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CATEGORICAL ABSTRACT ALGEBRAIC LOGIC: STRONG VERSION OF A PROTOALGEBRAIC π-INSTITUTION

A b s t r a c t. An analog of the strong version of a protoalgebraic logic, introduced by Font and Jansana, is presented for N-protoalgebraic π-institutions. Some properties of this strong version of an N-protoalgebraic π-institution are explored and they are related to the explicit definability of N-Leibniz theory systems. N-Leibniz theory systems were introduced in previous work by the author, also taking after the corresponding theory of Font and Jansana in the sentential framework.

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1. Introduction

The goal of the present work is to continue the study of N-Leibniz theory systems of a given N-protoalgebraic π-institution that was initiated in [24], along the lines of the theory of Leibniz filters of a protoalgebraic sentential logic of [10]. In [10], Font and Jansana studied the Leibniz filters of a protoalgebraic logic $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$. For such a logic, the Leibniz operator need not be injective on the collection of $\mathcal{S}$-filters on an $\mathcal{L}$-algebra $\mathbf{A}$. It is therefore reasonable to single out those $\mathcal{S}$-filters F on $\mathbf{A}$ that are included in every $\mathcal{S}$-filter on $\mathbf{A}$ having the same Leibniz congruence as F. These are the *Leibniz $\mathcal{S}$-filters* of $\mathbf{A}$.

The work of Font and Jansana may be split into two major parts. In the first part, the study of general properties of Leibniz filters is undertaken and, in the second, the study of properties of the sentential logic $\mathcal{S}^+$ defined by all those $\mathcal{S}$-matrices of the form $\langle \mathbf{A}, F \rangle$, with F Leibniz, is carried out. Font and Jansana called the logic $\mathcal{S}^+$ the *strong version* of the protoalgebraic logic $\mathcal{S}$. They were led to the introduction of $\mathcal{S}^+$ by their motivation to explain a phenomenon observed in a variety of specific examples, e.g., modal logic, quantum logic and many-valued logic, among others, in which logics come naturally in pairs, one stronger than the other, but with the same theorems. It turns out that in these examples the strongest logic of the pair is the strong version of the other logic in this specific formal sense. Jansana in [14] continues the study started in [10].

The author in [24], inspired by [10], started a similar study of the N-Leibniz theory systems of an N-protoalgebraic π-institution [22, 23]. [24] covered the analogous development for π-institutions of the first part of the work of [10]. In [25] some additional results were obtained on the basic theory. The present work has the role of extending the categorical theory along the lines of the second part of [10]. Thus, the aim is to introduce an analog of the strong version of a protoalgebraic sentential logic for π-institutions, study its properties and relate these results to the logical and the explicit definability of Leibniz theory systems in the context of π-institutions.

Before embarking on a brief review of the results presented in [24] on N-Leibniz theory systems, we remind the reader of the definition of a π-institution and of that of an N-protoalgebraic π-institution, which constitute the backbone of the framework in which the studies started in [24] and continued here are taking place. Recall from [8] (see also [12, 13]) that a π-*institution* $\mathcal{I}$ is a triple $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, such that

(i) $\mathbf{Sign}$ is a category, whose objects are called *signatures*;

(ii) $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, is a set-valued functor from the category $\mathbf{Sign}$ of signatures, called the *sentence functor* and giving, for each signature Σ, a set whose elements are called *sentences* over that signature Σ or Σ-*sentences*;

(iii) $C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \to \mathcal{P}(\mathrm{SEN}(\Sigma))$, for each $\Sigma \in |\mathbf{Sign}|$, is a mapping, called Σ-*closure*, such that

  (a) $A \subseteq C_\Sigma(A)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq \mathrm{SEN}(\Sigma)$,

  (b) $C_\Sigma(C_\Sigma(A)) = C_\Sigma(A)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq \mathrm{SEN}(\Sigma)$,

  (c) $C_\Sigma(A) \subseteq C_\Sigma(B)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq B \subseteq \mathrm{SEN}(\Sigma)$,

  (d) $\mathrm{SEN}(f)(C_{\Sigma_1}(A)) \subseteq C_{\Sigma_2}(\mathrm{SEN}(f)(A))$, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, $A \subseteq \mathrm{SEN}(\Sigma_1)$.

A subset $T_\Sigma \subseteq \mathrm{SEN}(\Sigma)$ is called a Σ-*theory* of $\mathcal{I}$ if it is closed, i.e., if $C_\Sigma(T_\Sigma) = T_\Sigma$. The collection of all Σ-theories of $\mathcal{I}$ is denoted by $\mathrm{Th}_\Sigma(\mathcal{I})$. A collection $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ of subsets $T_\Sigma \subseteq \mathrm{SEN}(\Sigma), \Sigma \in |\mathbf{Sign}|$, is called an *axiom family* of $\mathcal{I}$. It is called an *axiom system* of $\mathcal{I}$ if, in addition, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, $\mathrm{SEN}(f)(T_{\Sigma_1}) \subseteq T_{\Sigma_2}$. A collection $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ of subsets $T_\Sigma \subseteq \mathrm{SEN}(\Sigma), \Sigma \in |\mathbf{Sign}|$, is called a *theory family* of $\mathcal{I}$ if T_Σ is a Σ-theory, for all $\Sigma \in |\mathbf{Sign}|$. It is called a *theory system* of $\mathcal{I}$ if, in addition, it is an axiom system. The collection $\mathrm{ThFam}(\mathcal{I})$ of theory families of a π-institution $\mathcal{I}$ ordered by signature-wise inclusion $\leq$ forms a complete lattice $\mathbf{ThFam}(\mathcal{I}) = \langle \mathrm{ThFam}(\mathcal{I}), \leq\rangle$ and the same holds for the collection $\mathrm{ThSys}(\mathcal{I})$ of all theory systems under signature-wise inclusion. Moreover, it is not very hard to see that $\mathbf{ThSys}(\mathcal{I})$ forms a complete sublattice of $\mathbf{ThFam}(\mathcal{I})$.

Suppose, now, that $\mathbf{Sign}$ is a category and $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is an arbitrary functor. Recall from [22] (this is a corrected version of the original definition given in [16]) that the *clone of all natural transformations on* SEN is defined to be the locally small category with collection of objects $\{\mathrm{SEN}^\alpha : \alpha \text{ an ordinal}\}$ and collection of morphisms $\tau : \mathrm{SEN}^\alpha \to \mathrm{SEN}^\beta$ β-sequences of natural transformations $\tau_i : \mathrm{SEN}^\alpha \to \mathrm{SEN}$. Composition

$$\mathrm{SEN}^\alpha \xrightarrow{\langle\tau_i : i<\beta\rangle} \mathrm{SEN}^\beta \xrightarrow{\langle\sigma_j : j<\gamma\rangle} \mathrm{SEN}^\gamma$$

is defined by

$$\langle\sigma_j : j<\gamma\rangle \circ \langle\tau_i : i<\beta\rangle = \langle\sigma_j(\langle\tau_i : i<\beta\rangle) : j<\gamma\rangle.$$

A subcategory N of this category with objects all objects of the form SEN^k for $k<\omega$, and containing all projection morphisms $p^{k,i} : \mathrm{SEN}^k \to \mathrm{SEN}, i < k, k<\omega$, with $p^{k,i}_\Sigma : \mathrm{SEN}(\Sigma)^k \to \mathrm{SEN}(\Sigma)$ given by

$$p^{k,i}_\Sigma(\vec{\phi}) = \phi_i, \quad \text{for all} \quad \vec{\phi} \in \mathrm{SEN}(\Sigma)^k,$$

and such that, for every family $\{\tau_i : \mathrm{SEN}^k \to \mathrm{SEN} : i<l\}$ of natural transformations in N, the sequence $\langle\tau_i : i<l\rangle : \mathrm{SEN}^k \to \mathrm{SEN}^l$ is also in N, is referred to as *a category of natural transformations on* SEN.

Given a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, a collection $\theta = \{\theta_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$, such that θ_Σ is an equivalence relation on $\mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$, is called an *equivalence family on* SEN. If, in addition, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, θ satisfies $\mathrm{SEN}(f)^2(\theta_{\Sigma_1}) \subseteq \theta_{\Sigma_2}$, then θ is said to be an *equivalence system on* SEN. If N is a category of natural transformations on SEN and an equivalence system θ on SEN satisfies, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi_0, \psi_0, \ldots, \phi_{n-1}, \psi_{n-1} \in \mathrm{SEN}(\Sigma)$,

$$\langle\phi_i,\psi_i\rangle \in \theta_\Sigma,\ i<n, \quad \text{imply} \quad \langle\sigma_\Sigma(\phi_0,\ldots,\phi_{n-1}), \sigma_\Sigma(\psi_0,\ldots,\psi_{n-1}\rangle \in \theta_\Sigma,$$

then θ is a said to be an *N-congruence system on* SEN. An N-congruence system θ on SEN is said to be a *logical N-congruence system of* a π-institution $\mathcal{I} = \langle\mathbf{Sign}, \mathrm{SEN}, C\rangle$ if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle\phi,\psi\rangle \in \theta_\Sigma \quad \text{implies} \quad C_\Sigma(\phi) = C_\Sigma(\psi).$$

It was shown in [16] that, under the signature-wise inclusion $\leq$, the collection $\mathrm{Con}^N(\mathcal{I})$ of all logical N-congruence systems of $\mathcal{I}$ forms a complete lattice $\mathbf{Con}^N(\mathcal{I})$, whence, there exists a largest logical N-congruence system of $\mathcal{I}$, which is denoted by $\widetilde{\Omega}^N(\mathcal{I})$ and is called the *Tarski N-congruence system of $\mathcal{I}$*.

