Categorical abstract algebraic logic: The Diagram and the Reduction Operator Lemmas

George Voutsadakis*

School of Mathematics and Computer Science, Lake Superior State University,
Sault Sainte Marie, MI 49783, USA

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The study of structure systems, an abstraction of the concept of first-order structures, is continued. Structure systems have algebraic systems as their algebraic reducts and their relational component consists of a collection of relation systems on the underlying functors. An analog of the expansion of a first-order structure by constants is presented. Furthermore, analogs of the Diagram Lemma and the Reduction Operator Lemma from the theory of equality-free first-order structures are provided in the framework of structure systems.

1 Introduction

The recent rapid development of the theory of abstract algebraic logic (AAL) into a coherent theory created a common setting in which much of the previous work in the field of algebraic logic may be carried out. It also provided a unified general framework for the future study of the algebraization of a wide variety of new logical systems appearing in mathematical logic, in computer science, in linguistics and in other theoretical and applied fields of study. This development is due to the seminal work of Czelakowski, of Blok and Pigozzi, and of the Barcelona group under the leadership of Font and Jansana, among others. The interested reader may refer to the review article [18], the monograph [17], and the book [6] for an overview of the theory and its main results and applications.

The development of AAL brought again into the forefront older work of Bloom [2], which showed that its main objects of study, deductive systems, and their models, logical matrices, form a part of the theory of first-order logic without equality. More precisely, they may be studied in the framework of universal Horn logic (without equality) with a single unary relation symbol, the truth predicate, and of its ordinary first-order models.

This connection between sentential logics and equality-free first-order logic led Dellunde (partly in joint work with Casanovas and Jansana) and Elgueta (partly in joint work with Czelakowski and Jansana) to the study of equality-free first-order logic and its model theory from the point of view of AAL. Their work may be found in the series of papers [4, 8, 9, 10] and [11, 12, 13, 14, 15], respectively.

The main body of AAL, which is the one responsible for it becoming a mature and robust theory, deals with sentential logics. This framework is powerful enough to encompass most of the logics that had been studied in classical algebraic logic before, using a variety of case-specific techniques that were unified by AAL. There are logics, however, for which this framework is inadequate; most notably, logics with multiple signatures and quantifiers, but also logics whose syntax is not string-based. This led in [27] (see also [28, 29]) to the founding of the theory of categorical abstract algebraic logic (CAAL), whose main objects of study are π-institutions [16] (see also [19, 20]), structures more complex than sentential logics that allow handling successfully many of the logical systems that are not handled elegantly or cannot be handled at all by traditional methods in AAL.

* e-mail: gvoutsad@lssu.edu

In AAL, classes of universal algebras, more specifically, varieties and quasi-varieties of universal algebras, serve as the algebraic counterparts of sentential logics. In CAAL, recent work [30, 31, 32, 33] has revealed that the role of universal algebras is assumed by what were called algebraic systems in [32]. Moreover, the idea of using algebraic systems in place of universal algebras has proven fruitful in a different direction. It has helped critically in the development of an analog of the sub-theory of AAL, presented by Pałasinśka and Pigozzi [26] for handling algebraically the logical connective of implication, to the level of logics formalized as π-institutions [34, 35, 36, 37].

Because of these very recent developments and also because of some additional results obtained by the author in [38], it is very natural to consider lifting results from the model theory of equality-free first-order logic to the level of structures, abstracting first-order structures, that have as their underlying algebraic components algebraic systems rather than universal algebras. In fact, this is what has been started in the series of papers [39, 40, 41], following the work of Elgueta in [11] and inspired by both Dellunde's and Elgueta's work. These abstract structures, mimicking equality-free first-order models at this more abstract level, were introduced in [39] and called *structure systems* or simply *systems*. Basic aspects of the syntax and the semantics for the study of systems were also developed in [39] following leads from equality-free first-order logic and its models. Moreover, in [39], the reduced product construction is lifted in this setting and an analog of the well-known Ultraproduct Theorem of Łoś is obtained for structure systems. The second installment [40] studies Leibniz equality, a weak form of equality that stands in for genuine equality in the equality-free context of the theory, and provides analogs of the well-known Homomorphism Theorems of universal algebra in the context of systems. Finally, in [41], operators are introduced for classes of structure systems, and some of their properties when composed with one another are studied. A few of the concepts and the results of [39] and [40] that are critical for better understanding the developments in the present paper will be reviewed in Section 2.

The whole of this work and what is to follow has exactly the same goal in CAAL as the work of Elgueta did in AAL, i. e., to develop a more general framework for some of the results already known in a more restricted setting in CAAL and to place some of these results in a more natural context.

In the present work, this development is continued by formulating and proving analogs in the framework of structure systems of the Diagram Lemma and of the Reduction Operator Lemma of [11].

For general concepts and notation from category theory the reader is referred to any of [1, 3, 23]. Standard references on model theory are the books by Chang and Keisler [5], Hodges [21], and Marker [25].

2 Preliminaries

In this section, some concepts and some results that were introduced previously in the theory of CAAL, together with some of the basics of [39] and [40], will be recalled. This exposition of background information will, hopefully, facilitate the reading in the following sections.

Given a category $\mathbf{Sign}$ and a functor $\mathrm{SEN} : \mathbf{Sign} \longrightarrow \mathbf{Set}$, *the clone of all natural transformations on* SEN is defined to be the locally small category with collection of objects $\{\mathrm{SEN}^\alpha : \alpha \text{ an ordinal}\}$ and collection of morphisms $\tau : \mathrm{SEN}^\alpha \longrightarrow \mathrm{SEN}^\beta$ β-sequences of natural transformations $\tau_i : \mathrm{SEN}^\alpha \longrightarrow \mathrm{SEN}$ [33]. Composition

$$\mathrm{SEN}^\alpha \xrightarrow{\langle \tau_i \,:\, i<\beta\rangle} \mathrm{SEN}^\beta \xrightarrow{\langle \sigma_j \,:\, j<\gamma\rangle} \mathrm{SEN}^\gamma$$

is defined by

$$\langle \sigma_j : j < \gamma\rangle \circ \langle \tau_i : i < \beta\rangle = \langle \sigma_j(\langle \tau_i : i < \beta\rangle) : j < \gamma\rangle.$$

A subcategory $\boldsymbol{N}$ of this category consisting of *all objects* of the form SEN^k for $k < \omega$ and containing all projection morphisms $p^{k,i} : \mathrm{SEN}^k \longrightarrow \mathrm{SEN}$, $i < k$, $k < \omega$, with $p^{k,i}_\Sigma : \mathrm{SEN}(\Sigma)^k \longrightarrow \mathrm{SEN}(\Sigma)$ given by

$$p^{k,i}_\Sigma(\vec{\varphi}) = \varphi_i \quad \text{for all } \vec{\varphi} \in \mathrm{SEN}(\Sigma)^k,$$

and such that, for every family $\{\tau_i : \mathrm{SEN}^k \longrightarrow \mathrm{SEN} : i < l\}$ of natural transformations in $\boldsymbol{N}$, the sequence

$$\langle \tau_i : i < l\rangle : \mathrm{SEN}^k \longrightarrow \mathrm{SEN}^l$$

is also in $\boldsymbol{N}$, is referred to as a *category of natural transformations on* SEN.

This construction is very similar to that used in the formalization of Lawvere theories [22] (also [24, 1.5.35]) in categorical algebra, and generalizes the clone of operations on a given algebra from the theory of universal algebra.

The term *clone category* is used for a category $\boldsymbol{F}$ with all natural numbers as objects that is isomorphic to a category of natural transformations N on a given functor SEN via an isomorphism that preserves the projection operations and, as a consequence, also preserves objects (identifying SEN^k with k, $k \in \omega$).

A *system language* $\mathcal{L} = \langle \boldsymbol{F}, R, \varrho\rangle$ consists of a clone category $\boldsymbol{F}$, a nonempty set R of relation symbols, and a function $\varrho : R \longrightarrow \omega$ that assigns finite arities to the relation symbols in R. The set of *$\mathcal{L}$-terms* $\mathrm{Te}_{\mathcal{L}}$ is constructed by recursion using a fixed denumerable set of variables $V = \{x_k : k \in \omega\}$, which will sometimes be denoted, as usual, by the metavariables x, y, z, etc., by setting $x_k \in \mathrm{Te}_{\mathcal{L}}$ for all $k \in \omega$ and $\sigma(t_0, \dots, t_{n-1}) \in \mathrm{Te}_{\mathcal{L}}$ for all $\sigma \in \boldsymbol{F}(n, 1)$ and all $t_0, \dots, t_{n-1} \in \mathrm{Te}_{\mathcal{L}}$. An *atomic $\mathcal{L}$-formula* is an expression of the form $r(t_0, \dots, t_{n-1})$, where $r \in R$ with $\varrho(r) = n$ and $t_0, \dots, t_{n-1} \in \mathrm{Te}_{\mathcal{L}}$. Arbitrary *$\mathcal{L}$-formulas* are constructed using atomic $\mathcal{L}$-formulas and some set of adequate connectives and quantifiers in the ordinary first-order way.

Given a system language $\mathcal{L}$, an *$\mathcal{L}$-structure system* or, simply, *$\mathcal{L}$-system* $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \boldsymbol{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$ consists of

1. a functor $\mathrm{SEN}^{\mathfrak{A}} : \mathbf{Sign}^{\mathfrak{A}} \longrightarrow \mathbf{Set}$ with a category of natural transformations $\boldsymbol{N}^{\mathfrak{A}}$ on $\mathrm{SEN}^{\mathfrak{A}}$;
2. a surjective functor $F^{\mathfrak{A}} : \boldsymbol{F} \longrightarrow \boldsymbol{N}^{\mathfrak{A}}$ that preserves all projections $p^{k,i} : k \longrightarrow 1$, $k < \omega$, $i < k$;
3. a family $R^{\mathfrak{A}} = \{r^{\mathfrak{A}} : r \in R\}$ of relation systems on $\mathrm{SEN}^{\mathfrak{A}}$ indexed by R such that $r^{\mathfrak{A}}$ is n-ary if $\varrho(r) = n$.

$\mathcal{L}$-systems generalize both the ordinary matrix models of AAL and first-order structures, and form in the context of CAAL the natural models for theories over a fixed system language $\mathcal{L}$.

Let $t \in \mathrm{Te}_{\mathcal{L}}$ be an $\mathcal{L}$-term, $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \boldsymbol{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$ be an $\mathcal{L}$-system, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, and $\vec{\varphi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^{\omega}$. *The value of t at $\langle \Sigma, \vec{\varphi}\rangle$ in the system* $\mathfrak{A}$, denoted by $t^{\mathfrak{A}}_{\Sigma}(\vec{\varphi})$, is defined by recursion on the structure of t as follows: $x^{\mathfrak{A}}_{k_{\Sigma}}(\vec{\varphi}) = \varphi_k$ for all $k \in \omega$, and

$$\sigma(t_0, \dots, t_{n-1})^{\mathfrak{A}}_{\Sigma}(\vec{\varphi}) = F^{\mathfrak{A}}(\sigma)_{\Sigma}(t^{\mathfrak{A}}_{0_{\Sigma}}(\vec{\varphi}), \dots, t^{\mathfrak{A}}_{n-1_{\Sigma}}(\vec{\varphi}))$$

for all $\sigma \in \boldsymbol{F}(n, 1)$ and all $t_0, \dots, t_{n-1} \in \mathrm{Te}_{\mathcal{L}}$.

Finally, given an $\mathcal{L}$-formula α, an $\mathcal{L}$-system $\mathfrak{A}$, as above, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, and $\vec{\varphi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^{\omega}$, $\mathfrak{A}$ *satisfies* α *at* $\langle \Sigma, \vec{\varphi}\rangle$, written $\mathfrak{A} \models_{\Sigma} \alpha[\vec{\varphi}]$, is defined by recursion on the structure of the $\mathcal{L}$-formula α as follows:

1. If $\alpha = r(t_0, \dots, t_{n-1})$ is atomic, then $\mathfrak{A} \models_{\Sigma} r(t_0, \dots, t_{n-1})[\vec{\varphi}]$ iff $\langle t^{\mathfrak{A}}_{0_{\Sigma}}(\vec{\varphi}), \dots, t^{\mathfrak{A}}_{n-1_{\Sigma}}(\vec{\varphi})\rangle \in r^{\mathfrak{A}}_{\Sigma}$.
2. $\mathfrak{A} \models_{\Sigma} (\alpha_0 \wedge \alpha_1)[\vec{\varphi}]$ iff $\mathfrak{A} \models_{\Sigma} \alpha_0[\vec{\varphi}]$ and $\mathfrak{A} \models_{\Sigma} \alpha_1[\vec{\varphi}]$.
3. $\mathfrak{A} \models_{\Sigma} (\neg\beta)[\vec{\varphi}]$ iff $\mathfrak{A} \not\models_{\Sigma} \beta[\vec{\varphi}]$.
4. $\mathfrak{A} \models_{\Sigma} (\forall i)\beta[\vec{\varphi}]$ iff $\mathfrak{A} \models_{\Sigma} \beta[\vec{\psi}]$ for all $\vec{\psi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^{\omega}$ such that $\varphi_j = \psi_j$ for all $j \neq i$.

These conditions clearly define the semantics of all other connectives in the first-order model theory of $\mathcal{L}$-systems.

