REPORTS ON MATHEMATICAL LOGIC
41 (2006), 31–62

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CATEGORICAL ABSTRACT ALGEBRAIC LOGIC FULL MODELS, FREGE SYSTEMS AND METALOGICAL PROPERTIES

In memory of Willem Blok

A b s t r a c t. Font and Jansana studied the full models of sentential logics under the presence of a variety of metalogical properties. Their theory of full models was adapted, in recent work by the author, to cover the case of institutional logics. In the present work, the study of metalogical properties is carried out in the π-institution framework and the way they affect full models of π-institutions is investigated.

Received 17 September 2004
Keywords: abstract algebraic logic, deductive systems, institutions, equivalent deductive systems, algebraizable deductive systems, adjunctions, equivalent institutions, algebraizable institutions, Leibniz congruence, Tarski congruence, algebraizable sentential logics, full models, $\mathcal{S}$-algebras, Frege equivalence, congruence property, property of conjunction, deduction-detachment theorem, property of disjunction, reductio ad absurdum
1991 AMS Subject Classification: Primary: 03Gxx, Secondary: 18Axx, 68N05

1 Introduction

One of the main directions of research in Abstract Algebraic Logic has been the study of the interplay between metalogical properties, on the logic side, and algebraic properties of algebraizing classes of algebras, on the algebraic side. In [5, 9, 14], e.g., a detailed study of the deduction-detachment property for deductive systems is undertaken. Some other examples include [7], that studies the amalgamation property, and [8] on the Maehara interpolation property. In [14], Font and Jansana also studied, alongside the deduction detachment theorem, the congruence property, the properties of conjunction and disjunction, and that of reductio ad absurdum. They are interested, in particular, in how the presence of each of these, or combinations of these, properties affects, and interacts with, the full models of a sentential logic (see Section 2.4 of [14]).

In the context of π-institutions, Tarlecki [19] has been the first to have studied some metalogical properties. His studies were continued, but in a different context, by the author in [23]. This was following the introduction of algebraizable institutions in [20] (see also [21, 22]), which was, in turn, extending work of Blok and Pigozzi [3, 4] on algebraizable deductive systems. An overview of this and related work may be found in [10] and in [15].

The theory of Font and Jansana [14] has been adapted by the author in a series of papers to cover the case of institutional logics [24, 25, 26, 27, 28]. In light of this work, the results of Font and Jansana on metalogical properties are explored in the present paper in the context of π-institutions.

More precisely, in Section 2, a characterization is given of the $\langle F,\alpha\rangle$-min model of a finitary π-institution $\mathcal{I}$ for a surjective singleton translation $\langle F,\alpha\rangle : \mathcal{I} \to \mathrm{SEN}'$. Min models were introduced for sentential logics in [14] and the definition was adapted to the π-institution framework in [26]. The characterization, given here, parallels an analogous characterization of the $\mathcal{S}$-filter $\mathrm{Fi}^{\mathbf{A}}_{\mathcal{S}}(X)$ on an algebra $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}}\rangle$ generated by a subset $X \subseteq A$. It is the first time that finitary π-institutions are given special attention as a class of π-institutions. Virtually all deductive systems studied in the literature, however, are assumed to be finitary and this is a necessary assumption for some of the results that will be formulated in the sequel.

In Section 3, a special equivalence system of a π-institution, the Frege equivalence system, is introduced. It is then used to define the congruence property for a π-institution. Frege equivalence systems for sentential logics and the congruence property were studied in [14]. The general notion of an equivalence system for a π-institution was introduced in [24]. The notion of a Frege equivalence system, as far as the athor knows, appears in the institution framework for the first time in the present paper.

In Section 4, the property of conjunction is studied. A similar property, but from a slightly different point of view, was studied in [23]. In the present work, the property of conjunction is adapted from the sentential logic level in a way suitable to recapture analogs of some of the results of [14] for π-institutions and their models.

In Section 5, the property of deduction-detachment is studied. A similar version was introduced in [23]. Once more, the prototype for our work is the analogous property studied in the sentential logic level [14]. This property has been extensively studied before by many authors. Examples are the papers [5, 9, 14].

In Section 6, the property of disjunction is explored. Again, a similar form and some of its consequences were investigated in [23] and [14] contains a study in the sentential logic framework.

Finally, Section 7 studies the reductio ad absurdum. This property had not been introduced in [21]. [14], however, does contain an account of this property at the sentential logic framework. Some of the results of [14] are adapted in the π-institution level in this final section.

For bits and pieces of unexplained categorical notation, the reader is encouraged to consult any of [2, 6, 18].

2 $\langle F, \alpha\rangle$-Min Models for Surjective $\langle F, \alpha\rangle$

Recall from [16, 17] the definition of an institution and from [13] that of a π-institution. A **π-institution** $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ consists of

(i) a category **Sign** whose objects are called **signatures**,

(ii) a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, from the category **Sign** of signatures into the category **Set** of sets, called the **sentence functor** and giving, for each signature Σ, a set whose elements are called **sentences over** that signature Σ or **Σ-sentences** and

(iii) a mapping $C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \to \mathcal{P}(\mathrm{SEN}(\Sigma))$, for each $\Sigma \in |\mathbf{Sign}|$, called **$\Sigma$-closure,** such that

(a) $A \subseteq C_\Sigma(A)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq \mathrm{SEN}(\Sigma)$,

(b) $C_\Sigma(C_\Sigma(A)) = C_\Sigma(A)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq \mathrm{SEN}(\Sigma)$,

(c) $C_\Sigma(A) \subseteq C_\Sigma(B)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq B \subseteq \mathrm{SEN}(\Sigma)$,

(d) $\mathrm{SEN}(f)(C_{\Sigma_1}(A)) \subseteq C_{\Sigma_2}(\mathrm{SEN}(f)(A))$, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, $A \subseteq \mathrm{SEN}(\Sigma_1)$.

A collection $C = \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ satisfying properties (iii)(a)-(d) will be referred to as a **closure system on** SEN.

A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is said to be **finitary** if, for every $\Sigma \in |\mathbf{Sign}|$, C_Σ is a finitary closure operator on $\mathrm{SEN}(\Sigma)$ in the usual sense, i.e., if, for every $\Sigma \in |\mathbf{Sign}|, \Phi \subseteq \mathrm{SEN}(\Sigma), C_\Sigma(\Phi) = \bigcup_{\Psi \subseteq_\omega \Phi} C_\Sigma(\Psi)$, where $\subseteq_\omega$ denotes finite subset.

It is shown in the only, rather technical, result of the section that, if $\mathcal{I}$ is a finitary π-institution and $\langle F, \alpha\rangle : \mathcal{I} \to \mathrm{SEN}'$ is a surjective singleton translation, then the closure system $C'^{\min}$ of the $\langle F, \alpha\rangle$-min model $\mathcal{I}'^{\min} = \langle \mathbf{Sign}', \mathrm{SEN}', C'^{\min}\rangle$ of $\mathcal{I}$ on SEN' has an inductive characterization. It is analogous to the standard characterization of the $\mathcal{S}$-filter $\mathrm{Fi}^{\mathbf{A}}_{\mathcal{S}}(X)$ generated by a given subset $X \subseteq A$ of an algebra $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}}\rangle$, given in Lemma 1.18 of [14].

Lemma 2.1. *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a finitary π-institution and $\mathcal{I}'^{\min} = \langle \mathbf{Sign}', \mathrm{SEN}', C'^{\min}\rangle$ an $\langle F, \alpha\rangle$-min model of $\mathcal{I}$, where $\langle F, \alpha\rangle : \mathcal{I} \to \mathrm{SEN}'$ is a surjective singleton translation. Then, for all $\Sigma \in |\mathbf{Sign}|, \Phi \subseteq \mathrm{SEN}(\Sigma)$, $C'^{\min}_{F(\Sigma)}(\alpha_\Sigma(\Phi)) = \bigcup_{n \geq 0} X^n_{F(\Sigma)}(\alpha_\Sigma(\Phi))$, where*

$$X^0_{F(\Sigma)}(\alpha_\Sigma(\Phi)) = \alpha_\Sigma(\Phi)$$

and, for all $n \geq 0$,

$$X^{n+1}_{F(\Sigma)}(\alpha_\Sigma(\Phi)) =$$
$$\{\alpha_\Sigma(\chi) : (\exists\, Y \subseteq_\omega \mathrm{SEN}(\Sigma))(\chi \in C_\Sigma(Y) \text{ and } \alpha_\Sigma(Y) \subseteq X^n_{F(\Sigma)}(\alpha_\Sigma(\Phi))\}.$$

Proof. We first show that

$$\bigcup_{n=0}^{\infty} X^n_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \subseteq C'^{\min}_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

This is done by showing, by induction on $n \geq 0$, that

$$X^n_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \subseteq C'^{\min}_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

If $n = 0$, then it is obvious that $\alpha_\Sigma(\Phi) \subseteq C'^{\min}_{F(\Sigma)}(\alpha_\Sigma(\Phi))$.

Suppose $X^k_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \subseteq C'^{\min}_{F(\Sigma)}(\alpha_\Sigma(\Phi))$ and let $\phi' \in X^{k+1}_{F(\Sigma)}(\alpha_\Sigma(\Phi))$. Then, by definition, there exists $\phi \in \mathrm{SEN}(\Sigma), Y = \{y_0, \ldots, y_{m-1}\} \subseteq_\omega \mathrm{SEN}(\Sigma)$, such that

- $\phi' = \alpha_\Sigma(\phi)$,
- $\phi \in C_\Sigma(Y)$ and
- $\alpha_\Sigma(Y) \subseteq X^k_{F(\Sigma)}(\alpha_\Sigma(\Phi))$.

These, taken together, give

$$\begin{array}{rcl}\phi' & = & \alpha_\Sigma(\phi) \\ & \in & C'^{\min}_{F(\Sigma)}(\alpha_\Sigma(Y)) \\ & \subseteq & C'^{\min}_{F(\Sigma)}(X^k_{F(\Sigma)}(\alpha_\Sigma(\Phi))) \\ & \subseteq & C'^{\min}_{F(\Sigma)}(C'^{\min}_{F(\Sigma)}(\alpha_\Sigma(\Phi))) \\ & = & C'^{\min}_{F(\Sigma)}(\alpha_\Sigma(\Phi)).\end{array}$$

It now suffices to show that $C' = \{\bigcup_{n\geq 0} X^n_{F(\Sigma)}\}_{\Sigma\in|\mathbf{Sign}|}$ is a closure system on SEN'. Since $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ will then be, by the finitarity of $\mathcal{I}$, an $\langle F, \alpha\rangle$-model of $\mathcal{I}$ on SEN', we will be able to conclude that $C'^{\min} \leq C'$.

To show reflexivity, it suffices, by surjectivity, to show that, for all $\Sigma \in |\mathbf{Sign}|, \Phi \subseteq \mathrm{SEN}(\Sigma)$, $\alpha_\Sigma(\Phi) \subseteq C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$. But this is obvious by the definition of C'.

To show monotonicity, it suffices, by surjectivity, to show that, for all $\Sigma \in |\mathbf{Sign}|, \Phi \subseteq \Psi \subseteq \mathrm{SEN}(\Sigma)$, $C'_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \subseteq C'_{F(\Sigma)}(\alpha_\Sigma(\Psi))$. This can be accomplished by showing, by a routine induction on n, that $X^n_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \subseteq X^n_{F(\Sigma)}(\alpha_\Sigma(\Psi))$, for all $n \geq 0$. The details are omitted.

For idempotency, we need to show, by surjectivity, that, for all $\Sigma \in |\mathbf{Sign}|, \Phi \subseteq \mathrm{SEN}(\Sigma)$, $C'_{F(\Sigma)}(C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))) \subseteq C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$. We do this by first applying induction on n to show that, for all $n \geq 0$,

$$X_{F(\Sigma)}(X^n_{F(\Sigma)}(\alpha_\Sigma(\Phi))) \subseteq X^{n+1}_{F(\Sigma)}(\alpha_\Sigma(\Phi)). \tag{1}$$

Then, we use Equation (1) to show that

$$X^k_{F(\Sigma)}(X^l_{F(\Sigma)}(\alpha_\Sigma(\Phi))) \subseteq X^{k+l}_{F(\Sigma)}(\alpha_\Sigma(\Phi)). \tag{2}$$

Finally, Equation (2) is used to conclude

$$\begin{array}{rcl} X^n_{F(\Sigma)}(C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))) & = & X^n_{F(\Sigma)}(\bigcup_{i\geq 0} X^i_{F(\Sigma)}(\alpha_\Sigma(\Phi))) \\ & \subseteq & \bigcup_{i\geq 0} X^i_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \\ & = & C'_{F(\Sigma)}(\alpha_\Sigma(\Phi)), \end{array}$$

for all $n \geq 0$.

