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Categorical Abstract Algebraic Logic: Partially Ordered Algebraic Systems

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To Don Pigozzi and Kate Pałasińska.

Abstract. An extension of parts of the theory of partially ordered varieties and quasivarieties, as presented by Pałasińska and Pigozzi in the framework of abstract algebraic logic, is developed in the more abstract framework of categorical abstract algebraic logic. Algebraic systems, as introduced in previous work by the author, play in this more abstract framework the role that universal algebras play in the more traditional treatment. The aim here is to build the generalized framework and to formulate and prove abstract versions of the ordered homomorphism theorems in this framework.

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1. Introduction

This work falls in the intersection of two mathematical areas: Universal and categorical algebra, on the one hand, and categorical abstract algebraic logic, on the other. It abstracts parts of the theory of partially ordered algebras to the level of algebraic systems, as introduced in [19], thus creating the appropriate background framework for abstracting and further advancing the work of Pałasińska and Pigozzi (see [15]) on the theory of partially ordered varieties and quasi-varieties in the context of abstract algebraic logic. As noted by Pigozzi in [15], several other researchers have dealt with partially ordered classes of universal algebras and [4, 13] are but two older references on the subject. Moreover, the development of the theory of [15] is very closely related and falls, in much of its scope, under the general treatment of universal Horn logic without equality. [7–10] are some of the references in the recent literature in this direction from the point of view of abstract algebraic logic.

The research reported on in [15] is briefly reviewed here as background information. We then describe some elements of the theory of algebraic systems from categorical abstract algebraic logic [19], as related to the universal algebraic treatment, that partly motivated the present work.

The motivation for the work of Pałasińska and Pigozzi on partially ordered varieties and quasi-varieties of algebras stems from their desire to develop a theory of algebraizability centered around an abstraction of the notion of logical implication rather than that of logical equivalence. The paradigm would be the work of Blok and Pigozzi, first on protoalgebraic [2] and, later, on algebraizable logics [3], abstracting the concept of logical equivalence via the use of the Leibniz operator.

Given an algebraic language type $\mathcal{L}$ a polarity ρ is associated with $\mathcal{L}$ in such a way that to every n-ary function symbol λ in $\mathcal{L}$, each argument place of λ is assigned either a positive or a negative polarity. This assignment intends to model a situation arising in logical investigations with the presence of some logical connectives, like implication, whose application is monotone in one argument but antimonotone in another. Given such a polarity, an $\mathcal{L}$-algebra $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$, together with a quasi-ordering $\lesssim^{\mathbf{A}}$ on its universe, is said to form a ρ-qoalgebra if every operation $\lambda^{\mathbf{A}}$ of $\mathbf{A}$ is monotone in the arguments that have been assigned positive polarity by ρ and antimonotone in the arguments that have been assigned a negative polarity by ρ. Such an algebra is denoted by $\mathcal{A} = \langle \mathbf{A}, \lesssim^{\mathbf{A}} \rangle$. If $\lesssim^{\mathbf{A}}$ is a partial-ordering on A, then $\langle \mathbf{A}, \lesssim^{\mathbf{A}} \rangle$ is a ρ-poalgebra. Pigozzi [15] provides, next, a natural definition of an order homomorphism $h : \langle \mathbf{A}, \lesssim^{\mathbf{A}} \rangle \to \langle \mathbf{B}, \lesssim^{\mathbf{B}} \rangle$ between two ρ-poalgebras. Such a homomorphism is an $\mathcal{L}$-algebra homomorphism that, in addition, preserves the given partial orderings, i.e., such that $h(\lesssim^{\mathbf{A}}) \subseteq \lesssim^{\mathbf{B}}$. This definition is complemented by the definition of a quotient ρ-qoalgebra by a given congruence of the algebra that is compatible with the quasi-ordering on its universe. A congruence θ of $\mathbf{A}$ is said to be compatible with the ρ-quasi-ordering $\lesssim^{\mathbf{A}}$ if, for all $a_1, a_2, b_1, b_2 \in A$, $a_1 \theta b_1$, $a_2 \theta b_2$ and $a_1 \lesssim^{\mathbf{A}} a_2$ imply $b_1 \lesssim^{\mathbf{A}} b_2$. It turns out that, given such a congruence θ, the quotient $\lesssim^{\mathbf{A}}/\theta$ is a ρ-quasi-ordering on $\mathbf{A}/\theta$ and, therefore, one may consider the quotient ρ-qoalgebra $\langle \mathbf{A}/\theta, \lesssim^{\mathbf{A}}/\theta \rangle$. Two key notions leading up to the formulation of the order homomorphism theorems of [15] are the notion of a quasi-ordering on a ρ-poalgebra and the notion of the quotient of a ρ-poalgebra by one of its quasi-orderings. Given a ρ-poalgebra $\mathcal{A} = \langle \mathbf{A}, \lesssim^{\mathbf{A}} \rangle$, a ρ-quasi-ordering of $\mathcal{A}$ is a ρ-quasi-ordering $\lesssim$ on A, such that $\lesssim^{\mathbf{A}} \subseteq \lesssim$. The collection of all such quasi-orderings is denoted by $\mathrm{Qord}_\rho(\mathcal{A})$ and it is shown in Theorem 2.15 of [15] that it forms a complete algebraic lattice under inclusion, denoted by $\mathbf{Qord}_\rho(\mathcal{A}) = \langle \mathrm{Qord}_\rho(\mathcal{A}), \subseteq \rangle$. By $\sim = \lesssim \cap \gtrsim$ is denoted the symmetrization of the quasi-ordering $\lesssim$. If $\lesssim \in \mathrm{Qord}_\rho(\mathcal{A})$, then $\sim$ is a congruence on $\mathbf{A}$, that is compatible with $\lesssim$. Thus, given $\mathcal{A} = \langle \mathbf{A}, \lesssim^{\mathbf{A}} \rangle$ and $\lesssim \in \mathrm{Qord}_\rho(\mathcal{A})$, the quotient ρ-poalgebra of $\mathcal{A}$ by $\lesssim$ may be defined as the ρ-poalgebra $\langle \mathbf{A}/\sim, \lesssim/\sim \rangle$. This quotient ρ-poalgebra is denoted by $\mathcal{A}/\lesssim$. The ordinary Homomorphism, Isomorphism and Correspondence Theorems of Universal Algebra have now natural extensions to order counterparts. For instance, the Order Isomorphism Theorem says that if $h : \mathcal{A} \to \mathcal{B}$ is an order-epimorphism between two ρ-poalgebras and $\lesssim$ is the order-kernel of h (the pre-image under h of $\lesssim^{\mathbf{B}}$), then $\mathcal{A}/\lesssim$ is order-isomorphic to $\mathcal{B}$ and the Order Correspondence Theorem says that, given a ρ-poalgebra $\mathcal{A} = \langle \mathbf{A}, \lesssim^{\mathbf{A}} \rangle$ and $\lesssim \in \mathrm{Qord}_\rho(\mathcal{A})$, the complete lattice $\mathbf{Qord}_\rho(\mathcal{A}/\lesssim)$ is isomorphic to the principal filter of $\mathbf{Qord}_\rho(\mathcal{A})$ that is generated by the ρ-quasi-ordering $\lesssim$ under inclusion.

The motivation for lifting the theory reviewed above from the universal algebraic framework to a more abstract categorical framework stems from recent developments in categorical abstract algebraic logic, which have made it clear that the role that algebras play in the traditional treatment is assumed, in this context, by algebraic systems. As an example of this phenomenon, recall the Isomorphism Theorem 2.30 of Font and Jansana [11] and its abstract analog, Theorem 13 of [19]. Theorem 2.30 of [11] says that, given a sentential logic $\mathcal{S}$, the Tarski operator $\widetilde{\Omega}_{\mathbf{A}}$ on a given algebra $\mathbf{A}$ is an isomorphism between the ordered sets $\langle \mathrm{FMod}_{\mathcal{S}}(\mathbf{A}), \leq \rangle$ of full models of $\mathcal{S}$ on $\mathbf{A}$ and $\langle \mathrm{Con}_{\mathbf{Alg}\mathcal{S}}(\mathbf{A}), \subseteq \rangle$ of $\mathbf{Alg}\mathcal{S}$-congruences on $\mathbf{A}$. On the other hand, Theorem 13 of [19] says that given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on SEN, a functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, with N' a category of natural transformations on SEN', and $\langle F, \alpha \rangle : \mathcal{I} \to^{se} \mathrm{SEN}'$ a singleton (N, N')-epimorphic translation, the Tarksi operator $\widetilde{\Omega}^{\langle F,\alpha \rangle}_{\mathrm{SEN}'}$ is an order isomorphism between $\mathbf{FMod}^{\langle F,\alpha \rangle}_{\mathcal{I}}(\mathrm{SEN}')$ and $\mathbf{Con}^{\langle F,\alpha \rangle}_{\mathrm{Alg}^N(\mathcal{I})}(\mathrm{SEN}')$. If one compares the statements of these two theorems, taking into account the definitions of the full models in the two frameworks and of the corresponding definitions of congruences and of congruence systems, respectively, it is clear that the role of an $\mathcal{L}$-algebra, in the former framework, is assumed by the role of a sentence functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, endowed with a category N of natural transformations on SEN, in the latter framework. So it is important to obtain generalized versions of universal algebraic results in this framework. This is done for the Order Homomorphism, Order Isomorphism and Order Correspondence Theorems in the present work.

