is algebraizable with equivalent algebraic semantics the class BA of Boolean algebras (actually the two element Boolean algebra **2** suffices). In fact, if $\phi \to \psi, \phi \leftrightarrow \psi$ are defined by

$$\phi \to \psi := \neg\phi \vee \psi, \qquad \phi \leftrightarrow \psi := (\phi \to \psi) \wedge (\psi \to \phi),$$

then there are translations $\delta \approx \epsilon = \{\delta_0(v) \approx \epsilon_0(v)\}$, with $\delta_0(v) = v, \epsilon_0(v) = \top$ from $\mathcal{CPC}$ to $\mathcal{S}_{\mathrm{BA}}$ and $\Delta = \{\Delta_0(v, u)\}$, with $\Delta_0(v, u) = v \leftrightarrow u$ from $\mathcal{S}_{\mathrm{BA}}$ to $\mathcal{CPC}$. Note that this means that for all $\Phi \cup \psi \subseteq \mathrm{Fm}_{\mathcal{L}_{\mathrm{CPC}}}(V)$ and $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_{\mathcal{L}_{\mathrm{CPC}}}(V)$,

$$\Phi \vdash_{\mathrm{CPC}} \psi \quad \text{iff} \quad \{\phi \approx \top : \phi \in \Phi\} \models_{\mathrm{BA}} \psi \approx \top,$$
$$E \models_{\mathrm{BA}} \phi \approx \psi \quad \text{iff} \quad \{e_1 \leftrightarrow e_2 : e_1 \approx e_2 \in E\} \vdash_{\mathrm{CPC}} \{\phi \leftrightarrow \psi\},$$
$$\phi \dashv\vdash_{\mathrm{CPC}} \{\phi \leftrightarrow \top\} \quad \text{and} \quad \phi \approx \psi =\mid\models_{\mathrm{BA}} \{\phi \leftrightarrow \psi \approx \top\}.$$

This algebraization framework works well with all structural deductive systems. However, to algebraize multiple signature logics with quantifiers such as equational or first-order logic, it is necessary to first transform the original logic to a structural sentential counterpart. It is briefly reviewed here how this transformation was applied to equational logic in [9] and to first-order logic without terms in Appendix C of [4].

The Hyperequational Logic of Czelakowski and Pigozzi. Czelakowski and Pigozzi [9] view equational logic as a 2-deductive system $\mathcal{S}_{\mathrm{Q}}$ as follows: For a given language type $\mathcal{L} = \langle \Lambda, \rho \rangle$ and a given quasivariety Q over $\mathcal{L}$, $\mathcal{S}_{\mathrm{Q}}$ contains the axioms

- $v \approx v$,
- $\phi \approx \psi$, for all identities $\phi \approx \psi$ of Q,

and the following rules of inference

- $\frac{v \approx u}{u \approx v}$,
- $\frac{v \approx u, u \approx w}{v \approx w}$,
- $\frac{v_0 \approx u_0, \ldots, v_{n-1} \approx u_{n-1}}{\lambda(v_0, \ldots, v_{n-1}) \approx \lambda(u_0, \ldots, u_{n-1})}$, for all $\lambda \in \Lambda$, with $\rho(\lambda) = n$,
- $\frac{\phi_0 \approx \psi_0, \ldots, \phi_{n-1} \approx \psi_{n-1}}{\phi \approx \psi}$, for all quasi-identities $(\bigwedge_{i<n} \phi_i \approx \psi_i) \to \phi \approx \psi$ of Q.

The following 2-deductive system is based on the axiomatization of algebras of clones of infinitary operations given by Feldman [10] and later simplified by Cirulis [7]. It is the system HEQ_ω over the language $\mathcal{CL}$, containing an infinite sequence $\mathrm{S}^0, \mathrm{S}^1, \ldots$ of binary operation symbols and an infinite sequence $\mathrm{v}_0, \mathrm{v}_1, \ldots$ of nullary operation symbols, defined by the following axioms for all $n, m, l < \omega$,

- $\mathrm{S}^n_{\mathrm{v}_n}(u) \approx u$,
- $\mathrm{S}^n_u(\mathrm{v}_n) \approx u$,
- $\mathrm{S}^n_u(\mathrm{v}_m) \approx \mathrm{v}_m$, if $n \neq m$,
- $\mathrm{S}^n_u \mathrm{S}^n_w(z) \approx \mathrm{S}^n_{\mathrm{S}^n_u(w)}(z)$,
- $\mathrm{S}^n_{u(m/l)} \mathrm{S}^m_w(z) \approx \mathrm{S}^m_{\mathrm{S}^n_{u(m/l)}(w)} \mathrm{S}^n_{u(m/l)}(z)$, where $u(m/l) = \mathrm{S}^m_{\mathrm{v}_l}(u)$ and m, n, l are all distinct.

All algebras of type $\mathcal{CL}$ that satisfy these equations are called *substitution algebras* and form the variety SA. SA is an equivalent algebraic semantics for equational logic in the following sense: $\mathrm{HEQ}_\omega = \mathcal{S}_{\mathrm{SA}}$ and, conversely, every logic $\mathcal{S}_{\mathrm{V}}$, where V is a variety, can be viewed as a theory of HEQ_ω; more precisely, as the congruence of the formula algebra $\mathbf{Fm}_{\mathcal{CL}}(\{u_\lambda : \lambda \in \Lambda\})$ generated by imposing appropriate rank restrictions on the generators $u_\lambda, \lambda \in \Lambda$, depending on the rank $\rho(\lambda)$ of the operation symbol λ, and by postulating that all identities of V hold (see the appendix of [9] for more details).

The Structural Sentential Counterpart of First-Order Logic. This version of first-order logic is based on its algebraization via the class of cylindric algebras [14]. It is the deductive system PR_ω over the language consisting of

- the binary connectives $\vee, \wedge, \to$,
- the unary connective $\neg$,
- the nullary connectives $\top$ and $\bot$,
- the unary connectives $\mathrm{c}_0, \mathrm{c}_1, \ldots$, corresponding to existential quantifications, also known as cylindrifications, because of their role in the cylindric algebras of sets, and

- the nullary connectives $d_{00}, d_{01}, d_{10}, d_{20}, d_{11}, d_{02}, \ldots$, corresponding to equalities between variables, also known as diagonals, also because of their role in cylindric set algebras.

PR_ω is defined by the axioms given below, under the following conventions: $\exists v_n := c_n, v_m \approx v_n := d_{mn}, \forall v_n := \neg\exists v_n \neg$ and m, n, k natural numbers. Because of this alias use of the v's, x, y, z are now used to denote variables of this deductive system. The axioms are:

- ϕ for every classical tautology ϕ,
- $\forall v_n(x \to y) \to (\forall v_n x \to \forall v_n y)$,
- $\forall v_n x \to x$,
- $\forall v_n x \to \forall v_n \forall v_n x$,
- $\exists v_n x \to \forall v_n \exists v_n x$,
- $\forall v_m \forall v_n x \to \forall v_n \forall v_m x$,
- $v_n \approx v_n$,
- $\exists v_m(v_m \approx v_n)$,
- $v_m \approx v_n \wedge v_m \approx v_k \to v_n \approx v_k$,
- $(v_m \approx v_n \wedge \exists v_m(v_m \approx v_n \wedge x)) \to x$, if $m \neq n$,

and there are two rules of inference

- $\frac{x, x \to y}{y}$,
- $\frac{x}{\forall v_n x}$.

The study of all first-order theories in the standard sense can be reduced to the study of the structural sentential logic PR_ω. Its algebraization can be directly accomplished if one takes the class of cylindric algebras as the equivalent algebraic semantics (see Appendix C in [4] for more details). This shows that, in this case, construction of the algebraizing class precedes and paves the way for the transformation of the logic to its sentential counterpart, whereas intuition would demand the reverse to happen. The logic must naturally give rise to its algebraizing class of algebras.

The need for these artificial, ad-hoc transformations makes the use of this framework for the algebraization of multisignature logics with quantifiers rather unsatisfactory. This had been realized by Pigozzi soon after the appearance of [4]. At the same time, motivated by entirely different considerations, Goguen and Burstall [13] introduced the model-theoretic notion of *institution* to formalize the notion of a multisignature logic defined as a model-theoretic consequence relation. An institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models \rangle$ consists, roughly speaking, of an arbitrary category **Sign** of signatures, a functor SEN from **Sign** into the category **Set** of all small sets, giving, for each signature Σ, the set $\mathrm{SEN}(\Sigma)$ of Σ-sentences, a functor MOD from **Sign** into the opposite category $\mathbf{CAT}^{\mathrm{op}}$ of categories, giving, for each signature Σ, the category $\mathrm{MOD}(\Sigma)$ of Σ-models and, finally, for each signature Σ, a Σ-satisfaction relation $\models_\Sigma$ between Σ-models and Σ-sentences that satisfies the satisfaction condition, which can be summarized in the slogan "truth is invariant under change of notation" [13]. Both equational and first-order logic (without terms) can be formulated to fit the institution framework. In the first case signatures consist of function symbols and in the second of relation symbols. In the examples presented later in the paper, including equational and first-order logic, the signature category **Sign** is a Kleisli category of an algebraic theory **T** in some appropriately chosen category **C**. Informally, **C** will be called the "underlying category" of **Sign**. Fiadeiro and Sernadas [11] changed the formalism slightly, keeping in place multiple signatures but, at the same time, getting rid of the model-theoretic framework. Thus, π*-institutions* were defined, which provided an alternative structure to deductive systems, that can handle substitutions at the language level rather than the metalanguage level. A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ consists of an arbitrary category **Sign** of signatures, a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ (as before) and, for each signature Σ, a closure operator C_Σ on the set $\mathrm{SEN}(\Sigma)$ of Σ-sentences. The system $\{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ satisfies a generalized structurality condition. Structural k-deductive systems provide special examples of π-institutions and, also, every institution gives rise in a natural way to a π-institution by defining the Σ-consequence relation model-theoretically in the standard way. Following the previously adopted convention, if **Sign** is a category with structure over some category **C**, **C** will be said to be the "underlying category" of **Sign**. π-institutions will be used as the supporting structure in place of deductive systems in the generalization of Blok and Pigozzi's algebraizability framework to accomondate, besides classical deductive systems, logics with multiple signatures and quantifiers.

The formal definitions of a deductive system, of an institution and of a π-institution are presented in the next section. There, it is also shown how one can perceive a k-deductive system as a π-institution, therefore showing the reason why deductive systems are very simple special cases of the π-institution structure. Some more elaborate examples of institutions are also provided. First, the one that corresponds to equational logic over multiple signatures, followed by the one that corresponds to first-order logic without terms. In the first case, signatures consist of operation symbols whereas in the second they consist of relation symbols. As mentioned before, both of these logics can be algebraized under the deductive system framework, after their transformation to structural sentential counterparts. The last example in Section 2 provides an institutional logic which is diagram- rather than string-based. Its sentences will be arrows of a category and the consequence relation will correspond to the equational consequence relation induced by the class of all small categories.

In Section 3, some material pertaining to algebraic theories, also known as monads or triples, from categorical algebra, and their connection to adjunctions is reviewed. Since the notion of a π-institution is categorical in flavor, it is expected that the role of the equivalent algebraic semantics K in the present context will be taken by a subcategory of the category of algebras of some appropriately chosen algebraic theory. This theory is extracted by an adjunction between the signature category of the π-institution under consideration and its "underlying category". To all the examples of π-institutions that are given in Section 2 there is naturally associated an adjunction from the "underlying category" of their signature category to the signature category. This adjunction gives rise in a standard way (to be reviewed in Section 3) to an algebraic theory in this underlying category. It is easily shown in all cases that the signature category is the Kleisli category of the adjunction and that the original adjunction is the Kleisli adjunction of this theory.

Subsequently, a subcategory $\mathbf{Q}$ of the Eilenberg–Moore category of the theory will be chosen and, in Section 4, it will be shown how an institution is obtained based on this subcategory and a sentence functor EQ : $\mathbf{L} \to \mathbf{Set}$ on some full subcategory $\mathbf{L}$ of the Kleisli category of the algebraic theory, whose construction is based on a prespecified functor $\Xi : \mathbf{C} \to \mathbf{Set}$ on the category $\mathbf{C}$. This institution will be said to be an $\langle \mathbf{L}, \Xi, \mathbf{Q}\rangle$-*algebraic institution*. More precisely, let $\mathbf{C}$ be a category, $\mathbf{T}$ an algebraic theory in $\mathbf{C}$, $\mathbf{L}$ a full subcategory of the Kleisli category $\mathbf{C_T}$ of $\mathbf{T}$ in $\mathbf{C}$ and $\mathbf{Q}$ a subcategory of the Eilenberg–Moore category $\mathbf{C^T}$ of $\mathbf{T}$ in $\mathbf{C}$. Given a functor $\Xi : \mathbf{C} \to \mathbf{Set}$, giving, for each $C \in |\mathbf{C}|$, the set of C-formulas, an $\langle \mathbf{L}, \Xi, \mathbf{Q}\rangle$-algebraic institution $\mathcal{I}_{\langle \mathbf{L},\Xi,\mathbf{Q}\rangle} = \langle \mathbf{L}, \mathrm{EQ}, \mathrm{ALG}, \models\rangle$ has, for each $L \in |\mathbf{L}|$, as L-sentences pairs of C-formulas, as L-models $\mathbf{T}$-algebras in $\mathbf{Q}$ together with mappings from L to their underlying objects, "pinning down" the fundamental operations inside the clone of operations, and satisfaction of an equation by a model is determined as with ordinary algebras after C-formulas are interpreted according to the interpretation of their fundamental operations. Examples of institutions of this kind, based on the algebraic theories obtained by the adjunctions of the equational, first-order and equational categorical logic institutions, will also be provided in Section 4.

Finally, in Section 5, the notion of equivalent institutions introduced in [21] (see also [22]), which generalizes the notion of equivalence of deductive systems, will be used to define the notion of an *algebraizable institution*. Roughly speaking, an arbitrary institution $\mathcal{I}$ will be said to be algebraizable if there exists a category $\mathbf{C}$, and an algebraic theory $\mathbf{T}$ in $\mathbf{C}$, such that $\mathcal{I}$ is equivalent to an $\langle \mathbf{L}, \Xi, \mathbf{Q}\rangle$-algebraic institution for some full subcategory $\mathbf{L}$ of the Kleisli category $\mathbf{C_T}$, a functor Ξ : $\mathbf{L} \to \mathbf{Set}$ and a subcategory $\mathbf{Q}$ of the Eilenberg–Moore category $\mathbf{C^T}$ of $\mathbf{T}$-algebras in $\mathbf{C}$.

2. Institutional Logics

A *language type* $\mathcal{L} = \langle \Lambda, \rho\rangle$ is a pair whose first component is a set of operation symbols and whose second component $\rho : \Lambda \to \omega$ is a function giving the arity of each operation symbol in Λ. Given a language type $\mathcal{L}$ and a set of variables V, by $\mathrm{Fm}_{\mathcal{L}}(V)$ is denoted the set of all $\mathcal{L}$-*formulas* with variables in V. This set is the universe of the absolutely free $\mathcal{L}$-algebra, which will be denoted by $\mathbf{Fm}_{\mathcal{L}}(V)$. A mapping $\sigma : V \to \mathrm{Fm}_{\mathcal{L}}(V)$ is called an *assignment* (of formulas to variables), also denoted by $\sigma : V \to V$, and can be uniquely extended, by the freeness of $\mathbf{Fm}_{\mathcal{L}}(V)$, to a homomorphism $\sigma^* : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$, called a *substitution.*

An *$\mathcal{L}$-logic* $\mathcal{S}$ is a pair $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$, where $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)) \times \mathrm{Fm}_{\mathcal{L}}(V)$ is such that, for all $\Gamma, \Delta \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, $\phi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

(i) $\Gamma \vdash_{\mathcal{S}} \phi$, for all $\phi \in \Gamma$,
(ii) $\Gamma \vdash_{\mathcal{S}} \phi$ and $\Gamma \subseteq \Delta$ imply $\Delta \vdash_{\mathcal{S}} \phi$,
(iii) $\Gamma \vdash_{\mathcal{S}} \phi$ and $\Delta \vdash_{\mathcal{S}} \gamma$, for all $\gamma \in \Gamma$, imply $\Delta \vdash_{\mathcal{S}} \phi$.

The logic $\mathcal{S}$ is called *structural* if, in addition,

(iv) $\Gamma \vdash_{\mathcal{S}} \phi$ implies $\sigma(\Gamma) \vdash_{\mathcal{S}} \sigma(\phi)$, for all substitutions σ,

and it is called *finitary* if, in addition,

(v) $\Gamma \vdash_{\mathcal{S}} \phi$ implies $\Gamma' \vdash_{\mathcal{S}} \phi$, for some $\Gamma' \subseteq_{\mathrm{f}} \Gamma$.

The definition of an $\mathcal{S}$-logic may be restated in terms of closure operators. Given a set X, a *closure operator* C on X is a mapping $C : \mathcal{P}(X) \to \mathcal{P}(X)$ such that, for all $Y, Z \subseteq X$,

(i) $Y \subseteq C(Y)$,
(ii) $Y \subseteq Z$ implies $C(Y) \subseteq C(Z)$ and
(iii) $C(C(Y)) \subseteq C(Y)$.

A closure operator on $\mathrm{Fm}_{\mathcal{L}}(V)$ is called *structural* if, in addition, for all $\Gamma \cup \{\phi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

(iv) $\phi \in C(\Gamma)$ implies $\sigma(\phi) \in C(\sigma(\Gamma))$, for all substitutions σ,

and it is called *finitary* if, in addition,

(v) $\phi \in C(\Gamma)$ implies $\phi \in C(\Gamma')$, for some $\Gamma' \subseteq_{\mathrm{f}} \Gamma$.

Given an $\mathcal{L}$-logic $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$, denote by $C_{\mathcal{S}}$ the closure operator on $\mathrm{Fm}_{\mathcal{L}}(V)$ defined by

$$C_{\mathcal{S}}(\Gamma) = \{\phi \in \mathrm{Fm}_{\mathcal{L}}(V) : \Gamma \vdash_{\mathcal{S}} \phi\}$$

and, given a closure operator C on $\mathrm{Fm}_{\mathcal{L}}(V)$, define a logic $\mathcal{S}_C = \langle \mathcal{L}, \vdash_C \rangle$ by

$$\Gamma \vdash_C \phi \quad \text{iff} \quad \phi \in C(\Gamma).$$

Then the mappings $\mathcal{S} \mapsto C_{\mathcal{S}}$ and $C \mapsto \mathcal{S}_C$ define a bijective correspondence between logics and closure operators on $\mathrm{Fm}_{\mathcal{L}}(V)$, which restricts to a bijective correspondence between structural finitary logics and structural finitary closure operators. A structural finitary logic is usually referred to as a *sentential logic* or a *deductive system*.

In [4] the notion of an *algebraizable sentential logic* was introduced and a theory of algebraizability developed. The formalism of sentential logics cannot handle directly the case of multi-signature logics with quantifiers, like equational logic over multiple signatures and first-order logic. The general theory of algebraizable sentential logics may be applied to these logics but only after their transformation to sentential counterparts in a rather artificial and ad-hoc way. This transformation was carried out for equational logic in [9] and for first-order logic in Appendix C of [4] and was reviewed briefly in the Introduction. The main purpose of the present paper is to give a general formalism for handling directly the algebraization of multi-signature logics with quantifiers without the need to first perform this artificial transformation which essentially alters the spirit of the original logic.

To lay the foundations for this framework, the structure of a sentential logic must be replaced by a more general structure that can accomondate miltiple signatures. A structure that has proven to be very efficient in this respect is the model-theoretic structure of an institution, introduced in [13] and later modified to that of a π-institution in [11] to directly generalize structural logics and at the same time accomondate multiple signatures.