An N-congruence system θ on SEN is said to be *compatible with* the axiom family T, if, for all $\Sigma \in |\mathbf{Sign}|$, θ_Σ is compatible with T_Σ in the usual sense, i.e., if, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$, $\langle \phi, \psi \rangle \in \theta_\Sigma$ and $\phi \in T_\Sigma$ imply that $\psi \in T_\Sigma$. It was shown in Proposition 2.3 of [22] that there always exists a largest N-congruence system on SEN that is compatible with a given axiom family T. It is called the *Leibniz N-congruence system of T* and denoted by $\Omega^N(T)$.

In [22] Leibniz N-congruence systems are used to define the notion of an N-protoalgebraic π-institution. Namely, a π-institution

$$\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle,$$

with N a category of natural transformations on SEN, is said to be *N-protoalgebraic* if, for every theory family $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ of $\mathcal{I}$, every $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Omega^N_\Sigma(T) \quad \text{implies} \quad C_\Sigma(T_\Sigma \cup \{\phi\}) = C_\Sigma(T_\Sigma \cup \{\psi\}).$$

In Lemma 3.8 of [22], it is shown that a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on SEN, is N-protoalgebraic, if and only if, for all $T, T' \in \mathrm{ThFam}(\mathcal{I})$, we have that

$$T \leq T' \quad \text{implies} \quad \Omega^N(T) \leq \Omega^N(T'),$$

i.e., if and only if the N-Leibniz operator on the collection of all theory families of $\mathcal{I}$ is monotonic. Some other equivalent conditions are provided in [22], which also studies some additional properties of N-protoalgebraic π-institutions. A further study of N-protoalgebraicity is contained in [23] and the interested reader is advised to consult these two references.

In the remaining of this Introduction, some of the results presented in [24] on the N-Leibniz theory systems of an N-protoalgebraic π-institution are briefly reviewed. References to the corresponding results of [24] are provided, when appropriate, so that the interested reader may consult the proofs presented there.

Consider an N-protoalgebraic π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on SEN. A theory family $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ is said to be an *N-Leibniz theory family* of $\mathcal{I}$, if, for every theory family $T' = \{T'_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ of $\mathcal{I}$, such that $\Omega^N(T) = \Omega^N(T')$, we have $T \leq T'$. It was shown in Theorem 4.3 of [25] that, for $\mathcal{I}$ N-protoalgebraic, every N-Leibniz theory family is actually an N-Leibniz theory system.

Therefore, no need for the special notation $\mathrm{ThFam}^N(\mathcal{I})$ for N-Leibniz theory families arizes and the notation $\mathrm{ThSys}^N(\mathcal{I})$, introduced in [24], covers also N-Leibniz theory families.

It may be shown, as was done in Proposition 1 of [24] for theory systems, that, given any theory family T in an N-protoalgebraic π-institution, there exists a unique N-Leibniz theory family T^N, which by Theorem 4.3 of [25] is a theory system, such that $\Omega^N(T^N) = \Omega^N(T)$. For the proof of this result, consult the proof of Proposition 1 of [24] (see also Proposition 4.2 of [25]). As a consequence of the definition and of this fact, we have that $T^N \leq T$ and that an arbitrary theory family T is N-Leibniz if and only if $T^N = T$.

Suppose, now, that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is an N-protoalgebraic π-institution and that $\langle F, \alpha\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}'$ is a surjective (N, N')-epimorphic translation. It was shown in Proposition 5.22 of [22] that the $\langle F, \alpha\rangle$-min (N, N')-model $\mathcal{I}'^{\min}$ of $\mathcal{I}$ on SEN' is N'-protoalgebraic. In Theorem 3 of [24], an analog of Theorem 3 of [10] for π-institutions, it was shown that the N'-Leibniz operator is an isomorphism between the lattice of all N'-Leibniz theory systems of $\mathcal{I}'^{\min}$ and that of all $\mathrm{Alg}^N(\mathcal{I})$-$N'$-congruence systems θ on SEN', where θ is an $\mathrm{Alg}^N(\mathcal{I})$-$N'$-congruence system because the $\langle F, \pi^\theta_F\alpha\rangle$-min model of $\mathcal{I}$ on SEN'^θ is N'^θ-reduced. One obtains as a corollary of this result, combined with Theorem 4.3 of [25], that, given an N-protoalgebraic π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ and a theory family T of $\mathcal{I}$, T^N is the largest N-Leibniz theory system $\leq$-contained in T. Therefore, if $\mathcal{I}$ is an N-protoalgebraic π-institution, then, for all theory families T, T' of $\mathcal{I}$, such that $T \leq T'$, we obtain that $T^N \leq T'^N$.

In the final result of [24], it was shown that, if $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is an N-protoalgebraic π-institution, the collection $\mathrm{ThSys}^N(\mathcal{I})$ forms a join-complete subsemilattice of the complete lattice $\mathbf{ThSys}(\mathcal{I}) = \langle \mathrm{ThSys}(\mathcal{I}), \leq\rangle$ of all theory systems of $\mathcal{I}$. This result forms an analog of Proposition 13 of [10] in the context of π-institutions.

For all unexplained categorical terminology and notation the reader is referred to any of [1, 4, 15]. For background on the theory of abstract algebraic logic and discussion of the classes of the abstract algebraic hierarchy, including the class of protoalgebraic logics, the reader is referred to the review article [11], the monograph [9] and the book [6]. (Among the most important original sources are [2], [5], [7], [3].) More specifically, for the theory of the Leibniz filters of a protoalgebraic sentential logic see [10] and the sequel [14]. For a full account of the categorical theory, leading to the results presented in this paper, see [16]-[25] in the given order.

2. Strong Version $\mathcal{I}^{+_N}$ of $\mathcal{I}$

Given an N-protoalgebraic π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, a functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, with N' a category of natural transformations on SEN', and a surjective (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, the collection of all theory systems of the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN' will be used often in what follows. It will be denoted by $\mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$. On the other hand, by $\mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle^{N'}}(\mathrm{SEN}')$ will be denoted the collection of all N'-Leibniz filters of the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN'.

Also, given two functors $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ and $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, with categories of natural transformations N and N' on SEN and SEN', respectively, a prominent role will be played by the surjective (N, N')-epimorphic translations $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, with isomorphic functor components $F : \mathbf{Sign} \to \mathbf{Sign}'$. These translations will be called **isosurjective** (N, N')-epimorphic translations in the sequel.

Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, with N a category of natural transformations on SEN. Consider the following collection $\mathcal{F}$ of isosurjective (N, N')-epimorphic translations

$$\begin{aligned}\mathcal{F} \quad = \quad &\{\langle F, \alpha\rangle : \mathcal{I} \to^{se} \mathrm{SEN}' : \langle F, \alpha\rangle \text{ isosurjective } (N, N')\text{-epimorphic}\\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for some } \mathrm{SEN}', N'\}.\end{aligned}$$

Now let $\mathcal{T}'$ denote the collection of all N'-Leibniz theory systems of all min (N, N')-models of $\mathcal{I}$ via isosurjective (N, N')-logical morphisms:

$$\mathcal{T}' = \{T' \in \mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle^{N'}}(\mathrm{SEN}') : \langle F, \alpha\rangle \in \mathcal{F}\}.$$

Set

$$\begin{aligned}\mathcal{T} \quad &= \quad \{\alpha^{-1}(T') : T' \in \mathcal{T}'\}\\ &= \quad \{\alpha^{-1}(T') : T' \in \mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle^{N'}}(\mathrm{SEN}'), \langle F, \alpha\rangle \in \mathcal{F}\} \qquad (1)\end{aligned}$$

and note that every element in $\mathcal{T}$ is a theory system of $\mathcal{I}$. Therefore, it makes sense to consider the largest closure system C^{+_N} of $\mathcal{I}$, such that $\mathcal{T} \subseteq$

$\mathrm{ThSys}(C^{+_N})$. The corresponding smallest closed set system corresponding to C^{+_N} will be denoted by $\mathcal{C}^{+_N}$ and this notational convention, borrowed from [9], will be followed throughout for other closure systems as well.

Definition 2.1 *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$*, with* N *a category of natural transformations on* SEN*, is an* N*-protoalgebraic* π*-institution. Denote by* $\mathcal{I}^{+_N} = \langle \mathbf{Sign}, \mathrm{SEN}, C^{+_N}\rangle$ *the* π*-institution, whose closure system* C^{+_N} *is the closure system on* SEN *generated by the collection* $\mathcal{T}$ *of theory systems of* $\mathcal{I}$ *displayed in* (1)*.* $\mathcal{I}^{+_N}$ *will be called the* N-**strong version** *of the* N*-protoalgebraic* π*-institution* $\mathcal{I}$*.*

Sometimes, the N-strong version of $\mathcal{I}$ is just referred to as the *strong version*, when the category N of natural transformations is clear from context and there is no possibility of confusion.

It is more difficult in the case of N-protoalgebraic π-institutions than it is in the sentential logic framework, but one may still prove that, if $\mathcal{I}$ is N-weakly algebraizable (see [26] for an exposition on N-weak algebraizability), i.e., if Ω^N is also injective on theory systems, then $\mathrm{ThSys}(\mathcal{I}^{+_N}) = \mathrm{ThSys}(\mathcal{I})$. This is the content of the following proposition, only a partial analog of Proposition 16 of [10] for π-institutions.