For general concepts and notation from category theory the reader is referred to any of [1, 5, 14]. For an overview of the current state of affairs in abstract algebraic logic the review article [12], the monograph [11] and the book [6] are all excellent references. To follow recent developments on the categorical side of the subject the reader may refer to the series of papers [16–25] in the given order.

2. Polarities and Qosystems

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a functor. Recall that a collection $R = \{R_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, with $R_\Sigma \subseteq \mathrm{SEN}(\Sigma)^2$ a binary relation on $\mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$, is said to be a **relation system on** SEN, if, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$,

$$\mathrm{SEN}(f)^2(R_{\Sigma_1}) \subseteq R_{\Sigma_2}.$$

In case R_Σ is a quasi-ordering (qordering) on SEN(Σ), for all $\Sigma \in |\mathbf{Sign}|$, R will be referred to as a **qosystem on** SEN. Finally, in case R_Σ is a partial ordering (pordering), i.e., $\langle \mathrm{SEN}(\Sigma), R_\Sigma \rangle$ is a partially ordered set (poset), for all $\Sigma \in |\mathbf{Sign}|$, R will be said to be a **posystem on** SEN.

All the definitions concerning polarities on categories of natural transformations that follow generalize the framework of partially ordered universal algebras of Pigozzi and Pałasińska [15].

Let SEN : $\mathbf{Sign} \to \mathbf{Set}$ be a functor and N a category of natural transformations on SEN. Informally speaking, a *polarity for N* is an assignment of a polarity, either positive or negative, to each argument position of each natural transformation in N that respects composition of natural transformations. Thus, formally, if we denote (by abusing notation) by N the collection of morphisms of the category N, and by $r(\sigma) = \{0, 1, \ldots, r(\sigma) - 1\}$ the arity of the natural transformation $\sigma : \mathrm{SEN}^{r(\sigma)} \to \mathrm{SEN}$, then a **polarity** ρ **for** N is a function $\rho : (\bigcup_{\sigma \in N} \sigma \times r(\sigma)) \to \{+, -\}$ with the following **composition compatibility property:**

If $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}, \tau : \mathrm{SEN}^m \to \mathrm{SEN}$ are in N and k is fixed, $0 \leq k \leq m - 1$, the natural transformation $\omega : \mathrm{SEN}^{n+m-1} \to \mathrm{SEN}$ in N, defined by

$$\begin{aligned}\omega_\Sigma(\phi_0, \ldots, \phi_{m+n-2}) = \tau_\Sigma(&\phi_0, \ldots, \phi_{k-1},\\ &\sigma_\Sigma(\phi_k, \ldots, \phi_{k+n-1}), \phi_{k+n}, \ldots, \phi_{m+n-2}),\end{aligned}$$

for all $\Sigma \in |\mathbf{Sign}|, \phi_0, \ldots, \phi_{n+m-2} \in \mathrm{SEN}(\Sigma)$, must satisfy, for all $0 \leq j \leq m + n - 2$,

$$\rho(\omega, j) = \begin{cases} \rho(\tau, j), & \text{if } j < k \text{ or } j \geq k + n, \\ \rho(\sigma, j - k), & \text{if } \rho(\tau, k) = + \text{ and } k \leq j < k + n \\ -\rho(\sigma, j - k), & \text{if } \rho(\tau, k) = - \text{ and } k \leq j < k + n. \end{cases}$$

A natural transformation σ in N is said to be of **positive** or of **negative polarity at the** i**-th argument** (**with respect to** ρ) if $\rho(\sigma, i)$ is $+$ or $-$, respectively. We also set $\rho^+(\sigma) = \{i < r(\sigma) : \rho(\sigma, i) = +\}$ and $\rho^-(\sigma) = \{i < r(\sigma) : \rho(\sigma, i) = -\}$. Sometimes σ is said to be **monotone** or **antimonotone** in the i-th argument if it is of positive or negative, respectively, polarity at the i-th argument.

DEFINITION 1. Let SEN : $\mathbf{Sign} \to \mathbf{Set}$ be a functor, N a category of natural transformations on SEN and ρ a polarity for N. A qosystem $\lesssim = \{\lesssim_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ on

SEN is said to be a ρ-**qosystem** if, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, $\Sigma \in |\mathbf{Sign}|$, $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)^n$,

$$(\forall i \in \rho^+(\sigma))(\phi_i \lesssim_\Sigma \psi_i) \text{ and } (\forall j \in \rho^-(\sigma))(\phi_j \gtrsim_\Sigma \psi_j)$$
$$\text{imply} \quad \sigma_\Sigma(\vec{\phi}) \lesssim_\Sigma \sigma_\Sigma(\vec{\psi}). \tag{1}$$

Condition (1) is called ρ-**tonicity**. A posystem $\lesssim$ that satisfies ρ-tonicity is called a ρ-**posystem** and $\langle \mathrm{SEN}, \lesssim \rangle$ in that case is said to be a ρ-**pofunctor**.

If $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is a functor, N a category of natural transformations on SEN and ρ a polarity for N, such that $\rho(\sigma, i) = +$, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N and all $i < n$, then ρ is said to be **completely positive** or **completely monotone**. A **pofunctor** is a ρ-pofunctor $\langle \mathrm{SEN}, \lesssim \rangle$, with ρ a completely monotone polarity.

LEMMA 2. *Suppose that* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *is a functor,* N *a category of natural transformations on* SEN *and* ρ *a polarity for* N. *A qosystem* $\lesssim$ *on* SEN *is* ρ*-tonic if, for every* $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ *in* N, $i < n$, $\Sigma \in |\mathbf{Sign}|$, $\vec{\chi} \in \mathrm{SEN}(\Sigma)^n$ *and* $\phi, \psi \in \mathrm{SEN}(\Sigma)$, *such that* $\phi \lesssim_\Sigma \psi$, $i \in \rho^+(\sigma)$ *implies*

$$\sigma_\Sigma(\chi_0, \ldots, \chi_{i-1}, \phi, \chi_{i+1}, \ldots, \chi_{n-1}) \lesssim_\Sigma \sigma_\Sigma(\chi_0, \ldots, \chi_{i-1}, \psi, \chi_{i+1}, \ldots, \chi_{n-1}),$$

and $i \in \rho^-(\sigma)$ *implies*

$$\sigma_\Sigma(\chi_0, \ldots, \chi_{i-1}, \phi, \chi_{i+1}, \ldots, \chi_{n-1}) \gtrsim_\Sigma \sigma_\Sigma(\chi_0, \ldots, \chi_{i-1}, \psi, \chi_{i+1}, \ldots, \chi_{n-1}).$$

Proof. To demonstrate the basic idea behind the proof, only the case of a natural transformation $\sigma : \mathrm{SEN}^2 \to \mathrm{SEN}$ in N, such that $\rho(\sigma, 0) = +$ and $\rho(\sigma, 1) = -$, will be considered. To show that a qosystem $\lesssim$ on SEN is ρ-tonic if it satisfies the given hypotheses, let $\Sigma \in |\mathbf{Sign}|, \phi_0, \phi_1, \psi_0, \psi_1 \in \mathrm{SEN}(\Sigma)$, such that $\phi_0 \lesssim_\Sigma \psi_0$ and $\phi_1 \gtrsim_\Sigma \psi_1$. Then we have

$$\begin{aligned} \sigma_\Sigma(\phi_0, \phi_1) &\lesssim_\Sigma \sigma_\Sigma(\psi_0, \phi_1) && \text{(by the first hypothesis)} \\ &\lesssim_\Sigma \sigma_\Sigma(\psi_0, \psi_1) && \text{(by the second hypothesis).} \end{aligned}$$

Thus $\lesssim$ is indeed a ρ-qosystem. □

Given a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, by Δ^{SEN} is denoted the identity equivalence system on SEN, i.e., the equivalence system, such that, for all $\Sigma \in |\mathbf{Sign}|$, $\Delta^{\mathrm{SEN}}_\Sigma = \Delta_{\mathrm{SEN}(\Sigma)}$. Note that, for every category N of natural transformations on SEN and every polarity ρ for N, Δ^{SEN} is a ρ-posystem of SEN. Also note that, if $\lesssim$ is a ρ-posystem of SEN, then $\gtrsim$ is also a ρ-posystem of SEN. Therefore, if there exists a ρ-posystem of SEN different from Δ^{SEN}, then there exist at least three distinct ρ-pofunctors with the same underlying functor SEN : **Sign** $\rightarrow$ **Set** (barring the possibility of triviality of all collections of sentences).

3. QoSystem Quotients

Suppose that SEN : **Sign** $\rightarrow$ **Set** is a functor, N a category of natural transformations on SEN and ρ a polarity for N. Let $\lesssim$ be a ρ-qosystem. An N-congruence system θ on SEN is said to be **compatible** with the ρ-qosystem $\lesssim$ if, for all $\Sigma \in |\mathbf{Sign}|, \phi_0, \phi_1, \psi_0, \psi_1 \in \mathrm{SEN}(\Sigma)$,

$$\phi_0 \theta_\Sigma \psi_0 \quad \text{and} \quad \phi_1 \theta_\Sigma \psi_1 \quad \text{imply} \quad (\phi_0 \lesssim_\Sigma \phi_1 \quad \text{iff} \quad \psi_0 \lesssim_\Sigma \psi_1).$$

Compatibility of a given N-congruence system with a given ρ-qosystem has an easy but interesting characterization.

