George Voutsadakis

Categorical Abstract Algebraic Logic: Prealgebraicity and Protoalgebraicity

Abstract. Two classes of π-institutions are studied whose properties are similar to those of the protoalgebraic deductive systems of Blok and Pigozzi. The first is the class of N-protoalgebraic π-institutions and the second is the wider class of N-prealgebraic π-institutions. Several characterizations are provided. For instance, N-prealgebraic π-institutions are exactly those π-institutions that satisfy monotonicity of the N-Leibniz operator on theory systems and N-protoalgebraic π-institutions those that satisfy monotonicity of the N-Leibniz operator on theory families. Analogs of the correspondence property of Blok and Pigozzi for π-institutions are also introduced and their connections with pre- and protoalgebraicity are explored. Finally, relations of these two classes with the $(\mathcal{I}, N)$-algebraic systems, introduced previously by the author as an analog of the $\mathcal{S}$-algebras of Font and Jansana, and with an analog of the Suszko operator of Czelakowski for π-institutions are also investigated.

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1. Introduction

One of the most important contributions of the theory of *abstract algebraic logic*, as developed by Czelakowski, Blok and Pigozzi and Font and Jansana, among others, has been the use of the Leibniz operator [3] in classifying logics in different steps of a hierarchy that roughly reflects the extent to which a logic may be studied using algebraic methods. The most important steps in this hierarchy, known as the Leibniz hierarchy, are the classes of protoalgebraic [2] (see also [7]), equivalential [6] and algebraizable [3] logics. The book by Czelakowski [8] and the survey article by Font, Jansana and Pigozzi [14] provide overviews of the theory and the Leibniz hierarchy.

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A schematic representation of the different main levels is given below, where an arrow represents an inclusion relation between two classes.

$$\begin{tikzcd}
 & \text{finitely algebraizable} \arrow[dl] \arrow[dr] & & \\
\text{finitely equivalential} \arrow[dr] & & \text{algebraizable} \arrow[dl] \arrow[dr] & \\
 & \text{equivalential} \arrow[dr] & & \text{weakly algebraizable} \arrow[dl] \\
 & & \text{protoalgebraic} &
\end{tikzcd}$$

As shown in the diagram, the class of protoalgebraic logics is the widest of all classes and is located at the bottom of the Leibniz hierarchy. There is, by now, enough evidence in the literature to suggest, as Font, Jansana and Pigozzi put it in [14], that the protoalgebraic logics "constitute the main class of logics for which the advanced methods of universal algebra can be applied to their matrices to give strong and interesting results".

A taste of the strength of these results may be acquired by mentioning a few characterizing properties of protoalgebraic logics. First, by their original definition [2], a sentential logic $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ is protoalgebraic if, for every theory T of $\mathcal{S}$, and all $\mathcal{L}$-formulas $\phi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\langle \phi, \psi \rangle \in \Omega(T) \quad \text{implies} \quad T, \phi \vdash_{\mathcal{S}} \psi,$$

i.e., if two formulas are congruent modulo the Leibniz congruence $\Omega(T)$, then they are interderivable modulo the theory T, for every theory T of $\mathcal{S}$.

Another important property that turns out to be equivalent to protoalgebraicity is the monotonicity of the Leibniz operator on the theories of the logic. More precisely, $\mathcal{S}$ is protoalgebraic iff, for all theories T_1, T_2 of $\mathcal{S}$,

$$T_1 \subseteq T_2 \quad \text{implies} \quad \Omega(T_1) \subseteq \Omega(T_2).$$

Yet another characterization, closely related to the monotonicity of the Leibniz operator, is meet-continuity of the Leibniz operator on the theories of the logic. That is, $\mathcal{S}$ is protoalgebraic iff, for every collection $\{T_i\}_{i\in I}$ of theories of $\mathcal{S}$,

$$\bigcap_{i\in I}\Omega(T_i) = \Omega(\bigcap_{i\in I} T_i).$$

A more algebraic consequence of protoalgebraicity is that, for every $\mathcal{L}$-algebra $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}}\rangle$, the lattice of all $\mathcal{S}$-filters on $\mathbf{A}$ including a given $\mathcal{S}$-filter F is isomorphic to the lattice of all $\mathcal{S}$-filters on the quotient algebra $\mathbf{A}/\Omega_{\mathbf{A}}(F)$ including the $\mathcal{S}$-filter $F/\Omega_{\mathbf{A}}(F)$.

In terms of the Tarski operator $\widetilde{\Omega}$, introduced by Font and Jansana in [13], the protoalgebraicity of a sentential logic $\mathcal{S}$ may be characterized by the property that, for every theory T of $\mathcal{S}$,

$$\widetilde{\Omega}(\mathcal{S}^T) = \Omega(T),$$

where $\mathcal{S}^T = \langle \mathcal{L}, \vdash_{\mathcal{S}^T}\rangle$, is defined, for every $\Phi \cup \{\phi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, by

$$\Phi \vdash_{\mathcal{S}^T} \phi \quad \text{iff} \quad T, \Phi \vdash_{\mathcal{S}} \phi.$$

This characterization has the important consequence that $\mathcal{S}$ is protoalgebraic iff for any $\mathcal{L}$-algebra $\mathbf{A}$, every full model of $\mathcal{S}$ over $\mathbf{A}$ consists of a principal filter of the lattice of all $\mathcal{S}$-filters on $\mathbf{A}$ (Theorem 3.4 of [13]).

The present paper proposes an extension of the aforementioned results in order to cover the case of logical systems formalized as π-institutions [12] (see also [15, 16]). It is the first paper dealing directly with introducing an analog for the π-institution framework of a specific class of the existing Leibniz hierarchy of sentential logics. However, it is one of the concluding papers of a long series undertaken by the author with the goal of adapting general definitions, methods and results widely applied to the theory of algebraizability of deductive systems to π-institutions. In [18] (see also [19, 20, 21]) the first elements of the use of abstract algebraic logic methods for studying the algebraizability of institutional logics were presented. These included the notion of deductive equivalence between π-institutions, that parallels the notion of equivalence for k-deductive systems of [4], and a characterization of deductive equivalence using the theory categories of the π-institutions, also adapted by similar results of [3] on algebraizable deductive systems. It also included the notion of an algebraic π-institution, based on concepts borrowed from categorical algebra, and that of an algebraizable π-institution. However, no analog of the Leibniz or the Tarski operator had been introduced and this made the efforts for creating close analogs of the main results of the theory of algebraizable deductive systems for π-institutions impossible. The paper that did introduce such an analog for the first time was [22] (see also [24] for an attempt at generalizing). After the theory was equipped with a Tarski operator, a series of results unfolded, paralleling the development of the theory of general algebraic semantics for sentential logics of [13]. In [23], the main elements of the model theory of π-institutions were presented from the point of view of a categorical algebraizability theory for π-institutions. Very important among these, were the notions of a min and of a full model of a π-institution, both borrowed by [13] and adapted to the categorical theory. Min models were called basic full models by Font and Jansana and the name was only modified for mnemonic purposes, but the close analogy remains. In [25], the notion of an $(\mathcal{I}, N)$-algebraic system was introduced, paralleling the notion of an $\mathcal{S}$-algebra from the sentential logic framework. This made possible the formulation of a theorem relating full models with appropriately chosen congruence systems (Theorem 13 of [25]), adapting the main Isomorphism Theorem 2.30 of [13]. Finally, in [27], the full models of a π-institution and their relation with various metalogical properties were studied.

All the machinery developed in the theory so far is now put into motion in order to classify π-institutions into classes forming a categorical Leibniz hierarchy, paralleling the classification of deductive systems offered by abstract algebraic logic. In the present work, an analog of the lowest step, the class of protoalgebraic deductive systems, is introduced for π-institutions and several of the properties established for that class are either directly adapted and shown to be true for protoalgebraic π-institutions or are modified and explored inside the π-institution framework. There is good evidence from the results established here (combined with the evidence gathered over the last two to three decades in the deductive system framework) that this may be the right class to be placed at the bottom of an institutional Leibniz hierarchy.

Next, a brief outline of the contents of the paper is presented. In Section 2, the definition of the Tarski N-congruence system of a π-institution $\mathcal{I}$ [22] is adapted to obtain the *Leibniz N-congruence system* of $\mathcal{I}$ associated with a given axiom family of $\mathcal{I}$. These two are in a relation similar to the one between Tarski and Leibniz congruences of sentential logics. A characterization of the Leibniz N-congruence system of an axiom family is given, adapting the well-known characterization of the Leibniz congruence of Blok and Pigozzi [3] and the subsequent characterization of the Tarski operator of π-institutions of [22] that was inspired by it. In Section 3, the classes of *N-prealgebraic* and *N-protoalgebraic* π-institutions are defined. The definitions follow the definition of protoalgebraic logics of Blok and Pigozzi [2]. The reason why two classes, instead of a single class, are investigated in the π-institution context is that one may consider interderivability of two given sentences relative to all *theory families* (N-protoalgebraicity) or just relative to all *theory systems* (N-prealgebraicity), which form a subclass of the class of all theory families. Accordingly, N-protoalgebraicity is characterized by the monotonicity of the Leibniz N-congruence system operator on all theory families of $\mathcal{I}$, whereas N-prealgebraicity is characterized by the monotonicity of the Leibniz N-congruence system operator on the sublattice of theory systems. In Section 4, the correspondence property of Blok and Pigozzi [2] is adapted to π-institutions and the *family N-correspondence* and the *system N-correspondence properties* are introduced, together with their *strong versions*, also reflecting the aforementioned duality between families and systems. It is shown that the strong system N-correspondence property implies N-prealgebraicity, that the strong family N-correspondence property implies N-protoalgebraicity and, finally, that N-protoalgebraicity implies the family N-correspondence property in the interesting special case of a finitary N-rule based π-institution, a concept first introduced in [31]. Besides establishing this hierarchy of π-institutions, another interesting result of the investigations in Section 4 is that, given a finitary N-rule based N-protoalgebraic π-institution $\mathcal{I}$ and a theory system T of $\mathcal{I}$, the lattice of all theory families of $\mathcal{I}$ containing T is isomorphic to the lattice of all theory families of $\mathcal{I}^T/\Omega^N(T)$, an analog of a well-known property of protoalgebraic logics (see [2] and, also, the discussion on page 55 of [13]). In Section 5, a result of Font and Jansana relating the class $\mathrm{Alg}(\mathcal{S})$ of $\mathcal{S}$-algebras, i.e., algebra reducts of reduced full models, and the class $\mathrm{Alg}^*(\mathcal{S})$ of the algebra reducts of the reduced matrices of a given sentential logic $\mathcal{S}$, is explored in the π-institution framework. $(\mathcal{I},N)$-algebraic systems, introduced in [25] by the author as analogs of $\mathcal{S}$-algebras, come in handy, but an extension of the notion of an algebra in $\mathrm{Alg}^*(\mathcal{S})$ is also needed. This is introduced in the notion of an *$(\mathcal{I},N)^*$-algebraic system.* Finally, in Section 6, an analog of the *Suszko operator* of Czelakowski [8] is introduced and its relation to the Leibniz N-congruence system operator is explored for N-protoalgebraic π-institutions. This provides an analog of Theorem 1.5.4 of [8].

Finally, it is mentioned that further properties of N-protoalgebraic π-institutions are investigated in a companion paper [28]. There, the present line of research is continued with analogs of more syntactically oriented properties of sentential logics that are connected to protoalgebraicity in various ways. Following similar leads from the general theory of sentential logics, analogs of the classes of equivalential logics of Czelakowski [6] and of weakly algebraizable logics of Czelakowski and Jansana [11] for the π-institution framework are introduced and investigated in [30] and [29], respectively.

For all unexplained categorical notation, the reader is referred to any of the standard references [1, 5, 17].

2. Leibniz Congruence Systems

To introduce Leibniz N-congruence systems, which will play a crucial role in defining protoalgebraic π-institutions, we first recall the notion of a Tarski N-congruence system and its characterization from [22].

Let **Sign** be a category and SEN : **Sign** $\rightarrow$ **Set** a functor. The **clone of all natural transformations on** SEN is defined to be the locally small category with collection of objects $\{\mathrm{SEN}^\alpha : \alpha \text{ an ordinal}\}$ and collection of morphisms $\tau : \mathrm{SEN}^\alpha \rightarrow \mathrm{SEN}^\beta$ β-sequences of natural transformations $\tau_i : \mathrm{SEN}^\alpha \rightarrow \mathrm{SEN}$. Composition

$$\mathrm{SEN}^\alpha \xrightarrow{\langle \tau_i : i < \beta\rangle} \mathrm{SEN}^\beta \xrightarrow{\langle \sigma_j : j < \gamma\rangle} \mathrm{SEN}^\gamma$$

is defined by

$$\langle \sigma_j : j < \gamma\rangle \circ \langle \tau_i : i < \beta\rangle = \langle \sigma_j(\langle \tau_i : i < \beta\rangle) : j < \gamma\rangle.$$

A subcategory N of this category containing *all* objects of the form SEN^k for $k < \omega$, and all projection morphisms $p^{k,i} : \mathrm{SEN}^k \rightarrow \mathrm{SEN}, i < k, k < \omega$, with $p^{k,i}_\Sigma : \mathrm{SEN}(\Sigma)^k \rightarrow \mathrm{SEN}(\Sigma)$ given by

$$p^{k,i}_\Sigma(\vec{\phi}) = \phi_i, \quad \text{for all} \quad \vec{\phi} \in \mathrm{SEN}(\Sigma)^k,$$

and such that, for every family $\{\tau_i : \mathrm{SEN}^k \rightarrow \mathrm{SEN} : i < l\}$ of natural transformations in N, the sequence $\langle \tau_i : i < l\rangle : \mathrm{SEN}^k \rightarrow \mathrm{SEN}^l$ is also in N, is referred to as a **category of natural transformations on** SEN.