Proposition 2.2 *If* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$*, with* N *a category of natural transformations on* SEN*, is an* N*-weakly algebraizable* π*-institution, then* $\mathrm{ThSys}(\mathcal{I}^{+_N}) = \mathrm{ThSys}(\mathcal{I})$*.*

Proof. Since every $T \in \mathcal{T}$ is a theory system of $\mathcal{I}$, the least closure system generated by $\mathcal{T}$ is always included in $\mathcal{C}$. Hence $\mathrm{ThSys}(\mathcal{I}^{+_N}) \subseteq \mathrm{ThSys}(\mathcal{I})$ always holds.

It suffices, therefore, to show that, if $\mathcal{I}$ is N-weakly algebraizable, then $\mathrm{ThSys}(\mathcal{I}) \subseteq \mathrm{ThSys}(\mathcal{I}^{+_N})$. Since $\mathcal{I}$ is N-weakly algebraizable, $\mathrm{ThSys}^N(\mathcal{I}) = \mathrm{ThSys}(\mathcal{I})$. Therefore, considering the isosurjective (N, N)-epimorphic translation $\langle \mathrm{I}_{\mathbf{Sign}}, \iota\rangle : \mathcal{I} \rightarrow^{se} \mathrm{SEN}$, we get that $\mathrm{ThSys}(\mathcal{I}) = \mathrm{ThSys}^N(\mathcal{I}) \subseteq \mathcal{T} \subseteq \mathrm{ThSys}(\mathcal{I}^{+_N})$. $\square$

A technical lemma will now be presented that will help us show that the Leibniz theory system of every theory system of a min model of a π-institution via an isosurjective translation is a theory system of the min model of the strong version of the π-institution via the same translation.

The lemma asserts that, given an isosurjective translation $\langle F,\alpha\rangle$ from the sentence functor SEN of a π-institution $\mathcal{I}$ to a functor SEN$'$, and an axiom family T' on SEN$'$, if $\alpha^{-1}(T')$ is a theory family of $\mathcal{I}$, then T' is a theory family of the $\langle F,\alpha\rangle$-min model of $\mathcal{I}$ on SEN$'$.

Lemma 2.3 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution with N a category of natural transformations on* SEN. *Let* $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ *be a functor, with N' a category of natural transformations on* SEN$'$, $\langle F,\alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ *an isosurjective (N,N')-epimorphic translation and $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ the $\langle F,\alpha\rangle$-min (N,N')-model of $\mathcal{I}$ on* SEN$'$. *If* $\Sigma \in |\mathbf{Sign}|$ *and* $T'_{F(\Sigma)} \subseteq \mathrm{SEN}'(F(\Sigma))$, *such that* $\alpha_\Sigma^{-1}(T'_{F(\Sigma)}) \in \mathrm{Th}_\Sigma(\mathcal{I})$, *then* $T'_{F(\Sigma)} \in \mathrm{Th}_{F(\Sigma)}(\mathcal{I}')$.

Proof. Suppose that T' is an axiom family on SEN$'$, such that $\alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$. To see that $T' \in \mathrm{ThFam}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$, it suffices to show that there exists an (N,N')-model $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ of $\mathcal{I}$ on SEN$'$ via $\langle F,\alpha\rangle$, such that $T' \in \mathrm{ThFam}(\mathcal{I}')$, since, then, T' will also be a theory family of the $\langle F,\alpha\rangle$-min (N,N')-model of $\mathcal{I}$ on SEN$'$.

To this end, and taking into account both the surjectivity of $\langle F,\alpha\rangle$ and the fact that F is an isomorphism, let $\mathcal{C}' = \{\mathcal{C}'_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}'|}$, with $\mathcal{C}'_{F(\Sigma)} = \{T'_{F(\Sigma)} \subseteq \mathrm{SEN}'(F(\Sigma)) : \alpha_\Sigma^{-1}(T'_{F(\Sigma)}) \in \mathrm{Th}_\Sigma(\mathcal{I})\}$, for all $\Sigma \in |\mathbf{Sign}|$. It is easy to see that, for all $\Sigma \in |\mathbf{Sign}|$, $\mathcal{C}'_{F(\Sigma)}$ is a closed set system on $\mathrm{SEN}'(F(\Sigma))$. We have indeed that $\mathrm{SEN}'(F(\Sigma)) \in \mathcal{C}'_{F(\Sigma)}$ and, moreover, for all $T^i_{F(\Sigma)} \in \mathcal{C}'_{F(\Sigma)}, i \in I$, since $\alpha_\Sigma^{-1}(T^i_{F(\Sigma)}) \in \mathrm{Th}_\Sigma(\mathcal{I})$, for all $i \in I$, we get that $\alpha_\Sigma^{-1}(\bigcap_{i\in I} T^i_{F(\Sigma)}) = \bigcap_{i\in I} \alpha_\Sigma^{-1}(T^i_{F(\Sigma)}) \in \mathrm{Th}_\Sigma(\mathcal{I})$, whence $\bigcap_{i\in I} T^i_{F(\Sigma)} \in \mathcal{C}'_{F(\Sigma)}$ and $\mathcal{C}'_{F(\Sigma)}$ is a closed set system.

$\mathcal{C}'$ is also structural. To see this, suppose, using the surjectivity of $\langle F,\alpha\rangle$, that $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma,\Sigma')$ and $\Phi \subseteq \mathrm{SEN}(\Sigma)$. Then

$$\begin{aligned}
&\mathrm{SEN}'(F(f))(C'_{F(\Sigma)}(\alpha_\Sigma(\Phi)))\\
&\qquad = \mathrm{SEN}'(F(f))(\textstyle\bigcap\{T'_{F(\Sigma)} \in \mathcal{C}'_{F(\Sigma)} : \alpha_\Sigma(\Phi) \subseteq T'_{F(\Sigma)}\})\\
&\qquad \subseteq \textstyle\bigcap\{\mathrm{SEN}'(F(f))(T'_{F(\Sigma)}) : T'_{F(\Sigma)} \in \mathcal{C}'_{F(\Sigma)}, \alpha_\Sigma(\Phi) \subseteq T'_{F(\Sigma)}\}\\
&\qquad \subseteq \textstyle\bigcap\{C'_{F(\Sigma')}(\mathrm{SEN}'(F(f))(T'_{F(\Sigma)})) : T'_{F(\Sigma)} \in \mathcal{C}'_{F(\Sigma)}, \alpha_\Sigma(\Phi) \subseteq T'_{F(\Sigma)}\}\\
&\qquad \subseteq \textstyle\bigcap\{C'_{F(\Sigma')}(\mathrm{SEN}'(F(f))(\mathrm{SEN}'(F(f))^{-1}(T'_{F(\Sigma')}))) :\\
&\qquad\qquad\qquad T'_{F(\Sigma')} \in \mathcal{C}'_{F(\Sigma')}, \mathrm{SEN}'(F(f))(\alpha_\Sigma(\Phi)) \subseteq T'_{F(\Sigma')}\}\\
&\qquad \subseteq \textstyle\bigcap\{T'_{F(\Sigma')} \in \mathcal{C}'_{F(\Sigma')} : \mathrm{SEN}'(F(f))(\alpha_\Sigma(\Phi)) \subseteq T'_{F(\Sigma')}\}\\
&\qquad = C'_{F(\Sigma')}(\mathrm{SEN}'(F(f))(\alpha_\Sigma(\Phi))).
\end{aligned}$$

Thus $\mathcal{C}'$ is structural and, therefore, a closure system on SEN'.

Finally, it suffices to show that $\langle F, \alpha\rangle : \mathcal{I} \succ^{se} \mathcal{I}'$ is an (N, N')-logical morphism, i.e., that $\mathcal{I}'$ is an (N, N')-model of $\mathcal{I}$ via $\langle F, \alpha\rangle$. To this end, suppose that $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$. Thus, we have

$$\begin{array}{rcl}\phi & \in & C_\Sigma(\Phi)\\ & = & \bigcap\{T_\Sigma \in \mathrm{Th}_\Sigma(\mathcal{I}) : \Phi \subseteq T_\Sigma\}\\ & \subseteq & \bigcap\{\alpha_\Sigma^{-1}(T'_{F(\Sigma)}) : T'_{F(\Sigma)} \in \mathcal{C}'_{F(\Sigma)}, \Phi \subseteq \alpha_\Sigma^{-1}(T'_{F(\Sigma)})\}\\ & = & \alpha_\Sigma^{-1}(\bigcap\{T'_{F(\Sigma)} \in \mathcal{C}'_{F(\Sigma)} : \alpha_\Sigma(\Phi) \subseteq T'_{F(\Sigma)}\})\\ & = & \alpha_\Sigma^{-1}(C'_{F(\Sigma)}(\alpha_\Sigma(\Phi)),\end{array}$$

which shows that $\alpha_\Sigma(\phi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$ and concludes the proof that $\langle F, \alpha\rangle : \mathcal{I} \succ^{se} \mathcal{I}'$ is an (N, N')-logical morphism. Thus, $\mathcal{I}'$ is indeed an (N, N')-model of $\mathcal{I}$ via $\langle F, \alpha\rangle$.

Now, since, by hypothesis, $\alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$, we get, by the definition of $\mathcal{I}'$, that $T' \in \mathrm{ThFam}(\mathcal{I}')$, concluding the proof of the Lemma. $\square$

The result announced before Lemma 2.3 is now presented. It will prove useful in many of the proofs involving models of $\mathcal{I}$ and $\mathcal{I}^{+N}$ in what follows. It says that the Leibniz theory system of every theory system of a min model of $\mathcal{I}$ via an isosurjective translation is a theory system of the min model of $\mathcal{I}^{+N}$ via the same translation.

