Applied Categorical Structures (2005) 13: 265–280
DOI: 10.1007/s10485-005-5797-5

Categorical Abstract Algebraic Logic: $(\mathcal{I}, N)$-Algebraic Systems

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(Received: 23 July 2004; accepted: 18 April 2005)

Abstract. Algebraic systems play in the theory of algebraizability of π-institutions the role that algebras play in the theory of algebraizable sentential logics. In this same sense, $\mathcal{I}$-algebraic systems are to a π-institution $\mathcal{I}$ what $\mathcal{S}$-algebras are to a sentential logic $\mathcal{S}$. More precisely, an $(\mathcal{I}, N)$-algebraic system is the sentence functor reduct of an N'-reduced (N, N')-full model of a π-institution $\mathcal{I}$. Algebraic systems are formally introduced and their relationship with full models and with bilogical morphisms is investigated.

Mathematics Subject Classifications (2000): Primary: 03Gxx, secondary: 18Axx, 68N05.

Key words: abstract algebraic logic, deductive systems, institutions, equivalent deductive systems, algebraizable deductive systems, adjunctions, equivalent institutions, algebraizable institutions, Leibniz congruence, Tarski congruence, algebraizable sentential logics, $\mathcal{S}$-algebras.

1. Introduction

This paper continues the investigations on the possibility of abstracting results pertaining to the algebraization of deductive systems, as developed by Blok and Pigozzi, and to the algebraization of sentential logics, as developed by Font and Jansana, to the level of π-institutions. The motivation for this abstraction is two-fold. On the one hand, it stems from a desire to handle the algebraization of some well-known multi-signature logics with quantifiers in a way more natural than the one traditionally used in classical algebraic logic. This is explained in more details in the introductions of (Voutsadakis [23, 22]), the first two papers containing results on this abstraction program. It has led to the algebraization of equational logic (Voutsadakis [26, 24]) and to the algebraization of first-order logic without terms (Voutsadakis [27, 25]) using a novel categorical method. The second motivating factor comes from the hope that abstracting a framework that works well for some restricted classes of logics may help in the investigation of other, perhaps not so familiar, logics outside those classes. This direction parallels the motivation behind the development recently of abstract, institution-independent, model theory by Răzvan Diaconescu (see, e.g., [8–10]). The added generality, serving the purposes outlined above, on the logic side, is due to the fact that the π-institution framework can accommodate logics with multiple signatures and can incorporate substitutions in the object language and, on the algebraic side, due to the fact that categorical, instead of universal, algebras may be used for the algebraization process.

The theory of Blok and Pigozzi [3, 4] has as its primary object of study a deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$. It consists of a language type $\mathcal{L}$ together with a finitary and structural consequence relation $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)) \times \mathrm{Fm}_{\mathcal{L}}(V)$, where $\mathrm{Fm}_{\mathcal{L}}(V)$ denotes the set of all $\mathcal{L}$-formulas over a fixed denumerable set of variables V. A theory of $\mathcal{S}$ is a set T of $\mathcal{L}$-formulas that is closed under the $\mathcal{S}$-consequence, i.e., such that, for all $\phi \in \mathrm{Fm}_{\mathcal{L}}(V)$, if $T \vdash_{\mathcal{S}} \phi$, then $\phi \in T$. The main and most important tool of the theory is the Leibniz operator, which maps theories of the deductive system $\mathcal{S}$ to $\mathcal{L}$-congruences. It is formally defined by

$$\begin{aligned}\Omega(T) = \{\langle \alpha, \beta \rangle : {} & \phi(\alpha, \vec{\gamma}) \in T \text{ iff } \phi(\beta, \vec{\gamma}) \in T,\\ & \text{for all } \phi(p, \vec{q}) \in \mathrm{Fm}_{\mathcal{L}}(V), \vec{\gamma} \in \mathrm{Fm}_{\mathcal{L}}(V)^k\}.\end{aligned}$$

$\Omega(T)$ turns out to be the largest congruence on the formula algebra that is compatible with the theory T, in the sense that $\langle \alpha, \beta \rangle \in \Omega(T)$ and $\alpha \in T$ imply that $\beta \in T$. The Leibniz operator may be introduced, more generally, as an operator from the collection of $\mathcal{S}$-filters on an $\mathcal{L}$-algebra $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$ to the collection of congruences of the algebra. $F \subseteq A$ is an $\mathcal{S}$-filter on $\mathbf{A}$, if, for all $\Phi \cup \{\phi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$ and all homomorphisms $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$,

$$\Phi \vdash_{\mathcal{S}} \phi \quad \text{and} \quad h(\Phi) \subseteq F \quad \text{imply} \quad h(\phi) \in F.$$

The pair $\mathcal{A} = \langle \mathbf{A}, F \rangle$ is called an $\mathcal{S}$-matrix if F is an $\mathcal{S}$-filter on $\mathbf{A}$. In that case

$$\begin{aligned}\Omega_{\mathbf{A}}(F) = \{\langle a, b \rangle : {} & \phi^{\mathbf{A}}(a, \vec{c}) \in F \text{ iff } \phi^{\mathbf{A}}(b, \vec{c}) \in F,\\ & \text{for all } \phi(p, \vec{q}) \in \mathrm{Fm}_{\mathcal{L}}(V), \vec{c} \in A^k\}.\end{aligned}$$

Again $\Omega_{\mathbf{A}}(F)$ is the largest congruence on $\mathbf{A}$ that is compatible with F.

Based on this notion of Leibniz operator and several of the properties that it may or may not possess depending on the deductive system $\mathcal{S}$ under investigation, e.g., monotonicity, injectivity, continuity, etc., deductive systems are classified in several steps of an algebraic hierarchy, whose main classes are the protoalgebraic [2], the equivalential [20, 6] and the algebraizable [3] (see also [16–18]) deductive systems. The book by Czelakowski [7] and the survey article by Font, Jansana and Pigozzi [13] provide an overview of the theory and the resulting hierarchy.

After Blok and Pigozzi, the theory was elaborated on and further developed by the Barcelona Algebraic Logic group. Their work is detailed in the monograph of Font and Jansana [12]. One of the major modifications from the original model theory is the adoption of abstract logics as models of sentential logics in place of logical matrices. An abstract logic $\mathbb{L}$ over the signature $\mathcal{L}$ is a pair $\mathbb{L} = \langle \mathbf{A}, C \rangle$, where $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$ is an $\mathcal{L}$-algebra and C is a closure operator on A. In this case the abstract logic $\mathbb{L}$ is a model of the sentential logic $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ if, for all $\Phi \cup \{\phi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Phi \vdash_{\mathcal{S}} \phi \quad \text{implies} \quad h(\phi) \in C(h(\Phi)),$$

for every homomorphism $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$.

When one considers abstract logics, the place of the Leibniz operator, as it is applied on logical matrices, is taken by the Tarski operator $\widetilde{\Omega}$. It maps a closure system C over an algebra $\mathbf{A}$ to the greatest logical congruence of the abstract logic $\mathbb{L} = \langle \mathbf{A}, C \rangle$, i.e., the greatest congruence compatible with all closed sets of the closure system induced by C. If one divides out both the algebra $\mathbf{A}$ and the closure system C by the Tarski congruence of $\mathbb{L}$, then the reduction $\mathbb{L}^* = \langle \mathbf{A}^*, C^* \rangle$ of $\mathbb{L}$ is obtained. In case this reduction consists of the collection $\mathrm{Fi}_{\mathcal{S}}(\mathbf{A}^*)$ of all $\mathcal{S}$-filters on the algebra $\mathbf{A}^*$, $\mathbb{L}$ is said to be a full model of the sentential logic $\mathcal{S}$. Those models of $\mathcal{S}$ of the form $\mathbb{L} = \langle \mathbf{A}, \mathrm{Fi}_{\mathcal{S}}(\mathbf{A}) \rangle$, i.e., whose closure systems consist of the entire collection of $\mathcal{S}$-filters on the carrier algebra $\mathbf{A}$, are called the basic full models of the logic $\mathcal{S}$. Using that terminology, a full model of $\mathcal{S}$ is a model whose reduction is a basic full model on the quotient algebra. The collection of all full models of a sentential logic $\mathcal{S}$ on the algebra $\mathbf{A}$ is denoted by $\mathrm{FMod}_{\mathcal{S}}\, \mathbf{A}$ and it is ordered by the natural ordering $\leq$ on the corresponding closure operators, i.e.,

$$C \leq C' \quad \text{iff} \quad C(X) \subseteq C'(X), \quad \text{for all } X \subseteq A.$$

