CATEGORICAL ABSTRACT ALGEBRAIC LOGIC: OPERATORS ON CLASSES OF STRUCTURE SYSTEMS

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Abstract. The study of structure systems, an abstraction of the concept of first-order structures, is continued. Structure systems have algebraic systems, rather than universal algebras, as their algebraic reducts. Moreover, their relational component consists of a collection of relation systems on the underlying functors, rather than simply a system of relations on a single set. A variety of operators on classes of structure systems are introduced and studied, taking after similar work of Elgueta in the context of the model theory of equality-free first-order logic. Both Elgueta's and the present work are inspired by considerations arising in the study of the process of algebraization in abstract algebraic logic. The ways that these various class operators interact, when composed with one-another, are at the focus of current investigations.

1 Introduction The central role that classes of logical matrices play in the theory of abstract algebraic logic (see, e.g., [8]) together with the fact that logical matrices may be viewed as models of universal Horn logic without equality [3] (see also [8]) provided the motivation for the study of the model theory of equality-free first-order structures by Dellunde, Elgueta and their collaborators (see [9, 13, 14, 15, 16, 17] for Elgueta's work, some of which is joint with Czelakowski and some with Jansana, and [6, 10, 11, 12] for Dellunde's work, some of which is joint with Casanovas and Jansana). The idea was that equality-free first-order model theory, which, as contrasted with its counterpart with equality, was not as well studied, may benefit from results inspired by its interaction with abstract algebraic logic and that, conversely, some novel ideas in that theory may prove useful in the domain of the algebraization of sentential logics and the theory of logical matrices.

In recent work by the author [23, 24, 25] the theory of algebraizability of sentential logics has been abstracted to cover those logical systems that are formalized as π-institutions. The class of all these systems is wider than that of sentential logics since it includes logics with multiple signatures and quantifiers. Moreover, the π-institution presentation is, in some ways, more attractive from the metalogical point of view since it allows the treatment of substitutions in the object language rather than delegating them to the metalanguage. On the other hand, one has to pay the price that the added generality restricts, to a certain extent, both the quantity and the depth of the results obtained, since these apply now to a wider variety of logical systems. Ongoing investigations, however, allow optimism that the amount of results that one is still able to obtain is worth the effort and, also, that some of these results may prove fruitful in reconsidering or adding to the existing knowledge pertaining to the theory as applied specifically to sentential logics.

The concept of a logical matrix, when lifted to the π-institution framework, gives rise to that of a matrix system [26]. The concept of an abstract logic, which was used extensively by Font and Jansana in [18] as an alternative algebraic model for sentential logics, more

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faithfully respecting the properties of a logic in the algebraic domain than logical matrix models do, gives rise to that of a π-institution model of a given π-institution [24]. These analogies, together with that established between universal algebras and algebraic systems as algebraic models of sentential logics and of π-institutions, respectively, in [25], lead to the consideration of abstracting the model theory of equality-free first-order logic to structure systems. These are systems abstracting first-order structures and having algebraic systems, rather than algebras, as their algebraic components. The aim of this abstraction process is to obtain results paralleling those of Dellunde's and Elgueta's and thus, cross-fertilizing categorical abstract algebraic logic with the theory of equality-free first-order structure systems. This idea is further supported by some recent results providing an analog of Bloom's Theorem for the framework of π-institutions [29] and by the preceding two papers in this series [30, 31] that introduce structure systems and deal with aspects of their basic theory, inspired by Dellunde's and Elgueta's work.

In [30] both a syntactic and a semantic framework for the study of structure systems has been introduced. Since in the present work the focus is entirely on semantics, only the semantical aspects of [30] will be briefly reviewed in this introduction. A language type $\mathcal{L} = \langle \mathbf{F}, R, \rho \rangle$ consists of a category $\mathbf{F}$ of natural transformations on a sentence functor, a nonempty set R of relation symbols and an arity function $\rho : R \to \omega$. An $\mathcal{L}$-structure system $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$ consists of a sentence functor $\mathrm{SEN}^{\mathfrak{A}} : \mathbf{Sign}^{\mathfrak{A}} \to \mathbf{Set}$, a category $\mathbf{N}^{\mathfrak{A}}$ of natural transformations on $\mathrm{SEN}^{\mathfrak{A}}$, a surjective functor $F^{\mathfrak{A}} : \mathbf{F} \to \mathbf{N}^{\mathfrak{A}}$ preserving all projections and a set of relation systems on $\mathrm{SEN}^{\mathfrak{A}}$ of arities equal to the arities assigned to the relation symbols by ρ. Given two $\mathcal{L}$-structure systems $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$ and $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}} \rangle, R^{\mathfrak{B}} \rangle$, an $\mathcal{L}$-morphism $\langle F, \alpha \rangle : \mathfrak{A} \to \mathfrak{B}$ is an $(\mathbf{N}^{\mathfrak{A}}, \mathbf{N}^{\mathfrak{B}})$-epimorphic translation $\langle F, \alpha \rangle : \mathrm{SEN}^{\mathfrak{A}} \to^{se} \mathrm{SEN}^{\mathfrak{B}}$, such that $F^{\mathfrak{A}}(\sigma)$ and $F^{\mathfrak{B}}(\sigma)$ correspond under the $(\mathbf{N}^{\mathfrak{A}}, \mathbf{N}^{\mathfrak{B}})$-epimorphic property and such that, for all $r \in R$, with $\rho(r) = n$, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and all $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$,

$$\vec{\phi} \in r^{\mathfrak{A}}_{\Sigma} \quad \text{implies} \quad \alpha_{\Sigma}(\vec{\phi}) \in r^{\mathfrak{B}}_{F(\Sigma)}.$$

Using these basic definitions, the notions of a subsystem, a homomorphic image of an $\mathcal{L}$-system and those of a product, a reduced product and an ultraproduct are all rigorously defined in [30]. All of these definitions will become handy in the semantic investigations that are undertaken in the present work. The reader is therefore advised to revisit [30] and become familiar with them.

On the other hand, [31] introduces the notion of Leibniz equality for $\mathcal{L}$-systems, an abstraction of the Leibniz equality of [13], a weak version of equality that replaces genuine equality in the study of models of equality-free first-order logic. (See [2] for the origin of this notion.) That notion, in turn, is inspired by the role that Leibniz congruences play in the study of logical matrices and the relation, described above, between logical matrices and universal Horn logic without equality. Leibniz equality and the more general notion of a congruence system of an $\mathcal{L}$-system lead to a detailed study of quotients of $\mathcal{L}$-systems by congruence systems in [31] and to the formulation and proof of analogs of the Homomorphism, the Second Isomorphism and the Correspondence Theorems of Universal Algebra in the context of $\mathcal{L}$-systems.

In the present work the study of $\mathcal{L}$-systems, started in [30, 31] is continued. Class operators corresponding to the well-known operators of taking substructures, homomorphic images, reduced products, products, ultraproducts, subdirect products as well as filter extensions, reductions and expansions of first-order structures, are introduced for classes of structure systems. Several properties pertaining to the way these operators interact with one another when composed are investigated. These parallel corresponding properties introduced in [13] by Elgueta, but some are weaker in the present context, despite the fact that they all generalize the properties of [13]. The goal of this work, which is to be continued with additional results elsewhere, is to characterize classes of structure systems that are models of particular equality-free first-order theories in the spirit of Dellunde and Elgueta. These explorations will hopefully shed more light in the relationships between analogous concepts arising in the frameworks of the algebraizability of sentential logics and the algebraizability of π-institutions. Algebraizability remains, as before, the main framework in which our investigations take place.

For general concepts and notation from category theory the reader is referred to any of [1, 4, 21]. For an overview of the current state of affairs in abstract algebraic logic the reader is referred to the review article [19], the monograph [18] and the book [8]. To follow recent developments on the categorical side of the subject the reader may refer to the series of papers [23]-[26]. Finally, standard references on model theory are the books by Chang and Keisler [7], Hodges [20] and Marker [22].

