CATEGORICAL ABSTRACT ALGEBRAIC LOGIC: PROTOALGEBRAICITY AND LEIBNIZ THEORY SYSTEMS

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Abstract. Font and Jansana studied the Leibniz filters and the logic determined by the Leibniz filters of a given protoalgebraic sentential logic. A filter is Leibniz when it is the smallest among all the filters on the same algebra having the same Leibniz congruence. Inspired by their work, a study of the N-Leibniz theory systems of an N-protoalgebraic π-institution is initiated. A theory system is N-Leibniz if it is the smallest among all theory systems having the same Leibniz N-congruence system. In this study, some of the results of Font and Jansana on Leibniz filters are adapted to cover the case of N-protoalgebraic π-institutions. The N-Leibniz operator, used in the present setting, is the operator associating with a given theory family of a given π-institution the Leibniz N-congruence system of the theory family, as introduced in previous work by the author.

1 Introduction Josep Maria Font and Ramon Jansana in [7] study the Leibniz filters of a protoalgebraic logic $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$. Starting from the observation that the Leibniz operator need not be injective on the collection of $\mathcal{S}$-filters on an $\mathcal{L}$-algebra $\mathbf{A}$, they single out those $\mathcal{S}$-filters F on $\mathbf{A}$ that are included in every $\mathcal{S}$-filter on $\mathbf{A}$ having the same Leibniz congruence as F. These are the *Leibniz $\mathcal{S}$-filters* of $\mathbf{A}$.

The work of Font and Jansana may be split into two major parts. In the first part, they study general properties of Leibniz filters and, in the second, they introduce and study properties of the sentential logic $\mathcal{S}^+$ defined by all those $\mathcal{S}$-matrices of the form $\langle \mathbf{A}, F \rangle$, with F Leibniz. The logic $\mathcal{S}^+$ is called the *strong version* of the protoalgebraic logic $\mathcal{S}$. Font and Jansana were led to the introduction of $\mathcal{S}^+$ by their motivation to explain a phenomenon observed in a variety of specific examples, e.g., modal logic, quantum logic and many-valued logic, among others, in which logics come naturally in pairs, one stronger than the other, but with the same theorems. It turns out that in these examples the strongest logic of the pair is the strong version of the other logic in this specific formal sense. Jansana in [8] continues the study started in [7].

Our main interest here in the work of Font and Jansana focuses on the first part of their work, i.e., on the study of Leibniz filters and their basic properties. The first main property of Leibniz filters (Theorem 3 of [7]) is that the Leibniz operator $\Omega_{\mathbf{A}}$ on any algebra $\mathbf{A}$ is an isomorphism between $\langle \mathrm{Fi}^+_{\mathcal{S}}(\mathbf{A}), \subseteq \rangle$ and $\langle \mathrm{Con}_{\mathbf{Alg}\mathcal{S}}\mathbf{A}, \subseteq \rangle$, where by $\mathrm{Fi}^+_{\mathcal{S}}(\mathbf{A})$ is denoted the collection of all Leibniz $\mathcal{S}$-filters on the algebra $\mathbf{A}$, for a protoalgebraic logic $\mathcal{S}$. For the collection $\mathbf{Alg}\mathcal{S}$ of $\mathcal{S}$-algebras, the reader may consult Section 2.2 of [6]. The second property (Theorem 8 of [7]) is that a strict surjective homomorphism from an $\mathcal{S}$-matrix $\langle \mathbf{A}, F \rangle$ onto an $\mathcal{S}$-matrix $\langle \mathbf{B}, G \rangle$ induces a strict surjective homomorphism between the associated Leibniz matrices $\langle \mathbf{A}, F^+ \rangle$ and $\langle \mathbf{B}, G^+ \rangle$, where by F^+ is denoted the unique Leibniz $\mathcal{S}$-filter included

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in the $\mathcal{S}$-filter F and having the same Leibniz congruence as F, for a protoalgebraic logic $\mathcal{S}$, and similarly for G^+. Another important result is a characterization of the Leibniz $\mathcal{S}$-filters F on an algebra $\mathbf{A}$ as those filters for which the quotient $F/\Omega_{\mathbf{A}}(F)$ is the smallest $\mathcal{S}$-filter on the quotient algebra $\mathbf{A}/\Omega_{\mathbf{A}}(F)$. All three of these properties, and some additional results following from these properties, will be shown to have important counterparts in the π-institution framework.

In the present work, an analog of Leibniz filters is introduced for N-protoalgebraic π-institutions, based on the N-Leibniz operator, introduced in recent work by the author [16], after the original Leibniz operator of Blok and Pigozzi [3]. Several of the properties that were established in [7] for Leibniz filters are now studied in this setting for N-Leibniz theory systems. The study of the strong version of an N-protoalgebraic π-institution, an analog of the strong version of a protoalgebraic logic, is postponed for future work, since the theories of equivalential and weakly algebraizable π-institutions, whose analogs play key roles in the developments in [7] and [8], are still under development.