Let **Sign** be a category, SEN : **Sign** $\rightarrow$ **Set** be a functor and N be a category of natural transformations on SEN. Given $\Sigma \in |\mathbf{Sign}|$, an equivalence relation θ_Σ on $\mathrm{SEN}(\Sigma)$ is said to be an N-**congruence** if, for all $\sigma : \mathrm{SEN}^k \rightarrow \mathrm{SEN}$ in N and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)^k$,

$$\vec{\phi}\ \theta^k_\Sigma\ \vec{\psi} \quad \text{imply} \quad \sigma_\Sigma(\vec{\phi})\ \theta_\Sigma\ \sigma_\Sigma(\vec{\psi}).$$

A collection $\theta = \{\langle \Sigma, \theta_\Sigma\rangle : \Sigma \in |\mathbf{Sign}|\}$ is called an **equivalence system** of SEN if

- θ_Σ is an equivalence relation on $\mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$,
- $\mathrm{SEN}(f)^2(\theta_{\Sigma_1}) \subseteq \theta_{\Sigma_2}$, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$.

If, in addition, N is a category of natural transformations on SEN and θ_Σ is an N-congruence, for all $\Sigma \in |\mathbf{Sign}|$, then θ is said to be an N-**congruence system** of SEN. By $\mathrm{Con}^N(\mathrm{SEN})$ is denoted the collection of all N-congruence systems of SEN.

Let now $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}\rangle$ be a π-institution. An equivalence system θ of SEN is called a **logical equivalence system** of $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi, \psi\rangle \in \theta_\Sigma \quad \text{implies} \quad C_\Sigma(\phi) = C_\Sigma(\psi).$$

An N-congruence system of SEN is a **logical N-congruence system** of $\mathcal{I}$ if it is logical as an equivalence system of $\mathcal{I}$.

It is proven in [22] that the collection of all logical N-congruence systems of a given π-institution $\mathcal{I}$ forms a complete lattice under signature-wise inclusion and the largest element of the lattice is termed the **Tarski N-congruence system** of $\mathcal{I}$ and denoted by $\widetilde{\Omega}^N(\mathcal{I})$. Theorem 4 of [22] fully characterizes the Tarski N-congruence system of a π-institution.

THEOREM 1. [Theorem 4 of [22]] *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}\rangle$ be a π-institution, N a category of natural transformations on* SEN *and $\Sigma \in |\mathbf{Sign}|$. Then $\langle \phi, \psi\rangle \in \widetilde{\Omega}^N_\Sigma(\mathcal{I})$ if and only if, for all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, all natural transformations $\tau : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, all $\vec{\chi} = \langle \chi_0, \ldots, \chi_{k-1}\rangle \in \mathrm{SEN}(\Sigma')^k$, and all $i < k$,*

$$\begin{aligned} &C_{\Sigma'}(\tau_{\Sigma'}(\chi_0, \ldots, \chi_{i-1}, \mathrm{SEN}(f)(\phi), \chi_{i+1}, \ldots, \chi_{k-1})) = \\ &\qquad C_{\Sigma'}(\tau_{\Sigma'}(\chi_0, \ldots, \chi_{i-1}, \mathrm{SEN}(f)(\psi), \chi_{i+1}, \ldots, \chi_{k-1})). \end{aligned} \tag{i}$$

Equation (i) will be abbreviated to

$$C_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi})) = C_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})), \tag{ii}$$

with implicit the understanding that $\mathrm{SEN}(f)(\phi)$, on the left hand side, and $\mathrm{SEN}(f)(\psi)$, on the right hand side, may appear in a different than the first, but in the same, place in both sides of the equation.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution and N a category of natural transformations on SEN. Recall from [27] that a collection $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ of subsets $T_\Sigma \subseteq \mathrm{SEN}(\Sigma), \Sigma \in |\mathbf{Sign}|$, is called an **axiom system** of $\mathcal{I}$ if

$$\mathrm{SEN}(f)(T_{\Sigma_1}) \subseteq T_{\Sigma_2}, \quad \text{for all} \quad \Sigma_1, \Sigma_2 \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma_1, \Sigma_2). \tag{iii}$$

An axiom system is called a **theory system** [26] if, in addition, T_Σ is a Σ-theory, for every $\Sigma \in |\mathbf{Sign}|$.

To refer to a collection $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ of subsets $T_\Sigma \subseteq \mathrm{SEN}(\Sigma)$ that do not necessarily satisfy Condition (iii) we will use the term **axiom family** (and **theory family**, if T_Σ is a Σ-theory, for every $\Sigma \in |\mathbf{Sign}|$).

The collection ThFam($\mathcal{I}$) of theory families of a π-institution $\mathcal{I}$ ordered by signature-wise inclusion forms a complete lattice $\mathbf{ThFam}(\mathcal{I})$ and the same holds for the collection ThSys($\mathcal{I}$) of all theory systems under signature-wise inclusion. Moreover it is not very hard to see that $\mathbf{ThSys}(\mathcal{I})$ forms a complete sublattice of $\mathbf{ThFam}(\mathcal{I})$.

PROPOSITION 2.1. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution. The lattice $\mathbf{ThSys}(\mathcal{I}) = \langle \mathrm{ThSys}(\mathcal{I}), \leq\rangle$ of all theory systems of $\mathcal{I}$ ordered by signature-wise inclusion is a complete sublattice of the complete lattice $\mathbf{ThFam}(\mathcal{I}) = \langle \mathrm{ThFam}(\mathcal{I}), \leq\rangle$ of all theory families of $\mathcal{I}$ ordered by signature-wise inclusion.*

PROOF. It suffices to show that the signature-wise intersection

$$\{\bigcap_{i\in I} T^i_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$$

of a collection $\{T^i : i \in I\}$ of theory systems of $\mathcal{I}$ is also a theory system of $\mathcal{I}$ and that the maximum element of $\mathbf{ThFam}(\mathcal{I})$ is also in ThSys($\mathcal{I}$).

For the intersection, if $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$ and $\phi \in \mathrm{SEN}(\Sigma_1)$, with $\phi \in \bigcap_{i\in I} T^i_{\Sigma_1}$, we get $\phi \in T^i_{\Sigma_1}$, for all $i \in I$, whence, since T^i is a theory system, $\mathrm{SEN}(f)(\phi) \in T^i_{\Sigma_2}$, for all $i \in I$, whence $\mathrm{SEN}(f)(\phi) \in \bigcap_{i\in I} T^i_{\Sigma_2}$ and, therefore $\{\bigcap_{i\in I} T^i_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ is a theory system of $\mathcal{I}$.

As far as the maximum element of $\mathbf{ThFam}(\mathcal{I})$ is concerned, it is the theory family $\{\mathrm{SEN}(\Sigma)\}_{\Sigma\in|\mathbf{Sign}|}$, which is clearly a theory system of $\mathcal{I}$. ∎

An N-congruence system θ on SEN is said to be **compatible with** the axiom family T, if, for all $\Sigma \in |\mathbf{Sign}|$, θ_Σ is compatible with T_Σ in the usual sense, i.e., if, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle\phi, \psi\rangle \in \theta_\Sigma \quad \text{and} \quad \phi \in T_\Sigma \quad \text{implies} \quad \psi \in T_\Sigma.$$

The following proposition asserts that there always exists a largest N-congruence system on SEN that is compatible with a given axiom family T.

PROPOSITION 2.2. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ be a π-institution, $T = \{T_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ an axiom family of $\mathcal{I}$ and N a category of natural transformations on* SEN. *The collection $\mathrm{Con}^N(T)$ of all N-congruence systems of $\mathcal{I}$ that are compatible with T forms a complete lattice $\mathbf{Con}^N(T)$ under signature-wise inclusion. Moreover, $\mathbf{Con}^N(T)$ is a principal ideal of the complete lattice $\mathbf{Con}^N(\mathrm{SEN})$.*

PROOF. If $\theta^i, i \in I$, are in $\mathrm{Con}^N(T)$, then $\bigcap_{i\in I}\theta^i \in \mathrm{Con}^N(T)$, since, it is obviously an equivalence family of SEN, it is preserved by every morphism in **Sign** and by N, since every $\theta^i, i \in I$, is, and, finally, if $\langle\phi,\psi\rangle \in \bigcap_{i\in I}\theta^i_\Sigma$ and $\phi \in T_\Sigma$, then $\langle\phi,\psi\rangle \in \theta^i_\Sigma$, for all $i \in I$, whence, since θ^i is compatible with T, $\psi \in T_\Sigma$ as well. Therefore $\bigcap_{i\in I}\theta^i$ is also compatible with T.

It suffices, therefore, to show that $\mathrm{Con}^N(T)$ has a greatest element. The signature-wise union of every signature-wise directed subset of $\mathrm{Con}^N(T)$ is an upper bound for that subset in $\mathrm{Con}^N(\mathrm{SEN})$ and it is in $\mathrm{Con}^N(T)$, since every member of the subset is. So, by Zorn's Lemma, $\mathrm{Con}^N(T)$ has a maximal element. If $\theta \neq \theta'$ are two such maximal elements, then, it is not difficult to verify that their join η as N-congruence systems of SEN is compatible with T. This, however, contradicts the maximality of θ and θ', since, clearly, $\theta < \eta$ and $\theta' < \eta$. Therefore, the maximal element of $\mathrm{Con}^N(T)$ is a largest element.

That $\mathbf{Con}^N(T)$ is a principal ideal of the complete lattice $\mathbf{Con}^N(\mathrm{SEN})$ follows easily from the proof above. ■

The largest N-congruence system on SEN that is compatible with an axiom family T is called the **Leibniz N-congruence system** of T and is denoted by $\Omega^N(T)$.

The following result is an analog of Theorem 1 characterizing the Leibniz N-congruence system of an axiom family T of a π-institution $\mathcal{I}$. Recall the convention adopted in Equation (ii) as it will be in force in the statement of Proposition 2.3. Let it also be mentioned that the origin of all characterization theorems of this kind may be found in Blok and Pigozzi's characterization theorem of their Leibniz operator [3].

PROPOSITION 2.3. *Suppose $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, N a category of natural transformations on* SEN *and $T = \{T_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ an axiom family of $\mathcal{I}$. Then, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$, $\langle\phi,\psi\rangle \in \Omega^N_\Sigma(T)$ iff, for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma'), \sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N and $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{k-1}$,*

$$\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}) \in T_{\Sigma'} \quad \textit{iff} \quad \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi}) \in T_{\Sigma'}.$$

PROOF. Let $\theta = \{\theta_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ be defined, for all $\Sigma \in |\mathbf{Sign}|$, by

$$\begin{aligned}\theta_\Sigma = \{\langle\phi,\psi\rangle : {} & \sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}) \in T_{\Sigma'} \text{ iff } \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi}) \in T_{\Sigma'},\\ & \text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma'), \sigma : \mathrm{SEN}^k \to \mathrm{SEN} \text{ in } N,\\ & \vec{\chi} \in \mathrm{SEN}(\Sigma')^{k-1}\}\end{aligned}$$

θ_Σ is obviously an equivalence on $\mathrm{SEN}(\Sigma)$, for every $\Sigma \in |\mathbf{Sign}|$. Moreover, it is not difficult to see that it is an N-congruence, since, for all $\phi_0, \ldots, \phi_{n-1}, \psi_0, \ldots, \psi_{n-1} \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi_i, \psi_i \rangle \in \theta_\Sigma, i < n$, $\tau : \mathrm{SEN}^n \to \mathrm{SEN}$ in N and all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma'), \sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{k-1}$, we have

$$\begin{aligned} \sigma_{\Sigma'}(\mathrm{SEN}(f)(\tau_\Sigma(\vec{\phi})), \vec{\chi}) \in T_{\Sigma'} \quad & \text{iff} \quad \sigma_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}(f)^n(\vec{\phi})), \vec{\chi}) \in T_{\Sigma'} \\ & \text{iff} \quad \sigma_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}(f)^n(\vec{\psi})), \vec{\chi}) \in T_{\Sigma'} \\ & \text{iff} \quad \sigma_{\Sigma'}(\mathrm{SEN}(f)(\tau_\Sigma(\vec{\psi})), \vec{\chi}) \in T_{\Sigma'}. \end{aligned}$$

θ is an N-congruence system, since it is preserved, by definition, by every morphism in **Sign** and it is easily seen to be compatible with T. Therefore, since $\Omega^N(T)$ is the largest N-congruence system with this property, we get that $\theta \leq \Omega^N(T)$.