2 Operators on Classes of Structure Systems The following analogs of the operators $S, S_e, F, H, R, E, P, P_f, P_u$ and P_{sd} on classes of $\mathcal{L}$-structures, introduced in [13], will be considered in the present work: S is the operator of taking isomorphic copies of subsystems of a class of $\mathcal{L}$-structure systems:

$$\mathrm{S}(\mathtt{K}) = \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{C} \text{ and } \mathfrak{C} \subseteq \mathfrak{B} \text{ for some } \mathfrak{B} \in \mathtt{K}\}.$$

$\mathrm{S_e}$ is the operator of taking isomorphic copies of elementary subsystems of a class of $\mathcal{L}$-systems:

$$\mathrm{S_e}(\mathtt{K}) = \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{C} \text{ and } \mathfrak{C} \subseteq_e \mathfrak{B} \text{ for some } \mathfrak{B} \in \mathtt{K}\}.$$

F is the operator of taking isomorphic copies of filter extensions of a class of $\mathcal{L}$-systems:

$$\mathrm{F}(\mathtt{K}) = \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{C} \text{ and } \mathfrak{B} \sqsubseteq \mathfrak{C} \text{ for some } \mathfrak{B} \in \mathtt{K}\}.$$

H is the operator of taking isomorphic copies of $\mathcal{L}$-morphic images of a class of $\mathcal{L}$-systems;

$$\mathrm{H}(\mathtt{K}) = \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{C} \text{ and } h : \mathfrak{B} \twoheadrightarrow \mathfrak{C} \text{ for some } \mathfrak{B} \in \mathtt{K} \text{ and some } h\}.$$

R is the operator of taking isomorphic copies of reductions of a class of $\mathcal{L}$-systems:

$$\mathrm{R}(\mathtt{K}) = \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{C} \text{ and } h : \mathfrak{B} \twoheadrightarrow_s \mathfrak{C} \text{ for some } \mathfrak{B} \in \mathtt{K} \text{ and some } h\}.$$

E is the operator of taking isomorphic copies of expansions of a class of $\mathcal{L}$-systems:

$$\mathrm{E}(\mathtt{K}) = \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{C} \text{ and } h : \mathfrak{C} \twoheadrightarrow_s \mathfrak{B} \text{ for some } \mathfrak{B} \in \mathtt{K} \text{ and some } h\}.$$

We also use the operators $\mathrm{S_i}$, and $\mathrm{R_i}$ and $\mathrm{E_i}$ to denote simple subsystems of a class of $\mathcal{L}$-systems, i.e., subsystems on identical sentence functors, and reductions and expansions, respectively, of a class of $\mathcal{L}$-systems via reductive $\mathcal{L}$-morphisms with isomorphic functor components.

P is the operator of taking isomorphic copies of direct products of a class of $\mathcal{L}$-systems:

$$\mathrm{P}(\mathtt{K}) = \{\mathfrak{A} : \mathfrak{A} \cong \prod_{i \in I} \mathfrak{A}_i \text{ and } \mathfrak{A}_i \in \mathtt{K}, \text{ for all } i \in I\}.$$

$\mathrm{P_f}$ is the operator of taking isomorphic copies of reduced products via proper filters of a class of $\mathcal{L}$-systems:

$$\mathrm{P_f}(\mathtt{K}) = \{\mathfrak{A} : \mathfrak{A} \cong \prod_{i \in I} \mathfrak{A}_i/\mathcal{F}, \ \mathfrak{A}_i \in \mathtt{K}, \text{ for all } i \in I, \text{ and } \mathcal{F} \text{ is a proper filter on } I\}.$$

$\mathrm{P_u}$ is the operator of taking isomorphic copies of ultraproducts of a class of $\mathcal{L}$-systems:

$$\mathrm{P_u}(\mathtt{K}) = \{\mathfrak{A} : \mathfrak{A} \cong \prod_{i\in I} \mathfrak{A}_i/\mathcal{U},\ \mathfrak{A}_i \in \mathtt{K},\ \text{for all } i \in I,\ \text{and } \mathcal{U} \text{ is an ultrafilter on } I\}.$$

Finally, $\mathrm{P_{sd}}$ is the operator of taking isomorphic copies of subdirect products of a class of $\mathcal{L}$-systems:

$$\mathrm{P_{sd}}(\mathtt{K}) = \{\mathfrak{A} : h : \mathfrak{A} \rightarrowtail_{\mathrm{sd}} \prod_{i\in I} \mathfrak{A}_i,\ \text{for some } h,\ \mathfrak{A}_i \in \mathtt{K},\ \text{for all } i \in I\}.$$

For any two O, O' of these operators, we follow [13] in denoting their composition by OO', the property $\mathrm{O}(\mathtt{K}) \subseteq \mathrm{O}'(\mathtt{K})$ for every class $\mathtt{K}$, by $\mathrm{O} \leq \mathrm{O}'$ and the operator LO by O^*. Also following [13], we write $\overline{\mathrm{O}}$ for the operator $\mathrm{O} \in \{\mathrm{P}, \mathrm{P_f}, \mathrm{P_u}, \mathrm{P_{sd}}\}$ if the index sets over which the corresponding products are taken are assumed to be nonempty.

3 Idempotency

Lemma 1 *For any operator* $\mathrm{O} \in \{\mathrm{S}, \mathrm{S_e}, \mathrm{F}, \mathrm{H}, \mathrm{R}, \mathrm{E}\}$*, we have* $\mathrm{O}^2 := \mathrm{OO} = \mathrm{O}$.

Proof:
First, since all operators are inflationary, we have that $\mathrm{O} \leq \mathrm{O}^2$.

The reverse inequality must be shown case by case. The easy proofs are only sketched in the remainder of the proof.

For $\mathrm{O} = \mathrm{S}$, it is easy to see, by the relevant definitions, the fact that a subfunctor of a subfunctor is itself a subfunctor and by the subsystem condition on the relation systems of a subsystem that a subsystem of a subsystem is itself a subsystem. Therefore $\mathrm{S}^2 \leq \mathrm{S}$.

The same holds for the operator $\mathrm{S_e}$. It suffices here to observe that the property of being elementary as a subsystem transfers from a subsystem $\mathfrak{C}$ of a subsystem $\mathfrak{B}$ of an $\mathcal{L}$-system $\mathfrak{A}$ to $\mathfrak{C}$ being a subsystem of $\mathfrak{A}$. Hence $\mathrm{S_e^2} \leq \mathrm{S_e}$.

Analogous comments apply to the case of F. Thus $\mathrm{F}^2 \leq \mathrm{F}$.

Finally, for H, R and E we only need to observe that the composition of two surjective $\mathcal{L}$-morphisms is a surjective $\mathcal{L}$-morphism and that the composition of two strong and surjective $\mathcal{L}$-morphisms is also a strong and surjective $\mathcal{L}$-morphism. Therefore $\mathrm{H}^2 \leq \mathrm{H}, \mathrm{R}^2 \leq \mathrm{R}$ and $\mathrm{E}^2 \leq \mathrm{E}$.

This shows that all operators in the collection $\{\mathrm{S}, \mathrm{S_e}, \mathrm{F}, \mathrm{H}, \mathrm{R}, \mathrm{E}\}$ are idempotent. ■

Corollary 2 $\mathrm{E_i^2} = \mathrm{E_i}$ *and* $\mathrm{R_i^2} = \mathrm{R_i}$.

The same idempotency property holds also for the operators $\mathrm{P}, \mathrm{P_f}, \mathrm{P}_u$ and $\mathrm{P_{sd}}$. This is shown in the following lemma. The proof deals only with $\mathrm{P_f}$, since both direct products and ultraproducts are special cased of reduced products and, hence, the cases P and $\mathrm{P_u}$ follow from $\mathrm{P_f}$.

Lemma 3 *For any operator* $\mathrm{O} \in \{\mathrm{P}, \mathrm{P_f}, \mathrm{P_u}\}$*, we have* $\mathrm{O}^2 = \mathrm{O}$.

Proof:
Only the case $\mathrm{P_f^2} = \mathrm{P_f}$ will be shown in detail. Since products and ultraproducts are special cases of reduced products, the remaining two cases follow easily from this case. Obviously, we have that $\mathrm{P_f} \leq \mathrm{P_f^2}$.

For the reverse inclusion we are going to rely partly on the proof of Proposition 14 of [28]. That proof was modeled after the proof of the corresponding result from first-order logic structures that was presented in full detail as the proof of Lemma 2.22 of [5]. The reader is encouraged to consult that proof and compare the present setting with that of first-order logic.

Let J be a set, $I_j, j \in J$, a family of pairwise disjoint sets, $\mathfrak{A}_i = \langle \mathrm{SEN}^i, \langle \mathbf{N}^i, F^i \rangle, R^i \rangle$ be $\mathcal{L}$-systems, for all $i \in I_j, j \in J$, $\mathcal{F}$ a filter over J and, for all $j \in J$, $\mathcal{F}_j$ a filter over I_j. Define $I = \bigcup_{j \in J} I_j$ and

$$\hat{\mathcal{F}} = \{S \subseteq I : \{j \in J : S \cap I_j \in \mathcal{F}_j\} \in \mathcal{F}\}.$$

Then $\hat{\mathcal{F}}$ is a filter over I and it suffices to show that

$$\prod_{j \in J} (\prod_{i \in I_j} \mathfrak{A}_i / \mathcal{F}_j) / \mathcal{F} \cong \prod_{i \in I} \mathfrak{A}_i / \hat{\mathcal{F}}.$$

for all $\mathcal{L}$-systems $\mathfrak{A}_i = \langle \mathrm{SEN}^i, \langle \mathbf{N}^i, F^i \rangle, R^i \rangle$, $i \in I$.