In the remaining of the introduction we present some basic elements of the theory leading to the study of Leibniz theory systems and, then, provide a summary of the contents of the paper.

In [16] the notion of an N-protoalgebraic π-institution was introduced after protoalgebraic deductive systems [2]. Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution and N a category of natural transformations on SEN. Given a theory family T of $\mathcal{I}$, the Leibniz N-congruence system $\Omega^N(T)$ is the largest N-congruence system of $\mathcal{I}$ that is compatible with T (see Propositions 2.3 and 2.4 of [16]). $\mathcal{I}$ is said to be *N-protoalgebraic* (Section 3 of [16]) if the Leibniz N-congruence system operator Ω^N is monotone on all theory families of $\mathcal{I}$, i.e., if for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$,

$$T \leq T' \quad \text{implies} \quad \Omega^N(T) \leq \Omega^N(T').$$

Several properties of N-protoalgebraic π-institutions were presented in [16] and some more in [17]. Both [16] and [17] will provide important background information for the developments in the present work.

Based on this notion of N-protoalgebraicity and taking after the work of Font and Jansana [7], a theory system of a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, will be said to be N-Leibniz if it is included in all theory systems of $\mathcal{I}$ with the same Leibniz N-congruence system. It is shown that, similarly with the sentential logic situation, if $\mathcal{I}$ is N-protoalgebraic, then every theory system T of $\mathcal{I}$ includes a unique N-Leibniz theory system T^N of $\mathcal{I}$ with the same Leibniz N-congruence system. Furthermore, analogs of the properties reviewed above for Leibniz filters of a protoalgebraic sentential logic will be presented for Leibniz N-theory systems of an N-protoalgebraic π-institution. In Theorem 3, it is shown that, if $\langle F, \alpha\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}'$ is a surjective singleton (N, N')-epimorphic translation and by $\mathcal{I}'^{\min} = \langle \mathbf{Sign}', \mathrm{SEN}', C'^{\min}\rangle$ is denoted the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN', then the N'-Leibniz operator $\Omega^{N'}$ is an isomorphism between $\langle \mathrm{ThSys}^{N'}(\mathcal{I}'), \leq\rangle$ and $\langle \mathrm{Con}^{\langle F,\alpha\rangle}_{\mathrm{Alg}^N(\mathcal{I})}(\mathrm{SEN}'), \leq\rangle$ (see [17] for relevant definitions). In Theorem 8, it is shown that, if $\langle F, \alpha\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}'$ and $\langle G, \beta\rangle : \mathrm{SEN}' \rightarrow^{se} \mathrm{SEN}''$ are surjective singleton (N, N')- and (N', N'')-epimorphic translations, respectively, with $G : \mathbf{Sign}' \rightarrow \mathbf{Sign}''$ an isomorphism,

$$\begin{tikzcd}
& \mathcal{I} \arrow[dl, "\langle F,\alpha\rangle"'] \arrow[dr, "\langle GF,\beta_F\alpha\rangle"] & \\
\mathrm{SEN}' \arrow[rr, "\langle G,\beta\rangle"'] & & \mathrm{SEN}''
\end{tikzcd}$$

and T' and T'' are theory systems of the $\langle F,\alpha\rangle$-min (N,N')-model $\mathcal{I}'^{\min}$ of $\mathcal{I}$ on SEN' and of the $\langle GF,\beta_F\alpha\rangle$-min (N,N'')-model $\mathcal{I}''^{\min}$ of $\mathcal{I}$ on SEN'', respectively, such that $T'=\beta^{-1}(T'')$, then $T'^{N'}=\beta^{-1}(T''^{N''})$.

Finally, in the characterization Proposition 10, similar to Proposition 10 of [7], it is shown that, if $\langle F,\alpha\rangle:\mathrm{SEN}\rightarrow^{se}\mathrm{SEN}'$ is a surjective singleton (N,N')-epimorphic translation, a theory system T of the $\langle F,\alpha\rangle$-min (N,N')-model $\mathcal{I}'^{\min}$ of $\mathcal{I}$ on SEN' is N'-Leibniz if and only if $T/\Omega^{N'}(T)$ is the smallest theory system of the $\langle F,\pi_F^{\Omega^{N'}(T)}\alpha\rangle$-min $(N,N'^{\Omega^{N'}(T)})$-model of $\mathcal{I}$ on $\mathrm{SEN}'^{\Omega^{N'}(T)}$. All the results mentioned above presuppose that $\mathcal{I}$ is an N-protoalgebraic π-institution and parallel the corresponding results of [7] for protoalgebraic sentential logics.