Finally, for structurality, it may be shown, similarly, by induction on $n \geq 0$, that $\mathrm{SEN}(F(f))(X^n_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\Phi))) \subseteq X^n_{F(\Sigma_2)}(\alpha_{\Sigma_2}(\mathrm{SEN}(f)(\Phi)))$, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$ and $\Phi \subseteq \mathrm{SEN}(\Sigma_1)$. □

3 The Congruence Property

Let **Sign** be a category and SEN : **Sign** → **Set** be a functor. An **axiom system** T is a collection $T = \{T_\Sigma : \Sigma \in |\mathbf{Sign}|\}$, such that

- $T_\Sigma \subseteq \mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$, and
- $\mathrm{SEN}(f)(T_{\Sigma_1}) \subseteq T_{\Sigma_2}$, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$.

The collection of all axiom systems on SEN will be denoted by AxSys(SEN). For all axiom systems T^1, T^2 on SEN, define

$$T^1 \leq T^2 \quad \text{iff} \quad T^1_\Sigma \subseteq T^2_\Sigma, \text{ for all } \Sigma \in |\mathbf{Sign}|.$$

Let SEN : **Sign** → **Set** be a functor, C a closure system on SEN and T an axiom system on SEN. Define $C^T = \{C^T_\Sigma : \Sigma \in |\mathbf{Sign}|\}$ by

$$C^T_\Sigma(\Phi) = C_\Sigma(T_\Sigma \cup \Phi), \quad \text{for all } \Phi \subseteq \mathrm{SEN}(\Sigma).$$

Proposition 3.1. *Given a functor* SEN : **Sign** → **Set**, *a closure system C on* SEN *and an axiom system T on* SEN, *the collection $C^T = \{C^T_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ is a closure system on* SEN.

Proof. Properties (a) and (c) of a closure system are obvious. For (b), suppose $\phi \in C^T_\Sigma(C^T_\Sigma(\Phi))$. Then $\phi \in C_\Sigma(T_\Sigma \cup C^T_\Sigma(\Phi))$, whence

$$\begin{array}{rcl} \phi \in C_\Sigma(T_\Sigma \cup C_\Sigma(T_\Sigma \cup \Phi)) & = & C_\Sigma(C_\Sigma(T_\Sigma \cup \Phi)) \\ & = & C_\Sigma(T_\Sigma \cup \Phi) \\ & = & C^T_\Sigma(\Phi). \end{array}$$

Finally, for (d), let $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma_1, \Sigma_2), \Phi \subseteq \mathrm{SEN}(\Sigma_1)$. Then

$$\begin{array}{rcl} \mathrm{SEN}(f)(C^T_{\Sigma_1}(\Phi)) & = & \mathrm{SEN}(f)(C_{\Sigma_1}(T_{\Sigma_1} \cup \Phi)) \\ & \subseteq & C_{\Sigma_2}(\mathrm{SEN}(f)(T_{\Sigma_1} \cup \Phi)) \\ & = & C_{\Sigma_2}(\mathrm{SEN}(f)(T_{\Sigma_1}) \cup \mathrm{SEN}(f)(\Phi)) \\ & \subseteq & C_{\Sigma_2}(T_{\Sigma_2} \cup \mathrm{SEN}(f)(\Phi)) \\ & = & C^T_{\Sigma_2}(\mathrm{SEN}(f)(\Phi)). \end{array}$$

□

Let **Sign** be a category, SEN : **Sign** → **Set** be a functor and C a closure system on SEN. Define $\Lambda(C) = \{\Lambda_\Sigma(C) : \Sigma \in |\mathbf{Sign}|\}$, where

$$\Lambda_\Sigma(C) = \{\langle \phi, \psi \rangle \in \mathrm{SEN}(\Sigma)^2 : C_\Sigma(\phi) = C_\Sigma(\psi)\}.$$

Recall, before Proposition 3.2, the notion of an equivalence system on SEN : **Sign** → **Set** from [24].

Proposition 3.2. *Let* **Sign** *be a category,* SEN : **Sign** → **Set** *be a functor and C a closure system on* SEN. *Then $\Lambda(C)$ is an equivalence system on* SEN.

Proof. It is clear from the definition that the relation $\Lambda_\Sigma(C)$ is an equivalence relation, for all $\Sigma \in |\mathbf{Sign}|$. To show that $\Lambda(C)$ is an equivalence system, suppose that $\Sigma_1, \Sigma_2 \in \mathbf{Sign}, f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$ and $\langle \phi, \psi \rangle \in \Lambda_{\Sigma_1}(C)$. Then $C_{\Sigma_1}(\phi) = C_{\Sigma_1}(\psi)$, whence

$$\begin{array}{rcl} \mathrm{SEN}(f)(\phi) & \in & \mathrm{SEN}(f)(C_{\Sigma_1}(\phi)) \\ & = & \mathrm{SEN}(f)(C_{\Sigma_1}(\psi)) \\ & \subseteq & C_{\Sigma_2}(\mathrm{SEN}(f)(\psi)), \end{array}$$

and, therefore $C_{\Sigma_2}(\mathrm{SEN}(f)(\phi)) \subseteq C_{\Sigma_2}(\mathrm{SEN}(f)(\psi))$. Thus, by symmetry, we obtain $C_{\Sigma_2}(\mathrm{SEN}(f)(\phi)) = C_{\Sigma_2}(\mathrm{SEN}(f)(\psi))$. Hence, by the definition of $\Lambda_{\Sigma_2}(C)$, $\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi) \rangle \in \Lambda_{\Sigma_2}(C)$ and $\Lambda(C)$ is in fact an equivalence system on SEN. □

The equivalence system $\Lambda(C)$ is referred to as the **Frege equivalence system** of C on SEN. The Frege equivalence at the sentential logic level has been studied extensively in the past. For examples and more references on the subject see, for instance, [14, 11, 1].

The **Frege operator** of C on SEN is the mapping $\Lambda_C : \mathrm{AxSys}(\mathrm{SEN}) \to \mathrm{Eqv}(\mathrm{SEN})$ defined by

$$\Lambda_C(F) = \Lambda(C^F), \quad \text{for all } F = \{F_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} \in \mathrm{AxSys}(\mathrm{SEN}).$$

If $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, then $\Lambda(\mathcal{I})$ and $\Lambda_{\mathcal{I}}(F)$ will be alternative notations for $\Lambda(C)$ and $\Lambda_C(F)$, respectively.

An easy observation is that the Frege operator of C on SEN is monotonic.

Proposition 3.3. $\Lambda_C : \mathrm{AxSys}(\mathrm{SEN}) \to \mathrm{Eqv}(\mathrm{SEN})$ *is order preserving, i.e.,* $F \leq G$ *implies* $\Lambda_C(F) \leq \Lambda_C(G)$.

Proof. If $\langle \phi, \psi \rangle \in \Lambda_C(F)_\Sigma$, then we have $C^F_\Sigma(\phi) = C^F_\Sigma(\psi)$. Therefore $C_\Sigma(F_\Sigma \cup \{\phi\}) = C_\Sigma(F_\Sigma \cup \{\psi\})$. Thus $\phi \in C_\Sigma(F_\Sigma \cup \{\psi\})$, whence, since $F \leq G$, $\phi \in C_\Sigma(G_\Sigma \cup \{\psi\})$ and therefore $C_\Sigma(G_\Sigma \cup \{\phi\}) \subseteq C_\Sigma(G_\Sigma \cup \{\psi\})$. Now, by symmetry, it follows that $C_\Sigma(G_\Sigma \cup \{\phi\}) = C_\Sigma(G_\Sigma \cup \{\psi\})$. This proves that $\langle \phi, \psi \rangle \in \Lambda_C(G)_\Sigma$. Hence $\Lambda_C(F) \leq \Lambda_C(G)$. □

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ be a π-institution and N a category of natural transformations on SEN. Since a logical N-congruence system θ of $\mathcal{I}$ (see [24]) is defined to be an N-congruence system on SEN, such that, for all $\Sigma \in |\mathbf{Sign}|$,

$$\langle \phi, \psi \rangle \in \theta_\Sigma \quad \text{implies} \quad C_\Sigma(\phi) = C_\Sigma(\psi),$$

it is obvious that the collection of all logical N-congruences of $\mathcal{I}$ is the collection of all N-congruences on SEN that are signature-wise included in the Frege equivalence system $\Lambda(\mathcal{I})$.

Proposition 3.4. *Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ and N a category of natural transformations on* SEN,

$$\mathrm{LCon}^N(\mathcal{I}) = \{\theta \in \mathrm{Con}^N(\mathrm{SEN}) : \theta \leq \Lambda(\mathcal{I})\}.$$

As a consequence the Tarski N-congruence system $\widetilde{\Omega}^N(\mathcal{I})$ [24] is the largest N-congruence system on SEN under the signature-wise inclusion that is included in $\Lambda(\mathcal{I})$.

Corollary 3.5. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ be a π-institution and N a category of natural transformations on* SEN. *$\widetilde{\Omega}^N(\mathcal{I})$ is the $\leq$-greatest logical N-congruence of $\mathcal{I}$ included in $\Lambda(\mathcal{I})$.*

Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ and a category N of natural transformations on SEN, $\mathcal{I}$ is said to have the N-**congruence property** if $\Lambda(\mathcal{I}) \in \mathrm{Con}^N(\mathrm{SEN})$, i.e., if $\Lambda(\mathcal{I}) = \widetilde{\Omega}^N(\mathcal{I})$.

Thus, an N-reduced π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ has the N-congruence property if and only if, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$C_\Sigma(\phi) = C_\Sigma(\psi) \quad \text{implies} \quad \phi = \psi.$$

Bilogical morphisms between π-institutions carry Frege equivalence systems to Frege equivalence systems. This fact is formally expressed in the following lemma.

Lemma 3.6. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle, \mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C' \rangle$ be two π-institutions, N and N' categories of natural transformations on* SEN *and* SEN', *respectively, and $\langle F, \alpha \rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$ an (N, N')-bilogical morphism. Then, for all $\Sigma \in |\mathbf{Sign}|, \alpha_\Sigma(\Lambda_\Sigma(\mathcal{I})) = \Lambda_{F(\Sigma)}(\mathcal{I}')$.*

Proof. Suppose that $\langle \phi, \psi \rangle \in \Lambda_\Sigma(\mathcal{I})$. Then $C_\Sigma(\phi) = C_\Sigma(\psi)$. Hence, applying $\langle F, \alpha \rangle$, $\alpha_\Sigma(C_\Sigma(\phi)) = \alpha_\Sigma(C_\Sigma(\psi))$. But, then, by Corollary 16 of [24], we get $C'_{F(\Sigma)}(\alpha_\Sigma(\phi)) = C'_{F(\Sigma)}(\alpha_\Sigma(\psi))$. This means that $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \Lambda_{F(\Sigma)}(\mathcal{I}')$. Hence $\alpha_\Sigma(\Lambda_\Sigma(\mathcal{I})) \subseteq \Lambda_{F(\Sigma)}(\mathcal{I}')$.

Suppose, conversely, that $\langle \phi', \psi' \rangle \in \Lambda_{F(\Sigma)}(\mathcal{I}')$. Since $\langle F, \alpha \rangle$ is surjective, there exist $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\phi' = \alpha_\Sigma(\phi)$ and $\psi' = \alpha_\Sigma(\psi)$. Therefore $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \Lambda_{F(\Sigma)}(\mathcal{I}')$. Thus, by the definition of Λ, $C'_{F(\Sigma)}(\alpha_\Sigma(\phi)) = C'_{F(\Sigma)}(\alpha_\Sigma(\psi))$, whence, by Corollary 15 of [24], $C_\Sigma(\phi) = C_\Sigma(\psi)$, i.e., $\langle \phi, \psi \rangle \in \Lambda_\Sigma(\mathcal{I})$. Hence $\langle \phi', \psi' \rangle = \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \alpha_\Sigma(\Lambda_\Sigma(\mathcal{I}))$ and $\Lambda_{F(\Sigma)}(\mathcal{I}') \subseteq \alpha_\Sigma(\Lambda_\Sigma(\mathcal{I}))$. □

The following result expresses formally the fact that bilogical morphisms preserve the congruence property. This is the analog in the π-institution context of Proposition 2.40 of [14]. Its proof uses Lemma 3.6.