It is clear that, as categories, $\prod_{j \in J} (\prod_{i \in I_j} \mathbf{Sign}^i) \cong \prod_{i \in I} \mathbf{Sign}^i$, where an isomorphism $H : \prod_{i \in I} \mathbf{Sign}^i \cong \prod_{j \in J} (\prod_{i \in I_j} \mathbf{Sign}^i)$ is given at the object level by

$$H(\prod_{i \in I} \Sigma_i) = \prod_{j \in J} \prod_{i \in I_j} \Sigma_i,$$

and, similarly for morphisms. Next, a natural transformation

$$\gamma : (\prod_{i \in I} \mathrm{SEN}^i)^{\equiv^{\hat{\mathcal{F}}}} \to (\prod_{j \in J} (\prod_{i \in I_j} \mathrm{SEN}^i)^{\equiv^{\mathcal{F}_j}})^{\equiv^{\mathcal{F}}}$$

is constructed.

The following translations will be used in the construction: For all $j \in J$, the translation $\langle F^j, \alpha^j \rangle : \prod_{i \in I} \mathrm{SEN}^i \to \prod_{i \in I_j} \mathrm{SEN}^i$, given by

$$F^j(\prod_{i \in I} \Sigma_i) = \prod_{i \in I_j} \Sigma_i, \text{ for all } \Sigma_i \in |\mathbf{Sign}^i|, i \in I,$$

and, similarly for morphisms, and

$$\alpha^j_{\prod_{i \in I} \Sigma_i}(\vec{\phi}) = \vec{\phi} \restriction_{I_j}, \text{ for all } \vec{\phi} \in \prod_{i \in I} \mathrm{SEN}^i(\Sigma_i).$$

The natural projection translation $\langle \mathrm{I}^j, \pi^{\mathcal{F}_j} \rangle : \prod_{i \in I_j} \mathrm{SEN}^i \to (\prod_{i \in I_j} \mathrm{SEN}^i)^{\equiv^{\mathcal{F}_j}}$. The translation

$$\langle G, \beta \rangle := \prod_{j \in J} \langle \mathrm{I}^j, \pi^{\mathcal{F}_j} \rangle \langle F^j, \alpha^j \rangle : \prod_{i \in I} \mathrm{SEN}^i \to \prod_{j \in J} (\prod_{i \in I_j} \mathrm{SEN}^i)^{\equiv^{\mathcal{F}_j}}.$$

$$\begin{tikzcd}
\prod_{i \in I} \mathrm{SEN}^i \arrow[r, "\langle F^j, \alpha^j \rangle"] \arrow[rrd, "\langle G, \beta \rangle"'] & \prod_{i \in I_j} \mathrm{SEN}^i \arrow[r, "\langle \mathrm{I}^j, \pi^{\mathcal{F}_j} \rangle"] & (\prod_{i \in I_j} \mathrm{SEN}^i)^{\equiv^{\mathcal{F}_j}} \\
& & \prod_{j \in J} (\prod_{i \in I_j} \mathrm{SEN}^i)^{\equiv^{\mathcal{F}_j}} \arrow[u, "\langle \mathrm{I}^j, \pi^j \rangle"']
\end{tikzcd}$$

Consider also the natural projection translation

$$\langle \mathrm{I}, \pi^{\mathcal{F}} \rangle : \prod_{j\in J}(\prod_{i\in I_j} \mathrm{SEN}^i)^{\equiv^{\mathcal{F}_j}} \rightarrow^p (\prod_{j\in J}(\prod_{i\in I_j} \mathrm{SEN}^i)^{\equiv^{\mathcal{F}_j}})^{\equiv^{\mathcal{F}}}$$

and the natural projection translation $\langle \mathrm{I}, \pi^{\hat{\mathcal{F}}} \rangle : \prod_{i\in I} \mathrm{SEN}^i \rightarrow^p (\prod_{i\in I} \mathrm{SEN}^i)^{\equiv^{\hat{\mathcal{F}}}}$. Notice that we have, for all $\Sigma_i \in |\mathbf{Sign}^i|, \phi_i, \psi_i \in \mathrm{SEN}^i(\Sigma_i), i \in I$,

$$\begin{array}{ll} & \vec{\phi}\ \mathrm{Ker}_{\prod_{i\in I}\Sigma_i}(\langle \mathrm{I}, \pi^{\mathcal{F}}\rangle\langle G, \beta\rangle)\ \vec{\psi} \\ \text{iff} & \pi^{\mathcal{F}}_{G(\prod_{i\in I}\Sigma_i)}(\beta_{\prod_{i\in I}\Sigma_i}(\vec{\phi})) = \pi^{\mathcal{F}}_{G(\prod_{i\in I}\Sigma_i)}(\beta_{\prod_{i\in I}\Sigma_i}(\vec{\psi})) \\ \text{iff} & \{j \in J : \pi^j_{G(\prod_{i\in I}\Sigma_i)}(\beta_{\prod_{i\in I}\Sigma_i}(\vec{\phi})) = \pi^j_{G(\prod_{i\in I}\Sigma_i)}(\beta_{\prod_{i\in I}\Sigma_i}(\vec{\psi}))\} \in \mathcal{F} \\ \text{iff} & \{j \in J : \{i \in I_j : \phi_i = \psi_i\} \in \mathcal{F}_j\} \in \mathcal{F} \\ \text{iff} & \{i \in I : \phi_i = \psi_i\} \in \hat{\mathcal{F}} \\ \text{iff} & \vec{\phi} \equiv^{\hat{\mathcal{F}}}_{\prod_{i\in I}\Sigma_i} \vec{\psi}. \end{array}$$

Therefore, matching the hypothesis of the special case of the Order Isomorphism Theorem (Corollary 16 of [27]), in which all partial orderings involved are identity relations, we obtain an order isomorphism

$$\langle H, \gamma \rangle : (\prod_{i\in I} \mathrm{SEN}^i)^{\equiv^{\hat{\mathcal{F}}}} \rightarrow (\prod_{j\in J}(\prod_{i\in I_j} \mathrm{SEN}^i)^{\equiv^{\mathcal{F}_j}})^{\equiv^{\mathcal{F}}},$$

such that the following diagram commutes:

$$\begin{tikzcd} \prod_{i\in I} \mathrm{SEN}^i \arrow[r, "\langle \mathrm{I}, \pi^{\hat{\mathcal{F}}}\rangle"] \arrow[d, "\langle G, \beta\rangle"'] & (\prod_{i\in I} \mathrm{SEN}^i)^{\equiv^{\hat{\mathcal{F}}}} \arrow[d, "\langle H, \gamma\rangle"] \\ \prod_{j\in J}(\prod_{i\in I_j} \mathrm{SEN}^i)^{\equiv^{\hat{\mathcal{F}}}} \arrow[r, "\langle \mathrm{I}, \pi^{\mathcal{F}}\rangle"'] & (\prod_{j\in J}(\prod_{i\in I_j} \mathrm{SEN}^i)^{\equiv^{\mathcal{F}_j}})^{\equiv^{\mathcal{F}}} \end{tikzcd}$$

To conclude the proof, it suffices to show that

$$\langle H, \gamma \rangle : \prod_{i\in I} \mathfrak{A}_i/\hat{\mathcal{F}} \rightarrow \prod_{j\in J}(\prod_{i\in I_j} \mathfrak{A}_i/\mathcal{F}_j)/\mathcal{F}$$

is a strong $\mathcal{L}$-system morphism. To this end, suppose that $r \in R$, such that $\rho(r) = n$, and $\Sigma_i \in |\mathbf{Sign}^i|, \phi^0_i, \dots, \phi^{n-1}_i \in \mathrm{SEN}^i(\Sigma_i), i \in I$. We have