For necessary background on general categorical notions and notation, the reader is referred to any of [1, 4, 9]. For protoalgebraic deductive systems, the original source is [2], but also a good part of the book [5] by Czelakowski is devoted to the topic. For an overview of the theory of algebraic semantics of sentential logics that inspired the current developments on the categorical level, the monograph [6] is an excellent source. Finally, for background on all notions developed so far in the categorical theory of abstract algebraic logic, that made possible the transfer of many of the results on sentential logics of [6] to π-institutions, the reader may consult [10, 11, 12, 13, 14, 15, 16, 17] in that order.

2 Leibniz Theory Systems Consider an N-protoalgebraic π-institution $\mathcal{I}=\langle\mathbf{Sign},\mathrm{SEN},C\rangle$, with N a category of natural transformations on SEN. A theory system $T=\{T_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ is said to be an N-**Leibniz theory system** of $\mathcal{I}$, if, for every theory system $T'=\{T'_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ of $\mathcal{I}$, such that $\Omega^N(T)=\Omega^N(T')$, we have $T\leq T'$.

Recall that $\mathrm{ThSys}(\mathcal{I})$ denotes the collection of all theory systems of $\mathcal{I}$. Let, by analogy with the notation $\mathrm{Fi}^+_{\mathcal{S}}(\mathbf{A})$ of [7] for sentential logics, $\mathrm{ThSys}^N(\mathcal{I})$ denote the collection of all N-Leibniz theory systems of a π-institution $\mathcal{I}$, where the N in place of the $+$, stems from the intention of making the dependence on the category N of natural transformations transparent.

Call an N-protoalgebraic π-institution $\mathcal{I}$ N-**weakly algebraizable** iff, for all theory systems T,T' of $\mathcal{I}$, $\Omega^N(T)=\Omega^N(T')$ implies $T=T'$. Then it is easily seen that, if $\mathcal{I}$ is N-weakly algebraizable, then every theory system of $\mathcal{I}$ is an N-Leibniz theory system. In other words, if $\mathcal{I}$ is N-weakly algebraizable, then $\mathrm{ThSys}(\mathcal{I})=\mathrm{ThSys}^N(\mathcal{I})$.

The following proposition, an analog of Proposition 2 of [7], shows that, given any theory system T in an N-prealgebraic π-institution [16] , there exists a unique N-Leibniz theory system T^N, such that $\Omega^N(T^N)=\Omega^N(T)$.

Proposition 1 *Let $\mathcal{I}=\langle\mathbf{Sign},\mathrm{SEN},C\rangle$ be an N-prealgebraic π-institution. For every theory system T of $\mathcal{I}$, there exists a unique N-Leibniz theory system T^N of $\mathcal{I}$, such that $\Omega^N(T^N)=\Omega^N(T)$.*

Proof:
We show that $T^N=\{T^N_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$, defined, for all $\Sigma\in|\mathbf{Sign}|$, by

$$T^N_\Sigma=\bigcap\{T'_\Sigma : T'\in\mathrm{ThSys}(\mathcal{I})\text{ with }\Omega^N(T')=\Omega^N(T)\},$$

i.e., $T^N=\bigcap\{T'\in\mathrm{ThSys}(\mathcal{I}):\Omega^N(T')=\Omega^N(T)\}$, is such that $\Omega^N(T^N)=\Omega^N(T)$. Since, by Proposition 2.2 of [16], T^N is a theory system of $\mathcal{I}$, it will then be obvious that T^N is the smallest theory system of $\mathcal{I}$ with $\Omega^N(T^N)=\Omega^N(T)$ and is, therefore, an N-Leibniz theory system.

We have, indeed,

$$\begin{array}{rcl}\Omega^N(T^N) & = & \Omega^N(\bigcap\{T' \in \mathrm{ThSys}(\mathcal{I}) : \Omega^N(T') = \Omega^N(T)\}) \\ & = & \bigcap\{\Omega^N(T') : T' \in \mathrm{ThSys}(\mathcal{I}) : \Omega^N(T') = \Omega^N(T)\} \\ & = & \Omega^N(T),\end{array}$$

where the first equation follows from the definition of T^N and the second follows from the characterization of N-prealgebraicity given in Lemma 3.10 of [16]. ■

Proposition 1, together with Corollary 3.9 of [16] yield immediately

Corollary 2 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be an N-protoalgebraic π-institution. For every theory system T of $\mathcal{I}$, there exists a unique N-Leibniz theory system T^N of $\mathcal{I}$, such that $\Omega^N(T^N) = \Omega^N(T)$.*

As a consequence of the definition and of Proposition 1, we have that $T^N \leq T$ and that an arbitrary theory system T is N-Leibniz if and only if $T^N = T$.