To show the reverse system of inclusions, suppose $\Sigma \in |\mathbf{Sign}|, \langle \phi, \psi \rangle \in \Omega^N_\Sigma(T)$. Let $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma'), \sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N and $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{k-1}$. Then, since $\Omega^N(T)$ is, by definition, an N-congruence system of $\mathcal{I}$, we get that $\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi) \rangle \in \Omega^N_{\Sigma'}(T)$. Thus, since $\Omega^N_{\Sigma'}(T)$ is an N-congruence on $\mathrm{SEN}(\Sigma')$, we obtain

$$\langle \sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi}), \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}) \rangle \in \Omega^N_{\Sigma'}(T).$$

Therefore, since $\Omega^N(T)$ is compatible with T, we, finally, obtain

$$\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi}) \in T_{\Sigma'} \quad \text{iff} \quad \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}) \in T_{\Sigma'}.$$

Hence $\langle \phi, \psi \rangle \in \theta_\Sigma$ and $\Omega^N(T) \leq \theta$. ∎

Proposition 2.3, combined with Theorem 1, has the following interesting corollary that relates the Tarski N-congruence of a π-institution with the collection of all Leibniz N-congruences associated with its theory families. This fact was already established by Font and Jansana for sentential logics in Proposition 1.2 of [13] immediately following their introduction of the Tarski operator for sentential logics.

COROLLARY 2.4. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ be a π-institution and N a category of natural transformations on* SEN. *Then*

$$\widetilde{\Omega}^N(\mathcal{I}) = \bigcap_{T \in \mathrm{ThFam}(\mathcal{I})} \Omega^N(T).$$

3. N-Protoalgebraic π-Institutions

Recall from Section 3 of [26] (see also [27]) that, given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ and an axiom system $T = \{T_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ of $\mathcal{I}$, the closure system C^T on SEN is defined, for all $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, by

$$\phi \in C^T_\Sigma(\Phi) \quad \text{iff} \quad \phi \in C_\Sigma(T_\Sigma \cup \Phi).$$

The notation $\mathcal{I}^T$ is used for the π-institution $\mathcal{I}^T = \langle \mathbf{Sign}, \mathrm{SEN}, C^T\rangle$.

We now work towards characterizing the least theory system of a given π-institution $\mathcal{I}$ that includes a given theory system T of $\mathcal{I}$ and a given set of Σ_0-sentences Φ_0, for some distinguished signature $\Sigma_0 \in |\mathbf{Sign}|$ of $\mathcal{I}$.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, $T = \{T_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ a theory system of $\mathcal{I}$, $\Sigma_0 \in |\mathbf{Sign}|$ and $\Phi_0 \subseteq \mathrm{SEN}(\Sigma_0)$. Define, for all $\Sigma \in |\mathbf{Sign}|$,

$$T^{\langle\Sigma_0,\Phi_0\rangle}_\Sigma = C^T_\Sigma(\bigcup\{\mathrm{SEN}(f)(\Phi_0) : f \in \mathbf{Sign}(\Sigma_0, \Sigma)\}).$$

It is now shown that $T^{\langle\Sigma_0,\Phi_0\rangle} = \{T^{\langle\Sigma_0,\Phi_0\rangle}_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ is the least theory system of $\mathcal{I}$ that contains the theory system T and is such that $\Phi_0 \subseteq T^{\langle\Sigma_0,\Phi_0\rangle}_{\Sigma_0}$.

LEMMA 3.1. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, $T = \{T_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ a theory system of $\mathcal{I}$. For every $\Sigma_0 \in |\mathbf{Sign}|$ and $\Phi_0 \subseteq \mathrm{SEN}(\Sigma_0)$, $T^{\langle\Sigma_0,\Phi_0\rangle}$ is the least theory system of $\mathcal{I}$ such that*

1. *$T \leq T^{\langle\Sigma_0,\Phi_0\rangle}$ and*
2. *$\Phi_0 \subseteq T^{\langle\Sigma_0,\Phi_0\rangle}_{\Sigma_0}$.*

PROOF. First, it is clear that $T^{\langle\Sigma_0,\Phi_0\rangle}_\Sigma$ is a Σ-theory, for all $\Sigma \in |\mathbf{Sign}|$. Moreover, $T^{\langle\Sigma_0,\Phi_0\rangle}$ is a theory system of $\mathcal{I}$, since, for every $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, we have that

$$\begin{aligned}
\mathrm{SEN}(f)(T^{\langle\Sigma_0,\Phi_0\rangle}_{\Sigma_1}) = & \\
&= \mathrm{SEN}(f)(C^T_{\Sigma_1}(\bigcup\{\mathrm{SEN}(g)(\Phi_0) : g \in \mathbf{Sign}(\Sigma_0, \Sigma_1)\})) \\
&\subseteq C^T_{\Sigma_2}(\mathrm{SEN}(f)(\bigcup\{\mathrm{SEN}(g)(\Phi_0) : g \in \mathbf{Sign}(\Sigma_0, \Sigma_1)\})) \\
&= C^T_{\Sigma_2}(\bigcup\{\mathrm{SEN}(fg)(\Phi_0) : g \in \mathbf{Sign}(\Sigma_0, \Sigma_1)\}) \\
&\subseteq C^T_{\Sigma_2}(\bigcup\{\mathrm{SEN}(h)(\Phi_0) : h \in \mathbf{Sign}(\Sigma_0, \Sigma_2)\}) \\
&= T^{\langle\Sigma_0,\Phi_0\rangle}_{\Sigma_2}.
\end{aligned}$$

Since it is clear that $T \leq T^{\langle\Sigma_0,\Phi_0\rangle}$ and that $\Phi_0 \subseteq T^{\langle\Sigma_0,\Phi_0\rangle}_{\Sigma_0}$, it suffices, now, to show that, if T' is a theory system of $\mathcal{I}$ that satisfies Conditions 1 and 2, then $T^{\langle\Sigma_0,\Phi_0\rangle} \leq T'$.

We have, by Condition 2, $\Phi_0 \subseteq T'_{\Sigma_0}$. Thus, since T' is a theory system, for every $\Sigma \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_0, \Sigma)$, we have $\mathrm{SEN}(f)(\Phi_0) \subseteq T'_\Sigma$. Hence, by Condition 1, T'_Σ being a Σ-theory such that $T_\Sigma \subseteq T'_\Sigma$,

$$C^T_\Sigma(\bigcup\{\mathrm{SEN}(f)(\Phi_0) : f \in \mathbf{Sign}(\Sigma_0, \Sigma)\}) \subseteq T'_\Sigma.$$

Therefore, by the definition of $T^{\langle \Sigma_0, \Phi_0\rangle}$, $T^{\langle \Sigma_0, \Phi_0\rangle} \leq T'$. ■

In case $\Phi_0 = \{\phi_0\} \subseteq \mathrm{SEN}(\Sigma_0)$ is a singleton, then, in place of $T^{\langle \Sigma_0, \{\phi_0\}\rangle}$, the simplified notation $T^{\langle \Sigma_0, \phi_0\rangle}$ will be used for the least theory system of $\mathcal{I}$ containing the given theory system T and such that $\phi_0 \in T^{\langle \Sigma_0, \phi_0\rangle}_{\Sigma_0}$.

A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is said to be N-**prealgebraic** if, for every theory system $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ of $\mathcal{I}$, every $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi, \psi\rangle \in \Omega^N_\Sigma(T) \quad \text{implies} \quad T^{\langle \Sigma, \phi\rangle} = T^{\langle \Sigma, \psi\rangle}. \tag{iv}$$

On the other hand, a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is said to be N-**protoalgebraic** if, for every theory family $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ of $\mathcal{I}$, every $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi, \psi\rangle \in \Omega^N_\Sigma(T) \quad \text{implies} \quad C_\Sigma(T_\Sigma \cup \{\phi\}) = C_\Sigma(T_\Sigma \cup \{\psi\}). \tag{v}$$

By Theorem 1, a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, always satisfies the property $\widetilde{\Omega}^N(\mathcal{I}) \leq \Lambda(\mathcal{I})$, i.e., if, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi, \psi\rangle \in \widetilde{\Omega}^N_\Sigma(C) \quad \text{implies} \quad C_\Sigma(\phi) = C_\Sigma(\psi). \tag{vi}$$

Thus, Equation (v) may be viewed as a localized version of Property (vi), as applied to an arbitrary theory family T of $\mathcal{I}$.

The Leibniz operator Ω^N, perceived as an operator from the collection of all theory systems to the collection of all N-congruence systems, is monotonic iff $\mathcal{I}$ is N-prealgebraic. On the other hand, Ω^N, perceived as an operator from the collection of all theory families to the collection of all N-congruence systems, is monotonic iff $\mathcal{I}$ is N-protoalgebraic. This characterization of N-protoalgebraicity forms an analog of the well-known characterization of protoalgebraic logics by Blok and Pigozzi [2]. The characterizations of N-prealgebraicity and N-protoalgebraicity constitute the contents of the following two lemmas.

LEMMA 3.2. [Characterization of Prealgebraicity] *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *is a* π*-institution and* N *a category of natural transformations on* SEN. $\mathcal{I}$ *is* N*-prealgebraic iff, for all theory systems* T^1, T^2 *of* $\mathcal{I}$,

$$T^1 \leq T^2 \quad \textit{implies} \quad \Omega^N(T^1) \leq \Omega^N(T^2).$$

PROOF. Suppose, first, that $\mathcal{I}$ is N-prealgebraic and that $T^1 \leq T^2$. To show that $\Omega^N(T^1) \leq \Omega^N(T^2)$, it suffices to show, by the definition of Ω^N, that $\Omega^N(T^1)$ is compatible with T^2. To this end, let $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi\rangle \in \Omega^N_\Sigma(T^1)$ and $\phi \in T^2_\Sigma$. Then, by N-prealgebraicity, $T^{1\langle \Sigma,\phi\rangle}_\Sigma = T^{1\langle \Sigma,\psi\rangle}_\Sigma$, whence, we get

$$\begin{array}{lll}
\psi & \in & T^{1\langle \Sigma,\psi\rangle}_\Sigma \quad \text{(by the definition of } T^{1\langle \Sigma,\psi\rangle}\text{)}\\
 & = & T^{1\langle \Sigma,\phi\rangle}_\Sigma \quad \text{(by the hypothesis)}\\
 & \subseteq & T^{2\langle \Sigma,\phi\rangle}_\Sigma \quad \text{(by the hypothesis)}\\
 & = & T^2_\Sigma \quad \text{(by the definition of } T^{2\langle \Sigma,\phi\rangle}\text{, since } \phi \in T^2_\Sigma\text{)}.
\end{array}$$

Suppose, conversely, that, for every theory systems T^1, T^2 of $\mathcal{I}$, if $T^1 \leq T^2$, then $\Omega^N(T^1) \leq \Omega^N(T^2)$. To show that, in this case, $\mathcal{I}$ is N-prealgebraic, suppose that T is a theory system of $\mathcal{I}$ and that $\langle \phi, \psi\rangle \in \Omega^N_\Sigma(T)$. Then, by the monotonicity of Ω^N on theory systems, $\langle \phi, \psi\rangle \in \Omega^N_\Sigma(T^{\langle \Sigma,\phi\rangle})$. But, by definition of $T^{\langle \Sigma,\phi\rangle}$, $\phi \in T^{\langle \Sigma,\phi\rangle}_\Sigma$, whence, by the compatibility property of $\Omega^N(T^{\langle \Sigma,\phi\rangle})$, $\psi \in T^{\langle \Sigma,\phi\rangle}_\Sigma$. By symmetry $\phi \in T^{\langle \Sigma,\psi\rangle}_\Sigma$, whence $T^{\langle \Sigma,\phi\rangle} = T^{\langle \Sigma,\psi\rangle}$, by the minimality of $T^{\langle \Sigma,\phi\rangle}, T^{\langle \Sigma,\psi\rangle}$, as expressed by Lemma 3.1. Hence $\mathcal{I}$ is N-prealgebraic. ∎

Lemma 3.3 characterizes N-protoalgebraicity in terms of the monotonicity of the Leinbiz operator Ω^N on arbitrary theory families of a π-institution.