LEMMA 3. *Let* SEN : **Sign** $\rightarrow$ **Set** *be a functor, N a category of natural transformations on* SEN, *ρ a polarity for N and $\lesssim$ a ρ-qosystem. An N-congruence system θ is compatible with $\lesssim$ if and only if $\theta \leq \lesssim$, i.e., if and only if, for all* $\Sigma \in |\mathbf{Sign}|,\ \theta_\Sigma \subseteq \lesssim_\Sigma$.

Proof. Suppose, first, that θ is an N-congruence system compatible with $\lesssim$. Let $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$ be such that $\langle \phi, \psi \rangle \in \theta_\Sigma$. We also have the two relations $\langle \phi, \phi \rangle \in \theta_\Sigma$, since θ is an N-congruence system, and $\phi \lesssim_\Sigma \phi$, since $\lesssim$ is a qosystem. Therefore, by the compatibility of $\lesssim$ with θ, we get that $\phi \lesssim_\Sigma \psi$. Thus $\theta_\Sigma \subseteq \lesssim_\Sigma$, for all $\Sigma \in |\mathbf{Sign}|$, and, therefore, $\theta \leq \lesssim$.

Suppose, conversely, that $\theta \leq \lesssim$. Let $\Sigma \in |\mathbf{Sign}|, \phi_0, \phi_1, \psi_0, \psi_1 \in \mathrm{SEN}(\Sigma)$, such that $\phi_0 \theta_\Sigma \psi_0, \phi_1 \theta_\Sigma \psi_1$ and $\phi_0 \lesssim_\Sigma \phi_1$. Since $\phi_0 \theta_\Sigma \psi_0$ and θ is an N-congruence system, we get that $\psi_0 \theta_\Sigma \phi_0$. Therefore, since $\theta \leq \lesssim$, we now have the three relations $\psi_0 \lesssim_\Sigma \phi_0, \phi_0 \lesssim_\Sigma \phi_1$ and $\phi_1 \lesssim_\Sigma \psi_1$. Therefore, since $\lesssim$ is a qosystem, we obtain, by transitivity, $\psi_0 \lesssim_\Sigma \psi_1$ and, hence, θ is compatible with $\lesssim$. □

Let SEN : **Sign** $\rightarrow$ **Set** be a functor, N a category of natural transformations on SEN, ρ a polarity for N and $\lesssim$ a ρ-qosystem. If θ is an N-congruence system compatible with $\lesssim$, then one may define the **quotient ρ^θ-qosystem** $\lesssim/\theta$ on the quotient functor SEN^θ : **Sign** $\rightarrow$ **Set**, as defined in [16]. First, recall from [16] that the category N of natural transformations on SEN induces a category N^θ of natural transformations on SEN^θ. In turn, the polarity ρ on N induces a polarity ρ^θ on N^θ, defined by

$$\rho^\theta(\sigma^\theta, i) = \rho(\sigma, i), \quad \text{for all} \quad \sigma : \mathrm{SEN}^n \rightarrow \mathrm{SEN} \text{ in } N,\ i < n.$$

$\lesssim/\theta$ is defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, by the condition

$$\phi/\theta_\Sigma\ (\lesssim/\theta)_\Sigma\ \psi/\theta_\Sigma \quad \text{iff} \quad \phi \lesssim_\Sigma \psi. \tag{2}$$

Sometimes, we write $\lesssim_\Sigma/\theta_\Sigma$ in place of $(\lesssim/\theta)_\Sigma$.

Compatibility of θ with $\lesssim$ ensures that $\lesssim/\theta$, given by Condition (2), is well-defined, i.e., does not depend on the choice of the representatives for ϕ/θ_Σ or ψ/θ_Σ. The following proposition fully justifies the terminology *quotient ρ^θ-qosystem*. It forms an analog of Proposition 2.3 of [15].

PROPOSITION 4. *Let* SEN : **Sign** $\rightarrow$ **Set** *be a functor, N a category of natural transformations on* SEN, *ρ a polarity for N, $\lesssim$ a ρ-qosystem and θ an N-congruence system that is compatible with $\lesssim$. Then $\lesssim/\theta$ is a ρ^θ-qosystem of* SEN^θ.

Proof. Three conditions must be verified. First that $\lesssim_\Sigma/\theta_\Sigma$ is a quasi-ordering on $\mathrm{SEN}^\theta(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$, second that $\lesssim/\theta$ is a relation system on SEN^θ and, finally, that the ρ^θ-tonicity condition holds.

Reflexivity and transitivity of $\lesssim_\Sigma/\theta_\Sigma$ follow immediately from the reflexivity and the transitivity properties of $\lesssim_\Sigma$, for all $\Sigma \in |\mathbf{Sign}|$.

To see that $\lesssim/\theta$ is a relation system on SEN^θ, suppose that $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$ and $\phi, \psi \in \mathrm{SEN}(\Sigma_1)$, such that

$$\phi/\theta_{\Sigma_1} \lesssim_{\Sigma_1}/\theta_{\Sigma_1} \psi/\theta_{\Sigma_1}.$$

Then, by the definition of $\lesssim/\theta$, $\phi \lesssim_{\Sigma_1} \psi$. Therefore, since $\lesssim$ is a qosystem on SEN, $\mathrm{SEN}(f)(\phi) \lesssim_{\Sigma_2} \mathrm{SEN}(f)(\psi)$. Thus, again by the definition of $\lesssim/\theta$, $\mathrm{SEN}(f)(\phi)/\theta_{\Sigma_2} \lesssim_{\Sigma_2}/\theta_{\Sigma_2} \mathrm{SEN}(f)(\psi)/\theta_{\Sigma_2}$. But, by the definition of SEN^θ, this is equivalent to

$$\mathrm{SEN}^\theta(f)(\phi/\theta_{\Sigma_1}) \lesssim_{\Sigma_2}/\theta_{\Sigma_2} \mathrm{SEN}^\theta(f)(\psi/\theta_{\Sigma_1})$$

and, hence, $\lesssim/\theta$ is in fact a qosystem on SEN^θ.

Finally, to show the ρ^θ-tonicity condition, we again restrict to the case of a natural transformation $\sigma^\theta : (\mathrm{SEN}^\theta)^2 \rightarrow \mathrm{SEN}^\theta$ in N^θ, such that $\rho^\theta(\sigma^\theta, 0) = +$. The case of more arguments or of negative arguments may be treated similarly. Such a natural transformation in N^θ is induced by a natural transformation $\sigma : \mathrm{SEN}^2 \rightarrow \mathrm{SEN}$ in N, such that $\rho(\sigma, 0) = +$. So, using Lemma 2, suppose that $\Sigma \in |\mathbf{Sign}|, \phi, \psi, \chi \in \mathrm{SEN}(\Sigma)$, such that $\phi/\theta_\Sigma \lesssim_\Sigma/\theta_\Sigma \psi/\theta_\Sigma$. Thus, by the definition of $\lesssim/\theta$, $\phi \lesssim_\Sigma \psi$. Hence, since $\rho(\sigma, 0) = +$, $\sigma_\Sigma(\phi, \chi) \lesssim_\Sigma \sigma_\Sigma(\psi, \chi)$. Therefore, again by the definition of $\lesssim/\theta$, $\sigma_\Sigma(\phi, \chi)/\theta_\Sigma \lesssim_\Sigma/\theta_\Sigma \sigma_\Sigma(\psi, \chi)/\theta_\Sigma$. But, by the definition of σ^θ, this is equivalent to

$$\sigma^\theta_\Sigma(\phi/\theta_\Sigma, \chi/\theta_\Sigma) \lesssim_\Sigma/\theta_\Sigma \sigma^\theta_\Sigma(\psi/\theta_\Sigma, \chi/\theta_\Sigma).$$

Therefore $\lesssim/\theta$ satisfies ρ^θ-tonicity. $\square$

Let SEN : **Sign** $\rightarrow$ **Set** be a functor, N a category of natural transformations on SEN, ρ a polarity for N and $\lesssim$ a ρ-qosystem on SEN. Furthermore, let $\sim = \lesssim \cap \gtrsim$ be the equivalence system on SEN induced by the ρ-qosystem $\lesssim$ in the usual way. Then $\sim$ is an N-congruence system on SEN and is, in fact, the largest N-congruence system that is compatible with $\lesssim$.

PROPOSITION 5. *Let* SEN : **Sign** $\rightarrow$ **Set** *be a functor, N a category of natural transformations on* SEN, *ρ a polarity for N and $\lesssim$ a ρ-qosystem on* SEN.

1. *The family $\sim = \lesssim \cap \gtrsim$ is an N-congruence system on* SEN.
2. *$\sim$ is the largest N-congruence system that is compatible with $\lesssim$.*

Proof.