Suppose, now, that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is an N-protoalgebraic π-institution and that $\langle F, \alpha\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}'$ is a surjective singleton (N, N')-epimorphic translation. It was shown in Proposition 5.22 of [16] that the $\langle F, \alpha\rangle$-min (N, N')-model $\mathcal{I}'^{\min}$ of $\mathcal{I}$ on SEN' is N'-protoalgebraic. Next, it is shown that the N'-Leibniz operator is an isomorphism between the lattice of all N'-Leibniz theory systems of $\mathcal{I}'^{\min}$ and that of all $\mathrm{Alg}^N(\mathcal{I})$-$N'$congruence systems θ on SEN', where θ is an $\mathrm{Alg}^N(\mathcal{I})$-$N'$-congruence system because the $\langle F, \pi^\theta_F\alpha\rangle$-min model of $\mathcal{I}$ on SEN'^θ is N'^θ-reduced. This is an analog of Theorem 3 of [7] for π-institutions.

Theorem 3 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, be an N-protoalgebraic π-institution and $\langle F, \alpha\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}'$ a surjective singleton (N, N')-epimorphic translation and denote by $\mathcal{I}'^{\min} = \langle \mathbf{Sign}', \mathrm{SEN}', C'^{\min}\rangle$ the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN'. The N'-Leibniz operator $\Omega^{N'}$ is an isomorphism between the $\langle \mathrm{ThSys}^{N'}(\mathcal{I}'^{\min}), \leq\rangle$ and $\langle \mathrm{Con}^{\langle F,\alpha\rangle}_{\mathrm{Alg}^N(\mathcal{I})}(\mathrm{SEN}'), \leq\rangle$.*

Proof:

One can rely on Propositions 5.3 and 5.4 of [17].

To reveal more details, suppose that we know only that the N'-Leibniz operator $\Omega^{N'}$ is a mapping from the lattice $\langle \mathrm{ThSys}^{N'}(\mathcal{I}'^{\min}), \leq\rangle$ into $\langle \mathrm{Con}^{\langle F,\alpha\rangle}_{\mathrm{Alg}^N(\mathcal{I})}(\mathrm{SEN}'), \leq\rangle$. It is clearly one-to-one because of the definition of $\mathrm{ThSys}^{N'}(\mathcal{I}'^{\min})$. By Theorem 5.2 of [17], for every N'-congruence system $\theta \in \mathrm{Con}^{\langle F,\alpha\rangle}_{\mathrm{Alg}^N(\mathcal{I})}(\mathrm{SEN}')$, there exists a theory system $T \in \mathrm{ThSys}(\mathcal{I}'^{\min})$, such that $\theta = \Omega^{N'}(T)$. Therefore, we obtain $\theta = \Omega^{N'}(T^{N'})$ and $T^{N'} \in \mathrm{ThSys}^{N'}(\mathcal{I}'^{\min})$. So $\Omega^{N'}$ is also onto. It is clearly order-preserving by N-protoalgebraicity of $\mathcal{I}$ and the surjectivity of $\langle F, \alpha\rangle$ (Proposition 5.22 of [16]). To show that it is an order-isomorphism, suppose that T, T' are two theory systems in $\mathrm{ThSys}^{N'}(\mathcal{I}'^{\min})$, such that $\Omega^{N'}(T) \leq \Omega^{N'}(T')$. We have

$$\begin{array}{rcl}\Omega^{N'}((T \cap T')^{N'}) & = & \Omega^{N'}(T \cap T') \quad \text{(by the definition of } {}^{N'}\text{)} \\ & = & \Omega^{N'}(T) \cap \Omega^{N'}(T') \quad \text{(by } N'\text{-protoalgebraicity)} \\ & = & \Omega^{N'}(T) \quad \text{(by hypothesis).}\end{array}$$

Therefore, since T is an N'-Leibniz theory system, we get that $(T \cap T')^{N'} = T$, whence $T \leq T \cap T'$, i.e., $T \leq T'$. ■

Theorem 3 has the following two corollaries, analogs, respectively, of Corollaries 4 and 5 of [7].