Lemma 2.4 *Suppose $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, is an N-protoalgebraic π-institution. For every functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, with N' a category of natural transformations on SEN' and $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ an isosurjective (N, N')-epimorphic translation, if $T \in \mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$, then $T^{N'} \in \mathrm{ThSys}_{\mathcal{I}^{+N}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$.*

Proof. Suppose that $T \in \mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$. Then, since $\langle F, \alpha\rangle$ is isosusrjective, we obtain

$$T^{N'} \in \mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle^{N'}}(\mathrm{SEN}') \subseteq \mathcal{T}'.$$

Therefore $\alpha^{-1}(T^{N'}) \in \mathcal{T} \subseteq \mathrm{ThSys}(\mathcal{I}^{+N})$. Thus, by Lemma 2.3 and the isosurjectivity of $\langle F, \alpha\rangle$, we get that $T^{N'} \in \mathrm{ThSys}_{\mathcal{I}^{+N}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$. $\square$

The following Proposition, an analog of Proposition 18 of [10], exhibits some properties of $\mathcal{I}^{+_N}$ for an N-protoalgebraic π-institution $\mathcal{I}$. The last part of Proposition 18 of [10] has been separated and it is formulated as Proposition 2.6, since its statement is rather long in the present context and it may be easier to read independently.

Note the following variation of a notation first used in [22]. Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, an $(\mathcal{I}, N)$-*algebraic system* is a functor $\mathrm{SEN}' : \mathbf{Sign'} \to \mathbf{Set}$, equipped with a category N' of natural transformations on SEN', such that there exists an (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, such that the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN' is N'-reduced. The class of all $(\mathcal{I}, N)$-algebraic systems is denoted by $\mathrm{Alg}^N(\mathcal{I})$. Those $(\mathcal{I}, N)$-algebraic systems SEN' for which the (N, N')-epimorphic translation $\langle F, \alpha\rangle$ witnessing the membership in $\mathrm{Alg}^N(\mathcal{I})$ is surjective form the subclass $\mathrm{Alg}^N(\mathcal{I})^s$ of $\mathrm{Alg}^N(\mathcal{I})$ and those for which the (N, N')-epimorphic translation $\langle F, \alpha\rangle$ is isosurjective form the subclass $\mathrm{Alg}^N(\mathcal{I})^{is}$. Thus, we have that $\mathrm{Alg}^N(\mathcal{I})^{is} \subseteq \mathrm{Alg}^N(\mathcal{I})^s \subseteq \mathrm{Alg}^N(\mathcal{I})$.

A similar notational convention will be applied to the collection $\mathrm{Alg}^N(\mathcal{I})^*$, which was also introduced for the first time in [22]. This notation refers to the collection of all $(\mathcal{I}, N)^*$-*algebraic systems*. These are defined to be functors $\mathrm{SEN}' : \mathbf{Sign'} \to \mathbf{Set}$, equipped with a category N' of natural transformations on SEN', such that there exists an (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ and a theory family T' of the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN', such that $\Omega^{N'}(T') = \Delta^{\mathrm{SEN}'}$. The subcollection of all those $(\mathcal{I}, N)^*$-algebraic systems SEN' for which the (N, N')-epimorphic translation $\langle F, \alpha\rangle$ witnessing the membership in $\mathrm{Alg}^N(\mathcal{I})^*$ is isosurjective will be denoted by $\mathrm{Alg}^N(\mathcal{I})^{*is}$.

Proposition 2.5 *Suppose $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN, *is an N-protoalgebraic π-institution. Then*

1. $C \le C^{+_N}$ *or, equivalently,* $\mathcal{C}^{+_N} \le \mathcal{C}$;
2. $\mathrm{ThSys}^N(\mathcal{I}) \subseteq \mathrm{ThSys}(\mathcal{I}^{+_N}) \subseteq \mathrm{ThSys}(\mathcal{I})$;
3. $\mathrm{Alg}^N(\mathcal{I}^{+_N})^{is} = \mathrm{Alg}^N(\mathcal{I})^{is}$;
4. $\mathcal{I}^{+_N}$ *is N-protoalgebraic.*

Proof.

1. This part is clear, since $\mathcal{C}$ is a closure system on SEN, such that $\mathcal{T} \subseteq \mathrm{ThSys}(\mathcal{C})$, whence, by the minimality of $\mathcal{C}^{+_N}$, we get that $\mathcal{C}^{+_N} \leq \mathcal{C}$.

2. This is also clear from the definitions involved. The key is that $\mathcal{C}^{+_N}$ is generated by $\mathcal{T}$ and $\mathrm{ThSys}^N(\mathcal{I}) \subseteq \mathcal{T}$, which immediately yields the first inclusion, and, for the same reason, since $\mathcal{T} \subseteq \mathrm{ThSys}(\mathcal{I})$, the second inclusion follows.

3. Suppose that $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ is in $\mathrm{Alg}^N(\mathcal{I})^{is}$ via the isosurjective (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathcal{I} \to^{se} \mathrm{SEN}'$. Then, by an easy modification of Corollary 5.28 of [22], we get that $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ is in $\mathrm{Alg}^N(\mathcal{I})^{*is}$. Thus, there exists a theory system T of the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN', such that $\Omega^{N'}(T) = \Delta^{\mathrm{SEN}'}$. Therefore, since $\langle F, \alpha\rangle$ is isosurjective, we obtain, by Lemma 2.4, that $T^{N'}$ is a theory system of the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}^{+_N}$ on SEN', such that $\Omega^{N'}(T^{N'}) = \Delta^{\mathrm{SEN}'}$. This shows that SEN' is also in $\mathrm{Alg}^N(\mathcal{I}^{+_N})^{*is} \subseteq \mathrm{Alg}^N(\mathcal{I}^{+_N})^{is}$. The reverse inclusion is trivial, since the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}^{+_N}$ on SEN' is weaker than the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN'.

4. This immediately follows from the fact that $C \leq C^{+_N}$. □

In Proposition 2.6, an analog of the last part of Proposition 18 of [10] is formulated. It affirms that the process of passing from a π-institution to its strong version preserves theorem systems.

Proposition 2.6 *Suppose $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, is an N-protoalgebraic π-institution. For every functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, with N' a category of natural transformations on SEN', and $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ an isosurjective (N, N')-epimorphic translation, the theorem system of the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}^{+_N}$ on SEN' is equal to the theorem system of the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN'. In particular, $\mathcal{I}^{+_N}$ has the same theorem system as $\mathcal{I}$.*

Proof. Suppose that T is the theorem system of the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}^{+_N}$ on SEN'. Clearly, T is also a theory system of the $\langle F, \alpha\rangle$-min

(N, N')-model of $\mathcal{I}$ on SEN$'$. Suppose that T' is another theory system of the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN$'$, such that $T' \leq T$. Then we have that $T'^{N'} \leq T$, whence, since $\langle F, \alpha\rangle$ is isosurjective, by Lemma 2.4 and by the minimality of T, we get that $T'^{N'} = T$. This shows that $T \leq T'$ and, therefore, $T' = T$ and T is the theorem system of the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN$'$.

The last statement follows by considering the special case of the isosurjective (N, N)-epimorphic translation $\langle \mathrm{I}_{\mathbf{Sign}}, \iota\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}$. □

As in the sentential context, the second part of Proposition 2.5 motivates the question of when one has $\mathrm{ThSys}(\mathcal{I}^{+_N}) = \mathrm{ThSys}^N(\mathcal{I})$. We prove here an analog of Theorem 19 of [10] that answers this question.

Theorem 2.7 *Suppose $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, is an N-protoalgebraic π-institution. Then the following statements are equivalent:*

1. *For all functors $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ and isosurjective (N, N')-epimorphic translations $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, $\mathrm{ThSys}^{\langle F,\alpha\rangle}_{\mathcal{I}^{+_N}}(\mathrm{SEN}') = \mathrm{ThSys}^{\langle F,\alpha\rangle^{N'}}_{\mathcal{I}}(\mathrm{SEN}')$.*
2. *$\mathrm{ThSys}(\mathcal{I}^{+_N}) = \mathrm{ThSys}^N(\mathcal{I})$.*
3. *$\mathcal{I}^{+_N}$ is N-weakly algebraizable.*

Proof.

$1 \to 2$ Consider in Part 1 the special case of the isosurjective (N, N)-epimorphic translation $\langle \mathrm{I}_{\mathbf{Sign}}, \iota\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}$.

$2 \to 3$ Note that $\mathcal{I}^{+_N}$ is N-protoalgebraic by Part 4 of Proposition 2.5. Thus, it suffices to show that Ω^N is injective on the theory systems of $\mathcal{I}^{+_N}$. This, however, follows immediately by Part 2, together with Theorem 3 of [24].

$3 \to 1$ The inclusion $\mathrm{ThSys}^{\langle F,\alpha\rangle^{N'}}_{\mathcal{I}}(\mathrm{SEN}') \subseteq \mathrm{ThSys}^{\langle F,\alpha\rangle}_{\mathcal{I}^{+_N}}(\mathrm{SEN}')$ always holds, by Lemma 2.4, since $\langle F, \alpha\rangle$ is isosurjective.