Let $\mathrm{SEN} : \mathbf{Sign} \longrightarrow \mathbf{Set}$ be a functor and N be a category of natural transformations on SEN. In the sequel, the functor of N-terms with variables in an arbitrary set X that was presented in [37] will also be used. Given a set X, the collection $\mathrm{Te}^N(X)$ of *N-terms in the variables X* is defined recursively as follows:

1. $x \in \mathrm{Te}^N(X)$ for all $x \in X$;
2. $\sigma(t_0, \dots, t_{n-1}) \in \mathrm{Te}^N(X)$ for all $\sigma : \mathrm{SEN}^n \longrightarrow \mathrm{SEN}$ in N and all $t_0, \dots, t_{n-1} \in \mathrm{Te}^N(X)$.

Moreover, given sets X, Y and a mapping $f : X \longrightarrow Y$, f induces a mapping $\mathrm{Te}^N(f) : \mathrm{Te}^N(X) \longrightarrow \mathrm{Te}^N(Y)$, defined recursively on the structure of N-terms, by

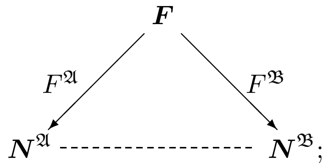
1. $\mathrm{Te}^N(f)(x) = f(x)$, for all $x \in X$;
2. for all $\sigma : \mathrm{SEN}^n \longrightarrow \mathrm{SEN}$ in N and all $t_0, \dots, t_{n-1} \in \mathrm{Te}^N(X)$,

$$\mathrm{Te}^N(f)(\sigma(t_0, \dots, t_{n-1})) = \sigma(\mathrm{Te}^N(f)(t_0), \dots, \mathrm{Te}^N(f)(t_{n-1})).$$

It is not difficult to see that, defined as above, $\mathrm{Te}^N : \mathbf{Set} \longrightarrow \mathbf{Set}$ is a functor, and that it is equipped with a category N^{t} of natural transformations that is compatible with N. By an *N-term* we will understand a member of $\mathrm{Te}^N(X)$ for some $X \in |\mathbf{Set}|$.

Let, now, $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \boldsymbol{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$ and $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \boldsymbol{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$ be two $\mathcal{L}$-structure systems. Recall from [39] that an $(\boldsymbol{N}^{\mathfrak{A}}, \boldsymbol{N}^{\mathfrak{B}})$-epimorphic translation $\langle F, \alpha\rangle : \mathrm{SEN}^{\mathfrak{A}} \longrightarrow^{\mathrm{se}} \mathrm{SEN}^{\mathfrak{B}}$ is said to be an *$\mathcal{L}$-morphism* $\langle F, \alpha\rangle : \mathfrak{A} \longrightarrow \mathfrak{B}$ if

1. the following triangle commutes:



the dashed line represents the two-way correspondence established by the $(\boldsymbol{N}^{\mathfrak{A}}, \boldsymbol{N}^{\mathfrak{B}})$-epimorphic property;

2. for all $r \in R$ with $\varrho(r) = n$, all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, and all $\vec{\varphi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$, $\vec{\varphi} \in r_{\Sigma}^{\mathfrak{A}}$ implies $\alpha_{\Sigma}(\vec{\varphi}) \in r_{F(\Sigma)}^{\mathfrak{B}}$.

Such an $\mathcal{L}$-morphism is called *strong* or *strict* if, in the last condition above, the displayed implication is replaced by an equivalence. A strict surjective $\mathcal{L}$-morphism is called *reductive* and is denoted by $\langle F, \alpha\rangle : \mathfrak{A} \longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{s}} \mathfrak{B}$.

Several results on $\mathcal{L}$-systems from both [39] and [40] will be useful in better understanding the results presented in the following sections. In [39, Lemma 6], given two $\mathcal{L}$-systems

$$\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \boldsymbol{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle \quad \text{and} \quad \mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \boldsymbol{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$$

and a strong $\mathcal{L}$-morphism $\langle F, \alpha\rangle : \mathfrak{A} \longrightarrow_{\mathrm{s}} \mathfrak{B}$, it is shown that if $\mathfrak{D}$ is an $\mathcal{L}$-subsystem of $\mathfrak{B}$, denoted $\mathfrak{D} \subseteq \mathfrak{B}$, then $\alpha^{-1}(\mathfrak{D}) \subseteq \mathfrak{A}$, and it is also shown that if $\mathfrak{C} \subseteq \mathfrak{A}$, $F : \mathbf{Sign}^{\mathfrak{A}} \longrightarrow \mathbf{Sign}^{\mathfrak{B}}$ is injective, and $F(\mathbf{Sign}^{\mathfrak{C}})$ is a subcategory of $\mathbf{Sign}^{\mathfrak{B}}$, then $\alpha(\mathfrak{C}) \subseteq \mathfrak{B}$. For the precise definitions of the pre-image $\alpha^{-1}(\mathfrak{D})$ and of the image $\alpha(\mathfrak{C})$ of $\mathcal{L}$-systems $\mathfrak{D}$ and $\mathfrak{C}$, respectively, along an $\mathcal{L}$-system morphism and related details, see [39, Section 3.2]. Moreover, in [39, Proposition 7], it is proven that every reductive $\mathcal{L}$-morphism is elementary. In [40], the notion of a *congruence system θ of an $\mathcal{L}$-system* $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \boldsymbol{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$ is defined. It is an $\boldsymbol{N}^{\mathfrak{A}}$-congruence system on $\mathrm{SEN}^{\mathfrak{A}}$, in the ordinary sense of CAAL, that, in addition, is compatible with all relation systems in $R^{\mathfrak{A}}$, i. e., such that for all $r \in R$ with $\varrho(r) = n$, all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, and all $\vec{\varphi}, \vec{\psi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$, $\vec{\varphi} \in r_{\Sigma}^{\mathfrak{A}}$ and $\vec{\varphi}\, \theta_{\Sigma}^{n} \vec{\psi}$ imply $\vec{\psi} \in r_{\Sigma}^{\mathfrak{A}}$. Furthermore, it is shown in [40, Proposition 1] that the collection $\mathrm{Con}^{\boldsymbol{N}^{\mathfrak{A}}}(\mathfrak{A})$ of all $\boldsymbol{N}^{\mathfrak{A}}$-congruence systems of $\mathfrak{A}$ forms a principal ideal of the complete lattice $\mathbf{Con}^{\boldsymbol{N}^{\mathfrak{A}}}(\mathrm{SEN}^{\mathfrak{A}})$ of all $\boldsymbol{N}^{\mathfrak{A}}$-congruence systems on $\mathrm{SEN}^{\mathfrak{A}}$, whence there always exists a largest $\boldsymbol{N}^{\mathfrak{A}}$-congruence system of $\mathfrak{A}$, which is termed *the Leibniz $\boldsymbol{N}^{\mathfrak{A}}$-congruence system of* $\mathfrak{A}$ and is denoted by $\Omega^{\boldsymbol{N}^{\mathfrak{A}}}(\mathfrak{A})$ or, more simply, $\Omega(\mathfrak{A})$. In [40, Theorem 6] a syntactic characterization is provided of the Leibniz congruence system of an $\mathcal{L}$-system $\mathfrak{A}$. A *Leibniz formula over $\mathcal{L}$* or a *Leibniz $\mathcal{L}$-formula* is a formula of the form $\beta(x, y)$ with two free variables such that for some atomic formula $\gamma(x, \vec{z})$ with at least one free variable x,

$$\beta(x, y) := (\forall \vec{z})(\gamma(x, \vec{z}) \leftrightarrow \gamma(y, \vec{z})),$$

where by $(\forall \vec{z})$ is denoted the string of universal quantifications $(\forall z_0) \ldots (\forall z_{k-1})$, where k is the length of the vector $\vec{z}$. It is shown in [40, Theorem 6] that, given an $\mathcal{L}$-system $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \boldsymbol{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, and $\varphi, \psi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$, $\langle \varphi, \psi\rangle \in \Omega_{\Sigma}^{\boldsymbol{N}^{\mathfrak{A}}}(\mathfrak{A})$ iff for all Leibniz $\mathcal{L}$-formulas $\beta(x, y)$,

$$\mathfrak{A} \vDash_{\Sigma'} \beta(x, y)[\mathrm{SEN}(f)(\varphi), \mathrm{SEN}(f)(\psi)] \quad \text{for all } \Sigma' \in |\mathbf{Sign}^{\mathfrak{A}}|,\ f \in \mathbf{Sign}^{\mathfrak{A}}(\Sigma, \Sigma').$$

Finally, in [40, Proposition 16], given two $\mathcal{L}$-systems

$$\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \boldsymbol{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle \quad \text{and} \quad \mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \boldsymbol{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$$

and a reductive system morphism $\langle F, \alpha\rangle : \mathfrak{A} \longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{s}} \mathfrak{B}$, where $F : \mathbf{Sign}^{\mathfrak{A}} \longrightarrow \mathbf{Sign}^{\mathfrak{B}}$ is an isomorphism, then the pair $\langle F^*, \alpha^*\rangle$ is defined by letting $F^* = F$ and

$$\alpha^* : \mathrm{SEN}^{\mathfrak{A}}/\Omega^{\boldsymbol{N}^{\mathfrak{A}}}(\mathfrak{A}) \longrightarrow \mathrm{SEN}^{\mathfrak{B}}/\Omega^{\boldsymbol{N}^{\mathfrak{B}}}(\mathfrak{B}) \circ F^*$$

be given, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, by

$$\alpha_{\Sigma}^*(\varphi^*) = \alpha_{\Sigma}(\varphi)^* \quad \text{for all } \varphi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma).$$

The following diagram illustrates the definition of $\langle F^*, \alpha^* \rangle$:

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\langle F, \alpha \rangle} & \mathfrak{B} \\ {\scriptstyle \langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \pi^{\boldsymbol{N}^{\mathfrak{A}}} \rangle} \downarrow & & \downarrow {\scriptstyle \langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{B}}}, \pi^{\boldsymbol{N}^{\mathfrak{B}}} \rangle} \\ \mathfrak{A}^* & \xrightarrow[\langle F^*, \alpha^* \rangle]{} & \mathfrak{B}^*. \end{array}$$

It is then proven that $\langle F^*, \alpha^* \rangle : \mathfrak{A}^* \longrightarrow \mathfrak{B}^*$ is an isomorphism of $\mathcal{L}$-systems. For the definition of quotient $\mathcal{L}$-systems and other related details, see [40, Section 4].

3 Expansion by constants

Suppose that $\mathcal{L} = \langle \boldsymbol{F}, R, \varrho \rangle$ is a system language and $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \boldsymbol{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$ is an $\mathcal{L}$-structure system. The analog in the context of structure systems of the expansion $\mathcal{L}_A$ of an equality-free first-order language $\mathcal{L}$ by a new set C_A of constant symbols c_a, $a \in A$, is *the expansion* $\mathcal{L}_{\mathfrak{A}}$ *of the system language* $\mathcal{L}$ *by the sentence functor* $\mathrm{SEN}^{\mathfrak{A}}$, i. e., the quadruple $\mathcal{L}_{\mathfrak{A}} = \langle \boldsymbol{F}, \mathrm{SEN}^{\mathfrak{A}}, R, \varrho \rangle$.

Using the notation of [37, Section 2] that was reviewed in the introduction, define a new sentence functor $\mathrm{Te}^{\mathfrak{A}} : \mathbf{Sign}^{\mathfrak{A}} \longrightarrow \mathbf{Set}$ as follows: At the object level, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$,

$$\mathrm{Te}^{\mathfrak{A}}(\Sigma) = \mathrm{Te}^{\boldsymbol{F}}(V \cup \mathrm{SEN}^{\mathfrak{A}}(\Sigma)).$$

Of course it is assumed that the variables in the denumerable collection V are disjoint from all sentences φ in $\mathrm{SEN}^{\mathfrak{A}}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$.

At the morphism level, for all $f \in \mathbf{Sign}^{\mathfrak{A}}(\Sigma, \Sigma')$, define $\mathrm{Te}^{\mathfrak{A}}(f) : \mathrm{Te}^{\mathfrak{A}}(\Sigma) \longrightarrow \mathrm{Te}^{\mathfrak{A}}(\Sigma')$ by recursion on the structure of $t \in \mathrm{Te}^{\mathfrak{A}}(\Sigma)$ as follows:

1. If $t = v \in V$, then $\mathrm{Te}^{\mathfrak{A}}(f)(v) = v$.
2. If $t = \varphi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$, then $\mathrm{Te}^{\mathfrak{A}}(f)(\varphi) = \mathrm{SEN}^{\mathfrak{A}}(f)(\varphi)$.
3. If $t = \sigma(t_0, \ldots, t_{n-1})$ for some $\sigma \in \boldsymbol{F}(n, 1)$ and $t_0, \ldots, t_{n-1} \in \mathrm{Te}^{\mathfrak{A}}(\Sigma)$, then

$$\mathrm{Te}^{\mathfrak{A}}(f)(\sigma(t_0, \ldots, t_{n-1})) = \sigma(\mathrm{Te}^{\mathfrak{A}}(f)(t_0), \ldots, \mathrm{Te}^{\mathfrak{A}}(f)(t_{n-1})).$$

An element $t \in \mathrm{Te}^{\mathfrak{A}}(\Sigma)$ will be referred to as a Σ*-term over* $\mathcal{L}_{\mathfrak{A}}$.