1. It must be shown that $\sim_\Sigma$ is an N-congruence relation, for all $\Sigma \in |\mathbf{Sign}|$, and that $\sim$ is an N-congruence system.
   Clearly, $\sim_\Sigma = \lesssim_\Sigma \cap \gtrsim_\Sigma$ is an equivalence relation on SEN(Σ) as the intersection of a quasi-ordering with its converse. To show that it is an N-congruence, suppose that $\sigma : \mathrm{SEN}^n \rightarrow \mathrm{SEN}$ in N, $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)^n$, such that $\phi_i \sim_\Sigma \psi_i$, for all $i < n$, i.e., such that $\phi_i \lesssim_\Sigma \psi_i$ and $\psi_i \lesssim_\Sigma \phi_i$, for all $i < n$. Therefore, regardless of the polarities that ρ assigns to the arguments of σ, we have that $\sigma_\Sigma(\vec{\phi}) \lesssim_\Sigma \sigma_\Sigma(\vec{\psi})$ and $\sigma_\Sigma(\vec{\psi}) \lesssim_\Sigma \sigma_\Sigma(\vec{\phi})$. Thus, $\sigma_\Sigma(\vec{\phi}) \sim_\Sigma \sigma_\Sigma(\vec{\psi})$ and $\sim_\Sigma$ is in fact an N-congruence.
   That $\sim$ is an N-congruence system, i.e., that it is preserved by all **Sign**-morphisms follows from the fact that it is the intersection of two relation systems.
2. Suppose that θ is an N-congruence system that is compatible with $\lesssim$. Then, by Lemma 3, $\theta \leq \lesssim$, whence, since θ is symmetric, $\theta \leq \gtrsim$. Therefore $\theta \leq \lesssim \cap \gtrsim = \sim$, i.e., $\sim$ is the largest N-congruence system on SEN compatible with $\lesssim$. □

It immediately follows that the only N-congruence system that is compatible with a given ρ-posystem on SEN is the identity congruence system.

COROLLARY 6. *Let* SEN : **Sign** $\rightarrow$ **Set** *be a functor, N a category of natural transformations on* SEN, *ρ a polarity for N and $\lesssim$ a ρ-posystem on* SEN. Δ^{SEN} *is the only N-congruence system on* SEN *that is compatible with $\lesssim$.*

The N-congruence system $\sim$ is termed the **symmetrization** of the qosystem $\lesssim$.

4. Order Translations

In this section we revisit the translations of [16] adding as a new feature preservation of existing specific qosystems on the two sentence functors that are related.

DEFINITION 7. Suppose that $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}, \mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ are two functors, N, N' categories of natural transformations on $\mathrm{SEN}, \mathrm{SEN}'$, respectively, and ρ, ρ' polarities for N, N', respectively. A singleton (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ is said to be a **polarity translation from** SEN **to** SEN$'$, denoted $\langle F, \alpha\rangle : \mathrm{SEN} \to^{p} \mathrm{SEN}'$, if, for all corresponding $\tau : \mathrm{SEN}^n \to \mathrm{SEN}$ in N and $\tau' : \mathrm{SEN}'^n \to \mathrm{SEN}'$ via the (N, N')-epimorphic property,

$$\rho'(\tau', i) = \rho(\tau, i), \quad \text{for all} \quad 0 \leq i \leq n - 1.$$

Given a ρ-posystem $\lesssim$ on SEN and a ρ'-posystem $\lesssim'$ on SEN$'$, a polarity translation $\langle F, \alpha\rangle : \mathrm{SEN} \to^{p} \mathrm{SEN}'$ is said to be an **order translation from** the ρ-pofunctor $\langle \mathrm{SEN}, \lesssim\rangle$ **to** the ρ'-pofunctor $\langle \mathrm{SEN}', \lesssim'\rangle$, denoted $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \lesssim\rangle \to^{p} \langle \mathrm{SEN}', \lesssim'\rangle$, if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\phi \lesssim_\Sigma \psi \quad \text{implies} \quad \alpha_\Sigma(\phi) \lesssim'_{F(\Sigma)} \alpha_\Sigma(\psi).$$

The next lemma provides a fairly simple characterization of order translations inside the class of polarity translations between two π-institutions.

LEMMA 8. *Suppose that* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *and* $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ *are two functors, N, N' categories of natural transformations on* $\mathrm{SEN}, \mathrm{SEN}'$, *respectively, ρ, ρ' polarities for N, N', respectively, $\lesssim, \lesssim'$, a ρ-posystem on* SEN *and a ρ'-posystem on* SEN$'$, *respectively. A polarity translation $\langle F, \alpha\rangle : \mathrm{SEN} \to^{p} \mathrm{SEN}'$ is an order translation if and only if, for all $\Sigma \in |\mathbf{Sign}|$, $\lesssim_\Sigma \subseteq \alpha_\Sigma^{-1}\ (\lesssim'_{F(\Sigma)})$, denoted $\lesssim \leq \alpha^{-1}(\lesssim')$.*

Proof. The condition $\lesssim_\Sigma \subseteq \alpha_\Sigma^{-1}(\lesssim'_{F(\Sigma)})$, for all $\Sigma \in |\mathbf{Sign}|$, is simply a restatement of the defining condition for an order translation. □

Definition 7 gives rise to a few strengthenings of the notion of an order translation. Keeping the same notation, a polarity translation $\langle F, \alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$ is said to be an **order monomorphism** if $F : \mathbf{Sign} \to \mathbf{Sign}'$ is an injection, $\alpha_\Sigma : \mathrm{SEN}(\Sigma) \to \mathrm{SEN}'(F(\Sigma))$ is an injection, for all $\Sigma \in |\mathbf{Sign}|$, and, finally, $\alpha_\Sigma(\lesssim_\Sigma) = \lesssim'_{F(\Sigma)} \cap \alpha_\Sigma(\mathrm{SEN}(\Sigma))^2$. A surjective order monomorphism is said to be an **order isomorphism**.

With these definitions at hand, Lemma 8 has the following consequence.

COROLLARY 9. *Suppose that* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}, \mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ *are two functors, N, N' categories of natural transformations on* $\mathrm{SEN}, \mathrm{SEN}'$, *respectively, ρ, ρ' polarities for N, N', respectively, $\lesssim, \lesssim'$, a ρ-posystem on* SEN *and a* ρ'-posystem on SEN$'$, *respectively. An isomorphic polarity translation $\langle F, \alpha\rangle : \mathrm{SEN} \cong^{p} \mathrm{SEN}'$ is an order isomorphism if and only if, for all $\Sigma \in |\mathbf{Sign}|$, $\lesssim_\Sigma = \alpha_\Sigma^{-1}(\lesssim'_{F(\Sigma)})$, also denoted by $\lesssim = \alpha^{-1}(\lesssim')$.*

Recall from [16] that, given a translation $\langle F, \alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$, by $\mathrm{Ker}(\langle F, \alpha\rangle)$ or, sometimes, $\theta^{\langle F, \alpha\rangle}$, is denoted the **kernel** of $\langle F, \alpha\rangle$, i.e., the equivalence system $\theta^{\langle F,\alpha\rangle} = \{\theta_\Sigma^{\langle F,\alpha\rangle}\}_{\Sigma\in|\mathbf{Sign}|}$, such that, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\phi\ \theta_\Sigma^{\langle F,\alpha\rangle}\ \psi \quad \text{if and only if} \quad \alpha_\Sigma(\phi) = \alpha_\Sigma(\psi).$$

It was shown in Proposition 26 of [16] that, if, in addition, $\langle F,\alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$ is (N,N')-epimorphic, then $\theta^{\langle F,\alpha\rangle}$ is an N-congruence system on SEN.

In the next result, given an order translation, the posystem of the domain functor is related to the inverse image under the translation of the posystem of the codomain functor. Lemma 10 is an analog of Lemma 2.5 of [15].

LEMMA 10. *Suppose that* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}, \mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ *are two functors,* N, N' *categories of natural transformations on* SEN, SEN$'$, *respectively,* ρ, ρ' *polarities for* N, N', *respectively, and* $\lesssim$ *and* $\lesssim'$ *a* ρ*-posystem on* SEN *and a* ρ'*-posystem on* SEN$'$, *respectively. If* $\langle F,\alpha\rangle : \langle \mathrm{SEN}, \lesssim\rangle \to^p \langle \mathrm{SEN}', \lesssim'\rangle$ *is an order translation, then* $\alpha^{-1}(\lesssim')$ *is a* ρ*-qosystem of* SEN*, such that* $\lesssim\, \leq \alpha^{-1}(\lesssim')$. *Moreover* $\theta^{\langle F,\alpha\rangle} = \alpha^{-1}(\lesssim') \cap \alpha^{-1}(\gtrsim')$.

Proof. First, it will be shown that $\alpha^{-1}(\lesssim')$ is a ρ-qosystem on SEN. To see that, for all $\Sigma \in |\mathbf{Sign}|$, $\alpha_\Sigma^{-1}(\lesssim'_{F(\Sigma)})$ is a quasi-ordering on $\mathrm{SEN}(\Sigma)$, let $\phi,\psi,\chi \in \mathrm{SEN}(\Sigma)$.