An *institution* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models\rangle$ is a quadruple consisting of

(i) a category **Sign** whose objects are called *signatures* and whose morphisms are called *assignments*,
(ii) a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ from the category of signatures to the category of small sets, giving, for each $\Sigma \in |\mathbf{Sign}|$, the set of Σ-*sentences* $\mathrm{SEN}(\Sigma)$ and mapping an assignment $f : \Sigma_1 \to \Sigma_2$ to a *substitution* $\mathrm{SEN}(f) : \mathrm{SEN}(\Sigma_1) \to \mathrm{SEN}(\Sigma_2)$,
(iii) a functor $\mathrm{MOD} : \mathbf{Sign} \to \mathbf{CAT}^{\mathrm{op}}$ from the category of signatures to the opposite of the category of categories giving, for each signature Σ, the category of Σ-*models* $\mathrm{MOD}(\Sigma)$,
(iv) for each signature Σ, a *satisfaction relation* $\models_\Sigma \subseteq |\mathrm{MOD}(\Sigma)| \times \mathrm{SEN}(\Sigma)$, such that, for all $f : \Sigma_1 \to \Sigma_2 \in \mathrm{Mor}(\mathbf{Sign}), \phi \in \mathrm{SEN}(\Sigma_1)$ and $m \in |\mathrm{MOD}(\Sigma_2)|$, the following *satisfaction condition* holds

$$\mathrm{MOD}(f)(m) \models_{\Sigma_1} \phi \quad \text{iff} \quad m \models_{\Sigma_2} \mathrm{SEN}(f)(\phi).$$

Pictorially, this condition may be illustrated as follows:

$$\begin{array}{ccccc} & \mathrm{MOD}(f)(m) & \models_{\Sigma_1} & \phi & \\ \mathrm{MOD}(f) & \uparrow & & \downarrow & \mathrm{SEN}(f) \\ & m & \models_{\Sigma_2} & \mathrm{SEN}(f)(\phi) & \end{array}$$

A π-*institution* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}\rangle$, on the other hand, is a triple with its first two components exactly the same as the first two components of an institution and, for every $\Sigma \in |\mathbf{Sign}|$, a closure operator $C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \to \mathcal{P}(\mathrm{SEN}(\Sigma))$, such that, for every $f : \Sigma_1 \to \Sigma_2 \in \mathrm{Mor}(\mathbf{Sign})$,

$$\mathrm{SEN}(f)(C_{\Sigma_1}(\Gamma)) \subseteq C_{\Sigma_2}(\mathrm{SEN}(f)(\Gamma)), \quad \text{for all } \Gamma \subseteq \mathrm{SEN}(\Sigma_1).$$

Given an institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models\rangle$, define, for all $\Sigma \in |\mathbf{Sign}|$, $\Gamma \subseteq \mathrm{SEN}(\Sigma)$, $M \subseteq |\mathrm{MOD}(\Sigma)|$,

$$\Gamma^* = \{m \in |\mathrm{MOD}(\Sigma)| : m \models_\Sigma \Gamma\} \quad \text{and} \quad M^* = \{\phi \in \mathrm{SEN}(\Sigma) : M \models_\Sigma \phi\}$$

and set $C_\Sigma(\Gamma) = \Gamma^{**}$, for all $\Sigma \in |\mathbf{Sign}|$, $\Gamma \subseteq \mathrm{SEN}(\Sigma)$. Then $\pi(\mathcal{I}) = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}\rangle$ is a π-institution, called the *π-institution associated with* the institution $\mathcal{I}$ and denoted by $\pi(\mathcal{I})$.

A few examples of institutions and π-institutions that will also serve to illustrate the theory in the following sections come next.

2.1. DEDUCTIVE INSTITUTIONS

Let $\mathcal{L}$ be a language type. An *$\mathcal{L}$-equation* is a pair $\langle \phi, \psi\rangle$ of $\mathcal{L}$-formulas, usually written $\phi \approx \psi$. With the motivation to incorporate equational logics into the sentential logic formalism, the notion of a k-deductive system was introduced in [5]. Given a positive integer k, a *k-deductive system* $\mathcal{S}$ over $\mathcal{L}$ is a pair $\mathcal{S} = \langle \mathcal{L}, \vdash_\mathcal{S}\rangle$, where, now, $\vdash_\mathcal{S} \subseteq \mathcal{P}(\mathrm{Fm}^k_\mathcal{L}(V)) \times \mathrm{Fm}^k_\mathcal{L}(V)$, satisfying all five conditions that a sentential logic must satisfy with k-tuples of formulas and sets of k-tuples of formulas taking the place of formulas and sets of formulas, respectively.

Let $\mathcal{S} = \langle \mathcal{L}, \vdash_\mathcal{S}\rangle$ be a k-deductive system over $\mathcal{L}$. We construct the π-institution $\mathcal{I}_\mathcal{S} = \langle \mathbf{Sign}_\mathcal{S}, \mathrm{SEN}_\mathcal{S}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_\mathcal{S}|}\rangle$ as follows:

(i) $\mathbf{Sign}_\mathcal{S}$ is the one-object category with object V and morphisms all assignments $f : V \rightharpoonup V$, i.e., set maps $f : V \to \mathrm{Fm}_\mathcal{L}(V)$. The identity morphism is the inclusion $i_V : V \to \mathrm{Fm}_\mathcal{L}(V)$. Composition $g \circ f$ of two assignments f and g is defined by $g \circ f = g^* f$, where, recall that $g^* : \mathbf{Fm}_\mathcal{L}(V) \to \mathbf{Fm}_\mathcal{L}(V)$ denotes the substitution extending the assignment g.

(ii) $\mathrm{SEN}_\mathcal{S} : \mathbf{Sign}_\mathcal{S} \to \mathbf{Set}$ maps V to $\mathrm{Fm}^k_\mathcal{L}(V)$ and $f : V \rightharpoonup V$ to $(f^*)^k : \mathrm{Fm}^k_\mathcal{L}(V) \to \mathrm{Fm}^k_\mathcal{L}(V)$. It is easy to see that $\mathrm{SEN}_\mathcal{S}$ is a functor.

(iii) Finally, $C_V : \mathcal{P}(\mathrm{Fm}^k_\mathcal{L}(V)) \to \mathcal{P}(\mathrm{Fm}^k_\mathcal{L}(V))$ is the standard closure operator $C_\mathcal{S} : \mathcal{P}(\mathrm{Fm}^k_\mathcal{L}(V)) \to \mathcal{P}(\mathrm{Fm}^k_\mathcal{L}(V))$ associated with the k-deductive system $\mathcal{S}$, i.e.,

$$C_V(\Phi) = \{\phi \in \mathrm{Fm}^k_\mathcal{L}(V) : \Phi \vdash_\mathcal{S} \phi\}, \quad \text{for all } \Phi \subseteq \mathrm{Fm}^k_\mathcal{L}(V).$$

C_V, defined in this way satisfies the conditions imposed in the definition of a π-institution. In fact, the last condition in the definition of a π-institution is, in this case, the structurality property of a deductive system that plays a central role in the classical theory of algebraizability. Abstracting structurality and incorporating substitutions in the object language, rather than handling them in the metalanguage, in the context of multi-signature logical systems is one of the basic motivations for the introduction of categorical abstract algebraic logic. $\mathcal{I}_\mathcal{S}$ is thus a π-institution. It will be called the *deductive π-institution associated with* the k-deductive system $\mathcal{S}$. Note that for $k = 1$ we obtain the deductive π-institutions associated with deductive systems in the sense of [4] and for $k = 2$ we obtain, among others, the π-institutions associated with the semantically defined equational 2-deductive systems $\mathcal{S}_K$ whose consequence relations $C_K : \mathcal{P}(\mathrm{Fm}^2_\mathcal{L}(V)) \to \mathcal{P}(\mathrm{Fm}^2_\mathcal{L}(V))$ are the equational consequence relations determined by some class K of $\mathcal{L}$-algebras.

2.2. EQUATIONAL INSTITUTION

An *ω-indexed set* or, simply, *ω-set* A is a family of sets $A = \{A_k : k \geq 1\}$, where it is assumed that $A_m \cap A_n = \emptyset$, for all $m, n \geq 1, m \neq n$. An *ω-indexed set morphism* or, simply, *ω-set morphism* $f : A \to B$, from an ω-set A to an ω-set B, is a collection of set maps $f = \{f_k : A_k \to B_k : k \geq 1\}$. Given two ω-set morphisms $f : A \to B, g : B \to C$, define their *composite* $gf : A \to C$ by $gf = \{g_k f_k : A_k \to C_k : k \geq 1\}$. With this composition, the collection of ω-sets with ω-set morphisms between them forms a category, called the *category of ω-sets* and denoted by Ω**Set**.

An ω-set $V = \{V_k : k \geq 1\}$, with $V_k = \{v_{ki} : i < k\}$, called *ω-set of variables*, is fixed in advance. Given an ω-set X, the *ω-set of X-terms* $\mathrm{Tm}_X(V) = \{\mathrm{Tm}_X(V)_k : k \geq 1\}$ is defined by letting $\mathrm{Tm}_X(V)_k$ be the smallest set with

- $v_{ki} \in \mathrm{Tm}_X(V)_k, i < k$,
- $x(t_0, \ldots, t_{n-1}) \in \mathrm{Tm}_X(V)_k$, for all $n \geq 1, x \in X_n, t_0, \ldots, t_{n-1} \in \mathrm{Tm}_X(V)_k$.

It is important for the reader to notice that X is not allowed to contain nullary operation symbols, since this would spoil the disjointness of the different levels of $\mathrm{Tm}_X(V)$, which is required for $\mathrm{Tm}_X(V)$ to be an ω-set. This, as is well known, does not harm the generality since nullary operations may always be replaced by constant unary operations.

Given $X, Y \in |\Omega\mathbf{Set}|, f : X \to \mathrm{Tm}_Y(V) \in \mathrm{Mor}(\Omega\mathbf{Set})$, let $f^* : \mathrm{Tm}_X(V) \to \mathrm{Tm}_Y(V)$ be the Ω**Set**-morphism such that $f^*_k(x(v_{k0}, \ldots, v_{k,k-1})) = f_k(x)$, for all $k \geq 1, x \in X_k$, and $f^*_k(t)$ is the Y-term obtained from t by recursively replacing each subterm $x(t_0, \ldots, t_{n-1})$ of t by $f_n(x)(f^*_k(t_0), \ldots, f^*_k(t_{n-1}))$. We write $f : X \rightharpoondown Y$ to denote an Ω**Set**-map $f : X \to \mathrm{Tm}_Y(V)$. Given two such morphisms $f : X \rightharpoondown Y$ and $g : Y \rightharpoondown Z$ their *composition* $g \circ f : X \rightharpoondown Z$ is defined to be the Ω**Set**-map $g \circ f = g^* f$. With this composition, the collection of ω-sets with the harpoon morphisms between them forms a category, denoted by **EQSig**. Identities in **EQSig** are the morphisms $j^{\mathrm{EQ}}_X : X \rightharpoondown X$, with $j^{\mathrm{EQ}}_{X_k}(x) = x(v_{k0}, \ldots, v_{k,k-1})$, for all $k \geq 1, x \in X_k$. The category **EQSig** will be the signature category of the institution for equational logic.

Next, define the sentence functor EQSEN : **EQSig** $\to$ **Set** by

$$\mathrm{EQSEN}(X) = \left(\bigcup_{k=1}^{\infty} \mathrm{Tm}_X(V)_k\right)^2, \quad \text{for every } X \in |\mathbf{EQSig}|,$$

and, given $f : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{EQSig})$, $\mathrm{EQSEN}(f) : \mathrm{EQSEN}(X) \to \mathrm{EQSEN}(Y)$ is given by

$$\mathrm{EQSEN}(f)(\langle s, t\rangle) = \langle f^*_k(s), f^*_l(t)\rangle, \quad \text{if } s \in \mathrm{Tm}_X(V)_k, t \in \mathrm{Tm}_X(V)_l,$$

for all $\langle s, t\rangle \in \mathrm{EQSEN}(X)$. EQSEN is well-defined, because $\mathrm{Tm}_X(V)_k \cap \mathrm{Tm}_X(V)_l = \emptyset$, for all $k, l \geq 1, k \neq l$. We call an $\langle s, t\rangle \in \mathrm{EQSEN}(X)$ an *X-equation* and denote it by $s \approx t$.

The model functor EQMOD : $\mathbf{EQSig} \to \mathbf{CAT}^{\mathrm{op}}$ of the equational institution is described next. Given a set A, by $\mathrm{Cl}(A)$ is denoted the ω-set whose k-th level $\mathrm{Cl}_k(A)$ consists of all functions $f : A^k \to A$. Given an ω-set X, an *X-algebra* $\mathbf{A} = \langle A, X^{\mathbf{A}} \rangle$ is a pair consisting of a set A together with an $\Omega\mathbf{Set}$-morphism $X^{\mathbf{A}} : X \to \mathrm{Cl}(A)$. If $x \in X_k$, following common usage, we write $x^{\mathbf{A}}$ for $X_k^{\mathbf{A}}(x) \in \mathrm{Cl}_k(A)$. Given two X-algebras $\mathbf{A}$ and $\mathbf{B}$, an *X-algebra homomorphism* $h : \mathbf{A} \to \mathbf{B}$ is a set map $h : A \to B$, such that, for all $n \geq 1, x \in X_n, \vec{a} \in A^n$,

$$h(x^{\mathbf{A}}(\vec{a})) = x^{\mathbf{B}}(h(\vec{a})).$$

X-algebras with X-algebra homomorphisms between them form a category, denoted by $\mathrm{EQMOD}(X)$. Given an X-algebra $\mathbf{A} = \langle A, X^{\mathbf{A}} \rangle$, define an $\Omega\mathbf{Set}$-morphism ${}^{\mathbf{A}} : \mathrm{Tm}_X(V) \to \mathrm{Cl}(A)$ by letting $v_{ki}^{\mathbf{A}} : A^k \to A$ be the i-th projection function in k variables and

$$x(t_0, \ldots, t_{n-1})^{\mathbf{A}} = x^{\mathbf{A}}(t_0^{\mathbf{A}}, \ldots, t_{n-1}^{\mathbf{A}}), \quad \text{for all}$$
$$n \geq 1, x \in X_n, t_0, \ldots, t_{n-1} \in \mathrm{Tm}_X(V)_k.$$

Then, it is not difficult to define EQMOD at the morphism level. To this end, let $f : X \to Y \in \mathrm{Mor}(\mathbf{EQSig})$. Then $\mathrm{EQMOD}(f)(\langle A, Y^{\mathbf{A}} \rangle) = \langle A, X^{\mathrm{EQMOD}(f)(\mathbf{A})} \rangle$, for all $\langle A, Y^{\mathbf{A}} \rangle \in |\mathrm{EQMOD}(Y)|$, where $x^{\mathrm{EQMOD}(f)(\mathbf{A})} = f_k(x)^{\mathbf{A}}$, for all $k \geq 1, x \in X_k$, and, if $h : \langle A, Y^{\mathbf{A}} \rangle \to \langle B, Y^{\mathbf{B}} \rangle \in \mathrm{Mor}(\mathrm{EQMOD}(Y))$, then

$$\mathrm{EQMOD}(f)(h) = h : \langle A, X^{\mathrm{EQMOD}(f)(\mathbf{A})} \rangle \to \langle B, X^{\mathrm{EQMOD}(f)(\mathbf{B})} \rangle.$$

h may be shown to be an X-algebra homomorphism and, hence, $\mathrm{EQMOD}(f)$ is well-defined at the morphism level.

Finally, for the satisfaction relation, we have, for every $X \in |\mathbf{EQSig}|$,

$$\mathbf{A} \models_X s \approx t \quad \text{iff} \quad s^{\mathbf{A}} = t^{\mathbf{A}},$$

for all $\mathbf{A} \in |\mathrm{EQMOD}(X)|, s \approx t \in \mathrm{EQSEN}(X)$. It is not very hard to verify that the satisfaction condition holds and that $\mathcal{EQ} = \langle \mathbf{EQSig}, \mathrm{EQSEN}, \mathrm{EQMOD}, \models \rangle$ is an institution.

Details of the constructions presented in this section will be given in the forthcoming [23].

2.3. INSTITUTION FOR FIRST-ORDER LOGIC WITHOUT TERMS

A *hierarchy of sets* or, simply, an *h-set* A is a family of sets $A = \{A_N : N \in \mathcal{P}_{\mathrm{f}}(\omega)\}$, where $\mathcal{P}_{\mathrm{f}}(\omega) = \{N : N \subseteq_{\mathrm{f}} \omega\}$, such that $A_{M \cap N} = A_M \cap A_N$, for all $M, N \subseteq_{\mathrm{f}} \omega$.

By a *morphism of h-sets* or an *h-set morphism* $f : A \to B$ from an h-set A to an h-set B we mean a family of set maps $f = \{f_N : A_N \to B_N : N \in \mathcal{P}_f(\omega)\}$, such that the following diagram commutes, for all $N \subseteq M \subseteq_f \omega$,

$$\begin{array}{ccc} A_N & \xrightarrow{f_N} & B_N \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ A_M & \xrightarrow[f_M]{} & B_M \end{array}$$

where by $i : A_N \to A_M$ and $i : B_N \to B_M$ are denoted the inclusion maps. Given two h-set morphisms $f : A \to B$ and $g : B \to C$, their *composite* $gf : A \to C$ is defined by $gf = \{g_N f_N : A_N \to B_N : N \subseteq_f \omega\}$. With this composition the collection of h-sets with h-set morphisms between them forms a category, called the *category of h-sets* and denoted by **HSet**.

By $\mathcal{L}$ in this section will be denoted the set of symbols $\{\neg, \wedge\} \cup \{\exists_k : k \in \omega\}$, used as connectives and quantifiers, respectively, in the construction of formulas. Given a set X, by $\overline{X}$ will be denoted an isomorphic copy of X constructed in some canonical way. $\overline{x}$ denotes then the copy of $x \in X$ in the set $\overline{X}$. Given an h-set X, the *h-set of X-formulas* $\mathrm{Fm}_{\mathcal{L}}(X) = \{\mathrm{Fm}_{\mathcal{L}}(X)_N : N \in \mathcal{P}_f(\omega)\} \in |\mathbf{HSet}|$ is defined by letting $\mathrm{Fm}_{\mathcal{L}}(X)_N$ be the smallest set with

- $\overline{x} \in \mathrm{Fm}_{\mathcal{L}}(X)_N$, for every $x \in X_N$,
- $\neg\phi, \phi_1 \wedge \phi_2 \in \mathrm{Fm}_{\mathcal{L}}(X)_N$, for all $\phi, \phi_1, \phi_2 \in \mathrm{Fm}_{\mathcal{L}}(X)_N$,
- $\exists_k\phi \in \mathrm{Fm}_{\mathcal{L}}(X)_N$, for every $\phi \in \mathrm{Fm}_{\mathcal{L}}(X)_{N\cup\{k\}}$.

Given $X, Y \in |\mathbf{HSet}|$, $f : X \to \mathrm{Fm}_{\mathcal{L}}(Y) \in \mathrm{Mor}(\mathbf{HSet})$, let $f^* : \mathrm{Fm}_{\mathcal{L}}(X) \to \mathrm{Fm}_{\mathcal{L}}(Y)$ be the **HSet**-morphism such that $f^*_N(\overline{x}) = f_N(x)$, for all $N \subseteq_f \omega, x \in X_N$, and $f^*_N(\phi)$ is the Y-formula obtained from ϕ by recursively replacing each subformula of ϕ by its image under f^*_N, except for subformulas following a quantifier $\exists_k$ which are replaced by their image under $f^*_{N\cup\{k\}}$. Write $f : X \rightharpoondown Y$ to denote an **HSet**-map $f : X \to \mathrm{Fm}_{\mathcal{L}}(Y)$. Given two such morphisms $f : X \rightharpoondown Y$ and $g : Y \rightharpoondown Z$ their *composition* $g \circ f : X \rightharpoondown Z$ is defined to be the **HSet**-map $g \circ f = g^* f$. With this composition the collection of h-sets with the harpoon morphisms between them forms a category, denoted by **FOSig**. Identities in this category are the morphisms $j^{\mathrm{FO}}_X : X \rightharpoondown X$, with $j^{\mathrm{FO}}_{X_N}(x) = \overline{x}$, for all $N \subseteq_f \omega, x \in X_N$. The category **FOSig** will serve as the signature category of the institution for first-order logic without terms.

