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Categorical Abstract Algebraic Logic: Structurality, protoalgebraicity, and correspondence

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The notion of an $\mathcal{I}$-matrix as a model of a given π-institution $\mathcal{I}$ is introduced. The main difference from the approach followed so far in Categorical Abstract Algebraic Logic (CAAL) and the one adopted here is that an $\mathcal{I}$-matrix is considered modulo the entire class of morphisms from the underlying N-algebraic system of $\mathcal{I}$ into its own underlying algebraic system, rather than modulo a single fixed (N, N')-logical morphism. The motivation for introducing $\mathcal{I}$-matrices comes from a desire to formulate a correspondence property for N-protoalgebraic π-institutions closer in spirit to the one for sentential logics than that considered in CAAL before. As a result, in the previously established hierarchy of syntactically protoalgebraic π-institutions, i. e., those with an implication system, and of protoalgebraic π-institutions, i. e., those with a monotone Leibniz operator, the present paper interjects the class of those π-institutions with the correspondence property, as applied to $\mathcal{I}$-matrices. Moreover, this work on $\mathcal{I}$-matrices enables us to prove many results pertaining to the local deduction-detachment theorems, paralleling classical results in Abstract Algebraic Logic formulated, first, by Czelakowski and Blok and Pigozzi. Those results will appear in a sequel to this paper.

1 Introduction

One of the most important classes of logics considered in Abstract Algebraic Logic is the class of protoalgebraic logics. These are the logics that are characterized by the monotonicity of the Leibniz operator on their theory lattices. They are believed to form the widest class of logics amenable to algebraic study techniques. Another very important characterizing property of protoalgebraic logics is the correspondence property. This property helps in establishing isomorphisms between appropriate filter lattices on logical matrices related by surjective matrix homomorphisms. In turn, the correspondence property lies at the heart of the work of Blok and Pigozzi [3], who provide a host of equivalent conditions for protoalgebraic logics to the property of having a form of the local deduction-detachment theorem. These facts constitute the motivating force behind the development of the theory presented in this paper. More details will be provided now on these connections. The goal is to help the reader position this work in the general landscape of Abstract Algebraic Logic and, in particular, realize its significance in the framework of protoalgebraic logics.

A *sentential logic* or *deductive system* $\mathcal{S}$ is a pair $\mathcal{S} = \langle \mathcal{L}, C_{\mathcal{S}} \rangle$, where $\mathcal{L}$ is an algebraic signature and

$$C_{\mathcal{S}} : \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)) \longrightarrow \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V))$$

is a finitary and structural consequence operator on the absolutely free $\mathcal{L}$-algebra $\mathbf{Fm}_{\mathcal{L}}(V)$ that is generated by a

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denumerable set of variables V. Being a *consequence operator* means that, for every $\Gamma \cup \Delta \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

1. $\varphi \in C_{\mathcal{S}}(\Gamma)$, for every $\varphi \in \Gamma$;
2. $\varphi \in C_{\mathcal{S}}(\Gamma)$ and $\gamma \in C_{\mathcal{S}}(\Delta)$, for every $\gamma \in \Gamma$, imply that $\varphi \in C_{\mathcal{S}}(\Delta)$.

$C_{\mathcal{S}}$ is *finitary* if, for every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, $\varphi \in C_{\mathcal{S}}(\Gamma)$ implies that there exists a finite $\Gamma' \subseteq \Gamma$ such that $\varphi \in C_{\mathcal{S}}(\Gamma')$. Finally, $C_{\mathcal{S}}$ being *structural* means that, for every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$ and every $\mathcal{L}$-endomorphism $h : \mathbf{Fm}_{\mathcal{L}}(V) \longrightarrow \mathbf{Fm}_{\mathcal{L}}(V)$, $\varphi \in C_{\mathcal{S}}(\Gamma)$ implies that $h(\varphi) \in C_{\mathcal{S}}(h(\Gamma))$.

The primary models used in the theory of sentential logics are logical matrices, which provide very flexible tools for characterizing various properties of classes of sentential logics. These characterizations include the correspondence property for protoalgebraic logics and various properties that are tantamount to having the local deduction-detachment theorem, which are the motivating forces behind the present study. Let $\boldsymbol{A} = \langle A, \mathcal{L}^{\boldsymbol{A}} \rangle$ be an $\mathcal{L}$-algebra. A subset $F \subseteq A$ of its universe A is called an *$\mathcal{S}$-filter on $\boldsymbol{A}$* if, for every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$ such that $\varphi \in C_{\mathcal{S}}(\Gamma)$ and all $h : \mathbf{Fm}_{\mathcal{L}}(V) \longrightarrow \boldsymbol{A}$, $h(\Gamma) \subseteq F$ implies $h(\varphi) \in F$. An *$\mathcal{S}$-matrix* $\mathfrak{A} = \langle \boldsymbol{A}, F_{\mathfrak{A}} \rangle$ consists of an $\mathcal{L}$-algebra $\boldsymbol{A} = \langle A, \mathcal{L}^{\boldsymbol{A}} \rangle$ and an $\mathcal{S}$-filter $F_{\mathfrak{A}}$ on $\boldsymbol{A}$. The collection of all $\mathcal{S}$-filters on $\boldsymbol{A}$ is denoted by $\mathrm{Fi}_{\mathcal{S}}(\boldsymbol{A})$. They form a complete lattice under inclusion, which is denoted by $\mathbf{Fi}_{\mathcal{S}}(\boldsymbol{A}) = \langle \mathrm{Fi}_{\mathcal{S}}(\boldsymbol{A}), \subseteq \rangle$. The collection of all $\mathcal{S}$-matrices, on the other hand, is denoted by $\mathrm{Mat}(\mathcal{S})$. The following facts are well-known from the theory of logical matrices. Given an $\mathcal{L}$-homomorphism $h : \boldsymbol{A} \longrightarrow \boldsymbol{B}$ and an $\mathcal{S}$-filter F on $\boldsymbol{B}$, the set $h^{-1}(F)$ is an $\mathcal{S}$-filter on $\boldsymbol{A}$. The collection of all $\mathcal{S}$-filters on the formula algebra $\mathbf{Fm}_{\mathcal{L}}(V)$ coincides with the set of all theories of the logic $\mathcal{S}$, i. e., $\mathrm{Fi}_{\mathcal{S}}(\mathbf{Fm}_{\mathcal{L}}(V)) = \mathrm{Th}(\mathcal{S})$. Finally, the class $\mathrm{Mat}(\mathcal{S})$ of all $\mathcal{S}$-matrices forms a complete semantics for $\mathcal{S}$ in the sense that, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, $\varphi \in C_{\mathcal{S}}(\Gamma)$ if and only if, for all $\mathfrak{A} = \langle \boldsymbol{A}, F_{\mathfrak{A}} \rangle \in \mathrm{Mat}(\mathcal{I})$ and every $\mathcal{L}$-homomorphism $h : \mathbf{Fm}_{\mathcal{L}}(V) \longrightarrow \boldsymbol{A}$, $h(\Gamma) \subseteq F_{\mathfrak{A}}$ implies that $h(\varphi) \in F_{\mathfrak{A}}$.

We turn now to the generation of $\mathcal{S}$-filters. Suppose, again, that $\boldsymbol{A} = \langle A, \mathcal{L}^{\boldsymbol{A}} \rangle$ is an $\mathcal{L}$-algebra and $X \subseteq A$ is a subset of its universe. Then *the $\mathcal{S}$-filter on $\boldsymbol{A}$ generated by X*, denoted by $\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(X)$, is the intersection of all $\mathcal{S}$-filters on $\boldsymbol{A}$ that contain X, i. e.,

$$\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(X) = \bigcap \{F \in \mathrm{Fi}_{\mathcal{S}}(\boldsymbol{A}) \,:\, X \subseteq F\}.$$

Moreover, if $\mathfrak{A} = \langle \boldsymbol{A}, F_{\mathfrak{A}} \rangle$ is an $\mathcal{S}$-matrix, we define

$$\mathrm{Fg}^{\mathfrak{A}}_{\mathcal{S}}(X) = \bigcap \{F \in \mathrm{Fi}_{\mathcal{S}}(\mathfrak{A}) \,:\, X \subseteq F\},$$

where

$$\mathrm{Fi}_{\mathcal{S}}(\mathfrak{A}) = \{F \in \mathrm{Fi}_{\mathcal{S}}(\boldsymbol{A}) \,:\, F_{\mathfrak{A}} \subseteq F\}.$$

It is shown in [3, Proposition 1.2.2] that, for every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, we have $C_{\mathcal{S}}(\Gamma, \varphi) = \mathrm{Fg}^{\mathfrak{A}}_{\mathcal{S}}(\varphi)$, where

$$\mathfrak{A} = \langle \mathbf{Fm}_{\mathcal{L}}(V), C_{\mathcal{S}}(\Gamma) \rangle.$$

A generalization of this property in the categorical framework will be proven in Section 4. Another property that will be abstracted in Section 4 asserts that each quotient of an $\mathcal{S}$-matrix $\mathfrak{A} = \langle \boldsymbol{A}, F_{\mathfrak{A}} \rangle$ by an $\mathcal{L}$-congruence θ that is compatible with $F_{\mathfrak{A}}$ is also an $\mathcal{S}$-matrix. More precisely, an $\mathcal{L}$-congruence θ on $\boldsymbol{A}$ is *compatible with* $F_{\mathfrak{A}}$ if, for all $a, b \in A$, $\langle a, b \rangle \in \theta$ and $a \in F_{\mathfrak{A}}$ imply that $b \in F_{\mathfrak{A}}$. Given such a congruence, denote by $\boldsymbol{A}/\theta$ the quotient algebra of $\boldsymbol{A}$ by θ and by $\mathfrak{A}/\theta$ the pair $\mathfrak{A}/\theta = \langle \boldsymbol{A}/\theta, F_{\mathfrak{A}}/\theta \rangle$. If $\mathfrak{A}$ is an $\mathcal{S}$-matrix, then this quotient $\mathfrak{A}/\theta$ is also an $\mathcal{S}$-matrix. Moreover, the projection homomorphism $\pi^\theta : \boldsymbol{A} \longrightarrow \boldsymbol{A}/\theta$ is a matrix homomorphism

$$\pi^\theta : \mathfrak{A} \longrightarrow \mathfrak{A}/\theta.$$

In general, given two $\mathcal{S}$-matrices $\mathfrak{A} = \langle \boldsymbol{A}, F_{\mathfrak{A}} \rangle$ and $\mathfrak{B} = \langle \boldsymbol{B}, F_{\mathfrak{B}} \rangle$, an $\mathcal{L}$-algebra homomorphism $h : \boldsymbol{A} \longrightarrow \boldsymbol{B}$ is an *$\mathcal{S}$-matrix homomorphism* $h : \mathfrak{A} \longrightarrow \mathfrak{B}$ if $h(F_{\mathfrak{A}}) \subseteq F_{\mathfrak{B}}$.