– Since $\lesssim'_{F(\Sigma)}$ is a quasi-ordering on $\mathrm{SEN}'(F(\Sigma))$, we have that $\alpha_\Sigma(\phi) \lesssim'_{F(\Sigma)} \alpha_\Sigma(\phi)$, whence we get $\phi\ \alpha_\Sigma^{-1}(\lesssim'_{F(\Sigma)})\ \phi$ and, therefore, $\alpha_\Sigma^{-1}(\lesssim'_{F(\Sigma)})$ is reflexive.
– For transitivity, suppose that $\Sigma \in |\mathbf{Sign}|, \phi,\psi,\chi \in \mathrm{SEN}(\Sigma)$, such that $\phi\ \alpha_\Sigma^{-1}(\lesssim'_{F(\Sigma)})\ \psi$ and $\psi\ \alpha_\Sigma^{-1}(\lesssim'_{F(\Sigma)})\ \chi$. Therefore $\alpha_\Sigma(\phi) \lesssim'_{F(\Sigma)} \alpha_\Sigma(\psi)$ and $\alpha_\Sigma(\psi) \lesssim'_{F(\Sigma)} \alpha_\Sigma(\chi)$. But $\lesssim'_{F(\Sigma)}$ is transitive, whence $\alpha_\Sigma(\phi) \lesssim'_{F(\Sigma)} \alpha_\Sigma(\chi)$. This proves that $\phi\ \alpha_\Sigma^{-1}(\lesssim'_{F(\Sigma)})\ \chi$ and, therefore $\alpha_\Sigma^{-1}(\lesssim'_{F(\Sigma)})$ is also transitive.

Next, to see that $\alpha^{-1}(\lesssim')$ is a qosystem, we need to show that, for every $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma_1,\Sigma_2)$, $\mathrm{SEN}(f)^2(\alpha_{\Sigma_1}^{-1}(\lesssim'_{F(\Sigma_1)})) \subseteq \alpha_{\Sigma_2}^{-1}(\lesssim'_{F(\Sigma_2)})$. To this end, suppose that $\phi,\psi \in \mathrm{SEN}(\Sigma_1)$, such that $\phi\ \alpha_{\Sigma_1}^{-1}(\lesssim'_{F(\Sigma_1)})\ \psi$. Then $\alpha_{\Sigma_1}(\phi) \lesssim'_{F(\Sigma_1)} \alpha_{\Sigma_1}(\psi)$. Therefore, since $\lesssim'$ is a posystem on SEN$'$, we get that

$$\mathrm{SEN}'(F(f))(\alpha_{\Sigma_1}(\phi)) \lesssim'_{F(\Sigma_2)} \mathrm{SEN}'(F(f))(\alpha_{\Sigma_1}(\psi)).$$

Thus

$$\begin{array}{ccc}
\mathrm{SEN}(\Sigma_1) & \xrightarrow{\alpha_{\Sigma_1}} & \mathrm{SEN}'(F(\Sigma_1)) \\
{\scriptstyle \mathrm{SEN}(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{SEN}'(F(f))} \\
\mathrm{SEN}(\Sigma_2) & \xrightarrow[\alpha_{\Sigma_2}]{} & \mathrm{SEN}'(F(\Sigma_2))
\end{array}$$

$\alpha_{\Sigma_2}(\mathrm{SEN}(f)(\phi)) \precsim'_{F(\Sigma_2)} \alpha_{\Sigma_2}(\mathrm{SEN}(f)(\psi))$, whence, finally, $\mathrm{SEN}(f)(\phi)\ \alpha_{\Sigma_2}^{-1}(\precsim'_{F(\Sigma_2)})\ \mathrm{SEN}(f)(\psi)$ and, therefore, $\alpha^{-1}(\precsim')$ is a qosystem.

To finish the proof of the first claim, we need to show ρ-tonicity of $\alpha^{-1}(\precsim')$. Once more we restrict to a natural transformation $\sigma : \mathrm{SEN}^2 \to \mathrm{SEN}$ in N, with $\rho(\sigma, 0) = +$, trusting that it adequately illustrates the main idea needed to handle the general case. σ corresponds, by the (N, N')-epimorphic property and the polarity of the translation, to a $\sigma' : \mathrm{SEN}'^2 \to \mathrm{SEN}'$, such that $\rho'(\sigma', 0) = +$. Therefore, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi, \chi \in \mathrm{SEN}(\Sigma)$, such that $\phi\ \alpha_\Sigma^{-1}(\precsim'_{F(\Sigma)})\ \psi$, we have $\alpha_\Sigma(\phi) \precsim'_{F(\Sigma)} \alpha_\Sigma(\psi)$, and, therefore, by Lemma 3, $\sigma'_{F(\Sigma)}(\alpha_\Sigma(\phi), \alpha_\Sigma(\chi)) \precsim'_{F(\Sigma)} \sigma'_{F(\Sigma)}(\alpha_\Sigma(\psi), \alpha_\Sigma(\chi))$. Thus, by the (N, N')-epimorphic property, $\alpha_\Sigma(\sigma_\Sigma(\phi, \chi)) \precsim'_{F(\Sigma)} \alpha_\Sigma(\sigma_\Sigma(\psi, \chi))$, which yields $\sigma_\Sigma(\phi, \chi)\ \alpha_\Sigma^{-1}(\precsim'_{F(\Sigma)})\sigma_\Sigma(\psi, \chi)$. This proves ρ-tonicity of $\alpha^{-1}(\precsim')$.

That $\precsim \leq \alpha^{-1}(\precsim')$ follows from Lemma 8.

Finally, for $\theta^{\langle F,\alpha\rangle} = \alpha^{-1}(\precsim') \cap \alpha^{-1}(\succsim)$, follow, for all $\Sigma \in |\mathbf{Sign}|, \phi,\ \psi \in \mathrm{SEN}(\Sigma)$, the following string of equivalences:

$$\begin{array}{lll} \phi\ \theta_\Sigma^{\langle F,\alpha\rangle}\ \psi & \text{iff} & \alpha_\Sigma(\phi) = \alpha_\Sigma(\psi) \\ & \text{iff} & \alpha_\Sigma(\phi) \precsim'_{F(\Sigma)} \alpha_\Sigma(\psi) \quad \text{and} \quad \alpha_\Sigma(\psi) \precsim'_{F(\Sigma)} \alpha_\Sigma(\phi) \\ & \text{iff} & \phi\ \alpha_\Sigma^{-1}(\precsim'_{F(\Sigma)})\ \psi \quad \text{and} \quad \psi\ \alpha_\Sigma^{-1}(\precsim'_{F(\Sigma)})\ \phi. \end{array}$$

□

The qosystem $\alpha^{-1}(\precsim')$ of Lemma 10 deserves a special name, assigned by the following definition.

DEFINITION 11. Suppose that $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}, \mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ are two functors, N, N' categories of natural transformations on $\mathrm{SEN}, \mathrm{SEN}'$, respectively, ρ, ρ' polarities for N, N', respectively, $\precsim, \precsim'$, a ρ-posystem on SEN and a ρ'-posystem on SEN', respectively. If $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \precsim\rangle \to^p \langle \mathrm{SEN}', \precsim'\rangle$ is an order translation, then $\alpha^{-1}(\precsim')$ is called the **order kernel** of $\langle F, \alpha\rangle$ and denoted by $\mathrm{OrdKer}(\langle F, \alpha\rangle)$.

With this terminology at hand, we may characterize those order translations that are order monomorphisms, providing an analog to Proposition 2.7 of [15].

PROPOSITION 12. *Suppose that* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}, \mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ *are two functors,* N, N' *categories of natural transformations on* $\mathrm{SEN}, \mathrm{SEN}'$, *respectively,* ρ, ρ' *polarities for* N, N', *respectively,* $\precsim, \precsim'$, *a* ρ*-posystem on* SEN *and a* ρ'*- posystem on* SEN', *respectively. If* $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \precsim\rangle \to^p \langle \mathrm{SEN}', \precsim'\rangle$ *is an order translation, with* $F : \mathbf{Sign} \to \mathbf{Sign}'$ *a monomorphism, then* $\langle F, \alpha\rangle$ *is an order monomorphism if and only if* $\mathrm{OrdKer}(\langle F, \alpha\rangle) = \precsim$.

Proof. Suppose, first, that $\mathrm{OrdKer}(\langle F, \alpha\rangle) = \precsim$, i.e., that $\alpha^{-1}(\precsim') = \precsim$. We need to show that $\alpha_\Sigma : \mathrm{SEN}(\Sigma) \to \mathrm{SEN}'(F(\Sigma))$ is an injection and that $\alpha_\Sigma(\precsim_\Sigma) = \precsim'_{F(\Sigma)} \cap \alpha_\Sigma(\mathrm{SEN}(\Sigma))^2$, for all $\Sigma \in |\mathbf{Sign}|$. To show injectivity of

$\alpha_\Sigma : \mathrm{SEN}(\Sigma) \to \mathrm{SEN}'(F(\Sigma))$, suppose $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. Then, since $\lesssim'_{F(\Sigma)}$ is a partial ordering, $\alpha_\Sigma(\phi) \lesssim'_{F(\Sigma)} \alpha_\Sigma(\psi)$ and $\alpha_\Sigma(\psi) \lesssim'_{F(\Sigma)} \alpha_\Sigma(\phi)$. Hence, $\phi\ \alpha_\Sigma^{-1}(\lesssim'_{F(\Sigma)})\ \psi$ and $\psi\ \alpha_\Sigma^{-1}(\lesssim'_{F(\Sigma)})\ \phi$. Thus, by the hypothesis, $\phi \lesssim_\Sigma \psi$ and $\psi \lesssim_\Sigma \phi$, which yields $\phi = \psi$. Therefore, α_Σ is injective. Finally, we have $\lesssim'_{F(\Sigma)} \cap\, \alpha_\Sigma(\mathrm{SEN}(\Sigma))^2 = \alpha_\Sigma(\alpha_\Sigma^{-1}(\lesssim'_{F(\Sigma)})) = \alpha_\Sigma(\lesssim_\Sigma)$

Suppose, conversely, that $\langle F, \alpha\rangle$ is an order monomorphism. Then we get, for all $\Sigma \in |\mathbf{Sign}|$, $\alpha_\Sigma(\alpha_\Sigma^{-1}(\lesssim'_{F(\Sigma)})) = \lesssim'_{F(\Sigma)} \cap\, \alpha_\Sigma(\mathrm{SEN}(\Sigma))^2 = \alpha_\Sigma(\lesssim_\Sigma)$. Therefore, by the injectivity of α_Σ, we obtain $\alpha_\Sigma^{-1}(\lesssim'_{F(\Sigma)}) = \lesssim_\Sigma$, i.e., $\mathrm{OrdKer}(\langle F, \alpha\rangle) = \lesssim$. □

5. Quotient PoFunctors and Homomorphism Theorems

The notion of a ρ-qosystem of a given ρ-pofunctor $\langle \mathrm{SEN}, \lesssim\rangle$ will now be introduced. It will be a key notion for the formulation of the order-homomorphism theorems that constitute the main target of this section. ρ-qosystems of $\langle \mathrm{SEN}, \lesssim\rangle$ form an analog, in the present setting, of the ρ-qorders of a ρ-poalgebra in the universal algebraic framework.