To show the reverse inclusion, suppose that $T \in \mathrm{ThSys}^{\langle F,\alpha\rangle}_{\mathcal{I}^{+_N}}(\mathrm{SEN}')$. Then also $T \in \mathrm{ThSys}^{\langle F,\alpha\rangle}_{\mathcal{I}}(\mathrm{SEN}')$. Now, considering $T^{N'}$, we get that $\Omega^{N'}(T) = \Omega^{N'}(T^{N'})$ and, since, again by Lemma 2.4, $T^{N'} \in \mathrm{ThSys}^{\langle F,\alpha\rangle}_{\mathcal{I}^{+_N}}(\mathrm{SEN}')$ as well and $\mathcal{I}^{+_N}$ is N-weakly algebraizable, we obtain that $T = T^{N'}$. Therefore $T \in \mathrm{ThSys}^{\langle F,\alpha\rangle^{N'}}_{\mathcal{I}}(\mathrm{SEN}')$. □

Furthermore, it can be shown that the three equivalent conditions of Theorem 2.7 imply closure of the collection of all N'-Leibniz theory systems under signature-wise intersections.

Proposition 2.8 *Suppose* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, *with* N *a category of natural transformations on* SEN, *is an* N*-protoalgebraic* π*-institution satisfying any of the equivalent statements of Theorem 2.7. Then, the following two equivalent statements hold:*

1. *For all functors* $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, *with* N' *a category of natural transformations on* SEN', *and isosurjective* (N, N')*-epimorphic translations* $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, $\mathrm{ThSys}^{\langle F,\alpha\rangle^{N'}}_{\mathcal{I}}(\mathrm{SEN}')$ *is closed under signature-wise intersections.*

2. *For all functors* $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, *with* N' *a category of natural transformations on* SEN', *and isosurjective* (N, N')*-epimorphic translations* $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, $\mathrm{ThSys}^{\langle F,\alpha\rangle^{N'}}_{\mathcal{I}}(\mathrm{SEN}')$ *forms a complete* $\leq$*-sublattice of the lattice* $\mathbf{ThSys}^{\langle F,\alpha\rangle}_{\mathcal{I}}(\mathrm{SEN}') = \langle \mathrm{ThSys}^{\langle F,\alpha\rangle}_{\mathcal{I}}(\mathrm{SEN}'), \leq\rangle$.

Proof. That the first statement of Theorem 2.7 implies Condition 1 follows from the fact that the collection $\mathrm{ThSys}^{\langle F,\alpha\rangle}_{\mathcal{I}^{+_N}}(\mathrm{SEN}')$ of all theory systems of the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}^{+_N}$ on SEN' is always closed under signature-wise intersections and the fact that, by the first statement of Theorem 2.7, the two collections $\mathrm{ThSys}^{\langle F,\alpha\rangle^{N'}}_{\mathcal{I}}(\mathrm{SEN}')$ and $\mathrm{ThSys}^{\langle F,\alpha\rangle}_{\mathcal{I}^{+_N}}(\mathrm{SEN}')$ are identical. To show the equivalence of Conditions 1 and 2, use Proposition 11 of [24]. □

Corollary 2.9 forms an analog of Proposition 20 of [10] for π-institutions. In particular, its last part provides an alternative characterization of the N'-Leibniz theory system $T^{N'}$ associated with a given theory system T in terms of the strong version $\mathcal{I}^{+_N}$ of the π-institution $\mathcal{I}$, in case $\mathcal{I}^{+_N}$ is N-weakly algebraizable.

Corollary 2.9 *Suppose $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, is an N-protoalgebraic π-institution that satisfies any of the equivalent statements of Theorem 2.7. Then, for all functors $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ and isosurjective (N, N')-epimorphic translations $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$,*

1. *if $\mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$ is closed under unions of directed subfamilies of theory systems, then $\mathrm{ThSys}_{\mathcal{I}^{+N}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$ has the same property;*
2. *for all $T \in \mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$, T is N'-Leibniz if and only if $T \in \mathrm{ThSys}_{\mathcal{I}^{+N}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$;*
3. *for all $T \in \mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$, $T^{N'}$ is the largest theory system in $\mathrm{ThSys}_{\mathcal{I}^{+N}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$ that is $\leq$-included in T.*

Proof.

1. By Proposition 11 of [24], if $\mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$ is closed under unions of directed subfamilies of theory systems, then so is $\mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle^{N'}}(\mathrm{SEN}')$. But, by Part 1 of Theorem 2.7, this last collection is equal to $\mathrm{ThSys}_{\mathcal{I}^{+N}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$.
2. Again by Part 1 of Theorem 2.7, we have $T \in \mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle^{N'}}(\mathrm{SEN}')$ if and only if $T \in \mathrm{ThSys}_{\mathcal{I}^{+N}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$.
3. By Corollary 4 of [24], $T^{N'}$ is the largest theory system in $\mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle^{N'}}(\mathrm{SEN}')$, $\leq$-included in T. But $\mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle^{N'}}(\mathrm{SEN}') = \mathrm{ThSys}_{\mathcal{I}^{+N}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$. □

For the needs of this paper, call a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, N-**equivalential** if there exists a subcollection E of natural transformations $\epsilon : \mathrm{SEN}^2 \to \mathrm{SEN}$ in N, such that E **explicitly defines N-Leibniz congruence systems of theory families of** $\mathcal{I}$, i.e., such that, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi, \psi\rangle \in \Omega_\Sigma^N(T) \quad \text{iff} \quad \begin{array}{l} E_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)) \subseteq T_{\Sigma'}, \\ \text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma'). \end{array} \tag{2}$$

The π-institutions that are called N-equivalential here were introduced for the first time in [27], where they were called syntactically N-equivalential.

Following similar conventions followed in [26], Condition (2) will be abbreviated to

$$\langle\phi,\psi\rangle \in \Omega^N_\Sigma(T) \quad \text{iff} \quad (\forall f)(E_{\Sigma'}(\mathrm{SEN}(f)(\phi),\mathrm{SEN}(f)(\psi)) \subseteq T_{\Sigma'}).$$

It can be seen that if a π-institution is N-equivalential, in this sense, then it is N-protoalgebraic.

Proposition 2.10 *Suppose that $\mathcal{I} = \langle\mathbf{Sign},\mathrm{SEN},C\rangle$, with N a category of natural transformations on* SEN*, is N-equivalential. Then $\mathcal{I}$ is N-protoalgebraic.*

Proof. Suppose that T,T' are theory families of $\mathcal{I}$, such that $T \leq T'$. Then, for all $\Sigma \in |\mathbf{Sign}|, \phi,\psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{rcl}\langle\phi,\psi\rangle \in \Omega^N_\Sigma(T) & \text{iff} & (\forall f)(E_{\Sigma'}(\mathrm{SEN}(f)(\phi),\mathrm{SEN}(f)(\psi)) \subseteq T_{\Sigma'}) \\ & \text{implies} & (\forall f)(E_{\Sigma'}(\mathrm{SEN}(f)(\phi),\mathrm{SEN}(f)(\psi)) \subseteq T'_{\Sigma'}) \\ & \text{iff} & \langle\phi,\psi\rangle \in \Omega^N_\Sigma(T'),\end{array}$$

i.e., Ω^N is monotone on theory families and, therefore, $\mathcal{I}$ is N-protoalgebraic, by Lemma 3.8 of [22]. □

Furthermore, a π-institution $\mathcal{I} = \langle\mathbf{Sign},\mathrm{SEN},C\rangle$, with N a category of natural transformations on SEN, will be said to be N-**algebraizable** if and only if $\mathcal{I}$ is N-equivalential and N-weakly algebraizable.

These two classes of π-institutions will be introduced and studied in more detail elsewhere.

Proposition 2.11 *Suppose $\mathcal{I} = \langle\mathbf{Sign},\mathrm{SEN},C\rangle$, with N a category of natural transformations on* SEN*, is an N-equivalential π-institution. Then $\mathcal{I}$ satisfies any of the equivalent statements of Theorem 2.7 if and only if $\mathcal{I}^{+_N}$ is N-algebraizable.*

Proof. Since, by Proposition 2.10, N-equivalential π-institutions are N-protoalgebraic, it suffices, by Part 1 of Proposition 2.5, to put together the definition of N-algebraizability and Part 3 of Theorem 2.7. □

3. Definability of N-Leibniz Theory Families

Definition 3.1 *A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, has* **explicitly definable N-Leibniz theory systems** *if there exists a collection X of natural transformations $\sigma : \mathrm{SEN} \to \mathrm{SEN}$ in N, such that, for all functors $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, all surjective (N, N')-epimorphic translations $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, all $T \in \mathrm{ThFam}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$ and all $\Sigma \in |\mathbf{Sign}'|$,*

$$\begin{aligned} T_\Sigma^{N'} \quad = \quad & \{\phi \in \mathrm{SEN}'(\Sigma) : X'_{\Sigma'}(\mathrm{SEN}'(f)(\phi)) \subseteq T_{\Sigma'}, \\ & \textit{for all } \Sigma' \in |\mathbf{Sign}'|, f \in \mathbf{Sign}'(\Sigma, \Sigma')\}, \end{aligned} \tag{3}$$

where by X' is denoted the collection of natural transformations on SEN' *corresponding to X via the (N, N')-epimorphic property and by $X'_\Sigma(\phi)$ is denoted the collection $X'_\Sigma(\phi) = \{\sigma'_\Sigma(\phi) : \sigma' \in X'\}$.*

Once more, using the same notational convention adopted previously, Equation (3) may be rewritten more compactly in the form

$$T_\Sigma^{N'} = \{\phi \in \mathrm{SEN}'(\Sigma) : (\forall f)(X'_{\Sigma'}(\mathrm{SEN}'(f)(\phi)) \subseteq T_{\Sigma'})\}.$$

Definition 3.2 *A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, has* **logically definable N-Leibniz theory systems** *if there exists a collection X of natural transformations $\sigma : \mathrm{SEN} \to \mathrm{SEN}$ in N, such that, for all functors $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, all surjective (N, N')-epimorphic translations $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, and all $T \in \mathrm{ThFam}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$, T is N'-Leibniz if and only if*

$$(\forall \Sigma \in |\mathbf{Sign}'|)(\forall \phi \in \mathrm{SEN}'(\Sigma))(\phi \in T_\Sigma \Rightarrow (\forall f)(X'_{\Sigma'}(\mathrm{SEN}'(f)(\phi)) \subseteq T_{\Sigma'})). \tag{4}$$

As was the case with Condition (3), the quantification $(\forall f)$ abbreviates the quantification $(\forall \Sigma' \in |\mathbf{Sign}'|)(\forall f \in \mathbf{Sign}'(\Sigma, \Sigma'))$ to save space. Hopefully no confusion will arise, since it is usually clear from context to which category $\mathbf{Sign}'$ it refers.