Proposition 3.7. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle, \mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C' \rangle$ be two π-institutions, N and N' categories of natural transformations on* SEN *and* SEN', *respectively, and $\langle F, \alpha \rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$ an (N, N')-bilogical morphism. Then $\mathcal{I}$ has the N-congruence property if and only if $\mathcal{I}'$ has the N'-congruence property.*

Proof. By Theorem 21 of [24] we have that

$$\widetilde{\Omega}^N_\Sigma(\mathcal{I}) = \alpha_\Sigma^{-1}(\widetilde{\Omega}^{N'}_{F(\Sigma)}(\mathcal{I}')), \quad \text{for every } \Sigma \in |\mathbf{Sign}|. \tag{3}$$

Suppose, first, that $\mathcal{I}'$ has the N'-congruence property and that $\langle \phi, \psi \rangle \in \Lambda_\Sigma(\mathcal{I})$. Then, by Lemma 3.6, $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \Lambda_{F(\Sigma)}(\mathcal{I}') = \widetilde{\Omega}^{N'}_{F(\Sigma)}(\mathcal{I}')$. Thus, by Equation (3), $\langle \phi, \psi \rangle \in \widetilde{\Omega}^N_\Sigma(\mathcal{I})$. Hence $\Lambda(\mathcal{I}) = \widetilde{\Omega}^N(\mathcal{I})$ and $\mathcal{I}$ has the N-congruence property.

Conversely, suppose that $\mathcal{I}$ has the N-congruence property and that $\langle \phi', \psi' \rangle \in \Lambda_{\Sigma'}(\mathcal{I}')$. Then, since $\langle F, \alpha \rangle$ is surjective, there exists $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $F(\Sigma) = \Sigma'$ and $\alpha_\Sigma(\phi) = \phi', \alpha_\Sigma(\psi) = \psi'$. Therefore $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \Lambda_{F(\Sigma)}(\mathcal{I}')$. Hence, again by Lemma 3.6, $\langle \phi, \psi \rangle \in \Lambda_\Sigma(\mathcal{I}) = \widetilde{\Omega}^N_\Sigma(\mathcal{I})$. Thus, by Equation (3),

$$\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \in \widetilde{\Omega}^{N'}_{F(\Sigma)}(\mathcal{I}'),$$

i.e., $\langle \phi', \psi' \rangle \in \widetilde{\Omega}^{N'}_{\Sigma'}(\mathcal{I}')$. Hence $\Lambda(\mathcal{I}') = \widetilde{\Omega}^{N'}(\mathcal{I}')$, which shows that $\mathcal{I}'$ has the N'-congruence property. $\square$

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ be a π-institution and N a category of natural transformations on SEN. $\mathcal{I}$ is said to be N-**self-extensional** if it has the N-congruence property, i.e., if $\Lambda(\mathcal{I}) = \widetilde{\Omega}^N(\mathcal{I})$. It is said to be **fully** N-**self-extensional** if, every (N, N')-full model of $\mathcal{I}$ has the N'-congruence property, i.e., if, for all $\mathcal{I}' \in \mathrm{FMod}^N(\mathcal{I})$, $\widetilde{\Omega}^{N'}(\mathcal{I}') = \Lambda(\mathcal{I}')$.

Proposition 3.8. *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ is a π-institution and N a category of natural transformations on* SEN. *Then $\mathcal{I}$ is N-self-extensional if and only if, for all $\Sigma \in |\mathbf{Sign}|, \sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N and all ϕ_i, ψ_i, $i = 0, \ldots, n-1$,*

$$C_\Sigma(\phi_i) = C_\Sigma(\psi_i),\ i < n, \quad \textit{imply} \quad \sigma_\Sigma(\vec{\psi}) \in C_\Sigma(\sigma_\Sigma(\vec{\phi})). \tag{4}$$

Proof. First, suppose that $\mathcal{I}$ is N-self-extensional and consider $\Sigma \in |\mathbf{Sign}|, \sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N and $\phi_i, \psi_i, i < n$, such that $C_\Sigma(\phi_i) = C_\Sigma(\psi_i)$, for all $i < n$. Then, by the definition of the Frege equivalence system, $\langle \phi_i, \psi_i \rangle \in \Lambda_\Sigma(\mathcal{I}) = \widetilde{\Omega}^N_\Sigma(\mathcal{I})$. Therefore, since $\widetilde{\Omega}^N(\mathcal{I})$ is an N-congruence system, we get $\langle \sigma_\Sigma(\vec{\phi}), \sigma_\Sigma(\vec{\psi}) \rangle \in \widetilde{\Omega}^N_\Sigma(\mathcal{I})$. Hence $C_\Sigma(\sigma_\Sigma(\vec{\phi})) = C_\Sigma(\sigma_\Sigma(\vec{\psi}))$. Thus, $\sigma_\Sigma(\vec{\psi}) \in C_\Sigma(\sigma_\Sigma(\vec{\phi}))$.

Suppose, conversely, that Condition (4) holds and that $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \Lambda_\Sigma(\mathcal{I})$. Then, if $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{n-1}$, we get, by Proposition 3.2,

$$C_{\Sigma'}(\mathrm{SEN}(f)(\phi)) = C_{\Sigma'}(\mathrm{SEN}(f)(\psi))$$

and, obviously, $C_{\Sigma'}(\chi_i) = C_{\Sigma'}(\chi_i)$, for all $i < n-1$. Hence, by Condition (4),

$$C_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi})) = C_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})).$$

Thus, by Theorem 4 of [24], $\langle \phi, \psi \rangle \in \widetilde{\Omega}^N_\Sigma(\mathcal{I})$, and, therefore, $\mathcal{I}$ is N-self-extensional. $\square$

Proposition 3.8 has the following corollary characterizing fully N-self-extensional π-institutions.

Corollary 3.9. *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ *is a* π*-institution and* N *a category of natural transformations on* SEN. $\mathcal{I}$ *is fully* N*-self-extensional if and only if every* (N, N')*-full model* $\mathcal{I}'$ *of* $\mathcal{I}$ *satisfies Condition (4) with* N *replaced by* N'.

Proposition 3.7 and Proposition 5.12 of [25] yield the following proposition, providing an alternative characterization of N-self-extensional π-institutions in terms of their min models.

Proposition 3.10. *A* π*-institution* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, *with* N *a category of natural transformations on* SEN, *is fully* N*-self-extensional if and only if every* (N, N')*-min model of* $\mathcal{I}$ *has the* N'*-congruence property.*

It is obvious that every fully N-self-extensional π-institution is N-self-extensional. For sentential logics, it was shown by Font and Jansana in [14] and by Czelakowski and Pigozzi in [11, 12] that, under a variety of diverse conditions, the converse is also true. But Babyonyshev [1] showed that the converse is not true in general.

The following proposition is a partial analog of Proposition 2.43 of [14]. It states, roughly, that, given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, a category N of natural transformations on SEN and two Σ-sentences ϕ and ψ of $\mathcal{I}$, ϕ and ψ are identified by the Lindenbaum-Tarski $(\mathcal{I}, N)$-algebraic system SEN^N of $\mathcal{I}$ (see [26]) if and only if they are deductively equivalent, i.e., they generate the same Σ-closed sets.

Proposition 3.11. *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, *with* N *a category of natural transformations on* SEN, *be an* N*-self-extensional* π*-institution. Then, for all* $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi, \psi \rangle \in \widetilde{\Omega}^N_\Sigma(\mathcal{I}) \quad \textit{iff} \quad C_\Sigma(\phi) = C_\Sigma(\psi).$$

Proof. That $\langle\phi,\psi\rangle \in \widetilde{\Omega}^N_\Sigma(\mathcal{I})$ implies $C_\Sigma(\phi) = C_\Sigma(\psi)$ is always true, by the definition of the Tarski N-congruence system of $\mathcal{I}$. If, conversely, $C_\Sigma(\phi) = C_\Sigma(\psi)$, then $\langle\phi,\psi\rangle \in \Lambda_\Sigma(\mathcal{I})$, whence, since $\mathcal{I}$ is N-self-extensional, $\langle\phi,\psi\rangle \in \widetilde{\Omega}^N_\Sigma(\mathcal{I})$. $\square$

4 The Property of Conjunction

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution and N a category of natural transformations on SEN. Recall from [23] that a natural transformation $\bigwedge : \mathcal{P}\mathrm{SEN}^2 \to \mathcal{P}\mathrm{SEN}$ is called a **conjunction for** $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}|, \Gamma, \Delta \subseteq \mathrm{SEN}(\Sigma)$,

$$C_\Sigma(\Gamma \cup \Delta) = C_\Sigma(\bigwedge_\Sigma(\Gamma,\Delta)).$$

$\mathcal{I}$ is said to **have conjunction** if there exists a conjunction for $\mathcal{I}$. It was shown in Theorem 4.23 of [23] that every π-institution has conjunction, where $\bigwedge : \mathcal{P}\mathrm{SEN}^2 \to \mathcal{P}\mathrm{SEN}$, given by

$$\bigwedge_\Sigma(\Gamma,\Delta) = \Gamma \cup \Delta, \quad \text{for all} \quad \Sigma \in |\mathbf{Sign}|, \Gamma, \Delta \subseteq \mathrm{SEN}(\Sigma),$$

is a conjunction for $\mathcal{I}$.

In the present context, $\mathcal{I}$ will be said to have the **property of** N-**conjunction**, or to have an N-**conjunction**, if there exists $\wedge : \mathrm{SEN}^2 \to \mathrm{SEN}$ in N, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi,\psi \in \mathrm{SEN}(\Sigma)$,

$$C_\Sigma(\phi,\psi) = C_\Sigma(\phi \wedge_\Sigma \psi),$$

where, of course, $\phi \wedge_\Sigma \psi := \wedge_\Sigma(\phi,\psi)$. In this case, $\mathcal{I}$ is said to have the N-conjunction **with respect to** $\wedge : \mathrm{SEN}^2 \to \mathrm{SEN}$.

The following lemmas lift some well-known properties of the property of conjunction from the abstract logic level to the π-institution level. The first says that the property of conjunction is preserved by bilogical morphisms.

Lemma 4.1. *Suppose* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle, \mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ *are* π*-institutions,* N, N' *categories of natural transformations on* $\mathrm{SEN}, \mathrm{SEN}'$*, respectively, and* $\langle F,\alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$ *an* (N,N')*-bilogical morphism from* $\mathcal{I}$ *to* $\mathcal{I}'$*. Then* $\mathcal{I}$ *has an* N*-conjunction iff* $\mathcal{I}'$ *has an* N'*-conjunction.*

Proof. Suppose that $\wedge : \mathrm{SEN}^2 \to \mathrm{SEN}$ is an N-conjunction for $\mathcal{I}$. Then, by the (N,N')-epimorphic property of $\langle F,\alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$, there exists a natural transformation $\wedge' : \mathrm{SEN}'^2 \to \mathrm{SEN}'$, such that, for all $\Sigma \in |\mathbf{Sign}|$,

$$\begin{tikzcd}
\mathrm{SEN}(\Sigma)^2 \arrow[r, "\wedge_\Sigma"] \arrow[d, "\alpha^2_\Sigma"'] & \mathrm{SEN}(\Sigma) \arrow[d, "\alpha_\Sigma"] \\
\mathrm{SEN}'(F(\Sigma))^2 \arrow[r, "\wedge'_{F(\Sigma)}"'] & \mathrm{SEN}'(F(\Sigma))
\end{tikzcd}$$

$$\alpha_\Sigma(\phi) \wedge'_{F(\Sigma)} \alpha_\Sigma(\psi) = \alpha_\Sigma(\phi \wedge_\Sigma \psi), \quad \text{for all } \phi,\psi \in \mathrm{SEN}(\Sigma)^2.$$

It will now be shown that $\wedge' : \mathrm{SEN}'^2 \to \mathrm{SEN}'$ is in fact an N'-conjunction for $\mathcal{I}'$. Let $\Sigma' \in |\mathbf{Sign}'|, \phi',\psi' \in \mathrm{SEN}'(\Sigma')$. Then, by the surjectivity of $\langle F,\alpha\rangle$, there exists $\Sigma \in |\mathbf{Sign}|, \phi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\Sigma' = F(\Sigma)$ and $\phi' = \alpha_\Sigma(\phi), \psi' = \alpha_\Sigma(\psi)$. Hence

$$\begin{aligned}
C'_{\Sigma'}(\phi' \wedge'_{\Sigma'} \psi') &= C'_{F(\Sigma)}(\alpha_\Sigma(\phi) \wedge'_{F(\Sigma)} \alpha_\Sigma(\psi)) \\
&= C'_{F(\Sigma)}(\alpha_\Sigma(\phi \wedge_\Sigma \psi)) \\
&= \alpha_\Sigma(C_\Sigma(\phi \wedge_\Sigma \psi)) \\
&= \alpha_\Sigma(C_\Sigma(\phi,\psi)) \\
&= C'_{F(\Sigma)}(\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)) \\
&= C'_{\Sigma'}(\phi',\psi').
\end{aligned}$$

The proof of the converse, i.e., that the existence of an N'-conjunction $\wedge'$ for $\mathcal{I}'$ implies the existence of an N-conjunction for $\mathcal{I}$ is very similar. □

Lemma 4.1 implies immediately the following result, yielding the equivalence of a π-institution and of its N-reduct with respect to the property of conjunction. For relevant definitions see [24].