Corollary 4 *Given an N-protoalgebraic π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ and a theory system T of $\mathcal{I}$, T^N is the largest N-Leibniz theory system $\leq$-contained in T.*

Proof:

Suppose that $T' \in \mathrm{ThSys}^N(\mathcal{I})$ and $T' \leq T$. Then, by N-protoalgebraicity, $\Omega^N(T') \leq \Omega^N(T)$ and, by the definition of ^{N}, $\Omega^N(T) = \Omega^N(T^N)$. Therefore $\Omega^N(T') \leq \Omega^N(T^N)$. Hence, by Theorem 3, we get that $T' \leq T^N$. ■

Now, Corollary 4 immediately yields

Corollary 5 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be an N-protoalgebraic π-institution. For all theory systems T, T' of $\mathcal{I}$, if $T \leq T'$, then $T^N \leq T'^N$.*

The following proposition characterizes those theory systems of a given N-protoalgebraic π-institution whose Leibniz N-congruence systems are the identity congruence systems.

Proposition 6 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be an N-protoalgebraic π-institution and $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ a surjective singleton (N, N')-epimorphic translation. The theory system T of the $\langle F, \alpha\rangle$-min (N, N')-model $\mathcal{I}'^{\min}$ of $\mathcal{I}$ on SEN' is such that $\Omega^{N'}(T) = \Delta_{\mathrm{SEN}'}$ if and only if SEN' is in $\mathrm{Alg}^N(\mathcal{I})^s$ via $\langle F, \alpha\rangle$ and $T^{N'}$ is the only N'-Leibniz theory system of $\mathcal{I}'$ $\leq$-included in T.*

Proof:

Suppose, first, that the theory system T of the $\langle F, \alpha\rangle$-min (N, N')-model $\mathcal{I}'^{\min}$ of $\mathcal{I}$ on SEN' is such that $\Omega^{N'}(T) = \Delta_{\mathrm{SEN}'}$. Then, by the definition of $\mathrm{Alg}^N(\mathcal{I})^{*s}$, we have $\mathrm{SEN}' \in \mathrm{Alg}^N(\mathcal{I})^{*s}$ and, since, by Corollary 5.23 of [16], $\mathrm{Alg}^N(\mathcal{I})^{*s} = \mathrm{Alg}^N(\mathcal{I})^s$, we obtain $\mathrm{SEN}' \in \mathrm{Alg}^N(\mathcal{I})^s$. Now, suppose that T' is an N'-Leibniz theory system of $\mathcal{I}'$, such that $T' \leq T$. Then by Corollary 5, $T' = T'^{N'} \leq T^{N'}$. Therefore, by N'-protoalgebraicity, $\Omega^{N'}(T') \leq \Omega^{N'}(T^{N'}) = \Omega^{N'}(T) = \Delta_{\mathrm{SEN}'}$. Hence $\Omega^{N'}(T') = \Delta_{\mathrm{SEN}'} = \Omega^{N'}(T^{N'})$. Theorem 3 now implies that $T' = T^{N'}$, whence $T^{N'}$ is the only N'-Leibniz theory system that is included in T.

Suppose, conversely, that SEN' is in $\mathrm{Alg}^N(\mathcal{I})^s$ via $\langle F, \alpha\rangle$ and $T^{N'}$ is the only N'-Leibniz theory system of $\mathcal{I}'$ $\leq$-included in T. Since, by Corollary 5.23 of [16], $\mathrm{Alg}^N(\mathcal{I})^{*s} = \mathrm{Alg}^N(\mathcal{I})^s$, there exists, by the definition of $\mathrm{Alg}^N(\mathcal{I})^{*s}$, a theory system T' of the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN', such that $\Omega^{N'}(T') = \Delta_{\mathrm{SEN}'}$. Then, we have that $\Omega^{N'}(T'^{N'}) = \Omega^{N'}(T') = \Delta_{\mathrm{SEN}'} \leq \Omega^{N'}(T^{N'})$, whence, by Theorem 3, we obtain $T'^{N'} \leq T^{N'} \leq T$. On the other hand, we have, by the hypothesis, that $T^{N'}$ is the only N'-Leibniz theory system $\leq$-included in T, whence $T'^{N'} = T^{N'}$ and, therefore, $\Omega^{N'}(T) = \Omega^{N'}(T^{N'}) = \Omega^{N'}(T'^{N'}) = \Delta_{\mathrm{SEN}'}$, as was to be shown. ■

We immediately obtain

Corollary 7 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be an N-protoalgebraic π-institution and $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ a surjective singleton (N, N')-epimorphic translation. If T, T' are theory systems of the $\langle F, \alpha\rangle$-min (N, N')-model $\mathcal{I}'^{\min}$ of $\mathcal{I}$ on SEN', such that $\Omega^{N'}(T) = \Omega^{N'}(T') = \Delta_{\mathrm{SEN}'}$, then $T^{N'} = T'^{N'}$.*

The next result is an analog of Theorem 8 of [7] for π-institutions. It says, roughly speaking, that, if a singleton surjective (N', N'')-epimorphic translation between two sentence functors SEN' and SEN'' preserves theory systems of min models on SEN' and SEN'', then it also preserves their corresponding Leibniz images.