DEFINITION 13. Suppose that $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is a functor, N a category of natural transformations on SEN, ρ a polarity for N and $\lesssim$ a ρ-posystem on SEN. A ρ-qosystem $\lesssim'$ is said to be a **ρ-qosystem of the ρ-pofunctor** $\langle \mathrm{SEN}, \lesssim\rangle$ if $\lesssim\, \leq\, \lesssim'$, i.e., if, for all $\Sigma \in |\mathbf{Sign}|, \lesssim_\Sigma \subseteq \lesssim'_\Sigma$. By $\mathrm{QoSys}_\rho(\langle \mathrm{SEN}, \lesssim\rangle)$ will be denoted the collection of all ρ-qosystems of $\langle \mathrm{SEN}, \lesssim\rangle$.

Recall from Lemma 3 that an N-congruence system θ on SEN is compatible with a qosystem $\lesssim$ on SEN if and only if $\theta \leq \lesssim$. If this is the case, then one may take the quotient ρ^θ-qosystem $\lesssim/\theta$ on the quotient functor SEN^θ. Now note that the symmetrization $\sim'$ of a ρ-qosystem $\lesssim'$ of a ρ-pofunctor $\langle \mathrm{SEN}, \lesssim\rangle$, as defined in Definition 13, is, by Lemma 3, compatible with $\lesssim'$. Therefore the following definition makes sense.

DEFINITION 14. Suppose that $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is a functor, N a category of natural transformations on SEN, ρ a polarity for N, $\lesssim$ a ρ-posystem on SEN and $\lesssim'$ a ρ-qosystem of $\langle \mathrm{SEN}, \lesssim\rangle$. The **quotient** $\rho^{\sim'}$ **− pofunctor of** $\langle \mathrm{SEN}, \lesssim\rangle$ **by** $\lesssim'$ is the $\rho^{\sim'}$-pofunctor $\langle \mathrm{SEN}/{\sim'}, \lesssim'/{\sim'}\rangle$, where, as usual, $\sim' = \lesssim' \cap \gtrsim'$. It will be denoted by $\langle \mathrm{SEN}, \lesssim\rangle/\lesssim'$ or by $\langle \mathrm{SEN}, \lesssim\rangle/{\sim'}$.

Quotient pofunctors and surjective order translations play in this, categorical, theory a role analogous to the role played by quotient algebras and surjective homomorphisms, respectively, in the context of universal algebra. Thus, it is no surprise that the following analogs of the well-known Homomorphism and Isomorphism Theorems hold in the present context. They extend the Order Homomorphism and Order Isomorphism Theorems for partially ordered algebras, given in [15].

THEOREM 15 (Order Homomorphism Theorem). *Suppose that* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *is a functor,* N *a category of natural transformations on* SEN, ρ *a polarity for* N, $\lesssim$ *a* ρ*-posystem on* SEN *and* $\lesssim'$ *a* ρ-qosystem of $\langle \mathrm{SEN}, \lesssim\rangle$.

1. The natural projection translation $\langle \mathrm{I}_{\mathbf{Sign}}, \pi^{\sim'}\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}^{\sim'}$ is an order epimorphism $\langle \mathrm{I}_{\mathbf{Sign}}, \pi^{\sim'}\rangle : \langle \mathrm{SEN}, \lesssim\rangle \to^{p} \langle \mathrm{SEN}^{\sim'}, \lesssim'/\sim'\rangle$ with $\mathrm{OrdKer}(\langle \mathrm{I}_{\mathbf{Sign}}, \pi^{\sim'}\rangle) = \lesssim'$.
2. Suppose that $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \lesssim\rangle \to^{p} \langle \mathrm{SEN}'', \lesssim''\rangle$ is an order translation from a ρ-pofunctor to a ρ''-pofunctor, such that $\lesssim' \leq \alpha^{-1}(\lesssim'')$. Then, there exists a unique order translation $\langle G, \beta\rangle : \langle \mathrm{SEN}^{\sim'}, \lesssim'/\sim'\rangle \to^{p} \langle \mathrm{SEN}'', \lesssim''\rangle$, such that $\langle F, \alpha\rangle = \langle G, \beta\rangle \circ \langle \mathrm{I}_{\mathbf{Sign}}, \pi^{\sim'}\rangle$.

$$\begin{tikzcd}
\langle \mathrm{SEN}, \lesssim\rangle \arrow[rr, "\langle \mathrm{I}_{\mathbf{Sign}}, \pi^{\sim'}\rangle"] \arrow[dr, "\langle F, \alpha\rangle"'] & & \langle \mathrm{SEN}^{\sim'}, \lesssim'/\sim'\rangle \arrow[dl, "\langle G, \beta\rangle"] \\
& \langle \mathrm{SEN}'', \lesssim''\rangle &
\end{tikzcd}$$

Moreover, for every $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$, $\beta_\Sigma(\phi/{\sim'_\Sigma}) = \alpha_\Sigma(\phi)$.

Proof.

1. Clearly, $\langle \mathrm{I}_{\mathbf{Sign}}, \pi^{\sim'}\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}^{\sim'}$ is surjective. Moreover, by the definition of $\lesssim'/\sim'$, we have, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

   $$\phi \lesssim_\Sigma \psi \quad \text{implies} \quad \phi \lesssim'_\Sigma \psi \quad \text{iff} \quad \phi/{\sim'_\Sigma} \lesssim'_\Sigma/{\sim'_\Sigma}\ \psi/{\sim'_\Sigma}.$$

   This relation shows that $\langle \mathrm{I}_{\mathbf{Sign}}, \pi^{\sim'}\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}^{\sim'}$ is an order epimorphism and, also, that $(\pi^{\sim'}_\Sigma)^{-1}(\lesssim'_\Sigma/{\sim'_\Sigma}) = \lesssim'_\Sigma$, i.e., that $\mathrm{OrdKer}(\langle \mathrm{I}_{\mathbf{Sign}}, \pi^{\sim'}\rangle) = \lesssim'$.
2. The proof of this part is very similar to the proof of Theorem 27 of [16]. We may define $G : \mathbf{Sign} \to \mathbf{Sign}''$ by $G = F$ and, for all $\Sigma \in |\mathbf{Sign}|, \beta_\Sigma : \mathrm{SEN}^{\sim'}(\Sigma) \to \mathrm{SEN}''(G(\Sigma))$, by

   $$\beta_\Sigma(\phi/{\sim'_\Sigma}) = \alpha_\Sigma(\phi), \quad \text{for all} \quad \phi \in \mathrm{SEN}(\Sigma).$$

   We show that β_Σ is well-defined and that $\langle G, \beta\rangle$ is an order translation. These will suffice since it is clear that, then, also $\langle F, \alpha\rangle = \langle G, \beta\rangle \circ \langle \mathrm{I}_{\mathbf{Sign}}, \pi^{\sim'}\rangle$.

If $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\phi \sim'_\Sigma \psi$, then $\phi \lesssim'_\Sigma \psi$ and $\psi \lesssim'_\Sigma \phi$. Therefore, by the hypothesis, $\phi\ \alpha_\Sigma^{-1}(\lesssim''_{F(\Sigma)})\ \psi$ and $\psi\ \alpha_\Sigma^{-1}(\lesssim''_{F(\Sigma)})\ \phi$. Therefore, we get that $\alpha_\Sigma(\phi) \lesssim''_{F(\Sigma)} \alpha_\Sigma(\psi)$ and $\alpha_\Sigma(\psi) \lesssim''_{F(\Sigma)} \alpha_\Sigma(\phi)$, which yields $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$ and, hence, β_Σ is indeed well-defined.
Finally, if $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\phi/{\sim'_\Sigma} \lesssim'_\Sigma/{\sim'_\Sigma}\ \psi/{\sim'_\Sigma}$, we obtain, by the definition of $\lesssim'/{\sim'}$, $\phi \lesssim'_\Sigma \psi$, whence $\phi\ \alpha_\Sigma^{-1}(\lesssim''_{F(\Sigma)})\ \psi$ and, thus, $\alpha_\Sigma(\phi) \lesssim''_{F(\Sigma)} \alpha_\Sigma(\psi)$ and, therefore, by the definition of β, $\beta_\Sigma(\phi/{\sim'_\Sigma}) \lesssim''_{F(\Sigma)} \beta_\Sigma(\psi/{\sim'_\Sigma})$. Thus, $\langle G, \beta\rangle$ is an order translation.