We define similarly the functor $\mathrm{Fm}^{\mathfrak{A}} : \mathbf{Sign}^{\mathfrak{A}} \longrightarrow \mathbf{Set}$ yielding Σ*-formulas over* $\mathcal{L}_{\mathfrak{A}}$ as follows: At the object level, $\mathrm{Fm}^{\mathfrak{A}}(\Sigma)$ is built recursively out of the Σ-terms over $\mathcal{L}_{\mathfrak{A}}$, the relation symbols in R, and some adequate collection of first-order connectives in the usual way. At the morphism level, the only change in a Σ-formula over $\mathcal{L}_{\mathfrak{A}}$ that is effected by $\mathrm{Fm}^{\mathfrak{A}}(f)$, where $f \in \mathbf{Sign}^{\mathfrak{A}}(\Sigma, \Sigma')$, is that every Σ-sentence $\varphi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$ appearing in a term in the formula is replaced by the Σ'-sentence $\mathrm{SEN}^{\mathfrak{A}}(f)(\varphi)$, i. e., every Σ-term t over $\mathcal{L}_{\mathfrak{A}}$ in the formula is replaced by the Σ'-term $\mathrm{Te}^{\mathfrak{A}}(f)(t)$.

An $\mathcal{L}_{\mathfrak{A}}$(*-structure*) *system* is a quadruple

$$\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \boldsymbol{N}^{\mathfrak{B}}, F^{\mathfrak{B}} \rangle, \langle C^{\mathfrak{B}}, \gamma^{\mathfrak{B}} \rangle, R^{\mathfrak{B}} \rangle,$$

where $\langle \mathrm{SEN}^{\mathfrak{B}}, \langle \boldsymbol{N}^{\mathfrak{B}}, F^{\mathfrak{B}} \rangle, R^{\mathfrak{B}} \rangle$ is an $\mathcal{L}$-system and $\langle C^{\mathfrak{B}}, \gamma^{\mathfrak{B}} \rangle : \mathrm{SEN}^{\mathfrak{A}} \longrightarrow^{\mathrm{s}} \mathrm{SEN}^{\mathfrak{B}}$ is a singleton translation.

Next, *the value* $t^{\mathfrak{B}}_{\Sigma}(\vec{\psi})$ *of a* Σ*-term* t *over* $\mathcal{L}_{\mathfrak{A}}$ *in an* $\mathcal{L}_{\mathfrak{A}}$*-system* $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \boldsymbol{N}^{\mathfrak{B}}, F^{\mathfrak{B}} \rangle, \langle C^{\mathfrak{B}}, \gamma^{\mathfrak{B}} \rangle, R^{\mathfrak{B}} \rangle$ *at the tuple* $\vec{\psi} \in \mathrm{SEN}^{\mathfrak{B}}(C^{\mathfrak{B}}(\Sigma))^{\omega}$ is defined by recursion on the structure of t as follows:

1. If $t = v_i \in V$, then $t^{\mathfrak{B}}_{\Sigma}(\vec{\psi}) = \psi_i$.
2. If $t = \varphi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$, then $t^{\mathfrak{B}}_{\Sigma}(\vec{\psi}) = \gamma^{\mathfrak{B}}_{\Sigma}(\varphi)$.
3. If $t = \sigma(t_0, \ldots, t_{n-1})$ for some n-ary σ in $\boldsymbol{F}$ and $t_0, \ldots, t_{n-1} \in \mathrm{Te}^{\mathfrak{A}}(\Sigma)$, then

$$t^{\mathfrak{B}}_{\Sigma}(\vec{\psi}) = \sigma^{\mathfrak{B}}_{C^{\mathfrak{B}}(\Sigma)}(t^{\mathfrak{B}}_{0_{\Sigma}}(\vec{\psi}), \ldots, t^{\mathfrak{B}}_{n-1_{\Sigma}}(\vec{\psi})).$$

Satisfaction of a Σ*-formula* $\alpha(\vec{v})$ *over* $\mathcal{L}_{\mathfrak{A}}$ *with free variables in the list* $\vec{v}$ *in an* $\mathcal{L}_{\mathfrak{A}}$*-system* $\mathfrak{B}$ *at the tuple* $\vec{\psi} \in \mathrm{SEN}^{\mathfrak{B}}(C^{\mathfrak{B}}(\Sigma))^{\omega}$ is denoted by $\mathfrak{B} \vDash_{\Sigma} \alpha[\vec{\psi}]$ and is defined by recursion on the structure of the Σ-formula α as follows:

1. If $\alpha = r(t_0, \dots, t_{n-1})$ is atomic, i. e., if $r \in R$ with $\varrho(r) = n$ and $t_0, \dots, t_{n-1} \in \mathrm{Te}^{\mathfrak{A}}(\Sigma)$, then

$$\mathfrak{B} \vDash_{\Sigma} r(t_0, \dots, t_{n-1})[\vec{\psi}] \quad \text{iff} \quad \langle t^{\mathfrak{B}}_{0_\Sigma}(\vec{\psi}), \dots, t^{\mathfrak{B}}_{n-1_\Sigma}(\vec{\psi})\rangle \in r^{\mathfrak{B}}_{C^{\mathfrak{B}}(\Sigma)}.$$

2. If α is a Boolean combination of Σ-formulas over $\mathcal{L}_{\mathfrak{A}}$ or results from a quantification of some other Σ-formula over $\mathcal{L}_{\mathfrak{A}}$, then the recursive step mimics the corresponding one from first-order logic.

We denote by $\mathrm{At}^{\mathfrak{A}}(\Sigma)$ the collection of all atomic Σ*-sentences over* $\mathcal{L}_{\mathfrak{A}}$.

Finally, we say that a Σ-formula α over $\mathcal{L}_{\mathfrak{A}}$ is *satisfied in an* $\mathcal{L}_{\mathfrak{A}}$*-system* $\mathfrak{B}$ if it is satisfied at every tuple in the system.

4 The Diagram Lemma

Let $\mathcal{L} = \langle \boldsymbol{F}, R, \varrho\rangle$ be a system language and $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \boldsymbol{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$ be an $\mathcal{L}$-system. *The diagram of* $\mathfrak{A}$, denoted by $\mathrm{Dg}\mathfrak{A} = \{\mathrm{Dg}_{\Sigma}\mathfrak{A}\}_{\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|}$, is defined by setting, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$,

$$\begin{aligned}\mathrm{Dg}_{\Sigma}\mathfrak{A} = \{&\alpha \in \mathrm{At}^{\mathfrak{A}}(\Sigma) : \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \boldsymbol{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, \langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \iota\rangle, R^{\mathfrak{A}}\rangle \vDash_{\Sigma} \alpha\}\\ &\cup \{\neg\alpha : \alpha \in \mathrm{At}^{\mathfrak{A}}(\Sigma), \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \boldsymbol{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, \langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \iota\rangle, R^{\mathfrak{A}}\rangle \vDash_{\Sigma} \neg\alpha\}.\end{aligned}$$

The Leibniz diagram of $\mathfrak{A}$, denoted by $\mathrm{Dgl}\mathfrak{A} = \{\mathrm{Dgl}_{\Sigma}\mathfrak{A}\}_{\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|}$, is defined by letting, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, $\mathrm{Dgl}_{\Sigma}\mathfrak{A}$ be the union of $\mathrm{Dg}_{\Sigma}\mathfrak{A}$ and all Σ-sentences over $\mathcal{L}_{\mathfrak{A}}$ of the form $\beta(t, t')$, for $\beta(x, y)$ a Leibniz $\mathcal{L}$-formula (see [40, Section 3] and, also, the introduction) and $t, t' \in \mathrm{Te}^{\mathfrak{A}}(\Sigma)$ closed Σ-terms over $\mathcal{L}_{\mathfrak{A}}$, i. e., containing no variables in V, such that $\langle t^{\mathfrak{A}}_{\Sigma}, t'^{\mathfrak{A}}_{\Sigma}\rangle \in \Omega_{\Sigma}(\mathfrak{A})$. More formally,

$$\begin{aligned}\mathrm{Dgl}_{\Sigma}\mathfrak{A} = \mathrm{Dg}_{\Sigma}\mathfrak{A} \cup \{\beta(t, t') : \;&\beta(x, y) \text{ a Leibniz } \mathcal{L}\text{-formula}, t, t' \in \mathrm{Te}^{\mathfrak{A}}(\Sigma) \text{ closed},\\ &\text{and } \langle t^{\mathfrak{A}}_{\Sigma}, t'^{\mathfrak{A}}_{\Sigma}\rangle \in \Omega_{\Sigma}(\mathfrak{A})\}.\end{aligned}$$

The elementary diagram $\mathrm{Dge}\mathfrak{A}$ *of* $\mathfrak{A}$ is defined as $\mathrm{Dge}\mathfrak{A} = \{\mathrm{Dge}_{\Sigma}\mathfrak{A}\}_{\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|}$, where for each $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, $\mathrm{Dge}_{\Sigma}\mathfrak{A}$ is the collection of all Σ-sentences over $\mathcal{L}_{\mathfrak{A}}$ which hold in $\langle \mathrm{SEN}^{\mathfrak{A}}, \langle \boldsymbol{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, \langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \iota\rangle, R^{\mathfrak{A}}\rangle$.

Note that, by [40, Theorem 6], we have that $\mathrm{Dgl}\mathfrak{A} \le \mathrm{Dge}\mathfrak{A}$, i. e., for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, $\mathrm{Dgl}_{\Sigma}\mathfrak{A} \subseteq \mathrm{Dge}_{\Sigma}\mathfrak{A}$.

The following lemma forms an analog for structure systems of the Diagram Lemma [11, Lemma 4.5]. Before its formulation, the reader should take notice of the construction of the singleton translation

$$\langle F^*, \alpha^*\rangle : \mathrm{SEN}^{\mathfrak{A}}/\Omega(\mathfrak{A}) \longrightarrow^{\mathrm{s}} \mathrm{SEN}^{\mathfrak{B}}/\Omega(\mathfrak{B})$$

out of the singleton translation $\langle F, \alpha\rangle : \mathrm{SEN}^{\mathfrak{A}} \longrightarrow^{\mathrm{s}} \mathrm{SEN}^{\mathfrak{B}}$ that was presented, under several appropriate hypotheses, in [40, Proposition 16] and reviewed in the introduction. The same construction, under different hypotheses, will be key in the proof of the Diagram Lemma.

Theorem 4.1 (Diagram Lemma) *Let* $\mathcal{L} = \langle \boldsymbol{F}, R, \varrho\rangle$ *be a system language,*

$$\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \boldsymbol{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle \quad \textit{and} \quad \mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \boldsymbol{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$$

be two $\mathcal{L}$*-systems, and* $\langle F, \alpha\rangle : \mathrm{SEN}^{\mathfrak{A}} \longrightarrow^{\mathrm{s}} \mathrm{SEN}^{\mathfrak{B}}$ *a singleton translation, where* $F : \mathbf{Sign}^{\mathfrak{A}} \longrightarrow \mathbf{Sign}^{\mathfrak{B}}$ *is a surjective functor.*

1. *If* $\langle \mathrm{SEN}^{\mathfrak{B}}, \langle \boldsymbol{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, \langle F, \alpha\rangle, R^{\mathfrak{B}}\rangle$ *is a model of* $\mathrm{Dgl}\mathfrak{A}$*, then* $\langle F^*, \alpha^*\rangle : \mathfrak{A}^* \longrightarrow_{\mathrm{s}} \mathfrak{B}^*$ *is a strong* $\mathcal{L}$*-morphism from* $\mathfrak{A}^*$ *into* $\mathfrak{B}^*$ *and* α^*_{Σ} *is injective for all* $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$.

2. *If* $\langle \mathrm{SEN}^{\mathfrak{B}}, \langle \boldsymbol{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, \langle F, \alpha\rangle, R^{\mathfrak{B}}\rangle$ *is a model of* $\mathrm{Dge}\mathfrak{A}$*, then* $\langle F^*, \alpha^*\rangle : \mathfrak{A}^* \longrightarrow_{\mathrm{e}} \mathfrak{B}^*$ *is an elementary* $\mathcal{L}$*-morphism from* $\mathfrak{A}^*$ *into* $\mathfrak{B}^*$ *and* α^*_{Σ} *is injective for all* $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$.

Moreover, implications become equivalences if, in addition, $\langle F, \alpha\rangle : \mathrm{SEN}^{\mathfrak{A}} \longrightarrow^{\mathrm{se}} \mathrm{SEN}^{\mathfrak{B}}$ *is an* $(\boldsymbol{N}^{\mathfrak{A}}, \boldsymbol{N}^{\mathfrak{B}})$*-epimorphic translation, where* $F : \mathbf{Sign}^{\mathfrak{A}} \longrightarrow \mathbf{Sign}^{\mathfrak{B}}$ *is a surjective functor.*

P r o o f.