Theorem 8 *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *is an* N*-protoalgebraic* π*-institution,* $\mathrm{SEN}' : \mathbf{Sign}' \rightarrow \mathbf{Set}$ *and* $\mathrm{SEN}'' : \mathbf{Sign}'' \rightarrow \mathbf{Set}$ *two functors and* N', N'' *categories of natural transformations on* $\mathrm{SEN}', \mathrm{SEN}''$*, respectively. Suppose that* $\langle F, \alpha\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}'$ *and* $\langle G, \beta\rangle : \mathrm{SEN}' \rightarrow^{se} \mathrm{SEN}''$ *are surjective singleton* (N, N')*- and* (N', N'')*-epimorphic translations, respectively, with* $G : \mathbf{Sign}' \rightarrow \mathbf{Sign}''$ *an isomorphism. Then, if* T' *and* T'' *are theory systems of the* $\langle F, \alpha\rangle$*-min* (N, N')*-model* $\mathcal{I}'^{\min}$ *of* $\mathcal{I}$ *on* SEN' *and of the* $\langle GF, \beta_F\alpha\rangle$*-min* (N, N'')*-model* $\mathcal{I}''^{\min}$ *of* $\mathcal{I}$ *on* SEN''*, respectively, such that* $T' = \beta^{-1}(T'')$*, then* $T'^{N'} = \beta^{-1}(T''^{N''})$*.*

$$\begin{tikzcd} & \mathcal{I} \arrow[dl, "\langle F,\alpha\rangle"'] \arrow[dr, "\langle GF,\beta_F\alpha\rangle"] & \\ \mathrm{SEN}' \arrow[rr, "\langle G,\beta\rangle"'] & & \mathrm{SEN}'' \end{tikzcd}$$

Proof:

The family $\beta^{-1}(T''^{N''})$ is a theory family of $\mathcal{I}'^{\min}$, since the closure system generated by $\langle G, \beta\rangle : \mathrm{SEN}' \rightarrow \mathcal{I}''^{\min}$ on SEN' is included in $\mathcal{I}'^{\min}$. Now, we have, using Lemma 5.21 of [16],

$$\begin{array}{rcl} \Omega^{N'}(\beta^{-1}(T''^{N''})) & = & \beta^{-1}(\Omega^{N''}(T''^{N''})) \quad \text{(by Lemma 5.21 of [16])} \\ & = & \beta^{-1}(\Omega^{N''}(T'')) \quad \text{(by the definition of } {}^{N''}) \\ & = & \Omega^{N'}(\beta^{-1}(T'')) \quad \text{(by Lemma 5.21 of [16])} \\ & = & \Omega^{N'}(T'). \quad \text{(by the hypothesis)} \end{array}$$

Therefore, by the definition of an N'-Leibniz theory system $T'^{N'} \leq \beta^{-1}(T''^{N''})$.

For the reverse inclusion, we first show, following the proof of Theorem 8 of [7], that

$$\beta^{-1}(\beta(T'^{N'})) = T'^{N'}. \tag{1}$$

The inclusion $T'^{N'}_{\Sigma'} \subseteq \beta^{-1}_{\Sigma'}(\beta_{\Sigma'}(T'^{N'}_{\Sigma'}))$, for all $\Sigma' \in |\mathbf{Sign}'|$, is set-theoretic. For the reverse inclusion, suppose that $\phi' \in \beta^{-1}_{\Sigma'}(\beta_{\Sigma'}(T'^{N'}_{\Sigma'}))$. Then, there exists $\psi' \in T'^{N'}_{\Sigma'}$, such that $\beta_{\Sigma'}(\phi') = \beta_{\Sigma'}(\psi')$. Therefore, since $\langle \beta_{\Sigma'}(\phi'), \beta_{\Sigma'}(\psi')\rangle \in \Omega^{N''}_{G(\Sigma')}(T'')$, we get that $\langle \phi', \psi'\rangle \in \beta^{-1}_{\Sigma'}(\Omega^{N''}_{G(\Sigma')}(T'')) = \Omega^{N'}_{\Sigma'}(T') = \Omega^{N'}_{\Sigma'}(T'^{N'})$. But $\psi' \in T'^{N'}_{\Sigma'}$ and $\Omega^{N'}(T'^{N'})$ is compatible with $T'^{N'}$, whence $\phi' \in T'^{N'}_{\Sigma'}$ as well. This concludes the proof of (1).