Font and Jansana go on to define the notion of an $\mathcal{S}$-algebra. An algebra $\mathbf{A}$ is an $\mathcal{S}$-algebra if the abstract logic consisting of all the $\mathcal{S}$-filters on $\mathbf{A}$ is reduced. This is tantamount to saying that $\mathbf{A}$ is the algebraic reduct of a reduced full model of $\mathcal{S}$. The class of all $\mathcal{S}$-algebras is denoted by $\mathbf{Alg}\, \mathcal{S}$. The collection of all $\mathbf{Alg}\, \mathcal{S}$-congruences on an algebra $\mathbf{A}$, i.e., congruences on $\mathbf{A}$ whose quotient algebras lie in $\mathbf{Alg}\, \mathcal{S}$, is denoted, as usual, by $\mathrm{Con}_{\mathbf{Alg}\, \mathcal{S}}\, \mathbf{A}$. In the Isomorphism Theorem 2.30 of [12], it is shown that, given an algebra $\mathbf{A}$, the Tarski operator is an order-isomorphism between $\langle \mathrm{FMod}_{\mathcal{S}}\, \mathbf{A}, \leq \rangle$ and $\langle \mathrm{Con}_{\mathbf{Alg}\, \mathcal{S}}\, \mathbf{A}, \subseteq \rangle$. This result will be the focus of the present work. More precisely, the concepts and results of [28] and [29], which abstract corresponding concepts and results from [12], will be used to provide an analog of Theorem 2.30 for the case of institutional logics. Some of the concepts and the results that are needed for what follows are presented in the remainder of this section.

First, recall the definitions of an institution [14, 15] and of a π-institution [11]. π-institutions play in the theory of categorical abstract algebraic logic [23, 22] the role that sentential logics play in the theory of Blok and Pigozzi and of Font and Jansana.

Let $\mathbf{Sign}$ be a category, $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ a functor and N a category of natural transformations on SEN, as defined in [28]. Given $\Sigma \in |\mathbf{Sign}|$, an equivalence relation θ_Σ on $\mathrm{SEN}(\Sigma)$ is said to be an N-congruence if, for all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)^k$,

$$\vec{\phi}\, \theta_\Sigma^k\, \vec{\psi} \quad \text{imply} \quad \sigma_\Sigma(\vec{\phi})\, \theta_\Sigma\, \sigma_\Sigma(\vec{\psi}).$$

A collection $\theta = \{\langle \Sigma, \theta_\Sigma \rangle : \Sigma \in |\mathbf{Sign}|\}$ is called an equivalence system of SEN if

– θ_Σ is an equivalence relation on $\mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$,

– $\mathrm{SEN}(f)^2(\theta_{\Sigma_1}) \subseteq \theta_{\Sigma_2}$, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$.

If, in addition, N is a category of natural transformations on SEN and θ_Σ is an N-congruence, for all $\Sigma \in |\mathbf{Sign}|$, then θ is said to be an N-congruence system of SEN.

Let now $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} \rangle$ be a π-institution. An equivalence system θ of SEN is called a logical equivalence system of $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi, \psi \rangle \in \theta_\Sigma \quad \text{implies} \quad C_\Sigma(\phi) = C_\Sigma(\psi).$$

An N-congruence system of SEN is a logical N-congruence system of $\mathcal{I}$ if it is logical as an equivalence system of $\mathcal{I}$.

It is proven in [28] that the collection of all logical N-congruence systems of a given π-institution $\mathcal{I}$ forms a complete lattice under signature-wise inclusion and the largest element of the lattice is termed the Tarski N-congruence system of $\mathcal{I}$ and denoted by $\widetilde{\Omega}^N(\mathcal{I})$. Theorem 4 of [28] fully characterizes the Tarski N-congruence system of a π-institution. Tarski N-congruence systems correspond in this framework to the Tarksi congruences of [12].

A π-institution $\mathcal{I}'$, in this context, is a model of a π-institution $\mathcal{I}$ if $\mathcal{I}$ is semi-interpretable in $\mathcal{I}'$, in symbols $\mathcal{I} \succ \mathcal{I}'$. If N, N' are categories of natural transformations on SEN, SEN$'$, respectively, then $\mathcal{I}'$ is said to be an (N, N')-model of $\mathcal{I}$ via $\langle F, \alpha \rangle : \mathcal{I} \to \mathcal{I}'$ if $\langle F, \alpha \rangle$ is an (N, N')-logical morphism, i.e., a singleton (N, N')-epimorphic semi-interpretation. It is said to be an (N, N')-full model of $\mathcal{I}$ via $\langle F, \alpha \rangle$ if the reduction $\mathcal{I}'^{N'}$ of $\mathcal{I}'$ via its Tarski N'-congruence system is the $(N, \overline{N'})$-model of $\mathcal{I}$ via $\langle F, \pi_F^{N'}\alpha \rangle$ with the least closure system, where $\langle \mathrm{I}_{\mathbf{Sign}'}, \pi^{N'} \rangle : \mathcal{I}' \vdash \mathcal{I}'^{N'}$ is the natural projection interpretation. This was called an $\langle F, \pi_F^{N'}\alpha \rangle$-min model of $\mathcal{I}$. Min models correspond to the basic full models in the sentential logic framework.

It is worth pausing here to add a parenthetical comment concerning a significant difference between the notions of model, min model and full model in this context and the corresponding ones of model, basic full model and full model, respectively, in the theory of sentential logics of [12]. In the theory of sentential logics, a model has to respect entailments under all possible translations (homomorphisms) from the formula algebra into the carrier algebra of the model. In the categorical theory, a model refers to one fixed translation from a π-institution to the one serving as its model. For an explanation of the difficulties and the intuitive plausibility that led to the adoption of this different approach in the categorical context the reader is referred to [29].

The development of the categorical theory is continued in the present paper by defining a notion of an $(\mathcal{I}, N)$-algebraic system, an analog of an $\mathcal{S}$-algebra, and proving an isomorphism theorem, analogous to Theorem 2.30 of [12], relating full models with logical congruence systems the algebraic reducts of whose quotients are $(\mathcal{I}, N)$-algebraic systems.

For general categorical notation where needed, the reader is referred to any of [1, 5, 19].

2. Algebraic Systems

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} \rangle$ be a π-institution and N a category of natural transformations on SEN. From the definitions of reduced and full models [29], it follows that the reduced full N-models of $\mathcal{I}$ are exactly those min (N, N')-models $\mathcal{I}'$ that are N'-reduced, for some category N' of natural transformations on the sentence functor SEN'. This fact partly motivates the following definition of an $(\mathcal{I}, N)$-algebraic system. $(\mathcal{I}, N)$-algebraic systems parallel, in the context of π-institutions, the concept of an $\mathcal{S}$-algebra of a sentential logic $\mathcal{S}$, in the theory of sentential logics of [12].

DEFINITION 1. If $\mathcal{I}$ is a π-institution, then a functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ is said to be an *$(\mathcal{I}, N)$-algebraic system* if and only if there exists a category N' of natural transformations on SEN' and a singleton (N, N')-epimorphic translation $\langle F, \alpha \rangle : \mathcal{I} \to^{se} \mathrm{SEN}'$, such that the $\langle F, \alpha \rangle$-min (N, N')-model $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C' \rangle$ of $\mathcal{I}$ on SEN' is N'-reduced, i.e., iff $\mathcal{I}'$ is a reduced (N, N')-full model of $\mathcal{I}$ via $\langle F, \alpha \rangle$.

Let $\mathrm{Alg}^N(\mathcal{I})$ denote the class of all $(\mathcal{I}, N)$-algebraic systems.