LEMMA 3.3. [Characterization of Protoalgebraicity] *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *be a* π*-institution and* N *a category of natural transformations on* SEN. $\mathcal{I}$ *is* N*-protoalgebraic iff, for all theory families* T^1, T^2 *of* $\mathcal{I}$,

$$T^1 \leq T^2 \quad \textit{implies} \quad \Omega^N(T^1) \leq \Omega^N(T^2).$$

PROOF. Suppose, first, that $\mathcal{I}$ is N-protoalgebraic and that $T^1 \leq T^2$. To show that $\Omega^N(T^1) \leq \Omega^N(T^2)$, it suffices to show, by the definition of Ω^N, that $\Omega^N(T^1)$ is compatible with T^2. To this end, let $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi\rangle \in \Omega^N_\Sigma(T^1)$ and $\phi \in T^2_\Sigma$. Then, by N-protoalgebraicity, we obtain

$C_\Sigma(T^1_\Sigma \cup \{\phi\}) = C_\Sigma(T^1_\Sigma \cup \{\psi\})$, whence, we get

$$\begin{array}{lll}\psi & \in & C_\Sigma(T^1_\Sigma \cup \{\psi\}) \\ & = & C_\Sigma(T^1_\Sigma \cup \{\phi\}) \quad \text{(by the hypothesis)} \\ & \subseteq & C_\Sigma(T^2_\Sigma \cup \{\phi\}) \quad \text{(by the hypothesis)} \\ & = & T^2_\Sigma \quad (\text{since } \phi \in T^2_\Sigma).\end{array}$$

Suppose, conversely, that, for all theory families T^1, T^2 of $\mathcal{I}$, if $T^1 \leq T^2$, then $\Omega^N(T^1) \leq \Omega^N(T^2)$. To show that, in this case, $\mathcal{I}$ is N-protoalgebraic, let T be a theory family of $\mathcal{I}$ and suppose that $\langle\phi, \psi\rangle \in \Omega^N_\Sigma(T)$. Consider the theory family $T^{[\langle\Sigma,\phi\rangle]}$ with $T^{[\langle\Sigma,\phi\rangle]} = \{T^{[\langle\Sigma,\phi\rangle]}_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}|}$, where

$$T^{[\langle\Sigma,\phi\rangle]}_{\Sigma'} = \begin{cases} T_{\Sigma'}, & \text{if } \Sigma' \neq \Sigma \\ C_\Sigma(T_\Sigma \cup \{\phi\}), & \text{if } \Sigma' = \Sigma \end{cases}$$

Then, since $\langle\phi, \psi\rangle \in \Omega^N_\Sigma(T)$, by the monotonicity of Ω^N on theory families, we obtain $\langle\phi, \psi\rangle \in \Omega^N_\Sigma(T^{[\langle\Sigma,\phi\rangle]})$. But, by definition of $T^{[\langle\Sigma,\phi\rangle]}$, $\phi \in T^{[\langle\Sigma,\phi\rangle]}_\Sigma$, whence, by the compatibility property of Ω^N, $\psi \in T^{[\langle\Sigma,\phi\rangle]}_\Sigma$, i.e., $\psi \in C_\Sigma(T_\Sigma \cup \{\phi\})$. By symmetry $\phi \in C_\Sigma(T_\Sigma \cup \{\psi\})$, whence $\mathcal{I}$ is N-protoalgebraic. ∎

The following corollary is now obtained, which one may also prove directly from the relevant definitions, since every theory system is a theory family and the action of the N-Leibniz operator on theory systems is the restriction of its action on theory families.

COROLLARY 3.4. *If a π-institution $\mathcal{I} = \langle\mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, is N-protoalgebraic, then it is N-prealgebraic.*

That is the following two-step hierarchy of sub-protoalgebraic π-institutions exists

$$\begin{array}{c} N\text{-Protoalgebraic} \\ \downarrow \\ N\text{-Prealgebraic} \end{array} \tag{vii}$$

It should be noted here that the prefix "$N-$" cannot be avoided when referring to the notions of N-prealgebraicity and N-protoalgebraicity, since N, which plays the role of the "algebraic part", depends on the π-institution but is not part of it. Hence the classification proposed is not a real classification of all π-institutions. In fact, on the one hand, for every **Set**-valued functor SEN and every category of natural transformations N on SEN, the sentence functor of all N-protoalgebraic and N-prealgebraic π-institutions is SEN. On the other hand, there can be two categories of natural transformations N and N' on SEN, such that $\mathcal{I}$ is N-protoalgebraic but not N'-protoalgebraic and similarly with prealgebraicity.

The monotonicity of the Leibniz operator on theory systems and its monotonicity on theory families are equivalent, respectively, to its meet-continuity with respect to collections of theory systems and to its meet-continuity with respect to collections of theory families.

LEMMA 3.5. *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *is a* π*-institution and* N *a category of natural transformations on* SEN. $\mathcal{I}$ *is*

1. N*-prealgebraic iff, for every collection* $\{T^i : i \in I\}$ *of theory systems of* $\mathcal{I}$, $\Omega^N(\bigcap_{i\in I} T^i) = \bigcap_{i\in I} \Omega^N(T^i)$.
2. N*-protoalgebraic iff, for every collection* $\{T^i : i \in I\}$ *of theory families of* $\mathcal{I}$, $\Omega^N(\bigcap_{i\in I} T^i) = \bigcap_{i\in I} \Omega^N(T^i)$.

PROOF. We only show the equivalence in Part 1. A similar argument establishes that of Part 2.

Suppose that $\mathcal{I}$ is N-prealgebraic and $\{T^i : i \in I\}$ is a collection of theory systems of $\mathcal{I}$. Then, we have $\bigcap_{i\in I} T^i \le T^i$, for every $i \in I$, whence, by Lemma 3.2, $\Omega^N(\bigcap_{i\in I} T^i) \le \Omega^N(T^i)$, for all $i \in I$, and, hence, $\Omega^N(\bigcap_{i\in I} T^i) \le \bigcap_{i\in I} \Omega^N(T^i)$. On the other hand, $\bigcap_{i\in I} \Omega^N(T^i)$ is an N-congruence system of SEN that is compatible with every T^i. Therefore, it is compatible with $\bigcap_{i\in I} T^i$. This, together with the definition of the Leibniz N-congruence system, yields that $\bigcap_{i\in I} \Omega^N(T^i) \le \Omega^N(\bigcap_{i\in I} T^i)$. ■

The reader is reminded that, given a π-institution $\mathcal{I}$, by Thm is denoted the theorem system of $\mathcal{I}$, i.e., $\mathrm{Thm} = \{\mathrm{Thm}_\Sigma\}_{\Sigma\in|\mathbf{Sign}|} = \{C_\Sigma(\emptyset)\}_{\Sigma\in|\mathbf{Sign}|}$.

Combining the Tarski N-congruence system of $\mathcal{I}$ together with the Leibniz N-congruence systems of its theory systems, N-prealgebraicity and N-protoalgebraicity are connected by the following proposition, providing an interesting partial analog of Proposition 3.1 of [13].

PROPOSITION 3.6. *Suppose* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *is a* π*-institution and* N *a category of natural transformations on* SEN. *The following conditions are related by* $1 \to 2 \leftrightarrow 3 \to 4$:

1. $\mathcal{I}$ *is* N*-protoalgebraic.*
2. *For every closure system* C' *on* SEN, *such that* $C \le C'$, $\widetilde{\Omega}^N(C') = \Omega^N(\mathrm{Thm}')$, *where* $\mathrm{Thm}' = \{C'_\Sigma(\emptyset) : \Sigma \in |\mathbf{Sign}|\}$.

3. *For all theory systems T of $\mathcal{I}$, $\widetilde{\Omega}^N(C^T) = \Omega^N(T)$.*
4. *$\mathcal{I}$ is N-prealgebraic.*

PROOF. $1 \to 2$: Assume that $\mathcal{I}$ is N-protoalgebraic. Suppose that C' is a closure system on SEN, such that $C \leq C'$, and let $\mathrm{Thm}' = \{C'_\Sigma(\emptyset) : \Sigma \in |\mathbf{Sign}|\}$. By the definition of the Leibniz and of the Tarski N-congruence systems, it is clear that $\widetilde{\Omega}^N(C') \leq \Omega^N(\mathrm{Thm}')$. For the reverse inclusion, note that for every theory family T' of C', we have that $\mathrm{Thm}' \leq T'$, whence, by the N-protoalgebraicity of $\mathcal{I}$ and Lemma 3.3, $\Omega^N(\mathrm{Thm}') \leq \Omega^N(T')$. Therefore $\Omega^N(\mathrm{Thm}') \leq \bigcap_{T'} \Omega^N(T') = \widetilde{\Omega}^N(C')$, the last equality holding by Corollary 2.4.

$2 \to 3$: This is obvious, since $C \leq C^T$ and T is the theorem system of C^T.

$3 \to 2$: This is also obvious, since, for all closure systems C' on SEN, such that $C \leq C'$, the theorem system Thm' of C' is a theory system of $\mathcal{I}$ and $C^{\mathrm{Thm}'} = C'$.

$3 \to 4$: Let T, T' be theory systems of $\mathcal{I}$, such that $T \leq T'$. Then T' is a theory family of $\mathcal{I}^T$, whence $\Omega^N(T) = \widetilde{\Omega}^N(\mathcal{I}^T) \leq \Omega^N(T')$, by hypothesis and by Corollary 2.4. Therefore, by Lemma 3.2, $\mathcal{I}$ is N-prealgebraic. ∎

4. The Correspondence Property

Protoalgebraicity allows the formulation of a correspondence between the theory families of a finitary N-rule based π-institution $\mathcal{I}$ containing a given theory system T and the theory families of the quotient of the induced π-institution $\mathcal{I}^T$ by its logical N-congruence system $\Omega^N(T)$. To formalize this result, we first remind the reader of some of the relevant definitions and results from previous work.

Recall, once more, from Proposition 2 of [27], that by $\mathcal{I}^T$ is denoted the π-institution $\mathcal{I}^T = \langle \mathbf{Sign}, \mathrm{SEN}, C^T \rangle$ induced by a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ and one of its axiom systems T. It is *important to note* that this construction is only valid if T is an axiom system but cannot be carried out if T is an arbitrary axiom family. Obviously, if T' is a theory family of $\mathcal{I}^T$, then $T \leq T'$. Therefore, if $\mathcal{I}$ is N-protoalgebraic, we get that $\Omega^N(T) \leq \Omega^N(T')$. This shows that $\Omega^N(T)$ is an N-congruence system of SEN that is compatible with C^T, i.e., $\Omega^N(T)$ is a logical N-congruence system of $\mathcal{I}^T$. This may also be seen directly by using the implication $1 \to 3$ of Proposition 3.6, since the Tarski N-congruence system of a π-institution is, by definition, a logical N-congruence system. Thus, by Proposition 22 of [22], for $\mathcal{I}$ N-protoalgebraic, it makes sense to define the logical quotient π-institution $\mathcal{I}^T/\Omega^N(T)$ of $\mathcal{I}^T$ by $\Omega^N(T)$. The possibility of carrying out this construction in the case of N-protoalgebraic π-institutions is the main motivation why they (and not the N-prealgebraic ones) are at the focus of our investigations in the present paper and its companion [28].

In Definition 2.3 and Theorem 2.4 of [2], Blok and Pigozzi gave a characterization of protoalgebraic deductive systems in terms of the *correspondence property.* Their definition is adapted here to the π-institution framework and, as a consequence, the *system* N-correspondence property, the *family* N-correspondence property and their *strong versions* are formulated for π-institutions. Inspired by the characterization theorem for protoalgebraic deductive systems in terms of the correspondence property, we formulate two results, one relating N-prealgebraic π-institutions to the system N-correspondence property and the other relating N-protoalgebraicity to the family N-correspondence property. These two results seem to reveal a refinement of the hierarchy for π-institutions, given in Diagram (vii), that partially contracts when one restricts attention to the π-institutions associated with sentential logics as in [22], Corollaries 5 and 6, since in that case there is no distinction between theory systems and theory families. However, it is worth noting that these π-institutions have the trivial one-element category as their signature category, i.e., they do not take into account the algebra endomorphisms of the algebra reduct of the logical system considered. If the endomorphisms are included in the signature category, then one immediately passes beyond the scope of the part of abstract algebraic logic based on the sentential logic framework (the traditional universal algebraic part). This is because, in the sentential logic framework, there is no mechanism for dealing with the endomorphisms in the object language.

It is worthwhile adding here that, if $\mathcal{I}_\mathcal{S}$ is the canonical π-institution associated with a sentential logic $\mathcal{S}$ in the sense of [19], Section 3, or [20], Section 2.1, then the notions of N-prealgebraicity and N-protoalgebraicity for π-institutions of this form do not coincide, since theory families of $\mathcal{I}_\mathcal{S}$ are just theories of the sentential logic $\mathcal{S}$ whereas theory systems of $\mathcal{I}_\mathcal{S}$ are theories of $\mathcal{S}$ that are closed under substitutions. For instance, it is known that the $\{\wedge,\vee\}$-fragment of classical propositional logic $\mathrm{CPC}_{\wedge\vee}$ is not protoalgebraic and hence is not N-protoalgebraic. But it is N-prealgebraic, since there are only two theory systems, $\{\emptyset\}$ and $\{\mathrm{Fm}_\mathcal{L}(V)\}$, which renders Ω^N monotonic on theory systems. Therefore, despite the fact that N-prealgebraicity is not as well behaved as N-protoalgebraicity, its study may contribute something new to the study of sentential logics.