Corollary 4.2. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution and N a category of natural transformations on* SEN. *$\mathcal{I}$ has an N-conjunction iff $\mathcal{I}^N$ has an $\overline{N}$-conjunction.*

Next, it is shown that, if a π-institution $\mathcal{I}$ has conjunction with respect to $\wedge$, then, necessarily, the Frege equivalence system of $\mathcal{I}$ is a $\{\wedge\}$-congruence system of $\mathcal{I}$.

Lemma 4.3. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution and N a category of natural transformations on* SEN. *If $\mathcal{I}$ has an N-conjunction*

$\wedge : \mathrm{SEN}^2 \to \mathrm{SEN}$, *then the Frege equivalence system* $\Lambda(\mathcal{I})$ *is a* $\{\wedge\}$*-congruence system and, for every axiom system* $F = \{F_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ *on* SEN, $\Lambda_C(F)$ *is also a* $\{\wedge\}$*-congruence system.*

Proof. Suppose that $\Sigma \in |\mathbf{Sign}|$ and that $\phi_0, \phi_1, \psi_0, \psi_1 \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi_0, \psi_0\rangle, \langle\phi_1, \psi_1\rangle \in \Lambda_\Sigma(\mathcal{I})$. Then, by the definition of the Frege equivalence system, we obtain $C_\Sigma(\phi_0) = C_\Sigma(\psi_0)$ and $C_\Sigma(\phi_1) = C_\Sigma(\psi_1)$. Therefore $C_\Sigma(\phi_0, \phi_1) = C_\Sigma(\psi_0, \psi_1)$. Hence, by the property of $\wedge$-conjunction, $C_\Sigma(\phi_0 \wedge_\Sigma \phi_1) = C_\Sigma(\psi_0 \wedge_\Sigma \psi_1)$. This proves that $\langle\phi_0 \wedge_\Sigma \phi_1, \psi_0 \wedge_\Sigma \psi_1\rangle \in \Lambda_\Sigma(\mathcal{I})$, whence $\Lambda(\mathcal{I})$ is indeed a $\{\wedge\}$-congruence system.

The proof for $\Lambda_C(F)$ is very similar and will be omitted. □

An alternative characterization of an N-conjunction for a π-institution $\mathcal{I}$ is provided by the following

Lemma 4.4. *A* π*-institution* $\mathcal{I}$ *has the* N*-conjunction property with respect to* $\wedge : \mathrm{SEN}^2 \to \mathrm{SEN}$ *in* N *if and only if, for all* $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\phi \wedge_\Sigma \psi \in C_\Sigma(\phi, \psi), \quad \phi \in C_\Sigma(\phi \wedge_\Sigma \psi) \quad \textit{and} \quad \psi \in C_\Sigma(\phi \wedge_\Sigma \psi).$$

Proof. Straightforward from the definition of the property of an N-conjunction $\wedge : \mathrm{SEN}^2 \to \mathrm{SEN}$. □

The property of having an N-conjunction is inherited by every model via a surjective logical morphism. The following lemma formalizes this result. Its proof is very similar to the proof of Lemma 4.1. We include it with a slight twist as an illustration of the alternate characterization of N-conjunction provided by Lemma 4.4.

Lemma 4.5. *Suppose that* $\mathcal{I} = \langle\mathbf{Sign}, \mathrm{SEN}, C\rangle$ *is a* π*-institution and* N *a category of natural transformations on* SEN*. If* $\mathcal{I}$ *has an* N*-conjunction* $\wedge$*, then every* (N, N')*-model* $\mathcal{I}'$ *of* $\mathcal{I}$ *via a surjective* (N, N')*-logical morphism* $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$ *has an* N'*-conjunction* $\wedge'$*.*

Proof. Suppose that $\wedge : \mathrm{SEN}^2 \to \mathrm{SEN}$ is an N-conjunction for $\mathcal{I}$. Then, by the (N, N')-epimorphic property of $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$, there exists a natural transformation $\wedge' : \mathrm{SEN}'^2 \to \mathrm{SEN}'$, such that, for all $\Sigma \in |\mathbf{Sign}|$,

$$\begin{tikzcd}
\mathrm{SEN}(\Sigma)^2 \arrow[r, "\wedge_\Sigma"] \arrow[d, "\alpha^2_\Sigma"'] & \mathrm{SEN}(\Sigma) \arrow[d, "\alpha_\Sigma"] \\
\mathrm{SEN}'(F(\Sigma))^2 \arrow[r, "\wedge'_{F(\Sigma)}"'] & \mathrm{SEN}'(F(\Sigma))
\end{tikzcd}$$

$$\alpha_\Sigma(\phi) \wedge'_{F(\Sigma)} \alpha_\Sigma(\psi) = \alpha_\Sigma(\phi \wedge_\Sigma \psi), \quad \text{for all } \phi, \psi \in \mathrm{SEN}(\Sigma)^2.$$

It suffices, by Lemma 4.4 and by symmetry, to show that, for all $\Sigma' \in |\mathbf{Sign}'|, \phi', \psi' \in \mathrm{SEN}'(\Sigma')$,

$$\phi' \wedge'_{\Sigma'} \psi' \in C'_{\Sigma'}(\phi', \psi') \quad \text{and} \quad \phi' \in C'_{\Sigma'}(\phi' \wedge'_{\Sigma'} \psi').$$

Since $\langle F, \alpha\rangle$ is surjective, there exists $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\Sigma' = F(\Sigma), \phi' = \alpha_\Sigma(\phi)$ and $\psi' = \alpha_\Sigma(\psi)$. Then, since $\wedge$ is an N-conjunction for $\mathcal{I}$, we have, by Lemma 4.4,

$$\phi \wedge_\Sigma \psi \in C_\Sigma(\phi, \psi) \quad \text{and} \quad \phi \in C_\Sigma(\phi \wedge_\Sigma \psi).$$

Since $\langle F, \alpha\rangle : \mathcal{I} \succ^{se} \mathcal{I}'$ is a semi-interpretation, these immediately yield

$$\alpha_\Sigma(\phi \wedge_\Sigma \psi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\phi), \alpha_\Sigma(\psi)) \quad \text{and} \quad \alpha_\Sigma(\phi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\phi \wedge_\Sigma \psi)).$$

Thus

$$\begin{aligned} \alpha_\Sigma(\phi) \wedge'_{F(\Sigma)} \alpha_\Sigma(\psi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\phi), \alpha_\Sigma(\psi)) \quad \text{and} & \\ \alpha_\Sigma(\phi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\phi) \wedge'_{F(\Sigma)} \alpha_\Sigma(\psi)).& \end{aligned}$$

Hence, we finally get

$$\phi' \wedge'_{\Sigma'} \psi' \in C'_{\Sigma'}(\phi', \psi') \quad \text{and} \quad \phi' \in C'_{\Sigma'}(\phi' \wedge'_{\Sigma'} \psi').$$

□

The following result is the analog of Proposition 2.46 of [14] for π-institutions. Recall that, given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ and a category N of natural transformations on SEN, by $\mathcal{I}^N$ is denoted the N-reduct of $\mathcal{I}$ and, for all $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$, by ϕ^N will be denoted the sentence $\phi/\widetilde{\Omega}^N_\Sigma(\mathcal{I}) \in \mathrm{SEN}^N(\Sigma)$.

Proposition 4.6. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution with an N-conjunction $\wedge : \mathrm{SEN}^2 \to \mathrm{SEN}$. Then, every finitary (N, N')-model $\mathcal{I}'$ of $\mathcal{I}$ via a surjective (N, N')-logical morphism $\langle F, \alpha\rangle : \mathcal{I} \succ^{se} \mathcal{I}'$ that has the N'-congruence property is an (N, N')-full model of $\mathcal{I}$.*

Proof. Suppose $\langle F, \alpha\rangle : \mathcal{I} \succ^{se} \mathcal{I}'$ is a surjective (N, N')-logical morphism onto a finitary $\mathcal{I}'$ that has the N'-congruence property. It suffices to show that the $\langle F, \pi_F^{N'}\alpha\rangle$-min $(N, \overline{N'})$-model of $\mathcal{I}$ on $\mathrm{SEN}'^{N'}$ $\mathcal{I}'^{\min} = \langle \mathbf{Sign}', \mathrm{SEN}'^{N'}, C'^{\min}\rangle$ is such that $C'^{\min} = C'^{N'}$.

Since $\mathcal{I}'^{N'}$ is an $(N, \overline{N'})$-model of $\mathcal{I}$ on $\mathrm{SEN}'^{N'}$ via $\langle F, \pi_F^{N'}\alpha\rangle$, we have that $C'^{\min} \leq C'^{N'}$.

Conversely, let $\Sigma \in |\mathbf{Sign}'|, \Phi' \cup \{\phi'\} \in \mathrm{SEN}'(\Sigma)$, such that $\phi'^{N'} \in C_\Sigma'^{N'}(\Phi'^{N'})$. Thus, by definition, $\phi' \in C'_\Sigma(\Phi')$. But $\mathcal{I}'$ is finitary, whence, there exist $\vec{\phi}' = \langle \phi'_0, \ldots, \phi'_{n-1}\rangle \in \mathrm{SEN}'(\Sigma)^n$, such that $\phi' \in C'_\Sigma(\vec{\phi}')$. Since $\mathcal{I}$ has the N-conjunction property with respect to some $\wedge : \mathrm{SEN}^2 \to \mathrm{SEN}$, by Lemma 4.5, $\mathcal{I}'$ has an N'-conjunction $\wedge' : \mathrm{SEN}'^2 \to \mathrm{SEN}'$. Hence $\phi' \in C'_\Sigma(\phi'_0 \wedge'_\Sigma \ldots \wedge'_\Sigma \phi'_{n-1})$. Thus

$$C'_\Sigma(\phi' \wedge'_\Sigma \phi'_0 \wedge'_\Sigma \ldots \wedge'_\Sigma \phi'_{n-1}) = C'_\Sigma(\phi'_0 \wedge'_\Sigma \ldots \wedge'_\Sigma \phi'_{n-1}).$$

Now $\mathcal{I}'$ has the N'-congruence property, whence, by Proposition 3.11 and Lemma 4.3,

$$\phi'^{N'} \wedge_\Sigma'^{N'} \phi_0'^{N'} \wedge_\Sigma'^{N'} \ldots \wedge_\Sigma'^{N'} \phi_{n-1}'^{N'} = \phi_0'^{N'} \wedge_\Sigma'^{N'} \ldots \wedge_\Sigma'^{N'} \phi_{n-1}'^{N'}.$$

But, now, by the property of conjunction, as inherited, by Lemma 4.5, by $\mathcal{I}'^{\min}$, we get that

$$\begin{array}{rcl}\phi'^{N'} & \in & C_\Sigma'^{\min}(\phi'^{N'} \wedge_\Sigma'^{N'} \phi_0'^{N'} \wedge_\Sigma'^{N'} \ldots \wedge_\Sigma'^{N'} \phi_{n-1}'^{N'}) \\ & = & C'^{\min}(\phi_0'^{N'} \wedge_\Sigma'^{N'} \ldots \wedge_\Sigma'^{N'} \phi_{n-1}'^{N'}) \\ & = & C_\Sigma'^{\min}(\phi_0'^{N'}, \ldots, \phi_{n-1}'^{N'}) \\ & \subseteq & C_\Sigma'^{\min}(\Phi'^{N'}).\end{array}$$

This shows that $C'^{N'} \leq C'^{\min}$ and therefore that $C'^{\min} = C'^{N'}$. Hence $\mathcal{I}'$ is an (N, N')-full model of $\mathcal{I}$. □

5 The Deduction-Detachment Theorem

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution and N a category of natural transformations on SEN. $\mathcal{I}$ is said to have the (**finitary uniterm**)

- N-**modus ponens** with respect to $\sigma : \mathrm{SEN}^2 \to \mathrm{SEN}$ in N if, for all $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi, \psi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\sigma_\Sigma(\phi, \psi) \in C_\Sigma(\Phi) \quad \text{implies} \quad \psi \in C_\Sigma(\Phi, \phi),$$

- N-**deduction theorem** with respect to $\sigma : \mathrm{SEN}^2 \to \mathrm{SEN}$ in N if, for all $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi, \psi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\psi \in C_\Sigma(\Phi, \phi) \quad \text{implies} \quad \sigma_\Sigma(\phi, \psi) \in C_\Sigma(\Phi),$$

- N-**deduction-detachment theorem** with respect to $\sigma : \mathrm{SEN}^2 \to \mathrm{SEN}$ in N if it has both the N-modus ponens and the N-deduction theorem with respect to $\sigma : \mathrm{SEN}^2 \to \mathrm{SEN}$ in N.