It will now be shown that the N-quotient functor $\mathrm{SEN}^N : \mathbf{Sign} \to \mathbf{Set}$ for a given π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, where N is a category of natural transformations on SEN, is an $(\mathcal{I}, N)$-algebraic system.

PROPOSITION 2. *Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ and a category N of natural transformations on* SEN, $\mathrm{SEN}^N : \mathbf{Sign} \to \mathbf{Set}$ *is an $(\mathcal{I}, N)$-algebraic system.*

Proof. Consider the category $\overline{N}$ of natural transformations on SEN^N induced by N. Then the N-reduct $\mathcal{I}^N = \langle \mathbf{Sign}, \mathrm{SEN}^N, C^N \rangle$ is the $\langle I_{\mathbf{Sign}}, \pi^N \rangle$-min $(N, \overline{N})$-model of $\mathcal{I}$ on SEN^N and it is $\overline{N}$-reduced. □

The $(\mathcal{I}, N)$-algebraic system SEN^N is called the *Lindenbaum–Tarski N-algebraic system* of $\mathcal{I}$.

Combining the definition of an $(\mathcal{I}, N)$-algebraic system together with the definitions of min and full models from [29], we obtain

PROPOSITION 3. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ be a π-institution and N a category of natural transformations on* SEN. *If $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C' \rangle$ is a π-institution and N' a category of natural transformations on* SEN', *then the following are equivalent:*

(1) *$\mathcal{I}'$ is an N'-reduced (N, N')-full model of $\mathcal{I}$ via $\langle F, \alpha \rangle$.*
(2) *$\mathcal{I}'$ is N'-reduced and C' is the $\langle F, \alpha \rangle$-min (N, N')-model of $\mathcal{I}$ on* SEN'.

(3) $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ *is a* $(\mathcal{I}, N)$*-algebraic system via* $\langle F, \alpha\rangle : \mathcal{I} \to^{se} \mathrm{SEN}'$ *and* C' *is an* $\langle F, \alpha\rangle$*-min* (N, N')*-model of* $\mathcal{I}$ *on* SEN'.

The definition of the class $\mathrm{Alg}^N(\mathcal{I})$ motivates the following definition of an $\mathrm{Alg}^N(\mathcal{I})$-congruence system of a given π-institution.

DEFINITION 4. Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ be a functor and N' a category of natural transformations on SEN'. An N'-congruence system θ on SEN' is an $\mathrm{Alg}^N(\mathcal{I})$-$N'$*-congruence system* if $\mathrm{SEN}'^{\theta} : \mathbf{Sign}' \to \mathbf{Set}$ is an $(\mathcal{I}, N)$-algebraic system via N'^{θ}.

The collection of all $\mathrm{Alg}^N(\mathcal{I})$-congruence systems on SEN' is denoted by $\mathrm{Con}_{\mathrm{Alg}^N(\mathcal{I})}(\mathrm{SEN}')$. By $\mathrm{Con}^{N'}_{\mathrm{Alg}^N(\mathcal{I})}(\mathrm{SEN}')$ is denoted the subcollection of all $\mathrm{Alg}^N(\mathcal{I})$-$N'$-congruence systems on SEN', for some fixed category N' of natural transformations on SEN'.

The fact that the N'-reduct of an (N, N')-full model of a π-institution $\mathcal{I}$ is, by definition, an $(N, \overline{N'})$-min model of $\mathcal{I}$ yields that the Tarski N'-congruence system of $\mathcal{I}'$ is an N'-logical congruence system of $\mathcal{I}'$ that is a member of $\mathrm{Con}_{\mathrm{Alg}^N(\mathcal{I})}(\mathrm{SEN}')$.

PROPOSITION 5. *Let* $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ *be a full* (N, N')*-model of a* π*-institution* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$. *Then* $\mathrm{SEN}'^{N'} : \mathbf{Sign}' \to \mathbf{Set}$ *is an* $(\mathcal{I}, N)$*-algebraic system and, therefore,* $\widetilde{\Omega}^{N'}(\mathcal{I}')$ *is an* N'*-logical congruence system in* $\mathrm{Con}_{\mathrm{Alg}^N(\mathcal{I})}(\mathrm{SEN}')$.

The following result provides a characterization of $(\mathcal{I}, N)$-algebraic systems without recourse to the notion of a full model. Roughly speaking, it says that the $(\mathcal{I}, N)$-algebraic systems are exactly the underlying sentence functors of the reduced models of $\mathcal{I}$. This is the analog of Proposition 2.19 of [12].

PROPOSITION 6. *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *be a* π*-institution and* N *a category of natural transformations on* SEN. *The class of all* $(\mathcal{I}, N)$*-algebraic systems is the class of all functors* $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, *such that, there exists a closure system* C' *on* SEN', *such that* $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ *is an* N'*-reduced* (N, N')*-model of* $\mathcal{I}$*, for some category* N' *of natural transformations on* SEN'.

Proof. Suppose that SEN' is an $(\mathcal{I}, N)$-algebraic system. Then, by definition, there exists a category N' of natural transformations on SEN', and a singleton (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathcal{I} \to \mathrm{SEN}'$, such that the $\langle F, \alpha\rangle$-min (N, N')-model $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ of $\mathcal{I}$ on SEN' is N'-reduced.

Conversely, let $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ be a functor, N' a category of natural transformations on SEN' and C' a closure system on SEN', such that $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ is an N'-reduced (N, N')-model of $\mathcal{I}$ via $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$. Then, if $C^{\min}$ is such that $\mathcal{I}^{\min} = \langle \mathbf{Sign}', \mathrm{SEN}', C^{\min}\rangle$ is the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN', we get $C^{\min} \le C'$, whence, by Corollary 9 of [28], $\widetilde{\Omega}^{N'}(\mathcal{I}^{\min}) \le$

$\widetilde{\Omega}^{N'}(\mathcal{I}')$. Thus, since $\mathcal{I}'$ is N'-reduced, so is $\mathcal{I}^{\min}$ and SEN$'$ is an $(\mathcal{I}, N)$-algebraic system. □

Suppose that $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ and $\mathrm{SEN}'' : \mathbf{Sign}'' \to \mathbf{Set}$ are $(\mathcal{I}, N)$-algebraic systems via the categories N', N'' of natural transformations, respectively. $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ and $\mathrm{SEN}'' : \mathbf{Sign}'' \to \mathbf{Set}$ are said to be *isomorphic* iff there exists a singleton (N', N'')-epimorphic translation $\langle F, \alpha\rangle : \mathrm{SEN}' \to^{se} \mathrm{SEN}''$ and a singleton (N'', N')-epimorphic translation $\langle G, \beta\rangle : \mathrm{SEN}'' \to^{se} \mathrm{SEN}'$ that are inverses of one another. $\langle F, \alpha\rangle$ and $\langle G, \beta\rangle$ will be said to be (N', N'')- and (N'', N')-*isomorphisms*, respectively, in that case.

The following proposition asserts that the class of all $(\mathcal{I}, N)$-algebraic systems $\mathrm{Alg}^N(\mathcal{I})$ is closed under isomorphisms.