Next, define the sentence functor FOSEN : **FOSig** $\to$ **Set** by

$$\mathrm{FOSEN}(X) = \mathrm{Fm}_{\mathcal{L}}(X)_\emptyset, \quad \text{for every } X \in |\mathbf{FOSig}|,$$

and, given $f : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{FOSig})$, $\mathrm{FOSEN}(f) : \mathrm{FOSEN}(X) \to \mathrm{FOSEN}(Y)$ is given by

$$\mathrm{FOSEN}(f)(\phi) = f^*_\emptyset(\phi), \quad \text{for all } \phi \in \mathrm{Fm}_{\mathcal{L}}(X)_\emptyset.$$

A $\phi \in \mathrm{Fm}_{\mathcal{L}}(X)_{\emptyset}$ is called an X-*sentence*.

The model functor FOMOD : $\mathbf{FOSig} \to \mathbf{CAT}^{\mathrm{op}}$ of the institution is described next. Given a set A, by $\mathrm{Rel}(A)$ is denoted the h-set whose N-th level $\mathrm{Rel}_N(A)$ consists of all relations $r \subseteq A^{\omega}$ that depend only on the individual variables indexed by the elements of N. Given $X \in |\mathbf{HSet}|$, an X-*structure* $\mathbf{A} = \langle A, X^{\mathbf{A}}\rangle$ is a pair consisting of a set A together with an **HSet**-morphism $X^{\mathbf{A}} : X \to \mathrm{Rel}(A)$. If $x \in X_N$, following common usage, we write $x^{\mathbf{A}}$ for $X_N^{\mathbf{A}}(x) \in \mathrm{Rel}_N(A)$. Given two X-structures $\mathbf{A}$ and $\mathbf{B}$, an X-*structure homomorphism* $h : \mathbf{A} \to \mathbf{B}$ is a surjective set map $h : A \to B$, such that, for all $N \subseteq_{\mathrm{f}} \omega, x \in X_N$,

$$\vec{a} \in x^{\mathbf{A}} \quad \text{iff} \quad h(\vec{a}) \in x^{\mathbf{B}}.$$

X-structures and X-structure homomorphisms between them form a category, denoted by $\mathrm{FOMOD}(X)$. Given an X-structure $\mathbf{A} = \langle A, X^{\mathbf{A}}\rangle$, define an **HSet**-morphism ${}^{\mathbf{A}} : \mathrm{Fm}_{\mathcal{L}}(X) \to \mathrm{Rel}(A)$ by letting

- $\overline{x}^{\mathbf{A}} = x^{\mathbf{A}}$, for all $x \in X_N$,
- $(\neg\phi)^{\mathbf{A}} = A^{\omega} - \phi^{\mathbf{A}}$, $(\phi_1 \wedge \phi_2)^{\mathbf{A}} = \phi_1^{\mathbf{A}} \cap \phi_2^{\mathbf{A}}$, for all $\phi, \phi_1, \phi_2 \in \mathrm{Fm}_{\mathcal{L}}(X)_N$,
- $(\exists_k \phi)^{\mathbf{A}} = \{\vec{b} : b_i = a_i \ \forall i \neq k \text{ and } \vec{a} \in \phi^{\mathbf{A}}\}$.

Now, it is not difficult to define FOMOD at the morphism level. Let $f : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{FOSig})$. Then

$$\mathrm{FOMOD}(f)(\langle A, Y^{\mathbf{A}}\rangle) = \langle A, X^{\mathrm{FOMOD}(f)(\mathbf{A})}\rangle,$$
$$\text{for all } \langle A, Y^{\mathbf{A}}\rangle \in |\mathrm{FOMOD}(Y)|,$$

where $x^{\mathrm{FOMOD}(f)(\mathbf{A})} = f_N(x)^{\mathbf{A}}$, for all $N \subseteq_{\mathrm{f}} \omega, x \in X_N$, and, if $h : \langle A, Y^{\mathbf{A}}\rangle \to \langle B, Y^{\mathbf{B}}\rangle \in \mathrm{Mor}(\mathrm{FOMOD}(Y))$, then $\mathrm{FOMOD}(f)(h) = h : \langle A, X^{\mathrm{FOMOD}(f)(\mathbf{A})}\rangle \to \langle B, X^{\mathrm{FOMOD}(f)(\mathbf{B})}\rangle$. h may be shown to be an X-structure homomorphism and therefore $\mathrm{FOMOD}(f)$ is well-defined at the morphism level.

Finally, for the satisfaction relation, we have, for every $X \in |\mathbf{FOSig}|$,

$$\mathbf{A} \models_X \phi \quad \text{iff} \quad \phi^{\mathbf{A}} = A^{\omega},$$

for all $\mathbf{A} \in |\mathrm{FOMOD}(X)|, \phi \in \mathrm{FOSEN}(X)$. It is not difficult to verify that the satisfaction condition holds and that $\mathcal{FO} = \langle \mathbf{FOSig}, \mathrm{FOSEN}, \mathrm{FOMOD}, \models\rangle$ is an institution.

Details of the constructions presented here will be given in the forthcoming [24].

2.4. INSTITUTION FOR THE EQUATIONAL LOGIC OF CATEGORIES

The previous two institutions, of equational and of first-order logic, provide examples of logics that, from the algebraic logic point of view, can also be treated in the sentential logic framework but only after they are first transformed to sentential counterparts. Later in the paper, their algebraization will be carried out directly without the need for this artificial transformation.

The next example, although "trivial" in some sense, has something unique in nature. The logic presented here is not a string-based logic, i.e., based on variables, connectives and well-formed formulas inductively formed by these, but rather a diagram-based logic. The role of signature objects is played by graphs and the role of formulas is played by the arrows of the category freely generated by a given graph. The example is "trivial" because it takes into account only the equational aspect of the logic of categories and disregards other very important aspects, like limits and colimits. The point to be made, however, is that this simple logic is algebraizable in the sense of categorical abstract algebraic logic although there is no direct way of expressing its algebraizability using the sentential logic framework of universal abstract algebraic logic.

A (*directed*) *graph* $G = \langle V, E, s, t\rangle$ consists of a set V of *nodes* or *vertices*, a set E of *edges*, and two functions $s, t : E \to V$, associating with each edge e its *source vertex* $s(e)$ and its *target vertex* $t(e)$, respectively. One writes $e : s(e) \to t(e)$ in this case. Sometimes the notation $V(G), E(G)$ is used to denote the sets of vertices, edges, respectively, of a graph G. Let $G = \langle V, E, s, t\rangle$ and $G' = \langle V', E', s', t'\rangle$ be graphs. A *graph morphism* $h : G \to G'$ is a pair $h = \langle h_1, h_2\rangle$, with $h_1 : V \to V'$ and $h_2 : E \to E'$ satisfying $s'(h_2(e)) = h_1(s(e))$ and $t'(h_2(e)) = h_1(t(e))$, for all $e \in E$. Usually, the subscript 1 or 2 is suppressed, since it is clear from context whether h is applied to a vertex or to an edge, respectively. Given two graph morphisms $f = \langle f_1, f_2\rangle : G_1 \to G_2$ and $g = \langle g_1, g_2\rangle : G_2 \to G_3$, define their *composite* componentwise, i.e., $gf = \langle g_1 f_1, g_2 f_2\rangle : G_1 \to G_3$. With this composition, graphs with graph morphisms between them form a category, denoted by **Gph**. A *path* p in a graph G is a sequence $p = (e_0, e_1, \ldots, e_{n-1})$ of edges in E, such that $t(e_i) = s(e_{i+1})$, for all $i = 0, \ldots, n-2$. Such a path is said to be *from* $s(e_0)$ *to* $t(e_{n-1})$. Two paths $p = (e_0, \ldots, e_{n-1})$ and $q = (f_0, \ldots, f_{m-1})$ are said to be *parallel* if they are from the same vertex to the same vertex, i.e., if $s(e_0) = s(f_0)$ and $t(e_{n-1}) = t(f_{m-1})$.

Given a graph G, one may construct the *free category* or *path category* $\mathbf{Pth}(G)$ on G. Its objects are the objects of G and, for all $v, u \in V$, the collection of its arrows from v to u is

$$\mathbf{Pth}(G)(v, u) = \{p : p \text{ is a path from } v \text{ to } u \text{ in } G\}.$$

Given two arrows $p = (e_0, \ldots, e_{n-1}) : u \to v$ and $q = (f_0, \ldots, f_{m-1}) : v \to w$ in $\mathbf{Pth}(G)$, the composite arrow $qp = (e_0, \ldots, e_{n-1}, f_0, \ldots, f_{m-1}) : u \to w$. Identities are the empty paths. By $\mathrm{Pth}(G)$ will be denoted the underlying graph of the category $\mathbf{Pth}(G)$.

Given $G_1, G_2 \in |\mathbf{Gph}|$, $f : G_1 \to \mathrm{Pth}(G_2) \in \mathrm{Mor}(\mathbf{Gph})$, let $f^* : \mathrm{Pth}(G_1) \to \mathrm{Pth}(G_2)$ be the **Gph**-morphism, that sends identities to identities, acts exacly like f on single-edge paths and extends f "by juxtaposition" on paths of more than unitary length. We denote a **Gph**-morphism $f : G_1 \to \mathrm{Pth}(G_2)$ by $f : G_1 \rightharpoondown G_2$. Given two such morphisms $f : G_1 \rightharpoondown G_2$ and $g : G_2 \rightharpoondown G_3$, their *composition* $g \circ f : G_1 \rightharpoondown G_3$ is defined to be the **Gph**-map $g \circ f = g^* f$. With this composition, the collection of graphs with the harpoon morphisms between them forms a category, denoted by **CSig**. Identities in **CSig** are the morphisms $j_G^{\mathrm{CT}} : G \rightharpoonup G$, with $j_{G_1}^{\mathrm{CT}}(v) = v$ and $j_{G_2}^{\mathrm{CT}}(e) = (e)$, for all $v \in V, e \in E$, where $G = \langle V, E, s, t\rangle \in |\mathbf{Gph}|$. The category **CSig** will be the signature category of the institution for the equational logic of categories.

Next, define the sentence functor CSEN : **CSig** $\rightarrow$ **Set** by letting CSEN(G) to be the set of all pairs of edges in Pth(G) for every $G \in |\mathbf{Gph}|$, i.e., the set of all pairs of paths in G, and, given $h : G_1 \rightharpoonup G_2 \in \mathrm{Mor}(\mathbf{CSig})$, CSEN$(f)$: CSEN$(G_1) \rightarrow$ CSEN(G_2) to be given by

$$\mathrm{CSEN}(h)(\langle p, q\rangle) = \langle h^*(p), h^*(q)\rangle,$$

for all edges p, q in Pth(G).

The model functor CMOD : **CSig** $\rightarrow$ $\mathbf{CAT}^{\mathrm{op}}$ of the institution is described next. Given a graph G, CMOD(G) is the category with objects all pairs $\langle \mathbf{C}, f\rangle$, where $\mathbf{C}$ is a small category and $f : G \rightarrow C \in \mathrm{Mor}(\mathbf{Gph})$, where C denotes the underlying graph of $\mathbf{C}$. Its morphisms $h : \langle \mathbf{C}, f\rangle \rightarrow \langle \mathbf{D}, g\rangle$ are functors $h : \mathbf{C} \rightarrow \mathbf{D}$, such that $hf = g$ as graph morphisms from G to D.

$$\begin{tikzcd} & G \arrow[dl, "f"'] \arrow[dr, "g"] & \\ C \arrow[rr, "h"'] & & D \end{tikzcd}$$

Furthermore, given a morphism $k : G_1 \rightharpoonup G_2 \in \mathrm{Mor}(\mathbf{CSig})$, CMOD$(k)$: CMOD$(G_2) \rightarrow$ CMOD(G_1) maps an object $\langle \mathbf{C}, f\rangle \in |\mathrm{CMOD}(G_2)|$ to the object $\langle \mathbf{C}, f^\dagger k\rangle$,

$$G_1 \xrightarrow[k]{} \mathrm{Pth}(G_2) \xrightarrow[f^\dagger]{} C$$

where $f^\dagger : \mathrm{Pth}(G_2) \rightarrow C$ is the natural extension of $f : G_2 \rightarrow C$ to paths "by composition", i.e., $f^\dagger(e_0, e_1, \ldots, e_{n-1}) = f(e_{n-1})f(e_{n-2})\ldots f(e_0)$, for all edges $(e_0, \ldots, e_{n-1})$ in Pth(G_2), and an arrow $h : \langle \mathbf{C}, f\rangle \rightarrow \langle \mathbf{D}, g\rangle$ to the arrow CMOD$(k)(h) : \langle \mathbf{C}, f^\dagger k\rangle \rightarrow \langle \mathbf{D}, g^\dagger k\rangle$, with CMOD$(k)(h) = h$. Note that, for all $e \in E_1$, if $k_2(e) = (e_0, \ldots, e_{n-1}) \in \mathrm{Pth}(G_2)$, then

$$\begin{aligned} hf^\dagger k(e) &= hf^\dagger(e_0, \ldots, e_{n-1}) \\ &= h(f(e_{n-1})\ldots f(e_0)) \\ &= h(f(e_{n-1}))\ldots h(f(e_0)) \\ &= g(e_{n-1})\ldots g(e_0) \\ &= g^\dagger(e_0, \ldots, e_{n-1}) \\ &= g^\dagger k(e), \end{aligned}$$

and, therefore, h is a well-defined morphism in CMOD(G_1).

Finally, for the satisfaction relation, we have, for all $G \in |\mathbf{CSig}|$,

$$\begin{aligned}&\langle \mathbf{C}, g\rangle \models_G (e_0, \ldots, e_{n-1}) \approx (f_0, \ldots, f_{m-1}) \quad \text{iff} \\ &g^{\dagger}(e_0, \ldots, e_{n-1}) = g^{\dagger}(f_0, \ldots, f_{m-1}),\end{aligned}$$

i.e., if and only if $g(e_{n-1}) \ldots g(e_0) = g(f_{m-1}) \ldots g(f_0)$, for all $\langle \mathbf{C}, g\rangle \in |\mathrm{CMOD}(G)|$, $\langle (e_0, \ldots, e_{n-1}), (f_0, \ldots, f_{m-1})\rangle \in \mathrm{CSEN}(G)$. It is relatively easy, in this case as well, to verify that $\mathcal{CL} = \langle \mathbf{CSig}, \mathrm{CSEN}, \mathrm{CMOD}, \models\rangle$ is an institution.

3. Adjunctions and Algebraic Theories

Let $\mathbf{C}$ be a category. An *algebraic theory* or *monad* or *triple* $\mathbf{T} = \langle T, \eta, \mu\rangle$ *in* $\mathbf{C}$ consists of a functor $T : \mathbf{C} \to \mathbf{C}$ and natural transformations $\eta : I_{\mathbf{C}} \to T$ and $\mu : TT \to T$, such that, for every $C \in |\mathbf{C}|$, the following diagrams commute:

$$\begin{tikzcd} T(C) \arrow[r, "\eta_{T(C)}"] \arrow[dr, "i_{T(C)}"'] & T(T(C)) \arrow[d, "\mu_C"] & T(C) \arrow[l, "T(\eta_C)"'] \arrow[dl, "i_{T(C)}"] \\ & T(C) & \end{tikzcd} \qquad \begin{tikzcd} T(T(T(C))) \arrow[r, "\mu_{T(C)}"] \arrow[d, "T(\mu_C)"'] & T(T(C)) \arrow[d, "\mu_C"] \\ T(T(C)) \arrow[r, "\mu_C"'] & T(C) \end{tikzcd}$$

The prototypical example of an algebraic theory is the *algebraic theory* $\mathbf{T}_{\mathcal{L}} = \langle T_{\mathcal{L}}, \eta_{\mathcal{L}}, \mu_{\mathcal{L}}\rangle$ in $\mathbf{Set}$ *associated with the variety of all* $\mathcal{L}$*-algebras* for some language type $\mathcal{L}$. $T_{\mathcal{L}} : \mathbf{Set} \to \mathbf{Set}$ sends a set X to the set $\mathrm{Fm}_{\mathcal{L}}(X)$, $\eta_{\mathcal{L}X} : X \to \mathrm{Fm}_{\mathcal{L}}(X)$ is the insertion-of-variables map and $\mu_{\mathcal{L}X} : \mathrm{Fm}_{\mathcal{L}}(\mathrm{Fm}_{\mathcal{L}}(X)) \to \mathrm{Fm}_{\mathcal{L}}(X)$ is the map $i^*_{\mathrm{Fm}_{\mathcal{L}}(X)}$ extending the identity map on $\mathrm{Fm}_{\mathcal{L}}(X)$ (see [18] and [19] for more details).

Given an algebraic theory $\mathbf{T} = \langle T, \eta, \mu\rangle$ in a category $\mathbf{C}$, a $\mathbf{T}$-*algebra* is a pair $\langle C, \xi\rangle$, where $C \in |\mathbf{C}|$ and $\xi : T(C) \to C \in \mathrm{Mor}(\mathbf{C})$, such that the following diagrams commute

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[dr, "i_C"'] & T(C) \arrow[d, "\xi"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} T(T(C)) \arrow[r, "\mu_C"] \arrow[d, "T(\xi)"'] & T(C) \arrow[d, "\xi"] \\ T(C) \arrow[r, "\xi"'] & C \end{tikzcd}$$

C is said to be the *underlying object* of $\langle C, \xi\rangle$ and ξ is its *structure map*.

Given two $\mathbf{T}$-algebras $\langle C, \xi\rangle$, $\langle D, \zeta\rangle$, a $\mathbf{T}$-*algebra homomorphism* $h : \langle C, \xi\rangle \to \langle D, \zeta\rangle$ *from* $\langle C, \xi\rangle$ *to* $\langle D, \zeta\rangle$ is a $\mathbf{C}$-morphism $h : C \to D$, such that the following diagram commutes

$$\begin{tikzcd} T(C) \arrow[r, "T(h)"] \arrow[d, "\xi"'] & T(D) \arrow[d, "\zeta"] \\ C \arrow[r, "h"'] & D \end{tikzcd}$$

$\mathbf{T}_{\mathcal{L}}$-algebras are in bijective correspondence with $\mathcal{L}$-algebras and $\mathbf{T}_{\mathcal{L}}$-algebra homomorphisms are the usual $\mathcal{L}$-algebra homomorphisms.

In general, $\mathbf{T}$-algebras with $\mathbf{T}$-algebra homomorphisms between them form a category $\mathbf{C}^{\mathbf{T}}$, called the *Eilenberg–Moore category* of the algebraic theory $\mathbf{T}$ in $\mathbf{C}$. For every object $C \in |\mathbf{C}|$, $\langle T(C), \mu_C\rangle$ is easily seen to be a $\mathbf{T}$-algebra. The full subcategory of $\mathbf{C}^{\mathbf{T}}$ with objects all $\mathbf{T}$-algebras of this form, i.e., $\langle T(C), \mu_C\rangle$, $C \in |\mathbf{C}|$, is called the *Kleisli category* of $\mathbf{T}$ in $\mathbf{C}$ and denoted by $\mathbf{C}_{\mathbf{T}}$. The Kleisli category has a better known description, which gives a category isomorphic to the one described above: its objects are the objects of $\mathbf{C}$, its morphisms $f : C \rightharpoonup D$ are $\mathbf{C}$-morphisms $f : C \to T(D)$ and composition of $f : C \rightharpoonup D$ and $g : D \rightharpoonup E$ is given by $g \circ f = \mu_E T(g) f$

$$\begin{tikzcd} C \arrow[r, "f"'] & T(D) \arrow[r, "T(g)"'] & T(T(E)) \arrow[r, "\mu_E"'] & T(E) \end{tikzcd}$$

Identities are the maps $\eta_C : C \rightharpoonup C$, $C \in |\mathbf{C}|$.