Next, it is shown, in preparation for further discussion, that inverse images of theory families of π-institutions under singleton semi-interpretations are also theory families and that inverse images of theory systems under singleton semi-interpretations are also theory systems. Recall from [22], Section 4, the definition of a singleton semi-interpretation.

LEMMA 4.1. *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle, \mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ are two π-institutions and $\langle F, \alpha\rangle : \mathcal{I} \succ^{s} \mathcal{I}'$ a singleton semi-interpretation.*

1. *If T' is a theory family of $\mathcal{I}'$, then $\alpha^{-1}(T') = \{\alpha_\Sigma^{-1}(T'_{F(\Sigma)})\}_{\Sigma\in|\mathbf{Sign}|}$ is a theory family of $\mathcal{I}$.*
2. *If T' is a theory system of $\mathcal{I}'$, then $\alpha^{-1}(T') = \{\alpha_\Sigma^{-1}(T'_{F(\Sigma)})\}_{\Sigma\in|\mathbf{Sign}|}$ is a theory system of $\mathcal{I}$.*

PROOF. 1. For this part, it suffices to show that, for all $\Sigma \in |\mathbf{Sign}|$, if $\phi \in C_\Sigma(\alpha_\Sigma^{-1}(T'_{F(\Sigma)}))$, then $\phi \in \alpha_\Sigma^{-1}(T'_{F(\Sigma)})$. Suppose, to this end, that $\phi \in C_\Sigma(\alpha_\Sigma^{-1}(T'_{F(\Sigma)}))$. Then, we have

$$\begin{array}{lll}\alpha_\Sigma(\phi) & \in & \alpha_\Sigma(C_\Sigma(\alpha_\Sigma^{-1}(T'_{F(\Sigma)}))) \\ & \subseteq & C'_{F(\Sigma)}(\alpha_\Sigma(\alpha_\Sigma^{-1}(T'_{F(\Sigma)}))) \quad (\text{since } \langle F,\alpha\rangle : \mathcal{I}\succ^{s}\mathcal{I}') \\ & \subseteq & C'_{F(\Sigma)}(T'_{F(\Sigma)}) \quad (\text{set-theoretic}) \\ & = & T'_{F(\Sigma)} \quad (T' \text{ a theory family}).\end{array}$$

Thus, we have indeed that $\phi \in \alpha_\Sigma^{-1}(T'_{F(\Sigma)})$.

2. That $\alpha_\Sigma^{-1}(T'_{F(\Sigma)})$ is a Σ-theory was shown in the first part. Thus, it remains to show that $\alpha^{-1}(T')$ is a theory system. To this end, let $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$. Then we have

$$\begin{array}{lll}\alpha_{\Sigma_2}(\mathrm{SEN}(f)(\alpha_{\Sigma_1}^{-1}(T'_{F(\Sigma_1)}))) & = & \mathrm{SEN}'(F(f))(\alpha_{\Sigma_1}(\alpha_{\Sigma_1}^{-1}(T'_{F(\Sigma_1)}))) \\ & & (\alpha \text{ a natural transformation}) \\ & \subseteq & \mathrm{SEN}'(F(f))(T'_{F(\Sigma_1)}) \ (\text{set-theory}) \\ & \subseteq & T'_{F(\Sigma_2)} \ (T' \text{ a theory system})\end{array}$$

whence, we get that $\mathrm{SEN}(f)(\alpha_{\Sigma_1}^{-1}(T'_{F(\Sigma_1)})) \subseteq \alpha_{\Sigma_2}^{-1}(T'_{F(\Sigma_2)})$ and $\alpha^{-1}(T')$ is indeed a theory system. ∎

Next, the system and the family N-correspondence properties and their strong counterparts are introduced.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution and N a category of natural transformations on SEN. $\mathcal{I}$ is said to have the **system N-correspondence**

property if, for every category $\mathbf{Sign}'$, every functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ and category of natural transformations N' on SEN', every surjective (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, if $C'^{\min}$ denotes the closure system of the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN', then for every theory system T of $\mathcal{I}$ and every $\Sigma \in |\mathbf{Sign}|$,

$$\alpha_\Sigma^{-1}(C'^{\min}_{F(\Sigma)}(\alpha_\Sigma(T_\Sigma))) = C_\Sigma(T_\Sigma \cup \alpha_\Sigma^{-1}(C'^{\min}_{F(\Sigma)}(\emptyset))).$$

$\mathcal{I}$ is said to have the **strong system N-correspondence property** if it has the system N-correspondence property and, in addition, for every theory system T of $\mathcal{I}$, if $C'^{\min}$ denotes the $\langle \mathrm{I}_{\mathbf{Sign}}, \pi^{\Omega^N(T)}\rangle$-min $(N, N^{\Omega^N(T)})$-model of $\mathcal{I}$ on $\mathrm{SEN}^{\Omega^N(T)}$, then

$$(\pi^{\Omega^N(T)})^{-1}(C'^{\min}(\emptyset)) \leq T.$$

$\mathcal{I}$ is said to have the **family N-correspondence property** if the same condition as in the system N-correspondence property holds with T ranging over arbitrary theory families of $\mathcal{I}$, i.e., if, for every category $\mathbf{Sign}'$, every functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ and category of natural transformations N' on SEN', every surjective (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, if $C'^{\min}$ denotes the closure system of the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN', then, for every theory family T of $\mathcal{I}$ and every $\Sigma \in |\mathbf{Sign}|$,

$$\alpha_\Sigma^{-1}(C'^{\min}_{F(\Sigma)}(\alpha_\Sigma(T_\Sigma))) = C_\Sigma(T_\Sigma \cup \alpha_\Sigma^{-1}(C'^{\min}_{F(\Sigma)}(\emptyset))).$$

$\mathcal{I}$ is said to have the **strong family N-correspondence property** if it has the family N-correspondence property and, in addition, for every theory family T of $\mathcal{I}$, if $C'^{\min}$ denotes the $\langle \mathrm{I}_{\mathbf{Sign}}, \pi^{\Omega^N(T)}\rangle$-min $(N, N^{\Omega^N(T)})$-model of $\mathcal{I}$ on $\mathrm{SEN}^{\Omega^N(T)}$, then

$$(\pi^{\Omega^N(T)})^{-1}(C'^{\min}(\emptyset)) \leq T.$$

We aim at showing that

- the strong system N-correspondence property implies N-prealgebraicity,
- the strong family N-correspondence property implies N-protoalgebraicity and
- in case $\mathcal{I}$ is finitary and N-rule based, then N-protoalgebraicity implies the family N-correspondence property.

For the proofs of these results, some auxiliary lemmas will be used. These lemmas help formulate partial analogs for π-institutions of the two directions of the equivalence presented in Lemma 1.11 (iii) of [2].

Given a singleton translation $\langle F, \alpha\rangle : \mathrm{SEN} \rightarrow^{s} \mathrm{SEN}'$, recall, before proceeding to the first lemma, the definition of the equivalence system $\theta^{\langle F,\alpha\rangle}$, from Section 6 of [22].

LEMMA 4.2. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN, *and suppose that $\langle F, \alpha\rangle : \mathcal{I} \rightarrow^{se} \mathrm{SEN}'$ is a surjective (N, N')-epimorphic translation. If $C'^{\min}$ denotes the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN', then, for every theory family T of $\mathcal{I}$,*

$$\alpha_{\Sigma}^{-1}(C'^{\min}_{F(\Sigma)}(\alpha_{\Sigma}(T_{\Sigma}))) = T_{\Sigma}, \quad \textit{for every } \Sigma \in |\mathbf{Sign}|,$$

implies that $\theta^{\langle F,\alpha\rangle}$ is compatible with T.

PROOF. Suppose that $T \in \mathrm{ThFam}(\mathcal{I})$, such that

$$\alpha_{\Sigma}^{-1}(C'^{\min}_{F(\Sigma)}(\alpha_{\Sigma}(T_{\Sigma}))) = T_{\Sigma}, \quad \text{for every } \Sigma \in |\mathbf{Sign}|.$$

To show the compatibility of $\theta^{\langle F,\alpha\rangle}$ with T, suppose that $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi\rangle \in \theta_{\Sigma}^{\langle F,\alpha\rangle}$ and $\phi \in T_{\Sigma}$. Then, we have, by the hypothesis, that $\alpha_{\Sigma}(\phi) \in \alpha_{\Sigma}(T_{\Sigma}) = C'^{\min}_{F(\Sigma)}(\alpha_{\Sigma}(T_{\Sigma}))$, whence, since $\langle \phi, \psi\rangle \in \theta_{\Sigma}^{\langle F,\alpha\rangle}$, we get that $\alpha_{\Sigma}(\psi) \in C'^{\min}_{F(\Sigma)}(\alpha_{\Sigma}(T_{\Sigma}))$, i.e., $\psi \in \alpha_{\Sigma}^{-1}(C'^{\min}_{F(\Sigma)}(\alpha_{\Sigma}(T_{\Sigma}))) = T_{\Sigma}$. Hence $\theta^{\langle F,\alpha\rangle}$ is compatible with T. ■

Recall, now, before proceeding to the second lemma, the definition of the closure system C^R induced by a collection R of N-rules of inference on a sentence functor SEN, where N is a category of natural transformations on SEN, defined via Proposition 2.2 of [31]. Recall also Definition 2.3 of [31], the definition of an N-rule based π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN.

LEMMA 4.3. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN, *be a finitary N-rule based π-institution, and R an N-rule base for $\mathcal{I}$. Suppose that $\langle F, \alpha\rangle : \mathcal{I} \rightarrow^{se} \mathrm{SEN}'$ is a surjective (N, N')-epimorphic translation. If $C'^{\min}$ denotes the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN', and R' corresponds to R via the (N, N')-epimorphic property, then $C'^{\min} \leq C^{R'}$.*

PROOF. Suppose that $C^{R'}$ is the closure system on SEN' induced by R' as in Proposition 2.2 of [31]. To show that $C'^{\min} \leq C^{R'}$, it suffices, by the minimality of $C'^{\min}$, to show that $C^{R'}$ is an (N, N')-model of $\mathcal{I}$ via $\langle F, \alpha\rangle$.

To this end, suppose that $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$. Thus, by the finitarity of $\mathcal{I}$, there exists $\Psi \subseteq_\omega \Phi$, such that $\phi \in C_\Sigma(\Psi)$. Now, by the N-rule basedness, there exists a finitary N-rule $\langle X, \sigma\rangle$ and $\vec{\phi} \in \mathrm{SEN}(\Sigma)^\omega$, such that $X_\Sigma(\vec{\phi}) \subseteq \Psi$ and $\sigma_\Sigma(\vec{\phi}) = \phi$. Therefore, $\alpha_\Sigma(X_\Sigma(\vec{\phi})) \subseteq \alpha_\Sigma(\Psi)$ and $\alpha_\Sigma(\sigma_\Sigma(\vec{\phi})) = \alpha_\Sigma(\phi)$. This implies that $X'_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})) \subseteq \alpha_\Sigma(\Psi)$ and $\sigma'_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})) = \alpha_\Sigma(\phi)$. Hence, since $\langle X', \sigma'\rangle \in R'$, $\alpha_\Sigma(\phi) \in C^{R'}_{F(\Sigma)}(\alpha_\Sigma(\Psi)) \subseteq C^{R'}_{F(\Sigma)}(\alpha_\Sigma(\Phi))$, which shows that $C^{R'}$ is a model of $\mathcal{I}$ via $\langle F, \alpha\rangle$ and, therefore, by the minimality of $C'^{\min}$, that $C'^{\min} \leq C^{R'}$. ∎

Finally, the third lemma in the series, whose proof depends on Lemma 4.3, is a partial converse to Lemma 4.2.

LEMMA 4.4. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a finitary N-rule based π-institution, where N is a category of natural transformations on* SEN, *and suppose that $\langle F, \alpha\rangle : \mathcal{I} \rightarrow^{se} \mathrm{SEN}'$ is a surjective (N, N')-epimorphic translation. If $C'^{\min}$ denotes the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on* SEN′, *then, for every theory family T of $\mathcal{I}$, if $\theta^{\langle F,\alpha\rangle}$ is compatible with T, then*

$$\alpha_\Sigma^{-1}(C'^{\min}_{F(\Sigma)}(\alpha_\Sigma(T_\Sigma))) = T_\Sigma, \quad \textit{for every } \Sigma \in |\mathbf{Sign}|.$$

PROOF. Suppose that $\theta^{\langle F,\alpha\rangle}$ is compatible with T. It will be shown, first, that $\alpha_\Sigma(T_\Sigma)$ is an $F(\Sigma)$-theory, for every $\Sigma \in |\mathbf{Sign}|$, i.e., that

$$C'^{\min}_{F(\Sigma)}(\alpha_\Sigma(T_\Sigma)) = \alpha_\Sigma(T_\Sigma). \qquad \text{(viii)}$$

To see this, we show that, if R is an N-rule base of $\mathcal{I}$, then $\alpha_\Sigma(T_\Sigma)$ is closed under the N'-rules R' and, then, use Lemma 4.3.