1. Assume that $\langle \mathrm{SEN}^{\mathfrak{B}}, \langle \boldsymbol{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, \langle F, \alpha\rangle, R^{\mathfrak{B}}\rangle$ is a model of $\mathrm{Dgl}\mathfrak{A}$. Define $F^* = F : \mathbf{Sign}^{\mathfrak{A}} \longrightarrow \mathbf{Sign}^{\mathfrak{B}}$ and $\alpha^* : \mathrm{SEN}^{\mathfrak{A}}/\Omega(\mathfrak{A}) \longrightarrow \mathrm{SEN}^{\mathfrak{B}}/\Omega(\mathfrak{B}) \circ F^*$ by letting, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$,

$$\alpha^*_\Sigma : \mathrm{SEN}^{\mathfrak{A}}(\Sigma)/\Omega_\Sigma(\mathfrak{A}) \longrightarrow \mathrm{SEN}^{\mathfrak{B}}(F(\Sigma))/\Omega_{F(\Sigma)}(\mathfrak{B})$$

be defined by

$$\alpha^*_\Sigma(\varphi^*) = \alpha_\Sigma(\varphi)^* \quad \text{for all } \varphi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma).$$

It must first be shown that α^* is a well-defined singleton translation from $\mathrm{SEN}^{\mathfrak{A}}/\Omega(\mathfrak{A})$ to $\mathrm{SEN}^{\mathfrak{B}}/\Omega(\mathfrak{B}) \circ F$. To this end, suppose that $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and that $\varphi, \varphi' \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$ are such that $\varphi^* = \varphi'^*$, i. e., $\langle \varphi, \varphi'\rangle \in \Omega_\Sigma(\mathfrak{A})$. By [40, Theorem 6], for all $\Sigma' \in |\mathbf{Sign}^{\mathfrak{A}}|$ and all $f \in \mathbf{Sign}^{\mathfrak{A}}(\Sigma, \Sigma')$, $\beta(\mathrm{SEN}^{\mathfrak{A}}(f)(\varphi), \mathrm{SEN}^{\mathfrak{A}}(f)(\varphi')) \in \mathrm{Dgl}_{\Sigma'}\mathfrak{A}$ for all Leibniz $\mathcal{L}$-formulas $\beta(x, y)$. Thus, since $\langle \mathrm{SEN}^{\mathfrak{B}}, \langle \boldsymbol{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, \langle F, \alpha\rangle, R^{\mathfrak{B}}\rangle$ is a model of $\mathrm{Dgl}\mathfrak{A}$, we get that

$$\mathfrak{B} \vDash_{F(\Sigma')} \beta(x, y)[\alpha_{\Sigma'}(\mathrm{SEN}^{\mathfrak{A}}(f)(\varphi)), \alpha_{\Sigma'}(\mathrm{SEN}^{\mathfrak{A}}(f)(\varphi'))].$$

This is equivalent, by the naturality of α, to

$$\mathfrak{B} \vDash_{F(\Sigma')} \beta(x, y)[\mathrm{SEN}^{\mathfrak{B}}(F(f))(\alpha_\Sigma(\varphi)), \mathrm{SEN}^{\mathfrak{B}}(F(f))(\alpha_\Sigma(\varphi'))].$$

Hence, by the surjectivity of F combined with [40, Theorem 6], we get that $\langle \alpha_\Sigma(\varphi), \alpha_\Sigma(\varphi')\rangle \in \Omega_{F(\Sigma)}(\mathfrak{B})$, i. e., that $\alpha_\Sigma(\varphi)^* = \alpha_\Sigma(\varphi')^*$. Therefore, α^*_Σ is a well-defined mapping for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$.

To see that $\alpha^* : \mathrm{SEN}^{\mathfrak{A}}/\Omega(\mathfrak{A}) \longrightarrow \mathrm{SEN}^{\mathfrak{B}}/\Omega(\mathfrak{B}) \circ F$ is a natural transformation, consider $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}^{\mathfrak{A}}|$, $f \in \mathbf{Sign}^{\mathfrak{A}}(\Sigma_1, \Sigma_2)$, and $\varphi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma_1)$. Then

$$\begin{array}{ccc}
\mathrm{SEN}^{\mathfrak{A}}(\Sigma_1)/\Omega_{\Sigma_1}(\mathfrak{A}) & \xrightarrow{\alpha^*_{\Sigma_1}} & \mathrm{SEN}^{\mathfrak{B}}(F(\Sigma_1))/\Omega_{F(\Sigma_1)}(\mathfrak{A}) \\
{\scriptstyle \mathrm{SEN}^{\mathfrak{A}}(f)/\Omega(\mathfrak{A})}\downarrow & & \downarrow{\scriptstyle \mathrm{SEN}^{\mathfrak{B}}(F(f))/\Omega(\mathfrak{B})} \\
\mathrm{SEN}^{\mathfrak{A}}(\Sigma_2)/\Omega_{\Sigma_2}(\mathfrak{A}) & \xrightarrow[\alpha^*_{\Sigma_2}]{} & \mathrm{SEN}^{\mathfrak{B}}(F(\Sigma_2))/\Omega_{F(\Sigma_2)}(\mathfrak{B})
\end{array}$$

and

$$\begin{aligned}
\alpha^*_{\Sigma_2}(\mathrm{SEN}^{\mathfrak{A}}(f)/\Omega(\mathfrak{A})(\varphi^*)) &= \alpha^*_{\Sigma_2}(\mathrm{SEN}^{\mathfrak{A}}(f)(\varphi)^*) \\
&= \alpha_{\Sigma_2}(\mathrm{SEN}^{\mathfrak{A}}(f)(\varphi))^* \\
&= \mathrm{SEN}^{\mathfrak{B}}(F(f))(\alpha_{\Sigma_1}(\varphi))^* \\
&= \mathrm{SEN}^{\mathfrak{B}}(F(f))/\Omega(\mathfrak{B})(\alpha_{\Sigma_1}(\varphi)^*) \\
&= \mathrm{SEN}^{\mathfrak{B}}(F(f))/\Omega(\mathfrak{B})(\alpha^*_{\Sigma_1}(\varphi^*)).
\end{aligned}$$

Next, we show that $\langle F, \alpha^*\rangle$ is $(\boldsymbol{N}^{\mathfrak{A}^*}, \boldsymbol{N}^{\mathfrak{B}^*})$-epimorphic. To this end, suppose that σ is n-ary in $\boldsymbol{F}$, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, and $\vec{\varphi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$. Then, since $\langle \sigma^{\mathfrak{A}}_\Sigma(\vec{\varphi}), \sigma^{\mathfrak{A}}_\Sigma(\vec{\varphi})\rangle \in \Omega_\Sigma(\mathfrak{A})$, by [40, Theorem 6] we get that for all Leibniz $\mathcal{L}$-formulas $\beta(x, y)$, all $\Sigma' \in |\mathbf{Sign}^{\mathfrak{A}}|$, and all $f \in \mathbf{Sign}^{\mathfrak{A}}(\Sigma, \Sigma')$,

$$\beta(\mathrm{SEN}^{\mathfrak{A}}(f)(\sigma^{\mathfrak{A}}_\Sigma(\vec{\varphi})), \sigma(\mathrm{SEN}^{\mathfrak{A}}(f)(\vec{\varphi}))) \in \mathrm{Dgl}_{\Sigma'}(\mathfrak{A}).$$

But, by the hypothesis, we have that $\langle \mathrm{SEN}^{\mathfrak{B}}, \langle \boldsymbol{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, \langle F, \alpha\rangle, R^{\mathfrak{B}}\rangle$ is a model of $\mathrm{Dgl}\mathfrak{A}$, whence we get that

$$\mathfrak{B} \vDash_{F(\Sigma')} \beta(x, y)[\alpha_{\Sigma'}(\mathrm{SEN}^{\mathfrak{A}}(f)(\sigma^{\mathfrak{A}}_\Sigma(\vec{\varphi}))), \sigma^{\mathfrak{B}}_{F(\Sigma')}(\alpha_\Sigma(\mathrm{SEN}^{\mathfrak{A}}(f)(\vec{\varphi})))],$$

whence

$$\mathfrak{B} \vDash_{F(\Sigma')} \beta(x, y)[\alpha_{\Sigma'}(\mathrm{SEN}^{\mathfrak{A}}(f)(\sigma^{\mathfrak{A}}_\Sigma(\vec{\varphi}))), \sigma^{\mathfrak{B}}_{F(\Sigma')}(\mathrm{SEN}^{\mathfrak{B}}(F(f))(\alpha_\Sigma(\vec{\varphi})))]$$

and, therefore,

$$\mathfrak{B} \vDash_{F(\Sigma')} \beta(x,y)[\mathrm{SEN}^{\mathfrak{B}}(F(f))(\alpha_\Sigma(\sigma^{\mathfrak{A}}_\Sigma(\vec{\varphi}))), \mathrm{SEN}^{\mathfrak{B}}(F(f))(\sigma^{\mathfrak{B}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\varphi})))].$$

Thus, by the surjectivity of F together with [40, Theorem 6], we obtain that

$$\langle \alpha_\Sigma(\sigma^{\mathfrak{A}}_\Sigma(\vec{\varphi})), \sigma^{\mathfrak{B}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\varphi}))\rangle \in \Omega_{F(\Sigma)}(\mathfrak{B}).$$

Thus,

(1) $$\alpha_\Sigma(\sigma^{\mathfrak{A}}_\Sigma(\vec{\varphi}))^* = \sigma^{\mathfrak{B}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\varphi}))^*.$$

Finally, we obtain

$$\begin{aligned}\alpha^*_\Sigma(\sigma^{\mathfrak{A}^*}_\Sigma(\vec{\varphi}^*)) &= \alpha^*_\Sigma(\sigma^{\mathfrak{A}}_\Sigma(\vec{\varphi})^*)\\ &= \alpha_\Sigma(\sigma^{\mathfrak{A}}_\Sigma(\vec{\varphi}))^*\\ &= \sigma^{\mathfrak{B}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\varphi}))^* \qquad \text{(by equation (1))}\\ &= \sigma^{\mathfrak{B}^*}_{F(\Sigma)}(\alpha_\Sigma(\vec{\varphi})^*)\\ &= \sigma^{\mathfrak{B}^*}_{F(\Sigma)}(\alpha^*_\Sigma(\vec{\varphi}^*)).\end{aligned}$$

This shows that $\langle F,\alpha^*\rangle : \mathrm{SEN}^{\mathfrak{A}}/\Omega(\mathfrak{A}) \longrightarrow^{\mathrm{se}} \mathrm{SEN}^{\mathfrak{B}}/\Omega(\mathfrak{B})$ is indeed an $(\boldsymbol{N}^{\mathfrak{A}^*}, \boldsymbol{N}^{\mathfrak{B}^*})$-epimorphic translation.

Next, it is shown that $\langle F,\alpha^*\rangle : \mathfrak{A}^* \longrightarrow \mathfrak{B}^*$ is an $\mathcal{L}$-system morphism and that it is strong. In fact, given $r \in R$ with $\varrho(r) = n$, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, and $\vec{\varphi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$, we get that

$$\begin{aligned}\vec{\varphi}^* \in r^{\mathfrak{A}^*}_\Sigma \quad &\text{iff} \quad \vec{\varphi} \in r^{\mathfrak{A}}_\Sigma\\ &\text{iff} \quad r(\vec{\varphi}) \in \mathrm{Dgl}_\Sigma\mathfrak{A}\\ &\text{iff} \quad \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \boldsymbol{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, \langle F,\alpha\rangle, R^{\mathfrak{B}}\rangle \vDash_\Sigma r(\vec{\varphi})\\ &\text{iff} \quad \alpha_\Sigma(\vec{\varphi}) \in r^{\mathfrak{B}}_{F(\Sigma)}\\ &\text{iff} \quad \alpha_\Sigma(\vec{\varphi})^* \in r^{\mathfrak{B}^*}_{F(\Sigma)}\\ &\text{iff} \quad \alpha^*_\Sigma(\vec{\varphi}^*) \in r^{\mathfrak{B}^*}_{F(\Sigma)}.\end{aligned}$$

This shows that $\langle F,\alpha^*\rangle : \mathfrak{A}^* \longrightarrow_{\mathrm{s}} \mathfrak{B}^*$ is a strong $\mathcal{L}$-system morphism.

Finally, by [40, Lemma 2], we get that $\mathrm{Ker}(\langle F,\alpha^*\rangle) \in \mathrm{Con}(\mathfrak{A}^*)$, whence $\mathrm{Ker}(\langle F,\alpha^*\rangle) = \Delta^{\mathrm{SEN}^{\mathfrak{A}^*}}$, since $\mathfrak{A}^*$ is reduced. Hence α^*_Σ is injective for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$.

Suppose, conversely, that $\langle F,\alpha\rangle : \mathrm{SEN}^{\mathfrak{A}} \longrightarrow^{\mathrm{se}} \mathrm{SEN}^{\mathfrak{B}}$, where $F : \mathbf{Sign}^{\mathfrak{A}} \longrightarrow \mathbf{Sign}^{\mathfrak{B}}$ is a surjective functor, is an $(\boldsymbol{N}^{\mathfrak{A}}, \boldsymbol{N}^{\mathfrak{B}})$-epimorphic translation, and that $\langle F^*,\alpha^*\rangle : \mathfrak{A}^* \longrightarrow_{\mathrm{s}} \mathfrak{B}^*$ is a strong $\mathcal{L}$-morphism from $\mathfrak{A}^*$ into $\mathfrak{B}^*$ and α^*_Σ is injective for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$. We must show that $\langle \mathrm{SEN}^{\mathfrak{B}}, \langle \boldsymbol{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, \langle F,\alpha\rangle, R^{\mathfrak{B}}\rangle$ is a model of $\mathrm{Dgl}\mathfrak{A}$.