In view of Theorem 9 of [25], we may obtain the following proposition and corollary providing characterizations of explicit and logical definability, respectively, in terms of the largest theory system $\overleftarrow{T}$ contained in a given theory family T in the context of N-protoalgebraicity. See [25] for the original definition of $\overleftarrow{T}$ and its significance in categorical abstract algebraic logic. We recall the definition here for the reader's convenience. Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution and $T = \{T_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ a theory family of $\mathcal{I}$. Then $\overleftarrow{T} = \{\overleftarrow{T}_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ is defined, for all $\Sigma \in |\mathbf{Sign}|$, by

$$\overleftarrow{T}_\Sigma = \bigcap\{\mathrm{SEN}(f)^{-1}(T_{\Sigma'}) : \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')\}.$$

It is shown in Proposition 2.1 of [25] that $\overleftarrow{T}$ is a theory system of $\mathcal{I}$, for all $T \in \mathrm{ThFam}(\mathcal{I})$, and in Proposition 2.2 of [25] that it is, in fact, the largest theory system of $\mathcal{I}$ that is $\leq$-included in the theory family T.

Proposition 3.3 *An N-protoalgebraic π-institution*

$$\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle,$$

with N a category of natural transformations on SEN*, has explicitly definable N-Leibniz theory systems if and only if there exists a collection X of natural transformations $\sigma : \mathrm{SEN} \to \mathrm{SEN}$ in N, such that, for all functors $\mathrm{SEN}' : \mathbf{Sign'} \to \mathbf{Set}$, all surjective (N, N')-epimorphic translations $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ and all $T \in \mathrm{ThFam}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$,*

$$T_\Sigma^{N'} = \{\phi \in \mathrm{SEN}'(\Sigma) : X'_\Sigma(\phi) \subseteq \overleftarrow{T}_\Sigma\}, \quad \textit{for all } \Sigma \in |\mathbf{Sign'}|. \tag{5}$$

Proof. Suppose, first, that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ has explicitly definable N-Leibniz theory systems via a collection X of natural transformations $\sigma : \mathrm{SEN} \to \mathrm{SEN}$ in N. Then, for all functors $\mathrm{SEN}' : \mathbf{Sign'} \to \mathbf{Set}$, all surjective (N, N')-epimorphic translations $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, all $T \in \mathrm{ThFam}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$ and all $\Sigma \in |\mathbf{Sign'}|, \phi \in \mathrm{SEN}'(\Sigma)$,

$$\begin{array}{rll} \phi \in T_\Sigma^{N'} & \text{iff} & (\forall f)(X'_{\Sigma'}(\mathrm{SEN}'(f)(\phi)) \subseteq T_{\Sigma'}) \\ & \text{iff} & (\forall f)(\mathrm{SEN}'(f)(X'_\Sigma(\phi)) \subseteq T_{\Sigma'}) \\ & \text{iff} & (\forall f)(X'_\Sigma(\phi) \subseteq \mathrm{SEN}'(f)^{-1}(T_{\Sigma'})) \\ & \text{iff} & X'_\Sigma(\phi) \subseteq \bigcap_f \mathrm{SEN}'(f)^{-1}(T_{\Sigma'}) \\ & \text{iff} & X'_\Sigma(\phi) \subseteq \overleftarrow{T}_\Sigma. \end{array}$$

Suppose, conversely, that there exists a collection X of natural transformations $\sigma : \mathrm{SEN} \to \mathrm{SEN}$ in N, such that, for all functors $\mathrm{SEN}' : \mathbf{Sign'} \to \mathbf{Set}$, all surjective (N, N')-epimorphic translations $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ and all $T \in \mathrm{ThFam}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$, Equation (5) holds. Then, for all functors $\mathrm{SEN}' : \mathbf{Sign'} \to \mathbf{Set}$, all surjective (N, N')-epimorphic translations

$\langle F, \alpha\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}'$, all $T \in \mathrm{ThFam}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$, all $\Sigma \in |\mathbf{Sign}'|$ and all $\phi \in \mathrm{SEN}'(\Sigma)$, follow the chain of equivalences

$$\begin{array}{rll} \phi \in T_{\Sigma}^{N'} & \text{iff} & X'_{\Sigma}(\phi) \subseteq \overleftarrow{T}_{\Sigma} \\ & \text{iff} & X'_{\Sigma}(\phi) \subseteq \bigcap_f \mathrm{SEN}'(f)^{-1}(T_{\Sigma'}) \\ & \text{iff} & (\forall f)(X'_{\Sigma}(\phi) \subseteq \mathrm{SEN}'(f)^{-1}(T_{\Sigma'})) \\ & \text{iff} & (\forall f)(\mathrm{SEN}'(f)(X'_{\Sigma}(\phi)) \subseteq T_{\Sigma'}) \\ & \text{iff} & (\forall f)(X'_{\Sigma'}(\mathrm{SEN}'(f)(\phi)) \subseteq T_{\Sigma'}). \end{array}$$

□

For logical definability, we have the following corollary of Proposition 3.3.

Corollary 3.4 *An N-protoalgebraic π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, has logically definable N-Leibniz theory systems if and only if there exists a collection X of natural transformations $\sigma : \mathrm{SEN} \rightarrow \mathrm{SEN}$ in N, such that, for all functors $\mathrm{SEN}' : \mathbf{Sign}' \rightarrow \mathbf{Set}$, all surjective (N, N')-epimorphic translations $\langle F, \alpha\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}'$ and all $T \in \mathrm{ThFam}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$, $T \in \mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle^{N'}}(\mathrm{SEN}')$ if and only if*

$$(\forall \Sigma \in |\mathbf{Sign}'|)(\forall \phi \in \mathrm{SEN}'(\Sigma))(\phi \in T_{\Sigma} \Longrightarrow X'_{\Sigma}(\phi) \subseteq \overleftarrow{T}_{\Sigma}). \qquad (6)$$

Proof. Use the relevant definitions together with Proposition 3.3. □

In the next proposition, an analog of Proposition 29 of [10] for π-institutions, it is shown that explicit definability implies logical definability and, furthermore, explicit definability implies closure of Leibniz theory systems under signature-wise intersections.

Proposition 3.5 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, is an N-protoalgebraic π-institution that has its N-Leibniz theory systems explicitly definable by a subcollection X of natural transformations in N. Then*

1. *$\mathcal{I}$ has its N-Leibniz theory systems logically definable by X and*

2. *the two equivalent conditions of Proposition 2.8 hold.*

Proof.

1. Suppose that $\mathcal{I}$ has its N-Leibniz theory systems explicitly definable by a subcollection X of natural transformations in N. Consider $T \in \mathrm{ThFam}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$. Then

$$T_\Sigma^{N'} = \{\phi \in \mathrm{SEN}'(\Sigma) : (\forall f)(X'_{\Sigma'}(\mathrm{SEN}'(f)(\phi)) \subseteq T_{\Sigma'})\},$$

for all $\Sigma \in |\mathbf{Sign}'|$. Thus, taking into account the fact that $T^{N'} \leq T$ always holds, we have that T is N'-Leibniz if and only if $T \leq T^{N'}$ if and only if, for all $\Sigma \in |\mathbf{Sign}'|, \phi \in \mathrm{SEN}'(\Sigma)$, $\phi \in T_\Sigma$ implies $(\forall f)(X'_{\Sigma'}(\mathrm{SEN}'(f)(\phi)) \subseteq T_{\Sigma'})$. This is Condition (4). Thus $\mathcal{I}$ has its N-Leibniz theory systems logically definable by X.