To see that $\alpha_\Sigma(T_\Sigma)$ is closed under the N'-rules R', suppose that

$$\langle\{\tau^0, \tau^1, \ldots, \tau^{n-1}\}, \sigma\rangle$$

is an N-rule in R and that $\vec{\psi} \in \mathrm{SEN}'(F(\Sigma))^\omega$, such that $\tau'^i_{F(\Sigma)}(\vec{\psi}) \in \alpha_\Sigma(T_\Sigma)$, for all $i < n$. Then, by surjectivity of $\langle F, \alpha\rangle$, there exists $\vec{\phi} \in \mathrm{SEN}(\Sigma)^\omega$, such that $\alpha_\Sigma(\vec{\phi}) = \vec{\psi}$. Therefore, $\tau'^i_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})) \in \alpha_\Sigma(T_\Sigma)$, for all $i < n$, whence, $\alpha_\Sigma(\tau^i_\Sigma(\vec{\phi})) \in \alpha_\Sigma(T_\Sigma)$, for all $i < n$. Now, using the compatibility of $\theta^{\langle F,\alpha\rangle}$ with T, we obtain that $\tau^i_\Sigma(\vec{\phi}) \in T_\Sigma$, for every $i < n$, whence, since $\langle\{\tau^0, \tau^1, \ldots, \tau^{n-1}\}, \sigma\rangle$ is an N-rule of $\mathcal{I}$, we get that $\sigma_\Sigma(\vec{\phi}) \in T_\Sigma$. Hence $\alpha_\Sigma(\sigma_\Sigma(\vec{\phi})) \in \alpha_\Sigma(T_\Sigma)$, i.e., $\sigma'_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})) \in \alpha_\Sigma(T_\Sigma)$, which yields that $\sigma'_{F(\Sigma)}(\vec{\psi}) \in \alpha_\Sigma(T_\Sigma)$. Thus, $\alpha_\Sigma(T_\Sigma)$ is closed under all N'-rules in R', showing that $\alpha_\Sigma(T_\Sigma)$ is an $F(\Sigma)$-theory of $C^{R'}$. But, by Lemma 4.3, $C'^{\min} \leq C^{R'}$, which implies that $\alpha_\Sigma(T_\Sigma)$ is also an $F(\Sigma)$-theory of $C'^{\min}$. This finishes the proof of Equation (viii).

Now we have

$$\begin{array}{rcl}\alpha_\Sigma^{-1}(C'^{\min}_{F(\Sigma)}(\alpha_\Sigma(T_\Sigma))) & = & \alpha_\Sigma^{-1}(\alpha_\Sigma(T_\Sigma)) \quad \text{(by Equation (viii))} \\ & = & T_\Sigma \text{ (by the compatibility of } \theta^{\langle F,\alpha\rangle} \text{ with } T).\end{array}$$

■

This result yields the following corollary, showing that, in the case of a finitary N-rule based π-institution $\mathcal{I}$ the ordinary and the strong versions of the system and of the family N-correspondence properties collapse to a single version.

COROLLARY 4.5. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN, *be a finitary N-rule based π-institution.*

1. *$\mathcal{I}$ has the strong system N-correspondence property if and only if it has the system N-correspondence property.*
2. *$\mathcal{I}$ has the strong family N-correspondence property if and only if it has the family N-correspondence property.*

PROOF. We will show that, if T is a theory family of $\mathcal{I}$ and if $C'^{\min}$ denotes the $\langle I_{\mathbf{Sign}}, \pi^{\Omega^N(T)}\rangle$-min $(N, N^{\Omega^N(T)})$-model of $\mathcal{I}$ on $\mathrm{SEN}^{\Omega^N(T)}$, then

$$(\pi_\Sigma^{\Omega^N(T)})^{-1}(C'^{\min}_\Sigma(\emptyset)) \subseteq T_\Sigma, \quad \text{for all } \Sigma \in |\mathbf{Sign}|.$$

We have that $\theta^{\langle I_{\mathbf{Sign}}, \pi^{\Omega^N(T)}\rangle} = \Omega^N(T)$, whence it is compatible with T and, thus, by Lemma 4.4, we have that, for all $\Sigma \in |\mathbf{Sign}|$,

$$(\pi_\Sigma^{\Omega^N(T)})^{-1}(C'^{\min}_\Sigma(\pi_\Sigma^{\Omega^N(T)}(T_\Sigma))) = T_\Sigma.$$

Therefore

$$\begin{array}{rcl}(\pi_\Sigma^{\Omega^N(T)})^{-1}(C'^{\min}_\Sigma(\emptyset)) & = & (\pi_\Sigma^{\Omega^N(T)})^{-1}(C'^{\min}_\Sigma(\pi_\Sigma^{\Omega^N(T)}(\emptyset))) \\ & \subseteq & (\pi_\Sigma^{\Omega^N(T)})^{-1}(C'^{\min}_\Sigma(\pi_\Sigma^{\Omega^N(T)}(T_\Sigma))) \\ & = & T_\Sigma.\end{array}$$

■

Next, it will be shown that if a π-institution $\mathcal{I}$ has the strong system N-correspondence property, then it is N-prealgebraic. This result forms a partial analog of part $(iii) \Rightarrow (i)$ of Theorem 2.4 of [2].

THEOREM 2. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution and N a category of natural transformations on* SEN. *If $\mathcal{I}$ has the strong system N-correspondence property then it is N-prealgebraic.*

PROOF. Suppose that $\mathcal{I}$ has the strong system N-correspondence property and consider theory systems T^1, T^2 of $\mathcal{I}$, such that $T^1 \leq T^2$. Let $C^{\min}$ denote the closure system of the $\langle \mathbf{I_{Sign}}, \pi^{\Omega^N(T^1)}\rangle$-min $(N, N^{\Omega^N(T^1)})$-model of $\mathcal{I}$ on $\mathrm{SEN}^{\Omega^N(T^1)}$. Then, by the strong system N-correspondence property, for all $\Sigma \in |\mathbf{Sign}|$,

$$(\pi_\Sigma^{\Omega^N(T^1)})^{-1}(C_\Sigma^{\min}(\pi_\Sigma^{\Omega^N(T^1)}(T_\Sigma^2))) = C_\Sigma(T_\Sigma^2 \cup (\pi_\Sigma^{\Omega^N(T^1)})^{-1}(C_\Sigma'^{\min}(\emptyset))),$$

and also $(\pi_\Sigma^{\Omega^N(T^1)})^{-1}(C_\Sigma'^{\min}(\emptyset)) = T_\Sigma^1$. Therefore, we have that

$$\begin{array}{rcl}(\pi_\Sigma^{\Omega^N(T^1)})^{-1}(C_\Sigma^{\min}(\pi_\Sigma^{\Omega^N(T^1)}(T_\Sigma^2))) & = & C_\Sigma(T_\Sigma^2 \cup (\pi_\Sigma^{\Omega^N(T^1)})^{-1}(C_\Sigma'^{\min}(\emptyset))) \\ & = & C_\Sigma(T_\Sigma^2 \cup T_\Sigma^1) \\ & = & C_\Sigma(T_\Sigma^2) \\ & = & T_\Sigma^2.\end{array}$$

This shows, by Lemma 4.2, that the Leibniz N-congruence system $\Omega^N(T^1)$ is compatible with T^2 and, therefore, that $\Omega^N(T^1) \leq \Omega^N(T^2)$, proving that $\mathcal{I}$ is indeed N-prealgebraic. ■

Combined with Corollary 4.5, Theorem 2 immediately yields

COROLLARY 4.6. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN, *be a finitary, N-rule based π-institution. If $\mathcal{I}$ has the system N-correspondence property, then it is N-prealgebraic.*

Now, it will be shown that every π-institution having the strong family N-correspondence property is N-protoalgebraic and that every N-protoalgebraic π-institution has the family N-correspondence property. This result forms a partial analog of $(i) \Leftrightarrow (iii)$ of Theorem 2.4 of [2].

THEOREM 3. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution and N a category of natural transformations on* SEN.

1. *If $\mathcal{I}$ has the strong family N-correspondence property then it is N-protoalgebraic.*
2. *If $\mathcal{I}$ is finitary, N-rule based and N-protoalgebraic, then $\mathcal{I}$ has the family N-correspondence property.*

Proof. The proof of the first part is carried out by following mutatis mutandis the steps in the proof of Theorem 2, but replacing system by family everywhere. So we will only prove carefully the second part.

Suppose, next, that $\mathcal{I}$ is a finitary, N-rule based, N-protoalgebraic π-institution. Let $\mathbf{Sign}'$ be a category, $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ a functor, N' a category of natural transformations on SEN' and $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ a surjective (N, N')-epimorphic translation. Let T be a theory system of $\mathcal{I}$. Denote by $C'^{\min}$ the closure system of the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN'. We do have, for all $\Sigma \in |\mathbf{Sign}|$, $T_\Sigma \subseteq \alpha_\Sigma^{-1}(C'^{\min}_{F(\Sigma)}(\alpha_\Sigma(T_\Sigma)))$, and since $C'^{\min}_{F(\Sigma)}(\emptyset) \subseteq C'^{\min}_{F(\Sigma)}(\alpha_\Sigma(T_\Sigma))$, we get that $\alpha_\Sigma^{-1}(C'^{\min}_{F(\Sigma)}(\emptyset)) \subseteq \alpha_\Sigma^{-1}(C'^{\min}_{F(\Sigma)}(\alpha_\Sigma(T_\Sigma)))$.
This shows that

$$C_\Sigma(T_\Sigma \cup \alpha_\Sigma^{-1}(C'^{\min}_{F(\Sigma)}(\emptyset))) \subseteq \alpha_\Sigma^{-1}(C'^{\min}_{F(\Sigma)}(\alpha_\Sigma(T_\Sigma))).$$

Suppose, on the other hand, that T' is a theory family of $\mathcal{I}$, such that $T, \alpha^{-1}(C'^{\min}(\emptyset)) \leq T'$ (recall that $\alpha^{-1}(C'^{\min}(\emptyset))$ is a theory family of $\mathcal{I}$ by Lemma 4.1). Then $\theta^{\langle F,\alpha\rangle}$ is compatible with $\alpha^{-1}(C'^{\min}(\emptyset))$. Thus, $\theta^{\langle F,\alpha\rangle} \leq \Omega^N(\alpha^{-1}(C'^{\min}(\emptyset)))$. On the other hand, $\Omega^N(\alpha^{-1}(C'^{\min}(\emptyset))) \leq \Omega^N(T')$, by N-protoalgebraicity. Therefore, $\theta^{\langle F,\alpha\rangle} \leq \Omega^N(T')$, and, since $\mathbf{Con}^N(T')$ is an ideal of $\mathbf{Con}^N(\mathrm{SEN})$ generated by $\Omega^N(T')$, we obtain that
$\theta^{\langle F,\alpha\rangle} \in \mathrm{Con}^N(T')$. Thus, $\theta^{\langle F,\alpha\rangle}$ is compatible with T', whence, we get, by Lemma 4.4, that $\alpha_\Sigma^{-1}(C'^{\min}_{F(\Sigma)}(\alpha_\Sigma(T'_\Sigma))) = T'_\Sigma$. Hence, $T \leq T'$ implies that $\alpha_\Sigma^{-1}(C'^{\min}_{F(\Sigma)}(\alpha_\Sigma(T_\Sigma))) \subseteq T'_\Sigma$. Therefore, $\alpha_\Sigma^{-1}(C'^{\min}_{F(\Sigma)}(\alpha_\Sigma(T_\Sigma)))$ is contained in all Σ-theories that contain both T_Σ and $\alpha_\Sigma^{-1}(C'^{\min}_{F(\Sigma)}(\emptyset))$ and, thus,

$$\alpha_\Sigma^{-1}(C'^{\min}_{F(\Sigma)}(\alpha_\Sigma(T_\Sigma))) \subseteq C_\Sigma(T_\Sigma \cup \alpha_\Sigma^{-1}(C'^{\min}_{F(\Sigma)}(\emptyset))). \qquad \blacksquare$$

It follows from Theorem 3, combined with Corollary 4.5, that N-protoalgebraicity and the family N-correspondence property are equivalent once attention is restricted to finitary N-rule based π-institutions. This result forms a perfect analog of $(i) \Leftrightarrow (iii)$ of Theorem 2.4 of [2].