First, suppose that $r \in R$ with $\varrho(r) = n$, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, and $\vec{\varphi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$. Since $\langle F,\alpha^*\rangle$ is strong, we obtain that

$$\begin{aligned}&\langle \mathrm{SEN}^{\mathfrak{A}}, \langle \boldsymbol{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, \langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \iota\rangle, R^{\mathfrak{A}}\rangle \vDash_\Sigma r(\vec{\varphi}) \quad \text{iff}\\ &\qquad\qquad \langle \mathrm{SEN}^{\mathfrak{B}^*}, \langle \boldsymbol{N}^{\mathfrak{B}^*}, F^{\mathfrak{B}^*}\rangle, \langle F,\alpha^*\rangle, R^{\mathfrak{B}^*}\rangle \vDash_\Sigma r(\vec{\varphi}).\end{aligned}$$

Thus, we obtain that

$$\langle \mathrm{SEN}^{\mathfrak{A}}, \langle \boldsymbol{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, \langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \iota\rangle, R^{\mathfrak{A}}\rangle \vDash_\Sigma r(\vec{\varphi}) \quad \text{iff} \quad \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \boldsymbol{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, \langle F,\alpha\rangle, R^{\mathfrak{B}}\rangle \vDash_\Sigma r(\vec{\varphi}).$$

Next, suppose that $\beta(x,y)$ is a Leibniz $\mathcal{L}$-formula, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, and $t(\vec{\varphi}), t'(\vec{\varphi})$ are Σ-terms over $\mathcal{L}_{\mathfrak{A}}$ such that $\beta(t,t') \in \mathrm{Dgl}_\Sigma\mathfrak{A}$. Then we have $\langle t^{\mathfrak{A}}_\Sigma(\vec{\varphi}), t'^{\mathfrak{A}}_\Sigma(\vec{\varphi})\rangle \in \Omega_\Sigma(\mathfrak{A})$. Thus, since $\langle F,\alpha\rangle : \mathrm{SEN}^{\mathfrak{A}} \longrightarrow^{\mathrm{se}} \mathrm{SEN}^{\mathfrak{B}}$ is an $(\boldsymbol{N}^{\mathfrak{A}}, \boldsymbol{N}^{\mathfrak{B}})$-epimorphic translation such that $\langle F,\alpha^*\rangle : \mathfrak{A}^* \longrightarrow_{\mathrm{s}} \mathfrak{B}^*$, we obtain that

$$\langle \alpha_\Sigma(t^{\mathfrak{A}}_\Sigma(\vec{\varphi})), \alpha_\Sigma(t'^{\mathfrak{A}}_\Sigma(\vec{\varphi}))\rangle \in \Omega_{F(\Sigma)}(\mathfrak{B}),$$

i. e., $\langle t^{\mathfrak{B}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\varphi})), t'^{\mathfrak{B}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\varphi}))\rangle \in \Omega_{F(\Sigma)}(\mathfrak{B})$. Hence, by [40, Theorem 6], we get that

$$\langle \mathrm{SEN}^{\mathfrak{B}}, \langle \boldsymbol{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, \langle F, \alpha\rangle, R^{\mathfrak{B}}\rangle \vDash_\Sigma \beta(t, t')$$

and, therefore, $\langle \mathrm{SEN}^{\mathfrak{B}}, \langle \boldsymbol{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, \langle F, \alpha\rangle, R^{\mathfrak{B}}\rangle$ is a model of $\mathrm{Dgl}_\Sigma\mathfrak{A}$. But $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ was arbitrary, which means that $\langle \mathrm{SEN}^{\mathfrak{B}}, \langle \boldsymbol{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, \langle F, \alpha\rangle, R^{\mathfrak{B}}\rangle$ is a model of $\mathrm{Dgl}\mathfrak{A}$, as was to be shown.

2. By 1., showing that the strong $\mathcal{L}$-system morphism $\langle F^*, \alpha^*\rangle : \mathfrak{A}^* \longrightarrow_{\mathrm{s}} \mathfrak{B}^*$ is also elementary is sufficient to prove the "only if" direction. To this end, suppose that $\gamma(\vec{x})$ is an $\mathcal{L}$-formula whose free variables are among those in the list $\vec{x}$ of length n, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, and $\vec{\varphi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$. Then we have

$$\begin{array}{lll}
\mathfrak{A}^* \vDash_\Sigma \gamma[\vec{\varphi}^*] & \text{iff} \quad \mathfrak{A} \vDash_\Sigma \gamma[\vec{\varphi}] & \text{(by [39, Proposition 7])}\\
& \text{iff} \quad \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \boldsymbol{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, \langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \iota\rangle, R^{\mathfrak{A}}\rangle \vDash_\Sigma \gamma(\vec{\varphi}) & \\
& \text{iff} \quad \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \boldsymbol{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, \langle F, \alpha\rangle, R^{\mathfrak{B}}\rangle \vDash_\Sigma \gamma(\vec{\varphi}) & \text{(by the hypothesis)}\\
& \text{iff} \quad \mathfrak{B} \vDash_{F(\Sigma)} \gamma[\alpha_\Sigma(\vec{\varphi})] & \\
& \text{iff} \quad \mathfrak{B}^* \vDash_{F(\Sigma)} \gamma[\alpha_\Sigma(\vec{\varphi})^*] & \text{(by [39, Proposition 7])}\\
& \text{iff} \quad \mathfrak{B}^* \vDash_{F(\Sigma)} \gamma[\alpha^*_\Sigma(\vec{\varphi}^*)] & \text{(by the definition of } \alpha^*_\Sigma).
\end{array}$$

For the converse, suppose that $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and that $\gamma(\vec{\varphi})$ is a Σ-sentence over $\mathcal{L}^{\mathfrak{A}}$. Then

$$\begin{array}{lll}
\multicolumn{3}{l}{\langle \mathrm{SEN}^{\mathfrak{A}}, \langle \boldsymbol{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, \langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \iota\rangle, R^{\mathfrak{A}}\rangle \vDash_\Sigma \gamma(\vec{\varphi})}\\
& \text{iff} \quad \mathfrak{A} \vDash_\Sigma \gamma[\vec{\varphi}] \quad (\gamma \text{ an } \mathcal{L}\text{-sentence}) & \\
& \text{iff} \quad \mathfrak{A}^* \vDash_\Sigma \gamma[\vec{\varphi}^*] & \text{(by [39, Proposition 7])}\\
& \text{iff} \quad \mathfrak{B}^* \vDash_{F(\Sigma)} \gamma[\alpha^*_\Sigma(\vec{\varphi}^*)] & \text{(by the hypothesis)}\\
& \text{iff} \quad \mathfrak{B} \vDash_{F(\Sigma)} \gamma[\alpha_\Sigma(\vec{\varphi})] & \text{(by [39, Proposition 7])}\\
& \text{iff} \quad \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \boldsymbol{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, \langle F, \alpha\rangle, R^{\mathfrak{B}}\rangle \vDash_\Sigma \gamma(\vec{\varphi}). & \square
\end{array}$$

Theorem 4.1 yields the following corollary, which forms an analog of [11, Corollary 4.6] for $\mathcal{L}$-systems. It says, roughly speaking, that an elementary morphism is strong, and if its functor component is an isomorphism, then the image of the domain is an elementary subsystem of the codomain. It also asserts that an elementary morphism between two $\mathcal{L}$-systems induces an elementary morphism between their Leibniz reductions and that, under this induced morphism, the image of the domain becomes an elementary subsystem of the codomain in case the functor component of the original elementary morphism is an isomorphism.

Corollary 4.2 *Let $\mathcal{L} = \langle \boldsymbol{F}, R, \varrho\rangle$ be a system language and*

$$\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \boldsymbol{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle \quad \textit{and} \quad \mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \boldsymbol{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$$

be two $\mathcal{L}$-systems.

1. *If $\langle F, \alpha\rangle : \mathfrak{A} \longrightarrow_{\mathrm{e}} \mathfrak{B}$ is an elementary $\mathcal{L}$-system morphism, then $\langle F, \alpha\rangle : \mathfrak{A} \longrightarrow_{\mathrm{s}} \mathfrak{B}$ is also strong, and if, in addition, $F : \mathbf{Sign}^{\mathfrak{A}} \longrightarrow \mathbf{Sign}^{\mathfrak{B}}$ is an isomorphism, then $\alpha(\mathfrak{A}) \subseteq_{\mathrm{e}} \mathfrak{B}$.*

2. *If $\langle F, \alpha\rangle : \mathfrak{A} \longrightarrow_{\mathrm{e}} \mathfrak{B}$ is an elementary $\mathcal{L}$-system morphism and F is surjective, then $\langle F, \alpha^*\rangle : \mathfrak{A}^* \longrightarrow_{\mathrm{e}} \mathfrak{B}^*$ is also elementary, and if, in addition, $F : \mathbf{Sign}^{\mathfrak{A}} \longrightarrow \mathbf{Sign}^{\mathfrak{B}}$ is an isomorphism, then $\alpha^*(\mathfrak{A}^*) \subseteq_{\mathrm{e}} \mathfrak{B}^*$.*

Proof.

1. It is clear from the definitions involved that $\langle F, \alpha\rangle : \mathfrak{A} \longrightarrow_{\mathrm{e}} \mathfrak{B}$ elementary implies that $\langle F, \alpha\rangle : \mathfrak{A} \longrightarrow_{\mathrm{s}} \mathfrak{B}$ is also strong. Now apply [39, Lemma 6, part 2] to obtain that $\alpha(\mathfrak{A}) \subseteq \mathfrak{B}$. However, since $\langle F, \alpha\rangle$ is elementary, we get that $\alpha(\mathfrak{A}) \subseteq_{\mathrm{e}} \mathfrak{B}$.

2. From the hypothesis, we obtain that $\langle \mathrm{SEN}^{\mathfrak{B}}, \langle \boldsymbol{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, \langle F, \alpha\rangle, R^{\mathfrak{B}}\rangle$ is a model of $\mathrm{Dge}\mathfrak{A}$. Thus, by Theorem 4.1, 2., we get that $\langle F^*, \alpha^*\rangle : \mathfrak{A}^* \longrightarrow_{\mathrm{e}} \mathfrak{B}^*$ is an elementary $\mathcal{L}$-system morphism. Therefore, with the help of [39, Lemma 6, part 2], we get that $\alpha^*(\mathfrak{A}^*) \subseteq_{\mathrm{e}} \mathfrak{B}^*$. $\square$

5 The Reduction Operator Lemmas

In this section an analog of [11, Reduction Operator Lemma 4.7] of Elgueta is formulated and proven for $\mathcal{L}$-systems. This result consists of three parts. All of them deal with equalities between operators on classes of $\mathcal{L}$-systems that are obtained by composing in various ways the Leibniz reduction operator L with other class operators. The first part shows that, if $\mathrm{S_i}$ denotes the operator of taking isomorphic copies of $\mathcal{L}$-subsystems with isomorphic functor components, then $\mathrm{LS_i} = \mathrm{LS_iL}$. The second part shows that, if $\mathrm{P}, \mathrm{P_f}, \mathrm{P_u}, \mathrm{P_{sd}}$ denote the operators of taking isomorphic copies of direct products, filtered products, ultraproducts, subdirect products, respectively, of $\mathcal{L}$-systems, then $\mathrm{LO} = \mathrm{LOL}$, where O stands for any of these product operators. Finally, in the last part, it is shown that, if $\mathrm{S_{ie}}$ is the operator of taking isomorphic copies of elementary $\mathcal{L}$-subsystems with isomorphic functor components of a class of $\mathcal{L}$-systems, then $\mathrm{LS_{ie}} = \mathrm{LS_{ie}L} = \mathrm{S_{ie}L}$. Our interest in this lemma lies with the fact that, when proving model-theoretic characterizations of different equality-free first-order definable classes and of their corresponding reductions, one starts with the abstract class and then applies the Reduction Operator Lemma to obtain the analogous result for the reduced class. See [11, Section 5] for more details. As a consequence, the present results open and pave the way for establishing in future work analogs of the equality-free class characterization theorems in the context of $\mathcal{L}$-systems, generalizing the corresponding results on equality-free first-order classes.

Recall from [39, Section 3] that an $\mathcal{L}$-system $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \boldsymbol{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$ with $\mathrm{SEN}^{\mathfrak{A}} : \mathbf{Sign}^{\mathfrak{A}} \longrightarrow \mathbf{Set}$ is said to be a *simple subsystem of an $\mathcal{L}$-system* $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \boldsymbol{N}^{\mathfrak{B}}, F^{\mathfrak{B}} \rangle, R^{\mathfrak{B}} \rangle$ with $\mathrm{SEN}^{\mathfrak{B}} : \mathbf{Sign}^{\mathfrak{B}} \longrightarrow \mathbf{Set}$ if it is a subsystem of $\mathfrak{B}$ such that their sentence functors $\mathrm{SEN}^{\mathfrak{A}} : \mathbf{Sign}^{\mathfrak{A}} \longrightarrow \mathbf{Set}$ and $\mathrm{SEN}^{\mathfrak{B}} : \mathbf{Sign}^{\mathfrak{B}} \longrightarrow \mathbf{Set}$ have the same domain, i. e., if $\mathbf{Sign}^{\mathfrak{A}} = \mathbf{Sign}^{\mathfrak{B}}$.