Eilenberg–Moore categories of algebraic theories in the category of sets are the same as varieties of single-sorted universal algebras, although with possibly infinitary operations (see, e.g., [19]). However, algebraic theories are *much* more general than universal algebra. They can be defined in arbitrary categories. They capture, for instance, multi-sorted universal algebras. Another example is provided by the category of topological groups, which is the Eilenberg–Moore category of an algebraic theory in the category of topological spaces.

Algebraic theories in a category $\mathbf{C}$ are very closely connected to adjunctions from $\mathbf{C}$. A brief overview of some of the connecting features is given here. Again the reader is advised to consult [18, 19] or [2] for more detailed accounts.

Let $\mathbf{C}$ and $\mathbf{D}$ be two categories. An *adjunction from* $\mathbf{C}$ to $\mathbf{D}$ is a quadruple $\langle F, U, \eta, \epsilon\rangle : \mathbf{C} \to \mathbf{D}$, where $F : \mathbf{C} \to \mathbf{D}$ and $U : \mathbf{D} \to \mathbf{C}$ are functors, $\eta : I_{\mathbf{C}} \to UF$ and $\epsilon : FU \to I_{\mathbf{D}}$ are natural transformations and the following triangles commute, for all $C \in |\mathbf{C}|$, $D \in |\mathbf{D}|$,

$$\begin{tikzcd} F(C) \arrow[r, "F(\eta_C)"] \arrow[dr, "i_{F(C)}"'] & F(U(F(C))) \arrow[d, "\epsilon_{F(C)}"] \\ & F(C) \end{tikzcd} \qquad \begin{tikzcd} U(D) \arrow[r, "\eta_{U(D)}"] \arrow[dr, "i_{U(D)}"'] & U(F(U(D))) \arrow[d, "U(\epsilon_D)"] \\ & U(D) \end{tikzcd}$$

F is said to be *left adjoint to* U, U *right adjoint to* F, η is called the *unit* of the adjunction and ϵ the *counit* of the adjunction.

Another equivalent formulation is that there exists an isomorphism $\phi_{C,D}$: $\mathbf{D}(F(C), D) \cong \mathbf{C}(C, U(D))$, for all $C \in |\mathbf{C}|, D \in |\mathbf{D}|$, that is natural in both C and D, i.e., for all $c \in \mathbf{C}(C_1, C_2), d \in \mathbf{D}(D_1, D_2)$, the following rectangle commutes

$$\begin{tikzcd} \mathbf{D}(F(C_2), D_1) \arrow[r, "\phi_{C_2,D_1}"] \arrow[d, "\mathbf{D}(F(c),d)"'] & \mathbf{C}(C_2, U(D_1)) \arrow[d, "\mathbf{C}(c,U(d))"] \\ \mathbf{D}(F(C_1), D_2) \arrow[r, "\phi_{C_1,D_2}"'] & \mathbf{C}(C_1, U(D_2)) \end{tikzcd}$$

where, for all $f \in \mathbf{D}(F(C_2), D_1)$, $\mathbf{D}(F(c), d)(f) = dfF(c)$ and, for all $g \in \mathbf{C}(C_2, U(D_1))$, $\mathbf{C}(c, U(d))(g) = U(d)gc$. Passage to $\phi_{C,D}$, given the adjunction, is accomplished by setting, for all $f : F(C) \to D \in \text{Mor}(\mathbf{D})$,

$$\phi_{C,D}(f) = U(f)\eta_C : C \to U(D).$$

This ϕ is natural and its converse is given, for all $g : C \to U(D) \in \text{Mor}(\mathbf{C})$, by

$$\psi_{C,D}(g) = \epsilon_D F(g) : F(C) \to D.$$

Conversely, if $\phi_{C,D} : \mathbf{D}(F(C), D) \to \mathbf{C}(C, U(D))$ is given, then one obtains $\eta_C : C \to U(F(C))$ by taking $\eta_C = \phi_{C,F(C)}(i_{F(C)})$, for all $C \in |\mathbf{C}|$, and ϵ_D : $F(U(D)) \to D$ by taking $\epsilon_D = \phi^{-1}_{U(D),D}(i_{U(D)})$, for all $D \in |\mathbf{D}|$.

Yet another equivalent formulation states that the functor $U : \mathbf{D} \to \mathbf{C}$ has a left adjoint if and only if, for all $C \in |\mathbf{C}|$, there exists a *free* $\mathbf{D}$*-structure* $F(C)$ *on* C *along* U. This means that there exists a map $\eta_C : C \to U(F(C))$ in $\mathbf{C}$ with the *universal mapping property*, i.e., for all other map $f : C \to U(D)$ in $\mathbf{C}$, there is a unique $\mathbf{D}$-map $f' : F(C) \to D$, such that the following triangle commutes

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[dr, "f"'] & U(F(C)) \arrow[d, "U(f')"] \\ & U(D) \end{tikzcd}$$

Given an algebraic theory $\mathbf{T} = \langle T, \eta, \mu \rangle$ in $\mathbf{C}$, there are two very important adjunctions that are associated with it. One is the *Eilenberg–Moore adjunction* $\langle F^{\mathbf{T}}, U^{\mathbf{T}}, \eta^{\mathbf{T}}, \epsilon^{\mathbf{T}} \rangle : \mathbf{C} \to \mathbf{C}^{\mathbf{T}}$ from $\mathbf{C}$ to the Eilenberg–Moore category of $\mathbf{T}$ in $\mathbf{C}$. $F^{\mathbf{T}}$ sends $C \in |\mathbf{C}|$ to the $\mathbf{T}$-algebra $\langle T(C), \mu_C \rangle$, $U^{\mathbf{T}}$ sends a $\mathbf{T}$-algebra to its underlying object, $\eta^{\mathbf{T}} = \eta$ and $\epsilon^{\mathbf{T}}_{\langle C,\xi \rangle} : \langle T(C), \mu_C \rangle \to \langle C, \xi \rangle$ is given by $\epsilon^{\mathbf{T}}_{\langle C,\xi \rangle} = \xi$.

In the case of the algebraic theory $\mathbf{T}_{\mathcal{L}} = \langle T_{\mathcal{L}}, \eta_{\mathcal{L}}, \mu_{\mathcal{L}} \rangle$ associated with the variety of all $\mathcal{L}$-algebras this adjunction is from $\mathbf{Set}$ to the category $\mathbf{Set}^{\mathbf{T}_{\mathcal{L}}}$, the functor $F^{\mathcal{L}}$ sending a set X to the algebra $\mathbf{Fm}_{\mathcal{L}}(X)$, the functor $U^{\mathcal{L}}$ sending an algebra $\mathbf{A}$ to its universe, $\eta^{\mathcal{L}}_X : X \to \mathrm{Fm}_{\mathcal{L}}(X)$ being the insertion-of-variables map and $\epsilon^{\mathcal{L}}_{\mathbf{A}} : \mathbf{Fm}_{\mathcal{L}}(A) \to \mathbf{A}$ being the map $i^{\dagger}_A$ that extends the identity on A to an algebra homomorphism, using the freeness of $\mathbf{Fm}_{\mathcal{L}}(A)$ over A.

The second adjunction associated with an algebraic theory $\mathbf{T} = \langle T, \eta, \mu\rangle$ in a category $\mathbf{C}$ is the *Kleisli adjunction* $\langle F_{\mathbf{T}}, U_{\mathbf{T}}, \eta_{\mathbf{T}}, \epsilon_{\mathbf{T}}\rangle : \mathbf{C} \to \mathbf{C}_{\mathbf{T}}$ from $\mathbf{C}$ to the Kleisli category of $\mathbf{T}$ in $\mathbf{C}$. $F_{\mathbf{T}}$ sends $C \in |\mathbf{C}|$ to itself and a map $f : C_1 \to C_2$ to the map $\eta_{C_2} f : C_1 \rightharpoonup C_2$. $U_{\mathbf{T}}$ sends C to $T(C)$ and a map $f : C_1 \rightharpoonup C_2$ to the map $\mu_{C_2} T(f) : T(C_1) \to T(C_2)$. $\eta_{\mathbf{T}} = \eta$ and, finally, $\epsilon_{\mathbf{T}_C} = i_{T(C)} : T(C) \rightharpoonup C$.

In the special case of $\mathbf{T}_{\mathcal{L}}$, i.e., of the adjunction $\langle F_{\mathcal{L}}, U_{\mathcal{L}}, \eta_{\mathcal{L}}, \epsilon_{\mathcal{L}}\rangle : \mathbf{Set} \to \mathbf{Set}_{\mathbf{T}_{\mathcal{L}}}$, $F_{\mathcal{L}}$ maps a set X to itself and a morphism $f : X \to Y$ to $F_{\mathcal{L}}(f) : X \to \mathrm{Fm}_{\mathcal{L}}(Y)$, which is f composed with the insertion-of-variables, $U_{\mathcal{L}}$ maps X to $\mathrm{Fm}_{\mathcal{L}}(X)$ and an assignment $f : X \to \mathrm{Fm}_{\mathcal{L}}(Y)$ to the substitution $f^* : \mathrm{Fm}_{\mathcal{L}}(X) \to \mathrm{Fm}_{\mathcal{L}}(Y)$ extending f, $\eta_{\mathcal{L}_X} : X \to \mathrm{Fm}_{\mathcal{L}}(X)$ is the insertion-of-variables and $\epsilon_{\mathcal{L}_X} : \mathrm{Fm}_{\mathcal{L}}(X) \to \mathrm{Fm}_{\mathcal{L}}(X)$ is the identity on $\mathrm{Fm}_{\mathcal{L}}(X)$.

Conversely, every adjunction $\langle F, U, \eta, \epsilon\rangle : \mathbf{C} \to \mathbf{D}$ gives rise to an algebraic theory $\mathbf{T} = \langle T, \eta, \mu\rangle$ in $\mathbf{C}$ by setting $T = UF$ and $\mu = U\epsilon_F$ (see, e.g., [18, p. 134]). Since adjunctions arise very naturally in many contexts, this way of constructing an algebraic theory is very popular. It will be exploited in the sequel to construct the three algebraic theories on which the algebraization of equational, first-order and categorical equational logics will be based.

3.1. THE ADJUNCTION OF EQUATIONAL LOGIC

The adjunction $\langle F_{\mathrm{EQ}}, U_{\mathrm{EQ}}, \eta_{\mathrm{EQ}}, \epsilon_{\mathrm{EQ}}\rangle : \Omega\mathbf{Set} \to \mathbf{EQSig}$ is defined as follows: $F_{\mathrm{EQ}} : \Omega\mathbf{Set} \to \mathbf{EQSig}$ sends an ω-set X to itself and an ω-set morphism $f : X \to Y$ to $F_{\mathrm{EQ}}(f) : X \rightharpoondown Y$, with $F_{\mathrm{EQ}}(f) = j^{\mathrm{EQ}}_Y f$. $U_{\mathrm{EQ}} : \mathbf{EQSig} \to \Omega\mathbf{Set}$ sends $X \in |\mathbf{EQSig}|$ to $\mathrm{Tm}_X(V)$ and a morphism $f : X \rightharpoondown Y$ to $f^* : \mathrm{Tm}_X(V) \to \mathrm{Tm}_Y(V)$. $\eta_{\mathrm{EQ}_X} = j^{\mathrm{EQ}}_X$, for all $X \in |\Omega\mathbf{Set}|$, and $\epsilon_{\mathrm{EQ}_X} = i_{\mathrm{Tm}_X(V)}$, for all $X \in |\mathbf{EQSig}|$.

This adjunction yields the algebraic theory $\mathbf{T}_{\mathrm{EQ}} = \langle T_{\mathrm{EQ}}, \eta_{\mathrm{EQ}}, \mu_{\mathrm{EQ}}\rangle$ in $\Omega\mathbf{Set}$, with $T_{\mathrm{EQ}} = U_{\mathrm{EQ}} F_{\mathrm{EQ}}$ and $\mu_{\mathrm{EQ}} = U_{\mathrm{EQ}} \epsilon_{\mathrm{EQ}_{F_{\mathrm{EQ}}}}$. Thus, T_{EQ} sends an ω-set X to $\mathrm{Tm}_X(V)$ and an ω-set morphism $f : X \to Y$ to the morphism $(j^{\mathrm{EQ}}_Y f)^* : \mathrm{Tm}_X(V) \to \mathrm{Tm}_Y(V)$ extending $j^{\mathrm{EQ}}_Y f : X \to \mathrm{Tm}_Y(V)$, $\eta_{\mathrm{EQ}_X} : X \to \mathrm{Tm}_X(V)$ is the insertion-of-variables and $\mu_{\mathrm{EQ}_X} : \mathrm{Tm}_{\mathrm{Tm}_X(V)}(V) \to \mathrm{Tm}_X(V)$ is the map extending the identity map on $\mathrm{Tm}_X(V)$. In this special case a $\mathbf{T}_{\mathrm{EQ}}$-algebra is a pair $\langle X, \xi\rangle$, where X is an ω-set and $\xi : \mathrm{Tm}_X(V) \to X$ is an ω-set morphism, such that the following diagrams commute

$$\begin{tikzcd} X \arrow[r, "\eta_{\mathrm{EQ}_X}"] \arrow[dr, "i_X"'] & \mathrm{Tm}_X(V) \arrow[d, "\xi"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Tm}_{\mathrm{Tm}_X(V)}(V) \arrow[r, "\mu_{\mathrm{EQ}_X}"] \arrow[d, "T_{\mathrm{EQ}}(\xi)"'] & \mathrm{Tm}_X(V) \arrow[d, "\xi"] \\ \mathrm{Tm}_X(V) \arrow[r, "\xi"'] & X \end{tikzcd}$$

The Kleisli category $\Omega\mathbf{Set}_{\mathbf{T}_{\mathrm{EQ}}}$ is exactly the category **EQSig** (its arrows were denoted by $\rightharpoondown$ anticipating this) and the adjunction $\langle F_{\mathrm{EQ}}, U_{\mathrm{EQ}}, \eta_{\mathrm{EQ}}, \epsilon_{\mathrm{EQ}}\rangle$ constructed above is the Kleisli adjunction corresponding to this algebraic theory. There is a special subclass of Eilenberg–Moore algebras of this algebraic theory that will be singled out because it will play a major role later in the algebraization of the equational institution. These are constructed as follows: Let A be a set. Denote by $\mathrm{Cl}(A)$ the ω-set with $\mathrm{Cl}_k(A)$ the set of all functions $f : A^k \to A$, for all $k \geq 1$. Now define $\xi_A : \mathrm{Tm}_{\mathrm{Cl}(A)}(V) \to \mathrm{Cl}(A)$ by induction on the structure of a $\mathrm{Cl}(A)$-term as follows:

- $\xi_{A_k}(v_{ki}) = p_{ki}$, for all $i < k$, where $p_{ki} : A^k \to A$ denotes the i-th projection function,
- $\xi_{A_k}(f(t_0, \ldots, t_{n-1})) = f(\xi_{A_k}(t_0), \ldots, \xi_{A_k}(t_{n-1}))$, for all $f \in \mathrm{Cl}_n(A)$, $t_0, \ldots, t_{n-1} \in \mathrm{Tm}_{\mathrm{Cl}(A)}(V)_k$.

$\langle \mathrm{Cl}(A), \xi_A\rangle$ is a $\mathbf{T}_{\mathrm{EQ}}$-algebra, for every set A, and the full subcategory of $\Omega\mathbf{Set}^{\mathbf{T}_{\mathrm{EQ}}}$ with objects all algberas of this form will be denoted by **FCln**, a shorthand for *full clone algebras*.

3.2. THE ADJUNCTION OF FIRST-ORDER LOGIC

In this section $\mathcal{L}$ will again denote the language type consisting of the connectives $\{\neg, \wedge\} \cup \{\exists_k : k \in \omega\}$ and $\mathrm{Fm}_{\mathcal{L}}(V)$ will denote the h-set of $\mathcal{L}$-formulas with variables in V. The adjunction $\langle F_{\mathrm{FO}}, U_{\mathrm{FO}}, \eta_{\mathrm{FO}}, \epsilon_{\mathrm{FO}}\rangle : \mathbf{HSet} \to \mathbf{FOSig}$ is defined as follows: $F_{\mathrm{FO}} : \mathbf{HSet} \to \mathbf{FOSig}$ sends an h-set X to itself and an h-set morphism $f : X \to Y$ to $F_{\mathrm{FO}}(f) : X \rightharpoondown Y$, with $F_{\mathrm{FO}}(f) = j_Y^{\mathrm{FO}} f$. $U_{\mathrm{FO}} : \mathbf{FOSig} \to \mathbf{HSet}$ sends $X \in |\mathbf{FOSig}|$ to $\mathrm{Fm}_{\mathcal{L}}(X)$ and a morphism $f : X \rightharpoondown Y$ to $f^* : \mathrm{Fm}_{\mathcal{L}}(X) \to \mathrm{Fm}_{\mathcal{L}}(Y)$. $\eta_{\mathrm{FO}_X} = j_X^{\mathrm{FO}}$, for all $X \in |\mathbf{HSet}|$, and $\epsilon_{\mathrm{FO}_X} = i_{\mathrm{Fm}_{\mathcal{L}}(X)}$, for all $X \in |\mathbf{FOSig}|$.