$$\begin{array}{ll} & \langle \vec{\phi^0}/\hat{\mathcal{F}}, \dots, \vec{\phi^{n-1}}/\hat{\mathcal{F}} \rangle \in r^{\prod_{i\in I}\mathfrak{A}_i/\hat{\mathcal{F}}}_{\prod_{i\in I}\Sigma_i} \\ \text{iff} & \{i \in I : \langle \phi^0_i, \dots, \phi^{n-1}_i \rangle \in r^{\mathfrak{A}_i}_{\Sigma_i}\} \in \hat{\mathcal{F}} \\ \text{iff} & \{j \in J : \{i \in I_j : \langle \phi^0_i, \dots, \phi^{n-1}_i \rangle \in r^{\mathfrak{A}_i}_{\Sigma_i}\} \in \mathcal{F}_j\} \in \mathcal{F} \\ \text{iff} & \{j \in J : \langle \pi^j_{G(\prod_{i\in I}\Sigma_i)}(\beta_{\prod_{i\in I}\Sigma_i}(\vec{\phi^0})), \dots, \\ & \qquad \pi^j_{G(\prod_{i\in I}\Sigma_i)}(\beta_{\prod_{i\in I}\Sigma_i}(\vec{\phi^{n-1}})) \rangle \in r^{\prod_{i\in I_j}\mathfrak{A}_i/\mathcal{F}_j}_{\prod_{i\in I_j}\Sigma_i}\} \in \mathcal{F} \\ \text{iff} & \langle \pi^{\mathcal{F}}_{G(\prod_{i\in I}\Sigma_i)}(\beta_{\prod_{i\in I}\Sigma_i}(\vec{\phi^0})), \dots, \pi^{\mathcal{F}}_{G(\prod_{i\in I}\Sigma_i)}(\beta_{\prod_{i\in I}\Sigma_i}(\vec{\phi^{n-1}})) \rangle \in r^{\prod_{j\in J}(\prod_{i\in I_j}\mathfrak{A}_i/\mathcal{F}_j)/\mathcal{F}}_{\prod_{j\in J}\prod_{i\in I_j}\Sigma_i} \\ \text{iff} & \langle \gamma_{\prod_{i\in I}\Sigma_i}(\pi^{\hat{\mathcal{F}}}_{\prod_{i\in I}\Sigma_i}(\vec{\phi^0})), \dots, \gamma_{\prod_{i\in I}\Sigma_i}(\pi^{\hat{\mathcal{F}}}_{\prod_{i\in I}\Sigma_i}(\vec{\phi^{n-1}})) \rangle \in r^{\prod_{j\in J}(\prod_{i\in I_j}\mathfrak{A}_i/\mathcal{F}_j)/\mathcal{F}}_{H(\prod_{i\in I}\Sigma_i)} \\ \text{iff} & \langle \gamma_{\prod_{i\in I}\Sigma_i}(\vec{\phi^0}/\hat{\mathcal{F}}), \dots, \gamma_{\prod_{i\in I}\Sigma_i}(\vec{\phi^{n-1}}/\hat{\mathcal{F}}) \rangle \in r^{\prod_{j\in J}(\prod_{i\in I_j}\mathfrak{A}_i/\mathcal{F}_j)/\mathcal{F}}_{H(\prod_{i\in I}\Sigma_i)}. \end{array}$$

■

Finally, we show that P_{sd} is also idempotent.

Lemma 4 $\mathrm{P}_{\mathrm{sd}}^2 = \mathrm{P}_{\mathrm{sd}}$.

Proof:
Continuing with the index notation from the proof of Lemma 3, consider the subdirect $\mathcal{L}$-system embeddings $\langle F, \alpha\rangle : \mathfrak{A} \rightarrowtail_{\mathrm{sd}} \prod_{j\in J} \mathfrak{A}_j$ and $\langle F^j, \alpha^j\rangle : \mathfrak{A}_j \rightarrowtail_{\mathrm{sd}} \prod_{i\in I_j} \mathfrak{A}_i$. From the proof of Lemma 3, we have that $\prod_{j\in J}(\prod_{i\in I_j} \mathfrak{A}_i) \cong \prod_{i\in I} \mathfrak{A}_i$. Thus, it suffices to show that there exists $\langle G, \beta\rangle : \mathfrak{A} \rightarrowtail_{\mathrm{sd}} \prod_{j\in J}(\prod_{i\in I_j} \mathfrak{A}_i)$. Simply define

$$\langle G, \beta\rangle = \prod_{j\in J}(\langle F^j, \alpha^j\rangle \circ \langle P^j, \pi^j\rangle \circ \langle F, \alpha\rangle).$$

$$\begin{tikzcd}
\mathrm{SEN} \arrow[r, "\langle F,\alpha\rangle"] \arrow[rrrd, "\prod_{j\in J}(\langle F^j,\alpha^j\rangle \circ \langle P^j,\pi^j\rangle \circ \langle F,\alpha\rangle)"'] & \prod_{j\in J} \mathrm{SEN}^j \arrow[r, "\langle P^j,\pi^j\rangle"] & \mathrm{SEN}^j \arrow[r, "\langle F^j,\alpha^j\rangle"] & \prod_{i\in I_j} \mathrm{SEN}^i \\
 & & & \prod_{j\in J}\prod_{i\in I_j} \mathrm{SEN}^i \arrow[u, "\langle P^j,\pi^j\rangle"']
\end{tikzcd}$$

It is not difficult to verify that $\langle G, \beta\rangle$ is a subdirect embedding of functors. To see that it is also a subdirect embedding of $\mathcal{L}$-systems, consider $r \in R$, with $\rho(r) = n$, $\Sigma \in |\mathbf{Sign}|$, $\vec{\phi} \in \mathrm{SEN}(\Sigma)^n$. Then, we have

$$\begin{aligned}
\vec{\phi} \in r_\Sigma^{\mathfrak{A}} \quad & \text{iff} \quad \alpha_\Sigma(\vec{\phi}) \in r_{F(\Sigma)}^{\prod_{j\in J}\mathfrak{A}_j} \\
& \text{iff} \quad \pi^j_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})) \in r^{\mathfrak{A}_j}_{P^j(F(\Sigma))}, \ \text{for all } j \in J \\
& \text{iff} \quad \alpha^j_{P^j(F(\Sigma))}(\pi^j_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi}))) \in r^{\prod_{i\in I_j}\mathfrak{A}_i}_{F^j(P^j(F(\Sigma)))}, \text{for all } j \in J \\
& \text{iff} \quad \pi^j_{G(\Sigma)}(\beta_\Sigma(\vec{\phi})) \in r^{\prod_{i\in I_j}\mathfrak{A}_i}_{P^j(G(\Sigma))}, \ \text{for all } j \in J \\
& \text{iff} \quad \beta_\Sigma(\vec{\phi}) \in r^{\prod_{j\in J}\prod_{i\in I_j}\mathfrak{A}_i}_{G(\Sigma)}.
\end{aligned}$$

■

The results proved so far are summarized in the following theorem, forming an analog of Lemma 4.1 of [13] for $\mathcal{L}$-systems.

Theorem 5 *For any operator* $\mathrm{O} \in \{\mathrm{S}, \mathrm{S_e}, \mathrm{F}, \mathrm{H}, \mathrm{R}, \mathrm{E}, \mathrm{P}, \mathrm{P_f}, \mathrm{P_u}, \mathrm{P_{sd}}\}$*, we have* $\mathrm{O}^2 = \mathrm{O}$.

Proof:
Use Lemmas 1, 3 and 4. ■

4 Properties of Expansions and Reductions We first prove three lemmas that collectively provide an analog for $\mathcal{L}$-systems of Part (i) of Lemma 4.2 of [13]. The first lemma shows that taking subsystems of expansions with isomorphic functor components is an operator lying below taking expansions with isomorphic functor components of subsystems. The same holds if, instead of taking subsystems, we consider taking elementary subsystems.

Lemma 6 $\mathrm{SE_i} \leq \mathrm{E_iS}$ *and* $\mathrm{S_eE_i} \leq \mathrm{E_iS_e}$.

Proof:

Let $\mathfrak{A}$ be an $\mathcal{L}$-system and $\mathtt{K}$ a class of $\mathcal{L}$-systems, such that $\mathfrak{A} \in \mathrm{SE_i}(\mathtt{K})$. Thus, there exist $\mathcal{L}$-systems $\mathfrak{B}$ and $\mathfrak{C}$ and a reductive $\mathcal{L}$-morphism $\langle F, \alpha \rangle : \mathfrak{C} \twoheadrightarrow_s \mathfrak{B}$, with F an isomorphism, such that $\mathfrak{B} \in \mathtt{K}$ and $\mathfrak{A} \subseteq \mathfrak{C}$. Now, by Part 2 of Lemma 5 of [30], $\alpha(\mathfrak{A}) \subseteq \mathfrak{B}$. Since the restriction $\langle F, \alpha \rangle \restriction_{\mathfrak{A}}$ of $\langle F, \alpha \rangle$ is a reductive $\mathcal{L}$-morphism $\langle F, \alpha \rangle \restriction_{\mathfrak{A}} : \mathfrak{A} \twoheadrightarrow_s \alpha\mathfrak{A}$ and it clearly has an isomorphic functor component, we now have that $\mathfrak{A}$ is an expansion of $\alpha(\mathfrak{A})$, which is, in turn, a subsystem of $\mathfrak{B} \in \mathtt{K}$, i.e., $\mathfrak{A} \in \mathrm{E_iS}(\mathtt{K})$. Therefore $\mathrm{SE_i} \leq \mathrm{E_iS}$ and the first statement of the lemma has been verified.