□

COROLLARY 16 (Order Isomorphism Theorem). *Suppose that* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ and $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ are two functors, N, N' categories of natural transformations on $\mathrm{SEN}, \mathrm{SEN}'$, respectively, ρ, ρ' polarities for N, N', respectively, $\lesssim, \lesssim'$, a ρ-posystem on SEN and a ρ'-posystem on SEN', respectively. Let $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \lesssim\rangle \to^p \langle \mathrm{SEN}', \lesssim'\rangle$ be a surjective order translation, with $F : \mathbf{Sign} \to \mathbf{Sign}'$ an isomorphism, and $\lesssim'' = \alpha^{-1}(\lesssim') = \mathrm{OrdKer}(\langle F, \alpha\rangle)$. Then $\langle \mathrm{SEN}, \lesssim\rangle/{\lesssim''} := \langle \mathrm{SEN}/{\sim''}, \lesssim''/{\sim''}\rangle \cong^p \langle \mathrm{SEN}', \lesssim'\rangle$. Moreover, the order isomorphism is realized by $\langle G, \beta\rangle : \langle \mathrm{SEN}/{\sim''}, \lesssim''/{\sim''}\rangle \cong^p \langle \mathrm{SEN}', \lesssim'\rangle$, such that

$$\beta_\Sigma(\phi/{\sim''_\Sigma}) = \alpha_\Sigma(\phi), \quad \text{for all} \quad \Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma).$$

Proof. $\langle F, \alpha\rangle$ is surjective and, by Theorem 15, the translation $\langle G, \beta\rangle : \mathrm{SEN}^{\sim''} \to \mathrm{SEN}'$, defined as in Theorem 15, by $G = F$ and, for all $\Sigma \in |\mathbf{Sign}|$, by

$$\beta_\Sigma(\phi/{\sim''_\Sigma}) = \alpha_\Sigma(\phi), \quad \text{for all} \quad \phi \in \mathrm{SEN}(\Sigma),$$

is an order translation $\langle G, \beta\rangle : \langle \mathrm{SEN}^{\sim''}, \lesssim''/{\sim''}\rangle \to \langle \mathrm{SEN}', \lesssim'\rangle$, such that $\langle F, \alpha\rangle = \langle G, \beta\rangle \circ \langle \mathrm{I}_{\mathbf{Sign}}, \pi^{\sim''}\rangle$. To see that this is in fact an order isomorphism, it suffices to show that it is an order monomorphism. According to Proposition 12 this may be achieved by showing that the order kernel of $\langle G, \beta\rangle$ is $\lesssim''/{\sim''}$. We do indeed have, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll} \phi/{\sim''_\Sigma}\ \beta_\Sigma^{-1}(\lesssim'_{F(\Sigma)})\ \psi/{\sim''_\Sigma} & \text{iff} & \beta_\Sigma(\phi/{\sim''_\Sigma}) \lesssim'_{F(\Sigma)} \beta_\Sigma(\psi/{\sim''_\Sigma}) \\ & \text{iff} & \alpha_\Sigma(\phi) \lesssim'_{F(\Sigma)} \alpha_\Sigma(\psi) \\ & \text{iff} & \phi\ \alpha_\Sigma^{-1}(\lesssim'_{F(\Sigma)})\ \psi \\ & \text{iff} & \phi \lesssim''_\Sigma \psi \\ & \text{iff} & \phi/{\sim''_\Sigma} \lesssim''_\Sigma/{\sim''_\Sigma}\ \psi/{\sim''_\Sigma}. \end{array}$$

□

Next, as a preparation for the Order Correspondence Theorem, it is shown that the collection of all ρ-qosystems on a given ρ-pofunctor $\langle \mathrm{SEN}, \lesssim\rangle$ forms a complete lattice under the signature-wise inclusion $\leq$. Meet is signature-wise intersection.

THEOREM 17. *Suppose that* SEN : **Sign** $\to$ **Set** *is a functor, N a category of natural transformations on* SEN, ρ *a polarity for N and* $\lesssim$ *a* ρ*-posystem on* SEN. *Then* $\langle \mathrm{QoSys}_\rho(\langle \mathrm{SEN}, \lesssim\rangle), \leq\rangle$ *is a complete algebraic lattice.*

Proof. To show that $\langle \mathrm{QoSys}_\rho(\langle \mathrm{SEN}, \lesssim\rangle), \leq\rangle$ is a complete lattice, it suffices to show that it is closed under arbitrary signature-wise intersections. Notice that, if $\mathcal{K}$ is a collection of ρ-qosystems on SEN containing $\lesssim$, then $\bigcap \mathcal{K}$ is also a ρ-qosystem on SEN as well and it also contains $\lesssim$. Therefore $\bigcap \mathcal{K} \in \mathrm{QoSys}_\rho(\langle \mathrm{SEN}, \lesssim\rangle)$.

Finally, to show algebraicity, consider an upward directed subfamily $\mathcal{K}$ of $\mathrm{QoSys}_\rho(\langle \mathrm{SEN}, \lesssim\rangle)$. Then, it is not difficult to see that $\bigcup \mathcal{K}$ is also a ρ-qosystem on SEN and it definitely contains $\lesssim$. Hence it is a member of $\mathrm{QoSys}_\rho(\langle \mathrm{SEN}, \lesssim\rangle)$ and $\langle \mathrm{QoSys}_\rho(\langle \mathrm{SEN}, \lesssim\rangle), \leq\rangle$ is algebraic. □

Joins in the complete lattice $\mathbf{QoSys}_\rho(\langle \mathrm{SEN}, \lesssim\rangle) = \langle \mathrm{QoSys}_\rho(\langle \mathrm{SEN}, \lesssim\rangle), \leq\rangle$ are given by signature-wise unions of arbitrary composites, as detailed by the following proposition, forming an analog of Theorem 2.16 of [15].

PROPOSITION 18. *Suppose that* SEN : **Sign** $\to$ **Set** *is a functor, N a category of natural transformations on* SEN, ρ *a polarity for N and* $\lesssim$ *a* ρ*-posystem on* SEN . *If* $\lesssim', \lesssim'' \in \mathrm{QoSys}_\rho(\langle \mathrm{SEN}, \lesssim\rangle)$, *then*

$$\lesssim' \vee \lesssim'' = \Big\{ \bigcup_{n<\omega} (\lesssim'_\Sigma ; \lesssim''_\Sigma)^n \Big\}_{\Sigma \in |\mathbf{Sign}|},$$

where $\vee$ *is the join in the complete algebraic lattice* $\mathbf{QoSys}_\rho(\langle \mathrm{SEN}, \lesssim\rangle)$.

Proof. It is clear that, for all $\Sigma \in |\mathbf{Sign}|$, $\bigcup_{n<\omega}(\lesssim'_\Sigma ; \lesssim''_\Sigma)^n$ is the least quasi-ordering on $\mathrm{SEN}(\Sigma)$ that contains both $\lesssim'_\Sigma$ and $\lesssim''_\Sigma$. Therefore the result will follow once it is shown that $\{\bigcup_{n<\omega}(\lesssim'_\Sigma ; \lesssim''_\Sigma)^n\}_{\Sigma \in |\mathbf{Sign}|}$ is a relation system and that it satisfies ρ-tonicity.

To show that $\{\bigcup_{n<\omega}(\lesssim'_\Sigma ; \lesssim''_\Sigma)^n\}_{\Sigma \in |\mathbf{Sign}|}$ is a relation system, consider $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$. Let $\phi, \psi \in \mathrm{SEN}(\Sigma_1)$, such that $\phi(\lesssim'_{\Sigma_1} ; \lesssim''_{\Sigma_1})^n \psi$. Thus, there exist $\chi_0, \ldots, \chi_{2n} \in \mathrm{SEN}(\Sigma_1)$, such that

$$\phi = \chi_0 \lesssim'_{\Sigma_1} \chi_1 \lesssim''_{\Sigma_1} \chi_2 \lesssim'_{\Sigma_1} \cdots \lesssim'_{\Sigma_1} \chi_{2n-1} \lesssim''_{\Sigma_1} \chi_{2n} = \psi.$$

But $\lesssim'$ and $\lesssim''$ are both qosystems on SEN, whence it follows that

$$\mathrm{SEN}(f)(\phi) \lesssim'_{\Sigma_2} \mathrm{SEN}(f)(\chi_1) \lesssim''_{\Sigma_2} \mathrm{SEN}(f)(\chi_2) \lesssim'_{\Sigma_2} \cdots \\ \cdots \lesssim'_{\Sigma_2} \mathrm{SEN}(f)(\chi_{2n-1}) \lesssim''_{\Sigma_2} \mathrm{SEN}(f)(\psi).$$

Therefore $\mathrm{SEN}(f)(\phi)\ (\lesssim'_{\Sigma_2} ; \lesssim''_{\Sigma_2})^n\ \mathrm{SEN}(f)(\psi)$ and, hence, the relation system $\{\bigcup_{n<\omega}(\lesssim'_\Sigma ; \lesssim''_\Sigma)^n\}_{\Sigma \in |\mathbf{Sign}|}$ is indeed a qosystem on SEN.