This adjunction yields the algebraic theory $\mathbf{T}_{\mathrm{FO}} = \langle T_{\mathrm{FO}}, \eta_{\mathrm{FO}}, \mu_{\mathrm{FO}}\rangle$ in **HSet**, with $T_{\mathrm{FO}} = U_{\mathrm{FO}} F_{\mathrm{FO}}$ and $\mu_{\mathrm{FO}} = U_{\mathrm{FO}}\epsilon_{\mathrm{FO}_{F_{\mathrm{FO}}}}$. Thus, T_{FO} sends an h-set X to $\mathrm{Fm}_{\mathcal{L}}(X)$ and an h-set morphism $f : X \to Y$ to the morphism $(j_Y^{\mathrm{FO}} f)^* : \mathrm{Fm}_{\mathcal{L}}(X) \to \mathrm{Fm}_{\mathcal{L}}(Y)$ extending $j_Y^{\mathrm{FO}} f : X \to \mathrm{Fm}_{\mathcal{L}}(Y)$, $\eta_{\mathrm{FO}_X} : X \to \mathrm{Fm}_{\mathcal{L}}(X)$ is the insertion-of-variables and $\mu_{\mathrm{FO}_X} : \mathrm{Fm}_{\mathcal{L}}(\mathrm{Fm}_{\mathcal{L}}(X)) \to \mathrm{Fm}_{\mathcal{L}}(X)$ is the map extending the identity map on $\mathrm{Fm}_{\mathcal{L}}(X)$. In this special case a $\mathbf{T}_{\mathrm{FO}}$-algebra is a pair $\langle X, \xi\rangle$, where X is an h-set and $\xi : \mathrm{Fm}_{\mathcal{L}}(X) \to X$ is an h-set morphism, such that the following diagrams commute

$$
\begin{tikzcd}
X \arrow[r, "\eta_{\mathrm{FO}_X}"] \arrow[dr, "i_X"'] & \mathrm{Fm}_{\mathcal{L}}(X) \arrow[d, "\xi"] \\
 & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Fm}_{\mathcal{L}}(\mathrm{Fm}_{\mathcal{L}}(X)) \arrow[r, "\mu_{\mathrm{FO}_X}"] \arrow[d, "T_{\mathrm{FO}}(\xi)"'] & \mathrm{Fm}_{\mathcal{L}}(X) \arrow[d, "\xi"] \\
\mathrm{Fm}_{\mathcal{L}}(X) \arrow[r, "\xi"'] & X
\end{tikzcd}
$$

The Kleisli category $\mathbf{HSet}_{\mathbf{T}_{\mathrm{FO}}}$ is exactly the category **FOSig** (its arrows were also denoted by $\rightharpoonup$ anticipating this) and the adjunction $\langle F_{\mathrm{FO}}, U_{\mathrm{FO}}, \eta_{\mathrm{FO}}, \epsilon_{\mathrm{FO}}\rangle$ constructed above is the Kleisli adjunction corresponding to this algebraic theory. There is, in this case as well, a special subclass of Eilenberg–Moore algebras of this algebraic theory that will be singled out because of its key role in the algebraization of the institution of first-order logic. These are constructed as follows: Let A be a set. Denote by $\mathrm{Rel}(A)$ the h-set with $\mathrm{Rel}_N(A)$ the set of all relations $R \subseteq A^\omega$ that depend only on the variables indexed by the elements of N, for all $N \subseteq_{\mathrm{f}} \omega$. Now define $\xi_A : \mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A)) \to \mathrm{Rel}(A)$ by induction on the structure of a $\mathrm{Rel}(A)$-formula as follows:

- $\xi_{A_N}(\overline{x}) = x$, for all $x \in \mathrm{Rel}_N(A)$,
- $\xi_{A_N}(\neg\phi) = A^\omega - \xi_{A_N}(\phi)$, for all $\phi \in \mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A))_N$,
- $\xi_{A_N}(\phi_1 \wedge \phi_2) = \xi_{A_N}(\phi_1) \cap \xi_{A_N}(\phi_2)$, for all $\phi_1, \phi_2 \in \mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A))_N$,
- $\xi_{A_N}(\exists_k\phi) = \{\vec{b} \in A^\omega : a_i = b_i \forall i \neq k \text{ and } \vec{a} \in \xi_{A_{N\cup\{k\}}}(\phi)\}$.

$\langle \mathrm{Rel}(A), \xi_A\rangle$ is a $\mathbf{T}_{\mathrm{FO}}$-algebra, for every set A, and the full subcategory of $\mathbf{HSet}^{\mathbf{T}_{\mathrm{FO}}}$ with objects all algebras of this form will be denoted by **FRln**, a shorthand for *full relation algebras*.

3.3. THE ADJUNCTION OF EQUATIONAL CATEGORICAL LOGIC

The adjunction $\langle F_{\mathrm{CT}}, U_{\mathrm{CT}}, \eta_{\mathrm{CT}}, \epsilon_{\mathrm{CT}}\rangle : \mathbf{Gph} \to \mathbf{CSig}$ is defined as follows: $F_{\mathrm{CT}} : \mathbf{Gph} \to \mathbf{CSig}$ sends a graph G to itself and a graph morphism $f : G_1 \to G_2$ to $F_{\mathrm{CT}}(f) : G_1 \rightharpoonup G_2$, with $F_{\mathrm{CT}}(f) = j^{\mathrm{CT}}_{G_2} f$. $U_{\mathrm{CT}} : \mathbf{CSig} \to \mathbf{Gph}$ sends $G \in |\mathbf{CSig}|$ to $\mathrm{Pth}(G)$ and a morphism $f : G_1 \rightharpoonup G_2$ to $f^* : \mathrm{Pth}(G_1) \to \mathrm{Pth}(G_2)$. $\eta_{\mathrm{CT}_G} = j^{\mathrm{CT}}_G$, for all $G \in |\mathbf{Gph}|$, and $\epsilon_{\mathrm{CT}_G} = i_{\mathrm{Pth}(G)}$, for all $G \in |\mathbf{CSig}|$. This is a well-known adjunction, better-known by the freeness of the path category $\mathbf{Pth}(G)$ over G along the forgetful functor from **Cat** to **Gph** associating with each category its underlying graph.

This adjunction yields the algebraic theory $\mathbf{T}_{\mathrm{CT}} = \langle T_{\mathrm{CT}}, \eta_{\mathrm{CT}}, \mu_{\mathrm{CT}}\rangle$ in **Gph**, with $T_{\mathrm{CT}} = U_{\mathrm{CT}} F_{\mathrm{CT}}$ and $\mu_{\mathrm{CT}} = U_{\mathrm{CT}} \epsilon_{\mathrm{CT}_{F_{\mathrm{CT}}}}$. Thus, T_{CT} sends a graph G to $\mathrm{Pth}(G)$ and a graph morphism $f : G_1 \to G_2$ to the morphism $(j^{\mathrm{CT}}_{G_2} f)^* : \mathrm{Pth}(G_1) \to \mathrm{Pth}(G_2)$ extending $j^{\mathrm{CT}}_{G_2} f : G_1 \to \mathrm{Pth}(G_2)$, $\eta_{\mathrm{CT}_G} : G \to \mathrm{Pth}(G)$ is the insertion-of-arrows and $\mu_{\mathrm{CT}_G} : \mathrm{Pth}(\mathrm{Pth}(G)) \to \mathrm{Pth}(G)$ is the map extending the identity map on $\mathrm{Pth}(G)$. In this special case a $\mathbf{T}_{\mathrm{CT}}$-algebra is a pair $\langle G, \xi\rangle$, where G is a graph and $\xi : \mathrm{Pth}(G) \to G$ is a graph morphism, such that the following diagrams commute

$$\begin{tikzcd} G \arrow[r, "\eta_{\mathrm{CT}_G}"] \arrow[dr, "i_G"'] & \mathrm{Pth}(G) \arrow[d, "\xi"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Pth}(\mathrm{Pth}(G)) \arrow[r, "\mu_{\mathrm{CT}_G}"] \arrow[d, "T_{\mathrm{CT}}(\xi)"'] & \mathrm{Pth}(G) \arrow[d, "\xi"] \\ \mathrm{Pth}(G) \arrow[r, "\xi"'] & G \end{tikzcd}$$

The Kleisli category $\mathbf{Gph}_{\mathbf{T}_{\mathrm{CT}}}$ is exactly the category **CSig** (its arrows were denoted by $\rightharpoonup$ following the same convention) and the adjunction $\langle F_{\mathrm{CT}}, U_{\mathrm{CT}}, \eta_{\mathrm{CT}}, \epsilon_{\mathrm{CT}}\rangle$ constructed above is the Kleisli adjunction corresponding to this algebraic theory. A special subclass of Eilenberg–Moore algebras of this algebraic theory will be singled out in this case too because of its key role in the algebraization of the institution of equational categorical logic. These are constructed as follows: Let $\mathbf{C}$ be a category. Denote by $G(\mathbf{C})$ the underlying graph of $\mathbf{C}$. Now define $\xi_{\mathbf{C}} : \mathrm{Pth}(G(\mathbf{C})) \to G(\mathbf{C})$ as follows:

$$\xi_{\mathbf{C}}((f_0, f_1, \ldots, f_{n-1})) = f_{n-1} f_{n-2} \ldots f_0,$$

for all $(f_0, f_1, \ldots, f_{n-1})$ in $\mathrm{Pth}(G(\mathbf{C}))$. $\langle G(\mathbf{C}), \xi_{\mathbf{C}}\rangle$ is a $\mathbf{T}_{\mathrm{CT}}$-algebra, for every category $\mathbf{C}$, and the full subcategory of $\mathbf{Gph}^{\mathbf{T}_{\mathrm{CT}}}$ with objects all algberas of this form will be denoted by **ACat**, a shorthand for *algebras of categories*.

Note that **ACat** is equivalent to **Cat**. This is the case since the forgetful functor from **Cat** to **Gph** is monadic, or, equivalently, since every Eilenberg–Moore algebra of that monad is isomorphic to an algebra of categories.

4. Algebraic Institutions

Let $\mathbf{C}$ be a category, $\mathbf{T} = \langle T, \eta, \mu\rangle$ an algebraic theory in monoid form in $\mathbf{C}$, $\mathbf{L}$ a full subcategory of $\mathbf{C}_{\mathbf{T}}$, $\Xi : \mathbf{C} \to \mathbf{Set}$ a functor and $\mathbf{Q}$ a subcategory of $\mathbf{C}^{\mathbf{T}}$. Define the $\langle \mathbf{L}, \Xi, \mathbf{Q}\rangle$-*algebraic institution* $\mathcal{I}_{\langle \mathbf{L}, \Xi, \mathbf{Q}\rangle} = \langle \mathbf{L}, \mathrm{EQ}, \mathrm{ALG}, \models\rangle$ as follows

(i) $\mathrm{EQ} : \mathbf{L} \to \mathbf{Set}$ is given by $\mathrm{EQ} = ((\Xi \circ U_{\mathbf{T}}) \restriction_{\mathbf{L}})^2$, i.e.,

$$\mathrm{EQ}(L) = \Xi(T(L))^2, \quad \text{for every } L \in |\mathbf{L}|,$$

and, given $f : L \rightharpoonup K \in \mathrm{Mor}(\mathbf{L})$,
$\mathrm{EQ}(f)(\langle s, t\rangle) = (\Xi(\mu_K T(f))(s), \Xi(\mu_K T(f))(t))$,
for all $\langle s, t\rangle \in \Xi(T(L))^2$,

$$\Xi(T(L)) \xrightarrow[\Xi(T(f))]{} \Xi(T(T(K))) \xrightarrow[\Xi(\mu_K)]{} \Xi(T(K)).$$

(ii) $\mathrm{ALG} : \mathbf{L} \to \mathbf{CAT}^{\mathrm{op}}$ is the functor that sends an object $L \in |\mathbf{L}|$ to the category $\mathrm{ALG}(L)$ with objects triples of the form $\langle\langle X, \xi\rangle, f\rangle$, $\langle X, \xi\rangle \in |\mathbf{Q}|$, $f : L \rightharpoonup X \in \mathrm{Mor}(\mathbf{C}_{\mathbf{T}})$, and morphisms $h : \langle\langle X, \xi\rangle, f\rangle \to \langle\langle Y, \zeta\rangle, g\rangle$ $\mathbf{Q}$-morphisms $h : \langle X, \xi\rangle \to \langle Y, \zeta\rangle$, such that $g = T(h) f$.

$$\begin{tikzcd} & L \arrow[dl, "f"'] \arrow[dr, "g"] & \\ T(X) \arrow[rr, "T(h)"'] & & T(Y) \end{tikzcd}$$

Moreover, given $k : L \rightharpoonup K \in \mathrm{Mor}(\mathbf{L})$, $\mathrm{ALG}(k) : \mathrm{ALG}(K) \to \mathrm{ALG}(L)$ is the functor that sends $\langle\langle X, \xi\rangle, f\rangle \in |\mathrm{ALG}(K)|$ to $\langle\langle X, \xi\rangle, f \circ k\rangle \in |\mathrm{ALG}(L)|$ and $h : \langle\langle X, \xi\rangle, f\rangle \to \langle\langle Y, \zeta\rangle, g\rangle \in \mathrm{Mor}(\mathrm{ALG}(K))$ to

$$\mathrm{ALG}(k)(h) = h : \langle\langle X, \xi\rangle, f \circ k\rangle \to \langle\langle Y, \zeta\rangle, g \circ k\rangle \in \mathrm{Mor}(\mathrm{ALG}(L)).$$

(iii) $\models_L \subseteq |\mathrm{ALG}(L)| \times \mathrm{EQ}(L)$ is defined by

$$\langle\langle X, \xi\rangle, f\rangle \models_L \langle s, t\rangle \quad \text{iff} \quad \Xi(\xi\mu_X T(f))(s) = \Xi(\xi\mu_X T(f))(t),$$

$$T(L) \xrightarrow[T(f)]{} T(T(X)) \xrightarrow[\mu_X]{} T(X) \xrightarrow[\xi]{} X$$

for all $\langle\langle X, \xi\rangle, f\rangle \in |\mathrm{ALG}(L)|$, $\langle s, t\rangle \in \mathrm{EQ}(L)$.

THEOREM 1. *$\mathcal{I}_{\langle \mathbf{L}, \Xi, \mathbf{Q}\rangle}$ is an institution.*

Proof. $\mathrm{EQ} : \mathbf{L} \to \mathbf{Set}$ is well-defined. Let $f : L \rightharpoonup K$, $g : K \rightharpoonup M \in \mathrm{Mor}(\mathbf{L})$. Then

$$\begin{aligned}
\mathrm{EQ}(g \circ f) &= \Xi(\mu_M T(g \circ f))^2 \quad \text{(by the definition of EQ)}\\
&= \Xi(\mu_M T(\mu_M T(g) f))^2 \quad (\text{since } g \circ f = \mu_M T(g) f)\\
&= \Xi(\mu_M T(\mu_M) T(T(g)) T(f))^2 \quad (\text{since } T \text{ is a functor})\\
&= \Xi(\mu_M \mu_{T(M)} T(T(g)) T(f))^2 \quad \text{(by commutativity of}
\end{aligned}$$

$$\begin{tikzcd}
T(T(T(M))) \arrow[r, "T(\mu_M)"] \arrow[d, "\mu_{T(M)}"'] & T(T(M)) \arrow[d, "\mu_M"] \\
T(T(M)) \arrow[r, "\mu_M"'] & T(M)
\end{tikzcd}\ )$$

$$= \Xi(\mu_M T(g) \mu_K T(f))^2 \quad \text{(by commutativity of}$$

$$\begin{tikzcd}
T(T(K)) \arrow[r, "T(T(g))"] \arrow[d, "\mu_K"] & T(T(T(M))) \arrow[d, "\mu_{T(M)}"] \\
T(K) \arrow[r, "T(g)"'] & T(T(M))
\end{tikzcd}\ )$$

$$\begin{aligned}
&= \Xi(\mu_M T(g))^2 \Xi(\mu_K T(f))^2 \quad (\text{since } \Xi \text{ is a functor})\\
&= \mathrm{EQ}(g)\mathrm{EQ}(f) \quad \text{(by the definition of EQ).}
\end{aligned}$$

$\mathrm{ALG} : \mathbf{L} \to \mathbf{CAT}^{\mathrm{op}}$ is well defined at the morphism level. Indeed, if $h : \langle\langle X, \xi\rangle, f\rangle \to \langle\langle Y, \zeta\rangle, g\rangle \in \mathrm{Mor}(\mathrm{ALG}(K))$, then

$$\begin{aligned}
T(h)(f \circ k) &= T(h)(\mu_X T(f) k) \quad (\text{since } f \circ k = \mu_X T(f) k)\\
&= \mu_Y T(T(h)) T(f) k \quad \text{(by commutativity of}
\end{aligned}$$

$$\begin{tikzcd} T(T(X)) \arrow[r, "T(T(h))"] \arrow[d, "\mu_X"] & T(T(Y)) \arrow[d, "\mu_Y"] \\ T(X) \arrow[r, "T(h)"'] & T(Y) \end{tikzcd} \quad)$$

$$\begin{aligned} &= \mu_Y T(T(h)f)k \quad \text{(since } T \text{ is a functor)} \\ &= \mu_Y T(g)k \quad \text{(since } h \in \mathrm{Mor}(\mathrm{ALG}(K))) \\ &= g \circ k. \end{aligned}$$

Thus, $h : \langle\langle X, \xi\rangle, f\circ k\rangle \to \langle\langle Y, \zeta\rangle, g\circ k\rangle \in \mathrm{Mor}(\mathrm{ALG}(L))$. Finally, the satisfaction condition holds, since, if $k : L \rightharpoonup K \in \mathrm{Mor}(\mathbf{L})$, $\langle s, t\rangle \in \mathrm{EQ}(L)$, $\langle\langle X, \xi\rangle, f\rangle \in |\mathrm{ALG}(K)|$,

$\mathrm{ALG}(k)(\langle\langle X, \xi\rangle, f\rangle) \models_L \langle s, t\rangle$

iff $\langle\langle X, \xi\rangle, f \circ k\rangle \models_L \langle s, t\rangle$, by definition of $\mathrm{ALG}(k)$,

iff $\Xi(\xi\mu_X T(f \circ k))(s) = \Xi(\xi\mu_X T(f \circ k))(t)$, by definition of $\models_L$,

iff $\Xi(\xi\mu_X T(\mu_X T(f)k))(s) = \Xi(\xi\mu_X T(\mu_X T(f)k))(t)$,
since $f \circ k = \mu_X T(f)k$,

iff $\Xi(\xi\mu_X T(\mu_X)T(T(f))T(k))(s) = \Xi(\xi\mu_X T(\mu_X)T(T(f))T(k))(t)$,
since T is a functor,

iff $\Xi(\xi\mu_X\mu_{T(X)}T(T(f))T(k))(s) = \Xi(\xi\mu_X\mu_{T(X)}T(T(f))T(k))(t)$,
by commutativity of

$$\begin{tikzcd} T(T(T(X))) \arrow[r, "T(\mu_X)"] \arrow[d, "\mu_{T(X)}"'] & T(T(X)) \arrow[d, "\mu_X"] \\ T(T(X)) \arrow[r, "\mu_X"'] & T(X) \end{tikzcd}$$

iff $\Xi(\xi\mu_X T(f)\mu_K T(k))(s) = \Xi(\xi\mu_X T(f)\mu_K T(k))(t)$,
by commutativity of

$$\begin{tikzcd} T(T(K)) \arrow[r, "T(T(f))"] \arrow[d, "\mu_K"'] & T(T(T(X))) \arrow[d, "\mu_{T(X)}"] \\ T(K) \arrow[r, "T(f)"'] & T(T(X)) \end{tikzcd}$$

iff $\Xi(\xi\mu_X T(f))(\Xi(\mu_K T(k))(s)) = \Xi(\xi\mu_X T(f))(\Xi(\mu_K T(k))(t))$,
since Ξ is a functor,

iff $\langle\langle X, \xi\rangle, f\rangle \models_K \Xi(\mu_K T(k))^2(\langle s, t\rangle)$, by the definition of $\models_K$,

iff $\langle\langle X, \xi\rangle, f\rangle \models_K \mathrm{EQ}(k)(\langle s, t\rangle)$, by the definition of $\mathrm{EQ}(k)$,

as required. □

Of particular interest are the following special cases:

(a) The category $\mathbf{C}$ has a terminal object 1 and the functor $\Xi : \mathbf{C} \to \mathbf{Set}$ is the covariant Hom-functor $\mathbf{C}(1, -)$. In this case

$$\mathrm{EQ}(L) = \mathbf{C}(1, T(L))^2, \quad \text{for all } L \in |\mathbf{L}|,$$

and, given $f : L \rightharpoonup K \in \mathrm{Mor}(\mathbf{L})$,

$$\mathrm{EQ}(f)(\langle s, t\rangle) = \mathbf{C}(1, \mu_K T(f))^2(\langle s, t\rangle) = \langle \mu_K T(f)s, \mu_K T(f)t\rangle.$$

Moreover, for $L \in |\mathbf{L}|$, $\langle\langle X, \xi\rangle, f\rangle \in |\mathrm{ALG}(L)|$, $\langle s, t\rangle \in \mathrm{EQ}(L)$, we have

$$\langle\langle X, \xi\rangle, f\rangle \models_L \langle s, t\rangle \quad \text{iff} \quad \xi\mu_X T(f)s = \xi\mu_X T(f)t.$$

Algebraic institutions of this form are exactly the ones introduced in [21]. It soon became apparent that they are far from sufficient for the algebraization of such well-known institutions, as the institution of equational logic over function symbols of fixed finite arities of Section 2 (see also [23]) and the institution of first-order logic without terms with relation symbols of fixed finite arities of Section 2 (see also [24]).