Finally, the last section of the paper discusses an abstraction of the correspondence property. A deductive system $\mathcal{S}$ is said to have *the correspondence property* if, for every surjective matrix homomorphism $h : \mathfrak{A} \longrightarrow \mathfrak{B}$ and each $\mathcal{S}$-filter F of $\mathfrak{A}$, we have that $h^{-1}(\mathrm{Fg}^{\mathfrak{B}}_{\mathcal{S}}(h(F))) = F \vee^{\mathbf{Fi}_{\mathcal{S}}(\mathfrak{A})} h^{-1}(F_{\mathfrak{B}})$. It is asserted in [3, Theorem 1.4.1] that $\mathcal{S}$ is protoalgebraic if and only if it has the correspondence property. A partial analog of this result for π-institutions will constitute the main result of Section 5 of the present work.

An outline of the contents of the paper is given next.

In Section 2 the notions of a finitary π-institution and of an N-rule-based π-institution are recalled. Some basic definitions pertaining to N-algebraic systems and N-morphisms are also reviewed. The notion of an N-structural π-institution, based on that of an N-morphism between two N-algebraic systems, is introduced, closing the section. This notion requires, roughly speaking, that the closure operator be invariant under all N-endomorphisms of the underlying N-algebraic system of the π-institution under consideration.

Section 3 starts with the definition of the notion of an $\mathcal{I}$-filter on a given N-algebraic system. This definition takes after the corresponding definition for sentential logics, but it is more involved in the categorical framework due to the presence of signature morphisms. Roughly speaking, an $\mathcal{I}$-filter must preserve inferences not only modulo N-morphisms but also modulo signature morphisms. As in the sentential logic framework, an $\mathcal{I}$-matrix is then defined as a pair consisting of an N-algebraic system together with an $\mathcal{I}$-filter on that system. In this section, three propositions are proven that show that $\mathcal{I}$-matrices satisfy many properties satisfied by ordinary logical matrices in the deductive system framework. This is the first indication that these definitions may be useful in further developing the categorical theory along the lines of the classical theory. In the first proposition, it is shown that the inverse images of $\mathcal{I}$-filters under N-morphisms are also $\mathcal{I}$-filters. The second proposition asserts that for N-structural π-institutions, the collection of $\mathcal{I}$-filters on the underlying N-algebraic system of the π-institution coincides with the collection of all theory families of the π-institution. Finally, in the third proposition, it is proven that the collection of all $\mathcal{I}$-matrices for an N-structural π-institution $\mathcal{I}$ forms a complete semantics for $\mathcal{I}$.

In Section 4, the discussion veers towards the filters that are generated by given collections of sentences of the π-institution under consideration. Notice that this operator is more complicated than the corresponding one for sentential logics. This is due to the fact that it takes into account the multiplicity of signatures. More precisely, not only is it possible to consider an $\mathcal{I}$-filter generated by a given set of Σ-sentences of an N-algebraic system, but, in addition, we may also consider the $\mathcal{I}$-filter generated by sets of Σ_i-sentences, for i in some set I, over different signatures of a given N-algebraic system. However, in the case where only sentences over a single signature are considered, a result paralleling the one for sentential logics is obtained. Namely, it is shown that $C_\Sigma(\Phi, \varphi)$ coincides with the $\mathcal{I}$-filter generated by the Σ-sentence φ on the $\mathcal{I}$-matrix generated by the set of Σ-sentences Φ. Section 4 closes with a detailed study of the property of lifting of N-quotients. This property is introduced to ensure that the quotient of a given $\mathcal{I}$-matrix modulo an N-congruence system that is compatible with the $\mathcal{I}$-filter of the $\mathcal{I}$-matrix is also an $\mathcal{I}$-matrix. While this property is obtained for free in the sentential logic framework, it cannot be proven for π-institutions without any special conditions. This shortcoming is the result of requiring that N-morphisms be natural.

Finally, in the last section of the paper, Section 5, the N-correspondence property as pertaining to $\mathcal{I}$-matrices is introduced and its relation with N-protoalgebraicity is studied in some detail. The section starts with a review of the definition of an N-protoalgebraic π-institution. Then the N-correspondence property is introduced. Similarly to the sentential logic framework, it is shown that, given a π-institution with the N-correspondence property, every surjective $\mathcal{I}$-morphism with an isomorphic functor component induces an isomorphism between corresponding $\mathcal{I}$-filter lattices. Moreover, it is proven that if an N-structural π-institution that admits lifting of N-quotients has the N-correspondence property, then it is N-protoalgebraic. Finally, the property of transferability of N-rules is introduced and, in another interesting result of this section, it is asserted that, for a π-institution $\mathcal{I}$ that has transferable N-rules and admits lifting of N-quotients, if $\mathcal{I}$ has an N-implication system, it has the N-correspondence property. Taken together, the results of Section 5 show that, under certain conditions on the π-institution $\mathcal{I}$, the property of having an N-implication system implies the N-correspondence property, which, in turn, implies N-protoalgebraicity. These three properties turn out to be equivalent properties in the deductive system framework. Despite the fact that no relevant counterexamples exist, it is conjectured that they are not, in general, equivalent properties in the categorical framework.

The current state-of-art in Abstract Algebraic Logic is detailed in the review article [7]. For more details the monograph [6] and the book [5] are recommended. For all unexplained categorical notation, the reader is referred to any of the standard references [1, 4, 8].

2 N-structurality and N-firs π-institutions

In this paper, we will sometimes deal with π-institutions $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with a category N of natural transformations on SEN, that are finitary, N-rule-based, and N-structural. The first two terms have been defined for π-institutions before and will be recalled below, while the last will be introduced in the sequel for the first time.

Recall that $\mathcal{I}$ being *finitary* means that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\varphi \in C_\Sigma(\Phi) \text{ implies } \varphi \in C_\Sigma(\Psi), \text{ for some finite } \Psi \subseteq \Phi.$$

This is also equivalent to the condition $C_\Sigma(\Phi) = \bigcup_{\Psi \subseteq_\omega \Phi} C_\Sigma(\Psi)$, for all $\Phi \subseteq \mathrm{SEN}(\Sigma)$ and $\Sigma \in |\mathbf{Sign}|$, where by $\subseteq_\omega$ is denoted the "finite subset" relation. Finitary π-institutions have been at the focus of the investigations in [9, Section 2].

For a functor SEN, with a category N of natural transformations on SEN, the collection $\mathrm{Te}^N(\mathrm{SEN})$ of *N-formulas over* SEN or, in accordance with the theory of sentential logics, *N-terms over* SEN is defined, in the present context, to be the collection of all natural transformations in N of the form $\sigma : \mathrm{SEN}^k \longrightarrow \mathrm{SEN}$, for some $k \in \omega$.

By an *N-rule of inference* or, simply, an *N-rule of* SEN [12] it is understood a member r of the Cartesian product $\mathcal{P}(\mathrm{Te}^N(\mathrm{SEN})) \times \mathrm{Te}^N(\mathrm{SEN})$. Such a rule is denoted $r = \langle X, \sigma\rangle$, where $X \subseteq \mathrm{Te}^N(\mathrm{SEN})$, $\sigma \in \mathrm{Te}^N(\mathrm{SEN})$. *The length of the N-rule* $r = \langle X, \sigma\rangle$ is the cardinal number $|r| = |X|^+$. The N-rule r is *axiomatic* if its length is equal to 1. Otherwise, it is a *proper* N-rule of inference. This means that if an N-rule $r = \langle X, \sigma\rangle$ is axiomatic, then $X = \emptyset$. Finally, the N-rule $r = \langle X, \sigma\rangle$ is *finitary* if $|r| < \omega$. A finitary π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is called *N-rule-based* if, for every $\Sigma \in |\mathbf{Sign}|$ and all $\varphi_0, \ldots, \varphi_{n-1}, \varphi \in \mathrm{SEN}(\Sigma)$, if $\varphi \in C_\Sigma(\varphi_0, \ldots, \varphi_{n-1})$, then there exist natural transformations $\sigma^{\langle\Sigma,\varphi_0\rangle}, \ldots, \sigma^{\langle\Sigma,\varphi_{n-1}\rangle}, \sigma^{\langle\Sigma,\varphi\rangle} : \mathrm{SEN}^k \longrightarrow \mathrm{SEN}$ in N and $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$ such that

1. $\langle\{\sigma^{\langle\Sigma,\varphi_0\rangle}, \ldots, \sigma^{\langle\Sigma,\varphi_{n-1}\rangle}\}, \sigma^{\langle\Sigma,\varphi\rangle}\rangle$ is an N-rule of $\mathcal{I}$;
2. $\sigma_\Sigma^{\langle\Sigma,\varphi_i\rangle}(\vec{\chi}) = \varphi_i$, $i < n$, and $\sigma_\Sigma^{\langle\Sigma,\varphi\rangle}(\vec{\chi}) = \varphi$.

Rule-basedness was first introduced in [12, Definition 3.3] (for not necessarily finitary π-institutions) with the goal of providing an analog of Bloom's well-known theorem for π-institutions.

Let $\mathrm{SEN} : \mathbf{Sign} \longrightarrow \mathbf{Set}$ be a functor and N be a category of natural transformations on SEN. An *N-algebraic system* is a triple $\boldsymbol{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, where $\mathrm{SEN}' : \mathbf{Sign}' \longrightarrow \mathbf{Set}$ is a set-valued functor on a category $\mathbf{Sign}'$, N' is a category of natural transformations on SEN', and $F' : N \longrightarrow N'$ is a surjective functor that preserves projections. Given two N-algebraic systems $\boldsymbol{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, $\boldsymbol{B} = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$, an *$N$-(algebraic) morphism* $\langle F, \alpha\rangle : \boldsymbol{A} \longrightarrow \boldsymbol{B}$ is a singleton (N', N'')-epimorphic translation $\langle F, \alpha\rangle : \mathrm{SEN}' \longrightarrow^{\mathrm{se}} \mathrm{SEN}''$ such that the following triangle commutes:

$$\begin{tikzcd} & N \arrow[dl, "F'"'] \arrow[dr, "F''"] & \\ N' \arrow[rr, dash, dashed] & & N'', \end{tikzcd}$$

where the dotted line represents the two-way correspondence established by the (N', N'')-epimorphic property of $\langle F, \alpha\rangle$.

A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is called *N-structural* if, for all N-(endo)morphisms $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \longrightarrow \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle$, $\langle F, \alpha\rangle : \mathcal{I} \succ^{\mathrm{se}} \mathcal{I}$ is an (N, N)-logical morphism, i. e., an (N, N)-epimorphic translation that is a semi-interpretation. More explicitly, this means that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)$, $\alpha_\Sigma(C_\Sigma(\Phi)) \subseteq C_{F(\Sigma)}(\alpha_\Sigma(\Phi))$.

Definition 2.1 A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is called *N-firs* if it is finitary, N-rule-based, and N-structural.

3 $\mathcal{I}$-matrices and completeness

Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, and an N-algebraic system $\boldsymbol{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, an axiom family T' of SEN', i. e., a family $T' = \{T'_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}'|}$ such that $T'_{\Sigma'} \subseteq \mathrm{SEN}'(\Sigma')$ for all $\Sigma' \in |\mathbf{Sign}'|$, is called an *$\mathcal{I}$-filter on $\boldsymbol{A}$* if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$ such that $\varphi \in C_\Sigma(\Phi)$,

$$\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T'_{F(\Sigma')} \text{ implies } \alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \in T'_{F(\Sigma')},$$

for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, and every N-morphism $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \longrightarrow \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$.