Let again $\mathfrak{A}$ be an $\mathcal{L}$-system and $\mathtt{K}$ a class of $\mathcal{L}$-systems, such that $\mathfrak{A} \in \mathrm{S_eE_i}(\mathtt{K})$. Thus, there exist $\mathcal{L}$-systems $\mathfrak{B}$ and $\mathfrak{C}$ and a reductive $\mathcal{L}$-morphism $\langle F, \alpha \rangle : \mathfrak{C} \twoheadrightarrow_s \mathfrak{B}$, with F an isomorphism, such that $\mathfrak{B} \in \mathtt{K}$ and $\mathfrak{A} \subseteq_{\mathrm{e}} \mathfrak{C}$. Again relying on Part 2 of Lemma 5 of [30], $\alpha(\mathfrak{A}) \subseteq \mathfrak{B}$. Since the restriction $\langle F, \alpha \rangle \restriction_{\mathfrak{A}}$ of $\langle F, \alpha \rangle$ is a reductive $\mathcal{L}$-morphism $\langle F, \alpha \rangle \restriction_{\mathfrak{A}} : \mathfrak{A} \twoheadrightarrow_s \alpha\mathfrak{A}$ and it clearly has an isomorphic functor component, we have, by Proposition 6 of [30], that it is elementary, whence $\alpha(\mathfrak{A}) \subseteq_{\mathrm{e}} \mathfrak{B} \in \mathtt{K}$ and, also, that $\mathfrak{A}$ is an expansion of $\alpha(\mathfrak{A})$, i.e., $\mathfrak{A} \in \mathrm{E_iS_e}(\mathtt{K})$. Therefore $\mathrm{S_eE_i} \leq \mathrm{E_iS_e}$ and the last statement of the lemma has been verified. ■

The next lemma shows that taking direct products of expansions of systems in some class is an operator smaller than that of taking expansions of direct products of systems in the class. The same holds if, instead of direct products, we consider taking arbitrary filtered products or ultraproducts.

Lemma 7 $\mathrm{PE} \leq \mathrm{EP}, \mathrm{P_fE} \leq \mathrm{EP_f}$ *and* $\mathrm{P_uE} \leq \mathrm{EP_u}$.

Proof:

Let $\mathfrak{A}$ be an $\mathcal{L}$-system and $\mathtt{K}$ a class of $\mathcal{L}$-systems, such that $\mathfrak{A} \in \mathrm{PE}(\mathtt{K})$. Thus, there exist a collection of $\mathcal{L}$-systems $\mathfrak{A}_i$ and $\mathfrak{B}_i, i \in I$, with $\mathfrak{B}_i \in \mathtt{K}$, for all $i \in I$, and a collection of reductive $\mathcal{L}$-morphisms $\langle F^i, \alpha^i \rangle : \mathfrak{A}_i \twoheadrightarrow_s \mathfrak{B}_i, i \in I$, such that $\mathfrak{A} \cong \prod_{i \in I} \mathfrak{A}_i$. Since $\prod_{i \in I} \mathfrak{B}_i \in \mathrm{P}(\mathtt{K})$, it suffices to show that, there exists $\langle F, \alpha \rangle : \prod_{i \in I} \mathfrak{A}_i \twoheadrightarrow_s \prod_{i \in I} \mathfrak{B}_i$, since then $\mathfrak{A} \cong \prod_{i \in I} \mathfrak{A}_i \in \mathrm{EP}(\mathtt{K})$.

Indeed, define $F : \prod_{i \in I} \mathbf{Sign}^{\mathfrak{A}_i} \to \prod_{i \in I} \mathbf{Sign}^{\mathfrak{B}_i}$, by $F = \prod_{i \in I} F^i$, i.e., as the unique functor making the following square commutative, for all $i \in I$:

$$\begin{tikzcd}
\prod_{i \in I} \mathbf{Sign}^{\mathfrak{A}_i} \arrow[r, "F"] \arrow[d, "P^i"'] & \prod_{i \in I} \mathbf{Sign}^{\mathfrak{B}_i} \arrow[d, "P^i"] \\
\mathbf{Sign}^{\mathfrak{A}_i} \arrow[r, "F^i"'] & \mathbf{Sign}^{\mathfrak{B}_i}
\end{tikzcd}$$

Define, also $\alpha : \prod_{i \in I} \mathrm{SEN}^{\mathfrak{A}_i} \to \prod_{i \in I} \mathrm{SEN}^{\mathfrak{B}_i} \circ \prod_{i \in I} F^i$, by letting, for all $\Sigma_i \in |\mathbf{Sign}^{\mathfrak{A}_i}|$ and all $\phi_i \in \mathrm{SEN}^{\mathfrak{A}_i}(\Sigma_i), i \in I$,

$$\alpha_{\prod_{i \in I} \Sigma_i}(\vec{\phi}) = \langle \alpha^i_{\Sigma_i}(\phi_i) : i \in I \rangle.$$

This is a surjective $\mathcal{L}$-system morphism $\langle F, \alpha \rangle : \prod_{i \in I} \mathfrak{A}_i \twoheadrightarrow \prod_{i \in I} \mathfrak{B}_i$. Thus, it suffices now to show that it is also strong.

To this end, suppose that $r \in R$, with $\rho(r) = n$, $\Sigma_i \in |\mathbf{Sign}^{\mathfrak{A}_i}|, i \in I$, and $\vec{\phi^0}, \ldots, \vec{\phi^{n-1}} \in \prod_{i \in I} \mathrm{SEN}^{\mathfrak{A}_i}(\Sigma_i)$. Then, we have

$$\begin{array}{lll}\langle \vec{\phi^0}, \ldots, \vec{\phi^{n-1}} \rangle \in r^{\prod_{i \in I} \mathfrak{A}_i}_{\prod_{i \in I} \Sigma_i} & \text{iff} & \langle \phi^0_i, \ldots, \phi^{n-1}_i \rangle \in r^{\mathfrak{A}_i}_{\Sigma_i}, \text{ for all } i \in I, \\ & \text{iff} & \langle \alpha^i_{\Sigma_i}(\phi^0_i), \ldots, \alpha^i_{\Sigma_i}(\phi^{n-1}_i) \rangle \in r^{\mathfrak{B}_i}_{F^i(\Sigma_i)}, \text{ for all } i \in I, \\ & \text{iff} & \langle \alpha_{\prod_{i \in I} \Sigma_i}(\vec{\phi^0}), \ldots, \alpha_{\prod_{i \in I} \Sigma_i}(\vec{\phi^{n-1}}) \rangle \in r^{\prod_{i \in I} \mathfrak{B}_i}_{F(\prod_{i \in I} \Sigma_i)}.\end{array}$$

The proof just given for $\mathrm{PE} \leq \mathrm{EP}$ may be adapted to provide a proof for both $\mathrm{P_f E} \leq \mathrm{EP_f}$ and $\mathrm{P_u E} \leq \mathrm{EP_u}$. The proof for $\mathrm{P_f E} \leq \mathrm{EP_f}$ will also be presented, but the one for $\mathrm{P_u E} \leq \mathrm{EP_u}$ is very similar and will be omitted.

Let $\mathfrak{A}$ be an $\mathcal{L}$-system and $\mathtt{K}$ a class of $\mathcal{L}$-systems, such that $\mathfrak{A} \in \mathrm{P_f E}(\mathtt{K})$. Thus, there exist a collection of $\mathcal{L}$-systems $\mathfrak{A}_i$ and $\mathfrak{B}_i, i \in I$, with $\mathfrak{B}_i \in \mathtt{K}$, for all $i \in I$, a proper filter $\mathcal{F}$ on I, and a collection of reductive $\mathcal{L}$-morphisms $\langle F^i, \alpha^i \rangle : \mathfrak{A}_i \twoheadrightarrow_s \mathfrak{B}_i, i \in I$, such that $\mathfrak{A} \cong \prod_{i \in I} \mathfrak{A}_i / \mathcal{F}$. In the same way, as before, since $\prod_{i \in I} \mathfrak{B}_i / \mathcal{F} \in \mathrm{P_f}(\mathtt{K})$, it suffices to show that, there exists $\langle F, \alpha \rangle : \prod_{i \in I} \mathfrak{A}_i / \mathcal{F} \twoheadrightarrow_s \prod_{i \in I} \mathfrak{B}_i / \mathcal{F}$, since then $\mathfrak{A} \cong \prod_{i \in I} \mathfrak{A}_i / \mathcal{F} \in \mathrm{EP_f}(\mathtt{K})$.