By the surjectivity of $\langle G, \beta\rangle$ and (1), we get that $\beta(T'^{N'})$ is a theory family of $\mathcal{I}''^{\min}$. So we obtain

$$\begin{array}{rcl} \beta^{-1}(\Omega^{N''}(\beta(T'^{N'}))) & = & \Omega^{N'}(\beta^{-1}(\beta(T'^{N'}))) \quad \text{(by Lemma 5.21 of [16])} \\ & = & \Omega^{N'}(T'^{N'}) \quad \text{(by (1))} \\ & = & \Omega^{N'}(T') \quad \text{(by the definition of } {}^{N'}) \\ & = & \Omega^{N'}(\beta^{-1}(T'')) \quad \text{(by the hypothesis)} \\ & = & \beta^{-1}(\Omega^{N''}(T'')). \quad \text{(by Lemma 5.21 of [16])} \end{array}$$

Thus, once more by the surjectivity of $\langle G, \beta\rangle$, we get that $\Omega^{N''}(\beta(T'^{N'})) = \Omega^{N''}(T'')$ and, hence, $T''^{N''} \leq \beta(T'^{N'})$, i.e., that $\beta^{-1}(T''^{N''}) \leq T'^{N'}$, which was to be shown. ■

An immediate corollary of Theorem 8 is that, under the same hypotheses as in the theorem, if, for a theory system T' of $\mathcal{I}'^{\min}$ and a theory system T'' of $\mathcal{I}''^{\min}$, it holds that $T' = \beta^{-1}(T'')$, then T' is N'-Leibniz if and only if T'' is N''-Leibniz.

Corollary 9 *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *is an* N*-protoalgebraic* π*-institution,* $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ *and* $\mathrm{SEN}'' : \mathbf{Sign}'' \to \mathbf{Set}$ *two functors and* N', N'' *categories of natural transformations on* $\mathrm{SEN}', \mathrm{SEN}''$*, respectively. Suppose that* $\langle F, \alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$ *and* $\langle G, \beta\rangle : \mathrm{SEN}' \to \mathrm{SEN}''$ *are surjective singleton* (N, N')*- and* (N', N'')*-epimorphic translations, respectively, with* $G : \mathbf{Sign}' \to \mathbf{Sign}''$ *an isomorphism. If* T' *and* T'' *are theory systems of the* $\langle F, \alpha\rangle$*-min* (N, N')*-model* $\mathcal{I}'^{\min}$ *of* $\mathcal{I}$ *on* SEN' *and of the* $\langle GF, \beta_F\alpha\rangle$*-min* (N, N'')*-model* $\mathcal{I}''^{\min}$ *of* $\mathcal{I}$ *on* SEN''*, respectively, such that* $T' = \beta^{-1}(T'')$*, then the theory system* T' *is* N'*-Leibniz in* $\mathcal{I}'^{\min}$ *if and only if the theory system* T'' *is* N''*-Leibniz in* $\mathcal{I}''^{\min}$*.*

With the help of Corollary 9, a characterization of N'-Leibniz theory systems of min (N, N')-models of an N-protoalgebraic π-institution via surjective singleton (N, N')-epimorphic translations may be given. Proposition 10 forms an analog of Proposition 10 of [7] in the π-institution framework.

Proposition 10 *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$*, with* N *a category of natural transformations on* SEN*, is an* N*-protoalgebraic* π*-institution. Let* $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ *be a surjective singleton* (N, N')*-epimorphic translation. A theory system* T *of the* $\langle F, \alpha\rangle$*-min* (N, N')*-model* $\mathcal{I}'^{\min}$ *of* $\mathcal{I}$ *on* SEN' *is* N'*-Leibniz if and only if* $T/\Omega^{N'}(T)$ *is the smallest theory system, i.e., the theorem system, of the* $\langle F, \pi_F^{\Omega^{N'}(T)}\alpha\rangle$*-min* $(N, N'^{\Omega^{N'}(T)})$*-model of* $\mathcal{I}$ *on* $\mathrm{SEN}'^{\Omega^{N'}(T)}$*.*

Proof:

Note that the following commutative triangle satisfies all the necessary conditions of the hypothesis of Theorem 8 and of Corollary 9.

$$\begin{tikzcd}
 & \mathcal{I} \arrow[dl, "\langle F,\alpha\rangle"'] \arrow[dr, "\langle F,\pi_F^{\Omega^{N'}(T)}\alpha\rangle"] & \\
\mathrm{SEN}' \arrow[rr, "\langle \mathrm{I}_{\mathbf{Sign}'},\pi^{\Omega^{N'}(T)}\rangle"'] & & \mathrm{SEN}'^{\Omega^{N'}(T)}
\end{tikzcd}$$