When the N-modus ponens, the N-deduction theorem or the N-deduction-detachment theorem are under consideration, the natural transformation $\sigma : \mathrm{SEN}^2 \to \mathrm{SEN}$ in N will be denoted by $\to$, following common practice in abstract algebraic logic and in analogy with the implication connective of classical and intuitionistic propositional logic, and, instead of the prefix notation $\to_\Sigma (\phi, \psi)$, the infix $\phi \to_\Sigma \psi$ will be usually used.

In [23], a form of the deduction-detachment property for π-institutions was considered but the notion was more general than the one considered here. It could be termed the infinitary multiterm deduction-detachment theorem in the sense that it allowed infinitely many sentences as arguments of the natural transformation and, also, infinitely many output sentences in place of the single sentence σ_Σ. Also the notion was not relativized to a given category N of natural transformations on SEN but was an arbitrary natural transformation from $\mathcal{P}\mathrm{SEN}^2$ to $\mathcal{P}\mathrm{SEN}$.

The following lemma provides a few basic properties of an N-deduction-detachment familiar from corresponding properties of the implication connective of classical propositional logic.

Lemma 5.1. *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, N a category of natural transformations on* SEN *and that $\mathcal{I}$ has the N-deduction-detachment theorem with respect to $\to: \mathrm{SEN}^2 \to \mathrm{SEN}$. Then, for all $\Sigma \in |\mathbf{Sign}|, \phi, \chi, \psi \in \mathrm{SEN}(\Sigma)$,*

1. $\phi \to_\Sigma \phi \in C_\Sigma(\emptyset)$, *2.* $\phi \to_\Sigma (\psi \to_\Sigma \phi) \in C_\Sigma(\emptyset)$,

3. $(\phi \to_\Sigma (\chi \to_\Sigma \psi)) \to_\Sigma ((\phi \to_\Sigma \chi) \to_\Sigma (\phi \to_\Sigma \psi)) \in C_\Sigma(\emptyset)$.

Proof.

1. Since, for all $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$, we have, by reflexivity, $\phi \in C_\Sigma(\phi)$, we get that $\phi \to_\Sigma \phi \in C_\Sigma(\emptyset)$.

2. Again by reflexivity, we have, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$, $\phi \in C_\Sigma(\phi, \psi)$, whence $\psi \to_\Sigma \phi \in C_\Sigma(\phi)$, and, therefore, $\phi \to_\Sigma (\psi \to_\Sigma \phi) \in C_\Sigma(\emptyset)$.

3. Note that, for all $\Sigma \in |\mathbf{Sign}|, \phi, \chi, \psi \in \mathrm{SEN}(\Sigma)$, we have, by the N-modus ponens with respect to $\to_\Sigma$, $\psi \in C_\Sigma(\phi, \phi \to_\Sigma \chi, \phi \to_\Sigma (\chi \to_\Sigma \psi))$. Therefore, by the deduction property with respect to $\to_\Sigma$, $\phi \to_\Sigma \psi \in C_\Sigma(\phi \to_\Sigma \chi, \phi \to_\Sigma (\chi \to_\Sigma \psi))$, whence $(\phi \to_\Sigma \chi) \to_\Sigma (\phi \to_\Sigma \psi) \in C_\Sigma(\phi \to_\Sigma (\chi \to_\Sigma \psi))$. Thus, finally, we have
$$(\phi \to_\Sigma (\chi \to_\Sigma \psi)) \to_\Sigma ((\phi \to_\Sigma \chi) \to_\Sigma (\phi \to_\Sigma \psi)) \in C_\Sigma(\emptyset).$$

$\square$

An alternative, simpler, characterization of the modus ponens property is provided in the following lemma. It parallels the well-known form of the modus ponens as an inference rule of classical propositional logic:
$$\frac{\phi, \phi \to \psi}{\psi}.$$

Lemma 5.2. *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution and N a category of natural transformations on* SEN. *$\mathcal{I}$ has the N-modus ponens with respect to $\to: \mathrm{SEN}^2 \to \mathrm{SEN}$ in N if and only if, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$, $\psi \in C_\Sigma(\phi, \phi \to_\Sigma \psi)$.*

Proof. Suppose, first, that, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$, $\psi \in C_\Sigma(\phi, \phi \to_\Sigma \psi)$ and that $\phi \to_\Sigma \psi \in C_\Sigma(\Phi)$. Then we have $\phi, \phi \to_\Sigma \psi \in C_\Sigma(\Phi, \phi)$, whence $\psi \in C_\Sigma(\phi, \phi \to_\Sigma \psi) \subseteq C_\Sigma(C_\Sigma(\Phi, \phi)) = C_\Sigma(\Phi, \phi)$.

Suppose, conversely, that $\mathcal{I}$ has the N-modus ponens and that $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$. Then, setting $\Phi = \{\phi \to_\Sigma \psi\}$, we get $\phi \to_\Sigma \psi \in C_\Sigma(\Phi)$, whence, by the N-modus ponens, $\psi \in C_\Sigma(\Phi, \phi)$, i.e., $\psi \in C_\Sigma(\phi, \phi \to_\Sigma \psi)$. $\square$

The property of having the N-modus ponens is preserved by surjective (N, N')-logical morphisms as was shown to be the case with the property of having an N-conjunction.

Lemma 5.3. *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *is a* π*-institution and* N *a category of natural transformations on* SEN. *If* $\mathcal{I}$ *has the* N*-modus ponens with respect to* $\rightarrow$: $\mathrm{SEN}^2 \rightarrow \mathrm{SEN}$ *in* N *then, every model* $\mathcal{I}'$ *of* $\mathcal{I}$ *via a surjective* (N, N')*-logical morphism* $\langle F, \alpha\rangle : \mathcal{I} \rangle\!\!-^{se} \mathcal{I}'$ *has the* N'*-modus ponens.*

Proof. Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ has the N-modus ponens with respect to $\rightarrow$ and that $\langle F, \alpha\rangle : \mathcal{I} \rangle\!\!-^{se} \mathcal{I}'$ is a surjective (N, N')-logical morphism from $\mathcal{I}$ to $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$. Let $\Sigma' \in |\mathbf{Sign}'|, \phi', \psi' \in \mathrm{SEN}'(\Sigma')$. By surjectivity, there exist $\Sigma \in |\mathbf{Sign}|$, such that $F(\Sigma) = \Sigma'$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\alpha_\Sigma(\phi) = \phi', \alpha_\Sigma(\psi) = \psi'$. Since $\mathcal{I}$ has the N-modus ponens with respect to $\rightarrow$, we get, by Lemma 5.2, $\psi \in C_\Sigma(\phi, \phi \rightarrow_\Sigma \psi)$. Thus, since $\langle F, \alpha\rangle$ is an (N, N')-logical morphism, $\alpha_\Sigma(\psi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\phi), \alpha_\Sigma(\phi \rightarrow_\Sigma \psi))$. Therefore, since $\rightarrow$ is in N, there exists, by the epimorphic property, a $\rightarrow'$: $\mathrm{SEN}'^2 \rightarrow \mathrm{SEN}'$ in N', such that $\psi' \in C'_{\Sigma'}(\phi', \phi' \rightarrow'_{\Sigma'} \psi')$. Hence, once more by Lemma 5.2, $\mathcal{I}'$ has the N'-modus ponens with respect to $\rightarrow'$. □

Finally, both the N-modus ponens and the N-deduction theorem are preserved by (N, N')-bilogical morphisms.

Lemma 5.4. *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *and* $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ *are* π*-institutions and* N, N' *categories of natural transformations on* SEN, SEN$'$, *respectively. If* $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$ *is an* (N, N')*-bilogical morphism, then* $\mathcal{I}$ *has the* N*-deduction-detachment theorem if and only if* $\mathcal{I}'$ *has the* N'*-deduction-detachment theorem.*

Proof. It was shown in Lemma 5.3 that the N-modus ponens is preserved by surjective logical morphisms, i.e., that if $\langle F, \alpha\rangle : \mathcal{I} \rangle\!\!-^{se} \mathcal{I}'$ is a surjective (N, N')-logical morphism and $\mathcal{I}$ has the N-modus ponens then $\mathcal{I}'$ has the N'-modus ponens. If $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$ is an (N, N')-bilogical morphism then, the same sequence of implications hold in the reverse direction showing that, if $\mathcal{I}'$ has the N'-modus ponens, then $\mathcal{I}$ has the N-modus ponens. It suffices, therefore, to show that the same holds with the property of having the N-deduction theorem.

Suppose, first, that $\mathcal{I}$ has the N-deduction theorem with respect to $\rightarrow$: $\mathrm{SEN}^2 \to \mathrm{SEN}$ and let $\Sigma' \in |\mathbf{Sign}'|, \Phi' \cup \{\phi', \psi'\} \subseteq \mathrm{SEN}'(\Sigma')$, such that $\psi' \in C'_{\Sigma'}(\Phi', \phi')$. By the surjectivity of $\langle F, \alpha\rangle$, we get that there exist $\Sigma \in |\mathbf{Sign}|$, such that $F(\Sigma) = \Sigma'$ and $\Phi \cup \{\phi, \psi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\alpha_\Sigma(\Phi) = \Phi', \alpha_\Sigma(\phi) = \phi'$ and $\alpha_\Sigma(\psi) = \psi'$. Hence, we obtain that $\alpha_\Sigma(\psi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi))$. But this implies that $\psi \in C_\Sigma(\Phi, \phi)$, whence, since $\mathcal{I}$ has the N-deduction theorem with respect to $\rightarrow$, we get $\phi \rightarrow_\Sigma \psi \in C_\Sigma(\Phi)$. Thus $\alpha_\Sigma(\phi \rightarrow_\Sigma \psi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$. Therefore, by the (N, N')-epimorphic property, there exists $\rightarrow'$: $\mathrm{SEN}'^2 \to \mathrm{SEN}'$, such that $\alpha_\Sigma(\phi) \rightarrow'_{F(\Sigma)} \alpha_\Sigma(\psi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$, but this is equivalent to $\phi' \rightarrow'_{\Sigma'} \psi' \in C'_{\Sigma'}(\Phi')$, which proves that $\mathcal{I}'$ has the N'-deduction theorem.

The converse is similar and will be omitted. □

Since $\langle \mathrm{I}_{\mathbf{Sign}}, \pi^N\rangle : \mathcal{I} \to \mathcal{I}^N$ is an $(N, \overline{N})$-bilogical morphism, Lemma 5.4 has the following

Corollary 5.5. *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *is a* π*-institution and* N *a category of natural transformations on* SEN. $\mathcal{I}$ *has the* N*-deduction detachment theorem if and only if* $\mathcal{I}^N$ *has the* $\overline{N}$*-deduction-detachment theorem.*

If a π-institution has the N-deduction-detachment theorem with respect to $\rightarrow$, then the Frege equivalence system $\Lambda(\mathcal{I})$ of $\mathcal{I}$ is a $\{\rightarrow\}$-congruence system of $\mathcal{I}$. The analogous property for an N-conjunction was given in Lemma 4.3.

Lemma 5.6. *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *is a* π*-institution and* N *a category of natural transformations on* SEN. *If* $\mathcal{I}$ *has the* N*-deduction-detachment theorem with respect to* $\rightarrow$: $\mathrm{SEN}^2 \to \mathrm{SEN}$ *in* N *then* $\Lambda(\mathcal{I})$ *is a* $\{\rightarrow\}$*-congruence system on* SEN *and* $\Lambda_C(F)$ *is a* $\{\rightarrow\}$*-congruence system on* SEN*, for all axiom systems* F *on* SEN.