PROPOSITION 7. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution and N a category of natural transformations on* SEN. *If* $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ *is an $(\mathcal{I}, N)$-algebraic system via N' and $\langle F, \alpha\rangle : \mathrm{SEN}' \to^{se} \mathrm{SEN}''$ is an (N', N'')-isomorphism, then* $\mathrm{SEN}'' : \mathbf{Sign}'' \to \mathbf{Set}$ *is also an $(\mathcal{I}, N)$-algebraic system via N''.*

Proof. Suppose that $\langle F, \alpha\rangle : \mathrm{SEN}' \to^{se} \mathrm{SEN}''$ is an (N', N'')-isomorphism and that C' is a closure system on SEN$'$, such that $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ is an N'-reduced (N, N')-min model of $\mathcal{I}$ via $\langle M, \mu\rangle : \mathcal{I} \succ^{se} \mathcal{I}'$. Let $\langle G, \beta\rangle : \mathrm{SEN}'' \to^{se} \mathrm{SEN}'$ be the inverse isomorphism to $\langle F, \alpha\rangle$.

$$\mathcal{I} \xrightarrow{\langle M,\mu\rangle} \mathcal{I}' \underset{\langle G,\beta\rangle}{\overset{\langle F,\alpha\rangle}{\rightleftarrows}} \mathcal{I}^{\langle G,\beta\rangle}$$

Then, by Proposition 5.3 of [29], $\mathcal{I}^{\langle G,\beta\rangle}$ is also an (N, N'')-min model of $\mathcal{I}$ via $\langle F, \alpha\rangle\langle M, \mu\rangle$ and, by Proposition 3.2 of [29], the two π-institutions $\mathcal{I}'$ and $\mathcal{I}^{\langle G,\beta\rangle}$ are isomorphic via $\langle F, \alpha\rangle : \mathcal{I}' \vdash^{se} \mathcal{I}^{\langle G,\beta\rangle}$ and $\langle G, \beta\rangle : \mathcal{I}^{\langle G,\beta\rangle} \vdash^{se} \mathcal{I}'$. Hence, by Theorem 21 of [28], since $\mathcal{I}'$ is N'-reduced, $\mathcal{I}^{\langle G,\beta\rangle}$ is also N''-reduced and, therefore, an $(\mathcal{I}, N)$-algebraic system. □

The following proposition brings together several key definitions introduced in the theory so far. In particular, it points out some of the connections between full models, min models and algebraic systems.

PROPOSITION 8. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution and N a category of natural transformations on* SEN. *Suppose that $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ is also a π-institution and N' a category of natural transformations on* SEN$'$. *Then the following are equivalent:*

(1) *$\mathcal{I}'$ is a full (N, N')-model of $\mathcal{I}$ via $\langle F, \alpha\rangle : \mathcal{I} \succ^{se} \mathcal{I}'$.*
(2) $\mathrm{SEN}'^{N'} : \mathbf{Sign}' \to \mathbf{Set}$ *is an $(\mathcal{I}, N)$-algebraic system via $\langle F, \pi_F^{N'}\alpha\rangle$ and $\mathcal{I}'^{N'}$ is an $\langle F, \pi_F^{N'}\alpha\rangle$-min $(N, \overline{N'})$-model of $\mathcal{I}$ on* $\mathrm{SEN}'^{N'}$.

(3) *There exists an* (N', N'')*-bilogical morphism* $\langle G, \beta\rangle$*, with* G *an isomorphism, between* $\mathcal{I}'$ *and a* π*-institution* $\mathcal{I}''$*, such that* $\mathrm{SEN}'' : \mathbf{Sign}'' \to \mathbf{Set}$ *is an* $(\mathcal{I}, N)$*-algebraic system via* $\langle GF, \beta_F\alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}''$ *and* $\mathcal{I}''$ *is a* $\langle GF, \beta_F\alpha\rangle$*-min* (N, N'')*-model of* $\mathcal{I}$ *on* SEN''.

Proof. This proof consists of putting together previously introduced definitions and results on min models, full models and algebraic systems.

(1) $\Rightarrow$ (2) Suppose that $\mathcal{I}'$ is a full (N, N')-model of $\mathcal{I}$ via $\langle F, \alpha\rangle : \mathcal{I} \rangle\!\!-^{se} \mathcal{I}'$. Then, by definition, $\mathcal{I}'^{N'}$ is an $\langle F, \pi_F^{N'}\alpha\rangle$-min $(N, \overline{N'})$-model on $\mathrm{SEN}'^{N'}$ and, since it is obviously $\overline{N'}$-reduced, $\mathrm{SEN}'^{N'} : \mathbf{Sign}' \to \mathbf{Set}$ is an $(\mathcal{I}, N)$-algebraic system via $\langle F, \pi_F^{N'}\alpha\rangle$.

(2) $\Rightarrow$ (3) Now, suppose that $\mathrm{SEN}'^{N'} : \mathbf{Sign}' \to \mathbf{Set}$ is an $(\mathcal{I}, N)$-algebraic system via $\langle F, \pi_F^{N'}\alpha\rangle$ and $\mathcal{I}'^{N'}$ is an $\langle F, \alpha\rangle$-min $(N, \overline{N'})$-model of $\mathcal{I}$ on $\mathrm{SEN}'^{N'}$. Then $\langle \mathrm{I}_{\mathbf{Sign}'}, \pi^{N'}\rangle : \mathcal{I}' \vdash^{se} \mathcal{I}'^{N'}$ is an $(N', \overline{N'})$-bilogical morphism, with $\mathrm{I}_{\mathbf{Sign}'}$ an isomorphism, and, by hypothesis, $\mathrm{SEN}'^{N'} : \mathbf{Sign}' \to \mathbf{Set}$ is an $(\mathcal{I}, N)$-algebraic system and $\mathcal{I}'^{N'}$ is an $\langle F, \alpha\rangle$-min $(N, \overline{N'})$-model on $\mathrm{SEN}'^{N'}$.

(3) $\Rightarrow$ (1) Finally, suppose there exists an (N', N'')-bilogical morphism $\langle G, \beta\rangle$, with G an isomorphism, between $\mathcal{I}'$ and a π-institution $\mathcal{I}''$, such that $\mathrm{SEN}'' : \mathbf{Sign}'' \to \mathbf{Set}$ is an $(\mathcal{I}, N)$-algebraic system via $\langle GF, \beta_F\alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}''$ and $\mathcal{I}''$ is a $\langle GF, \beta_F\alpha\rangle$-min (N, N'')-model of $\mathcal{I}$ on SEN''. Then, by Proposition 5.12 of [29], $\mathcal{I}'$ is a full (N, N')-model of $\mathcal{I}$ via $\langle F, \alpha\rangle : \mathcal{I} \rangle\!\!-^{se} \mathcal{I}'$. □

The following completeness theorem for π-institutions with respect to the classes of full, min and reduced full models is the analog of the Completeness Theorem 2.22 of [12] for sentential logics.

THEOREM 9 (Completeness Theorem). *Every* π*-institution* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$*, with* N *a category of natural transformations on* SEN*, is complete with respect to the following classes of* π*-institutions:*

(1) *The class of all* (N, N')*-full models of* $\mathcal{I}$.
(2) *The class of all* (N, N')*-min models of* $\mathcal{I}$.
(3) *The class of all* N'*-reduced* (N, N')*-full models of* $\mathcal{I}$.

Proof. All three classes contain the model $\mathcal{I}^N$ via $\langle \mathrm{I}_{\mathbf{Sign}}, \pi^N\rangle : \mathcal{I} \vdash^{se} \mathcal{I}^N$. $\mathcal{I}$ is therefore complete with respect to all three by Proposition 4.9 of [29]. □

Finally, a monotonicity theorem for the classes of algebraic systems of two π-institutions with the same sentence functor is presented. Roughly speaking, it is shown that finer closure systems have more algebraic systems. This is the analog of Proposition 2.27 of [12].

PROPOSITION 10. *Let* $\mathbf{Sign}$ *be a category,* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a functor,* N *a category of natural transformations on* SEN *and* C^1, C^2 *two closure systems on* SEN*, such that* $C^1 \le C^2$*. If* $\mathcal{I}^1 = \langle \mathbf{Sign}, \mathrm{SEN}, C^1\rangle$ *and* $\mathcal{I}^2 = \langle \mathbf{Sign}, \mathrm{SEN}, C^2\rangle$*, then* $\mathbf{Alg}^N(\mathcal{I}^2) \subseteq \mathbf{Alg}^N(\mathcal{I}^1)$.