Suppose that T is N'-Leibniz. Then, since $T = (\pi^{\Omega^{N'}(T)})^{-1}(T/\Omega^{N'}(T))$, we obtain, by Corollary 9, that $T/\Omega^{N'}(T)$ is also $N'^{\Omega^{N'}(T)}$-Leibniz. Let T'' be a theory system of the $\langle F, \pi_F^{\Omega^{N'}(T)}\alpha\rangle$-min $(N, N'^{\Omega^{N'}(T)})$-model of $\mathcal{I}$ on $\mathrm{SEN}'^{\Omega^{N'}(T)}$. Consider the theory system $T'' \cap T/\Omega^{N'}(T)$. We have

$$(T'' \cap T/\Omega^{N'}(T))^{N'^{\Omega^{N'}(T)}} \leq T'' \cap T/\Omega^{N'}(T) \leq T/\Omega^{N'}(T),$$

whence, by N-protoalgebraicity and Proposition 5.22 of [16],

$$\Omega^{N'^{\Omega^{N'}(T)}}((T'' \cap T/\Omega^{N'}(T))^{N'^{\Omega^{N'}(T)}}) \leq \Omega^{N'^{\Omega^{N'}(T)}}(T/\Omega^{N'}(T)) = \Delta_{\mathrm{SEN}'^{\Omega^{N'}(T)}}$$

and, therefore, $\Omega^{N'^{\Omega^{N'}(T)}}((T''\cap T/\Omega^{N'}(T))^{N'^{\Omega^{N'}(T)}}) = \Omega^{N'^{\Omega^{N'}(T)}}(T/\Omega^{N'}(T)) = \Delta_{\mathrm{SEN}'^{\Omega^{N'}(T)}}$. But both $T/\Omega^{N'}(T)$ and $(T'' \cap T/\Omega^{N'}(T))^{N'^{\Omega^{N'}(T)}}$ are $N'^{\Omega^{N'}(T)}$-Leibniz and, therefore, $T/\Omega^{N'}(T) = (T'' \cap T/\Omega^{N'}(T))^{N'^{\Omega^{N'}(T)}}$. Thus $T/\Omega^{N'}(T) = T'' \cap T/\Omega^{N'}(T)$, which yields $T/\Omega^{N'}(T) \leq T''$. Thus $T/\Omega^{N'}(T)$ is the smallest theory system of the $\langle F, \pi_F^{\Omega^{N'}(T)}\alpha\rangle$-min $(N, N'^{\Omega^{N'}(T)})$-model of $\mathcal{I}$ on $\mathrm{SEN}'^{\Omega^{N'}(T)}$.

Suppose, conversely, that $T/\Omega^{N'}(T)$ is the smallest theory system of the $\langle F, \pi_F^{\Omega^{N'}(T)}\alpha\rangle$-min $(N, N'^{\Omega^{N'}(T)})$-model of $\mathcal{I}$ on $\mathrm{SEN}'^{\Omega^{N'}(T)}$. Since $\Omega^{N'}(T) = \Omega^{N'}(T^{N'})$, it makes sense to consider the two quotients $T/\Omega^{N'}(T)$ and $T^{N'}/\Omega^{N'}(T)$. Obviously, since $T^{N'} \leq T$, we have that $T^{N'}/\Omega^{N'}(T) \leq T/\Omega^{N'}(T)$, whence, since both are theory systems of the $\langle F, \pi_F^{\Omega^{N'}(T)}\alpha\rangle$-min $(N, N'^{\Omega^{N'}(T)})$-model of $\mathcal{I}$ on $\mathrm{SEN}'^{\Omega^{N'}(T)}$, we obtain, by the hypothesis, that $T^{N'}/\Omega^{N'}(T) = T/\Omega^{N'}(T)$, whence, by Corollary 4.16 of [16], we obtain that $T^{N'} = T$ and T is N'-Leibniz. ■

Proposition 11 *For every N-protoalgebraic π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, the collection* $\mathrm{ThSys}^N(\mathcal{I})$ *forms a join-complete subsemilattice of the complete lattice* $\mathbf{ThSys}(\mathcal{I}) = \langle \mathrm{ThSys}(\mathcal{I}), \leq\rangle$ *of all theory systems of $\mathcal{I}$.*

Proof:

Suppose that $\{T^i : i \in I\}$ is a collection of theory systems in $\mathrm{ThSys}^N(\mathcal{I})$. Then $T^i = (T^i)^N \leq (\bigvee_{i\in I} T^i)^N$, by Corollary 5. Therefore, $\bigvee_{i\in I} T^i \leq (\bigvee_{i\in I} T^i)^N$ and, since the reverse inclusion always holds, $(\bigvee_{i\in I} T^i)^N = \bigvee_{i\in I} T^i$. Hence $\bigvee_{i\in I} T^i$ is indeed an N-Leibniz theory system. ■

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