Proof. We only prove the first statement. The second may be proved similarly. Let $\Sigma \in |\mathbf{Sign}|, \phi_0, \phi_1, \psi_0, \psi_1 \in \mathrm{SEN}(\Sigma)$, such that

$$\langle \phi_0, \phi_1\rangle, \langle \psi_0, \psi_1\rangle \in \Lambda_\Sigma(\mathcal{I}).$$

Then $C_\Sigma(\phi_0) = C_\Sigma(\phi_1)$ and $C_\Sigma(\psi_0) = C_\Sigma(\psi_1)$. These imply that

$$C_\Sigma(\psi_1) = C_\Sigma(\psi_0) \subseteq C_\Sigma(\phi_0, \phi_0 \rightarrow_\Sigma \psi_0) = C_\Sigma(\phi_1, \phi_0 \rightarrow_\Sigma \psi_0).$$

Hence $\psi_1 \in C_\Sigma(\phi_1, \phi_0 \rightarrow_\Sigma \psi_0)$, and, therefore, $\phi_1 \rightarrow_\Sigma \psi_1 \in C_\Sigma(\phi_0 \rightarrow_\Sigma \psi_0)$. By symmetry, we also have that $\phi_0 \rightarrow_\Sigma \psi_0 \in C_\Sigma(\phi_1 \rightarrow_\Sigma \psi_1)$, whence, we get $\langle \phi_0 \rightarrow_\Sigma \psi_0, \phi_1 \rightarrow_\Sigma \psi_1 \rangle \in \Lambda_\Sigma(\mathcal{I})$. □

Theorem 5.7. *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ is a finitary π-institution and N a category of natural transformations on* SEN. *If $\mathcal{I}$ has the N-deduction-detachment theorem with respect to $\rightarrow : \mathrm{SEN}^2 \rightarrow \mathrm{SEN}$, then, every (N, N')-full model $\mathcal{I}'$ of $\mathcal{I}$ via a surjective (N, N')-logical morphism $\langle F, \alpha \rangle : \mathcal{I} \rangle\!\!-^{se} \mathcal{I}'$ has the N'-deduction-detachment theorem.*

Proof. First, by Lemma 5.3, the N'-modus ponens has been taken care of. So, it suffices to prove the N'-deduction theorem. Note that, in view of Lemma 5.4, the price of replacing an interpretation by a semi-interpretation is imposing the stronger requirements of $\mathcal{I}$ being finitary and of $\mathcal{I}'$ being a full (N, N')-model of $\mathcal{I}$.

Suppose that $\mathcal{I}$ has the N-deduction-detachment theorem with respect to $\rightarrow : \mathrm{SEN}^2 \rightarrow \mathrm{SEN}$. It suffices to show that if $\langle F, \alpha \rangle : \mathcal{I} \rangle\!\!-^{se} \mathcal{I}'$ is a surjective (N, N')-logical morphism and $\mathcal{I}'$ is an $\langle F, \alpha \rangle$-min (N, N')-model of $\mathcal{I}$, then $\mathcal{I}'$ has the N'-deduction theorem with respect to the $\rightarrow' : \mathrm{SEN}'^2 \rightarrow \mathrm{SEN}'$ of Lemma 5.3. We will take advantage in this proof of Lemma 2.1 of Section 2. Let $\Sigma' \in |\mathbf{Sign}'|, \Phi' \cup \{\phi', \psi'\} \subseteq \mathrm{SEN}'(\Sigma')$, such that $\psi' \in C'_{\Sigma'}(\Phi', \phi')$. Then, by surjectivity, there exists $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi, \psi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\Sigma' = F(\Sigma)$ and $\Phi' = \alpha_\Sigma(\Phi), \phi' = \alpha_\Sigma(\phi), \psi' = \alpha_\Sigma(\psi)$. Hence, we get $\alpha_\Sigma(\psi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi))$. Our goal is to show that

$$\alpha_\Sigma(\phi) \rightarrow'_{F(\Sigma)} \alpha_\Sigma(\psi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

To this aim, we use Lemma 2.1, which shows that $C'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi)) = \bigcup_{n=0}^{\infty} X_n$, where $X_0 = \alpha_\Sigma(\Phi) \cup \alpha_\Sigma(\phi)$ and, for all $n \geq 0$,

$$X_{n+1} = \{\alpha_\Sigma(\chi) : (\exists\, Y \subseteq_\omega \mathrm{SEN}(\Sigma))(\chi \in C_\Sigma(Y) \text{ and } \alpha_\Sigma(Y) \subseteq X_n)\}.$$

This characterization allows us to use induction on n to show that, if $\alpha_\Sigma(\psi) \in X_n$, then $\alpha_\Sigma(\phi) \rightarrow'_{F(\Sigma)} \alpha_\Sigma(\psi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$.

If $n = 0$, $\alpha_\Sigma(\psi) \in \alpha_\Sigma(\Phi)$ or $\alpha_\Sigma(\psi) = \alpha_\Sigma(\phi)$.

- If $\alpha_\Sigma(\psi) \in \alpha_\Sigma(\Phi)$, then we get $\alpha_\Sigma(\psi) \rightarrow'_{F(\Sigma)} (\alpha_\Sigma(\phi) \rightarrow'_{F(\Sigma)} \alpha_\Sigma(\psi)) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$ since, by Lemma 5.1, $\psi \rightarrow_\Sigma (\phi \rightarrow_\Sigma \psi) \in C_\Sigma(\Phi)$. Hence, by N'-modus ponens, $\alpha_\Sigma(\phi) \rightarrow'_{F(\Sigma)} \alpha_\Sigma(\psi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$.

- If, on the other hand, $\alpha_\Sigma(\psi) = \alpha_\Sigma(\phi)$, then $\alpha_\Sigma(\phi) \to'_{F(\Sigma)} \alpha_\Sigma(\psi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$, since, again by Lemma 5.1, $\phi \to_\Sigma \phi \in C_\Sigma(\Phi)$.

Assume, as the induction hypothesis, that, for all $n \leq k$, if $\alpha_\Sigma(\psi) \in X_n$, then $\alpha_\Sigma(\phi) \to'_{F(\Sigma)} \alpha_\Sigma(\psi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$.

Suppose that $\alpha_\Sigma(\psi) \in X_{k+1}$. Then, there exist $x \in \mathrm{SEN}(\Sigma)$ and a finite $Y=\{y_0, y_1, \ldots, y_{m-1}\} \subseteq \mathrm{SEN}(\Sigma)$, such that $x \in C_\Sigma(y_0, \ldots, y_{m-1}), \alpha_\Sigma(x) = \alpha_\Sigma(\psi)$ and $\alpha_\Sigma(y_i) \in X_k, i = 0, \ldots, m-1$.

- If $Y = \emptyset$, we get $\alpha_\Sigma(\psi) \in C'_{F(\Sigma)}(\emptyset)$, whence, since $\alpha_\Sigma(\psi) \to'_{F(\Sigma)} (\alpha_\Sigma(\phi) \to'_{F(\Sigma)} \alpha_\Sigma(\psi)) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$, we obtain that $\alpha_\Sigma(\phi) \to'_{F(\Sigma)} \alpha_\Sigma(\psi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$, by N'-modus ponens.
- If $Y \neq \emptyset$, then, by the induction hypothesis, $\alpha_\Sigma(\phi) \to'_{F(\Sigma)} \alpha_\Sigma(y_i) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$, for all $i = 0, \ldots, m-1$. But $x \in C_\Sigma(y_0, \ldots, y_{m-1})$ implies, by the N-deduction-detachment theorem, that $\phi \to_\Sigma x \in C_\Sigma(\phi \to_\Sigma y_0, \ldots, \phi \to_\Sigma y_{m-1})$, whence

$$\begin{aligned} &\alpha_\Sigma(\phi) \to'_{F(\Sigma)} \alpha_\Sigma(\psi) \\ &\qquad \in C'_{F(\Sigma)}(\alpha_\Sigma(\phi) \to'_{F(\Sigma)} \alpha_\Sigma(y_0), \ldots, \alpha_\Sigma(\phi) \to'_{F(\Sigma)} \alpha_\Sigma(y_{m-1})) \\ &\qquad \subseteq C'_{F(\Sigma)}(\alpha_\Sigma(\Phi)). \end{aligned}$$

□

Finally, it is shown that, for a given π-institution $\mathcal{I}$ with the deduction-detachment property, every finitary model of $\mathcal{I}$, possessing the deduction theorem and the congruence property, is a full model of $\mathcal{I}$.

Proposition 5.8. *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ is a π-institution, N a category of natural transformations on* SEN, *and that $\mathcal{I}$ has the N-deduction-detachment theorem with respect to $\to$: $\mathrm{SEN}^2 \to \mathrm{SEN}$ in N. Then every finitary (N, N')-model $\mathcal{I}'$ via a surjective (N, N')-logical morphism $\langle F, \alpha \rangle : \mathcal{I} \succ^{se} \mathcal{I}'$ with the N'-deduction theorem with respect to $\to'$ and the N'-congruence property is a full (N, N')-model of $\mathcal{I}$.*

Proof. Suppose $\langle F, \alpha \rangle : \mathcal{I} \succ^{se} \mathcal{I}'$ is a surjective (N, N')-logical morphism and $\mathcal{I}'$ has the N'-deduction theorem and the N'-congruence property. It suffices to show that the $\langle F, \pi_F^{N'} \alpha \rangle$-min $(N, \overline{N'})$-model of $\mathcal{I}$ on $\mathrm{SEN}'^{N'}$ $\mathcal{I}'^{\min} = \langle \mathbf{Sign}', \mathrm{SEN}'^{N'}, C'^{\min} \rangle$ is such that $C'^{\min} = C'^{N'}$.

Since $\mathcal{I}'^{N'}$ is an $(N, \overline{N'})$-model of $\mathcal{I}$ on $\mathrm{SEN}'^{N'}$ via $\langle F, \pi_F^{N'}\alpha\rangle$, we have that $C'^{\min} \leq C'^{N'}$.

Conversely, let $\Sigma \in |\mathbf{Sign}'|, \Phi' \cup \{\phi'\} \in \mathrm{SEN}'(\Sigma)$, such that $\phi'^{N'} \in C'^{N'}_\Sigma(\Phi'^{N'})$. Thus, by definition, $\phi' \in C'_\Sigma(\Phi')$. But $\mathcal{I}'$ is finitary, whence, there exist $\vec{\phi}' = \langle \phi'_0, \ldots, \phi'_{n-1}\rangle \in \mathrm{SEN}'(\Sigma)^n$, such that $\phi' \in C'_\Sigma(\vec{\phi}')$. Since $\mathcal{I}'$ has the N'-deduction theorem with respect to $\rightarrow'$: $\mathrm{SEN}'^2 \to \mathrm{SEN}'$, $\phi'_0 \rightarrow'_\Sigma (\phi'_1 \rightarrow'_\Sigma (\ldots(\phi'_{n-1} \rightarrow'_\Sigma \phi')\ldots)) \in C'_\Sigma(\emptyset) = C'_\Sigma(\phi' \rightarrow'_\Sigma \phi')$. Thus

$$C'_\Sigma(\phi'_0 \rightarrow'_\Sigma (\phi'_1 \rightarrow'_\Sigma (\ldots(\phi'_{n-1} \rightarrow'_\Sigma \phi')\ldots))) = C'_\Sigma(\phi' \rightarrow'_\Sigma \phi').$$

Now $\mathcal{I}'$ has the N'-congruence property, whence, by Proposition 3.11 and Lemma 5.6,

$$\phi'^{N'}_0 \rightarrow'^{N'}_\Sigma (\phi'^{N'}_1 \rightarrow'^{N'}_\Sigma (\ldots(\phi'^{N'}_{n-1} \rightarrow'^{N'}_\Sigma \phi'^{N'})\ldots)) = \phi'^{N'} \rightarrow'^{N'}_\Sigma \phi'^{N'}.$$

But, now, by the $\overline{N'}$-modus ponens with respect to $\rightarrow'^{\overline{N'}}$, as inherited by $\mathcal{I}'^{\min}$, according to Lemma 5.3, we get that

$$\begin{array}{lll}\phi'^{N'} & \in & C'^{\min}_\Sigma(\phi'^{N'}_0, \ldots, \phi'^{N'}_{n-1}, \phi'^{N'}_1 \rightarrow'^{N'}_\Sigma (\ldots(\phi'^{N'}_{n-1} \rightarrow'^{N'}_\Sigma \phi'^{N'})\ldots)) \\ & = & C'^{\min}_\Sigma(\phi'^{N'}_0, \ldots, \phi'^{N'}_{n-1}, \phi'^{N'} \rightarrow'^{N'}_\Sigma \phi'^{N'}) \\ & = & C'^{\min}_\Sigma(\phi'^{N'}_0, \ldots, \phi'^{N'}_{n-1}) \\ & \subseteq & C'^{\min}_\Sigma(\Phi'^{N'}).\end{array}$$

This shows that $C'^{N'} \leq C'^{\min}$ and therefore that $C'^{\min} = C'^{N'}$. Hence $\mathcal{I}'$ is an (N, N')-full model of $\mathcal{I}$. □

6 The Property of Disjunction

A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, has a (**finitary uniterm**) N-**disjunction** $\vee : \mathrm{SEN}^2 \to \mathrm{SEN}$ in N if, for all $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi, \psi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$C_\Sigma(\Phi, \phi \vee_\Sigma \psi) = C_\Sigma(\Phi, \phi) \cap C_\Sigma(\Phi, \psi).$$

An alternate characterization of the property of N-disjunction is provided by the following lemma.