Notice the universal quantification over the N-morphism $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \longrightarrow \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$. This gives the distinctive flavor, mentioned in the abstract and in the introduction, to the current investigations, as contrasted to previous ones in CAAL. In all previous contexts, matrices, which were referred to as matrix systems, were considered over a fixed (N, N')-logical morphism.

The collection of all $\mathcal{I}$-filters on $\boldsymbol{A}$ is denoted by $\mathrm{Fi}^{\mathcal{I}}(\boldsymbol{A})$. It is easy to see that

$$\nabla^{\mathrm{SEN}'} = \{\mathrm{SEN}'(\Sigma')\}_{\Sigma' \in |\mathbf{Sign}'|} \in \mathrm{Fi}^{\mathcal{I}}(\boldsymbol{A})$$

and that $\mathrm{Fi}^{\mathcal{I}}(\boldsymbol{A})$ is closed under signature-wise intersections. Therefore, the pair $\mathbf{Fi}^{\mathcal{I}}(\boldsymbol{A}) := \langle \mathrm{Fi}^{\mathcal{I}}(\boldsymbol{A}), \leq\rangle$ is a complete lattice, where $\leq$ denotes signature-wise inclusion.

Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, an *$\mathcal{I}$-matrix* is a pair $\mathfrak{A} = \langle \boldsymbol{A}, T'\rangle$, where

1. $\boldsymbol{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$ is an N-algebraic system,
2. T' is an $\mathcal{I}$-filter on $\boldsymbol{A}$.

The collection of all $\mathcal{I}$-matrices will be denoted by $\mathrm{Mat}(\mathcal{I})$.

Three well-known results from the theory of sentential logics will now be formulated in the context of π-institutions. The first result states that, given a sentential logic $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}}\rangle$, for every $\mathcal{L}$-algebra homomorphism $h : \boldsymbol{A} \longrightarrow \boldsymbol{B}$ and every $\mathcal{S}$-filter $G \in \mathrm{Fi}_{\mathcal{S}}(\boldsymbol{B})$ on $\boldsymbol{B}$, the set $h^{-1}(G)$ is also an $\mathcal{S}$-filter on $\boldsymbol{A}$, i. e., $\mathcal{S}$-filters are preserved under inverse homomorphic images. The second states that the collection of all $\mathcal{S}$-filters on the formula algebra $\mathbf{Fm}_{\mathcal{L}}(V)$ coincides with the theories of the logic, i. e., that $\mathrm{Fi}_{\mathcal{S}}(\mathbf{Fm}_{\mathcal{L}}(V)) = \mathrm{Th}(\mathcal{S})$. Finally, the third states that the class $\mathrm{Mat}(\mathcal{S})$ of all $\mathcal{S}$-matrices forms a complete semantics for the logic $\mathcal{S}$.

Given

1. two sentence functors $\mathrm{SEN} : \mathbf{Sign} \longrightarrow \mathbf{Set}$ and $\mathrm{SEN}' : \mathbf{Sign}' \longrightarrow \mathbf{Set}$,
2. a translation $\langle F, \alpha\rangle : \mathrm{SEN} \longrightarrow \mathrm{SEN}'$,
3. an axiom family $T' = \{T'_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}'|}$ of SEN′,

recall that by $\alpha^{-1}(T')$ is denoted the axiom family of SEN defined by $\alpha^{-1}(T') = \{\alpha_\Sigma^{-1}(T'_{F(\Sigma)})\}_{\Sigma \in |\mathbf{Sign}|}$.

Proposition 3.1 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution and N be a category of natural transformations on* SEN. *Let $\boldsymbol{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, $\boldsymbol{B} = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$ be two N-algebraic systems, $\langle F, \alpha\rangle : \boldsymbol{A} \longrightarrow \boldsymbol{B}$ be an N-morphism, and $T'' \in \mathrm{Fi}^{\mathcal{I}}(\boldsymbol{B})$. Then $\alpha^{-1}(T'') \in \mathrm{Fi}^{\mathcal{I}}(\boldsymbol{A})$.*

P r o o f. Suppose that $\langle G, \beta\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \longrightarrow \boldsymbol{A}$ is an N-morphism, $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\varphi \in C_\Sigma(\Phi)$, $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, such that $\beta_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq \alpha^{-1}_{G(\Sigma')}(T''_{F(G(\Sigma'))})$. Then we have that

$$\alpha_{G(\Sigma')}(\beta_{\Sigma'}(\mathrm{SEN}(f)(\Phi))) \subseteq T''_{F(G(\Sigma'))}.$$

Hence, since $T'' \in \mathrm{Fi}^{\mathcal{I}}(\boldsymbol{B})$ and $\langle FG, \alpha_G\beta\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \longrightarrow \boldsymbol{B}$ is an N-morphism, we obtain that

$$\alpha_{G(\Sigma')}(\beta_{\Sigma'}(\mathrm{SEN}(f)(\varphi))) \in T''_{F(G(\Sigma'))},$$

i. e., $\beta_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \in \alpha^{-1}_{G(\Sigma')}(T''_{F(G(\Sigma'))})$, showing that $\alpha^{-1}(T'') \in \mathrm{Fi}^{\mathcal{I}}(\boldsymbol{A})$. □

Next, it is shown that, for an N-structural π-institution $\mathcal{I}$, the collection of all theory families $\mathrm{ThFam}(\mathcal{I})$ coincides with the collection of all $\mathcal{I}$-filters on the N-algebraic system $\langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle$.

Proposition 3.2 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN, *is an N-structural π-institution. Then $\mathrm{ThFam}(\mathcal{I}) = \mathrm{Fi}^{\mathcal{I}}(\langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle)$.*

P r o o f. For simplicity of notation, let $\boldsymbol{A} = \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle$.

Suppose, first, that $T \in \mathrm{ThFam}(\mathcal{I})$. To show that $T \in \mathrm{Fi}^{\mathcal{I}}(\boldsymbol{A})$, let $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$ such that $\varphi \in C_\Sigma(\Phi)$. Let, also, $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, and let $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \longrightarrow \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle$ be an N-endomorphism such that $\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T_{F(\Sigma')}$. Then we have

$$\begin{aligned}
\alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) &\in \alpha_{\Sigma'}(\mathrm{SEN}(f)(C_\Sigma(\Phi))) && \text{(since } \varphi \in C_\Sigma(\Phi)) \\
&\subseteq \alpha_{\Sigma'}(C_{\Sigma'}(\mathrm{SEN}(f)(\Phi))) && \text{(since } C \text{ is a closure system on SEN)} \\
&\subseteq C_{\Sigma'}(\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi))) && \text{(by } N\text{-structurality)} \\
&\subseteq C_{\Sigma'}(T_{F(\Sigma')}) && \text{(by hypothesis)} \\
&= T_{F(\Sigma')} && \text{(by hypothesis).}
\end{aligned}$$

Thus, $T \in \mathrm{Fi}^{\mathcal{I}}(\boldsymbol{A})$. Note the crucial role that N-structurality played in this part of the proof.

Suppose, conversely, that $T \in \mathrm{Fi}^{\mathcal{I}}(\boldsymbol{A})$. To see that $T \in \mathrm{ThFam}(\mathcal{I})$, suppose $\Sigma \in |\mathbf{Sign}|$, $\varphi \in \mathrm{SEN}(\Sigma)$ such that $\varphi \in C_\Sigma(T_\Sigma)$. Then, since $\langle \mathrm{I}, \iota\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \longrightarrow \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle$ is an N-endomorphism such that

$$\iota_\Sigma(T_\Sigma) = T_\Sigma \subseteq T_\Sigma \in \mathrm{Fi}_{\mathcal{I}}(\boldsymbol{A}) \quad \text{and} \quad \varphi \in C_\Sigma(T_\Sigma),$$

we must have $\varphi = \iota_\Sigma(\varphi) \in T_\Sigma$. Hence $C_\Sigma(T_\Sigma) \subseteq T_\Sigma$ and, therefore, $T \in \mathrm{ThFam}(\mathcal{I})$. □

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, be a π-institution and

$$\mathsf{M} = \{\langle \boldsymbol{A}^i, T^i\rangle \,:\, i \in I\}$$

be a class of $\mathcal{I}$-matrices, with $\boldsymbol{A}^i = \langle \mathrm{SEN}^i, \langle N^i, F^i\rangle\rangle$, $i \in I$. Define the triple $\mathcal{I}^{\mathsf{M}} = \langle \mathbf{Sign}, \mathrm{SEN}, C^{\mathsf{M}}\rangle$ by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\begin{aligned}
\varphi \in C^{\mathsf{M}}_\Sigma(\Phi) \quad \text{iff} \quad & \alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T^i_{F(\Sigma')} \text{ implies } \alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \in T^i_{F(\Sigma')}, \\
& \text{for all } \Sigma' \in |\mathbf{Sign}|,\ f \in \mathbf{Sign}(\Sigma, \Sigma'), \text{ and every } N\text{-morphism} \\
& \langle F, \alpha\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \longrightarrow \langle \mathrm{SEN}^i, \langle N^i, F^i\rangle\rangle,\ i \in I.
\end{aligned}$$

It is shown next that the triple $\mathcal{I}^{\mathsf{M}} = \langle \mathbf{Sign}, \mathrm{SEN}, C^{\mathsf{M}}\rangle$ is a π-institution.

Proposition 3.3 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be a π-institution and M be a class of $\mathcal{I}$-matrices. Then $\mathcal{I}^{\mathsf{M}} = \langle \mathbf{Sign}, \mathrm{SEN}, C^{\mathsf{M}}\rangle$ is a π-institution.*

P r o o f. First, it is shown that, for every $\Sigma \in |\mathbf{Sign}|$, $C^{\mathsf{M}}_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \longrightarrow \mathcal{P}(\mathrm{SEN}(\Sigma))$ is a closure operator on $\mathrm{SEN}(\Sigma)$. Since reflexivity and monotonicity are obvious, we only check idempotency.

Suppose that $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$ such that $\varphi \in C^{\mathsf{M}}_\Sigma(C^{\mathsf{M}}_\Sigma(\Phi))$. This means that, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, all $i \in I$, and all $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \longrightarrow \langle \mathrm{SEN}^i, \langle N^i, F^i\rangle\rangle$, we have that

$$\alpha_{\Sigma'}(\mathrm{SEN}(f)(C^{\mathsf{M}}_\Sigma(\Phi))) \subseteq T^i_{F(\Sigma')} \text{ implies } \alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \in T^i_{F(\Sigma')}. \tag{1}$$

Hence, if $\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T^i_{F(\Sigma')}$, then, by the definition of C^{M}, we get that

$$\alpha_{\Sigma'}(\mathrm{SEN}(f)(C^{\mathsf{M}}_\Sigma(\Phi))) \subseteq T^i_{F(\Sigma')},$$

which, by condition (1), implies that $\alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \in T^i_{F(\Sigma')}$, showing that $\varphi \in C^{\mathsf{M}}_\Sigma(\Phi)$. Thus,

$$C^{\mathsf{M}}_\Sigma(C^{\mathsf{M}}_\Sigma(\Phi)) \subseteq C^{\mathsf{M}}_\Sigma(\Phi),$$

which, together with monotonicity, yields idempotency.