The following lemma is a technical result that will be useful in the proofs of the first and of the third part of the Reduction Operator Lemma.

Lemma 5.1 *Let $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \boldsymbol{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$ be a simple subsystem of $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \boldsymbol{N}^{\mathfrak{B}}, F^{\mathfrak{B}} \rangle, R^{\mathfrak{B}} \rangle$, and let $\theta = \{\theta_\Sigma\}_{\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|} \in \mathrm{Con}(\mathfrak{B})$. Define $\theta^{\mathfrak{A}} = \{\theta^{\mathfrak{A}}_\Sigma\}_{\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|}$ by $\theta^{\mathfrak{A}}_\Sigma = \theta_\Sigma \cap \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^2$ for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$. Define also $\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \eta \rangle : \mathfrak{A}/\theta^{\mathfrak{A}} \longrightarrow \mathfrak{B}/\theta$ by letting, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, $\eta_\Sigma : \mathrm{SEN}^{\mathfrak{A}}(\Sigma)/\theta^{\mathfrak{A}}_\Sigma \longrightarrow \mathrm{SEN}^{\mathfrak{B}}(\Sigma)/\theta_\Sigma$ be given by*

$$\eta_\Sigma(\varphi/\theta^{\mathfrak{A}}_\Sigma) = \varphi/\theta_\Sigma \quad \textit{for all } \varphi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma).$$

Then $\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \eta \rangle : \mathfrak{A}/\theta^{\mathfrak{A}} \rightarrowtail_{\mathrm{s}} \mathfrak{B}/\theta$ defines a strong embedding from $\mathfrak{A}/\theta^{\mathfrak{A}}$ into $\mathfrak{B}/\theta$.

Proof. First, note that for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, η_Σ is well-defined. Indeed, if for $\varphi, \varphi' \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$ we have that $\varphi/\theta^{\mathfrak{A}}_\Sigma = \varphi'/\theta^{\mathfrak{A}}_\Sigma$, then, by the definition of $\theta^{\mathfrak{A}}$, it follows immediately that $\varphi/\theta_\Sigma = \varphi'/\theta_\Sigma$. That

$$\eta : \mathrm{SEN}^{\mathfrak{A}}/\theta^{\mathfrak{A}} \longrightarrow \mathrm{SEN}^{\mathfrak{B}}/\theta$$

is a natural transformation is easy to see. The same goes for the $(\boldsymbol{N}^{\mathfrak{A}^{\theta^{\mathfrak{A}}}}, \boldsymbol{N}^{\mathfrak{B}^{\theta}})$-epimorphic property, whence

$$\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \eta \rangle : \mathrm{SEN}^{\mathfrak{A}}/\theta^{\mathfrak{A}} \longrightarrow \mathrm{SEN}^{\mathfrak{B}}/\theta$$

is an $(\boldsymbol{N}^{\mathfrak{A}^{\theta^{\mathfrak{A}}}}, \boldsymbol{N}^{\mathfrak{B}^{\theta}})$-epimorphic translation. To see that $\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \eta \rangle : \mathfrak{A}/\theta^{\mathfrak{A}} \longrightarrow \mathfrak{B}/\theta$ is an $\mathcal{L}$-morphism, suppose that $r \in R$ with $\varrho(r) = n$, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, and $\vec{\varphi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$. Then

$$\vec{\varphi}/\theta^{\mathfrak{A}}_\Sigma \in r^{\mathfrak{A}/\theta^{\mathfrak{A}}}_\Sigma \quad \text{iff} \quad \vec{\varphi} \in r^{\mathfrak{A}}_\Sigma \quad \text{iff} \quad \vec{\varphi} \in r^{\mathfrak{B}}_\Sigma \quad \text{iff} \quad \vec{\varphi}/\theta_\Sigma \in r^{\mathfrak{B}/\theta}_\Sigma.$$

The above equivalences also show that $\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \eta \rangle$ is a strong $\mathcal{L}$-morphism. Finally, to see that it is an injection, suppose that $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and $\varphi, \varphi' \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$ such that $\varphi/\theta_\Sigma = \varphi'/\theta_\Sigma$. Thus,

$$\langle \varphi, \varphi' \rangle \in \theta_\Sigma \cap \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^2 = \theta^{\mathfrak{A}}_\Sigma.$$

Hence $\varphi/\theta^{\mathfrak{A}}_\Sigma = \varphi'/\theta^{\mathfrak{A}}_\Sigma$ and η_Σ is in fact injective, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$. □

Having Lemma 5.1 at hand, we now proceed with the first part of the Reduction Operator Lemma showing that $\mathrm{LS_i} = \mathrm{LS_iL}$, where $\mathrm{S_i}$ denotes the operator of taking isomorphic copies of $\mathcal{L}$-subsystems with isomorphic functor components. L is the Leibniz reduction operator that maps a class of $\mathcal{L}$-systems to the class of all isomorphic copies of the Leibniz reductions of its members. For the formal definitions of those operators, see [39] and, specifically for the Leibniz reduction of an $\mathcal{L}$-system, see [40].

Lemma 5.2 (Reduction Operator Lemma I) $\mathrm{LS_i} = \mathrm{LS_iL}$.

P r o o f. We begin by showing that $\mathrm{LS_i} \le \mathrm{LS_iL}$. Suppose, to this end, that $\mathfrak{A}$ is an $\mathcal{L}$-system and K a class of $\mathcal{L}$-systems such that $\mathfrak{A} \in \mathrm{LS_i}(\mathsf{K})$. Thus, there exist $\mathfrak{B}, \mathfrak{C}$ with $\mathfrak{B} \in \mathsf{K}$ such that $\mathfrak{A} \cong \mathfrak{C}^*$ and $\mathfrak{C} \subseteq_{\mathrm{i}} \mathfrak{B}$. Then by Lemma 5.1, there is a strong embedding $\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{C}}}, \eta\rangle : \mathfrak{C}/\Omega(\mathfrak{B})^{\mathfrak{C}} \rightarrowtail_{\mathrm{s}} \mathfrak{B}^*$ with an isomorphic functor component. Thus, by [39, Lemma 6, part 2], $\mathfrak{C}/\Omega(\mathfrak{B})^{\mathfrak{C}}$ is isomorphic to a subsystem of $\mathfrak{B}^*$. But, by [40, Proposition 16], we also get that $\mathfrak{C}^* \cong (\mathfrak{C}/\Omega(\mathfrak{B})^{\mathfrak{C}})^*$, whence $\mathfrak{A} \cong (\mathfrak{C}/\Omega(\mathfrak{B})^{\mathfrak{C}})^*$. This shows that $\mathfrak{A}$ is isomorphic to the Leibniz reduction of the $\mathcal{L}$-system $\mathfrak{C}/\Omega(\mathfrak{B})^{\mathfrak{C}}$, which is, in turn, isomorphic to a subsystem of $\mathfrak{B}^*$ which is the Leibniz reduction of the $\mathcal{L}$-system $\mathfrak{B} \in \mathsf{K}$. Thus, $\mathfrak{A} \in \mathrm{LS_iL}(\mathsf{K})$ and, therefore, $\mathrm{LS_i} \le \mathrm{LS_iL}$.

For the reverse inclusion, suppose that $\mathfrak{A}$ is an $\mathcal{L}$-system and K a class of $\mathcal{L}$-systems, such that $\mathfrak{A} \in \mathrm{LS_iL}(\mathsf{K})$. Thus, there exist $\mathcal{L}$-systems $\mathfrak{B}, \mathfrak{C}$ with $\mathfrak{B} \in \mathsf{K}$ such that $\mathfrak{A} \cong \mathfrak{C}^*$ and $\mathfrak{C} \subseteq_{\mathrm{i}} \mathfrak{B}^*$. Consider the reductive $\mathcal{L}$-morphism $\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{B}}}, \pi^{\Omega(\mathfrak{B})}\rangle : \mathfrak{B} \twoheadrightarrow_{\mathrm{s}} \mathfrak{B}^*$. By [39, Lemma 6, part 1], we have that $\pi^{\Omega(\mathfrak{B})^{-1}}(\mathfrak{C}) \subseteq_{\mathrm{i}} \mathfrak{B}$. Also, the restriction $\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{B}}}, \pi^{\Omega(\mathfrak{B})}\rangle \restriction_{\pi^{\Omega(\mathfrak{B})^{-1}}(\mathfrak{C})} : \pi^{\Omega(\mathfrak{B})^{-1}}(\mathfrak{C}) \twoheadrightarrow_{\mathrm{s}} \mathfrak{C}$ is a reductive $\mathcal{L}$-morphism, whence $\mathfrak{A} \cong \mathfrak{C}^*$ is isomorphic to a reduction via

$$\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{C}}}, \pi^{\Omega(\mathfrak{C})}\rangle \circ \langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{B}}}, \pi^{\Omega(\mathfrak{B})}\rangle \restriction_{\pi^{\Omega(\mathfrak{B})^{-1}}(\mathfrak{C})} : \pi^{\Omega(\mathfrak{B})^{-1}}(\mathfrak{C}) \twoheadrightarrow_{\mathrm{s}} \mathfrak{C}^*$$

of a subsystem $\pi^{\Omega(\mathfrak{B})^{-1}}(\mathfrak{C}) \subseteq_{\mathrm{i}} \mathfrak{B} \in \mathsf{K}$. Therefore $\mathfrak{A} \in \mathrm{LS_i}(\mathsf{K})$ and, hence, $\mathrm{LS_iL} \le \mathrm{LS_i}$. □

Next, we turn to the second part of the Reduction Operator Lemma. In this part, it is shown that $\mathrm{LO} = \mathrm{LOL}$, where O is any of the operators P of taking isomorphic copies of direct products of $\mathcal{L}$-systems, $\mathrm{P_f}$ of taking isomorphic copies of filtered products of $\mathcal{L}$-systems, $\mathrm{P_u}$ of taking isomorphic copies of ultraproducts of $\mathcal{L}$-systems, and $\mathrm{P_{sd}}$ of taking isomorphic copies of subdirect products of $\mathcal{L}$-systems.

Lemma 5.3 (Reduction Operator Lemma II) *For all* $\mathrm{O} \in \{\mathrm{P}, \mathrm{P_f}, \mathrm{P_u}, \mathrm{P_{sd}}\}$, $\mathrm{LO} = \mathrm{LOL}$.

P r o o f. We proceed in two parts. First, it is shown that $\mathrm{LP_f} = \mathrm{LP_fL}$, and second, that $\mathrm{LP_{sd}} = \mathrm{LP_{sd}L}$. The cases covering products and ultraproducts follow directly from the case covering reduced products.

For reduced products, it suffices to show that if $\mathfrak{A}_i = \langle \mathrm{SEN}^i, \langle \boldsymbol{N}^i, F^i\rangle, R^i\rangle$, $i \in I$, is a collection of $\mathcal{L}$-systems and $\mathcal{F}$ is a proper filter over I, then

$$(\textstyle\prod_{i\in I} \mathfrak{A}_i/\mathcal{F})^* \cong (\prod_{i\in I} \mathfrak{A}_i^*/\mathcal{F})^*.$$

This is because, in that case, if $\mathfrak{A}$ is an $\mathcal{L}$-system and K a class of $\mathcal{L}$-systems, then on the one hand $\mathfrak{A} \in \mathrm{LP_f}(\mathsf{K})$ iff $\mathfrak{A} \cong (\prod_{i\in I} \mathfrak{A}_i/\mathcal{F})^*$ for some $\mathfrak{A}_i \in \mathsf{K}$, $i \in I$, and on the other hand $\mathfrak{A} \in \mathrm{LP_fL}(\mathsf{K})$ iff $\mathfrak{A} \cong (\prod_{i\in I} \mathfrak{A}_i^*/\mathcal{F})^*$ for some $\mathfrak{A}_i \in \mathsf{K}$, $i \in I$.