Lemma 6.1. *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution and N a category of natural transformations on* SEN*. $\mathcal{I}$ has an N-disjunction $\vee$ if and only if, for all $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi, \psi, \phi_1, \phi_2\} \subseteq \mathrm{SEN}(\Sigma)$,*

1. $\phi \vee_\Sigma \psi \in C_\Sigma(\phi)$,
2. $\phi \vee_\Sigma \psi \in C_\Sigma(\psi)$ *and*
3. $\psi \in C_\Sigma(\Phi, \phi_1)$ *and* $\psi \in C_\Sigma(\Phi, \phi_2)$ *imply* $\psi \in C_\Sigma(\Phi, \phi_1 \vee_\Sigma \phi_2)$.

Proof. Suppose that $\mathcal{I}$ has an N-disjunction $\vee : \mathrm{SEN}^2 \to \mathrm{SEN}$. Then, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$, by taking $\Phi = \emptyset$ in the defining condition of the N-disjunction, we get $C_\Sigma(\phi \vee_\Sigma \psi) = C_\Sigma(\phi) \cap C_\Sigma(\psi)$. Therefore $\phi \vee_\Sigma \psi \in C_\Sigma(\phi)$ and $\phi \vee_\Sigma \psi \in C_\Sigma(\psi)$. Finally, for arbitrary $\Phi \cup \{\phi_1, \phi_2, \psi\} \subseteq \mathrm{SEN}(\Sigma)$, since $C_\Sigma(\Phi, \phi_1 \vee_\Sigma \phi_2) = C_\Sigma(\Phi, \phi_1) \cap C_\Sigma(\Phi, \phi_2)$, we get that, if $\psi \in C_\Sigma(\Phi, \phi_1)$ and $\psi \in C_\Sigma(\Phi, \phi_2)$, then $\psi \in C_\Sigma(\Phi, \phi_1 \vee_\Sigma \phi_2)$.

Suppose, conversely, that the three conditions of the statement are satisfied. Let $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi, \psi\} \subseteq \mathrm{SEN}(\Sigma)$. Since $\phi \vee_\Sigma \psi \in C_\Sigma(\phi)$ and $\phi \vee_\Sigma \psi \in C_\Sigma(\psi)$, we get that $C_\Sigma(\Phi, \phi \vee_\Sigma \psi) \subseteq C_\Sigma(\Phi, \phi) \cap C_\Sigma(\Phi, \psi)$. The reverse inclusion is the third condition in the statement. $\square$

The N-disjunction, when it exists, is "commutative" and "idempotent".

Lemma 6.2. *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ *is a* π*-institution and* N *a category of natural transformations on* SEN. *If* $\mathcal{I}$ *has an* N*-disjunction* $\vee$*, then, for all* $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$C_\Sigma(\phi \vee_\Sigma \psi) = C_\Sigma(\psi \vee_\Sigma \phi) \quad \text{and} \quad C_\Sigma(\phi) = C_\Sigma(\phi \vee_\Sigma \phi).$$

Proof. Applying the defining condition for an N-disjunction, taking $\Phi = \emptyset$, we get $C_\Sigma(\phi \vee_\Sigma \psi) = C_\Sigma(\phi) \cap C_\Sigma(\psi) = C_\Sigma(\psi) \cap C_\Sigma(\phi) = C_\Sigma(\psi \vee_\Sigma \phi)$ and $C_\Sigma(\phi \vee_\Sigma \phi) = C_\Sigma(\phi) \cap C_\Sigma(\phi) = C_\Sigma(\phi)$. $\square$

As was the case with the property of having an N-conjunction and of having the N-deduction-detachment theorem, the property of having an N-disjunction is preserved under bilogical morphisms.

Lemma 6.3. *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle, \mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C' \rangle$ *are* π*-institutions,* N, N' *categories of natural transformations on* SEN, SEN$'$, *respectively, and* $\langle F, \alpha \rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$ *an* (N, N')*-bilogical morphism.* $\mathcal{I}$ *has an* N*-disjunction if and only if* $\mathcal{I}'$ *has an* N'*-disjunction.*

Proof. Both directions are very similar to the corresponding directions of Lemmas 4.1 and 5.4 and will be omitted. $\square$

Applying Lemma 6.3 to the $(N, \overline{N})$-bilogical morphism $\langle \mathrm{I}_{\mathbf{Sign}}, \pi^N \rangle : \mathcal{I} \vdash^{se} \mathcal{I}^N$, we immediately obtain

Corollary 6.4. *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *is a* π*-institution and* N *a category of natural transformations on* SEN*.* $\mathcal{I}$ *has an* N*-disjunction if and only if* $\mathcal{I}^N$ *has an* $\overline{N}$*-disjunction.*

The property of N-disjunction may be applied to a collection of disjunctions involving a single sentence ψ and a finite collection of sentences $\phi_1, \ldots, \phi_n$. This extension of the defining property of disjunction is accomplished by applying induction on the number n of sentences.

Lemma 6.5. *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *is a* π*-institution and* N *a category of natural transformations on* SEN*. If* $\mathcal{I}$ *has an* N*-disjunction* $\vee$*, then, for all* $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi_1, \ldots, \phi_n, \psi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$C_\Sigma(\Phi, \phi_1 \vee_\Sigma \psi, \ldots, \phi_n \vee_\Sigma \psi) = C_\Sigma(\Phi, \phi_1, \ldots, \phi_n) \cap C_\Sigma(\Phi, \psi).$$

Proof. The proof is by induction on $n \geq 1$. For $n = 1$, the equality reduces to the defining equality of an N-disjunction $\vee$. Assume that the statement is true for $n = k$, i.e., that, for all $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi_1, \ldots, \phi_k, \psi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$C_\Sigma(\Phi, \phi_1 \vee_\Sigma \psi, \ldots, \phi_k \vee_\Sigma \psi) = C_\Sigma(\Phi, \phi_1, \ldots, \phi_k) \cap C_\Sigma(\Phi, \psi).$$

Then, for $n = k + 1$, we get

$$\begin{array}{ll} \multicolumn{2}{l}{C_\Sigma(\Phi, \phi_1 \vee_\Sigma \psi, \ldots, \phi_{k+1} \vee_\Sigma \psi) =} \\ = & C_\Sigma(\Phi, \phi_1 \vee_\Sigma \psi, \ldots, \phi_k \vee_\Sigma \psi, \phi_{k+1}) \cap C_\Sigma(\Phi, \phi_1 \vee_\Sigma \psi, \ldots, \phi_k \vee_\Sigma \psi, \psi) \\ = & C_\Sigma(\Phi, \phi_1, \ldots, \phi_{k+1}) \cap C_\Sigma(\Phi, \phi_{k+1}, \psi) \cap \\ & \qquad C_\Sigma(\Phi, \phi_1, \ldots, \phi_k, \psi) \cap C_\Sigma(\Phi, \psi) \\ = & C_\Sigma(\Phi, \phi_1, \ldots, \phi_{k+1}) \cap C_\Sigma(\Phi, \psi). \end{array}$$

□

Furthermore, if a given sentence ψ is a consequence of finitely many sentences $\phi_1, \ldots, \phi_n$, then the disjunction of ψ with any other sentence is also a consequence of the disjunctions of $\phi_1, \ldots, \phi_n$ with that same sentence.

Lemma 6.6. *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *is a* π*-institution and* N *a category of natural transformations on* SEN*. If* $\mathcal{I}$ *has an* N*-disjunction* $\vee$*, then, for all* $\Sigma \in |\mathbf{Sign}|$, $\phi_1, \ldots, \phi_n, \psi, \chi \in \mathrm{SEN}(\Sigma)$,

$$\psi \in C_\Sigma(\phi_1, \ldots, \phi_n) \quad \textit{implies} \quad \psi \vee_\Sigma \chi \in C_\Sigma(\phi_1 \vee_\Sigma \chi, \ldots, \phi_n \vee_\Sigma \chi).$$

Proof. Suppose that $\psi \in C_\Sigma(\phi_1, \dots, \phi_n)$. Recall that, by Lemma 6.1, $\psi \vee_\Sigma \chi \in C_\Sigma(\psi)$ and $\psi \vee_\Sigma \chi \in C_\Sigma(\chi)$. Therefore, we get

$$\begin{array}{rcl} \psi \vee_\Sigma \chi & \in & C_\Sigma(\psi) \cap C_\Sigma(\chi) \\ & \subseteq & C_\Sigma(\phi_1, \dots, \phi_n) \cap C_\Sigma(\chi) \\ & = & C_\Sigma(\phi_1 \vee_\Sigma \chi, \dots, \phi_n \vee_\Sigma \chi), \end{array}$$

the last equality holding by Lemma 6.5. $\square$

Finally, it is shown that the property of disjunction for finitary π-institutions is inherited by full models via surjective logical morphisms.

Theorem 6.7. *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ is a finitary π-institution and N a category of natural transformations on* SEN*. If $\mathcal{I}$ has an N-disjunction $\vee$, then, every (N, N')-full model $\mathcal{I}'$ of $\mathcal{I}$ via a surjective (N, N')-logical morphism $\langle F, \alpha \rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$ has an N'-disjunction.*

Proof. Suppose that $\mathcal{I}$ has the N-disjunction $\vee : \mathrm{SEN}^2 \to \mathrm{SEN}$. It suffices, by Lemma 6.3, to show that if $\langle F, \alpha \rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$ is a surjective (N, N')-logical morphism onto the $\langle F, \alpha \rangle$-min (N, N')-model $\mathcal{I}'$ of $\mathcal{I}$, then $\mathcal{I}'$ has an N'-disjunction. We will take advantage in the proof, once more, of Lemma 2.1 of Section 2. Let $\Sigma' \in |\mathbf{Sign}'|, \Phi' \cup \{\phi', \psi'\} \subseteq \mathrm{SEN}'(\Sigma')$. By surjectivity, there exist $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi, \psi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\Sigma' = F(\Sigma)$ and $\Phi' = \alpha_\Sigma(\Phi), \phi' = \alpha_\Sigma(\phi), \psi' = \alpha_\Sigma(\psi)$. Our goal is to show that

$$C'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi) \vee'_{F(\Sigma)} \alpha_\Sigma(\psi)) = \\ C'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi)) \cap C'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\psi)).$$

By the N-disjunction property for $\mathcal{I}$, we get that $C_\Sigma(\Phi, \phi \vee_\Sigma \psi) = C_\Sigma(\Phi, \phi)$ $\cap C_\Sigma(\Phi, \psi)$. Thus $C_\Sigma(\Phi, \phi \vee_\Sigma \psi) \subseteq C_\Sigma(\Phi, \phi)$ and $C_\Sigma(\Phi, \phi \vee_\Sigma \psi) \subseteq C_\Sigma(\Phi, \psi)$, whence $C'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi) \vee'_{F(\Sigma)} \alpha_\Sigma(\psi)) \subseteq C'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi))$ and $C'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi) \vee'_{F(\Sigma)} \alpha_\Sigma(\psi)) \subseteq C'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\psi))$, which yield that

$$C'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi) \vee'_{F(\Sigma)} \alpha_\Sigma(\psi)) \subseteq \\ C'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi)) \cap C'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\psi)).$$

For the converse, it may be shown, first, using a similar induction as in the proof of Theorem 5.7, that, for all $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi, \chi, \psi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\alpha_\Sigma(\psi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi)) \quad \text{implies}$$

$$\alpha_\Sigma(\psi) \vee'_{F(\Sigma)} \alpha_\Sigma(\chi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\psi) \vee'_{F(\Sigma)} \alpha_\Sigma(\chi)).$$