Finally, it remains to show that C^{M} is a closure system on SEN, i. e., that we have, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, $\mathrm{SEN}(f)(C^{\mathsf{M}}_{\Sigma_1}(\Phi)) \subseteq C^{\mathsf{M}}_{\Sigma_2}(\mathrm{SEN}(f)(\Phi))$, for all $\Phi \subseteq \mathrm{SEN}(\Sigma_1)$.

Suppose, to this end, that $\varphi \in C^{\mathsf{M}}_{\Sigma_1}(\Phi)$. Let $\Sigma' \in |\mathbf{Sign}|$, $g \in \mathbf{Sign}(\Sigma_2, \Sigma')$, and

$$\langle F, \alpha\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \longrightarrow \langle \mathrm{SEN}^i, \langle N^i, F^i\rangle\rangle \text{ such that } \alpha_{\Sigma'}(\mathrm{SEN}(g)(\mathrm{SEN}(f)(\Phi))) \subseteq T^i_{F(\Sigma')}.$$

$$\begin{tikzcd} \Sigma_1 \arrow[rr, "f"] \arrow[dr] & & \Sigma_2 \arrow[dl, "g"] \\ & \Sigma'. & \end{tikzcd}$$

This implies that $\alpha_{\Sigma'}(\mathrm{SEN}(gf)(\Phi)) \subseteq T^i_{F(\Sigma')}$, whence, since $\varphi \in C^{\mathsf{M}}_{\Sigma_1}(\Phi)$, we get $\alpha_{\Sigma'}(\mathrm{SEN}(gf)(\varphi)) \in T^i_{F(\Sigma')}$. Therefore $\alpha_{\Sigma'}(\mathrm{SEN}(g)(\mathrm{SEN}(f)(\varphi))) \in T^i_{F(\Sigma')}$, giving that $\mathrm{SEN}(f)(\varphi) \in C^{\mathsf{M}}_{\Sigma_2}(\mathrm{SEN}(f)(\Phi))$ and concluding the argument that C^{M} is also structural. □

Now, the proposition asserting the completeness of the class of all $\mathcal{I}$-matrices as a semantics for an N-structural π-institution $\mathcal{I}$ takes the following form.

Proposition 3.4 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, is an N-structural π-institution. Then $C = C^{\mathrm{Mat}(\mathcal{I})}$.*

Proof. Let $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$.

Suppose, first, that $\varphi \in C_\Sigma(\Phi)$. Consider $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, and

$$\langle F, \alpha\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \longrightarrow \langle \mathrm{SEN}', \langle N', F'\rangle\rangle, \text{ with } \langle\langle \mathrm{SEN}', \langle N', F'\rangle\rangle, T'\rangle \in \mathrm{Mat}(\mathcal{I}),$$

such that $\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T'_{F(\Sigma')}$. Then, since

$$\langle\langle \mathrm{SEN}', \langle N', F'\rangle\rangle, T'\rangle \in \mathrm{Mat}(\mathcal{I}),$$

we have that $T' \in \mathrm{Fi}^{\mathcal{I}}(\langle \mathrm{SEN}', \langle N', F'\rangle\rangle)$, whence we obtain that $\alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \in T'_{F(\Sigma')}$, which shows that $\varphi \in C^{\mathrm{Mat}(\mathcal{I})}_\Sigma(\Phi)$.

Suppose, conversely, that $\varphi \in C^{\mathrm{Mat}(\mathcal{I})}_\Sigma(\Phi)$. Thus, for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, and all

$$\langle F, \alpha\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \longrightarrow \langle \mathrm{SEN}', \langle N', F'\rangle\rangle, \text{ with } \langle\langle \mathrm{SEN}', \langle N', F'\rangle\rangle, T'\rangle \in \mathrm{Mat}(\mathcal{I}),$$

we have that

$$\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T'_{F(\Sigma')} \text{ implies } \alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \in T'_{F(\Sigma')}.$$

Therefore, if $T \in \mathrm{ThFam}(\mathcal{I})$ such that $\Phi \subseteq T_\Sigma$, then, since $T \in \mathrm{Fi}^{\mathcal{I}}(\langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle)$ by Proposition 3.2, we have that $\langle\langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle, T\rangle \in \mathrm{Mat}(\mathcal{I})$, which, together with the fact that $\iota_\Sigma(\mathrm{SEN}(i_\Sigma)(\Phi)) = \Phi \subseteq T_{\mathrm{I}(\Sigma)}$, yields that $\varphi = \iota_\Sigma(\mathrm{SEN}(i_\Sigma)(\varphi)) \in T_\Sigma$. Thus $\varphi \in C_\Sigma(\Phi)$ and, therefore, $C^{\mathrm{Mat}(\mathcal{I})} = C$. □

4 $\mathcal{I}$-filter generation and lifting of quotients

Consider a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ and a category N of natural transformations on SEN.

Let $\boldsymbol{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$ be an N-algebraic system and $\mathfrak{A} = \langle \boldsymbol{A}, T'\rangle \in \mathrm{Mat}(\mathcal{I})$ be an $\mathcal{I}$-matrix. A collection $U = \{U_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}'|}$ is called an *$\mathcal{I}$-filter of* $\mathfrak{A}$ if U is an $\mathcal{I}$-filter on $\boldsymbol{A}$ such that $T' \leq U$. The collection of all $\mathcal{I}$-filters of $\mathfrak{A}$, denoted by $\mathrm{Fi}^{\mathcal{I}}\mathfrak{A}$, is closed under arbitrary signature-wise intersections and contains $\nabla^{\mathrm{SEN}'}$, whence it forms a complete lattice $\mathbf{Fi}^{\mathcal{I}}\mathfrak{A} = \langle \mathrm{Fi}^{\mathcal{I}}\mathfrak{A}, \leq\rangle$.

Given $\Sigma \in |\mathbf{Sign}'|$ and $\Phi \subseteq \mathrm{SEN}'(\Sigma)$, denote by $\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(\langle \Sigma, \Phi\rangle)$ *the $\mathcal{I}$-filter of* $\mathfrak{A}$ *generated by* $\langle \Sigma, \Phi\rangle$, defined by

$$\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(\langle \Sigma, \Phi\rangle) = \bigcap\{U \in \mathrm{Fi}^{\mathcal{I}}\mathfrak{A} : \Phi \subseteq U_\Sigma\},$$

where, of course, intersection is taken signature-wise. More generally, given a collection of pairs

$$X = \{\langle \Sigma_i, \Phi_i \rangle : i \in I\},$$

with $\Sigma_i \in |\mathbf{Sign}'|$ and $\Phi_i \subseteq \mathrm{SEN}'(\Sigma_i)$, $i \in I$, denote by $\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(X)$ *the* $\mathcal{I}$*-filter of* $\mathfrak{A}$ *generated by* X, defined by

$$\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(X) = \bigcap\{U \in \mathrm{Fi}^{\mathcal{I}}\mathfrak{A} : \Phi_i \subseteq U_{\Sigma_i}, \text{ for all } i \in I\}.$$

To formulate the following proposition, which forms an analog of [3, Proposition 1.2.2] in the context of π-institutions, it is useful to recall from [10], given $\Sigma \in |\mathbf{Sign}|$ and $\Phi \subseteq \mathrm{SEN}(\Sigma)$, the definition of the least theory family $T^{[\langle \Sigma, \Phi \rangle]}$ containing a given theory family T and such that $\Phi \subseteq T_\Sigma^{[\langle \Sigma, \Phi \rangle]}$. This is the theory family

$$T^{[\langle \Sigma, \Phi \rangle]} = \{T_{\Sigma'}^{[\langle \Sigma, \Phi \rangle]}\}_{\Sigma' \in |\mathbf{Sign}|}$$

defined, for all $\Sigma' \in |\mathbf{Sign}|$, by

$$T_{\Sigma'}^{[\langle \Sigma, \Phi \rangle]} = \begin{cases} C_\Sigma(T_\Sigma, \Phi) & \text{if } \Sigma' = \Sigma, \\ T_{\Sigma'} & \text{otherwise.} \end{cases}$$

Also, recall that $\mathrm{Thm} = \{\mathrm{Thm}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ denotes the theorem system of $\mathcal{I}$, i. e., it is defined by

$$\mathrm{Thm}_\Sigma = C_\Sigma(\emptyset), \text{ for all } \Sigma \in |\mathbf{Sign}|.$$

Proposition 4.1 *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$*, with* N *a category of natural transformations on* SEN*, is an* N*-structural* π*-institution,* $\Sigma \in |\mathbf{Sign}|$*,* $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$*,* $\boldsymbol{A} = \langle \mathrm{SEN}, \langle N, \mathrm{I} \rangle \rangle$*,* $\mathfrak{A} = \langle \boldsymbol{A}, \mathrm{Thm}^{[\langle \Sigma, \Phi \rangle]} \rangle$*. Then* $C_\Sigma(\Phi, \varphi) = \mathrm{Fg}_\Sigma^{\mathcal{I},\mathfrak{A}}(\langle \Sigma, \varphi \rangle)$*.*

Proof. In fact, we have that

$$\begin{aligned} C_\Sigma(\Phi, \varphi) &= \bigcap\{T_\Sigma : T \in \mathrm{ThFam}(\mathcal{I}), \Phi \cup \{\varphi\} \subseteq T_\Sigma\} && \text{(definition of } C_\Sigma) \\ &= \bigcap\{T_\Sigma : T \in \mathrm{Fi}^{\mathcal{I}}\boldsymbol{A}, \Phi \cup \{\varphi\} \subseteq T_\Sigma\} && \text{(Proposition 3.2)} \\ &= \bigcap\{T_\Sigma : T \in \mathrm{Fi}^{\mathcal{I}}\mathfrak{A}, \varphi \in T_\Sigma\} && \text{(definition of } \mathrm{Fi}^{\mathcal{I}}\mathfrak{A}) \\ &= \mathrm{Fg}_\Sigma^{\mathcal{I},\mathfrak{A}}(\langle \Sigma, \varphi \rangle) && \text{(definition of } \mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(\langle \Sigma, \varphi \rangle)). \end{aligned}$$ □

Let

1. $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on SEN, be a π-institution,
2. $\boldsymbol{A} = \langle \mathrm{SEN}', \langle N', F' \rangle \rangle$ be an N-algebraic system,
3. $\mathfrak{A} = \langle \boldsymbol{A}, T' \rangle$ be an $\mathcal{I}$-matrix,
4. $U \in \mathrm{Fi}^{\mathcal{I}}\mathfrak{A}$.

Recall from [10, Section 2] that an N'-congruence system θ on SEN' is called *compatible with* U if $\langle \varphi, \psi \rangle \in \theta_\Sigma$ and $\varphi \in U_\Sigma$ imply that $\psi \in U_\Sigma$, for all $\Sigma \in |\mathbf{Sign}'|$ and all $\varphi, \psi \in \mathrm{SEN}'(\Sigma)$. Recall also [10, Proposition 2.2] that there always exists a largest N'-congruence system on SEN' compatible with U that is denoted by $\Omega^{N'}(U)$ and called *the Leibniz* N'*-congruence system of* U.