2. Suppose that $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ is a functor, with N' a category of natural transformations on SEN', and $\langle F,\alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ an isosurjective (N,N')-epimorphic translation. Then, for all $T^i \in \mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}'), i \in I$, and all $\Sigma \in |\mathbf{Sign}'|$, we have that

$$\overleftarrow{\bigcap_{i\in I} T^i_\Sigma} = \bigcap_{i\in I} \overleftarrow{T^i_\Sigma}.$$

Indeed,

$$\begin{aligned}
\overleftarrow{\bigcap_{i\in I} T^i_\Sigma} &= \textstyle\bigcap_{f\in\mathbf{Sign}'(\Sigma,\Sigma')} \mathrm{SEN}'(f)^{-1}(\bigcap_{i\in I} T^i_{\Sigma'}) \\
&= \textstyle\bigcap_{f\in\mathbf{Sign}'(\Sigma,\Sigma')} \bigcap_{i\in I} \mathrm{SEN}'(f)^{-1}(T^i_{\Sigma'}) \\
&= \textstyle\bigcap_{i\in I} \bigcap_{f\in\mathbf{Sign}'(\Sigma,\Sigma')} \mathrm{SEN}'(f)^{-1}(T^i_{\Sigma'}) \\
&= \textstyle\bigcap_{i\in I} \overleftarrow{T^i_\Sigma}.
\end{aligned}$$

Now suppose that $T^i \in \mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle^{N'}}(\mathrm{SEN}'), i \in I$. Then, by Part 1, since N-Leibniz theory systems are logically definable by X, we obtain, by Corollary 3.4, that, for all $\Sigma \in |\mathbf{Sign}'|$ and all $\phi \in \mathrm{SEN}'(\Sigma)$, $\phi \in T^i_\Sigma$ implies $X'_\Sigma(\phi) \subseteq \overleftarrow{T^i_\Sigma}$. This yields that, for all $\Sigma \in |\mathbf{Sign}'|$ and all $\phi \in \mathrm{SEN}'(\Sigma)$, $\phi \in \bigcap_{i\in I} T^i_\Sigma$ implies $X'_\Sigma(\phi) \subseteq \bigcap_{i\in I} \overleftarrow{T^i_\Sigma}$, which, by what was shown above, implies that $X'_\Sigma(\phi) \subseteq \overleftarrow{\bigcap_{i\in I} T^i_\Sigma}$. Thus, again by the hypothesis, Part 1, and Corollary 3.4, we get that $\bigcap_{i\in I} T^i \in \mathrm{ThSys}_{\mathcal{I}}^{\langle F,\alpha\rangle^{N'}}(\mathrm{SEN}')$. □

The following proposition shows that, under the assumptions of N-protoalgebraicity and explicit definability of N-Leibniz theory systems, N-equivalentiality of $\mathcal{I}^{+_N}$ implies N-equivalentiality of $\mathcal{I}$. This is an analog of Proposition 30 of [10].

Proposition 3.6 *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, *with* N *a category of natural transformations on* SEN, *is an* N*-protoalgebraic* π*-institution that has its* N*-Leibniz theory systems explicitly definable by a subcollection* X *of natural transformations in* N*. If* $\mathcal{I}^{+_N}$ *is* N*-equivalential, then* $\mathcal{I}$ *is also* N*-equivalential.*

Proof. Suppose that X explicitly defines N-Leibniz theory systems of $\mathcal{I}$ and that E is a collection of natural transformations on SEN, that defines N-Leibniz congruence systems of theory families of $\mathcal{I}^{+_N}$. Suppose that $T \in \mathrm{ThFam}(\mathcal{I}), \Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$. Then

$$\begin{aligned}
\langle \phi,\psi\rangle &\in \Omega^N_\Sigma(T)\\
&\text{iff}\quad \langle \phi,\psi\rangle \in \Omega^N_\Sigma(T^N) \quad (\text{since } \Omega^N(T) = \Omega^N(T^N))\\
&\text{iff}\quad (\forall f)(E_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)) \subseteq T^N_{\Sigma'})\\
&\qquad (\text{by hypothesis, since } T^N \in \mathrm{ThSys}(\mathcal{I}^{+_N}))\\
&\text{iff}\quad (\forall f)(\forall g)(X_{\Sigma''}(\mathrm{SEN}(g)(E_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)))) \subseteq T_{\Sigma''})\\
&\qquad (\text{by hypothesis, since } X \text{ defines } N\text{-Leibniz theory systems})\\
&\text{iff}\quad (\forall f)(\forall g)(X_{\Sigma''}(E_{\Sigma''}(\mathrm{SEN}(gf)(\phi), \mathrm{SEN}(gf)(\psi))) \subseteq T_{\Sigma''})
\end{aligned}$$

$$\begin{tikzcd}
\mathrm{SEN}(\Sigma')^2 \arrow[r, "E_{\Sigma'}"] \arrow[d, "\mathrm{SEN}(g)^2"'] & \mathrm{SEN}(\Sigma') \arrow[d, "\mathrm{SEN}(g)"] \\
\mathrm{SEN}(\Sigma'')^2 \arrow[r, "E_{\Sigma''}"'] & \mathrm{SEN}(\Sigma'')
\end{tikzcd}$$

$$\text{iff}\quad (\forall f)(X_{\Sigma'}(E_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi))) \subseteq T_{\Sigma'}).$$

Therefore $\mathcal{I}$ is also N-equivalential, with $X \circ E$ being a set of natural transformations in N that define N-Leibniz congruences of theory families of $\mathcal{I}$. □

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution. Given $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, we use the following abbreviation

$$\Phi \models^{\mathrm{ThSys}^N(\mathcal{I})}_\Sigma \phi \quad \text{iff} \quad (\forall T \in \mathrm{ThSys}^N(\mathcal{I}))(\Phi \subseteq T_\Sigma \Longrightarrow \phi \in T_\Sigma).$$

The reader is cautioned that $\models^{\mathrm{ThSys}^N(\mathcal{I})}$ is not necessarily structural, i.e., it does not necessarily hold, given $\Sigma, \Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma, \Sigma')$, that

$$\Phi \models_{\Sigma}^{\mathrm{ThSys}^N(\mathcal{I})} \phi \quad \text{implies} \quad \mathrm{SEN}(f)(\Phi) \models_{\Sigma'}^{\mathrm{ThSys}^N(\mathcal{I})} \mathrm{SEN}(f)(\phi).$$

In the next lemma, a few properties that were given in the sentential context in Lemma 31 of [10] are now adapted to the present context. As pointed out by Czelakowski and Jansana in [10], these properties abstract some of the properties of modal operators in modal logic.

Lemma 3.7 *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, *with* N *a category of natural transformations on* SEN, *is an* N*-protoalgebraic* π*-institution that has its* N*-Leibniz theory systems explicitly definable by a subcollection* X *of natural transformations in* N*. Then the following conditions hold, for all* $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

1. *If* $\phi \in C_\Sigma(\Phi)$*, then, for every* $T \in \mathrm{ThFam}(\mathcal{I})$,
$$(\forall f)(\mathrm{SEN}(f)(X_\Sigma(\Phi)) \subseteq T_{\Sigma'}) \Longrightarrow (\forall f)(\mathrm{SEN}(f)(X_\Sigma(\phi)) \subseteq T_{\Sigma'}).$$
2. *For all* $T \in \mathrm{ThFam}(\mathcal{I})$, $(\forall f)(\mathrm{SEN}(f)(X_\Sigma(\phi)) \subseteq T_{\Sigma'}) \Longrightarrow \phi \in T_\Sigma$.
3. $\phi \models_{\Sigma}^{\mathrm{ThSys}^N(\mathcal{I})} X_\Sigma(\phi)$.
4. *For all* $T \in \mathrm{ThFam}(\mathcal{I})$,
$$(\forall f)(\mathrm{SEN}(f)(X_\Sigma(\phi)) \subseteq T_{\Sigma'}) \Longrightarrow (\forall f)(\mathrm{SEN}(f)(X_\Sigma(X_\Sigma(\phi))) \subseteq T_{\Sigma'}).$$

Proof.

1. Suppose that $\phi \in C_\Sigma(\Phi)$ and let $T \in \mathrm{ThFam}(\mathcal{I})$, such that $(\forall f)(\mathrm{SEN}(f)(X_\Sigma(\Phi)) \subseteq T_{\Sigma'})$. Then, by explicit definability, $\Phi \subseteq T_\Sigma^N$, whence, since $\phi \in C_\Sigma(\Phi)$, $\phi \in T_\Sigma^N$, which yields, by explicit definability, $(\forall f)(\mathrm{SEN}(f)(X_\Sigma(\phi)) \subseteq T_{\Sigma'})$.
2. Let $T \in \mathrm{ThFam}(\mathcal{I})$, such that $(\forall f)(\mathrm{SEN}(f)(X_\Sigma(\phi)) \subseteq T_{\Sigma'})$. Then, by explicit definability, $\phi \in T_\Sigma^N \subseteq T_\Sigma$.
3. Given $T \in \mathrm{ThSys}^N(\mathcal{I})$, we have $\phi \in T_\Sigma$ implies $\phi \in T_\Sigma^N$, since $T = T^N$, for all $T \in \mathrm{ThSys}^N(\mathcal{I})$, whence, by explicit definability, $(\forall f)(\mathrm{SEN}(f)(X_\Sigma(\phi)) \subseteq T_{\Sigma'})$. Therefore, by specializing to $f = i_\Sigma$, $X_\Sigma(\phi) \subseteq T_\Sigma$ and $\phi \models_{\Sigma}^{\mathrm{ThSys}^N(\mathcal{I})} X_\Sigma(\phi)$.

4. Suppose that $T \in \mathrm{ThFam}(\mathcal{I})$, such that $(\forall f)(\mathrm{SEN}(f)(X_\Sigma(\phi)) \subseteq T_{\Sigma'})$. Then, by explicit definability, $\phi \in T^N_\Sigma$, whence, by Part 3, $X_\Sigma(\phi) \subseteq T^N_\Sigma$ and, therefore, again by explicit definability, $(\forall f)(\mathrm{SEN}(f)(X_\Sigma(X_\Sigma(\phi))) \subseteq T_{\Sigma'})$. □

An analog of Theorem 32 of [10] is undertaken next. It provides a characterization of explicit definability in terms of the closure of the collection of all N-Leibniz theory systems under signature-wise intersections and of two simple properties of the defining family of the natural transformations in N.