Proof. Suppose that $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ is in $\mathbf{Alg}^N(\mathcal{I}^2)$. Then, by Proposition 6, there exists a closure system C' on SEN', such that $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C' \rangle$ is an N'-reduced (N, N')-model of $\mathcal{I}^2$ via $\langle F, \alpha \rangle : \mathcal{I}^2 \vdash^{se} \mathcal{I}'$, for some category N' of natural transformations on SEN'. But then

$$\mathcal{I}^1 \xrightarrow{\langle \mathrm{I}_{\mathbf{Sign}}, \iota \rangle} \mathcal{I}^2 \xrightarrow{\langle F, \alpha \rangle} \mathcal{I}'$$

$\mathcal{I}'$ is an N'-reduced (N, N')-model of $\mathcal{I}^1$ via $\langle F, \alpha \rangle : \mathcal{I}^1 \vdash^{se} \mathcal{I}'$, which, again by Proposition 6, yields that $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ is in $\mathbf{Alg}^N(\mathcal{I}^1)$. □

3. Full Models and Algebraic Systems

In this section, the Isomorphism Theorem of Font and Jansana (Theorem 2.30 of [12]) between the lattice of full models $\langle \mathrm{FMod}_{\mathcal{S}}(\mathbf{A}), \leq \rangle$ of a sentential logic $\mathcal{S}$ over an algebra $\mathbf{A}$ and that of the $\mathbf{Alg}(\mathcal{S})$-congruences of $\mathbf{A}$ $\langle \mathrm{Con}_{\mathbf{Alg}(\mathcal{S})}(\mathbf{A}), \subseteq \rangle$ is lifted to the π-institution level. Before describing the corresponding result intuitively, it may be useful to introduce some notation based on the concepts that have already been discussed in [29] and in the previous section.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ be a π-institution and N a category of natural transformations on SEN. Let also $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ be a functor, N' a category of natural transformations on SEN' and $\langle F, \alpha \rangle : \mathcal{I} \to^{se} \mathrm{SEN}'$ a singleton (N, N')-epimorphic translation. Denote by $\mathrm{FMod}_{\mathcal{I}}^{\langle F, \alpha \rangle}(\mathrm{SEN}')$ the collection of all (N, N')-full models of $\mathcal{I}$ on SEN' via $\langle F, \alpha \rangle$. Also denote by $\mathrm{Con}_{\mathrm{Alg}^N(\mathcal{I})}^{\langle F, \alpha \rangle}(\mathrm{SEN}')$ the collection of all $\mathrm{Alg}^N(\mathcal{I})$-$N'$-congruence systems θ on SEN', where SEN'^{θ} is an $(\mathcal{I}, N)$-algebraic system because the $\langle F, \pi_F^{\theta}\alpha \rangle$-min model $\mathcal{I}'$ on SEN'^{θ} is N'^{θ}-reduced. These will be referred to as $\mathrm{Alg}^N(\mathcal{I})$-$N'$-*congruence systems* on SEN' *via* $\langle F, \alpha \rangle$. Also by $\widetilde{\Omega}_{\mathrm{SEN}'}^{\langle F, \alpha \rangle}(C')$ will be denoted the Tarski N'-congruence system $\widetilde{\Omega}^{N'}(C')$, where $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C' \rangle$ is an (N, N')-full model of $\mathcal{I}$ via $\langle F, \alpha \rangle$. This use of the Tarksi congruence system symbol will be perceived as a Tarski operator from the collection $\mathrm{FMod}_{\mathcal{I}}^{\langle F, \alpha \rangle}(\mathrm{SEN}')$ into $\mathrm{Con}_{\mathrm{Alg}^N(\mathcal{I})}^{\langle F, \alpha \rangle}(\mathrm{SEN}')$.

Using the notation of the previous paragraph, it will be shown that the collection $\mathrm{Con}_{\mathrm{Alg}^N(\mathcal{I})}^{\langle F, \alpha \rangle}(\mathrm{SEN}')$ of all $\mathrm{Alg}^N(\mathcal{I})$-$N'$-congruence systems on SEN' via $\langle F, \alpha \rangle$ forms a lattice $\mathbf{Con}_{\mathbf{Alg}^N(\mathcal{I})}^{\langle F, \alpha \rangle}(\mathrm{SEN}')$ which is isomorphic with the lattice $\mathbf{FMod}_{\mathcal{I}}^{\langle F, \alpha \rangle}(\mathrm{SEN}')$ formed by all the (N, N')-full models of $\mathcal{I}$ on SEN' via $\langle F, \alpha \rangle$.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ be a π-institution, N a category of natural transformations on SEN, $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ a functor, N' a category of natural transformations on SEN' and $\langle F, \alpha \rangle : \mathcal{I} \to^{se} \mathrm{SEN}'$ a singleton (N, N')-epimorphic translation. For all $\theta \in \mathrm{Con}_{\mathrm{Alg}^N(\mathcal{I})}^{\langle F, \alpha \rangle}(\mathrm{SEN}')$, define

$$\widetilde{H}_{\mathrm{SEN}'}^{\langle F, \alpha \rangle}(\theta) = \langle \mathbf{Sign}', \mathrm{SEN}', C'^{\leftarrow \theta} \rangle,$$

where $C'^{\leftarrow\theta}$ is the closure system on SEN' generated by $\langle \mathrm{I}_{\mathbf{Sign}'}, \pi^\theta\rangle : \mathrm{SEN}' \to \mathcal{I}'$, where $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}'^\theta, C'\rangle$ is the $\langle F, \pi^\theta_F\alpha\rangle$-min model of $\mathcal{I}$ on SEN'^θ (for closure system generation see Section 3 of [29]).

With this notation, Proposition 3.2 of [29] yields immediately

PROPOSITION 11. *$\langle \mathrm{I}_{\mathbf{Sign}'}, \pi^\theta\rangle : \widetilde{H}^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}(\theta) \vdash^{se} \mathcal{I}'$ is an (N, N')-bilogical morphism.*

Some properties of $\widetilde{H}^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}$ viewed as an operator from the collection $\mathrm{Con}^{\langle F,\alpha\rangle}_{\mathrm{Alg}^N(\mathcal{I})}(\mathrm{SEN}')$ into $\mathrm{FMod}^{\langle F,\alpha\rangle}_{\mathcal{I}}(\mathrm{SEN}')$ are given in the following lemma, which is an analog for π-institutions of Lemma 2.29 of [12].

LEMMA 12. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, N a category of natural transformations on* SEN, $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ *a functor, N' a category of natural transformations on* SEN' *and $\langle F, \alpha\rangle : \mathcal{I} \to^{se} \mathrm{SEN}'$ a singleton (N, N')-epimorphic translation. Suppose $\theta \in \mathrm{Con}^{\langle F,\alpha\rangle}_{\mathrm{Alg}^N(\mathcal{I})}(\mathrm{SEN}')$ and let $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}'^\theta, C'\rangle$ be the $\langle F, \pi^\theta_F\alpha\rangle$-min model of $\mathcal{I}$ on SEN'^θ. Then*

(1) *θ is a logical N'-congruence system of $\widetilde{H}^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}(\theta)$.*
(2) $\widetilde{H}^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}(\theta)/\theta = \langle \mathbf{Sign}', \mathrm{SEN}'^\theta, C'\rangle$.
(3) $\widetilde{H}^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}(\theta) \in \mathrm{FMod}^{\langle F,\alpha\rangle}_{\mathcal{I}}(\mathrm{SEN}')$.
(4) *The mapping $\theta \mapsto \widetilde{H}^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}(\theta)$ is order preserving, i.e., if $\theta^1 \leq \theta^2$, then* $\widetilde{H}^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}(\theta^1) \leq \widetilde{H}^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}(\theta^2)$.

Proof. (1) θ is, by definition, an N'-congruence system on SEN'. Therefore, it suffices to show that θ is logical, i.e., that, for all $\Sigma \in |\mathbf{Sign}'|$, $\phi, \psi \in \mathrm{SEN}'(\Sigma)$,

$$\langle \phi, \psi\rangle \in \theta_\Sigma \quad \text{implies} \quad C'^{\leftarrow\theta}_\Sigma(\phi) = C'^{\leftarrow\theta}_\Sigma(\psi).$$

Suppose $\langle \phi, \psi\rangle \in \theta_\Sigma$. Then we have $\pi^\theta_\Sigma(\phi) = \pi^\theta_\Sigma(\psi)$ and, hence, $C'_\Sigma(\pi^\theta_\Sigma(\phi)) = C'_\Sigma(\pi^\theta_\Sigma(\psi))$. But, then, by the definition of $C'^{\leftarrow\theta}$, we get that $C'^{\leftarrow\theta}_\Sigma(\phi) = C'^{\leftarrow\theta}_\Sigma(\psi)$.