Next, a desideratum that seems necessary so that a satisfactory theory of $\mathcal{I}$-matrices, paralleling that pertaining to sentential logics, may be developed will be discussed. A reason will be given as to why this desideratum may fail under our running hypotheses. Then additional assumptions will be made on the π-institution $\mathcal{I}$ to force the desideratum to hold and discussion will be continued in this more restricted framework, where several results that are known to hold for sentential logics, including a form of the correspondence theorem for protoalgebraic deductive systems, will be lifted to the present framework.

Let

1. $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on SEN, be a π-institution,
2. $\boldsymbol{A} = \langle \mathrm{SEN}', \langle N', F' \rangle \rangle$ be an N-algebraic system,
3. $\mathfrak{A} = \langle \boldsymbol{A}, T' \rangle$ be an $\mathcal{I}$-matrix.

In the present context, we would like to have the tuple

$$\mathfrak{A}/\Omega^{N'}(T') = \langle\langle \mathrm{SEN}'/\Omega^{N'}(T'), \langle N'^{\Omega^{N'}(T')}, F'^{\Omega^{N'}(T')}\rangle\rangle, T'/\Omega^{N'}(T')\rangle$$

to also be an $\mathcal{I}$-matrix. Unfortunately, this may fail due to the fact that, given an N-morphism

$$\langle F, \alpha\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \longrightarrow \langle \mathrm{SEN}'/\Omega^{N'}(T'), \langle N'^{\Omega^{N'}(T')}, F'^{\Omega^{N'}(T')}\rangle\rangle,$$

there may not exist an N-morphism $\langle F, \beta\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \longrightarrow \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$ making the following diagram commute:

$$\begin{tikzcd}
 & \mathrm{SEN} \arrow[dl, "\langle F,\beta\rangle"'] \arrow[dr, "\langle F,\alpha\rangle"] & \\
\mathrm{SEN}' \arrow[rr, "\langle \mathrm{I}, \pi^{\Omega^{N'}(T')}\rangle"'] & & \mathrm{SEN}'/\Omega^{N'}(T').
\end{tikzcd}$$

The reason is that $\beta : \mathrm{SEN} \longrightarrow \mathrm{SEN}' \circ F$ needs to be a natural transformation. To fix this shortcoming we stipulate that $\langle \mathrm{SEN}, N\rangle$ admit lifting of quotients according to the following

Definition 4.2 Let $\mathrm{SEN} : \mathbf{Sign} \longrightarrow \mathbf{Set}$ be a functor and N be a category of natural transformations on SEN. The functor SEN is said to *admit lifting of N-quotients* if, for every N-algebraic system $\boldsymbol{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, every N'-congruence system θ on SEN', and every N-morphism $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \longrightarrow \boldsymbol{A}/\theta$, there exists an N-morphism $\langle F, \beta\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \longrightarrow \boldsymbol{A}$ that makes the following diagram commute:

$$\begin{tikzcd}
\mathrm{SEN} \arrow[rr, "\langle F,\alpha\rangle"] \arrow[dr, "\langle F,\beta\rangle"'] & & \mathrm{SEN}'/\theta \\
 & \mathrm{SEN}'. \arrow[ur, "\langle \mathrm{I}, \pi^{\theta}\rangle"'] &
\end{tikzcd}$$

A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, will be said to *admit lifting of N-quotients* if the functor SEN admits lifting of N-quotients. An N-firs π-institution $\mathcal{I}$ will be said to be *N-qfirs* if it admits lifting of N-quotients.

An equivalent characterization of this property is provided by the following proposition.

Proposition 4.3 *Let*

$$\mathrm{SEN} : \mathbf{Sign} \longrightarrow \mathbf{Set}$$

be a functor and N be a category of natural transformations on SEN*. The functor* SEN *admits lifting of N-quotients if and only if, for all N-algebraic systems $\boldsymbol{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, $\boldsymbol{B} = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$, every surjective N-morphism $\langle F, \alpha\rangle : \boldsymbol{A} \longrightarrow \boldsymbol{B}$, with F an isomorphism, and every N-morphism $\langle G, \beta\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \longrightarrow \boldsymbol{B}$, there exists an N-morphism $\langle H, \gamma\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \longrightarrow \boldsymbol{A}$ such that the following diagram commutes:*

$$\begin{tikzcd}
 & \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \arrow[dl, "\langle H,\gamma\rangle"'] \arrow[dr, "\langle G,\beta\rangle"] & \\
\boldsymbol{A} \arrow[rr, "\langle F,\alpha\rangle"'] & & \boldsymbol{B}.
\end{tikzcd}$$

P r o o f. The fact that the given condition implies that $\mathcal{I}$ admits lifting of N-quotients is obvious, because the given condition is clearly stronger than that defining lifting of N-quotients. So it suffices to show that if $\mathcal{I}$ admits lifting of N-quotients, then the condition of the statement holds. For simplicity we show that, for all N-algebraic systems $\boldsymbol{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, $\boldsymbol{B} = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$, with $\mathrm{SEN}', \mathrm{SEN}'' : \mathbf{Sign}' \longrightarrow \mathbf{Set}$, every surjective N-morphism $\langle \mathrm{I}, \alpha\rangle : \boldsymbol{A} \longrightarrow \boldsymbol{B}$, and every N-morphism $\langle G, \beta\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \longrightarrow \boldsymbol{B}$, there is an N-morphism $\langle G, \gamma\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \longrightarrow \boldsymbol{A}$ such that the following diagram commutes:

$$\begin{tikzcd}
 & \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \arrow[dl, "\langle G, \gamma\rangle"'] \arrow[dr, "\langle G, \beta\rangle"] & \\
\boldsymbol{A} \arrow[rr, "\langle \mathrm{I}, \alpha\rangle"'] & & \boldsymbol{B}.
\end{tikzcd}$$

To see that this holds, it suffices to notice that the N-morphism $\langle \mathrm{I}, \alpha\rangle : \boldsymbol{A} \longrightarrow \boldsymbol{B}$ admits a decomposition

$$\mathrm{SEN}' \xrightarrow{\langle \mathrm{I}, \pi^{\langle \mathrm{I},\alpha\rangle}\rangle} \mathrm{SEN}'/\theta^{\langle \mathrm{I},\alpha\rangle} \xrightarrow{\langle \mathrm{I}, \alpha^*\rangle} \mathrm{SEN}'',$$

where

$$\langle \mathrm{I}, \pi^{\langle \mathrm{I},\alpha\rangle}\rangle : \mathrm{SEN}' \longrightarrow \mathrm{SEN}'/\theta^{\langle \mathrm{I},\alpha\rangle}$$

denotes the canonical projection N-morphism from the functor SEN' onto the quotient functor $\mathrm{SEN}'/\theta^{\langle \mathrm{I},\alpha\rangle}$ by the kernel N-congruence system $\theta^{\langle \mathrm{I},\alpha\rangle}$ of $\langle \mathrm{I}, \alpha\rangle$, and $\langle \mathrm{I}, \alpha^*\rangle : \mathrm{SEN}'/\theta^{\langle \mathrm{I},\alpha\rangle} \longrightarrow \mathrm{SEN}''$ is defined, for all $\Sigma \in |\mathbf{Sign}'|$ and all $\varphi \in \mathrm{SEN}'(\Sigma)$, by

$$\alpha^*_\Sigma(\varphi/\theta^{\langle \mathrm{I},\alpha\rangle}_\Sigma) = \alpha_\Sigma(\varphi).$$

It is not very hard to see that $\langle \mathrm{I}, \alpha^*\rangle$ is well-defined and is, indeed, an N-isomorphism. Thus, the conclusion follows by applying lifting of N-quotients to the left triangle of the following diagram to obtain the N-morphism $\langle G, \gamma\rangle : \mathrm{SEN} \longrightarrow \mathrm{SEN}'$:

$$\begin{tikzcd}
 & \mathrm{SEN} \arrow[dl, "\langle G, \gamma\rangle"'] \arrow[d, "\langle G, {\alpha^*_G}^{-1}\beta\rangle" description] \arrow[dr, "\langle G, \beta\rangle"] & \\
\mathrm{SEN}' \arrow[r, "\langle \mathrm{I}, \pi^{\langle \mathrm{I},\alpha\rangle}\rangle"'] & \mathrm{SEN}'/\theta^{\langle \mathrm{I},\alpha\rangle} \arrow[r, "\langle \mathrm{I}, \alpha^*\rangle"'] & \mathrm{SEN}''.
\end{tikzcd}$$

□

The following proposition shows that Definition 4.2 satisfies the desideratum presented at the beginning of the section and, therefore, justifies the introduction of the "lifting of N-quotients" property.

Proposition 4.4 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be a π-institution that admits lifting of N-quotients, $\boldsymbol{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$ be an N-algebraic system, $\mathfrak{A} = \langle \boldsymbol{A}, T'\rangle$ be an $\mathcal{I}$-matrix, and θ be an N'-congruence system on* SEN' *that is compatible with T'. If $\boldsymbol{A}/\theta = \langle {\mathrm{SEN}'}^\theta, \langle {N'}^\theta, {F'}^\theta\rangle\rangle$, then the pair $\mathfrak{A}/\theta = \langle \boldsymbol{A}/\theta, T'/\theta\rangle$ is also an $\mathcal{I}$-matrix.*

P r o o f. It is to show that T'/θ is an $\mathcal{I}$-filter on $\boldsymbol{A}/\theta$. To this end, suppose $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$ such that $\varphi \in C_\Sigma(\Phi)$ and $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \longrightarrow \boldsymbol{A}/\theta$ are such that

$$\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T'_{F(\Sigma')}/\theta_{F(\Sigma')}.$$

Since $\mathcal{I}$ admits lifting of N-quotients, SEN admits lifting of N-quotients, whence there exists

$$\langle F, \beta\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \longrightarrow \boldsymbol{A}$$

such that the following triangle commutes:

$$\begin{tikzcd}
\mathrm{SEN} \arrow[rr, "\langle F,\alpha\rangle"] \arrow[dr, "\langle F,\beta\rangle"'] & & \mathrm{SEN}'/\theta \\
& \mathrm{SEN}'. \arrow[ur, "\langle \mathrm{I},\pi^{\theta}\rangle"'] &
\end{tikzcd}$$

Thus, we have that $\pi^{\theta}_{F(\Sigma')}(\beta_{\Sigma'}(\mathrm{SEN}(f)(\Phi))) \subseteq T'_{F(\Sigma')}/\theta_{F(\Sigma')}$. This, however, is equivalent to

$$\beta_{\Sigma'}(\mathrm{SEN}(f)(\Phi))/\theta_{F(\Sigma')} \subseteq T'_{F(\Sigma')}/\theta_{F(\Sigma')},$$

which, using the postulated compatibility of θ with T', yields $\beta_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T'_{F(\Sigma')}$. Since $\mathfrak{A} = \langle \boldsymbol{A}, T'\rangle$ is an $\mathcal{I}$-matrix, T' is an $\mathcal{I}$-filter on $\boldsymbol{A}$, which, taken together with $\varphi \in C_{\Sigma}(\Phi)$, now gives that

$$\beta_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \in T'_{F(\Sigma')}.$$

Therefore $\pi^{\theta}_{F(\Sigma')}(\beta_{\Sigma'}(\mathrm{SEN}(f)(\varphi))) \in T'_{F(\Sigma')}/\theta_{F(\Sigma')}$, i. e., $\alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \in T'_{F(\Sigma')}/\theta_{F(\Sigma')}$, and T'/θ is indeed an $\mathcal{I}$-filter on $\boldsymbol{A}/\theta$. □

Consider, once more, a π-institution

$$\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle,$$

with N a category of natural transformations on SEN. Let $\boldsymbol{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, $\boldsymbol{B} = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$ be two N-algebraic systems, $\mathfrak{A} = \langle \boldsymbol{A}, T'\rangle$, $\mathfrak{B} = \langle \boldsymbol{B}, T''\rangle$ be two $\mathcal{I}$-matrices. An N-morphism $\langle F, \alpha\rangle : \boldsymbol{A} \longrightarrow \boldsymbol{B}$ is said to be an $\mathcal{I}$-(*matrix*) *morphism from* $\mathfrak{A}$ *to* $\mathfrak{B}$, written

$$\langle F, \alpha\rangle : \mathfrak{A} \longrightarrow \mathfrak{B},$$

if, for all $\Sigma \in |\mathbf{Sign}'|$, $\alpha_{\Sigma}(T'_{\Sigma}) \subseteq T''_{F(\Sigma)}$ or, equivalently, if $T'_{\Sigma} \subseteq \alpha^{-1}_{\Sigma}(T''_{F(\Sigma)})$, for all $\Sigma \in |\mathbf{Sign}'|$, which may also be written as $T' \leq \alpha^{-1}(T'')$.