We show that the pair $\langle \mathrm{I}, \alpha\rangle : \prod_{i\in I} \mathfrak{A}_i^*/\mathcal{F} \longrightarrow (\prod_{i\in I} \mathfrak{A}_i/\mathcal{F})^*$ is a reductive $\mathcal{L}$-morphism, where $\langle \mathrm{I}, \alpha\rangle$ is defined by letting $\mathrm{I} := \mathrm{I}_{\prod_{i\in I} \mathbf{Sign}^i}$, and for all $\Sigma_i \in |\mathbf{Sign}^i|$, $i \in I$, letting

$$\alpha_{\prod_{i\in I} \Sigma_i} : (\textstyle\prod_{i\in I} \mathrm{SEN}^i(\Sigma_i)/\Omega_{\Sigma_i}(\mathfrak{A}_i))/\mathcal{F} \longrightarrow (\prod_{i\in I} \mathrm{SEN}^i(\Sigma_i)/\mathcal{F})/\Omega_{\prod_{i\in I} \Sigma_i}(\prod_{i\in I} \mathfrak{A}_i/\mathcal{F})$$

be defined by

$$\alpha_{\prod_{i\in I} \Sigma_i}(\vec{\varphi}^{\,*}/\mathcal{F}) = (\vec{\varphi}/\mathcal{F})^* \quad \text{for all } \vec{\varphi} \in \textstyle\prod_{i\in I} \mathrm{SEN}^i(\Sigma_i).$$

First, it is shown that α is well-defined. To this end, suppose that $\Sigma_i \in |\mathbf{Sign}^i|$ and $\varphi_i, \psi_i \in \mathrm{SEN}^i(\Sigma_i)$, $i \in I$, are such that

$$\vec{\varphi}^{\,*}/{\equiv^{\mathcal{F}}_{\prod_{i\in I} \Sigma_i}} = \vec{\psi}^{\,*}/{\equiv^{\mathcal{F}}_{\prod_{i\in I} \Sigma_i}}.$$

Therefore, we have $\{i \in I : \varphi_i^* = \psi_i^*\} \in \mathcal{F}$. To see that $(\vec{\varphi}/\mathcal{F})^* = (\vec{\psi}/\mathcal{F})^*$, use [40, Theorem 6]. To this end, let $\gamma(x, z_1, \ldots, z_k)$ be an atomic $\mathcal{L}$-formula, $\Sigma_i' \in |\mathbf{Sign}^i|$, $f_i \in \mathbf{Sign}^i(\Sigma_i, \Sigma_i')$, $i \in I$, and

$$\vec{\chi}_1/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i'}}, \ldots, \vec{\chi}_k/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i'}} \in \prod_{i\in I}\mathrm{SEN}^i(\Sigma_i')/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i'}}.$$

Then, by [39, Theorem 8], we get that

$$\begin{aligned}\prod_{i\in I}\mathfrak{A}_i/\mathcal{F} \models_{\prod_{i\in I}\Sigma_i'} \gamma[\textstyle\prod_{i\in I}\mathrm{SEN}^i(f_i)(\vec{\varphi})/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i'}}, \vec{\chi}_1/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i'}}, \ldots, \vec{\chi}_k/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i'}}] \quad \text{iff} \\ \{i \in I : \mathfrak{A}_i \models_{\Sigma_i'} \gamma[\mathrm{SEN}^i(f_i)(\varphi_i), \chi_{1i}, \ldots, \chi_{ki}]\} \in \mathcal{F}.\end{aligned}$$

But we also have

$$\begin{aligned}&\{i \in I : \varphi_i^* = \psi_i^*\} \cap \{i \in I : \mathfrak{A}_i \models_{\Sigma_i'} \gamma[\mathrm{SEN}^i(f_i)(\varphi_i), \chi_{1i}, \ldots, \chi_{ki}]\} \\ &\quad \subseteq \{i \in I : \mathfrak{A}_i \models_{\Sigma_i'} \gamma[\mathrm{SEN}^i(f_i)(\psi_i), \chi_{1i}, \ldots, \chi_{ki}]\},\end{aligned}$$

whence, by the filter property of $\mathcal{F}$, we get that

$$\{i \in I : \mathfrak{A}_i \models_{\Sigma_i'} \gamma[\mathrm{SEN}^i(f_i)(\psi_i), \chi_{1i}, \ldots, \chi_{ki}]\} \in \mathcal{F},$$

which, again by [39, Theorem 8], yields that

$$\prod_{i\in I}\mathfrak{A}_i/\mathcal{F} \models_{\prod_{i\in I}\Sigma_i'} \gamma[\textstyle\prod_{i\in I}\mathrm{SEN}^i(f_i)(\vec{\psi})/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i'}}, \vec{\chi}_1/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i'}}, \ldots, \vec{\chi}_k/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i'}}].$$

Hence, if $\vec{\varphi}^*/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i}} = \vec{\psi}^*/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i}}$, we get that

$$\begin{aligned}\prod_{i\in I}\mathfrak{A}_i/\mathcal{F} \models_{\prod_{i\in I}\Sigma_i'} \gamma(x, \vec{z}) \to \gamma(y, \vec{z})[&\textstyle\prod_{i\in I}\mathrm{SEN}^i(f_i)(\vec{\varphi})/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i'}}, \\ &\textstyle\prod_{i\in I}\mathrm{SEN}^i(f_i)(\vec{\psi})/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i'}}, \\ &\vec{\chi}_1/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i'}}, \ldots, \vec{\chi}_k/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i'}}].\end{aligned}$$

The converse implication may be shown to hold by symmetry, whence, since γ was an arbitrary atomic $\mathcal{L}$-formula and $\vec{\chi}_1, \ldots, \vec{\chi}_k$ arbitrary tuples in $\prod_{i\in I}\mathrm{SEN}^i(\Sigma_i')$, we get, by [40, Theorem 6], that

$$(\vec{\varphi}/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i}})^* = (\vec{\psi}/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i}})^*.$$

The facts that α is a natural transformation, that $\langle \mathrm{I}, \alpha\rangle$ is an $\mathcal{L}$-system morphism, and that it is strong and surjective all follow relatively easy from the definitions of the reduced products involved.

To finish off this part of the proof, it suffices now to rely on [40, Proposition 16] to conclude that

$$\langle \mathrm{I}, \alpha^*\rangle : (\textstyle\prod_{i\in I}\mathfrak{A}_i^*/\mathcal{F})^* \cong (\prod_{i\in I}\mathfrak{A}_i/\mathcal{F})^*.$$

We turn now to the proof of $\mathrm{LP_{sd}} = \mathrm{LP_{sd}L}$. We show first that $\mathrm{LP_{sd}} \leq \mathrm{LP_{sd}L}$. Suppose that $\mathfrak{A}$ is an $\mathcal{L}$-system and K a class of $\mathcal{L}$-systems, such that $\mathfrak{A} \in \mathrm{P_{sd}}(\mathsf{K})$ so that $\mathfrak{A}^* \in \mathrm{LP_{sd}}(\mathsf{K})$. Then, there exist $\mathfrak{A}_i \in \mathsf{K}$, $i \in I$, and $\langle F, \alpha\rangle : \mathfrak{A} \rightarrowtail_{\mathrm{sd}} \prod_{i\in I}\mathfrak{A}_i$. Let $\langle P^i, \pi^i\rangle : \prod_{i\in I}\mathfrak{A}_i \longrightarrow \mathfrak{A}_i$, $i \in I$, and $\langle P'^i, \pi'^i\rangle : \prod_{i\in I}\mathfrak{A}_i^* \longrightarrow \mathfrak{A}_i^*$, $i \in I$, be the projection $\mathcal{L}$-morphisms. Define the pair $\langle G, \beta\rangle : \mathfrak{A} \longrightarrow \prod_{i\in I}\mathfrak{A}_i^*$ as follows: let $G = F$, and let, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, $\beta_\Sigma : \mathrm{SEN}^{\mathfrak{A}}(\Sigma) \longrightarrow \prod_{i\in I}\mathrm{SEN}^i(P^i(G(\Sigma))/\Omega_{P^i(G(\Sigma))}(\mathfrak{A}^i)$ be given by

$$\beta_\Sigma(\varphi) = \textstyle\prod_{i\in I}\pi^i_{F(\Sigma)}(\alpha_\Sigma(\varphi))^* \quad \text{for all } \varphi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma).$$

It is not difficult to check that $\langle G, \beta\rangle$ is a strong $\mathcal{L}$-morphism such that for all $i \in I$, $\langle P'^i, \pi'^i\rangle \circ \langle G, \beta\rangle$ is surjective. To show that $\langle G, \beta\rangle$ is strong, consider $r \in R$ with $\varrho(r) = n$. Let $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and $\vec{\varphi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$. Then

$$\begin{aligned}\vec{\varphi} \in r^{\mathfrak{A}}_\Sigma \quad &\text{iff} \quad \alpha_\Sigma(\vec{\varphi}) \in r^{\prod_{i\in I}\mathfrak{A}_i}_{F(\Sigma)} \\ &\text{iff} \quad (\forall i \in I)(\pi^i_{F(\Sigma)}(\alpha_\Sigma(\vec{\varphi})) \in r^{\mathfrak{A}_i}_{P^i(F(\Sigma))}) \\ &\text{iff} \quad (\forall i \in I)(\pi^i_{F(\Sigma)}(\alpha_\Sigma(\vec{\varphi}))^* \in r^{\mathfrak{A}_i^*}_{P^i(F(\Sigma))}) \\ &\text{iff} \quad \textstyle\prod_{i\in I}\pi^i_{F(\Sigma)}(\alpha_\Sigma(\vec{\varphi}))^* \in r^{\prod_{i\in I}\mathfrak{A}_i^*}_{F(\Sigma)} \\ &\text{iff} \quad \beta_\Sigma(\vec{\varphi}) \in r^{\prod_{i\in I}\mathfrak{A}_i^*}_{F(\Sigma)}.\end{aligned}$$

Surjectivity of $\langle P'^i, \pi'^i\rangle \circ \langle G, \beta\rangle$ follows easily from the fact that $\langle P^i, \pi^i\rangle \circ \langle F, \alpha\rangle$ is surjective for every $i \in I$.

Hence, applying [39, Lemma 6, part 2] and [40, Homomorphism Theorem 10], we obtain that there exists

$$\langle H, \gamma\rangle : \mathfrak{A}/\mathrm{Ker}(\langle G, \beta\rangle) \rightarrowtail_{\mathrm{sd}} \prod_{i\in I} \mathfrak{A}_i^*.$$

Therefore, $(\mathfrak{A}/\mathrm{Ker}(\langle G, \beta\rangle))^* \in \mathrm{LP_{sd}L}(\mathsf{K})$. But, by [40, Proposition 16], we have that $(\mathfrak{A}/\mathrm{Ker}(\langle G, \beta\rangle))^* \cong \mathfrak{A}^*$, which completes this part of the proof.

For the converse inclusion, suppose that K is a class of $\mathcal{L}$-systems and $\mathfrak{A} \in \mathrm{LP_{sd}L}(\mathsf{K})$. Therefore, there exist $\mathfrak{B}_i \in \mathsf{K}$, $i \in I$, and a subdirect product $\mathfrak{C} \subseteq_{\mathrm{sd}} \prod_{i\in I} \mathfrak{B}_i^*$ such that $\mathfrak{A} \cong \mathfrak{C}^*$. Now consider the reductive $\mathcal{L}$-morphism

$$\prod_{i\in I} \mathfrak{B}_i \xrightarrow{\langle \mathrm{I}, \pi\rangle := \prod_{i\in I}\langle \mathrm{I}^i, \pi^i\rangle} \prod_{i\in I} \mathfrak{B}_i^*$$

Since $\mathfrak{C}$ is an $\mathcal{L}$-subsystem of $\prod_{i\in I} \mathfrak{B}_i^*$, by [39, Lemma 6, part 1], we get that $\pi^{-1}(\mathfrak{C}) \subseteq \prod_{i\in I} \mathfrak{B}_i$. Moreover, since $\mathfrak{C} \subseteq_{\mathrm{sd}} \prod_{i\in I} \mathfrak{B}_i^*$, we obtain that $\pi^{-1}(\mathfrak{C}) \subseteq_{\mathrm{sd}} \prod_{i\in I} \mathfrak{B}_i$. Now, since $\langle \mathrm{I}, \pi\rangle : \pi^{-1}(\mathfrak{C}) \twoheadrightarrow_{\mathrm{s}} \mathfrak{C}$ is a reductive $\mathcal{L}$-morphism, we obtain, by [40, Proposition 16], that $\mathfrak{C}^* \cong \pi^{-1}(\mathcal{C})^*$. Thus, we now have $\mathfrak{A} \cong \pi^{-1}(\mathfrak{C})^*$ and $\pi^{-1}(\mathfrak{C}) \subseteq_{\mathrm{sd}} \prod_{i\in I} \mathfrak{B}_i$ with $\mathfrak{B}_i \in \mathsf{K}$ for all $i \in I$, which yields that $\mathfrak{A} \in \mathrm{LP_{sd}}(\mathsf{K})$. □

Finally, the third part of the Reduction Operator Lemma is presented. It shows that $\mathrm{LS_{ie}} = \mathrm{LS_{ie}L} = \mathrm{S_{ie}L}$, where $\mathrm{S_{ie}}$ is the operator of taking isomorphic copies of elementary subsystems of $\mathcal{L}$-systems with isomorphic functor components.

Lemma 5.4 (Reduction Operator Lemma III) $\mathrm{LS_{ie}} = \mathrm{LS_{ie}L} = \mathrm{S_{ie}L}$.

P r o o f. We first show that $\mathrm{LS_{ie}} = \mathrm{LS_{ie}L}$ in a manner very similar to the one used for the proof of Lemma 5.2, and then we show that $\mathrm{LS_{ie}L} = \mathrm{S_{ie}L}$.