(2) Let $\Sigma \in |\mathbf{Sign}'|$, $\Phi \cup \phi \subseteq \mathrm{SEN}'(\Sigma)$. Then

$$\begin{aligned} \phi/\theta_\Sigma \in C'^{\leftarrow\theta^\theta}_\Sigma(\Phi/\theta_\Sigma) \quad &\text{iff} \quad \phi \in C'^{\leftarrow\theta}_\Sigma(\Phi) \\ &\text{iff} \quad \phi/\theta_\Sigma \in C'_\Sigma(\Phi/\theta_\Sigma). \end{aligned}$$

Therefore $\widetilde{H}^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}(\theta)/\theta = \mathcal{I}'$.

(3) By hypothesis, $\langle F, \alpha\rangle : \mathcal{I} \to^{se} \mathrm{SEN}'$ is a singleton (N, N')-epimorphic translation. So, it suffices to show that $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \langle \mathbf{Sign}', \mathrm{SEN}', C'^{\leftarrow\theta}\rangle$ is a semi-interpretation and that $\langle \mathbf{Sign}', \mathrm{SEN}', C'^{\leftarrow\theta}\rangle$ is full. Let $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$. If $\phi \in C_\Sigma(\Phi)$, then

$$\pi^\theta_{F(\Sigma)}(\alpha_\Sigma(\phi)) \in C'_{F(\Sigma)}(\pi^\theta_{F(\Sigma)}(\alpha_\Sigma(\Phi))),$$

whence $\alpha_\Sigma(\phi) \in C'^{\leftarrow\theta}_{F(\Sigma)}(\alpha_\Sigma(\Phi))$ and $\widetilde{H}^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}(\theta)$ is an (N, N')-model of $\mathcal{I}$ via $\langle F, \alpha\rangle$. It is full, by Proposition 5.10 of [29], since, by Proposition 5.8 of [29], $\mathcal{I}'$ is a full model of $\mathcal{I}$ via $\langle F, \pi_F^\theta \alpha\rangle$, and, by Proposition 11, $\langle \mathrm{I}_{\mathbf{Sign}'}, \pi^\theta\rangle : \widetilde{H}_{\mathrm{SEN}'}^{\langle F,\alpha\rangle}(\theta) \vdash^{se} \mathcal{I}'$ is an (N, N')-bilogical morphism.

(4) Suppose that $\theta^1 \leq \theta^2$ are two $\mathrm{Alg}^N(\mathcal{I})$-$N'$-congruence systems on SEN' via $\langle F, \alpha\rangle$. Let $\mathcal{I}'^1 = \langle \mathbf{Sign}', \mathrm{SEN}'^{\theta^1}, C'^1\rangle$ and $\mathcal{I}'^2 = \langle \mathbf{Sign}', \mathrm{SEN}'^{\theta^2}, C'^2\rangle$ be the $\langle F, \pi_F^{\theta^1}\alpha\rangle$-min and $\langle F, \pi_F^{\theta^2}\alpha\rangle$-min models of $\mathcal{I}$ on SEN'^{θ^1} and SEN'^{θ^2}, respectively. Note that, if $\langle \mathrm{I}_{\mathbf{Sign}'}, \eta\rangle : \mathrm{SEN}'^{\theta^1} \to^{se} \mathrm{SEN}'^{\theta^2}$ is defined, for all $\Sigma \in |\mathbf{Sign}'|$, $\phi \in \mathrm{SEN}'(\Sigma)$, by

$$\eta_\Sigma(\phi/\theta_\Sigma^1) = \phi/\theta_\Sigma^2,$$

$$\begin{tikzcd}
\mathrm{SEN}' \arrow[r, "\langle \mathrm{I}_{\mathbf{Sign}'}, \pi^{\theta^1}\rangle"] \arrow[dr, "\langle \mathrm{I}_{\mathbf{Sign}'}, \pi^{\theta^2}\rangle"'] & \mathrm{SEN}'^{\theta^1} \arrow[d, "\langle \mathrm{I}_{\mathbf{Sign}'}, \eta\rangle"] \\
& \mathrm{SEN}'^{\theta^2}
\end{tikzcd}$$

we have, for all $\Sigma \in |\mathbf{Sign}'|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}'(\Sigma)$,

$$\phi/\theta_\Sigma^1 \in C_\Sigma'^1(\Phi/\theta_\Sigma^1) \quad \text{implies} \quad \phi/\theta_\Sigma^2 \in C_\Sigma'^2(\Phi/\theta_\Sigma^2).$$

Therefore, for all $\Sigma \in |\mathbf{Sign}'|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}'(\Sigma)$,

$$\begin{array}{lll}
\phi \in C_\Sigma'^{\leftarrow\theta^1}(\Phi) & \text{iff} & \pi_\Sigma^{\theta^1}(\phi) \in C_\Sigma'^1(\pi_\Sigma^{\theta^1}(\Phi)) \\
& \text{implies} & \eta_\Sigma(\pi_\Sigma^{\theta^1}(\phi)) \in C_\Sigma'^2(\eta_\Sigma(\pi_\Sigma^{\theta^1}(\Phi))) \\
& \text{iff} & \pi_\Sigma^{\theta^2}(\phi) \in C_\Sigma'^2(\pi_\Sigma^{\theta^2}(\Phi)) \\
& \text{iff} & \phi \in C_\Sigma'^{\leftarrow\theta^2}(\Phi).
\end{array}$$

Therefore $\widetilde{H}_{\mathrm{SEN}'}^{\langle F,\alpha\rangle}(\theta^1) \leq \widetilde{H}_{\mathrm{SEN}'}^{\langle F,\alpha\rangle}(\theta^2)$. □

It will now be shown that the poset $\mathbf{Con}_{\mathbf{Alg}^N(\mathcal{I})}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$ of all $\mathrm{Alg}^N(\mathcal{I})$-$N'$-congruence systems on SEN' via $\langle F, \alpha\rangle$ under the $\leq$-ordering is isomorphic to the poset $\mathbf{FMod}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$ of all (N, N')-full models of $\mathcal{I}$ on SEN' via $\langle F, \alpha\rangle$ under the $\leq$-ordering. These two orderings were formally defined in [28] (see Theorem 3 and Corollary 9 therein for more details). The Isomorphism Theorem that follows is a π-institution analog of the corresponding Theorem 2.30 of [12] for sentential logics. It should be emphasized, once more, that it is an *analog and not a direct generalization* of Theorem 2.30 of [12] due to the difference in the definitions of full models and of algebraic systems between the sentential logic and the π-institution frameworks. In sentential logics, full models and $\mathcal{S}$-algebras are homomorphism-independent, whereas in π-institutions, full models and $(\mathcal{I}, N)$-algebraic systems are translation-specific. This difference was discussed in the comments towards the end of Section 1. However, it will be shown at the end of this section how one comes close to Theorem 2.30 of [12] using Theorem 13.

THEOREM 13 (The Isomorphism Theorem). *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *be a* π*-institution and* N *a category of natural transformations on* SEN. *Let also* $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ *be a functor,* N' *a category of natural transformations on* SEN' *and* $\langle F, \alpha\rangle : \mathcal{I} \to^{se} \mathrm{SEN}'$ *a singleton* (N, N')*-epimorphic translation. The Tarksi operator* $\widetilde{\Omega}^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}$ *is an order isomorphism between* $\mathbf{FMod}^{\langle F,\alpha\rangle}_{\mathcal{I}}(\mathrm{SEN}')$ *and* $\mathbf{Con}^{\langle F,\alpha\rangle}_{\mathrm{Alg}^N(\mathcal{I})}(\mathrm{SEN}')$ *and* $\widetilde{H}^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}$ *is its inverse operator.*

Proof. By Proposition 5, if $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ is an (N, N')-full model of $\mathcal{I}$ via $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$, then $\widetilde{\Omega}^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}(C') = \widetilde{\Omega}^{N'}(\mathcal{I}')$ is an $\mathrm{Alg}^N(\mathcal{I})$-$N'$-congruence system on SEN' via $\langle F, \alpha\rangle$. By Lemma 12, if the N'-congruence system $\theta \in \mathrm{Con}^{\langle F,\alpha\rangle}_{\mathrm{Alg}^N(\mathcal{I})}(\mathrm{SEN}')$, then

$$\widetilde{H}^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}(\theta) \in \mathrm{FMod}^{\langle F,\alpha\rangle}_{\mathcal{I}}(\mathrm{SEN}').$$

Therefore, the two mappings $\widetilde{\Omega}^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}$ and $\widetilde{H}^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}$ are well-defined. It suffices, therefore, to show that they are inverses of one another and order-preserving.