Let $\mathcal{I}$ be a π-institution that admits lifting of N-quotients. The next proposition asserts that, given an $\mathcal{I}$-matrix $\mathfrak{A} = \langle \boldsymbol{A}, T'\rangle$, with $\boldsymbol{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, and an N'-congruence system θ on SEN' compatible with T', then the N-morphism $\langle \mathrm{I}, \pi^{\theta}\rangle : \boldsymbol{A} \longrightarrow \boldsymbol{A}/\theta$ is an $\mathcal{I}$-morphism $\langle \mathrm{I}, \pi^{\theta}\rangle : \mathfrak{A} \longrightarrow \mathfrak{A}/\theta$ and, moreover, it satisfies

$$T = (\pi^{\theta})^{-1}(T'/\theta).$$

Its proof, given Proposition 4.4, is straightforward and will, therefore, be omitted.

Proposition 4.5 *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$*, with* N *a category of natural transformations on* SEN*, be a* π*-institution that admits lifting of* N*-quotients,* $\boldsymbol{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$ *be an* N*-algebraic system,* $\mathfrak{A} = \langle \boldsymbol{A}, T'\rangle$ *be an* $\mathcal{I}$*-matrix, and* θ *be an* N'*-congruence system on* SEN' *that is compatible with* T'*. Then* $\langle \mathrm{I}, \pi^{\theta}\rangle : \mathfrak{A} \longrightarrow \mathfrak{A}/\theta$ *is an* $\mathcal{I}$*-morphism and, moreover,* $T' = (\pi^{\theta})^{-1}(T'/\theta)$*.*

In particular, in the special case where $\theta = \Omega^{N'}(T')$, we obtain the following corollary.

Corollary 4.6 *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$*, with* N *a category of natural transformations on* SEN*, be a* π*-institution that admits lifting of* N*-quotients,* $\boldsymbol{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$ *be an* N*-algebraic system, and* $\mathfrak{A} = \langle \boldsymbol{A}, T'\rangle$ *be an* $\mathcal{I}$*-matrix. Then* $\langle \mathrm{I}, \pi^{\Omega^{N'}(T')}\rangle : \mathfrak{A} \longrightarrow \mathfrak{A}/\Omega^{N'}(T')$ *is an* $\mathcal{I}$*-morphism and, moreover, it holds that*

$$T' = (\pi^{\Omega^{N'}(T')})^{-1}(T'/\Omega^{N'}(T')).$$

In the sequel, common practice in Abstract Algebraic Logic will be followed in denoting by $\mathfrak{A}^*$ the $\mathcal{I}$-matrix $\mathfrak{A}^* = \langle \boldsymbol{A}/\Omega^{N'}(T'), T'/\Omega^{N'}(T')\rangle$, where $\boldsymbol{A}/\Omega^{N'}(T') = \langle \mathrm{SEN}'^{\Omega^{N'}(T')}, \langle N'^{\Omega^{N'}(T')}, F'^{\Omega^{N'}(T')}\rangle\rangle$.

5 Protoalgebraicity and the correspondence property

The definition of an N-protoalgebraic π-institution from [10] is now recalled. The goal is to use the notion of an $\mathcal{I}$-matrix to study the relation between N-protoalgebraicity and a version of the correspondence property that is closer in spirit to the original version of Blok and Pigozzi [2]. The version of the correspondence property used in [10] refers to a fixed surjective logical morphism. On the other hand, dealing with $\mathcal{I}$-matrices, in the sense of the present work, allows one to reason over the entire collection of N-morphisms, bringing this framework closer in spirit to the original theory of logical matrices.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, be a π-institution. $\mathcal{I}$ is said to be N-*protoalgebraic* if, for every theory family $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}|$, and all $\varphi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle \varphi, \psi\rangle \in \Omega^N_\Sigma(T) \text{ implies } C_\Sigma(T_\Sigma, \varphi) = C_\Sigma(T_\Sigma, \psi).$$

This condition turns out to be equivalent [10, Lemma 3.3] to the monotonicity of the N-Leibniz operator on the theory families of $\mathcal{I}$, i. e., the condition that, for all $T^1, T^2 \in \mathrm{ThFam}(\mathcal{I})$,

$$T^1 \leq T^2 \text{ implies } \Omega^N(T^1) \leq \Omega^N(T^2).$$

Let $\boldsymbol{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, $\boldsymbol{B} = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$ be two N-algebraic systems, $\mathfrak{A} = \langle \boldsymbol{A}, T'\rangle$, $\mathfrak{B} = \langle \boldsymbol{B}, T''\rangle$ be two $\mathcal{I}$-matrices, and $\langle F, \alpha\rangle : \mathfrak{A} \longrightarrow \mathfrak{B}$ be an $\mathcal{I}$-morphism from $\mathfrak{A}$ to $\mathfrak{B}$. Given an $\mathcal{I}$-filter U of $\mathfrak{A}$, consider the collection $\alpha(U) = \{\langle F(\Sigma), \alpha_\Sigma(U_\Sigma)\rangle\}_{\Sigma \in |\mathbf{Sign}'|}$. Note that this tuple may, on the one hand, omit some signatures $\Sigma'' \in |\mathbf{Sign}''|$ and, on the other, may associate two sets with a specific signature $\Sigma'' \in |\mathbf{Sign}''|$. Define

$$\alpha^*(U) = \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\alpha(U)).$$

Thus $\alpha^* : \mathrm{Fi}^{\mathcal{I}}(\mathfrak{A}) \longrightarrow \mathrm{Fi}^{\mathcal{I}}(\mathfrak{B})$, for every $\mathcal{I}$-morphism $\langle F, \alpha\rangle : \mathfrak{A} \longrightarrow \mathfrak{B}$.

Definition 5.1 A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is said to have *the N-correspondence property* if, for every surjective $\mathcal{I}$-morphism $\langle F, \alpha\rangle : \mathfrak{A} \longrightarrow \mathfrak{B}$, with F an isomorphism,

$$\alpha^{-1}(\alpha^*(U)) = U \vee^{\mathbf{Fi}^{\mathcal{I}}(\mathfrak{A})} \alpha^{-1}(T''),$$

for every $U \in \mathrm{Fi}^{\mathcal{I}}(\mathfrak{A})$, where $\mathfrak{A} = \langle \boldsymbol{A}, T'\rangle$, $\mathfrak{B} = \langle \boldsymbol{B}, T''\rangle$, $\boldsymbol{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, $\boldsymbol{B} = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$.

Prior to exploring the relationship between the N-correspondence property and N-protoalgebraicity, we obtain an important consequence of the N-correspondence property. It states that every surjective $\mathcal{I}$-morphism with an isomorphic functor component induces a certain isomorphism between lattices of $\mathcal{I}$-filters. This result generalizes to π-institutions a well-known result from the theory of protoalgebraic deductive systems (see, e. g., [3, p. 82]).

Proposition 5.2 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be a π-institution having the N-correspondence property. Let, also, $\boldsymbol{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, $\boldsymbol{B} = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$ be two N-algebraic systems and $\mathfrak{A} = \langle \boldsymbol{A}, T'\rangle$, $\mathfrak{B} = \langle \boldsymbol{B}, T''\rangle$ be two $\mathcal{I}$-matrices. If $\langle F, \alpha\rangle : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a surjective $\mathcal{I}$-morphism, with F an isomorphism, then α^* induces an isomorphism between the sublattice of $\mathbf{Fi}^{\mathcal{I}}\mathfrak{A}$ with universe $\{U \in \mathrm{Fi}^{\mathcal{I}}\mathfrak{A} : U \geq \alpha^{-1}(T'')\}$ and $\mathbf{Fi}^{\mathcal{I}}\mathfrak{B}$.*

P r o o f. First, note that if $U \in \mathrm{Fi}^{\mathcal{I}}\mathfrak{A}$, then $\alpha^*(U) = \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\alpha(U))$ belongs to $\mathrm{Fi}^{\mathcal{I}}\mathfrak{B}$, whence the given mapping is well-defined.

Next, it is shown that α^* is injective. Let $U, V \in \mathrm{Fi}^{\mathcal{I}}\mathfrak{A}$ such that $\alpha^{-1}(T'') \leq U$ and $\alpha^{-1}(T'') \leq V$, and assume that $\alpha^*(U) = \alpha^*(V)$. Then $\alpha^{-1}(\alpha^*(U)) = \alpha^{-1}(\alpha^*(V))$, whence, by the N-correspondence property,

$$U \vee^{\mathbf{Fi}^{\mathcal{I}}\mathfrak{A}} \alpha^{-1}(T'') = V \vee^{\mathbf{Fi}^{\mathcal{I}}\mathfrak{A}} \alpha^{-1}(T'').$$

But, by the hypothesis, $\alpha^{-1}(T'') \leq U$ and $\alpha^{-1}(T'') \leq V$, whence $U = V$ and α^* is indeed injective.