Define $F : \prod_{i \in I} \mathbf{Sign}^{\mathfrak{A}_i} \to \prod_{i \in I} \mathbf{Sign}^{\mathfrak{B}_i}$ exactly as before, i.e., by $F = \prod_{i \in I} F^i$. Define, also $\alpha : \prod_{i \in I} \mathrm{SEN}^{\mathfrak{A}_i} / \mathcal{F} \to \prod_{i \in I} \mathrm{SEN}^{\mathfrak{B}_i} / \mathcal{F} \circ F$, by letting, for all $\Sigma_i \in |\mathbf{Sign}^{\mathfrak{A}_i}|$ and all $\phi_i \in \mathrm{SEN}^{\mathfrak{A}_i}(\Sigma_i), i \in I$,

$$\alpha_{\prod_{i \in I} \Sigma_i}(\vec{\phi} / \mathcal{F}) = \langle \alpha^i_{\Sigma_i}(\phi_i) : i \in I \rangle / \mathcal{F}.$$

It may be shown that $\langle F, \alpha \rangle$ is well-defined, and that it is a surjective $\mathcal{L}$-system morphism $\langle F, \alpha \rangle : \prod_{i \in I} \mathfrak{A}_i / \mathcal{F} \twoheadrightarrow \prod_{i \in I} \mathfrak{B}_i / \mathcal{F}$. Thus, it suffices now to show that it is also strong.

To this end, suppose that $r \in R$, with $\rho(r) = n$, $\Sigma_i \in |\mathbf{Sign}^{\mathfrak{A}_i}|, i \in I$, and $\vec{\phi^0}, \ldots, \vec{\phi^{n-1}} \in \prod_{i \in I} \mathrm{SEN}^{\mathfrak{A}_i}(\Sigma_i)$. Then, we have

$$\begin{array}{ll} & \langle \vec{\phi^0} / \mathcal{F}, \ldots, \vec{\phi^{n-1}} / \mathcal{F} \rangle \in r^{\prod_{i \in I} \mathfrak{A}_i / \mathcal{F}}_{\prod_{i \in I} \Sigma_i} \\ \text{iff} & \{ i \in I : \langle \phi^0_i, \ldots, \phi^{n-1}_i \rangle \in r^{\mathfrak{A}_i}_{\Sigma_i} \} \in \mathcal{F} \\ \text{iff} & \{ i \in I : \langle \alpha^i_{\Sigma_i}(\phi^0_i), \ldots, \alpha^i_{\Sigma_i}(\phi^{n-1}_i) \rangle \in r^{\mathfrak{B}_i}_{F^i(\Sigma_i)} \} \in \mathcal{F} \\ \text{iff} & \langle \langle \alpha^i_{\Sigma_i}(\phi^0_i) : i \in I \rangle / \mathcal{F}, \ldots, \langle \alpha^i_{\Sigma_i}(\phi^{n-1}_i) : i \in I \rangle / \mathcal{F} \rangle \in r^{\prod_{i \in I} \mathfrak{B}_i / \mathcal{F}}_{\prod_{i \in I} F^i(\Sigma_i)} \\ \text{iff} & \langle \alpha_{\prod_{i \in I} \Sigma_i}(\vec{\phi^0} / \mathcal{F}), \ldots, \alpha_{\prod_{i \in I} \Sigma_i}(\vec{\phi^{n-1}} / \mathcal{F}) \rangle \in r^{\prod_{i \in I} \mathfrak{B}_i / \mathcal{F}}_{F(\prod_{i \in I} \Sigma_i)}. \end{array}$$

■

Finally, it is shown that an analogous property to that of Lemma 7 holds when subdirect products of $\mathcal{L}$-systems are considered in place of direct products. More precisely, it is shown that taking subdirect products of expansions of systems via reductive morphisms with isomorphic functor components is an operator smaller than that of taking expansions with isomorphic functor components of subdirect products of $\mathcal{L}$-systems.

Lemma 8 $\mathrm{P_{sd} E_i} \leq \mathrm{E_i P_{sd}}$.

Proof:

Let $\mathfrak{A}$ be an $\mathcal{L}$-system and $\mathtt{K}$ a class of $\mathcal{L}$-systems, such that $\mathfrak{A} \in \mathrm{P_{sd} E_i}(\mathtt{K})$. Then, there exist $\mathcal{L}$-systems $\mathfrak{A}_i$ and $\mathfrak{B}_i$, $i \in I$, such that $\mathfrak{B}_i \in \mathtt{K}$, for all $i \in I$, and reductive $\mathcal{L}$-morphisms $\langle F^i, \alpha^i \rangle : \mathfrak{A}_i \twoheadrightarrow_s \mathfrak{B}_i, i \in I$, with isomorphic functor components, such that $\langle G, \beta \rangle : \mathfrak{A} \rightarrowtail_{\mathrm{sd}} \prod_{i \in I} \mathfrak{A}_i$. Define the $\mathcal{L}$-morphism

$$\langle H, \gamma \rangle = \prod_{i \in I} \langle F^i, \alpha^i \rangle \circ \langle G, \beta \rangle : \mathfrak{A} \to \prod_{i \in I} \mathfrak{B}_i$$

by using the following commutative diagram:

$$\begin{tikzcd}
& \mathfrak{A} \arrow[dl, "\langle G,\beta\rangle"'] \arrow[dr, "\langle H,\gamma\rangle"] & \\
\prod_{i\in I}\mathfrak{A}_i \arrow[rr, "{\langle F,\alpha\rangle := \prod_{i\in I}\langle F^i,\alpha^i\rangle}"'] \arrow[d, "\langle P^i,\pi^i\rangle"'] & & \prod_{i\in I}\mathfrak{B}_i \arrow[d, "\langle P^i,\pi^i\rangle"] \\
\mathfrak{A}_i \arrow[rr, "\langle F^i,\alpha^i\rangle"'] & & \mathfrak{B}_i
\end{tikzcd}$$

First, note that, because F^i is injective, for all $i \in I$, and because G is injective, we have that H is also injective. Thus, by Part 2 of Lemma 5 of [30], $\gamma(\mathfrak{A})$ is a subsystem of $\prod_{i\in I}\mathfrak{B}_i$.

Now, because $\langle F^i,\alpha^i\rangle, i \in I$, are all strong, the $\mathcal{L}$-morphism $\langle F,\alpha\rangle$ is also strong, and, hence, since $\langle G,\beta\rangle$ is strong, we conclude that $\langle H,\gamma\rangle = \langle F,\alpha\rangle \circ \langle G,\beta\rangle$ is strong. Therefore $\langle H,\gamma\rangle : \mathfrak{A} \twoheadrightarrow_s \gamma(\mathfrak{A})$ is a reductive $\mathcal{L}$-morphism. Now it suffices to show that $\gamma(\mathfrak{A}) \rightarrowtail \prod_{i\in I}\mathfrak{B}_i$ is a subdirect embedding, because, then, $\mathfrak{A}$ would be an expansion of a subdirect product of the $\mathfrak{B}_i$'s and $\mathfrak{B}_i \in \mathtt{K}$, for all $i \in I$. This is however true, since

$$\langle P^i,\pi^i\rangle \circ \langle H,\gamma\rangle = \langle F^i,\alpha^i\rangle \circ \langle P^i,\pi^i\rangle \circ \langle G,\beta\rangle$$

and $\langle P^i,\pi^i\rangle \circ \langle G,\beta\rangle$ is surjective, since $\langle G,\beta\rangle$ is also a subdirect embedding. ■

In the last two lemmas of the section, the focus is switched from the expansion operator to the reduction operator. In the first, it is shown that taking subsystems of reductions is an operator smaller than that of taking reductions of subsystems and, similarly, that taking elementary subsystems of reductions is an operator smaller than that of taking reductions of elementary subsystems.

Lemma 9 $\mathrm{SR} \leq \mathrm{RS}$ *and* $\mathrm{S_eR} \leq \mathrm{RS_e}$.

Proof:
Let $\mathfrak{A}$ be an $\mathcal{L}$-system and $\mathtt{K}$ a class of $\mathcal{L}$-systems and suppose that $\mathfrak{A} \in \mathrm{SR}(\mathtt{K})$. Thus, there exist $\mathfrak{B}$ and $\mathfrak{C}$ and a reductive $\mathcal{L}$-morphism $\langle F,\alpha\rangle : \mathfrak{B} \twoheadrightarrow_s \mathfrak{C}$, with $\mathfrak{B} \in \mathtt{K}$, such that $\mathfrak{A} \subseteq \mathfrak{C}$. Now, by Part 1 of Lemma 5 of [30], we have that $\alpha^{-1}(\mathfrak{A}) \subseteq \mathfrak{B}$ and we also have that $\langle F,\alpha\rangle \restriction_{\alpha^{-1}(\mathfrak{A})}: \alpha^{-1}(\mathfrak{A}) \twoheadrightarrow_s \mathfrak{A}$ is also a reductive $\mathcal{L}$-morphism. Therefore $\mathfrak{A}$ is a reduction of $\alpha^{-1}(\mathfrak{A})$, which is a subsystem of $\mathfrak{B} \in \mathtt{K}$, which shows that $\mathfrak{A} \in \mathrm{RS}(\mathtt{K})$. Hence $\mathrm{SR} \leq \mathrm{RS}$.