(b) (A subcase of (a)) Let $\mathbf{C} = \mathbf{Set}$, $\Xi : \mathbf{Set} \to \mathbf{Set}$ the functor $\mathbf{Set}(1, -)$, which is isomorphic to the identity functor. Then

$$\mathrm{EQ}(L) = T(L)^2, \quad \text{for every } L \in |\mathbf{Set}|,$$

and, given $f : L \rightharpoonup K \in \mathrm{Mor}(\mathbf{L})$,

$$\mathrm{EQ}(f)(\langle s, t\rangle) = \langle \mu_K T(f)(s), \mu_K T(f)(t)\rangle.$$

Further, if $\langle\langle X, \xi\rangle, f\rangle \in |\mathrm{ALG}(L)|$, $\langle s, t\rangle \in \mathrm{EQ}(L)$,

$$\langle\langle X, \xi\rangle, f\rangle \models_L \langle s, t\rangle \quad \text{iff} \quad \xi\mu_X T(f)(s) = \xi\mu_X T(f)(t).$$

By the $\langle \mathbf{L}, \Xi, \mathbf{Q}\rangle$-*algebraic* π-*institution*, we will understand the π-institution (also denoted by $\mathcal{I}_{\langle \mathbf{L}, \Xi, \mathbf{Q}\rangle}$) associated with the institution $\mathcal{I}_{\langle \mathbf{L}, \Xi, \mathbf{Q}\rangle}$ in the sense of Section 2 (see also [22]). (There, it was denoted by $\pi(\mathcal{I}_{\langle \mathbf{L}, \Xi, \mathbf{Q}\rangle})$. The π is omitted to simplify the notation, since it is usually clear from context which structure is under discussion.)

4.1. ALGEBRAIC 2-DEDUCTIVE SYSTEMS

Let $\mathcal{L} = \langle \Lambda, \rho\rangle$ be a propositional language and V a countable set of variables. Recall that $\mathrm{Fm}_{\mathcal{L}}(V)$ denotes the set of formulas constructed by recursion using variables in V and connectives in $\mathcal{L}$ in the usual way. Recall also that an assignment of formulas to variables is a mapping $f : V \to \mathrm{Fm}_{\mathcal{L}}(V)$, which is also denoted by $f : V \rightharpoondown V$, and that such an assignment can be extended uniquely to a substitution, i.e., an endomorphism of the formula algebra $\mathbf{Fm}_{\mathcal{L}}(V)$, denoted by $f^* : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$.

Let K be a class of $\mathcal{L}$-algebras (in the usual universal algebraic sense). By $\mathcal{S}_K = \langle \mathcal{L}, \models_K \rangle$ we denote the 2-deductive system associated with K in the sense of [5], i.e., $\mathcal{S}_K$ is the system whose formulas are the pairs $\langle \phi, \psi \rangle$, $\phi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$, and, for all $E \cup \{\langle \phi, \psi \rangle\} \subseteq \mathrm{Fm}^2_{\mathcal{L}}(V)$, $E \models_K \langle \phi, \psi \rangle$, if and only if, for all $\mathbf{A} \in K$, $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$, if $h(e_1) = h(e_2)$, for all $\langle e_1, e_2 \rangle \in E$, then $h(\phi) = h(\psi)$.

Now, let $\mathcal{V}_{\mathcal{L}}$ be the variety of all $\mathcal{L}$-algebras. With this variety there is associated the algebraic theory $\mathbf{T}_{\mathcal{L}} = \langle T_{\mathcal{L}}, \eta_{\mathcal{L}}, \mu_{\mathcal{L}} \rangle$ in monoid form in the category **Set** (see the discussion in the previous section and also [19]). Let $\mathbf{F}_{\mathcal{L}}$ denote the full subcategory of $\mathbf{Set}_{\mathbf{T}_{\mathcal{L}}}$ with the single object V. (Its morphisms are the assignments $f : V \rightharpoonup V$ and composition is the Kleisli composition.) Let $\mathbf{Q}_K$ be the full subcategory of $\mathbf{Set}^{\mathbf{T}_{\mathcal{L}}}$ with objects the $\mathbf{T}_{\mathcal{L}}$-algebras corresponding to the $\mathcal{L}$-algebras in K. Then, the $\langle \mathbf{F}_{\mathcal{L}}, I_{\mathbf{Set}}, \mathbf{Q}_K \rangle$-algebraic institution $\mathcal{I}_{\langle \mathbf{F}_{\mathcal{L}}, I_{\mathbf{Set}}, \mathbf{Q}_K \rangle}$ will be called the *algebraic institution associated with* $\mathcal{S}_K$.

4.2. EQUATIONAL ALGEBRA

Let $\mathbf{T}_{\mathrm{EQ}} = \langle T_{\mathrm{EQ}}, \eta_{\mathrm{EQ}}, \mu_{\mathrm{EQ}} \rangle$ be the algebraic theory in $\Omega\mathbf{Set}$ that was described in the previous section, $\Omega\mathbf{Set}_{\mathbf{T}_{\mathrm{EQ}}}$ the Kleisli category of $\mathbf{T}_{\mathrm{EQ}}$ in $\Omega\mathbf{Set}$ and **FCln** the full subcategory of $\Omega\mathbf{Set}^{\mathbf{T}_{\mathrm{EQ}}}$ with collection of objects $\langle \mathrm{Cl}(A), \xi_A \rangle$, as described before.

Define the functor $\Xi_{\mathrm{EQ}} : \Omega\mathbf{Set} \to \mathbf{Set}$ as follows:

$$\Xi_{\mathrm{EQ}}(X) = \bigcup_{i \geq 1} X_i, \quad \text{for all } X \in |\Omega\mathbf{Set}|,$$

and, given $f : X \to Y \in \mathrm{Mor}(\Omega\mathbf{Set})$, $\Xi_{\mathrm{EQ}}(f) : \bigcup_{i \geq 1} X_i \to \bigcup_{i \geq 1} Y_i$ is defined by

$$\Xi_{\mathrm{EQ}}(f)(x) = f_n(x), \quad \text{for all } x \in X_n.$$

Let $\mathcal{I}_{\langle \Omega\mathbf{Set}_{\mathbf{T}_{\mathrm{EQ}}}, \Xi_{\mathrm{EQ}}, \mathbf{FCln} \rangle}$ be the $\langle \Omega\mathbf{Set}_{\mathbf{T}_{\mathrm{EQ}}}, \Xi_{\mathrm{EQ}}, \mathbf{FCln} \rangle$-algebraic institution and denote by $\mathcal{EA}$ the $\langle \Omega\mathbf{Set}_{\mathbf{T}_{\mathrm{EQ}}}, \Xi_{\mathrm{EQ}}, \mathbf{FCln} \rangle$-algebraic π-institution, i.e., the π-institution associated with $\mathcal{I}_{\langle \Omega\mathbf{Set}_{\mathbf{T}_{\mathrm{EQ}}}, \Xi_{\mathrm{EQ}}, \mathbf{FCln} \rangle}$.

Briefly $\mathcal{EA} = \langle \Omega\mathbf{Set}_{\mathbf{T}_{\mathrm{EQ}}}, \mathrm{EQ}_{\mathrm{EA}}, \{C_{\mathrm{EA}_X}\}_{X \in |\Omega\mathbf{Set}_{\mathbf{T}_{\mathrm{EQ}}}|} \rangle$ has the following description:

1. its signature category is the Kleisli category $\Omega\mathbf{Set}_{\mathbf{T}_{\mathrm{EQ}}}$,
2. its sentence functor sends an ω-set X to the set of all equations $s \approx t$, with $s, t \in \bigcup_{i \geq 1} \mathrm{Tm}_X(V)_i$ and an ω-set morphism $f : X \rightharpoonup Y$ to the morphism $\mathrm{EQ}_{\mathrm{EA}}(f) : (\bigcup_{i \geq 1} \mathrm{Tm}_X(V)_i)^2 \to (\bigcup_{i \geq 1} \mathrm{Tm}_Y(V)_i)^2$, that maps an equation

$$s \approx t \text{ in } \left(\bigcup_{i \geq 1} \mathrm{Tm}_X(V)_i \right)^2, \quad \text{with } s \in \mathrm{Tm}_X(V)_m \text{ and } t \in \mathrm{Tm}_X(V)_n$$

to the equation $f^*_m(s) \approx f^*_n(t)$ in $(\bigcup_{i \geq 1} \mathrm{Tm}_Y(V)_i)^2$, and, finally,

3. its sentence closure operator

$$C_{\mathrm{EA}_X} : \mathcal{P}\left(\left(\bigcup_{i\geq 1}\mathrm{Tm}_X(V)_i\right)^2\right) \to \mathcal{P}\left(\left(\bigcup_{i\geq 1}\mathrm{Tm}_X(V)_i\right)^2\right)$$

is given by

$$s \approx t \in C_{\mathrm{EA}_X}(E) \quad \text{iff}$$

for all $\langle \mathrm{Cl}(A), \xi_A\rangle \in |\mathbf{FCln}|$, $f : X \rightharpoonup \mathrm{Cl}(A)$,

$$(\xi_A \mu_{\mathrm{EQ}_{\mathrm{Cl}(A)}} T_{\mathrm{EQ}}(f))^*(e_1) = (\xi_A \mu_{\mathrm{EQ}_{\mathrm{Cl}(A)}} T_{\mathrm{EQ}}(f))^*(e_2), \quad \text{for all } e_1 \approx e_2 \in E,$$

implies

$$(\xi_A \mu_{\mathrm{EQ}_{\mathrm{Cl}(A)}} T_{\mathrm{EQ}}(f))^*(s) = (\xi_A \mu_{\mathrm{EQ}_{\mathrm{Cl}(A)}} T_{\mathrm{EQ}}(f))^*(t).$$

4.3. FIRST-ORDER ALGEBRA

Let $\mathbf{T}_{\mathrm{FO}} = \langle T_{\mathrm{FO}}, \eta_{\mathrm{FO}}, \mu_{\mathrm{FO}}\rangle$ be the algebraic theory in **HSet**, described in the preceding section, $\mathbf{HSet}_{\mathbf{T}_{\mathrm{FO}}}$ the Kleisli category of $\mathbf{T}_{\mathrm{FO}}$ in **HSet** and **FRln** the full subcategory of $\mathbf{HSet}^{\mathbf{T}_{\mathrm{FO}}}$ with collection of objects all full relation algebras $\langle \mathrm{Rel}(A), \xi_A\rangle$, also defined in Section 3.

Define the functor $\Xi_{\mathrm{FO}} : \mathbf{HSet} \to \mathbf{Set}$ as follows:

$$\Xi_{\mathrm{FO}}(X) = X_\emptyset, \quad \text{for all } X \in |\mathbf{HSet}|,$$

and, given $f : X \to Y \in \mathrm{Mor}(\mathbf{HSet})$, $\Xi_{\mathrm{FO}} : X_\emptyset \to Y_\emptyset$ is defined by

$$\Xi_{\mathrm{FO}}(f)(x) = f_\emptyset(x), \quad \text{for all } x \in X_\emptyset.$$

Let $\mathcal{I}_{\langle \mathbf{HSet}_{\mathbf{T}_{\mathrm{FO}}}, \Xi_{\mathrm{FO}}, \mathbf{FRln}\rangle}$ be the $\langle \mathbf{HSet}_{\mathbf{T}_{\mathrm{FO}}}, \Xi_{\mathrm{FO}}, \mathbf{FRln}\rangle$-algebraic institution and denote by $\mathcal{FOA}$ the $\langle \mathbf{HSet}_{\mathbf{T}_{\mathrm{FO}}}, \Xi_{\mathrm{FO}}, \mathbf{FRln}\rangle$-algebraic π-institution, i.e., the π-institution associated with $\mathcal{I}_{\langle \mathbf{HSet}_{\mathbf{T}_{\mathrm{FO}}}, \Xi_{\mathrm{FO}}, \mathbf{FRln}\rangle}$.

Briefly, $\mathcal{FOA} = \langle \mathbf{HSet}_{\mathbf{T}_{\mathrm{FO}}}, \mathrm{EQ}_{\mathrm{FA}}, \{C_{\mathrm{FA}_X}\}_{X\in|\mathbf{HSet}_{\mathbf{T}_{\mathrm{FO}}}|}\rangle$ has the following description:

1. its signature category is the Kleisli category $\mathbf{HSet}_{\mathbf{T}_{\mathrm{FO}}}$,
2. its sentence functor sends an h-set X to the set of all equations $s \approx t$, with $s, t \in \mathrm{Fm}_{\mathcal{L}}(X)_\emptyset$, where, again, $\mathcal{L}$ consists of the connectives $\{\neg, \wedge\} \cup \{\exists_k : k \in \omega\}$, and an h-set morphism $f : X \rightharpoonup Y$ to the morphism $\mathrm{EQ}_{\mathrm{FA}}(f) : \mathrm{Fm}_{\mathcal{L}}(X)^2_\emptyset \to \mathrm{Fm}_{\mathcal{L}}(Y)^2_\emptyset$, that maps an equation $s \approx t$ in $\mathrm{Fm}_{\mathcal{L}}(X)^2_\emptyset$, to the equation $f^*_\emptyset(s) \approx f^*_\emptyset(t)$ in $\mathrm{Fm}_{\mathcal{L}}(Y)^2_\emptyset$ and, finally,

3. its sentence closure operator

$$C_{\mathrm{FA}_X} : \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(X)^2_{\emptyset}) \to \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(X)^2_{\emptyset})$$

is given by

$$s \approx t \in C_{\mathrm{FA}_X}(E) \quad \text{iff}$$

for all $\langle \mathrm{Rel}(A), \xi_A \rangle \in |\mathbf{FRln}|$, $f : X \rightharpoondown \mathrm{Rel}(A)$,

$$(\xi_A \mu_{\mathrm{FO}_{\mathrm{Rel}(A)}} T_{\mathrm{FO}}(f))^*(e_1) = (\xi_A \mu_{\mathrm{FO}_{\mathrm{Rel}(A)}} T_{\mathrm{FO}}(f))^*(e_2), \quad \text{for all } e_1 \approx e_2 \in E,$$

implies

$$(\xi_A \mu_{\mathrm{FO}_{\mathrm{Rel}(A)}} T_{\mathrm{FO}}(f))^*(s) = (\xi_A \mu_{\mathrm{FO}_{\mathrm{Rel}(A)}} T_{\mathrm{FO}}(f))^*(t).$$

4.4. ALGEBRA OF CATEGORIES

Now, let $\mathbf{T}_{\mathrm{CT}} = \langle T_{\mathrm{CT}}, \eta_{\mathrm{CT}}, \mu_{\mathrm{CT}} \rangle$ be the algebraic theory in **Gph**, described in Section 3, $\mathbf{Gph}_{\mathbf{T}_{\mathrm{CT}}}$ its Kleisli category in **Gph** and **ACat** the full subcategory of its Eilenberg–Moore category $\mathbf{Gph}^{\mathbf{T}_{\mathrm{CT}}}$ with collection of objects all algebras of categories $\langle G(\mathbf{C}), \xi_{\mathbf{C}} \rangle$.

Define the functor $\Xi_{\mathrm{CT}} : \mathbf{Gph} \to \mathbf{Set}$ as follows:

$$\Xi_{\mathrm{CT}}(G) = E(\mathrm{Pth}(G)), \quad \text{for all } G \in |\mathbf{Gph}|,$$

where, by $E(G)$ is denoted the collection of edges of G, and, given $f : G_1 \to G_2 \in \mathrm{Mor}(\mathbf{Gph})$, $\Xi_{\mathrm{CT}}(f) : E(\mathrm{Pth}(G_1)) \to E(\mathrm{Pth}(G_2))$ is defined by

$$\Xi_{\mathrm{CT}}(f)((e_0, e_1, \ldots, e_{n-1})) = (f(e_0), f(e_1), \ldots, f(e_{n-1})),$$

for all $(e_0, \ldots, e_{n-1}) \in E(\mathrm{Pth}(G_1))$.

Let $\mathcal{I}_{\langle \mathbf{Gph}_{\mathbf{T}_{\mathrm{CT}}}, \Xi_{\mathrm{CT}}, \mathbf{ACat} \rangle}$ be the $\langle \mathbf{Gph}_{\mathbf{T}_{\mathrm{CT}}}, \Xi_{\mathrm{CT}}, \mathbf{ACat} \rangle$-algebraic institution and denote by $\mathcal{CA}$ the $\langle \mathbf{Gph}_{\mathbf{T}_{\mathrm{CT}}}, \Xi_{\mathrm{CT}}, \mathbf{ACat} \rangle$-algebraic π-institution, i.e., the π-institution associated with $\mathcal{I}_{\langle \mathbf{Gph}_{\mathbf{T}_{\mathrm{CT}}}, \Xi_{\mathrm{CT}}, \mathbf{ACat} \rangle}$. A brief description of $\mathcal{CA} = \langle \mathbf{Gph}_{\mathbf{T}_{\mathrm{CT}}}, \mathrm{EQ}_{\mathrm{CA}}, \{C_{\mathrm{CA}_G}\}_{G \in |\mathbf{Gph}_{\mathbf{T}_{\mathrm{CT}}}|} \rangle$ follows:

1. its signature category is the Kleisli category $\mathbf{Gph}_{\mathbf{T}_{\mathrm{CT}}}$,
2. its sentence functor sends a graph G to the set of all equations

$$(e_0, \ldots, e_{n-1}) \approx (f_0, \ldots, f_{m-1}) \in E(\mathrm{Pth}(G))^2,$$

and a graph morphism $h : G_1 \rightharpoondown G_2$ to the morphism $\mathrm{EQ}_{\mathrm{CA}}(h) : E(\mathrm{Pth}(G_1)) \to E(\mathrm{Pth}(G_2))$ that maps an equation $(e_0, \ldots, e_{n-1}) \approx (f_0, \ldots, f_{m-1})$ in $E(\mathrm{Pth}(G_1))^2$ to the equation $(h(e_0), \ldots, h(e_{n-1})) \approx (h(f_0), \ldots, h(f_{m-1}))$ in $E(\mathrm{Pth}(G_2))^2$ and, finally,

3. its sentence closure operator

$$C_{\mathrm{CA}_G} : \mathcal{P}(E(\mathrm{Pth}(G))^2) \to \mathcal{P}(E(\mathrm{Pth}(G))^2)$$

is given by

$$(e_0, \ldots, e_{n-1}) \approx (f_0, \ldots, f_{m-1}) \in C_{\mathrm{CA}_G}(E) \quad \text{iff}$$

for all $\langle G(\mathbf{C}), \xi_{\mathbf{C}} \rangle \in |\mathbf{ACat}|$, $f : G \rightharpoonup G(\mathbf{C})$,

$$(\xi_{\mathbf{C}} \mu_{\mathrm{CT}_{G(\mathbf{C})}} T_{\mathrm{CT}}(f))^*(p) = (\xi_{\mathbf{C}} \mu_{\mathrm{CT}_{G(\mathbf{C})}} T_{\mathrm{CT}}(f))^*(q), \quad \text{for all } p \approx q \in E,$$

implies

$$(\xi_{\mathbf{C}} \mu_{\mathrm{CT}_{G(\mathbf{C})}} T_{\mathrm{CT}}(f))^*(e_0, \ldots, e_{n-1}) = (\xi_{\mathbf{C}} \mu_{\mathrm{CT}_{G(\mathbf{C})}} T_{\mathrm{CT}}(f))^*(f_0, \ldots, f_{m-1}).$$

5. Algebraizable Institutions

Recall that, given a positive integer k, a k-deductive system over a language type $\mathcal{L}$ is defined in [5] to be a pair $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$, where $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(\mathrm{Fm}^k_{\mathcal{L}}(V)) \times \mathrm{Fm}^k_{\mathcal{L}}(V)$ is a structural and finitary consequence relation on the set of k-formulas $\mathrm{Fm}^k_{\mathcal{L}}(V)$. In [6], given a k-deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ and an l-deductive system $\mathcal{T} = \langle \mathcal{L}, \vdash_{\mathcal{T}} \rangle$ over the same language type $\mathcal{L}$, a *k–l-translation* $\tau : \mathcal{S} \to \mathcal{T}$ is defined to be a set $\tau = \{\tau^i(v_0, \ldots, v_{k-1}) : i < n\}$ of n l-formulas $\tau^i(v_0, \ldots, v_{k-1}) = \langle \tau^i_j(v_0, \ldots, v_{k-1}) : j < l \rangle$, in k variables $v_0, \ldots, v_{k-1}$. A k–l-translation is said to be a *k–l-interpretation* if, for all $\Phi \cup \{\phi\} \subseteq \mathrm{Fm}^k_{\mathcal{L}}(V)$,

$$\Phi \vdash_{\mathcal{S}} \phi \quad \text{if and only if} \quad \tau(\Phi) \vdash_{\mathcal{T}} \tau(\phi). \tag{1}$$

The two systems $\mathcal{S}$ and $\mathcal{T}$ are called *equivalent* if there exists a k–l-interpretation $\tau : \mathcal{S} \to \mathcal{T}$ and an l–k-interpretation $\sigma : \mathcal{T} \to \mathcal{S}$, such that, for all $\phi \in \mathrm{Fm}^k_{\mathcal{L}}(V)$ and $\psi \in \mathrm{Fm}^l_{\mathcal{L}}(V)$,

$$\sigma(\tau(\phi)) \dashv\vdash_{\mathcal{S}} \phi \quad \text{and} \quad \tau(\sigma(\psi)) \dashv\vdash_{\mathcal{T}} \psi. \tag{2}$$

In [22] these notions were extended to cover the case of institutional logics. Let $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_1|} \rangle$, $\mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_2|} \rangle$ be two π-institutions. A translation of $\mathcal{I}_1$ in $\mathcal{I}_2$ is a pair $\langle F, \alpha \rangle : \mathcal{I}_1 \to \mathcal{I}_2$ consisting of a functor $F : \mathbf{Sign}_1 \to \mathbf{Sign}_2$ and a natural transformation $\alpha : \mathrm{SEN}_1 \to \mathcal{P}\mathrm{SEN}_2 F$.