First, to see that $\mathrm{LS_{ie}} \le \mathrm{LS_{ie}L}$, let $\mathfrak{A}$ be an $\mathcal{L}$-system and K a class of $\mathcal{L}$-systems, such that $\mathfrak{A} \in \mathrm{LS_{ie}}(\mathsf{K})$. Thus, there exist $\mathfrak{B}, \mathfrak{C}$ with $\mathfrak{B} \in \mathsf{K}$ such that $\mathfrak{A} \cong \mathfrak{C}^*$ and $\mathfrak{C} \subseteq_{\mathrm{ie}} \mathfrak{B}$. Then, by Lemma 5.1, there exists a strong embedding $\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{C}}}, \eta\rangle : \mathfrak{C}/\Omega(\mathfrak{B})^{\mathfrak{C}} \rightarrowtail_{\mathrm{s}} \mathfrak{B}^*$ with an isomorphic functor component. But, in this case, we also have that $\mathfrak{C}/\Omega(\mathfrak{B})^{\mathfrak{C}} \equiv \mathfrak{C}$ and that $\mathfrak{B}^* \equiv \mathfrak{B}$, and, hence, $\mathfrak{C}/\Omega(\mathfrak{B})^{\mathfrak{C}} \equiv \mathfrak{B}^*$, whence the embedding

$$\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{C}}}, \eta\rangle : \mathfrak{C}/\Omega(\mathfrak{B})^{\mathfrak{C}} \rightarrowtail_{\mathrm{s}} \mathfrak{B}^*$$

is also elementary. Again, using [39, Lemma 6, part 2] and [40, Homomorphism Theorem 10], $\mathfrak{C}/\Omega(\mathfrak{B})^{\mathfrak{C}}$ is isomorphic to an elementary subsystem of $\mathfrak{B}^*$. Now [40, Proposition 16] is invoked to complete the argument.

For the reverse inclusion, suppose that $\mathfrak{A}$ is an $\mathcal{L}$-system and K a class of $\mathcal{L}$-systems such that $\mathfrak{A} \in \mathrm{LS_{ie}L}(\mathsf{K})$. Thus, there exist $\mathcal{L}$-systems $\mathfrak{B}, \mathfrak{C}$ with $\mathfrak{B} \in \mathsf{K}$ such that $\mathfrak{A} \cong \mathfrak{C}^*$ and $\mathfrak{C} \subseteq_{\mathrm{ie}} \mathfrak{B}^*$. Consider the reductive $\mathcal{L}$-morphism $\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{B}}}, \pi^{\Omega(\mathfrak{B})}\rangle : \mathfrak{B} \twoheadrightarrow_{\mathrm{s}} \mathfrak{B}^*$. By [39, Lemma 6, part 1], we get $\pi^{\Omega(\mathfrak{B})^{-1}}(\mathfrak{C}) \subseteq_{\mathrm{i}} \mathfrak{B}$. But, since $\mathfrak{C} \subseteq_{\mathrm{ie}} \mathfrak{B}^*$, we obtain, in this case, the additional fact that $\pi^{\Omega(\mathfrak{B})^{-1}}(\mathfrak{C}) \subseteq_{\mathrm{ie}} \mathfrak{B}$. Indeed, we have that for all $\mathcal{L}$-formulas γ, all $\Sigma \in |\mathbf{Sign}^{\pi^{\Omega(\mathfrak{B})^{-1}}(\mathfrak{C})}|$, and all $\vec{\varphi} \in \mathrm{SEN}^{\pi^{\Omega(\mathfrak{B})^{-1}}(\mathfrak{C})}(\Sigma)$,

$$\begin{array}{llll} \pi^{\Omega(\mathfrak{B})^{-1}}(\mathfrak{C}) \models_\Sigma \gamma[\vec{\varphi}] & \text{iff} & \mathfrak{C} \models_\Sigma \gamma[\vec{\varphi}^*] & (\text{by the definition of } \pi^{\Omega(\mathfrak{B})^{-1}}(\mathfrak{C})) \\ & \text{iff} & \mathfrak{B}^* \models_\Sigma \gamma[\vec{\varphi}^*] & (\text{since } \mathfrak{C} \subseteq_{ie} \mathfrak{B}^*) \\ & \text{iff} & \mathfrak{B} \models_\Sigma \gamma[\vec{\varphi}] & (\text{by the definition of } \mathfrak{B}^*). \end{array}$$

Again, the remaining of the argument matches now the argument given to conclude the reverse inclusion. Therefore, $\mathrm{LS_{ie}} = \mathrm{LS_{ie}L}$ has now been established.

To see now that $\mathrm{LS_{ie}L} = \mathrm{S_{ie}L}$, it suffices to show that if $\mathfrak{A}, \mathfrak{B}$ are two $\mathcal{L}$-systems such that $\mathfrak{A} \subseteq_{\mathrm{ie}} \mathfrak{B}$ and $\mathfrak{B}$ is Leibniz reduced, then $\mathfrak{A}$ is also Leibniz reduced. To prove this, let $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, $\varphi, \psi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$ such that $\langle \varphi, \psi\rangle \in \Omega_\Sigma(\mathfrak{A})$. This holds, by [40, Theorem 6], iff for all Leibniz $\mathcal{L}$-formulas $\beta(x, y)$, all $\Sigma' \in |\mathbf{Sign}^{\mathfrak{A}}|$, and all $f \in \mathbf{Sign}^{\mathfrak{A}}(\Sigma, \Sigma')$,

$$\mathfrak{A} \models_{\Sigma'} \beta(x, y)[\mathrm{SEN}^{\mathfrak{A}}(f)(\varphi), \mathrm{SEN}^{\mathfrak{A}}(f)(\psi)],$$

which, in view of $\mathfrak{A} \subseteq_{\mathrm{ie}} \mathfrak{B}$, is equivalent to

$$\mathfrak{B} \models_{\Sigma'} \beta(x, y)[\mathrm{SEN}^{\mathfrak{B}}(f)(\varphi), \mathrm{SEN}^{\mathfrak{B}}(f)(\psi)].$$

Therefore, again by an application of [40, Theorem 6], we get that $\langle\varphi,\psi\rangle \in \Omega_\Sigma(\mathfrak{B})$. (Note the necessity of $\subseteq_{\rm i}$ for this argument!) Thus, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, we have $\Omega_\Sigma(\mathfrak{A}) \subseteq \Omega_\Sigma(\mathfrak{B}) = \Delta_\Sigma^{\mathrm{SEN}^{\mathfrak{B}}}$. Therefore $\Omega(\mathfrak{A}) = \Delta_\Sigma^{\mathrm{SEN}^{\mathfrak{A}}}$ and $\mathfrak{A}$ is Leibniz reduced. □

The Reduction Operator Theorem, the promised analog of the homonymous result of [11], summarizes the results presented in Lemmas 5.2, 5.3, and 5.4.

Theorem 5.5 (Reduction Operator Theorem)

1. $\mathrm{LS_i} = \mathrm{LS_iL}$.
2. *For all* $\mathrm{O} \in \{\mathrm{P}, \mathrm{P_f}, \mathrm{P_u}, \mathrm{P_{sd}}\}$, $\mathrm{LO} = \mathrm{LOL}$.
3. $\mathrm{LS_{ie}} = \mathrm{LS_{ie}L} = \mathrm{S_{ie}L}$.

We intend to continue the work presented in this paper with the goal of abstracting several of Elgueta's results to the present framework. Elgueta's results generalize well-known results of the theory of models of first-order logic to the equality-free context. The present framework leads to further generalization of these results to a multi-signature equality-free context.

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References

[1] M. Barr and C. Wells, Category Theory for Computing Science, Third Edition (Les Publications CRM, 1999).
[2] S. L. Bloom, Some theorems on structural consequence operations. Studia Logica **34**, 1 – 9 (1975).
[3] F. Borceux, Handbook of Categorical Algebra. Encyclopedia of Mathematics and Its Applications 50 (Cambridge University Press, 1994).
[4] E. Casanovas, P. Dellunde, and R. Jansana, On elementary equivalence for equality-free logic. Notre Dame J. Formal Logic **37**, 506 – 522 (1996).
[5] C. C. Chang and H. J. Keisler, Model Theory (Elsevier, 1990).
[6] J. Czelakowski, Protoalgebraic Logics (Kluwer Academic Publishers, 2001).
[7] J. Czelakowski and R. Elgueta, Local characterization theorems for some classes of structures. Acta Scientarium Mathematicarum **65**, 19 – 32 (1999).
[8] P. Dellunde, Equality-free logic: the method of diagrams and preservation theorems. Logic J. IGPL **7**, 717 – 732 (1999).
[9] P. Dellunde, On definability of the equality in classes of algebras with an equivalence relation. Studia Logica **64**, 345 – 353 (2000).
[10] P. Dellunde and R. Jansana, Some characterization theorems for infinitary universal Horn logic without equality. J. Symbolic Logic **61**, 1242 – 1260 (1996).
[11] R. Elgueta, Characterizing classes defined without equality. Studia Logica **58**, 357 – 394 (1997).
[12] R. Elgueta, Subdirect representation theory for classes without equality. Algebra Universalis **40**, 201 – 246 (1998).
[13] R. Elgueta, Algebraic characterizations for universal fragments of logic. Math. Logic Quart. **45**, 385 – 398 (1999).
[14] R. Elgueta, Freeness in classes without equality. J. Symbolic Logic **64**, 1159 – 1194 (1999).
[15] R. Elgueta and R. Jansana, Definability of Leibniz equality. Studia Logica **63**, 223 – 243 (1999).
[16] J. Fiadeiro and A. Sernadas, Structuring theories on consequence. In: Recent Trends in Data Type Specification (D. Sannella and A. Tarlecki, eds.). Lecture Notes in Computer Science 332, pp. 44 – 72 (Springer, 1988).
[17] J. M. Font and R. Jansana, A General Algebraic Semantics for Sentential Logics. Lecture Notes in Logic 7 (Springer, 1996).
[18] J. M. Font, R. Jansana, and D. Pigozzi, A survey of abstract algebraic logic. Studia Logica **74**, 13 – 97 (2003).
[19] J. A. Goguen and J. M. Burstall, Introducing institutions. In: Proceedings of the Logic of Programming Workshop (E. Clarke and D. Kozen, eds.). Lecture Notes in Computer Science 164, pp. 221 – 256 (Springer, 1984).
[20] J. A. Goguen and J. M. Burstall, Institutions: abstract model theory for specification and programming. J. ACM **39**, 95 – 146 (1992).
[21] W. Hodges, Model Theory (Cambridge University Press, 1993).

[22] F. W. Lawvere, Functorial semantics of algebraic theories. Doctoral dissertation, Columbia University, New York, 1963.
[23] S. Mac Lane, Categories for the Working Mathematician (Springer, 1971).
[24] E. G. Manes, Algebraic Theories (Springer, 1976).
[25] D. Marker, Model Theory: An Introduction (Springer, 2002).
[26] D. Pigozzi, Partially ordered varieties and quasivarieties. Preprint.
Available at http://www.math.iastate.edu/dpigozzi/.
[27] G. Voutsadakis, Categorical abstract algebraic logic. Doctoral dissertation, Iowa State University, Ames, Iowa, 1998.
[28] G. Voutsadakis, Categorical abstract algebraic logic: Equivalent institutions. Studia Logica **74** (1 – 2), 275 – 311 (2003).
[29] G. Voutsadakis, Categorical abstract algebraic logic: Algebraizable institutions. Applied Categorical Structures **10**, 531 – 568 (2002).
[30] G. Voutsadakis, Categorical abstract algebraic logic: Tarski congruence systems, logical morphisms and logical quotients. Submitted to the Annals of Pure and Applied Logic.
Preprint available at http://www.voutsadakis.com/RESEARCH/papers.html.
[31] G. Voutsadakis, Categorical abstract algebraic logic: Models of π-institutions. Notre Dame J. Formal Logic **46**, 439 – 460 (2005).
[32] G. Voutsadakis, Categorical abstract algebraic logic: $(\mathcal{I}, N)$-algebraic systems. Applied Categorical Structures **13**, 265 – 280 (2005).
[33] G. Voutsadakis, Categorical abstract algebraic logic: Prealgebraicity and protoalgebraicity. To appear in Studia Logica. Preprint available at http://www.voutsadakis.com/RESEARCH/papers.html.
[34] G. Voutsadakis, Categorical abstract algebraic logic: Partially ordered algebraic systems. Applied Categorical Structures **14**, 81 – 98 (2006).
[35] G. Voutsadakis, Categorical abstract algebraic logic: Closure operators on classes of pofunctors. Submitted to Algebra Universalis. Preprint available at http://www.voutsadakis.com/RESEARCH/papers.html.
[36] G. Voutsadakis, Categorical abstract algebraic logic: Subdirect representation of pofunctors. To appear in Communications in Algebra. Preprint available at http://www.voutsadakis.com/RESEARCH/papers.html.
[37] G. Voutsadakis, Categorical abstract algebraic logic: Ordered equational logic and algebraizable povarieties. To appear in Order. Preprint available at http://www.voutsadakis.com/RESEARCH/papers.html.
[38] G. Voutsadakis, Categorical abstract algebraic logic: Bloom's theorem for rule-based π-institutions. Submitted to the Logic Journal of the IGPL.
Preprint available at http://www.voutsadakis.com/RESEARCH/papers.html.
[39] G. Voutsadakis, Categorical abstract algebraic logic: Structure systems and Łoś' theorem. Far East J. Mathematical Sciences **24**, 73 – 98 (2007).
[40] G. Voutsadakis, Categorical abstract algebraic logic: Leibniz equality and homomorphism theorems. Applied Categorical Structures **14**, 357 – 376 (2006).
[41] G. Voutsadakis, Categorical abstract algebraic logic: Operators on classes of structure systems. Scientiae Mathematicae Japonicae **19**, 1009 – 1023 (2006).