Working along the same line, suppose that $\mathfrak{A} \in \mathrm{S_eR}(\mathtt{K})$. Thus, there exist $\mathfrak{B}$ and $\mathfrak{C}$ and a reductive $\mathcal{L}$-morphism $\langle F,\alpha\rangle : \mathfrak{B} \twoheadrightarrow_s \mathfrak{C}$, with $\mathfrak{B} \in \mathtt{K}$, such that $\mathfrak{A} \subseteq_{\mathrm{e}} \mathfrak{C}$. For the same reasons, as above, $\alpha^{-1}(\mathfrak{A}) \subseteq \mathfrak{B}$ and we also have that $\langle F,\alpha\rangle \restriction_{\alpha^{-1}(\mathfrak{A})}: \alpha^{-1}(\mathfrak{A}) \twoheadrightarrow_s \mathfrak{A}$ is a reductive $\mathcal{L}$-morphism. Thus, it suffices now to show that $\alpha^{-1}(\mathfrak{A}) \subseteq_{\mathrm{e}} \mathfrak{B}$. Suppose that $\gamma(\vec{x})$ is an $\mathcal{L}$-formula, $\Sigma \in |F^{-1}(\mathbf{Sign}^{\mathfrak{A}})|$ and $\vec{\phi} \in \mathrm{SEN}^{\alpha^{-1}(\mathfrak{A})}(\Sigma)^n$. Then, we have

$$\begin{array}{lll}
\alpha^{-1}(\mathfrak{A}) \models_\Sigma \beta[\vec{\phi}] & \text{iff} & \mathfrak{A} \models_{F(\Sigma)} \beta[\alpha_\Sigma(\vec{\phi})] \quad (\text{since } \langle F,\alpha\rangle : \alpha^{-1}(\mathfrak{A}) \twoheadrightarrow_s \mathfrak{A}) \\
& \text{iff} & \mathfrak{C} \models_{F(\Sigma)} \beta[\alpha_\Sigma(\vec{\phi})] \quad (\text{since } \mathfrak{A} \subseteq_{\mathrm{e}} \mathfrak{C}) \\
& \text{iff} & \mathfrak{B} \models_\Sigma \beta[\vec{\phi}] \quad (\text{since } \langle F,\alpha\rangle : \mathfrak{B} \twoheadrightarrow_s \mathfrak{C}).
\end{array}$$

Hence $\alpha^{-1}(\mathfrak{A}) \subseteq_{\mathrm{e}} \mathfrak{B}$. ■

In the last lemma, it is proven that taking direct products of reductions is an operator smaller than that of taking reductions of direct products and, similarly, if, instead of direct products, either reduced products or ultraproducts are considered.

Lemma 10 $\mathrm{PR} \leq \mathrm{RP}, \mathrm{P_f R} \leq \mathrm{RP_f}$ *and* $\mathrm{P_u R} \leq \mathrm{RP_u}$.

Proof:

Let $\mathfrak{A}$ be an $\mathcal{L}$-system and $\mathtt{K}$ a class of $\mathcal{L}$-systems, such that $\mathfrak{A} \in \mathrm{PR}(\mathtt{K})$. Then, there exist $\mathfrak{B}_i$ and $\mathfrak{A}_i, i \in I$, with $\mathfrak{A} \cong \prod_{i\in I} \mathfrak{A}_i$, and $\langle F^i, \alpha^i\rangle : \mathfrak{B}_i \twoheadrightarrow_s \mathfrak{A}_i$, such that $\mathfrak{B}_i \in \mathtt{K}$, for all $i \in I$. Now, let, as in the proof of Lemma 7, $\langle F, \alpha\rangle := \prod_{i\in I}\langle F^i, \alpha^i\rangle : \prod_{i\in I} \mathfrak{B}_i \to \prod_{i\in I} \mathfrak{A}_i$. Since $\prod_{i\in I} \mathfrak{B}_i \in \mathrm{P}(\mathtt{K})$, it suffices to show that $\langle F, \alpha\rangle$ is a reductive $\mathcal{L}$-morphism. Both the fact that it is surjective and the fact that it is strong follow directly from the corresponding properties possessed by each of the $\langle F^i, \alpha^i\rangle$, $i \in I$.

The two remaining inequalities follow along similar lines and the detailed proofs will be omitted. ∎

5 The Filter Extension and the Reduction Operators In this final section of the paper, we deal with a lemma that relates the filter extension with the expansion and the reduction operators as well as with one that relates the reduction and the expansion operator via reductive $\mathcal{L}$-morphisms with isomorphic functor components to the operator of taking Leibniz reductions of $\mathcal{L}$-systems.

In the first lemma, it is shown that taking expansions of filter extensions is an operator smaller than taking filter extensions of expansions, that taking filter extensions of reductions is an operator smaller than taking reductions of filter extensions and, finally, that taking filter extensions of subsystems with isomorphic functor components is an operator smaller than taking subsystems with isomorphic functor components of filter extensions.

Lemma 11

1. $\mathrm{EF} \leq \mathrm{FE}$
2. $\mathrm{FR} \leq \mathrm{RF} = \mathrm{H}$
3. $\mathrm{FS_i} \leq \mathrm{S_i F}$

Proof:

1. Let $\mathfrak{A}$ be an $\mathcal{L}$-system and $\mathtt{K}$ a class of $\mathcal{L}$-systems, such that $\mathfrak{A} \in \mathrm{EF}(\mathtt{K})$. Thus, there exist $\mathfrak{B}, \mathfrak{C}$, with $\mathfrak{B} \in \mathtt{K}$, and $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{C}$, such that $\mathfrak{B} \sqsubseteq \mathfrak{C}$. But now we have that
$$\mathfrak{A} \sqsupseteq \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, \alpha^{-1}(R^{\mathfrak{B}})\rangle \overset{\langle F,\alpha\rangle}{\twoheadrightarrow_s} \mathfrak{B} \in \mathtt{K},$$
whence $\mathfrak{A} \in \mathrm{FE}(\mathtt{K})$.

2. Let $\mathfrak{A}$ be an $\mathcal{L}$-system and $\mathtt{K}$ a class of $\mathcal{L}$-systems, such that $\mathfrak{A} \in \mathrm{FR}(\mathtt{K})$. Thus, there exist $\mathfrak{B}, \mathfrak{C}$, with $\mathfrak{B} \in \mathtt{K}$, and $\langle F, \alpha\rangle : \mathfrak{B} \twoheadrightarrow_s \mathfrak{C}$, such that $\mathfrak{C} \sqsubseteq \mathfrak{A}$. But now we have, using Part 1 of Lemma 5 of [30], that
$$\mathtt{K} \ni \mathfrak{B} \sqsubseteq \alpha^{-1}(\mathfrak{A}) \overset{\langle F,\alpha\rangle}{\twoheadrightarrow_s} \mathfrak{A},$$
whence $\mathfrak{A} \in \mathrm{RF}(\mathtt{K})$.

   For the equality of Part 2, suppose, first, that $\mathfrak{A}$ is an $\mathcal{L}$-system and that $\mathtt{K}$ is a class of $\mathcal{L}$-systems, such that $\mathfrak{A} \in \mathrm{H}(\mathtt{K})$. Thus, there exists an $\mathcal{L}$-system $\mathfrak{B} \in \mathtt{K}$ and a surjective

$\mathcal{L}$-morphism $\langle F,\alpha\rangle : \mathfrak{B} \twoheadrightarrow \mathfrak{A}$. But then we have, again using Part 1 of Lemma 5 of [30],

$$\mathtt{K} \ni \mathfrak{B} \sqsubseteq \alpha^{-1}(\mathfrak{A}) \overset{\langle F,\alpha\rangle}{\twoheadrightarrow_s} \mathfrak{A},$$

whence $\mathfrak{A} \in \mathrm{RF}(\mathtt{K})$. If, conversely, $\mathfrak{A} \in \mathrm{RF}(\mathtt{K})$, then, there exist $\mathfrak{B}, \mathfrak{C}$, with $\mathfrak{B} \in \mathtt{K}$, and $\langle F,\alpha\rangle : \mathfrak{C} \twoheadrightarrow_s \mathfrak{A}$, such that $\mathfrak{B} \sqsubseteq \mathfrak{C}$. But then $\langle F,\alpha\rangle : \mathfrak{B} \twoheadrightarrow \mathfrak{A}$ is a surjective $\mathcal{L}$-morphism, whence, since $\mathfrak{B} \in \mathtt{K}$, $\mathfrak{A} \in \mathrm{H}(\mathtt{K})$.