Finally, α^* is shown to be surjective. Suppose, to this end, that $W \in \mathrm{Fi}^{\mathcal{I}}\mathfrak{B}$. Then $W \in \mathrm{Fi}^{\mathcal{I}}\boldsymbol{B}$, with $T'' \leq W$. Hence $\alpha^{-1}(T'') \leq \alpha^{-1}(W)$ and, therefore, taking into account Proposition 3.1,

$$\alpha^{-1}(W) \vee^{\mathbf{Fi}^{\mathcal{I}}\mathfrak{A}} \alpha^{-1}(T'') = \alpha^{-1}(W).$$

Now, using the N-correspondence property, we get that $\alpha^{-1}(\alpha^*(\alpha^{-1}(W))) = \alpha^{-1}(W)$. Thus, by the surjectivity of $\langle F, \alpha\rangle$, $\alpha^*(\alpha^{-1}(W)) = W$. Hence α^* is also surjective and, therefore, an isomorphism between the sublattice of $\mathbf{Fi}^{\mathcal{I}}\mathfrak{A}$ with universe $\{U \in \mathrm{Fi}^{\mathcal{I}}\mathfrak{A} : U \geq \alpha^{-1}(T'')\}$ and $\mathbf{Fi}^{\mathcal{I}}\mathfrak{B}$. □

Next, the relationship between N-protoalgebraicity and the N-correspondence property is explored. The work starts with a proposition showing that if an N-structural π-institution $\mathcal{I}$, admitting lifting of N-quotients, has the N-correspondence property, then it is N-protoalgebraic.

Proposition 5.3 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, is an N-structural π-institution that admits lifting of N-quotients and has the N-correspondence property. Then it is N-protoalgebraic.*

P r o o f. Suppose that $\mathcal{I}$ is N-structural, admits lifting of N-quotients, and has the N-correspondence property, and let $T^1, T^2 \in \mathrm{ThFam}(\mathcal{I})$ such that $T^1 \leq T^2$. Consider the N-algebraic systems $\boldsymbol{A} = \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle$ and $\boldsymbol{A}/\Omega^N(T^1) = \langle \mathrm{SEN}/\Omega^N(T^1), \langle N^{\Omega^N(T^1)}, F^{\Omega^N(T^1)}\rangle\rangle$. Then the tuple $\mathfrak{A} = \langle \boldsymbol{A}, T^1\rangle$ is an $\mathcal{I}$-matrix, by Proposition 3.2, since $\mathcal{I}$ is N-structural. The tuple $\mathfrak{A}^* = \langle \boldsymbol{A}/\Omega^N(T^1), T^1/\Omega^N(T^1)\rangle$ is also an $\mathcal{I}$-matrix, by Proposition 3.2 and Corollary 4.6, since $\mathcal{I}$ is N-structural and admits lifting of N-quotients. Hence, the natural projection morphism $\langle \mathrm{I}, \pi^{\Omega^N(T^1)}\rangle : \mathfrak{A} \longrightarrow \mathfrak{A}^*$ is an $\mathcal{I}$-morphism. Since, also by N-structurality, $T^2 \in \mathrm{Fi}^{\mathcal{I}}(\mathfrak{A})$, we get, using the N-correspondence property, that

$$(\pi^{\Omega^N(T^1)})^{-1}((\pi^{\Omega^N(T^1)})^*(T^2)) = T^2 \vee^{\mathbf{Fi}^{\mathcal{I}}\mathfrak{A}} (\pi^{\Omega^N(T^1)})^{-1}(T^1/\Omega^N(T^1)),$$

i. e., that

$$(\pi^{\Omega^N(T^1)})^{-1}((\pi^{\Omega^N(T^1)})^*(T^2)) = T^2.$$

This yields that $\Omega^N(T^1)$ is compatible with T^2, i. e., that $\Omega^N(T^1) \leq \Omega^N(T^2)$, and proves that $\mathcal{I}$ is N-protoalgebraic. □

Next we provide an alternative characterization of the N-correspondence property.

Definition 5.4 Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is a π-institution. Let $\boldsymbol{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, $\boldsymbol{B} = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$ be two N-algebraic systems and $\mathfrak{A} = \langle \boldsymbol{A}, T'\rangle$, $\mathfrak{B} = \langle \boldsymbol{B}, T''\rangle$ be two $\mathcal{I}$-matrices. An $\mathcal{I}$-morphism $\langle F, \alpha\rangle : \mathfrak{A} \longrightarrow \mathfrak{B}$ is *strict* if (in addition to $T' \leq \alpha^{-1}(T'')$, it also satisfies) $T'^{\mathrm{c}} \leq \alpha^{-1}(T''^{\mathrm{c}})$, where $^{\mathrm{c}}$ is used to denote signature-wise complementation.

The π-institution $\mathcal{I}$ has *the strict N-correspondence property* if, for each strict $\mathcal{I}$-morphism $\langle F, \alpha\rangle : \mathfrak{A} \longrightarrow \mathfrak{B}$, with F an isomorphism, $U = \alpha^{-1}(\alpha(U))$, for every $U \in \mathrm{Fi}^{\mathcal{I}}(\mathfrak{A})$.

In Proposition 5.6, it will be shown that the N-correspondence property and the strict N-correspondence property are equivalent properties for a large class of π-institutions. We first present a lemma to the effect that, informally speaking, under the presence of lifting of quotients, strict surjective $\mathcal{I}$-morphisms, with isomorphic functor components, and $\mathcal{I}$-filter generation commute with one another.

Lemma 5.5 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be a π-institution that admits lifting of N-quotients, let $\boldsymbol{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, $\boldsymbol{B} = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$ be two N-algebraic systems, and let $\mathfrak{A} = \langle \boldsymbol{A}, T'\rangle$, $\mathfrak{B} = \langle \boldsymbol{B}, T''\rangle$ be two $\mathcal{I}$-matrices. If $\langle F, \alpha\rangle : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a surjective strict $\mathcal{I}$-morphism, with F an isomorphism, then $\alpha(\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(U)) = \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\alpha(U))$, for all $U \in \mathrm{AxFam}(\mathrm{SEN})$.*

P r o o f. We have $\alpha(U) \leq \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\alpha(U))$, whence $U \leq \alpha^{-1}(\mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\alpha(U)))$. But, by Proposition 3.1, the collection $\alpha^{-1}(\mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\alpha(U)))$ is an $\mathcal{I}$-filter of $\mathfrak{A}$, whence we get that $\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(U) \leq \alpha^{-1}(\mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\alpha(U)))$. This entails that $\alpha(\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(U)) \leq \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\alpha(U))$.

To show the reverse inclusion $\mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\alpha(U)) \leq \alpha(\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(U))$, it is enough, given that $\alpha(U) \leq \alpha(\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(U))$, to show that $\alpha(\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(U)) \in \mathrm{Fi}^{\mathcal{I}}\mathfrak{B}$. In turn, since $\langle F, \alpha\rangle$ is surjective and strict, it suffices to show that

$$\alpha(\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(U)) \in \mathrm{Fi}^{\mathcal{I}}\boldsymbol{B}.$$

To this end, suppose that $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$ such that $\varphi \in C_\Sigma(\Phi)$, $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, and $\langle G, \beta\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \longrightarrow \boldsymbol{B}$ are such that

$$\begin{tikzcd} & \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \arrow[d, "{\langle G,\beta\rangle}"] \\ \boldsymbol{A} \arrow[r, "{\langle F,\alpha\rangle}"'] & \boldsymbol{B}, \end{tikzcd} \qquad \beta_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq \alpha_{F^{-1}(G(\Sigma'))}(\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}_{F^{-1}(G(\Sigma'))}(U)).$$

As $\mathcal{I}$ admits lifting of N-quotients, there is, by Proposition 4.3, an N-morphism $\langle H, \gamma\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \longrightarrow \boldsymbol{A}$ such that the following triangle commutes:

$$\begin{tikzcd} & \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \arrow[dl, "{\langle H,\gamma\rangle}"'] \arrow[dr, "{\langle G,\beta\rangle}"] & \\ \boldsymbol{A} \arrow[rr, "{\langle F,\alpha\rangle}"'] & & \boldsymbol{B}. \end{tikzcd}$$

Therefore $\alpha_{H(\Sigma')}(\gamma_{\Sigma'}(\mathrm{SEN}(f)(\Phi))) \subseteq \alpha_{F^{-1}(G(\Sigma'))}(\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}_{F^{-1}(G(\Sigma'))}(U))$. But $\langle F, \alpha\rangle$ is strict, whence we obtain that $\gamma_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq \mathrm{Fg}^{\mathcal{I},\mathfrak{A}}_{H(\Sigma')}(U)$. Therefore, since $\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(U) \in \mathrm{Fi}^{\mathcal{I}}\mathfrak{A}$, we get that

$$\gamma_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \in \mathrm{Fg}^{\mathcal{I},\mathfrak{A}}_{H(\Sigma')}(U).$$

This yields that $\alpha_{H(\Sigma')}(\gamma_{\Sigma'}(\mathrm{SEN}(f)(\varphi))) \in \alpha_{F^{-1}(G(\Sigma'))}(\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}_{F^{-1}(G(\Sigma'))}(U))$, which implies that

$$\beta_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \in \alpha_{F^{-1}(G(\Sigma'))}(\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}_{F^{-1}(G(\Sigma'))}(U)).$$

Thus, we get that $\alpha(\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(U)) \in \mathrm{Fi}^{\mathcal{I}}\boldsymbol{B}$. □

The following proposition shows that a π-institution $\mathcal{I}$, admitting lifting of N-quotients, has the N-correspondence property if and only if it has the strict N-correspondence property.

Proposition 5.6 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be a π-institution that admits lifting of N-quotients. $\mathcal{I}$ has the N-correspondence property if and only if $\mathcal{I}$ has the strict N-correspondence property.*

P r o o f. Suppose, first, that $\mathcal{I}$ has the N-correspondence property and let $\langle F, \alpha\rangle : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a strict $\mathcal{I}$-morphism, with F an isomorphism, and $U \in \mathrm{Fi}^{\mathcal{I}}\mathfrak{A}$. Since it is clear that $U \le \alpha^{-1}(\alpha(U))$, it is sufficient to show the reverse inclusion. To take advantage of the N-correspondence property, we consider the surjective $\mathcal{I}$-morphism $\langle F, \alpha\rangle : \mathfrak{A} \longrightarrow \alpha(\mathfrak{A})$, where $\alpha(\mathfrak{A}) = \langle \alpha(\boldsymbol{A}), T'' \cap \alpha(\mathrm{SEN}')\rangle$. Then, by the N-correspondence property applied to this new $\langle F, \alpha\rangle$ and the filter $U \in \mathrm{Fi}^{\mathcal{I}}\mathfrak{A}$, we get that

$$\alpha^{-1}(\alpha(U)) \le \alpha^{-1}(\alpha^*(U)) = U \vee^{\mathbf{Fi}^{\mathcal{I}}\mathfrak{A}} \alpha^{-1}(T'' \cap \alpha(\mathrm{SEN}')) = U \vee^{\mathbf{Fi}^{\mathcal{I}}\mathfrak{A}} T' = U.$$

Suppose, conversely, that $\mathcal{I}$ has the strict N-correspondence property, and let $\langle F, \alpha\rangle : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a surjective $\mathcal{I}$-morphism, with F an isomorphism, and $U \in \mathrm{Fi}^{\mathcal{I}}\mathfrak{A}$. Note that $U \vee^{\mathbf{Fi}^{\mathcal{I}}\mathfrak{A}} \alpha^{-1}(T'') \le \alpha^{-1}(\alpha^*(U))$ since $U \le \alpha^{-1}(\alpha(U)) \le \alpha^{-1}(\alpha^*(U))$ and also $T'' \le \alpha^*(U)$, whence $\alpha^{-1}(T'') \le \alpha^{-1}(\alpha^*(U))$. Thus, it suffices to show that $\alpha^{-1}(\alpha^*(U)) \le U \vee^{\mathbf{Fi}^{\mathcal{I}}\mathfrak{A}} \alpha^{-1}(T'')$. In fact, we proceed to show full equality. To take advantage of the strict N-correspondence property we consider the strict surjective $\mathcal{I}$-morphism $\langle F, \alpha\rangle : \langle \boldsymbol{A}, \alpha^{-1}(T'')\rangle \longrightarrow \mathfrak{B}$.