Finally, for ρ-tonicity, let us again restrict to a $\sigma : \mathrm{SEN}^2 \to \mathrm{SEN}$ in N, such that $\rho(\sigma, 0) = +$. The general case of a possibly negative polarity and of more arguments may be handled similarly. Suppose that $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\phi(\precsim'_\Sigma; \precsim''_\Sigma)^n\psi$. Then, there exist, as before, $\chi_0, \ldots, \chi_{2n} \in \mathrm{SEN}(\Sigma)$, such that

$$\phi = \chi_0 \precsim'_\Sigma \chi_1 \precsim''_\Sigma \chi_2 \precsim'_\Sigma \cdots \precsim'_\Sigma \chi_{2n-1} \precsim''_\Sigma \chi_{2n} = \psi.$$

Therefore, for all $\chi \in \mathrm{SEN}(\Sigma)$,

$$\begin{aligned}\sigma_\Sigma(\phi, \chi) \precsim'_\Sigma \sigma_\Sigma(\chi_1, \chi) \precsim''_\Sigma \sigma_\Sigma(\chi_2, \chi) \precsim'_\Sigma \cdots \\ \precsim'_\Sigma \sigma_\Sigma(\chi_{2n-1}, \chi) \precsim''_\Sigma \sigma_\Sigma(\psi, \chi).\end{aligned}$$

Thus $\sigma_\Sigma(\phi, \chi)\ (\precsim'_\Sigma; \precsim''_\Sigma)^n\ \sigma_\Sigma(\psi, \chi)$, which, according to Lemma 2, verifies ρ-tonicity (in the first argument). □

The ground has now been prepared for the final result of the present work, the Order Correspondence Theorem, establishing an isomorphism between the qosystems of the quotient of a pofunctor with the qosystems of the pofunctor containing the qosystem over which the quotient is taken. Theorem 19 abstracts Theorem 2.17 of [15].

THEOREM 19 (Order Correspondence Theorem). *Suppose that* SEN : **Sign** → **Set** *is a functor, N a category of natural transformations on* SEN, *ρ a polarity for N*, *$\precsim$ a ρ-posystem on* SEN *and*

$$\precsim' \in \mathrm{QoSys}_\rho(\langle \mathrm{SEN}, \precsim\rangle)$$

a ρ-qosystem of $\langle \mathrm{SEN}, \precsim\rangle$. *Then, for every $\rho^{\sim'}$-qosystem $\preceq''$ of* $\langle \mathrm{SEN}/{\sim'}, \precsim'/{\sim'}\rangle$, *there exists a unique* $\precsim'' \in \mathrm{QoSys}_\rho(\langle \mathrm{SEN}, \precsim\rangle)$, *such that* $\preceq'' = \precsim''/{\sim'}$.

Moreover, the mapping $\preceq'' \mapsto \precsim''$ is a lattice isomorphism between $\mathbf{QoSys}_{\rho^{\sim'}}(\langle \mathrm{SEN}/{\sim'}, \precsim'/{\sim'}\rangle)$ *and the principal filter of* $\mathbf{QoSys}_\rho(\langle \mathrm{SEN}, \precsim\rangle)$ *generated by $\precsim'$.*

Proof. Suppose that $\preceq'' \in \mathrm{QoSys}_{\rho^{\sim'}}(\langle \mathrm{SEN}/{\sim'}, \precsim'/{\sim'}\rangle)$, i.e., $\preceq''$ is a $\rho^{\sim'}$-qosystem on $\mathrm{SEN}/{\sim'}$, such that $\precsim'/{\sim'} \leq \preceq''$. Define the relation family $\precsim'' = \{\precsim''_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\precsim''_\Sigma = \{\langle \phi, \psi\rangle \in \mathrm{SEN}(\Sigma)^2 : \phi/{\sim'_\Sigma} \preceq''_\Sigma \psi/{\sim'_\Sigma}\}.$$

It is not difficult to verify that $\precsim''$ is a ρ-qosystem of $\langle \mathrm{SEN}, \precsim'\rangle$. One needs to check that $\precsim''_\Sigma$ is a quasi-ordering on $\mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$, that $\precsim''$ is a relation system on SEN and that $\precsim' \leq \precsim''$.

Since $\preceq'' \in \mathrm{QoSys}_{\rho^{\sim'}}(\langle \mathrm{SEN}/{\sim'}, \precsim'/{\sim'}\rangle)$, we have, for all $\Sigma \in |\mathbf{Sign}|$, $\phi \in \mathrm{SEN}(\Sigma)$, $\phi/{\sim'_\Sigma} \preceq''_\Sigma \phi/{\sim'_\Sigma}$, whence $\phi \precsim''_\Sigma \phi$ and $\precsim''$ is reflexive. Moreover, if $\Sigma \in |\mathbf{Sign}|, \phi, \psi, \chi \in \mathrm{SEN}(\Sigma)$, such that $\phi \precsim''_\Sigma \psi$ and $\psi \precsim''_\Sigma \chi$, then we have that $\phi/{\sim'_\Sigma} \preceq''_\Sigma \psi/{\sim'_\Sigma}$ and $\psi/{\sim'_\Sigma} \preceq''_\Sigma \chi/{\sim'_\Sigma}$ and, therefore, $\phi/{\sim'_\Sigma} \preceq''_\Sigma \chi/{\sim'_\Sigma}$, whence $\phi \precsim''_\Sigma \chi$, which shows that $\precsim''$ is also transitive.

To see that $\lesssim''$ is a qosystem, suppose that $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$ and $\phi, \psi \in \mathrm{SEN}(\Sigma_1)$, such that $\phi \lesssim''_{\Sigma_1} \psi$. Then $\phi/{\sim'_{\Sigma_1}} \preceq''_{\Sigma_1} \psi/{\sim'_{\Sigma_1}}$, whence, we obtain $\mathrm{SEN}^{\sim'}(f)(\phi/{\sim'_{\Sigma_1}}) \preceq''_{\Sigma_2} \mathrm{SEN}^{\sim'}(f)(\psi/{\sim'_{\Sigma_1}})$, i.e., $\mathrm{SEN}(f)(\phi)/{\sim'_{\Sigma_2}} \preceq''_{\Sigma_2} \mathrm{SEN}(f)(\psi)/{\sim'_{\Sigma_2}}$. Therefore, $\mathrm{SEN}(f)(\phi) \lesssim''_{\Sigma_2} \mathrm{SEN}(f)(\psi)$ and $\lesssim''$ is a qosystem on SEN.

Finally, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$, we have

$$\begin{array}{rcl} \phi \lesssim'_{\Sigma} \psi & \text{iff} & \phi/{\sim'_{\Sigma}} \lesssim'_{\Sigma}/{\sim'_{\Sigma}} \psi/{\sim'_{\Sigma}} \\ & \text{implies} & \phi/{\sim'_{\Sigma}} \preceq''_{\Sigma} \psi/{\sim'_{\Sigma}} \\ & \text{iff} & \phi \lesssim''_{\Sigma} \psi. \end{array}$$

Thus, $\lesssim'' \in \mathrm{QoSys}_{\rho}(\langle \mathrm{SEN}, \lesssim' \rangle)$.

Next, let $\lesssim'' \in \mathrm{QoSys}_{\rho}(\langle \mathrm{SEN}, \lesssim' \rangle)$. Then $\sim'$ is compatible with $\lesssim''$ and, therefore, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$, $\phi/{\sim'_{\Sigma}} \lesssim''_{\Sigma}/{\sim'_{\Sigma}} \psi/{\sim'_{\Sigma}}$ if and only if $\phi \lesssim''_{\Sigma} \psi$. Thus, $\lesssim''/{\sim'} = \{\langle \phi/{\sim'_{\Sigma}}, \psi/{\sim'_{\Sigma}} \rangle : \langle \phi, \psi \rangle \in \lesssim''_{\Sigma}\}_{\Sigma \in |\mathbf{Sign}|}$ is a $\rho^{\sim'}$-qosystem on $\mathrm{SEN}/{\sim'}$ and $\lesssim'/{\sim'} \leq \lesssim''/{\sim'}$. If $\lesssim'', \lesssim''' \in \mathrm{QoSys}_{\rho}(\langle \mathrm{SEN}, \lesssim \rangle)$, such that $\lesssim''/{\sim'} = \lesssim'''/{\sim'}$, then, obviously, $\lesssim'' = \lesssim'''$. Hence, the mapping $\preceq'' \mapsto \lesssim''$ is a bijection between $\mathrm{QoSys}_{\rho}(\langle \mathrm{SEN}/{\sim'}, \lesssim'/{\sim'} \rangle)$ and the subcollection of all qosystems $\lesssim'' \in \mathrm{QoSys}_{\rho}(\langle \mathrm{SEN}, \lesssim \rangle)$, such that $\lesssim' \leq \lesssim''$. Note that this bijection and its inverse are obviously $\leq$-order-preserving. $\square$

We intend to continue the work reported in this paper to cover additional issues concerning the algebraization of π-institutions via algebraic systems focusing on an abstraction of logical implication rather than of logical equivalence. This has been the major motivation for the work presented in [15], as was elaborated on in the Introduction.

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