3. Suppose, finally, that $\mathfrak{A}$ is an $\mathcal{L}$-system and $\mathtt{K}$ a class of $\mathcal{L}$-systems, such that $\mathfrak{A} \in \mathrm{FS_i}(\mathtt{K})$. Thus, assume, without loss of generality, that, there exist $\mathfrak{B}, \mathfrak{C}$, with $\mathfrak{B} \in \mathtt{K}$, such that $\mathfrak{C} \subseteq_{\mathrm{i}} \mathfrak{B}$ and $\mathfrak{C} \sqsubseteq \mathfrak{A}$. Define the triple

$$\mathfrak{D} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}} \cup R^{\mathfrak{A}}\rangle,$$

i.e., for all $r \in R$, with $\rho(r) = n$, set, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{B}}|$, and all $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{B}}(\Sigma)^n$,

$$\vec{\phi} \in r^{\mathfrak{D}}_{\Sigma} \quad \text{iff} \quad \vec{\phi} \in r^{\mathfrak{B}}_{\Sigma} \quad \text{or} \quad \vec{\phi} \in r^{\mathfrak{A}}_{\Sigma}.$$

Now it is not difficult to verify that $\mathfrak{B} \sqsubseteq \mathfrak{D}$ and also that $\mathfrak{A} \subseteq_{\mathrm{i}} \mathfrak{D}$, whence, since $\mathfrak{B} \in \mathtt{K}$, we get that $\mathfrak{A} \in \mathrm{S_iF}(\mathtt{K})$.

■

The last lemma relates the operators of taking reductions and expansions with isomorphic functor components and the operator L of taking Leibniz reductions of $\mathcal{L}$-systems.

Lemma 12 *1.* $\mathrm{R_iE_i} \leq \mathrm{E_iR_i} = \mathrm{E_iL}$

2. $\mathrm{LE_i} = \mathrm{LR_i} = \mathrm{R_iL} = \mathrm{L} \leq \mathrm{E_iL}$

Proof:

1. It is first shown that $\mathrm{R_iE_i} \leq \mathrm{E_iR_i}$. Suppose, to this end, that $\mathfrak{A}$ is an $\mathcal{L}$-system and $\mathtt{K}$ a class of $\mathcal{L}$-systems, such that $\mathfrak{A} \in \mathrm{R_iE_i}(\mathtt{K})$. Then, there exist $\mathfrak{B}$ and $\mathfrak{C}$, with $\mathfrak{B} \in \mathtt{K}$ and reductive $\mathcal{L}$-morphisms $\langle F,\alpha\rangle : \mathfrak{C} \twoheadrightarrow_s \mathfrak{A}$ and $\langle G,\beta\rangle : \mathfrak{C} \twoheadrightarrow_s \mathfrak{B}$, with isomorphic functor components. By Corollary 17 of [31], we have that $\mathfrak{A}^* \cong \mathfrak{B}^* \cong \mathfrak{C}^*$, whence $\mathfrak{A}$ is an expansion of $\mathfrak{C}^*$ via a reductive $\mathcal{L}$-morphism with an isomorphic functor component and $\mathfrak{C}^*$ is a reduction of $\mathfrak{B}$ via a reductive $\mathcal{L}$-morphism with an isomorphic functor component. Therefore, since $\mathfrak{B} \in \mathtt{K}$, $\mathfrak{A} \in \mathrm{E_iR_i}(\mathtt{K})$.

   For the equality, $\mathrm{E_iL} \leq \mathrm{E_iR_i}$ is trivial, since every Leibniz reduction is a reduction via a reductive $\mathcal{L}$-morphism with an isomorphic functor component, in fact an identity functor component. For the reverse inclusion, suppose that $\mathfrak{A}$ is an $\mathcal{L}$-system and $\mathtt{K}$ a class of $\mathcal{L}$-systems, such that $\mathfrak{A} \in \mathrm{E_iR_i}(\mathtt{K})$. Thus, there exist $\mathfrak{B}, \mathfrak{C}$, with $\mathfrak{B} \in \mathtt{K}$, and reductive $\mathcal{L}$-morphisms $\langle F,\alpha\rangle : \mathfrak{C} \twoheadrightarrow_s \mathfrak{A}, \langle G,\beta\rangle : \mathfrak{B} \twoheadrightarrow_s \mathfrak{C}$ with isomorphic functor components. But then $\mathfrak{A}$ is obviously an expansion of $\mathfrak{C}^*$ via a reductive $\mathcal{L}$-morphism with an isomorphic functor component and, by Corollary 17 of [31], $\mathfrak{C}^* \cong \mathfrak{B}^*$ is a Leibniz reduct of $\mathfrak{B} \in \mathtt{K}$. Therefore, we obtain that $\mathfrak{A} \in \mathrm{E_iL}(\mathtt{K})$.

2. That $\mathrm{L} \leq \mathrm{E_iL}$ is trivial.

   We proceed, now, to show that $\mathrm{LE_i} = \mathrm{L}$. The inclusion $\mathrm{L} \leq \mathrm{LE_i}$ is obvious. For the reverse inclusion, suppose that $\mathfrak{A}$ is an $\mathcal{L}$-system and $\mathtt{K}$ a class of $\mathcal{L}$-systems, such that $\mathfrak{A} \in \mathrm{LE_i}(\mathtt{K})$. Thus, there exist $\mathfrak{B}, \mathfrak{C}$, with $\mathfrak{B} \in \mathtt{K}$ and a reductive $\mathcal{L}$-morphism

$\langle F, \alpha \rangle : \mathfrak{C} \twoheadrightarrow_s \mathfrak{B}$ with an isomorphic functor component, such that $\mathfrak{A} \cong \mathfrak{C}^*$. But, by Lemma 17 of [31], we also have that $\mathfrak{B}^* \cong \mathfrak{C}^*$ and, therefore, $\mathfrak{A} \cong \mathfrak{B}^*$, whence, since $\mathfrak{B} \in \mathtt{K}$, $\mathfrak{A} \in \mathrm{L}(\mathtt{K})$.

Finally, we show the string of inequalities

$$\mathrm{LR}_\mathrm{i} \leq \mathrm{L} \leq \mathrm{R}_i\mathrm{L} \leq \mathrm{R}_\mathrm{i} \leq \mathrm{LR}_i,$$

which will conclude the proof of Part 2.

For $\mathrm{LR}_\mathrm{i} \leq \mathrm{L}$, suppose that $\mathfrak{A}$ is an $\mathcal{L}$-system and $\mathtt{K}$ a class of $\mathcal{L}$-systems, such that $\mathfrak{A} \in \mathrm{LR}_\mathrm{i}(\mathtt{K})$. Then, there exist $\mathfrak{B}, \mathfrak{C}$, such that $\mathfrak{B} \in \mathtt{K}$ and a reductive $\mathcal{L}$-morphism $\langle F, \alpha \rangle : \mathfrak{B} \twoheadrightarrow_s \mathfrak{C}$, with isomorphic functor component, such that $\mathfrak{A} \cong \mathfrak{C}^*$. But then, using Lemma 17 of [31], we get that $\mathfrak{A} \cong \mathfrak{C}^* \cong \mathfrak{B}^*$ and, since $\mathfrak{B} \in \mathtt{K}$, $\mathfrak{A} \in \mathrm{L}(\mathtt{K})$.

$\mathtt{L} \leq \mathrm{R}_\mathrm{i}\mathrm{L}$ is trivial as is $\mathrm{R}_\mathrm{i} \leq \mathrm{LR}_i$. Thus, it suffices now to show that $\mathrm{R}_i\mathrm{L} \leq \mathrm{R}_\mathrm{i}$. Suppose that $\mathfrak{A}$ is an $\mathcal{L}$-system and $\mathtt{K}$ a class of $\mathcal{L}$-systems, such that $\mathfrak{A} \in \mathrm{R}_\mathrm{i}\mathrm{L}(\mathtt{K})$. Thus, there exist $\mathfrak{B}, \mathfrak{C}$, with $\mathfrak{B} \in \mathtt{K}$, and $\langle F, \alpha \rangle : \mathfrak{C} \twoheadrightarrow_s \mathfrak{A}$, such that $\mathfrak{C} \cong \mathfrak{B}^*$. Thus, we have that

$$\mathfrak{B} \stackrel{\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{B}}}, \pi^{\Omega(\mathfrak{B})} \rangle}{\twoheadrightarrow} \mathfrak{B}^* \cong \mathfrak{C} \twoheadrightarrow_s \mathfrak{A}$$

is a reductive $\mathcal{L}$-morphism from $\mathfrak{B}$ to $\mathfrak{A}$ with an isomorphic functor component, and, hence, since $\mathfrak{B} \in \mathtt{K}$, we have $\mathfrak{A} \in \mathrm{R}_\mathrm{i}(\mathtt{K})$.

■

We intend to continue the work presented in this paper with the goal of abstracting several of Elgueta's results to the present framework. Elgueta's results generalize well-known results of the theory of models of first-order logic to the equality-free context. The present framework leads to further generalization of these results to a multi-signature equality-free context.

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