Clearly $U \vee^{\mathbf{Fi}^{\mathcal{I}}\mathfrak{A}} \alpha^{-1}(T'') \in \mathrm{Fi}^{\mathcal{I}}(\langle \boldsymbol{A}, \alpha^{-1}(T'')\rangle)$. Therefore, by the strict N-correspondence property, we get

$$\begin{aligned} U \vee^{\mathbf{Fi}^{\mathcal{I}}\mathfrak{A}} \alpha^{-1}(T'') &= \alpha^{-1}(\alpha(U \vee^{\mathbf{Fi}^{\mathcal{I}}\mathfrak{A}} \alpha^{-1}(T''))) \\ &= \alpha^{-1}(\alpha(\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(U, \alpha^{-1}(T'')))) \\ &= \alpha^{-1}(\alpha(\mathrm{Fg}^{\mathcal{I},\langle \boldsymbol{A},\alpha^{-1}(T'')\rangle}(U))) \\ &= \alpha^{-1}(\mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\alpha(U))) && \text{(by Lemma 5.5 and the hypothesis)} \\ &= \alpha^{-1}(\alpha^*(U)). \end{aligned}$$
□

An additional condition on the π-institutions is introduced now that will allow us to formulate a result to the effect that, provided this condition and the property of lifting of quotients, the existence of an implication system implies the correspondence property. The need for this condition stems from the fact that in the present framework there do not necessarily exist surjective translations from the sentence functor of a given π-institution to the underlying functors of each of its matrices. However, notice that this is always the case for deductive systems and their countable matrices due to the freeness of the underlying formula algebra of the deductive system.

Definition 5.7 Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, be a π-institution. $\mathcal{I}$ is said to have *transferable N-rules* if, for every N-rule $r = \langle \{\sigma^0, \dots, \sigma^{n-1}\}, \tau\rangle$ of $\mathcal{I}$, the rule r holds in all $\mathcal{I}$-matrices. This means that, for every $\mathcal{I}$-matrix $\mathfrak{A} = \langle \boldsymbol{A}, T'\rangle$, with $\boldsymbol{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, all $\Sigma \in |\mathbf{Sign}'|$, $\vec{\chi} \in \mathrm{SEN}'(\Sigma)^k$, ${\sigma'}^0_\Sigma(\vec{\chi}), \dots, {\sigma'}^{n-1}_\Sigma(\vec{\chi}) \subseteq T'_\Sigma$ imply that $\tau'_\Sigma(\vec{\chi}) \in T'_\Sigma$.

An N-(q)firs π-institution $\mathcal{I}$ with transferable N-rules will be called *N-(q)first*.

Before stating and proving Proposition 5.8, we remind the reader of the notion of an N-implication system for a π-institution $\mathcal{I}$. N-implication systems for π-institutions were introduced in [11], taking after the corresponding notion for deductive systems (see, e. g., [5, Section 1.1]).

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution and N be a category of natural transformations on SEN. A collection $E = \{\varepsilon^i : i \in I\}$ of natural transformations $\varepsilon^i : \mathrm{SEN}^2 \longrightarrow \mathrm{SEN}$ in N is said to be an *N-implication system for $\mathcal{I}$* if, for every $\Sigma \in |\mathbf{Sign}|$ and all $\varphi, \psi \in \mathrm{SEN}(\Sigma)$,

1. $\varepsilon^i_\Sigma(\varphi, \varphi) \in C_\Sigma(\emptyset)$, for all $i \in I$ (E-reflexivity),
2. $\psi \in C_\Sigma(\{\varepsilon^i_\Sigma(\varphi, \psi) : i \in I\} \cup \{\varphi\})$ (E-modus ponens).

Usually, these two properties will be abbreviated, respectively, as $E_\Sigma(\varphi, \varphi) \subseteq C_\Sigma(\emptyset)$, $\psi \in C_\Sigma(E_\Sigma(\varphi, \psi), \varphi)$. Proposition 5.8 asserts that, for a π-institution $\mathcal{I}$, with transferable N-rules, that admits lifting of N-quotients, the property of having an N-implication system implies the N-correspondence property. Those π-institutions that have an N-implication system are termed *syntactically N-protoalgebraic*. It is known that syntactic N-protoalgebraicity implies N-protoalgebraicity and, moreover, it has been conjectured, but still remains an open problem, that the converse does not hold in general.

Proposition 5.8 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be a π-institution with transferable N-rules that admits lifting of N-quotients. If $\mathcal{I}$ has an N-implication system, then $\mathcal{I}$ has the N-correspondence property.*

Proof. Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is a π-institution with transferable N-rules that admits lifting of N-quotients and $E = \{\varepsilon^i : i \in I\}$ is an N-implication system for $\mathcal{I}$. By Proposition 5.6, it suffices to show that $\mathcal{I}$ has the strict N-correspondence property. To this end, let $\boldsymbol{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, $\boldsymbol{B} = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$ be two N-algebraic systems, $\mathfrak{A} = \langle \boldsymbol{A}, T'\rangle$, $\mathfrak{B} = \langle \boldsymbol{B}, T''\rangle$ be two $\mathcal{I}$-matrices, $\langle F, \alpha\rangle : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a strict $\mathcal{I}$-morphism, with F an isomorphism, and $U \in \mathrm{Fi}^{\mathcal{I}}\mathfrak{A}$. We have to show that $U = \alpha^{-1}(\alpha(U))$. Since $U \leq \alpha^{-1}(\alpha(U))$ always holds, it is sufficient to show that $\alpha^{-1}(\alpha(U)) \leq U$. To this end, let $\Sigma \in |\mathbf{Sign}'|$ and $\varphi \in \mathrm{SEN}'(\Sigma)$ such that $\varphi \in \alpha_\Sigma^{-1}(\alpha_\Sigma(U_\Sigma))$. Then we have $\alpha_\Sigma(\varphi) \in \alpha_\Sigma(U_\Sigma)$. Thus, there exists $\psi \in \mathrm{SEN}'(\Sigma)$, with $\psi \in U_\Sigma$, such that $\alpha_\Sigma(\varphi) = \alpha_\Sigma(\psi)$. Hence, we obtain that

$$\begin{aligned} \alpha_\Sigma(E'_\Sigma(\psi, \varphi)) &= E''_{F(\Sigma)}(\alpha_\Sigma(\psi), \alpha_\Sigma(\varphi)) && \text{(since } E \text{ is in } N) \\ &\subseteq T''_{F(\Sigma)} && \text{(since } \mathcal{I} \text{ has tranferable } N\text{-rules).} \end{aligned}$$

Therefore, since $\langle F, \alpha\rangle$ is strict, $E'_\Sigma(\psi, \varphi) \subseteq T'_\Sigma \subseteq U_\Sigma$. Now, using the modus ponens property with respect to E and the fact that $\mathcal{I}$ has transferable N-rules, we get $\varphi \in U_\Sigma$, thus $\alpha^{-1}(\alpha(U)) \leq U$, as was to be shown. □

Theorem 5.9 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be a π-institution.*

1. *Under the hypotheses that $\mathcal{I}$ has transferable N-rules and admits lifting of N-quotients, if $\mathcal{I}$ has an N-implication system, then $\mathcal{I}$ has the N-correspondence property.*

2. *Under the hypotheses that $\mathcal{I}$ is N-structural and admits lifting of N-quotients, if $\mathcal{I}$ has the N-correspondence property, then it is N-protoalgebraic.*

P r o o f. The first part is given by Proposition 5.8 and the second by Proposition 5.3. □

Thus, for N-qfirst π-institutions, Theorem 5.9 gives a refinement of the N-protoalgebraic hierarchy as follows:

$$\begin{tikzcd}
\text{syntactic } N\text{-protoalgebraicity} \arrow[d] \\
N\text{-correspondence property} \arrow[d] \\
N\text{-protoalgebraicity.}
\end{tikzcd}$$

Finally, the paper concludes with a proposition providing an equivalent condition to the N-correspondence property. This characterization forms an analog of the one presented in [3, Theorem 1.4.2] for deductive systems with some obvious modifications due to the finitarity of the deductive system framework.

Proposition 5.10 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be a π-institution. $\mathcal{I}$ has the N-correspondence property if and only if, for all $\mathcal{I}$-matrices $\mathfrak{A} = \langle \boldsymbol{A}, T'\rangle$ and $\mathfrak{B} = \langle \boldsymbol{B}, T''\rangle$, with $\boldsymbol{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, $\boldsymbol{B} = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$, every surjective $\mathcal{I}$-morphism $\langle F, \alpha\rangle : \mathfrak{A} \longrightarrow \mathfrak{B}$, with F an isomorphism, and every axiom family $X = \{X_\Sigma\}_{\Sigma\in|\mathbf{Sign}'|}$ of SEN', we have*

$$\alpha^{-1}(\mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\alpha(X))) = \mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(X) \vee^{\mathbf{Fi}^{\mathcal{I}}\mathfrak{A}} \alpha^{-1}(T''). \tag{2}$$

P r o o f. If X is an $\mathcal{I}$-filter of $\mathfrak{A}$, then one immediately obtains from equation (2) the N-correspondence property for $\mathcal{I}$. Thus, it only suffices to show that the N-correspondence property implies equation (2). To this end, let $X = \{X_\Sigma\}_{\Sigma\in|\mathbf{Sign}'|}$ be an axiom family on SEN' and set $V = \mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(X) \in \mathrm{Fi}^{\mathcal{I}}(\mathfrak{A})$. Applying the N-correspondence property with $U = V$, we get that $\alpha^{-1}(\alpha^*(\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(X))) = \mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(X) \vee^{\mathbf{Fi}^{\mathcal{I}}\mathfrak{A}} \alpha^{-1}(T'')$, i. e.,

$$\alpha^{-1}(\mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\alpha(\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(X)))) = \mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(X) \vee^{\mathbf{Fi}^{\mathcal{I}}\mathfrak{A}} \alpha^{-1}(T'').$$

Thus, it suffices to show that $\mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\alpha(X)) = \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\alpha(\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(X)))$.

The left-to-right inclusion is obvious, since $X \leq \mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(X)$.

For the right-to-left inclusion, we have that $\alpha(X) \leq \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\alpha(X))$, whence $X \leq \alpha^{-1}(\mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\alpha(X)))$. Thus, since, by Proposition 3.1, $\alpha^{-1}(\mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\alpha(X))) \in \mathrm{Fi}^{\mathcal{I}}(\mathfrak{A})$, we get that $\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(X) \leq \alpha^{-1}(\mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\alpha(X)))$, which implies that $\alpha(\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(X)) \leq \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\alpha(X))$. Therefore, $\mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\alpha(\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(X))) \leq \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\alpha(X))$. □

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