A translation is called an interpretation if, in addition, for all $\Sigma_1 \in |\mathbf{Sign}_1|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}_1(\Sigma_1)$,

$$\phi \in C_{\Sigma_1}(\Phi) \quad \text{if and only if} \quad \alpha_{\Sigma_1}(\phi) \subseteq C_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\Phi)).$$

This condition corresponds to condition (1) in the case of a k–l-translation for deductive systems. $\mathcal{I}_1$ and $\mathcal{I}_2$ are called deductively equivalent if there exist interpretations $\langle F, \alpha \rangle : \mathcal{I}_1 \to \mathcal{I}_2$ and $\langle G, \beta \rangle : \mathcal{I}_2 \to \mathcal{I}_1$, such that

1. $\langle F, G, \eta, \epsilon \rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$ is an adjoint equivalence,
2. for all $\Sigma_1 \in |\mathbf{Sign}_1|$, $\Sigma_2 \in |\mathbf{Sign}_2|$, $\phi \in \mathrm{SEN}_1(\Sigma_1)$, $\psi \in \mathrm{SEN}_2(\Sigma_2)$,

$$C_{G(F(\Sigma_1))}(\mathrm{SEN}_1(\eta_{\Sigma_1})(\phi)) = C_{G(F(\Sigma_1))}(\beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\phi))),$$
$$C_{\Sigma_2}(\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\psi)))) = C_{\Sigma_2}(\psi).$$

These conditions correspond to (2), but they are more complex reflecting their ability to handle the more intricate context of multiple signature logics. Two institutions are *deductively equivalent* if their associated π-institutions are equivalent in the sense described above.

Blok and Pigozzi call a k-deductive system *algebraizable* in [4] if it is equivalent to an algebraic 2-deductive system, i.e., one whose consequence relation is the equational consequence relation corresponding to some class of $\mathcal{L}$-algebras. Since equivalence of deductive systems corresponds to deductive equivalence of π-institutions and algebraic 2-deductive systems correspond to algebraic π-institutions, the following definition naturally generalizes the definition for deductive systems.

DEFINITION 2. A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ is *algebraizable* if it is deductively equivalent to an $\langle \mathbf{L}, \Xi, \mathbf{Q}\rangle$-algebraic π-institution. Similarly, an institution $\mathcal{I}$ is *algebraizable* if it is deductively equivalent to an $\langle \mathbf{L}, \Xi, \mathbf{Q}\rangle$-algebraic institution.

The examples of institutions provided in Section 2 will now be revisited and it will be sketched how one establishes that the corresponding algebraic institutions obtained in Section 4 can be used as their algebraic counterparts.

5.1. ALGEBRAIZABLE DEDUCTIVE SYSTEMS

Let $\mathcal{L}$ be a language type and $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}}\rangle$ a k-deductive system over $\mathcal{L}$. Recall from Section 2 that to $\mathcal{S}$ there is associated the deductive π-institution $\mathcal{I}_{\mathcal{S}} = \langle \mathbf{Sign}_{\mathcal{S}}, \mathrm{SEN}_{\mathcal{S}}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_{\mathcal{S}}|}\rangle$. Suppose, next, that $\mathcal{S}$ is algebraizable in the sense of [4]. Thus, there exists a class K of $\mathcal{L}$-algebras, such that $\mathcal{S}$ is interpretable in the equational deductive system $\mathcal{S}_K = \langle \mathcal{L}, \models_K\rangle$ via an interpretation $\tau : \mathcal{S} \to \mathcal{S}_K$, i.e., for all k-formulas $\langle \phi_0, \dots, \phi_{k-1}\rangle$ and all $\Phi \subseteq \mathrm{Fm}^k_{\mathcal{L}}(V)$,

$$\Phi \vdash_{\mathcal{S}} \langle \phi_0, \dots, \phi_{k-1}\rangle \quad \text{iff} \quad \tau(\Phi) \models_K \tau(\langle \phi_0, \dots, \phi_{k-1}\rangle),$$

$\mathcal{S}_K$ is interpretable in $\mathcal{S}$ via an interpretation $\sigma : \mathcal{S}_K \to \mathcal{S}$, i.e., for all equations $\phi \approx \psi$ and all $E \subseteq \mathrm{Fm}^2_{\mathcal{L}}(V)$,

$$E \models_K \phi \approx \psi \quad \text{iff} \quad \sigma(E) \vdash_{\mathcal{S}} \sigma(\phi, \psi),$$

and, for all k-formulas $\langle \phi_0, \dots, \phi_{k-1}\rangle$ and all equations $\phi \approx \psi$,

$$\langle \phi_0, \dots, \phi_{k-1}\rangle \dashv\vdash_{\mathcal{S}} \sigma(\tau(\langle \phi_0, \dots, \phi_{k-1}\rangle)) \quad \text{and} \quad \phi \approx \psi =\!\!| \models_K \tau(\sigma(\phi, \psi)). \quad (3)$$

To the equational deductive system $\mathcal{S}_K = \langle \mathcal{L}, \models_K\rangle$, considered as a 2-deductive system, there is also associated in exactly the same way a π-institution $\mathcal{I}_K =$

$\mathcal{I}_{\mathcal{S}_K} = \langle \mathbf{Sign}_K, \mathrm{SEN}_K, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_K|}\rangle$ and it is obvious, by the two constructions, that $\mathbf{Sign}_{\mathcal{S}} = \mathbf{Sign}_K$.

Define the translation $\langle F, \alpha\rangle : \mathcal{I}_{\mathcal{S}} \to \mathcal{I}_K$ by $F = I_{\mathbf{Sign}_{\mathcal{S}}}$ and $\alpha : \mathrm{SEN}_{\mathcal{S}} \to \mathcal{P}\mathrm{SEN}_K$ by $\alpha_V : \mathrm{Fm}^k_{\mathcal{L}}(V) \to \mathcal{P}(\mathrm{Fm}^2_{\mathcal{L}}(V))$, with

$$\alpha_V(\phi_0, \ldots, \phi_{k-1}) = \tau(\phi_0, \ldots, \phi_{k-1}), \quad \text{for all } \langle \phi_0, \ldots, \phi_{k-1}\rangle \in \mathrm{Fm}^k_{\mathcal{L}}(V),$$

and the translation $\langle G, \beta\rangle : \mathcal{I}_K \to \mathcal{I}_{\mathcal{S}}$ by $G = I_{\mathbf{Sign}_K}$ and $\beta : \mathrm{SEN}_K \to \mathcal{P}\mathrm{SEN}_{\mathcal{S}}$ by $\beta_V : \mathrm{Fm}^2_{\mathcal{L}}(V) \to \mathcal{P}(\mathrm{Fm}^k_{\mathcal{L}}(V))$, with

$$\beta_V(\phi, \psi) = \sigma(\phi, \psi), \quad \text{for all } \phi \approx \psi \in \mathrm{Fm}^2_{\mathcal{L}}(V).$$

Then $\langle F, \alpha\rangle : \mathcal{I}_{\mathcal{S}} \to \mathcal{I}_K$ and $\langle G, \beta\rangle : \mathcal{I}_K \to \mathcal{I}_{\mathcal{S}}$ are interpretations because τ and σ are interpretations in the sense of [6] and it is true that

$$\sigma(\tau(\phi_0, \ldots, \phi_{k-1}))^{c_{\mathcal{S}}} = \langle \phi_0, \ldots, \phi_{k-1}\rangle^{c_{\mathcal{S}}} \quad \text{and} \quad \tau(\sigma(\phi, \psi))^{c_K} = \langle \phi, \psi\rangle^{c_K},$$

for all $\langle \phi_0, \ldots, \phi_{k-1}\rangle \in \mathrm{Fm}^k_{\mathcal{L}}(V)$ and $\phi \approx \psi \in \mathrm{Fm}^2_{\mathcal{L}}(V)$, in $\mathcal{I}_{\mathcal{S}}$ and $\mathcal{I}_K$, respectively, since these conditions are, respectively, equivalent to the conditions in (3). Thus $\mathcal{I}_{\mathcal{S}}$ and $\mathcal{I}_K$ are equivalent π-institutions in the present sense. It is not difficult to see that $\mathcal{I}_K$ is the algebraic institution $\mathcal{I}_{\langle \mathbf{F}_{\mathcal{L}}, I_{\mathbf{Set}}, \mathbf{Q}_K\rangle}$ associated with $\mathcal{S}_K$, whose description was given in Section 4. Thus, if $\mathcal{S}$ is algebraizable in the sense of [4], $\mathcal{I}_{\mathcal{S}}$ is equivalent to an algebraic π-institution and is, therefore, algebraizable in the present sense.

The question remains open of whether $\mathcal{S}$ must be algebraizable in the sense of [4] (or even [15]) if $\mathcal{I}_{\mathcal{S}}$ is algebraizable in the present sense. If this question is answered to the negative, then the current notion of algebraizability, when applied to the deductive π-institutions $\mathcal{I}_{\mathcal{S}}$, has the potential of changing the meaning of algebraizability for deductive systems, giving rise to a properly wider class of algebraizable deductive systems than the ones described in [4] and [15].

5.2. EQUATIONAL LOGIC

Recall from Section 2 the definition of the institution $\mathcal{EQ} = \langle \mathbf{EQSig}, \mathrm{EQSEN}, \mathrm{EQMOD}, \models\rangle$ that represents the system of equational logic with multiple signatures. Recall also from Section 4 the definition of the $\langle \Omega\mathbf{Set}_{\mathbf{T}_{\mathrm{EQ}}}, \Xi_{\mathrm{EQ}}, \mathbf{FCln}\rangle$-algebraic π-institution $\mathcal{EA} = \langle \Omega\mathbf{Set}_{\mathbf{T}_{\mathrm{EQ}}}, \mathrm{EQ}_{\mathrm{EA}}, \{C_{\mathrm{EA}_X}\}_{X \in |\Omega\mathbf{Set}_{\mathbf{T}_{\mathrm{EQ}}}|}\rangle$ corresponding to the $\langle \Omega\mathbf{Set}_{\mathbf{T}_{\mathrm{EQ}}}, \Xi_{\mathrm{EQ}}, \mathbf{FCln}\rangle$-algebraic institution $\mathcal{I}_{\langle \Omega\mathbf{Set}_{\mathbf{T}_{\mathrm{EQ}}}, \Xi_{\mathrm{EQ}}, \mathbf{FCln}\rangle}$. As was pointed out in Section 3, $\mathbf{EQSig} = \Omega\mathbf{Set}_{\mathbf{T}_{\mathrm{EQ}}}$ and therefore the following translations $\langle F, \alpha\rangle : \mathcal{EQ} \to \mathcal{EA}$ and $\langle G, \beta\rangle : \mathcal{EA} \to \mathcal{EQ}$ may be legitimately defined: $F = I_{\mathbf{EQSig}}$ and $\alpha : \mathrm{EQSEN} \to \mathcal{P}\mathrm{EQ}_{\mathrm{EA}}$ is given, for all $X \in |\mathbf{EQSig}|$, by $\alpha_X : \mathrm{EQSEN}(X) \to \mathcal{P}(\mathrm{EQ}_{\mathrm{EA}}(X))$, with

$$\alpha_X(s \approx t) = \{s \approx t\}, \quad \text{for all } s \approx t \in \left(\bigcup_{k=1}^{\infty} \mathrm{Tm}_X(V)_k\right)^2.$$

Similarly, $G = I_{\Omega\mathbf{Set}_{\mathbf{T}_{\mathrm{EQ}}}}$ and $\beta : \mathrm{EQ}_{\mathrm{EA}} \to \mathscr{P}\mathrm{EQSEN}$ is given, for all $X \in |\Omega\mathbf{Set}_{\mathbf{T}_{\mathrm{EQ}}}|$, by $\beta_X : \mathrm{EQ}_{\mathrm{EA}}(X) \to \mathscr{P}(\mathrm{EQSEN}(X))$, with

$$\beta_X(s \approx t) = \{s \approx t\}, \quad \text{for all } s \approx t \in \left(\bigcup_{k=1}^{\infty} \mathrm{Tm}_X(V)_k\right)^2.$$

It is trivial to check that α and β are natural transformations. It is also trivial to check that the invertibility conditions

$$\beta_X(\alpha_X(s \approx t))^{c_{\mathcal{EQ}}} = \{s \approx t\}^{c_{\mathcal{EQ}}} \quad \text{and} \quad \alpha_X(\beta_X(s \approx t))^{c_{\mathcal{EA}}} = \{s \approx t\}^{c_{\mathcal{EA}}}$$

are satisfied, for all $X \in |\mathbf{EQSig}|$, $s \approx t \in (\bigcup_{k=1}^{\infty} \mathrm{Tm}_X(V)_k)^2$, since

$$\beta_X(\alpha_X(s \approx t)) = \alpha_X(\beta_X(s \approx t)) = \{s \approx t\}.$$

However, it is not trivial to show that α and β are interpretations, i.e., that, for all $X \in |\mathbf{EQSig}|$, $E \cup \{s \approx t\} \subseteq (\bigcup_{k=1}^{\infty} \mathrm{Tm}_X(V)_k)^2$,

$$s \approx t \in E^{c_{\mathcal{EQ}}} \quad \text{iff} \quad s \approx t \in E^{c_{\mathcal{EA}}}.$$

We sketch the steps of this proof below. Details will be provided in [23].

Let $X \in |\mathbf{EQSig}|$, $k \geq 1$, $t \in \mathrm{Tm}_X(V)_k$. Then, for all X-algebras $\mathbf{A} = \langle A, X^{\mathbf{A}}\rangle \in |\mathrm{EQMOD}(X)|$,

$$t^{\mathbf{A}} = \xi_{A_k}((\eta_{\mathrm{Cl}(A)} X^{\mathbf{A}})^*_k(t)). \tag{4}$$

This situation is depicted as follows:

$$X \xrightarrow[X^{\mathbf{A}}]{} \mathrm{Cl}(A) \xrightarrow[\eta_{\mathrm{Cl}(A)}]{} \mathrm{Tm}_{\mathrm{Cl}(A)}(V),$$

$$\mathrm{Tm}_X(V) \xrightarrow[(\eta_{\mathrm{Cl}(A)} X^{\mathbf{A}})^*]{} \mathrm{Tm}_{\mathrm{Cl}(A)}(V) \xrightarrow[\xi_A]{} \mathrm{Cl}(A).$$

Similarly, for all $\mathbf{T}_{\mathrm{EQ}}$-algebras $\langle \mathrm{Cl}(A), \xi_A\rangle \in |\mathbf{FCln}|$, and all $f : X \rightharpoonup \mathrm{Cl}(A)$,

$$t^{\mathbf{A}} = \xi_{A_k}(f^*_k(t)), \tag{5}$$

which is illustrated by

$$\mathrm{Tm}_X(V) \xrightarrow[f^*]{} \mathrm{Tm}_{\mathrm{Cl}(A)}(V) \xrightarrow[\xi_A]{} \mathrm{Cl}(A).$$

Suppose, now, that for $E \cup \{s \approx t\} \subseteq (\bigcup_{k=1}^{\infty} \mathrm{Tm}_X(V)_k)^2$, $s \approx t \in E^{c_{\mathcal{EQ}}}$. We need to show that $s \approx t \in E^{c_{\mathcal{EA}}}$. Since $s \approx t \in E^{c_{\mathcal{EQ}}}$, for all $\mathbf{A} = \langle A, X^{\mathbf{A}}\rangle \in |\mathrm{EQMOD}(X)|$, $\vec{a} \in A^{\omega}$, $E^{\mathbf{A}}(\vec{a})$ implies $s^{\mathbf{A}}(\vec{a}) = t^{\mathbf{A}}(\vec{a})$. But, then, if $\langle\langle A, \xi\rangle, f\rangle \in |\mathrm{EAMOD}(X)|$, such that $\xi(f^*(e_0)) = \xi(f^*(e_1))$, for all $e_0 \approx e_1 \in E$, we have, by (5), that $E^{\mathbf{A}}$ holds, whence $s^{\mathbf{A}} = t^{\mathbf{A}}$ and, by following the reverse reasoning, $\xi(f^*(s)) = \xi(f^*(t))$, i.e., $s \approx t \in E^{c_{\mathcal{EA}}}$, as was to be shown. The converse follows along the same lines except that (4) is used instead of (5).

Thus $\pi(\mathcal{EQ})$ and $\mathcal{EA}$ are equivalent π-institutions and, since $\mathcal{EA}$ is an algebraic π-institution, $\pi(\mathcal{EQ})$ is an algebraizable π-institution. Thus the institution of equational logic $\mathcal{EQ}$ is algebraizable.

5.3. FIRST-ORDER LOGIC WITHOUT TERMS

Recall from Section 2 the definition of the institution $\mathcal{FOL} = \langle \mathbf{FOSig}, \mathrm{FOSEN}, \mathrm{FOMOD}, \models\rangle$ that represents the system of first-order logic without terms over multiple (relational) signatures. Recall also from Section 4 the definition of the $\langle \mathbf{HSet}_{\mathbf{T}_{\mathrm{FO}}}, \Xi_{\mathrm{FO}}, \mathbf{FRln}\rangle$-algebraic π-institution $\mathcal{FOA} = \langle \mathbf{HSet}_{\mathbf{T}_{\mathrm{FO}}}, \mathrm{EQ}_{\mathrm{FA}}, \{C_{\mathrm{FA}_X}\}_{X\in|\mathbf{HSet}_{\mathbf{T}_{\mathrm{FO}}}|}\rangle$ corresponding to the $\langle \mathbf{HSet}_{\mathbf{T}_{\mathrm{FO}}}, \Xi_{\mathrm{FA}}, \mathbf{FRln}\rangle$-algebraic institution $\mathcal{I}_{\langle \mathbf{HSet}_{\mathbf{T}_{\mathrm{FO}}}, \Xi_{\mathrm{FO}}, \mathbf{FRln}\rangle}$.