Corollary 4.7. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN, *be a finitary N-rule based π-institution. $\mathcal{I}$ has the family N-correspondence property if and only if it is N-protoalgebraic.*

Theorems 2 and 3 reveal the existence of the following refinement of the hierarchy of Diagram (vii) for classes of π-institutions:

$$
\begin{tikzcd}[column sep=small]
& \text{Strong } N\text{-Family Correspondence} \arrow[dl] \arrow[dr] & \\
\text{Strong } N\text{-System Correspondence} \arrow[dd] & & N\text{-Protoalgebraic} \arrow[ddll] \arrow[dd, "\text{Finitary, } N\text{-Rule Based}"] \\
& & \\
N\text{-Prealgebraic} & & N\text{-Family Correspondence}
\end{tikzcd}
\tag{ix}
$$

Now, the following correspondence between theory family lattices may be established, which adapts Theorem 2.5 of [2] to π-institutions.

COROLLARY 4.8. *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on* SEN, *is a finitary, N-rule based, N-protoalgebraic π-institution and T a theory system of $\mathcal{I}$. Then $T' \mapsto T'/\Omega^N(T)$ establishes a lattice isomorphism between the lattice of all theory families of $\mathcal{I}$ $\le$-containing T and the lattice of all theory families of $\mathcal{I}^T/\Omega^N(T)$.*

PROOF. By the N-protoalgebraicity of $\mathcal{I}$ it follows that $T' \mapsto T'/\Omega^N(T)$ is well-defined, since, by Theorem 3, $\mathcal{I}$ has the family N-correspondence property, which yields that $T'/\Omega^N(T)$ is a theory family of $\mathcal{I}^T/\Omega^N(T)$. It is straightforward to see that $T' \mapsto T'/\Omega^N(T)$ is order preserving. The mapping $T'' \mapsto \{\phi : \phi/\Omega^N_\Sigma(T) \in T''\}$, for all theory families T'' of $\mathcal{I}^T/\Omega^N(T)$, is its inverse mapping and it is also order preserving. ■

The isomorphism of Corollary 4.8 yields also an isomorphism between the lattice of all theory systems of $\mathcal{I}$ $\le$-containing T and the lattice of all theory systems of $\mathcal{I}^T/\Omega^N(T)$.

COROLLARY 4.9. *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ is an N-protoalgebraic π-institution and T a theory system of $\mathcal{I}$. Then $T' \mapsto T'/\Omega^N(T)$ establishes a lattice isomorphism between the lattice of all theory systems of $\mathcal{I}$ $\le$-containing T and the lattice of all theory systems of $\mathcal{I}^T/\Omega^N(T)$.*

5. The Tarski Operator and Algebraic Systems

Recall from [25] that, given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ and a category N of natural transformations on SEN, an $(\mathcal{I}, N)$-algebraic system is a functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, such that there exists a singleton (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, such that the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN' is N'-reduced. The class of all $(\mathcal{I}, N)$-algebraic systems was denoted by $\mathrm{Alg}^N(\mathcal{I})$.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution and N a category of natural transformations on SEN. A functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ is said to be an $(\mathcal{I}, N)^*$**-algebraic system** if there exist a category N' of natural transformations on SEN', an (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ and a theory family T' of the $\langle F, \alpha\rangle$-min (N, N')-model $\mathcal{I}'^{\min}$ of $\mathcal{I}$ on SEN', such that $\Omega^{N'}(T') = \Delta_{\mathrm{SEN}'}$. The collection of all $(\mathcal{I}, N)^*$-algebraic systems is denoted by $\mathrm{Alg}^N(\mathcal{I})^*$.

Those functors SEN' in $\mathrm{Alg}^N(\mathcal{I})$ and in $\mathrm{Alg}^N(\mathcal{I})^*$, such that an $\langle F, \alpha\rangle$-min (N, N')-model $\mathcal{I}'^{\min}$ witnessing the membership is N'-protoalgebraic are called $(\mathcal{I}, N)^p$- and $(\mathcal{I}, N)^{*p}$-algebraic systems, respectively. The collection of all $(\mathcal{I}, N)^p$-algebraic systems is denoted by $\mathrm{Alg}^N(\mathcal{I})^p$ and the collection of all $(\mathcal{I}, N)^{*p}$-algebraic systems by $\mathrm{Alg}^N(\mathcal{I})^{*p}$.

Note that there could possibly exist two categories of natural transformations N' and N'' on SEN', an (N, N')-epimorphic translation $\langle F, \alpha\rangle$: $\mathrm{SEN} \to^{se} \mathrm{SEN}'$ and an (N, N'')-epimorphic translation $\langle G, \beta\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, such that both the $\langle F, \alpha\rangle$-min (N, N')-model and the $\langle G, \beta\rangle$-min (N, N'')-model of $\mathcal{I}$ on SEN' witness the membership of SEN' to $\mathrm{Alg}^N(\mathcal{I})$ (or $\mathrm{Alg}^N(\mathcal{I})^*$), the first being N'-protoalgebraic and the second not being N''-protoalgebraic. That is the reason why in the preceding paragraph the indefinite article "an" was used in reference to the witnessing model.

It is now shown that, for every π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, the class of all $(\mathcal{I}, N)^*$-algebraic systems is a subclass of the class of all $(\mathcal{I}, N)$-algebraic systems. Thus, in particular, the class of all $(\mathcal{I}, N)^{*p}$-algebraic systems is a subclass of the class of all $(\mathcal{I}, N)^p$-algebraic systems. On the other hand, if SEN' is an $(\mathcal{I}, N)^p$-algebraic system, then it is also an $(\mathcal{I}, N)^{*p}$-algebraic system. Thus, the class of all $(\mathcal{I}, N)^p$-algebraic systems coincides with the class of all $(\mathcal{I}, N)^{*p}$-algebraic systems. This is an analog of Proposition 3.2 of [13] for π-institutions. (See also Theorem 3.4 of [14].) Note, however, the *difference between the two cases.* Roughly speaking, in the model theory of sentential logics every model of a protoalgebraic sentential logic is itself protoalgebraic, whereas this is not necessarily true in the model theory of π-institutions.

It is true, however, for models via surjective logical morphisms as will be shown in Proposition 5.5.

This result is proved, following [13], by first showing that, for any π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, the class of all $(\mathcal{I}, N)^*$-algebraic systems is contained in the class of all $(\mathcal{I}, N)$-algebraic systems. This is the analog of Proposition 2.24 of [13]. Then, it is shown that N'-protoalgebraicity forces the reverse inclusion.

PROPOSITION 5.1. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution and N a category of natural transformations on* SEN. *Then* $\mathrm{Alg}^N(\mathcal{I})^* \subseteq \mathrm{Alg}^N(\mathcal{I})$.

PROOF. Let $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ be a functor in $\mathrm{Alg}^N(\mathcal{I})^*$. Suppose that $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ is an (N, N')-epimorphic translation and that T' is a theory family of the $\langle F, \alpha\rangle$-min (N, N')-model $\mathcal{I}'^{\min}$ of $\mathcal{I}$ on SEN', such that $\Omega^{N'}(T') = \Delta_{\mathrm{SEN}'}$. We obviously have $\widetilde{\Omega}^{N'}(\mathcal{I}'^{\min}) \leq \Omega^{N'}(T')$ and, thus, we obtain $\widetilde{\Omega}^{N'}(\mathcal{I}'^{\min}) = \Delta_{\mathrm{SEN}'}$. Therefore, $\mathcal{I}'^{\min}$ is N'-reduced and, by the definition of $\mathrm{Alg}^N(\mathcal{I})$, SEN' is in $\mathrm{Alg}^N(\mathcal{I})$. ■

Proposition 5.1 immediately yields

COROLLARY 5.2. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution and N a category of natural transformations on* SEN. *Then* $\mathrm{Alg}^N(\mathcal{I})^{*p} \subseteq \mathrm{Alg}^N(\mathcal{I})^p$.

The two classes compared in Corollary 5.2 are actually identical.

PROPOSITION 5.3. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution and N a category of natural transformations on* SEN. *Then* $\mathrm{Alg}^N(\mathcal{I})^{*p} = \mathrm{Alg}^N(\mathcal{I})^p$.

PROOF. Since, by Corollary 5.2, we have $\mathrm{Alg}^N(\mathcal{I})^{*p} \subseteq \mathrm{Alg}^N(\mathcal{I})^p$, it suffices to show that $\mathrm{Alg}^N(\mathcal{I})^p \subseteq \mathrm{Alg}^N(\mathcal{I})^{*p}$. To this end, let $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ be in $\mathrm{Alg}^N(\mathcal{I})^p$. Then, there exists an (N, N')-epimorphic translation $\langle F, \alpha\rangle$: $\mathrm{SEN} \to^{se} \mathrm{SEN}'$, such that the $\langle F, \alpha\rangle$-min (N, N')-model $\mathcal{I}'^{\min}$ of $\mathcal{I}$ on SEN' is N'-reduced and N'-protoalgebraic. Thus, on the one hand, $\widetilde{\Omega}^{N'}(\mathcal{I}'^{\min}) = \Delta_{\mathrm{SEN}'}$ and, on the other, by Proposition 3.6, $\Omega^{N'}(\mathrm{Thm}') = \widetilde{\Omega}^{N'}(\mathcal{I}'^{\min})$, where Thm' is the theorem system of $\mathcal{I}'^{\min}$. Hence $\Omega^{N'}(\mathrm{Thm}') = \Delta_{\mathrm{SEN}'}$ and SEN' is in $\mathrm{Alg}^N(\mathcal{I})^{*p}$. ■

Next, a proposition (Proposition 5.5) is formulated to the effect that every (N, N')-model $\mathcal{I}'$ of an N-protoalgebraic π-institution $\mathcal{I}$ via a surjective (N, N')-logical morphism is N'-protoalgebraic. Proposition 5.5 needs for its proof Lemma 5.4, which shows, roughly speaking, that the Leibniz operator commutes with inverse surjective logical morphisms. Lemma 5.5 will provide the monotonicity of the Leibniz operator $\Omega^{N'}$ on the theory families of an (N, N')-model $\mathcal{I}'$ of a π-institution $\mathcal{I}$ via a surjective (N, N')-logical morphism, given the monotonicity of Ω^N on the theory families of $\mathcal{I}$.

LEMMA 5.4. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, N a category of natural transformations on* SEN *and $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ an (N, N')-model of $\mathcal{I}$ via a surjective (N, N')-logical morphism $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$. Then, for every theory family T' of $\mathcal{I}'$ and every $\Sigma \in |\mathbf{Sign}|$, $\Omega^N_\Sigma(\alpha^{-1}(T')) = \alpha^{-1}_\Sigma(\Omega^{N'}_{F(\Sigma)}(T'))$.*

PROOF. First, note that, by Lemma 4.1, $\alpha^{-1}(T')$ is a theory family of $\mathcal{I}$, whence $\Omega^N_\Sigma(\alpha^{-1}(T'))$ is well-defined.

For all $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$, we have that, for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma'), \sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N and $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{k-1}$,

$$\begin{aligned}
&\alpha_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi})) \in T'_{F(\Sigma')}\\
&\qquad\text{iff}\quad \sigma'_{F(\Sigma')}(\alpha^k_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi})) \in T'_{F(\Sigma')}\\
&\qquad\text{iff}\quad \sigma'_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)), \alpha^{k-1}_{\Sigma'}(\vec{\chi})) \in T'_{F(\Sigma')}\\
&\qquad\text{iff}\quad \sigma'_{F(\Sigma')}(\mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi)), \alpha^{k-1}_{\Sigma'}(\vec{\chi})) \in T'_{F(\Sigma')}
\end{aligned}$$

iff, by surjectivity and by the (N, N')-epimorphic property of $\langle F, \alpha\rangle$, for all $\Sigma'' \in |\mathbf{Sign}'|, f' \in \mathbf{Sign}'(F(\Sigma), \Sigma''), \sigma' : \mathrm{SEN}'^k \to \mathrm{SEN}'$ in N', $\vec{\chi}' \in \mathrm{SEN}'(\Sigma'')^{k-1}$, $\sigma'_{\Sigma''}(\mathrm{SEN}'(f')(\alpha_\Sigma(\phi)), \vec{\chi}') \in T'_{\Sigma''}$.

Now, using the characterization of Leibniz N-congruence systems, given in Proposition 2.3, and the string of equivalences presented above, we get that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $\langle \phi, \psi\rangle \in \Omega^N_\Sigma(\alpha^{-1}(T'))$ if and only if, for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma'), \sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N and $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{k-1}$,

$$\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi}) \in \alpha^{-1}_{\Sigma'}(T'_{F(\Sigma')}) \quad\text{iff}\quad \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}) \in \alpha^{-1}_{\Sigma'}(T'_{F(\Sigma')}),$$

which holds if and only if, for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma'), \sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N and $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{k-1}$,

$$\alpha_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi})) \in T'_{F(\Sigma')} \quad\text{iff}\quad \alpha_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})) \in T'_{F(\Sigma')},$$

which is equivalent to, for all $\Sigma'' \in |\mathbf{Sign}'|, f' \in \mathbf{Sign}'(F(\Sigma), \Sigma''), \sigma' : \mathrm{SEN}'^k \to \mathrm{SEN}'$ in N', $\vec{\chi}' \in \mathrm{SEN}'(\Sigma'')^{k-1}$,

$$\sigma'_{\Sigma''}(\mathrm{SEN}'(f')(\alpha_\Sigma(\phi)), \vec{\chi}') \in T'_{\Sigma''} \quad\text{iff}\quad \sigma'_{\Sigma''}(\mathrm{SEN}'(f')(\alpha_\Sigma(\psi)), \vec{\chi}') \in T'_{\Sigma''},$$

i.e., if and only if $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in \Omega^{N'}_{F(\Sigma)}(T')$. ■

Now Lemma 5.4 is used to show, roughly speaking, that protoalgebraicity is preserved by surjective logical morphisms, by showing that monotonicity of the Leibniz operator is transferred to models via surjective logical morphisms.