Suppose, first, that $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ is an (N, N')-full model of $\mathcal{I}$ via $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$. Then, by Proposition 5, $\mathrm{SEN}'^{N'}$ is an $(\mathcal{I}, N)$-algebraic system and $\widetilde{\Omega}^{N'}(\mathcal{I}')$ is an $\mathrm{Alg}^N(\mathcal{I})$-$N'$-congruence system on SEN'. Moreover, the closure system C' on SEN' is generated by $\langle I_{\mathbf{Sign}'}, \pi^{N'}\rangle : \mathrm{SEN}' \to^{se} \mathcal{I}''$, where $\mathcal{I}'' = \langle \mathbf{Sign}', \mathrm{SEN}'^{N'}, C''\rangle$ is the $\langle F, \pi^{N'}_F\alpha\rangle$-min $(N, \overline{N'})$-model of $\mathcal{I}$ on $\mathrm{SEN}'^{N'}$. Therefore we obtain $\mathcal{I}' = \widetilde{H}^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}(\widetilde{\Omega}^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}(\mathcal{I}'))$, by the definition of $\widetilde{H}^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}$.

Next, suppose that $\theta \in \mathrm{Con}^{\langle F,\alpha\rangle}_{\mathrm{Alg}^N(\mathcal{I})}(\mathrm{SEN}')$. Then, if $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}'^{\theta}, C'\rangle$ is the $\langle F, \pi^{\theta}_F\alpha\rangle$-min (N, N'^{θ})-model of $\mathcal{I}$ on SEN'^{θ}, we conclude that $\widetilde{\Omega}^{N'^{\theta}}(\mathcal{I}') = \Delta^{\mathrm{SEN}'^{\theta}}$. Thus, by Theorem 21 of [28] and Proposition 11,

$$\begin{aligned}
\widetilde{\Omega}^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}(\widetilde{H}^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}(\theta)) &= \widetilde{\Omega}^{N'}(C'^{\leftarrow\theta}) \\
&= \pi^{\theta^{-1}}(\widetilde{\Omega}^{N'^{\theta}}(\mathcal{I}')) \\
&= \pi^{\theta^{-1}}(\Delta^{\mathrm{SEN}'^{\theta}}) \\
&= \theta.
\end{aligned}$$

This concludes the proof that $\widetilde{\Omega}^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}$ and $\widetilde{H}^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}$ are inverse bijections. Both $\widetilde{\Omega}^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}$ and $\widetilde{H}^{\langle F,\alpha\rangle}_{\mathrm{SEN}'}$ are order-preserving by Corollary 9 of [28] and by Lemma 12, respectively, whence they are order-isomorphisms between $\mathbf{FMod}^{\langle F,\alpha\rangle}_{\mathcal{I}}(\mathrm{SEN}')$ and $\mathbf{Con}^{\langle F,\alpha\rangle}_{\mathrm{Alg}^N(\mathcal{I})}(\mathrm{SEN}')$. □

It will now be shown that the poset $\mathbf{Con}^{\langle F,\alpha\rangle}_{\mathrm{Alg}^N(\mathcal{I})}(\mathrm{SEN}')$ of all $\mathrm{Alg}^N(\mathcal{I})$-$N'$-congruence systems on SEN' via $\langle F, \alpha\rangle$ with the $\leq$-ordering is a complete lattice. Its meet is signature-wise intersection.

THEOREM 14. *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *be a* π*-institution and* N *a category of natural transformations on* SEN. *Let also* $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ *be a functor,*

N' a category of natural transformations on SEN$'$ *and* $\langle F, \alpha\rangle : \mathcal{I} \rightarrow^{se}$ SEN$'$ *a singleton* (N, N')*-epimorphic translation. Then* $\mathbf{Con}^{\langle F,\alpha\rangle}_{\mathrm{Alg}^N(\mathcal{I})}(\mathrm{SEN}')$ *is a complete lattice under signature-wise inclusion, where meet is signature-wise intersection.*

Proof. Suppose that $\{\theta^i\}_{i\in I}$ is a non-empty family of $\mathrm{Alg}^N(\mathcal{I})$-$N'$-congruence systems on SEN$'$ via $\langle F, \alpha\rangle$. Let $\theta = \bigcap_{i\in I} \theta^i$, i.e., $\theta_\Sigma = \bigcap_{i\in I} \theta^i_\Sigma$, for all $\Sigma \in |\mathbf{Sign}'|$. It will be shown that θ is also an $\mathrm{Alg}^N(\mathcal{I})$-$N'$-congruence system on SEN$'$ via $\langle F, \alpha\rangle$. Since θ^i is an $\mathrm{Alg}^N(\mathcal{I})$-$N'$-congruence system on SEN$'$ via $\langle F, \alpha\rangle$, the $\langle F, \pi^{\theta^i}_F \alpha\rangle$-min (N, N'^{θ^i})-model $\mathcal{I}'^i = \langle \mathbf{Sign}', \mathrm{SEN}'^{\theta^i}, C^i\rangle$ is N'^{θ^i}-reduced. Now define $\langle \mathrm{I}_{\mathbf{Sign}'}, \beta^i\rangle : \mathrm{SEN}'/\theta \rightarrow \mathrm{SEN}'/\theta^i$, by

$$\beta^i_\Sigma(\phi/\theta_\Sigma) = \phi/\theta^i_\Sigma, \quad \text{for all } \Sigma \in |\mathbf{Sign}'|, \phi \in \mathrm{SEN}'(\Sigma).$$

Since $\langle \mathrm{I}_{\mathbf{Sign}'}, \beta^i\rangle$ is a surjective $(N'^\theta, N'^{\theta^i})$-logical morphism from the $\langle F, \pi^\theta_F \alpha\rangle$-min (N, N'^θ)-model $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}'^\theta, C'\rangle$ to $\mathcal{I}'^i$, we have, by Proposition 8 of [28],

$$\begin{aligned}\widetilde{\Omega}^{N'^\theta}(\mathcal{I}') &\subseteq \beta^{i^{-1}}(\widetilde{\Omega}^{N'^{\theta^i}}(\mathcal{I}'^i))\\ &= \beta^{i^{-1}}(\Delta^{\mathrm{SEN}'^{\theta^i}}).\end{aligned}$$

Therefore, for all $\Sigma \in |\mathbf{Sign}'|$, $\phi, \psi \in \mathrm{SEN}'(\Sigma)$, if $\langle \phi/\theta_\Sigma, \psi/\theta_\Sigma\rangle \in \widetilde{\Omega}^{N'^\theta}_\Sigma(\mathcal{I}')$, then $\phi/\theta^i_\Sigma = \psi/\theta^i_\Sigma$, for all $i \in I$, and, hence $\phi/\theta_\Sigma = \psi/\theta_\Sigma$, i.e., $\mathcal{I}'$ is N'^θ-reduced. Hence θ is also an $\mathrm{Alg}^N(\mathcal{I})$-$N'$-congruence system on SEN$'$ via $\langle F, \alpha\rangle$.