Having this at hand, we now obtain, for all $\chi \in \mathrm{SEN}(\Sigma)$,

$$\alpha_\Sigma(\chi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi)) \cap C'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\psi))$$

implies $\alpha_\Sigma(\chi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi))$ and $\alpha_\Sigma(\chi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\psi))$, whence $\alpha_\Sigma(\chi) \vee'_{F(\Sigma)} \alpha_\Sigma(\psi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi) \vee'_{F(\Sigma)} \alpha_\Sigma(\psi))$ and $\alpha_\Sigma(\chi) \vee'_{F(\Sigma)} \alpha_\Sigma(\chi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\psi) \vee'_{F(\Sigma)} \alpha_\Sigma(\chi))$. Therefore, we, finally, obtain

$$\begin{array}{rcl}
\alpha_\Sigma(\chi) & \in & C'_{F(\Sigma)}(\alpha_\Sigma(\chi) \vee'_{F(\Sigma)} \alpha_\Sigma(\chi)) \\
& \subseteq & C'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\psi) \vee'_{F(\Sigma)} \alpha_\Sigma(\chi)) \\
& = & C'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\chi) \vee'_{F(\Sigma)} \alpha_\Sigma(\psi)) \\
& \subseteq & C'_{F(\Sigma)}(\alpha_\Sigma(\Phi), \alpha_\Sigma(\phi) \vee'_{F(\Sigma)} \alpha_\Sigma(\psi)).
\end{array}$$

□

7 Reductio ad Absurdum

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution and N a category of natural transformations on SEN. $\mathcal{I}$ has the

- N-**Intuitionistic Reductio ad Absurdum** with respect to a $\neg : \mathrm{SEN} \to \mathrm{SEN}$ in N if, for all $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\neg_\Sigma \phi \in C_\Sigma(\Phi) \quad \text{iff} \quad C_\Sigma(\Phi, \phi) = \mathrm{SEN}(\Sigma),$$

- N-**Reductio ad Absurdum** with respect to $\neg : \mathrm{SEN} \to \mathrm{SEN}$ in N if, for all $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\phi \in C_\Sigma(\Phi) \quad \text{iff} \quad C_\Sigma(\Phi, \neg_\Sigma \phi) = \mathrm{SEN}(\Sigma).$$

The following lemma provides a fundamental connection between the two properties of the N-intuitionistic reductio ad absurdum and of the N-reductio ad absurdum.

Lemma 7.1. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution and N a category of natural transformations on* SEN. *$\mathcal{I}$ has the N-reductio ad absurdum with respect to $\neg : \mathrm{SEN} \to \mathrm{SEN}$ if and only if it has the N-intuitionistic reductio ad absurdum with respect to $\neg$ and, for all $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$, $\phi \in C_\Sigma(\neg_\Sigma \neg_\Sigma \phi)$.*

Proof. Suppose, first, that $\mathcal{I}$ has the N-reductio ad absurdum with respect to $\neg$. Then, for all $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$, $\neg_\Sigma\phi \in C_\Sigma(\neg_\Sigma\phi)$, whence $C_\Sigma(\neg_\Sigma\phi, \neg_\Sigma\neg_\Sigma\phi) = \mathrm{SEN}(\Sigma)$ and, therefore, $\phi \in C_\Sigma(\neg_\Sigma\neg_\Sigma\phi)$. Next, let $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$. To prove that $\neg_\Sigma\phi \in C_\Sigma(\Phi)$ iff $C_\Sigma(\Phi, \phi) = \mathrm{SEN}(\Sigma)$, it suffices to show that $C_\Sigma(\Phi, \neg_\Sigma\neg_\Sigma\phi) = C_\Sigma(\Phi, \phi)$. By what was just proven, it suffices, in turn, to show that $C_\Sigma(\Phi, \neg_\Sigma\neg_\Sigma\phi) \subseteq C_\Sigma(\Phi, \phi)$. In fact, we have

$$\begin{array}{rll} \phi \in C_\Sigma(\phi) & \text{iff} & C_\Sigma(\phi, \neg_\Sigma\phi) = \mathrm{SEN}(\Sigma) \\ & \text{implies} & C_\Sigma(\phi, \neg_\Sigma\neg_\Sigma\neg_\Sigma\phi) = \mathrm{SEN}(\Sigma) \\ & \text{iff} & \neg_\Sigma\neg_\Sigma\phi \in C_\Sigma(\phi). \end{array}$$

Suppose, conversely, that $\mathcal{I}$ has the N-intuitionistic reductio ad absurdum with respect to $\neg$ and, for all $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$, $\phi \in C_\Sigma(\neg_\Sigma\neg_\Sigma\phi)$. To show that $\phi \in C_\Sigma(\Phi)$ iff $C_\Sigma(\Phi, \neg_\Sigma\phi) = \mathrm{SEN}(\Sigma)$ it suffices to show that $\neg_\Sigma\neg_\Sigma\phi \in C_\Sigma(\Phi)$ iff $\phi \in C_\Sigma(\Phi)$. The implication from left to right is an immediate consequence of $\phi \in C_\Sigma(\neg_\Sigma\neg_\Sigma\phi)$. For the converse, we have

$$\begin{array}{rll} \neg_\Sigma\phi \in C_\Sigma(\neg_\Sigma\phi) & \text{iff} & C_\Sigma(\phi, \neg_\Sigma\phi) = \mathrm{SEN}(\Sigma) \\ & \text{iff} & \neg_\Sigma\neg_\Sigma\phi \in C_\Sigma(\phi). \end{array}$$

□

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution and N a category of natural transformations on SEN. A natural transformation $\perp : \mathrm{SEN} \to \mathrm{SEN}$ in N is said to be an N-**inconsistent element** if, for all $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$, $C_\Sigma(\perp_\Sigma\phi) = \mathrm{SEN}(\Sigma)$.

Under the presence of the deduction-detachment theorem, the intuitionistic reductio ad absurdum turns out to be equivalent to the presence of an inconsistent element.

Lemma 7.2. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, N a category of natural transformations on* SEN *and suppose that $\mathcal{I}$ has the N-deduction-detachment theorem with respect to* $\to: \mathrm{SEN}^2 \to \mathrm{SEN}$. *Then $\mathcal{I}$ has he N-intuitionistic reductio ad absurdum with respect to* $\neg : \mathrm{SEN} \to \mathrm{SEN}$ *if and only if it has an N-inconsistent element* $\perp : \mathrm{SEN} \to \mathrm{SEN}$. *Moreover, in this case, for all* $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$, $C_\Sigma(\neg_\Sigma\phi) = C_\Sigma(\phi \to_\Sigma \perp_\Sigma\phi)$.

Proof. Suppose, first, that $\mathcal{I}$ has the N-intuitionistic reductio ad absurdum with respect to $\neg$. Then, for all $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll} \phi \in C_\Sigma(\phi) & \text{iff} & \phi \rightarrow_\Sigma \phi \in C_\Sigma(\emptyset) \\ & \text{implies} & \neg_\Sigma\neg_\Sigma(\phi \rightarrow_\Sigma \phi) \in C_\Sigma(\emptyset) \\ & \text{iff} & C_\Sigma(\neg_\Sigma(\phi \rightarrow_\Sigma \phi)) = \mathrm{SEN}(\Sigma). \end{array}$$

Now let $\perp_\Sigma\phi = \neg_\Sigma(\phi \rightarrow_\Sigma \phi)$, for all $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$.

Suppose, conversely, that $\perp : \mathrm{SEN} \rightarrow \mathrm{SEN}$ is an N-inconsistent element. Define, for all $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$, $\neg_\Sigma\phi = \phi \rightarrow_\Sigma \perp_\Sigma\phi$. We then have

$$\begin{array}{lll} \neg_\Sigma\phi \in C_\Sigma(\Phi) & \text{iff} & \phi \rightarrow_\Sigma \perp_\Sigma\phi \in C_\Sigma(\Phi) \\ & \text{iff} & \perp_\Sigma\phi \in C_\Sigma(\Phi, \phi) \\ & \text{iff} & C_\Sigma(\Phi, \phi) = \mathrm{SEN}(\Sigma). \end{array}$$

Therefore $\mathcal{I}$ has the N-intuitionistic reductio ad absurdum with respect to $\neg$.

Finally, suppose that $\mathcal{I}$ has the N-deduction-detachment theorem with respect to $\rightarrow$, the N-intuitionistic reductio ad absurdum with respect to $\neg$ and the N-inconsistent element $\perp$. Then we have, for all $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$, $\perp_\Sigma\phi \in C_\Sigma(\phi, \phi \rightarrow_\Sigma \perp_\Sigma\phi)$, whence $C_\Sigma(\phi, \phi \rightarrow_\Sigma \perp_\Sigma\phi) = \mathrm{SEN}(\Sigma)$ and hence $\neg_\Sigma\phi \in C_\Sigma(\phi \rightarrow_\Sigma \perp_\Sigma\phi)$. Therefore, $C_\Sigma(\neg_\Sigma\phi) \subseteq C_\Sigma(\phi \rightarrow_\Sigma \perp_\Sigma\phi)$. On the other hand, $\neg_\Sigma\phi \in C_\Sigma(\neg_\Sigma\phi)$, whence $\perp_\Sigma\phi \in \mathrm{SEN}(\Sigma) = C_\Sigma(\phi, \neg_\Sigma\phi)$ and, therefore, $\phi \rightarrow_\Sigma \perp_\Sigma\phi \in C_\Sigma(\neg_\Sigma\phi)$. Thus $C_\Sigma(\phi \rightarrow_\Sigma \perp_\Sigma\phi) \subseteq C_\Sigma(\neg_\Sigma\phi)$. □

The existence of an inconsistent element is a property inherited by all models via surjective logical morphisms.

Lemma 7.3. *If a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN, *has an N-inconsistent element $\perp : \mathrm{SEN} \rightarrow \mathrm{SEN}$, then every (N, N')-model $\mathcal{I}'$ of $\mathcal{I}$ via a surjective (N, N')-logical morphism $\langle F, \alpha\rangle : \mathcal{I} \succ^{se} \mathcal{I}'$ has an N'-inconsistent element.*

Proof. Suppose that $\perp : \mathrm{SEN} \rightarrow \mathrm{SEN}$ is an N-inconsistent element of $\mathcal{I}$ and that $\mathcal{I}'$ is an (N, N')-model of $\mathcal{I}$ via a surjective (N, N')-logical morphism $\langle F, \alpha\rangle : \mathcal{I} \succ^{se} \mathcal{I}'$. Then, for all $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$, we have $C_\Sigma(\perp_\Sigma\phi) = \mathrm{SEN}(\Sigma)$, whence $\alpha_\Sigma(C_\Sigma(\perp_\Sigma\phi)) = \alpha_\Sigma(\mathrm{SEN}(\Sigma))$ and, thus, using surjectivity and the fact that $\langle F, \alpha\rangle$ is a logical morphism, $C'_{F(\Sigma)}(\alpha_\Sigma(\perp_\Sigma\phi)) = \mathrm{SEN}'(F(\Sigma))$. Since $\langle F, \alpha\rangle$ is (N, N')-epimorphic, there exists $\perp' : \mathrm{SEN}' \to \mathrm{SEN}'$, such that $C'_{F(\Sigma)}(\perp'_{F(\Sigma)}\alpha_\Sigma(\phi)) = \mathrm{SEN}'(F(\Sigma))$. Therefore, since $\langle F, \alpha\rangle$ is surjective, for all $\Sigma' \in |\mathbf{Sign}'|, \phi' \in \mathrm{SEN}'(\Sigma')$, we get $C'_{\Sigma'}(\perp'_{\Sigma'}\phi') = \mathrm{SEN}'(\Sigma')$. This implies that $\perp' : \mathrm{SEN}' \to \mathrm{SEN}'$ is an N'-inconsistent element of $\mathcal{I}'$. $\square$

Corollary 7.4. *If a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, satisfies the N-deduction-detachment theorem with respect to $\to$: $\mathrm{SEN}^2 \to \mathrm{SEN}$ in N and the N-intuitionistic reductio ad absurdum with respect to $\neg : \mathrm{SEN} \to \mathrm{SEN}$, then every (N, N')-full model $\mathcal{I}'$ of $\mathcal{I}$ via a surjective (N, N')-logical morphism $\langle F, \alpha\rangle : \mathcal{I} \succ^{se} \mathcal{I}'$ satisfies the N'-deduction-detachment theorem and the N'-intuitionistic reductio ad absurdum.*

Proof. This is a direct consequence of Theorem 5.7 and Lemmas 7.2 and 7.3. $\square$

Acknowledgements

The author wishes to thank Don Pigozzi, Charles Wells, Giora Slutzki and Josep Maria Font for their encouragement and support.

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