Given $X \in |\mathbf{FOSign}|$, $N \subseteq_{\mathrm{f}} \omega$ and $\phi, \psi \in \mathrm{Fm}_{\mathcal{L}}(X)_N$, define

$$T(\phi) = \neg(\phi \wedge \neg\phi), \quad \phi \to \psi = \neg(\phi \wedge \neg\psi) \quad \text{and}$$
$$\phi \leftrightarrow \psi = (\phi \to \psi) \wedge (\psi \to \phi).$$

Then, since as was pointed out in Section 3, $\mathbf{FOSig} = \mathbf{HSet}_{\mathbf{T}_{\mathrm{FO}}}$, the following translations $\langle F, \alpha\rangle : \mathcal{FOL} \to \mathcal{FOA}$ and $\langle G, \beta\rangle : \mathcal{FOA} \to \mathcal{FOL}$ may be legitimately defined: $F = I_{\mathbf{FOSig}}$ and $\alpha : \mathrm{FOSEN} \to \mathcal{P}\mathrm{EQ}_{\mathrm{FA}}$ is given, for all $X \in |\mathbf{FOSig}|$, by $\alpha_X : \mathrm{FOSEN}(X) \to \mathcal{P}(\mathrm{EQ}_{\mathrm{FA}}(X))$, with

$$\alpha_X(\phi) = \{\phi \approx T(\phi)\}, \quad \text{for all } \phi \in \mathrm{Fm}_{\mathcal{L}}(X)_{\emptyset}.$$

Similarly, $G = I_{\mathbf{HSet}_{\mathbf{T}_{\mathrm{FO}}}}$ and $\beta : \mathrm{EQ}_{\mathrm{FA}} \to \mathcal{P}\mathrm{FOSEN}$ is given, for all $X \in |\mathbf{HSet}_{\mathbf{T}_{\mathrm{FO}}}|$, by $\beta_X : \mathrm{EQ}_{\mathrm{FA}}(X) \to \mathcal{P}(\mathrm{FOSEN}(X))$, with

$$\beta_X(\phi \approx \psi) = \{\phi \leftrightarrow \psi\}, \quad \text{for all } \phi \approx \psi \in \mathrm{Tm}_X(V)^2_{\emptyset}.$$

It is trivial to check that α and β are natural transformations. According to [22] we only need to show that α is an interpretation, i.e., that, for all $X \in |\mathbf{HSet}|$, $\Phi \cup \{\phi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(X)_{\emptyset}$,

$$\phi \in \Phi^{C\mathcal{FOL}} \quad \text{iff} \quad \phi \approx T(\phi) \in \{\psi \approx T(\psi) : \psi \in \Phi\}^{C\mathcal{FOA}} \tag{6}$$

and that the second of the invertibility conditions holds, i.e., that, for all $X \in |\mathbf{HSet}|$, $\phi, \psi \in \mathrm{Fm}_{\mathcal{L}}(X)_{\emptyset}$,

$$\{\phi \approx \psi\}^{C\mathcal{FOA}} = \{\phi \leftrightarrow \psi \approx T(\phi \leftrightarrow \psi)\}^{C\mathcal{FOA}}. \tag{7}$$

The proofs are based on the following fact, which, together with other details, will be elaborated on further in [24]: Let $X \in |\mathbf{HSet}|$, $N \subseteq_{\mathrm{f}} \omega$, $\phi \in \mathrm{Fm}_{\mathcal{L}}(X)_N$. Then, for all $\langle \mathrm{Rel}(A), \xi_A\rangle \in |\mathbf{FRln}|$ and $f : X \rightharpoonup \mathrm{Rel}(A)$, the X-structure $\mathbf{A} = \langle \mathrm{Rel}(A), \xi_A f\rangle$ satisfies

$$\phi^{\mathbf{A}} = \xi_{A_N}(f^*_N(\phi)). \tag{8}$$

This situation pictorially is as follows:

$$\mathrm{Fm}_{\mathcal{L}}(X) \xrightarrow[f^*]{} \mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A)) \xrightarrow[\xi_A]{} \mathrm{Rel}(A).$$

To prove (6), let $X \in |\mathbf{HSet}|$, $\Phi \cup \{\phi\} \subseteq \mathrm{FOSEN}(X)$. If $\phi \in \Phi^{c\mathcal{FOL}}$, then, for all $\mathbf{A} \in |\mathrm{FOMOD}(X)|$,

$$\mathbf{A} \models_X \Phi \quad \text{implies} \quad \mathbf{A} \models_X \phi.$$

Suppose that $\langle \mathrm{Rel}(A), \xi_A \rangle \in |\mathbf{FRln}|$ and $f : X \rightharpoonup \mathrm{Rel}(A)$ are such that $\xi_A(f^*(\psi)) = \xi_A(f^*(T(\psi)))$, for all $\psi \in \Phi$. Then, by (8), if $\mathbf{A} = \langle \mathrm{Rel}(A), \xi_A f \rangle$, $\psi^{\mathbf{A}} = T(\psi)^{\mathbf{A}}$, for all $\psi \in \Phi$. Therefore $\mathbf{A} \models_X \psi$ for all $\psi \in \Phi$, and, hence, $\mathbf{A} \models_X \phi$. Reversing the steps above then yields $\xi_A(f^*(\phi)) = \xi_A(f^*(T(\phi)))$, i.e., $\phi \approx T(\phi) \in \{\psi \approx T(\psi) : \psi \in \Phi\}^{c\mathcal{FOA}}$, as required. The reverse implication may be proved similarly.

For (7) we have, for all $\langle\langle \mathrm{Rel}(A), \xi_A \rangle, f\rangle \in |\mathrm{FAMOD}(X)|$,

$$\begin{aligned}
&\langle\langle \mathrm{Rel}(A), \xi_A \rangle, f\rangle \models_X \phi \leftrightarrow \psi \approx T(\phi \leftrightarrow \psi)\\
&\quad \text{iff} \quad \xi_A(f^*(\phi \leftrightarrow \psi)) = \xi_A(f^*(T(\phi \leftrightarrow \psi)))\\
&\quad \text{iff} \quad (\phi \leftrightarrow \psi)^{\mathbf{A}} = T(\phi \leftrightarrow \psi)^{\mathbf{A}} \quad \text{(by (8))}\\
&\quad \text{iff} \quad \phi^{\mathbf{A}} = \psi^{\mathbf{A}} \quad (\text{since } T(\phi \leftrightarrow \psi)^{\mathbf{A}} = A^{\omega})\\
&\quad \text{iff} \quad \xi_A(f^*(\phi)) = \xi_A(f^*(\psi)) \quad \text{(by (8))}\\
&\quad \text{iff} \quad \langle\langle \mathrm{Rel}(A), \xi_A \rangle, f\rangle \models_X \phi \approx \psi.
\end{aligned}$$

Thus $\pi(\mathcal{FOL})$ and $\mathcal{FOA}$ are equivalent π-institutions and, since $\mathcal{FOA}$ is an algebraic π-institution, $\pi(\mathcal{FOL})$ is an algebraizable π-institution. Thus the institution of first-order logic $\mathcal{FOL}$ is algebraizable.

5.4. EQUATIONAL CATEGORICAL LOGIC

Recall from Section 2 the definition of the institution $\mathcal{CL} = \langle \mathbf{CSig}, \mathrm{CSEN}, \mathrm{CMOD}, \models\rangle$ that represents the system of categorical equational logic, i.e., the logic that governs derivations of arrow equalities from other valid arrow equalities in categories. Recall also from Section 4 the definition of the $\langle \mathbf{Gph}_{\mathbf{T}_{\mathrm{CT}}}, \Xi_{\mathrm{CT}}, \mathbf{ACat}\rangle$-algebraic π-institution $\mathcal{CA} = \langle \mathbf{Gph}_{\mathbf{T}_{\mathrm{CT}}}, \mathrm{EQ}_{\mathrm{CA}}, \{C_{\mathrm{CA}_G}\}_{G \in |\mathbf{Gph}_{\mathbf{T}_{\mathrm{CT}}}|}\rangle$ corresponding to the $\langle \mathbf{Gph}_{\mathbf{T}_{\mathrm{CT}}}, \Xi_{\mathrm{CT}}, \mathbf{ACat}\rangle$-algebraic institution $\mathcal{I}_{\langle \mathbf{Gph}_{\mathbf{T}_{\mathrm{CT}}}, \Xi_{\mathrm{CT}}, \mathbf{ACat}\rangle}$.

Since, as was pointed out in Section 3, $\mathbf{CSig} = \mathbf{Gph}_{\mathbf{T}_{\mathrm{CT}}}$, the following translations $\langle F, \alpha\rangle : \mathcal{CL} \to \mathcal{CA}$ and $\langle G, \beta\rangle : \mathcal{CA} \to \mathcal{CL}$ may be legitimately defined: $F = I_{\mathbf{CSig}}$ and $\alpha : \mathrm{CSEN} \to \mathcal{P}\mathrm{EQ}_{\mathrm{CA}}$ is given, for all $G \in |\mathbf{CSig}|$, by $\alpha_G : \mathrm{CSEN}(G) \to \mathcal{P}(\mathrm{EQ}_{\mathrm{CA}}(G))$, with

$$\alpha_X(\langle (e_0, \ldots, e_{n-1}), (f_0, \ldots, f_{m-1})\rangle) = \{(e_0, \ldots, e_{n-1}) \approx (f_0, \ldots, f_{m-1})\},$$

for all edges $(e_0, \ldots, e_{n-1}), (f_0, \ldots, f_{m-1})$ of $\mathrm{Pth}(G)$. Similarly, $G = I_{\mathbf{Gph}_{\mathbf{T}_{\mathrm{CT}}}}$ and $\beta : \mathrm{EQ}_{\mathrm{CA}} \to \mathcal{P}\mathrm{CSEN}$ is given, for all $G \in |\mathbf{Gph}_{\mathbf{T}_{\mathrm{CT}}}|$, by $\beta_G : \mathrm{EQ}_{\mathrm{CA}}(G) \to \mathcal{P}(\mathrm{CSEN}(G))$, with

$$\beta_G(p \approx q) = \{\langle p, q\rangle\}, \quad \text{for all } p \approx q \in E(\mathrm{Pth}(G))^2.$$

It is trivial to check that α and β are natural transformations. According to [22] we only need to show that α is an interpretation, i.e., that, for all $G \in |\mathbf{Gph}|$, $P \cup \{\langle (e_0, \ldots, e_{n-1}), (f_0, \ldots, f_{m-1})\rangle\} \subseteq \mathrm{Mor}(\mathbf{Pth}(\mathrm{Pth}(G))^2$,

$$\begin{aligned} &\langle (e_0, \ldots, e_{n-1}), (f_0, \ldots, f_{m-1})\rangle \in P^{c\mathcal{C}\mathcal{L}} \quad \text{iff} \\ &(e_0, \ldots, e_{n-1}) \approx (f_0, \ldots, f_{m-1}) \in \{(p_0, \ldots, p_{k-1}) \approx (q_0, \ldots, q_{l-1}) : \\ &\langle (p_0, \ldots, p_{k-1}), (q_0, \ldots, q_{l-1})\rangle \in P\}^{c\mathcal{C}\mathcal{A}} \end{aligned} \tag{9}$$

and that the second of the invertibility conditions holds, i.e., that, for all $X \in |\mathbf{Gph}|$, $p, q \in E(\mathrm{Pth}(G))$,

$$\{p \approx q\}^{c\mathcal{C}\mathcal{A}} = \{p \approx q\}^{c\mathcal{C}\mathcal{A}}. \tag{10}$$

The last condition is trivial whereas the proof of (9) is based on the following fact: Let $G \in |\mathbf{Gph}|$, $\mathbf{C}$ a small category, with C its underlying graph, and $g : G \rightharpoondown C \in \mathrm{Mor}(\mathbf{CSig})$. Then, for all paths $(e_0, \ldots, e_{n-1})$ in G,

$$g^\dagger(e_0, \ldots, e_{n-1}) = (\xi_{\mathbf{C}}\mu_{\mathrm{CT}_C}T_{\mathrm{CT}}(g))^*(e_0, \ldots, e_{n-1}). \tag{11}$$

This situation is pictorially as follows:

$$\mathrm{Pth}(G) \xrightarrow[T_{\mathrm{CT}}(g)]{} \mathrm{Pth}(\mathrm{Pth}(C)) \xrightarrow[\mu_{\mathrm{CT}_C}]{} \mathrm{Pth}(C) \xrightarrow[\xi_{\mathbf{C}}]{} C.$$

To prove (9), let $G \in |\mathbf{Gph}|$, $P \cup \{\langle (e_0, \ldots, e_{n-1}), (f_0, \ldots, f_{m-1})\rangle\} \subseteq \mathrm{CSEN}(X)$. If $\langle (e_0, \ldots, e_{n-1}), (f_0, \ldots, f_{m-1})\rangle \in P^{c\mathcal{C}\mathcal{L}}$, then, for all $\langle \mathbf{C}, g\rangle \in |\mathrm{CMOD}(X)|$,

$$\langle \mathbf{C}, g\rangle \models_G P \quad \text{implies} \quad \langle \mathbf{C}, g\rangle \models_G \langle (e_0, \ldots, e_{n-1}), (f_0, \ldots, f_{m-1})\rangle.$$

Suppose that $\langle C, \xi_{\mathbf{C}}\rangle \in |\mathbf{ACat}|$ and $g : G \rightharpoondown C$ are such that

$$(\xi_{\mathbf{C}}\mu_{\mathrm{CT}_C}T_{\mathrm{CT}}(g))^*(p) = (\xi_{\mathbf{C}}\mu_{\mathrm{CT}_C}T_{\mathrm{CT}}(g))^*(q), \quad \text{for all } p, q \in P.$$

Then, by (11), $g^\dagger(p) = g^\dagger(q)$, whence $g^\dagger(e_0, \ldots, e_{n-1}) = g^\dagger(f_0, \ldots, f_{m-1})$. Reversing the steps above then yields

$$(\xi_{\mathbf{C}}\mu_{\mathrm{CT}_C}T_{\mathrm{CT}}(g))^*(e_0, \ldots, e_{n-1}) = (\xi_{\mathbf{C}}\mu_{\mathrm{CT}_C}T_{\mathrm{CT}}(g))^*(f_0, \ldots, f_{m-1}),$$

i.e., $(e_0, \ldots, e_{n-1}) \approx (f_0, \ldots, f_{m-1}) \in P^{c\mathcal{C}\mathcal{A}}$, as required. The reverse implication may be proved similarly.

Thus $\pi(\mathcal{C}\mathcal{L})$ and $\mathcal{C}\mathcal{A}$ are equivalent π-institutions and, since $\mathcal{C}\mathcal{A}$ is an algebraic π-institution, $\pi(\mathcal{C}\mathcal{L})$ is an algebraizable π-institution. Thus the institution of the equational logic of categories $\mathcal{C}\mathcal{L}$ is algebraizable.

An interesting open problem concerning the algebraizability of the institution $\mathcal{C}\mathcal{L}$ of the equational logic of categories is whether it is algebraizable via an algebraic institution based on an algebraic theory in the category of sets. This question naturally arises from the well-known facts that categories are models (Eilenberg–Moore algebras) of an algebraic theory in the category of graphs but not of an algebraic theory in the category of sets.

Acknowledgements

I am deeply in debt to Don Pigozzi, who has been watching and encouraging my every step. The moral support of the following people over the last few years is also greatly appreciated: Cliff Bergman, Roger Maddux, Jonathan Smith and Sung Song of the Mathematics and Giora Slutzki, Gary Leavens and Jack Lutz of the Computer Science at Iowa State and my post-doc advisor Charles Wells at Case Western Reserve.

Finally, thanks to the anonymous referee for several useful comments.

References

1. Barbour, G. and Raftery, J.: Unifying algebraizability and full congruence regularity (Abstract), *Workshop on Abstract Algebraic Logic*, Centre de Recerca Matemàtica, Quaderns, Núm. 10/Gener, 1998.
2. Barr, M. and Wells, C.: *Toposes, Triples and Theories*, Springer-Verlag, New York, 1985.
3. Blok, W. J. and Pigozzi, D.: Protoalgebraic logics, *Studia Logica* **45** (1986), 337–369.
4. Blok, W. J. and Pigozzi, D.: Algebraizable logics, *Mem. Amer. Math. Soc.* **77**(396) (1989).
5. Blok, W. J. and Pigozzi, D.: Algebraic semantics for universal Horn logic without equality, in A. Romanowska and J. D. H. Smith (eds.), *Universal Algebra and Quasigroup Theory*, Heldermann Verlag, Berlin, 1992.
6. Blok, W. J. and Pigozzi, D.: Abstract algebraic logic and the deduction theorem, *Bull. Symbolic Logic*, to appear.
7. Cirulis, J.: An algebraization of first-order logic with terms, in H. Andréka, J. D. Monk and I. Németi (eds.), *Algebraic Logic* (Proc. Conf. Budapest, 1988), Colloq. Math. Soc. J. Bolyai, North-Holland, Amsterdam, 1991, pp. 125–146.
8. Czelakowski, J.: Equivalential logics I, II, *Studia Logica* **40** (1981), 227–236, 355–372.
9. Czelakowski, J. and Pigozzi, D.: Amalgamation and interpolation in abstract algebraic logic. Models, algebras and proofs, in *Lecture Notes in Pure Appl. Math.* **203**, Dekker, New York, 1999, pp. 187–265.
10. Feldman, N.: Axiomatization of polynomial substitution algebras, *J. Symbolic Logic* **47** (1982), 481–492.
11. Fiadeiro, J. and Sernadas, A.: Structuring theories on consequence, in D. Sannella and A. Tarlecki (eds.), *Recent Trends in Data Type Specification*, Lecture Notes in Comput. Sci. 332, Springer-Verlag, New York, 1988, pp. 44–72.
12. Font, J. M. and Jansana, R.: *A General Algebraic Semantics for Sentential Logics*, Lecture Notes in Logic 7, Springer-Verlag, Berlin, 1996.
13. Goguen, J. A. and Burstall, R. M.: Institutions: Abstract model theory for specification and programming, *J. Assoc. Comput. Mach.* **39**(1) (1992), 95–146.
14. Henkin, L., Monk, J. D. and Tarski, A.: *Cylindric Algebras Part I,* North-Holland, Amsterdam, 1971.
15. Herrmann, B.: Equivalential logics and definability of truth, Dissertation, Freie Universität Berlin, Berlin, 1993.
16. Herrmann, B.: Equivalential and algebraizable logics, *Studia Logica* **57** (1996), 419–436.
17. Herrmann, B.: Characterizing equivalential and algebraizable logics by the Leibniz operator, *Studia Logica* **58** (1997), 305–323.
18. Mac Lane, S.: *Category Theory for the Working Mathematician*, Springer-Verlag, New York, 1971.
19. Manes, E. G.: *Algebraic Theories*, Springer-Verlag, New York, 1976.

20. Pynko, A. P.: Definitional equivalence and algebraizability of generalized logical systems, *Ann. Pure Appl. Logic* **98** (1999), 1–68.
21. Voutsadakis, G.: Categorical abstract algebraic logic, Doctoral Dissertation, Iowa State University, Ames, IA, 1998.
22. Voutsadakis, G.: Categorical abstract algebraic logic: Equivalent institutions, *Studia Logica*, to appear.
23. Voutsadakis, G.: Algebraizing the equational institution, Preprint.
24. Voutsadakis, G.: Algebraizing the institution of first-order logic without terms, Preprint.