Finally, it suffices to show that $\mathbf{Con}^{\langle F,\alpha\rangle}_{\mathrm{Alg}^N(\mathcal{I})}(\mathrm{SEN}')$ has a largest element. But it is easy to see that $\nabla^{\mathrm{SEN}'}$ is an N'-congruence system on SEN$'$ and that the $\langle F, \pi^{\nabla^{\mathrm{SEN}'}}_F \alpha\rangle$-min $(N, N'^{\nabla^{\mathrm{SEN}'}})$-model of $\mathcal{I}$ on $\mathrm{SEN}'^{\nabla^{\mathrm{SEN}'}}$ is $\nabla^{\mathrm{SEN}'}$-reduced. □

Putting together the Isomorphism Theorem 13 and Theorem 14 we obtain

COROLLARY 15. *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *be a* π*-institution and* N *a category of natural transformations on* SEN. *Let also* $\mathrm{SEN}' : \mathbf{Sign}' \rightarrow \mathbf{Set}$ *be a functor,* N' *a category of natural transformations on* SEN$'$ *and* $\langle F, \alpha\rangle : \mathcal{I} \rightarrow^{se}$ SEN$'$ *a singleton* (N, N')*-epimorphic translation.* $\mathbf{FMod}^{\langle F,\alpha\rangle}_{\mathcal{I}}(\mathrm{SEN}')$ *is a complete lattice and the Tarski operator is a lattice isomorphism between* $\mathbf{FMod}^{\langle F,\alpha\rangle}_{\mathcal{I}}(\mathrm{SEN}')$ *and the complete lattice* $\mathbf{Con}^{\langle F,\alpha\rangle}_{\mathrm{Alg}^N(\mathcal{I})}(\mathrm{SEN}')$.

Finally, a result is presented on the relation of bilogical morphisms between models of a given π-institution $\mathcal{I}$ with the corresponding $\mathrm{Alg}^N(\mathcal{I})$-congruence system lattices. Theorem 16 is the analog for π-institutions of Proposition 2.33 of [12].

THEOREM 16. *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *be a* π*-institution and* N *a category of natural transformations on* SEN. *Further, let* $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ *be an*

(N, N')-full model of $\mathcal{I}$ via $\langle M, \mu\rangle$ and $\mathcal{I}'' = \langle \mathbf{Sign}'', \mathrm{SEN}'', C''\rangle$ be an (N, N'')-full model of $\mathcal{I}$ via $\langle N, \nu\rangle$, such that $\langle F, \alpha\rangle : \mathcal{I}' \vdash^{se} \mathcal{I}''$ is an (N', N'')-bilogical morphism, with F an isomorphism and $\langle F, \alpha\rangle\langle M, \mu\rangle = \langle N, \nu\rangle$.

$$\begin{tikzcd} & \mathcal{I} \arrow[dl, "\langle M,\mu\rangle"'] \arrow[dr, "\langle N,\nu\rangle"] & \\ \mathcal{I}' \arrow[rr, "\langle F,\alpha\rangle"'] & & \mathcal{I}'' \end{tikzcd}$$

Then the mapping $\mathcal{C}^\bullet \mapsto \alpha^{-1}(\mathcal{C}^\bullet)$ is an isomorphism between the lattice of all (N, N')-full models of $\mathcal{I}$ on SEN' via $\langle M, \mu\rangle$ extending $\mathcal{I}'$ and the lattice of all (N, N'')-full models of $\mathcal{I}$ on SEN'' via $\langle N, \nu\rangle$ extending $\mathcal{I}''$. Moreover the principal ideals of the lattices $\mathbf{Con}^{\langle M,\mu\rangle}_{\mathbf{Alg}^N(\mathcal{I})}(\mathrm{SEN}')$ and $\mathbf{Con}^{\langle N,\nu\rangle}_{\mathbf{Alg}^N(\mathcal{I})}(\mathrm{SEN}'')$ determined by the Tarski congruence systems $\widetilde{\Omega}^{\langle M,\mu\rangle}_{\mathrm{SEN}'}(C')$ and $\widetilde{\Omega}^{\langle N,\nu\rangle}_{\mathrm{SEN}''}(C'')$, respectively, are isomorphic.

Proof. That the mapping $\mathcal{C}^\bullet \mapsto \alpha^{-1}(\mathcal{C}^\bullet)$ is an isomorphism between the lattice of all (N, N')-models of $\mathcal{I}$ on SEN' via $\langle M, \mu\rangle$ and the lattice of all (N, N'')-models of $\mathcal{I}$ on SEN'' via $\langle N, \nu\rangle$ is a consequence of Corollary 18 of [28]. Each of these models is a full model of $\mathcal{I}$ if and only if the other is, by Proposition 5.10 of [29]. Finally, the last part of the theorem follows from the Isomorphism Theorem 13 and Corollary 15. □

In conclusion, it is shown, as a demonstration of the fact that Theorem 13 encompasses some general results from both Universal Algebra and Abstract Algebraic Logic, how one may obtain the formula algebra case of Theorem 2.30 of [12] as a consequence of Theorem 13. Therefore, this also shows that the Isomorphism Theorem 5.1 of [3] with $\mathbf{A} = \mathbf{Fm}_{\mathcal{L}}(V)$ is also a special case of the Isomorphism Theorem 13.

Consider the statement of Theorem 13. Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be the π-institution with $\mathbf{Sign}$ the trivial one-element category, with object, e.g., a set of denumerable propositional variables V and $\mathrm{SEN}(V) = \mathrm{Fm}_{\mathcal{L}}(V)$, where $\mathcal{L}$ is a fixed but arbitrary universal algebraic signature. Suppose, also, that C is a structural closure operator on $\mathrm{Fm}_{\mathcal{L}}(V)$. Therefore $\mathcal{I}$ corresponds to a sentential logic in the sense of Font and Jansana [12]. Take the category N of natural transformations on SEN to be the clone of all algebraic operations generated by the signature $\mathcal{L}$. In that case an N-congruence system on SEN coincides with a universal algebraic $\mathcal{L}$-congruence on $\mathrm{Fm}_{\mathcal{L}}(V)$. Let now $\mathrm{SEN}' = \mathrm{SEN}$, $N' = N$ and $\langle F, \alpha\rangle = \langle \mathrm{I}_{\mathbf{Sign}}, \iota\rangle$ be the identity surjective singleton (N, N)-epimorphic translation. Then Theorem 13 yields that the Tarksi operator $\widetilde{\Omega}^{\langle \mathrm{I}_{\mathbf{Sign}},\iota\rangle}_{\mathrm{SEN}}$ is an order isomorphism between $\mathbf{FMod}^{\langle \mathrm{I}_{\mathbf{Sign}},\iota\rangle}_{\mathcal{I}}(\mathrm{SEN})$ and $\mathbf{Con}^{\langle \mathrm{I}_{\mathbf{Sign}},\iota\rangle}_{\mathrm{Alg}^N(\mathcal{I})}(\mathrm{SEN})$ and $\widetilde{H}^{\langle \mathrm{I}_{\mathbf{Sign}},\iota\rangle}_{\mathrm{SEN}}$ is its inverse operator. But, in this context, a full model in $\mathrm{FMod}^{\langle \mathrm{I}_{\mathbf{Sign}},\iota\rangle}_{\mathcal{I}}(\mathrm{SEN})$ coincides with a full model according to [12] on the formula algebra $\mathbf{Fm}_{\mathcal{L}}(V)$ and an $\mathrm{Alg}^N(\mathcal{I})$-$N$-congruence coincides with an Alg $\mathcal{S}$-congruence on the formula algebra. Therefore, with this special setup, we obtain the special case of the Isomorphism Theorem 2.30 of [12] with $\mathbf{A} = \mathbf{Fm}_{\mathcal{L}}(V)$.

Acknowledgements

This paper continues work by the author started in [28, 29] (see also the related [30]). This line of research generalizes and adapts results of Font and Jansana [12] from the framework of sentential logics to that of π-institutions. Both the categorical theory and the results of Font and Jansana have their origins in the pioneering work of Czelakowski [6] and Blok and Pigozzi [2, 3]. In particular, the categorical theory originated with the author's doctoral dissertation [21] (see also [23, 22]), written at Iowa State under Don Pigozzi's supervision. Don's guidance and support is gratefully acknowledged.

Finally, many thanks to an anonymous referee for useful comments and suggestions.

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