Theorem 3.8 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN, *is an N-protoalgebraic π-institution. $\mathcal{I}$ has its N-Leibniz theory systems explicitly definable by a subcollection X of natural transformations in N if and only if*

- *any of the two equivalent conditions of Proposition 2.8 hold,*
- *for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\{\phi \in \mathrm{SEN}(\Sigma) : (\forall f)(\mathrm{SEN}(f)(X_\Sigma(\phi)) \subseteq T_{\Sigma'})\}$ is a Σ-theory, for all $\Sigma \in |\mathbf{Sign}|$, and*
- *$\mathcal{I}$ satisfies, for all $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,*

$$\begin{aligned}\Phi \models^{\mathrm{ThSys}^N(\mathcal{I})}_\Sigma \phi \quad & \textit{iff} \\ & (\forall T \in \mathrm{ThFam}(\mathcal{I}))((\forall f)(\mathrm{SEN}(f)(X_\Sigma(\Phi)) \subseteq T_{\Sigma'}) \Longrightarrow \\ & \qquad (\forall f)(\mathrm{SEN}(f)(\phi) \in T_{\Sigma'})).\end{aligned} \tag{7}$$

Proof. Suppose, first, that $\mathcal{I}$ has its N-Leibniz theory systems explicitly definable by a subcollection X of natural transformations in N. Then, the two equivalent conditions of Proposition 2.8 hold, by Part 2 of Proposition 3.5, and the second condition of the statement is satisfied, since, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}|$, $\{\phi \in \mathrm{SEN}(\Sigma) : (\forall f)(\mathrm{SEN}(f)(X_\Sigma(\phi)) \subseteq T_{\Sigma'})\} = T^N_\Sigma$, by explicit definability. Therefore, it suffices to show that the Equivalence (7) holds.

For the left-to-right implication, assume that $\Phi \models^{\mathrm{ThSys}^N(\mathcal{I})}_\Sigma \phi$ and that $T \in \mathrm{ThFam}(\mathcal{I})$, such that $(\forall f)(\mathrm{SEN}(f)(X_\Sigma(\Phi)) \subseteq T_{\Sigma'})$. Then, by explicit definability, $\Phi \subseteq T^N_\Sigma$. Hence, by hypothesis, since $T^N \in \mathrm{ThSys}^N(\mathcal{I})$, $\phi \in T^N_\Sigma$, which yields that, for all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\mathrm{SEN}(f)(\phi) \in T^N_{\Sigma'} \subseteq T_{\Sigma'}$, which verifies the left-to-right implication in Condition (7).

For the right-to-left implication, suppose that

$$(\forall T \in \mathrm{ThFam}(\mathcal{I}))((\forall f)(\mathrm{SEN}(f)(X_\Sigma(\Phi)) \subseteq T_{\Sigma'}) \Longrightarrow$$

$$(\forall f)(\mathrm{SEN}(f)(\phi) \in T_{\Sigma'}))$$

and consider $T \in \mathrm{ThSys}^N(\mathcal{I})$, such that $\Phi \subseteq T_\Sigma$. Since T is N-Leibniz, we have $T = T^N$, whence $\Phi \subseteq T^N_\Sigma$. Thus, by explicit definability,

$$(\forall f)(\mathrm{SEN}(f)(X_\Sigma(\Phi)) \subseteq T_{\Sigma'}).$$

Hence, by the hypothesis, $(\forall f)(\mathrm{SEN}(f)(\phi) \in T_{\Sigma'})$, implying $\phi \in T_\Sigma$, which yields that $\Phi \models^{\mathrm{ThSys}^N(\mathcal{I})}_\Sigma \phi$.

Suppose, conversely, that the equivalent conditions of Proposition 2.8 hold, that for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\{\phi \in \mathrm{SEN}(\Sigma) : (\forall f)(\mathrm{SEN}(f)(X_\Sigma(\phi)) \subseteq T_{\Sigma'})\}$ is a Σ-theory, for all $\Sigma \in |\mathbf{Sign}|$, and that Equivalence (7) holds. It follows from Equivalence (7) that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\phi \models^{\mathrm{ThSys}^N(\mathcal{I})}_\Sigma X_\Sigma(\phi). \tag{8}$$

Now, given $T \in \mathrm{ThFam}(\mathcal{I})$, it suffices to show, by Corollary 4 of [24], that the collection $T^\circ = \{T^\circ_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, with

$$T^\circ_\Sigma = \{\phi \in \mathrm{SEN}(\Sigma) : (\forall f)(\mathrm{SEN}(f)(X_\Sigma(\phi)) \subseteq T_{\Sigma'})\},$$

for all $\Sigma \in |\mathbf{Sign}|$, is the largest N-Leibniz theory system $\leq$-contained in T. This part of the proof is broken into five distinct steps. Namely, it is shown, in order, that

- T°_Σ is a Σ-theory, for all $\Sigma \in |\mathbf{Sign}|$,
- T° is a theory system,
- T° is an N-Leibniz theory system,
- $T^\circ \leq T$ and, finally,
- if $T' \in \mathrm{ThSys}^N(\mathcal{I})$, such that $T' \leq T$, then $T' \leq T^\circ$.

The first part is exactly the second condition in the hypothesis.

For the second part, suppose that $\Sigma, \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\phi \in T^\circ_\Sigma$. Then, by definition $(\forall g)(\mathrm{SEN}(g)(X_\Sigma(\phi)) \subseteq T_{\Sigma''})$.

$$\begin{tikzcd} \Sigma \arrow[rr, "f"] \arrow[dr, "g"'] & & \Sigma' \arrow[dl, "k"] \\ & \Sigma'' & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{SEN}(\Sigma) \arrow[r, "X_\Sigma"] \arrow[d, "\mathrm{SEN}(f)"'] & \mathrm{SEN}(\Sigma) \arrow[d, "\mathrm{SEN}(f)"] \\ \mathrm{SEN}(\Sigma') \arrow[r, "X_{\Sigma'}"'] & \mathrm{SEN}(\Sigma') \end{tikzcd}$$

This implies $(\forall k)(\mathrm{SEN}(kf)(X_\Sigma(\phi)) \subseteq T_{\Sigma''})$. Therefore

$$(\forall k)(\mathrm{SEN}(k)(X_{\Sigma'}(\mathrm{SEN}(f)(\phi))) \subseteq T_{\Sigma''}),$$

which, by definition, yields $\mathrm{SEN}(f)(\phi) \in T^\circ_{\Sigma'}$. So T° is a theory system.

For the third part, suppose to the contrary that T° is not an N-Leibniz theory system. Then, by the hypothesis and Proposition 2.8, there exists $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\Phi \models^{\mathrm{ThSys}^N(\mathcal{I})}_\Sigma \phi, \Phi \subseteq T^\circ_\Sigma$ and $\phi \notin T^\circ_\Sigma$. Therefore, we get, by (8), $\Phi \models^{\mathrm{ThSys}^N(\mathcal{I})}_\Sigma X_\Sigma(\phi), \Phi \subseteq T^\circ_\Sigma, \phi \notin T^\circ_\Sigma$. Hence, by the hypothesis, we have

$$(\forall T \in \mathrm{ThFam}(\mathcal{I}))((\forall f)(\mathrm{SEN}(f)(X_\Sigma(\Phi)) \subseteq T_{\Sigma'}) \Longrightarrow$$

$$(\forall f)(\mathrm{SEN}(f)(X_\Sigma(\phi)) \subseteq T_{\Sigma'})),$$

$(\forall f)(\mathrm{SEN}(f)(X_\Sigma(\Phi)) \subseteq T_{\Sigma'})$ and $\neg(\forall f)(\mathrm{SEN}(f)(X_\Sigma(\phi)) \subseteq T_{\Sigma'})$, which is a contradiction. Therefore, T° is in fact an N-Leibniz theory system.

For the fourth part, suppose that $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in T^\circ_\Sigma$. Then $(\forall f)(\mathrm{SEN}(f)(X_\Sigma(\phi)) \subseteq T_{\Sigma'})$. Now, by the Equivalence (7), since $\phi \models^{\mathrm{ThSys}^N(\mathcal{I})}_\Sigma \phi$, we get that $(\forall f)(\mathrm{SEN}(f)(\phi) \in T_{\Sigma'})$. But this implies that $\phi \in T_\Sigma$, whence $T^\circ_\Sigma \subseteq T_\Sigma$ and, Σ being arbitrary, we obtain $T^\circ \leq T$.

Finally, suppose that $T' \in \mathrm{ThSys}^N(\mathcal{I})$, such that $T' \leq T$, and $\Sigma \in |\mathbf{Sign}|, \phi \in T'_\Sigma$. Then, by Relation (8), $X_\Sigma(\phi) \in T'_\Sigma$, whence, for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}(f)(X_\Sigma(\phi)) \subseteq T'_{\Sigma'} \subseteq T_{\Sigma'}.$$

But this shows that $\phi \in T^\circ_\Sigma$, i.e., $T'_\Sigma \subseteq T^\circ_\Sigma$ and, Σ being arbitrary, $T' \leq T^\circ$, as was to be shown. $\square$

Font and Jansana showed in Theorem 33 of [10] that, if a sentential logic has its Leibniz filters logically definable by a collection Y of formulas satisfying the analog in that setting of Condition 1 of Lemma 3.7, then it has its filters explicitly definable by a collection X of formulas that contains all iterates of the formulas in Y. Unlike the situation in the sentential logic framework, it was not possible to show that logical definability in the context of N-protoalgebraic π-institutions implies explicit definability. It is, moreover, conjectured that, in the framework of the present work, explicit definability is a properly stronger property than logical definability. No example to verify this is available at the present time.

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