PROPOSITION 5.5. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, N a category of natural transformations on* SEN *and $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ an (N, N')-model of $\mathcal{I}$ via a surjective (N, N')-logical morphism $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$. If $\mathcal{I}$ is N-protoalgebraic, then $\mathcal{I}'$ is N'-protoalgebraic.*

PROOF. Suppose that T', T'' are theory families of $\mathcal{I}'$, such that $T' \leq T''$. Consider the theory families $\alpha^{-1}(T'), \alpha^{-1}(T'')$ of $\mathcal{I}$. We obviously have $\alpha^{-1}(T') \leq \alpha^{-1}(T'')$, whence, by the N-protoalgebraicity of $\mathcal{I}$,

$$\Omega^N(\alpha^{-1}(T')) \leq \Omega^N(\alpha^{-1}(T'')).$$

Thus, Lemma 5.4, combined with the surjectivity of $\langle F, \alpha\rangle$, yields that $\Omega^{N'}(T') \leq \Omega^{N'}(T'')$ and $\mathcal{I}'$ is indeed N'-protoalgebraic. ■

Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ and N a category of natural transformations on SEN, and, in view of Propositions 5.5 and 5.3, those functors SEN$'$ in $\mathrm{Alg}^N(\mathcal{I})$ and in $\mathrm{Alg}^N(\mathcal{I})^*$, such that an $\langle F, \alpha\rangle$-min (N, N')-model $\mathcal{I}'^{\min}$ witnessing the membership is via a surjective (N, N')-logical morphism are called $(\mathcal{I}, N)^s$- and $(\mathcal{I}, N)^{*s}$-**algebraic systems**, respectively. The collection of $(\mathcal{I}, N)^s$-algebraic systems is denoted by $\mathrm{Alg}^N(\mathcal{I})^s$ and of $(\mathcal{I}, N)^{*s}$-algebraic systems by $\mathrm{Alg}^N(\mathcal{I})^{*s}$.

The reader is advised to look back at the remarks made about the use of the indefinite article "an" towards the beginning of this section, when $\mathrm{Alg}^N(\mathcal{I})^p$ and $\mathrm{Alg}^N(\mathcal{I})^{*p}$ were defined. The same remarks apply in these case as well.

Using these definitions, together with Proposition 5.5, and carrying out a proof almost identical to the proof of Proposition 5.3, we obtain

COROLLARY 5.6. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be an N-protoalgebraic π-institution. Then*

$$\mathrm{Alg}^N(\mathcal{I})^{*s} = \mathrm{Alg}^N(\mathcal{I})^s.$$

Note that, unlike the models referred to in Proposition 5.3, whose properties are not intrinsically related to the properties of $\mathcal{I}$, those referred to in Corollary 5.6 are in fact models that can be determined based entirely on the π-institution $\mathcal{I}$ at hand. This makes the result of Corollary 5.6 more attractive than that of Proposition 5.3.

6. Suszko Congruence Systems

In the case of deductive systems, the equality of the Suszko operator with the Leibniz operator provides a necessary and sufficient condition for protoalgebraicity. This is the content of Theorem 1.5.4 of [8]. An analog of this theorem, providing a characterization of N-protoalgebraic π-institutions in terms of the N-Suszko operator, is provided in Proposition 6.4. The elements of the theory of Section 1.5 of [8] that are needed to describe this partial analog of Theorem 1.5.4 are presented in this section.

Before the exposition, a *notational clarification*: Czelakowski [8] uses the symbol Σ to denote the Suszko operator. Since Σ is heavily used in the present context to denote an arbitrary signature in $|\mathbf{Sign}|$, we will denote the institutional operator corresponding to the Suszko operator of Czelakowski with Θ instead of Σ. Also recall, once more, the notational convention in effect from Equation (ii), since it will be once more in effect in the formulation of the definition of the institutional Suszko operator.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, N a category of natural transformations on SEN and $T = \{T_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ a theory family of $\mathcal{I}$. Define the family of binary relations $\Theta^N(T) = \{\Theta^N_\Sigma(T)\}_{\Sigma\in|\mathbf{Sign}|}$ by letting, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{aligned}&\langle\phi,\psi\rangle \in \Theta^N_\Sigma(T) \quad \text{iff}\\ &\quad C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi})\}) = C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})\}),\end{aligned} \tag{x}$$

for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma'), \sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N and $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{k-1}$.

It turns out that $\Theta^N(T)$, as defined above, is an N-congruence system on SEN that is compatible with the theory family T.

PROPOSITION 6.1. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, N a category of natural transformations on* SEN *and $T = \{T_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ a theory family of $\mathcal{I}$. Then $\Theta^N(T)$ is an N-congruence system on* SEN *that is compatible with the theory family T.*

PROOF. First, it is clear that, for all $\Sigma \in |\mathbf{Sign}|$, $\Theta^N_\Sigma(T)$ is an equivalence relation on $\mathrm{SEN}(\Sigma)$. It is not very difficult to see that $\Theta^N(T)$ is invariant under signature morphisms, whence $\Theta^N(T)$ is an equivalence system on SEN. To show that it is an N-congruence system on SEN, let $\tau : \mathrm{SEN}^n \to \mathrm{SEN}$ be in N, $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)^n$. If $\langle\phi_i, \psi_i\rangle \in \Theta^N_\Sigma(T), i < n$, then, for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma'), \sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{k-1}$, and all $i < n$,

$$C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi_i), \vec{\chi})\}) = C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi_i), \vec{\chi})\}).$$

Hence, we have

$$\begin{array}{l} C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\tau_\Sigma(\vec{\phi})), \vec{\chi})\}) = \\ \begin{array}{cl} = & C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}(f)^n(\vec{\phi})), \vec{\chi})\}) \\ = & C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\tau_{\Sigma'}(\mathrm{SEN}(f)(\psi_0), \mathrm{SEN}(f)(\phi_1), \ldots, \\ & \qquad\qquad \mathrm{SEN}(f)(\phi_{n-1})), \vec{\chi})\}) \\ = & \ldots \\ = & C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\tau_\Sigma(\vec{\psi})), \vec{\chi})\}) \end{array} \end{array}$$

and, therefore, $\langle\tau_\Sigma(\vec{\phi}), \tau_\Sigma(\vec{\psi})\rangle \in \Theta^N_\Sigma(T)$ and $\Theta^N(T)$ is an N-congruence system on SEN.

Finally, to show compatibility with T, suppose that $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi, \psi\rangle \in \Theta^N_\Sigma(T)$, and $\phi \in T_\Sigma$. Taking $\sigma = \iota$, the identity natural transformation, and $f = i_\Sigma$, the identity signature morphism, in Condition (x), we get that, $C_\Sigma(T_\Sigma \cup \{\phi\}) = C_\Sigma(T_\Sigma \cup \{\psi\})$, whence, since $\phi \in T_\Sigma$ and T_Σ is a Σ-theory, we get that $C_\Sigma(T_\Sigma \cup \{\psi\}) = T_\Sigma$. Therefore $\psi \in T_\Sigma$, which shows that $\Theta^N(T)$ is compatible with T. ∎

$\Theta^N(T)$ will be called the **Suszko N-congruence system** corresponding to the theory family T. Θ^N, when viewed as an operator on the theory families of the π-institution $\mathcal{I}$, is called the **N-Suszko operator** and, sometimes denoted by $\Theta^N_\mathcal{I}$ or Θ^N_C if the closure system on SEN needs to be made transparent.

Proposition 6.1 and the definition of the Leibniz N-congruence system $\Omega^N(T)$ of the theory family T immediately yield the following

Corollary 6.2. *Let $\mathcal{I} = \langle\mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, N a category of natural transformations on* SEN. *Then $\Theta^N(T) \leq \Omega^N(T)$, for every theory family $T = \{T_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ of $\mathcal{I}$.*

The Suszko operator, unlike the Leibniz operator, and similarly with the deductive system framework, is always monotone on theory families of a π-institution.

Proposition 6.3. *Let $\mathcal{I} = \langle\mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, N a category of natural transformations on* SEN. *Then $\Theta^N(T^1) \leq \Theta^N(T^2)$, for all theory families T^1, T^2 of $\mathcal{I}$, such that $T^1 \leq T^2$.*

PROOF. Let $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$ and suppose that $\langle \phi, \psi \rangle \in \Theta_\Sigma^N(T^1)$. Then

$$C_{\Sigma'}(T^1_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi})\}) = C_{\Sigma'}(T^1_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})\}),$$

for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma'), \sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N and $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{k-1}$. Therefore,

$$\begin{array}{rcl} \sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi}) & \in & C_{\Sigma'}(T^1_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})\}) \\ & \subseteq & C_{\Sigma'}(T^2_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})\}). \end{array}$$

This proves that

$$C_{\Sigma'}(T^2_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi})\}) \subseteq C_{\Sigma'}(T^2_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})\}).$$

The reverse inclusion follows by symmetry. ■

Similarly with Theorem 1.5.3 of [8], the N-Suszko operator $\Theta_{\mathcal{I}}^N$ of a π-institution $\mathcal{I}$ may be characterized as the operator yielding the largest N-congruence system, such that, when it identifies two Σ-sentences ϕ and ψ, then ϕ and ψ satisfy $C_\Sigma(T_\Sigma \cup \{\phi\}) = C_\Sigma(T_\Sigma \cup \{\psi\})$.

THEOREM 4. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ be a π-institution, N a category of natural transformations on* SEN. *Suppose that, for every theory family T of $\mathcal{I}$, $\mathcal{O}^N(T)$ is an N-congruence system on* SEN, *such that, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$,*

$$\langle \phi, \psi \rangle \in \mathcal{O}_\Sigma^N(T) \quad \textit{implies} \quad C_\Sigma(T_\Sigma \cup \{\phi\}) = C_\Sigma(T_\Sigma \cup \{\psi\}). \tag{xi}$$

Then $\mathcal{O}^N(T) \leq \Theta^N(T)$, for all theory families T of $\mathcal{I}$.

PROOF. Suppose that $\mathcal{O}^N(T)$ is an N-congruence system on SEN, satisfying Condition (xi), and that $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \mathcal{O}_\Sigma^N(T)$. Since $\mathcal{O}^N(T)$ is an equivalence system on SEN, we have that, for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\langle \mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi) \rangle \in \mathcal{O}_{\Sigma'}^N(T)$. Since $\mathcal{O}^N(T)$ is an N-congruence system on SEN, we get, for all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{k-1}$,

$$\langle \sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi}), \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}) \rangle \in \mathcal{O}_{\Sigma'}^N(T).$$

Therefore, by the condition of the hypothesis, we obtain that

$$C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi), \vec{\chi})\}) = C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})\}),$$

i.e., that $\langle \phi, \psi \rangle \in \Theta_\Sigma^N(T)$. Therefore $\mathcal{O}^N(T) \leq \Theta^N(T)$. ■

And, finally, the analog of Theorem 1.5.4 of [8] for π-institutions is now presented.

PROPOSITION 6.4. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, N a category of natural transformations on* SEN. *$\mathcal{I}$ is N-protoalgebraic if and only if, for every theory family T of $\mathcal{I}$, $\Theta^N(T) = \Omega^N(T)$.*

PROOF. The right-to-left implication is an easy consequence of Lemma 3.3 and Proposition 6.3.

Suppose, conversely, that $\mathcal{I}$ is N-protoalgebraic. We have, by Corollary 6.2, that $\Theta^N(T) \leq \Omega^N(T)$, for all theory families T of $\mathcal{I}$. So it suffices to prove that reverse system of inclusions. To this end, following the proof of Theorem 1.10 of [9], Theorem 4 will be used. According to Theorem 4, given a theory family T of $\mathcal{I}$, to show that $\Omega^N(T) \leq \Theta^N(T)$, it suffices to show that, for every theory family T of $\mathcal{I}$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $\langle \phi, \psi\rangle \in \Omega^N(T)$ implies $C_\Sigma(T_\Sigma \cup \{\phi\}) = C_\Sigma(T_\Sigma \cup \{\psi\})$. But this is exactly the condition in the definition of N-protoalgebraicity. ∎

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GEORGE VOUTSADAKIS
School of Mathematics and Computer Science
Lake Superior State University
Sault Sainte Marie, MI 49783
USA
`gvoutsad@lssu.edu`
`http://voutsadakis.com/main.html`